Notes on Petri Nets and Attack Trees

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Abstract

Cybersecurity professionals often use attack trees to visualize the various possible paths of attack on a resource. Such visualizations provide important tools for communication and threat analysis. However, they can quickly become complex and unwieldy, and must rely on humans for error-checking and the tracing of possible paths. We are working to translate attack trees into Petri nets, mathematical graphical models that can be analyzed and verified much more quickly and accurately. In their original form, Petri nets are constructed of places and transitions between places, with a flow relation indicating how resources can flow through the net, or what sequence of steps is needed to get from one point to another in the net. We have begun development of a chainable Petri net, which facilitates the construction of complex nets using simple logical operations already found in attack trees. Once constructed, these nets can be analyzed using the powerful tools native to the Petri net model. This will enable cybersecurity professionals to check two nets for equivalence or hierarchy, incorporate one net into another, compute possibilities under given circumstances, and complete many other tasks with greater speed and accuracy than is possible under the current model.

1 Notes on Attack Trees and Petri Nets

Definition 1. *A **finite multiset** over a set S can be defined as a formal sum denoted by $n_1 a_1 \oplus \cdots \oplus n_i a_i$, where $n_1, \ldots, n_i \in \mathbb{N}$, $a_1, \ldots, a_i \in S$, and na denotes that a occurs n times in the multiset, such that the following properties hold:*

- $na_i \oplus ma_j = ma_j \oplus na_i$,
- $na \oplus ma = (m + n)a$,
- $na \oplus 0 = na = 0 \oplus na$.

Definition 2. *The **free commutative monoid** generated by the set A is the set of all finite multisets drawn from A, where the monoidal operation is the formal sum of multisets, and the monoidal unit is $\emptyset$.*

Definition 3. *A **generalized graph**, $(V, T, \mathsf{src}, \mathsf{tar}, \_^*, I, \otimes)$, consists of the following structure:*

- *A set of nodes V,*
- *A set of edges T,*

- *A functor* $\_^* : \mathsf{Set} \longrightarrow \mathsf{Set}$*, where* $(V^*, I, \otimes)$ *is a monoid,*
- *A source function* $\mathsf{src} : T \longrightarrow V^*$*,*
- *A target function* $\mathsf{tar} : T \longrightarrow V^*$*.*

Definition 4. *A* ***graph morphism****,* $h : G \longrightarrow G'$*, between graphs is a pair* (f, g) *where:*

- $f : T \longrightarrow T'$ *is a function,*
- $g : V \longrightarrow V'$ *is a function.*

Definition 5. *A* ***monoid homomorphism*** *between two monoids* $(M_1, \oplus_1, 0_1)$ *and* $(M_2, \oplus_2, 0_2)$ *is a function* $f : M_1 \longrightarrow M_2$ *such that:*

- $f(a_1 \oplus_1 a_2) = f(a_1) \oplus f(a_2)$
- $f(0_1) = 0_2$

Definition 6. *A* ***generalized graph morphism*** *between two generalized graphs* $(V_1, T_1, \mathsf{src}_1, \mathsf{tar}_1, \_^{*_1}, I_1, \otimes_1)$ *and* $(V_2, T_2, \mathsf{src}_2, \mathsf{tar}_2, \_^{*_2}, I_2, \otimes_2)$ *is a graph morphism* (f, g) *where* $f : T_1 \longrightarrow T_2$ *is a function and* $g : V_1^{*_1} \longrightarrow V_2^{*_2}$ *is a monoid homomorphism.*

Definition 7. *Original defn: A* ***Petri net****,* (P, T, F, M_0)*, consists of the following structure:*

- *A finite set of places P,*
- *A finite set of transitions T that is disjoint from P,*
- *A causal dependency relation* $F : (P \times T) + (T \times P) \longrightarrow \mathbb{N}^+$*,*
- *An initial marking* M_0*.*

Definition 8. *A* ***Petri net****,* $(S, T, \mathsf{src}, \mathsf{tar}, \_^{\oplus}, \emptyset, \oplus)$*, is a generalized graph. We call the elements of the set S places and the elements of T transitions. Furthermore, we call the generalized graph homomorphisms between Petri nets* ***Petri net homomorphisms****.*

We will denote a Petri net $(S, T, \mathsf{src}, \mathsf{tar}, \_^{\oplus}, \emptyset, \oplus)$ by the pair $(S^{\oplus}, T)$ for readability. Thus, take as objects Petri nets $(S^{\oplus}, T)$ and as morphisms Petri net morphisms. This defines a category Petri.

