# Photogrammetry II

## Triangulation and Absolute Orientation

**Cyrill Stachniss**

The slides have been created by Cyrill Stachniss.

# Motivation

Given the relative orientation of two images, compute the points in 3D

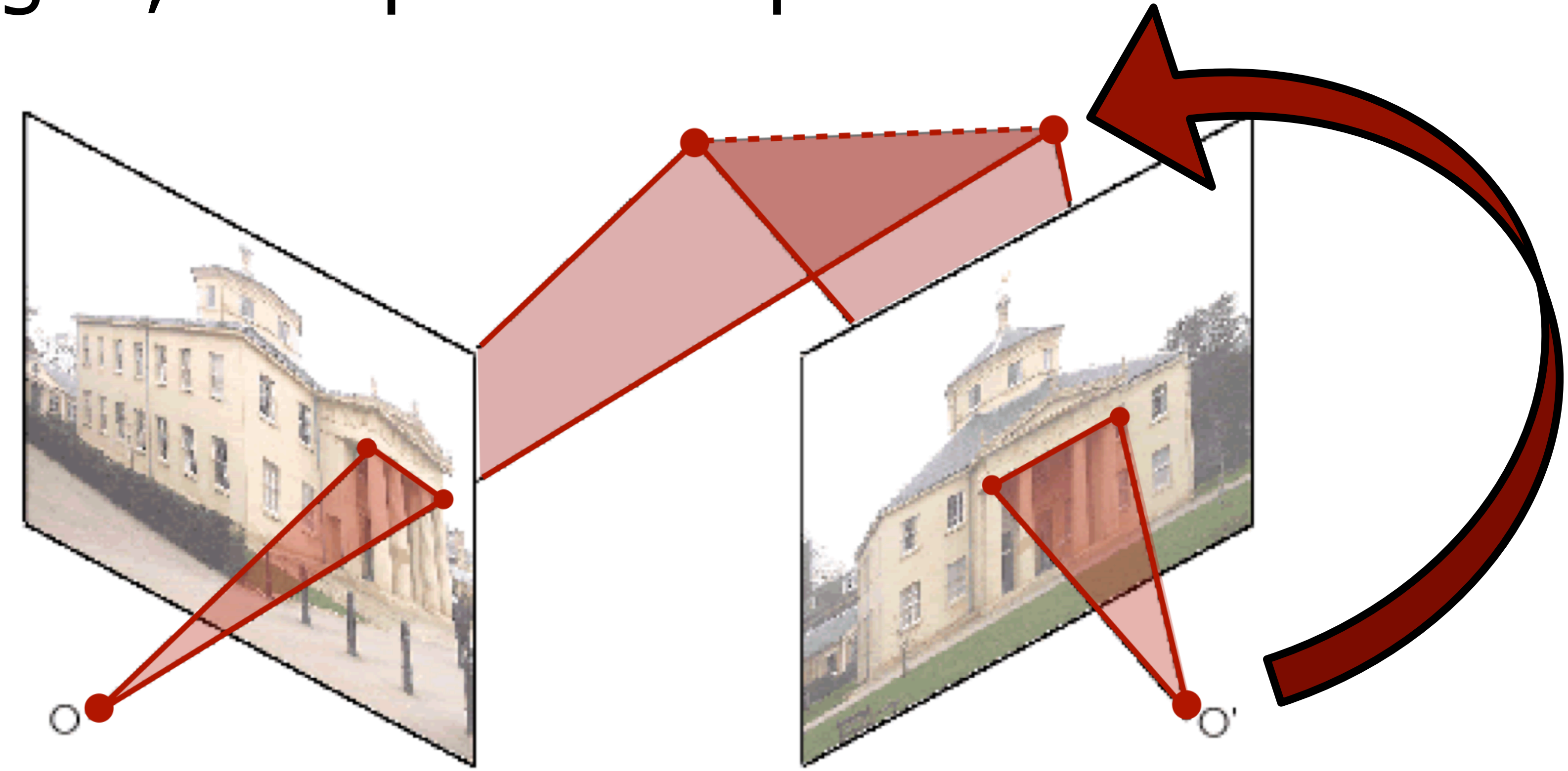


Image courtesy: Schindler

# Topic

## Last lectures

Computing the relative orientation of two images

## Today

Given the relative orientation of the images, compute the **3D location of corresponding points**

EN: Triangulation / forward intersection
DE: Räumlicher Vorwärtsschnitt

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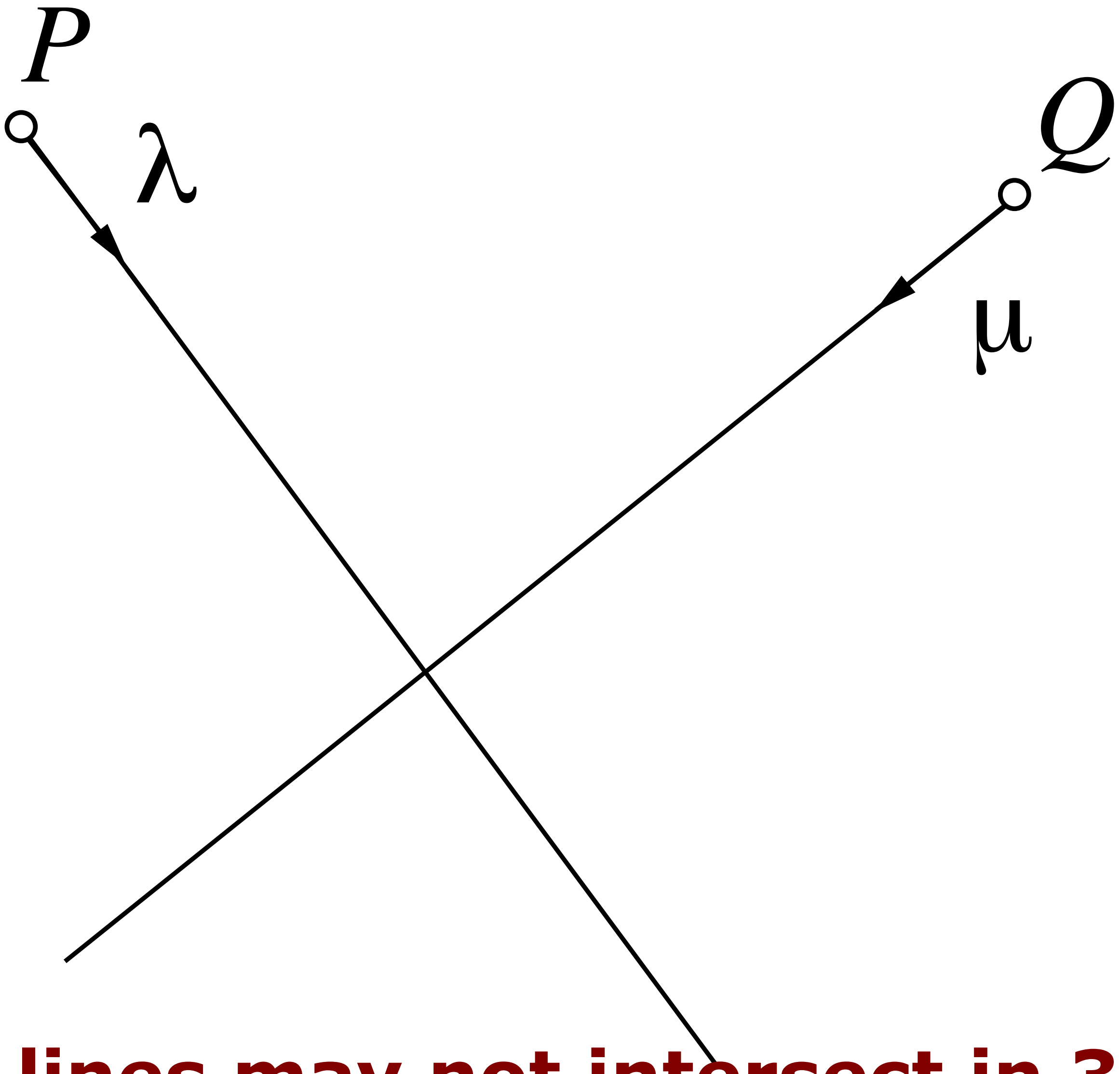
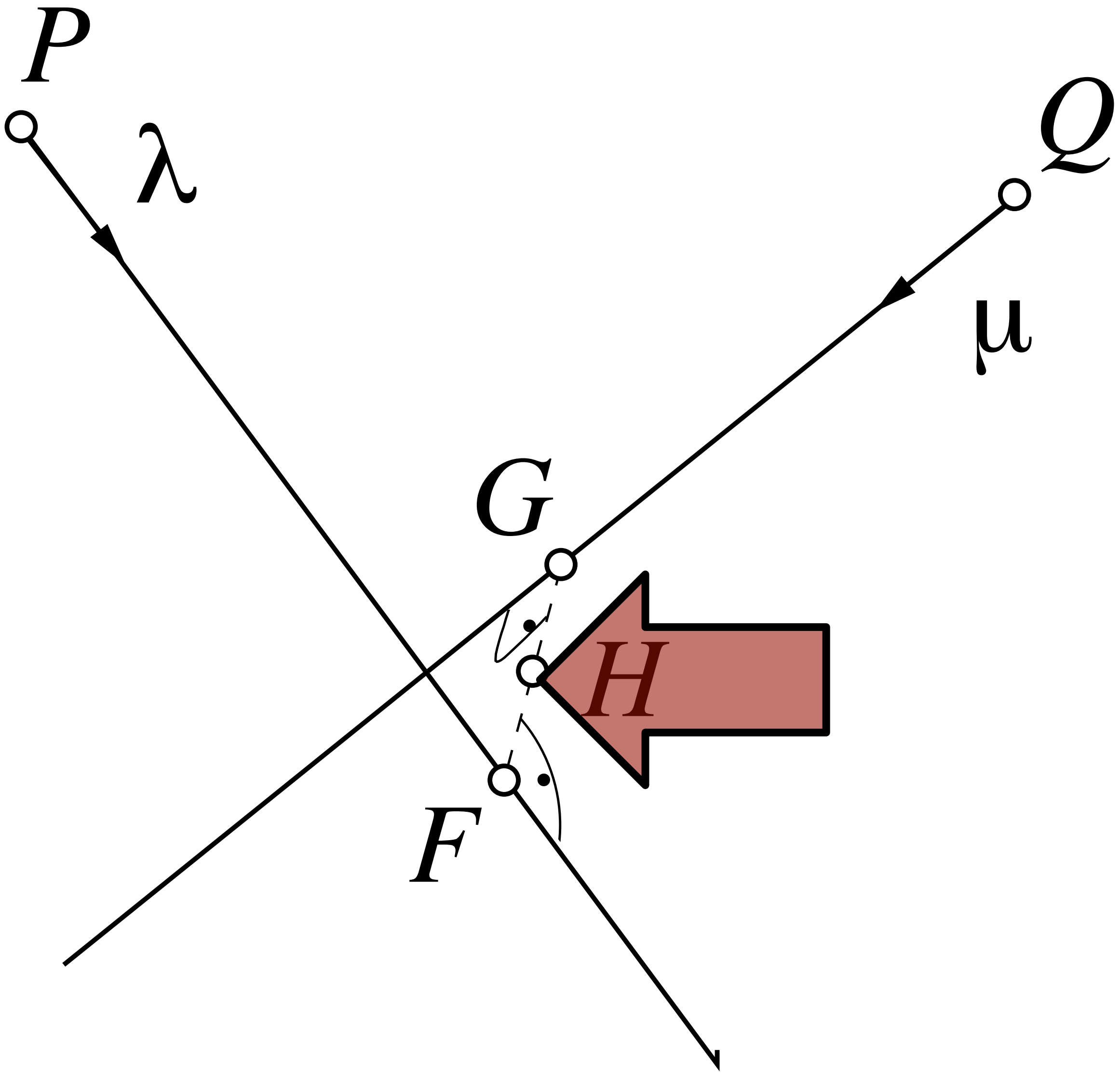
# 1.
# Geometric Approach

# The Problem




**The lines may not intersect in 3D!**

# Find the Point H



# Geometric Solution

- Equation for two lines in 3D

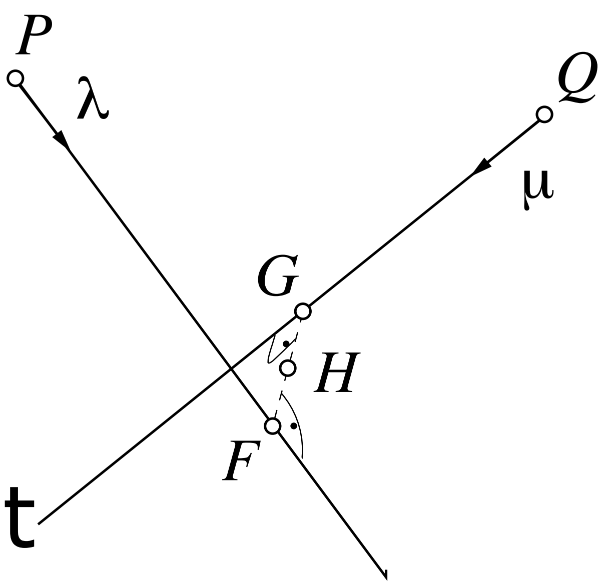
$$\boldsymbol{f} = \boldsymbol{p} + \lambda \boldsymbol{r} \qquad \boldsymbol{g} = \boldsymbol{q} + \mu \boldsymbol{s}$$

- with the points $\boldsymbol{p} = \boldsymbol{X}_{O'} \quad \boldsymbol{q} = \boldsymbol{X}_{O''}$
- and the directions (calibrated camera)

$$\boldsymbol{r} = R'^{\mathsf{T}}\ {}^{k}\mathbf{x}' \qquad \boldsymbol{s} = R''^{\mathsf{T}}\ {}^{k}\mathbf{x}''$$

- with ${}^{k}\mathbf{x}' = (x', y', c)^{\mathsf{T}} \quad {}^{k}\mathbf{x}'' = (x'', y'', c)^{\mathsf{T}}$

# Geometric Solution




- The shortest connection requires that FG is orthogonal to both lines
- This leads to the constraint

$$(\boldsymbol{f}-\boldsymbol{g})\cdot\boldsymbol{r}=0 \qquad (\boldsymbol{f}-\boldsymbol{g})\cdot\boldsymbol{s}=0$$

which directly leads to

$$\begin{aligned}(\boldsymbol{q}+\lambda\boldsymbol{s}-\boldsymbol{p}-\mu\boldsymbol{r})\cdot\boldsymbol{s} &= 0\\ (\boldsymbol{q}+\lambda\boldsymbol{s}-\boldsymbol{p}-\mu\boldsymbol{r})\cdot\boldsymbol{r} &= 0\end{aligned}$$

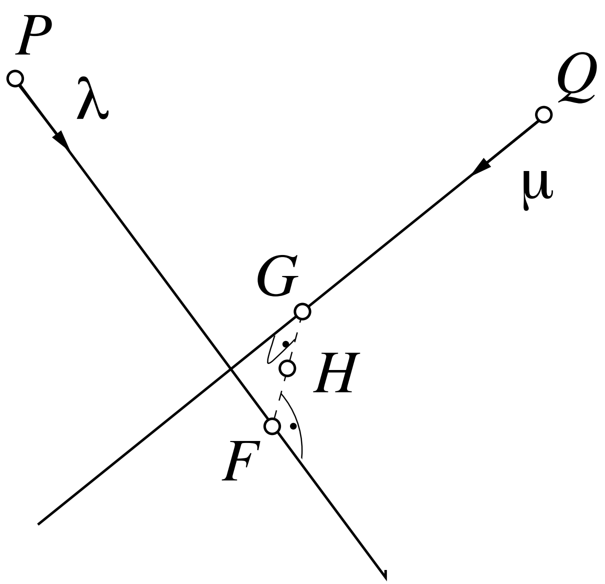
- Two equations, two unknowns

# Geometric Solution

- By solving the equations

$$\begin{aligned}(\boldsymbol{q} + \lambda \boldsymbol{s} - \boldsymbol{p} - \mu \boldsymbol{r}) \cdot \boldsymbol{s} &= 0 \\ (\boldsymbol{q} + \lambda \boldsymbol{s} - \boldsymbol{p} - \mu \boldsymbol{r}) \cdot \boldsymbol{r} &= 0\end{aligned}$$

  we obtain $\lambda, \mu$
- $\lambda, \mu$ directly yield F and G
- We compute H as the middle of the line connecting F and G



# 2.
# For the Stereo Normal Case

Solution already computed in
Photogrammetry I, Ch. 10-Matching

# Stereo Normal Case

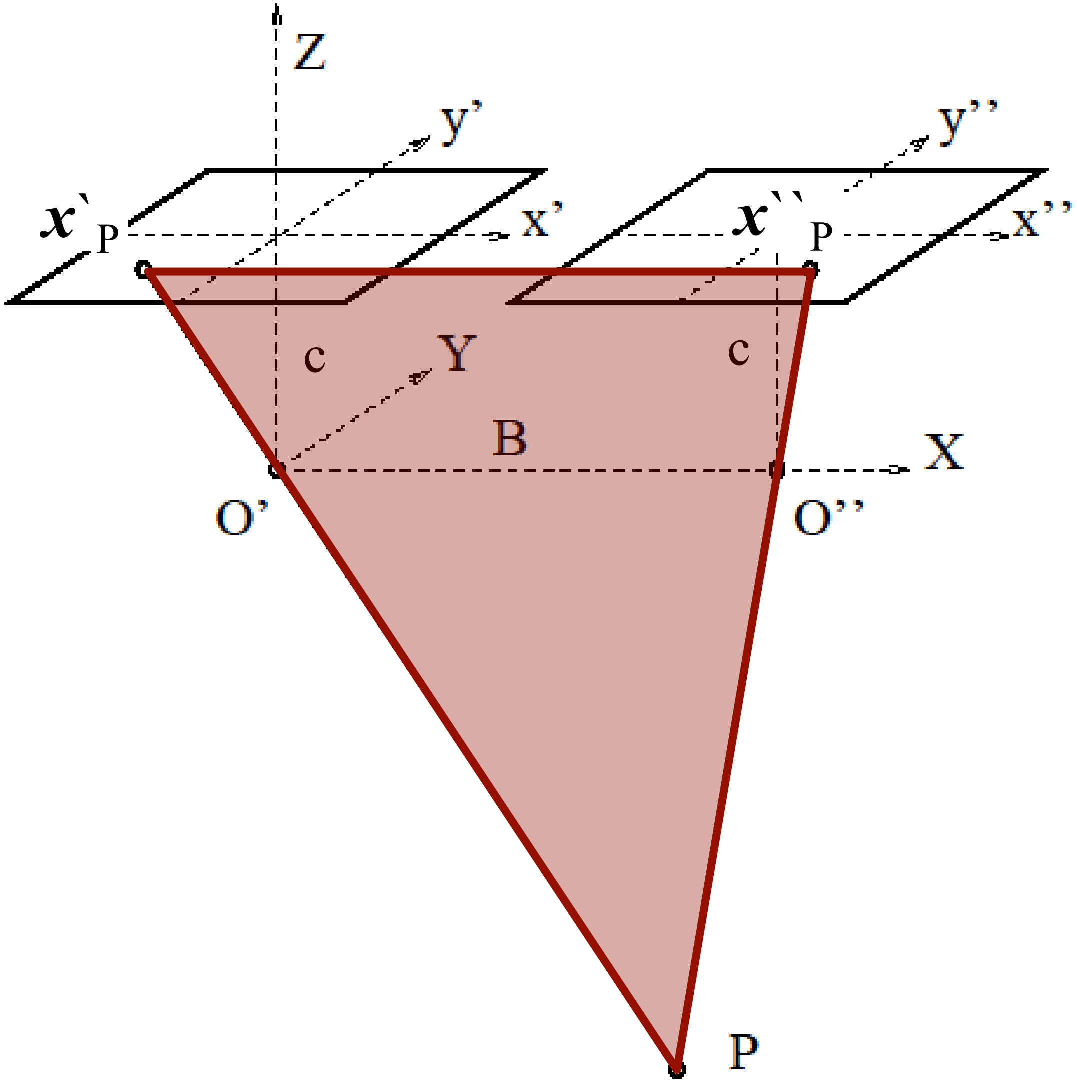



Image courtesy: Förstner

# Stereo Normal: Intersection

1. $Z$-coordinate from intercept theorem

$$\frac{Z}{c} = \frac{B}{-\underbrace{(x'' - x')}_{p_x}}$$

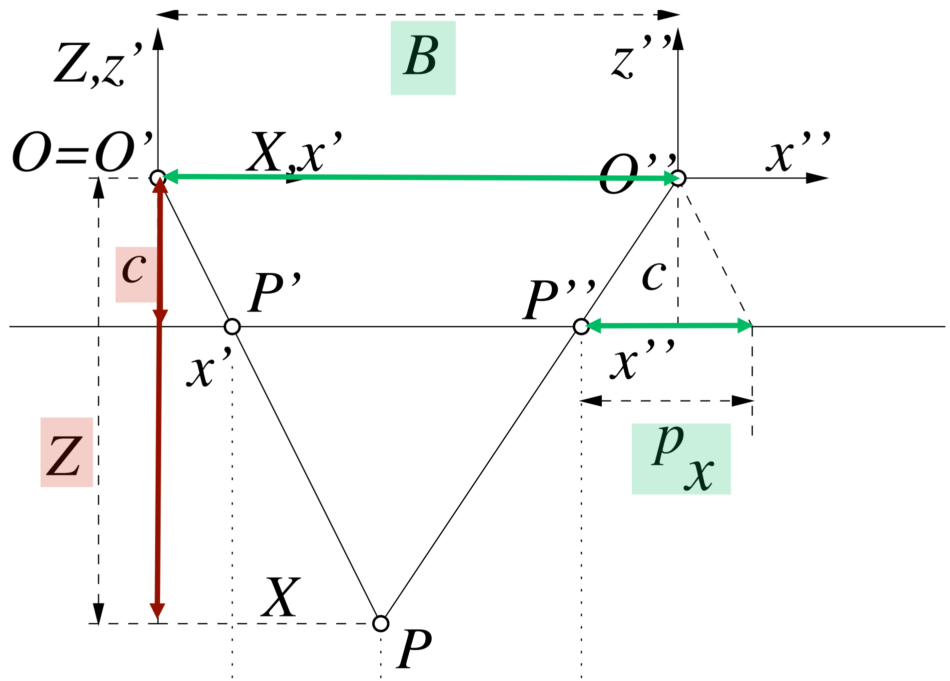



Image courtesy: Förstner

# Stereo Normal: Intersection

1. $Z$-coordinate from intercept theorem

$$\frac{Z}{c} = \frac{B}{\underbrace{-(x'' - x')}_{p_x}}$$

$$\Rightarrow \quad Z = c\frac{B}{-(x'' - x')}$$

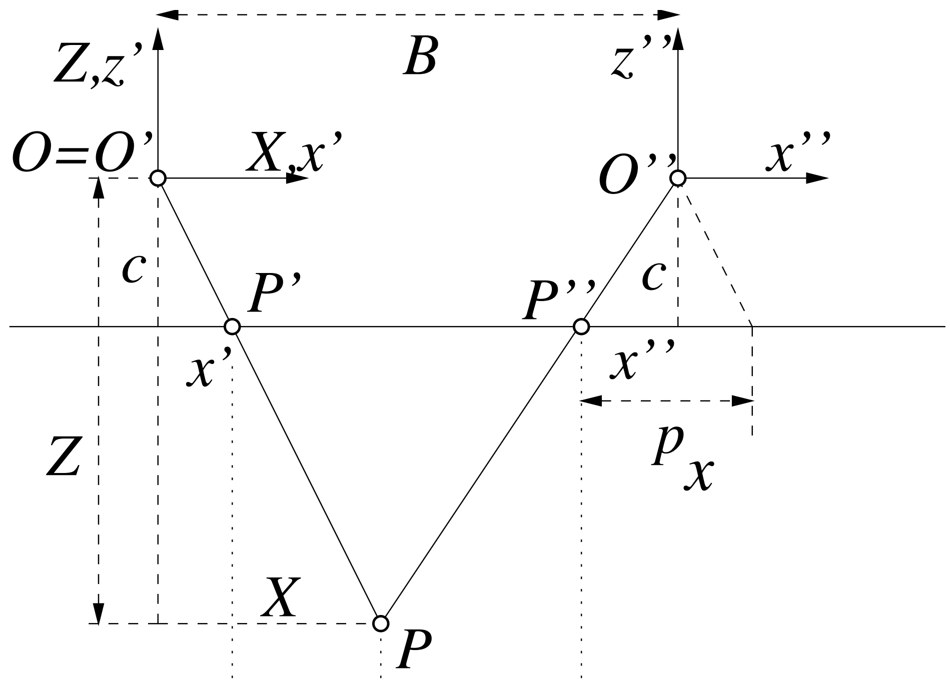



Image courtesy: Förstner

# Stereo Normal: Intersection

1. Z-coordinate from intercept theorem

$$\frac{Z}{c} = \frac{B}{-(x'' - x')}$$

2. X-coordinate

$$\frac{X}{x'} = \frac{Z}{c}$$

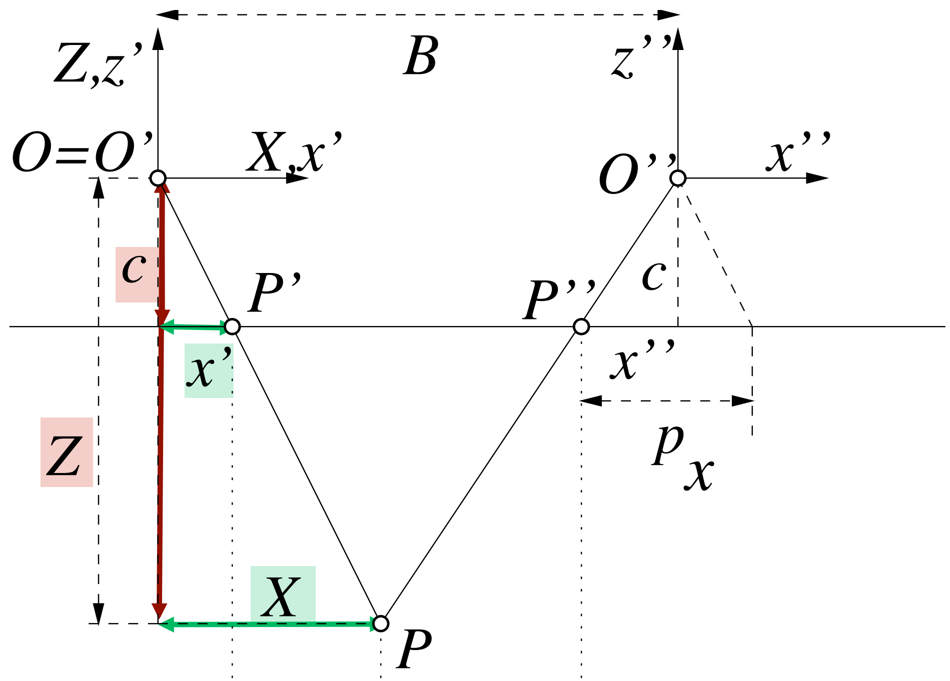



Image courtesy: Förstner

# Stereo Normal: Intersection

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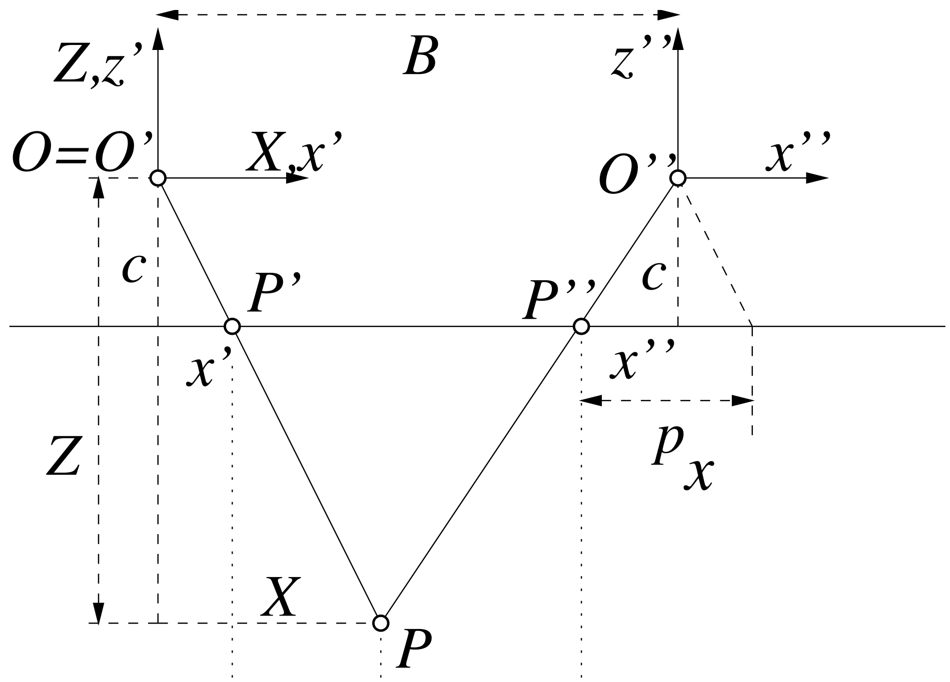


$$\Rightarrow \quad X = x' \frac{B}{-(x'' - x')}$$

Image courtesy: Förstner

# Stereo Normal: Intersection

1. Z-coordinate from intercept theorem

$$\frac{Z}{c} = \frac{B}{-(x'' - x')}$$

2. X-coordinate

$$\frac{X}{x'} = \frac{Z}{c}$$

3. Y-coordinate by mean

$$\frac{Y}{X} = \frac{\frac{y'+y''}{2}}{x'}$$

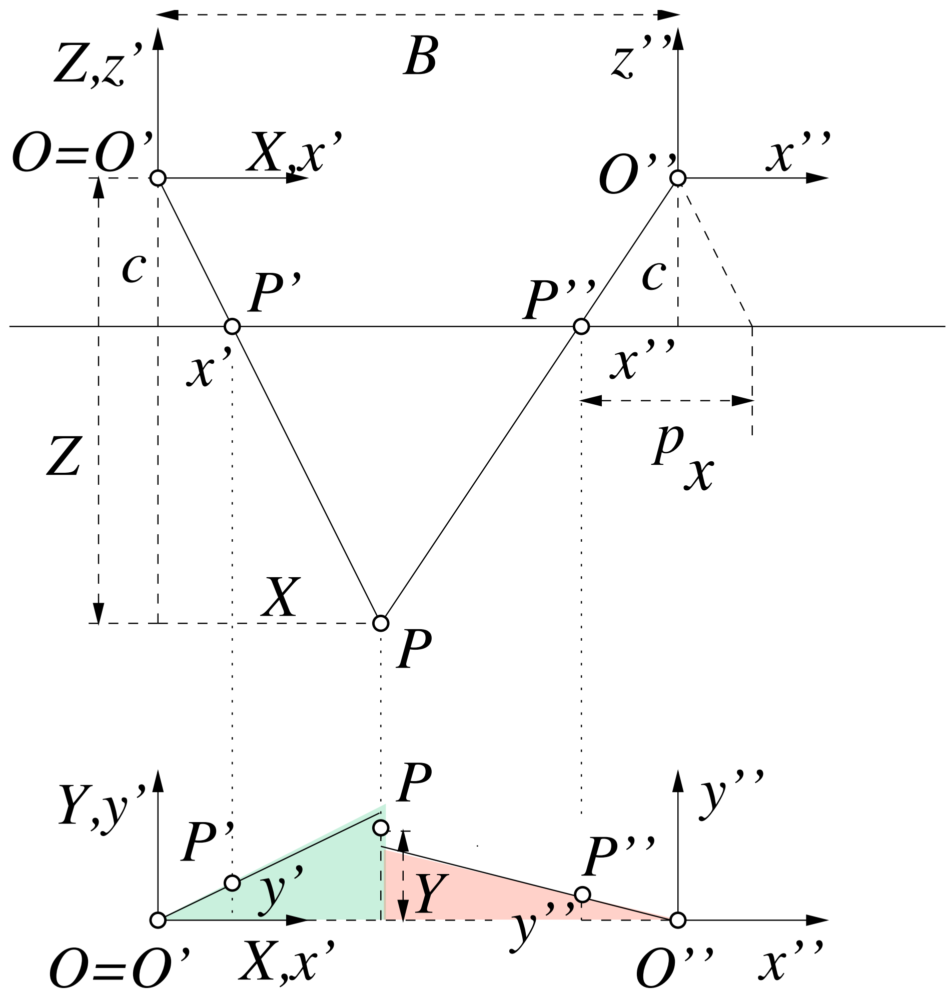


Image courtesy: Förstner

# Stereo Normal: Intersection

1. $Z$-coordinate from intercept theorem

$$\frac{Z}{c} = \frac{B}{-(x'' - x')}$$

2. $X$-coordinate

$$\frac{X}{x'} = \frac{Z}{c}$$

3. $Y$-coordinate by mean

$$\frac{Y}{X} = \frac{\frac{y'+y''}{2}}{x'}$$

$$\Rightarrow \quad Y = \frac{y' + y''}{2} \frac{B}{-(x'' - x')}$$

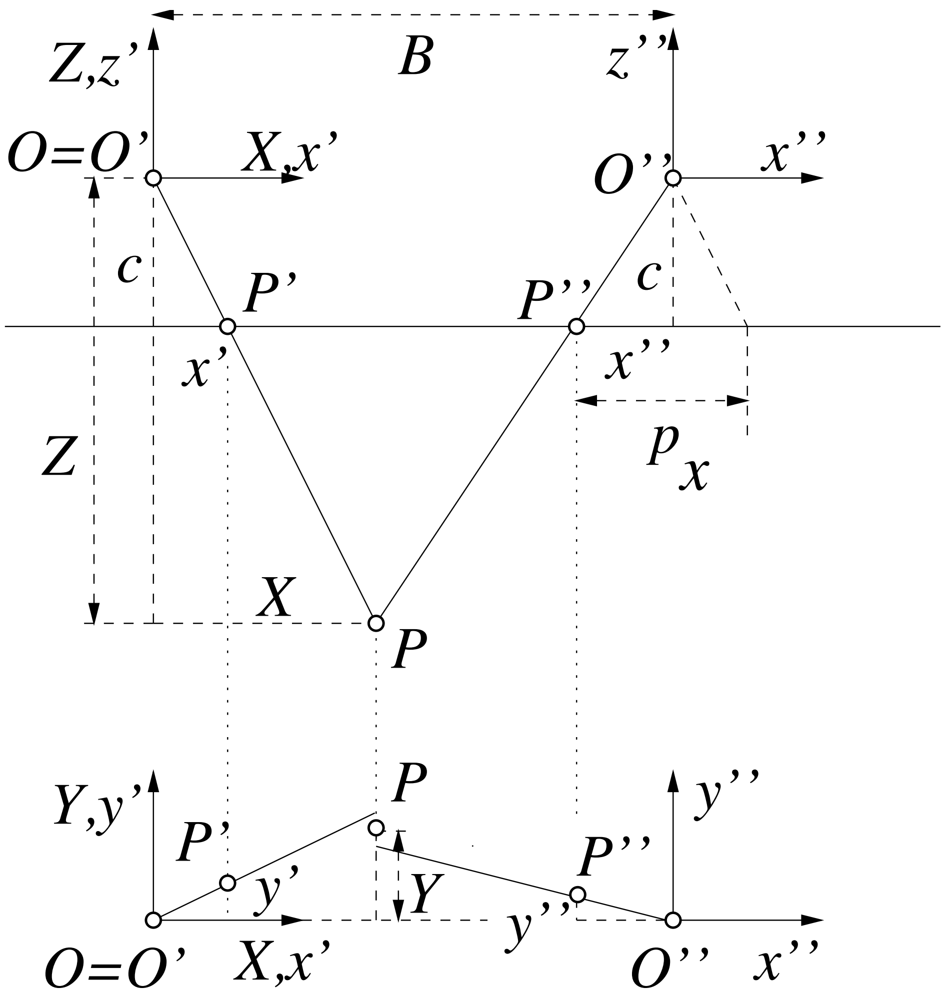


Image courtesy: Förstner

# Intersection of Two Rays for the Stereo Normal Case

$$X = x' \frac{B}{-(x'' - x')} \qquad Y = \frac{y' + y''}{2} \frac{B}{-(x'' - x')} \qquad Z = c \frac{B}{-(x'' - x')}$$

- $x$-parallax $p_x = x'' - x'$ corresponds to depth $Z$
- $y$-parallax $p_y = y'' - y'$ corresponds to the consistency of image points and should be small (due to stereo normal case)
- The parallax is also called disparity (DE: Parallaxe, Disparität)

# X-Parallax (X-Disparity)

- The x-parallax is a key element

$$
\begin{aligned}
X &= x' \frac{B}{-(x'' - x')} \\
Y &= \frac{y' + y''}{2} \frac{B}{-(x'' - x')} \\
Z &= c \frac{B}{-(x'' - x')}
\end{aligned}
$$

# X-Parallax and Scale Number

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$$
\begin{aligned}
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image scale number

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Z &= c \boxed{\frac{B}{-(x''-x')}}
\end{aligned}
$$

image scale number

- Image scale number: $M = \frac{-B}{x''-x'} = \frac{Z}{c}$

$$
\boxed{X = Mx' \qquad Y = M\frac{y'+y''}{2} \qquad Z = Mc}
$$

# Intersection of Two Rays for the Stereo Normal Case

- If the y-parallax is zero, we obtain

$$X = x' \frac{B}{-p_x} \qquad Y = y' \frac{B}{-p_x} \qquad Z = c \frac{B}{-p_x}$$

# Intersection of Two Rays for the Stereo Normal Case

- If the y-parallax is zero, we obtain

$$X = x' \frac{B}{-p_x} \qquad Y = y' \frac{B}{-p_x} \qquad Z = c \frac{B}{-p_x}$$

- We can write this as

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} -\frac{B}{p_x} & 0 & 0 \\ 0 & -\frac{B}{p_x} & 0 \\ 0 & 0 & -\frac{B}{p_x} \end{bmatrix} \begin{bmatrix} x' \\ y' \\ c \end{bmatrix}$$

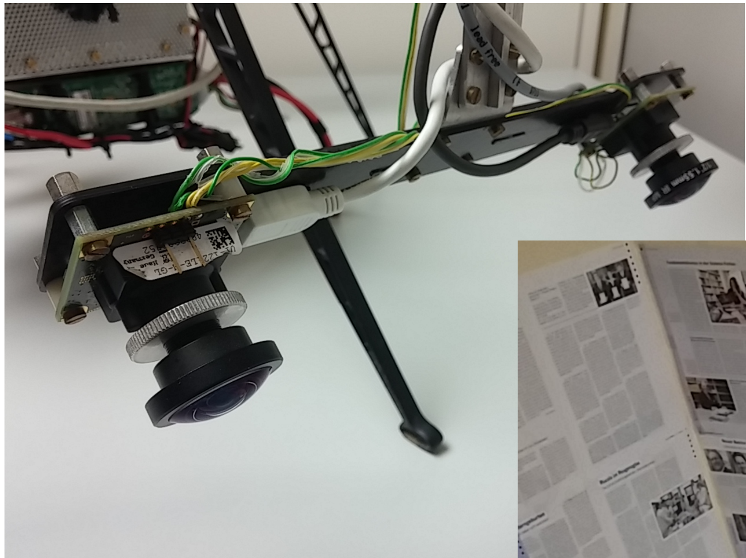
## Parallax Map

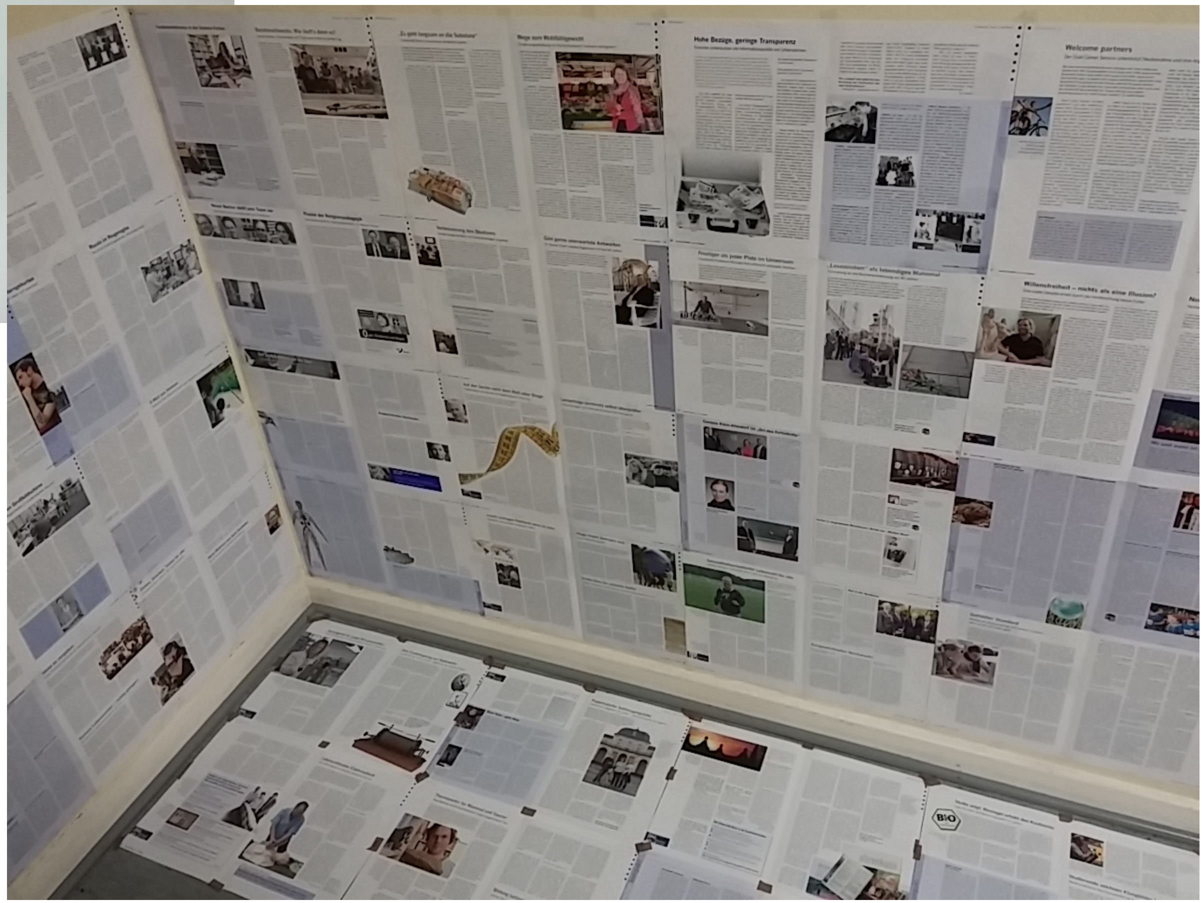
- Using H.C. and the parallax as input

$$\begin{bmatrix} U \\ V \\ W \\ T \end{bmatrix} = \begin{bmatrix} B & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & Bc & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ 1 \\ p_x \end{bmatrix}$$

- For a set of points $\{x', y'\}$ in the first image, $\{x', y', p_x\}$ is called **parallax map**
- **The parallax map directly leads to the 3D coordinates of the point**

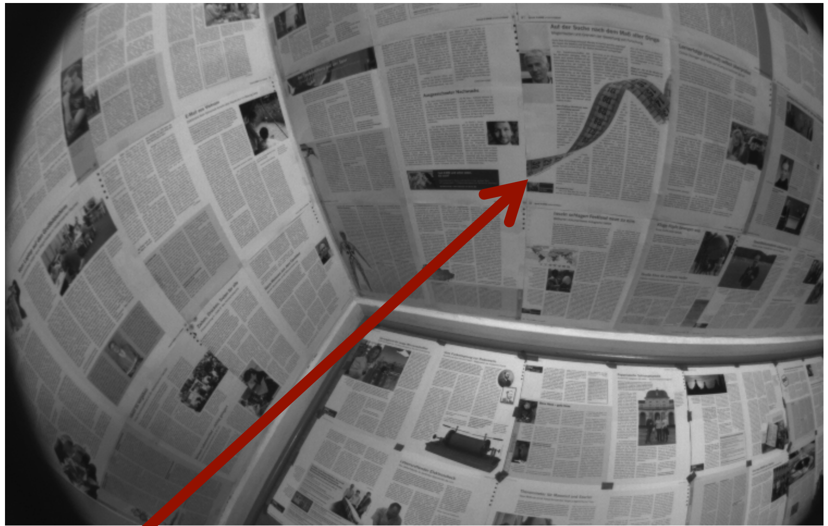
# Example – Setup





# Example – Image Pair

# Example – Parallax Map

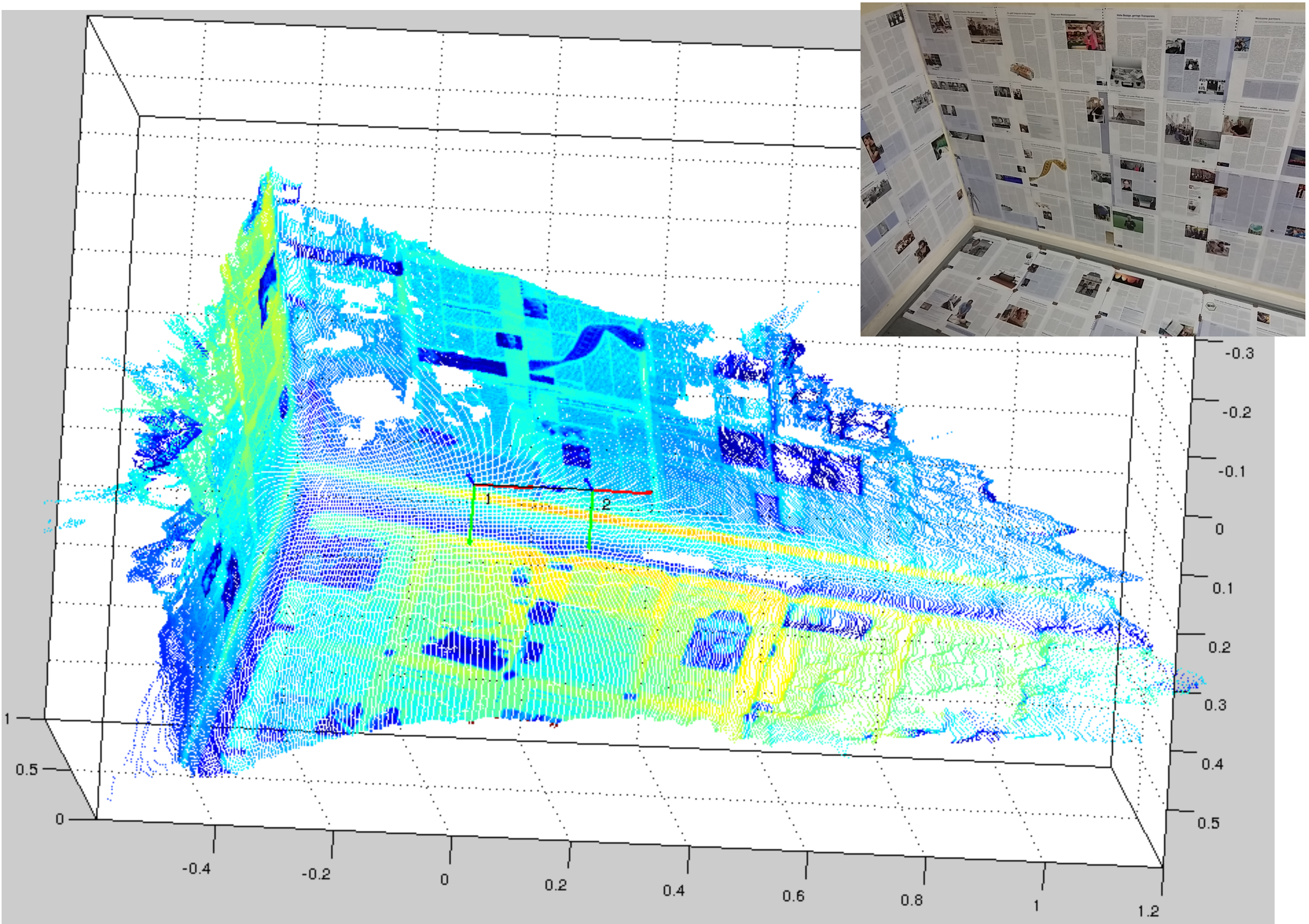


parallax map $\{x', y', p_x\}_1 \ldots \{x', y', p_x\}_N$

# Example – Parallax Map

parallax map $\{x', y', p_x\}_1 \ldots \{x', y', p_x\}_N$

# Example – 3D Point Cloud



# 3.
# Quality of the 3D Points

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What influences the quality of the 3D points obtained in the normal case?

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A. Quality of the orientation parameters

B. Quality of the measured image coordinates

# Quality of the 3D Points

What influences the quality of the 3D points obtained in the normal case?

A. Quality of the orientation parameters

**B. Quality of the measured image coordinates**

# Quality of the 3D Points

- Assuming that we measure the image coordinates in x/y with $\sigma_{x'} = \sigma_{y'}$
- Starting from

$$X = Mx' \qquad Y = M\frac{y' + y''}{2}$$

- Directly yields the uncertainty in x/y

$$\boxed{\sigma_X = M\sigma_{x'} = \frac{Z}{c}\sigma_{x'}} \qquad \boxed{\sigma_Y = \frac{\sqrt{2}}{2}M\sigma_{y'} = \frac{\sqrt{2}}{2}\frac{Z}{c}\sigma_{y'}}$$

# Quality of the 3D Points

- For the Z coordinate, we start with

$$Z = Mc = -\frac{B}{p_x} c \qquad \to \qquad Z\, p_x = -B\, c$$

- and obtain for the relative precision (DE: Relativgenauigkeit)

$$\boxed{\frac{\sigma_Z}{Z} = \frac{\sigma_{p_x}}{p_x}}$$

**The relative precision of the height is the relative precision of the x-parallax**

# Height/Depth Precision

- Starting from $\frac{\sigma_Z}{Z} = \frac{\sigma_{p_x}}{p_x}$ we obtain:

$$\sigma_Z = \frac{Z}{p_x}\sigma_{p_x} = \frac{cB}{p_x^2}\sigma_{p_x} = \frac{Z^2}{cB}\sigma_{p_x} = \frac{Z}{c\,B/Z}\sigma_{p_x}$$

$$Z = \frac{cB}{p_x} \qquad \frac{1}{p_x} = \frac{Z}{cB} \qquad Z = \frac{1}{1/Z}$$

# Height/Depth Precision

- Starting from $\frac{\sigma_Z}{Z} = \frac{\sigma_{p_x}}{p_x}$ we obtain:

$$\sigma_Z = \frac{Z}{p_x}\sigma_{p_x} = \frac{cB}{p_x^2}\sigma_{p_x} = \frac{Z^2}{cB}\sigma_{p_x} = \frac{Z}{c\ B/Z}\sigma_{p_x}$$

Standard deviation of Z depends

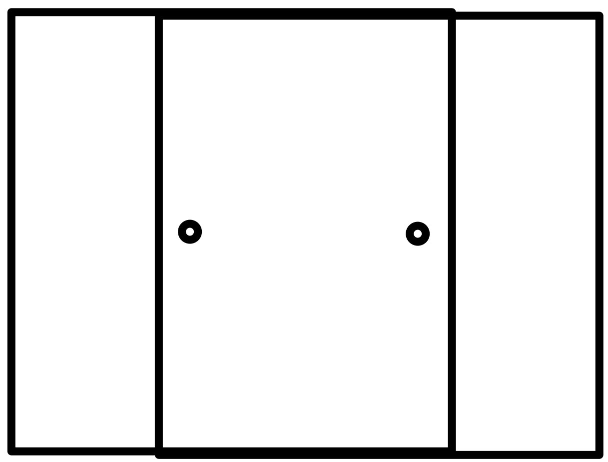
- on the **x-parallax standard deviation**
- inverse quadratically on the **x-parallax**
- quadratically on the **depth**
- inversely on the **base/depth ratio**

# Example: Aerial Image Analysis

- Typical values

1. $\sigma_{x'} = 7\ \mu\text{m}, \rightarrow \sigma_{p_x} \approx 10\ \mu\text{m}$
2. $p_x \approx b = 0.4 \times 23\ \text{cm} = 92\ \text{mm}$

➡ $\sigma_Z = ?$



60% overlap results in an avg. parallax of ~0.4x23cm

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$$\Rightarrow \quad \sigma_Z = Z\frac{\sigma_{p_x}}{p_x} = Z\frac{10\ \mu\text{m}}{92\ \text{mm}} \approx \frac{1}{10.000}\,Z$$

# Example: Aerial Image Analysis

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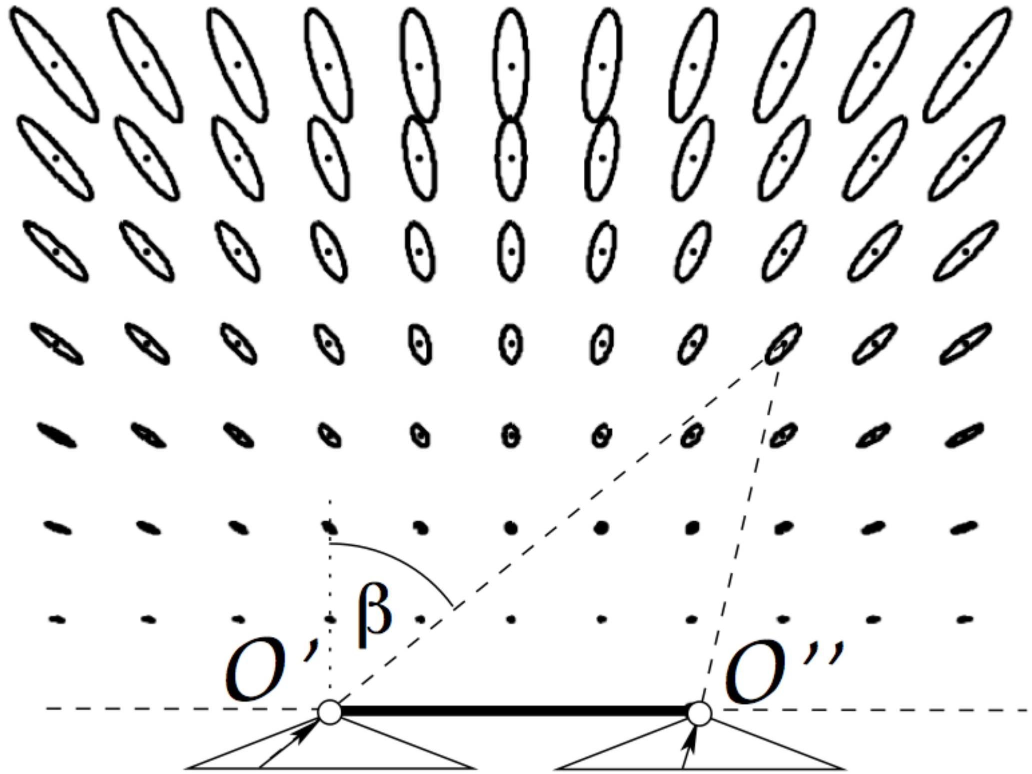
**The precision of the elevation from an aerial stereo image is approx. 1:10000 of the flight altitude**

# Stereo Uncertainty Field




$$x', x''$$

$$\sigma^2_{x'} = \sigma^2_{x''}$$

Image courtesy: Förstner and Wrobel

# Summary - Triangulation

- We can estimate 3D point locations (in the camera frame) given corresponding points and orientation parameters through triangulation
- Geometric approach
- Triangulation for the stereo normal case
- Quality of the 3D Points for the stereo normal case

# Part II
# Absolute Orientation

"Where are the points in the world?"

# Relative Orientation

- The result of the R.O. is the so-called **photogrammetric model**
- It contains the
  - parameters of the relative orientation of both cameras
  - 3D coordinates of N points in a local coordinate frame

$$ {}^{m}\boldsymbol{X}_n = ({}^{m}X_n, {}^{m}Y_n, {}^{m}Z_n)^{\mathsf{T}} \qquad n = 1, \cdots, N $$

- Known up to a similarity transform (for calibrated cameras)

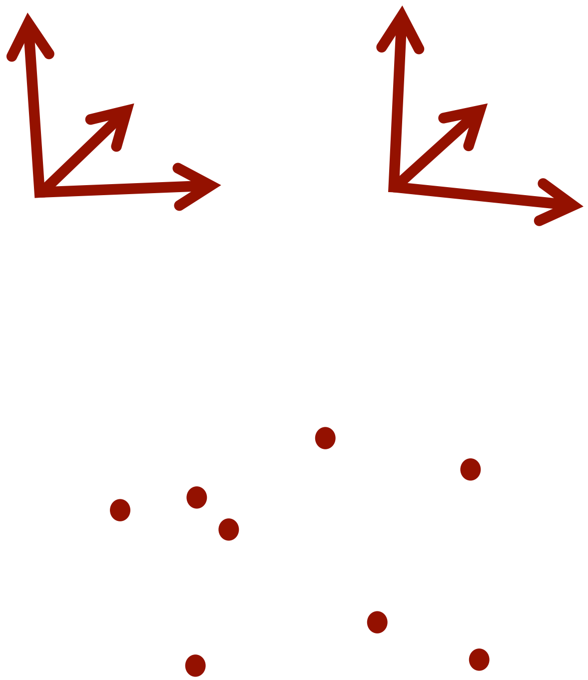
# Absolute Orientation

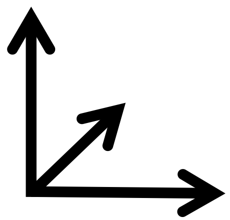
- A similarity transform maps the photogrammetric model into the object reference frame

$$ {}^{o}\boldsymbol{X}_{n} = \underline{\lambda R}\, {}^{m}\boldsymbol{X}_{n} + \underline{\boldsymbol{T}} $$

- 7 DoF for the similarity transform (3 rotation, 3 translation, 1 scale)
- **Control points (DE: Passpunkte)** are required

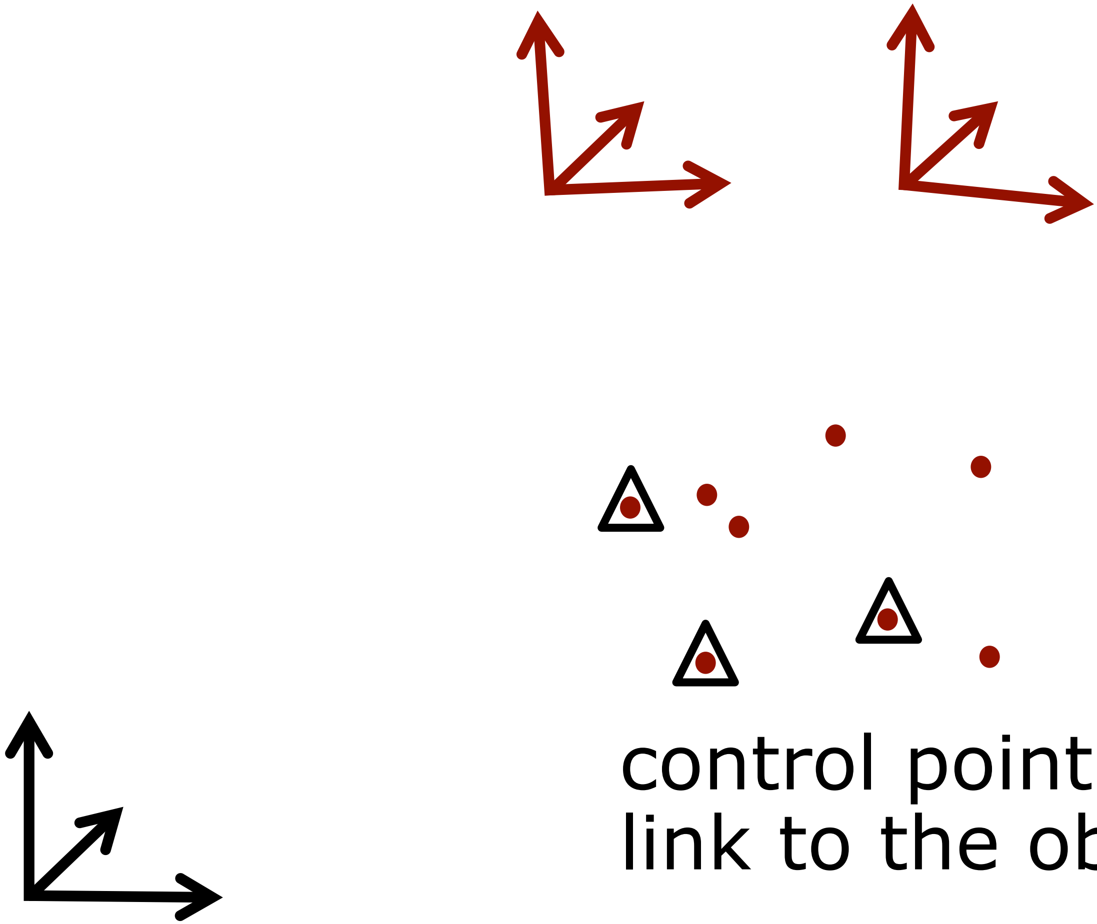
# Photogrammetric Model





missing

# Object Reference Frame



control points create the link to the object frame

# Least Squares Solution

- Non-linear least squares solution
- At least 3 control points (X,Y,Z known)
- Solution has been discussed in detail in the lecture 3D Coordinate Systems
- See: Förstner, Photo II, Chapter 1.5

# Sketch of the Solution

- Points in object and local system

$$\boldsymbol{y}_n = \lambda R \boldsymbol{x}_n - \boldsymbol{T} \qquad n = 1, \cdots, N$$

- Can be written as

$$\underbrace{\lambda^{-\frac{1}{2}}(\boldsymbol{y}_n - \boldsymbol{y}_0)}_{\boldsymbol{b}_n} = R\,\underbrace{\lambda^{\frac{1}{2}}(\boldsymbol{x}_n - \boldsymbol{x}_0)}_{\boldsymbol{a}_n}$$

- Function to minimize (w/ weights $p_n$)

$$\Phi(\boldsymbol{x}_0, \lambda, R) = \sum [\boldsymbol{b}_n - R\boldsymbol{a}_n]^{\mathsf{T}} [\boldsymbol{b}_n - R\boldsymbol{a}_n] p_n$$

# Minimize $\Phi(\boldsymbol{x}_0, \lambda, R)$

- Computing the first derivatives yields

$$\frac{\partial \Phi}{\partial \boldsymbol{x}_0} = 0 \quad \rightarrow \quad \boldsymbol{x}_0 = \frac{\sum \boldsymbol{x}_n p_n}{\sum p_n}$$

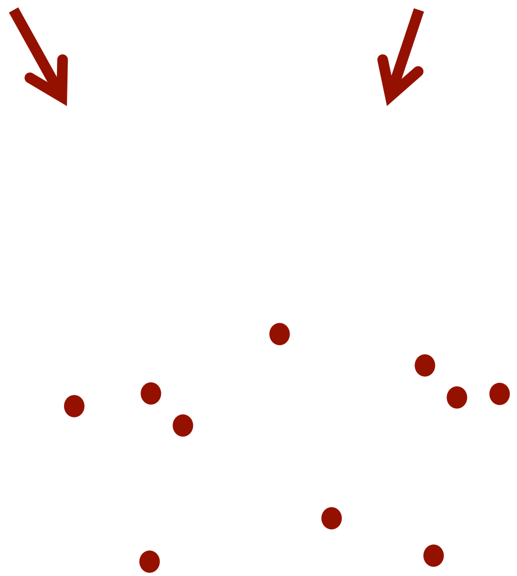
$$\frac{\partial \Phi}{\partial \lambda} = 0 \quad \rightarrow \quad \lambda^2 = \frac{\sum (\boldsymbol{y}_n - \boldsymbol{y}_0)^{\mathsf{T}} (\boldsymbol{y}_n - \boldsymbol{y}_0) p_n}{\sum (\boldsymbol{x}_n - \boldsymbol{x}_0)^{\mathsf{T}} (\boldsymbol{x}_n - \boldsymbol{x}_0) p_n}$$

- 3D rotation via SVD

$$H = \sum_{i=1}^{k} (\boldsymbol{a}_n \boldsymbol{b}_n^T) p_n \ , \ \ \mathrm{svd}(H) = UDV^T \quad \rightarrow \quad R = VU^T$$

- Details, see "3D Coordinate Systems"

# Overview – Initial Stage

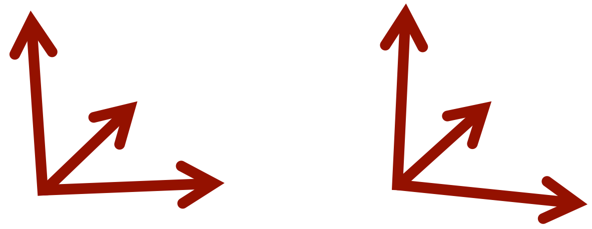


two cameras

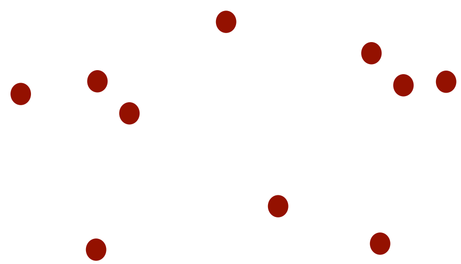
+

corresponding points in both images

# Overview – 1st Step

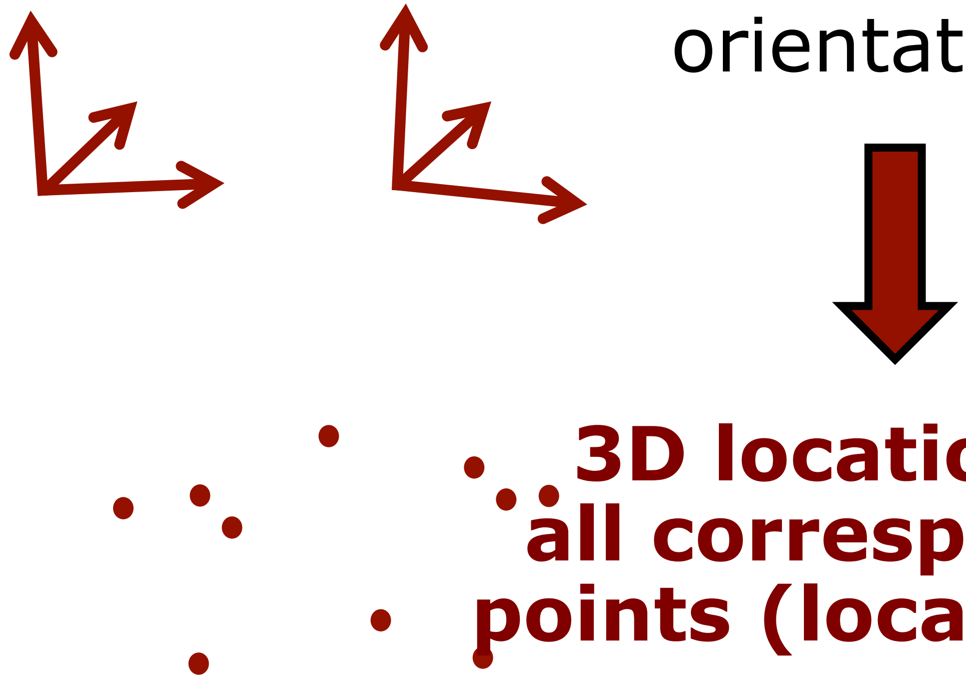


**relative orientation**



corresponding points in both images

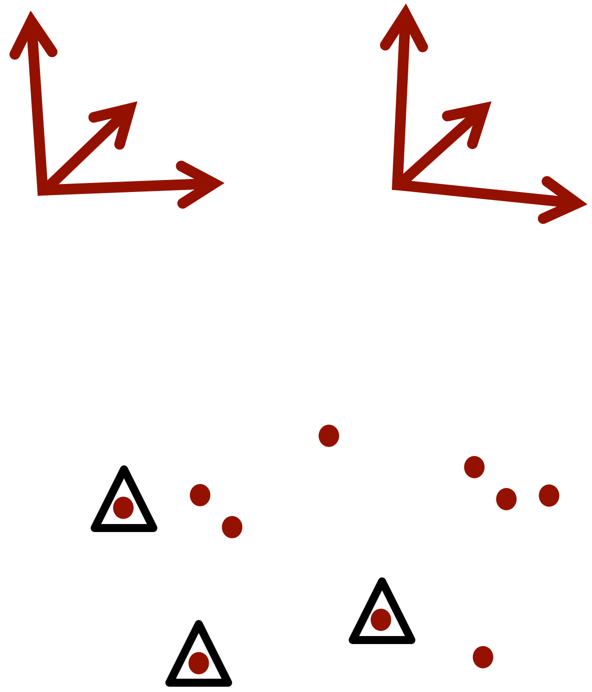
# Overview – 1st Step



relative
orientation

**3D locations of all corresponding points (local frame)**

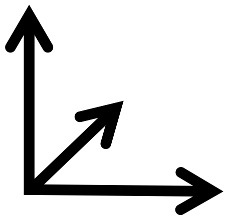
# Overview – 2nd Step



relative
orientation

+

control
points



**3D location of all corresponding points in the object frame**

## 2-Step Solution

- By combining all techniques from the Photogrammetry II course, we obtain
- **Relative orientation** without control points and 3D location of correspond. points in a local frame
- **Absolute orientation** of cameras and corresponding points through control points

# 2-Step Solution

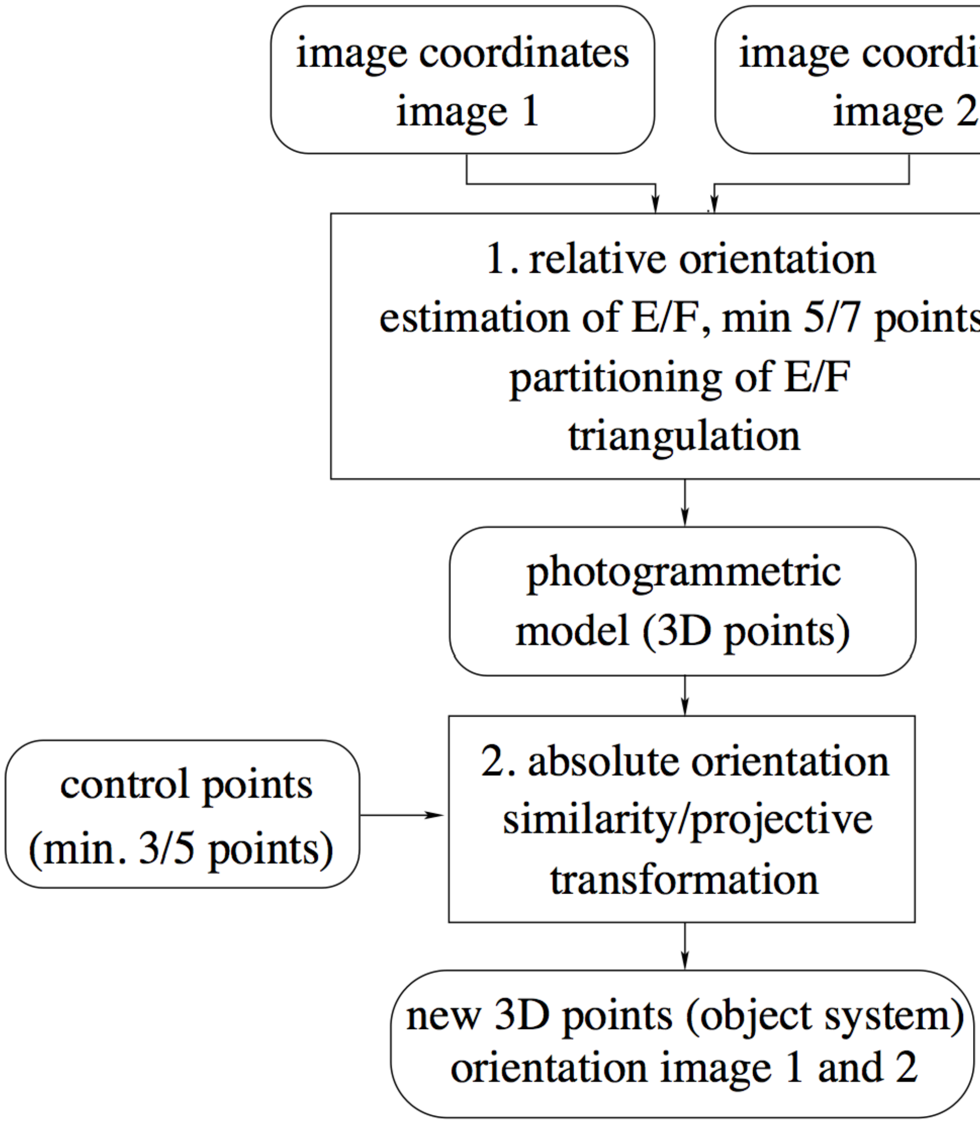



Image courtesy: Förstner and Wrobel

# Control Points

- Full control points: X, Y, Z known
  (DE: Vollpasspunkte)

- Planimetric control points: X, Y known
  (DE: Lagepasspunkte)

- Height control points: Z known
  (DE: Höhenpasspunkte)

# Discussion:

# Which Other Orientation Approaches Do We Know?

# Photo I & II

- Direct linear transform (DLT)
- Spatial Resection (P3P, RRS)
- Relative orientation
- Triangulation
- Absolute orientation

# Photo I & II

- Direct linear transform (DLT)
- Spatial Resection (P3P, RRS)
- Relative orientation
- Triangulation
- Absolute orientation

**How could we achieve the same using combinations of the techniques listed above?**

# Other Possibilities

**Option 1**

- DLT for each camera using control pts
- Triangulation for all corresponding pts

**Option 2**

- P3P for each camera using control pts
- Triangulation for all corresponding pts

**Option 3**

- One big least squares approach (bundle adjustment)

# Option 1 & 2 – DLT / P3P

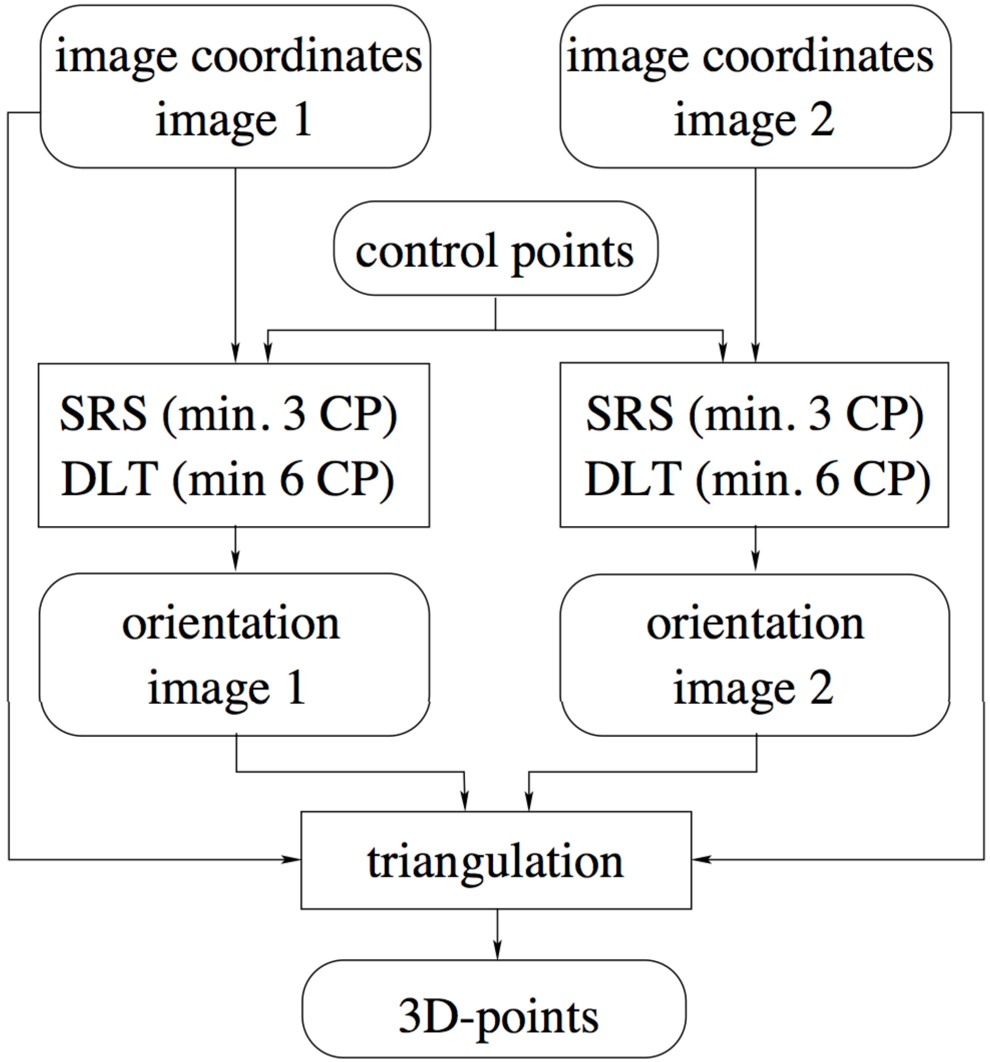



Image courtesy: Förstner and Wrobel

# Option 3 – Bundle Adjustment

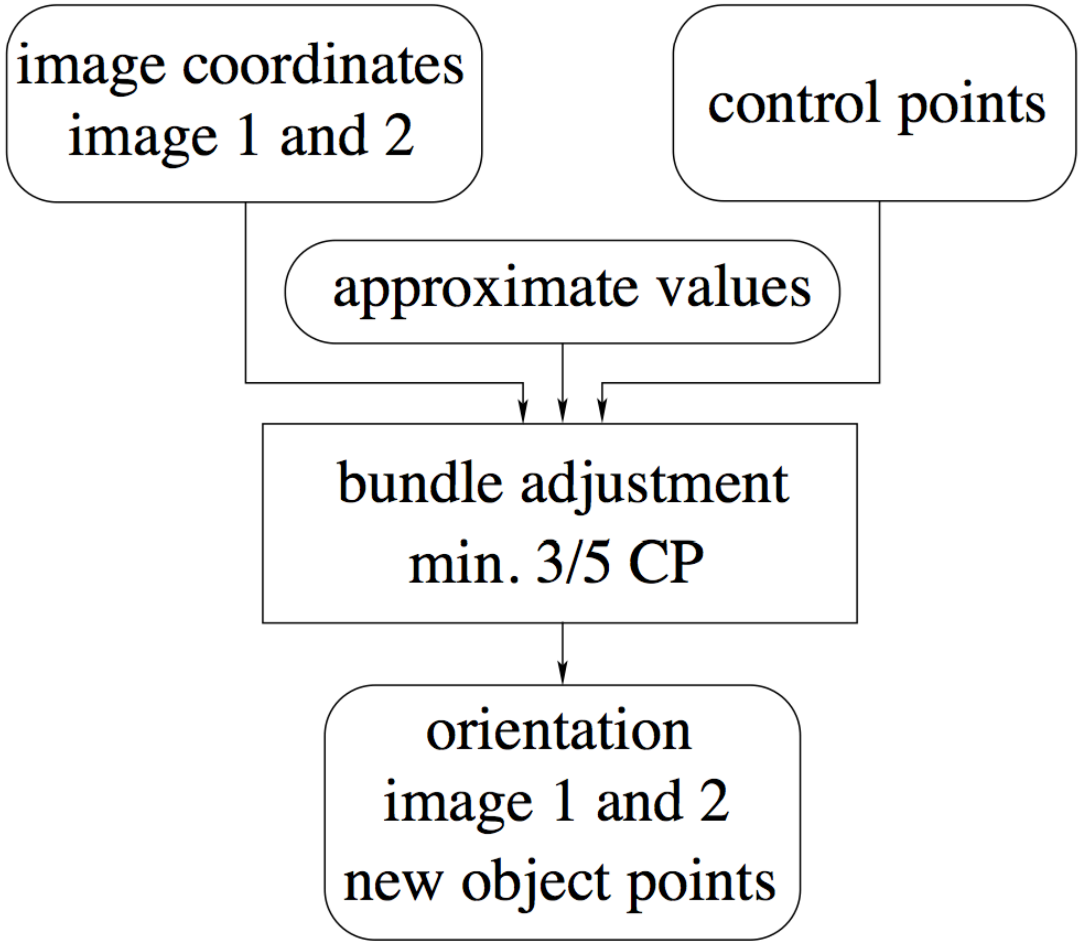



Image courtesy: Förstner and Wrobel

# Which Solution is the Best One?

# Relevant Properties

- Is the solution **statistically optimal**? (the precision of the estimated parameters, the object coordinates and orientation)
- **Check-ability** of the observations
  - Check-ability depends on the redundancy
  - Redundancy R depends on the unknowns (U), observations (N), and constraints (H)
  - $R = N - U + H$

# Points

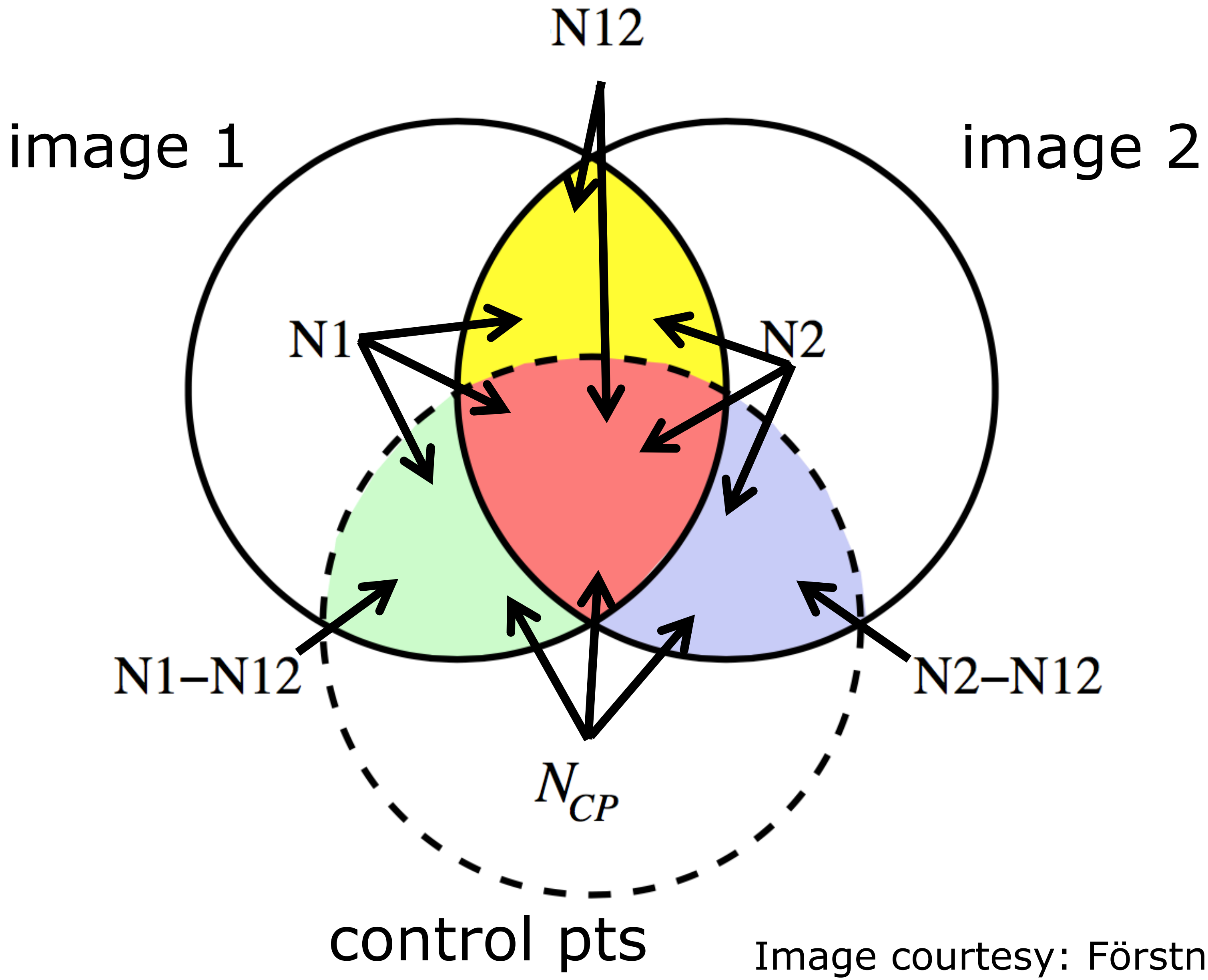


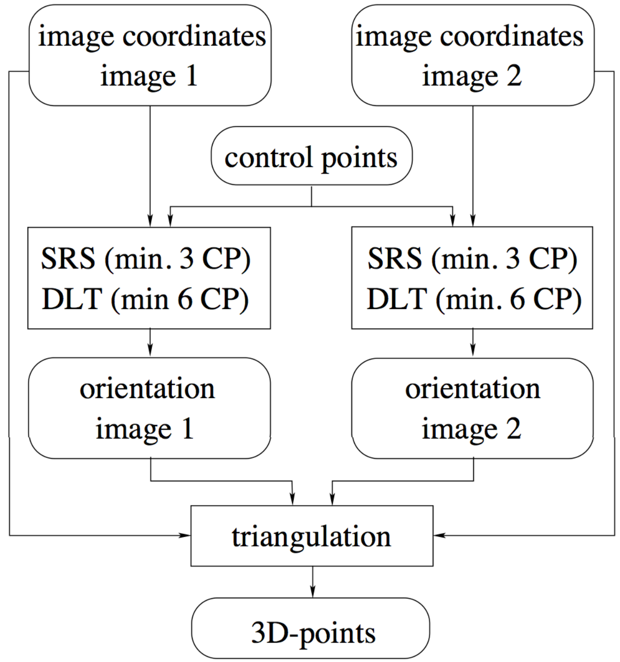

Image courtesy: Förstner

# Observations

- $N_{CP}$: visible control points
- $N_1$: usable points in image 1
- $N_2$: usable points in image 2
- $N_{12}$: points in image 1 and 2
- $N_{NP}$: new points visible in both images (and no control points)
- $N_{FCP}$: visible full controls points
- $N_{PCP}$: visible planimetric controls points
- $N_{HCP}$: visible height controls points

# DLT

- Solution optimal?
- Can we exploit all information?
- Can we exploit new points?
- Can we exploit partially known parameters?



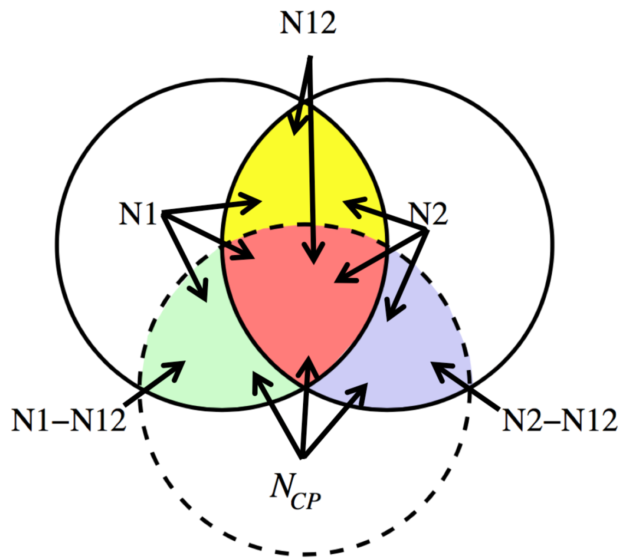




Image courtesy: Förstner and Wrobel

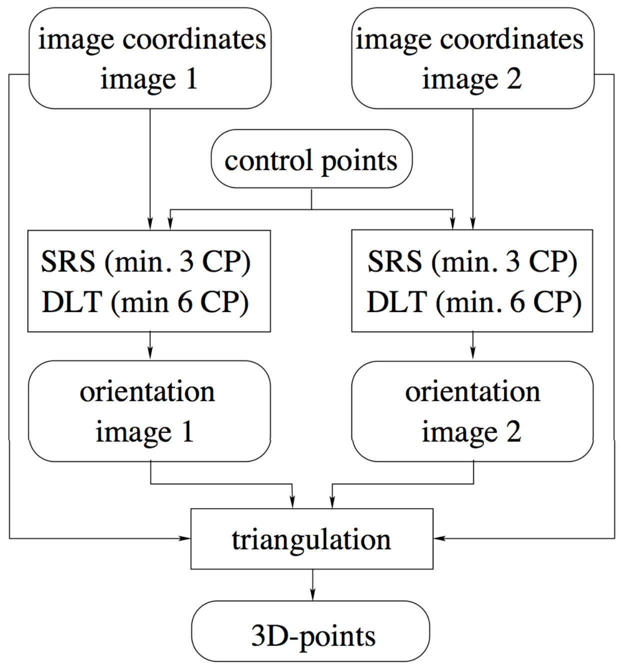
# DLT

- Solution is **not optimal**
- Cannot exploit new points or partially known parameters
- Observations

$$N = 2 \cdot (N_1 + N_2 - 2N_{NP})$$

- Redundancy

$$R = 2 \cdot (N_1 + N_2 - 2N_{NP}) - 22$$



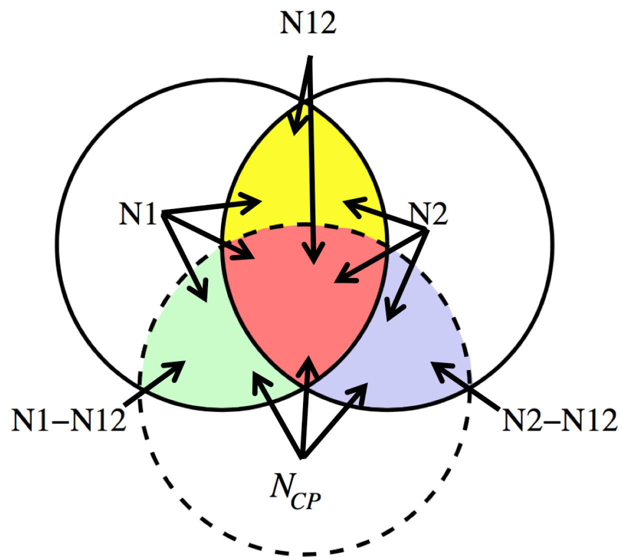
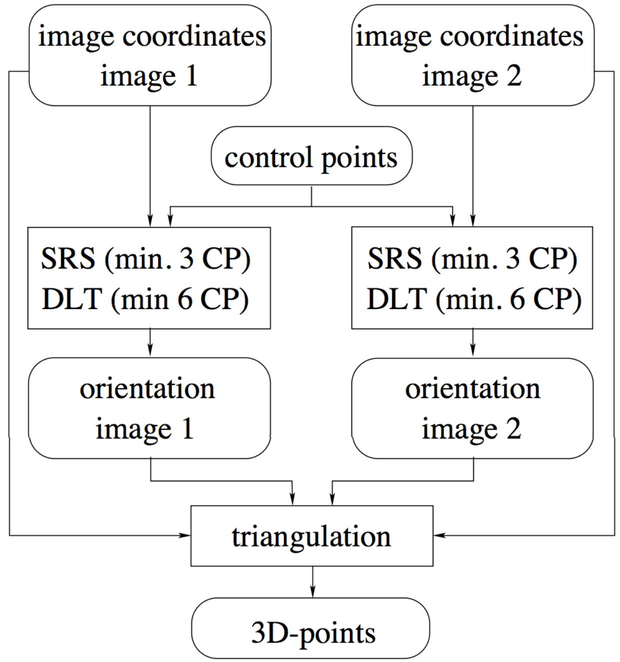




Image courtesy: Förstner and Wrobel

# P3P/SRS/RRS

- Solution optimal?
- Can we exploit all information?
- Can we exploit new points?
- Can we exploit partially known parameters?



