# FORMAL METHODS

# LECTURE V: CTL MODEL CHECKING

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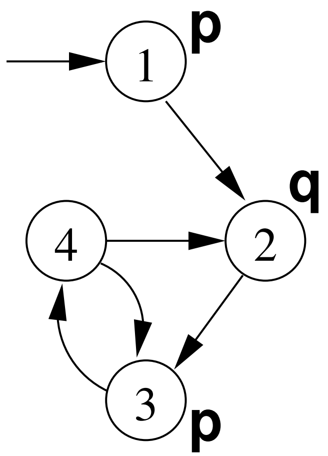
## Summary of Lecture V

- CTL Model Checking: General Ideas.
- CTL Model Checking: The Labeling Algorithm.
- Labeling Algorithm in Details.
- CTL Model Checking: Theoretical Issues.

# CTL Model Checking

CTL Model Checking is a formal verification technique s.t.

- The system is represented as a Kripke Model $\mathcal{KM}$:



- The property is expressed as a CTL formula $\varphi$, e.g.:

$$\boxed{P}\,\Box(p \Rightarrow \boxed{P}\,\Diamond q)$$

- The algorithm checks whether **all** the initial states, $s_0$, of the Kripke model satisfy the formula ($\mathcal{KM}, s_0 \models \varphi$).

## CTL M.C. Algorithm: General Ideas

The algorithm proceeds along two macro-steps:

1. Construct the set of states where the formula holds:
   $[\![\varphi]\!] := \{s \in S : \mathcal{KM}, s \models \varphi\}$
   ($[\![\varphi]\!]$ is called the denotation of $\varphi$);

2. Then compare the denotation with the set of initial states:
   $I \subseteq [\![\varphi]\!]$ ?

# CTL M.C. Algorithm: General Ideas (Cont.)

To compute $[\![\varphi]\!]$ proceed "bottom-up" on the structure of the formula, computing $[\![\varphi_i]\!]$ for each subformula $\varphi_i$ of $\varphi$.

For example, to compute $[\![\boxed{\text{P}}\ \Box(p \Rightarrow \boxed{\text{P}}\ \Diamond q)]\!]$ we need to compute:

- $[\![q]\!]$,
- $[\![\boxed{\text{P}}\ \Diamond q]\!]$,
- $[\![p]\!]$,
- $[\![p \Rightarrow \boxed{\text{P}}\ \Diamond q]\!]$,
- $[\![\boxed{\text{P}}\ \Box(p \Rightarrow \boxed{\text{P}}\ \Diamond q)]\!]$

## CTL M.C. Algorithm: General Ideas (Cont.)

To compute each $[\![\varphi_i]\!]$ for generic subformulas:

- Handle boolean operators by standard set operations;
- Handle temporal operators $\boxed{\mathrm{P}}\bigcirc$, $\langle\mathrm{P}\rangle\bigcirc$ by computing pre-images;
- Handle temporal operators $\boxed{\mathrm{P}}\Box$, $\langle\mathrm{P}\rangle\Box$, $\boxed{\mathrm{P}}\Diamond$, $\langle\mathrm{P}\rangle\Diamond$, $\boxed{\mathrm{P}}\,\mathcal{U}$ , $\langle\mathrm{P}\rangle\,\mathcal{U}$ , by applying fixpoint operators.

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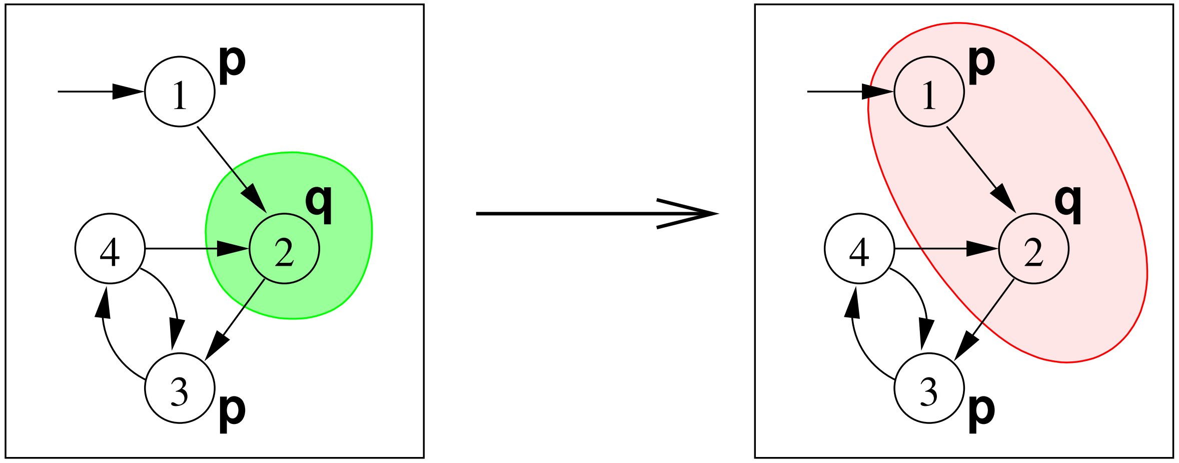
## The Labeling Algorithm: General Idea

