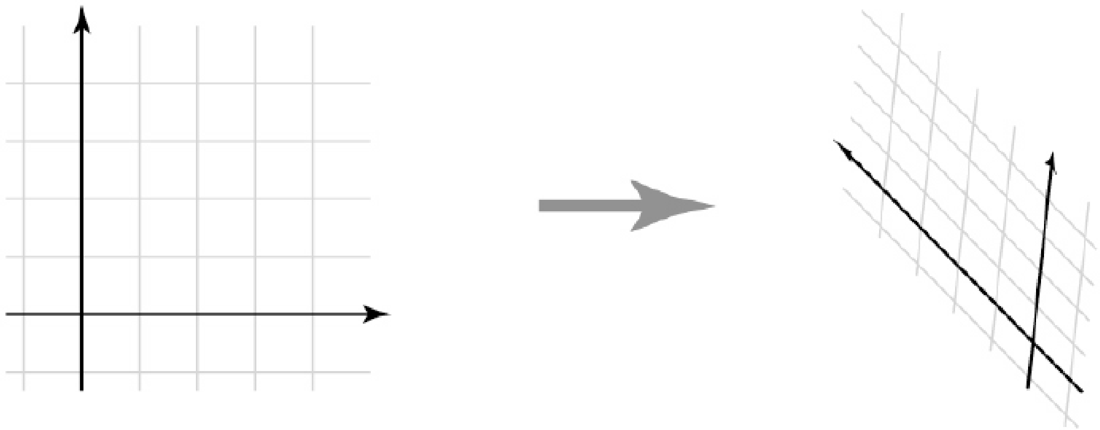
Transformations

COMP 557

Affine transformations

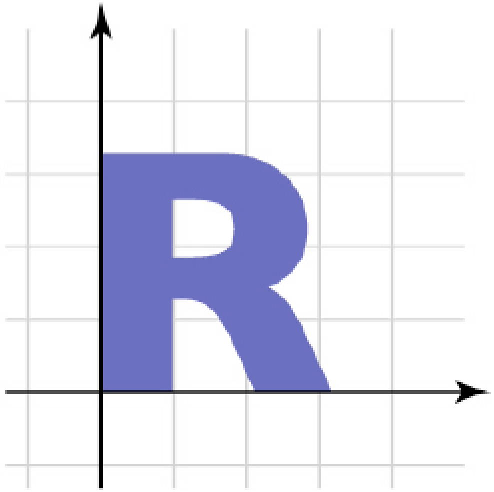
- The set of transformations we have been looking at, linear transformations combined with translation, is known as affine transformations
  - Straight lines are preserved
  - Parallel lines are preserved
  - Ratios of lengths along lines are preserved (e.g., midpoints)

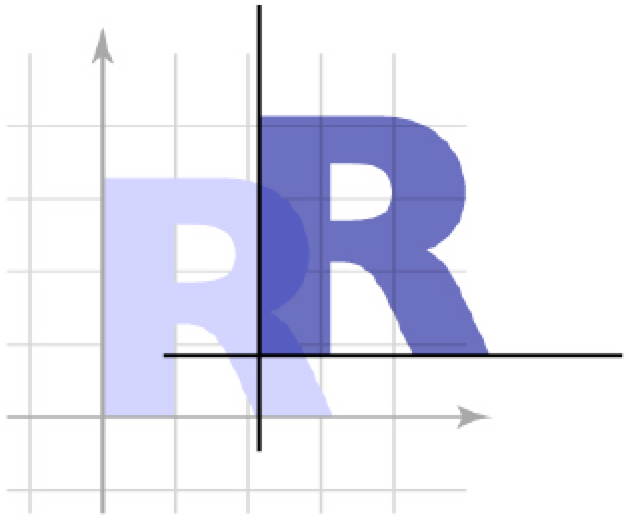


Affine transformation examples in homogeneous coordinates

$$\begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix}, \text{ for example, } \begin{pmatrix} 1 & 0 & 2.15 \\ 0 & 1 & 0.85 \\ 0 & 0 & 1 \end{pmatrix}$$

