Transformations

1. Let $\mathbf{p} = (1, 3, 4)$ and $\mathbf{q} = (5, 2, 7)$ be two vectors in $\mathbb{R}^3$. Calculate $\mathbf{p} \times \mathbf{q}$. Calculate $\mathbf{p} \cdot \mathbf{q}$. Verify for yourself that $\mathbf{p} \times \mathbf{q}$ is perpendicular to both $\mathbf{p}$ and $\mathbf{q}$.

2. Let $\mathbf{p}$ and $\mathbf{q}$ be two points in $\mathbb{R}^3$ and let $\mathbf{p}' = (2, 7, 4, 3)$ and $\mathbf{q}' = (1, 3, 5, 2)$ be a representation of these points in homogeneous coordinates. Compute the $\mathbb{R}^3$ representation of these points. While cross products and dot products do not make sense for points, they do make sense for vectors. Compute vectors $\mathbf{a} = \mathbf{p} - 0$ and $\mathbf{b} = \mathbf{q} - 0$, where 0 is the point at the origin, and then calculate $\mathbf{a} \times \mathbf{b}$ and $\mathbf{a} \cdot \mathbf{b}$.

3. Give a homogeneous representation of the point $p = (6, 4, 3)^T$. Give a homogeneous representation of the vector $v = (6, 4, 3)^T$. Give a non-homogeneous representation of the point expressed in homogeneous coordinates as $(2, 3, 5, 7)^T$.

   Answer:

   $$p = (6, 4, 3, 1)^T \qquad v = (6, 4, 3, 0)^T \qquad \left(\frac{2}{7}, \frac{3}{7}, \frac{5}{7}\right)$$

4. Give a homogeneous representation of the point $p = (2, 5, 7)^T$. Give a homogeneous representation of the vector $v = (2, 5, 7)^T$. Give a non-homogeneous representation of the point expressed in homogeneous coordinates as $(2, 5, 7, 7)^T$.

5. Write each of the following transformations as a sequence of 4×4 matrices:

   (a) Scale the scene by a factor 3 in the y direction, while leaving the position of the point $(x, y, z) = (4, -1, 1)$ unchanged.

   (b) Rotate the scene θ degrees, such that the axis of rotation is the vector $(1, 1, 0)$ and the origin $(0, 0, 0)$ stays fixed. Define the direction of positive rotation to be clockwise when one is looking towards the origin in the direction opposite to the vector, namely one is looking in direction $(-1, -1, 0)$. Assume (as always) a right handed coordinate system.

6. Give a transformation that maps the plane $y = f_0$ to the plane $y = 0$ and that maps the plane $y = f_1$ to the plane $y = 1$. Hint: use a translation and a scaling.

7. Let p_A be a point, v_A a vector, and n_A a normal vector. If these points and vectors are expressed in homogeneous coordinates of local reference frame A, and the 4×4 matrix $M_{W \leftarrow A}$ is a transformation from frame A to the world frame W, then explain how can one compute the point, vector, and normal in world coordinates?

   Answer:

   $$\begin{aligned} p_W &= M_{W \leftarrow A}\, p_A, \\ v_W &= M_{W \leftarrow A}\, v_A, \\ n_W &= M_{W \leftarrow A}^{-T}\, n_A, \end{aligned}$$

   where $M_{W \leftarrow A}^{-T}$ is the inverse transpose of $M_{W \leftarrow A}$.

8. Describe the main difference between affine transformations and linear transformations.

9. List two kinds of 3D transformations that are not possible using linear transformations (i.e., 3x3 matrix multiplication) and non-homogeneous representation of points.

   Answer: Translation and perspective projection.

10. Show than any sequence of rotations and translations can be expressed as a single translation followed by a single rotation. Similarly, show that any sequence can also be written as a rotation followed by a single translation. You can assume that any two rotations will multiply to give another rotation. Consider how you might swap the order of a rotation and translation without changing the result!

Answer: Let R be a 3×3 rotation matrix, and I be the $3x3$ identity, and let t be a vector represeting a translation. Given a rotation multiplied by a translation, we can write it instead as a translation multiplied by a rotation:

$$\begin{aligned}\begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} I & t \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} R & Rt \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} I & Rt \\ 0 & 1 \end{pmatrix}\begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix}.\end{aligned}$$

Likewise, given a translation multiplied by a rotaiton, we can write it instead as a rotation multiplied by a tranlsation:

$$\begin{aligned}\begin{pmatrix} I & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} I & R^T t \\ 0 & 1 \end{pmatrix}.\end{aligned}$$

Given an arbitrary sequence, we can perform swaps to bring all the rotations to the left, or likewise all the rotations to the right. Since rotations combine to give a rotation, and translations combine to give a translation, we can write any arbitrary sequence as a single translation followed by a single rotation, or vice versa.

11. Show than any sequence of non-zero non-uniform scales and translations can be expressed as a single translation followed by a single non-zero non-uniform scale. Similarly show the same sequence as a single non-zero non-uniform scale followed by a single translation.

