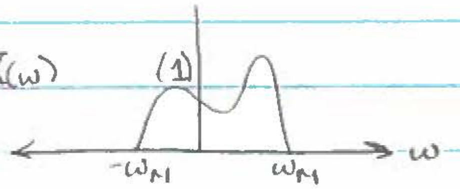
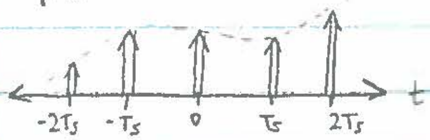
Sig Sys - Problem set 8
Mika Ichiki-Welches

1. x(t)

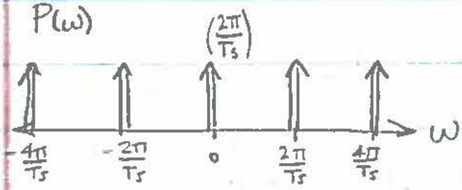
X(ω)



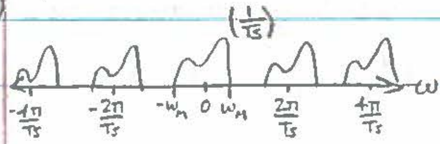
a) $x_p(t)$



b) P(ω)



c) $X_p(\omega) = \frac{1}{2\pi} X * P(\omega)$

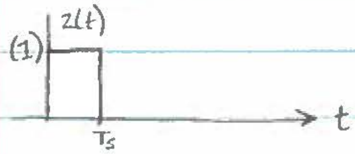


d) $\frac{2\pi}{T_s} > 2 \cdot \omega_M$

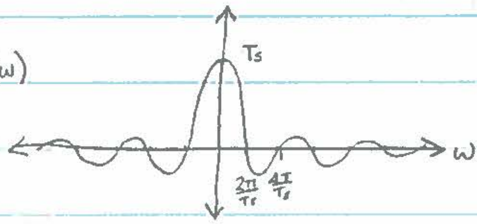
e) sum the sinc functions (h(t)'s) at each impulse in $x_p(t)$

$x(t) = x_p * h(t)$

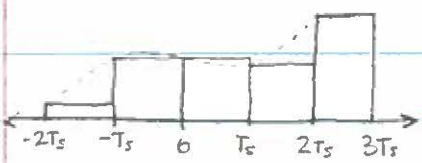
f) z(t)



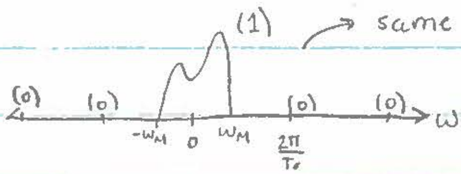
Z(ω)



g) $x_z(t) = x_p * z(t)$

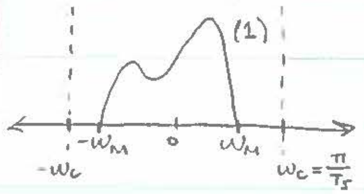


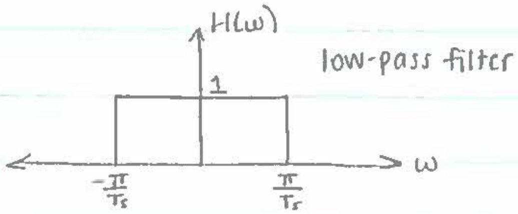
h) $X_z(\omega) = X_p(\omega) Z(\omega)$



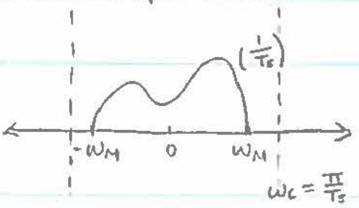
same as X(ω)?