- The Labeling Algorithm:
  - Input: Kripke Model and a CTL formula;
  - Output: set of states satisfying the formula.
- Main Idea: Label the states of the Kripke Model with the subformulas of $\varphi$ satisfied there.

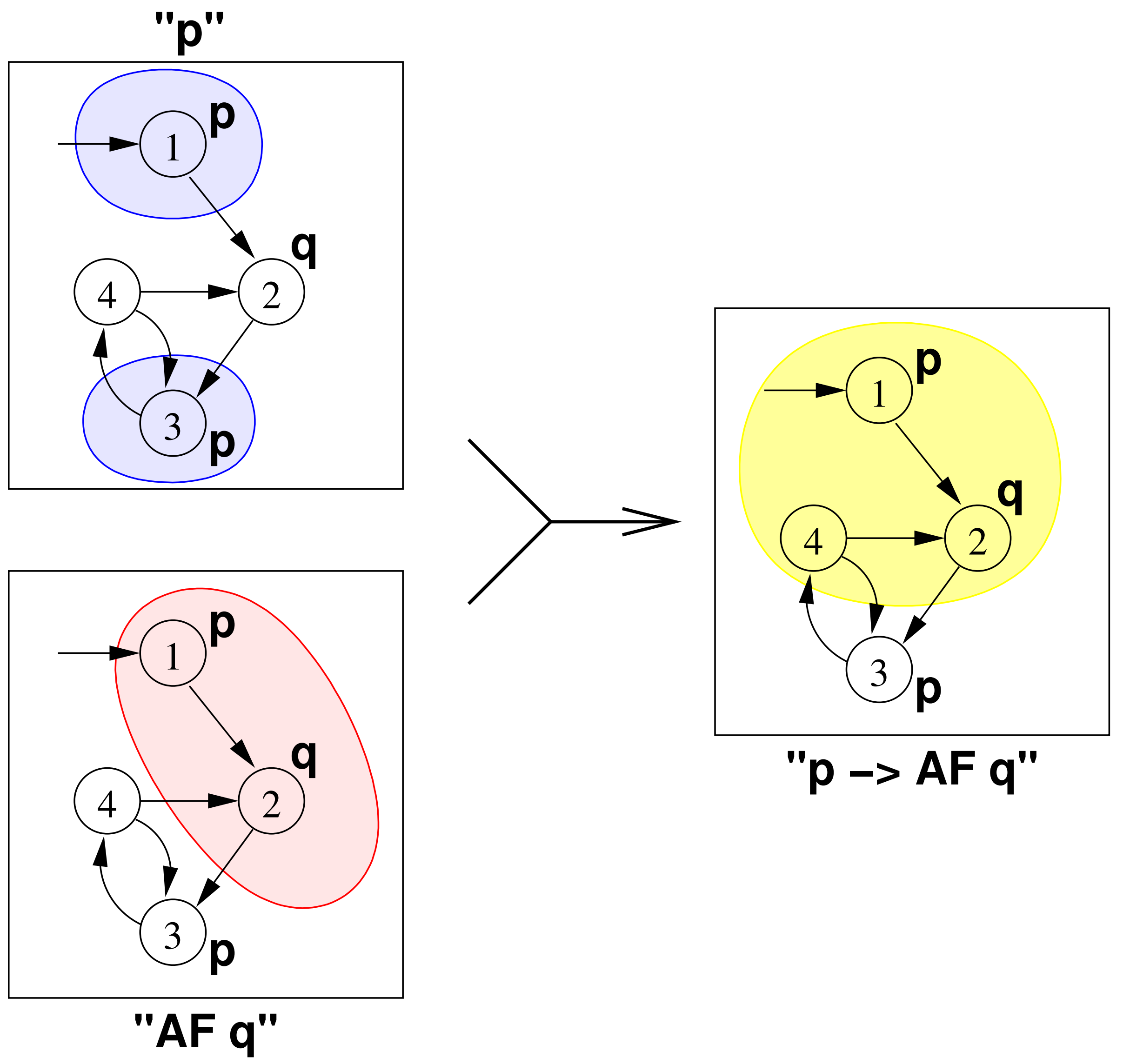
# The Labeling Algorithm: An Example



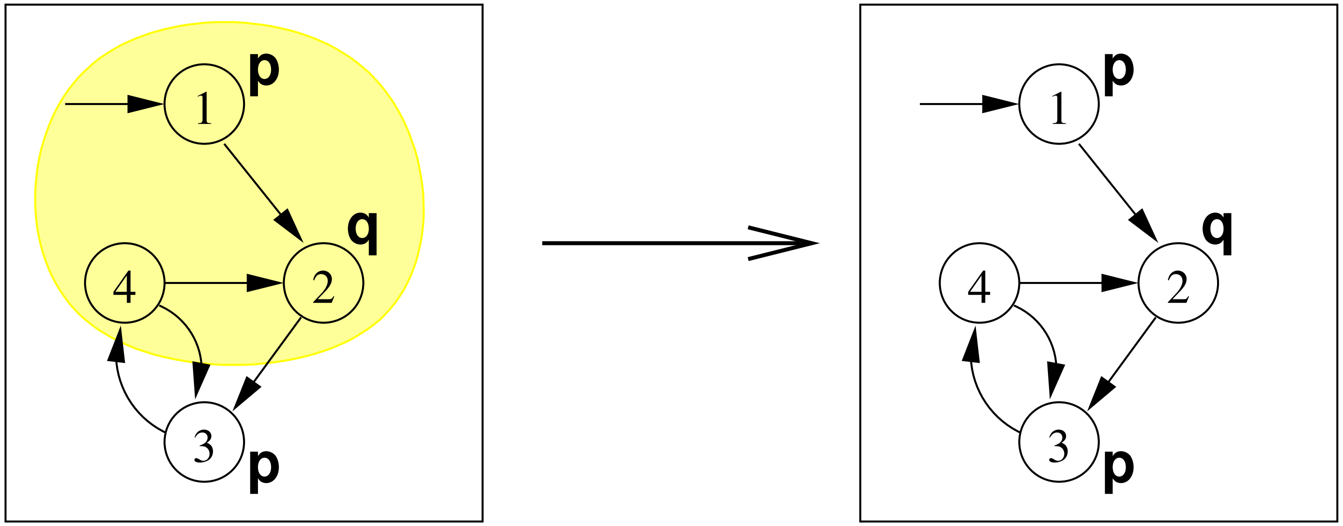
"q" "AF q"

- $\boxed{P}\Diamond q \equiv (q \vee \boxed{P}\bigcirc(\boxed{P}\Diamond q))$
- $[\![\boxed{P}\Diamond$q$]\!]$ can be computed as the union of:
  - $[\![q]\!] = \{2\}$
  - $[\![q \vee \boxed{P}\bigcirc q]\!] = \{2\} \cup \{1\} = \{1,2\}$
  - $[\![q \vee \boxed{P}\bigcirc(q \vee \boxed{P}\bigcirc q)]\!] = \{2\} \cup \{1\} = \{1,2\}$ (fixpoint).

## The Labeling Algorithm: An Example (Cont.)



# The Labeling Algorithm: An Example (Cont.)



"p –> AF q" "AG(p –> AF q)"

- ▷ $\boxed{P}\ \Box\varphi \equiv (\varphi \wedge \boxed{P} \bigcirc (\boxed{P}\ \Box\varphi))$
- ▷ $[[\boxed{P}\ \Box\varphi]]$ can be computed as the intersection of:
  - $[[\varphi]] = \{1,2,4\}$
  - $[[\varphi \wedge \boxed{P} \bigcirc \varphi]] = \{1,2,4\} \cap \{1,3\} = \{1\}$
  - $[[\varphi \wedge \boxed{P} \bigcirc (\varphi \wedge \boxed{P} \bigcirc \varphi)]] = \{1,2,4\} \cap \{\} = \{\}$ (fixpoint)

## The Labeling Algorithm: An Example (Cont.)

- ▷ The set of states where the formula holds is empty, thus:
  - The initial state does not satisfy the property;
  - $\mathcal{KM} \not\models \boxed{\text{P}} \Box (p \Rightarrow \boxed{\text{P}} \Diamond q)$.

