

# Lecture 9: Epipolar Geometry

Professor Fei-Fei Li

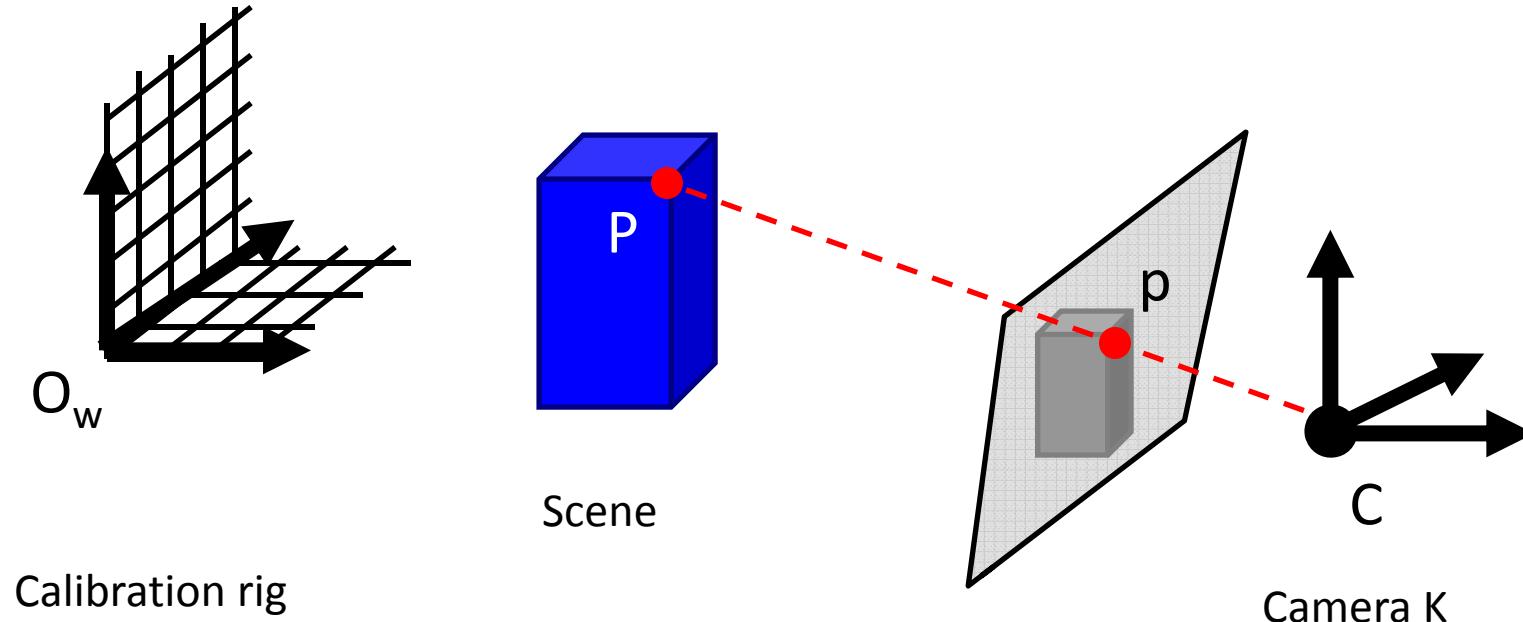
Stanford Vision Lab

# What we will learn today?

- Why is stereo useful?
- Epipolar constraints
- Essential and fundamental matrix
- Estimating F (**Problem Set 2 (Q2)**)
- Rectification

**Reading:**  
[HZ] Chapters: 4, 9, 11  
[FP] Chapters: 10

# Recovering structure from a single view



Why is it so difficult?

Intrinsic ambiguity of the mapping from 3D to image (2D)

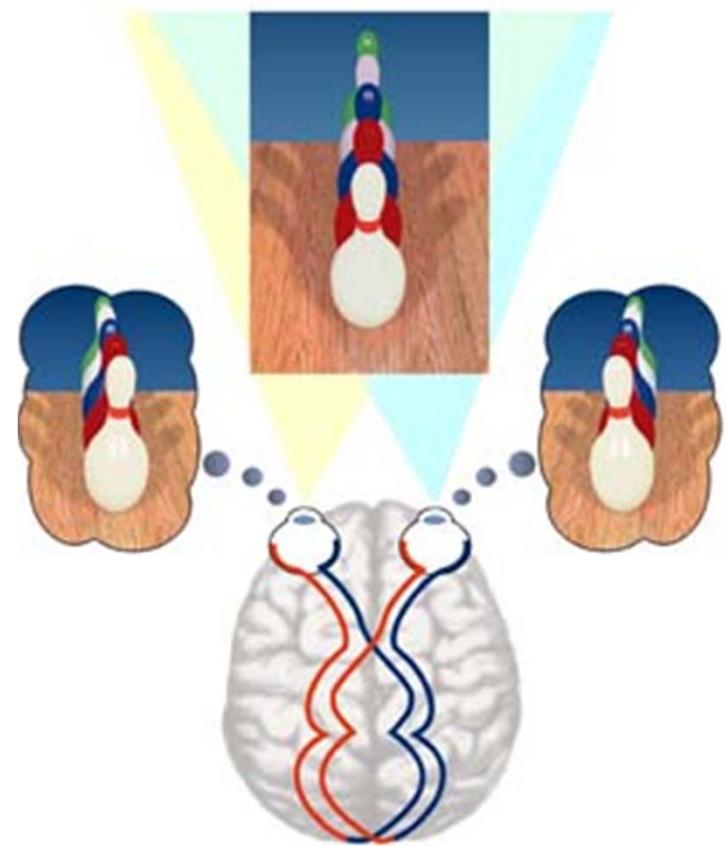
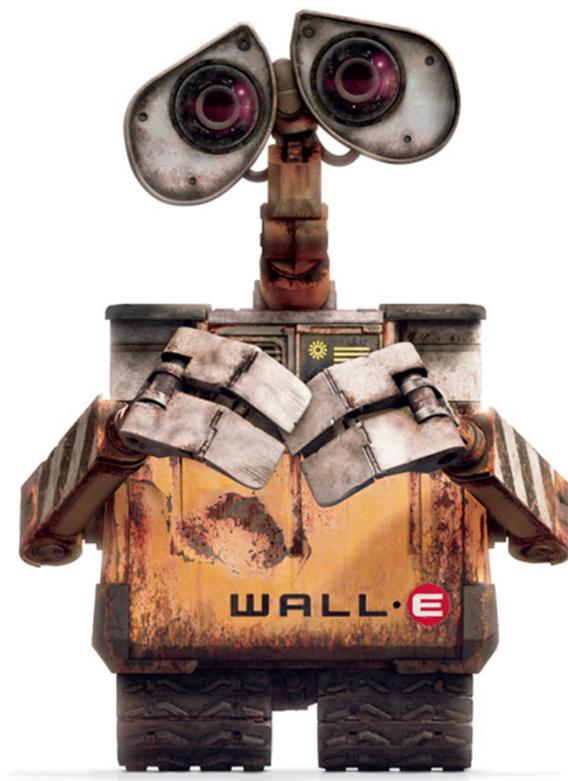
# Recovering structure from a single view



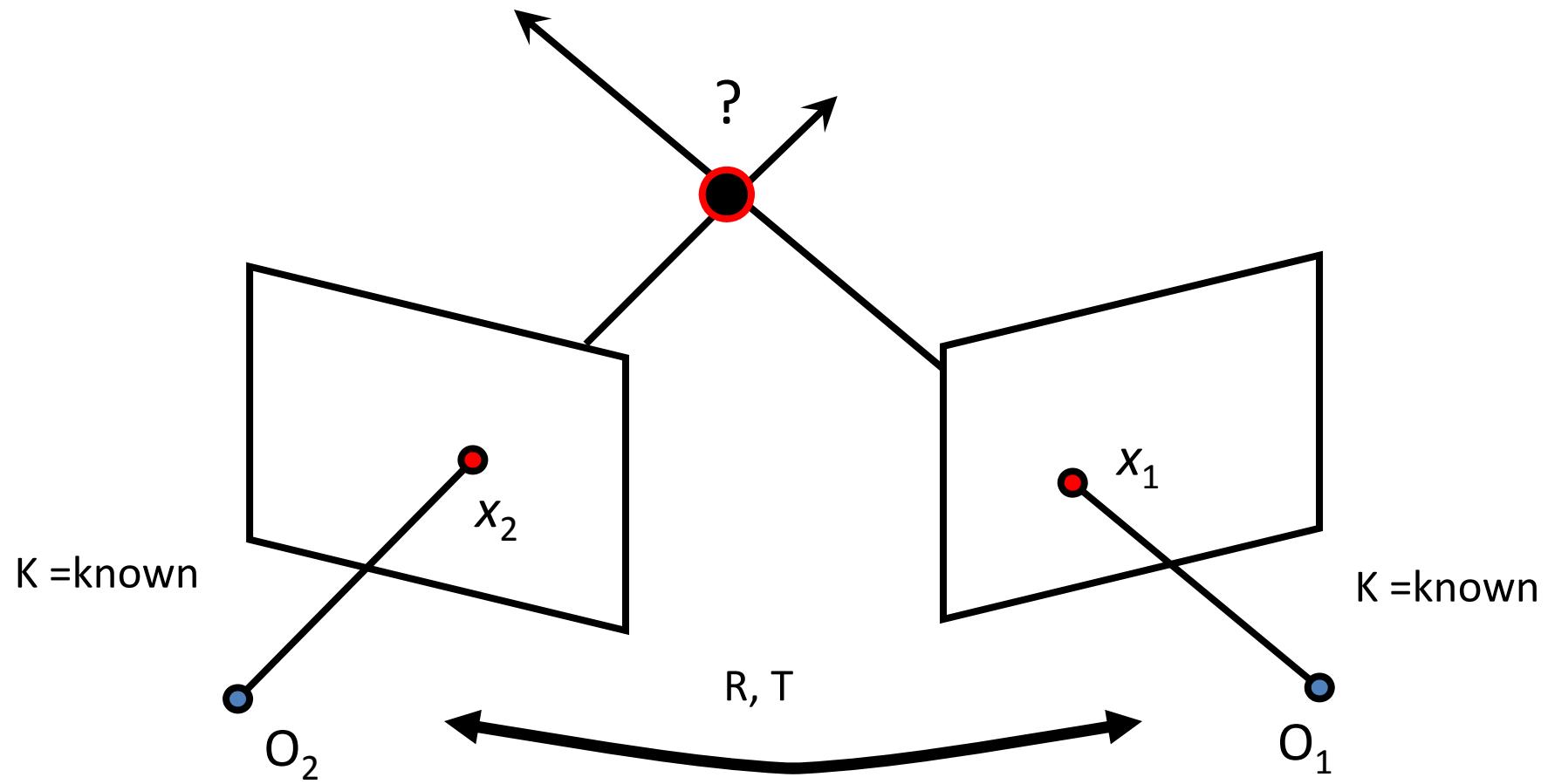
Courtesy slide S. Lazebnik

Intrinsic ambiguity of the mapping from 3D to image (2D)

# Two eyes help!



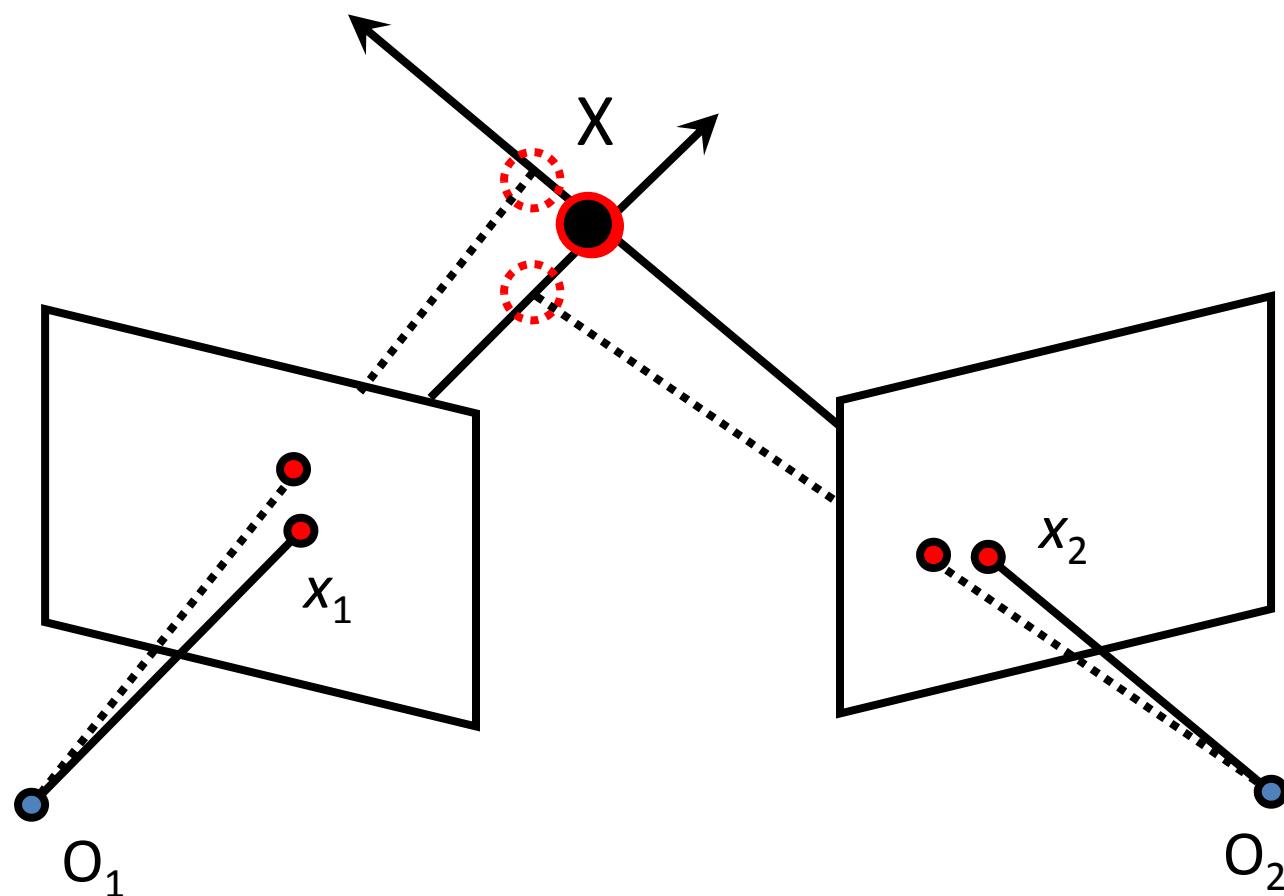
# Two eyes help!



This is called **triangulation**

# Triangulation

- Find  $X$  that minimizes  $d^2(x_1, P_1 X) + d^2(x_2, P_2 X)$



# Stereo-view geometry

- **Correspondence:** Given a point in one image, how can I find the corresponding point  $x'$  in another one?
- **Camera geometry:** Given corresponding points in two images, find camera matrices, position and pose.
- **Scene geometry:** Find coordinates of 3D point from its projection into 2 or multiple images.

This lecture (#9)

# Stereo-view geometry

Next lecture (#10)

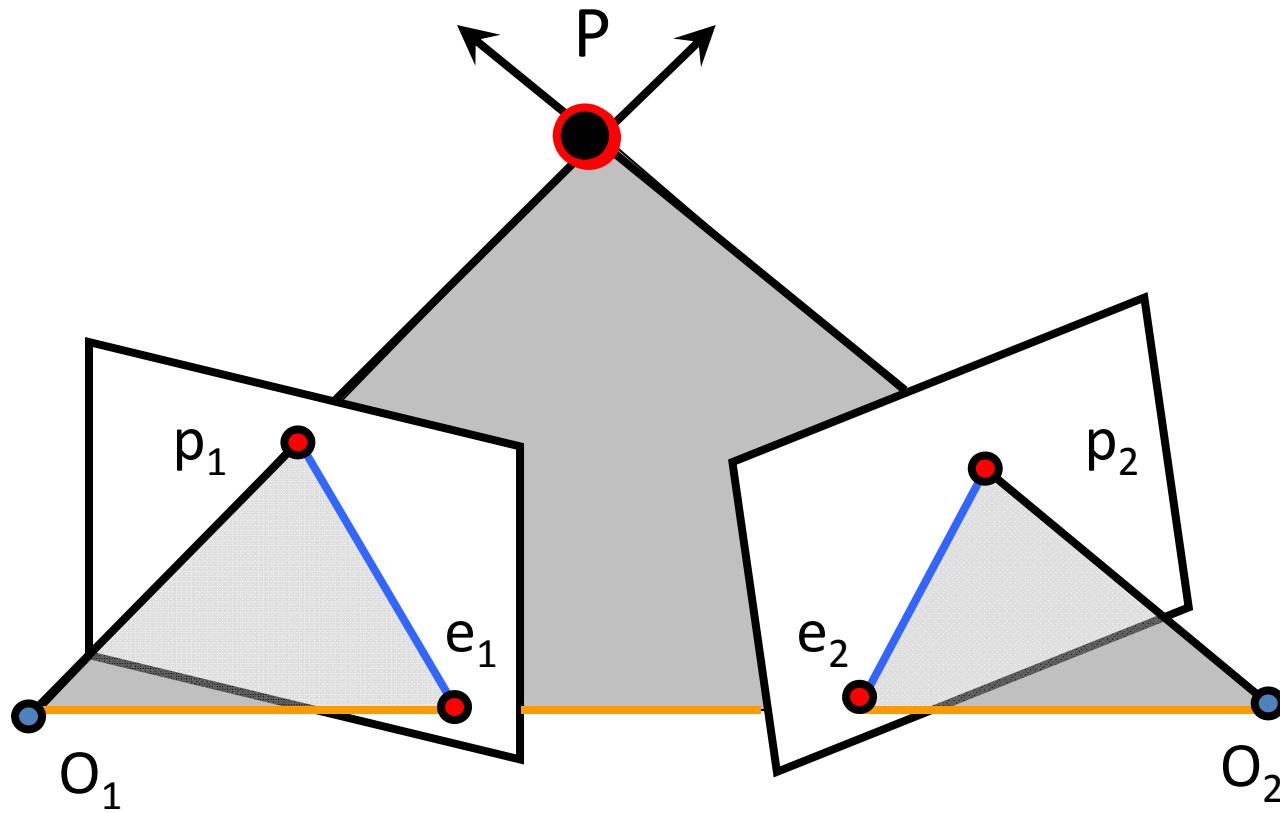
- **Correspondence:** Given a point in one image, how can I find the corresponding point  $x'$  in another one?
- **Camera geometry:** Given corresponding points in two images, find camera matrices, position and pose.
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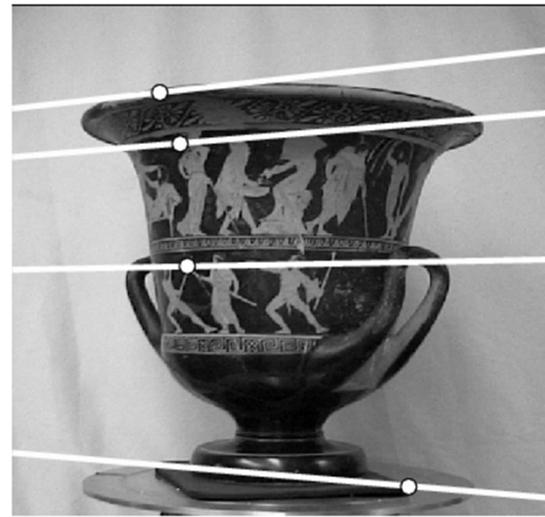
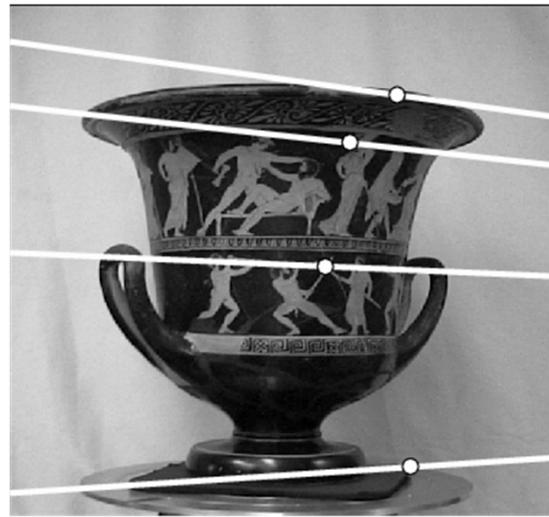
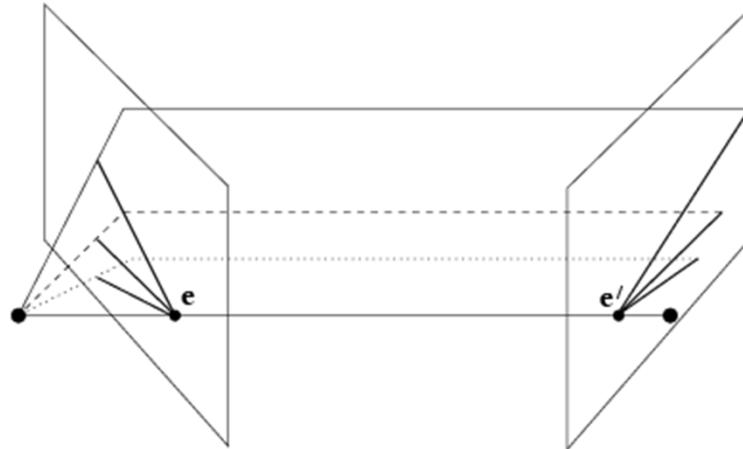
**Reading:**  
[HZ] Chapters: 4, 9, 11  
[FP] Chapters: 10

# Epipolar geometry

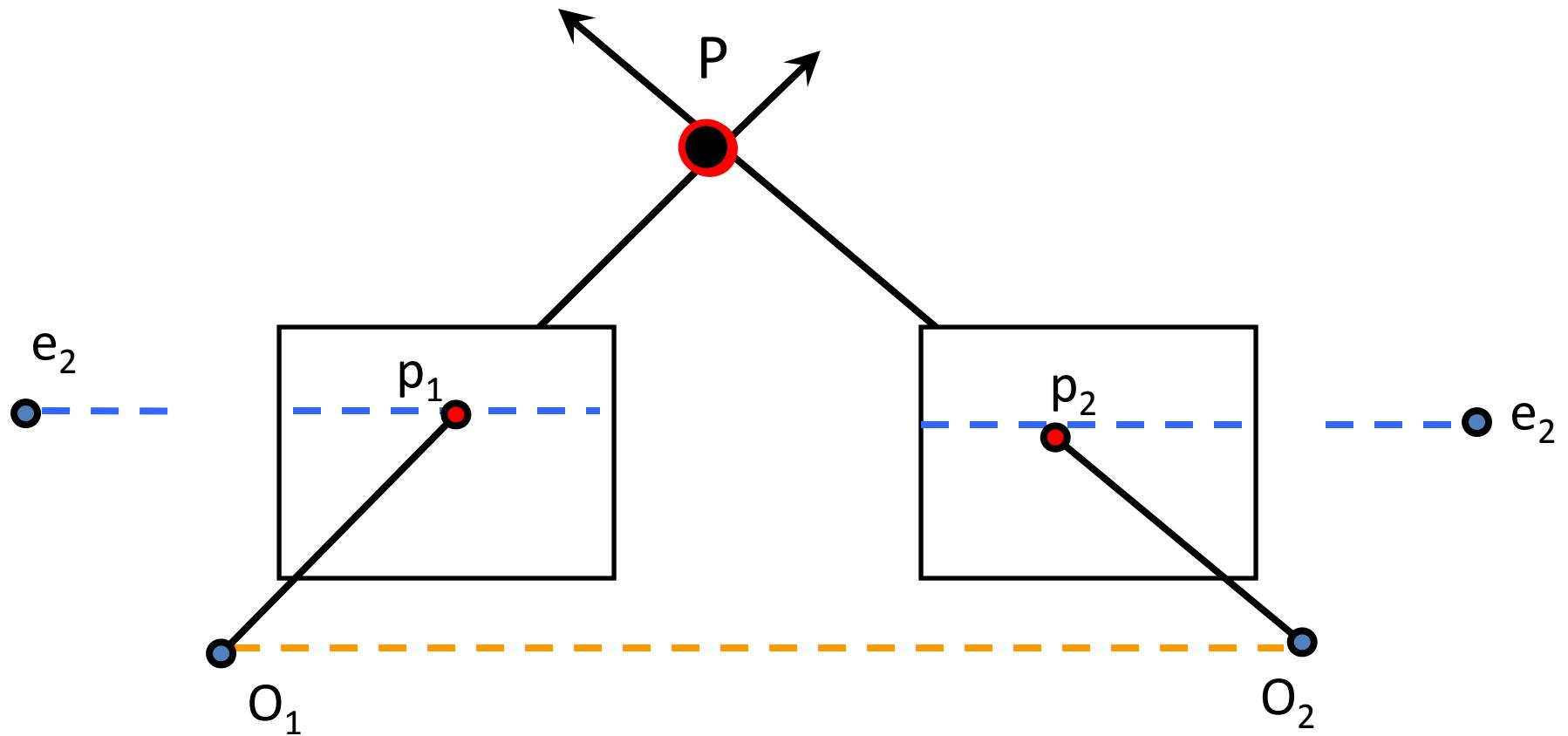


- Epipolar Plane
- Baseline
- Epipolar Lines
- Epipoles  $e_1, e_2$ 
  - = intersections of baseline with image planes
  - = projections of the other camera center
  - = vanishing points of camera motion direction

# Example: Converging image planes

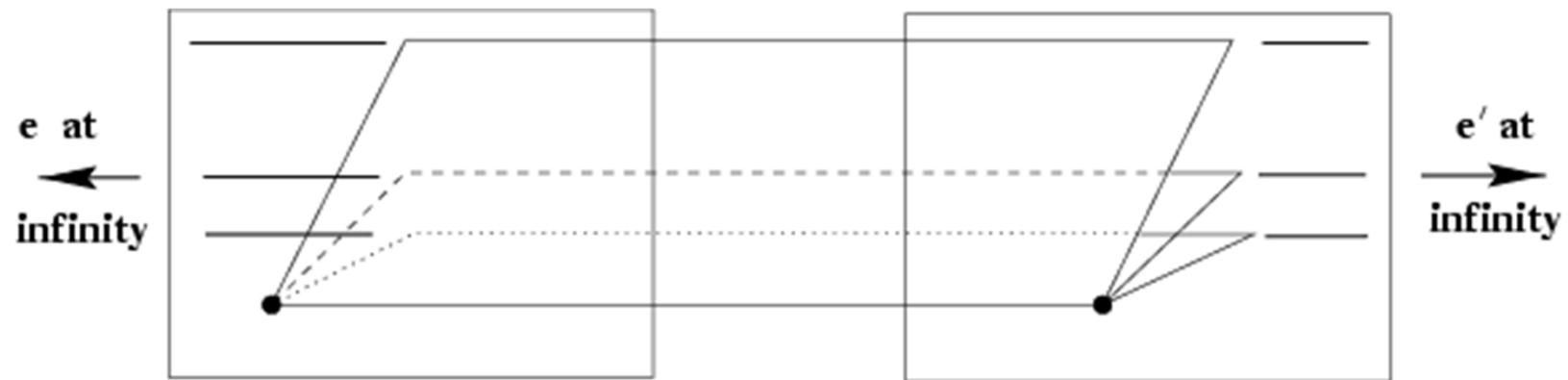


# Example: Parallel image planes

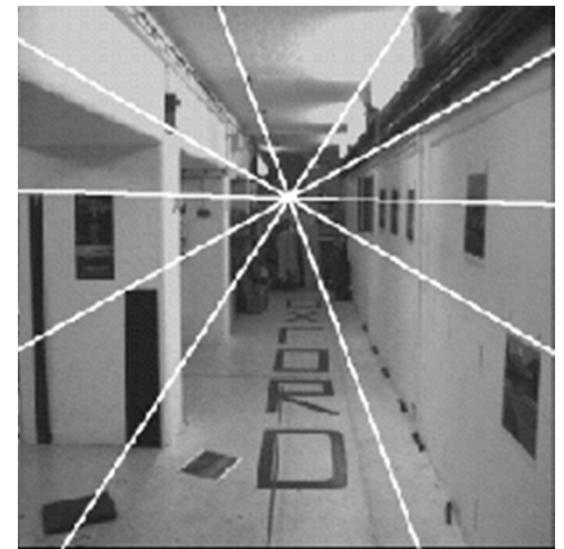
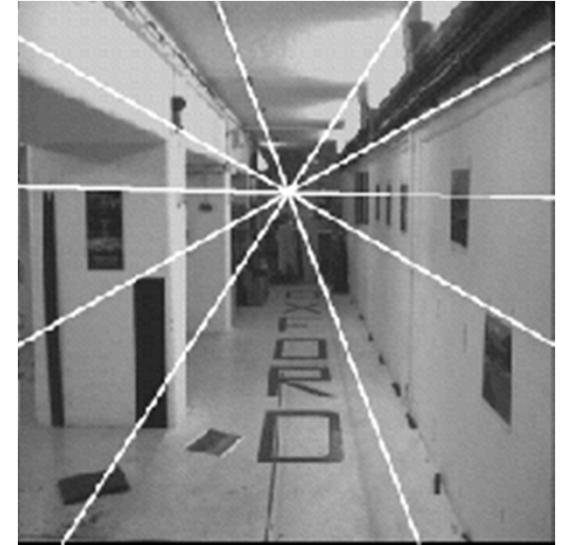
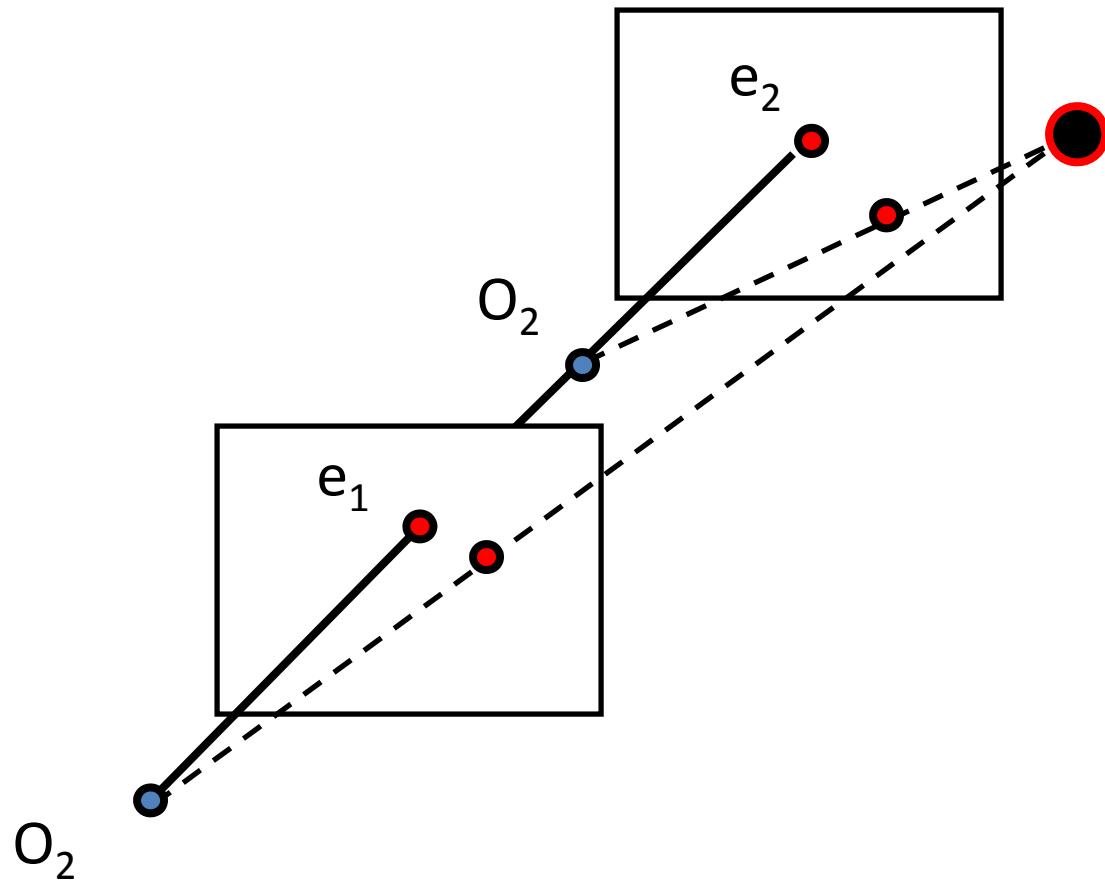


- Baseline intersects the image plane at infinity
- Epipoles are at infinity
- Epipolar lines are parallel to x axis

# Example: Parallel image planes

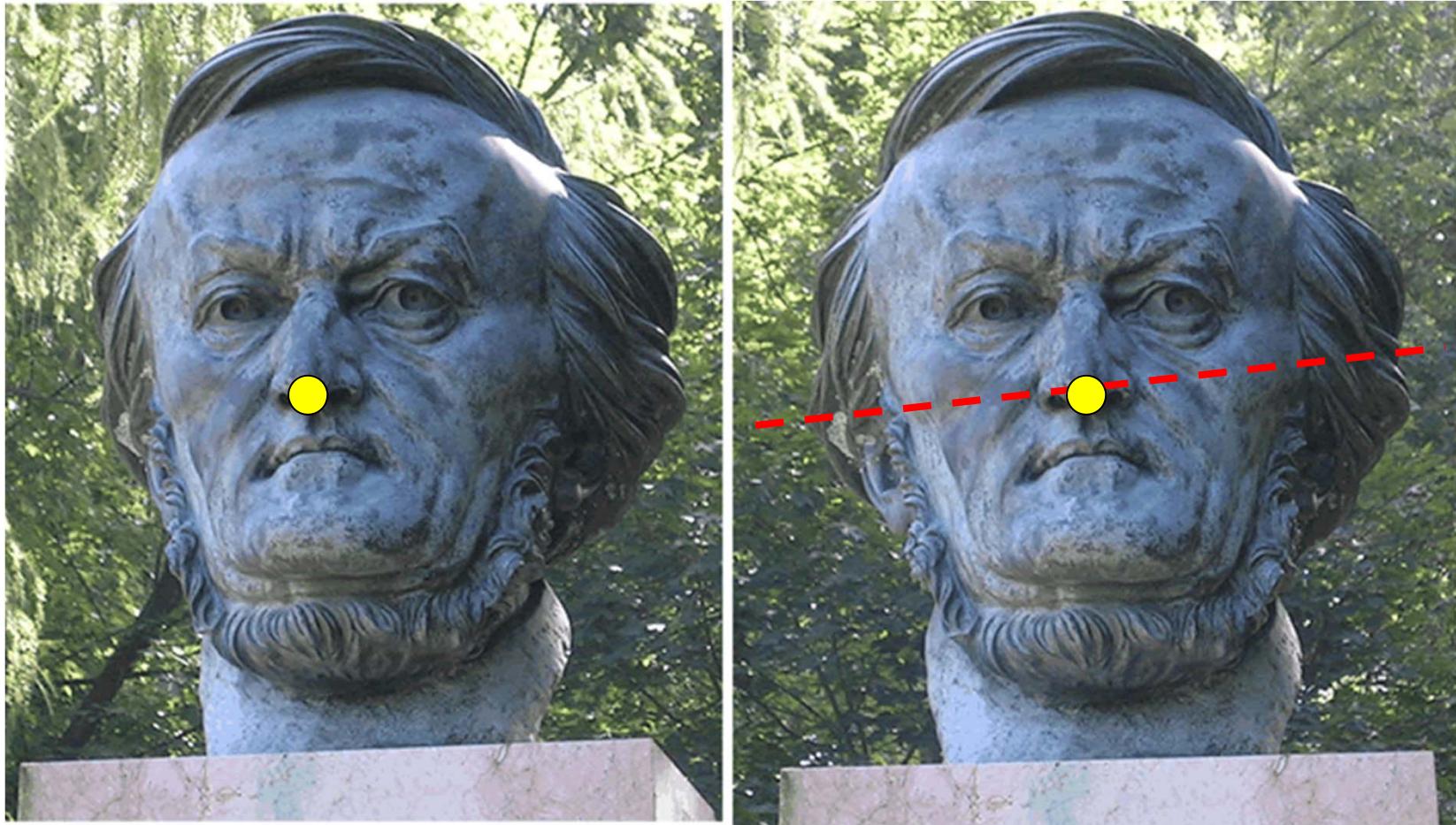


# Example: Forward translation



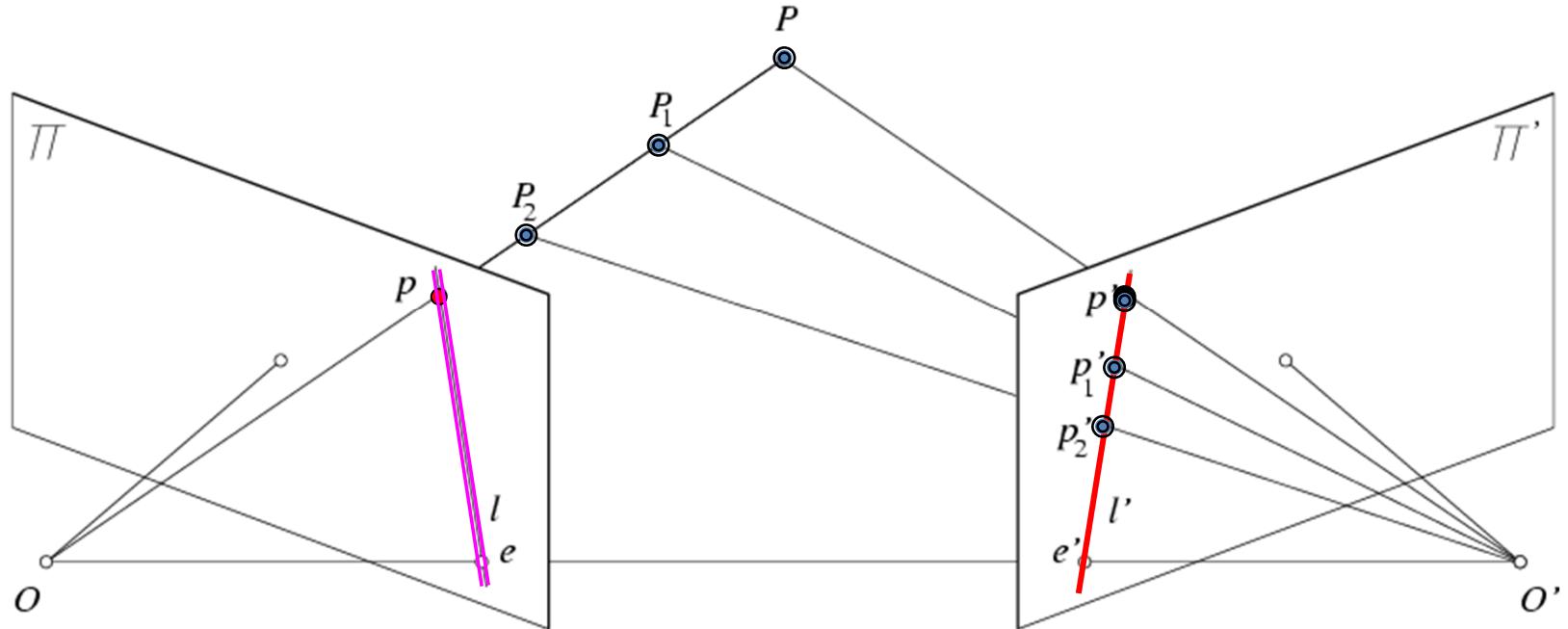
- The epipoles have same position in both images
- Epipole called FOE (focus of expansion)

# Epipolar Constraint



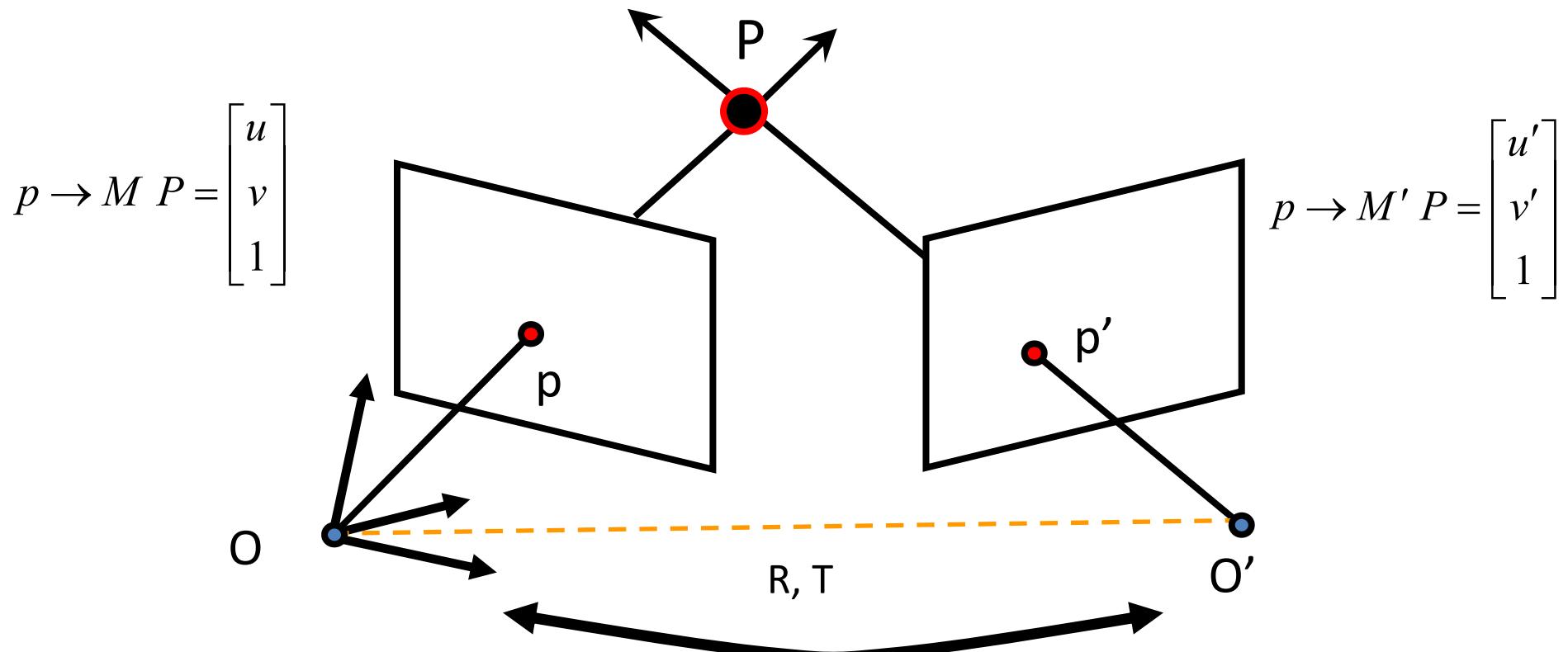
- Two views of the same object
- Suppose I know the camera positions and camera matrices
- Given a point on left image, how can I find the corresponding point on right image?

# Epipolar Constraint



- Potential matches for  $p$  have to lie on the corresponding epipolar line  $l'$ .
- Potential matches for  $p'$  have to lie on the corresponding epipolar line  $l$ .

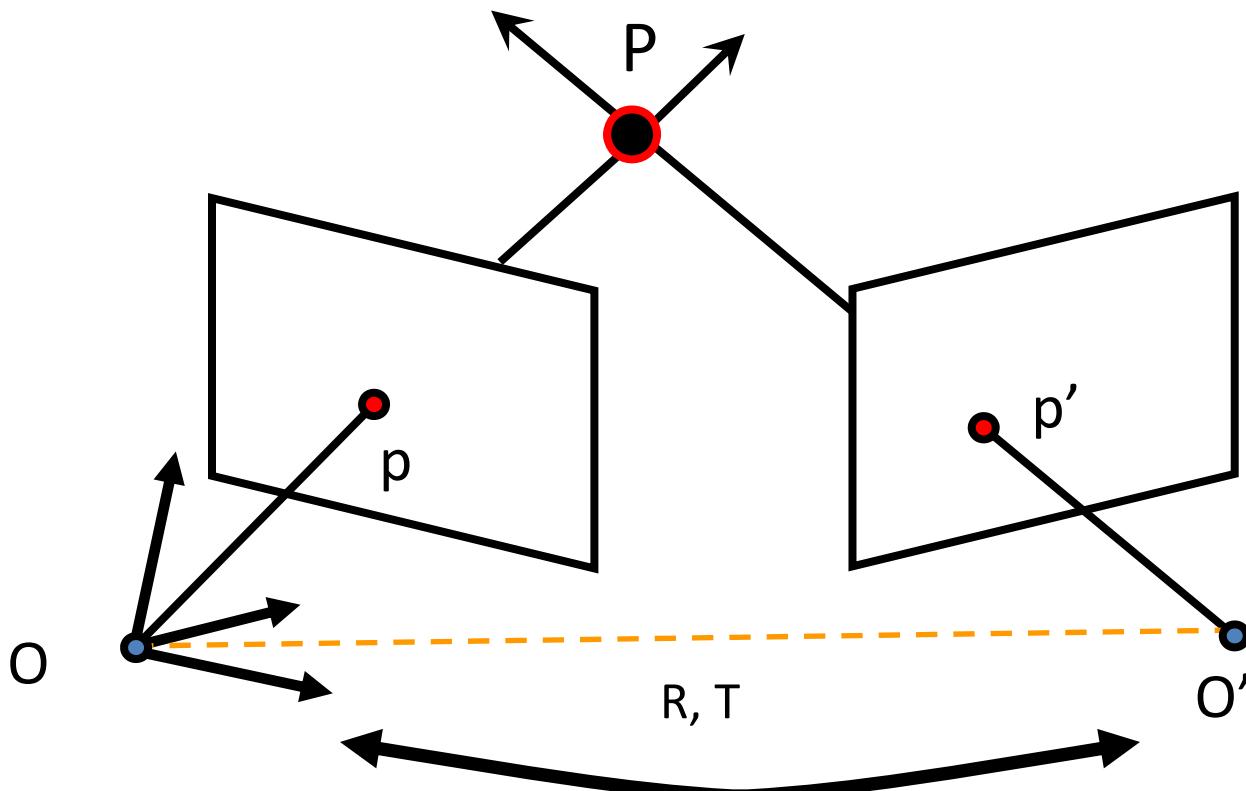
# Epipolar Constraint



$$M = K \begin{bmatrix} I & 0 \end{bmatrix}$$

$$M' = K \begin{bmatrix} R & T \end{bmatrix}$$

# Epipolar Constraint



$$M = K \begin{bmatrix} I & 0 \end{bmatrix}$$

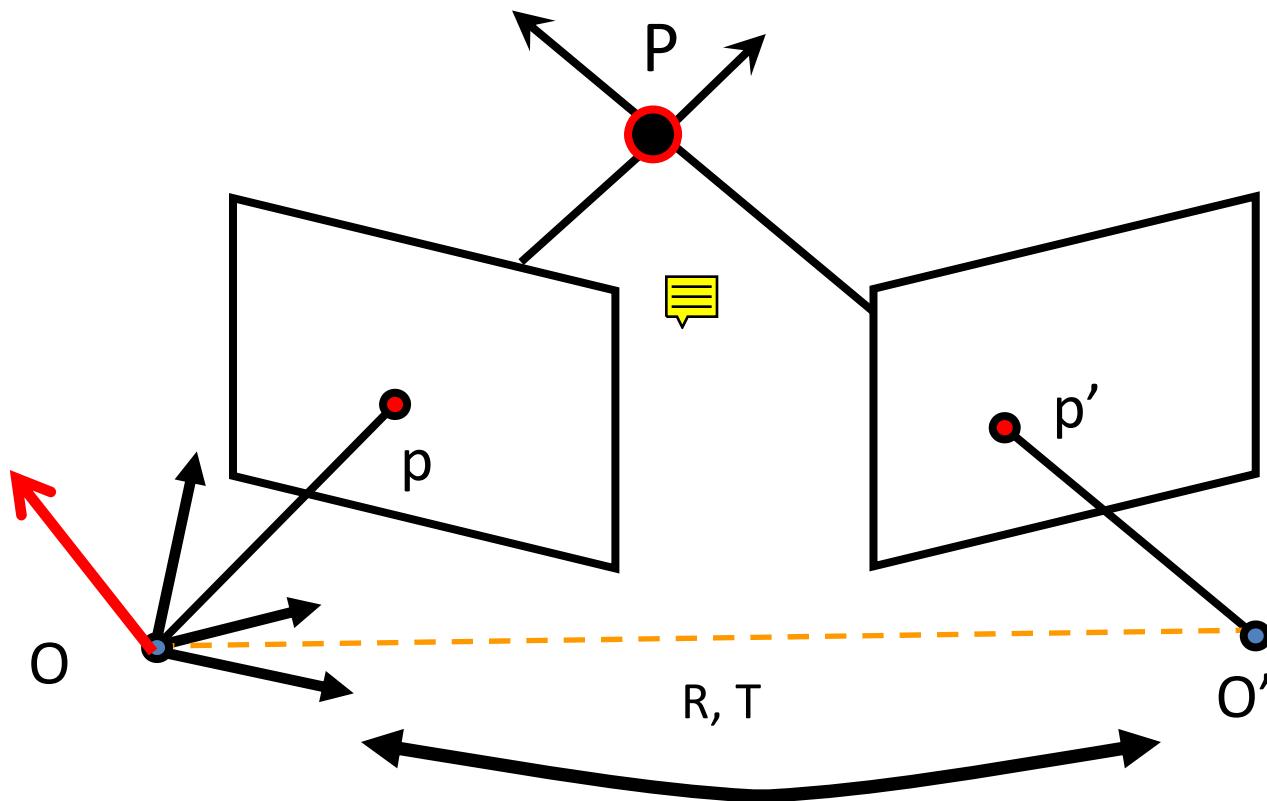
$K_1$  and  $K_2$  are known  
(calibrated cameras)

$$M = \begin{bmatrix} I & 0 \end{bmatrix}$$

$$M' = K \begin{bmatrix} R & T \end{bmatrix}$$

$$M' = \begin{bmatrix} R & T \end{bmatrix}$$

# Epipolar Constraint



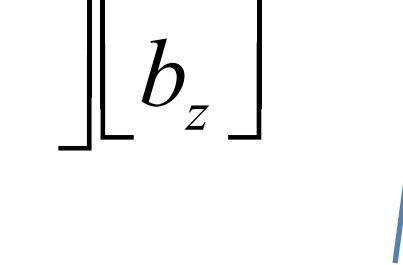
$$T \times (R p')$$

Perpendicular to epipolar plane

$$p^T \cdot [T \times (R p')] = 0$$

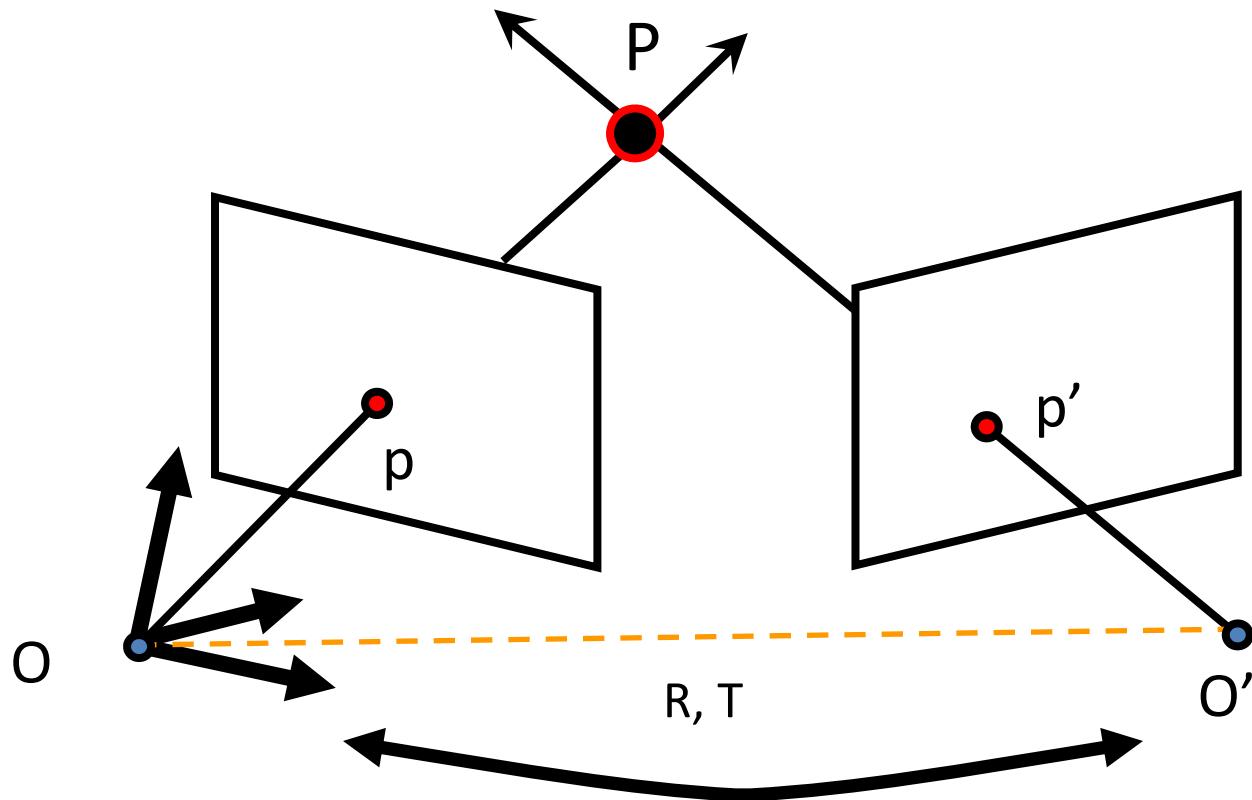
# Cross product as matrix multiplication

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = [\mathbf{a}_\times] \mathbf{b}$$



“skew symmetric matrix”

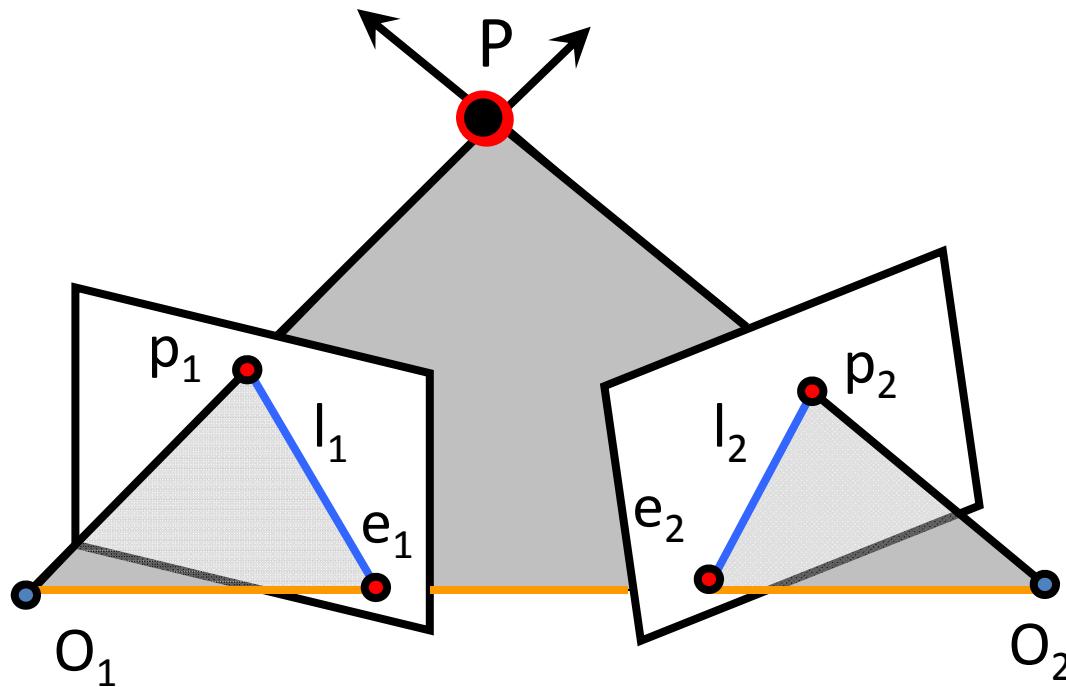
# Triangulation



$$p^T \cdot [T \times (R p')] = 0 \rightarrow p^T \cdot [T_x] \cdot R p' = 0$$

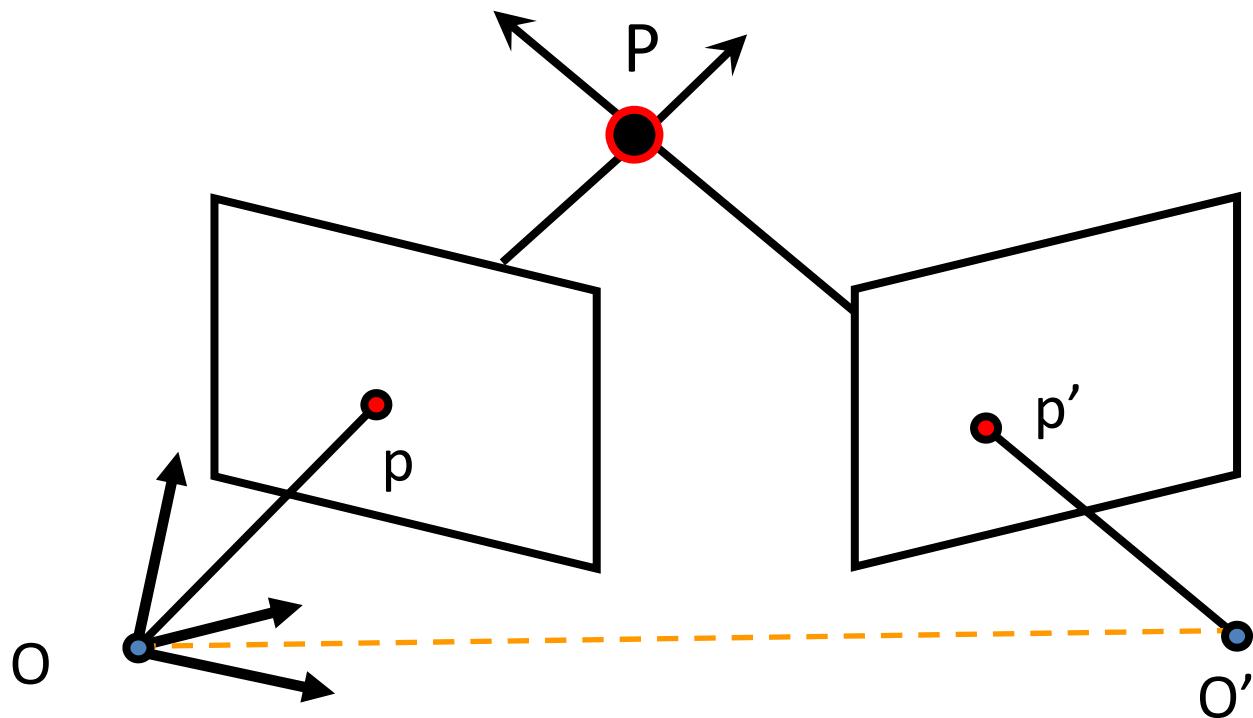
(Longuet-Higgins, 1981)  $T_x$   $E = \text{essential matrix}$