Lemma 9. *Suppose we have two Petri nets,* $(S_1^{\oplus}, T_1)$ *and* $(S_2^{\oplus}, T_2)$*.*

$$(S_1^{\oplus} \times S_2^{\oplus}) \cong (S_1 + S_2)^{\oplus}$$

Proof. To show this, we need to define two homomorphisms between them, which we will call F and G.

It is important to note that the $\oplus$ operation is commutative. This commutativity allows the elements of the resulting free commutative monoid (FCM) to be rearranged, which will help us in our proof by allowing FCMs to be sorted in the following way:

Consider an arbitrary $s \in (S_1 + S_2)^{\oplus}$. Since the generator is the disjoint union of S_1 and S_2, the elements of s are drawn from those two sets, but are not grouped according to their origin set. Using commutativity, we can move all $y_i \in S_1$ to the front, and all

$z_j \in S_2$ to the end. Then the form of s is $(\oplus_i y_i) \oplus (\oplus_j z_j)$, where the elements are grouped according to their origin set. Any element of $(S_1 + S_2)^{\oplus}$ can be similarly sorted. For the purposes of this proof, assume that all such elements are so sorted.

Now, let us define the function $F : (S_1 + S_2)^{\oplus} \longrightarrow (S_1^{\oplus} \times S_2^{\oplus})$.
Let k be an arbitrary element of $(S_1 + S_2)^{\oplus}$. We know that $k \in (S_1 + S_2)^{\oplus}$ must have the form $(\oplus_i y_i) \oplus (\oplus_j z_j)$, where $y_i \in S_1$ and $z_i \in S_2$ as described above. Furthermore, since the + operator is disjoint union, each element includes a marker indicating its origin set, which we can use to locate the boundary between elements of S_1 and elements of S_2. Therefore we can define F as follows:

$$F((\oplus_i y_i \in S_1) \oplus (\oplus_j z_j \in S_2)) = ((\oplus_i y_i \in S_1), (\oplus_j z_j \in S_2))$$

$$F((S_1, \oplus) \oplus (S_2, \oplus)) = ((S_1, \oplus), (S_2, \oplus))$$

This allows an element of $(S_1 + S_2)^{\oplus}$ to be matched with an element of $(S_1^{\oplus} \times S_2^{\oplus})$, since the generator of $(S_1 + S_2)^{\oplus}$ contains the generator for both $S_1^{\oplus}$ and $S_2^{\oplus}$.

Another important feature of FCMs arises from lifting the coproducts of sets to FCMs. For the coproduct of sets, we know that there exist the following functions:

$$inj_1 : S_1 \longrightarrow (S_1 + S_2)$$

$$inj_2 : S_2 \longrightarrow (S_1 + S_2)$$

Likewise, for the coproduct of FCMs, there are the functions:

$$inj_1^{\oplus} : S_1^{\oplus} \longrightarrow (S_1 + S_2)^{\oplus}$$

$$inj_1^{\oplus}(\oplus_i n_i x_i) = \oplus_i n_i (inj_1 x_i)$$

$$inj_2^{\oplus} : S_2^{\oplus} \longrightarrow (S_1 + S_2)^{\oplus}$$

$$inj_2^{\oplus}(\oplus_i n_i x_i) = \oplus_i n_i (inj_2 x_i)$$

Using this property, let us now define $G : (S_1^{\oplus} \times S_2^{\oplus}) \longrightarrow (S_1 + S_2)^{\oplus}$.
Let k be an arbitrary element of $(S_1^{\oplus} \times S_2^{\oplus})$. We know k must have the form (M_1, M_2) where M_i is a multiset of the form $(n_1 z_1 \oplus ... \oplus n_b z_b)$, and all $z \in S_i$ for $i \in \{1, 2\}$ (all n being natural number counters for the multiset).
Therefore we define the function G as follows:

