1b) $\|A-B\|_F = \|U\Sigma V^T - B\|_F$

$B \in O(n)$ $\quad = \|\Sigma - \underbrace{U^T B V}_{\in O(n)}\|_F$

$$\Rightarrow \min_{B \in O(n)} \|A-B\|_F = \min_{Q \in O(n)} \|\Sigma - Q\|_F$$

$$\|\Sigma - Q\|_F^2 = tr\left((\Sigma-Q)^T(\Sigma - Q)\right)$$

$$= \underbrace{tr(\Sigma^T\Sigma)}_{=\text{const}} - 2\,\underline{tr(\Sigma^T Q)} + \underbrace{tr(Q^T Q)}_{=n\ =\text{const}}$$

$$\operatorname{tr}(\Sigma^T Q) = \sum_{i=1}^{n} \underbrace{\sigma_i}_{\geq 0} \underbrace{q_{ii}}_{\in[-1,1]}$$

$$\xrightarrow{\max} \quad q_{ii} = 1 \quad \forall i$$

$$\Rightarrow \quad Q = I = U^T B V \qquad \Rightarrow \quad B = U V^T$$

$$2)\ \|A-A_k\|_2 = \|U\underbrace{(\Sigma-\Sigma_k)}_{=\begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & \sigma_{k+1} & \\ 0 & & & & \ddots \ \sigma_p \end{pmatrix}}V^T\|_2$$

$$\overset{1a)}{=} \sigma_{k+1}$$

$$\|A-B\|_2 \geq \sigma_{k+1} \qquad \forall\ rk(B)=k$$

$$B = \tilde{U}\tilde{\Sigma}\tilde{V}^T \qquad \tilde{\Sigma} = \begin{pmatrix} \gamma_1 & & & & \\ & \ddots & & & \\ & & \gamma_k & & \\ & & & 0 & \\ & & & & \ddots \ 0 \end{pmatrix}$$

$$= XY^T \qquad X \in \mathbb{R}^{m\times k},\ Y \in \mathbb{R}^{n\times k}$$

$$\|A-B\|_2 \underset{\|w\|_2=1}{\geq} \|(A-B)w\|_2 = \underbrace{\|U\Sigma V^T w - XY^T w\|_2}_{\overset{!}{\geq}\ \sigma_{k+1}}$$

good: $Y^T w = 0 \Rightarrow w \in \mathcal{N}(Y^T)$, $Y \in \mathbb{R}^{n \times k}$

$V^T w = e_{k+1}$

$V \in \mathcal{O}(n) \Rightarrow \langle v_j, v_k \rangle = \delta_{jk}$

$$\left.\begin{aligned} \dim \mathcal{N}(Y^T) &\geq n-k \\ \dim \operatorname{span}\{v_1, \ldots, v_{k+1}\} &= k+1 \end{aligned}\right\} \Rightarrow \mathcal{N}(Y^T) \cap \operatorname{span}\{v_1, \ldots, v_{k+1}\} \neq \{0\}$$

$\Rightarrow \exists w \in \mathbb{R}^n : Y^T w = 0$

$$w = c_1 v_1 + \ldots + c_{k+1} v_{k+1}$$

wlog $\quad c_1^2 + \ldots + c_{k+1}^2 = 1$

$$\Rightarrow \|(A-B)w\|_2^2 = \|U \Sigma V^T w\|_2^2$$

$$= \|\Sigma V^T w\|_2^2$$

$$= c_1^2 \sigma_1^2 + \ldots + \sigma_{k+1}^2 c_{k+1}^2$$

$$\geq \sigma_{k+1}^2 \underbrace{\left(c_1^2 + \ldots + c_{k+1}^2\right)}_{=1} = \sigma_{k+1}^2$$

$$\|A-B\|_F^2 = \operatorname{tr}\left((A-B)^T(A-B)\right)$$

$$= \underbrace{\operatorname{tr}(A^TA)}_{=\sum_{j=1}^{p}\sigma_j^2} - 2\underbrace{\operatorname{tr}(A^TB)}_{\leq\sum_{j=1}^{p}\sigma_j\gamma_j} + \underbrace{\operatorname{tr}(B^TB)}_{=\sum_{j=1}^{p}\gamma_j^2}$$

$$\geq \sum_{j=1}^{p} \underbrace{\sigma_j^2 - 2\sigma_j\gamma_j + \gamma_j^2}_{=(\sigma_j-\gamma_j)^2}$$

