Lecture 15 (14 was the midterm)

- midterm exam:
   solutions, common mistakes

- linear algebra review:

   - rotations versus change of coordinates

open book exams

vs.

closed book exams

Q1 Consider a 2D space (x,y) whose points are represented using homogeneous coordinates. Give a product of matrices that performs the following. Rotate the scene by theta degrees clockwise around the point (x,y) = (2,3). That is, the given point stays fixed and all other points are moved.
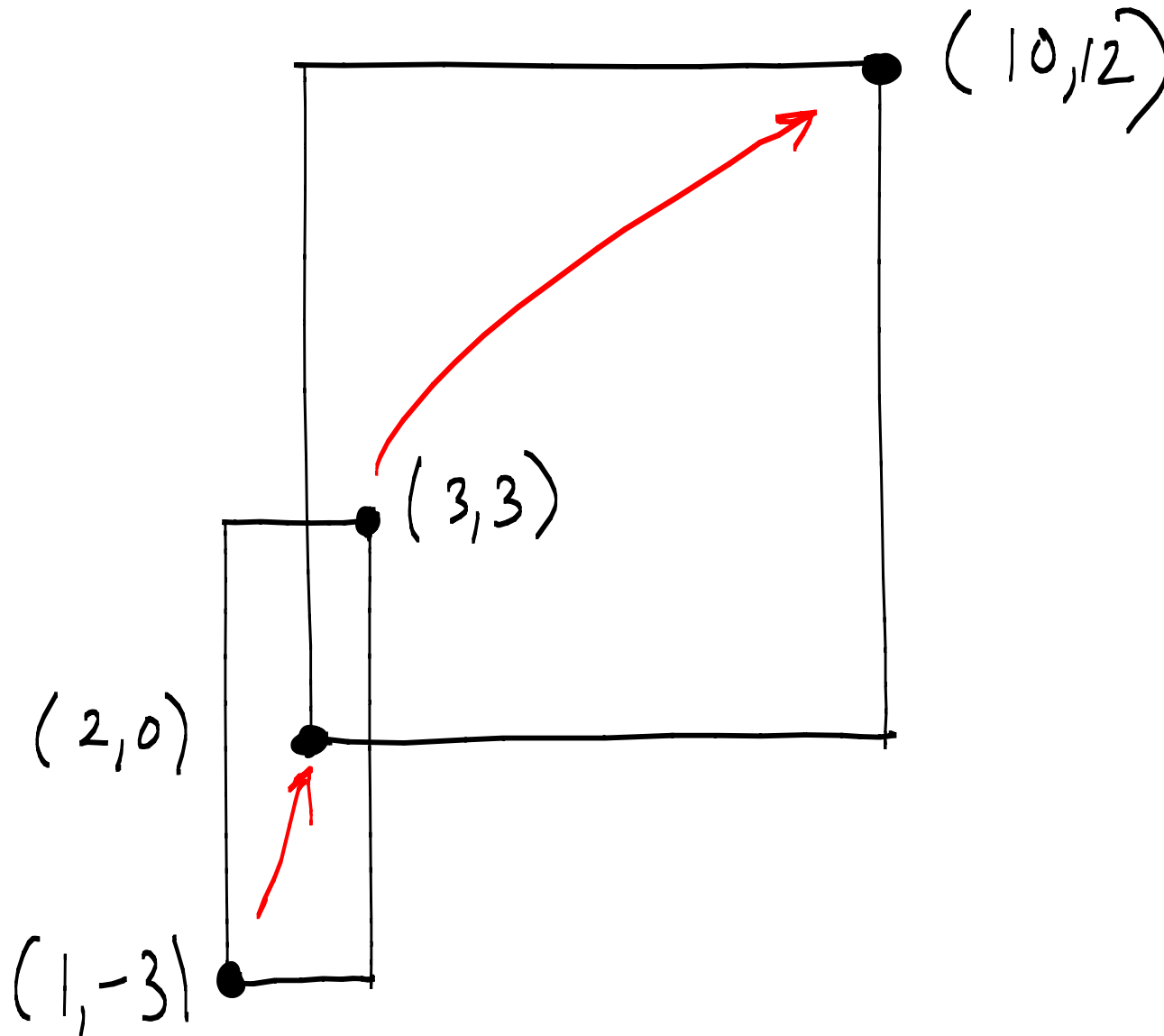
$$y \qquad \qquad (2, 3)$$

$$x$$
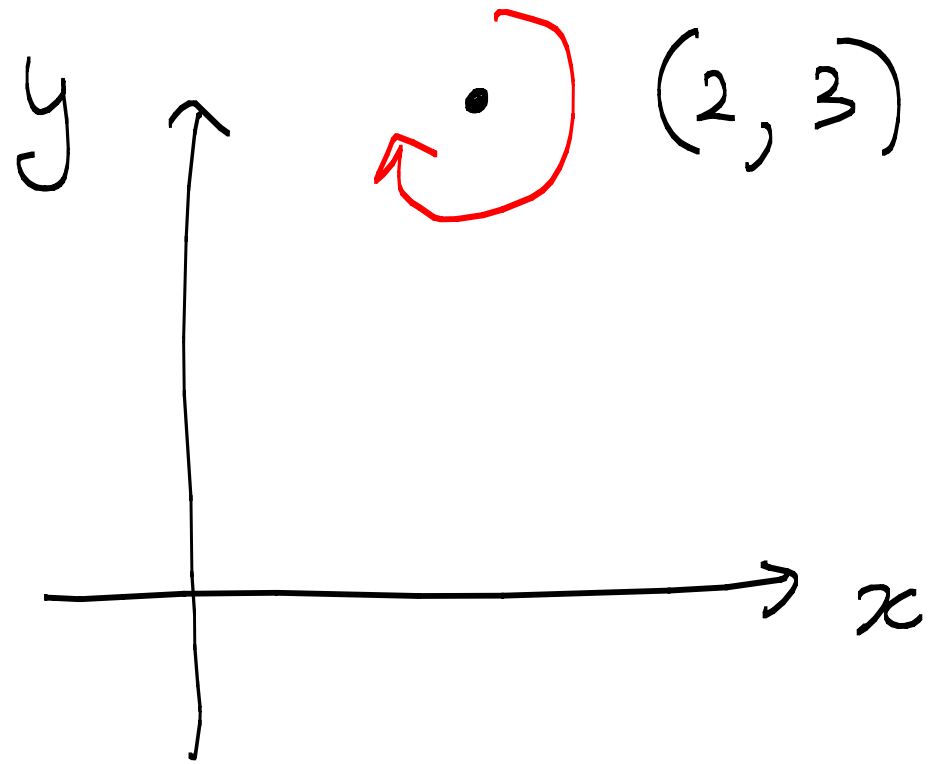
Q1 solution

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

Common mistakes:

- Only one translation, or switching ordering.
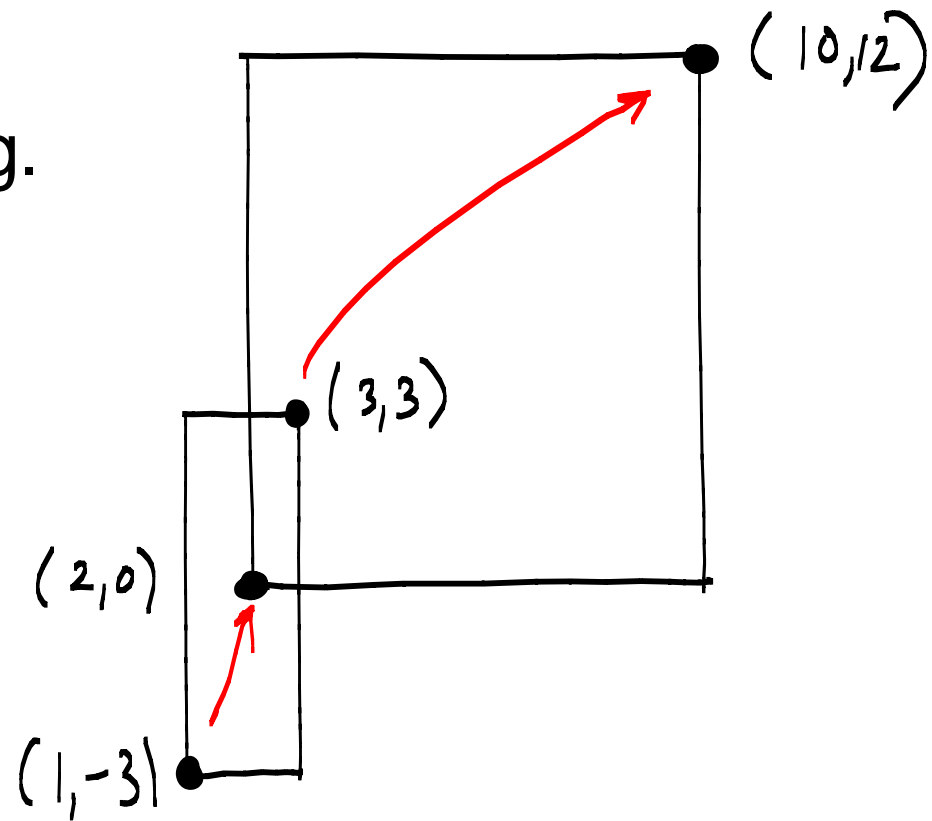- Signs on the sines (no penalty)

Q2 Consider the same situation as Q1, but now give a product of matrices that maps a rectangle having opposite corners (1,-3) and (3,3) to a rectangle having opposite corners (2,0) and (10,12).