$$G(M_1, M_2) \quad = \quad (inj_1^{\oplus} M_1) \oplus (inj_2^{\oplus} M_2)$$

By this transformation, we can match an element of $(S_1^{\oplus} \times S_2^{\oplus})$ with an element of $(S_1 + S_2)^{\oplus}$, since by the nature of coproducts, any s $\in S_1^{\oplus} \in (S_1 + S_2)^{\oplus}$, and similarly any s $\in S_2^{\oplus} \in (S_1 + S_2)^{\oplus}$.

Now that we have defined a homomorphism in both directions, we must prove that $F;G = Id$ and $G;F = Id$. Take an arbitrary multiset M of the form $(n_1z_1 \oplus ... \oplus n_bz_b)$, with some elements drawn from a set S_a and some drawn from S_b and each $n_i \in \mathbb{N}$. Sorting this multiset will yield the form $(\oplus_a n_a y_a) \oplus (\oplus_b n_b z_b)$, where $y_a \in S_a$ and $z_b \in S_b$. Putting this into the functions, we get:

$$\begin{aligned} G(F((\oplus_a n_a y_a) \oplus (\oplus_b n_b z_b))) &= G((\oplus_a n_a y_a), (\oplus_b n_b z_b)) \\ &= (\oplus_a n_a y_a) \oplus (\oplus_b n_b z_b) \end{aligned}$$

Thus, $F;G = Id$.

Now let us check for identity in the opposite direction, proving G;F = Id.
Take arbitrary sets S_a and S_b. Generate the FCM of each and then their product: $(S_a^{\oplus} \times S_b^{\oplus})$. The result will have the form $((\oplus_a n_a y_a), (\oplus_b n_b z_b))$, where $y_a \in S_a$ and $z_b \in S_b$. Putting this into the functions, we get:

$$\begin{aligned} G((\oplus_a n_a y_a), (\oplus_b n_b z_b)) &= ((\oplus_a n_a y_a) \oplus (\oplus_b n_b z_b)) \\ F((\oplus_a n_a y_a) \oplus (\oplus_b n_b z_b)) &= ((\oplus_a n_a y_a), (\oplus_b n_b z_b)) \end{aligned}$$

Thus, G;F = Id. Since $(S_1^{\oplus} \times S_2^{\oplus}) \leftrightarrow (S_1 + S_2)^{\oplus}$, we can conclude that $(S_1^{\oplus} \times S_2^{\oplus}) \cong (S_1 + S_2)^{\oplus}$.

□

Definition 10. *A **chainable petri net**, $(S^{\oplus}, T, i, start, f, end)$ is a Petri net with the following additional features:*

- *An element of S, $i \in S$, called the initial place. This is the starting point of the net.*
- *An element of T, $start \in T$, called the starting transition of the net. If the initial marking is $(s_1 \oplus s_2 ... \oplus s_n)$, then $\mathsf{src}(start) = i$, and $\mathsf{tar}(start) = (s_1 \oplus s_2 ... \oplus s_n)$, with elements not duplicated: $s_j \neq s_k$ when $j \neq k$.*
- *An element of S, $f \in S$, called the final place. This marks the completion of the net.*
- *An element of T, $end \in T$, that is the end transition of the net. If the final marking is $(s_1 \oplus s_2 ... \oplus s_n)$, $\mathsf{src}(end) = (s_1 \oplus s_2 ... \oplus s_n)$, and $\mathsf{tar}(end) = f$.*
- *Morphisms for chainable nets must preserve both start and end, and their associated places i and f. So, a morphism $< a, b >: (N, start, end) \longrightarrow (N', start', end')$ is an ordinary net morphism that preserves the markings $b(start) = start'$ and $b(end) = end'$.*

BA: does the preservation of i and f need to be explicit or is it implied already?

The chainable petri net, or CNET, is a petri net that tracks both its initial and its final places, to facilitate chaining the petri net together with other petri nets at either end. This enables the use of operators such as OR, SEQ, & AND.
Suppose we have two CNETs, $(S_1^{\oplus}, T_1, start_1, end_1)$ and $(S_2^{\oplus}, T_2, start_2, end_2)$.

Example 11. *First let us look at a simple operation, the sequential ordering (SEQ) of two Petri nets. This is a common operation that requires one task to be completed*

before another. This means that the operation cannot be commutative, since the order matters. In a simple Petri net, this is modeled graphically as two (or more) places arranged along a non-branching line. Clearly in such an arrangement the first place must be activated in order to activate the second place, and so on. Only when the last node is activated has the SEQ operation completed.

Definition 12. *Given two Petri nets* $N_1 = (S_1^{\oplus}, T_1, start_1, end_1)$ *and* $N_2 = (S_2^{\oplus}, T_2, start_2, end_2)$*, the* ***sequentitial composition*** *is* $N_1; N_2$*, where:*

BA: Should I make this a new symbol since it has special properties?

- *The initial place* $i = start_1$.
- *The final place* $f = end_2$.
- $S = S_1 + S_2$.
- $T = T_1 + T_2 + (end_1 \longrightarrow start_2)$.

Order is key here. Making the first net's start the composition's start ensures that the composition begins with the net that is first in the sequence. Similarly, making the second net's end the composition's end ensures that the composition finishes upon the completion of the second in the sequence. Finally, the new transition between the end of the first net and the start of the second net provides the necessary connection between the two. Thus, while this kind of composition retains associativity, it is modified so that it is no longer symmetric, thus becoming sequential.

Lemma 13. *The SEQ operation for two CNETs,* $A; B$*, is not symmetric.*

Proof. Let $A = (P_A, T_A, \mathsf{src_A}, \mathsf{tar_A}, i_A, f_A)$ and $B = (P_B, T_B, \mathsf{src_B}, \mathsf{tar_B}, i_B, f_B)$. Symmetry would imply that $A; B = B; A$. Let us compute the components of each to test for symmetry.
The components of $A; B$ *are* :

- $P_{A;B} = P_A + P_B$.
- $T_{A;B} = T_A + T_B + (f_A \longrightarrow i_B)$.
- $i_{A;B} = i_A$.
- $f_{A;B} = f_B$.

The components of $B; A$ *are* :

- $P_{B;A} = P_B + P_A$.
- $T_{B;A} = T_B + T_A + (f_B \longrightarrow i_A)$.
- $i_{B;A} = i_B$.
- $f_{B;A} = f_A$.

Relying on the symmetry of disjoint union, we see that these two compositions have identical places, $P_A + P_B = P_B + P_A$. However, whereas the construction of $A; B$ includes $(f_A \longrightarrow i_B) \in T_{A;B}$, the construction of $B; A$ includes $(f_B \longrightarrow i_A) \in T_{B;A}$. Furthermore, the initial and final places are not the same. Thus, $A; B \neq B; A$ and The operation is not symmetric. □

Lemma 14. *The SEQ operation for CNETs, A;B;C, is associative.*

Proof. Let $A = (P_A, T_A, \mathsf{src_A}, \mathsf{tar_A}, i_A, f_A), B = (P_B, T_B, \mathsf{src_B}, \mathsf{tar_B}, i_B, f_B)$ and $C = (P_C, T_C, \mathsf{src_C}, \mathsf{tar_C}, i_C, f_C)$. Composing the three, $A; B; C$, can be done one of two ways: $(A; B); C$ or $A; (B; C)$. For the operation to be associative, the following must be true: $(A; B); C = A; (B; C)$. Let us compute each to test for equality.
First, we compute $(A; B); C$ by taking $A; B$:

- $P_{A;B} = P_A + P_B$.
- $T_{A;B} = T_A + T_B + (f_A \longrightarrow i_B)$.
- $i_{A;B} = i_A$.
- $f_{A;B} = f_B$.

