Evaluation of separation routines for some classes of inequalities of the Clique Partitioning polytope

Colloquium “Großer Beleg”

Maximilian Moeller

21.11.2023

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Agenda

1. Introduction to Clique Partitioning
2. Separating Inequalities
3. Data
4. Empirical Results
5. Conclusion

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Introduction

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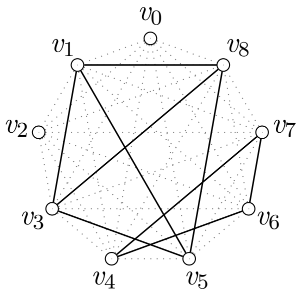
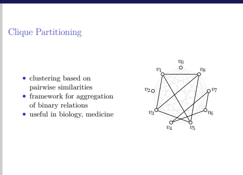
└─Introduction

Introduction

Clique Partitioning

- clustering based on pairwise similarities
- framework for aggregation of binary relations
- useful in biology, medicine





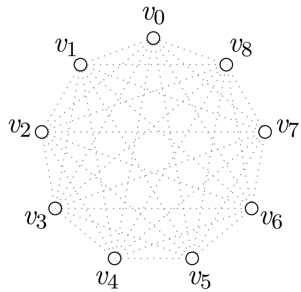
1. biology: taxonomy of animals (*e.g.*, whales and dolphins)
2. medicine: clustering of organoids in light microscopy images
3. probably also operations research

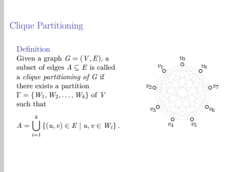
Clique Partitioning

Definition

Given a graph $G = (V, E)$, a subset of edges $A \subseteq E$ is called a *clique partitioning of* G if there exists a partition $\Gamma = \{W_1, W_2, \dots, W_k\}$ of V such that

$$A = \bigcup_{i=1}^{k} \{(u, v) \in E \mid u, v \in W_i\}.$$





1. show which partition is induced
2. typically only complete graphs considered (indicates for every pair of nodes)
3. then one-to-one corresponds to an equivalence relation
4. induces complete subgraphs *clique* partitioning

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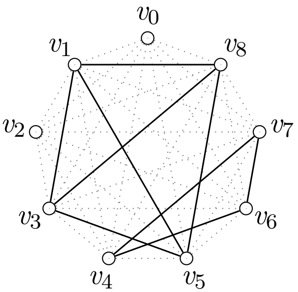

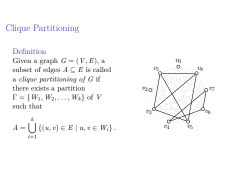

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The Clique Partitioning Problem

Input: complete graph $K_n = (V_n, E_n)$ and edge costs $w \in \mathbb{R}^{E_n}$

Compute: a clique partitioning of minimum weight

$$\begin{aligned} \min \quad & \sum_{e \in E_n} w_e x_e \\ \text{s.t.} \quad & \forall e \in E_n \colon x_e \in \{0,1\} \\ & x \text{ characterizes a clique partitioning} \end{aligned}$$

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1. characterize clique partitionings by binary vectors
2. 1 means it is in the clique partitioning and 0 means it is not in it
3. exactly opposite to multicut (equivalent problem)

The Clique Partitioning Problem

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2. interesting instances need to have ‘conflicts’

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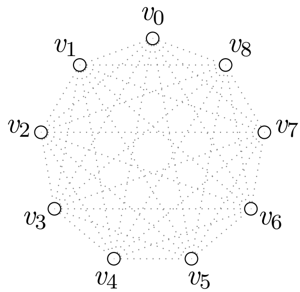

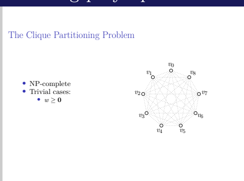

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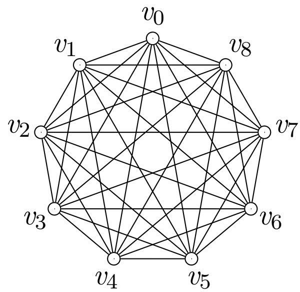

