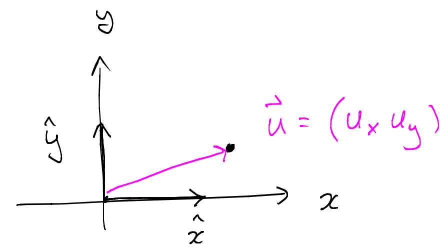
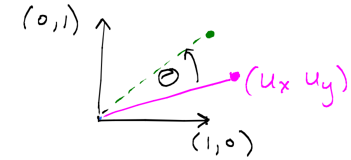
lecture 2

- model transformations (rotations, scaling, translation)
- intro to homogeneous coordinates



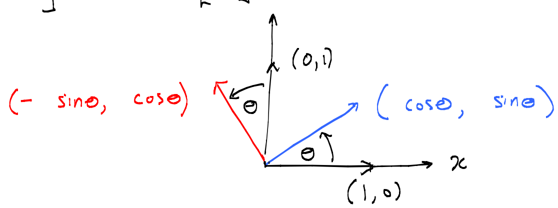
2D Rotation



$$\begin{bmatrix} u_x' \\ u_y' \end{bmatrix} = R_\theta \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

$\uparrow$ 2×2 matrix

$$\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = R_\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



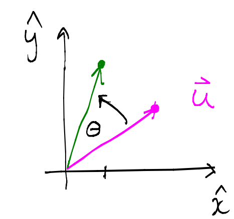
$$\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = R_\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}_{2\times 2}$$

Two ways to think about R.

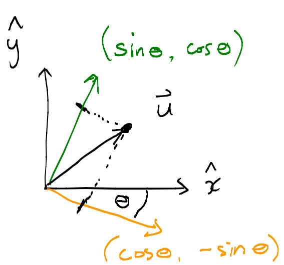
1) R rotates points within a fixed coordinate frame ("world coordinates")

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



2) R maps to a new coordinate system by projecting onto new axes.

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



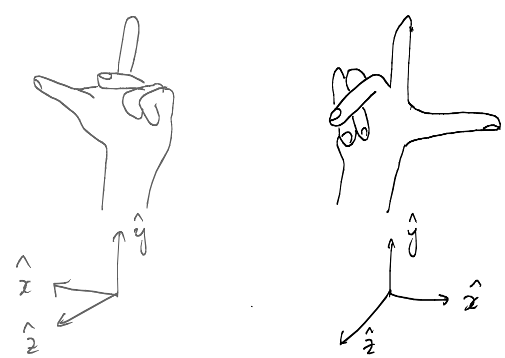
How will rotations be used?

1) re-orient an object ("model")

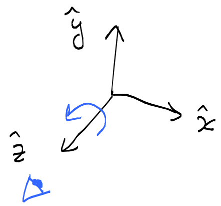
2) map from world coordinates to camera coordinates ("view")

3D Rotations

Left vs. Right Hand Coordinates

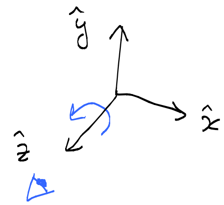


Example: rotate about z axis

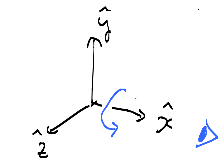


counter-clockwise
(assuming eye is looking in the -z direction and the coordinates are righthanded)

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



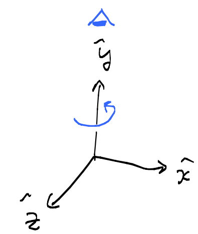
$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$



Counter-clockwise

note signs

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$



General 3D rotation

$R_{3\times 3}$ such that

- $R^T R = I \longleftarrow$ identity matrix

  that is, $R^{-1} = R^T$

- determinant of R is 1.

Claim: Rotation matrices preserve dot product.