- ▷ Counterexample: A lazo-shaped path: $1, 2, \{3, 4\}^{\omega}$ (satisfying $\Diamond\!\!\!\!\!\text{P}\ \Diamond (p \wedge \Diamond\!\!\!\!\!\text{P}\ \Box \neg q))$

## Summary

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## The Labeling Algorithm: General Schema

▷ Assume $\varphi$ written in terms of $\neg$, $\wedge$, $\langle P\rangle\bigcirc$, $\langle P\rangle\mathcal{U}$, $\langle P\rangle\Box$ – minimal set of CTL operators

▷ The Labeling algorithm takes a CTL formula and a Kripke Model as input and returns the set of states satisfying the formula (i.e., the *denotation* of $\varphi$):

1. For every $\varphi_i \in Sub(\varphi)$, find $[\![\varphi_i]\!]$;
2. Compute $[\![\varphi]\!]$ starting from $[\![\varphi_i]\!]$;
3. Check if $I \subseteq [\![\varphi]\!]$.

▷ Subformulas $Sub(\varphi)$ of $\varphi$ are checked bottom-up

▷ To compute each $[\![\varphi_i]\!]$: if the main operator of $\varphi_i$ is a

- *Boolean Operator*: apply standard set operations;
- *Temporal Operator*: apply recursive rules until a fixpoint is reached.

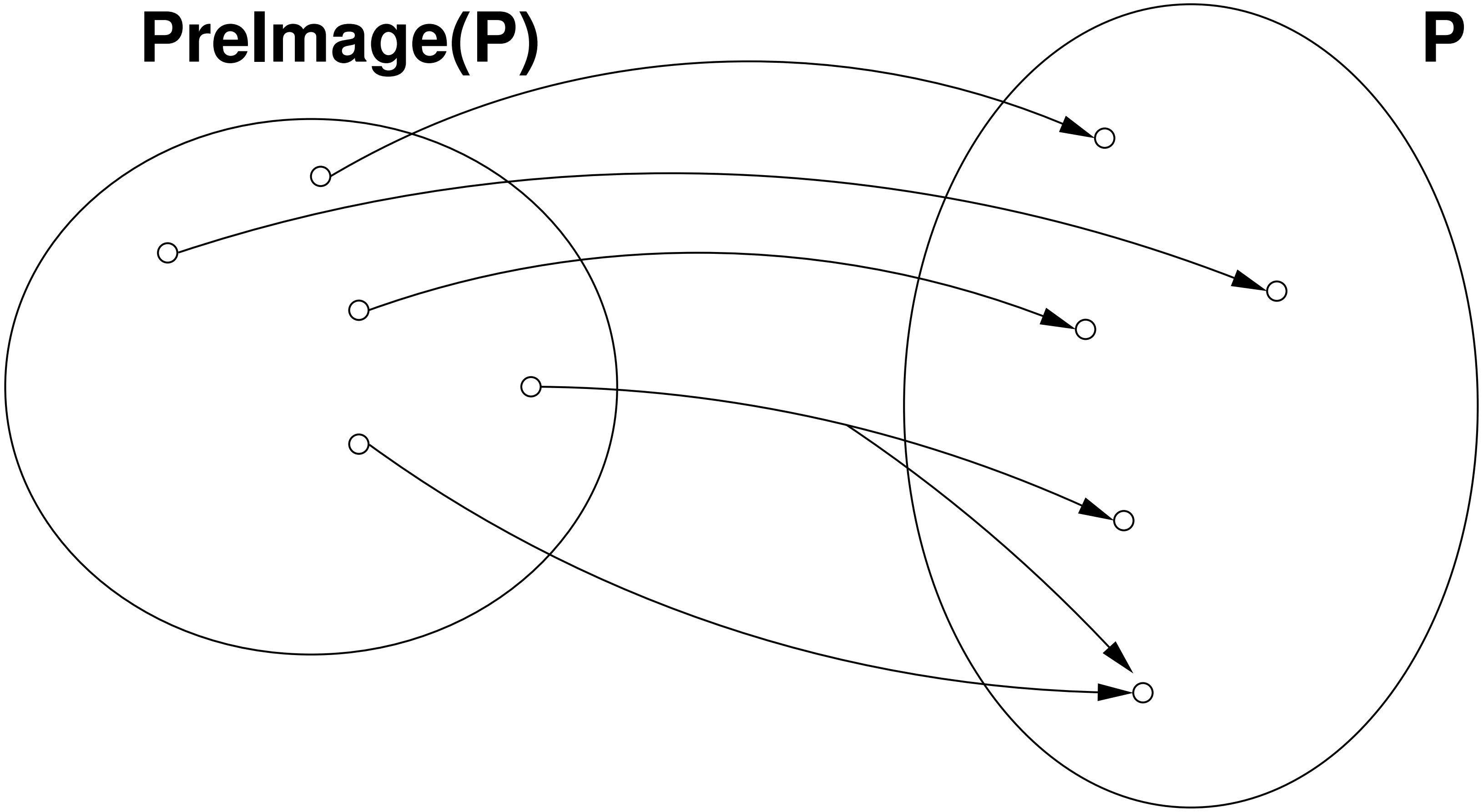
## Denotation of Formulas: The Boolean Case