Now we add C:

- $P_{(A;B);C} = P_A + P_B + P_C$.
- $T_{(A;B);C} = T_A + T_B + (f_A \longrightarrow i_B) + (f_B \longrightarrow i_C)$.
- $i_{(A;B);C} = i_A$.
- $f_{(A;B);C} = f_C$.

Now let us follow the second path, computing $A; (B; C)$ by first taking $B; C$:

- $P_{B;C} = P_B + P_C$.
- $T_{B;C} = T_B + T_C + (f_B \longrightarrow i_C)$.
- $i_{B;C} = i_B$.
- $f_{B;C} = f_C$.

Now we add A:

- $P_{A;(B;C)} = P_A + P_B + P_C$.
- $T_{A;(B;C)} = T_A + T_B + (f_A \longrightarrow i_B) + (f_B \longrightarrow i_C)$.
- $i_{A;(B;C)} = i_A$.
- $f_{A;(B;C)} = f_C$.

These clearly give the same result, showing that $(A; B); C = A; (B; C)$.
In sequential composition, the first CNET's initial place i always becomes the composition's initial place. Then, the final place of a preceding net is joined to the initial place of the subsequent net by adding to T for the composition. The final place of the last net in the sequence becomes the final place f of the composition. Thus, the operation is associative for any number of nets. □

Example 15. *Next let us examine disjunction (OR) in CNETs, starting with a basic example. Consider an ordinary Petri net $(S^{\oplus}, T)$. A simple model of the disjunction s_2 OR s_3 is given by the following relations:*

- *Let $S = \{s_1, s_2, s_3\}$, and $\mathsf{src_x}, \mathsf{tar_x}$: $(s_1) \longrightarrow (s_3)$, and $(s_2) \longrightarrow (s_3)$.*
- *Given this relation, as long as s_1, s_2, or both are activated, s_3 will be activated.*
- *If neither s_1 nor s_2 are activated, s_3 will also remain inactive.*

Within a single Petri net, we can see that disjunction is modeled graphically by two transition arcs leading to one place. If either transition fires, the place will be activated, enabling its subsequent transition(s) to fire. The disjunction of two CNETs will result in a similar pattern: if at least one of the CNET disjuncts reaches its final marking, then the disjunction itself will reach its final marking.

Definition 16. *Given two CNETs, $N_1 = (S_1^{\oplus}, T_1, start_1, end_1)$ and $N_2 = (S_2^{\oplus}, T_2, start_2, end_2)$, their **disjunction** is $N_1 + N_2$, where:*

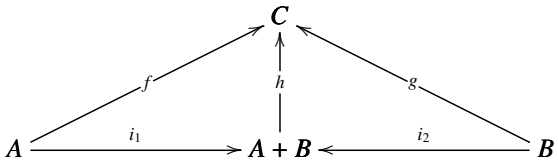
- *The initial place $i = (i_1, i_2)$.*
- *The final place f_3 is a new place.*
- $P = ((P_1 + P_2) + \{(i_1, i_2)\} + \{f_1\} + \{f_2\} + \{f_3\})$.
- $T = (T_1 + T_2 + (f_1 \longrightarrow f_3) + (f_2 \longrightarrow f_3) + (f_3 \longrightarrow \emptyset))$.
- $[\mathsf{src}_1, \mathsf{src}_2], [\mathsf{tar}_1, \mathsf{tar}_2] : T \longrightarrow P^{\oplus}$.

HE: I do not know how to read this definition.

