Questions

1. Suppose that a shiny ground plane $y = 0$ is illuminated by sunlight. Let the direction of the sun be $(1, 2, 2)$. If a viewer is at $(x, y, z) = (4, 6, 7)$, determine the position on the ground plane at which the peak of the highlight occurs. Assume that the peak occurs where a mirror reflection takes a ray from the source and reflects it directly to the viewer position.

2. Suppose that a shiny plane, $2x + y + 2z + 17 = 0$, is illuminated by sunlight. Let the direction of the sun be $\mathbf{l} = (0, 1, 0)$. Assuming the viewer is at the origin, $(x, y, z) = (0, 0, 0)$, determine the position on this shiny plane at which the peak of the highlight occurs.

3. Consider a triangle in the image plane. Suppose the intensities $I(x, y)$ have been computed to be

$$I_1 = I(40, 14) = 10, \quad I_2 = I(60, 34) = 90, \quad I_3 = I(80, 24) = 140$$

   Using linear interpolation, calculate the intensity $I(50, 19)$.

4. (a) How would you use convex combinations of (x, y, I) to answer the previous question?
   (b) How would you solve the same problem by fitting a plane to the three points in (x, y, I) space? (This is at the level of MATH 133.)

5. How to interpolate the normals in Phong shading ? Assume a triangle mesh.

6. The Blinn-Phong model is cheaper to use than the Phong model in the case that the light source is at infinity and the viewer is far from the polygon (in relation to the size of the polygon). Why?

Answers

1. We calculate $\mathbf{r}$ using the formula for mirror reflection given in class.

$$\mathbf{l} + \mathbf{r} = 2(\mathbf{n} \cdot \mathbf{l})\mathbf{n} \ .$$

This formula assumes $\mathbf{l}$ has been normalized to unit length, so

$$\mathbf{l} = (\frac{1}{3}, \frac{2}{3}, \frac{2}{3}) \ .$$

Since the ground plane is $y = 0$, the surface normal to the ground plane is $\mathbf{n} = (0,1,0)$. Thus, $\mathbf{n} \cdot \mathbf{l} = \frac{2}{3}$ which we can plug into the formula for $\mathbf{r}$. We find that the mirror reflection direction is $\mathbf{r} = (-\frac{1}{3}, \frac{2}{3}, -\frac{2}{3})$.

According to the Phong model, the peak of the highlight occurs for a ray that is in direction $\mathbf{r}$. This ray must pass through the camera which is at (4,6,7). Hence this ray belongs to a line with parametric equation:

$$(x, y, z) = (4, 6, 7) + t \ (-\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}) \ .$$

We need to compute where this line intersects the $y = 0$ plane. This occurs at $0 = 6 + \frac{2}{3}t$, or $t = -9$. Thus, the peak of the highlight occurs at

$$(x, y, z) \ = \ (7, 0, 13).$$

2. From the equation of the plane, we get the unit normal to be $\mathbf{n} = (\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$. Plugging this into the formula

$$\mathbf{r} = 2(\mathbf{n} \cdot \mathbf{l})\mathbf{n} - \mathbf{l}.$$

gives us $\mathbf{r} = (\frac{4}{9}, \frac{-7}{9}, \frac{4}{9})$.

The peak of the highlight occurs from a ray that is in direction $\mathbf{r}$. This ray must pass through the viewer who is at $(0, 0, 0)$. Hence this ray belongs to a line with parametric equation:

$$(x, y, z) = (0, 0, 0) + t(\frac{4}{9}, \frac{-7}{9}, \frac{4}{9}).$$

We are interested in where this line intersects the plane $2x + y + 2z + 17 = 0$. This occurs at $t = -17$. Thus the point of this peak highlight ray occurs on the ground plane at:

$$(x, y, z) = (-\frac{68}{9}, \frac{119}{9}, -\frac{68}{9})$$

3. The formula we use is:

$$I(x,y) = I(x_0,y_0) + (I(x_1,y_1) - I(x_0,y_0))(\frac{x - x_0}{x_1 - x_0})$$

We will need to interpolate three times. First we interpolate between I_1 and I_2:

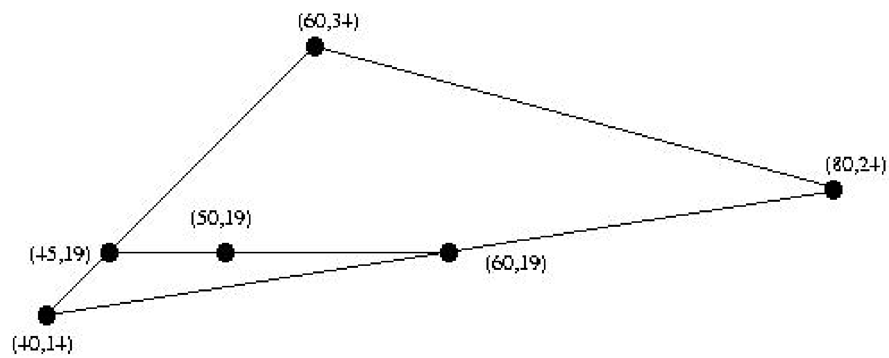
$$I(45,19) = 10 + (90 - 10)(\frac{45-40}{60-40}) = 10 + 20 = 30$$