Q2 solution (4)

All involve scaling.

Scale x by $\dfrac{10-2}{3-1} = \dfrac{8}{2} = 4$

Scale y by $\dfrac{12-0}{3-(-3)} = \dfrac{12}{6} = 2$
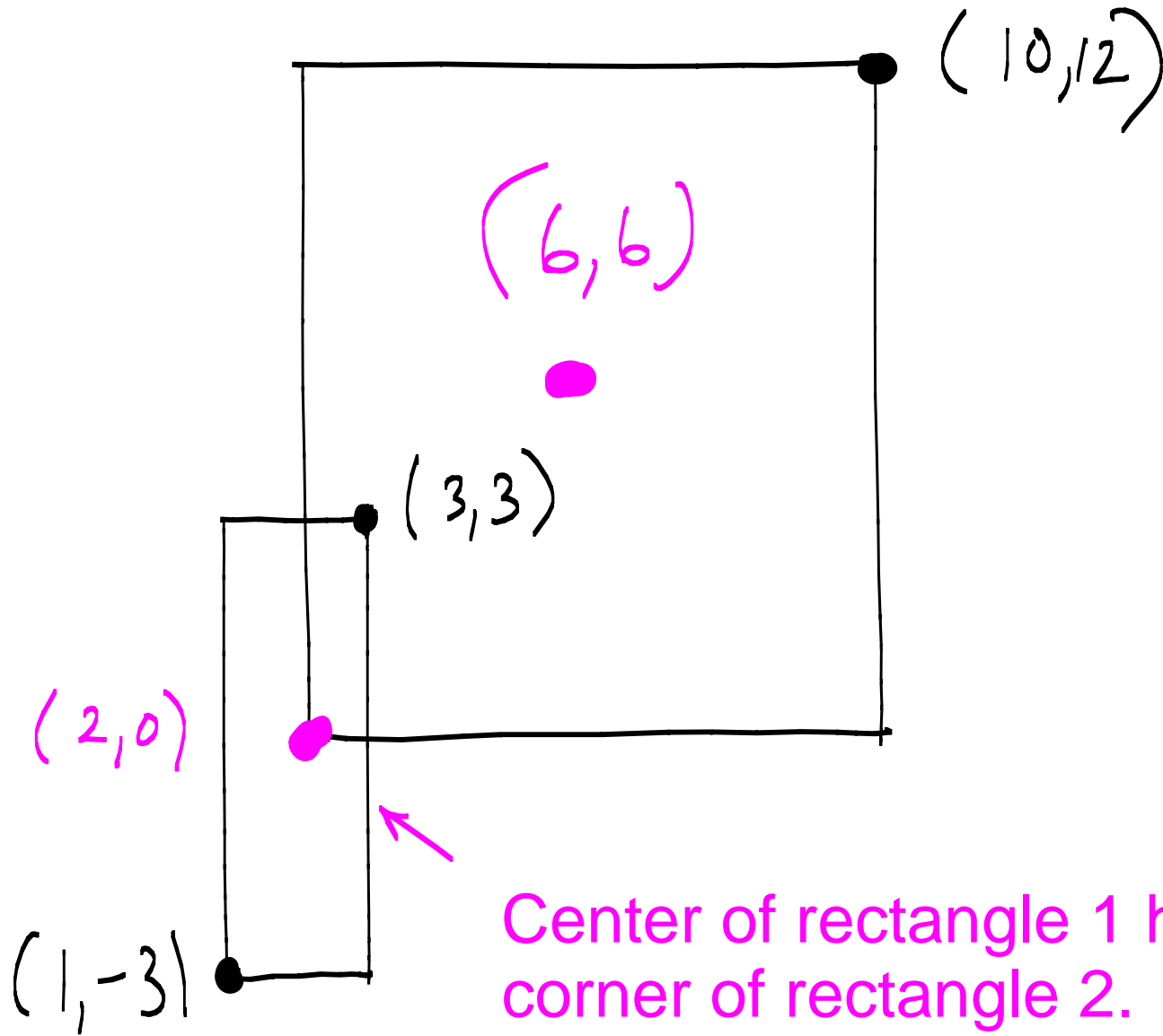
Q2 solution #1

Translate bottom left corner (1,-3) to origin.
Scale.
Translate origin to (2, 0)

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Q2 solution #2: Work with centers of the rectangles.

$(10,12)$

$(6,6)$

$(3,3)$

$(2,0)$

$(1,-3)$

Center of rectangle 1 happens to be a corner of rectangle 2. (Accident)

Q2 solution # 2

Translate center of rectangle 1 to origin.
Scale.
Translate origin to center of rectangle 2.

$$\begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Q2 solution # 3

Scale.
Translate (new) bottom left corner of rectangle 1 to bottom left corner of rectangle 2.
[Alternatively work with top right corner.]

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Q2 solution # 4 (not recommeded, but some did it)

Translate. How much ?
Scale.

$$\vec{x}_{\text{bottom left}} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -0.5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \vec{x}_{\text{bottom left}}$$

2

bottom left

1

solve for this

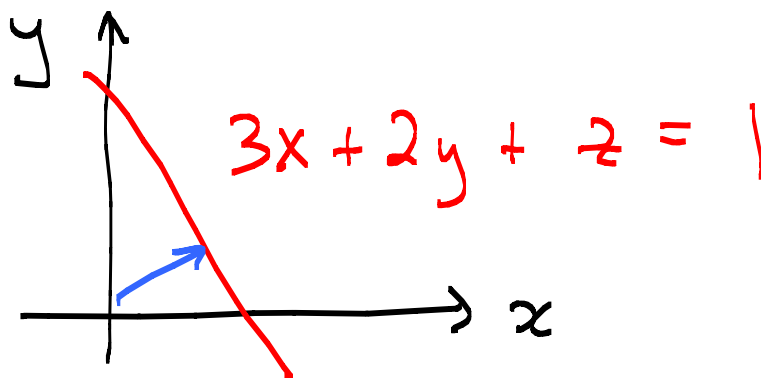multiply by inverse of scaling matrix

Q3

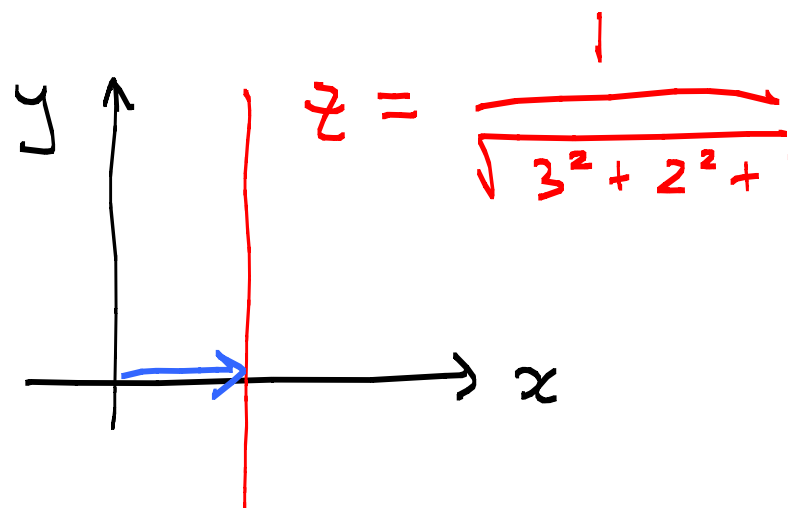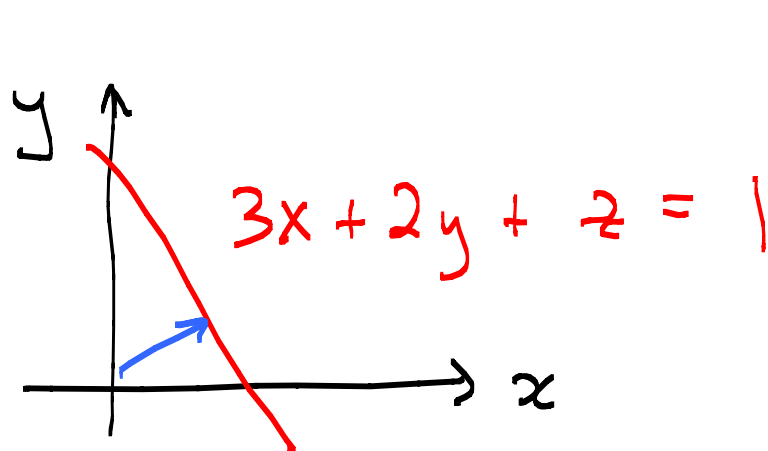3. Describe how you would obtain a 4×4 projection matrix that maps $\Re^3$ to the plane

$$3x + 2y + z = 1.$$

Assume the center of projection *i.e.* eye is $(0,0,0)$.

