SEMESTER III
MODULE 2

FOURIER SERIES

Fourier series

Definition

A series of sines and cosines of an angle and its multiples of the form.

$$\frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \ldots + a_n \cos nx + \ldots$$

$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots + b_n \sin nx + \ldots$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

is called the *Fourier series, where* $a_0, a_1, a_2, \ldots a_n, \ldots b_1, b_2, b_3 \ldots b_n \ldots$ are constants.

Dirichlet's Conditions:

Consider a function $f(x)$ in the interval (a,a+2l) satisfying the following conditions.

1. $f(x)$ is single valued in (a,a+2l)
2. $f(x)$ is continuous or has discontinuities finite in number in the interval (a,a+2l)
3. $f(x)$ has no maxima or minima or has maxima and minima finite in number in the interval (a,a+2l)
4. $f(x)$ is periodic with period 2l in the interval (a,a+2l)

These conditions are called Dirichlet's Conditions.

Then $f(x)$ in the interval (a,a+2l) can be written in terms of an infinite series as follows.

$$f(x) = a_0 + \sum_{n=1}^{\infty}[a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}]$$

Where $a_0 = \frac{1}{2l}\int_a^{a+2l} f(x)dx$

$$a_n = \frac{1}{l}\int_a^{a+2l} f(x)\cos\frac{n\pi x}{l}dx$$

$$b_n = \frac{1}{l}\int_a^{a+2l} f(x)\sin\frac{n\pi x}{l}dx$$

Above formulae are called Euler's formulae.

Note that 1) $\sin^{-1} x$ can not be expressed as a Fourier series as it is not a single valued function.

2) $\tan x$ can not be expressed as a Fourier series in $(0,2\pi)$ as it is not defined at $x = \frac{\pi}{2}$ and $3\frac{\pi}{2}$ which are the points of interval $(0,2\pi)$.

Advantages of Fourier series

1. Discontinuous function can be represented by Fourier series. Although derivatives of the discontinuous functions do not exist. (This is not true for Taylor's series).

2. The Fourier series is useful in expanding the periodic functions since outside the closed interval, there exists a periodic extension of the function.

3. Expansion of an oscillating function by Fourier series gives all modes of oscillation (fundamental and all overtones) which is extremely useful in physics.

4. Fourier series of a discontinuous function is not uniformly convergent at all points.

5. Term by term integration of a convergent Fourier series is always valid, and it may be valid if the series is not convergent. However, term by term, differentiation of a Fourier series is not valid in most cases.

Useful Integrals

The following integrals are useful in Fourier Series.

(*i*) $\int_0^{2\pi} \sin nx\, dx = 0$ (*ii*) $\int_0^{2\pi} \cos nx\, dx = 0$

(*iii*) $\int_0^{2\pi} \sin^2 nx\, dx = \pi$ (*iv*) $\int_0^{2\pi} \cos^2 nx\, dx = \pi$

(*v*) $\int_0^{2\pi} \sin nx \cdot \sin mx\, dx = 0$ (*vi*) $\int_0^{2\pi} \cdot \cos nx \cos mx\, dx = 0$

(*vii*) $\int_0^{2\pi} \sin nx \cdot \cos mx\, dx = 0$ (*viii*) $\int_0^{2\pi} \sin nx \cdot \cos nx\, dx = 0$

(*ix*) $[uv] = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots$

where $v_1 = \int v\, dx,\ v_2 = \int v_1\, dx$ and so on. $u' = \frac{du}{dx},\ u'' = \frac{d^2u}{dx^2}$ and so on

(*x*) $\sin n\pi = 0,\ \cos n\pi = (-1)^n$ where $n \in I$

- SOME REQUIRED FORMULAE:

1) $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$ if $f(x)$ is even

$= 0$ if $f(x)$ is odd.