12. Write the homogeneous 4×4 transformation matrix representing a reflection in the xz plane.

13. Derive a matrix for a reflection in the plane with normal n going through the origin.

14. Derive a matrix for a reflection in the plane with normal n and going through point p_0.

15. Suppose you have a method `drawCylinder` which draws a cylinder of diameter 1 and length 1 with the axis of the cylinder along the z axis, and the base of the cylinder on the xy plane. Give a sequence of transformations that will let you draw the cylinder in world coordinates centered at $(1, 0, 5)$, with radius 2, and a height of 10, with the axis of the cylinder along the y axis.

16. Let $p = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)^T$ be a point in non-homogeneous coordinates on a sphere with center at the origin and with radius equal to 1. Let $n = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)^T$ be the normal vector in non-homogeneous coordinates of the sphere at point p. Given transformations, T, S, R,

$$T = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

   (a) What is the position of the point p on the sphere after the sphere is transformed by the product TSR? Show your work by appling each transformation separately rather than computing the product of the three matrices.

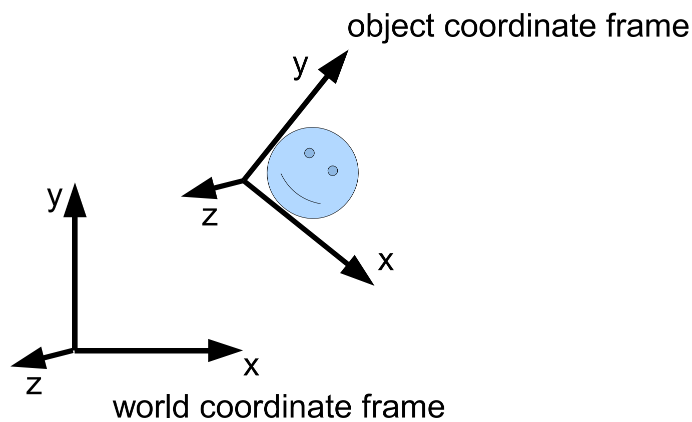
   (b) What is the normal of the point p on the sphere after the sphere is transformed by the product TSR? Show your work by appling each transformation separately.

17. Let $p = (a, b, c)$ be a point on a sphere of radius 1 centered at the origin. Note that $n = (a, b, c)$ is the normal of that point. Given transformations, A, B, C, where

$$A = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where is the position p after the sphere is transformed by the product ABC? What is the normal of this point after the same transformation?

18. Consider the following picture, and let matrix M specify a transformation from object coordinates to world coordinates. That is, the object coordinate frame and the happy face are drawn in world coordinates by first transforming by a matrix M.



Let $R_Z(\theta)$ be a rotation about the z axis by θ, that is,

$$R_Z(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(a) Draw a sketch and describe in words what happens if we replace M with $R_z(\theta)M$, for $\theta = \pi/2$.

(b) Draw a sketch and describe in words what happens if we replace M with $MR_z(\theta)$, for $\theta = \pi/2$.

19. Rewrite the following matrix as a product of a scaling, a rotation, and a translation (not necessarily in that order):

$$\begin{bmatrix} 0 & -3 & 0 & 10 \\ 2 & 0 & 0 & 11 \\ 0 & 0 & 4 & 12 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Hint: note that the rotation matrix should have determinant 1, and the scaling matrix should have non-negative elements only.

20. Show that scale and translation do not commute.

21. Give a short proof or counter example for the following.

(a) scaling matrices commute, $\mathbf{S}_1\mathbf{S}_2 = \mathbf{S}_2\mathbf{S}_1$ (trivial)

(b) translation matrices commute, $\mathbf{T}_1\mathbf{T}_2 = \mathbf{T}_2\mathbf{T}_1$ (trivial)

(c) rotation matrices typically do *not* commute, $\mathbf{R}_1\mathbf{R}_2 = \mathbf{R}_2\mathbf{R}_1$ (though they do commuate always, if rotation is about a common axis)

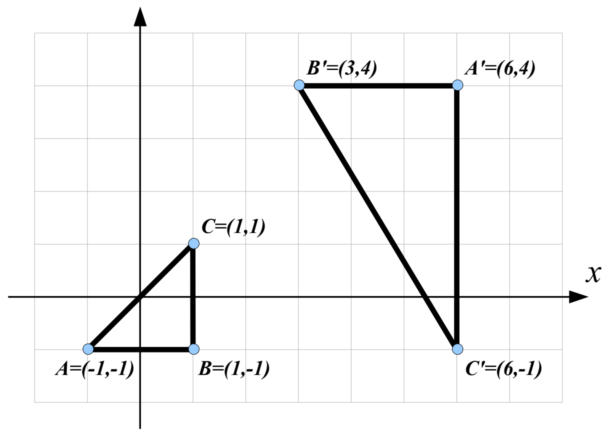
(d) rotation and translation typically do *not* commute, $\mathbf{RT} \neq \mathbf{TR}$

(e) scaling and translation typically do *not* commute, $\mathbf{ST} \neq \mathbf{TS}$

(f) scaling and rotation typically do *not* commute, $\mathbf{SR} \neq \mathbf{RS}$

22. List two transformations that cannot be represented using non-homogeneous linear transformations.

23. Consider the following figure:



Give a matrix, or product of matrices, that will transform the triangle ABC into the triangle $A'B'C'$.