You do not need to provide numerical values for the matrix (or matrices), but be sure to specify how you *would* obtain such values.

Q3 solution # 1 (what I had in mind)

$3x + 2y + z = 1$

$$z = \frac{1}{\sqrt{3^2 + 2^2 + 1}}$$

Apply a rotation to bring plane normal (3, 2, 1) to z axis.
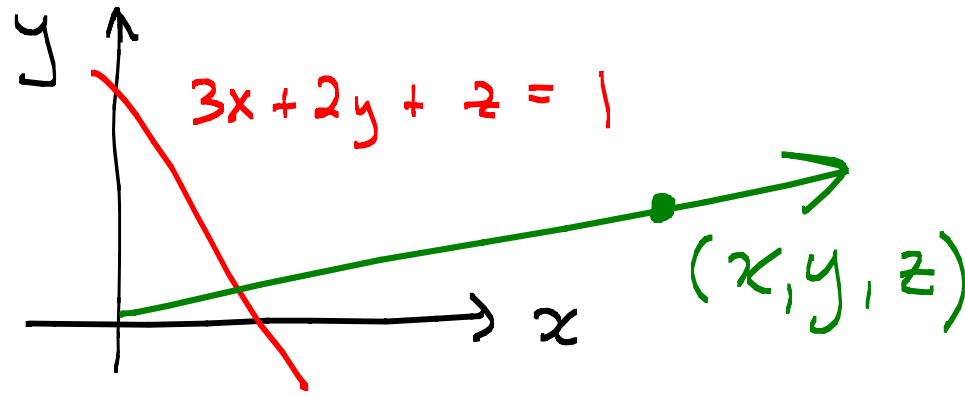
Apply perspective projection to the plane

$$z = \frac{1}{\sqrt{3^2 + 2^2 + 1}}$$

Rotate back.

So, $R^T \, M_{\text{projection}} \, R$
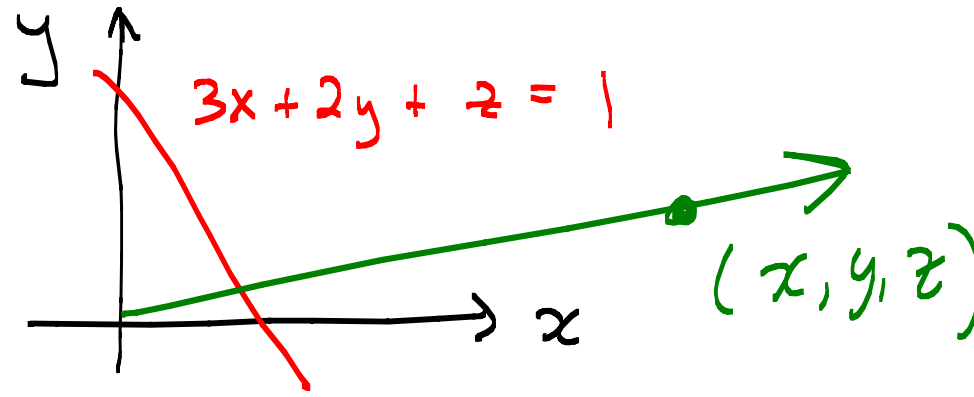
Q3 solution #2 (several students did this)

y

$3x + 2y + z = 1$

(x, y, z)

x

Take ray, p(t) = (0,0,0) + t (x, y, z), and compute where it intersects the given plane.

Then what ? (Question asks for a matrix representation.)

Q3 solution #2   (several students did this)

Take ray, p(t) = (0,0,0) + t (x, y, z), and compute where it intersects the given plane.

Plugging in: $3 (t x) + 2 (t y) + 1 (t z) = 1.$

Thus, $t = 1/ (3x + 2y + z).$ Thus, point of intersection is

$(x / (3x + 2y + z), y / (3x + 2y + z), z / (3x + 2y + z)).$

Write the point of intersection in homogeneous coords:

$(x\ /\ (3x + 2y + z),\ \ y\ /\ (3x + 2y + z),\ \ z\ /\ (3x + 2y + z),\ \ 1\ )$

$\equiv$ $(x,\ \ y,\ \ z,\ \ 3x + 2y + z\ )$

Thus,

$$
\begin{bmatrix} x \\ y \\ z \\ 3x + 2y + z \end{bmatrix}
=
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}
\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}
$$

More general: Project onto plane ax + by + cz = 1.

$$
\begin{bmatrix} x \\ y \\ z \\ ax + by + cz \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & b & c & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}
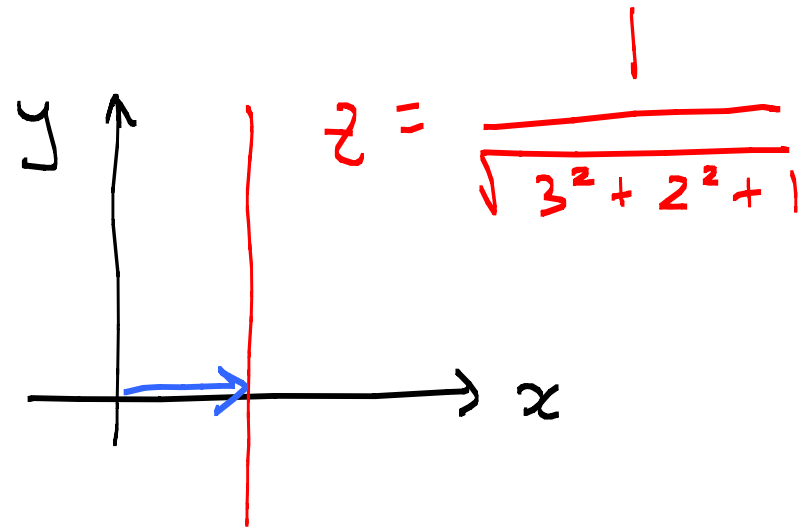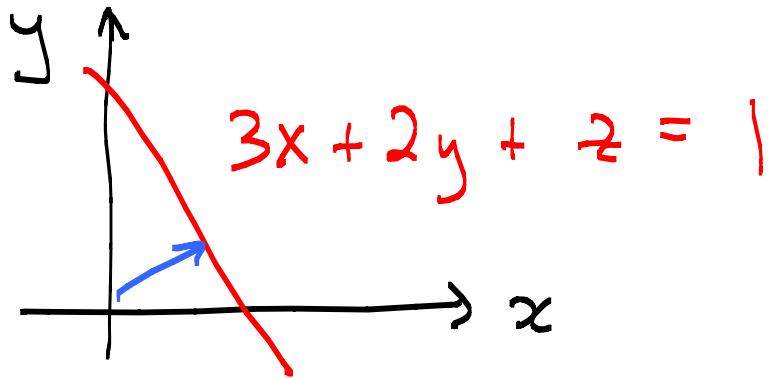$$

The example in the lecture was (a, b, c) = (0, 0, 1/f).

Let's verify that the above two solutions are identical, *and at the same time review some basic linear algebra that some of you seem to be rusty on.*

Problem: Project (x, y, z) onto plane $ax + by + cz = 1$

and let center of projection (eye) be $(0, 0, 0)$.

Solution 1 (from 4 slides ago): $R \; M_{projection} \; R^T$

What is matrix R ?

y

$3x + 2y + z = 1$

x

y

$z = \dfrac{1}{\sqrt{3^2 + 2^2 + 1}}$

x

R rotates the plane's unit normal

$$n = (a, b, c) \Big/ \sqrt{a^2 + b^2 + c^2}$$

to the z axis.

R rotates the plane's unit normal

$$n = (a, b, c) \ / \ \sqrt{a^2 + b^2 + c^2}$$

to the z axis.

Upper left 3x3 submatrix of R has orthonormal rows.

$$
\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \longleftrightarrow & & & 0 \\ \longleftrightarrow & & & 0 \\ \longleftarrow n \longrightarrow & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \\ 1 \end{bmatrix}
$$

$$R_{4 \times 4}$$

Solution 1: $\qquad R^T \quad M_{projection} \quad R$

$$\frac{1}{f} = \sqrt{a^2 + b^2 + c^2}$$

Does this give the same matrix as solution 2 ?

Claim: (verify for yourself, see next few slides)

$$\begin{bmatrix} \uparrow & \uparrow & \uparrow & 0 \\ & & & 0 \\ \downarrow & \downarrow & n & 0 \\ & \downarrow & \downarrow & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & * & 1 \end{bmatrix} \begin{bmatrix} \longleftrightarrow & & & 0 \\ \longleftarrow & & & a \\ \longleftarrow & n & \longrightarrow & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sqrt{a^2 + b^2 + c^2}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & b & c & 0 \end{bmatrix}$$

ASIDE (basic fact from linear algebra)

If the <u>rows</u> of a 3x3 matrix R are <u>orthonormal</u>, then:

$$R^T_{3\times 3} \; R_{3\times 3} = I$$

This is not obvious !   How to prove it ?

