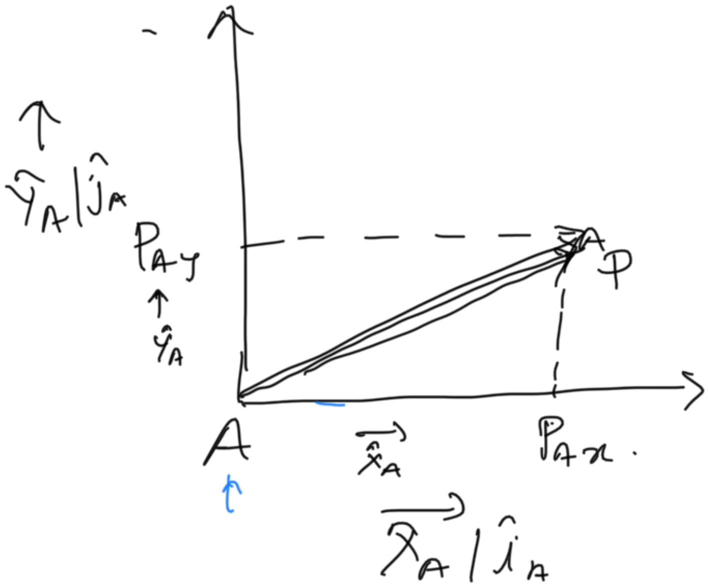
LECTURE-2

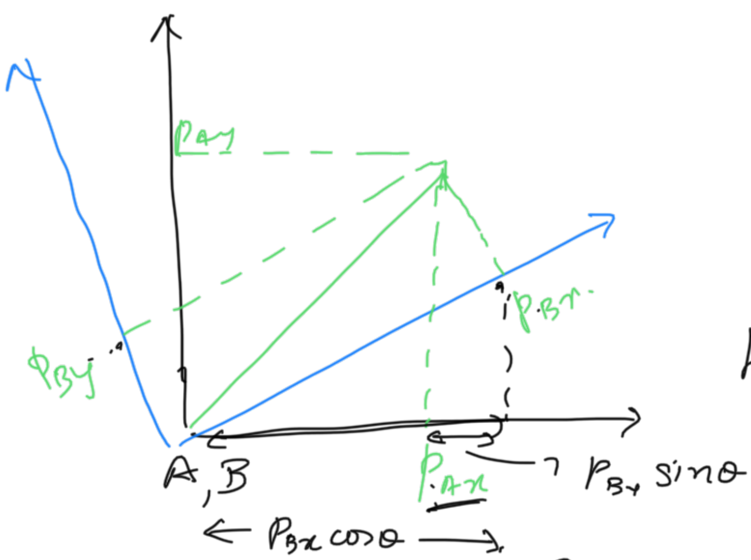
Coordinate Transform:



$${}^{A}\vec{P} = \begin{bmatrix} P_{Ax} \\ P_{Ay} \\ P_{Az} \end{bmatrix} \longrightarrow (1)$$

$$= \underbrace{\begin{bmatrix} \hat{X}_A & \hat{Y}_A & \hat{Z}_A \end{bmatrix}}_{\text{unit vectors}} \begin{bmatrix} P_{Ax} \\ P_{Ay} \\ P_{Az} \end{bmatrix} \longrightarrow (2)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} P_{Ax} \\ P_{Ay} \\ P_{Az} \end{bmatrix}$$



$${}^{A}P = \begin{bmatrix} P_{Ax} & P_{Ay} & P_{Az} \end{bmatrix}^T$$

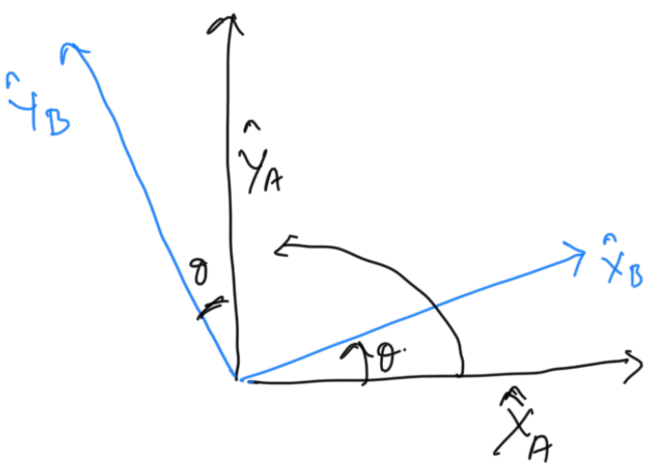
$${}^{B}P = \begin{bmatrix} P_{Bx} & P_{By} & P_{Bz} \end{bmatrix}^T$$

