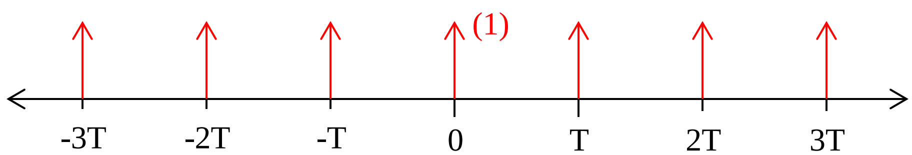
Signals and Systems – Problem Set 7

Mika Ichiki-Welches

1. Fourier Series Representation of Unit Impulses

a.



$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

b. Fourier series representation of p(t):

$$C_k(t = 0) = \frac{1}{T} e^{j\frac{2\pi}{T}k(0)}$$

$$p(t) = \sum_{k=-\infty}^{\infty} C_k e^{j\frac{2\pi}{T}kt}$$

$$\boldsymbol{p(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{j\frac{2\pi}{T}kt}}$$

c. X(ω), in terms of Ck:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$X(\omega) = \int_{-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} C_k e^{j\frac{2\pi}{T}kt} \right] e^{-j\omega t} dt$$

$$X(\omega) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} C_k e^{j\frac{2\pi}{T}kt} e^{-j\omega t} dt$$

Because $e^{j\omega_0 t}$ in the time domain pairs to $2\pi\delta(\omega - \omega_0)$ in the frequency domain, this becomes:

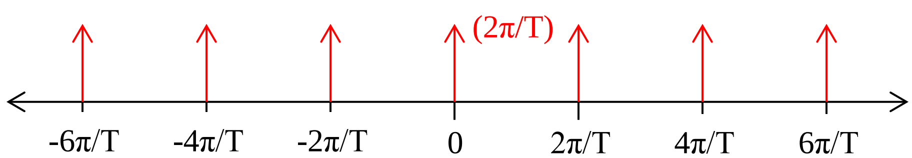
$$\boldsymbol{X(\omega) = \sum_{k=-\infty}^{\infty} C_k 2\pi\delta(\omega - \frac{2\pi}{T} k)}$$

d. Using your answer to the previous two parts, find P(ω).

$$P(\omega) = \sum_{k=-\infty}^{\infty} C_k 2\pi\delta(\omega - \frac{2\pi}{T}k)$$

$$\boldsymbol{P(\omega) = \sum_{k=-\infty}^{\infty} \frac{1}{T} 2\pi\delta(\omega - \frac{2\pi}{T}k)}$$

e. P(ω):



With a large fundamental period T, the unit impulses in p(t) would be more widely spread apart, while the impulses in P(ω) would be closer together and shorter. This makes sense, as a larger period would result in a smaller frequency.

2. Low-Pass LTI System

a.

$$h(t) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi}\frac{1}{jt}\left[e^{j\omega t}\right]_{-\omega_c}^{\omega_c}$$

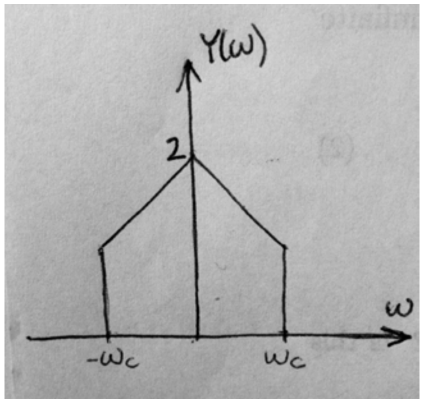
$$= \frac{1}{2\pi}\frac{1}{jt}(e^{j\omega_c t} - e^{-j\omega_c t})$$

$$= \frac{1}{\pi t}(\frac{e^{j\omega_c t} - e^{-j\omega_c t}}{2j})$$

$$= \boldsymbol{\frac{\sin(\omega_c t)}{\pi t}}$$

Which is a sinc function equivalent to $\frac{\omega_c}{\pi} sinc(\frac{t}{\pi}\omega_c)$.