# Triangulation



- $E p_2$  is the epipolar line associated with  $p_2$  ( $l_1 = E p_2$ )
- $E^T p_1$  is the epipolar line associated with  $p_1$  ( $l_2 = E^T p_1$ )
- $E$  is singular (rank two)
- $E e_2 = 0$  and  $E^T e_1 = 0$
- $E$  is 3x3 matrix; 5 DOF

# Triangulation

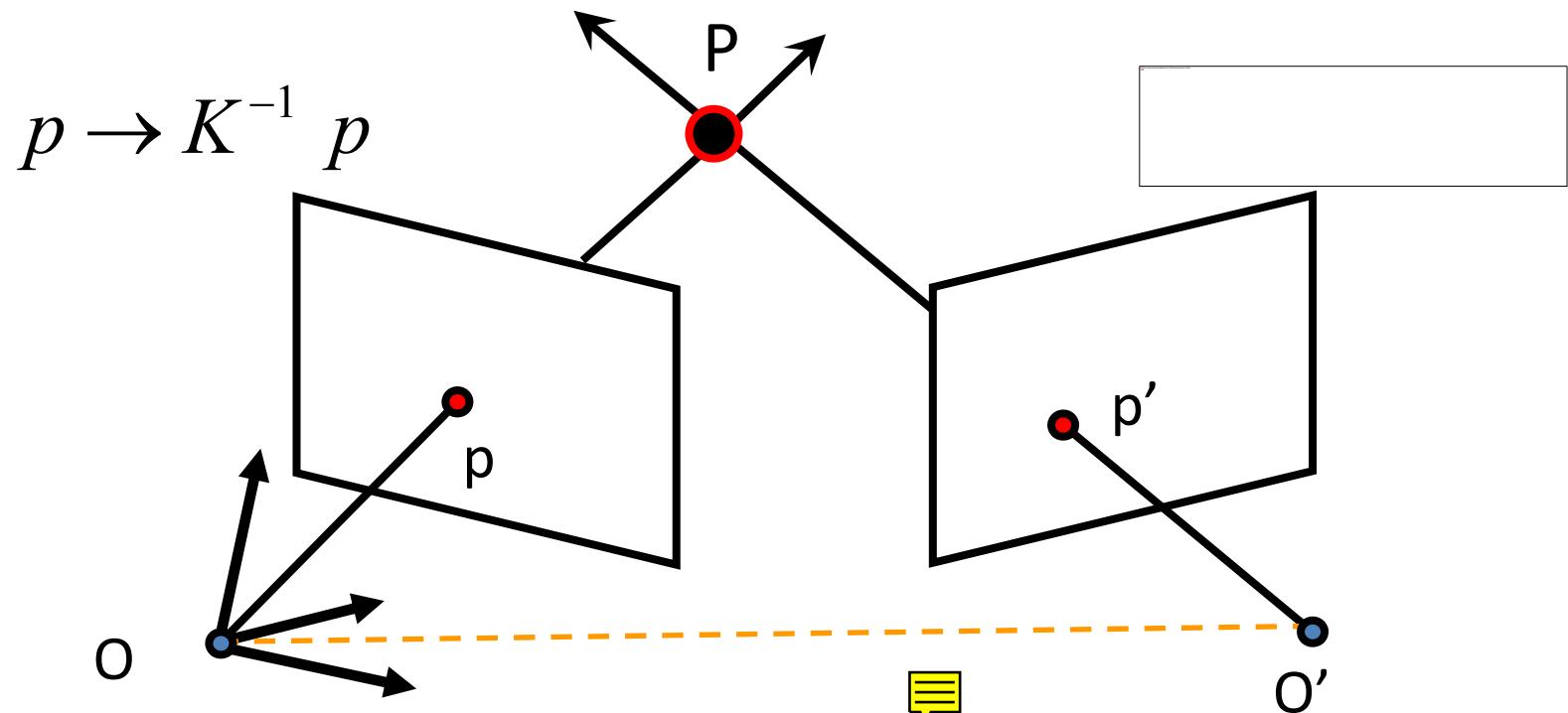


$$P \rightarrow M P \rightarrow p = \begin{bmatrix} u \\ v \end{bmatrix}$$

$$M = \boxed{K} \begin{bmatrix} I & 0 \end{bmatrix}$$

unknown

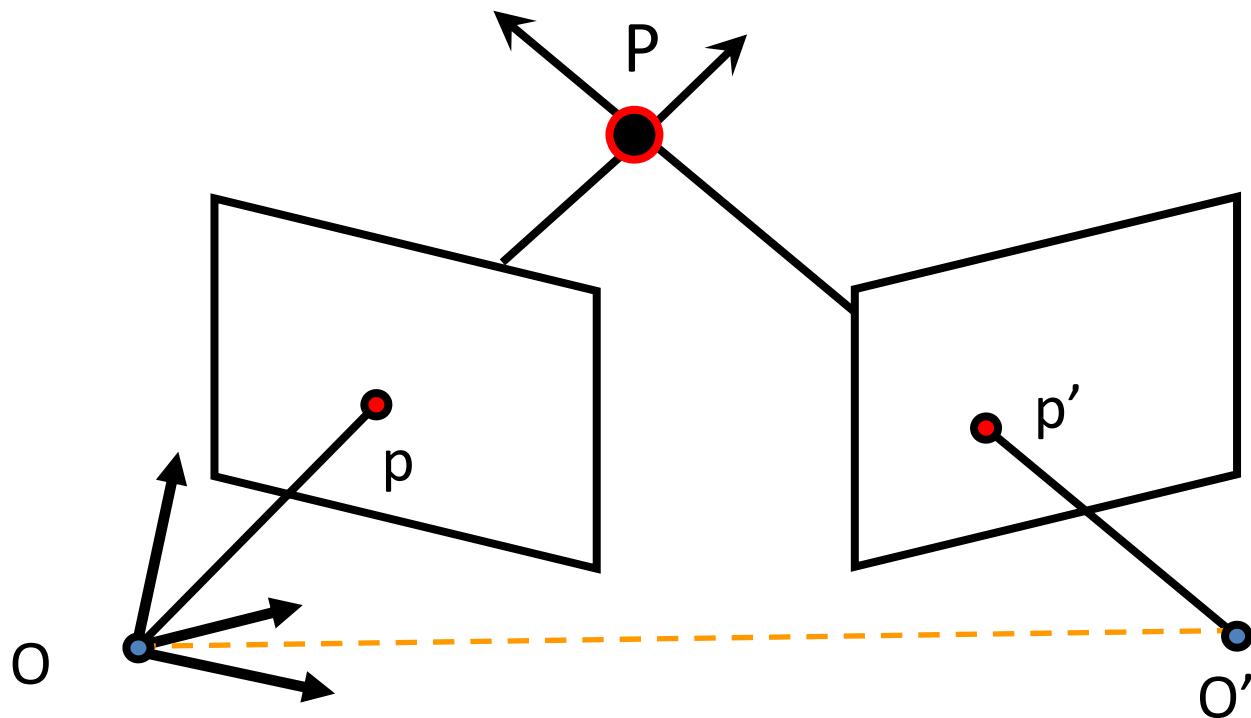
# Triangulation



$$p^T \cdot [T_x] \cdot R p' = 0 \rightarrow (K^{-1} p)^T \cdot [T_x] \cdot R K'^{-1} p' = 0$$

$$p^T [K^{-T} \cdot [T_x] \cdot R K'^{-1}] p' = 0 \rightarrow p^T [F] p' = 0$$

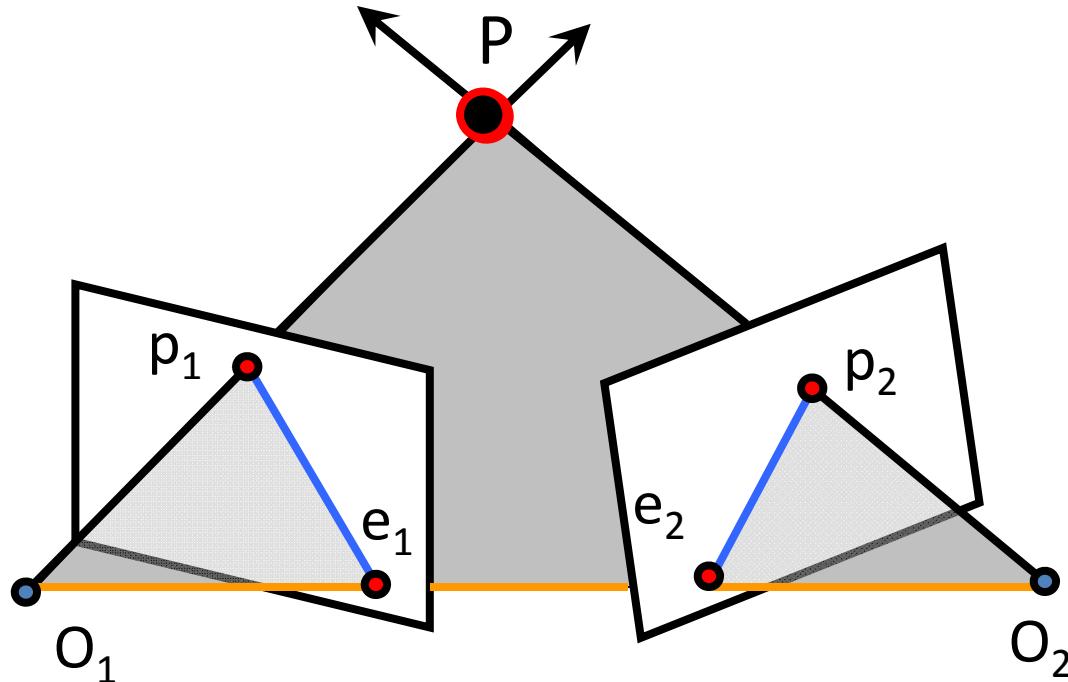
# Triangulation



$$p^T F p' = 0$$

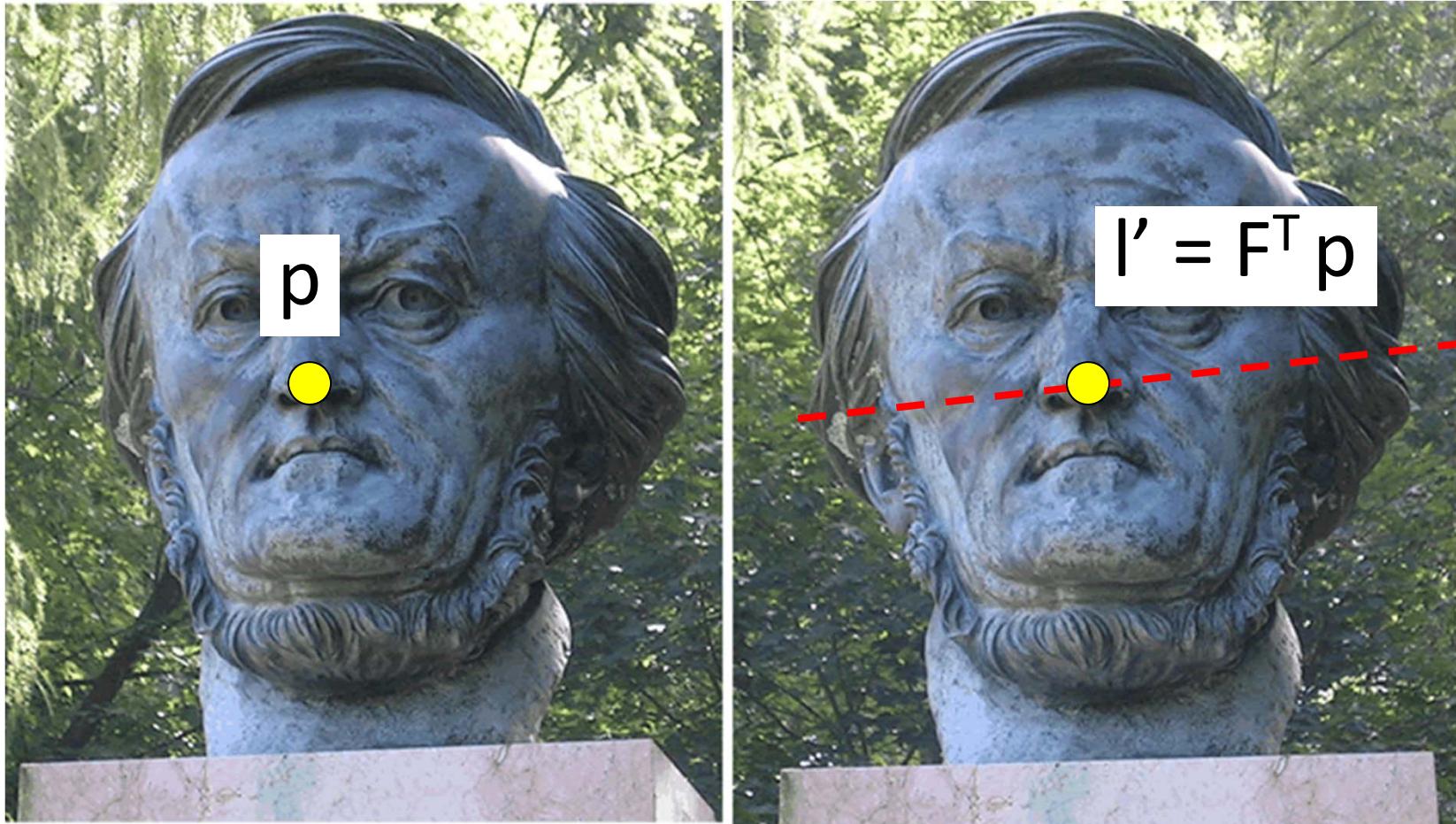
**F = Fundamental Matrix**  
(Faugeras and Luong, 1992)

# Triangulation



- $F p_2$  is the epipolar line associated with  $p_2$  ( $l_1 = F p_2$ )
- $F^T p_1$  is the epipolar line associated with  $p_1$  ( $l_2 = F^T p_1$ )
- $F$  is singular (rank two)
- $F e_2 = 0$  and  $F^T e_1 = 0$
- $F$  is  $3 \times 3$  matrix; 7 DOF

# Why is F useful?



- Suppose  $F$  is known
- No additional information about the scene and camera is given
- Given a point on left image, how can I find the corresponding point on right image?

# Why is F useful?

- $F$  captures information about the epipolar geometry of 2 views + camera parameters
- **MORE IMPORTANTLY:**  $F$  gives constraints on how the scene changes under view point transformation (without reconstructing the scene!)
- Powerful tool in:
  - 3D reconstruction
  - Multi-view object/scene matching

# What we will learn today?

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- Estimating F (**Problem Set 2 (Q2)**)
- Rectification

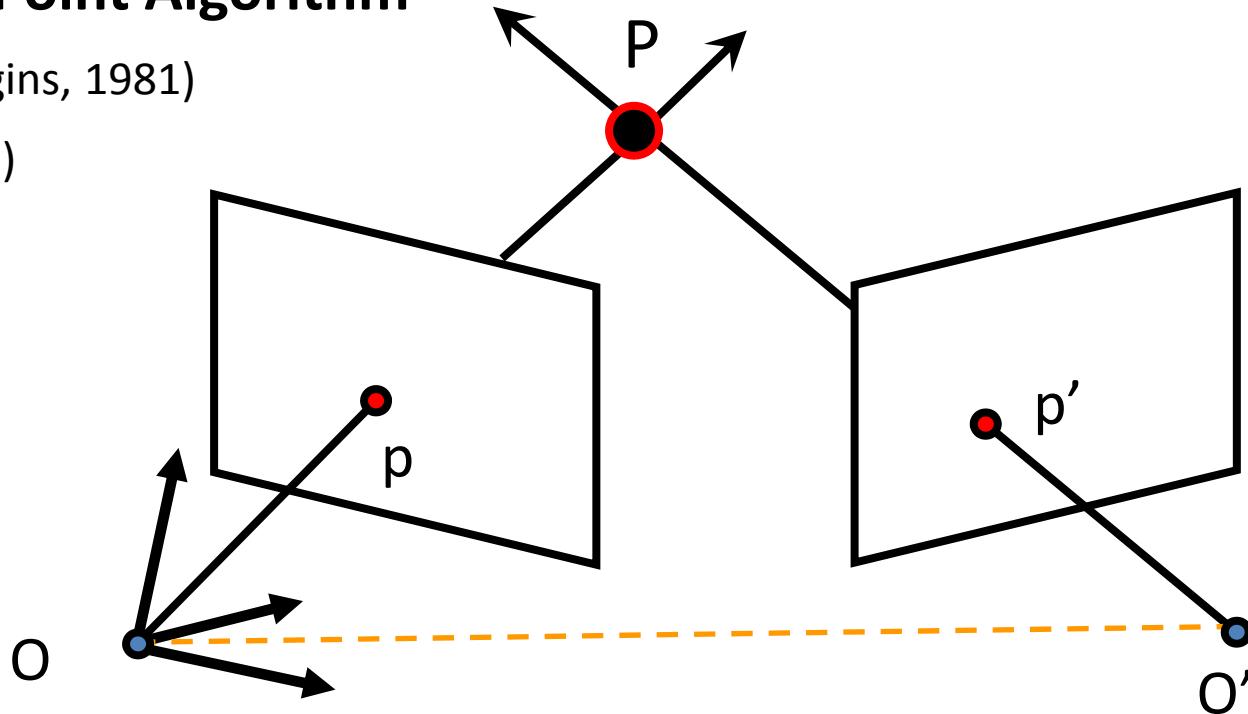
**Reading:**  
[HZ] Chapters: 4, 9, 11  
[FP] Chapters: 10

# Estimating F

## The Eight-Point Algorithm

(Longuet-Higgins, 1981)

(Hartley, 1995)



$$P \rightarrow p = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

$$P \rightarrow p' = \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix}$$

$$p^T F p' = 0$$

# Estimating F

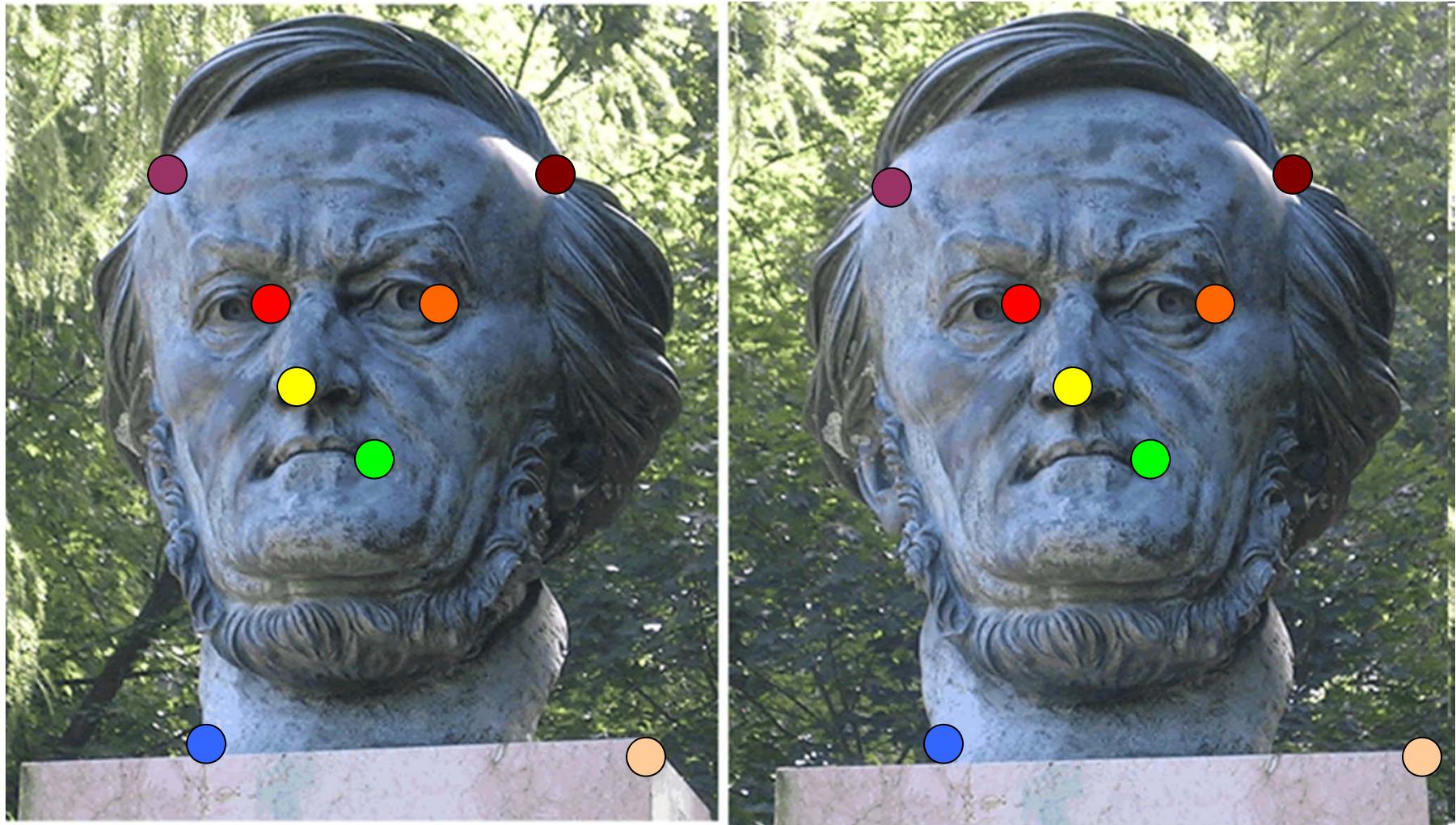
$$p^T F p' = 0 \quad \rightarrow$$

$$(u, v, 1) \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix} \begin{pmatrix} u' \\ v' \\ 1 \end{pmatrix} = 0$$

$$\rightarrow (uu', uv', u, vu', vv', v, u', v', 1) \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0$$