What is ${}^{B}P$ in A?

$${}^{B}P = \begin{bmatrix} \hat{X}_B & \hat{Y}_B & \hat{Z}_B \\ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} P_{Bx} \\ P_{By} \\ P_{Bz} \end{bmatrix}$$

Then $^{A}P = \begin{bmatrix} {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B & {}^{A}\hat{Z}_B \end{bmatrix} \begin{bmatrix} p_{Bx} \\ p_{By} \\ p_{Bz} \end{bmatrix} \longrightarrow (3)$

$$= \underbrace{\begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix}}_{R_B^A} \begin{bmatrix} p_{Bx} \\ p_{By} \\ p_{Bz} \end{bmatrix} \longrightarrow (4)$$



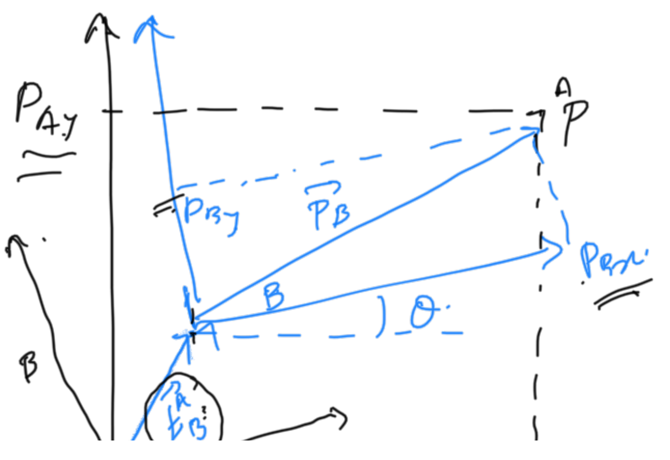
R_B^A or rotation of frame B with respect to A.

R_B^A is also the ROTATION MATRIX of B wrt A.

$$R_B^A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

→ can also be obtained from trigonometry.

In presence of both Rotation and Translation



$$\underbrace{{}^{A}\vec{P}}_{3\times1} = \underbrace{R_B^A \vec{P}_B}_{3\times1} + \vec{t}_B^A \longrightarrow (5)$$

[origin of B's frame in A.]

${}^{A}t_{B}^{A}$ is the translation of B wrt A. ${}^{A}p = {}^{A}\vec{t}_{B}^{A}$

$$\boxed{\underset{4\times1}{{}^{A}P} = \underset{4\times4}{T_B^A}\ \underset{4\times1}{{}^{B}P} \longrightarrow (6)}$$

$${}^{A}\vec{P} = T_B^A\ {}^{B}\vec{P}$$

T_B^A is the homogenous transform matrix

$$\underset{4\times4}{T_B^A} = \begin{bmatrix} R^A_{B\,3\times3} & \vec{t}^{\,A}_{B\,3\times1} \\ 0_{1\times3} & 1_{1\times1} \end{bmatrix} \longrightarrow (7)$$

$[0\ 0\ 0\ 1]$

$$T_B^A = \begin{bmatrix} C\theta & -S\theta & 0 & t^A_{Bx} \\ S\theta & C\theta & 0 & t^A_{By} \\ 0 & 0 & 1 & t^A_{Bz} \\ 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow (8)$$

$C\theta = \cos(\theta)$
$S\theta = \sin(\theta)$

$$\underbrace{{}^{A}\vec{P} = \begin{bmatrix} P_{Ax} \\ P_{Ay} \\ P_{Az} \\ 1 \end{bmatrix} \quad {}^{B}\vec{P} = \begin{bmatrix} P_{Bx} \\ P_{By} \\ P_{Bz} \\ 1 \end{bmatrix}}_{\text{Homogenous Coordinates}}$$

$\}\ \mathbb{R}^3$

$\mathbb{R}^3 \hookrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$M: \mathbb{R}^3 \rightarrow P^3$