i) $\bar{X}(\omega) = X_z(\omega)H(\omega)$





$\hat{X}(\omega) = X_p(\omega)H(\omega)$

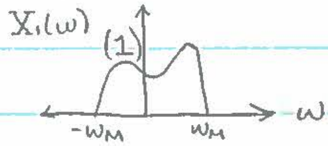


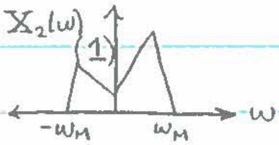
j) $\bar{X}(\omega)$ has an area of 1, like $X(\omega)$, while $\hat{X}(\omega)$ is scaled to an area of $\frac{1}{T_s}$ by the transform $\frac{2\pi}{T}\sum_{-\infty}^{\infty}\delta(\omega - \frac{2\pi}{T}k)$ and the property $x(t)h(t) \longrightarrow \frac{1}{2\pi}X * H(\omega)$. $\bar{X}(\omega)$ got an area of 1 because $X_z(\omega)$ is the product of $X_p(\omega)$, which has an area of $\frac{1}{T_s}$ as I just explained, and the sinc function $Z(\omega)$, with area T_s.

k) $T_s : 1$

2. $y(t) = x_1(t)\cos(\omega_1 t) + x_2(t)\cos(\omega_2 t)$

$\omega_1, \omega_2 >> \omega_M \quad , \quad \omega_1 + 2\omega_M < \omega_2$





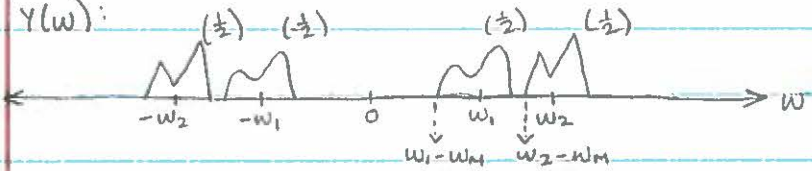
a) $\cos(\omega_0 t) \longrightarrow \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$

$x_1(t)\cos(\omega_1 t) \longrightarrow \frac{1}{2\cancel{\pi}} X_1(\omega) * [\cancel{\pi}\delta(\omega-\omega_1) + \pi\delta(\omega+\omega_1)]$

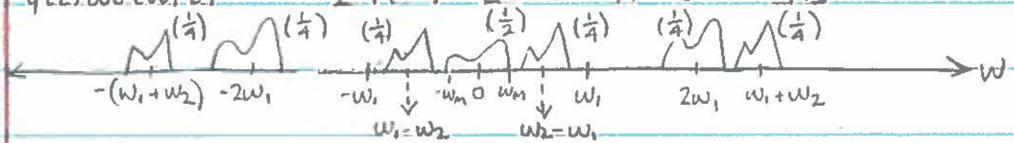
$= \frac{1}{2} X_1(\omega) * [\delta(\omega-\omega_1) + \delta(\omega+\omega_1)]$

$x_2(t)\cos(\omega_2 t) \longrightarrow \frac{1}{2} X_2(\omega) * [\delta(\omega-\omega_2) + \delta(\omega+\omega_2)]$

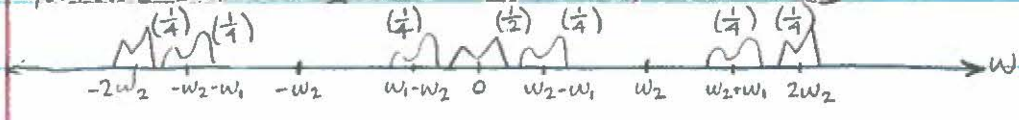
Y(ω):



b) $y(t)\cos(\omega_1 t) \longrightarrow \frac{1}{2} Y(\omega) * [\delta(\omega-\omega_1) + \delta(\omega+\omega_1)]$

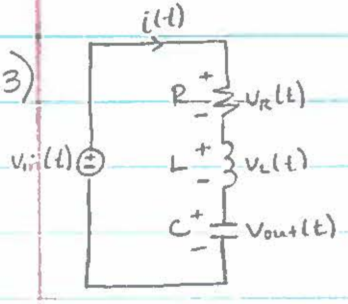


$y(t)\cos(\omega_2 t) \longrightarrow \frac{1}{2} Y(\omega) * [\delta(\omega-\omega_2) + \delta(\omega+\omega_2)]$



c) Find the Fourier transform of $y(t)\cos(\omega_1 t)$ and $y(t)\cos(\omega_2 t)$, and put a low-pass filter on each of them with a cutoff frequency of $|\omega_M|$, to extract the Fourier transforms of $x_1(t)$ and $x_2(t)$. Take the inverse Fourier transforms to get $x_1(t)$ and $x_2(t)$.