- Translation

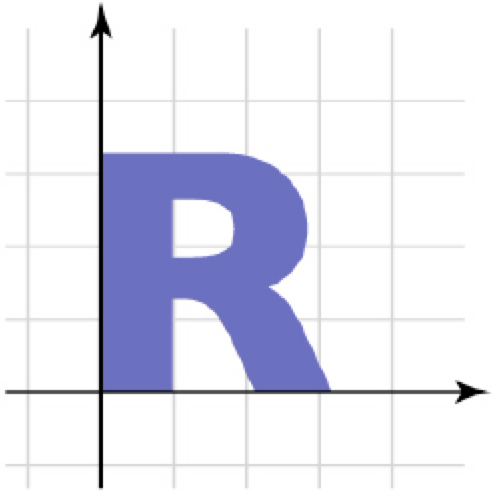


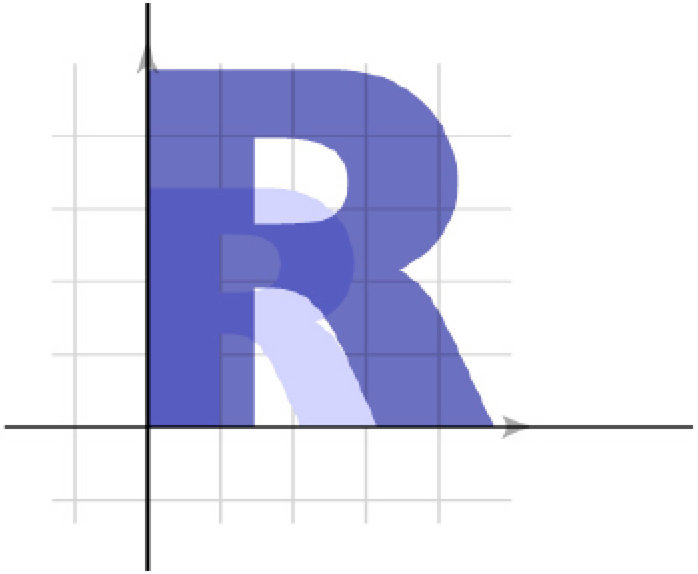


Affine transformation examples in homogeneous coordinates

$$\begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{pmatrix}\text{, for example, }\begin{pmatrix} 1.5 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Uniform scale

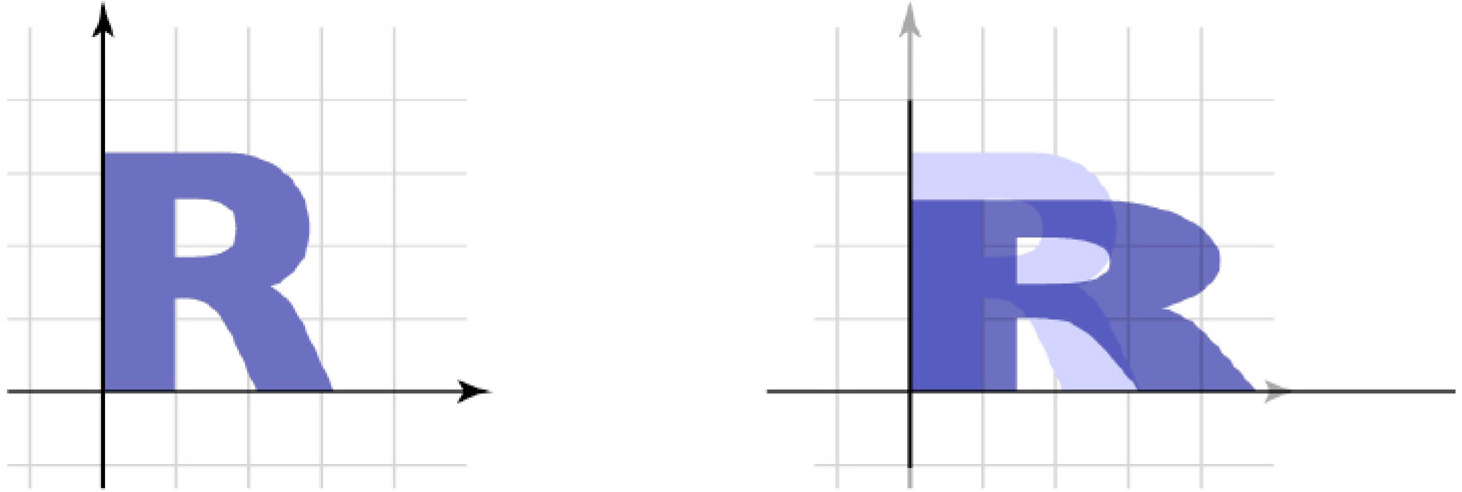




Affine transformation examples in homogeneous coordinates

$$\begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ for example, } \begin{pmatrix} 1.5 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