Let $\mathcal{KM} = \langle S, I, R, L, \Sigma \rangle$ be a Kripke Model.

$$
\begin{array}{lcl}
[\![false]\!] & = & \{\} \\
[\![true]\!] & = & S \\
[\![p]\!] & = & \{s \mid p \in L(s)\} \\
[\![\neg\varphi_1]\!] & = & S \setminus [\![\varphi_1]\!] \\
[\![\varphi_1 \wedge \varphi_2]\!] & = & [\![\varphi_1]\!] \cap [\![\varphi_2]\!]
\end{array}
$$

# Denotation of Formulas: The $\Diamond_P \bigcirc$ Case

- $[\![\Diamond_P \bigcirc \varphi]\!] = \{s \in S \mid \exists s'.\langle s, s'\rangle \in R \text{ and } s' \in [\![\varphi]\!]\}$
- $[\![\Diamond_P \bigcirc \varphi]\!]$ is said to be the Pre-image of $[\![\varphi]\!]$ $(\textsc{Pre}([\![\varphi]\!]))$.
- Key step of every CTL M.C. operation.



## Denotation of Formulas: The $\Diamond_P\,\Box$ Case

- From the semantics of the $\Box$ temporal operator:

$$\Box\varphi \equiv \varphi \wedge \bigcirc(\Box\varphi)$$

- Then, the following equivalence holds:

$$\Diamond_P\,\Box\varphi \equiv \varphi \wedge \Diamond_P \bigcirc(\Diamond_P\,\Box\varphi)$$

- To compute $[\![\Diamond_P\,\Box\varphi]\!]$ we can apply the following recursive definition:

$$[\![\Diamond_P\,\Box\varphi]\!] = [\![\varphi]\!] \cap \textsc{Pre}([\![\Diamond_P\,\Box\varphi]\!])$$

## Denotation of Formulas: The $\overset{\mathrm{P}}{\Diamond}\Box$Case (Cont

- We can compute $X := [\![\overset{\mathrm{P}}{\Diamond}\Box\varphi]\!]$ inductively as follows:

$$\begin{array}{lcl} X_1 & := & [\![\varphi]\!] \\ X_2 & := & X_1 \cap \textsc{Pre}(X_1) \\ \ldots & & \\ X_{j+1} & := & X_j \cap \textsc{Pre}(X_j) \end{array}$$

- When $X_n = X_{n+1}$ we reach a fixpoint and we stop.
- **Termination.** Since $X_{j+1} \subseteq X_j$ for every $j \geq 0$, thus a fixed point always exists (Knaster-Tarski's theorem).

