

Lecture 9: Epipolar Geometry

Professor Fei-Fei Li

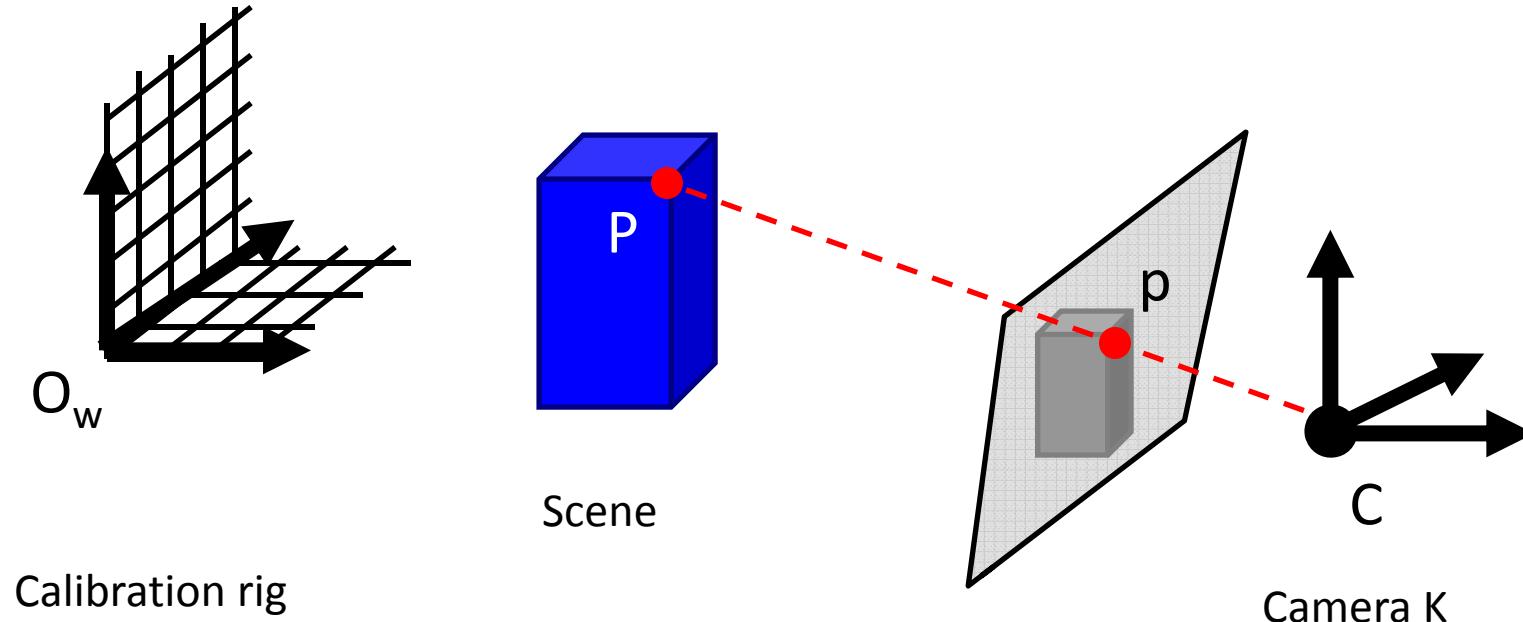
Stanford Vision Lab

What we will learn today?

- Why is stereo useful?
- Epipolar constraints
- Essential and fundamental matrix
- Estimating F (**Problem Set 2 (Q2)**)
- Rectification

Reading:
[HZ] Chapters: 4, 9, 11
[FP] Chapters: 10

Recovering structure from a single view



Why is it so difficult?

Intrinsic ambiguity of the mapping from 3D to image (2D)

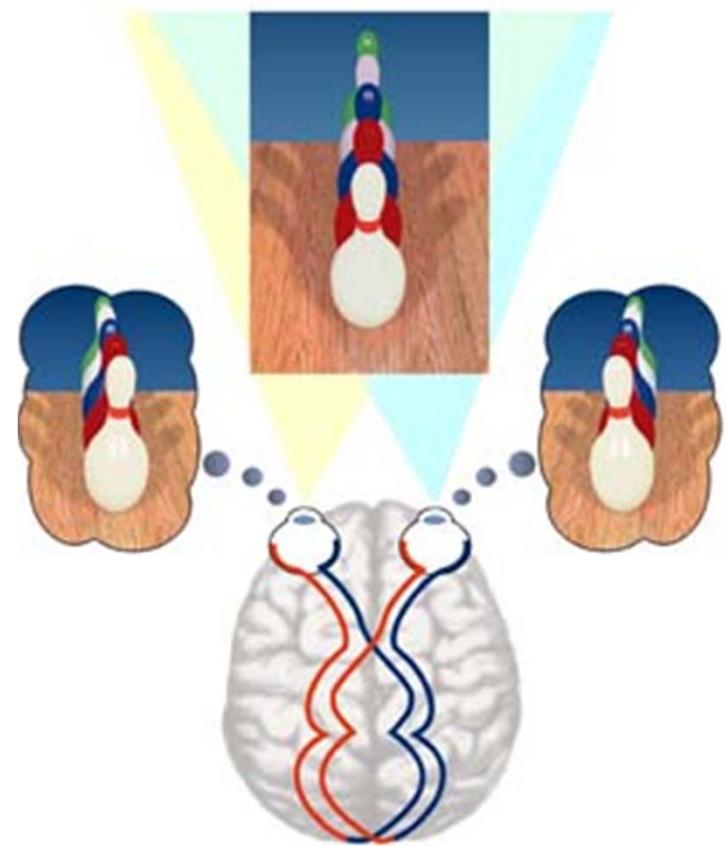
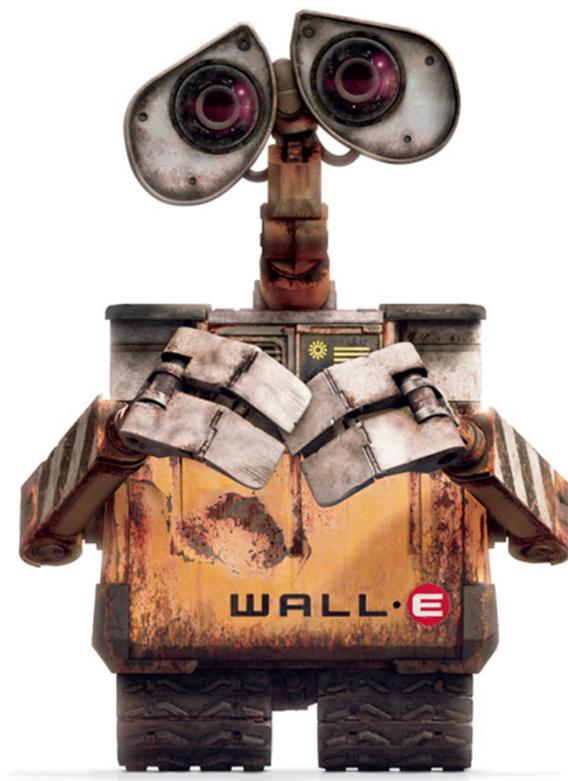
Recovering structure from a single view



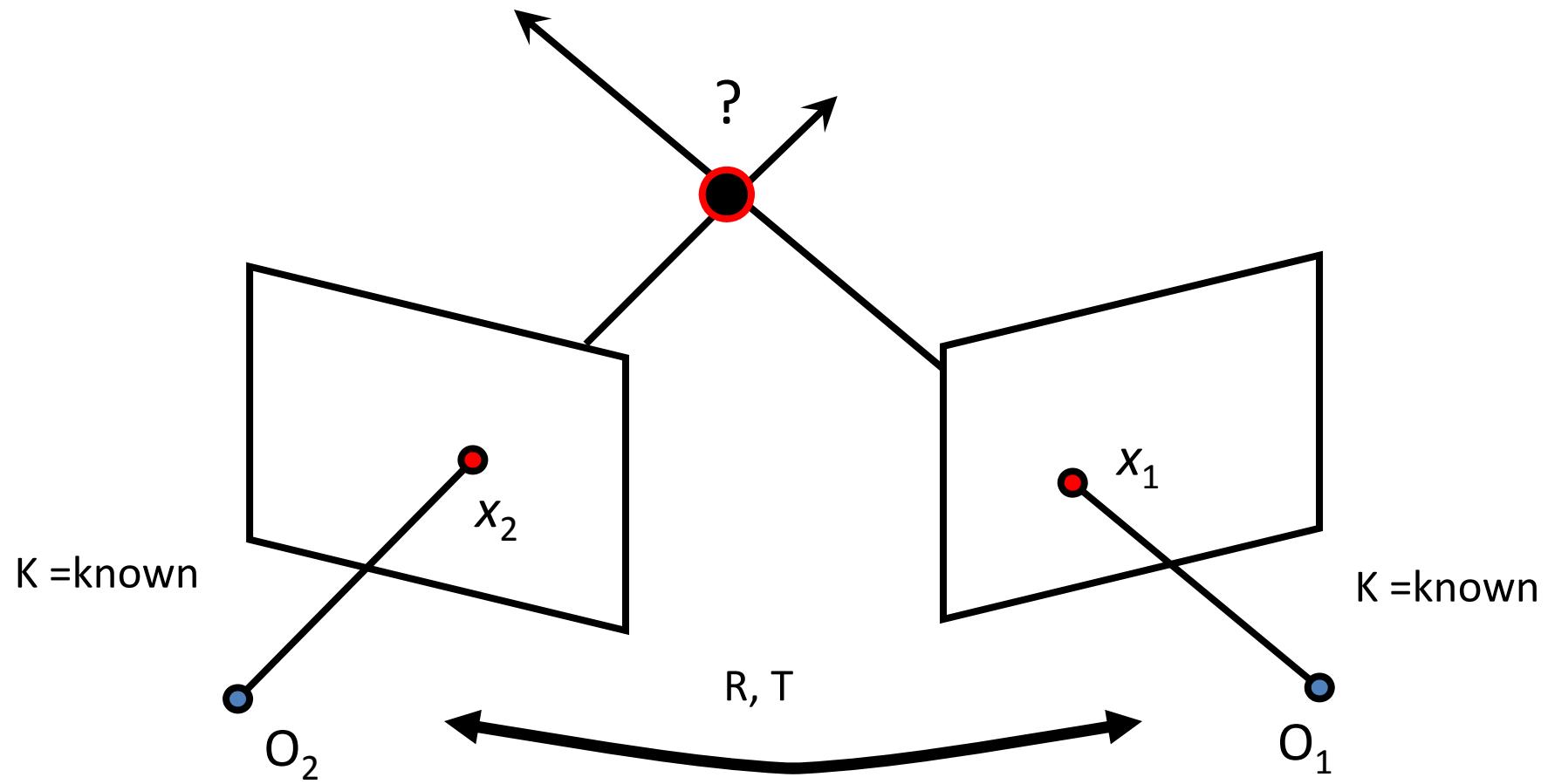
Courtesy slide S. Lazebnik

Intrinsic ambiguity of the mapping from 3D to image (2D)

Two eyes help!



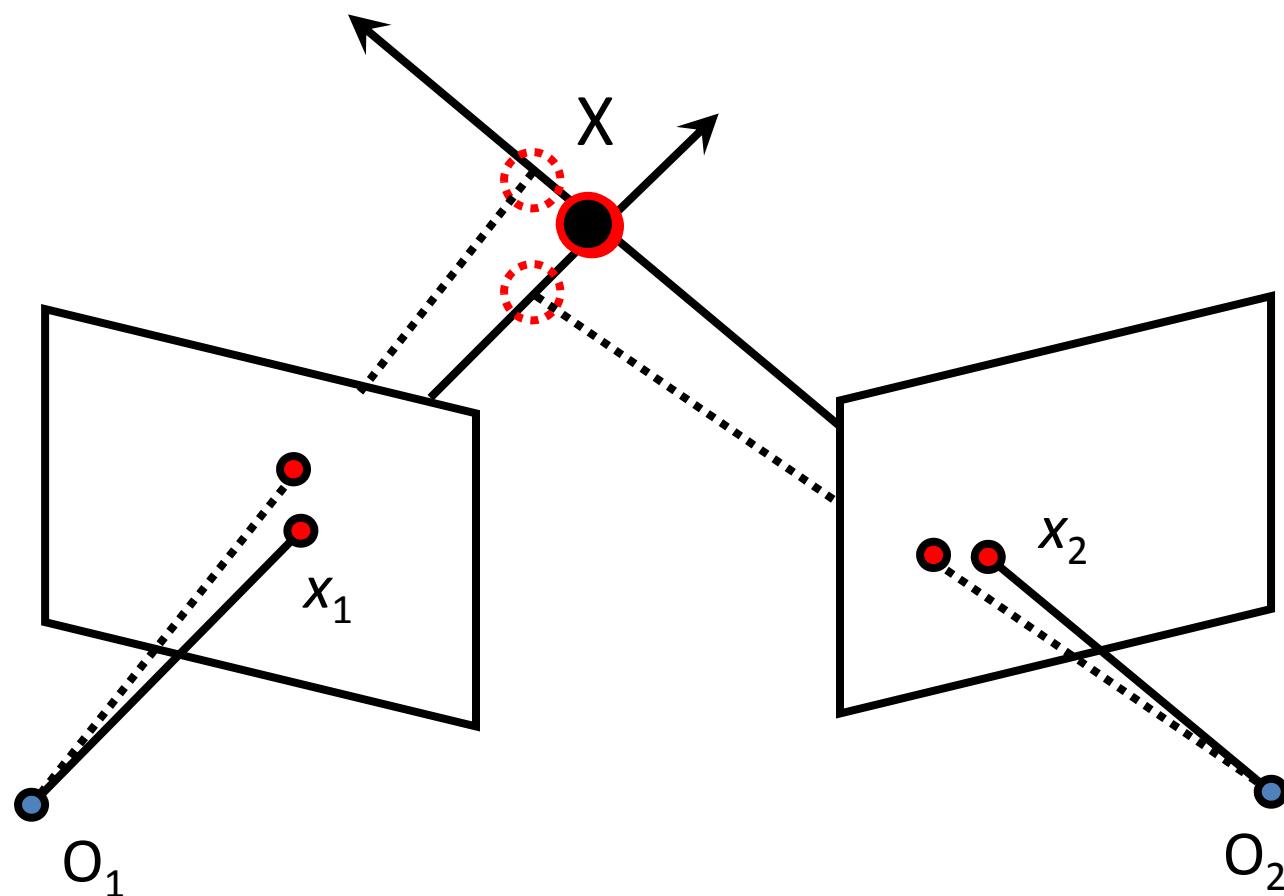
Two eyes help!



This is called **triangulation**

Triangulation

- Find X that minimizes $d^2(x_1, P_1 X) + d^2(x_2, P_2 X)$



Stereo-view geometry

- **Correspondence:** Given a point in one image, how can I find the corresponding point x' in another one?
- **Camera geometry:** Given corresponding points in two images, find camera matrices, position and pose.
- **Scene geometry:** Find coordinates of 3D point from its projection into 2 or multiple images.

This lecture (#9)

Stereo-view geometry

Next lecture (#10)

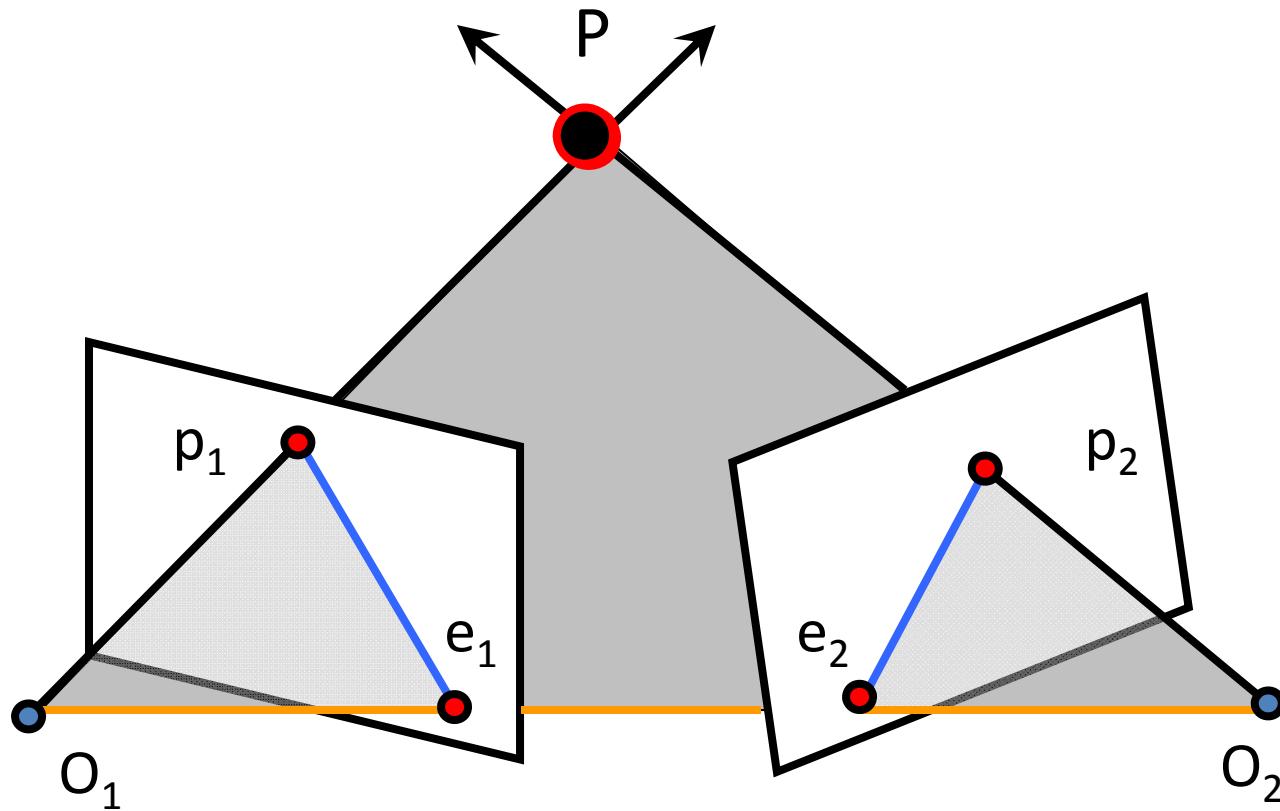
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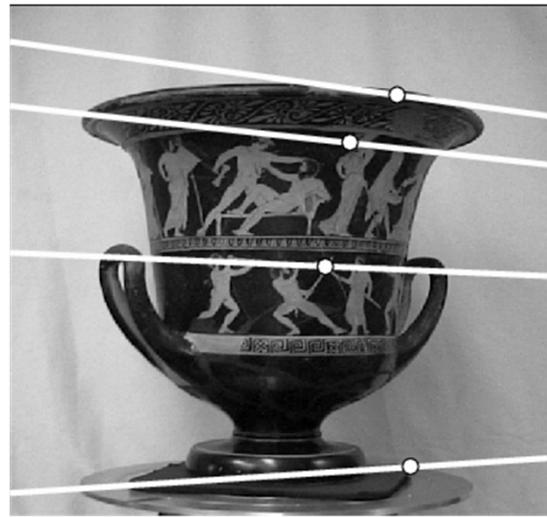
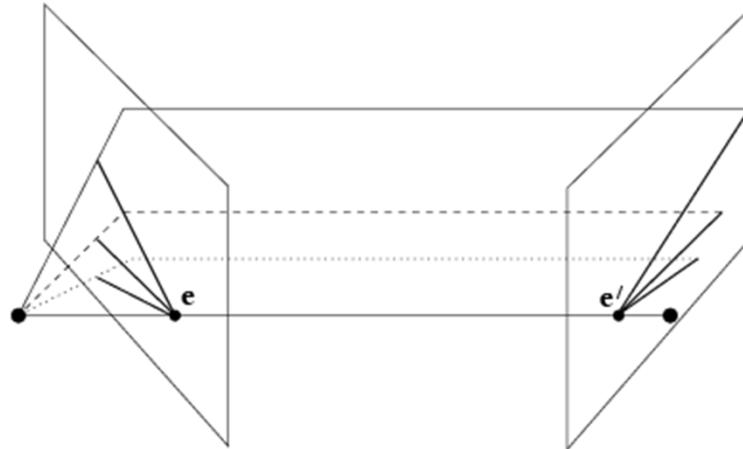
Reading:
[HZ] Chapters: 4, 9, 11
[FP] Chapters: 10