We form the initial place of the disjunction by pairing the initial places of the two disjuncts, so that each branch of the disjunction can be reached initially. The final place is a new place that is the target of the final place of each disjunct, so that the completion of either branch (or both) can trigger the disjunction's final place. The disjunction's places gather these elements together, along with the places of each branch. Finally, the src and tar functions appropriate to each branch are used to compute runs, with the addition of the new transitions for f.

Lemma 17. *The disjunction of two CNETs, A + B, is a coproduct.*

Proof. $N_1 + N_2$ is a coproduct if it has morphisms $i_1 : A \longrightarrow A + B$ and $i_2 : B \longrightarrow A + B$ such that for any object C with morphisms $f : A \longrightarrow C$ and $g : B \longrightarrow C$, there is a unique morphism $h : A + B \longrightarrow C$ such that $f = i_1; h$ and $g = i_2; h$, meaning that this diagram commutes:



Let $A = (P_A, T_A, \mathsf{src_A}, \mathsf{tar_A}, i_A, f_A)$ and $B = (P_B, T_B, \mathsf{src_B}, \mathsf{tar_B}, i_B, f_B)$, and take A + B. Let us define i_0 and i_1, which are injections for Petri nets. We will take it by parts for clarity.

$P_{A+B} = (P_A + P_B + \{(i_A, i_B)\} + \{f_A\} + \{f_B\})$

The coproduct of sets of places has the injective functions already, which covers all the constituents except the initial places. For the initial places, $i_A \longrightarrow i_A \times i_B$ and $i_B \longrightarrow i_A \times i_B$ just maps the individual initial place to the pair of initial places, using the position in the pair to indicate whether it is i_A or i_B.

$T_{A+B} = T_A + T_B$, so again we can use the coproduct of sets to obtain the injective functions for this part of the net.

$[\mathsf{src}_1, \mathsf{src}_2], [\mathsf{tar}_1, \mathsf{tar}_2] : T \longrightarrow P^{\oplus}$. These functions already respect set membership, and so their injective function is also clear. Since these composite pieces are all coproducts, we can rely on their injections to yield the injections of the larger structure.

Now, suppose we have another CNET C and morphisms $f : A \longrightarrow C$ and $g : B \longrightarrow C$; we define h as follows:

$$h(x) = \begin{cases} f(x), & \text{if x is from A} \\ g(x), & \text{if x is from B} \end{cases} \tag{1}$$

Given this definition of h, it is clear that $f = i_1; h$ and $g = i_2; h$, since h(x) applies f(x) if $x \in A$ and g(x) if $x \in B$.
To show that this satisfies the uniqueness requirement, let us consider an arbitrary function $k : (A + B) \longrightarrow C$, where $i_1; k = f$ and $i_2; k = g$. For $a \in A$ and $b \in B$, the following equalities hold:

$$i_1(a); k(a) = f(a) = h(a) \tag{2}$$

$$i_2(b); k(b) = g(b) = h(b) \tag{3}$$

Thus, $h = k$, showing that h is unique. This provides a morphism as required, showing that the CNET disjunction is a coproduct.

□

Definition 18. *Given two CNETs,* $N_1 = (P_1, T_1, \mathsf{src}_1, \mathsf{tar}_1, i_1, f_1)$ *and* $N_2 = (P_2, T_2, \mathsf{src}_2, \mathsf{tar}_2, i_2, f_2)$*, their* ***conjunction*** *is* $N_1 \times N_2$*, where:*

- *The initial place* $i = (i_1, i_2)$*.*
- *The final place* $f = (f_1, f_2)$ *is a new place.*
- $P = ((P_1 \times P_2) + \{(i_1, i_2)\} + \{(f_1, f_2)\})$*.*
- $T = (T_1 \times T_2)$*.*
- $[\mathsf{src}_1, \mathsf{src}_2], [\mathsf{tar}_1, \mathsf{tar}_2] : T \longrightarrow P^{\oplus}$*.*

We form the initial and final places of the conjunction by pairing the initial and final places of the two conjuncts, ensuring each can be reached initially, and requiring that both complete in order to complete the conjunction of the two branches. The conjunction's places gather these elements together, along with the places of each branch. Finally, the src and tar functions appropriate to each branch are used to compute runs. Transitions are paired, making these two branches run in parallel.