Let's take 8 corresponding points

# Estimating F



# Estimating F

$$\mathbf{W} \begin{pmatrix} u_1u'_1 & u_1v'_1 & u_1 & v_1u'_1 & v_1v'_1 & v_1 & u'_1 & v'_1 & 1 \\ u_2u'_2 & u_2v'_2 & u_2 & v_2u'_2 & v_2v'_2 & v_2 & u'_2 & v'_2 & 1 \\ u_3u'_3 & u_3v'_3 & u_3 & v_3u'_3 & v_3v'_3 & v_3 & u'_3 & v'_3 & 1 \\ u_4u'_4 & u_4v'_4 & u_4 & v_4u'_4 & v_4v'_4 & v_4 & u'_4 & v'_4 & 1 \\ u_5u'_5 & u_5v'_5 & u_5 & v_5u'_5 & v_5v'_5 & v_5 & u'_5 & v'_5 & 1 \\ u_6u'_6 & u_6v'_6 & u_6 & v_6u'_6 & v_6v'_6 & v_6 & u'_6 & v'_6 & 1 \\ u_7u'_7 & u_7v'_7 & u_7 & v_7u'_7 & v_7v'_7 & v_7 & u'_7 & v'_7 & 1 \\ u_8u'_8 & u_8v'_8 & u_8 & v_8u'_8 & v_8v'_8 & v_8 & u'_8 & v'_8 & 1 \end{pmatrix} \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = \mathbf{0}$$

$\mathbf{f}$

- Homogeneous system  $\mathbf{W f} = 0$
- Rank 8  $\rightarrow$  A non-zero solution exists (unique)
- If  $N > 8$   $\rightarrow$  Lsq. solution by SVD!

$$\hat{\mathbf{F}} \quad \|\mathbf{f}\| = 1$$

# Estimating F

$$p^T \hat{F} p' = 0$$

The estimated  $\hat{F}$  may have full rank ( $\det(\hat{F}) \neq 0$ )  
(F should have rank=2 instead)

Find F that minimizes

$$\|F - \hat{F}\| = 0$$

Frobenius norm (\*)

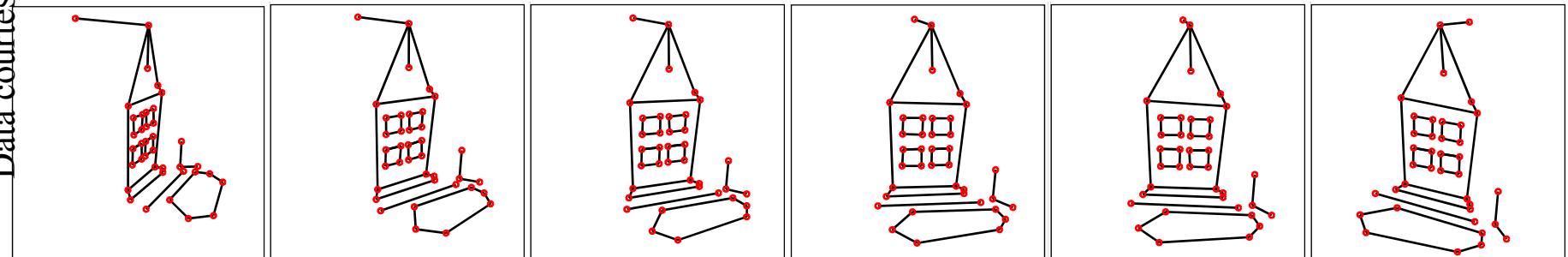
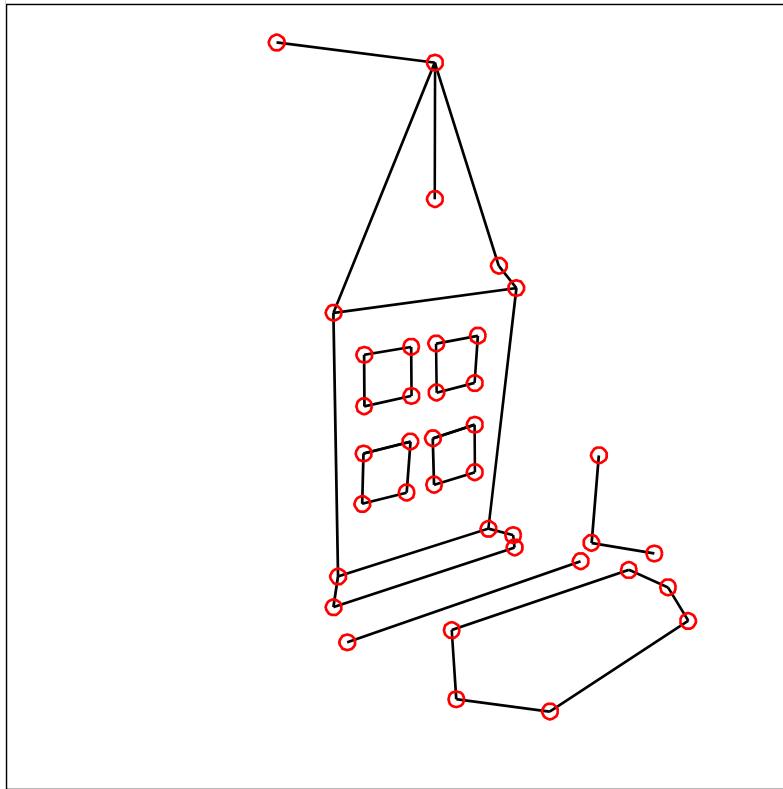
Subject to  $\det(F)=0$

SVD (again!) can be used to solve this problem

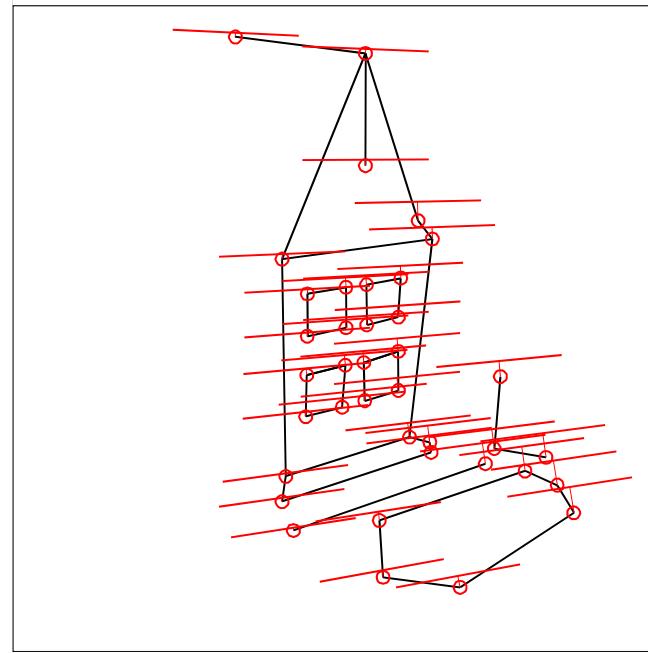
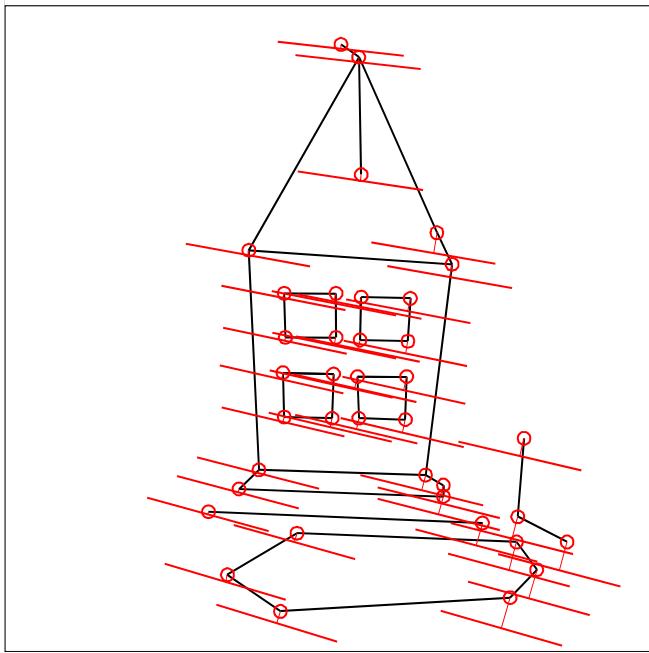
(\*) Sqrt root of the sum pf squares of all entries

# Example

Data courtesy of R. Mohr and B. Boufama.

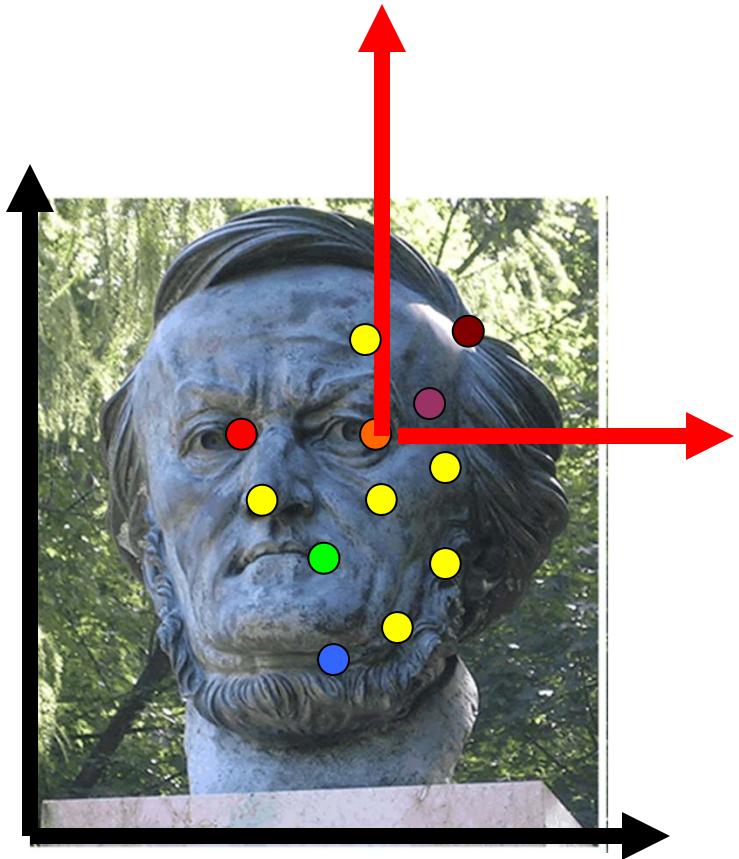


# Example



Mean errors:  
10.0pixel  
9.1pixel

# Normalization



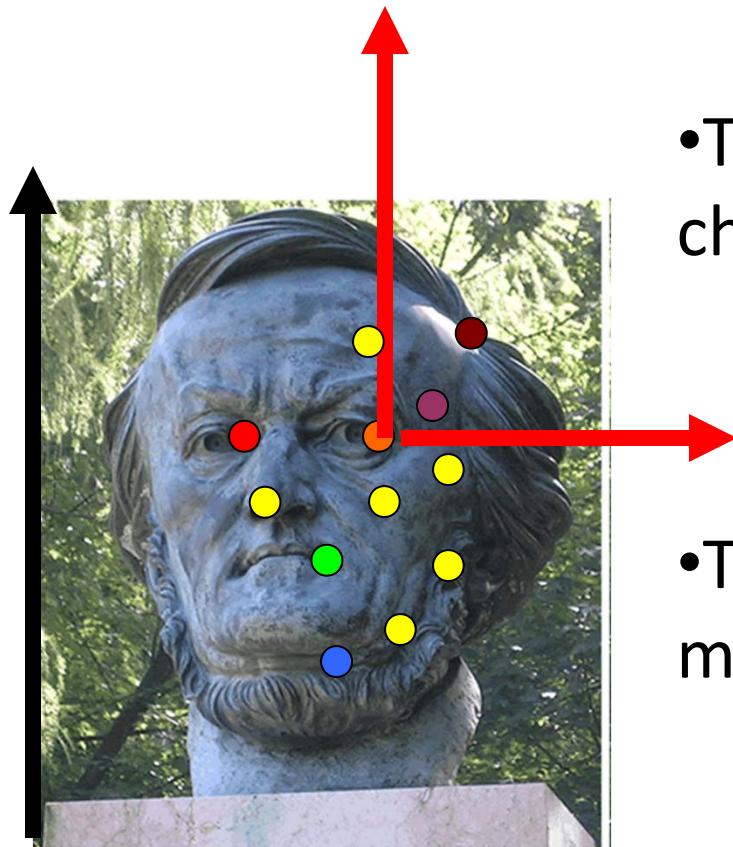
Is the accuracy in estimating  $F$  function of the ref. system in the image plane?

E.g. under similarity transformation ( $T = \text{scale} + \text{translation}$ ):

$$q_i = T_i p_i \quad q'_i = T'_i p'_i$$

Does the accuracy in estimating  $F$  change if a transformation  $T$  is applied?

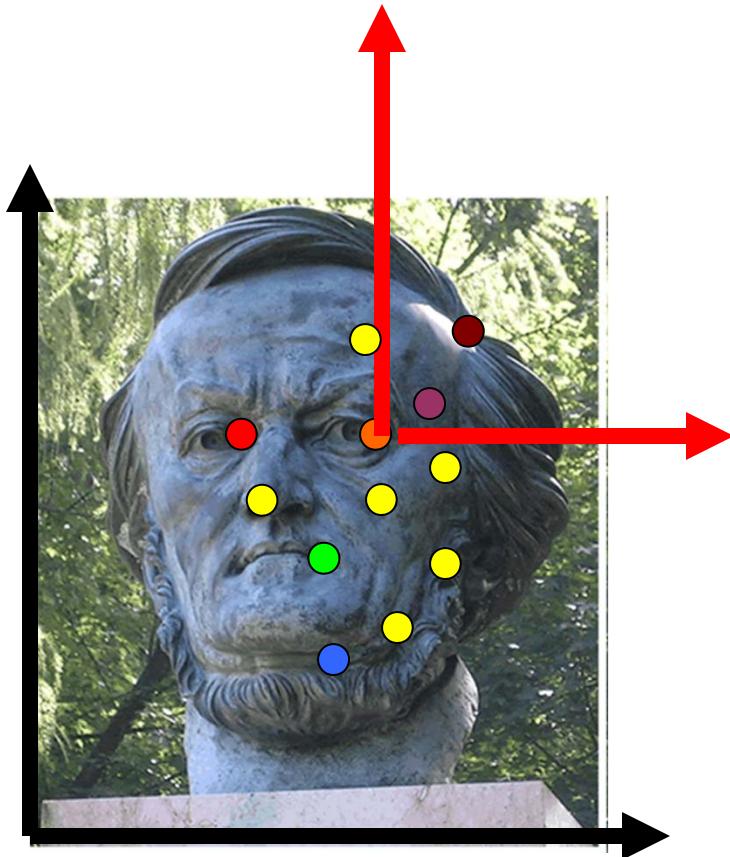
# Normalization



- The accuracy in estimating  $F$  does change if a transformation  $T$  is applied
- There exists a  $T$  for which accuracy is maximized

Why?