2) $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]$

3) $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]$

4) If $f(x)$ is discontinuous at $x = c$ then the value of $f(x)$ $x = c$ is given by $\frac{1}{2}[\lim_{x\to c^-} f(x) + \lim_{x\to c^+} f(x)]$

5) Parseval's identity for the function $f(x)$ in the interval (a, a+2l) is given by $\frac{1}{2l}\int_{a}^{a+2l} f^2(x)\,dx = {a_0}^2 + \frac{1}{2}\sum_{n=1}^{\infty} {a_n}^2 + {b_n}^2$

If $f(x)$ is defined in the interval (a,a+2l) Fourier series is $f(x) = a_0 + \sum_{n=1}^{\infty}[a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}]$ where

$a_0 = \frac{1}{2l}\int_a^{a+2l} f(x)dx$; $a_n = \frac{1}{l}\int_a^{a+2l} f(x)\cos\frac{n\pi x}{l}dx$

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$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx$$

$a_n = \frac{1}{\pi}\int_a^{2\pi} f(x)\cos nx\, dx$

$b_n = \frac{1}{\pi}\int_a^{2\pi} f(x)\sin nx\, dx$

FOURIER SERIES IN THE INTERVAL $(0,2\pi)$.

EX 1 Obtain the Fourier expansion of $f(x) = (\frac{\pi - x}{2})^2$ with $f(x+\pi) = f(x)$ .Also deduce that

(i) $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots\ldots\ldots$

(ii) $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots\ldots\ldots$

(iii) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\ldots\ldots\ldots$

(iv) $\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots\ldots\ldots\ldots$

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Solution:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots\ldots\ldots (1)$$

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$$\therefore \quad a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} \frac{(\pi - x)^2}{4}\,dx$$

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$$\therefore \quad a_0 = \frac{1}{8\pi}\left[\frac{(\pi - x)^3}{-3}\right]_0^{2\pi} = -\frac{1}{24\pi}[-\pi^3 - \pi^3] = \frac{\pi^2}{12} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (A)$$

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$$= \frac{1}{4\pi}\left[(\pi - x)^2\left(\frac{\sin nx}{n}\right) - 2(\pi - x)(-1)\left(-\frac{\cos nx}{n^2}\right) + 2(-1)(-1)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

(By generalised rule of integration by parts.)

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$$= \frac{1}{4\pi}\left[\left(0 + 2\pi\frac{\cos 2n\pi}{n^2} - 0\right) - \left(0 - \frac{2\pi}{n^2} - 0\right)\right]$$

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$$\therefore\ a_n = \frac{1}{4\pi}\left[\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right] = \frac{1}{n^2} \qquad [\because\ \cos 2n\pi = 1] \qquad \text{.....................(B)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} \frac{(\pi - x)^2}{4} \bullet \sin nx\,dx$$

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$$= \frac{1}{4\pi}\left[\left(-\frac{\pi^2 \cos 2n\pi}{n} + 0 + \frac{2\cos 2n\pi}{n^3}\right) - \left(-\frac{\pi^2}{n} + 0 + \frac{2}{n^3}\right)\right]$$

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Putting these values in (1), we get

$$\left(\frac{\pi - x}{2}\right)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2}\cos nx$$

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$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots \qquad \ldots\ldots\ldots (3)$$

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(ii) Again, put $x = \pi$ in (2).

$$\therefore \quad 0 = \frac{\pi^2}{12} - \frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} + \ldots$$

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(iii) To get the last result add (3) and (4).

$$\therefore \quad \frac{\pi^2}{6} + \frac{\pi^2}{12} = \frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} + \frac{2}{7^2} + \ldots$$

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(iii) To get the last result add (3) and (4).

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$$\therefore \quad \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots$$

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$$\frac{1}{2\pi}\int_0^{2\pi}[f(x)]^2 dx = a_0^2 + \frac{1}{2}\sum(a_n^2 + b_n^2) \qquad \text{.....................} (5)$$

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Now, $\frac{1}{2\pi}\int_0^{2\pi}[f(x)]^2 dx = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)^4 dx$

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Deduction : Putting $x = 0$, we get $\displaystyle \frac{3}{4} = \sum_{n=2}^{\infty} \frac{1}{n^2 - 1}.$

EX 3 Find the Fourier series for $f(x) = e^{-x}$ in $0 < x < 2\pi$. Deduce the value of the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1}$ .Also derive the series for $\operatorname{csch} \pi$

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EX 4. Find Fourier expansion of $\cos px$ in $(0,2\pi)$ where p is not an integer .