Use the following:

If the <u>rows</u> of a 3x3 matrix R are <u>orthonormal</u>, then:

$$\begin{bmatrix} R \\ \xleftarrow{\quad} n \xrightarrow{\quad} \end{bmatrix} \begin{bmatrix} R^T \\ n \end{bmatrix} = I$$

This *should be* obvious, if you understand what it means for vectors to be orthonormal and you understand how matrix multiplication works.

Let's use the above to prove the claim on the previous slide.

Proof:
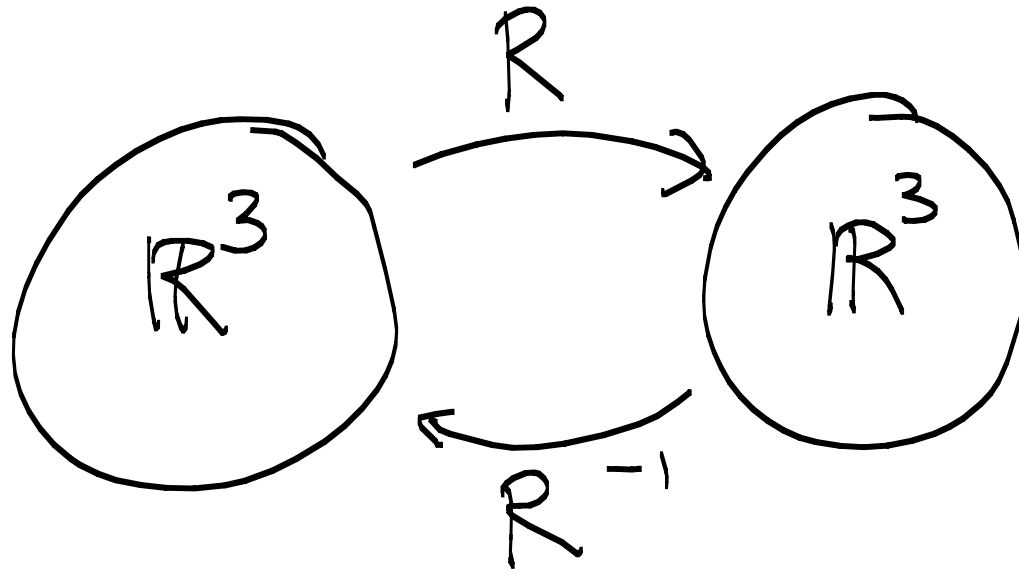
rows of R are orthonormal

$\Rightarrow$

$$R R^T = I$$

$\Rightarrow\Rightarrow$

$$R^T = R^{-1}$$

$\Rightarrow\Rightarrow\Rightarrow$

$$R^T R = I$$

$\Rightarrow$ columns of R are orthonormal

4. Consider the projective transformation:

$$\begin{bmatrix} f_0 & 0 & 0 & 0 \\ 0 & f_0 & 0 & 0 \\ 0 & 0 & f_0 + f_1 & -f_0 f_1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

(a) Which points in $\Re^3$ get mapped to points at infinity?

(b) Which points at infinity get mapped to points in $\Re^3$?

Q4   solution :

(a)   Which points in R^3 map to infinity ?

For a general scene point, we have:

$$\begin{bmatrix} f_0 & 0 & 0 & 0 \\ 0 & f_0 & 0 & 0 \\ 0 & 0 & f_0 + f_1 & -f_0 f_1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} f_0 x \\ f_0 y \\ (f_0 + f_1)z - f_0 f_1 \\ z \end{bmatrix}$$

Thus,   points for which z = 0   get mapped to infinity.

Q4 (b) solution and common mistake

Which points at infinity map to R^3 ?

$$\begin{bmatrix} f_0 & 0 & 0 & 0 \\ 0 & f_0 & 0 & 0 \\ 0 & 0 & f_0+f_1 & -f_0f_1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix} = \begin{bmatrix} f_0 x \\ f_0 y \\ (f_0+f_1)z \\ z \end{bmatrix}$$
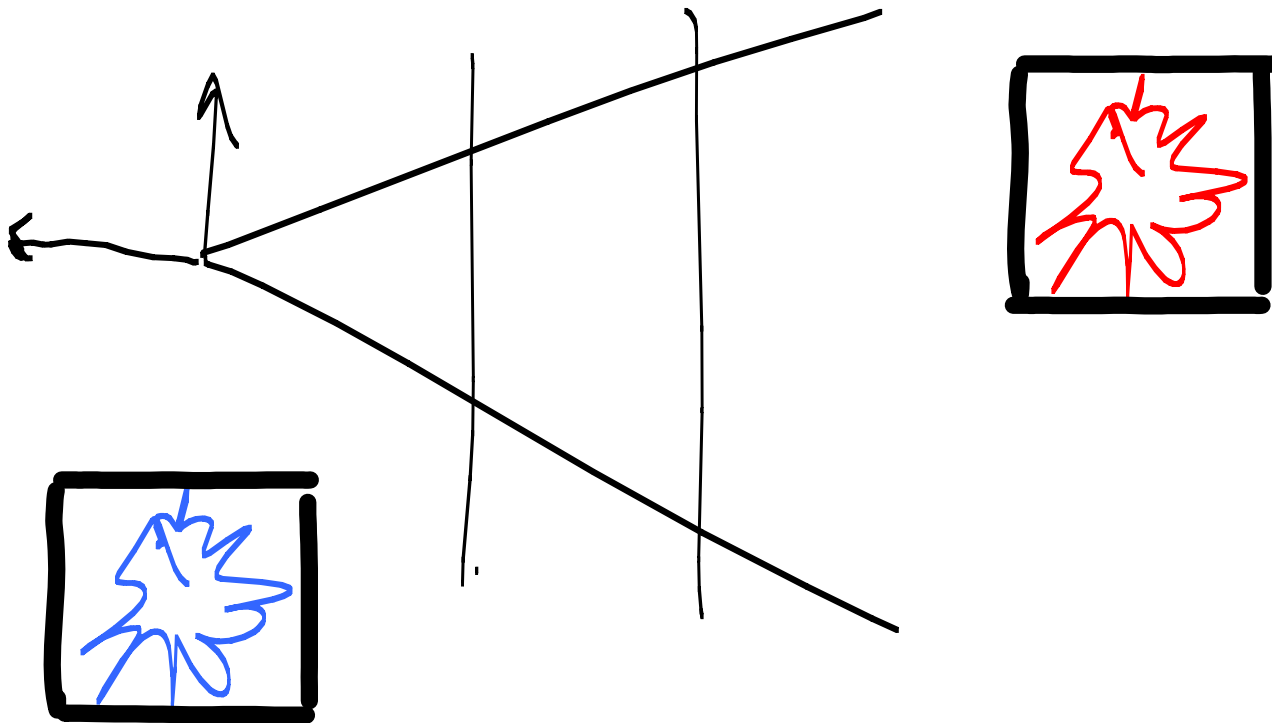
@ ∞: $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

@ ∞: $\begin{bmatrix} f_0 x \\ 0 \\ 0 \\ 0 \end{bmatrix}$

All points at infinity get mapped to finite points
(except if z = 0)

Q5

How could bounding volumes be used to speed up clipping?