Epipolar geometry

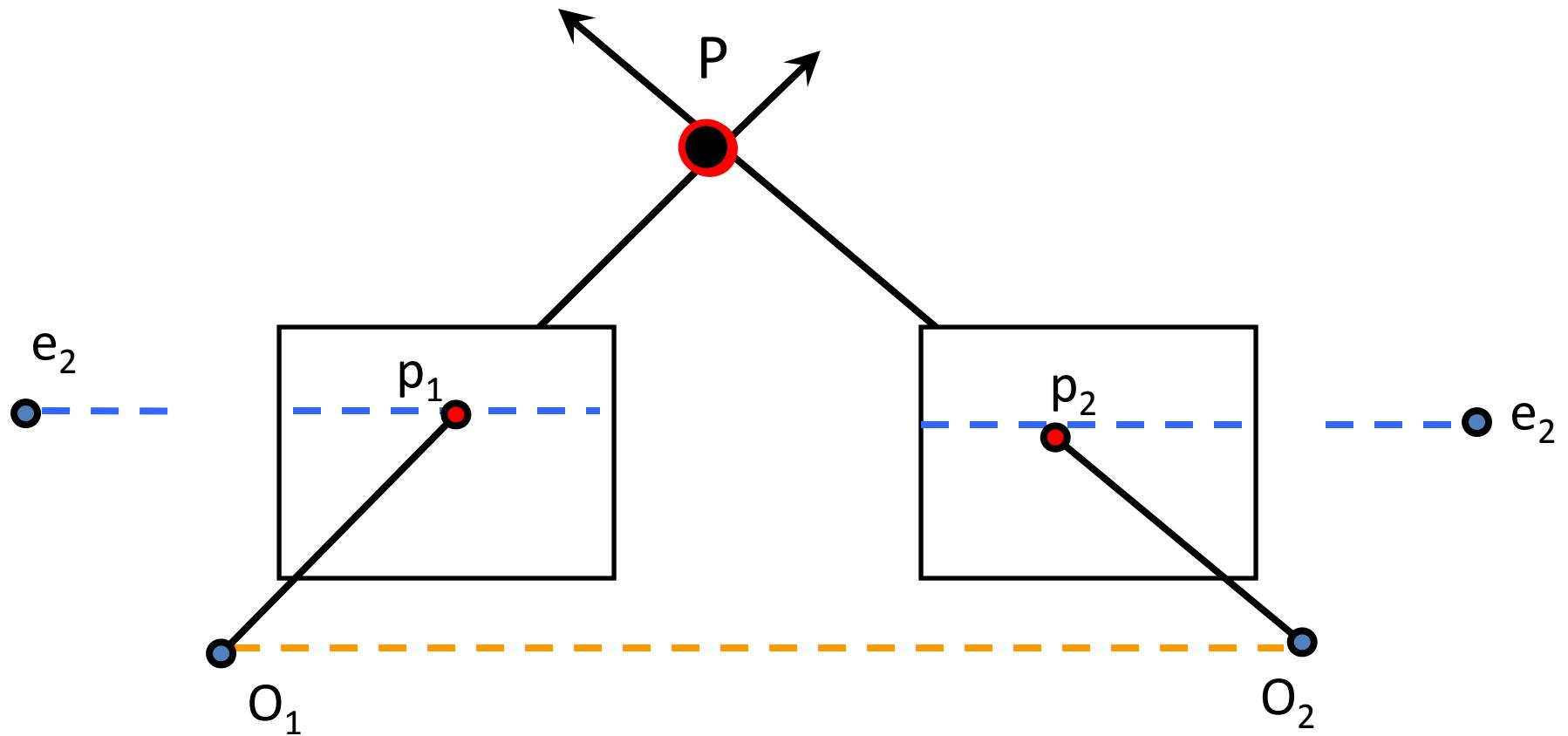


- Epipolar Plane
- Baseline
- Epipolar Lines
- Epipoles e_1, e_2
 - = intersections of baseline with image planes
 - = projections of the other camera center
 - = vanishing points of camera motion direction

Example: Converging image planes

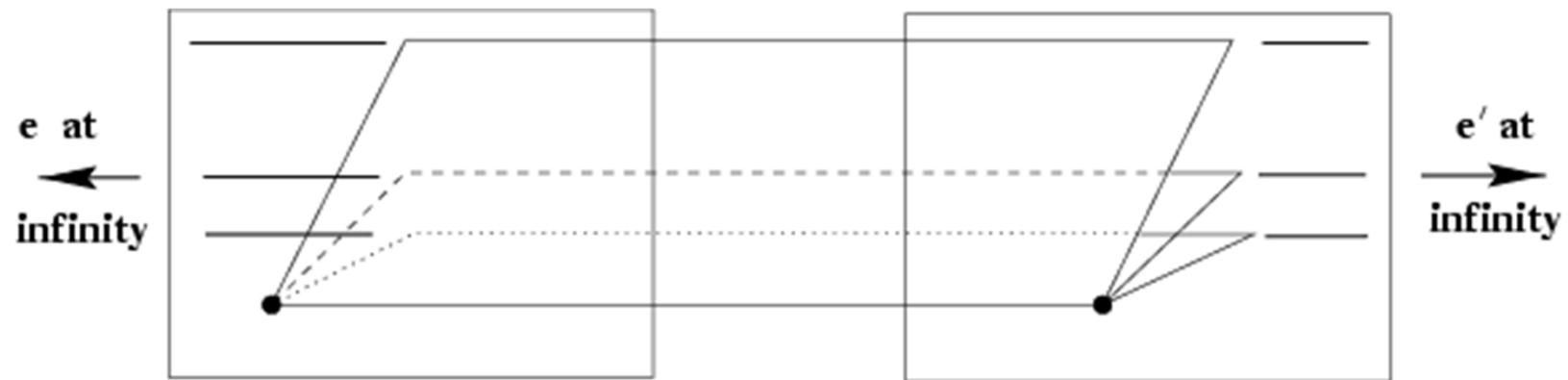


Example: Parallel image planes

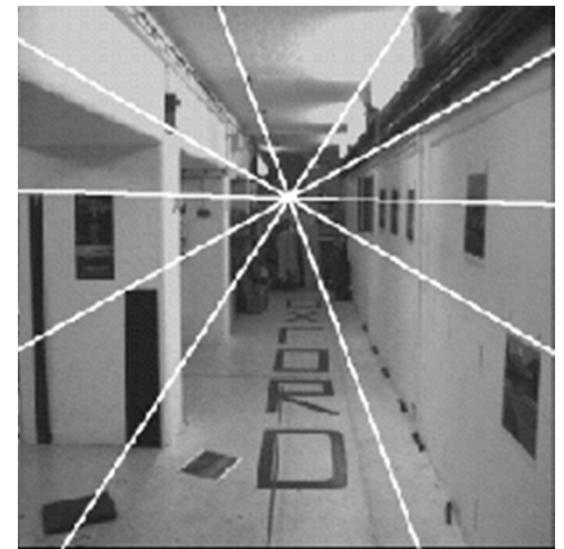
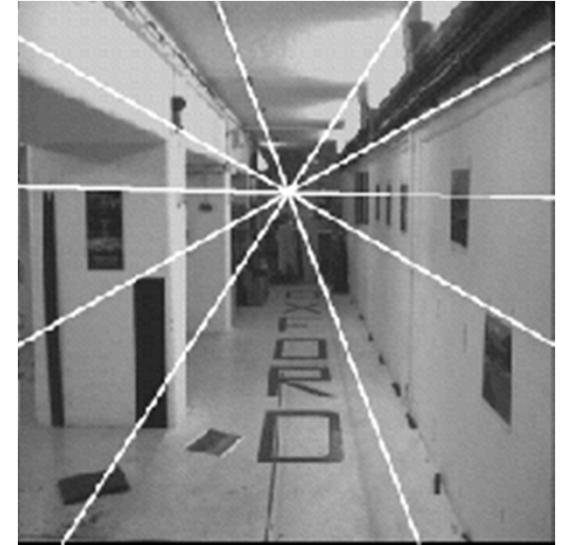
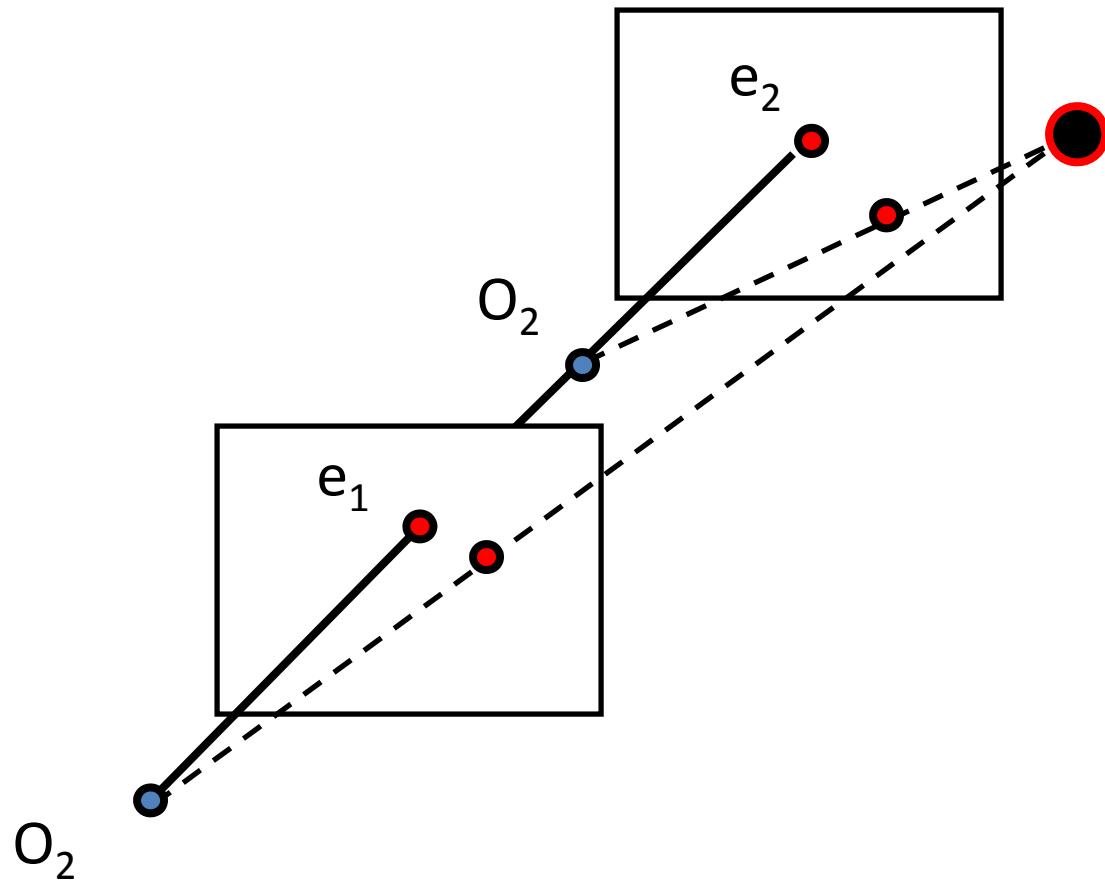


- Baseline intersects the image plane at infinity
- Epipoles are at infinity
- Epipolar lines are parallel to x axis

Example: Parallel image planes

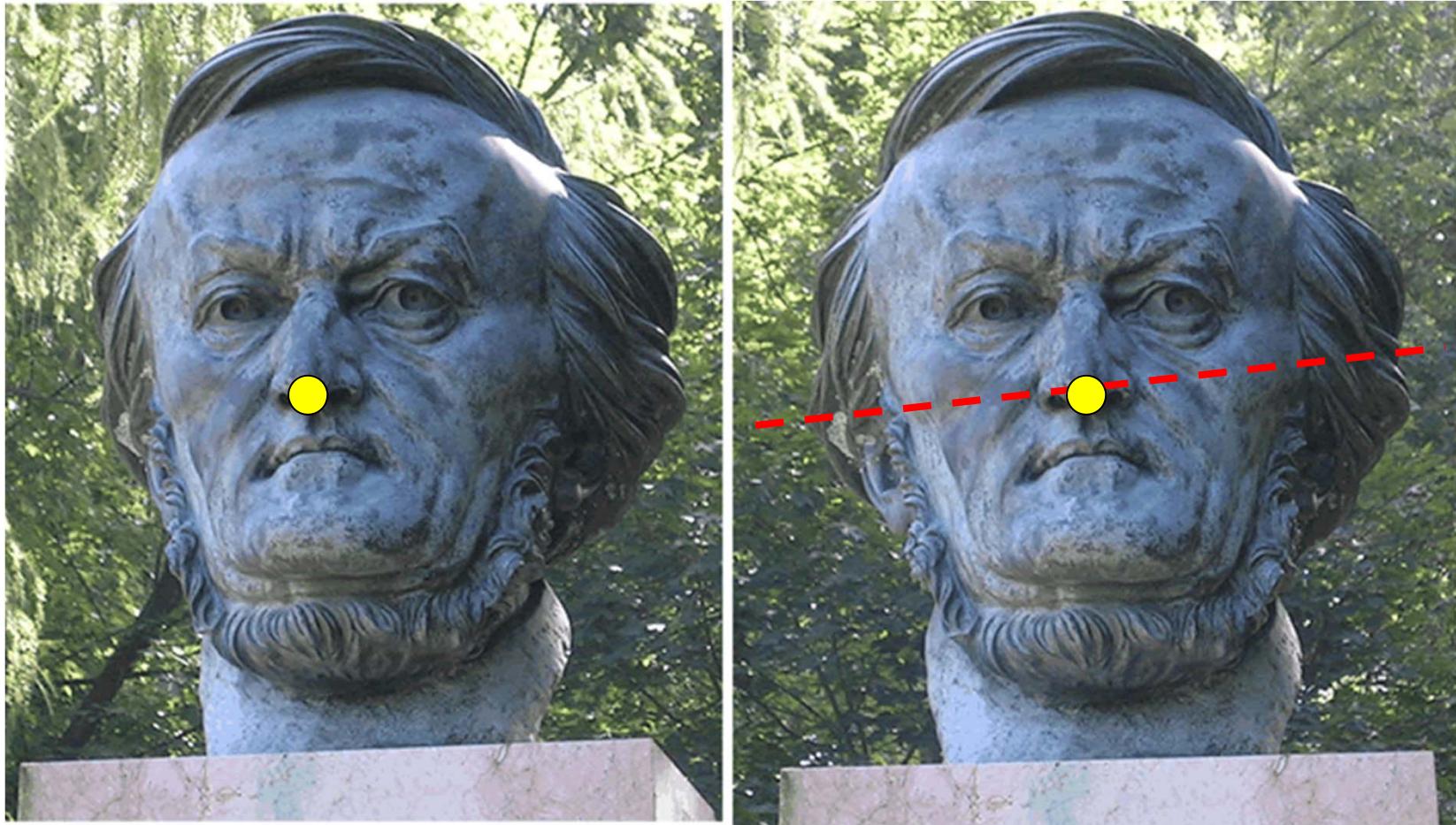


Example: Forward translation



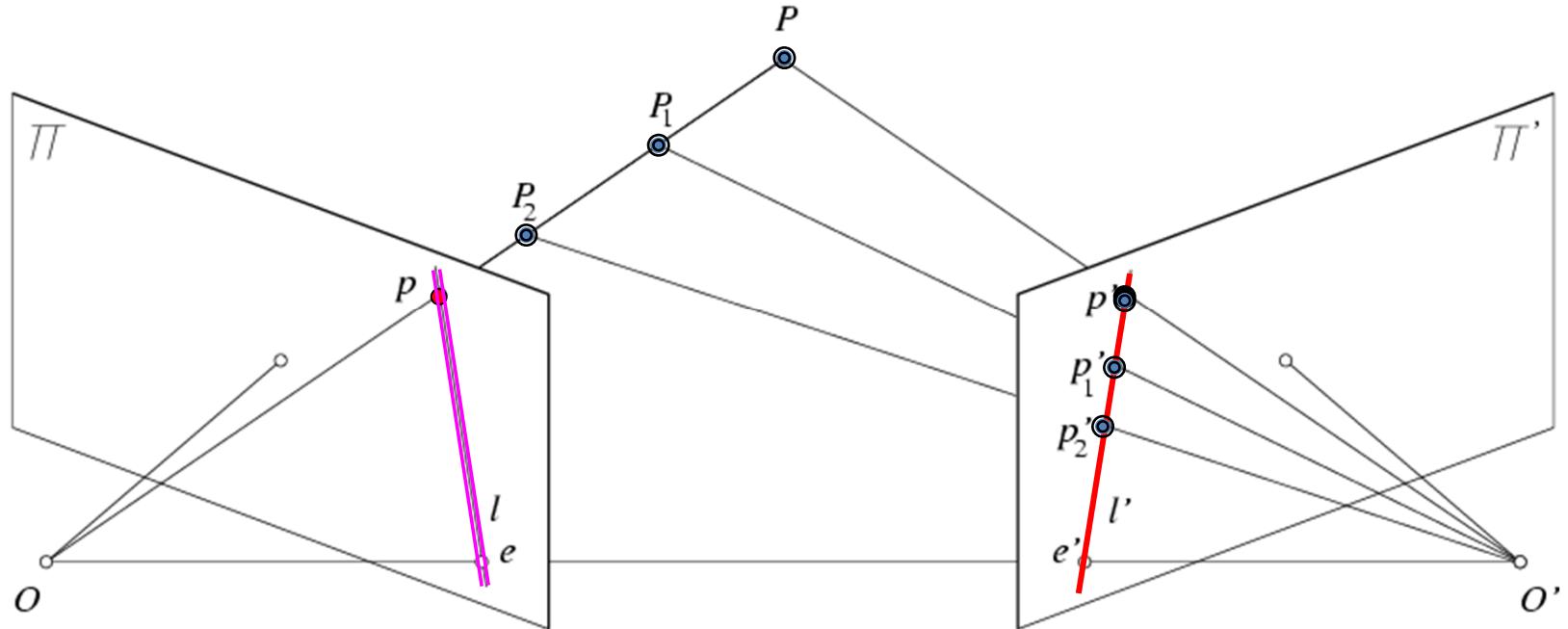
- The epipoles have same position in both images
- Epipole called FOE (focus of expansion)

Epipolar Constraint



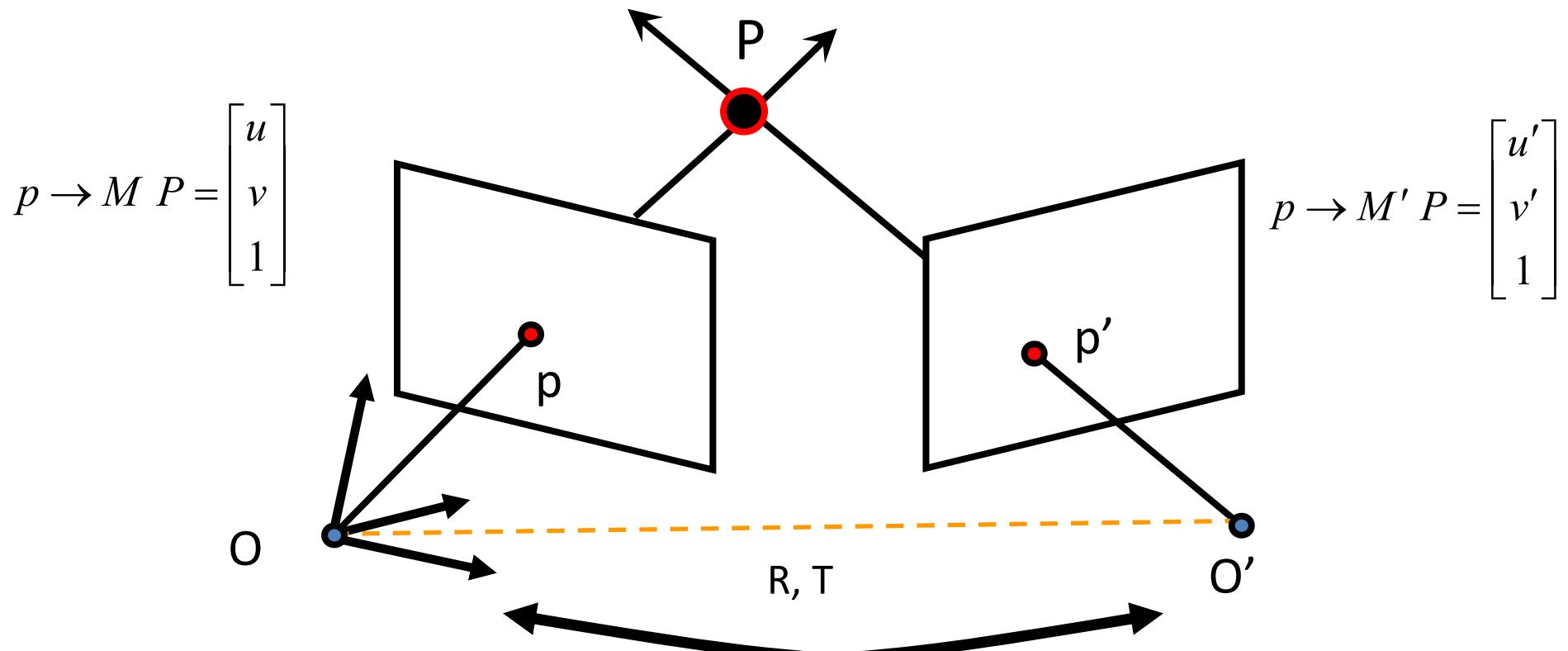
- Two views of the same object
- Suppose I know the camera positions and camera matrices
- Given a point on left image, how can I find the corresponding point on right image?