Hence deduce that $\pi \operatorname{csch} p\pi = \frac{1}{p} + \sum_{n=1}^{\infty}(-1)^n\left[\frac{1}{p+n} + \frac{1}{p-n}\right]$

Also deduce that $\pi \cot 2p\pi = \frac{1}{2p} + p\sum_{n=1}^{\infty}\frac{1}{p^2-n^2}$

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Putting these values in (1), we get,

$$\cos px = \frac{\sin 2p\pi}{2p\pi} + \frac{p\sin 2p\pi}{\pi}\sum_{n=1}^{\infty}\frac{\cos nx}{p^2-n^2} - \frac{(1-\cos 2p\pi)}{\pi}\sum_{n=1}^{\infty}\frac{n\sin nx}{p^2-n^2} \qquad \text{..........} (2)$$

To deduce the first result, put $x = \pi$ in (2).

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Fourier series in the interval $(-\pi, \pi)$

Period of f(x) is 2π

$f(x) = a_0 + \sum a_n \cos nx + \sum b_n \sin nx$, where

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EX1 Find the Fourier series for a periodic function

$$f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

State the value of $f(x)$ at $x = 0$ and hence deduce that

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$$+\left[3\cdot\frac{\sin x}{1} - \frac{1}{2}\sin 2x + \sin 3x - \ldots\right] \qquad \ldots\ldots\ldots\ (2)$$

deduction

State the value of $f(x)$ at $x = 0$ and hence deduce that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

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EX 2 Find the Fourier series for $f(x) = e^{-|x|}$ in $(-\pi, \pi)$

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$$\therefore \quad a_n = \frac{2}{\pi(1+n^2)}\left[1 - (-1)^n e^{-\pi}\right]$$

$$\therefore \quad f(x) = \frac{1}{\pi}(1 - e^{-\pi}) + \frac{2}{\pi}\sum_{1}^{\infty}\left[\frac{1-(-1)^n e^{-\pi}}{1+n^2}\right]\cdot \cos nx$$

EX 3 Obtain Fourier expansion of $f(x) = |\cos x|$ in $(0,2\pi)$

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Here $f(x)$ is an even function because $f(-x) = |\cos(-x)| = f(x)$. $[\therefore b_n = 0]$

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$$\therefore \quad a_n = \begin{cases} -\dfrac{4}{\pi(n^2-1)}\cos\left(\dfrac{n\pi}{2}\right) & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd and } n \neq 1 \end{cases} \qquad \text{.....................} (4)$$

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Putting these values from (2) and (4) in (1), we get,

$$|\cos x| = \frac{2}{\pi} + \frac{4}{\pi}\left[\frac{\cos 2x}{3} - \frac{\cos 4x}{15} + \ldots\right]$$

EX 4.Obtain Fourier series for $f(x) = \begin{cases} x + \frac{\pi}{2}, & -\pi < x < 0 \\ \frac{\pi}{2} - x, & 0 < x < \pi \end{cases}$

Hence deduce that (i) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \ldots\ldots\ldots\ldots\ldots$

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$$\therefore \quad f(x) = \frac{4}{\pi}\left[\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right] \quad \text{..........} (2)$$

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Hence, from (1), (2) and (3),

$$\frac{\pi^2}{12} = \frac{1}{2}\cdot\frac{16}{\pi^2}\left[\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\ldots\right]$$

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$$\left. -(2-2x)\left(-\frac{9}{4n^2\pi^2}\cdot\cos\frac{2n\pi x}{3}\right) + (-2)\left(-\frac{27}{8n^3\pi^3}\cdot\sin\frac{2n\pi x}{3}\right)\right]_0^3$$

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EX 2 If $f(x) = \begin{cases} \pi x, & 0 < x < 1 \\ \pi(2-x), & 1 < x < 2 \end{cases}$ With period 2

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$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ -\dfrac{4}{n^2\pi} & \text{if } n \text{ is odd} \end{cases}$$

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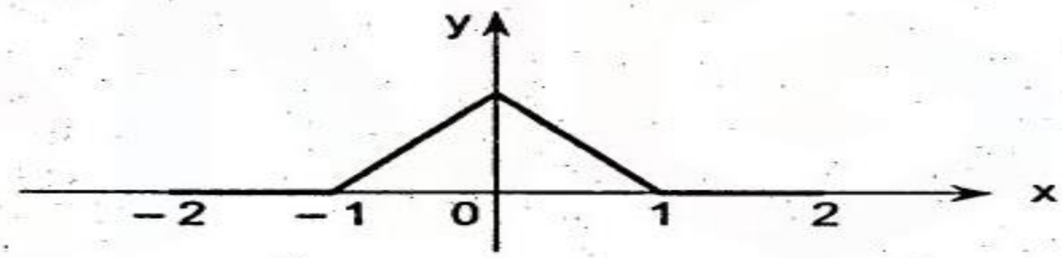
Solution: The graph of the function is shown below,