Answer: For convenience, we'll first translate A to the origin. Translate by $(1,1)$, shear in x by -1, rotate by 180, scale by 1.5 in x and 2.5 in y, translate by (6,4). In homogeneous coordinates,

$$\begin{pmatrix} 1 & 0 & 6 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1.5 & 0 & 0 \\ 0 & 2.5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Alternatively, we could set up a system of equations for the 6 degrees of freedom of a 2D homogeneous affine transformation matrix and solve.

Viewing and Projections

24. Write an expression for the projection of a point p onto a plane $Ax + By + Cz + D = 0$ with a light source at infinity in the direction (d_x, d_y, d_z). Can you write the expression as a matrix multiply? Note that this computation will let you determine the location of the point's shadow.

25. Write an expression for the projection of a point p onto a plane $Ax + By + Cz + D = 0$ with a point light source at point l. Can you write the expression as a matrix multiply? Note that this computation will let you determine the location of the point's shadow.

26. What characterizes an orthographic projection? Why might an orthographic projection be useful?

27. What characterizes an oblique projection? Why might an oblique projection be useful?

28. Let $p_0 = (1, 1, 9)$, $p_1 = (3, 1, 15)$, $p_2 = (1, 4, 15)$ define a triangle within a plane

$$Ax + By + Cz + D = 0.$$

(a) Compute the coefficients A, B, C, D of the plane. Hint: first compute the surface normal.

(b) Consider a ray from the origin and in direction $(1, 1, 7)$. Where does this ray intersect the plane?

29. Consider a camera at $(3, 4, 3)$ which lies in the plane

$$x - 2y + z + 2 = 0.$$

Define the V_{UP} direction to be $(1, -2, 0)$. Give a transformation that projects the scene *orthographically* onto this plane, i.e., the projection is in the direction normal to the plane, all points in $\mathbb{R}^3$ are projected onto the plane.

30. Consider a 2D image space whose points are represented in homogeneous coordinates by (wx, wy, w). For each of the two transformations below, give a product of 3×3 matrices that performs this transformation.

    (a) Rotate the scene by θ degrees clockwise around the point $(x, y) = (2, 3)$, that is, this point is not moved but all other points are moved.

    (b) Scale and translate a rectangle that has opposite corners $(1, 4)$ and $(3, 5)$ to a rectangle that has opposite corners $(-1, 0)$ and $(8, 6)$.

31. Let P be an invertible homogeneous matrix representing a perspective projection that preserves the z value of the $z = f_0$ near plane and $z = f_1$ far plane, i.e.,

$$P = \begin{pmatrix} f_0 & 0 & 0 & 0 \\ 0 & f_0 & 0 & 0 \\ 0 & 0 & f_0 + f_1 & -f_0 f_1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Where do points behind the viewer go to after perspective projection?

32. Not all planar projections are perspective projections. Describe and name the two types of non perspective planar projections.

Answer: Non-perspective planar projections involve projecting points onto a plane with a given projection direction. When the projection direction is orthogonal to the plane, it is called an *Orthographic* projection, and is commonly used in CAD drawings (e.g., top, left, front) to show the profile of an object or a floor plan. When the projection direction is not orthogonal to the projection plane, the projection is called an *oblique* projection. Such projections show the profile of an object, much like an orthographic projection, but depth is also visible in the image. Examples include cabinet and cavalier projections.

33. Let P be an invertible homogeneous matrix representing a perspective projection that preserves the z value of the $z = n$ near plane and $z = f$ far plane, i.e.,

$$P = \begin{pmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n + f & -nf \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

What happens to vectors when transformed by this perspective projection? What happens to points in the plane at the eye (i.e., z=0) after applying this perspective projection? Your answers should give both the result of the matrix multiplication, and a short written description.

Answer:

$$\begin{pmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n + f & -nf \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 0 \end{pmatrix} = \begin{pmatrix} nx \\ ny \\ (n+f)z \\ z \end{pmatrix} = \begin{pmatrix} nx/z \\ ny/z \\ n+f \\ 1 \end{pmatrix}$$

Thus, vectors map to points on the plane $z = n + f$.

$$\begin{pmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n + f & -nf \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} nx \\ ny \\ -nf \\ 0 \end{pmatrix}$$

Thus, points on the plane at the eye map to vectors.

34. Consider the following perspective projection,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}.$$

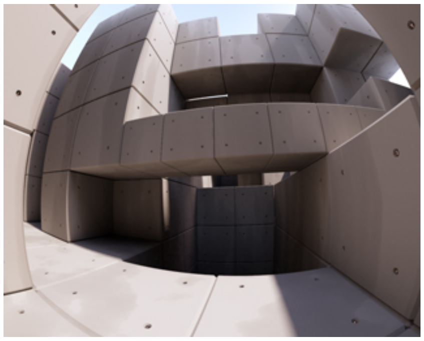
What is the projection plane?