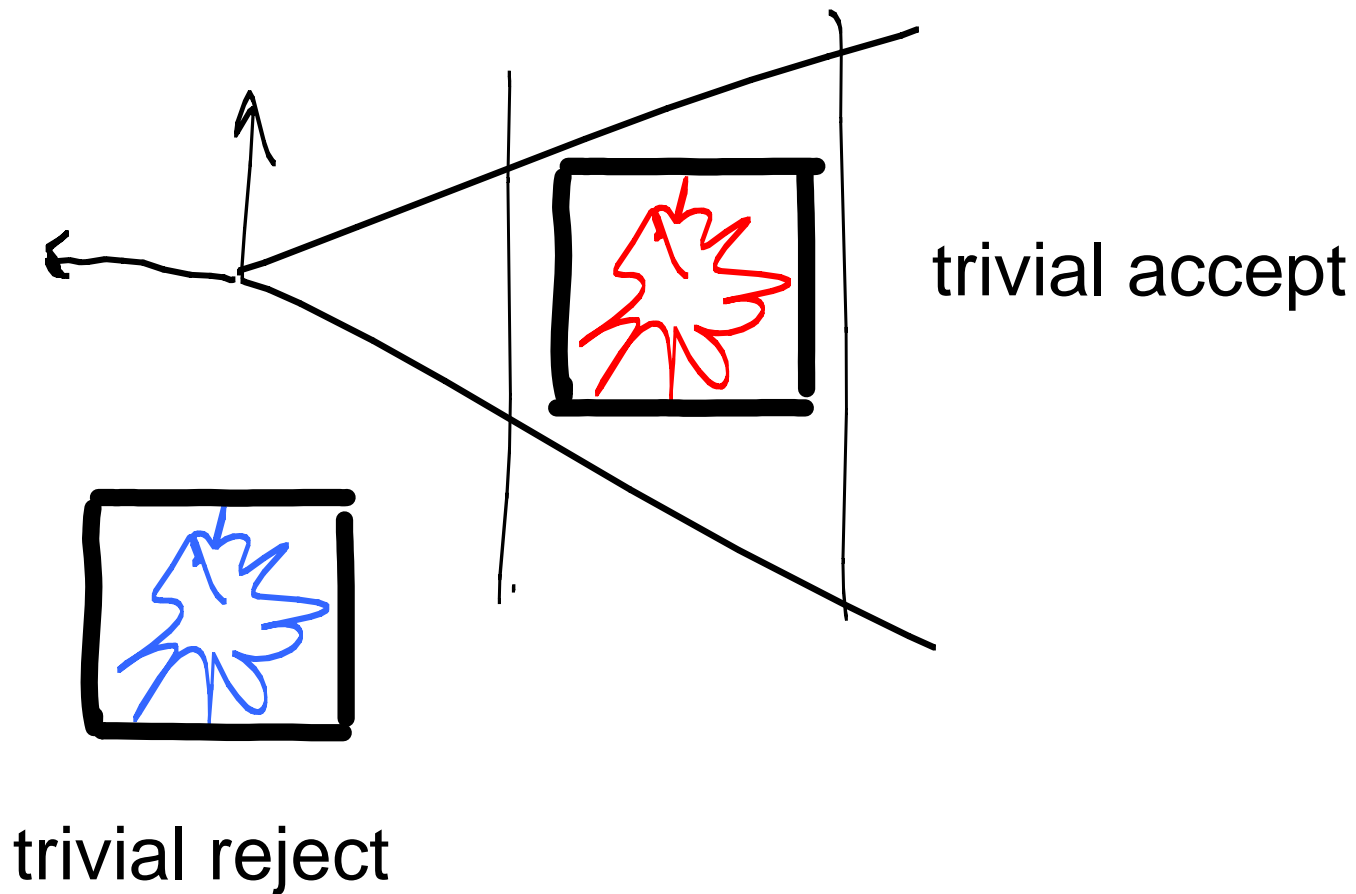At what stage of the pipeline should this clipping occur? Justify your answer.

Q5 (solution)

Common mistake: " ... cast a ray ..... "

In lecture 8, I discussed how bounding volumes can be used in ray casting.   But the question is not asking about ray casting.   Rather it is about clipping.

Q5a   How can BV be used for clipping?
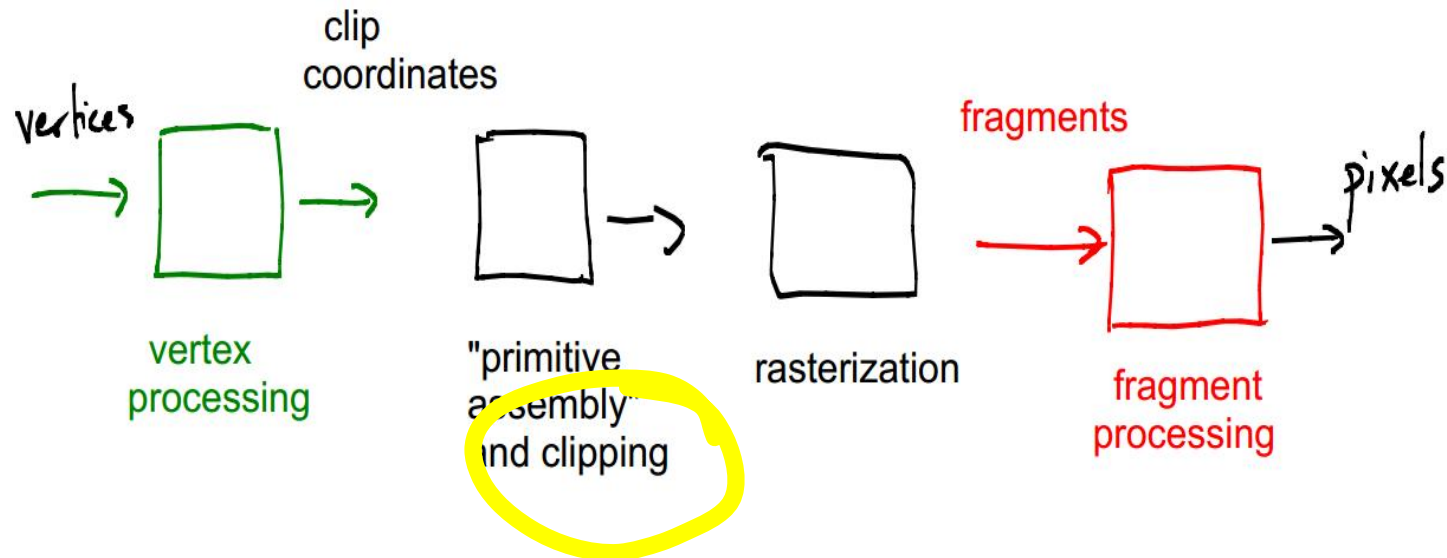
A:

trivial accept

trivial reject

Q5b : Where should clipping occur in pipeline? Justify your answer.

Common answers that received 0 (if not enough justification) :

"... at the clipping stage..."

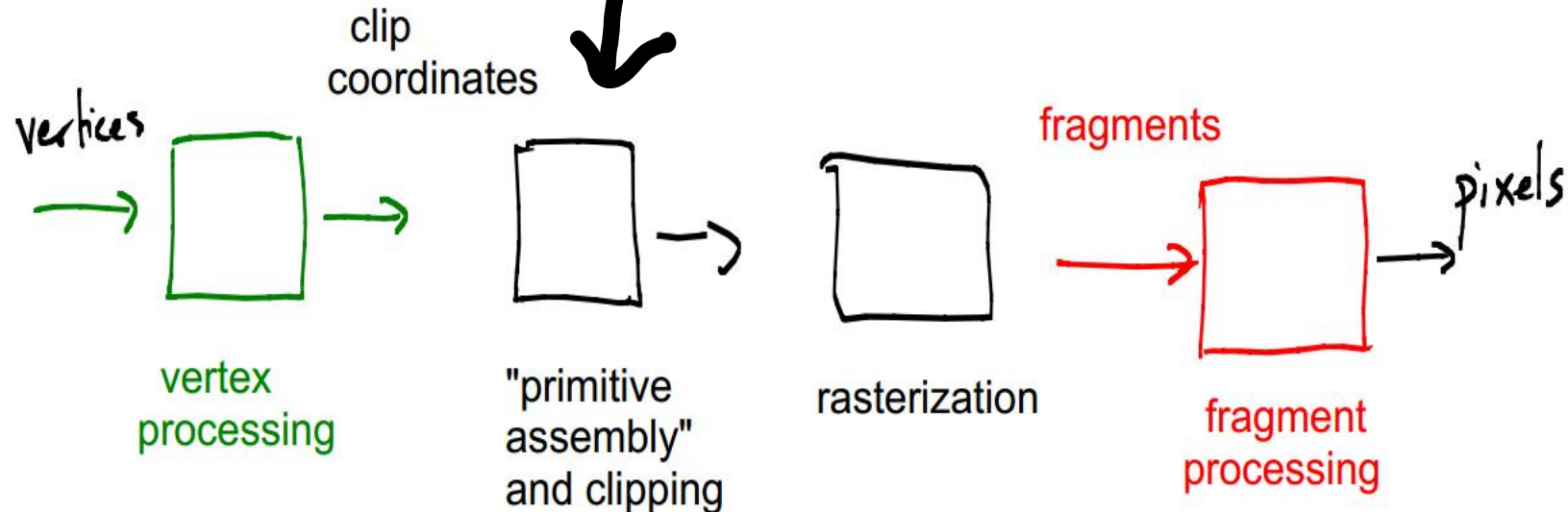"... after the vertex processing..."

"... before rasterization... "

vertices

clip
coordinates

fragments

pixels

vertex
processing

"primitive
assembly
and clipping

rasterization

fragment
processing

We were looking for one of two answers:

Trivial rejects can be done <u>before the vertex stage </u>(on the CPU).

Trivial accepts can be done in the pipeline <u>at the "clipping" stage</u>: If BV is trivially accepted, then all surfaces within BV can be trivially accepted too.   Otherwise, test surfaces (or sub-BV's).

clip
coordinates

fragments

vertices

pixels

vertex
processing

"primitive
assembly"
and clipping

rasterization

fragment
processing

Q6:

Claim: "If a scene contains quadric surfaces, then the depth buffer method (hidden surface removal) can only be applied if these quadric surfaces are first discretized into polygons." Is this claim true or false? Briefly explain.