BA: parallelization a good thing or bad thing?

Example nets: K =
$S_K : \{a, b, c\}$
$T_K : \{t\}$
$F_K(a, t) = 2$
$F_K(b, t) = 1$
$F_K(t, c) = 2$
$F_K(else) = 0$
$S_K^{\oplus} : \{2a \oplus 1b \oplus 2c\}$
$t = \{2a \oplus 1b \longrightarrow 2c\}$
$\delta_{0K}(t) = 2a \oplus 1b\}$
$\delta_{1K}(t) = 2c\}$
M =
$S_M : \{d, e\}$
$T_M : \{t'\}$
$F_M(d, t') = 2$
$F_M(t', e) = 1$
$F_M(else) = 0$
$S_M^{\oplus} : \{2d \oplus 1e\}$
$t = \{2d \longrightarrow 1e\}$
$\delta_{0M}(t') = 2d\}$
$\delta_{1M}(t') = 1e\}$
$S_K^{\oplus} \times S_M^{\oplus} = (2a \oplus 1b, 2d) \longrightarrow (2c, 1e)$
$(S_K + S_M)^{\oplus} = (\{1\} \times S_K) \cup (\{2\} \times S_M)^{\oplus}$
$((1, 2a), (1, 1b), (2, 2d))^{\oplus} =$
$S_{KM}^{\oplus} : \{(1, 2a), (1, 1b), (2, 2d)\}$
$T_{KM} : \{t_1, t_2\}$
$t_1 = ((1, 2a) \oplus (1, 1b)) \longrightarrow (1, 2c)$
$t_2 = (2, 2d) \longrightarrow (2, 1e)$
$\delta_{0KM}(t_1) = (1, 2a) \oplus (1, 1b)\}$
$\delta_{0KM}(t_2) = (2, 2d)\}$
$\delta_{1KM}(t_1) = (1, 2c)\}$
$\delta_{1KM}(t_2) = (2, 1e)\}$
$(S_1^{\oplus} \times S_2^{\oplus})$ and $(S_1 + S_2)^{\oplus}$ are isomorphic.
Suppose there are two Petri nets $S_1 and S_2$, with sets of places $M_1 and M_2$ and sets of arcs $T_1 and T_2$.
Let F be the homomorphism from $(S_1 + S_2)^{\oplus}$ to $(S_1^{\oplus} \times S_2^{\oplus})$.
$F((S_1 + S_2)^{\oplus}) = (S_1^{\oplus} + S_2^{\oplus})$
where $S_1 + S_2 = (\{1\} \times S_1) \cup (\{2\} \times S_2)$
For F:
Calculating the set of places of the codomain:
$S_1 = \bigcup_{y \in domain(x,y)} | x = 1)$
$S_2 = \bigcup_{y \in domain(x,y)} | x = 2)$
Note: I mean to separate out the two sets using the disjoint markers here.
The set of places of the codomain = $(S_1^{\oplus} \times S_2^{\oplus})$
The arrows of the codomain maintain the structure defined in the domain, disregarding

the origin markers {1} and {2} added in by taking the disjoint union.
Since the function preserves the structure between the objects, only changing the names of the objects, this is a homomorphism.
Now, let G be the homomorphism from $(S_1^{\oplus} \times S_2^{\oplus})$ to $(S_1 + S_2)^{\oplus}$.
The objects or places of the codomain are the objects of the domain modified in the following way:
where objects in the domain have the form (x, y), $(\{1\} \times x) \cup (\{2\} \times y)$.
The arrows of the codomain maintain the structure of the domain, adding in origin markers {1} and {2} by position as described above.
Since G preserves the group's structure and only modifies objects' names, it is a homomorphism. Clearly,$G \circ F = F \circ G$, since F removes the origin markings {1} and {2} and G puts them back. Structure remains unchanged in either direction.

[1]

References

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