Epipolar Constraint



- Potential matches for p have to lie on the corresponding epipolar line l' .
- Potential matches for p' have to lie on the corresponding epipolar line l .

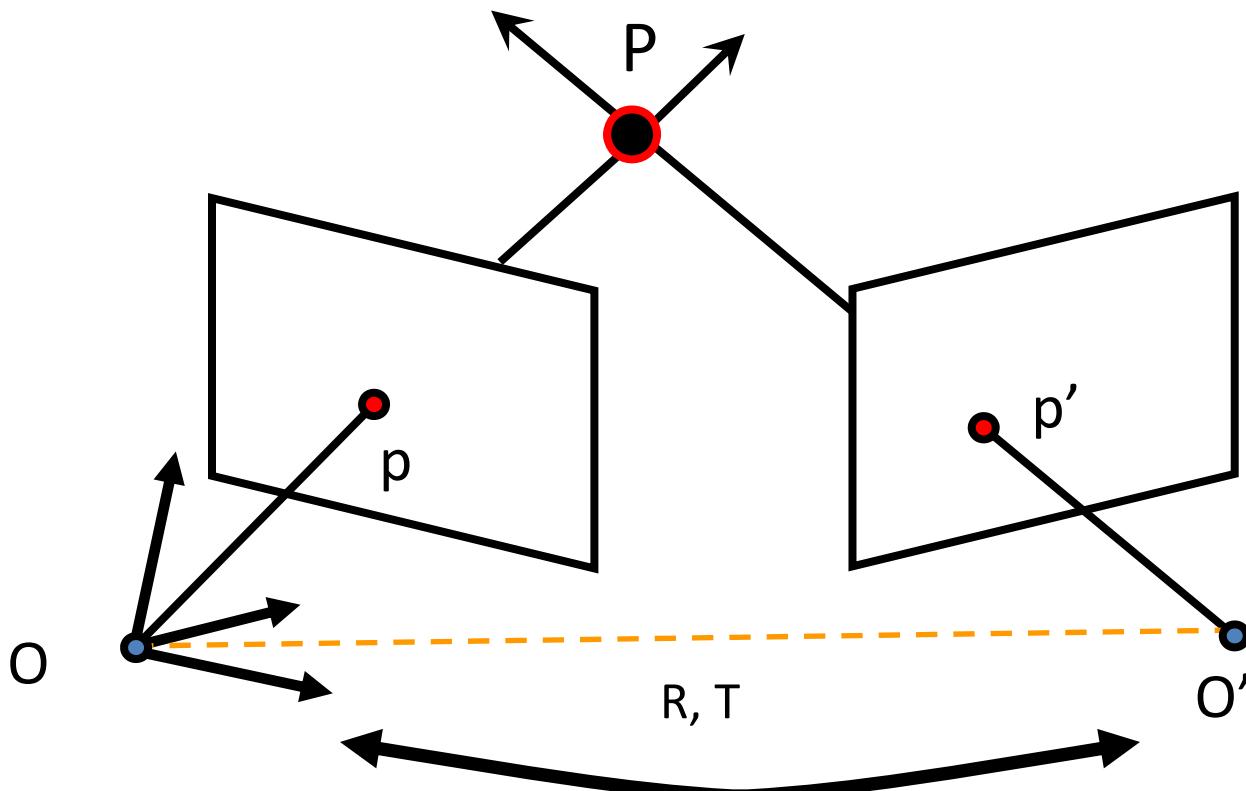
Epipolar Constraint



$$M = K \begin{bmatrix} I & 0 \end{bmatrix}$$

$$M' = K \begin{bmatrix} R & T \end{bmatrix}$$

Epipolar Constraint



$$M = K \begin{bmatrix} I & 0 \end{bmatrix}$$



$$M = \begin{bmatrix} I & 0 \end{bmatrix}$$

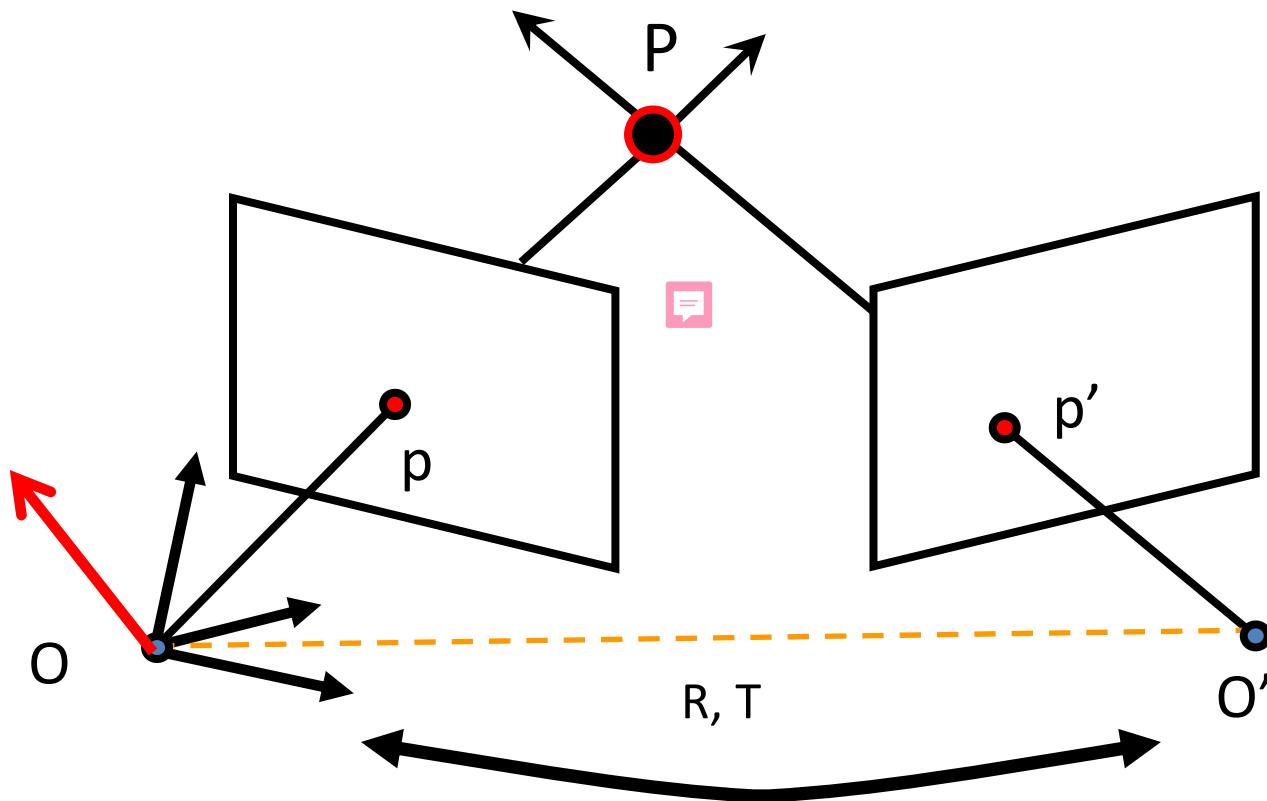
K_1 and K_2 are known
(calibrated cameras)

$$M' = K \begin{bmatrix} R & T \end{bmatrix}$$



$$M' = \begin{bmatrix} R & T \end{bmatrix}$$

Epipolar Constraint



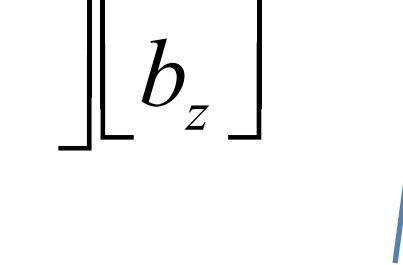
$$T \times (R \ p')$$

Perpendicular to epipolar plane

$$p^T \cdot [T \times (R \ p')] = 0$$

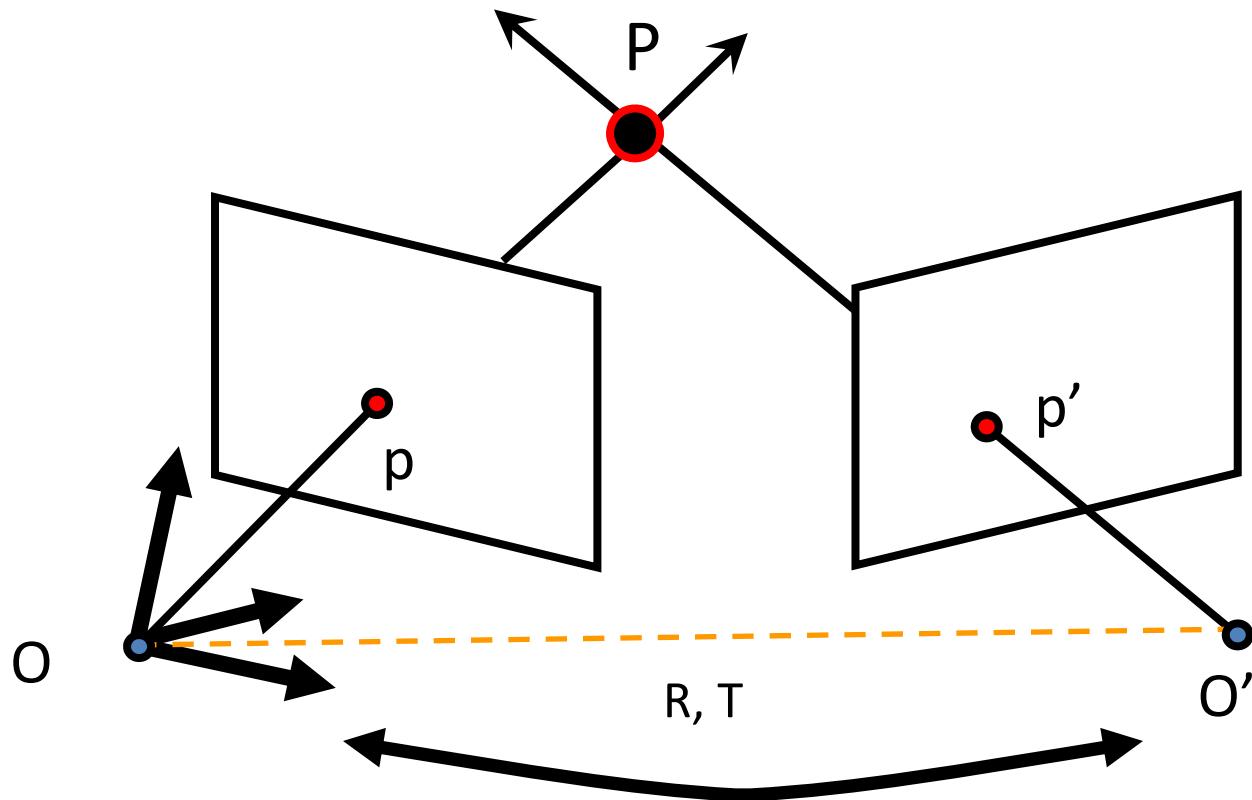
Cross product as matrix multiplication

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = [\mathbf{a}_\times] \mathbf{b}$$



“skew symmetric matrix”

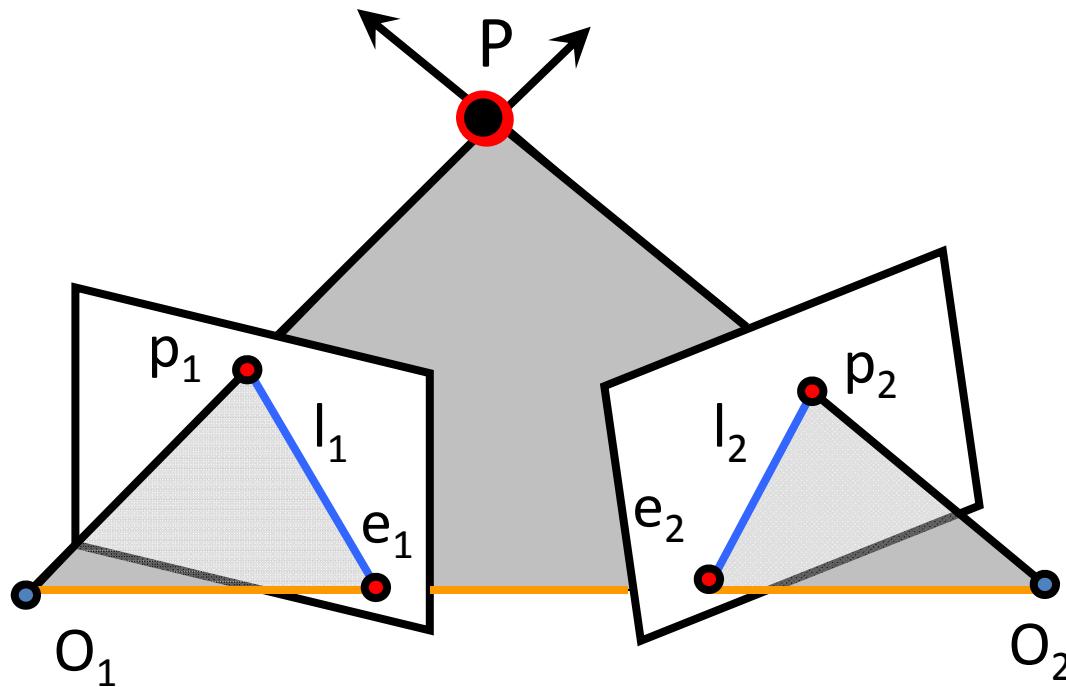
Triangulation



$$p^T \cdot [T \times (R p')] = 0 \rightarrow p^T \cdot [T_x] \cdot R p' = 0$$

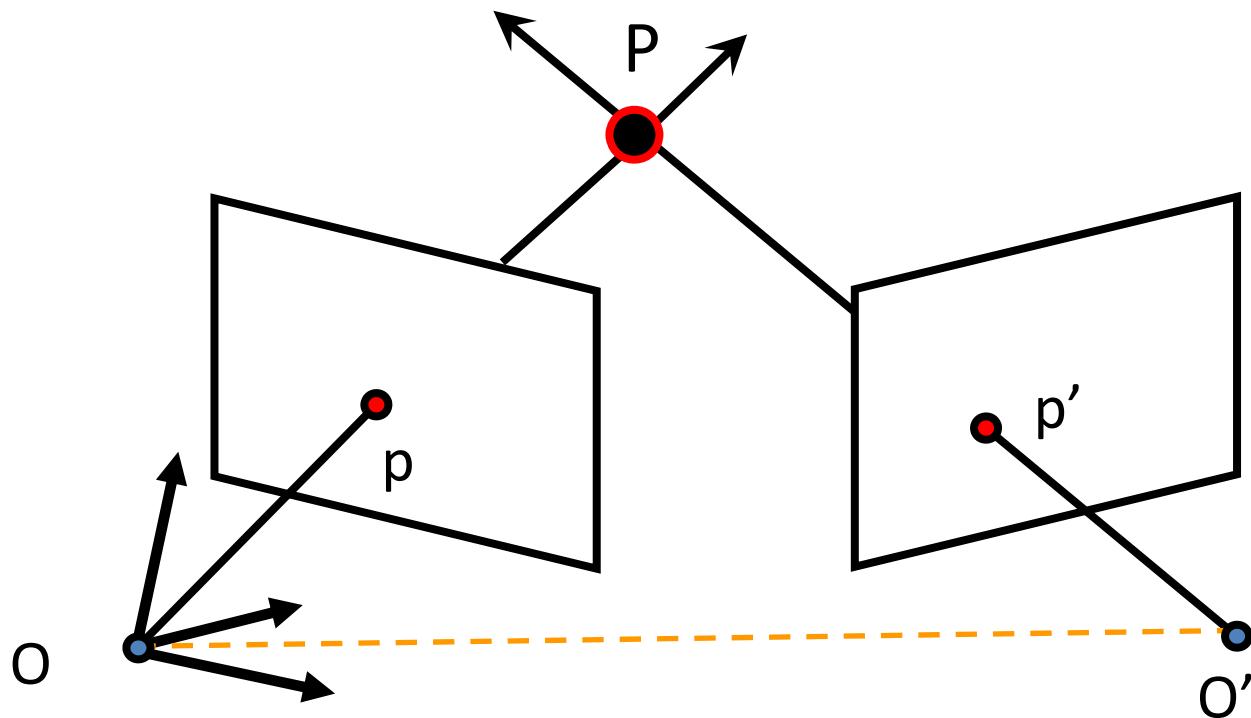
(Longuet-Higgins, 1981) T_x $E = \text{essential matrix}$

Triangulation



- $E p_2$ is the epipolar line associated with p_2 ($l_1 = E p_2$)
- $E^T p_1$ is the epipolar line associated with p_1 ($l_2 = E^T p_1$)
- E is singular (rank two)
- $E e_2 = 0$ and $E^T e_1 = 0$
- E is 3x3 matrix; 5 DOF