Common mistake: "True because .... "

```
for each polygon
    for each pixel in the image projection of the polygon
        z := Ax + By + C
            // equation of polygon's plane *in screen coordinates*
        if z < z(x,y)
            compute RGB(x,y)
            z(x,y) := z
```

Q6 (solution):

object (quadric, polygon, ...)

for each polygon

    for each pixel in the image projection of the polygon

       z := Ax + By + C   compute z(x,y)

       // equation of polygon's plane *in screen coordinates*

   if z < z(x,y)

          compute RGB(x,y)

          z(x,y) := z

It is a bit more difficult to find the image projection of a quadric,   but not impossible .   e.g. I showed in lecture 7 how to check if a ray intersects a quadric.

Q7: Construct a BSP tree for a 2D scene below. For each subtree, choose the edge from the list in alphabetical order. In particular, the root of the tree is edge a.

What is the order of edges drawn for a viewer that is located between edges and b ? (Ignore back face culling.)
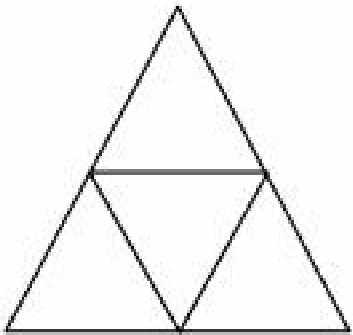
Q7 (solution)

c   b   a   e   d
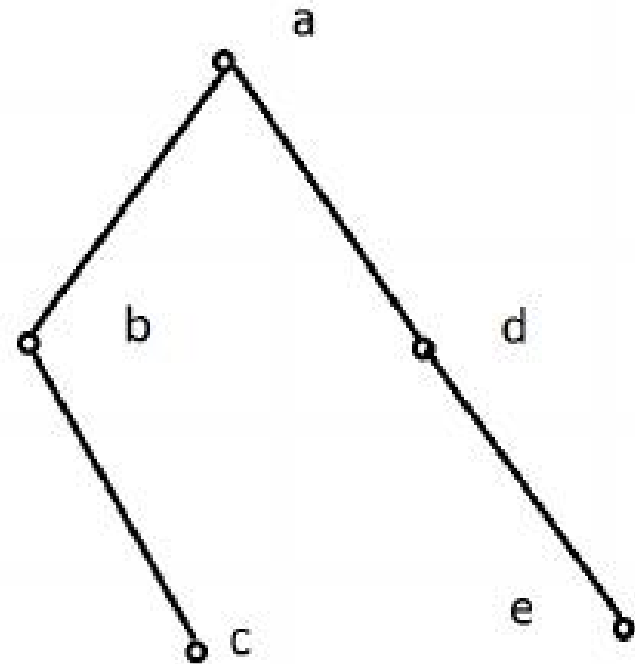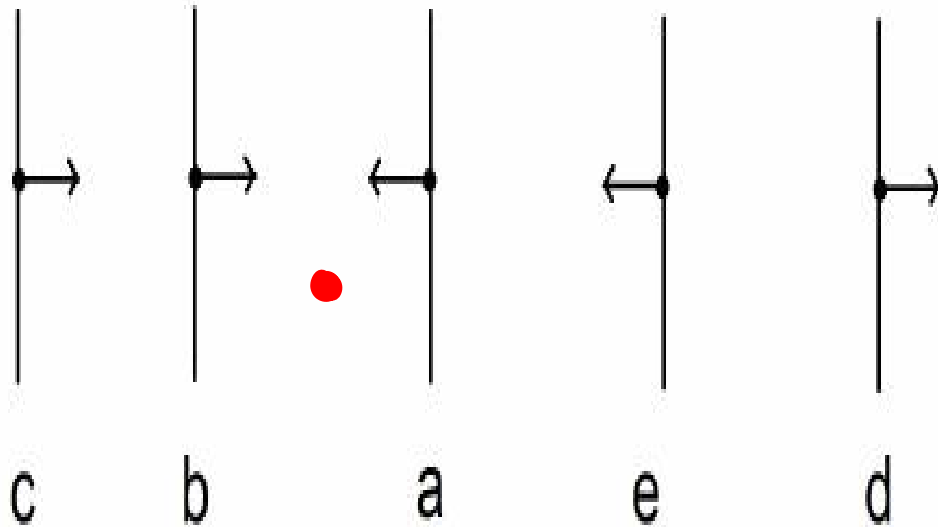
Q8:

Consider an equilateral triangle. Partition it into four equilateral subtriangles as sketched below.

Then delete the central subtriangle leaving three subtriangles. Repeating this recursively, infinitely many times, gives a fractal.

a) Show that the surface area of this fractal is 0.
b) Give an expression for the dimension of this fractal.
   Hint: C = S^D

Solution:

a) (3/4)^n goes to 0 as n goes to infinity.
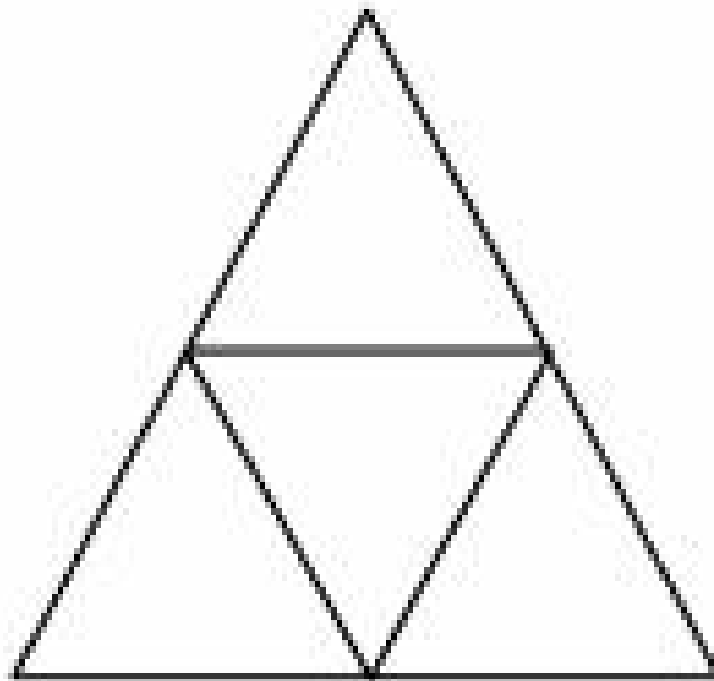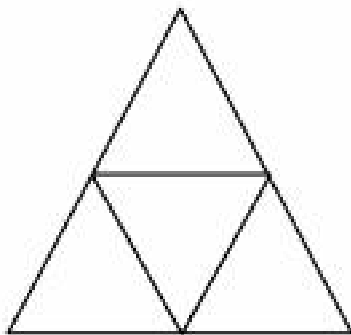
b) S = 2, C = 3

Q9: Suppose you would like to fit a cubic curve p(t) = (x(t), y(t), z(t)) to two given 3D points p(0) and p(1) and suppose p'(1) and p"(1) are also given.

Show how to find the coefficients of this cubic curve.

SOLUTION:

$$\mathbf{p}(t) = \quad = \begin{bmatrix} a_x & b_x & c_x & d_x \\ a_y & b_y & c_y & d_y \\ a_z & b_z & c_z & d_z \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

Taking derivatives with respect to t and substituting $t = 0$ and $t = 1$ gives:

$$[\,\mathbf{p}(0) \quad \mathbf{p}(1) \quad \mathbf{p}'(1) \quad \mathbf{p}''(1)\,] = \quad = \begin{bmatrix} a_x & b_x & c_x & d_x \\ a_y & b_y & c_y & d_y \\ a_z & b_z & c_z & d_z \end{bmatrix} \begin{bmatrix} 0 & 1 & 3 & 6 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Some students wrote:

$$x(t) = a\,t^3 + b\,t^2 + c\,t + d$$

$$x'(t) = 3\,a\,t^2 + 2\,b\,t + c$$

$$x''(t) = 6\,a\,t + 2\,b$$

Substituting $t = 0$ and 1 gives:

$$x(0) = d$$

$$x(1) = a + b + c + d$$
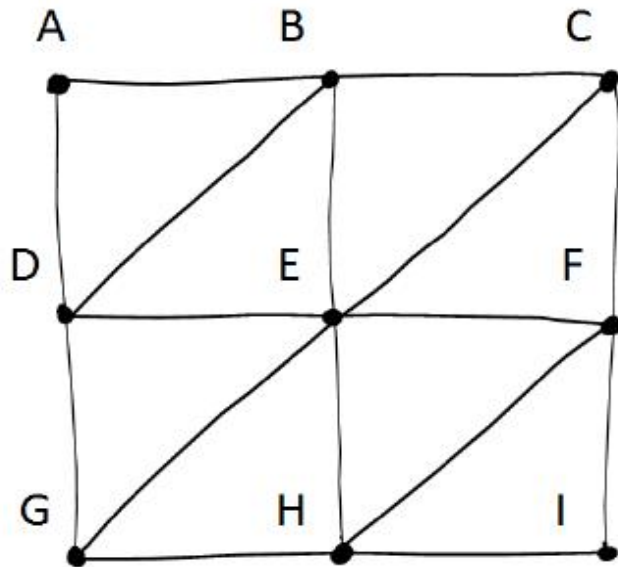
$$x'(1) = 3\,a + 2\,b + c$$