35. Show that an invertible 4×4 projection matrix $\mathbf{M}$ maps planes to planes. Show that the transformation of a vector normal to the plane is not necessarily normal to the transfomration of the plane.

36. Show that an invertible 4×4 projection matrix $\mathbf{M}$ maps lines to lines.

37. Prove that an affine transformation preserves parallel lines.

38. The image at right is composed entirely of cube shaped geometry (Alex McLeod (American, contemporary), *An Architectural Space?*, c. 2003, digital image rendered with Brazil r/s software). Why can this image not have been produced with a planar perspective projection?



Subdivision

39. Whats the difference between even and odd vertices in a subdivision scheme?

    Answer: The even vertices correspond to existing vertices at the higher subdivision level, while the odd vertices are new vertices.

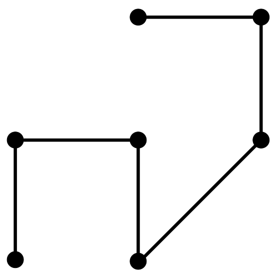
40. What is the valence of the new vertices (also called odd vertices) that are created when subdividing a triangle mesh with the Loop subdivision scheme.

41. What characterizes an extraordinary vertex in the Loop subdivision scheme?

42. What characterizes an extraordinary vertex in the Catmull-Clark subdivision scheme?

    Answer: The Loop scheme works on triangles, so extraordinary vertices have degree not equal to 6. The Catmull-Clark scheme works on quadrilaterals, so an extraordinary vertex will be one that has degree equal to something other than 4.

43. Consider the following figure consisting of a piecewise linear curve used to define a smooth subdivision curve. Draw a diagram that clearly shows one level of subdivision of the curve using the subdivision mask $(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$ (i.e., the Lane-Riensfeld algorithm for producing quadratic B-spline curves). Assume that endpoints stay fixed.



44. Apply the subdivision mask $(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$ twice to the control polygon in the previous question (draw each stage clearly!), and assume that the endpoints stay fixed (i.e., special rules for the endpoints / boundary).

45. This question concerns subdivision curves using the subdivision mask $(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$ (i.e., the Lane-Riensfeld algorithm for $n = 2$). Assume the additional rule for the boundary that leaves the endpoints fixed. Write Java style pseudo-code that performs one level of subdivision and returns the result.

```
List<Point> subdivide( List<Point> L ) {
    // TODO
}
```

Answer:

We can first insert the midpoints, creating a temporary list of points, and then create the final list by applying the averaging mask $(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$.

```
List<Point> subdivide( List<Point> L ) {
    List<Point> tmp = new List<Point>();
    tmp.add( L.get(0) );
    for ( int i = 1; i < L.size(); i++ ) {
        tmp.add( 1/2 * L.get(i-1) + 1/2 * L.get(i) );
        tmp.add( new Point( L.get(i) ) );
    }
    List<Point> S = new List<Point>();
    S.add( tmp.get(0) );
    for ( int i = 1; i < tmp.size()-1; i++ ) {
        S.add( 1/4 * tmp.get(i-1) + 1/2 * L.get(i) + 1/4 * tmp.get(i+1) );
    }
    S.add( tmp.get( tmp.size()-1 ) );
    return S;
}
```

Alternatively, if we work out the even and odd masks, we find that odd vertices are simply the average of their neighbours, while even vertices are weighted $(\frac{1}{8}, \frac{6}{8}, \frac{1}{8})$. Thus, we can build the subdivided point list directly.

```
List<Point> subdivide( List<Point> L ) {
    List<Point> S = new List<Point>();
    S.add( L.get(0) );
    for ( int i = 1; i < L.size()-1; i++ ) {
        Point odd = 1/2 * L.get(i-1) + 1/2 * L.get(i);
        Point even = 1/8 * L.get(i-1) + 6/8 * L.get(i) + 1/8 * L.get(i+1);
        S.add( odd );
        S.add( even );
    }
    // add the last odd vertex
    S.add( 1/2 * L.get(L.size()-2) + 1/2 * L.get(L.size()-1) );
    S.add( L.get(L.size()-1) );
    return S;
}
```

46. For the corner cutting Chaikin scheme, compute even and odd subdivision rules by combining the midpoint edge division step and averaging step.

47. For the corner cutting Chaikin scheme, write down the subdivision matrix, and use its eigenvalue decomposition to compute the averaging weights that is used to "push" a vertex to its position in the limit. Use the same decomposition to compute the tangent of the limit curve at a given vertex.

Meshes

48. Describe a method for converting an arbitrary mesh containing convex planar polygons of any number of sides to a triangle mesh. Draw a diagram to help explain your method.

49. Describe a method for converting an arbitrary mesh containing convex planar polygons of any number of sides to a quadrilateral mesh. Draw a diagram to help explain your method.

