Answers to midterm questions

[4] 1. Answer the following short questions. **Keep your answers very short and to the point!**

(a) Give a homogeneous representation of the point $p = (3, 1, 4)^T$.
$(3w, 1w, 4w, w)$ for $w \neq 0$

(b) In 3D, under what conditions will translation commute with rotation?
Translation and rotation in 3D commute when the rotation axis is in the same direction as the translation.

(c) Given a 3x3 matrix R, what properties must R have for it to represent a 3D rotation?
$\det(R) = 1, R^T R = I$

(d) Let $R_x(\theta_x)$ be a rotation in 3D about the x axis by θ_x degrees, and similarly let R_y and R_z be rotations about y and z respectively. Give example values for θ_x,θ_y, and θ_z which produce gimbal lock in the Euler angle rotation $R_x(\theta_x)R_y(\theta_y)R_z(\theta_z)$.
$R_x(\theta_x)R_y(90°)R_z(\theta_z)$

[2] 2. Draw a sketch to describe the order of the pipeline of transformations. Use labels to decribe both the different coordinate spaces, and the transformations between them.
See slides on viewing, page 6.

[2] 3. Given eye point $\boldsymbol{e}$, look at point $\boldsymbol{l}$, and up vector $\boldsymbol{a}$, write down the formulas necessary to compute the viewing transformation. Draw and label a diagram of the world and camera frames to accompany your formulas.

$$\boldsymbol{w} = \frac{\boldsymbol{e} - \boldsymbol{l}}{\|\boldsymbol{e} - \boldsymbol{l}\|}, \quad \boldsymbol{u} = \frac{\boldsymbol{a} \times \boldsymbol{w}}{\|\boldsymbol{a} \times \boldsymbol{w}\|}, \quad \boldsymbol{v} = \boldsymbol{w} \times \boldsymbol{u}$$

and the viewing transform V can be written

$$\begin{pmatrix} \boldsymbol{u} & \boldsymbol{v} & \boldsymbol{w} & \boldsymbol{e} \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1}$$

See the diagram on page 7 of the viewing slides, and note that the camera axes should be drawn as well in the explanation diagram.

[4] 4. Answer the following questions related to the perspective projection matrix we derived in class, with near plane at distance n and far plane at distance f along the negative axis.

$$P = \begin{pmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n+f & nf \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

(a) Work out what happens when you transform the vector $(x, y, z)^T$ by P. Provide a short interpretation of the result in non-homogeneous coordinates.
See Slide 36 of the viewing slides. The vector, in homogeneous coordinates is $(x, y, z, 0)^T$, and multiplied by P one obtains $(nx, ny, (n + f)z, -z)^T$. Thus, the vectors become points on a plane in front of the camera at distance $n + f$, i.e., coordinate $(n + f)z/(-z) = -n - f$. All vectors in 3D map to this 2D because the points on the plane have different representations in homogeneous coordinates, i.e., the negative of the z component of the vector is the w coordinate of the homogeneous point after projection.

(b) Work out what P does to a point (x, y, z) with $z > 0$, i.e., a point behind the camera. Your short interpretation of the result in non-homogeneous coordinates should make use of the fact that $z > 0$.
Points behind the camera go in front of the camera. That is,

$$P\ (x, y, z, 1)^T = (nx, ny, (n + f)z + nf, -z)^T,$$

and one can note that the z coordinate of the normalized result is $\frac{(n+f)z+nf}{-z}$, which is negative because we are told $z > 0$, and n and f are positive distances to the near and far plane along the negative z-axis.

[2] 5. The projection matrix we derived in class preserves z values on the near and far plane. Can you modify the projection matrix so that it instead projects points onto the near plane? Either write down the matrix or explain why you can't.
We want the modified matrix to project the z value exactly in the same way as the x and y, thus,

$$P = \begin{pmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

and this will take point $(x, y, z, 1)^T$ to $(nx, ny, nz, -z)^T$. After dividing by the w coordinate we have $(-nx/w, -ny/z - n, 1)^T$, which is the desired point on the near plane.

[2] 6. Suppose you have a sphere of radius one at the origin, a point light source at position $(10, 10, 10)^T$, and you are viewing the sphere from position $(0, 0, 10)^T$, with both positions given in world coordinates. If the sphere is drawn with Lambertian shading, which point on the sphere will appear brightest? Write your answer in world coordinates.
The brightness of a point on the sphere will only depend on the light direction and the normal, thus, does not depend on the viewing position. The brightest point will have a normal that points toward the light, which will be the point $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})^T$.

[4] 7. What point is brightest on a glossy plane with plane equation $z = 0$ (i.e., the xy plane) drawn with Lambertian and Blinn-Phong shading using a directional light?

With all quantities and expressions in world coordinates, let the eye position be $\boldsymbol{e}$ and let $\boldsymbol{l}$ be the vector pointing towards the light. First, derive a formula for the reflected light direction $\boldsymbol{r}$. Then, solve for the point on the plane that appears brightest, writing your answer in terms of $\boldsymbol{e}$ and $\boldsymbol{r}$. *Hint: You may want to solve for the point of intersection between a parametric line equation and an implicit plane equation.*

The directional light will illuminate the flat plane identically everywhere, thus, the brightest point is determined entirely by the specular reflection. Given unit length normal $\boldsymbol{n}$ and light direction $\boldsymbol{l}$ The reflected direction is can be computed with the formula

$$\boldsymbol{r} = (2\boldsymbol{n}\boldsymbol{n}^T - I)\boldsymbol{l}.$$

In this case, $\boldsymbol{r} = (l_x, l_y, -l_z)^T$ because the plane normal is in the z direction. Defining parametric line $\boldsymbol{e} + t\boldsymbol{r}$, and substituting into plane equation $z = 0$, gives $\boldsymbol{e}_z + t\boldsymbol{r}_z = 0$, or $t = \boldsymbol{e}_z/\boldsymbol{r}_z$. Thus the brightest point is $(\boldsymbol{e}_x + \boldsymbol{r}_x\boldsymbol{e}_z/\boldsymbol{r}_z, \boldsymbol{e}_y + \boldsymbol{r}_y\boldsymbol{e}_z/\boldsymbol{r}_z, 0)^T$.