## Denotation of Formulas: The $\Diamond_{P}\ \mathcal{U}$ Case

- From the semantics of the $\mathcal{U}$ temporal operator:

$$\varphi\,\mathcal{U}\,\psi \equiv \psi \vee (\varphi \wedge \bigcirc(\varphi\,\mathcal{U}\,\psi))$$

- Then, the following equivalence holds:

$$\Diamond_{P}(\varphi\,\mathcal{U}\,\psi) \equiv \psi \vee (\varphi \wedge \Diamond_{P}\bigcirc\Diamond_{P}(\varphi\,\mathcal{U}\,\psi))$$

- To compute $[\![\Diamond_{P}(\varphi\,\mathcal{U}\,\psi)]\!]$ we can apply the following recursive definition:

$$[\![\Diamond_{P}(\varphi\,\mathcal{U}\,\psi)]\!] = [\![\psi]\!] \cup ([\![\varphi]\!] \cap \textsc{Pre}([\![\Diamond_{P}(\varphi\,\mathcal{U}\,\psi)]\!]))$$

## Denotation of Formulas: The $\Diamond_{\mathrm{P}}\ \mathcal{U}$ Case (Cont

- We can compute $X := [\![\Diamond_{\mathrm{P}} (\varphi\, \mathcal{U}\, \psi)]\!]$ inductively as follows:

$$
\begin{aligned}
X_1 &:= [\![\psi]\!] \\
X_2 &:= X_1 \cup ([\![\varphi]\!] \cap \textsc{Pre}(X_1)) \\
&\ldots \\
X_{j+1} &:= X_j \cup ([\![\varphi]\!] \cap \textsc{Pre}(X_j))
\end{aligned}
$$

- When $X_n = X_{n+1}$ we reach a fixpoint and we stop.
- **Termination.** Since $X_{j+1} \supseteq X_j$ for every $j \geq 0$, thus a fixed point always exists (Knaster-Tarski's theorem).

## The Pseudo-Code

We assume the Kripke Model to be a global variable:

```
FUNCTION Label($\varphi$) {
    case $\varphi$ of
        $true$:                       return $S$;
        $false$:                      return $\{\}$;
        an atom $p$:                  return $\{s \in S \mid p \in L(s)\}$;
        $\neg\varphi_1$:              return $S\setminus$Label($\varphi_1$);
        $\varphi_1 \wedge \varphi_2$: return Label($\varphi_1$)$\cap$Label($\varphi_2$);
        $\langle P \rangle \bigcirc \varphi_1$:   return PRE(Label($\varphi_1$));
        $\langle P \rangle (\varphi_1\, \mathcal{U}\, \varphi_2)$: return Label_EU(Label($\varphi_1$),Label($\varphi_2$));
        $\langle P \rangle \Box \varphi_1$:       return Label_EG(Label($\varphi_1$));
    end case
}
```

## PreImage

$$[\![\langle P \rangle \bigcirc \varphi]\!] = \textsc{Pre}([\![\varphi]\!]) = \{s \in S \mid \exists s'. \langle s, s' \rangle \in R \text{ and } s' \in [\![\varphi]\!]\}$$

```
FUNCTION PRE($[\![\varphi]\!]$){
    var $X$;
    $X := \{\}$;
    for each $s' \in [\![\varphi]\!]$ do
        for each $s \in S$ do
            if $\langle s, s' \rangle \in R$ then
                $X := X \cup \{s\}$;
    return $X$
}
```

## Label_EG

$$[\![ \Diamond_P \Box \varphi ]\!] = [\![ \varphi ]\!] \cap \textsc{Pre}([\![ \Diamond_P \Box \varphi ]\!])$$

```
FUNCTION LABEL_EG($[\![\varphi]\!]$){
    **var** $X, OLD\text{-}X$;
    $X := [\![\varphi]\!]$;
    $OLD\text{-}X := \emptyset$;
    **while** $X \neq OLD\text{-}X$
    **begin**
        $OLD\text{-}X := X$;
        $X := X \cap \textsc{Pre}(X)$
    **end**
    **return** $X$
}
```

## Label_EU

$$[\![ \langle P \rangle (\varphi\, \mathcal{U}\, \psi)]\!] = [\![\psi]\!] \cup ([\![\varphi]\!] \cap \textsc{Pre}([\![ \langle P \rangle (\varphi\, \mathcal{U}\, \psi)]\!]))$$

```
FUNCTION LABEL_EU($[\![\varphi]\!]$,$[\![\psi]\!]$){
    **var** $X, OLD\text{-}X$;
    $X := [\![\psi]\!]$;
    $OLD\text{-}X := S$;
    **while** $X \neq OLD\text{-}X$
    **begin**
        $OLD\text{-}X := X$;
        $X := X \cup ([\![\varphi]\!] \cap \textsc{Pre}(X))$
    **end**
    **return** $X$
}
```

## Summary

- CTL Model Checking: General Ideas.
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## Correctness and Termination

- The Labeling algorithm works recursively on the structure φ.
- For most of the logical constructors the algorithm does the correct things according to the semantics of CTL.
- To prove that the algorithm is *Correct* and *Terminating* we need to prove the correctness and termination of both $\langle P \rangle \Box$ and $\langle P \rangle \mathcal{U}$ operators.