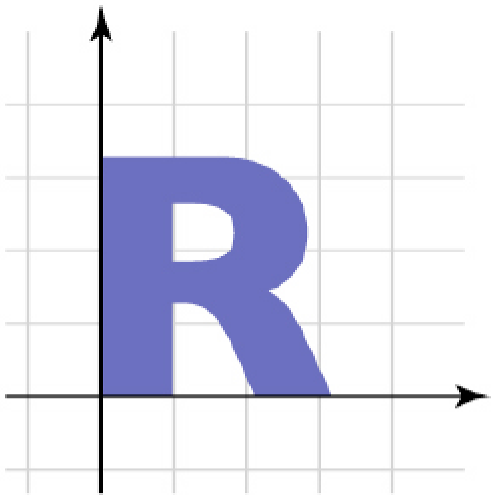
- Non-uniform scale

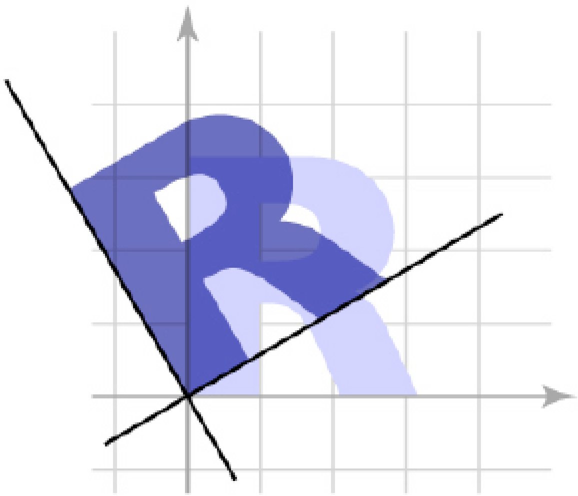


Affine transformation examples in homogeneous coordinates

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ for example, } \begin{pmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Rotation

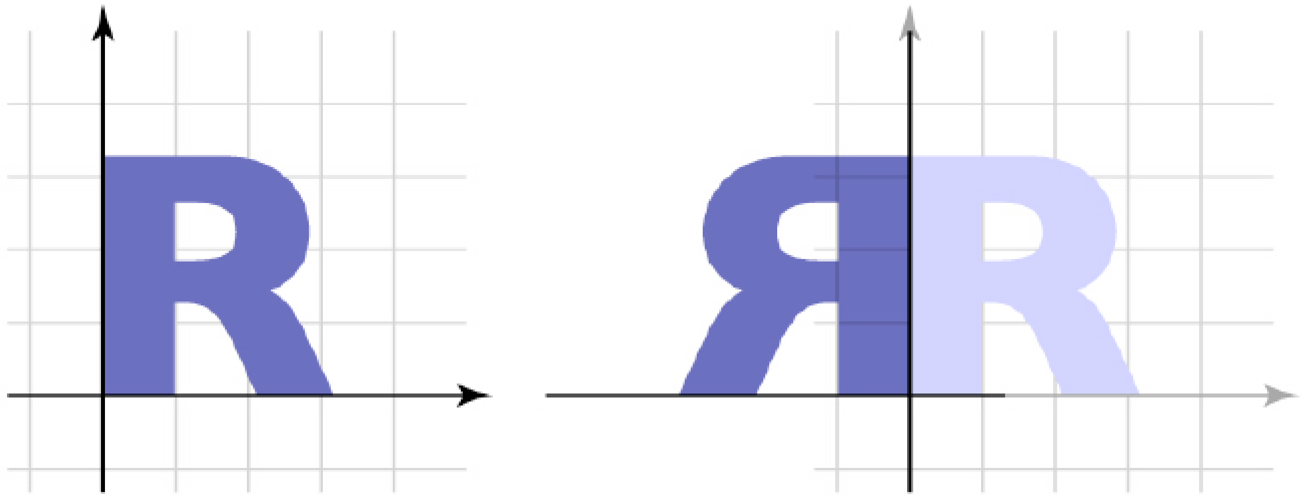




Affine transformation examples in homogeneous coordinates

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

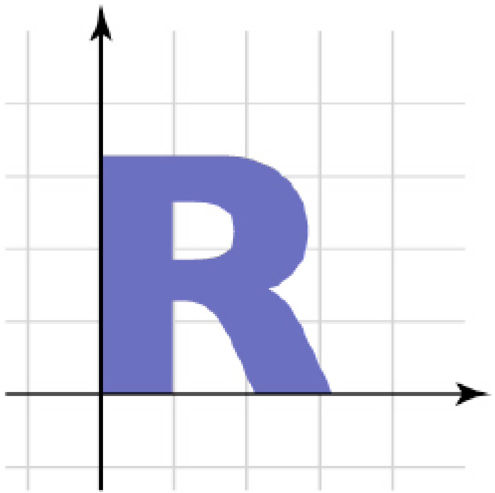
- Reflection (about y axis)