50. A classic soccer ball has only pentagons and hexagons, and each vertex has degree three. How many pentagons does a soccer ball have? How many hexagons can a soccer ball have? Hint: assume that there are x pentagons and y hexagons and use the *Euler Characteristic*.

51. Consider a mesh, which is a closed orientable manifold (i.e., no boundary), and is defined entirely with quadrilaterals using 100 vertices, with each vertex having exactly degree 4.

    (a) How many edges are there?
    Answer: Because each vertex has degree 4, it is adjacent to 4 edges, so we can count 4 edges for each vertex but we divide by 2 because each edge has two vertices, that is, $100 \times 4/2 = 200$.

    (b) How many faces are there?
    Answer: Each vertex is adjacent to 4 faces, so we count 4 faces for each vertex but we divide by 4 so we can count 4 edges for each vertex but we divide by four because each face is a quadrilateral and has 4 vertices, that is, $100 \times 4/4 = 100$.

    (c) What is the genus of the mesh (i.e., how many holes does it have)? Hint: recall the Euler characteristic, $F - E + V + 2g + \#\partial = 2$.
    Answer: $100 - 200 + 100 + 2g + 0 = 2$, thus, the genus is 2.

52. Is it possible to tile the sphere with only hexagons using degree 3 vertices? A harder question, can you tile the sphere using only hexagons and vertices of any degree greater or equal than three?

53. Suppose you're given a triangle soup (vertex list, and a set of polygons that are described as vertex index lists). Given that the mesh is manifold (or manifold with boundary), describe an algorithm or write pseudo code to check if the manifold is orientable.

54. Given only the triangle soup, describe an algorithm (or write pseudo code) to check if the mesh is manifold without boundary.

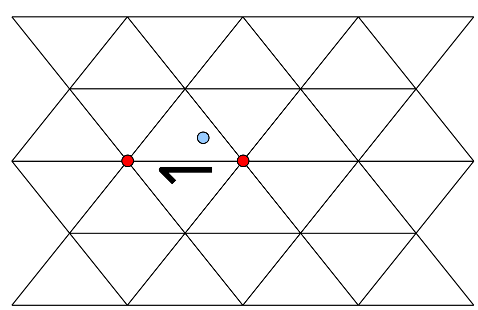
55. Given only the triangle soup, describe an algorithm (or write pseudo code) to check if the mesh is manifold with boundary.

56. Write pseudocode that performs an edge collapse. The method should have the signature shown below next to the figure. Note that the method takes as parameter a half edge e to tell which edge should be collapsed, and a new vertex v at the desired new position of the combined endpoint vertices.

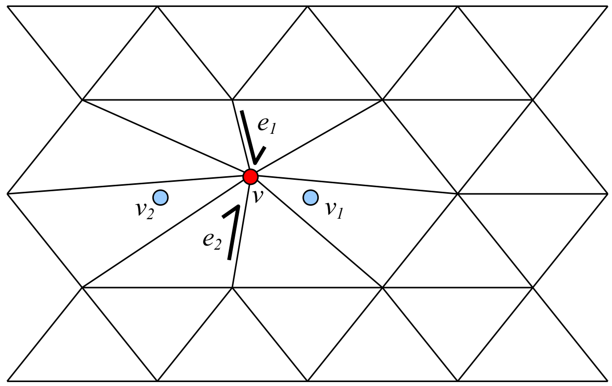
    Be sure to correctly set all half edges arriving at the vertices incident to the initial edge to have the same head pointer for the new vertex. The diagram below shows one possible input mesh. Draw a diagram to show the local structure of the mesh after the collapse.

    You should include high level comments in your pseudo code to describe what you are doing. Feel free to draw an annotated diagram along with your comments. Assume the following definition of a half edge class.

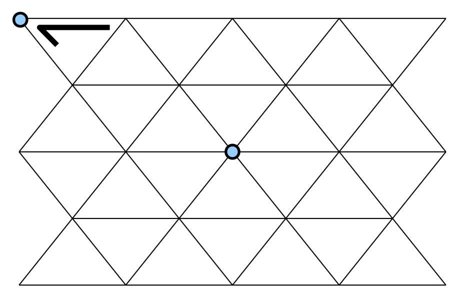
```
class HE {
    HE twin; // twin half edge
    HE next; // next half edge
    Vertex v; // head vertex
}

void CollapseEdge( HE e, Vertex v ) {
    // TODO
}
```



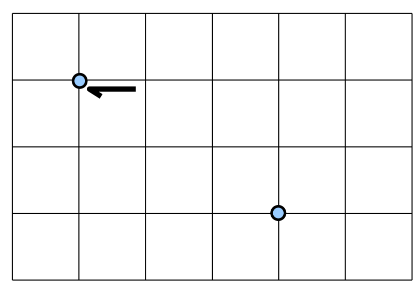
57. Write pseudocode for a method called `vertexSplit` that performs a vertex split on a half edge data structure. Your method should take as parameters two half edges, e_1 and e_2 (that both have as head the same vertex v), and two vertex positions v_1 and v_2 (see the diagram below). The new edge between v_1 and v_2 has two incident faces that should be inserted between e_1 and its twin, and e_2 and its twin. Be sure to also correctly change the head vertex of *all* the half edges that were originally arriving at v. Draw a diagram to show the local structure of the mesh before and after to convince yourself that your pseudocode is correct. How many half edges do you need to create? Write comments in your pseudo code to describe exactly what is happening.