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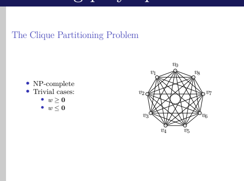

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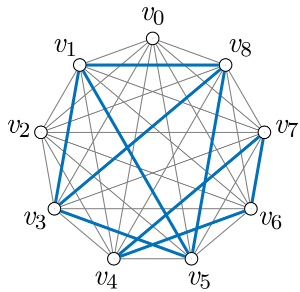

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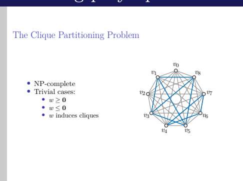

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An ILP-formulation of Clique Partitioning

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2. explain support graph: left hand side can be drawn as graph, dotted negative, solid positive
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4. for the same i, j, k there are actually three triangles (show)

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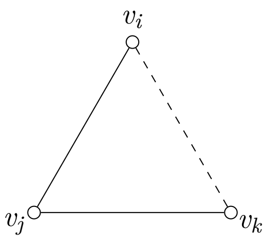
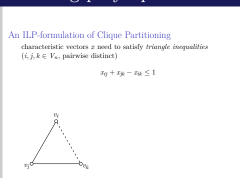
$$x_{ij} + x_{jk} - x_{ik} \leq 1$$





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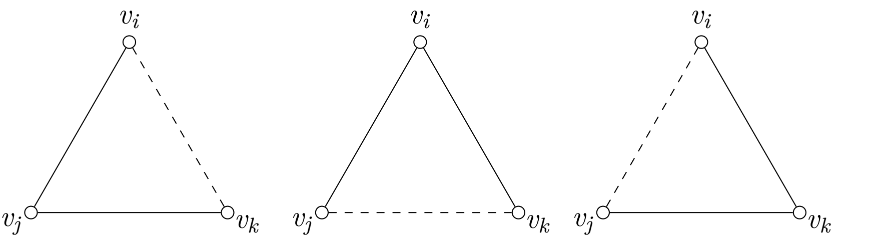

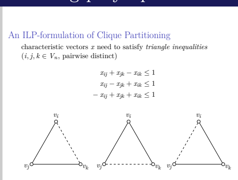
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1. original task: branch-and-cut algorithm, *i.e.*, solving the ILP
2. consists of multiple parts: **cutting planes for bounds**, finding good solutions and branch management
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abbreviation	`maxcut`	`var_once`
Δ	400	–
Δ_∞	∞	–
$\Delta^{\leq 1}$	400	✓
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2-Partition Inequalities (st)

Let $S, T \subseteq V_n$ disjoint.

2-*partition inequality* ([S, T]-inequality)

$$\sum_{s \in S} \sum_{t \in T} x_{st} - \sum_{\substack{s,s' \in S \\ s \neq s'}} x_{ss'} - \sum_{\substack{t,t' \in T \\ t \neq t'}} x_{tt'} \leq \min(|S|, |T|)$$

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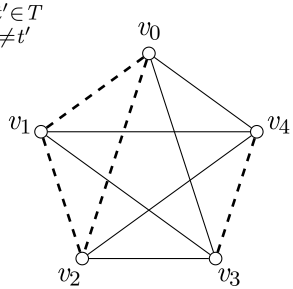

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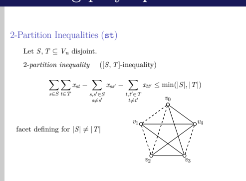
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- `cetacea` (36)

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- Normal `r_normal_[size]_[mu]_[sigma]`
  $\texttt{size} \in \{25, 50\}$
  $(\texttt{mu}, \texttt{sigma}) \in \{0, 0.5, 2\} \times \{0.5, 1, 2\}$

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Empirical Results

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