# Normalization



$$W f = 0,$$

$$\|f\| = 1$$

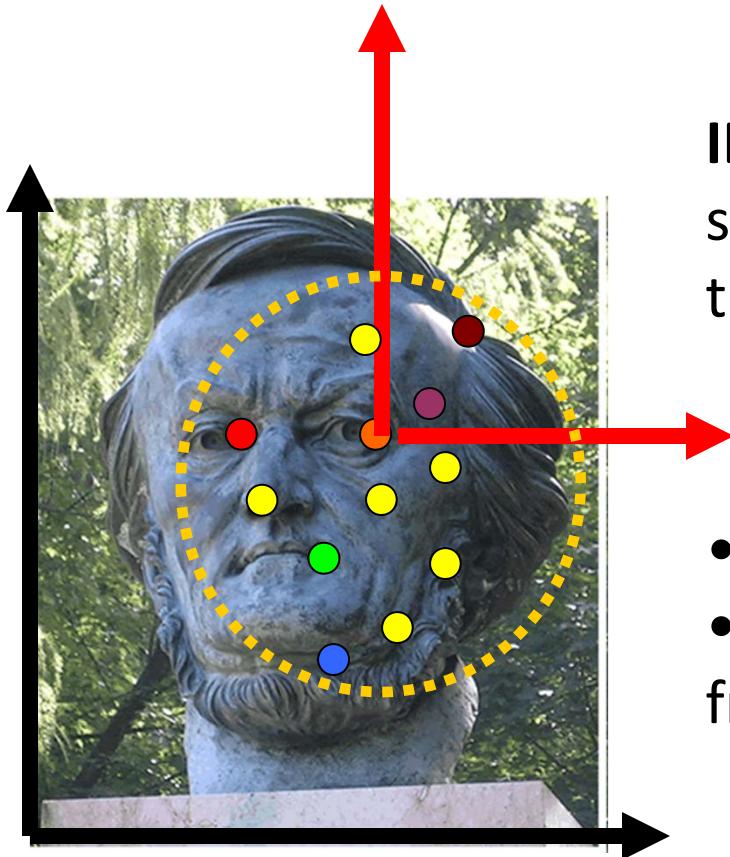
Lsq solution  
by SVD

$$\longrightarrow F$$

- SVD enforces  $\text{Rank}(W)=8$
- Recall the structure of  $W$ :  
Highly un-balance  
(not well conditioned)
- Values of  $W$  must have similar magnitude

More details HZ pag 108

# Normalization



**IDEA:** Transform image coordinate system ( $T = \text{translation} + \text{scaling}$ ) such that:

- Origin = centroid of image points
- Mean square distance of the data points from origin is 2 pixels

$$q_i = T_i p_i \quad q'_i = T'_i p'_i \quad (\text{normalization})$$

# The Normalized Eight-Point Algorithm

0. Compute  $T_i$  and  $T'_i$

1. Normalize coordinates:

$$q_i = T_i p_i \quad q'_i = T'_i p'_i$$

2. Use the eight-point algorithm to compute  $F'_q$  from the points  $q_i$  and  $q'_i$ .

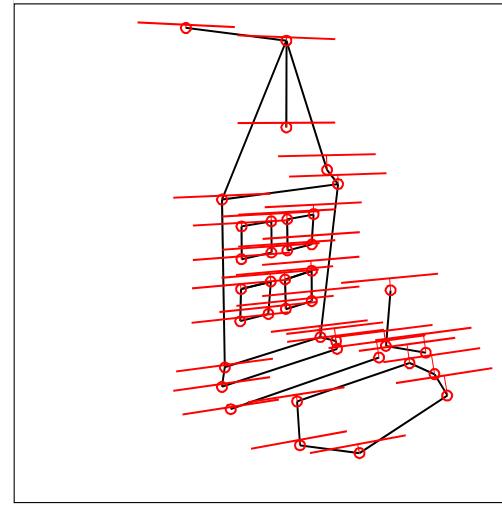
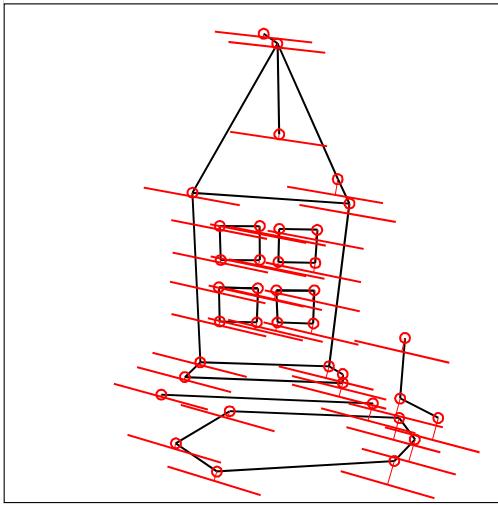
3. Enforce the rank-2 constraint.

$$\rightarrow F_q \quad \begin{cases} q^T F_q q' = 0 \\ \det(F_q) = 0 \end{cases}$$

4. De-normalize  $F_q$ :  $F = {T'}^T F_q T$

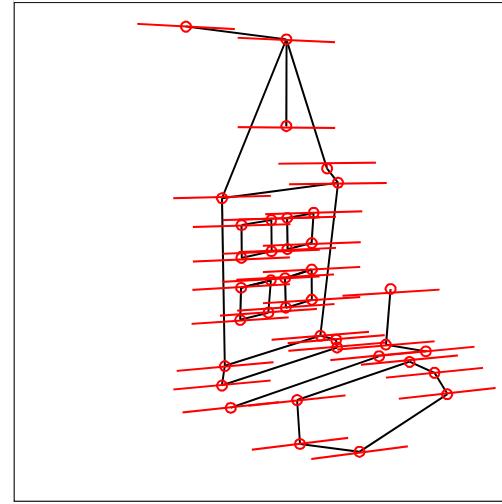
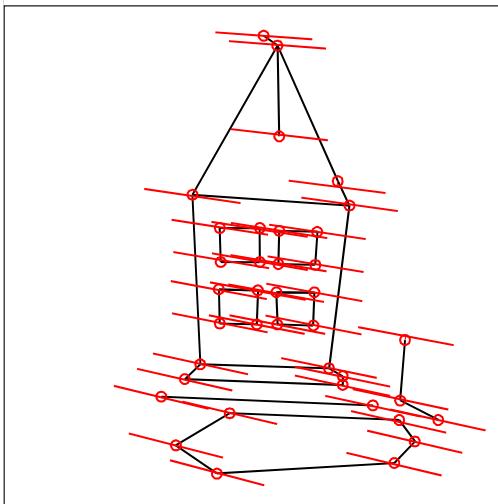
# Example

Without transformation



Mean errors:  
10.0pixel  
9.1pixel

With transformation



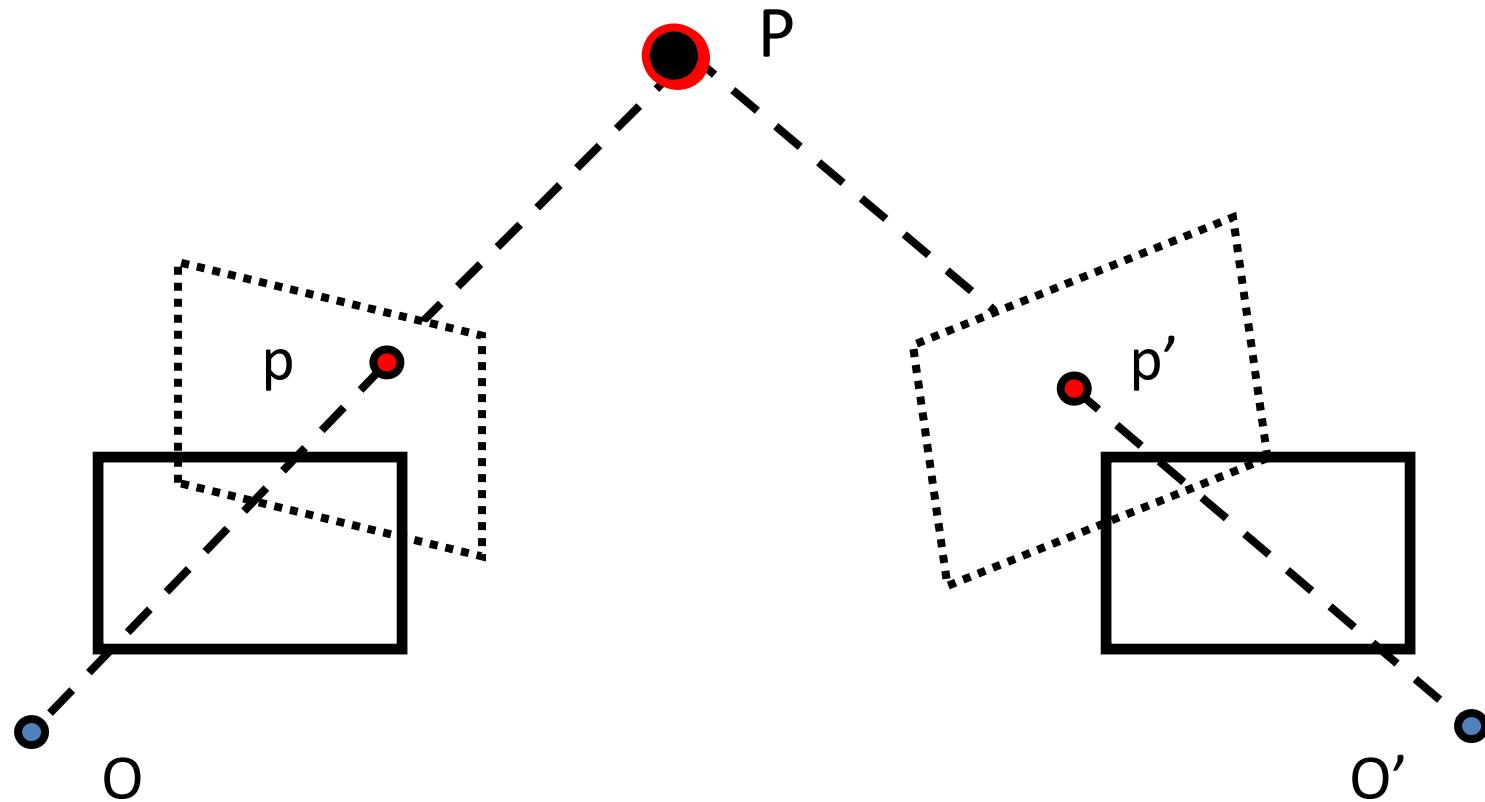
Mean errors:  
1.0pixel  
0.9pixel

# What we will learn today?

- Why is stereo useful?
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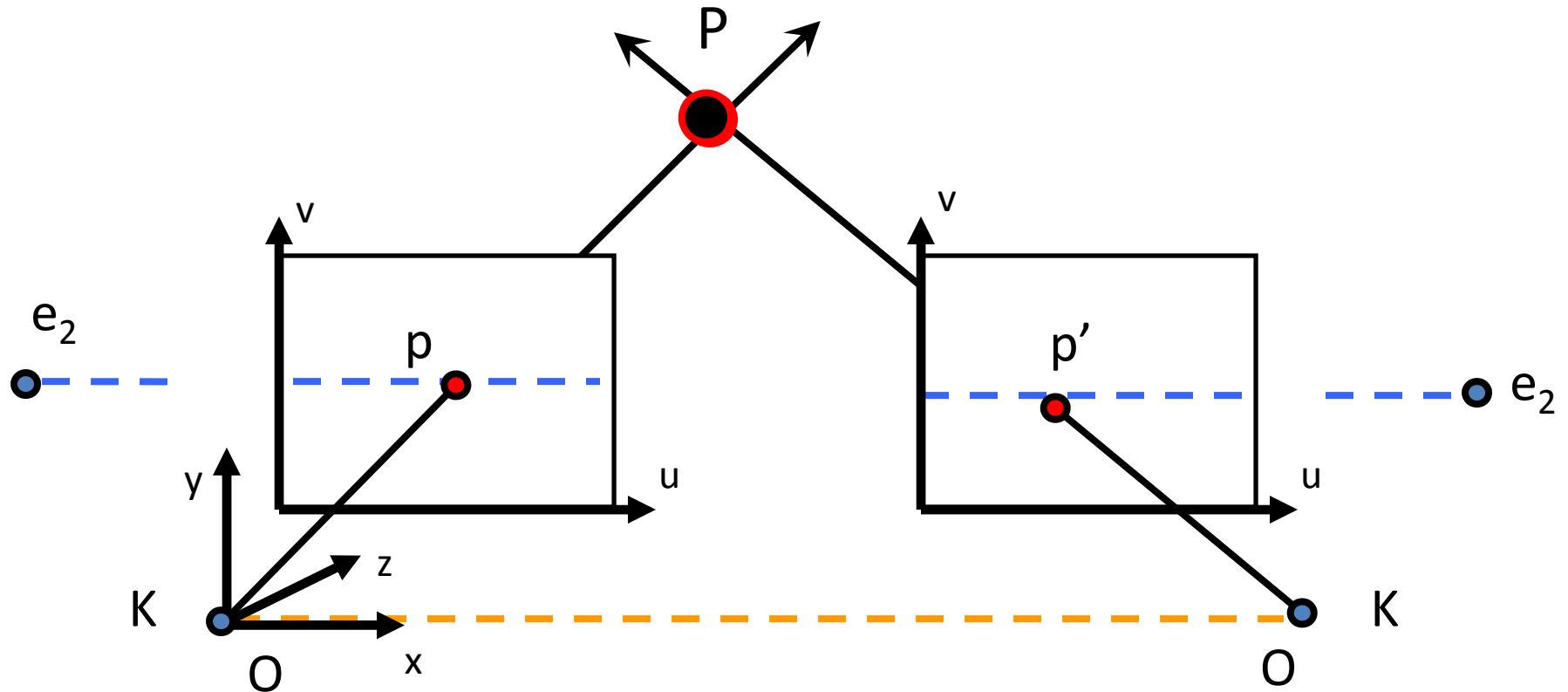
**Reading:**  
[HZ] Chapters: 4, 9, 11  
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# Rectification



- Make two camera images “parallel”
  - Correspondence problem becomes easier

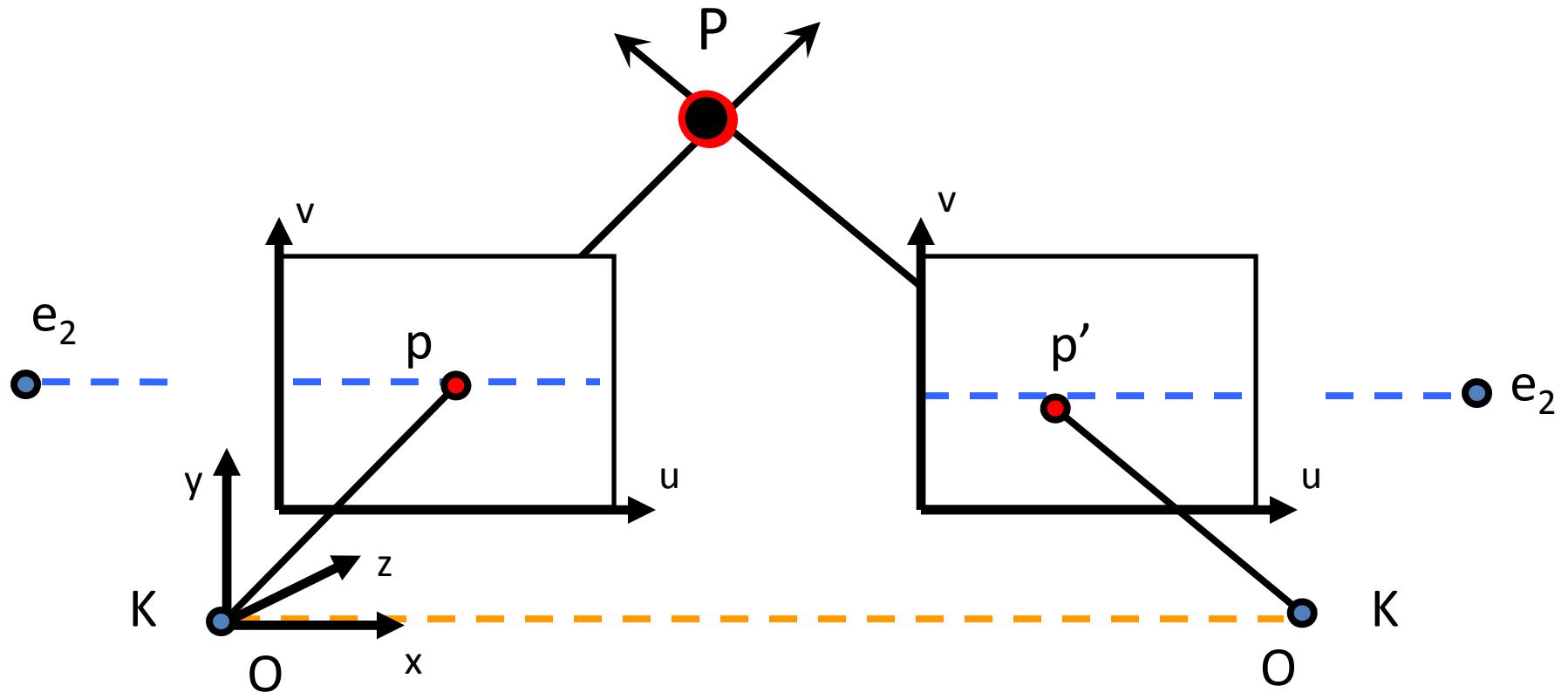
# Rectification



- Parallel epipolar lines
- Epipoles at infinity
- $v = v'$

Let's see why....

# Rectification



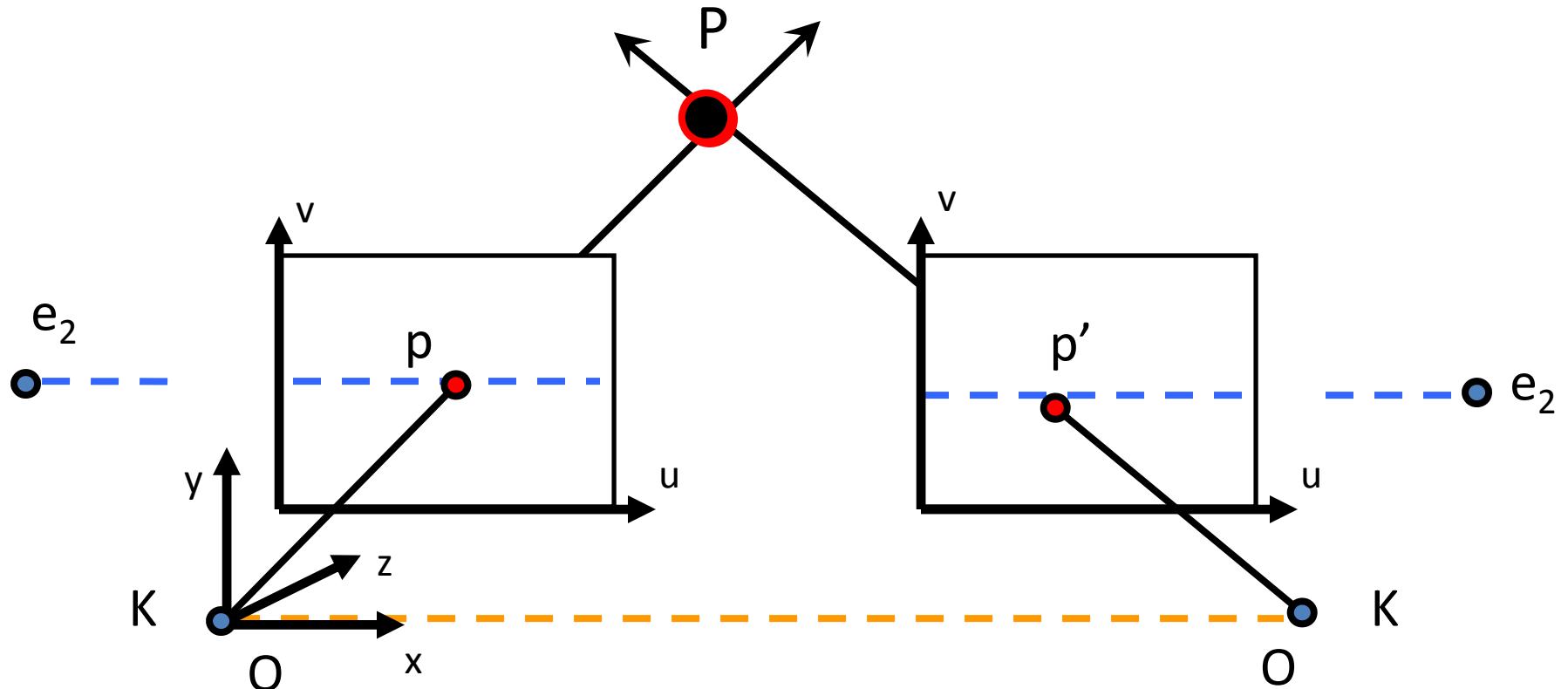
$K_1 = K_2 = \text{known}$   
 $x$  parallel to  $O_1O_2$

$$E = [t_x]R$$

# Cross product as matrix multiplication

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = [\mathbf{a}_\times] \mathbf{b}$$