$$x''(1) = 6\,a + 2\,b$$

and solve for a, b, c, d.
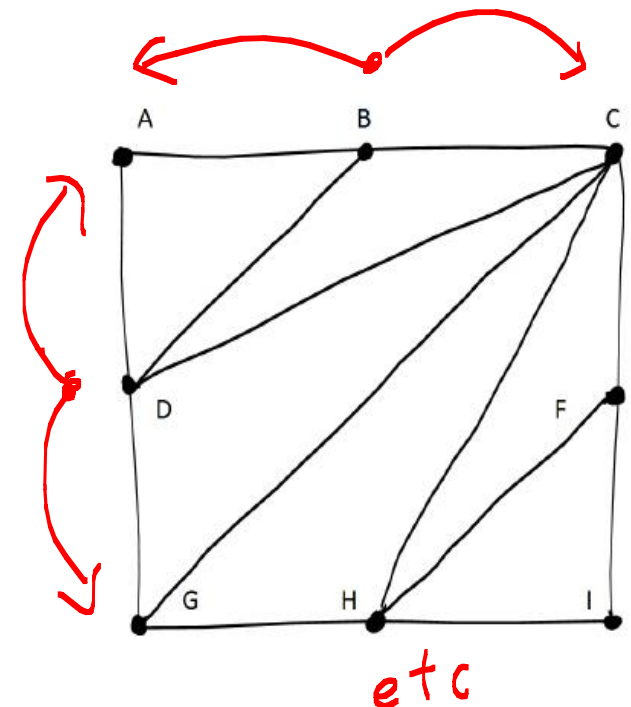
This is fine. It is essentially the same solution as above.

Q10:

Simplify the mesh below by collapsing the edge EC onto vertex C. Draw the simplified mesh.

Is it possible to have more edge collapses and also to maintain the overall square shape?   If so, which edges can be collapsed (and onto which vertices) ?

Solution:

etc

Lecture 15 (14 was the midterm)

- midterm exam:
    solutions, common mistakes

- linear algebra review:

    - rotations versus change of coordinates

Rotation matrices

$$\begin{bmatrix} \leftarrow\; u \;\rightarrow \\ \leftarrow\; v \;\rightarrow \\ \leftarrow\; w \;\rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ u & v & w \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R^T \qquad R \;=\; I$$

How to interpret ?

$$R = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ u & v & w \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$$
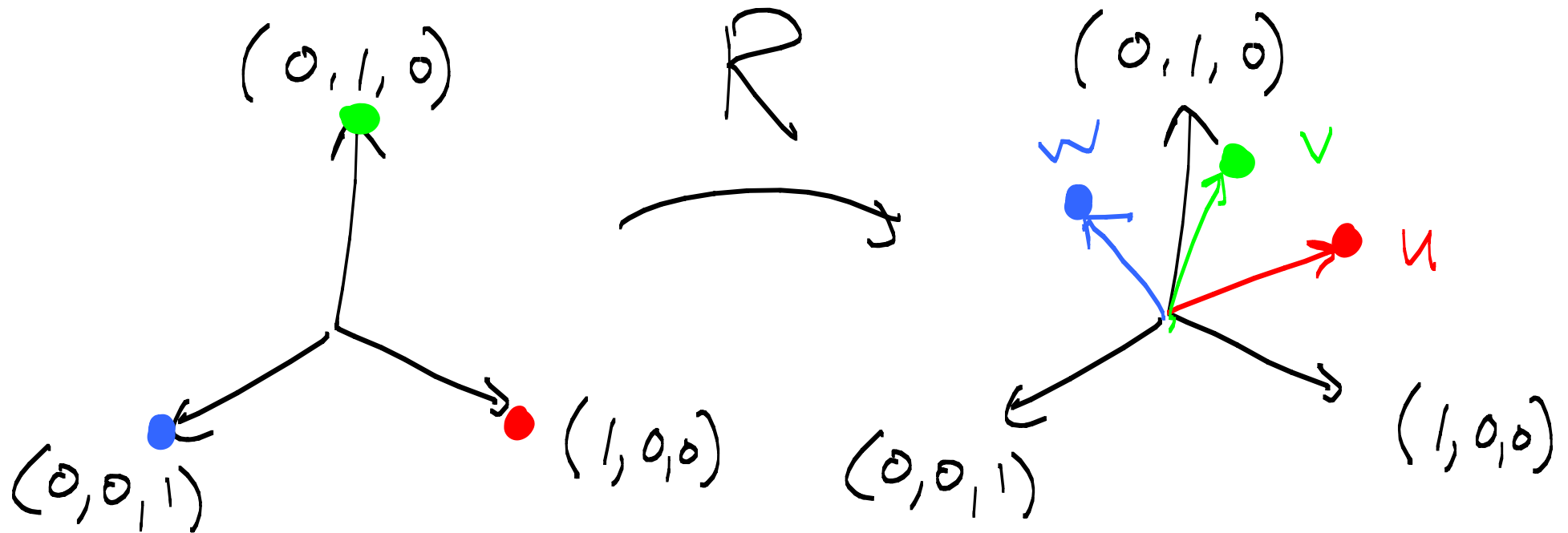
How does it map:
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \; ?$$

$$\begin{bmatrix} \color{red}{\uparrow \mathbf{u} \downarrow} & \color{green}{\uparrow \mathbf{v} \downarrow} & \color{blue}{\uparrow \mathbf{w} \downarrow} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \color{red}{\mathbf{u}}$$

$$\begin{bmatrix} \color{red}{\uparrow \mathbf{u} \downarrow} & \color{green}{\uparrow \mathbf{v} \downarrow} & \color{blue}{\uparrow \mathbf{w} \downarrow} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \color{green}{\mathbf{v}}$$

$$\begin{bmatrix} \color{red}{\uparrow \mathbf{u} \downarrow} & \color{green}{\uparrow \mathbf{v} \downarrow} & \color{blue}{\uparrow \mathbf{w} \downarrow} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \color{blue}{\mathbf{w}}$$

$(0, 1, 0)$ R $(0, 1, 0)$

$(0, 0, 1)$ $(1, 0, 0)$ $(0, 0, 1)$ $(1, 0, 0)$
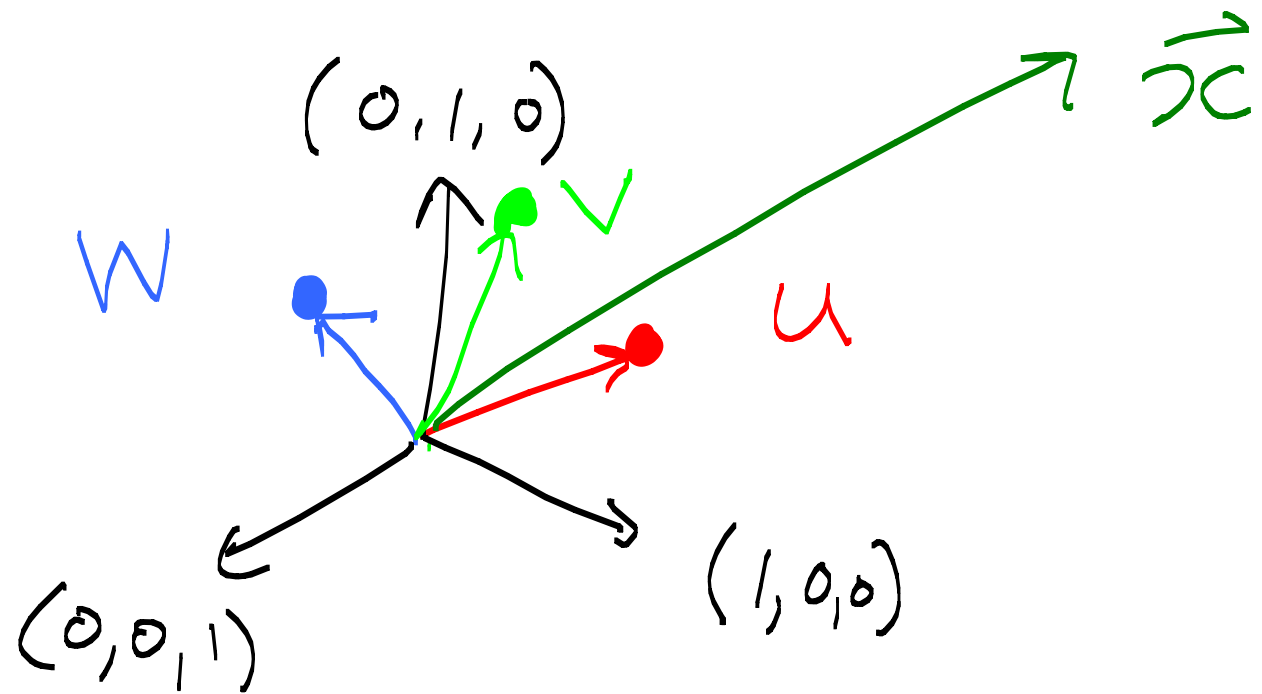
w v u

In this interpretation, we rotate R^3 in some fixed coordinate system.

