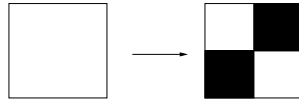


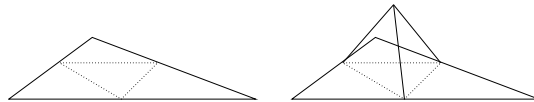
Questions

1. Consider an object that is defined recursively as follows. Begin with a unit square; partition this square into four equal subsquares; delete the two off-diagonal subsquares, as shown below. Repeat recursively on the remaining subsquares.



Sketch the object that one obtains after the third iteration (the first iteration is shown above), and calculate the fractal dimension D of the object that one obtains in the limit.

2. Consider a Koch-like surface defined by the recursive production rule illustrated in the figure. A equilateral triangle is partitioned into 4 subtriangles, and the central one is replaced by a 3 sided pyramid. Repeating this recursive procedure infinitely many times, what is the dimension of the resulting surface ?



3. This is for students who some basic background in statistics. Consider a sum of n independent identically distributed random variables

$$Y = \sum_{i=1}^n X_i$$

where each X_i has mean 0 and variance σ^2 . Let this sum Y be the position of a drunken sailor after n steps. Using the fact that the variance of a sum of independent random variables is equal to the sum of the variances, it is easy to show that the variance of Y is $n\sigma^2$.

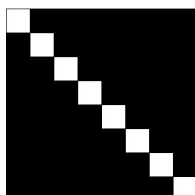
Now consider another drunken sailor that takes twice as many steps ($2n$ instead of n) but the standard deviation of each step is only half as big, namely $\frac{\sigma}{2}$ instead of σ . What is the variance of the position of this sailor after the $2n$ steps? (Before doing the calculation, ask yourself if you think the variance the same, greater, or less than that of the first sailor.)

4. Following the previous question, what should be the standard deviation of the second sailor's steps, so that the variance of position after $2n$ steps is the same as the variance of the first sailor's position after n steps?

Solutions

1.

In this case, $S = 2$ (we're scaling x, y by a factor of $\frac{1}{2}$) and $C = 2$ (we're getting twice as many squares). So when we plug this into the formula we get: $D = \frac{\log C}{\log S} = \frac{\log 2}{\log 2} = 1$. This is just the dimensionality of a line which is what we get in the limit.



2. $C = 6$, $S = 2$, hence $D = \frac{\log C}{\log S} \approx 2.58$

3. For independent random variables, the variance of the sum is equal to the sum of the variance. For the second drunken sailor, the variance of each step is $(\frac{\sigma}{2})^2$ and there are $2n$ steps, so the variance of the sum of the $2n$ steps is $2n(\frac{\sigma}{2})^2$ or $\frac{n}{2}\sigma^2$. This variance is *half* than that of the original drunken sailor.

4. For the second drunken sailor, suppose the standard deviation is $r\sigma$ where r (roughness) is a scale factor. Then the variance of each step is $(r\sigma)^2$ and there are $2n$ steps, so the variance of the sum of the $2n$ steps is $2n(r\sigma)^2$ or $2nr^2\sigma^2$. This variance is $n\sigma^2$ when $r = \frac{1}{\sqrt{2}}$. This is essentially the reason why the midpoint displacement algorithm given in class uses a roughness parameter of $\frac{1}{\sqrt{2}}$.