3)



$i(t) = C \frac{d}{dt} V_{out}(t)$

$V_L(t) = L \frac{d}{dt} i(t)$

$= LC \frac{d^2}{dt^2} V_{out}(t)$

a) $V_{in}(t) = V_R(t) + V_L(t) + V_{out}(t)$

$V_{in}(t) = R\,i(t) + L \frac{d}{dt} i(t) + V_{out}(t)$

$$\boxed{V_{in}(t) = RC \frac{d}{dt} V_{out}(t) + LC \frac{d^2}{dt^2} V_{out}(t) + V_{out}(t)}$$

b) $V_{in}(\omega) = RCj\omega V_{out}(\omega) + LCj^2\omega^2 V_{out}(\omega) + V_{out}(t)$

$$H(\omega) = \frac{V_{out}(\omega)}{V_{in}(\omega)} = \boxed{\frac{1}{RCj\omega + -LC\cancel{j^2}\omega^2 + 1}}$$

c) $$|H(\omega)| = \frac{1}{|RCj\omega - LC\omega^2 + 1|} = \boxed{\frac{1}{\sqrt{\omega^2(RC)^2 + (1 - LC\omega^2)^2}}}$$ magnitude of frequency respons

d) $\frac{d}{d\omega}\left(\omega^2(RC)^2 + (1-LC\omega^2)^2\right) = 2\omega(RC)^2 + 2(1-LC\omega^2)\cdot -2LC\omega$

first derivative $\Rightarrow$ $= 2\omega(RC)^2 - 4LC\omega(1 - LC\omega^2)$

$0 = \omega(RC)^2 - 2LC\omega + 2L^2C^2\omega^3$

$0 = ((RC)^2 - 2LC)\omega + (2L^2C^2)\omega^3$

$\underline{\omega = 0}$, $0 = (RC)^2 - 2LC + 2L^2C^2\omega^2$

$2LC - (RC)^2 = 2L^2C^2\omega^2$

$\omega^2 = \frac{2LC - (RC)^2}{2L^2C^2}$

second derivative

$\omega = \sqrt{\frac{2L - R^2C}{2L^2C}} = \underline{\underline{\sqrt{\frac{1}{LC} - \frac{R^2}{2L^2}}}}$

$\frac{d^2}{d\omega^2}('') = \underline{R^2C^2 - 2LC + 6L^2C^2\omega^2}$

$0 \overset{?}{>} R^2C^2 - 2LC + 6L^2C^2\left(\frac{1}{LC} - \frac{R^2}{2L^2}\right)$

$0 \overset{?}{>} R^2C^2 - 2LC + 6LC - 3R^2C^2$

$0 > 4LC - 2R^2C^2$ (concave down)

$\boxed{\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{2L^2}}}$ makes H(w) at a maximum.

1

e. i. $C = 10^{-7} F,\ L = 10^{-2} H,\ R = 400\Omega$

magnitude:

$$|H(\omega)| = \frac{1}{\sqrt{\omega^2(400 \cdot 10^{-7})^2 + (1 - 10^{-2} \cdot 10^{-7}\omega^2)^2}}$$

$$= \frac{1}{\sqrt{1.6 \times 10^{-9}\omega^2 + (1 - 10^{-9}\omega^2)^2}} = \frac{1}{\sqrt{1.6 \times 10^{-9}\omega^2 + 1 - 2 \cdot 10^{-9}\omega^2 + 10^{-18}\omega^4}}$$