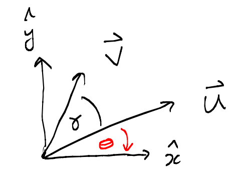
ie. For any vectors u, v

$$\vec{u}\cdot\vec{v} = (R\vec{u})\cdot(R\vec{v})$$

Proof

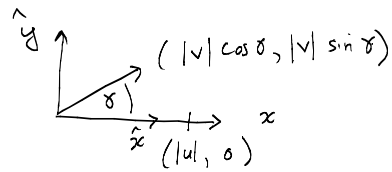
$$(R\vec{u})^T R\vec{v} = \vec{u}^T R^T R \vec{v} = \vec{u}^T \vec{v} = \vec{u}\cdot\vec{v}$$

Example (2D)



$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\gamma$

Rotate $\vec{u}$ to x gives:



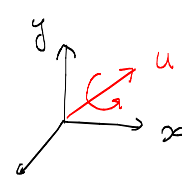
Rotation versus Reflection

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ reflection about $x=0$ plane

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ reflection about $x=y$ plane

For these examples, determinant is -1 (not a rotation).

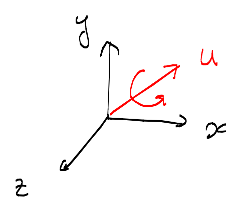
A rotation matrix R always defines an axis of rotation and an angle of rotation.



Given R, what is axis and angle?

$$R\vec{u} = \vec{u}$$

$\vec{u}$ is axis of rotation (eigenvector with eigenvalue 1)



Exercise:

What is angle?

Example Problem 1

Given a unit vector $\vec{p}$, find a 3D rotation matrix that maps $\hat{z}$ to $\vec{p}$.

$$R\,\hat{z} = \vec{p}.$$

Assume $\vec{p} \neq \hat{z}$ since in that case the problem is trivial.

Step 1

Observe the 3^{RD} column of R must be the vector p.

Why?

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \underset{3\times 3}{\begin{bmatrix} & R & \end{bmatrix}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Step 2:

The first two columns of R must be orthonormal to p.

Since $\vec{p} \neq \hat{z}$, we can use:

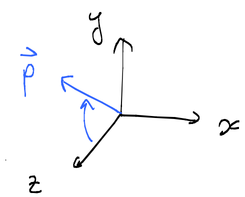
$$R = \begin{bmatrix} p' \times p & \underset{\uparrow \, p'}{\dfrac{p \times z}{|p \times z|}} & \vec{p} \end{bmatrix}$$

We just need to check that the determinant is 1 (not -1).

Recall: Example Problem 1

We have found R such that

$$\vec{p} = R\,\hat{z}.$$

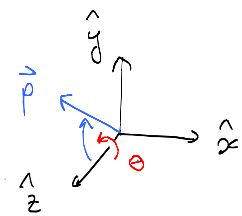


Is R unique?

No, because

$$\vec{p} = \underbrace{R\,R_z(\theta)}\,\hat{z}$$

is another solution for any θ.



Problem Example 2

Find a rotation that maps a unit vector $\vec{p}$ to $\hat{z}$.

(Easy.)

$$\hat{z} = R^T \vec{p}$$

Problem Example 3

Find a rotation matrix that rotates by θ around an axis $\vec{p}$.

Step 1: rotate p to z axis.

Step 2: rotate by θ around $\hat{z}$.

Step 3: rotate z axis to p.

$$R \;\; R_z(\theta) \;\; R^T$$

Problem Example 4

Find a rotation matrix that rotates by θ around an axis ***p*** and that is composed of a sequence of rotations *only* around axes ***x, y, z.***

Example solution: (think this through for yourself)

1. Rotate around x axis to bring p to the xy plane.
2. Rotate around z axis to bring p to the y axis.
3. Rotate by θ around y axis.
4. Apply inverse rotation of 2.
5. Apply inverse rotation of 1.