# Monotone Functions and Fixpoints

**Definition.** Let $S$ be a set and $F$ a function, $F : 2^S \rightarrow 2^S$, then:

1. $F$ is monotone iff $X \subseteq Y$ then $F(X) \subseteq F(Y)$;
2. A subset $X$ of $S$ is called a fixpoint of $F$ iff $F(X) = X$;
3. $X$ is a least fixpoint (LFP) of $F$, written $\mu X.F(X)$, iff, for every other fixpoint $Y$ of $F$, $X \subseteq Y$
4. $X$ is a greatest fixpoint (GFP) of $F$, written $\nu X.F(X)$, iff, for every other fixpoint $Y$ of $F$, $Y \subseteq X$

**Example.** Let $S = \{s_0, s_1\}$ and $F(X) = X \cup \{s_0\}$.

# Knaster-Tarski Theorem

**Notation:** $F^i(X)$ means applying $F$ $i$-times, i.e., $F(F(\ldots F(X)\ldots))$.

**Theorem[Knaster-Tarski].** Let $S$ be a finite set with $n+1$ elements. If $F : 2^S \rightarrow 2^S$ is a monotone function then:

1. $\mu X.F(X) \equiv F^{n+1}(\emptyset)$;
2. $\nu X.F(X) \equiv F^{n+1}(S)$.

**Proof.** (See the textbook “Logic in CS” pg.241)

## Correctness and Termination: $\Diamond_{P}\,\Box$ Case

The function LABEL_EG computes:

$$[\![\Diamond_{P}\,\Box\varphi]\!] = [\![\varphi]\!] \cap \textsc{Pre}([\![\Diamond_{P}\,\Box\varphi]\!])$$

applying the semantic equivalence:

$$\Diamond_{P}\,\Box\varphi \equiv \varphi \wedge \Diamond_{P}\bigcirc(\Diamond_{P}\,\Box\varphi)$$

Thus, $[\![\Diamond_{P}\,\Box\varphi]\!]$ is the fixpoint of the function:

$$F(X) = [\![\varphi]\!] \cap \textsc{Pre}(X)$$

## Correctness and Termination: $\Diamond_{\mathrm{P}}\,\Box$Case (Con

**Theorem.** Let $F(X) = [\![\varphi]\!] \cap \textsc{Pre}(X)$, and let $S$ have $n+1$ elements. Then:

1. $F$ is monotone;
2. $[\![\Diamond_{\mathrm{P}}\,\Box\varphi]\!]$ is the greatest fixpoint of $F$.

**Proof.** (See the textbook “Logic in CS” pg.242)

## Correctness and Termination: $\Diamond_{P}\ \mathcal{U}$ Case

The function LABEL_EU computes:

$$[\![\Diamond_{P}(\varphi\,\mathcal{U}\,\psi)]\!] = [\![\psi]\!] \cup ([\![\varphi]\!] \cap \textsc{Pre}([\![\Diamond_{P}(\varphi\,\mathcal{U}\,\psi)]\!]))$$

applying the semantic equivalence:

$$\Diamond_{P}(\varphi\,\mathcal{U}\,\psi) \equiv \psi \vee (\varphi \wedge \Diamond_{P}\bigcirc\Diamond_{P}(\varphi\,\mathcal{U}\,\psi))$$

Thus, $[\![\Diamond_{P}(\varphi\,\mathcal{U}\,\psi)]\!]$ is the fixpoint of the function:

$$F(X) = [\![\psi]\!] \cup ([\![\varphi]\!] \cap \textsc{Pre}(X))$$

## Correctness and Termination: $\Diamond_{P}\ \mathcal{U}$ Case (Con

**Theorem.** Let $F(X) = [\![\psi]\!] \cup ([\![\varphi]\!] \cap \textsc{Pre}(X))$, and let $S$ have $n+1$ elements. Then:

1. $F$ is monotone;
2. $[\![\Diamond_{P}(\varphi\,\mathcal{U}\,\psi)]\!]$ is the least fixpoint of $F$.

**Proof.** (See the textbook “Logic in CS” pg.243)

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