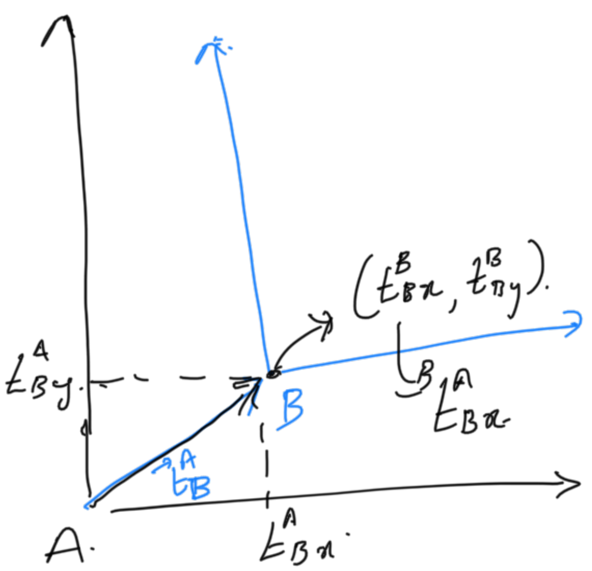
$\downarrow \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$

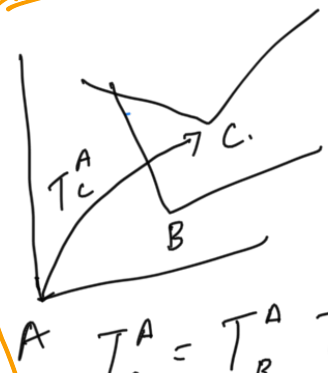
$$T_C^A = T_B^A\ T_C^B \longrightarrow (9)$$

Concatenate frames elegantly.

The non homogenous equations lead to messy concatenation of frames.

Q: Given $\vec{t}^{A}_{B}$ what is $\vec{t}^{B}_{A}$ in terms of $\vec{t}^{A}_{B}$ and R^{A}_{B}





$$T^{A}_{C} = T^{A}_{B}\, T^{B}_{C}$$

$\vec{t}^{A}_{B}$ is a vector in A.

${}^{B}\vec{t}^{A}_{B}$ is the same vector in B.

$$\underline{\underline{{}^{B}\vec{t}^{A}_{B}}} = T^{B}_{A}\, \underset{4\times 1}{\vec{t}^{A}_{B}} \longrightarrow (10)$$

$$= \begin{bmatrix} R^{B}_{A} & t^{B}_{A} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \underset{3\times 1}{\vec{t}^{A}_{B}} \\ 1 \end{bmatrix} \longrightarrow (11)$$

$$\xrightarrow[\text{homogenous}]{\text{non}} \underline{R^{B}_{A}\, \vec{t}^{A}_{B}} + \underline{\vec{t}^{B}_{A}} \longrightarrow (12)$$

But $\underline{\underline{{}^{B}\vec{t}^{A}_{B} = 0}}$ Why??

$$\underset{3\times 1}{0} = R^{B}_{A}\, \vec{t}^{A}_{B} + \vec{t}^{B}_{A} \longrightarrow (13)$$

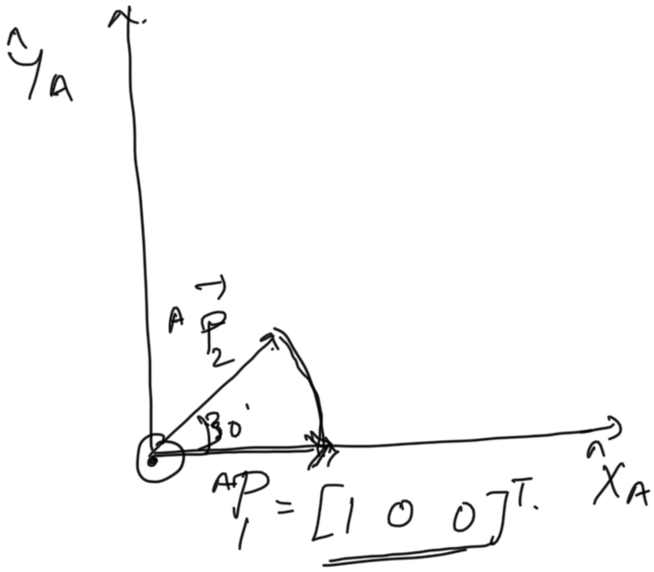
$$t_A^B = -R_A^B \, t_B^A = -R_B^{A\,T} \, t_B^A \quad (14)$$

$$R_A^B = [R_B^A]^{-1} = [R_B^A]^T \quad \text{Why?}$$

$$R_A^B \, R_A^{B\,T} = I_{3\times 3}$$

$\boxed{\text{J.J. Craig's Book, 2nd Chapter}}$

The Equivalence of Operators and Transforms:



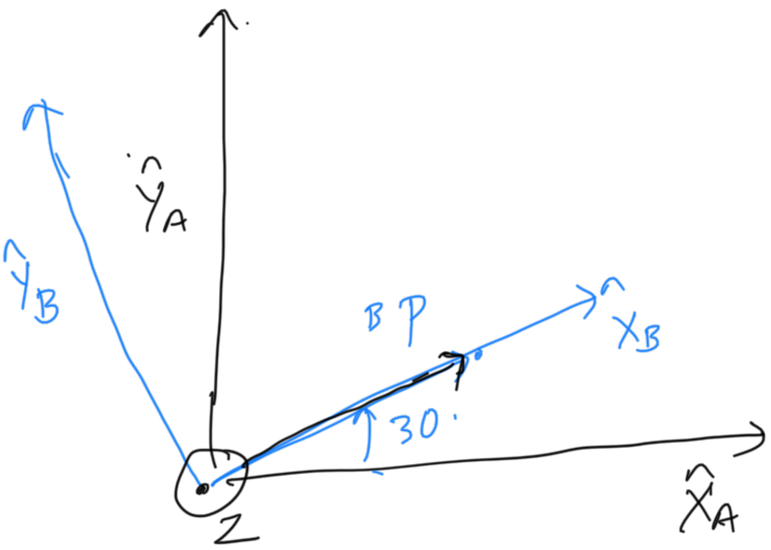
$$ {}^A P_2 = R_{\hat{Z}_A}(\theta)\, {}^A P_1 $$

$$ = \begin{bmatrix} C30 & -S30 \\ S30 & C30 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} $$

$$ = \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix} $$

Here $R_{\hat{Z}_A}(\theta)$ is a rotation operator that operate on ${}^A\vec{P}_1$ and takes it to ${}^A\vec{P}_2$

This is same as:



$${}^BP = [1 \quad 0 \quad \cancel{0}]^T$$

$${}^AP = R_B^A \, {}^BP$$

$$= \begin{bmatrix} C30 & -S30 \\ S30 & C30 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= [\sqrt{3}/2 \quad 1/2]^T$$

Representing Rotations as Euler Angle:

Z-Y-X Euler Angles:

$R_1 \quad R_2 \quad R_3$

$\|R_1\|^2 = 1 \quad R_1 . R_2 = 0$

→ Description of B wrt A. as a Sequence of Euler Angle rotation.

→ Initially A and B coincide.

1). Rotate B about $\hat{Z}_A$ or $\hat{Z}_B$ by an angle α to get intermediate frame B1.

$$R_{B1}^B \text{ or } R_{B1}^A = R_Z(\alpha)$$

$$= \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow (1).$$

2). Now rotate B1 about $\hat{Y}_{B1}$ by β

to obtain B2

$$R_{B2}^{B1} = R_Y(\beta)$$

$$= \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \longrightarrow (2).$$

3). Now rotate B2 about X_{B2} by an angle γ to obtain B3.

$$R_{B3}^{B2} = R_X(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} \longrightarrow (3).$$

Then $R_B^A = R_{B1}^A \, R_{B2}^{B1} \, R_{B3}^{B2} \longrightarrow (4).$

$$= \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} \longrightarrow (5)$$

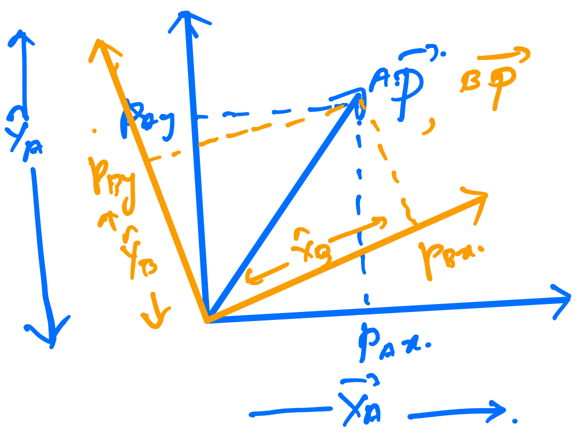
∴ Any Rotation matrix that describes a frame wrt another such as R_B^A can be decomposed into a SEQUENCE of 3 rotation about the moving cartesian axes.

$^{A}\vec{P}$ → vector in A.

R_B^A → rotation of {B} wrt {A}.

T_B^A → ~~Ho~~ Homogenous transform of B wrt A.

$\vec{t}_B^{A}$ → translation of B wrt A and this is a vector in A. and if you want you can write it as $^{A}\vec{t}_B^{A}$ or just $^{A}\vec{t}_B$.