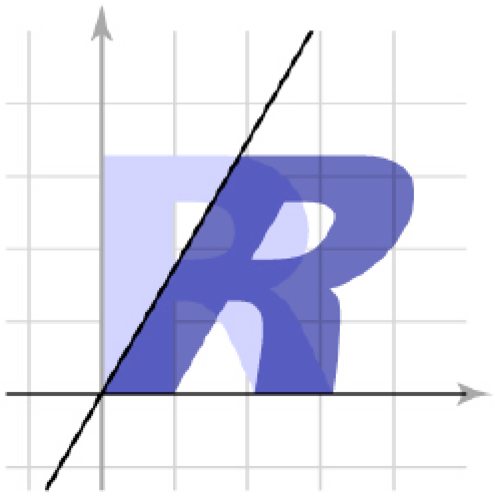


Affine transformation examples in homogeneous coordinates

$$\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ for example, } \begin{pmatrix} 1 & 0.5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Shear





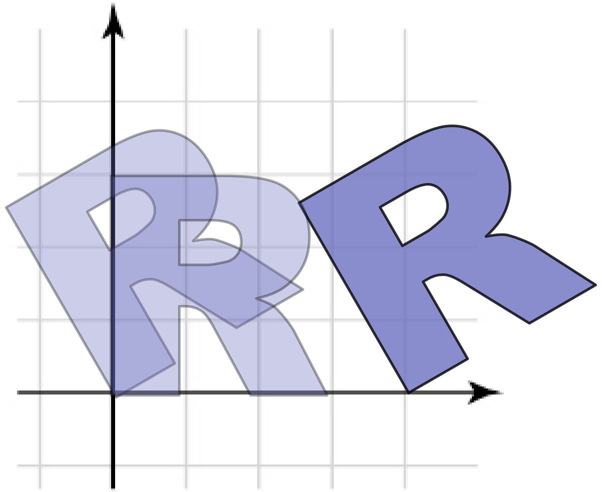
General affine transformations

- The previous slides show “canonical” examples of the different types of affine transformations
- In general, an affine transformations can be different than any single canonical example
  - Often define them as a product of canonical transforms
  - Sometimes define them more directly based on desired properties (e.g., creating a change of coordinates)

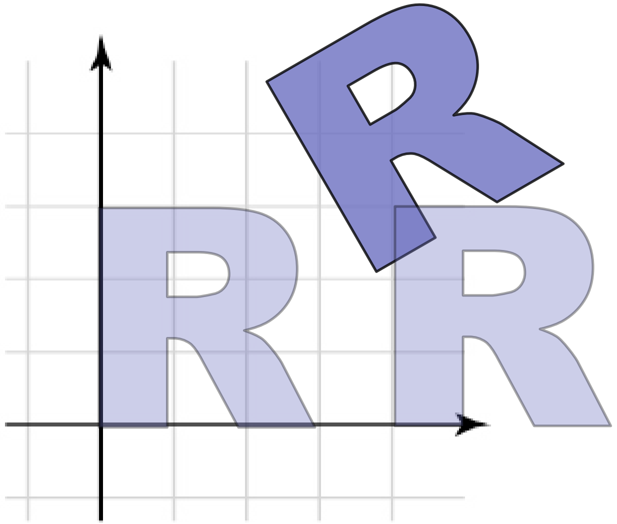
Composite affine transformation

- In general, non commutative: order matters!

rotate then translate



translate then rotate



Rigid motions

- A transformation made up of only translation and rotation is a rigid motion
  - Also called a rigid body transformation

$$E = \begin{pmatrix} R & \mathbf{u} \\ 0 & 1 \end{pmatrix}$$

- The inverse is easy to write, because the rotation R is orthonormal with $R^{-1} = R^T$, thus,

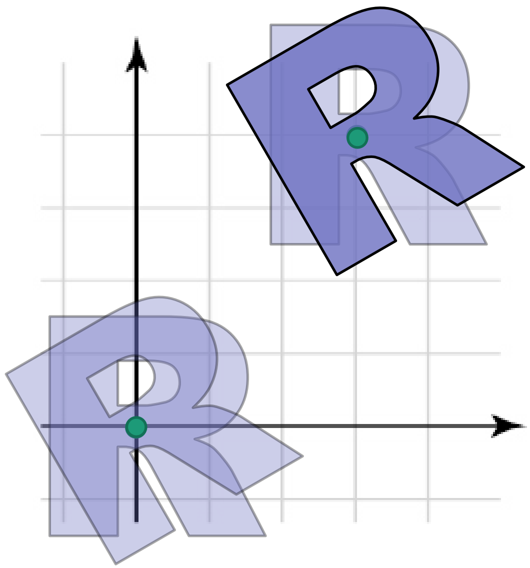
$$E^{-1} = \begin{pmatrix} R^T & -R^T\mathbf{u} \\ 0 & 1 \end{pmatrix}$$

$$E^{-1}E = \begin{pmatrix} R^T & -R^T\mathbf{u} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R & \mathbf{u} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R^T R & R^T\mathbf{u} - R^T\mathbf{u} \\ 0 & 1 \end{pmatrix}$$

