1) if $\alpha > 2$:

$$|f(x) - f(y)| \le c \, \|x-y\|_2^{\overbrace{\alpha/2}^{>1}} \quad \forall y \in B_r(x)$$

$y = x + h$ with $h \in B_r(0)$

$$\frac{|f(x+h) - f(x)|}{\|h\|_2} \le c \, \|h\|_2^{\overbrace{\beta}^{>0}} \xrightarrow{\|h\|_2 \downarrow 0} 0$$

$\Rightarrow \; Df(x) = 0 \quad \forall x$

$\Rightarrow \; f \equiv \text{const}$ $\lightning$ not true for all $f \in \mathcal{F}_\alpha$

2)

$$S_X = \mathrm{span}\{K(\cdot, x_1), \dots, K(\cdot, x_N)\}$$

$$\mathcal{F} = S_X \oplus \underbrace{S_X^\perp}$$

$$= \{f \in \mathcal{F} \mid f_X = 0\}$$

$$= \{f \in \mathcal{F} \mid (f, s)_K = 0 \ \forall s \in S_X\}$$

$$\Rightarrow s \in \mathcal{F} : \quad s = \underbrace{\sum_{j=1}^{N} c_j K(\cdot, x_j)}_{\tilde{s}} + v$$

$$v_X = 0$$

$$\|s_X - f_X\|_2^2 = \|(\tilde{s}+v)_X - f_X\|_2^2$$

$$= \|\tilde{s}_X - f_X\|_2^2 \quad \text{independent of } v$$

$$\|s\|_K^2 = (\tilde{s}+v, \tilde{s}+v)_K = \|\tilde{s}\|_K^2 + \|v\|_K^2$$

$$\geq \|\tilde{s}\|_K^2 \quad \text{with "=" iff } v=0$$

$\hookrightarrow$ $s \in \mathcal{F}$ min $\Rightarrow$ $v=0$

$$\Rightarrow s = \tilde{s} = \sum_{j=1}^{N} c_j K(\cdot, x_j) \in S_X$$

$$s = \sum_{j=1}^{N} c_j K(\cdot, x_j) \Rightarrow s(x_i) = \sum_{j=1}^{N} c_j K(x_i, x_j)$$

$$\| s_X - f_X \|_2^2 = \| A_{K,X} c - f_X \|_2^2$$

$$A_{K,X} = \left(K(x_j, x_k) \right)_{j,k=1,\dots,N}$$

$$\| s \|_K^2 = c^T A_{K,X} c$$

$$\hookrightarrow \min_{c \in \mathbb{R}^N} \frac{1}{N} \| A_{K,X} c - f_X \|_2^2 + \alpha c^T A_{K,X} c$$

$$\rightsquigarrow \left(\frac{1}{N} A_{K,X}^T A_{K,X} + \alpha A_{K,X} \right) c = \frac{1}{N} A_{K,X}^T f_X$$

3\] $R(h) = \mathbb{E}[(h(x) - Y)^2]$

$$\overset{!}{\geq} \mathbb{E}[(f(X) - Y)^2] = R(f)$$

$$\mathbb{E}[(h(x) - Y)^2] = \mathbb{E}[((h(x) - f(x)) + (f(X) - Y))^2]$$

$$= \mathbb{E}[(h(x) - f(x))^2 + 2(h(X) - f(X)) + (f(X) - Y)^2]$$

$$= \underbrace{\mathbb{E}[(h(x) - f(x))^2]}_{\geq 0} + \mathbb{E}[(f(X) - Y)^2]$$

$$+ 2 \underbrace{\mathbb{E}[(h(x) - f(x))(f(X) - Y)]}_{= 0}$$

- $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X \mid Y]]$
- $\mathbb{E}[f(Y) X \mid Y] = f(Y)\, \mathbb{E}[X \mid Y]$

$$\mathbb{E}[(h(x) - f(x))(f(x) - y)]$$

$$= \mathbb{E}\Big[\underbrace{\mathbb{E}[(h(x) - f(x))(f(X) - Y) \mid X]} \Big]$$

$$= (h(X) - f(x))\, \underbrace{\mathbb{E}[f(X) - Y \mid X]}$$

$$= f(X) - \mathbb{E}[Y \mid X]$$

$$= f(x) - f(x) = 0$$

$$= 0$$

4) $\mathcal{Y} = [a, b]$

$$\Rightarrow \underbrace{L(\underbrace{h(X_i)}_{\in \mathcal{Y}}, \underbrace{Y_i}_{\in \mathcal{Y}})}_{= (h(X_i) - Y_i)^2} \in [0, (b-a)^2]$$

$$Z_i = \frac{1}{\underline{(b-a)^2}} L(h(X_i), Y_i) \in [0, 1]$$

$\hookrightarrow$ Hoeffding's inequality