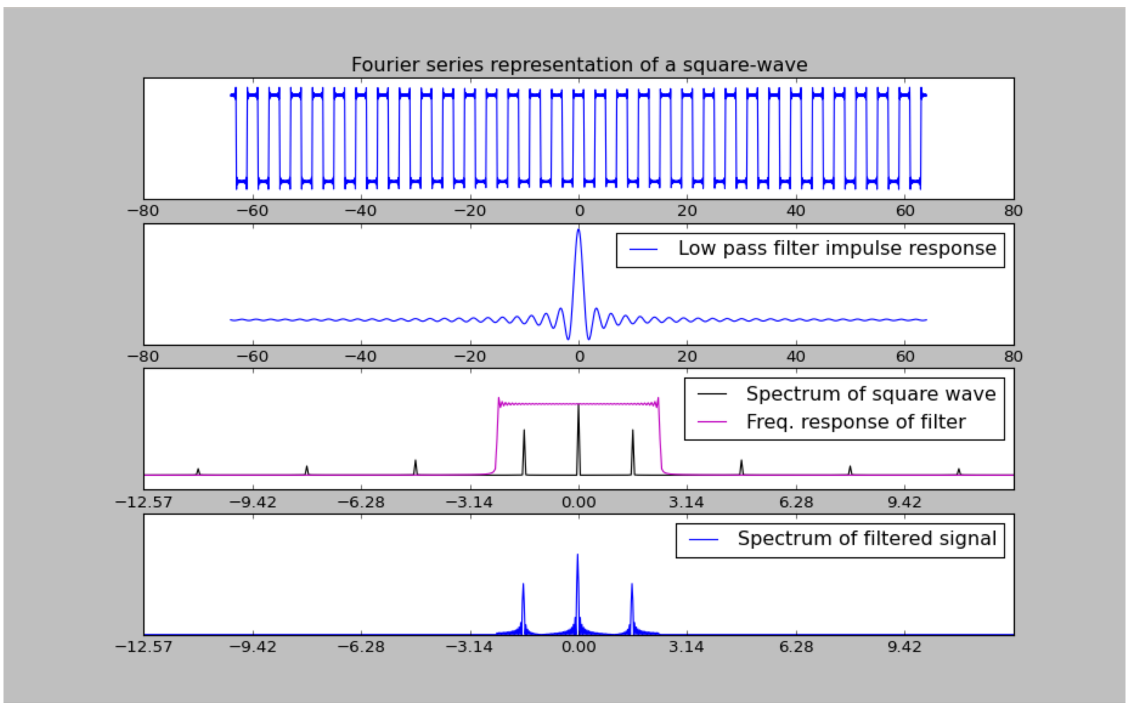
b. Due to the convolution theorem, $y(t) = x * h(t)$ is the time domain equivalent to $Y(\omega) = X(\omega)H(\omega)$ in the frequency domain. Therefore, Y(ω) should look like a low-passed version of X(ω), cut off at – ω_c and ω_c:



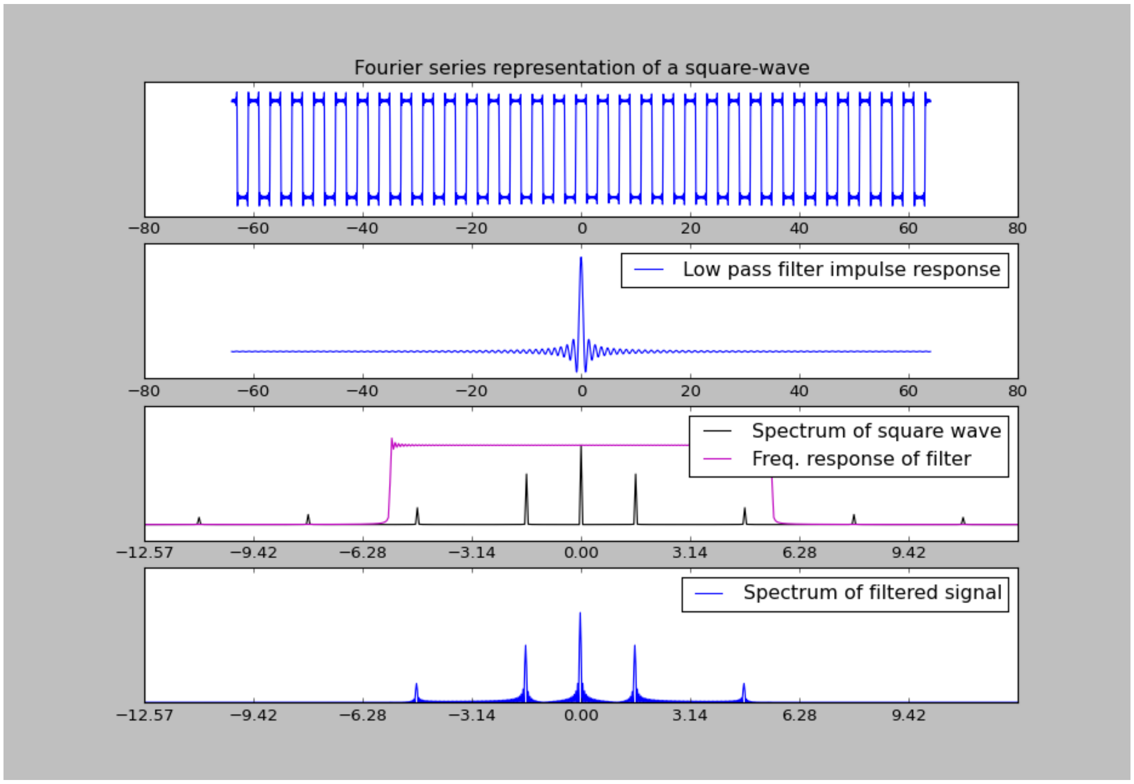
c. The rectangular function H(ω) passes only a band of frequencies from – ω_c to ω_c. In the time domain, the sinc function takes the impulse and acts as a filter that removes all frequency components above the cut-off frequency ω_c.

d. Square Wave Filter Exercise – implemented with cut-off frequencies 0.75π and 1.75π:

ω_c = 0.75π:



ω_c = 1.75π:



You can tell from the above figures that as expected, the Fourier series representation of a square wave with cut-off frequency of $\omega_c = 1.75\pi$ passed more frequencies than did the square wave with $\omega_c = 0.75\pi$.

3. The signal $\cos(\omega_0 t)$ in the time domain is equivalent to $\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$ in the frequency domain. Taking into account the Fourier transform property that $x(t)h(t)$ equates to $\frac{1}{2\pi} X * H(\omega)$, Y(ω) will look like X(ω) convoluted to the right and left of zero by ω_c: