lecture 10

- cubic curves
- cubic splines
- bicubic surfaces

We want to define smooth curves:

- for defining paths of cameras or objects

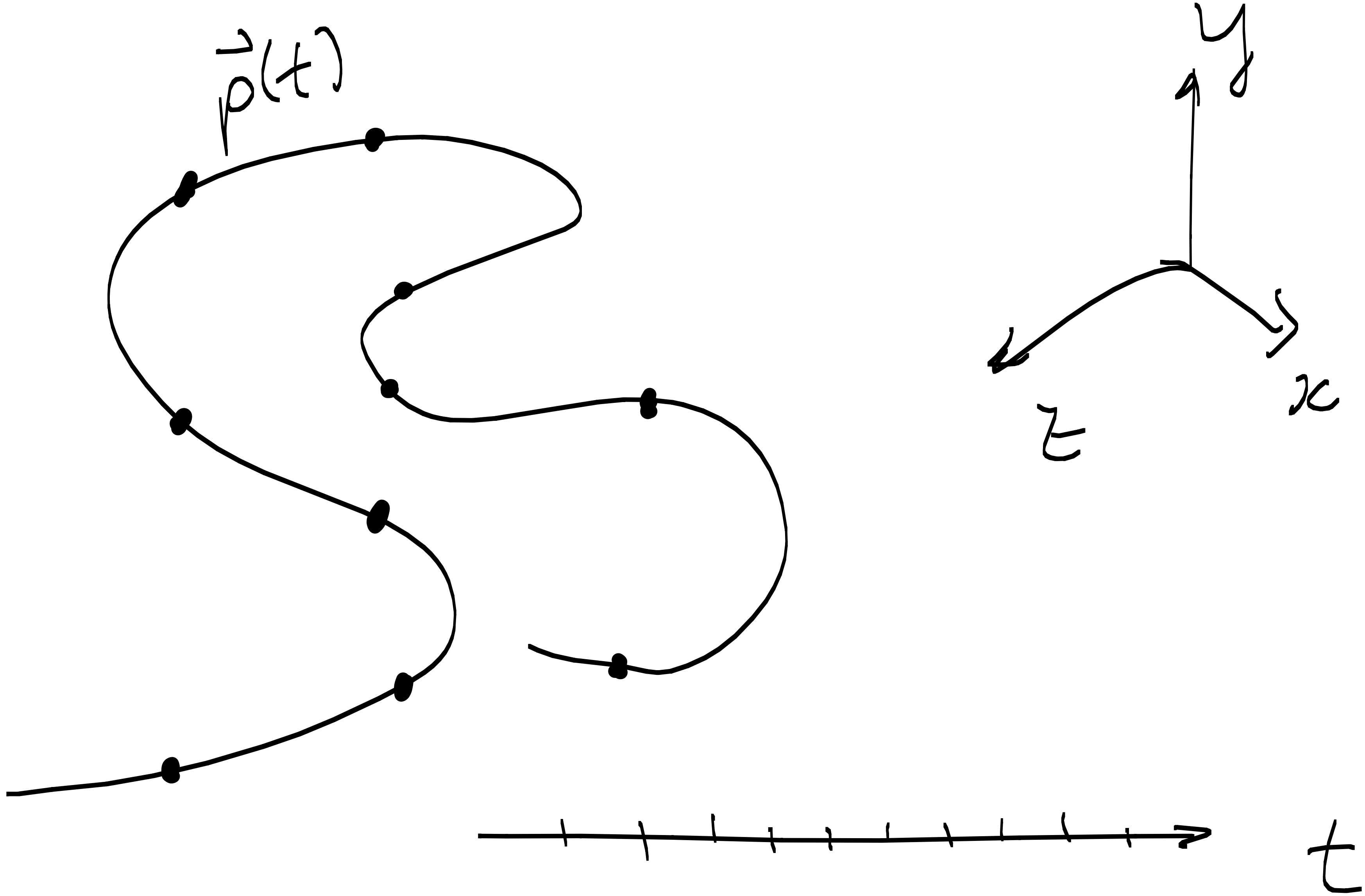
- for defining 1D shapes of objects

We want to define smooth *surfaces* too.

Parametric Equation of a Line

$$\begin{aligned}\mathbf{p}(t) &= \mathbf{p}(0) + t\,(\mathbf{p}(1) - \mathbf{p}(0)) \\ &= (1 - t)\,\mathbf{p}(0) + t\,\mathbf{p}(1)\end{aligned}$$

The curve is a linear combination of two points. How to generalize this?



The curve will be a linear combination of the points.   How ?

Cubic Curves

$$(x(t), y(t), z(t))$$

Each is a polynomial of degree 3 and defined over all t.

$$\begin{aligned} x(t) &= a_x t^3 + b_x t^2 + c_x t + d_x \\ y(t) &= a_y t^3 + b_y t^2 + c_y t + d_y \\ z(t) &= a_z t^3 + b_z t^2 + c_z t + d_z \end{aligned}$$

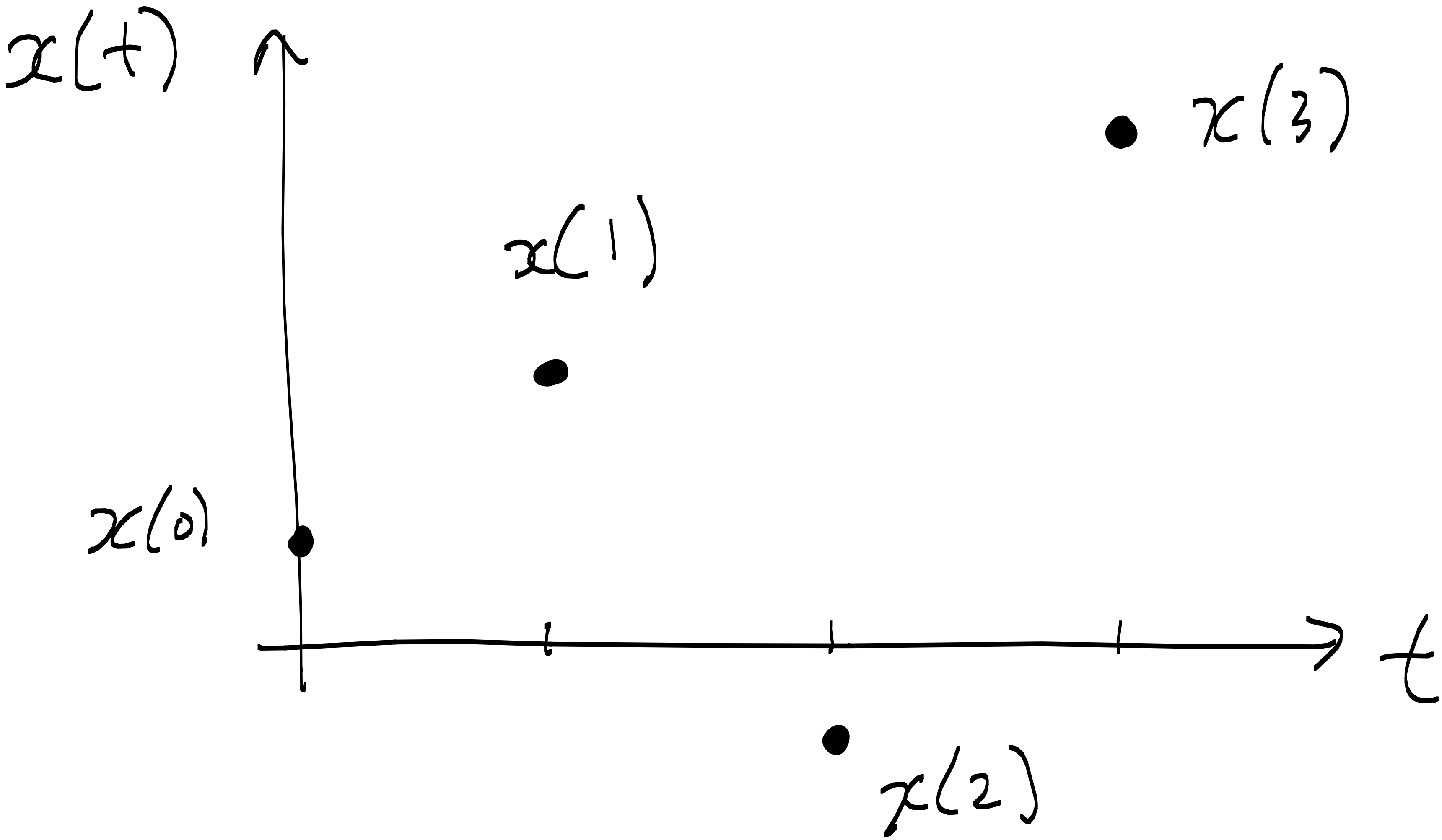
$$x(t) = a_x t^3 + b_x t^2 + c_x t + d_x$$

$$\begin{bmatrix} x(t) \end{bmatrix} = \begin{bmatrix} a_x & b_x & c_x & d_x \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

Choosing the coefficients gives us a curve, but how do we choose the coefficients ?

Fitting a cubic function.

(Why are 4 points needed ?)



$$\begin{bmatrix} x(t) \end{bmatrix} = \begin{bmatrix} a_x & b_x & c_x & d_x \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

$t = 0, 1, 2, 3$

$$\begin{bmatrix} x(0) & x(1) & x(2) & x(3) \end{bmatrix} = \begin{bmatrix} a_x & b_x & c_x & d_x \end{bmatrix} \begin{bmatrix} 0 & 1 & 8 & 27 \\ 0 & 1 & 4 & 9 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x(0) & x(1) & x(2) & x(3) \end{bmatrix} = \begin{bmatrix} a_x & b_x & c_x & d_x \end{bmatrix} \begin{bmatrix} 0 & 1 & 8 & 27 \\ 0 & 1 & 4 & 9 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{B} \equiv \begin{bmatrix} 0 & 1 & 8 & 27 \\ 0 & 1 & 4 & 9 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}^{-1}$$

Multiplying by B gives solution for coefficients a, b, c, d.

