Calculus Review

Types of func^n

① Univariate single-valued $f: \mathbb{R} \to \mathbb{R}$

② multivariate single valued $f: \mathbb{R}^n \to \mathbb{R}$

③ Multivariate multivalued $f: \mathbb{R}^n \to \mathbb{R}^m$

gradient

① Univariate single valued:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

② multivariate single valued

$$\frac{\delta f(\bar{x})}{\delta x_1} = \lim_{h \to 0} \frac{f(x_1+h, x_2, \ldots x_n) - f(\bar{x})}{h}$$

$$\text{gradient} = \nabla f = \begin{bmatrix} \frac{\delta f}{\delta x_1} \\ \vdots \\ \frac{\delta f}{\delta x_n} \end{bmatrix} \Bigg\} \text{simply stack all partial der.}$$

nabla

③ Multivariate multivalued

→ given by Jacobian

can be rep^r as a stack of multivariate s.v. funcn $\rightarrow$ $f(\vec{x}) = \begin{bmatrix} f_1(\vec{x}) \\ \vdots \\ f_m(\vec{x}) \end{bmatrix}_{m\times 1}$

$$\boxed{J_f = \begin{bmatrix} \nabla f_1^T \\ \vdots \\ \nabla f_m^T \end{bmatrix}_{m\times n} = \begin{bmatrix} \frac{\delta f_1}{\delta x_1} & \cdots & \frac{\delta f_1}{\delta x_n} \\ & \vdots & \\ \frac{\delta f_m}{\delta x_1} & \cdots & \frac{\delta f_m}{\delta x_n} \end{bmatrix}}$$

$$\boxed{(J_f)_{ij} = \frac{\delta f_i}{\delta x_j}}$$

$\rightarrow$ Double derivative

$\rightarrow$ Hessian

$$H_f = \nabla^2 f$$ $\rightarrow$ multivariate single valued

$$= \nabla \begin{bmatrix} \frac{\delta f}{\delta x_1} \\ \vdots \\ \frac{\delta f}{\delta x_n} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\delta^2 f}{\delta x_1^2} & \frac{\delta^2 f}{\delta x_1 x_2} & \cdots & \frac{\delta^2 f}{\delta x_1 x_n} \\ & \vdots & & \\ \frac{\delta^2 f}{\delta x_n x_1} & \cdots & & \frac{\delta^2 f}{\delta x_n^2} \end{bmatrix}$$

$$\boxed{(H_f)_{ij} = \frac{\delta^2 f}{\delta x_i x_j}}$$ $\Rightarrow$ symmetric

→ Ex

$$f(x) = xy + z$$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} y \\ x \\ 1 \end{bmatrix}$$

NOTE

$$H_f = \nabla^2 f = \nabla \begin{bmatrix} y \\ x \\ 1 \end{bmatrix}$$

faster to compute this first.

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

→ Laplacian

$$\boxed{\Delta f = \underbrace{\nabla^2 f}_{\text{just notation}} = \sum \frac{\partial^2 f}{\partial x_i^2}}$$

just notation (see next note)

sum of all unmixed partial derivatives

Note:

$$H = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix}$$

$$L = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}$$