Triangulation

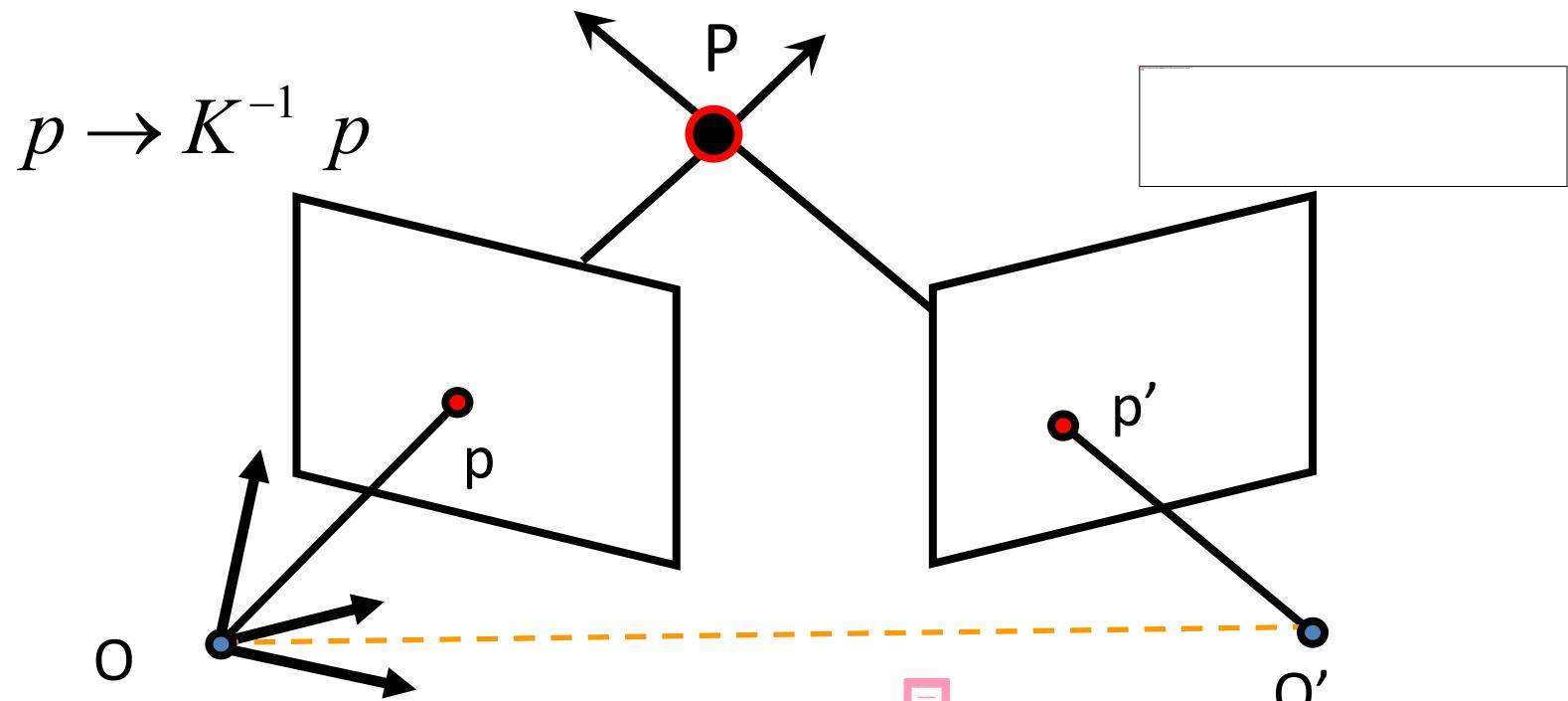


$$P \rightarrow M P \rightarrow p = \begin{bmatrix} u \\ v \end{bmatrix}$$

$$M = \boxed{K} \begin{bmatrix} I & 0 \end{bmatrix}$$

unknown

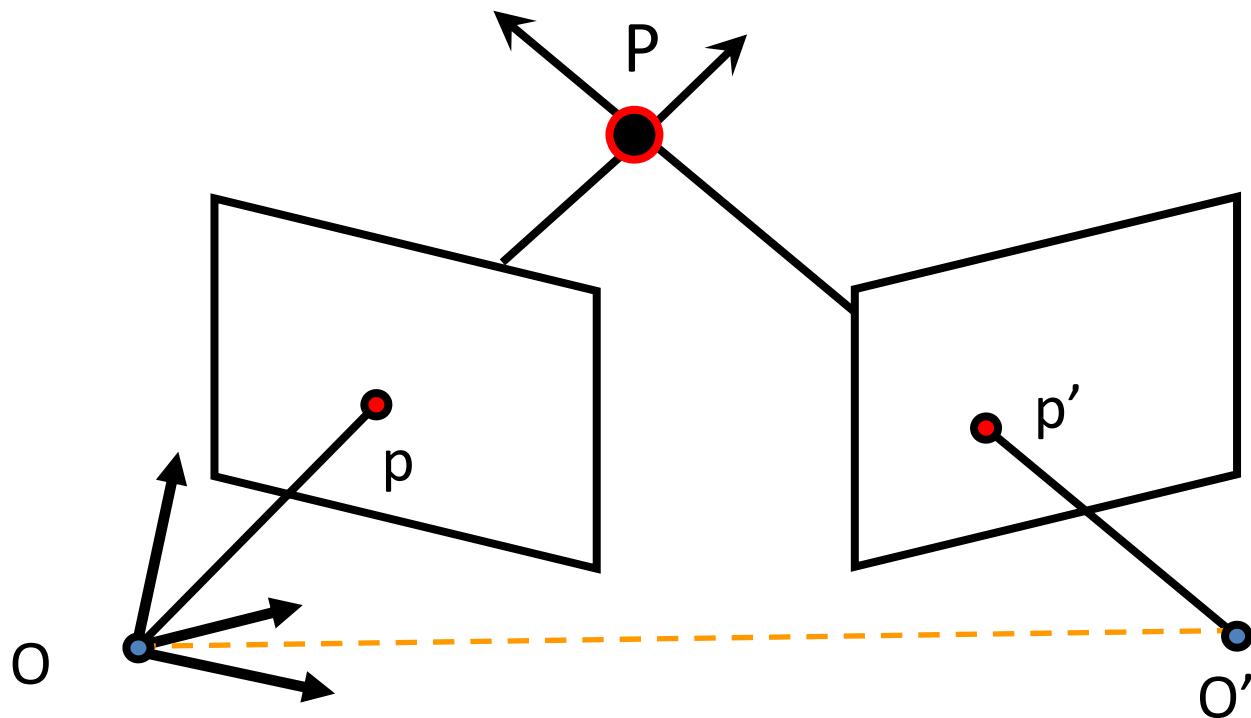
Triangulation



$$p^T \cdot [T_x] \cdot R p' = 0 \rightarrow (K^{-1} p)^T \cdot [T_x] \cdot R K'^{-1} p' = 0$$

$$p^T [K^{-T} \cdot [T_x] \cdot R K'^{-1}] p' = 0 \rightarrow p^T [F] p' = 0$$

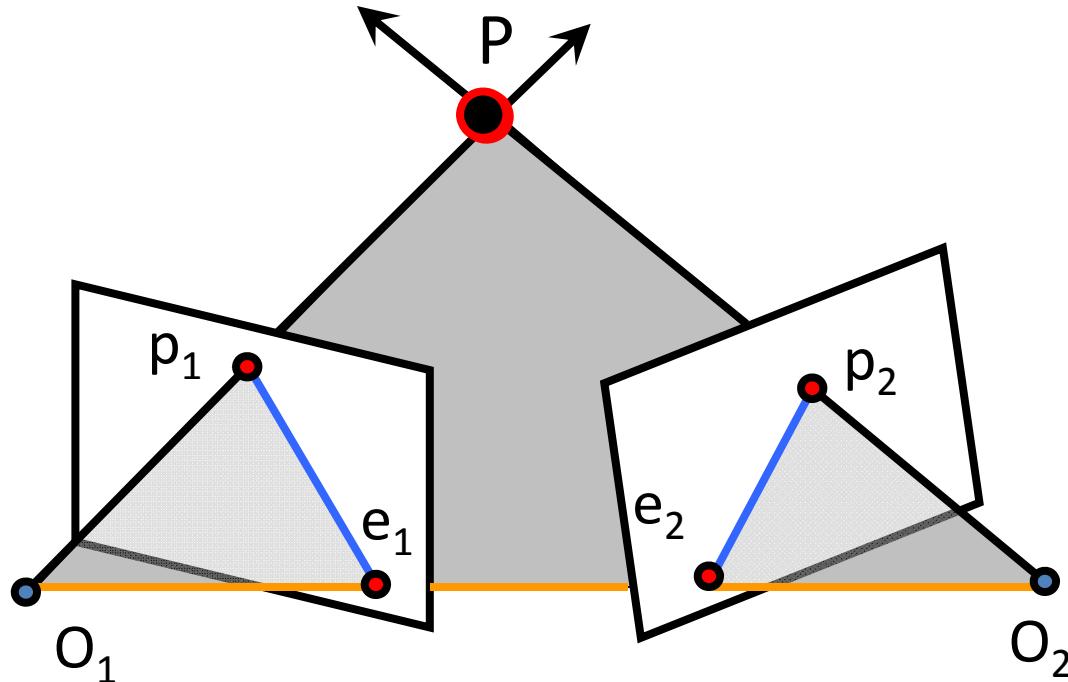
Triangulation



$$p^T F p' = 0$$

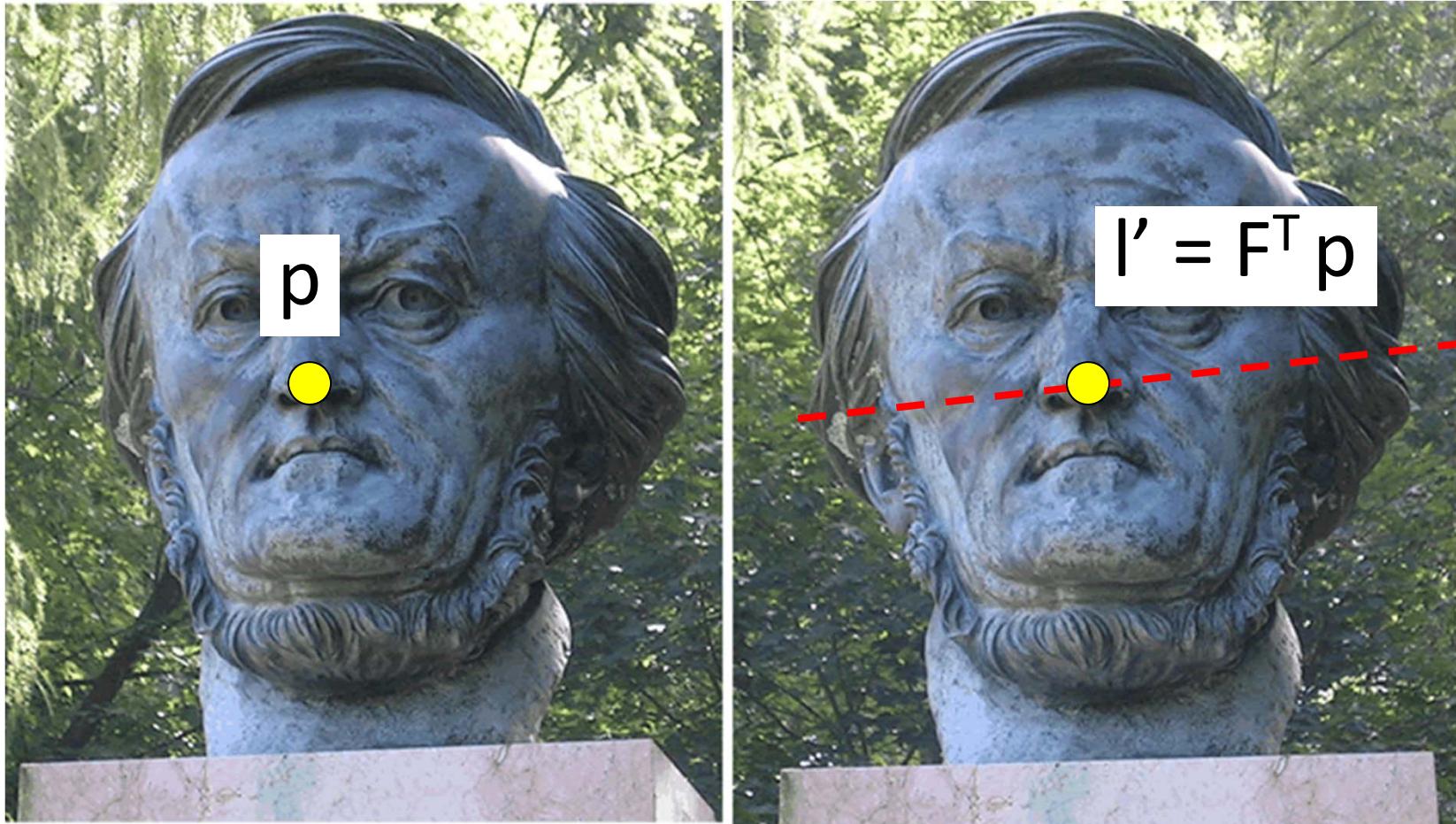
F = Fundamental Matrix
(Faugeras and Luong, 1992)

Triangulation



- $F p_2$ is the epipolar line associated with p_2 ($l_1 = F p_2$)
- $F^T p_1$ is the epipolar line associated with p_1 ($l_2 = F^T p_1$)
- F is singular (rank two)
- $F e_2 = 0$ and $F^T e_1 = 0$
- F is 3×3 matrix; 7 DOF

Why is F useful?



- Suppose F is known
- No additional information about the scene and camera is given
- Given a point on left image, how can I find the corresponding point on right image?

Why is F useful?

- F captures information about the epipolar geometry of 2 views + camera parameters
- **MORE IMPORTANTLY:** F gives constraints on how the scene changes under view point transformation (without reconstructing the scene!)
- Powerful tool in:
 - 3D reconstruction
 - Multi-view object/scene matching

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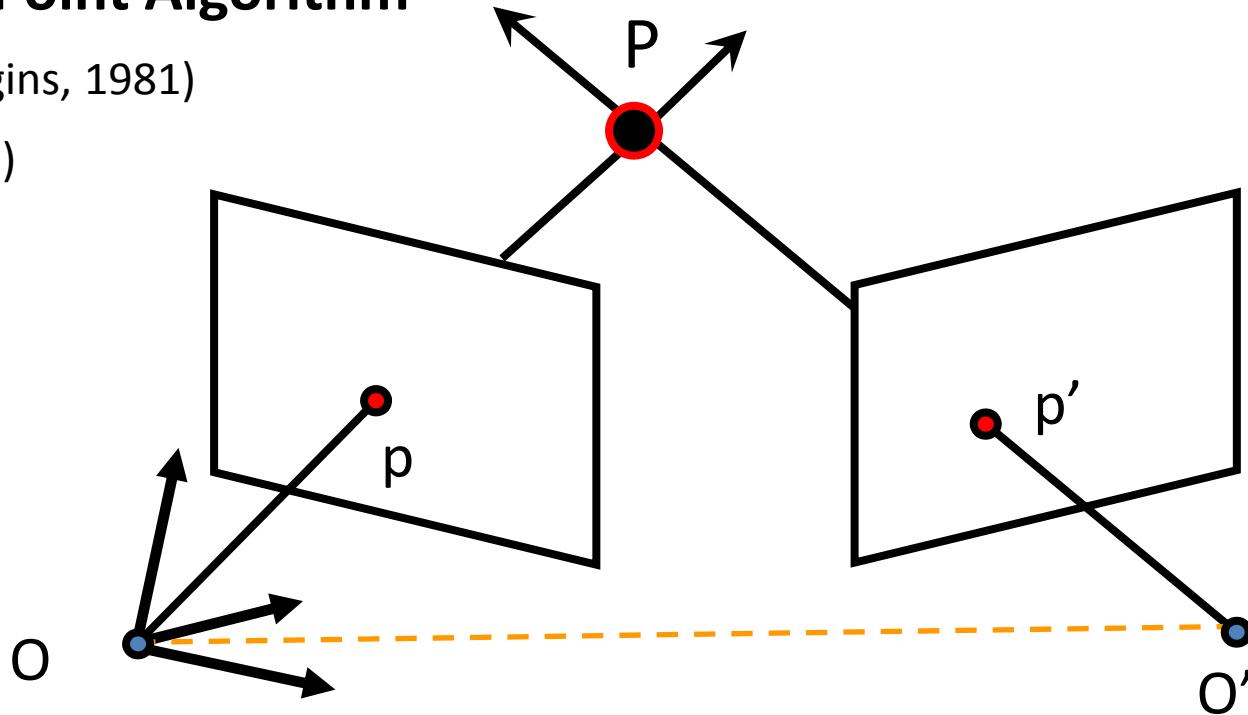
Reading:
[HZ] Chapters: 4, 9, 11
[FP] Chapters: 10

Estimating F

The Eight-Point Algorithm

(Longuet-Higgins, 1981)

(Hartley, 1995)



$$P \rightarrow p = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

$$P \rightarrow p' = \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix}$$

$$p^T F p' = 0$$

Estimating F

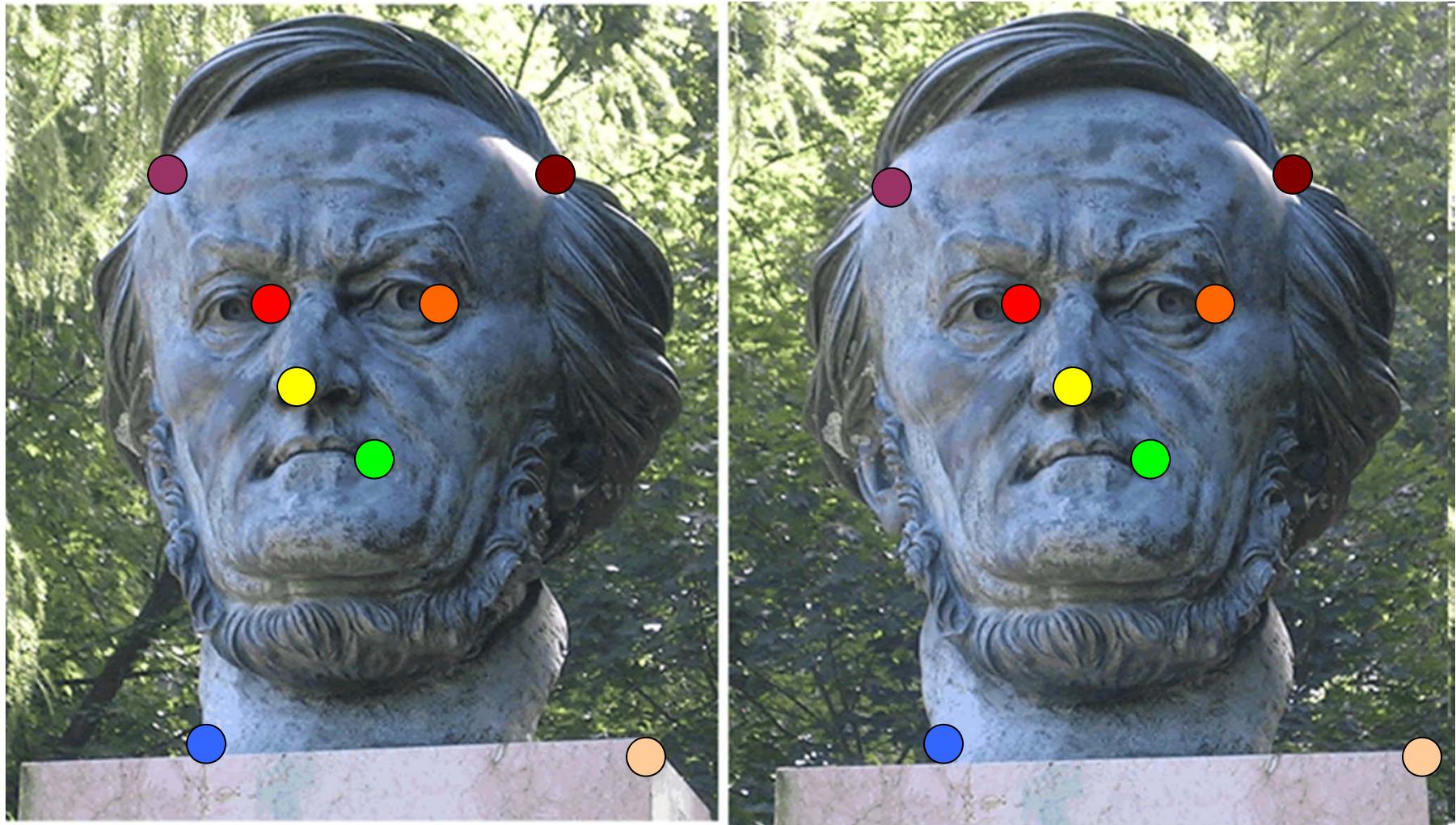
$$p^T F p' = 0 \quad \rightarrow$$

$$(u, v, 1) \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix} \begin{pmatrix} u' \\ v' \\ 1 \end{pmatrix} = 0$$

$$\rightarrow (uu', uv', u, vu', vv', v, u', v', 1) \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0$$

