$$\sum_{i,j=1}^{n} c_i c_j k(x_i, x_j) = \left(\sum_{i=1}^{n} c_i \Phi(x_i), \sum_{j=1}^{n} c_j \Phi(x_j) \right)_{\mathcal{H}} \geq 0$$

$\Rightarrow$:

$$\Phi : \mathcal{X} \longrightarrow \mathbb{R}^{\mathcal{X}}$$

$$x \longmapsto \Phi(x) = k(\cdot, x)$$

$$S = \left\{ s = \sum_{i=1}^{n} \alpha_i \, k(\cdot, x_i) \;\middle|\; n \in \mathbb{N}, \alpha_i \in \mathbb{R}, x_i \in \mathcal{X} \right\}$$

$$\left(\sum_{i=1}^{n_f} \alpha_i k(\cdot, x_i), \sum_{j=1}^{n_g} \beta_j k(\cdot, y_j) \right)_S = \sum_{i=1}^{n_f} \sum_{j=1}^{n_g} \alpha_i \beta_j k(x_i, y_j)$$

$$f = \sum_{i=1}^{n_f} \alpha_i K(\cdot, x_i) = \sum_{i=1}^{\tilde{n}_f} \tilde{\alpha}_i K(\cdot, \tilde{x}_i)$$

$$\left(\sum_{i=1}^{\tilde{n}_f} \tilde{\alpha}_i K(\cdot, \tilde{x}_i), g \right)_S \overset{!}{=} \left(\sum_{i=1}^{n_f} \alpha_i K(\cdot, x_i), g \right)_S$$

$$g = \sum_{j=1}^{n_g} \beta_j K(\cdot, y_j)$$

$$\left(\sum_{i=1}^{\tilde{n}_f} \tilde{\alpha}_i K(\cdot, \tilde{x}_i), g \right)_S = \sum_{j=1}^{n_g} \beta_j \underbrace{\sum_{i=1}^{\tilde{n}_f} \tilde{\alpha}_i K(y_j, \tilde{x}_i)}_{= f(y_j)}$$

$$= \left(\sum_{i=1}^{n_f} \alpha_i K(\cdot, x_i), g \right)_S \quad \checkmark$$

$$\mathcal{H} = \overline{S}$$

by construction: $\Big(\underbrace{\Phi(x)}_{=k(\cdot,x)}, \underbrace{\Phi(y)}_{=k(\cdot,y)} \Big)_{\mathcal{H}} = k(x,y)$

2] $X = (x_1, \dots, x_m) \in \mathbb{R}^{n \times m}$, $\Phi : \mathbb{R}^n \to \mathcal{H}$

$C : \mathcal{H} \to \mathcal{H}$, $\varphi \longmapsto C\varphi = \sum_{j=1}^{m} (\varphi, \psi_j)_{\mathcal{H}} \, \psi_j$

$$\psi_j = \Phi(x_j) - \overline{\Phi}$$

$$G = \left(\langle \psi_i, \psi_j \rangle_{\mathcal{H}} \right)_{1 \le i,j \le m} \in \mathbb{R}^{m \times m}$$

$$G_{i,j} = \langle \Phi(x_i) - \overline{\Phi}, \Phi(x_j) - \overline{\Phi} \rangle_{\mathcal{H}}$$

$$= \langle \Phi(x_i), \Phi(x_j) \rangle_{\mathcal{H}} - \langle \Phi(x_i), \overline{\Phi} \rangle_{\mathcal{H}} - \langle \Phi(x_j), \overline{\Phi} \rangle_{\mathcal{H}} + \langle \overline{\Phi}, \overline{\Phi} \rangle_{\mathcal{H}}$$

$$= K(x_i, x_j) - \frac{1}{m} \sum_{s=1}^{m} \left(k(x_i, x_s) + k(x_j, x_s) \right) + \frac{1}{m^2} \sum_{s,t=1}^{m} k(x_s, x_t)$$

$$\mathbb{1}_m = \frac{1}{m} \begin{pmatrix} 1 & - & 1 \\ 1 & \ddots & 1 \\ 1 & - & 1 \end{pmatrix} \in \mathbb{R}^{m \times m}$$

$$G = A - \mathbb{1}_m A - A \mathbb{1}_m + \mathbb{1}_m A \mathbb{1}_m$$

- Let $\lambda \neq 0$ EW of G with EV $\|v\| = 1$

  $\hookrightarrow \varphi = \frac{1}{\sqrt{\lambda}} \sum_{i=1}^{m} v_i \Psi_i$ is normalized EV of C corresponding to λ

- Let $\lambda \neq 0$ EW of C with EV $\|\Psi\|_{\mathcal{H}} = 1$

  $v_j = \frac{1}{\lambda} \langle \Psi_j, \varphi \rangle_{\mathcal{H}}$ ist normalized EV of G corresp. to λ

$$\beta_j = \begin{pmatrix} (u_1, \psi_j)_{\mathcal{H}} \\ \vdots \\ (u_p, \psi_j)_{\mathcal{H}} \end{pmatrix} \in \mathbb{R}^p$$

$$(u_\ell, \psi_j)_{\mathcal{H}} = \frac{1}{\sqrt{\lambda_\ell}} \sum_{i=1}^{m} v_{\ell,i} \underbrace{(\psi_i, \psi_j)_{\mathcal{H}}}_{= G_{i,j}}$$

3] $\rho(z) = \| \underline{\Phi}(z) - P_{\underline{\Phi}}(x) \|_{\mathcal{H}}^2$

$$= \underbrace{\left(\underline{\Phi}(z), \underline{\Phi}(z) \right)_{\mathcal{H}}}_{= k(z,z) = 1} - 2 \left(\underline{\Phi}(z), P_{\underline{\Phi}}(x) \right)_{\mathcal{H}} + C$$

$\hookrightarrow \max_z \underbrace{\left(\underline{\Phi}(z), P_{\underline{\Phi}}(x) \right)_{\mathcal{H}}}_{= \sum_{j=1}^{m} \gamma_j \underbrace{k(z, x_j)}_{= \exp(-c \| z - x_j \|_2^2)}} =: \tilde{\rho}(z)$

$\hookrightarrow \nabla_z \tilde{\rho}(z) = 0$

$\boxed{4}$ $$L(x,\lambda,\mu) = \frac{1}{2} x^T Q x + c^T x + \lambda^T (Ax - a) + \mu^T (Bx - b)$$

$$g(\lambda,\mu) = \inf_{x \in \mathbb{R}^n} L(x,\lambda,\mu)$$

dual problem $\max_{\lambda,\mu} g(\lambda,\mu)$ s.t. $\lambda \geq 0$

$$g(\lambda,\mu) = \frac{1}{2} \begin{pmatrix} \lambda \\ \mu \end{pmatrix}^T P \begin{pmatrix} \lambda \\ \mu \end{pmatrix} + d^T \begin{pmatrix} \lambda \\ \mu \end{pmatrix} - \frac{1}{2} c^T Q^{-1} c$$

with

$$P = \begin{pmatrix} AQ^{-1}A^T & AQ^{-1}B^T \\ BQ^{-1}A^T & BQ^{-1}B^T \end{pmatrix}$$

$$d = \begin{pmatrix} AQ^{-1}c + a \\ BQ^{-1}c + b \end{pmatrix}$$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}^T P \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$$

$\alpha \in \ker(A^T)$ s.t. $(\alpha, \beta) \neq 0$

$\beta \in \ker(B^T)$