We apply the same technique for y(t) and z(t) .

I will walk through the derivation again, but this time solve for all x(t), y(t), z(t).

$$\mathbf{p}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} a_x & b_x & c_x & d_x \\ a_y & b_y & c_y & d_y \\ a_z & b_z & c_z & d_z \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

Trying to solve for this.

$$\mathbf{p}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} a_x & b_x & c_x & d_x \\ a_y & b_y & c_y & d_y \\ a_z & b_z & c_z & d_z \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

$t = 0, 1, 2, 3$

$$\mathbf{G} \equiv \begin{bmatrix} \mathbf{p}(0) & \mathbf{p}(1) & \mathbf{p}(2) & \mathbf{p}(3) \end{bmatrix}$$

$$
\begin{bmatrix} \mathbf{p}(0) & \mathbf{p}(1) & \mathbf{p}(2) & \mathbf{p}(3) \end{bmatrix} = \begin{bmatrix} a_x & b_x & c_x & d_x \\ a_y & b_y & c_y & d_y \\ a_z & b_z & c_z & d_z \end{bmatrix} \begin{bmatrix} 0 & 1 & 8 & 27 \\ 0 & 1 & 4 & 9 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}
$$

Now what ?

$$\begin{bmatrix} \mathbf{p}(0) & \mathbf{p}(1) & \mathbf{p}(2) & \mathbf{p}(3) \end{bmatrix} = \begin{bmatrix} a_x & b_x & c_x & d_x \\ a_y & b_y & c_y & d_y \\ a_z & b_z & c_z & d_z \end{bmatrix} \begin{bmatrix} 0 & 1 & 8 & 27 \\ 0 & 1 & 4 & 9 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} *$$

*

$$\mathbf{B} \equiv \begin{bmatrix} 0 & 1 & 8 & 27 \\ 0 & 1 & 4 & 9 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}^{-1}$$

This gives us the 4x4 coefficient matrix (a,b,c,d).

$$\mathbf{p}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \mathbf{G}\,\mathbf{B} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

geometry $$\mathbf{G} \equiv \begin{bmatrix} \mathbf{p}(0) & \mathbf{p}(1) & \mathbf{p}(2) & \mathbf{p}(3) \end{bmatrix}$$

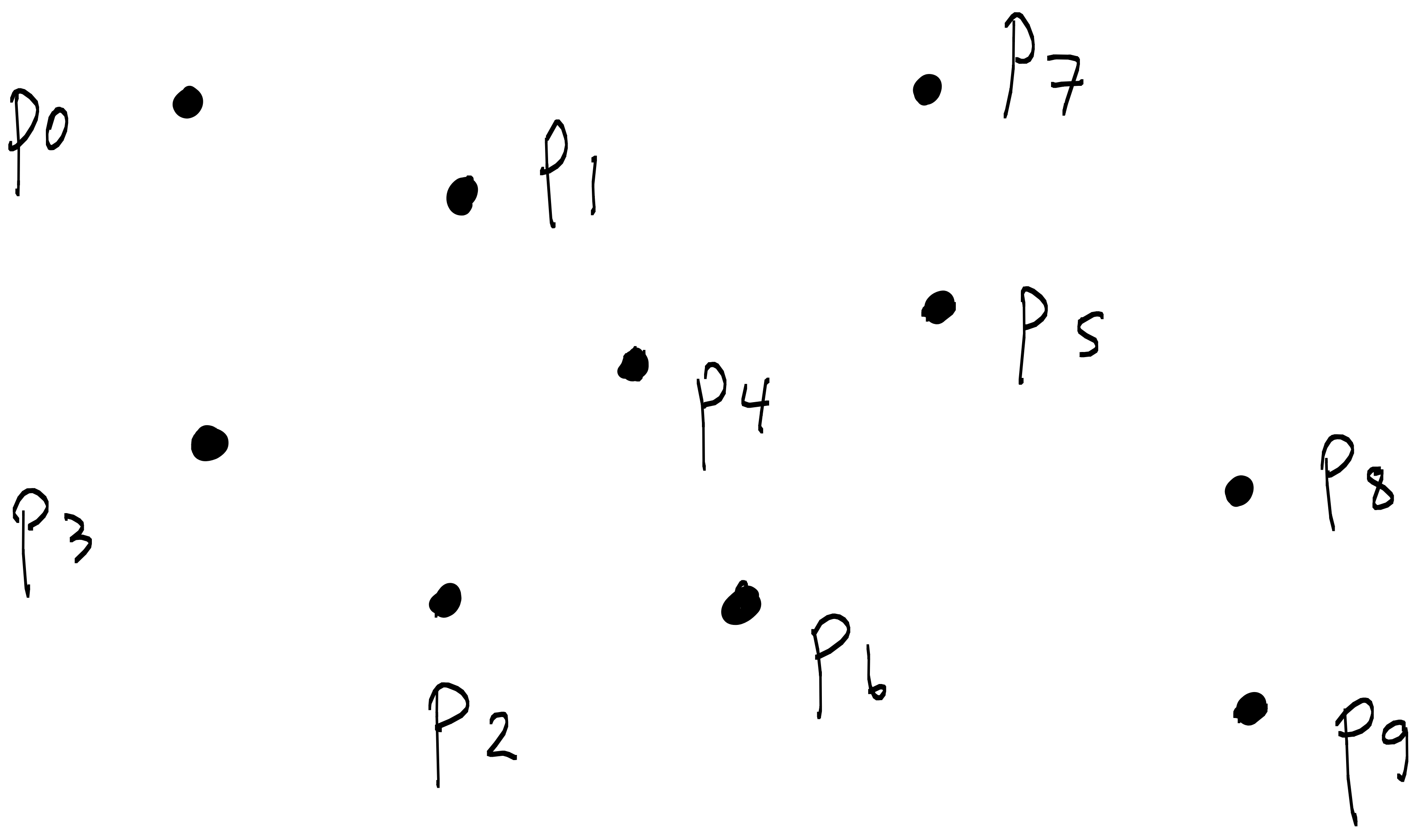
blending $$\mathbf{B} \equiv \begin{bmatrix} 0 & 1 & 8 & 27 \\ 0 & 1 & 4 & 9 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}^{-1}$$

$$\mathbf{B} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

A blending matrix is 4 x 1.
Each row is a polynomial of degree 3 in t.

The blending matrix **B** is independent of the geometry matrix ! It can be precomputed once and stored.

Suppose we are given n >> 4 points and we would like a smooth curve through them in given order.   How?



We could take

[p0, p1, p2, p3]

[p3, p4, p5, p6]

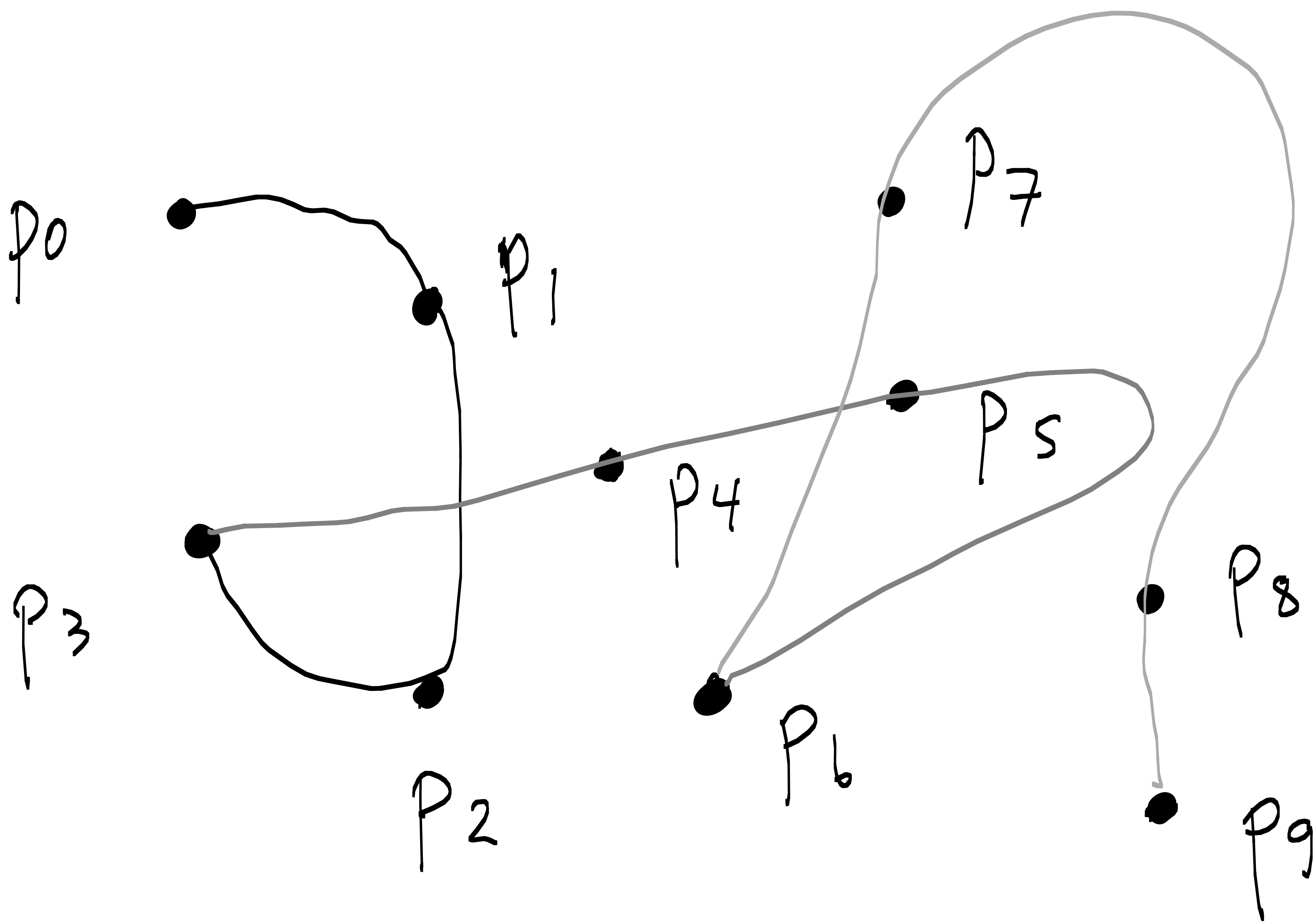
[p6, p7, p8, p9]

and make cubic curve segments for each, and put them together at "knot" points.

Q: Does that work ?

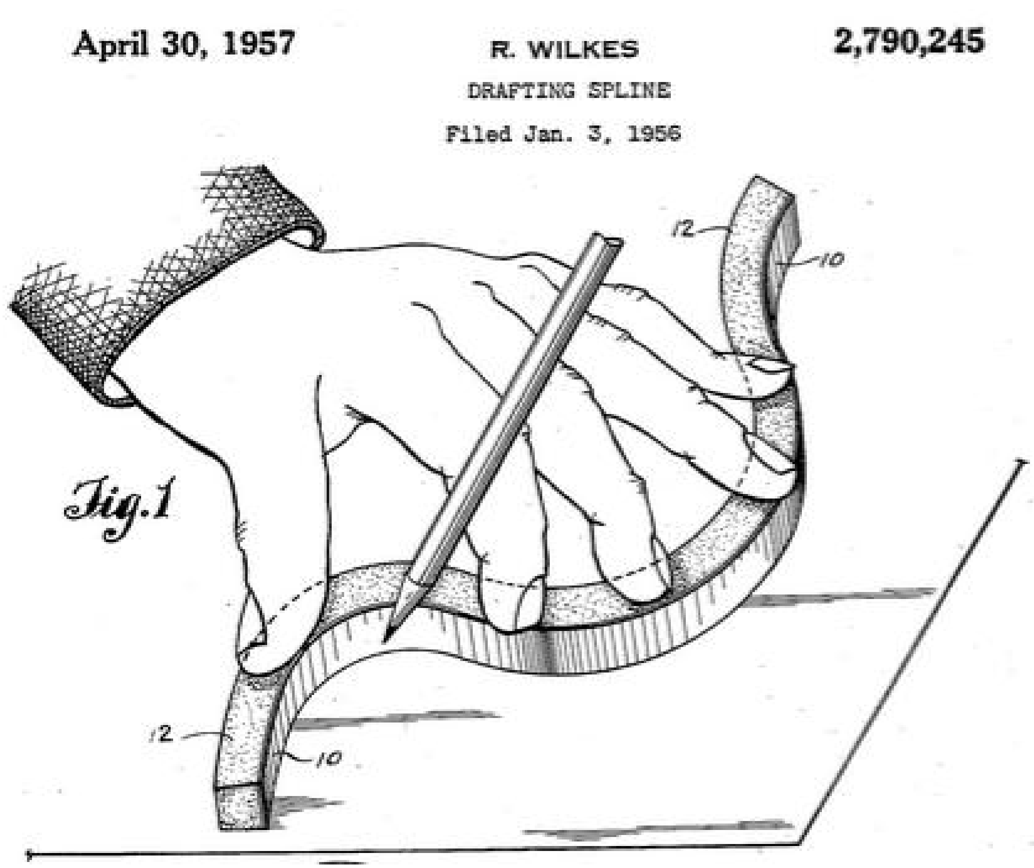
A:

There is no reason why the resulting curve should be smooth at the knots.



Splines

In general, curves used for interpolating between points are called 'splines'.

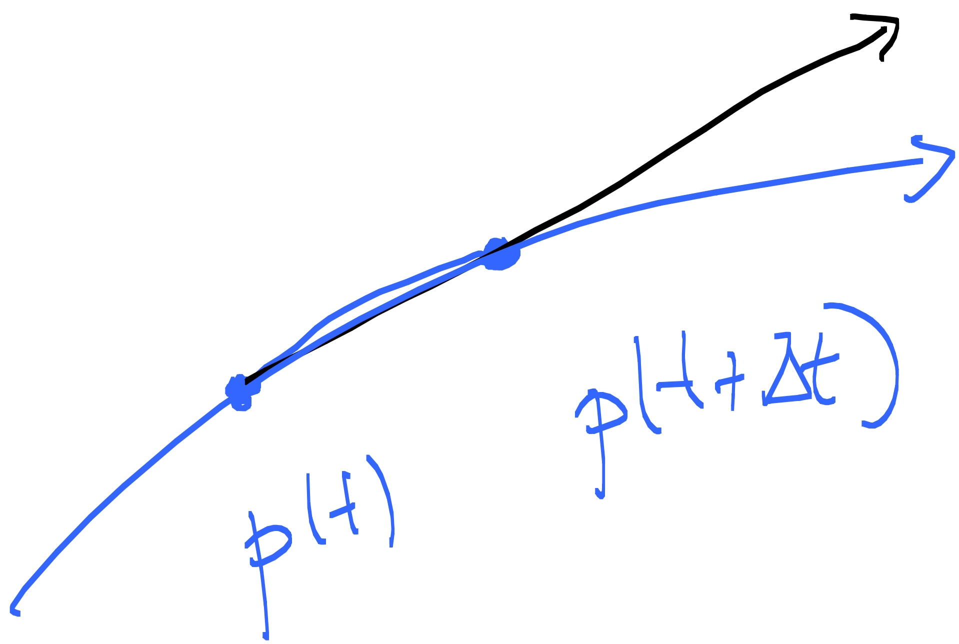


In mathematics, a **spline** is a piecewise polynomial function.

It should satisfy some continuity condition at the places (called "knots") where the polynomial pieces join.
An example is that the "tangent" is continuous ...

Tangent Vector

$$\mathbf{p}'(t) \equiv \frac{d\mathbf{p}(t)}{dt} \equiv lim_{\delta t \to 0} \frac{\mathbf{p}(t+\delta t) - \mathbf{p}(t)}{\delta t}$$



How to reformulate the problem to consider tangents?

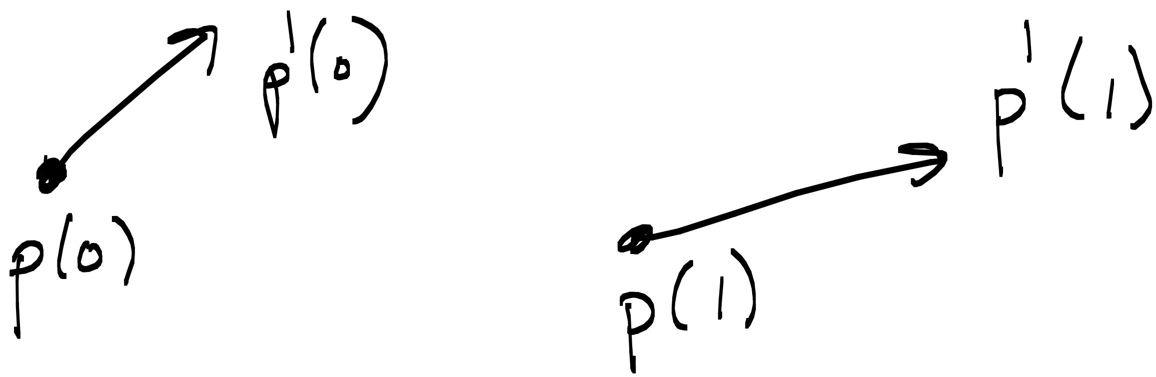
$$\mathbf{p}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} a_x & b_x & c_x & d_x \\ a_y & b_y & c_y & d_y \\ a_z & b_z & c_z & d_z \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

$$\mathbf{p}'(t) = \begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} a_x & b_x & c_x & d_x \\ a_y & b_y & c_y & d_y \\ a_z & b_z & c_z & d_z \end{bmatrix} \begin{bmatrix} 3t^2 \\ 2t \\ 1 \\ 0 \end{bmatrix}$$

Hermite Curve

http://en.wikipedia.org/wiki/Charles_Hermite

Given this:

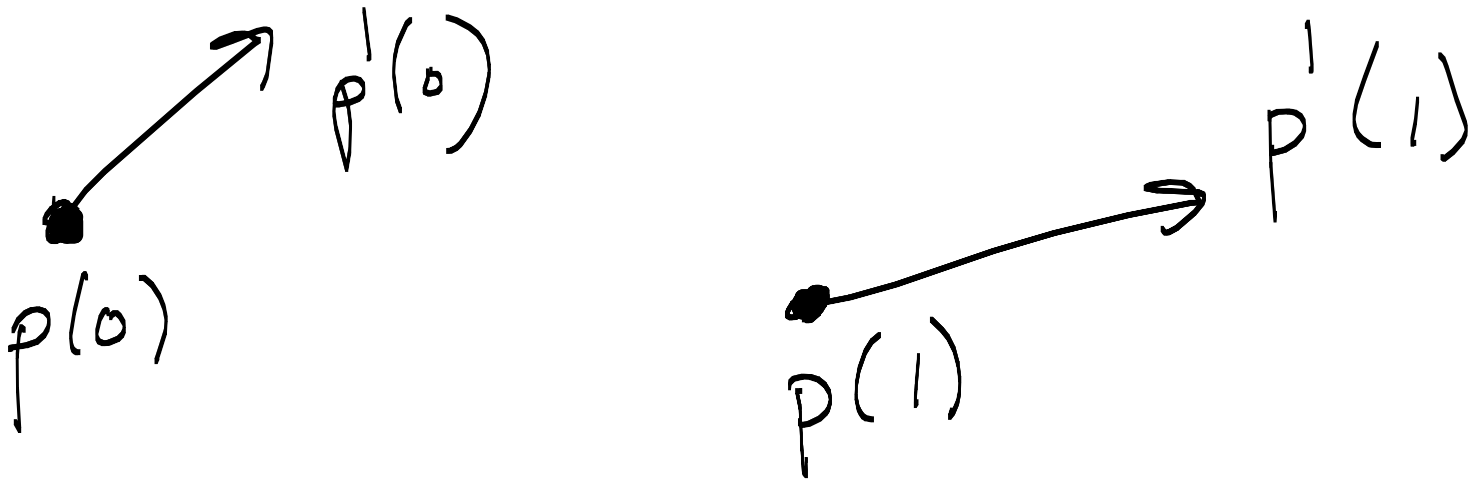


How can we use these data to choose matrices as follows ?

$$\mathbf{p}(t) \;=\; \mathbf{G}_{Hermite}\,\mathbf{B}_{Hermite}\begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

Hermite Curve

http://en.wikipedia.org/wiki/Charles_Hermite



$$
\begin{bmatrix} \mathbf{p}(0) & \mathbf{p}(1) & \mathbf{p}'(0) & \mathbf{p}'(1) \end{bmatrix} = \begin{bmatrix} a_x & b_x & c_x & d_x \\ a_y & b_y & c_y & d_y \\ a_z & b_z & c_z & d_z \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}
$$

$$\mathbf{G}_{Hermite}$$

Multiply both sides by the following.

$$\mathbf{B}_{Hermite} \equiv \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}^{-1}$$

This gives us the coefficient matrix.

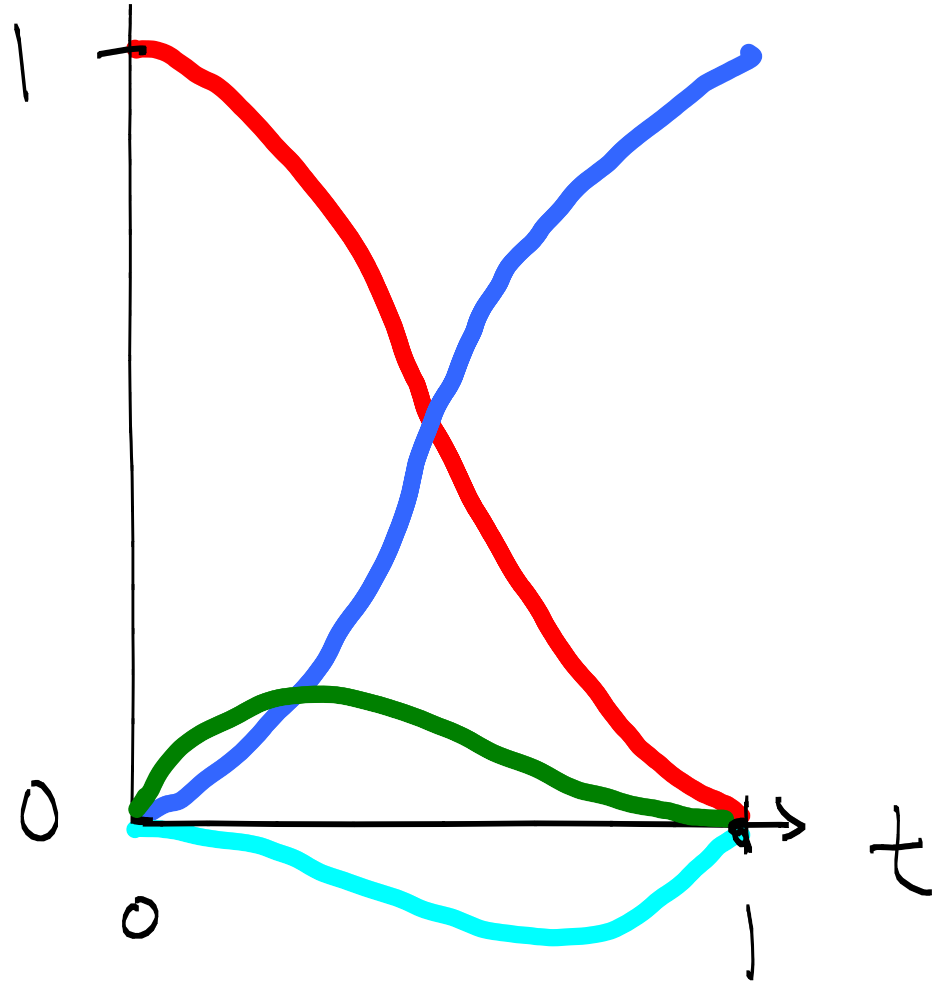
$$\mathbf{p}(t) = \mathbf{G}_{Hermite}\,\mathbf{B}_{Hermite} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

$$\mathbf{G}_{Hermite} \equiv \begin{bmatrix} \mathbf{p}(0) & \mathbf{p}(1) & \mathbf{p}'(0) & \mathbf{p}'(1) \end{bmatrix}$$

$$\mathbf{B}_{Hermite} \equiv \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}^{-1}$$

Q: What are the weights at time t = 0, 1 ?

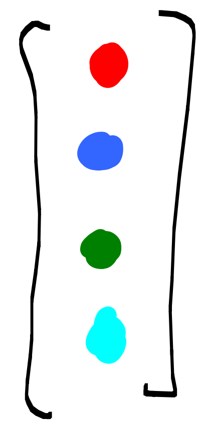
A:



$$\mathbf{p}(t) = \mathbf{G}_{Hermite}\,\mathbf{B}_{Hermite}\begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

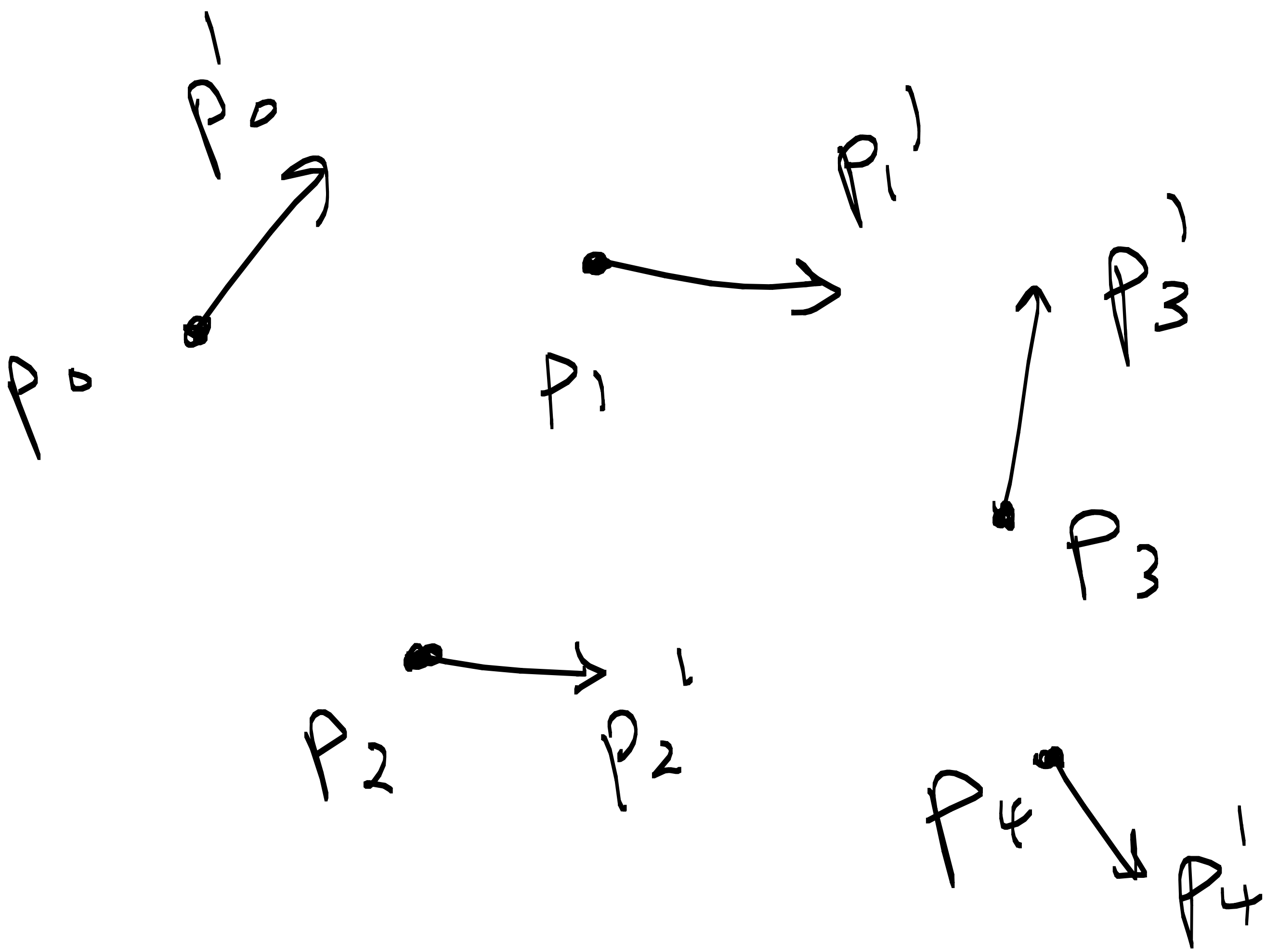
$$\mathbf{G}_{Hermite} \equiv \left[\; \mathbf{p}(0) \quad \mathbf{p}(1) \quad \mathbf{p}'(0) \quad \mathbf{p}'(1) \;\right]$$

$$\mathbf{B}_{Hermite}\begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix} =$$

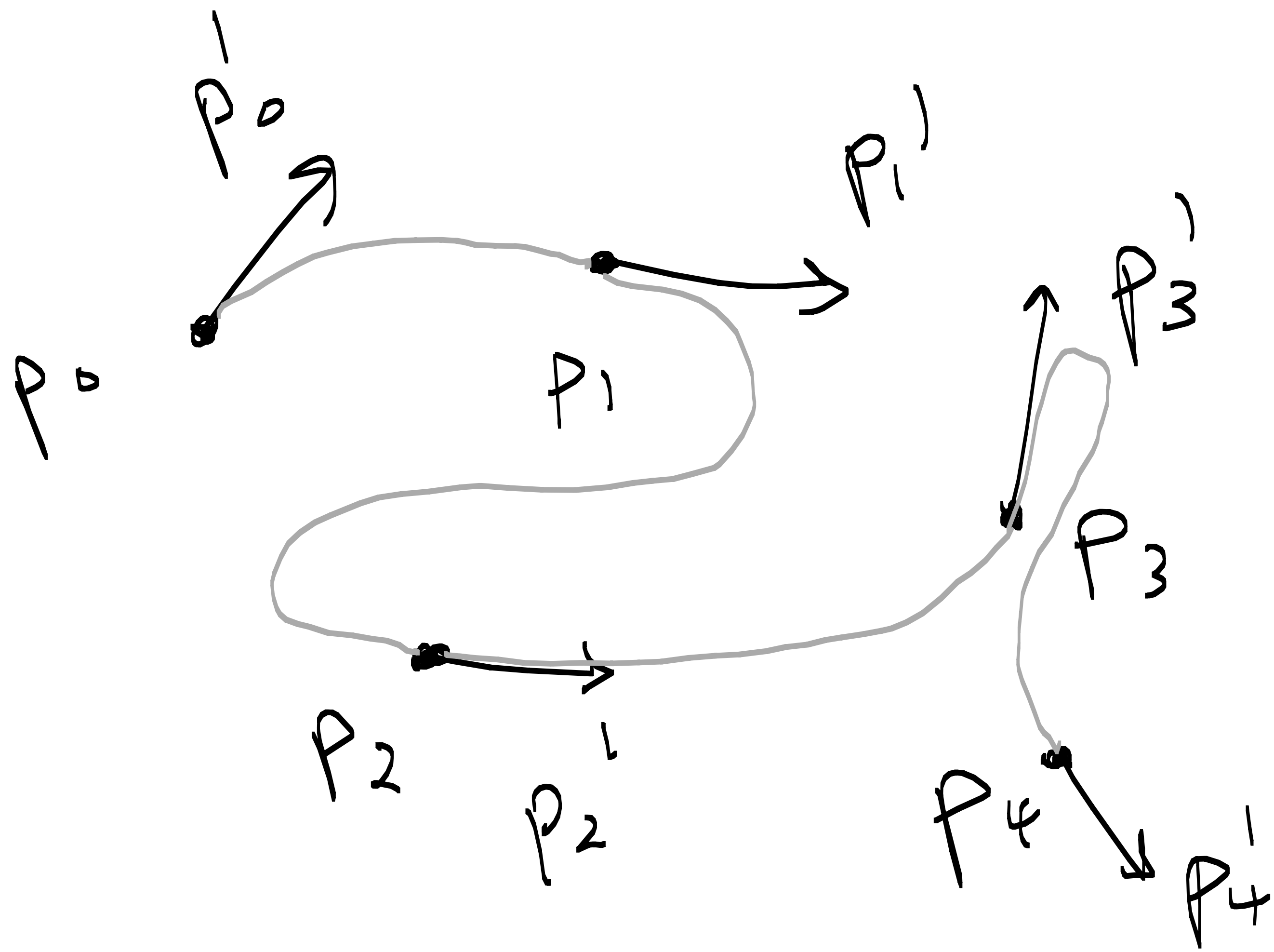


Hermite Cubic Spline

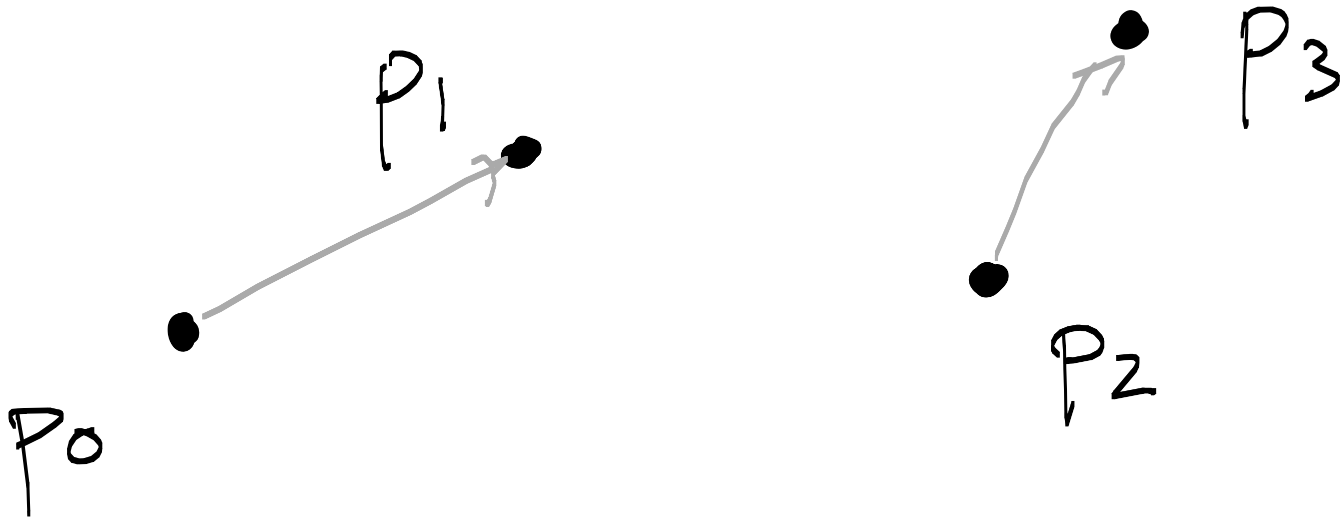
How do we fit a curve to n point + tangent pairs ?



The tangents are equal at the knots (good).



Bézier Curve



$$
\begin{aligned}
\mathbf{q}(0) &= \mathbf{p}_0 \\
\mathbf{q}(1) &= \mathbf{p}_3 \\
\mathbf{q}'(0) &= \beta(\mathbf{p}_1 - \mathbf{p}_0) \\
\mathbf{q}'(1) &= \beta(\mathbf{p}_3 - \mathbf{p}_2)
\end{aligned}
$$

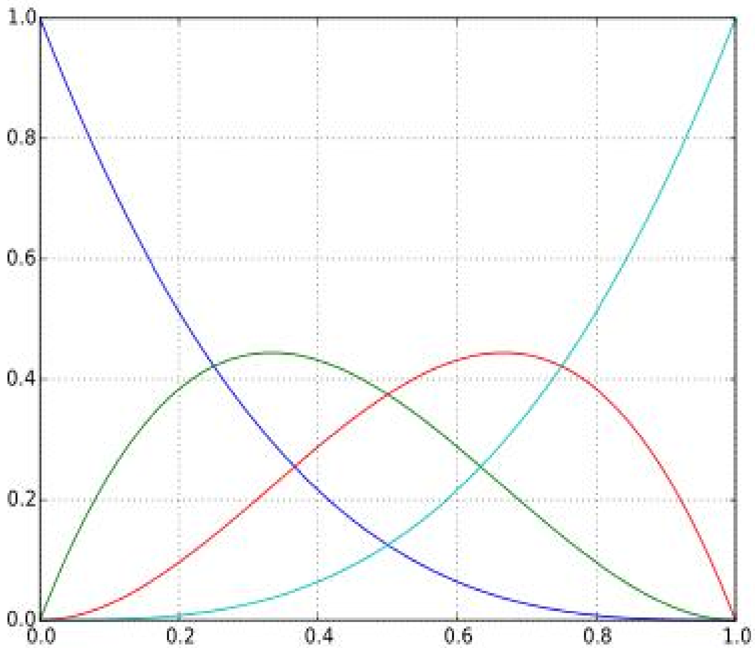
use $\beta = 3$

$$\mathbf{G}_{Hermite} = \begin{bmatrix} \mathbf{q}(0) & \mathbf{q}(1) & \mathbf{q}'(0) & \mathbf{q}'(1) \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 3 \end{bmatrix}$$

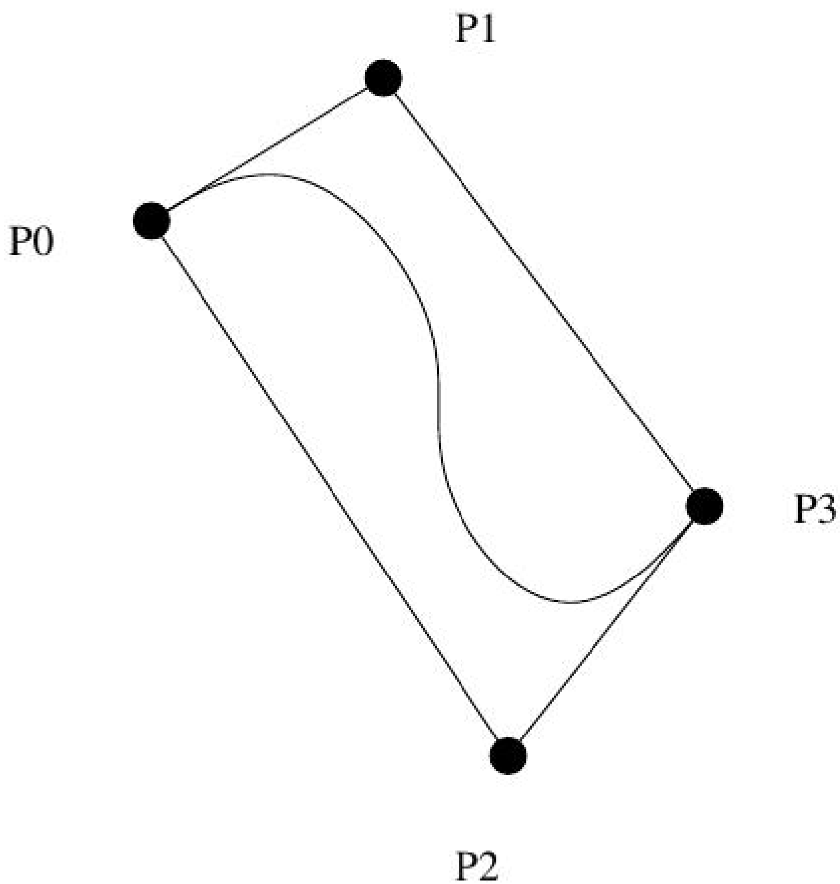
$$\mathbf{G}_{Hermite}\,\mathbf{B}_{Hermite} = \mathbf{G} \underbrace{\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 3 \end{bmatrix} \mathbf{B}_{Hermite}}_{\mathbf{B}_{Bezier}}$$

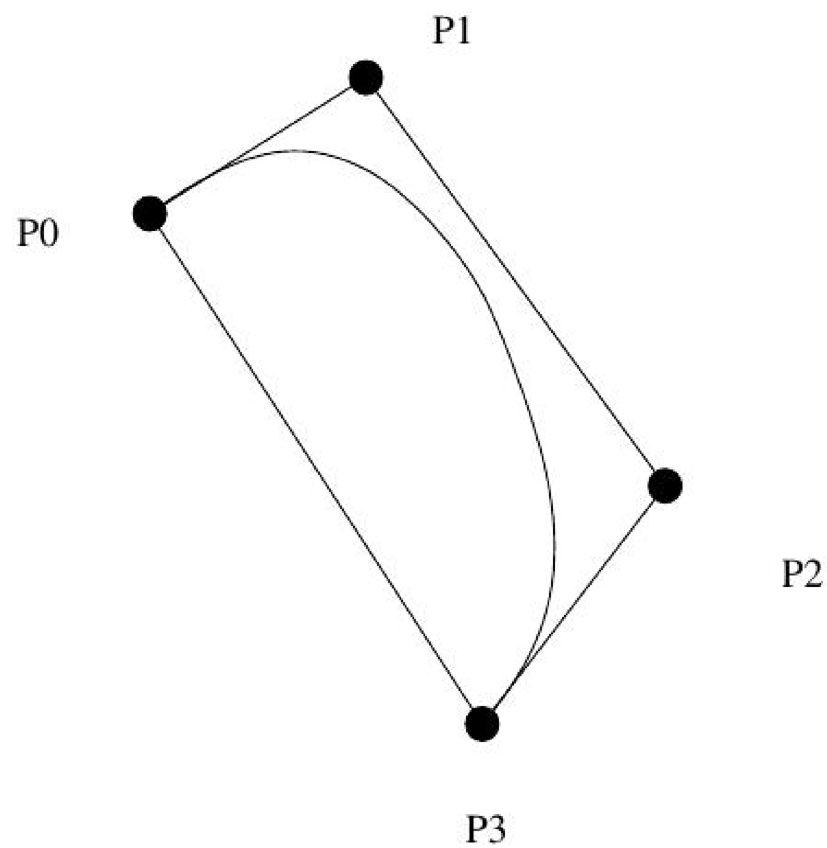
$$\mathbf{B}_{Bezier} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix} = \begin{bmatrix} (1-t)^3 \\ 3t(1-t)^2 \\ 3t^2(1-t) \\ t^3 \end{bmatrix} =$$



"Bernstein polynomials"

for t in [0, 1],   they lie in [0,1]
and they sum to 1.
(convexity property)

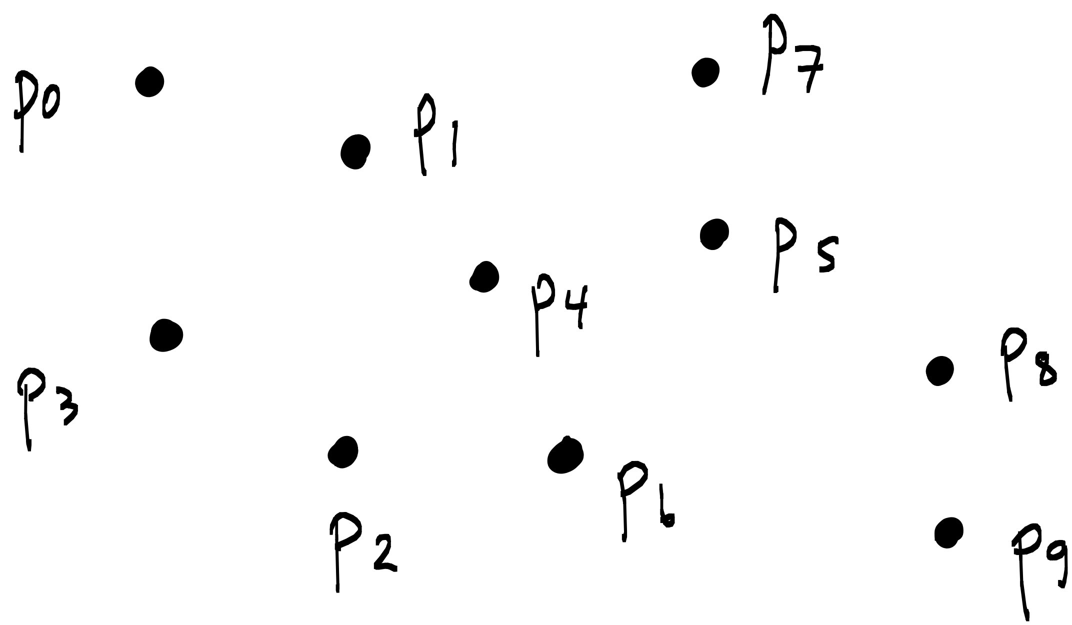




Convexity property can be useful for clipping and intersections e.g. bounding volume.

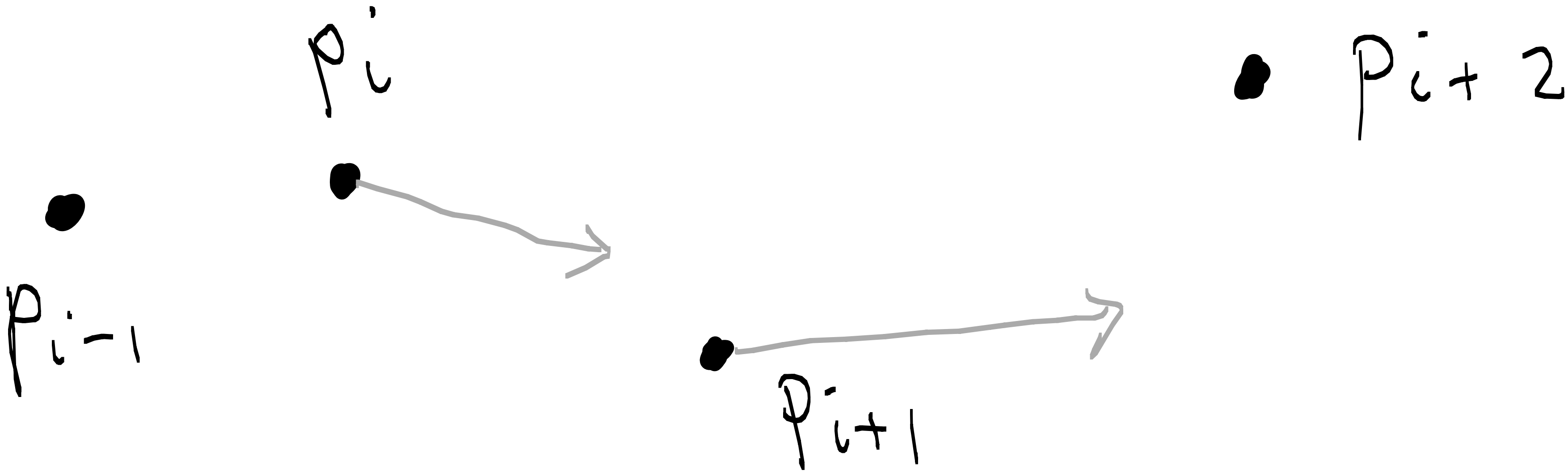
Catmull-Rom Spline (Motivation)

With Bezier, we avoided specifying tangent vectors but now the curve passes through only half the points. (Essentially, we are still specifying tangents.)



We would like to interpolate ALL points (unlike Bezier !) but also have the nice Hermite property of continuous tangent vector at knots.

Catmull-Rom Spline



$$\mathbf{p}_i' = \frac{1}{2}(\mathbf{p}_{i+1} - \mathbf{p}_{i-1}) \qquad \mathbf{p}_{i+1}' = \frac{1}{2}(\mathbf{p}_{i+2} - \mathbf{p}_i)$$

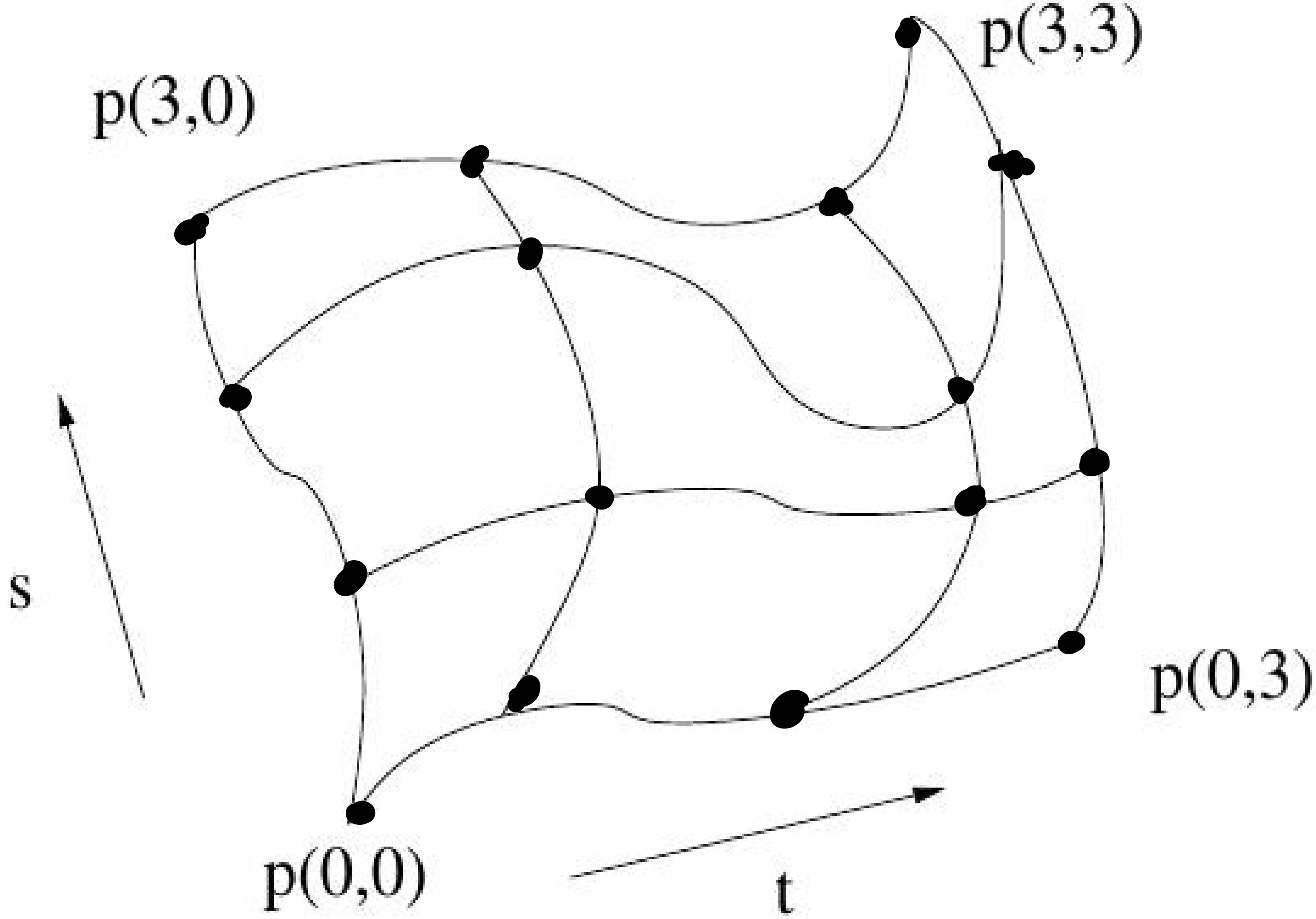
Exercise: what are the geometry and blending matrices?

lecture 10

- cubic curves

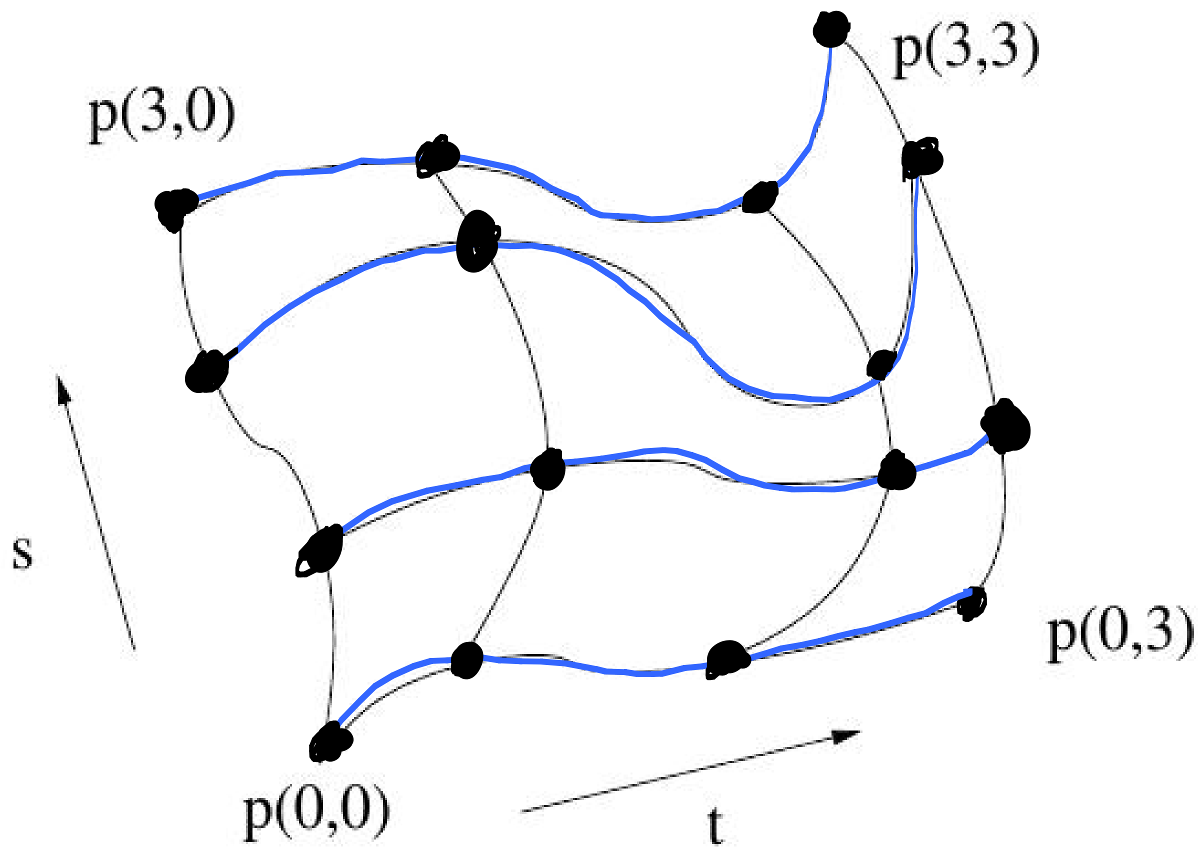
- bicubic surfaces

Suppose we are given a 4x4 "grid" of 3D points.



Start by fitting 4 cubic curves, corresponding to
s = 0, 1, 2, 3. Use the vanilla method at start of lecture.

Then, for each t, fit a cubic curve as a function of s.



The solution is:

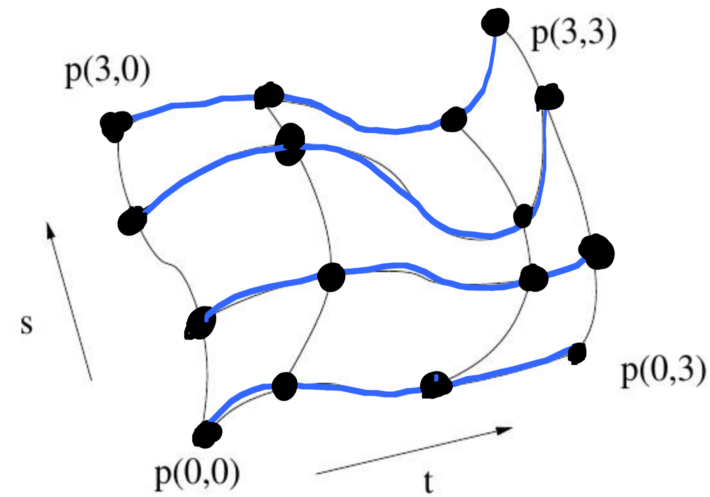
$$x(s,t) = \begin{bmatrix} s^3 & s^2 & s & 1 \end{bmatrix} \mathbf{B}^T \, \mathbf{G}_x \, \mathbf{B} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

... and similarly for y(s,t) and z(s,t).

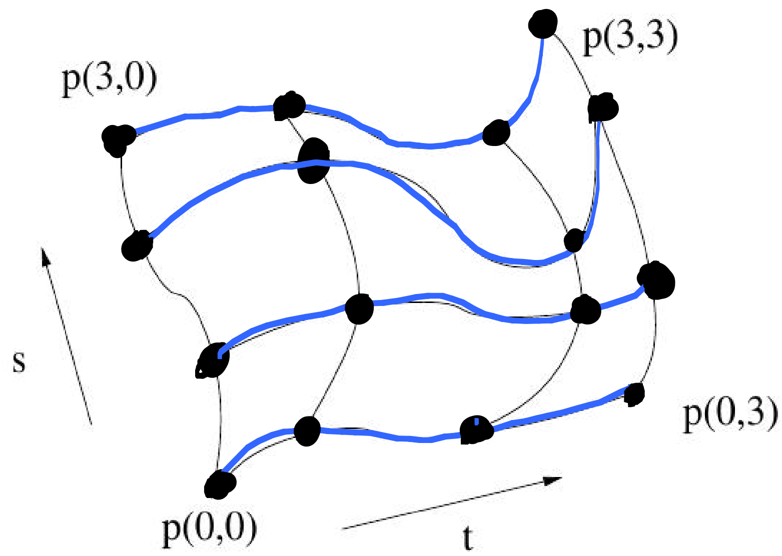
I will derive it over the next several slides.