Let's take 8 corresponding points

Estimating F



Estimating F

$$\mathbf{W} \begin{pmatrix} u_1u'_1 & u_1v'_1 & u_1 & v_1u'_1 & v_1v'_1 & v_1 & u'_1 & v'_1 & 1 \\ u_2u'_2 & u_2v'_2 & u_2 & v_2u'_2 & v_2v'_2 & v_2 & u'_2 & v'_2 & 1 \\ u_3u'_3 & u_3v'_3 & u_3 & v_3u'_3 & v_3v'_3 & v_3 & u'_3 & v'_3 & 1 \\ u_4u'_4 & u_4v'_4 & u_4 & v_4u'_4 & v_4v'_4 & v_4 & u'_4 & v'_4 & 1 \\ u_5u'_5 & u_5v'_5 & u_5 & v_5u'_5 & v_5v'_5 & v_5 & u'_5 & v'_5 & 1 \\ u_6u'_6 & u_6v'_6 & u_6 & v_6u'_6 & v_6v'_6 & v_6 & u'_6 & v'_6 & 1 \\ u_7u'_7 & u_7v'_7 & u_7 & v_7u'_7 & v_7v'_7 & v_7 & u'_7 & v'_7 & 1 \\ u_8u'_8 & u_8v'_8 & u_8 & v_8u'_8 & v_8v'_8 & v_8 & u'_8 & v'_8 & 1 \end{pmatrix} \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = \mathbf{0}$$

\mathbf{f}

- Homogeneous system $\mathbf{W f} = 0$
- Rank 8 \rightarrow A non-zero solution exists (unique)
- If $N > 8$ \rightarrow Lsq. solution by SVD!

$$\hat{\mathbf{F}} \quad \|\mathbf{f}\| = 1$$

Estimating F

$$p^T \hat{F} p' = 0$$

The estimated \hat{F} may have full rank ($\det(\hat{F}) \neq 0$)
(F should have rank=2 instead)

Find F that minimizes

$$\|F - \hat{F}\| = 0$$

Frobenius norm (*)

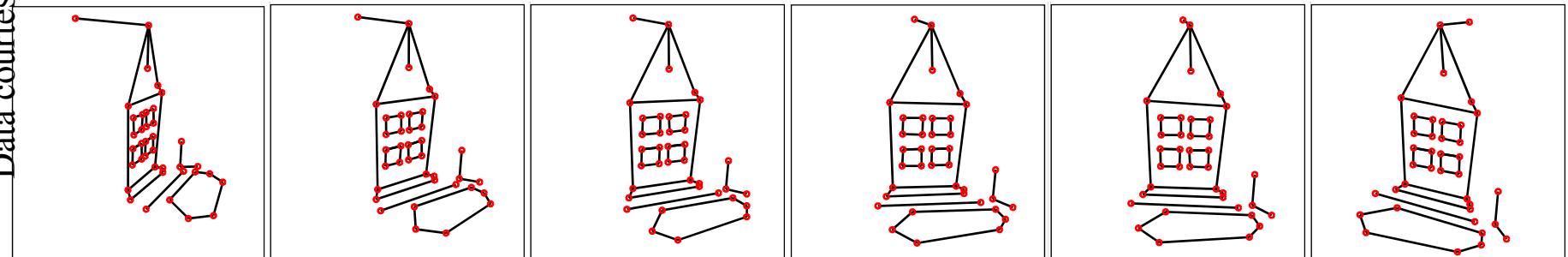
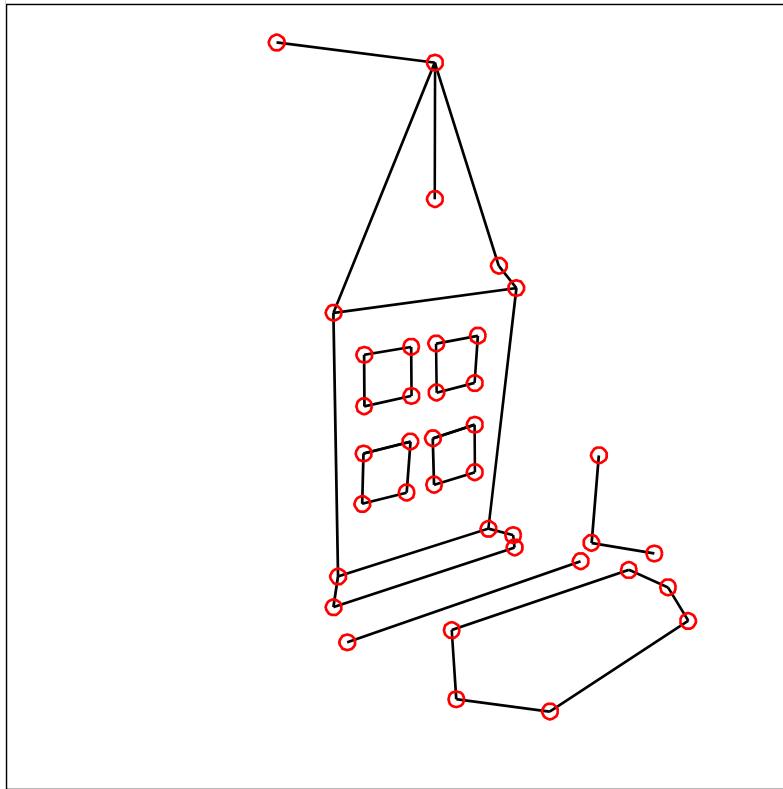
Subject to $\det(F)=0$

SVD (again!) can be used to solve this problem

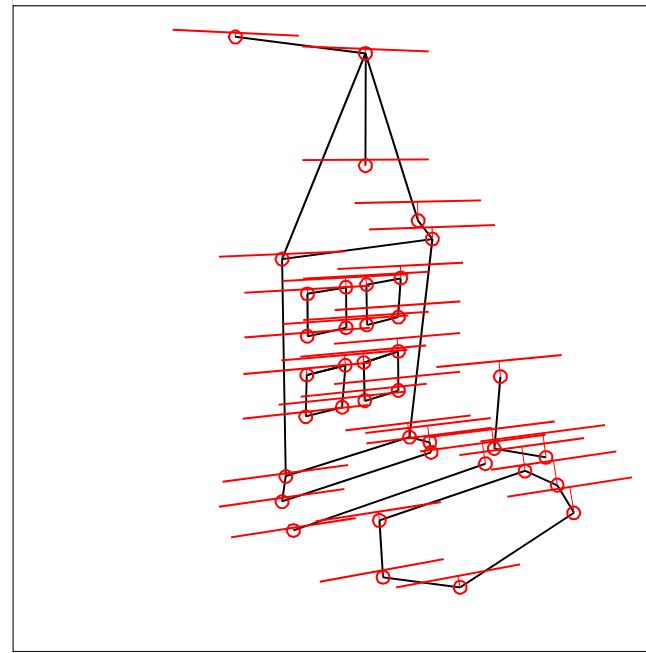
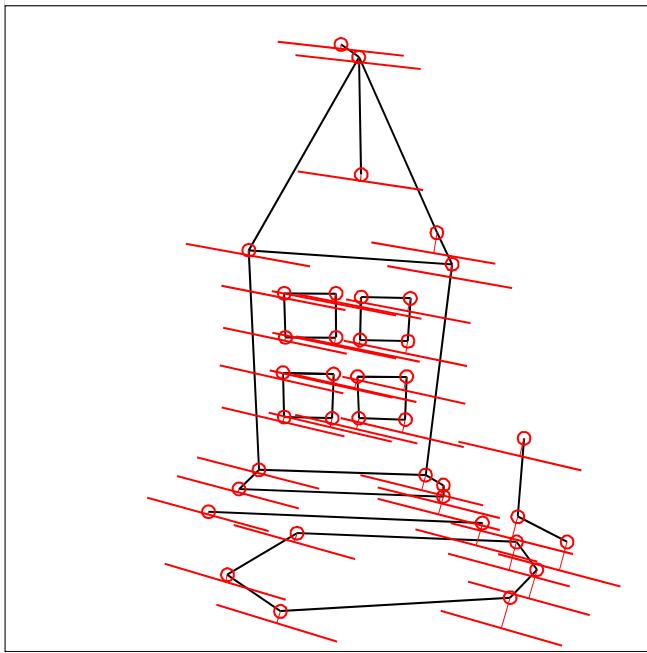
(*) Sqrt root of the sum pf squares of all entries

Example

Data courtesy of R. Mohr and B. Boufama.

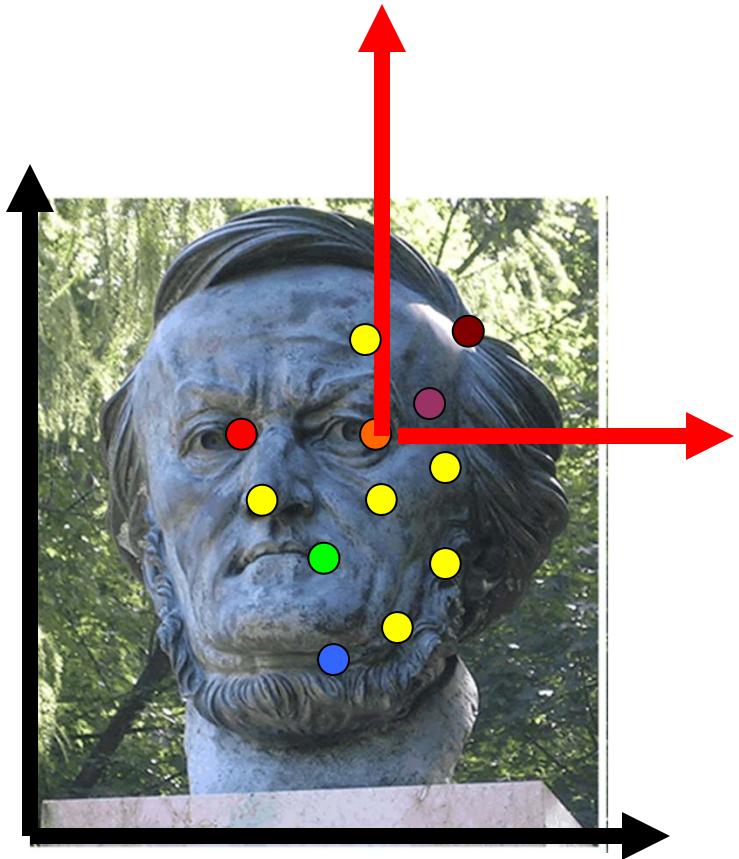


Example



Mean errors:
10.0pixel
9.1pixel

Normalization



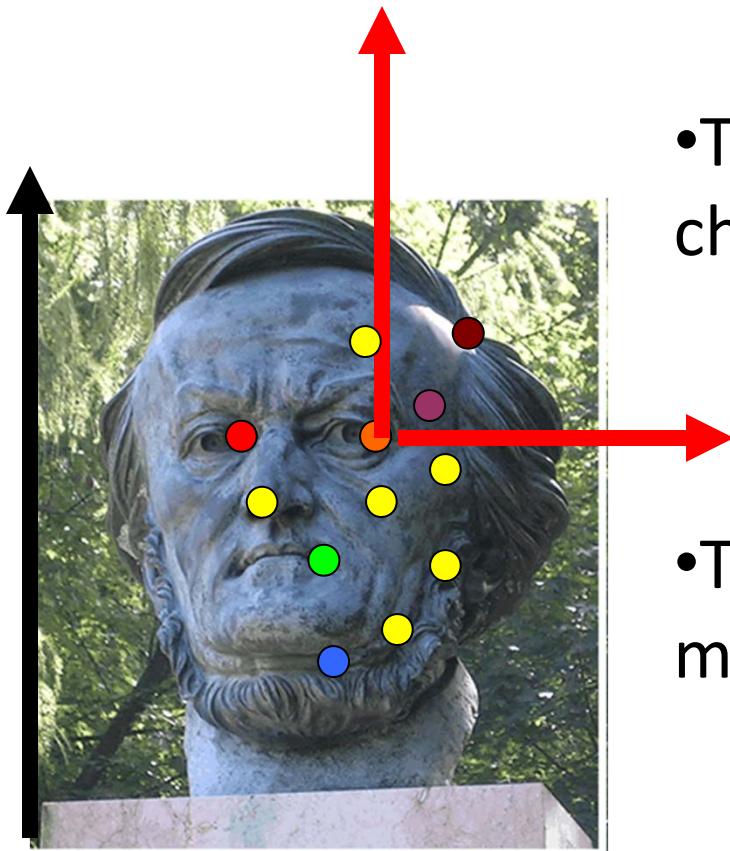
Is the accuracy in estimating F function of the ref. system in the image plane?

E.g. under similarity transformation ($T = \text{scale} + \text{translation}$):

$$q_i = T_i p_i \quad q'_i = T'_i p'_i$$

Does the accuracy in estimating F change if a transformation T is applied?

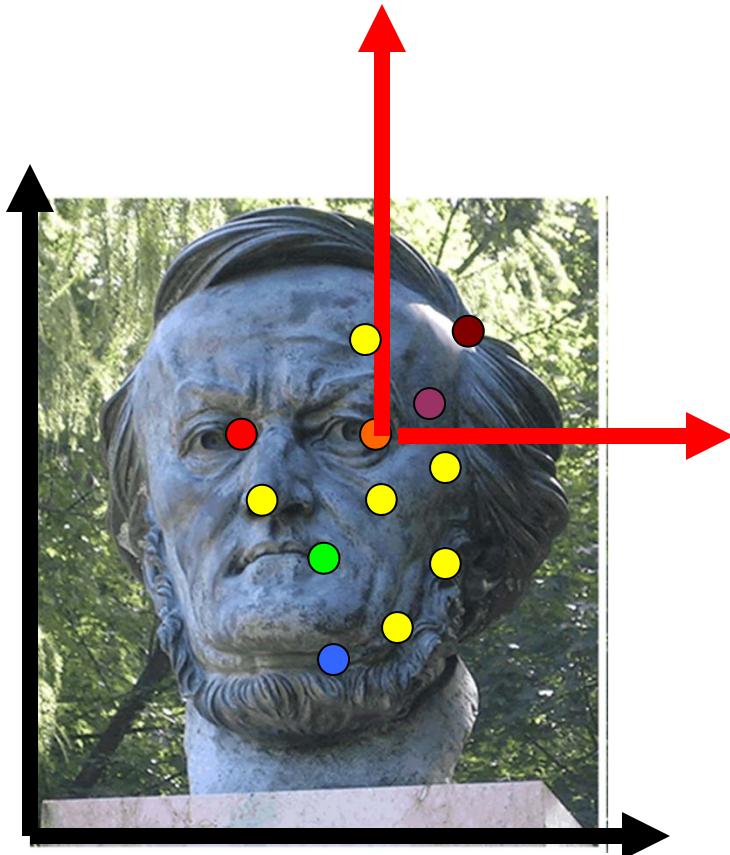
Normalization



- The accuracy in estimating F does change if a transformation T is applied
- There exists a T for which accuracy is maximized

Why?

Normalization



$$W f = 0,$$

$$\|f\| = 1$$

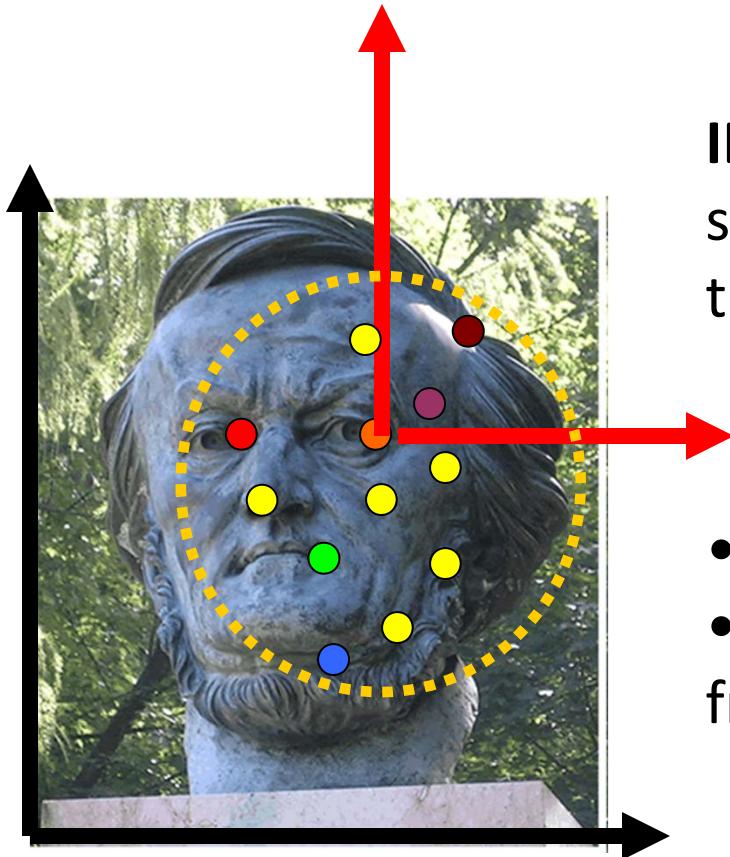
Lsq solution
by SVD

$$\longrightarrow F$$

- SVD enforces $\text{Rank}(W)=8$
- Recall the structure of W :
Highly un-balance
(not well conditioned)
- Values of W must have similar magnitude

More details HZ pag 108

Normalization



IDEA: Transform image coordinate system ($T = \text{translation} + \text{scaling}$) such that:

- Origin = centroid of image points
- Mean square distance of the data points from origin is 2 pixels

$$q_i = T_i p_i \quad q'_i = T'_i p'_i \quad (\text{normalization})$$

The Normalized Eight-Point Algorithm

0. Compute T_i and T'_i

1. Normalize coordinates:

$$q_i = T_i p_i \quad q'_i = T'_i p'_i$$

2. Use the eight-point algorithm to compute F'_q from the points q_i and q'_i .

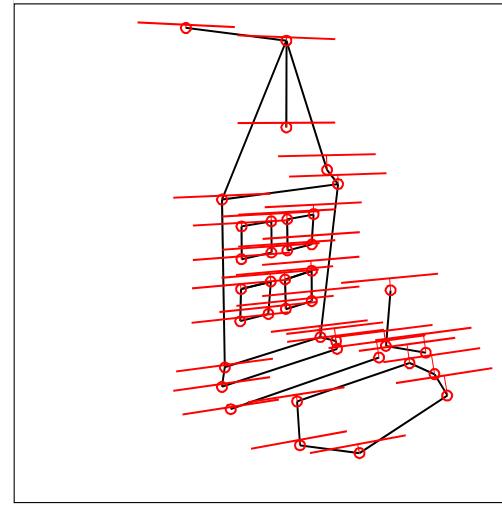
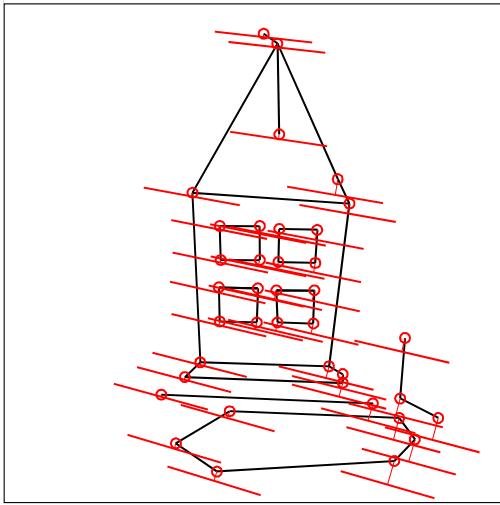
3. Enforce the rank-2 constraint.