$\Rightarrow$ choose $\sigma_j = \gamma_j$ for $j = 1, \dots, k$

3) "$\Rightarrow$" KKT conditions

$$L(x,\mu) = \|Ax-b\|_2^2 + \mu^T(d-Cx)$$

$$\Rightarrow \nabla_x L(\hat{x},\mu) = 0$$

$$\nabla_\mu L(\hat{x},\mu) = 0 \quad \Rightarrow C\hat{x} = d$$

$$\nabla_x L(x,\mu) = 2A^TAx - 2A^Tb - C^T\mu = 0$$

$$A^TAx - A^Tb + C^T\underbrace{\left(-\frac{\mu}{2}\right)}_{=z} = 0$$

"$\Leftarrow$" to show: $\|Ax-b\|_2^2 \geq \|A\hat{x}-b\|_2^2$

$\forall x: \underline{Cx=d}$

$$\|Ax-b\|_2^2 = \|A(x-\hat{x})+A\hat{x}-b\|_2^2$$

$$= \|A(x-\hat{x})\|_2^2 + 2(x-\hat{x})^T \overbrace{A^T(A\hat{x}-b)}^{=-C^Tz}$$

$$+ \underline{\|A\hat{x}-b\|_2^2}$$

$$= \underbrace{\|A(x-\hat{x})\|_2^2}_{\geq 0} - 2\underbrace{(x-\hat{x})^T C^T z}_{=(Cx-C\hat{x})^T z = 0} + \|A\hat{x}-b\|_2^2$$

$$\geq \|A\hat{x}-b\|_2^2$$

$\hat{x}$ not unique $\Rightarrow$ $A(x-\hat{x}) = 0$

$\wedge\ C(x-\hat{x}) = 0$

$$\Rightarrow \begin{pmatrix} A \\ C \end{pmatrix} (x-\hat{x}) = 0$$

$\begin{pmatrix} A \\ C \end{pmatrix}$ has lin. ind. columns

$$\Rightarrow \ker \begin{pmatrix} A \\ C \end{pmatrix} = \{0\}$$

$$\Rightarrow x - \hat{x} = 0 \Rightarrow x = \hat{x} \ \lightning$$

4) $S \in \mathcal{C}^m$ $\quad p_K(t) = a_{K,0} + a_{K,1} t + \dots + a_{K,n} t^n$

on $[\underline{t_K}, t_{K+1}]$: $p_K \in \mathcal{P}_n$

$$\underbrace{p_{K-1}^{(j)}(t_K)} = \underbrace{p_K^{(j)}(t_K)} \quad \forall\, j = 0, \dots, m$$

$j=0$: $a_{K-1,0} + a_{K-1,1} t_K + \dots + a_{K-1,n} t_K^n$ $\quad a_{K,0} + a_{K,1} t_K + \dots + a_{K,n} t_K^n$ $\quad K = 1, \dots, M$

$$C = \begin{pmatrix} C_1 & -C_1 & & 0 \\ & \ddots & \ddots & \\ 0 & & C_M & -C_M \end{pmatrix} \qquad C_K = \begin{pmatrix} 1 & t_K & \dots & t_K^n \\ 0 & 1 & 2t_K \dots & n t_K^{n-1} \\ \vdots & & & \\ 0 & 0 & \dots & \frac{n!}{(n-m)!} t_K^{n-m} \end{pmatrix}$$