58. Given a half edge data structure and a half edge pointing at a given vertex, as show in in the grid at right, write a sequence of pointer dereferences that will bring you to any half edge with the destination vertex as head. Use `n` for next half edge, and `t` for the twin half edge, so if the destination was at the head of the next half edge's twin, then your answer would be `he = current.n.t;`



59. Given a half edge data structure and a half edge pointing at a given vertex, as showin in the grid below, write a sequence of pointer dereferences that will bring you to any half edge with the destination vertex as head. Use `n` for next half edge, and `t` for the twin half edge, so if the destination was at the head of the next half edge's twin, then your answer would be `he = current.n.t;`



Spline Curves and Surfaces

60. Describe the advantages and disadvantages of the parallel transport frame.

61. Describe one advantage and one disadvantage of Frenet Frames.

62. When can you convert a quadratic Bezier to a Cubic Hermite curve?

63. When can you write a cubic Hermite curve as a quadratic Bezier?

64. Let $g_0 = (0,0)^T$, $g_1 = (1,1)^T$, $g_2 = (-1,1)^T$ be the control points for a quadratic Bezier curve

$$c(t) = \sum_{i=0}^{2} B_{i,2}(t) g_i.$$

(a) Evaluate the Bernstein basis functions at $t = 0.5$, and compute the position of the point $c(0.5)$.

(b) What is the derivative of the curve at $t = 0.5$? Show your work.

(c) Make a sketch of the control polygon and apply Decaslejau's algorithm to draw the position of $c(0.5)$. Does your picture agree with the answer to the first two parts of this question?

65. Describe the steps of Decaslejau's algorithm.

66. Describe an algorithm that draws a Bézier cubic curve to a specified tolerance using straight lines. Make use of the convex hull property and de Caseljau's algorithm.

Answer: Suppose the Bézier curve is degree n, and the control points are p_i for $i = 0...n$. The check the distance between control each point p_i and the line created by p_0 and p_n for $i = 1...n-1$. This distance can be found by computing the normalized direction of the vector connecting the endpoints, subtracting $p_i - p_0$ projected onto this vector from $p_i - p_0$, and computing the length of the resulting vector. That is,

$$\begin{aligned} v &= \frac{p_n - p_0}{||p_n - p_0||}, \\ a_i &= p_i - p_0, \\ d_i &= ||(I - vv^T)a_i||, \end{aligned}$$

and if d_i is less than the tolerance for all $i = 1..n-1$, then by the convex hull property we know the entire curve will lie within this tolerance of the line connecting p_0 and p_n.

If the distance is greater than the tolerance for some i, then use de Caseljau's algorithm to divide the curve into two pieces and repeat. This process will eventually terminate because the curve is locally flat provided we are looking at a small enough piece of the curve (another good exercise would be to show that the tolerance violation must decrease for a subdivided curve).

67. Using the convex hull property, compute axis aligned bounds for the Bezier curve with control points (-1,-1) (1,1) (2,1) (-3,1). How might you compute a better (i.e., smaller) axis aligned bounding box for the curve?

Answer: The lower left corner is (-3,-1) and the upper right corner is (2,1). We could possibly compute a smaller axis aligned bounding box by subdividing the curve with de Caslejau's algorithm and taking the minimum and maximum of the x and y coordinates of the two parts.

68. (a) Given three points $(x(0), y(0))$, $(x(1), y(1))$, $(x(2), y(2))$ in the (x, y) plane, show how to compute a matrix $\mathbf{C}$ such that

$$\mathbf{q}(t) \;=\; (x(t), y(t)) \;=\; \mathbf{C} \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}$$

and the curve $\mathbf{q}(t)$ passes through the three given points at $t = 0, 1, 2$. Note that the curve $\mathbf{q}(t)$ is *quadratic*, rather than cubic. Be sure to specify the size of the matrix $\mathbf{C}$ in your answer.

(b) Give a formula for the tangent vector $\mathbf{q}'(t)$ to this curve (not necessily of unit length). Your formula should be similar to the one above.

(c) Suppose we are given only two points $\mathbf{q}(0)$ and $\mathbf{q}(1)$ in the plane. How could one define a *quadratic* curve through these points by choosing a suitable tangent vector, or a set of tangent vectors?

69. How many real scalar values are necessary to describe the shape of a rational Bezier curve? Suppose the curve is in d dimensions and the degree of the Bezier is n. What is your answer when $d = 2$ and $n = 3$.