$$\rightarrow F_q \quad \begin{cases} q^T F_q q' = 0 \\ \det(F_q) = 0 \end{cases}$$

4. De-normalize F_q : $F = {T'}^T F_q T$

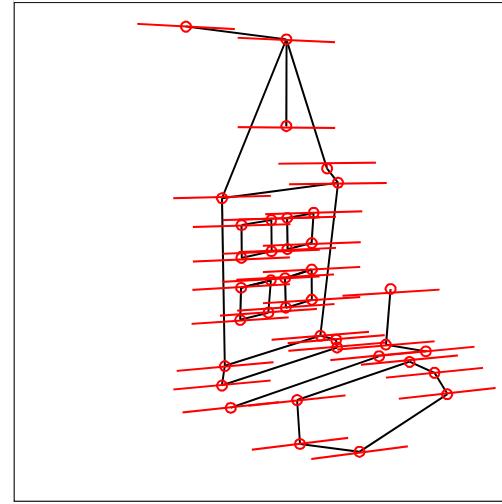
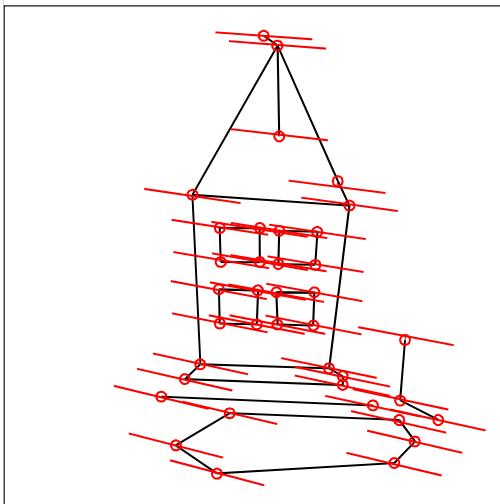
Example

Without transformation



Mean errors:
10.0pixel
9.1pixel

With transformation



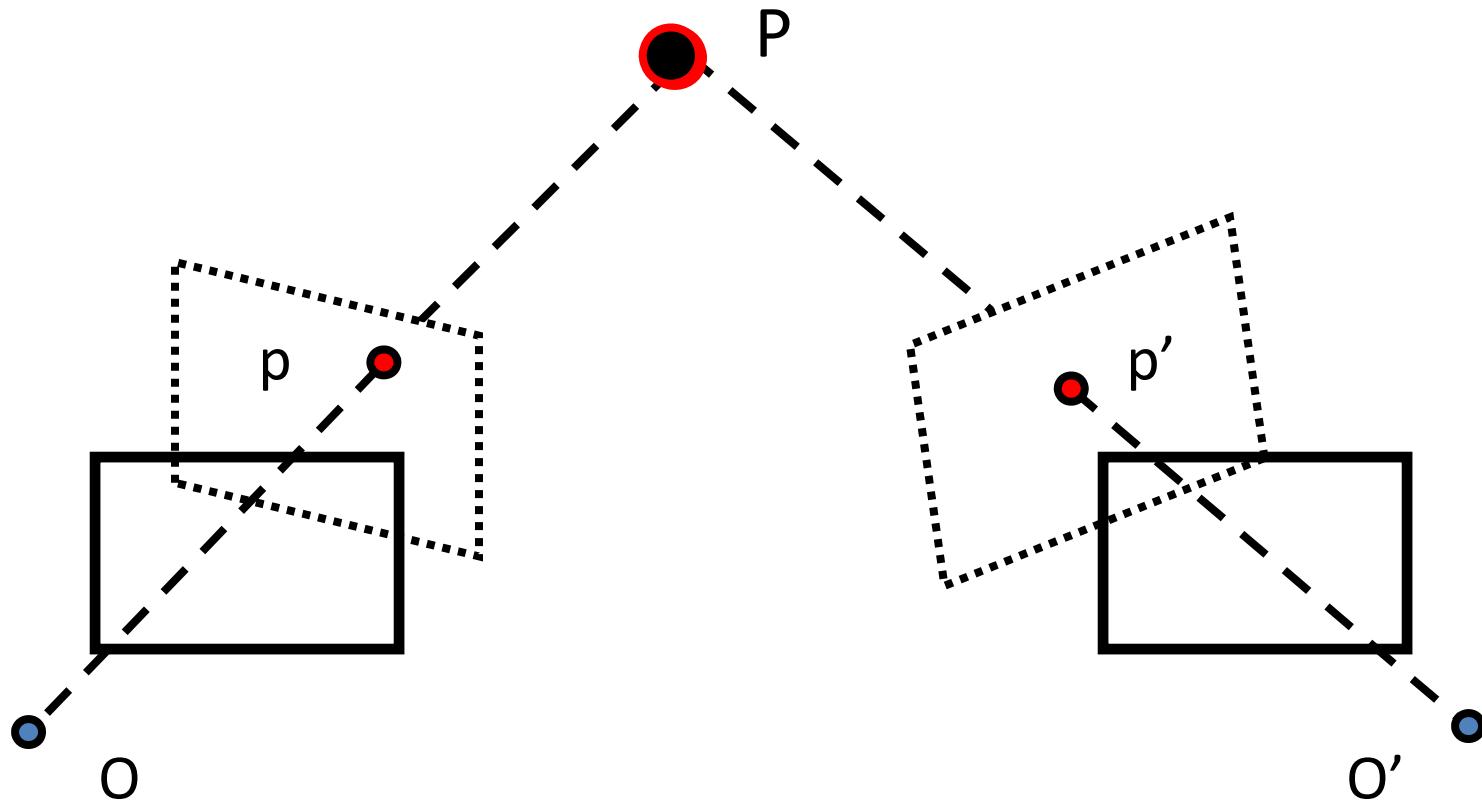
Mean errors:
1.0pixel
0.9pixel

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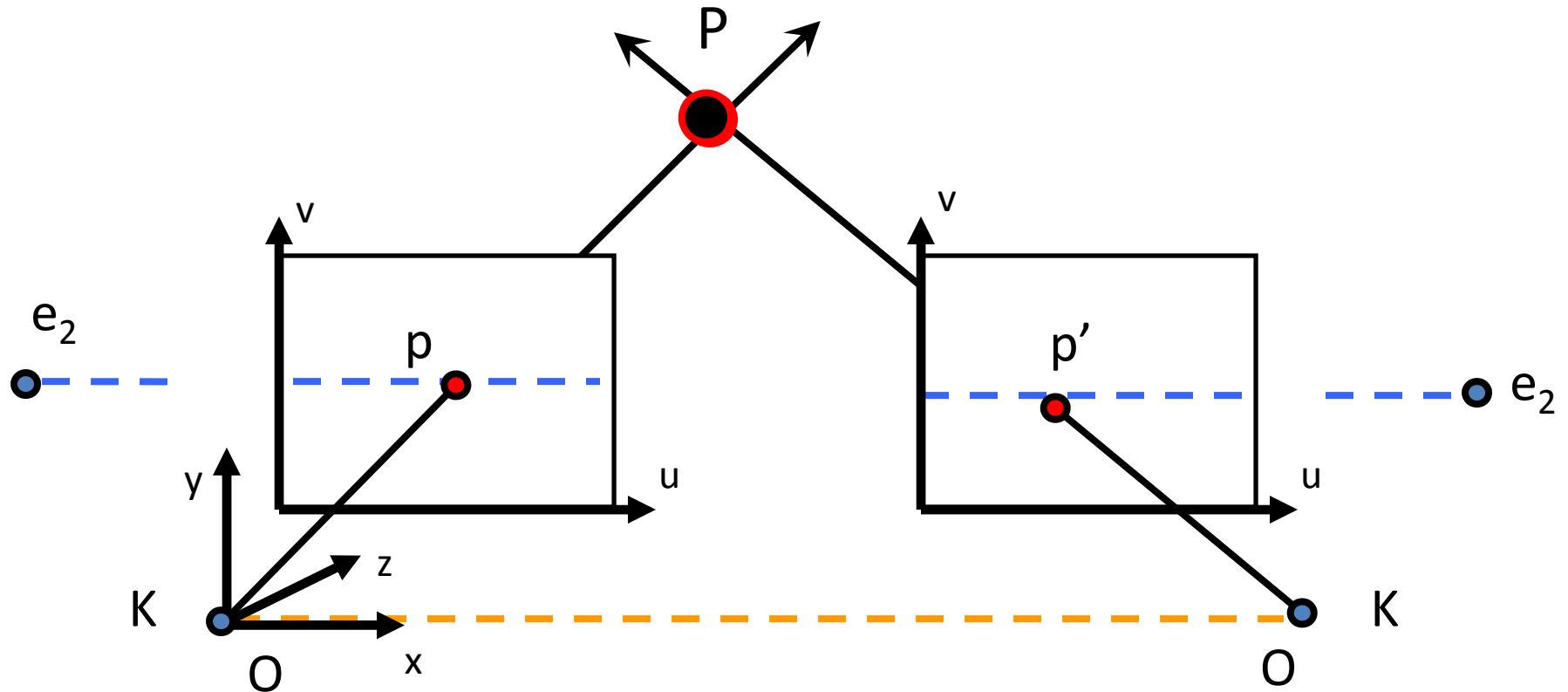
Reading:
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Rectification



- Make two camera images “parallel”
 - Correspondence problem becomes easier

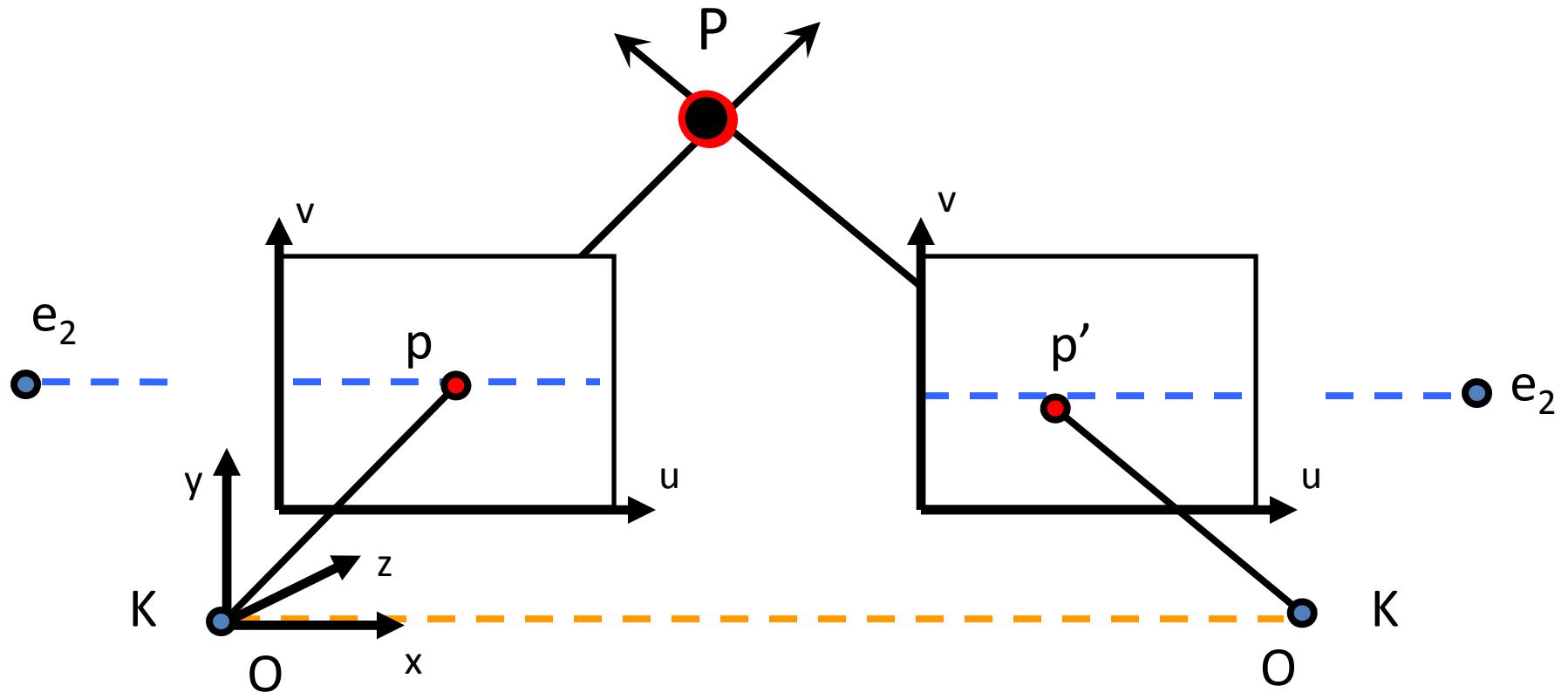
Rectification



- Parallel epipolar lines
- Epipoles at infinity
- $v = v'$

Let's see why....

Rectification



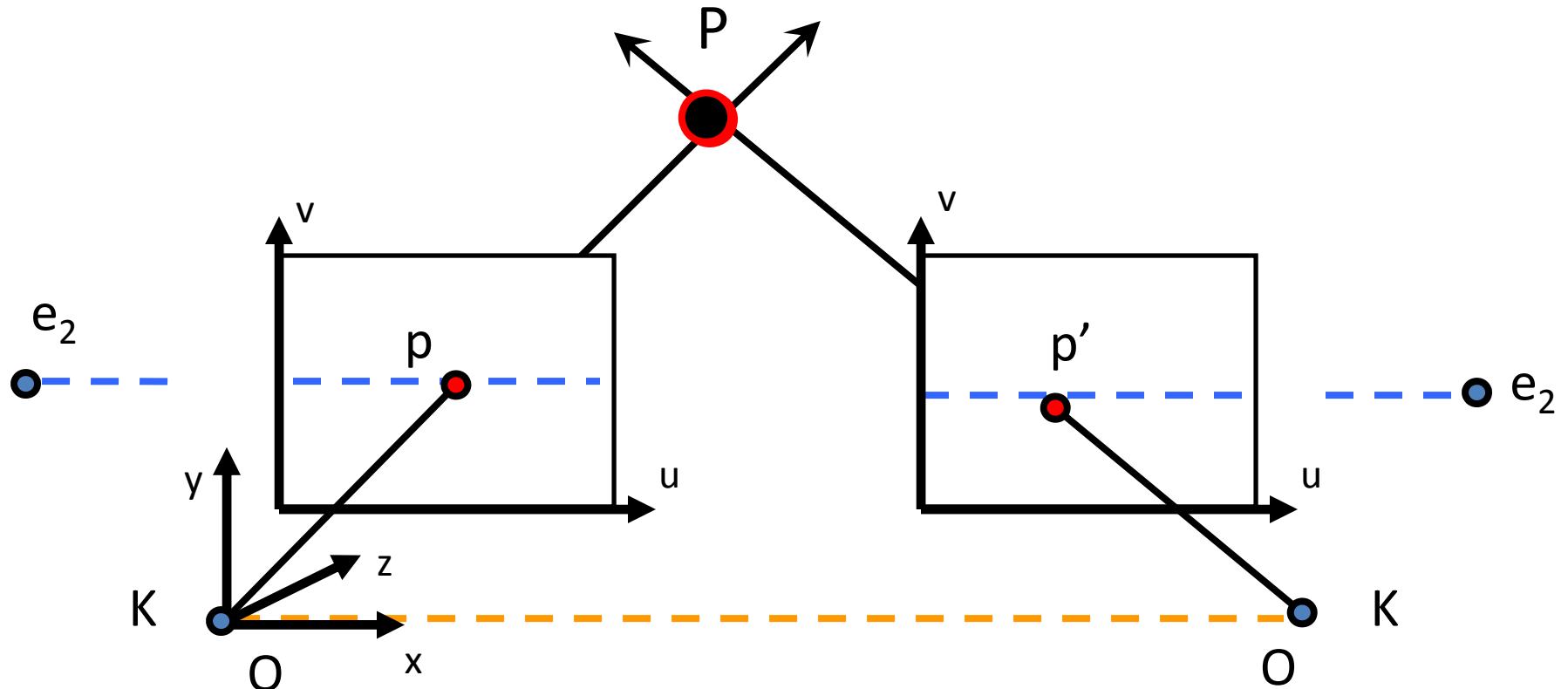
$K_1 = K_2 = \text{known}$
 x parallel to O_1O_2

$$E = [t_x]R$$

Cross product as matrix multiplication

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = [\mathbf{a}_\times] \mathbf{b}$$

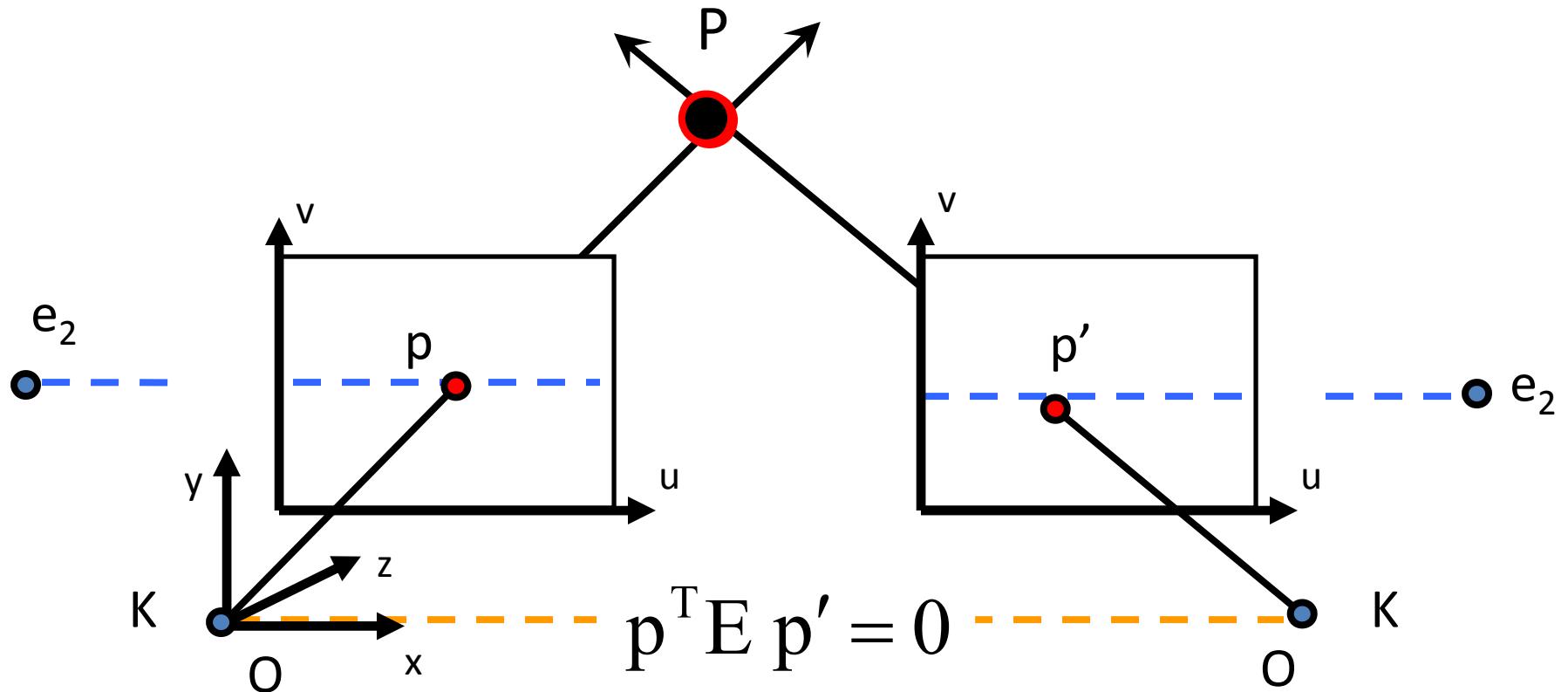
Rectification



$K_1 = K_2 = \text{known}$
 $x \text{ parallel to } O_1O_2$

$$E = [t_x]R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -T \\ 0 & T & 0 \end{bmatrix} \rightarrow v = v'?$$