$$= \boxed{\frac{1}{\sqrt{10^{-18}\omega^4 - 0.4 \cdot 10^{-9}\omega^2 + 1}}}$$

phase:

$$\measuredangle H(\omega) = \measuredangle 1 - \measuredangle\left[-LC\omega^2 + RCj\omega + 1\right]$$

$$= 0 - \tan^{-1}\left(\frac{RCj\omega}{1 - LC\omega^2}\right)$$

$$= \boxed{-\tan^{-1}\left(\frac{4 \cdot 10^{-5} j\omega}{1 - 10^{-9}\omega^2}\right)}$$

ii. $C = 10^{-7} F,\ L = 10^{-2} H,\ R = 50\Omega$

magnitude:

$$|H(\omega)| = \frac{1}{\sqrt{\omega^2(50 \cdot 10^{-7})^2 + (1 - 10^{-2} \cdot 10^{-7}\omega^2)^2}}$$

$$= \frac{1}{\sqrt{2.5 \times 10^{-11}\omega^2 + 1 - 2 \cdot 10^{-9}\omega^2 + 10^{-18}\omega^4}}$$

$$= \boxed{\frac{1}{\sqrt{(2.5 \times 10^{-11} - 2 \times 10^{-9})\omega^2 + 10^{-18}\omega^4 + 1}}}$$

phase:

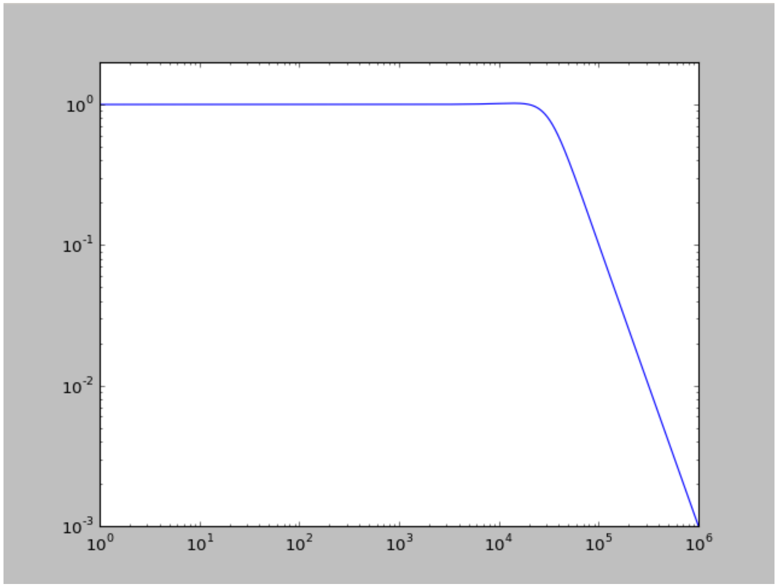
$$\measuredangle H(\omega) = \measuredangle 1 - \measuredangle\left[-LC\omega^2 + RCj\omega + 1\right]$$

$$= 0 - \tan^{-1}\left(\frac{RCj\omega}{1 - LC\omega^2}\right)$$

$$= \boxed{-\tan^{-1}\left(\frac{5 \cdot 10^{-6} j\omega}{1 - 10^{-9}\omega^2}\right)}$$

i.

magnitude:



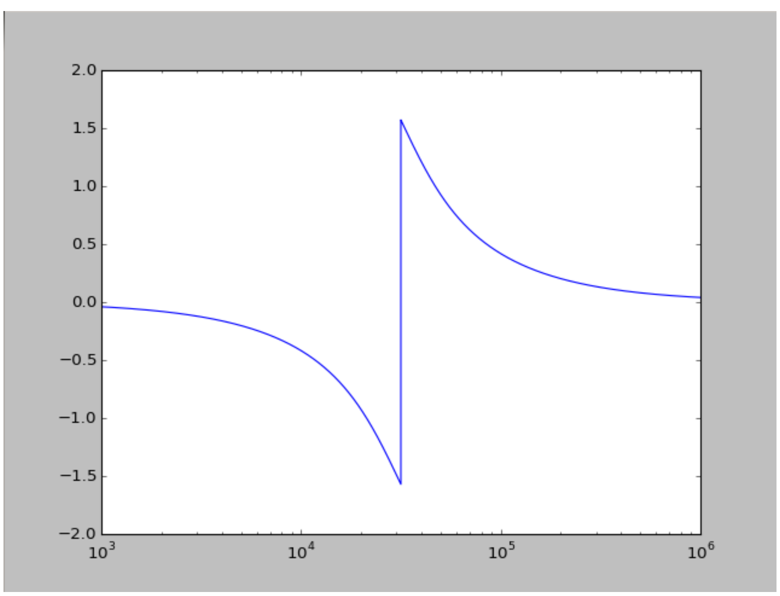
This is about the shape I expected to generate for the magnitude.

phase:

Initially, I calculated phase using arctan, with the code:

```
omega = -np.arctan(4e-5*w/(1-1e-9*w**2))
```

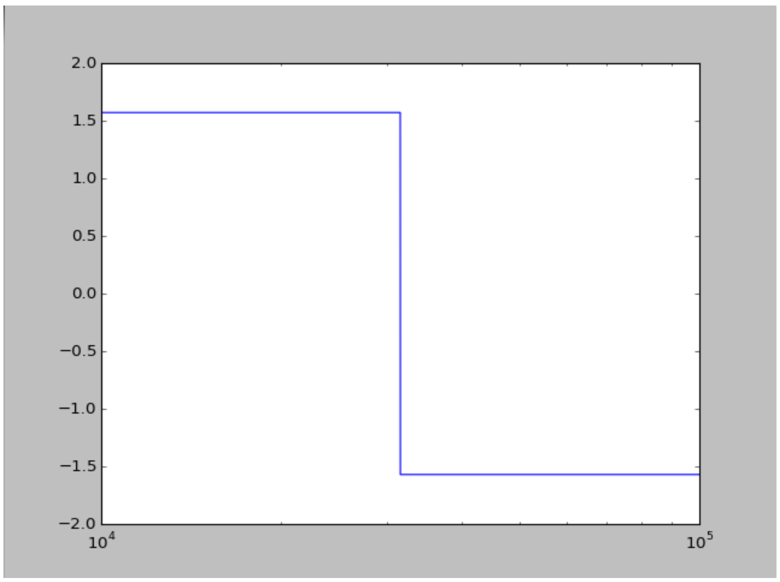
This, however, produced a plot like this:



I was recommended by Cynthia Chen to use cmath.phase, instead. I wrote a for loop for the phase:

```
omegas = []
for W in w:
    omega = cmath.phase(4e-5*W*1j/(1-1e-9*W**2))
    omegas.append(omega)
```

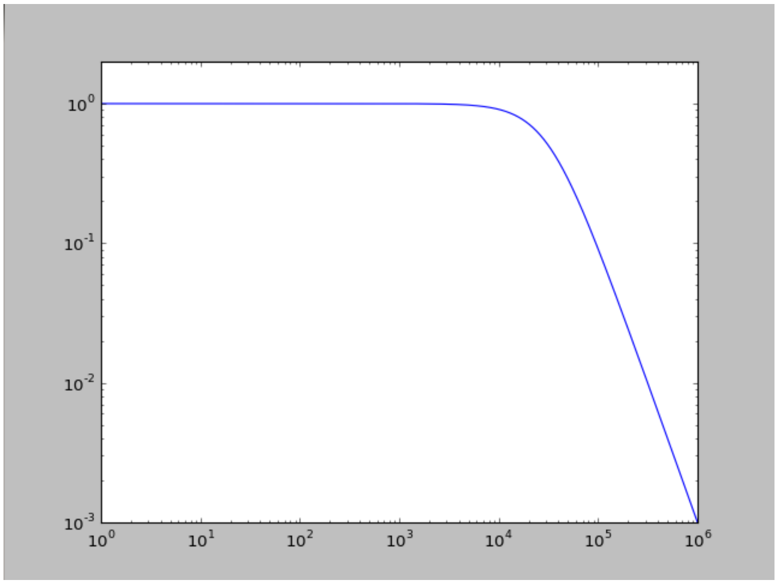
This gave me the plot:



This is closer to what I expected to see for phase, but much boxier. I tried fixing this by adding more elements to my ω array, but even at the limit, 1e7, before the code would just run forever, I was only able to generate a wave like the one above.

ii.