Next we interpolate between I_1 and I_3:

$$I(60,19) = 10 + (140 - 10)(\frac{60-40}{80-40}) = 10 + 65 = 75$$

Finally, we interpolate between $I(45,19)$ and $I(60,19)$:

$$I(50,19) = 30 + (75 - 30)(\frac{50-45}{60-45}) = 30 + 15 = 45$$



4. (a) Consider the (x,y,I) to be $\Re^3$. We want to express any point $\mathbf{p} = (x,y,I)$ as a convex combination of the three given vertices + intensities of the triangle $\mathbf{p}_i, i = 1,2,3$.

$$\mathbf{p} = \mathbf{p}_1 + s(\mathbf{p}_2 - \mathbf{p}_1) + t(\mathbf{p}_3 - \mathbf{p}_1)$$

or

$$\mathbf{p} = (1 - s - t)\mathbf{p}_1 + s\mathbf{p}_2 + t\mathbf{p}_3.$$

Now, for a given (x,y), we want to solve for (s,t) and then use that (s,t) to find $I(x,y)$. There are three equations and two unknowns so, for a given x,y, we can solve for s,t using standard linear algebra. We write the first two components of the first equation above as follows (although standard row reduction for two linear equations works fine too).

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 60-40 & 80-40 & 40 \\ 34-14 & 24-14 & 14 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \\ 1 \end{bmatrix}$$

and so for any (x,y) – such as the $(50,19)$ in the question – we can find the (s,t) by:

$$\begin{bmatrix} s \\ t \\ 1 \end{bmatrix} = \begin{bmatrix} 20 & 40 & 40 \\ 20 & 10 & 14 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

With the (s,t) value computed, we can find the corresponding I value by

$$I(s,t) = I_0 + s(I_1 - I_0) + t(I_2 - I_0).$$

(b) Let $\mathbf{p}_1 = (40, 14, 10), \mathbf{p}_2 = (60, 34, 90), \mathbf{p}_3 = (80, 24, 140)$. To find the normal $(\mathbf{n_x}, \mathbf{n_y}, \mathbf{n_I})$ to a plane through these points, compute a cross product, say $\mathbf{n} = (\mathbf{p}_2 - \mathbf{p}_1) \times (\mathbf{p}_3 - \mathbf{p}_1)$. This will not a unit vector, but we don't care. To fit a plane, we plug one of our point into

$$\mathbf{n}_x x + \mathbf{n}_y x + \mathbf{n}_I I = d$$

to get d. Then for a new (x,y), we can find the I by plugging in.

Note that this method is arguably better for computation (especially by hand!) than the one in (a) in that it doesn't require a matrix inversion.

5. If we have a triangular mesh, then we can linearly interpolate the surface normals. For example, we interpolate the normal along the edge joining (x_0, y_0) and (x_2, y_2) as follows:

$$\mathbf{n}(x', y) := \mathbf{n}(x_0, y_0)(\frac{y_2 - y}{y_2 - y_0}) + \mathbf{n}(x_2, y_2)\frac{(y - y_0)}{(y_2 - y_0)}$$

and then normalize $\mathbf{n}(x', y)$ so that it is of unit length. We would do the same to interpolate the normal along the edge joining (x_2, y_2) and (x_1, y_1). Finally, we interpolate along the dotted line to obtain the normal at an interior point (x, y). We then use this interpolated normal vector to compute the intensity at pixel (x, y). Notice that the normalization requires computing a square root, so it is not cheap.

6. The Blinn-Phong model requires that you compute $\mathbf{H} \cdot \mathbf{n}$ where $\mathbf{H}$ depends on $\mathbf{l}$ and $\mathbf{v}$. If the light source is at infinity and the viewer is far from the polygon, then the $\mathbf{l}$ vector is constant and the $\mathbf{v}$ vector is approximately constant. In this case, the $\mathbf{H}$ vector is constant. In this case, the Blinn-Phong model requires just dot product between $\mathbf{H}$ and $\mathbf{n}$ where only the latter varies across the polygon.

   The Phong model requires that $\mathbf{r}$ is computed for each point on the polygon and then $\mathbf{v} \cdot \mathbf{r}$ is computed. Note that approximating $\mathbf{r}$ as a constant doesn't work, because $\mathbf{v}$ is essentially constant in the case we are considering and so $\mathbf{v} \cdot \mathbf{r}$ would be constant too, which wouldn't produce the desired spatial variations in intensity across the polygon.

   The Phong model requires you to compute the reflection vector $\mathbf{r}$ which is a bit more expensive to compute since it requires reflecting the light vector $\mathbf{l}$ about the surface normal vector