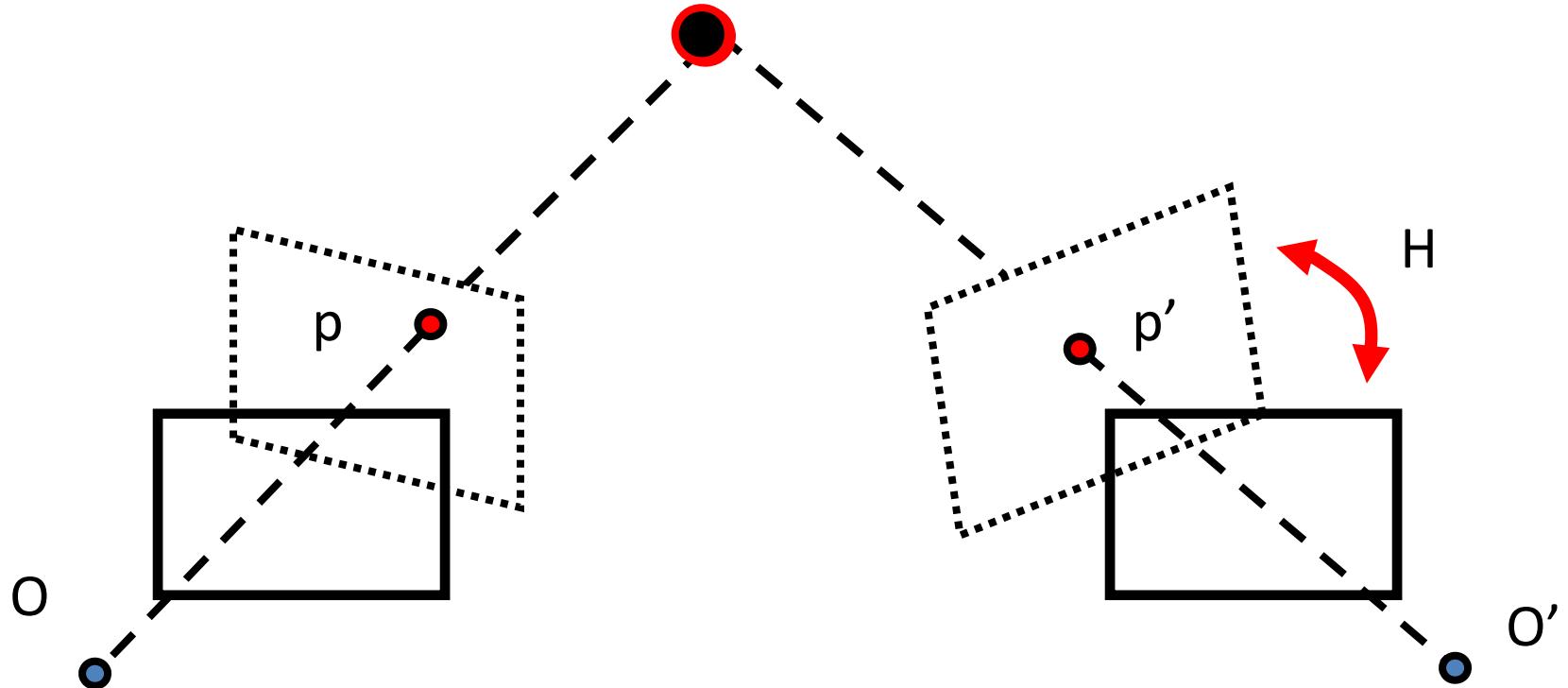
Rectification



$$p^T E p' = 0$$

$$(u \quad v \quad 1) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -T \\ 0 & T & 0 \end{bmatrix} \begin{pmatrix} u' \\ v' \\ 1 \end{pmatrix} = 0 \quad (u \quad v \quad 1) \begin{pmatrix} 0 \\ -T \\ Tv' \end{pmatrix} = 0 \quad Tv = Tv' \rightarrow v = v'$$

Rectification



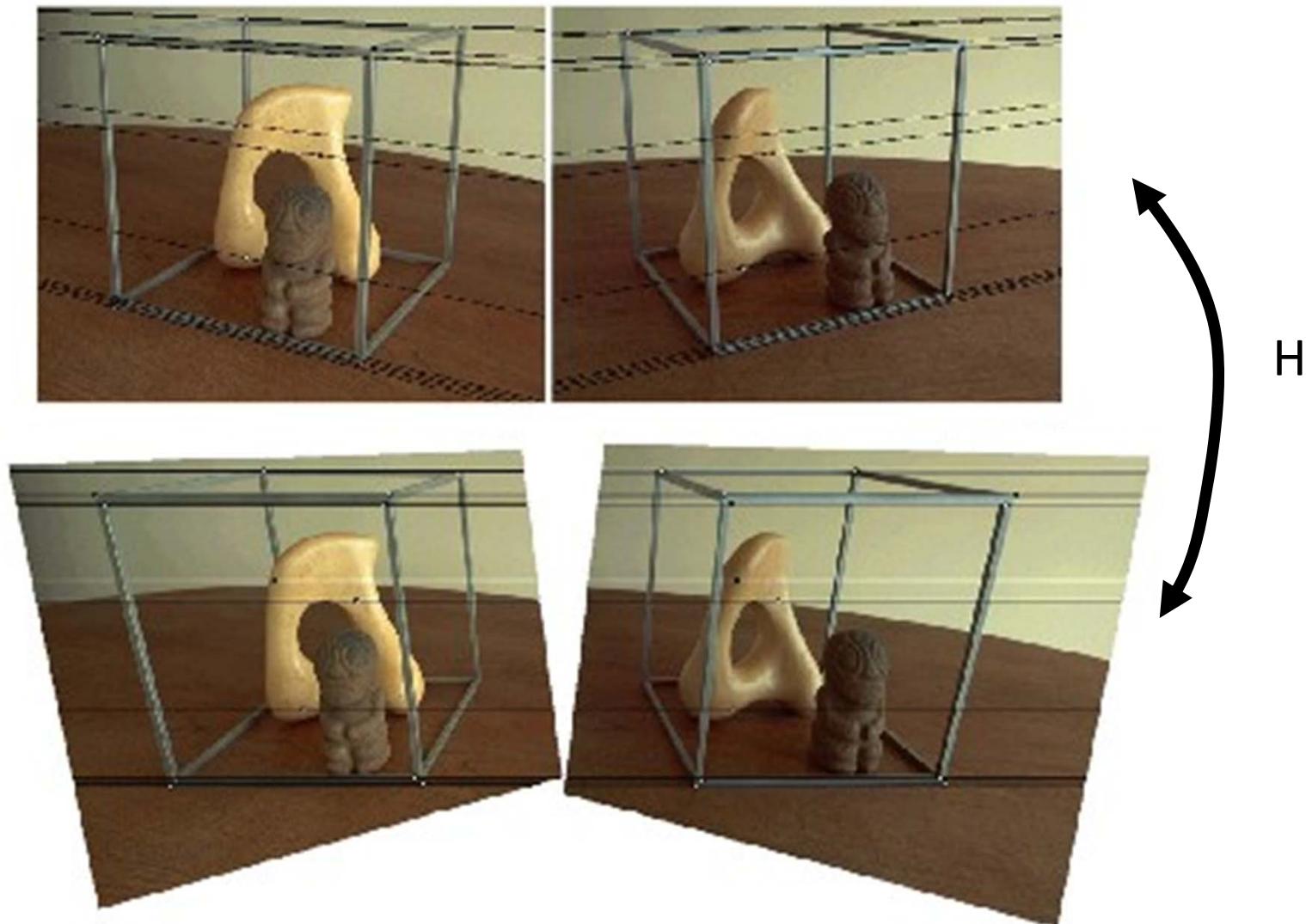
GOAL of rectification : Estimate a perspective transformation H that makes images parallel

Impose $v' = v$

- This leaves degrees of freedom for determining H
- If not appropriate H is chosen, severe projective distortions on image take place
- We impose a number of restriction while computing H

[HZ] Chapters: 11 (sec. 11.12)

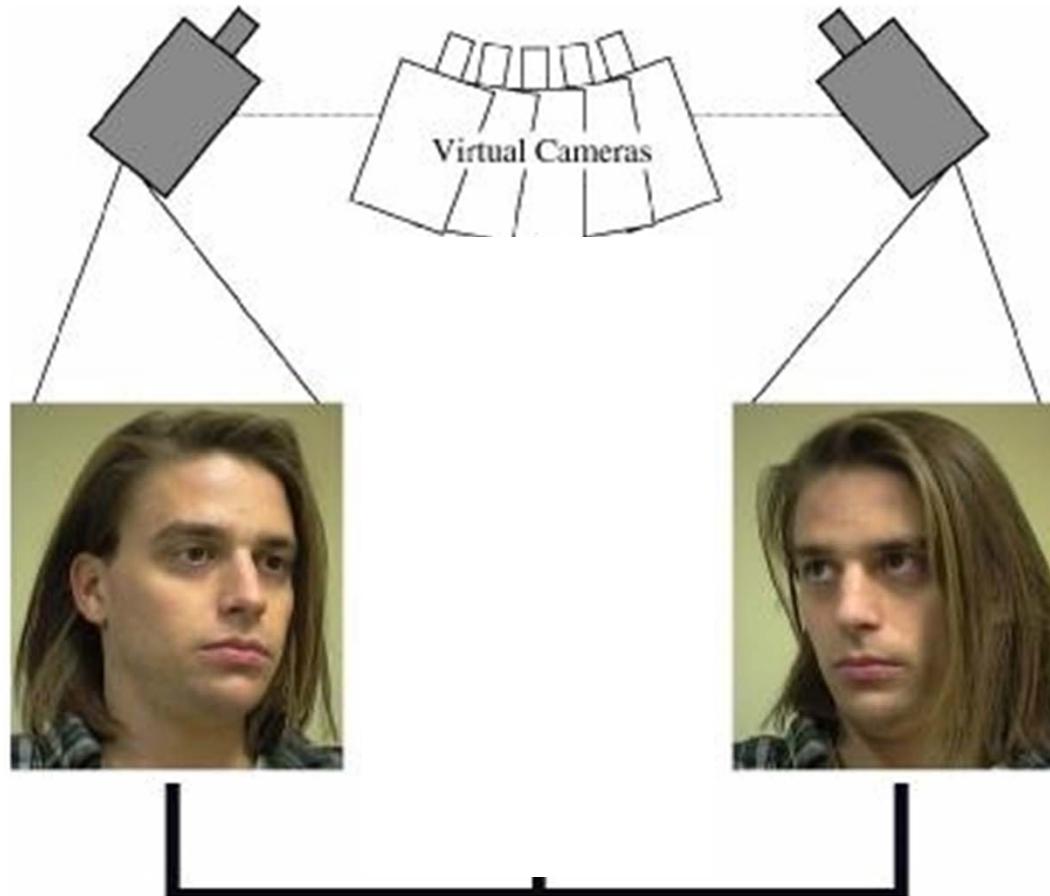
Rectification



Courtesy figure S. Lazebnik

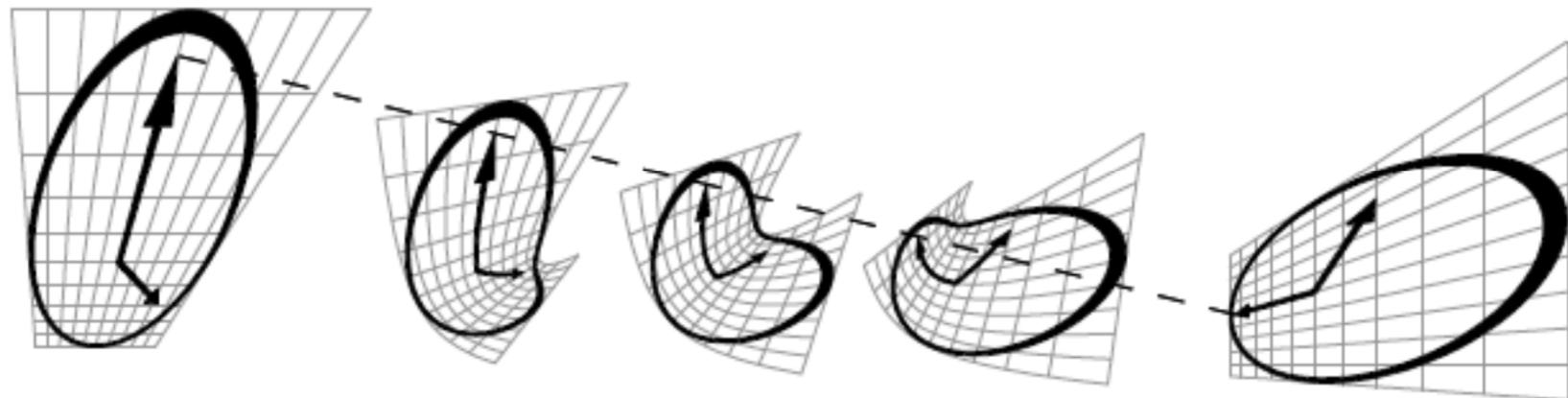
Application: view morphing

S. M. Seitz and C. R. Dyer, *Proc. SIGGRAPH 96*, 1996, 21-30

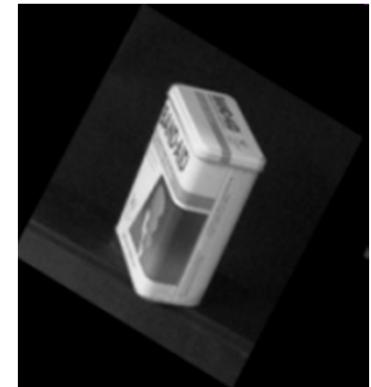
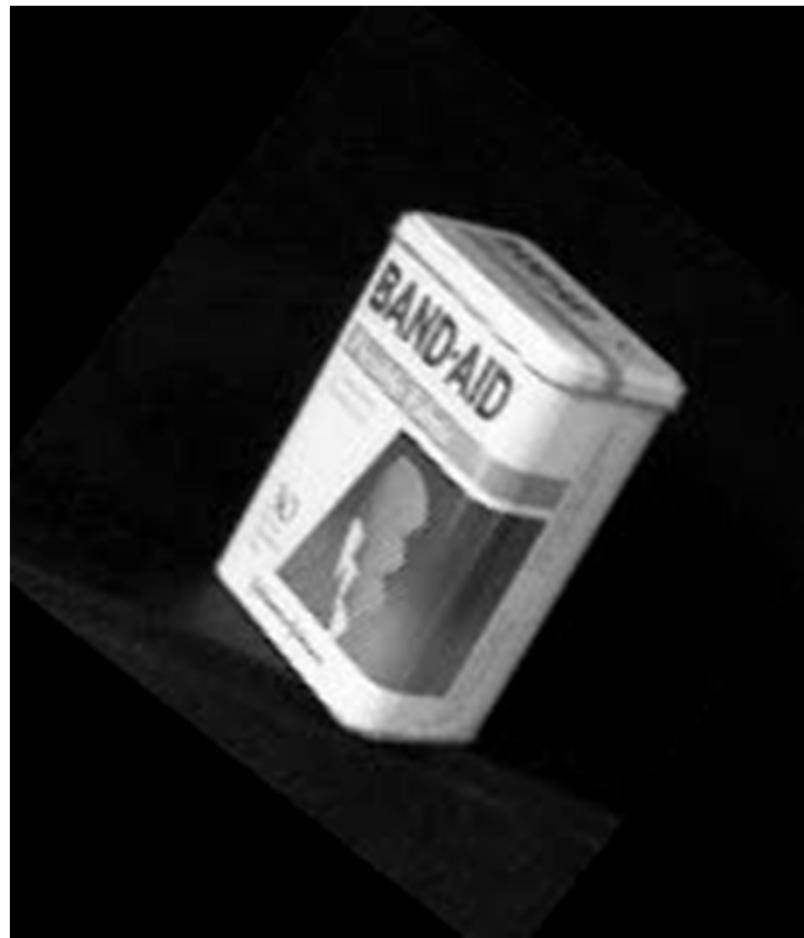
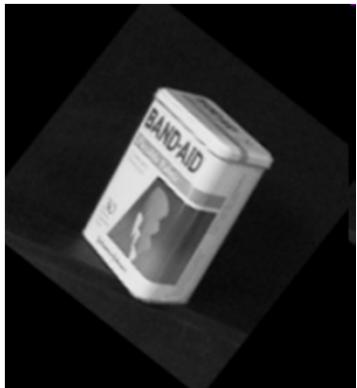


Application: view morphing

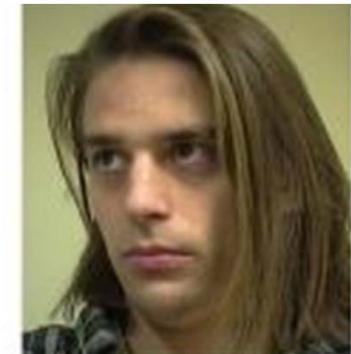
If rectification is not applied, the morphing procedure does not generate geometrically correct interpolations