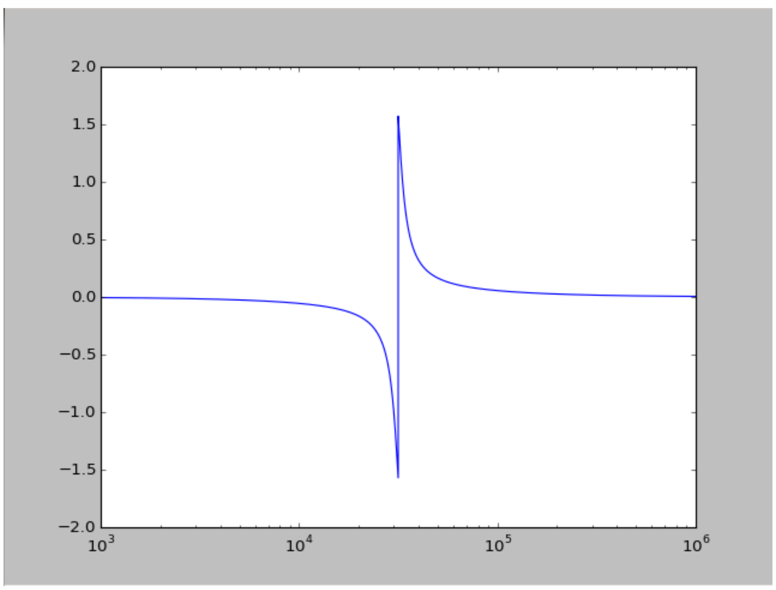
magnitude:



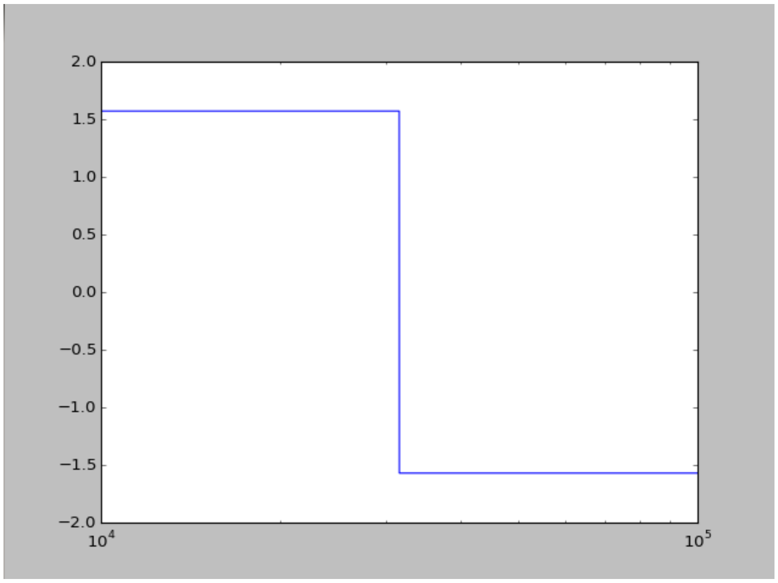
Likewise, this is around the shape I expected for the magnitude when R=50Ω instead, although it was supposed to produce a slight raise in magnitude (a bump) before it decreased linearly.

phase:

As before, arctan produced a strange looking phase for me:



So I tried again with cmath.phase, and came up with a very similar plot to the phase plot from part i.



This again looks about right, though more boxy, than I expected to see from phase. I tried again with increasing the number of elements, with the same limit.