Parte 4

Resolução de Sistemas Lineares - 2

CI1164 - Introdução à Computação Científica

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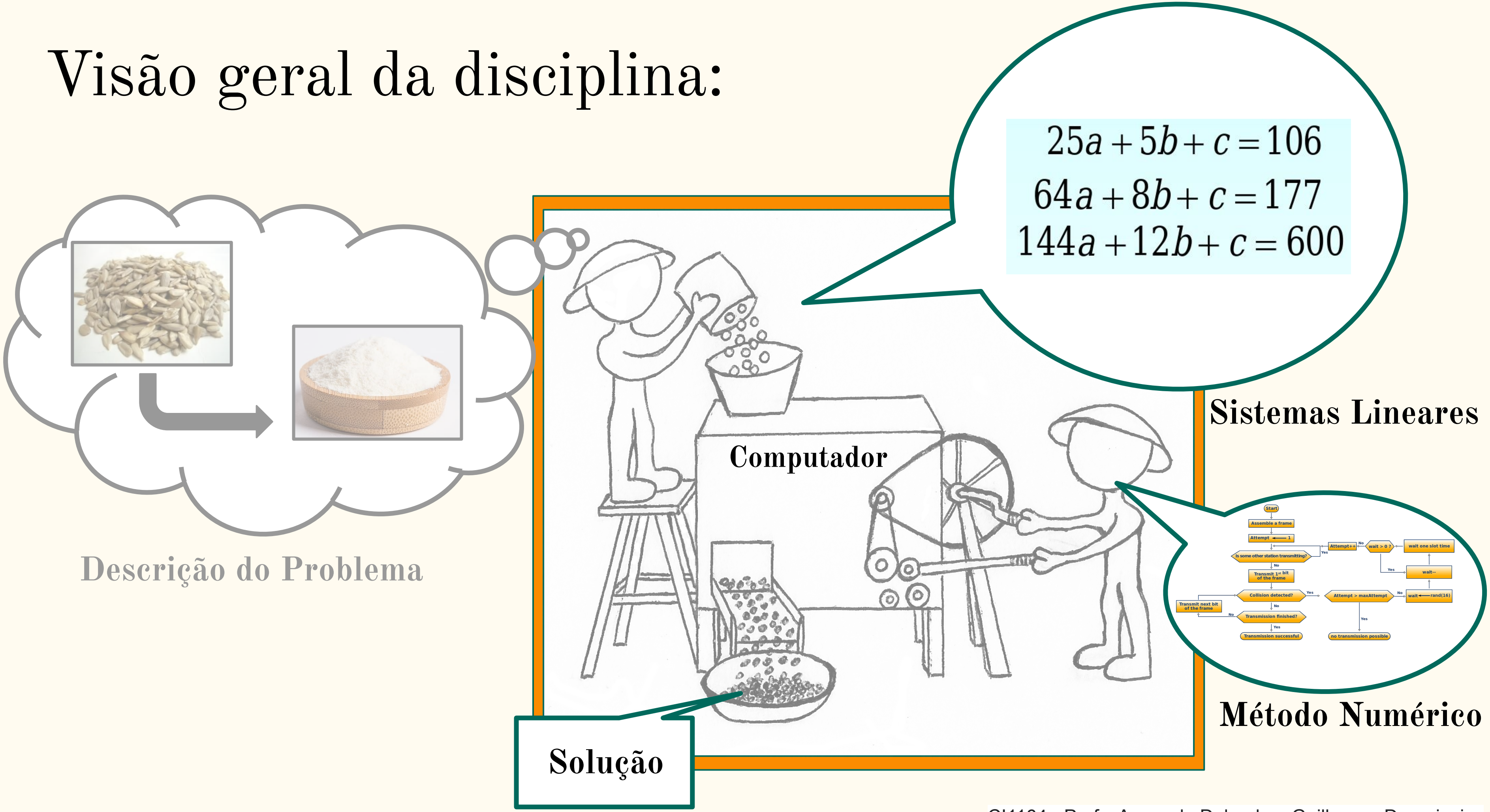
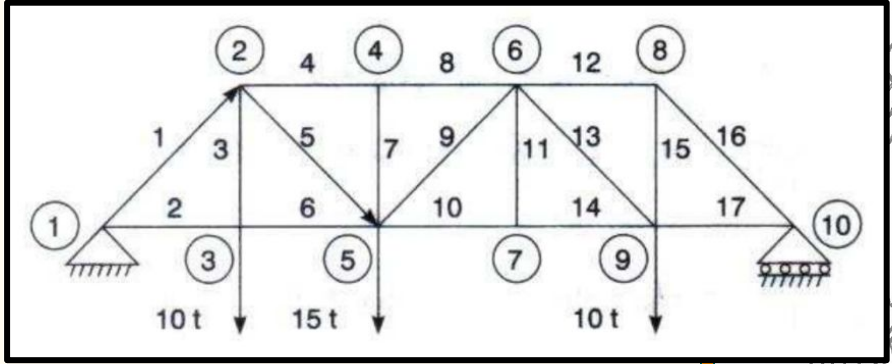
Departamento de Informática - UFPR