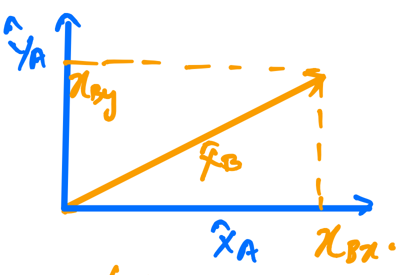
$\vec{P}_A$ is a l.c of the unit vectors that make {A} or

$$\vec{P}_A = \begin{bmatrix}\hat{X}_A & 0\\ 0 & \hat{Y}_A\end{bmatrix}\begin{bmatrix}P_{Ax}\\ P_{Ay}\end{bmatrix}$$

$$= P_{Ax}\hat{X}_A + P_{Ay}\hat{Y}_A \to (1)$$

$$\vec{P}_B = \begin{bmatrix}\hat{X}_B & 0\\ 0 & \hat{Y}_B\end{bmatrix}\begin{bmatrix}P_{Bx}\\ P_{By}\end{bmatrix}$$

$$\vec{P}_B = \begin{bmatrix}\hat{X}_B\\ 0\end{bmatrix}P_{Bx} + \begin{bmatrix}0\\ \hat{Y}_B\end{bmatrix}P_{By} \to (2).$$



(fig2)

$$\hat{X}_B = \begin{bmatrix}\hat{X}_A & 0\\ 0 & \hat{Y}_A\end{bmatrix}\begin{bmatrix}x_{Bx}\\ x_{By}\end{bmatrix} = x_{Bx}\begin{bmatrix}\hat{X}_A\\ 0\end{bmatrix} + x_{By}\begin{bmatrix}0\\ \hat{Y}_A\end{bmatrix}$$

Then $\begin{bmatrix}\hat{X}_B\\ 0\end{bmatrix}P_{Bx} = \left(x_{Bx}\begin{bmatrix}\hat{X}_A\\ 0\end{bmatrix} + x_{By}\begin{bmatrix}0\\ \hat{Y}_A\end{bmatrix}\right)P_{Bx}$

Repeat again: (Use figures above)

$$\vec{P}_B = \begin{bmatrix}1\\ 0\end{bmatrix}P_{Bx} + \begin{bmatrix}0\\ 1\end{bmatrix}P_{By} = \begin{bmatrix}P_{Bx}\\ P_{By}\end{bmatrix} \to (1).$$

where $\hat{x}_B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\hat{y}_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \longrightarrow (2)$.

or $\vec{P_B} = P_{Bx}\,\hat{x}_B + P_{By}\,\hat{y}_B \longrightarrow (3)$

Now ${}^A\hat{x}_B = \begin{bmatrix} x_{Bx} \\ x_{By} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_{Bx} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_{By} \longrightarrow (4)$

see fig2

where $\hat{x}_A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \hat{y}_A$

or $\vec{P_B} = P_{Bx}\left(\begin{bmatrix} x_{Bx} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_{By} \end{bmatrix} \right) + P_{By}\,\hat{y}_B \longrightarrow (5)$

${}^A\hat{y}_B = \begin{bmatrix} y_{Bx} \\ y_{By} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} y_{Bx} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} y_{By} \longrightarrow (6)$.

Then $\vec{P_B} = P_{Bx} \begin{bmatrix} x_{Bx} \\ x_{By} \end{bmatrix} + P_{By} \begin{bmatrix} y_{Bx} \\ y_{By} \end{bmatrix} \longrightarrow (7)$

$\longrightarrow$ similar to fig2 component of $\hat{y}_B$ in $\{A\}$

$$= \begin{bmatrix} x_{Bx} & y_{Bx} \\ x_{By} & y_{By} \end{bmatrix} \begin{bmatrix} P_{Bx} \\ P_{By} \end{bmatrix} \longrightarrow (8)$$

$$= \begin{bmatrix} \hat{x}_B \cdot \hat{x}_A & \hat{y}_B \cdot \hat{x}_A \\ \hat{x}_B \cdot \hat{y}_A & \hat{y}_B \cdot \hat{y}_A \end{bmatrix} \begin{bmatrix} P_{Bx} \\ P_{By} \end{bmatrix} \longrightarrow (9)$$

$$= R_B^A \vec{P_B} \longrightarrow (10)$$

The LHS of (10) is actually P_B as represented in A or actually P_A.

Here $R_B^A \in SO(2)$. (in (10))

can be extended to 3 dimensional case

as
$$^{A}P = \begin{bmatrix} p_{Ax} \\ p_{Ay} \\ p_{Az} \end{bmatrix} = \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix} \begin{bmatrix} p_{Bx} \\ p_{By} \\ p_{Bz} \end{bmatrix}$$

or
$$\boxed{^{A}\vec{P} = R_B^A \vec{P_B}}.$$