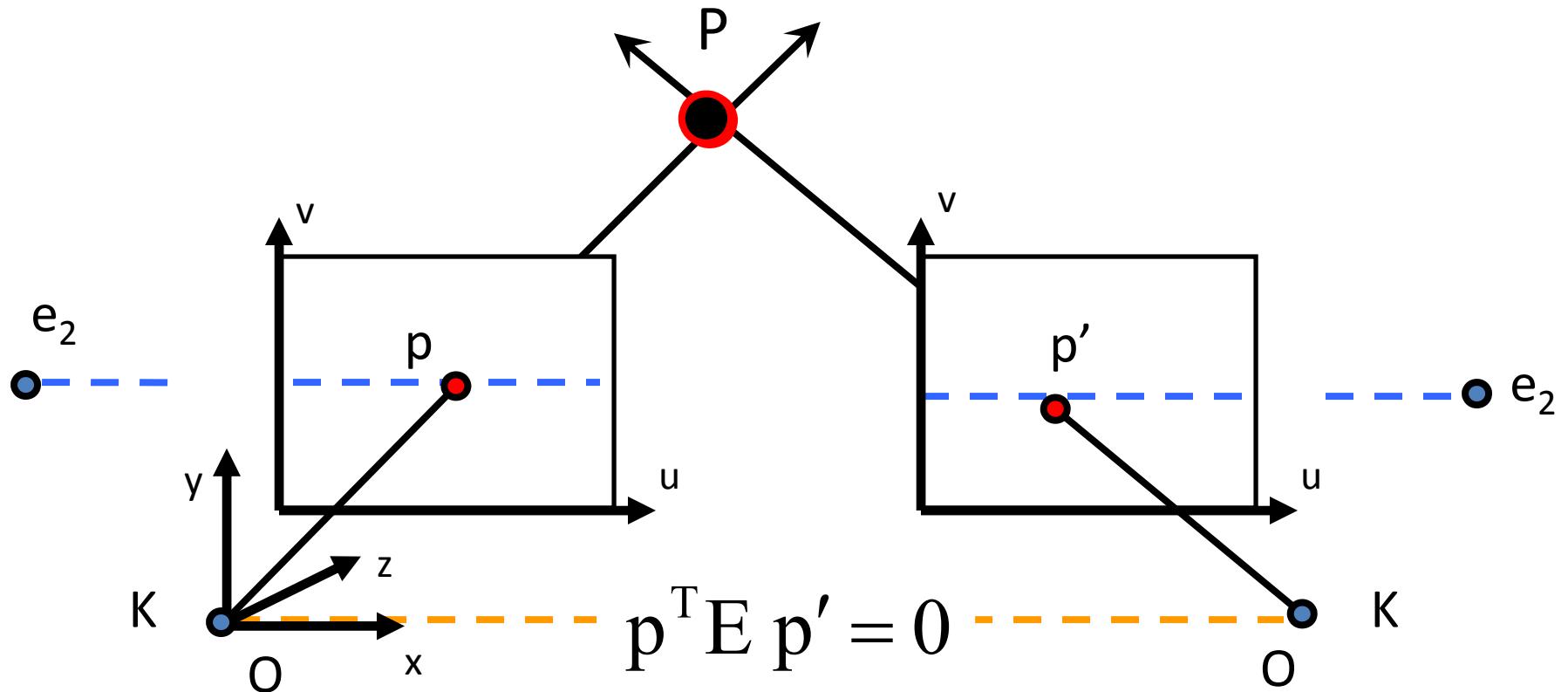
# Rectification



$K_1 = K_2 = \text{known}$   
 $x$  parallel to  $O_1O_2$

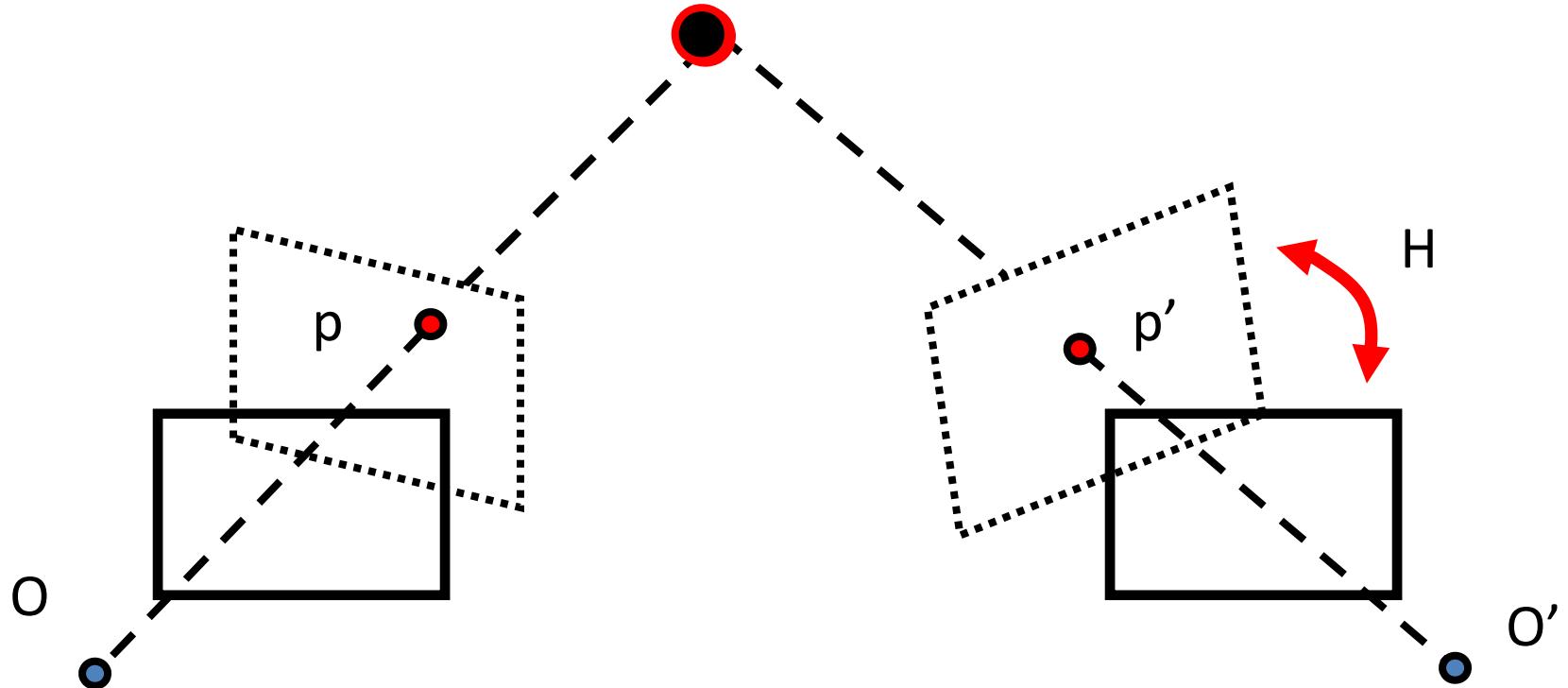
$$E = [t_x]R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -T \\ 0 & T & 0 \end{bmatrix} \rightarrow v = v'?$$

# Rectification



$$(u \quad v \quad 1) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -T \\ 0 & T & 0 \end{bmatrix} \begin{pmatrix} u' \\ v' \\ 1 \end{pmatrix} = 0 \quad (u \quad v \quad 1) \begin{pmatrix} 0 \\ -T \\ Tv' \end{pmatrix} = 0 \quad Tv = Tv' \rightarrow v = v'$$

# Rectification



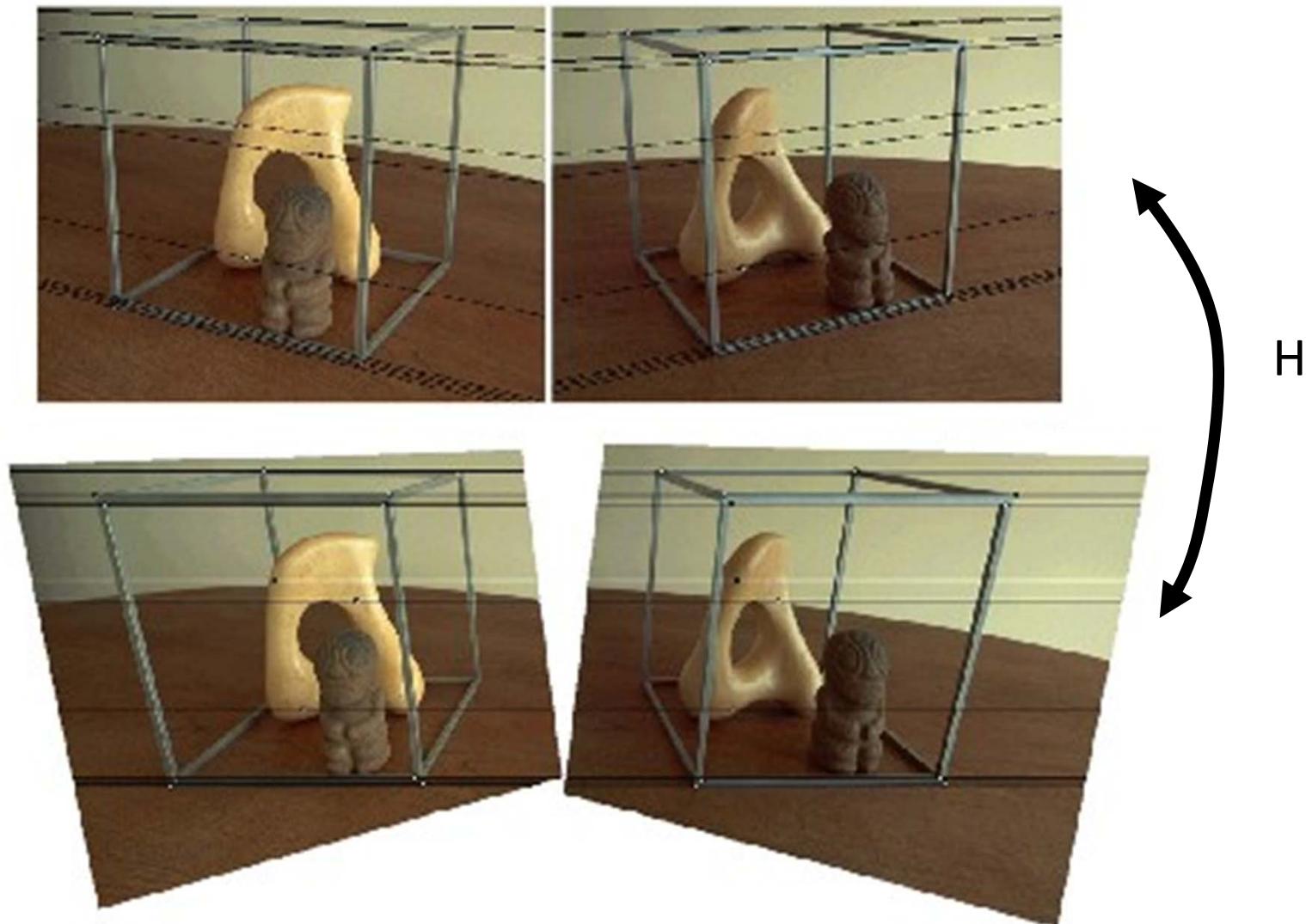
**GOAL of rectification :** Estimate a perspective transformation  $H$  that makes images parallel

Impose  $v' = v$

- This leaves degrees of freedom for determining  $H$
- If not appropriate  $H$  is chosen, severe projective distortions on image take place
- We impose a number of restriction while computing  $H$

[HZ] Chapters: 11 (sec. 11.12)

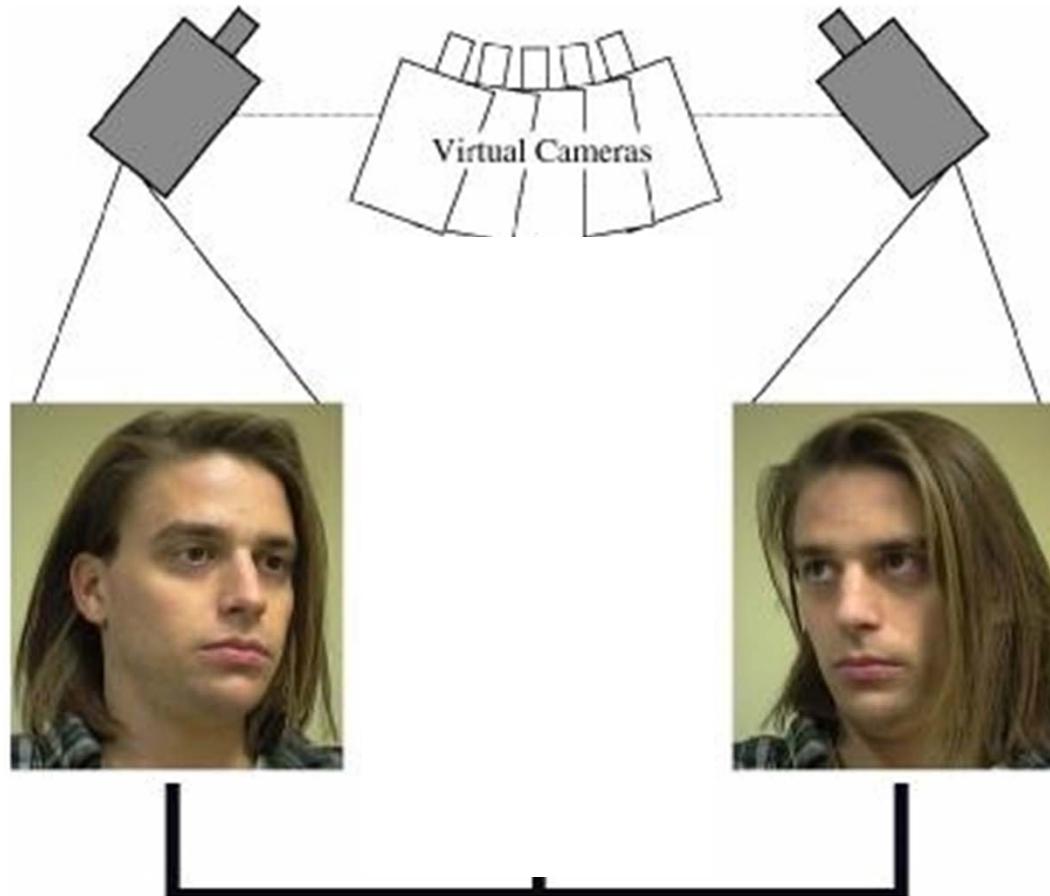
# Rectification



Courtesy figure S. Lazebnik

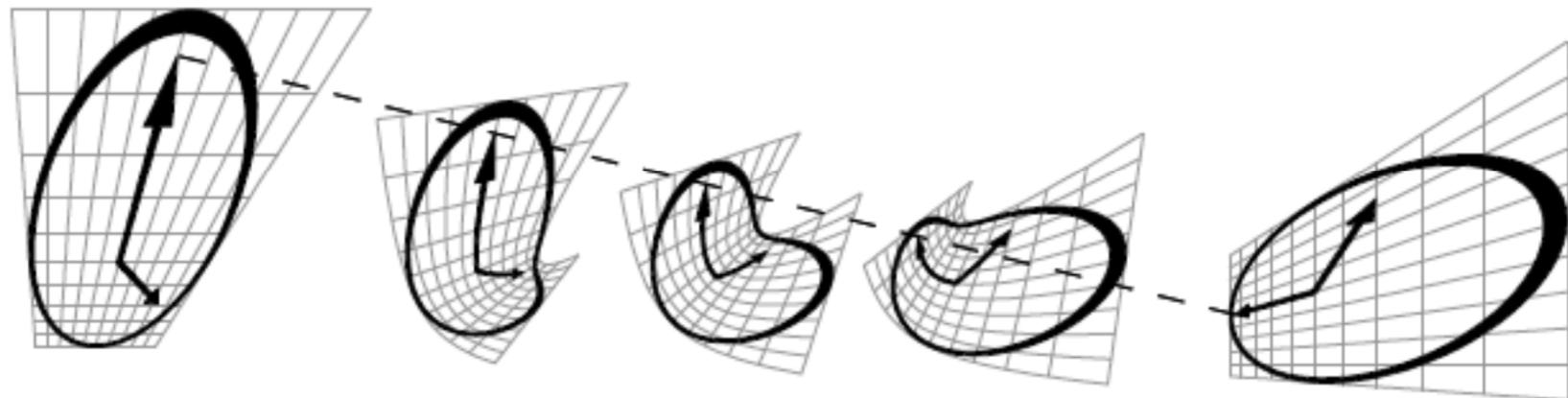
# Application: view morphing

S. M. Seitz and C. R. Dyer, *Proc. SIGGRAPH 96*, 1996, 21-30

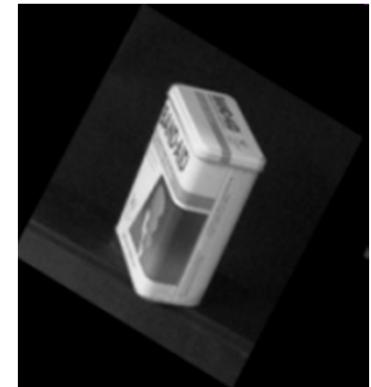
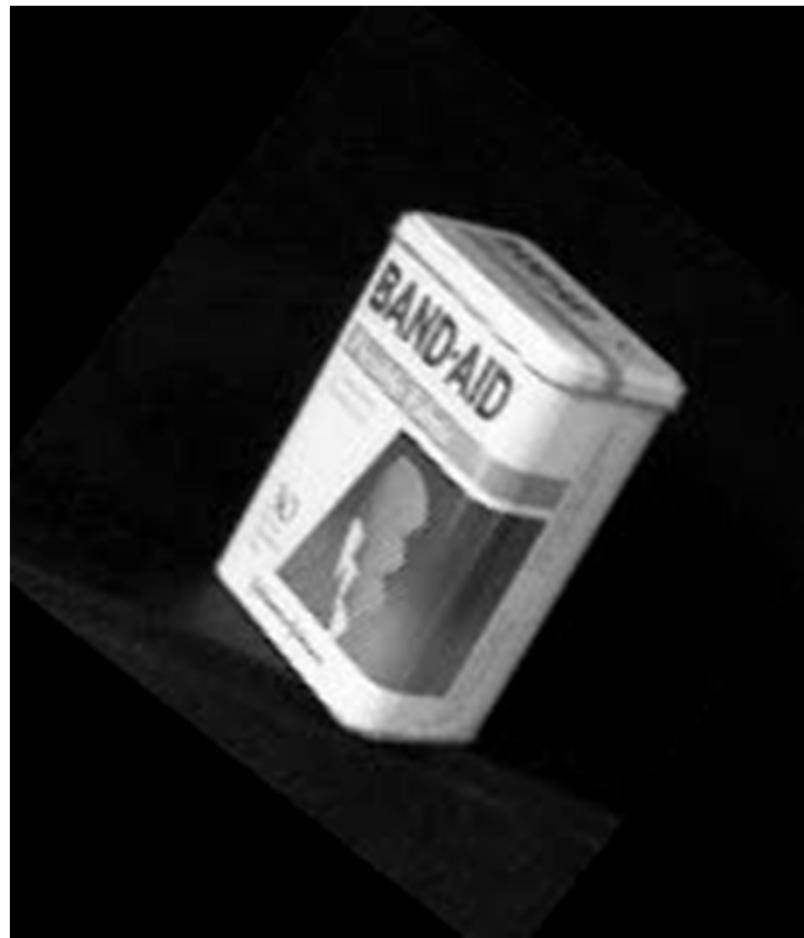


# Application: view morphing

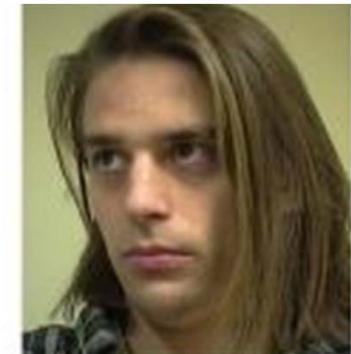
If rectification is not applied, the morphing procedure does not generate geometrically correct interpolations