Composing to change axes

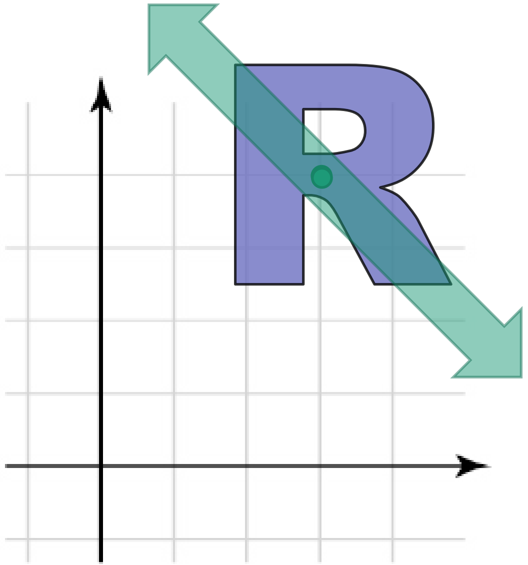
- Want to rotate about a particular point in space?
  - Easily done by composing transformations
  - We know how to rotate about the origin
  - Translate that point to origin, rotate, and translate back!

$$M = T^{-1}\,R\,T$$



Composing to change axes

- Want to scale along particular axis and point?
  - We know how to scale about y axis at origin
  - Suppose we want to stretch by 1.5 times in the direction and at the point shown below



$$M = T^{-1}\, R^{-1} S\, R\, T$$

What are the contents of S R and T?

Transforming points and vectors

- Note distinction between points and vectors
  - Vectors are offsets (differences between points)
  - Points have a location
    - Represented by vector offset from a fixed origin
- Points and vectors transform differently
  - Points respond to translation; vectors do not
  - Consider

$$v = p - q$$
$$T(x) = Mx + t$$

Transforming points and vectors

$$
\begin{aligned}
v &= p - q \\
T(x) &= Mx + t
\end{aligned}
$$

- T is an affine transformation
- T is not a linear transformation

$$
\begin{aligned}
T(x+y) &\neq T(x) + T(y) \\
&= M(x+y) + t + t
\end{aligned}
$$

$$
\begin{aligned}
T(p) - T(q) &= Mp + t - (Mq + t) \\
&= M(p-q) + (t-t) \\
&= Mv
\end{aligned}
$$

- Vectors are only transformed by linear part

Transforming points and vectors

- In homogeneous coordinates, vectors have $w = 0$
- Translation of an affine transformation is excluded for vectors in homogeneous coordinates

$$\begin{pmatrix} M & \mathbf{t} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ 1 \end{pmatrix} = \begin{pmatrix} M\mathbf{p} + \mathbf{t} \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} M & \mathbf{t} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ 0 \end{pmatrix} = \begin{pmatrix} M\mathbf{v} \\ 0 \end{pmatrix}$$

- Preview of what is to come
  - Last coordinate need not always be 0 or 1
  - Can change last row in matrix to do perspective projection

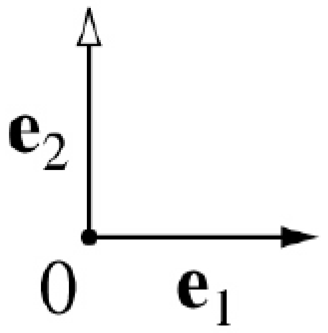
Coordinate systems

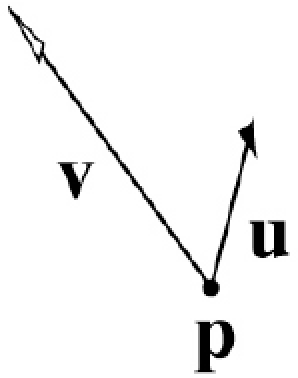
- Can express vectors with respect to different bases
- Linear transformation is a change of basis
- This leads to an alternate view
  - We were previously thinking of modeling transformations (e.g., “make it bigger”)
  - Can instead think of them as a means of using different reference frames, or different units (e.g., cm to inches)