For each

s = 0, 1, 2, 3



$$x(s,t) = \begin{bmatrix} x(s,0) & x(s,1) & x(s,2) & x(s,3) \end{bmatrix} \mathbf{B} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$



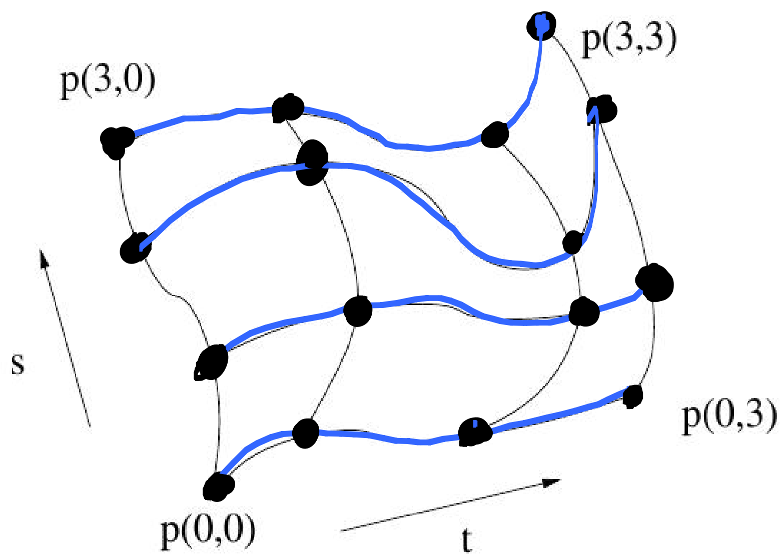
Stacking them gives:

$$\begin{bmatrix} x(0,t) \\ x(1,t) \\ x(2,t) \\ x(3,t) \end{bmatrix} = \begin{bmatrix} x(0,0) & x(0,1) & x(0,2) & x(0,3) \\ x(1,0) & x(1,1) & x(1,2) & x(1,3) \\ x(2,0) & x(2,1) & x(2,2) & x(2,3) \\ x(3,0) & x(3,1) & x(3,2) & x(3,3) \end{bmatrix} \mathbf{B} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

For any t, we have four points which we write as a row vector. This gives us our vanilla problem.

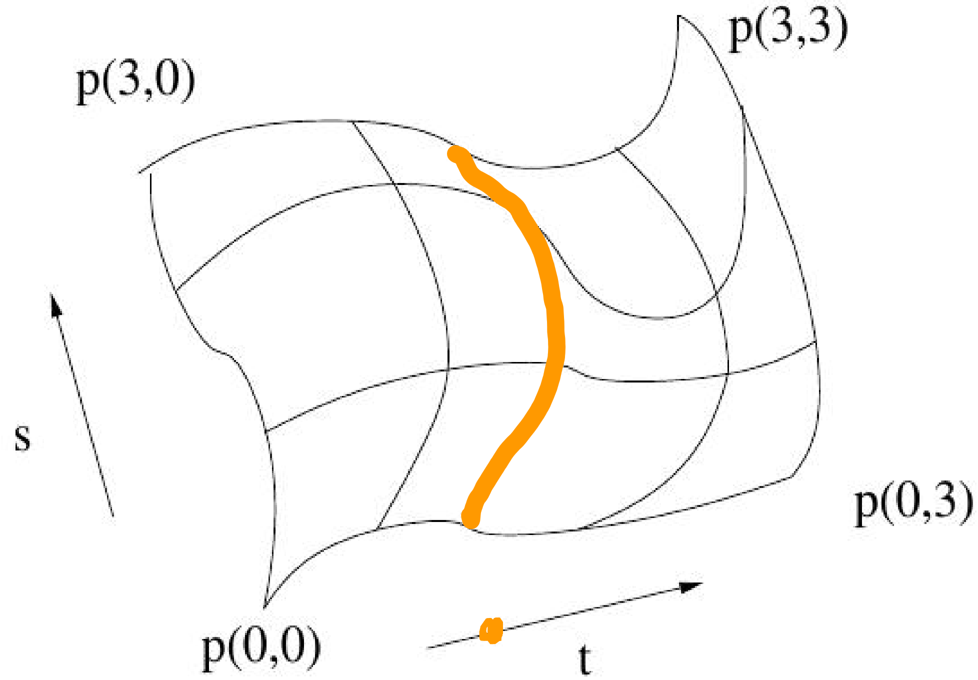
$$\begin{bmatrix} x(0,t) & x(1,t) & x(2,t) & x(3,t) \end{bmatrix}$$

Note: We're only keeping track of x values here, but there are y and z values too.



For any t, we fit a **cubic of parameter s.**

$$x(s,t) = \begin{bmatrix} x(0,t) & x(1,t) & x(2,t) & x(3,t) \end{bmatrix} \mathbf{B} \begin{bmatrix} s^3 \\ s^2 \\ s \\ 1 \end{bmatrix}$$



Taking the transpose...

$$x(s,t) = \begin{bmatrix} s^3 & s^2 & s & 1 \end{bmatrix} \mathbf{B}^T \begin{bmatrix} x(0,t) \\ x(1,t) \\ x(2,t) \\ x(3,t) \end{bmatrix}$$

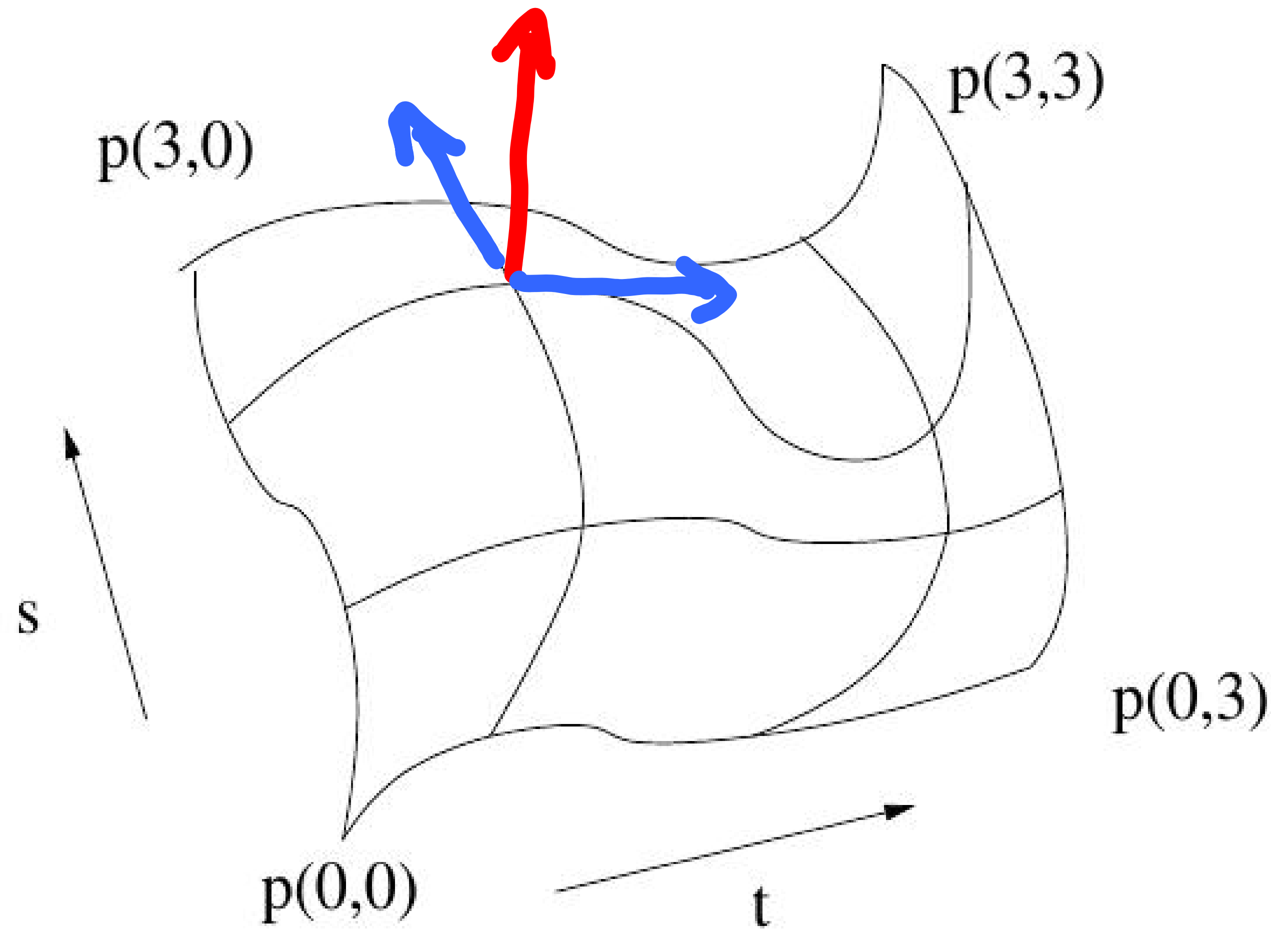
... and replacing the right column with ...

$$\begin{bmatrix} x(0,t) \\ x(1,t) \\ x(2,t) \\ x(3,t) \end{bmatrix} = \begin{bmatrix} x(0,0) & x(0,1) & x(0,2) & x(0,3) \\ x(1,0) & x(1,1) & x(1,2) & x(1,3) \\ x(2,0) & x(2,1) & x(2,2) & x(2,3) \\ x(3,0) & x(3,1) & x(3,2) & x(3,3) \end{bmatrix} \mathbf{B} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

... gives the answer.

$$x(s,t) = \begin{bmatrix} s^3 & s^2 & s & 1 \end{bmatrix} \mathbf{B}^T \mathbf{G}_x \mathbf{B} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

Surface tangents and surface normals



surface tangents

$$\frac{\partial}{\partial t} x(s,t) = \begin{bmatrix} s^3 & s^2 & s & 1 \end{bmatrix} \mathbf{B}^T \, \mathbf{G}_x \, \mathbf{B} \begin{bmatrix} 3t^2 \\ 2t \\ 1 \\ 0 \end{bmatrix}$$

$$\frac{\partial}{\partial s} x(s,t) = \begin{bmatrix} 3s^2 & 2s & 1 & 0 \end{bmatrix} \mathbf{B}^T \, \mathbf{G}_x \, \mathbf{B} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

Any linear combination of these is also a tangent.

surface tangents

$$\frac{\partial}{\partial t}\begin{bmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{bmatrix} \qquad \frac{\partial}{\partial s}\begin{bmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{bmatrix}$$

surface normal

$$\frac{\partial}{\partial s}\mathbf{p}(s,t) \times \frac{\partial}{\partial t}\mathbf{p}(s,t)$$

Summary

- cubic curve vs cubic spline
- 'vanilla', Hermite, Bézier, Catmull-Rom
- geometry (control points) versus blending
- defining tangents in terms of differences of point positions
- lots of algebra that you should understand but not memorize