70. Let $g_0 = (-1, -2)$, $g_1 = (-1, 0)$, $g_2 = (1, 0)$, $g_3 = (1, 2)$ be the control points for a cubic Bezier curve in the 2D plane $c(t) = \sum_{i=0}^{3} B_{i,3}(t) g_i$.

(a) Evaluate the Bernstein basis functions at $t = 1/3$, and compute the position of the point Where is the point $c(1/3)$.

(b) Make a sketch of the control polygon and apply Decaslejau's algorithm to draw the position of $c(1/3)$.

(c) What is the derivative of the curve at $t = 1/3$?

71. Show that a Bernstein basis polynomail $B_{i,n}(t)$ attains only one maximum on $[0, 1]$ and does so at $t = i/n$.

72. Let M be an affine 3D transformation $M(x) = Ax+v$, where A is a 3×3 matrix, and v is a 3D vector, and let g_i for $i = 1..n$ be 3D control points of a degree n Bezier curve. show that $M(\sum_{i=0}^{n} B_{i,n}(t)g_i) = \sum_{i=0}^{n} B_{i,n}(t)M(g_i)$. That is, show that Bezier curves have affine invariance.

73. Given the bicubic Bezier patch $p(s,t) = \sum_{i=0}^{3}\sum_{j=0}^{3} B_{i,3}(s)B_{j,3}(t)g_{ij}$, $s \in [0, 1]$, $t \in [0, 1]$, with $g_{ij} \in \mathbb{R}^3$, write a simple expression for a vector that is parallel to the surface normal at $p(0, 0)$.

74. Describe the difference between $C1$ and $G1$ continuity.

75. How many real values are necessary to describe the shape of a cubic rational Bezier curve in three dimensional space? Provide a short justification for your answer.

76. Given a polynomial $p(t) = \sum_{i=0}^{n} \alpha_i\, t^i$, for $t \in [0, 1]$ describe in words an easy method for computing bounds on the values obtained by this polynomial curve, i.e., l and h such that $p(t) \in [l, h]$ for all $t \in [0, 1]$. Explain why your method works.

77. How would you obtain a degree 4 polynomial

$$p(x) = ax^4 + bx^3 + cx^2 + dx + e$$

with given values $p(0), p(1), p(2), p(3), p(4)$?

78. This question concerns a rational Bezier curve.

(a) Given the quadratic Bezier curve $p(t) = \sum_{i=0}^{2} B_{i,2}(t)g_i$, compute an expression for the point at the center of the curve, and its derivative (i.e., $p(t)$ and $p'(t)$ at $t = 1/2$) with respect to its control points.

(b) Suppose we have the rational quadratic Bezier curve

$$p(t) = \frac{N(t)}{D(t)} \qquad \text{where} \qquad N(t) = \sum_{i=0}^{2} B_{i,2}(t)w_i g_i \qquad D(t) = \sum_{i=0}^{2} B_{i,2}(t)w_i$$

Let $g_0 = (1, 0)$, $g_1 = (1, 1)$, $g_2 = (0, 1)$ and $w_0 = 1$, $w_1 = 1$, $w_2 = 2$ be the nonhomogeneous control points and weights of this rational quadratic Bezier curve.

Use the formula from the previous part of this question to compute the value and derivative of the numerator and the denominator at the center of the curve, that is, $N(t), N'(t), D(t), D'(t)$, for $t = 1/2$. Keep your answer in the form of a reduced fraction.

(c) Combine these values you compute the position of the curve and the tangent of the curve at $t = 0.5$. Recall, the quotient rule: $(f/g)' = (f'g - fg')/g^2$. Keep your answer in the form of a reduced fraction.

(d) The curve represents a circular arc centered at (0,0) with radius 1. Compute the distance between the point you computed and the center of this circle to show that your answer agrees with this.

(e) Draw a labeled diagram showing the circle, the arc, the control points, and the tangent vector you computed.

Quadrics and Simplification

79. Describe 3 common shapes that can be defined by a quadric.

80. What kind of shape do you get when you apply an invertible projection transformation to a quadric?

81. Suppose three faces of a cube meet at a vertex v_0 at the origin with normals pointing in each of the positive axis directions.

(a) For each plane, write a simple expression of the form $q^T K_p q$ for the squared distance of a point q to the plane.

(b) Write the quadric error function for the vertex where these three planes meet. That is, give Q_{v_0}. What can you say about the shape of this quadric?

(c) Suppose that 3 other faces of this cube meet at the vertex v_1, which is at the point $(-1, 0, 0, 1)^T$ in homogeneous coordinates. The faces have normals in positive y, positive z, and negative x directions. Write the quadric error function for this vertex. That is, give Q_{v_1}. What can you say about the shape of this quadric?

(d) What can you say about the shape of the quadric defined by the sum $Q_{v_0} + Q_{v_1}$.

82. How can you decide if two triangles intersect? Write pseudocode to outline your intersection test.

83. Describe two different scenarios where mesh simplification is important.

Rasterization and Texture

84. What does Bresenham's algorithm do and why is it fast?

85. Suppose you have a texture on a square in 3D space. When mapped to screen space with a perspective projection, suppose the square looks like a trapezoid. Explain why bilinear interpolation of the texture coordinates in screen space is a bad idea.