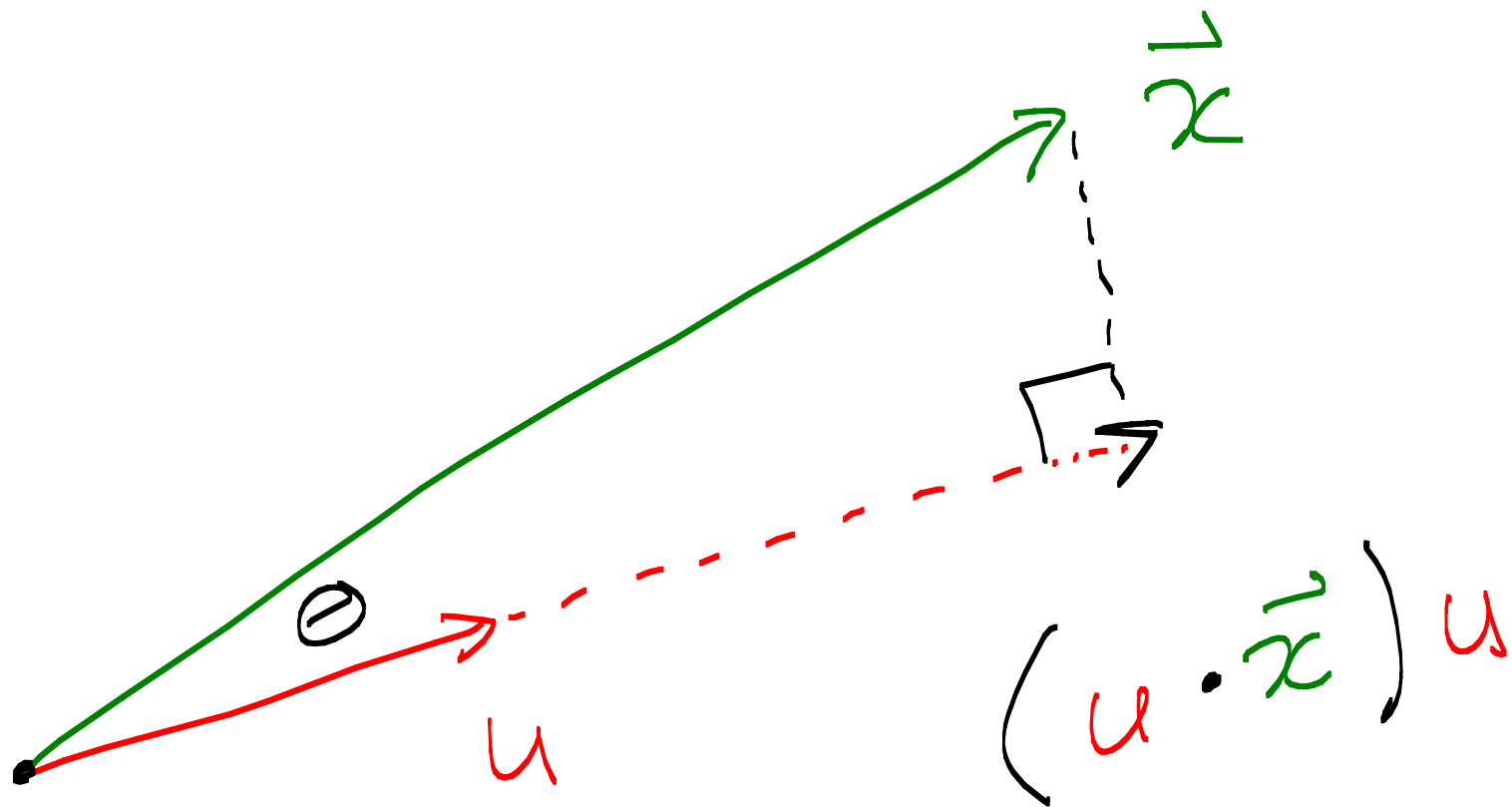There is no change in coordinate systems here.

How to interpret ?

$$R^T = \begin{bmatrix} \leftarrow u \rightarrow \\ \leftarrow v \rightarrow \\ \leftarrow w \rightarrow \end{bmatrix}$$
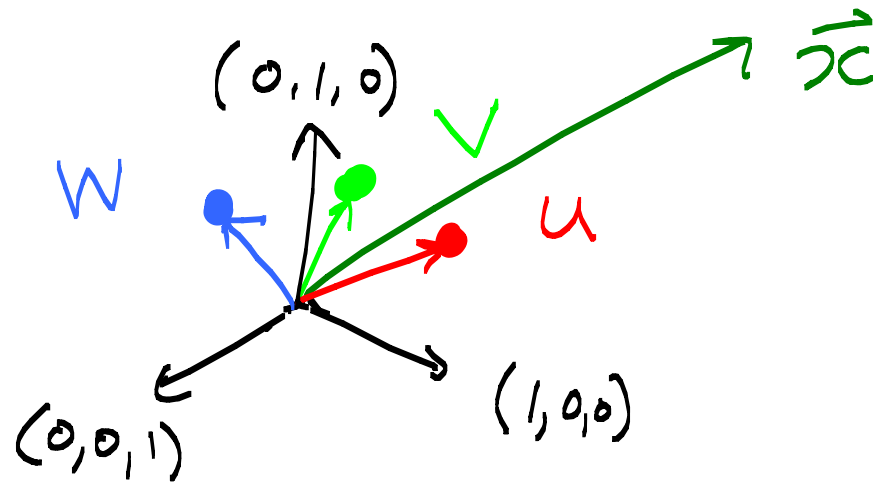
Claim: R^T is a change in coordinate system.

$$R^T \vec{x} = \begin{bmatrix} \leftarrow u \rightarrow \\ \leftarrow v \rightarrow \\ \leftarrow w \rightarrow \end{bmatrix} \vec{x}$$

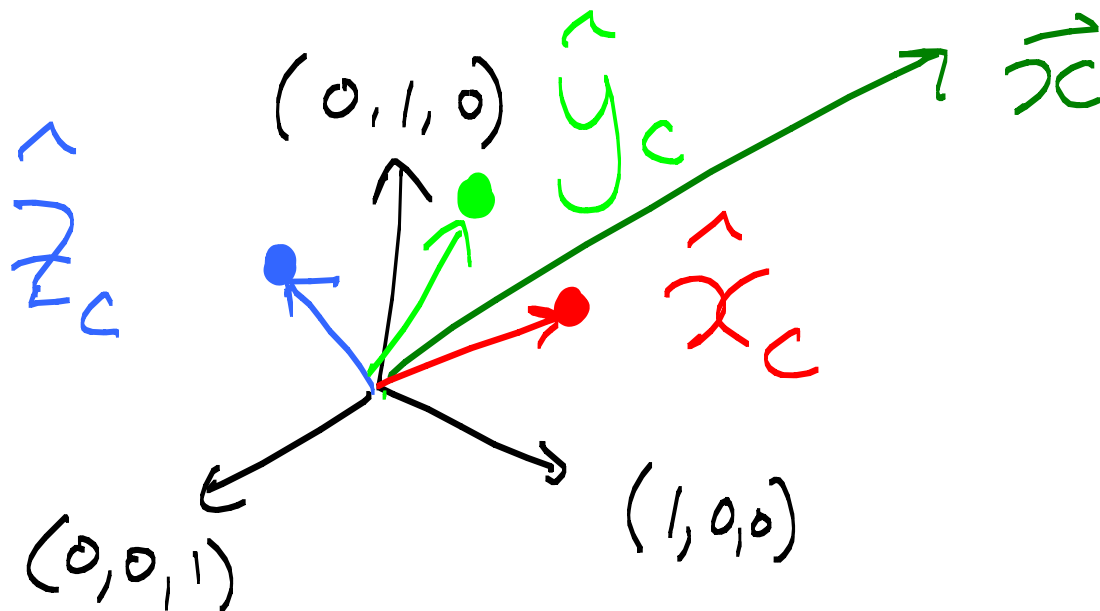$$u \cdot \vec{x} = |\vec{x}| \cos \theta$$

$\vec{x}$

θ

u

$(u \cdot \vec{x}) u$

change in coordinate system

Example: suppose camera is at (0, 0, 0).

$$R_{camera \leftarrow world} \qquad \vec{x} = \begin{bmatrix} \leftarrow & x_c & \rightarrow \\ \leftarrow & y_c & \rightarrow \\ \leftarrow & z_c & \rightarrow \end{bmatrix} \vec{x}$$

Announcements   (A2)

-   3D plants (not 2D as in starter code) and significantly different from what is given in starter code.   Let's see evidence that you've experimented a bit.

-   "Submit the entire directory as a zip or tar file to the myCourses Assignment/A2 folder. The file should be named FirstnameLastname.zip (or .tar) and should unpack into a directory with that same name."

-   For Q5,   for the tiled pathway,   the surface normal of the tile should be parallel to the normal of the bicubic surface.