ASIDE: Representations of rotations

(very important for Computer Animation)

1) Axis-Angle -> OpenGL's glRotate()

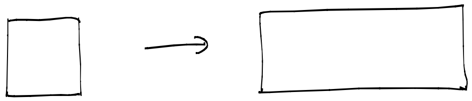
2) Euler angles (Rz Rx Ry)

3) Quaternions

https://www.youtube.com/watch?v=syQnn_xuB8U&list=PL2y2aRaUaygU2zXme_Z11GyJUslwgaeUD

https://www.youtube.com/watch?v=zc8b2Jo7mno

Scaling



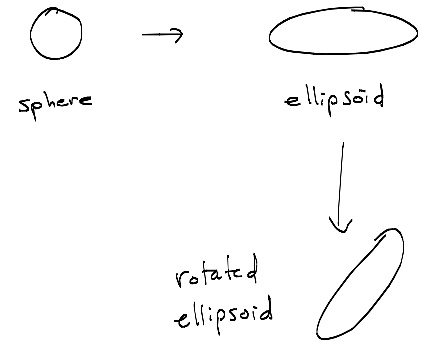
e.g. stretch in x direction

$$(x, y, z) \rightarrow (sx, y, z)$$

Scaling

$$\begin{bmatrix} s_x x \\ s_y y \\ s_z z \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

This transformation stretches $(s > 1)$ or compresses $(s < 1)$ the scene in each of the canonical directions.



Translation by $\vec{t}$

$$T : (x, y, z) \longrightarrow (x + t_x, y + t_y, z + t_z)$$

But this is not a linear transformation.

Why not?

$$\underbrace{T(\vec{u} + \vec{v})}_{\vec{u} + \vec{v} + \vec{t}} \neq \underbrace{T\vec{u} + T\vec{v}}_{\vec{u} + \vec{v} + 2\vec{t}}$$

So we cannot represent T by a 3×3 matrix.

Trick: use a 4th coordinate.

$$\begin{bmatrix} x + t_x \\ y + t_y \\ z + t_z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

This is called a "homogeneous coordinates" representation.

In computer graphics, we always use a 4D representation to transform points.

$$\left[\begin{array}{c|c} R & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 & 0 \end{matrix} & 1 \end{array}\right] \qquad \left[\begin{array}{c|c} S & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 & 0 \end{matrix} & 1 \end{array}\right]$$

rotation scaling

Homogeneous Coordinates

We represent (x,y, z) by (x, y, z, 1).

Now define an equivalence:

$(x, y, z, 1) \equiv (wx, wy, wz, w)$ for any $w \neq 0$.

This takes each line { (wx, wy, wz, w) } in R^4 and associates it with the 3D point (x, y, z).

Careful:

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} + \begin{bmatrix} a' \\ b' \\ c' \\ d' \end{bmatrix} \not\equiv \begin{bmatrix} a + a' \\ b + b' \\ c + c' \\ d + d' \end{bmatrix}$$

The above is an abuse of notation.
It is meant to express that:

$$\begin{bmatrix} a/d \\ b/d \\ c/d \end{bmatrix} + \begin{bmatrix} a'/d' \\ b'/d' \\ c'/d' \end{bmatrix} \neq \begin{bmatrix} (a+a')/(d+d') \\ (b+b')/(d+d') \\ (c+c')/(d+d') \end{bmatrix}$$

Points at infinity

Take (x,y,z) and consider

$$\lim_{s \to \infty} (sx, sy, sz).$$

This can be expressed using homogeneous coordinates:

$$(sx, sy, sz, 1) \equiv \left(x, y, z, \frac{1}{s}\right)$$

Letting $s \to \infty$ gives $(x, y, z, 0)$.

Called a "point at infinity" (or "direction vector")

How do points at infinity behave under:

- rotation
- translation
- scaling

?

$$\begin{bmatrix} R\begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ 0 \end{bmatrix} = \left[\begin{array}{ccc|c} & R & & 0 \\ & & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} S_x x \\ S_y y \\ S_z z \\ 0 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix}$$

Scaling stretches / compresses the axes.

What does (x, y, z, ε) represent

as $\varepsilon \to 0$ from positive side

versus $\varepsilon \to 0$ from negative side?