86. What is the purpose of Gouraud shading? What problem can arise when using Gouraud shading?

87. What is the purpose of Phong shading? Is Phong shading useful when drawing a cube?

88. What problem can arise when using the painter's algorithm? Draw a diagram to demonstrate this problem.

89. When does magnification and minification occur in texture mapping? Sketch an example of each.

90. Given an arbitrary mesh, describe a reasonable method for assigning a texture coordinate function to the surface.

91. A spherical environment map can be used to render reflections on surfaces. What happens if we want to change the viewing direction? How does a cube map alleviate this problem.

92. What is the visual difference between bump mapping and displacement mapping?

93. Assume that you have an algorithm that can fill 3D triangles with a constant colour.

    (a) Explain what additional information and additions to the algorithm are required to Gouraud shade the triangles.

    (b) Given the algorithm in (a), explain what additional information and additions to the algorithm are required to texture map the triangles using bilinear interpolation, including an explanation of how the bilinear interpolation is done.

    (c) Explain the advantages and disadvantages of using nearest-neighbour interpolation compared with bilinear interpolation.

    (d) Explain why a MIPmap would be useful for texture mapping and how one could be incorporated into the algorithm from (b).

94. What is aliasing? Give examples. Describe methods for reducing aliasing problems when ray tracing a scene.

95. What effects can be produced with the accumulation buffer to improve the visual realism of a rendered scene?

96. What is the difference between the painter's algorithm and the use of a z-buffer for hidden surface removal. What problems can arise when using the painter's algorithm? What problems can arise when using the z-buffer?

97. What is back face culling and when can it be used?

Lighting and Shading

98. Describe the Blinn-Phong illumination model, what it is good for, and what are its drawbacks.

99. What is a Lambertian surface?

100. Suppose that a shiny ground plane $y = 0$ is illuminated by sunlight. Let the sun be in direction $(0, 1, 1)$ and let the camera be at position $(x, y, z) = (3, 4, 5)$. Determine the position on the ground plane at which the peak of the highlight occurs. *Hint:* recall that for a mirror, $\mathbf{l} + \mathbf{r} = 2(\mathbf{n} \cdot \mathbf{l})\mathbf{n}$, when all vectors are normal length.

101. Suppose we are computing the illumination of a point $p = (1, -2, 0)$ with normal $n = (1, 1, 0)$. Let the eye location be $e = (0, 0, 10)$ and the light location be $L = (0, 10, 0)$, and let the diffuse material properties be $k_d = (1, 0, 0)$, and specular material properties be $k_s = (1, 1, 1)$, light intensity $I = 1$, and specular exponent $n = 64$. Assuming only diffuse and specular light model (i.e., no ambient term), what is the illumination of this point?

Ray Tracing

102. Given a triangle with points A, B, and C, and a ray $R(t) = P + Vt$, outline the steps and computations necessary to find the point of intersection if it exists.

103. Given a sphere of radius r centered at point C, and a ray $R(t) = P + Vt$, outline the steps and computations necessary to find the point of intersection if it exists.

104. Ray tracing is typically slower than forward rendering. Why is this? Give an example of a typical optimization which can improve the speed at which images can be generated with ray tracing.

Colour

105. In the standard OpenGL rendering pipeline, and in the ray tracing assignment, light colour and intensity are given by an RGB tuple, and diffuse reflectance properties are also specified by an RGB tuple. Describe a real world scenario where this model of of light and reflectance would not be correct.

106. In the context of colorimetry, what are complementary colours?

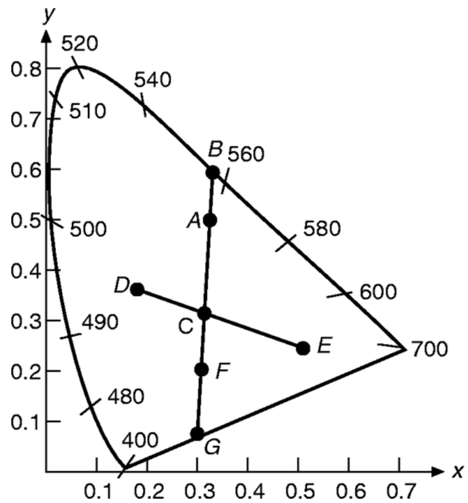
107. Why are the opposite ends of the spectrum in the CIE horse shoe diagram connected by a straight line?

108. What property of the eye allows us to represent a color with 3 values?

109. Describe the colour matching experiment that lead to the definition of the CIE standard colour space.

110. Why is the CIE chromaticity diagram two dimensional?

111. The following questions refer to the CIE chromaticity diagram. Assume that point C is the white point.



(a) Can illuminants D and E be mixed to produce white?

(b) What is the approximate purity of illuminant F?

(c) Does F have a dominant spectral wavelength?

(d) What is the purity of illuminant A?

(e) What is the dominant spectral wavelength of illuminant A?