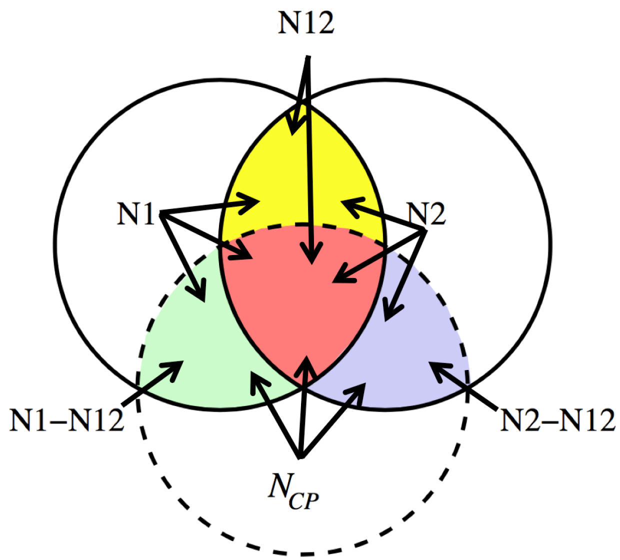




Image courtesy: Förstner and Wrobel

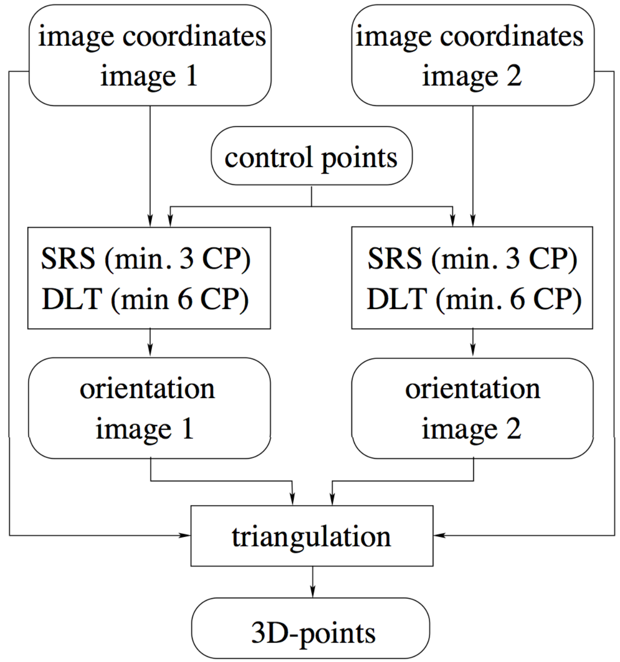
# P3P/SRS/RRS

- Solution is **not optimal**
- Cannot exploit new points or partially known parameters
- Observations

$$N = 2 \cdot (N_1 + N_2 - 2N_{NP})$$

- Redundancy

$$R = 2 \cdot (N_1 + N_2 - 2N_{NP}) - 12$$



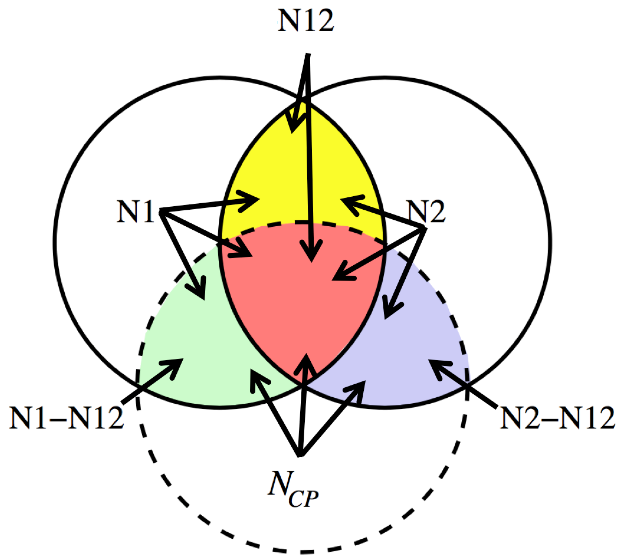
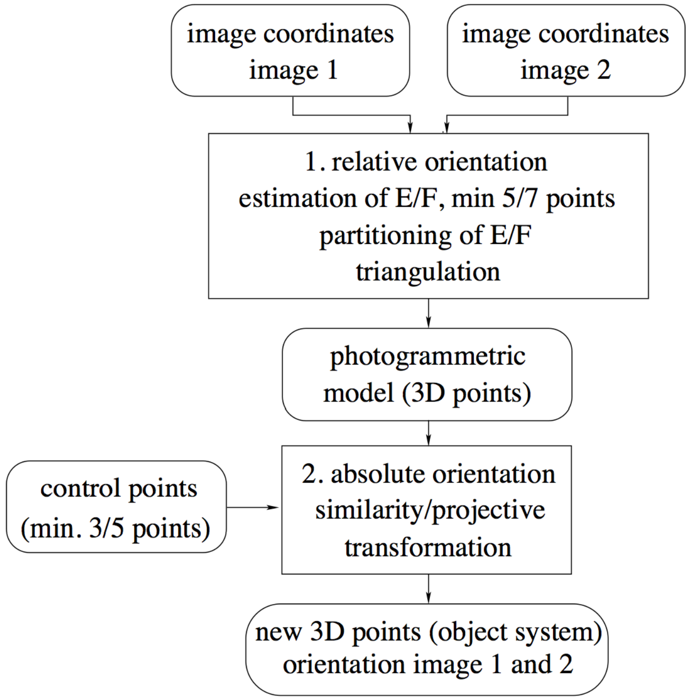




Image courtesy: Förstner and Wrobel

# 2-Step Solution

- Solution optimal?
- Can we exploit all information?
- Can we exploit new points?
- Can we exploit correlations?



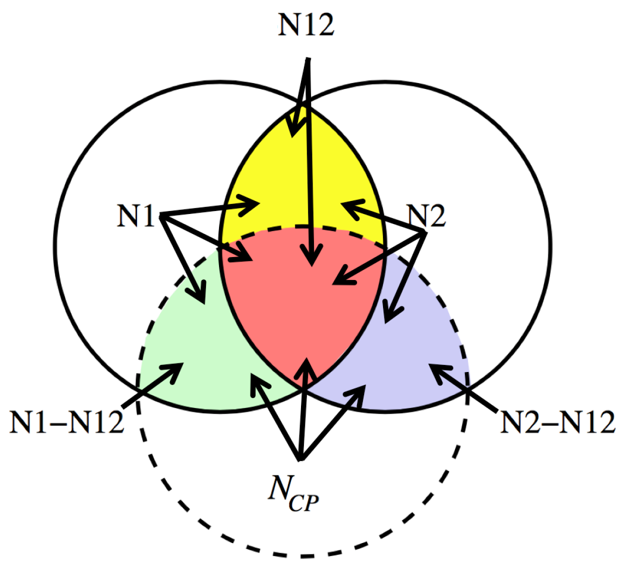
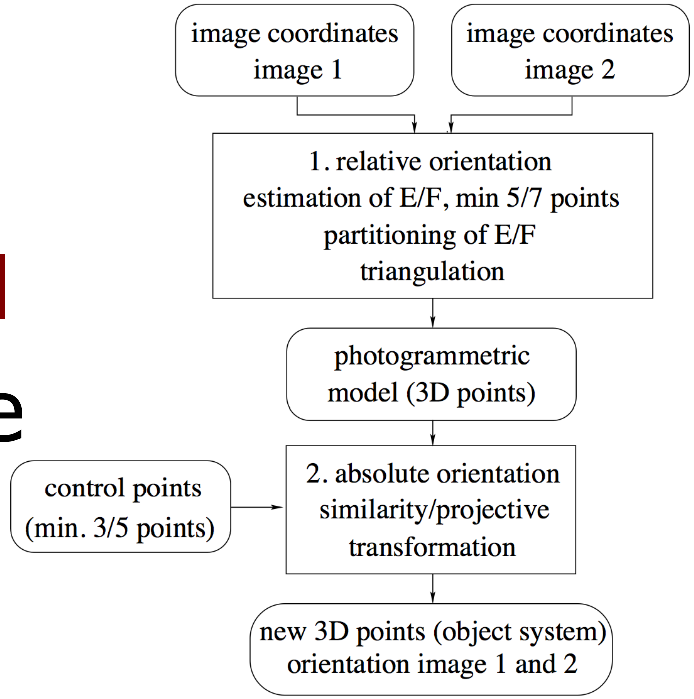




Image courtesy: Förstner and Wrobel

# 2-Step Solution




- Solution is **not optimal**
- Correlation of points are neglected (AO)
- R.O. ignores the known control point positions
- Observations

$$N = N_{12} \qquad\qquad N = N_{CP}$$

- Redundancy

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$$R = N_{12} - U_{RO} \qquad\qquad R = 3 \cdot N_{CP} - U_{AO}$$

Image courtesy: Förstner and Wrobel

# Bundle Adjustment

- Solution optimal?
- Can we exploit all information?

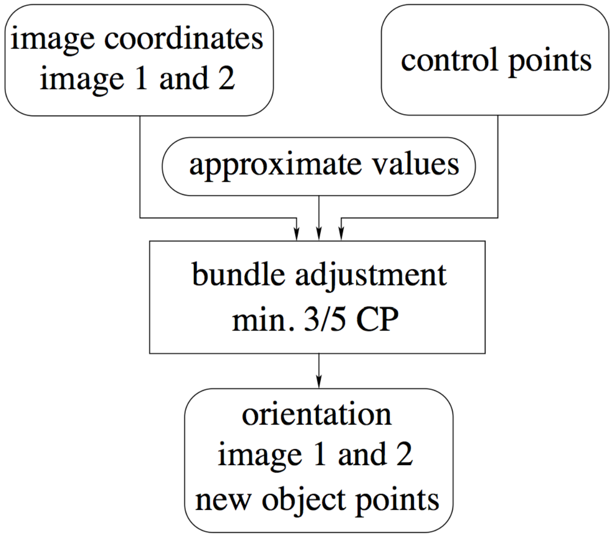


Image courtesy: Förstner and Wrobel

# Bundle Adjustment

- **Optimal solution**
- Requires initial guess
- Observations

$$N = 2{\cdot}(N_1 + N_2)$$

- Redundancy

$$R = 2{\cdot}(N_1 + N_2) - (U_{EO} + U_{IO} + 3{\cdot}N_{NP} + 2{\cdot}N_{HCP} + N_{PCP})$$

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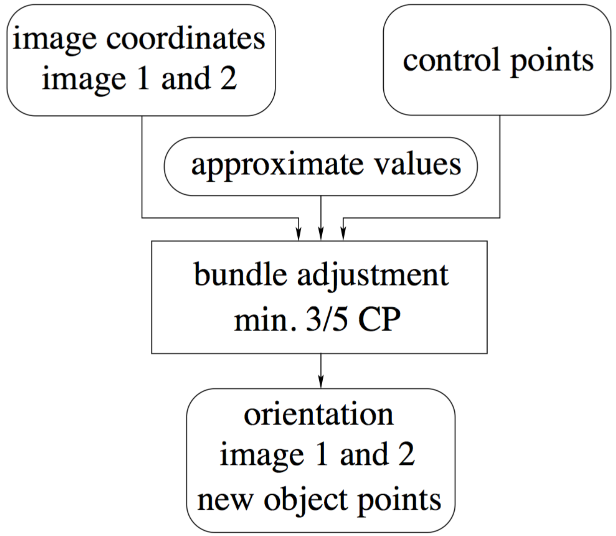


Image courtesy: Förstner and Wrobel

# Discussion – Bundle Adjustment

- Two-image bundle adjustment (BA)
- BA leads to the statistically optimal solution (given convergence)
- BA can deal with a moderate amount of gross errors
- BA requires an approximate solution as an initial guess

# Discussion – 2-Step Solution

- Two step solution allows for checking the photogrammetric and geodetic measurements separately (corresponding points vs. control points)
- Can handle gross errors for R/N>0.5
- Serves as an initial guess for BA

# Discussion – 2 x P3P

- Only applicable of both images observe at least 3 (4) full control pts
- Direct approach can be used to find gross errors
- Less accurate that the 2-step solution in case of large sets of new points

# Discussion – 2 x DLT

- Only applicable of both images observe at least 6 full control pts
- Point cannot lie on one plane
- Initial guess for BA in case the calibration parameters are unknown

# Summary

- Absolute Orientation transforms the photog. model to the object frame
- Different ways for orienting points absolutely (DLT, P3P, 2-Step, BA)
- Bundle adjustment is the optimal solution in a statistical sense but requires a (good) initial guess

# Literature

- Förstner, Skript Photogrammetrie II, Chapter 1.4 – 1.6
- Förstner, Wrobel: Photogrammetric Computer Vision,  Ch. 12.4 - 12.6

# Slide Information

- The slides have been created by Cyrill Stachniss as part of the photogrammetry and robotics courses.
- I tried to acknowledge all people from whom I used images or videos. In case I made a mistake or missed someone, please let me know.
- The photogrammetry material heavily relies on the very well written lecture notes by Wolfgang Förstner and the Photogrammetric Computer Vision book by Förstner & Wrobel.
- Parts of the robotics material stems from the great Probabilistic Robotics book by Thrun, Burgard and Fox.
- If you are a university lecturer, feel free to use the course material. If you adapt the course material, please make sure that you keep the acknowledgements to others and please acknowledge me as well. To satisfy my own curiosity, please send me email notice if you use my slides.

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