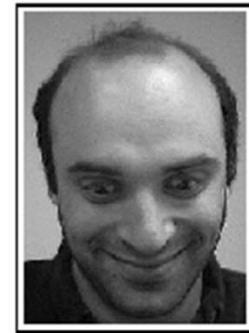
# Application: view morphing



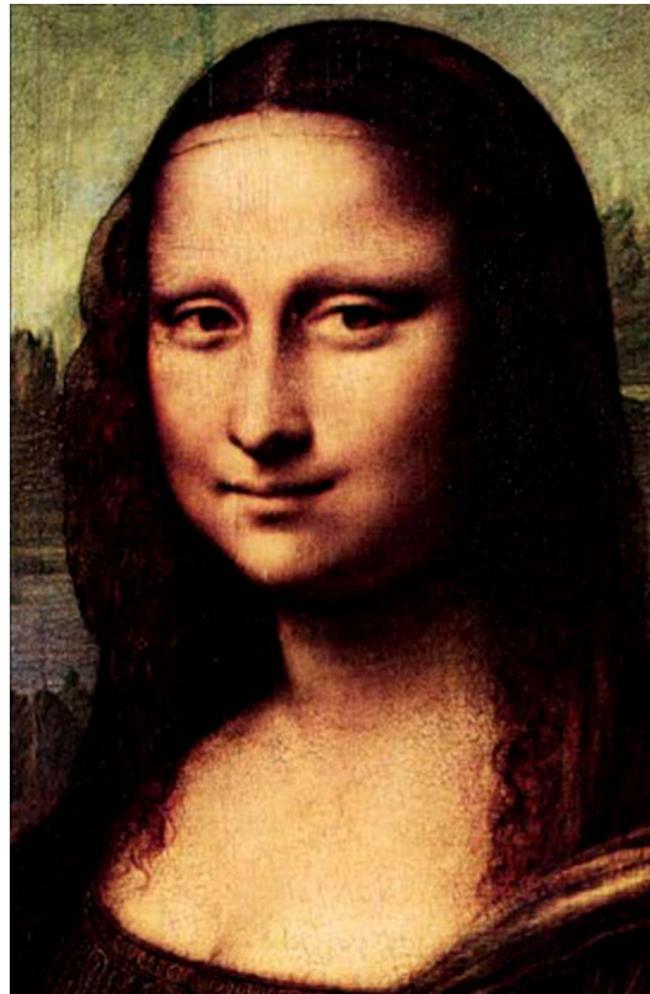
# Application: view morphing



# Application: view morphing



# Application: view morphing



# The Fundamental Matrix Song

<http://danielwedge.com/fmatrix/>

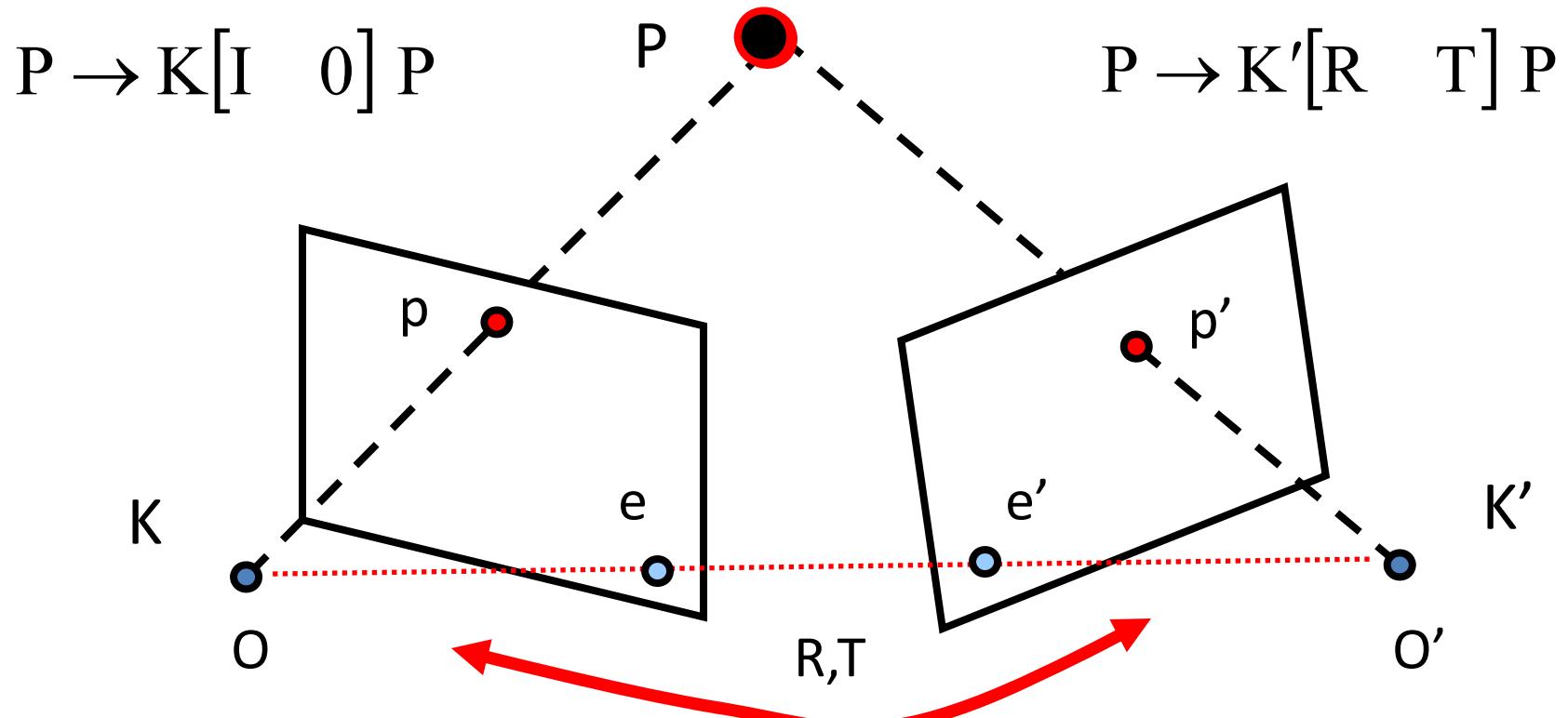
# What we have learned today?

- Why is stereo useful?
- Epipolar constraints
- Essential and fundamental matrix
- Estimating F (**Problem Set 2 (Q2)**)
- Rectification

**Reading:**  
[HZ] Chapters: 4, 9, 11  
[FP] Chapters: 10

# Supplementary materials

# Making image planes parallel

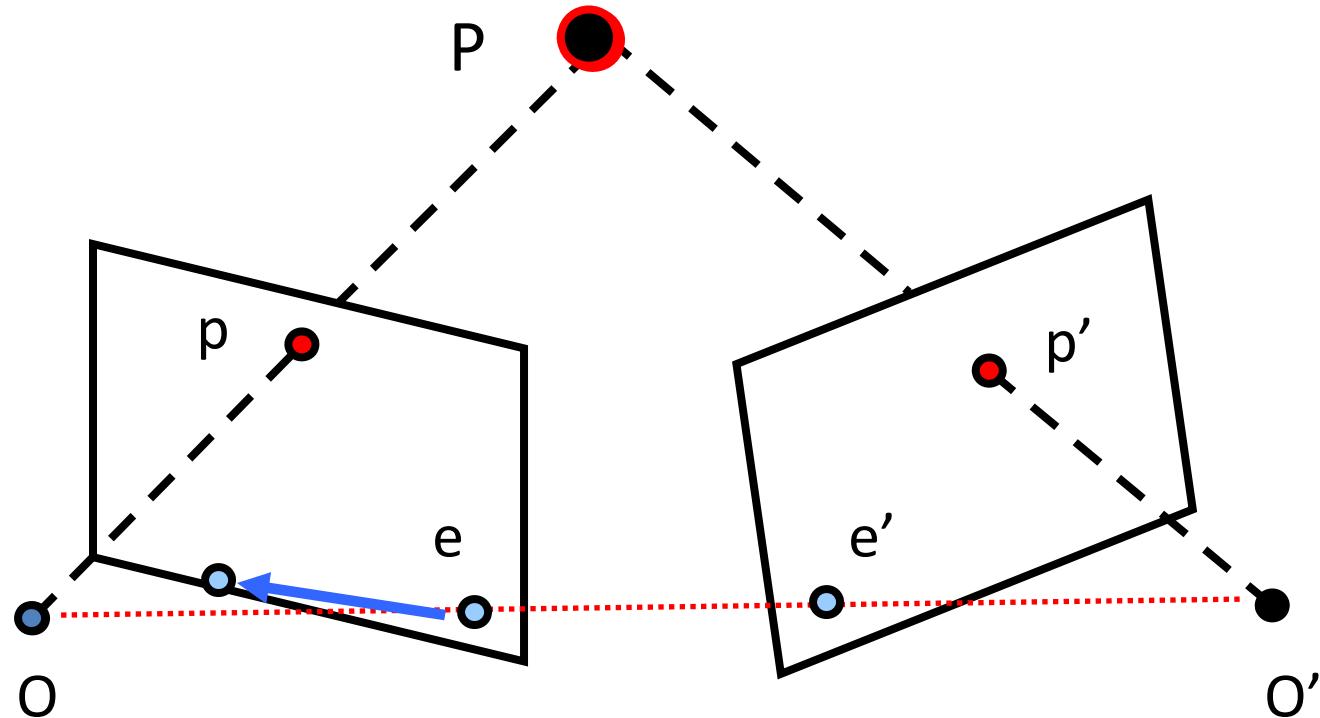


0. Compute epipoles

$$e = K R^T \quad T = [e_1 \quad e_2 \quad 1]^T$$

$$e' = K' T$$

# Making image planes parallel



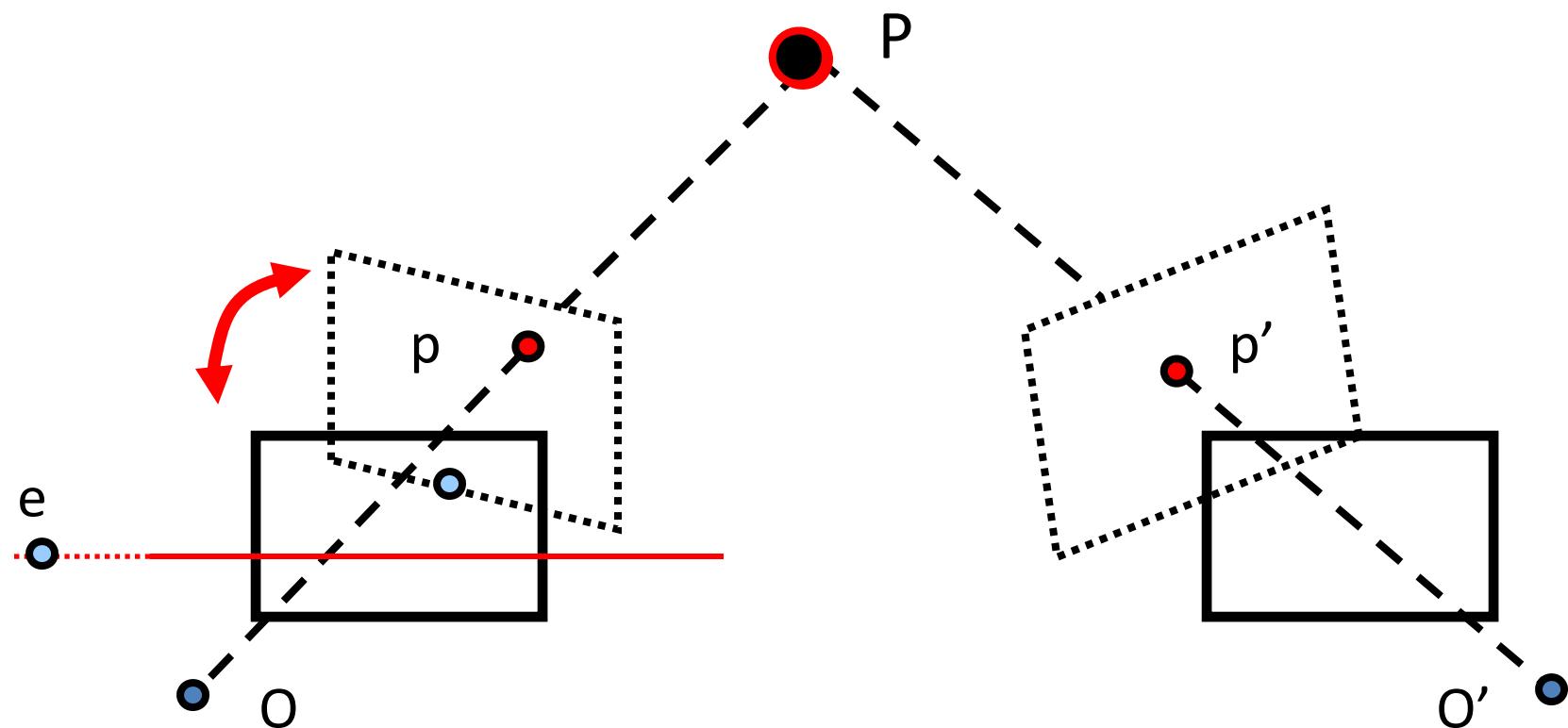
1. Map  $e$  to the x-axis at location  $[1,0,1]^T$  (normalization)

$$e = [e_1 \ e_2 \ 1]^T \rightarrow$$

$$[1 \ 0 \ 1]^T$$

$$H_1 = R_H T_H$$

# Making image planes parallel



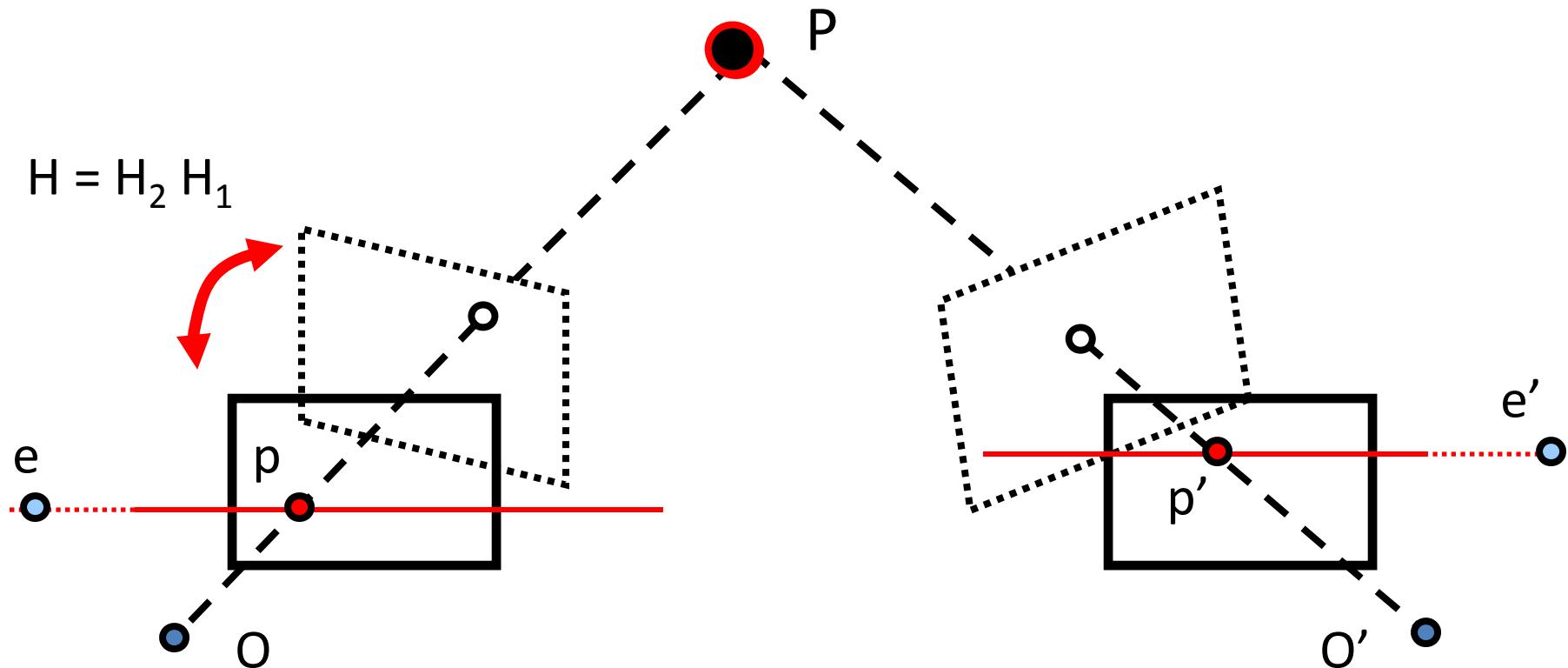
2. Send epipole to infinity:

$$e = [1 \ 0 \ 1]^T \rightarrow [1 \ 0 \ 0]^T$$

$$H_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Minimizes the distortion in a neighborhood (approximates id. mapping)

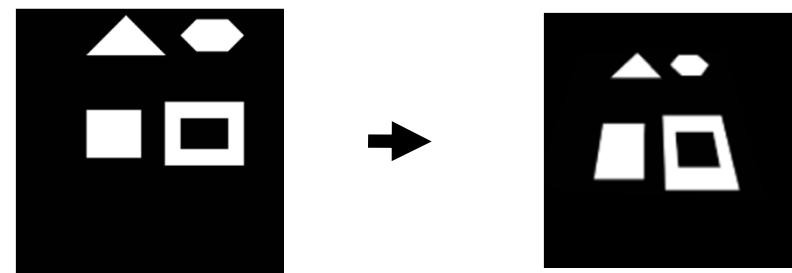
# Making image planes parallel



3. Define:  $H = H_2 H_1$
4. Align epipolar lines

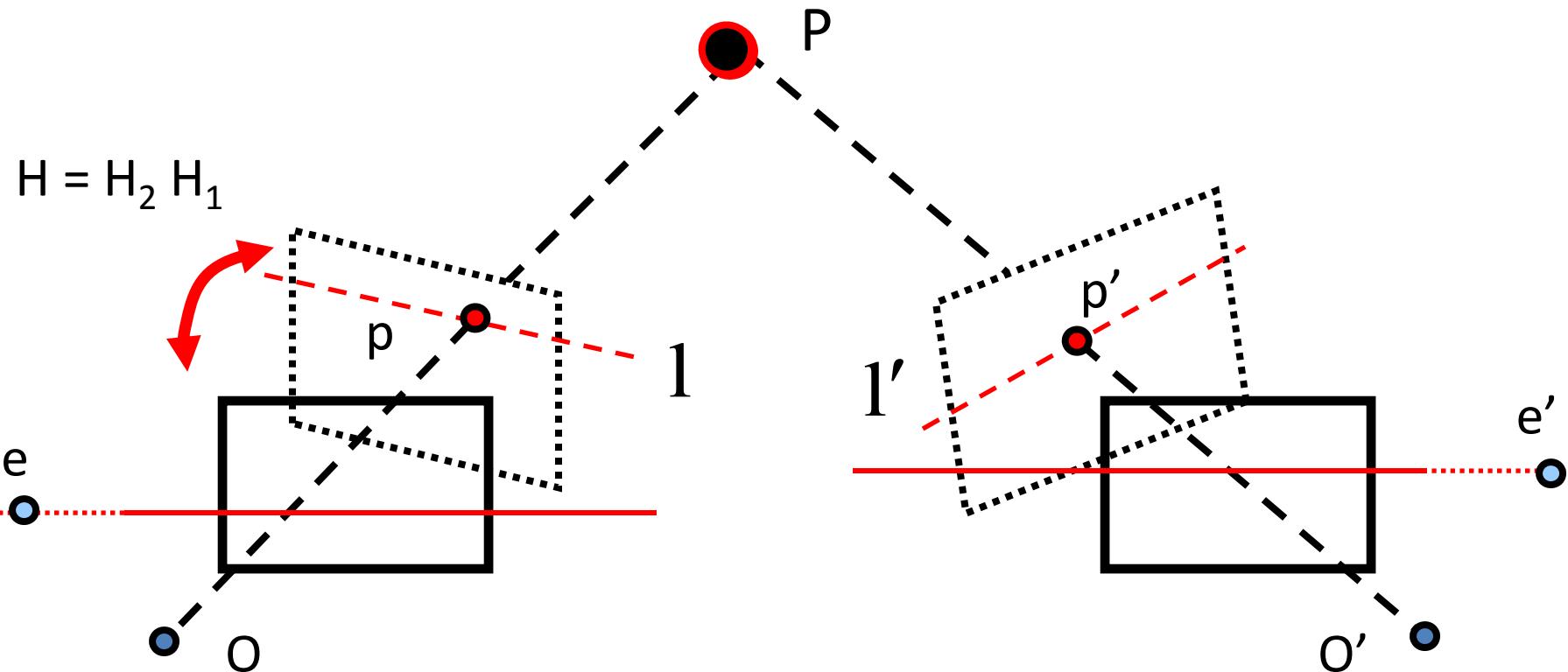
# Projective transformation of a line (in 2D)

$$H = \begin{bmatrix} A & t \\ v & b \end{bmatrix}$$



$$l \rightarrow H^{-T} l$$

# Making image planes parallel



3. Define:  $H = H_2 H_1$

$$\overline{H'}^{-T} l' = \overline{H}^{-T} l$$

4. Align epipolar lines

These are called **matched pair** of transformation

[HZ] Chapters: 11 (sec. 11.12)