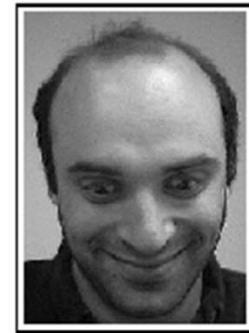
Application: view morphing



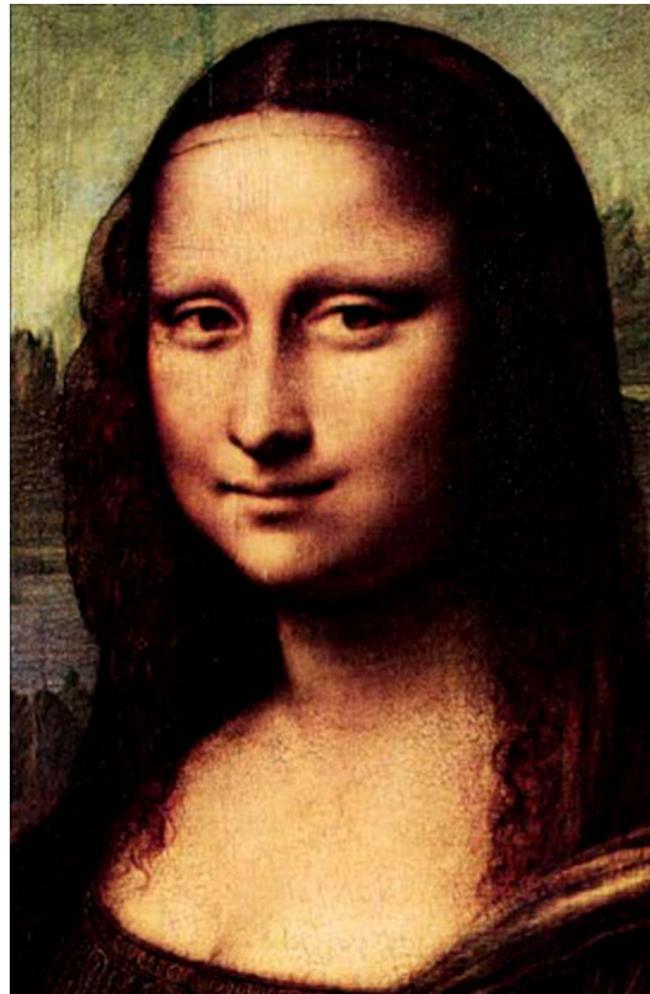
Application: view morphing



Application: view morphing



Application: view morphing



The Fundamental Matrix Song

<http://danielwedge.com/fmatrix/>

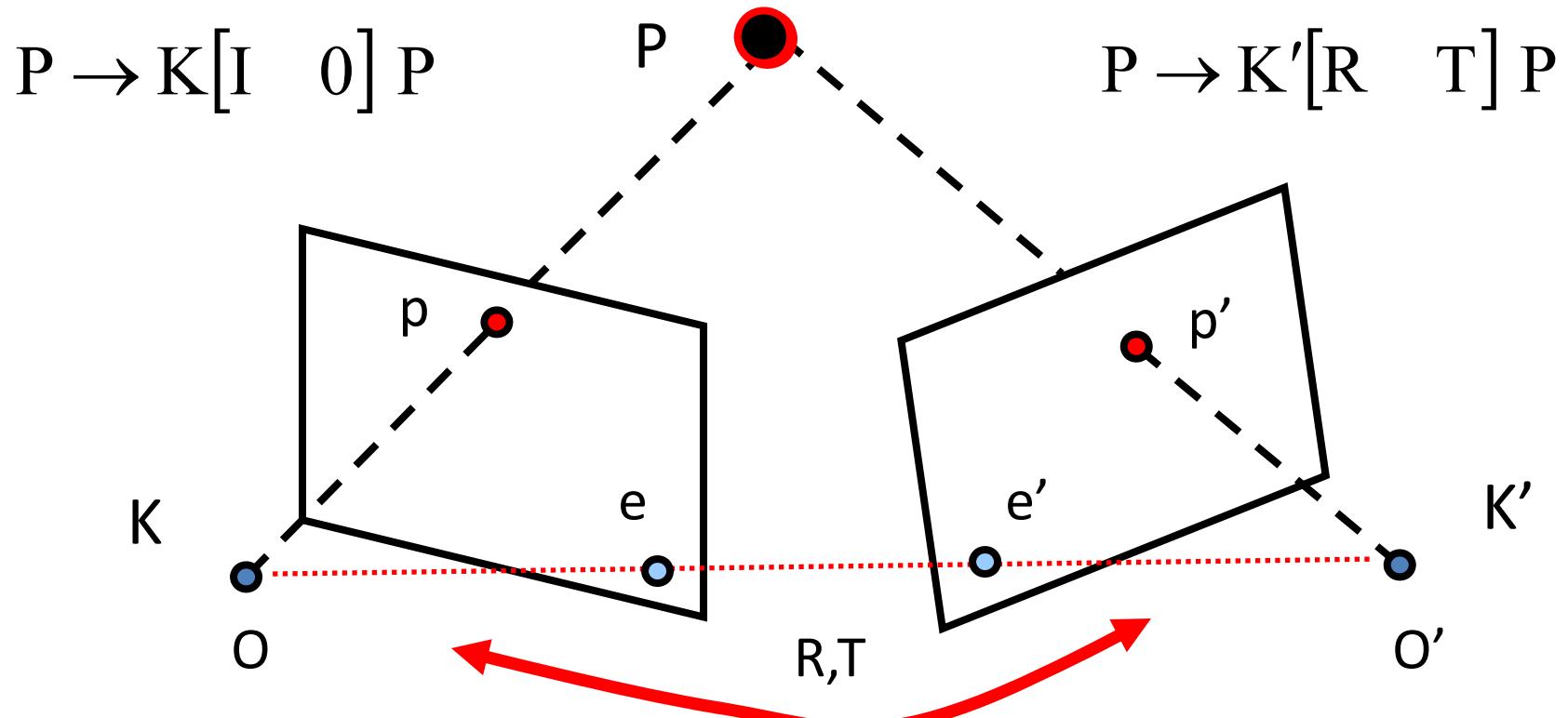
What we have learned today?

- Why is stereo useful?
- Epipolar constraints
- Essential and fundamental matrix
- Estimating F (**Problem Set 2 (Q2)**)
- Rectification

Reading:
[HZ] Chapters: 4, 9, 11
[FP] Chapters: 10

Supplementary materials

Making image planes parallel

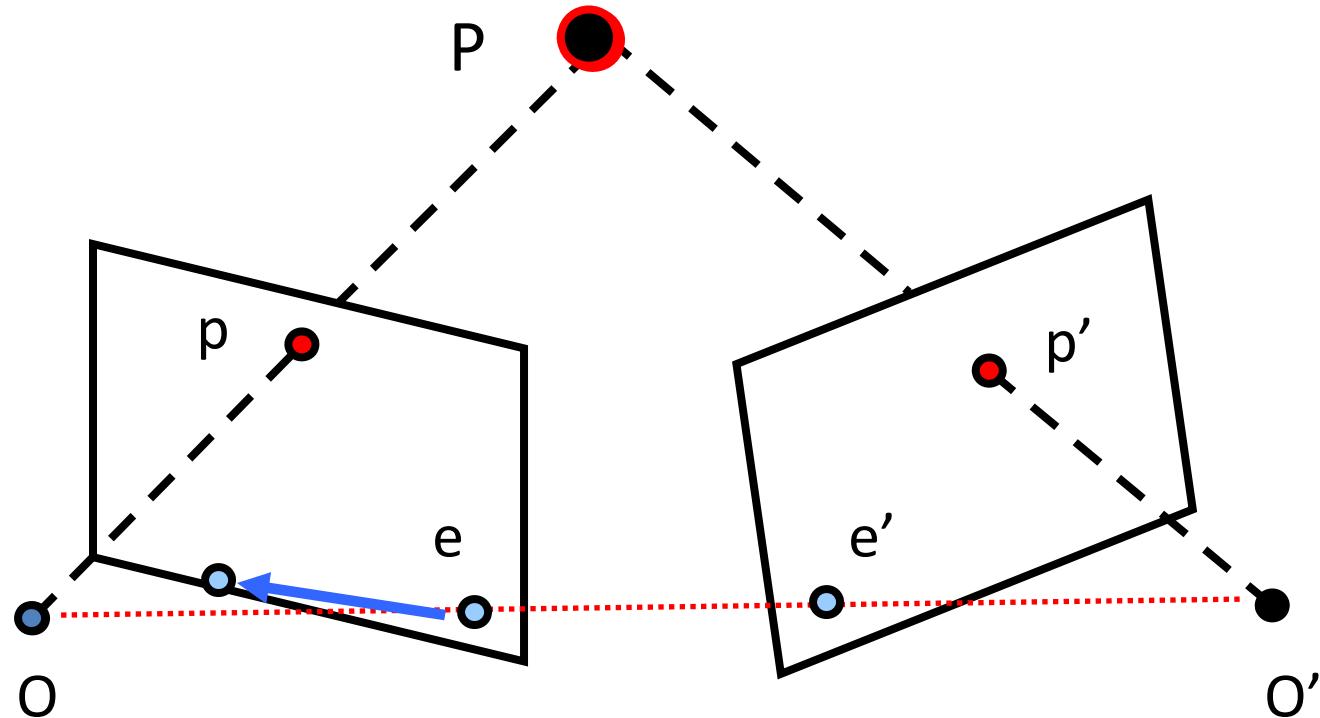


0. Compute epipoles

$$e = K R^T \quad T = [e_1 \quad e_2 \quad 1]^T$$

$$e' = K' T$$

Making image planes parallel



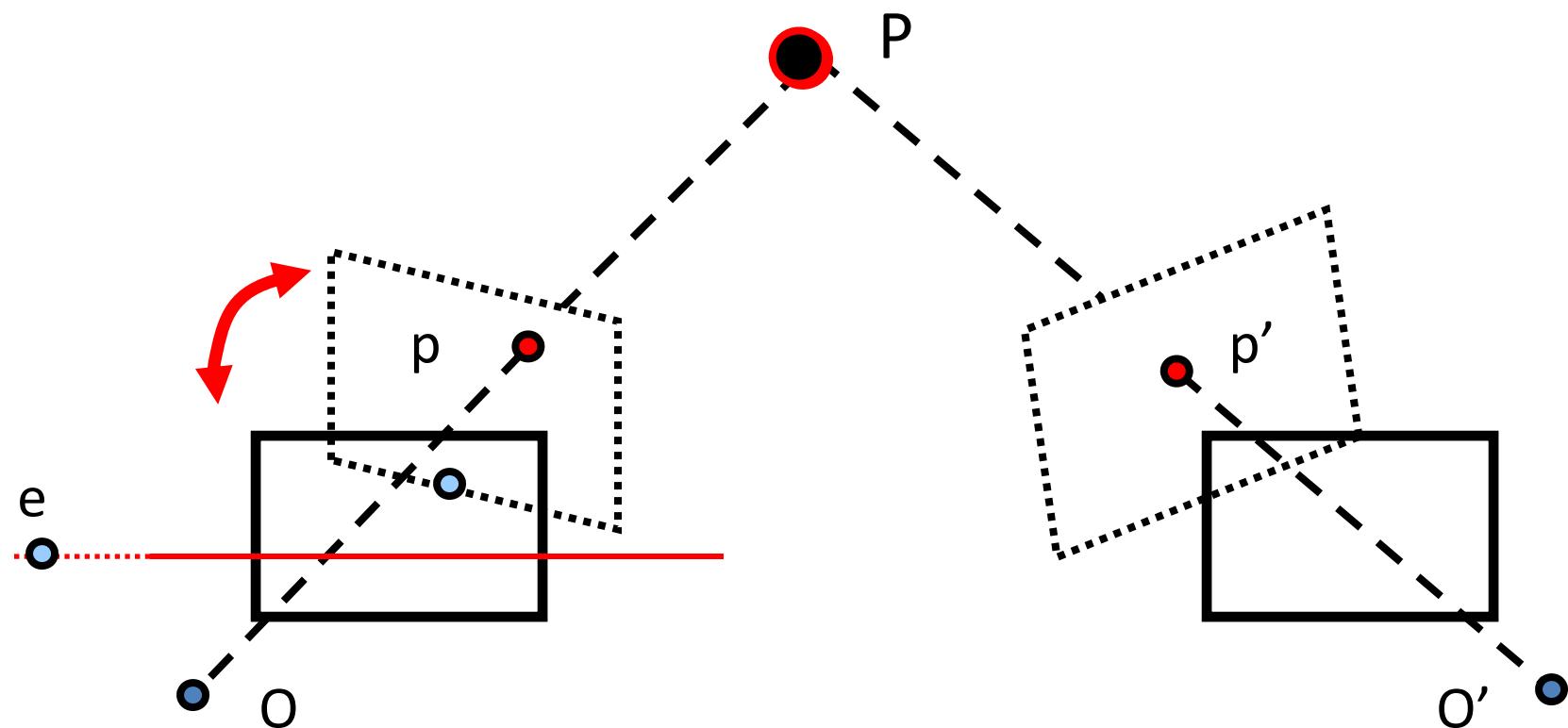
1. Map e to the x -axis at location $[1,0,1]^T$ (normalization)

$$e = [e_1 \ e_2 \ 1]^T \rightarrow$$

$$[1 \ 0 \ 1]^T$$

$$H_1 = R_H T_H$$

Making image planes parallel



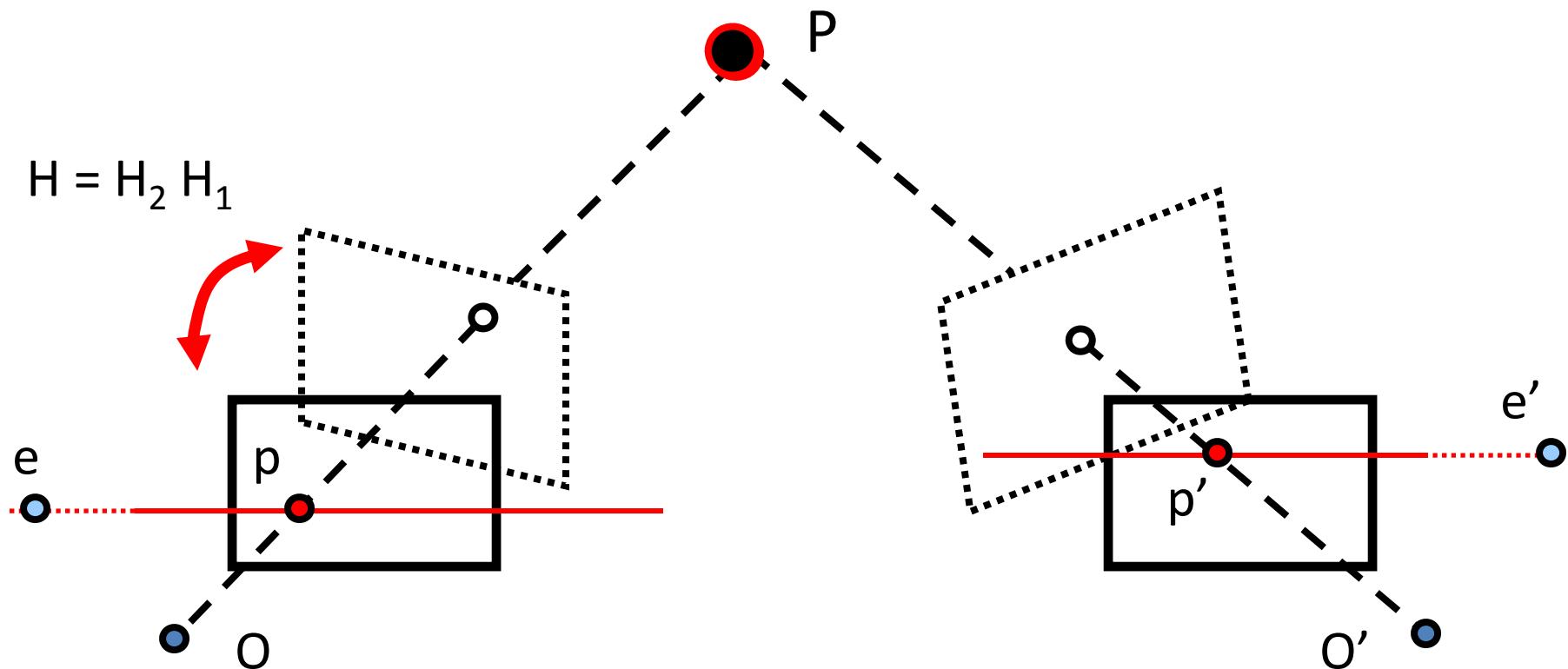
2. Send epipole to infinity:

$$e = [1 \ 0 \ 1]^T \rightarrow [1 \ 0 \ 0]^T$$

$$H_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Minimizes the distortion in a neighborhood (approximates id. mapping)

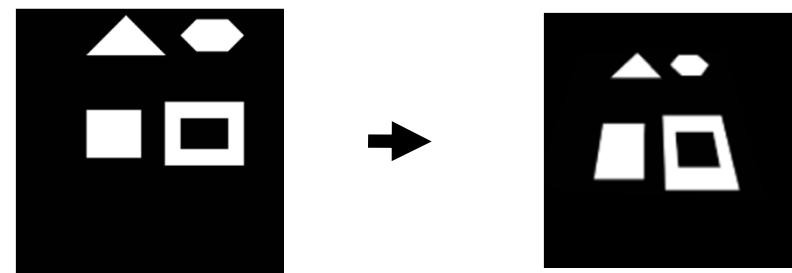
Making image planes parallel



3. Define: $H = H_2 H_1$
4. Align epipolar lines

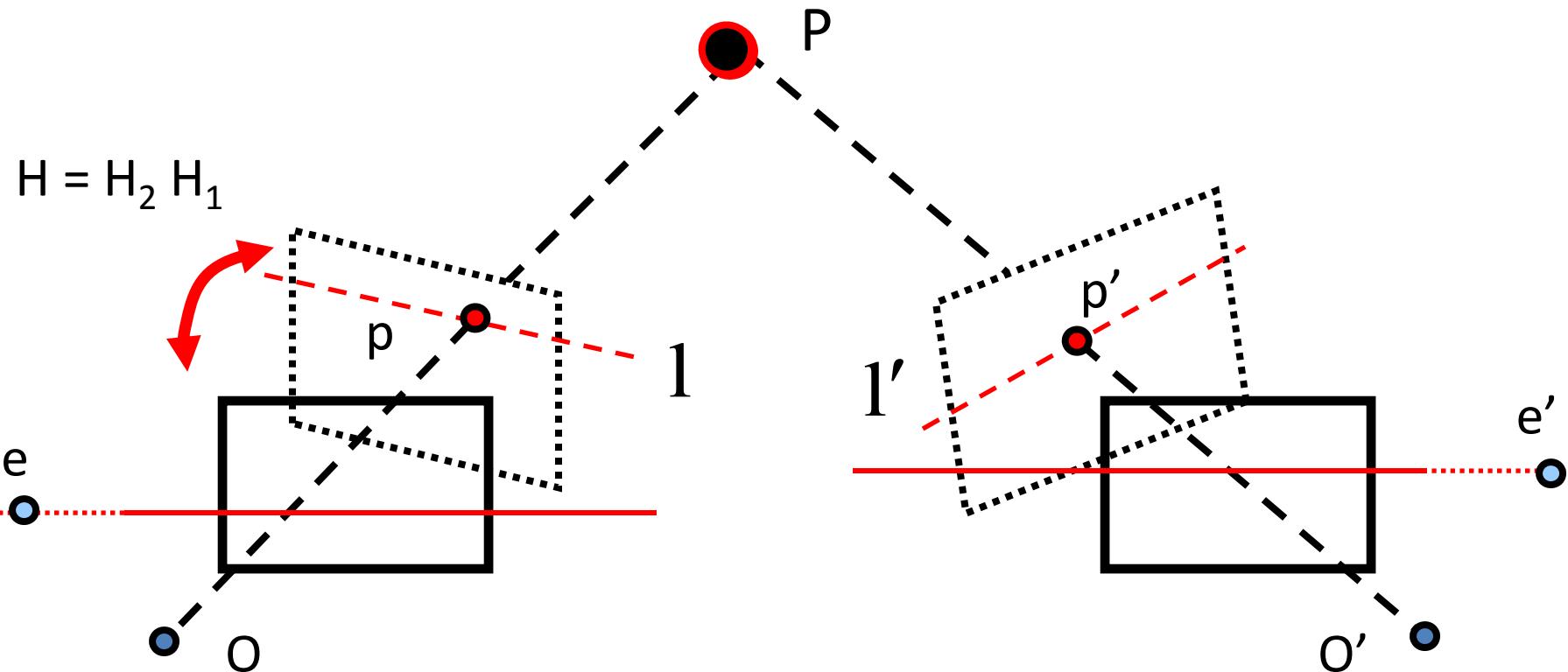
Projective transformation of a line (in 2D)

$$H = \begin{bmatrix} A & t \\ v & b \end{bmatrix}$$



$$l \rightarrow H^{-T} l$$

Making image planes parallel



3. Define: $H = H_2 H_1$
4. Align epipolar lines

[HZ] Chapters: 11 (sec. 11.12)

$$\overline{H'}^{-T} l' = \overline{H}^{-T} l$$

These are called **matched pair** of transformation