Affine change of coordinates in 2D

- There are 6 degrees of freedom

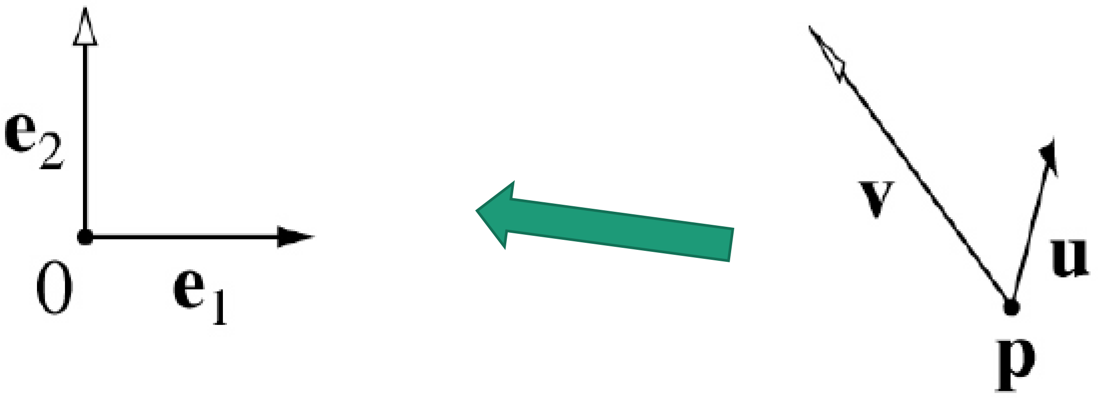
$$\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \mathbf{u} & \mathbf{v} & \mathbf{p} \\ 0 & 0 & 1 \end{pmatrix}$$





Affine change of coordinates

- Coordinate frame: point and a basis
- Interpretation is that the transformation changes the representation of point (or vector) from one basis to another
  - Frame to canonical matrix has frame in columns
  - Takes points in coordinates of the frame
  - Gives points in coordinates of the canonical basis

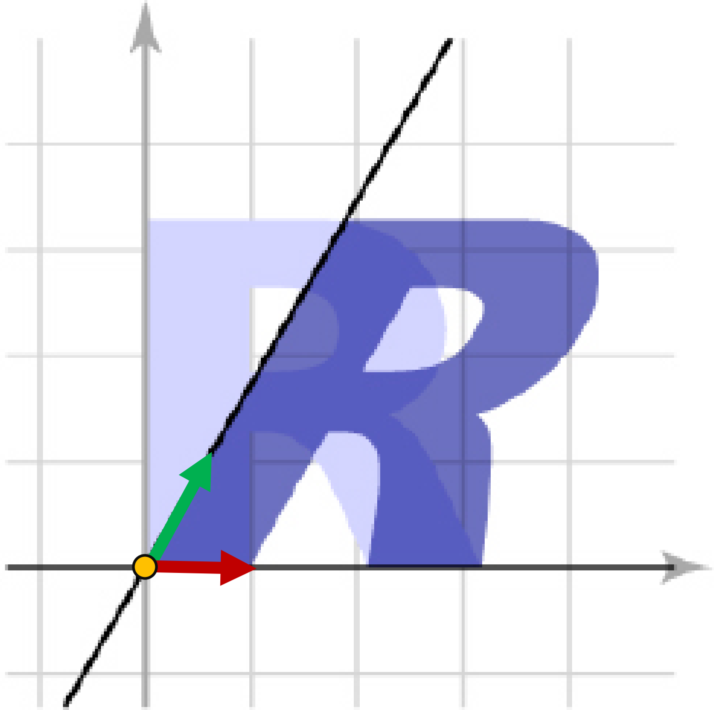


$$\begin{pmatrix}\mathbf{u} & \mathbf{v} & \mathbf{p}\\ 0 & 0 & 1\end{pmatrix}$$

Affine change of coordinates

- A new way to “read off” the matrix
  - e.g., shear from earlier
  - Observe how the matrix columns correspond to the change of shape

$$\begin{array}{ccc} \boldsymbol{u} & \boldsymbol{v} & \boldsymbol{p} \\ \end{array}$$
$$\begin{pmatrix} 1 & 0.5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Affine change of coordinates

- When we move an object to the origin to apply a transformation, we are really changing coordinates
  - Easy to express the transformation in the object's frame
  - Define it there and transform it

$$T_e = F\, T_F\, F^{-1}$$

  - T_e is the transform expressed with respect to $\mathbf{e}_1\ \mathbf{e}_2$ canonical basis
  - T_F is the transformation expressed in the natural frame
  - F is the frame-to-canonical matrix $(\mathbf{u} \quad \mathbf{v} \quad \mathbf{p})$, here in block form with homogeneous representation for basis vectors $\mathbf{u}$ and $\mathbf{v}$ and origin point $\mathbf{p}$
  - This is a similarity transformation