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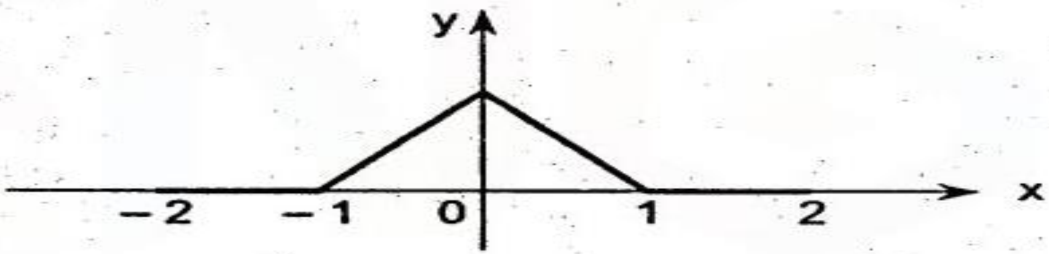

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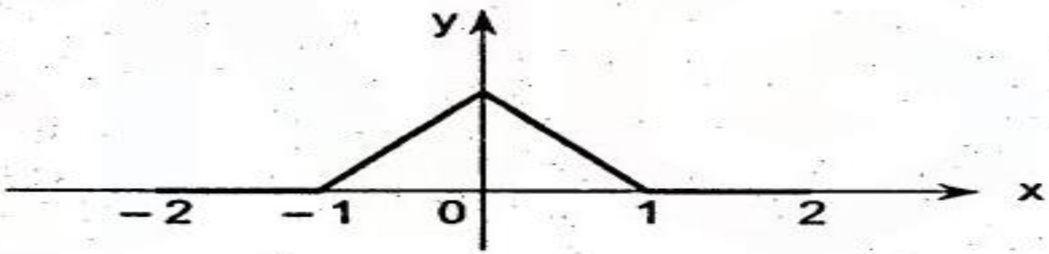

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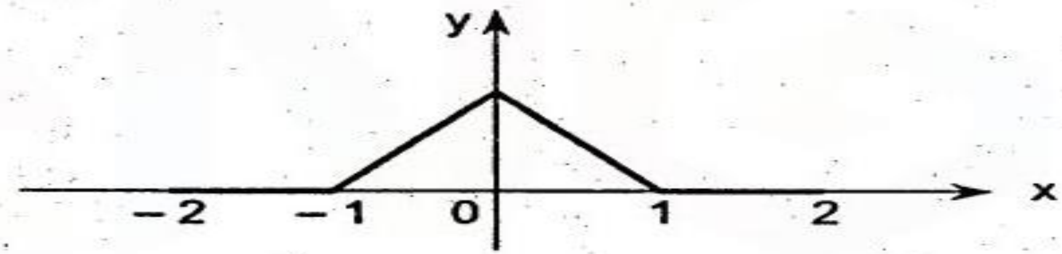

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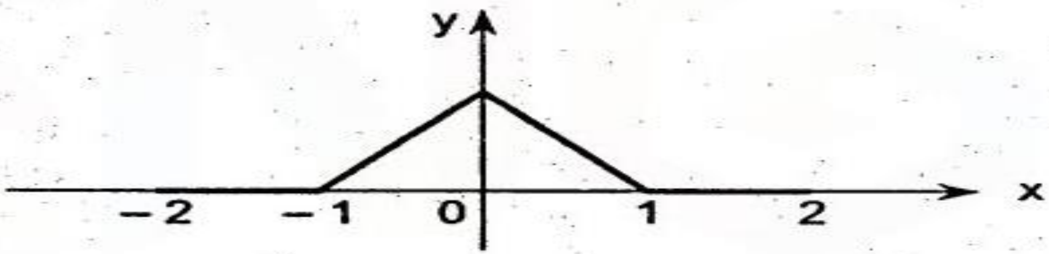

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$\therefore$ $f(x)$ is an even function.

$\therefore$ $b_n = 0$ and $l = 2$.

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$$= \frac{1}{2}\left[x - \frac{x^2}{2}\right]_0^1 = \frac{1}{4}$$

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