Coordinate frame summary

- Frame is a point plus a basis
- Frame matrix (frame to canonical) is

$$F = \begin{pmatrix} \mathbf{u} & \mathbf{v} & \mathbf{p} \\ 0 & 0 & 1 \end{pmatrix}$$

- Change coordinates of point a to and from frame by multiplying by F and F^{-1}

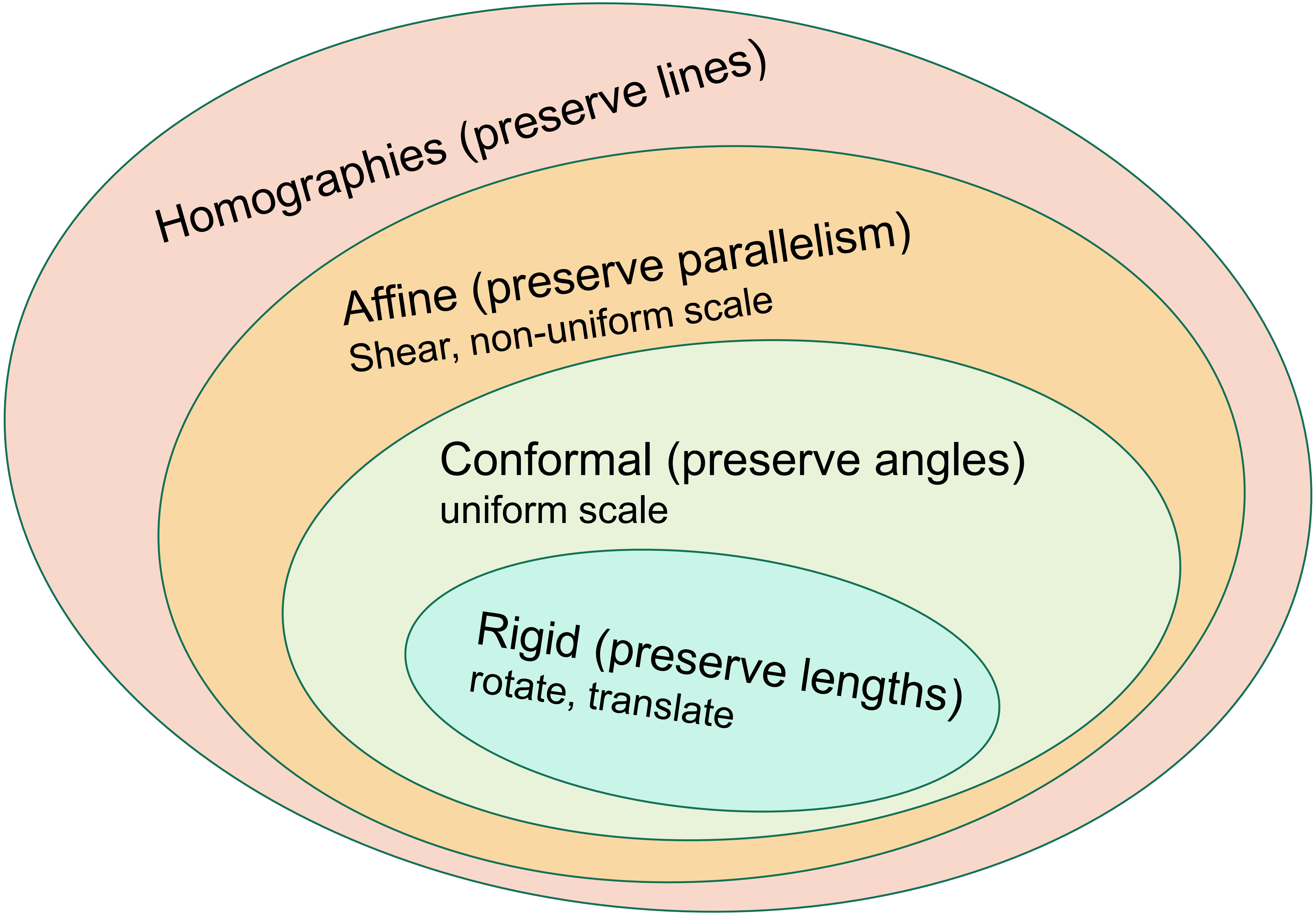
$$a_e = F a_F \qquad a_F = F^{-1} a_e$$

- "Move" transformations using similarity transform

$$T_e = F\, T_F\, F^{-1} \qquad T_F = F^{-1}\, T_e\, F$$

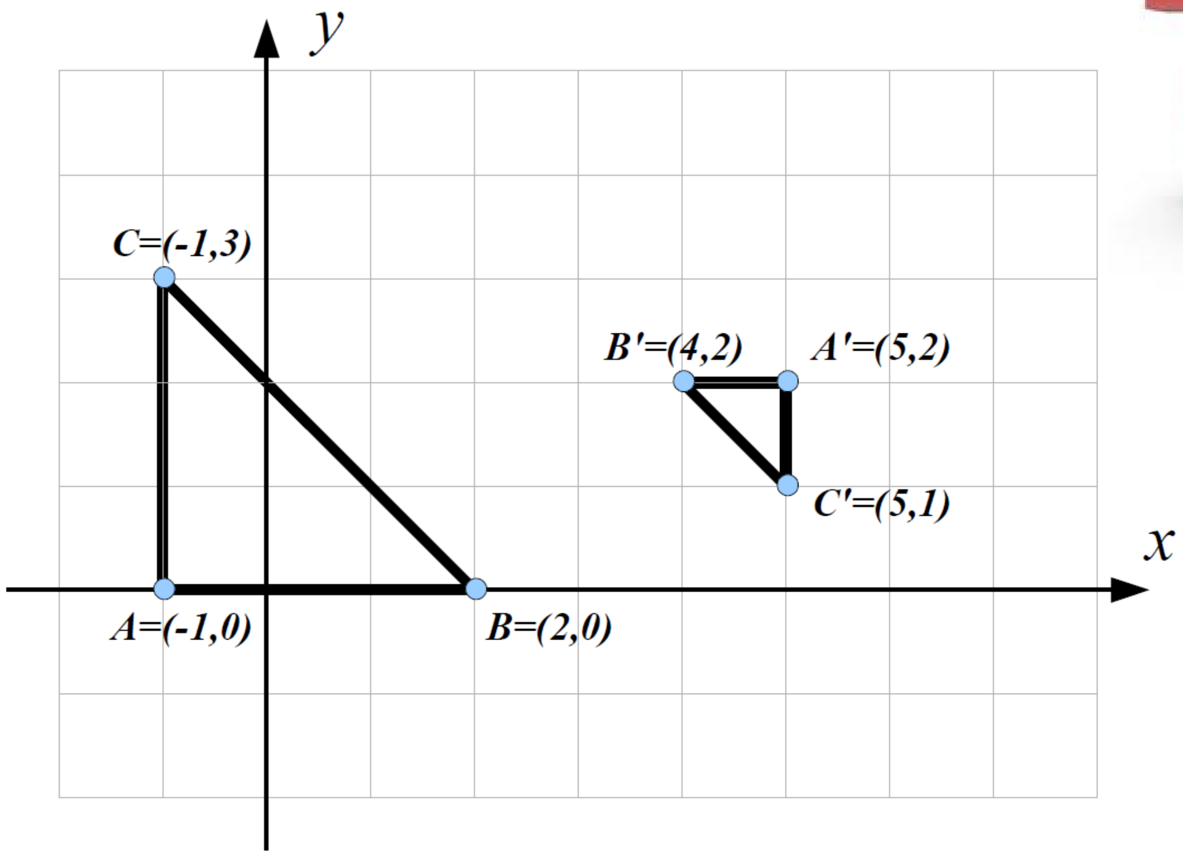
Classes of Transformations

- The set of 3x3 matrices we can use to transform homogenous points have a variety of properties
  - Most generally, they map lines to lines.
    - These are called **Homographies** provided they are invertible (this set include perspective projection)
  - A more restrictive set also preserves parallel lines
    - These are called **Affine** transforms
  - A more restrictive set also preserves angles
    - These are called **Conformal**
  - A more restrictive set also preserves lengths
    - These are called **Rigid**
- Where do translate, rotate, and scale fit?



Homographies (preserve lines)
Affine (preserve parallelism)
Shear, non-uniform scale
Conformal (preserve angles)
uniform scale
Rigid (preserve lengths)
rotate, translate

Question



Give a homogeneous transformation matrix, or product of matrices, that will transform triangle ABC to triangle A'B'C'.

Review and more information

- Review
  - CGPP Chapter 7 up to and not including 7.7
    - Essential Math and Geometry of 2D and 3D
  - FCG Chapter 2 and Chapter 5
- CGPP Chapter 10 up to and including 10.10, or FCG Chapter 5
  - 2D transformations
  - Points and Vectors
  - Skip the SVD (e.g., CGPP 10.3.8 and 10.3.9)
  - Inverses of transformation matrices
  - Coordinate Transformations