Visão geral da disciplina:

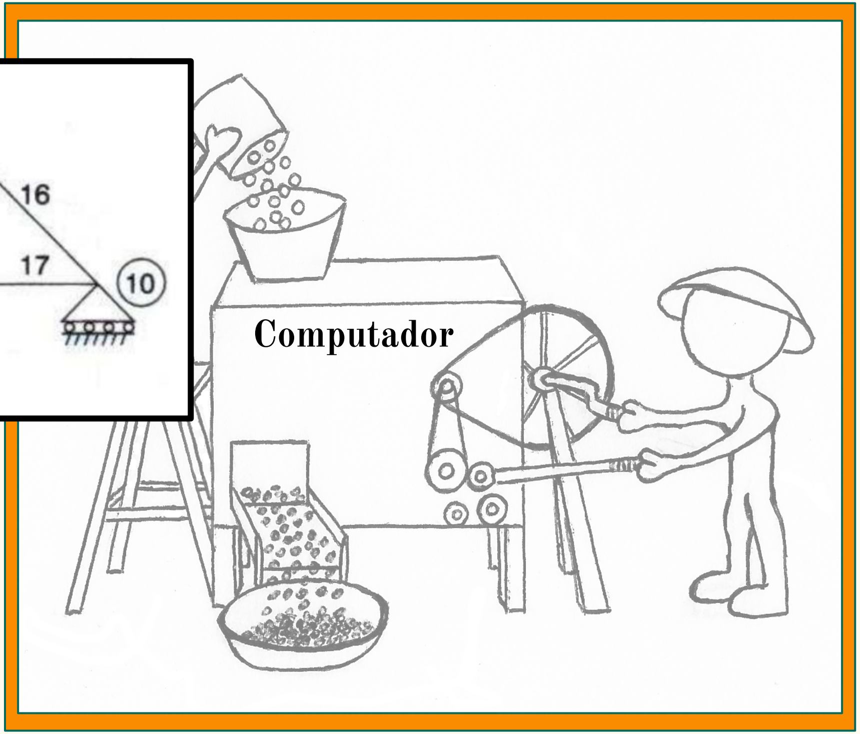


Resolução de Sistemas Lineares

Fonte: Cálculo Numérico (Ruggiero & Lopes)

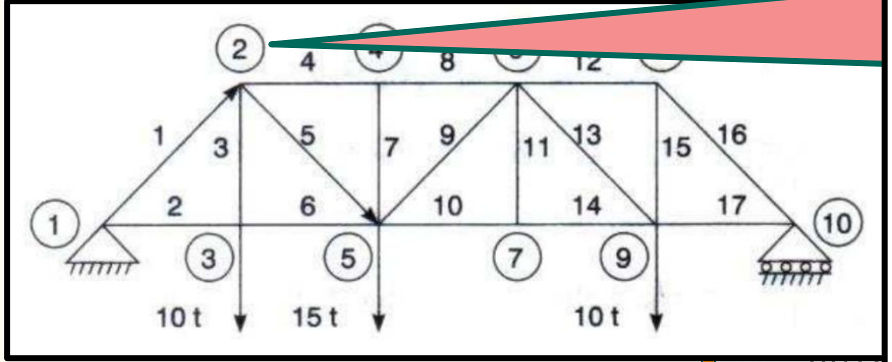


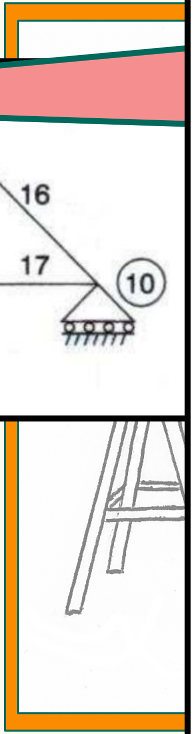
Descrição do Problema



Resolução de Sistemas Lineares

Fonte: Cálculo Numérico (Ruggiero & Lopes)





Descrição do Problema

$$
\begin{cases}
-\alpha f_1 + f_4 + \alpha f_5 = 0 \\
-\alpha f_1 - f_3 - \alpha f_5 = 0 \\
-f_2 + f_6 = 0 \\
f_3 - 10 = 0 \\
-f_4 + f_8 = 0 \\
-f_7 = 0 \\
-\alpha f_5 - f_6 + \alpha f_9 + f_{10} = 0 \\
\alpha f_5 + f_7 + \alpha f_9 - 15 = 0 \\
-f_8 - \alpha f_9 + f_{12} + \alpha f_{13} = 0 \\
-\alpha f_9 - f_{11} - \alpha f_{13} = 0 \\
-f_{10} + f_{14} = 0 \\
f_{11} = 0 \\
-f_{12} + \alpha f_{16} = 0 \\
-f_{15} - \alpha f_{16} = 0 \\
-\alpha f_{13} - f_{14} + f_{17} = 0 \\
\alpha f_{13} + f_{15} - f_{10} = 0 \\
-\alpha f_{16} - f_{17} = 0
\end{cases}
$$

$\alpha f_5 = 0$

$\alpha f_5 = 0$

Resolução de Sistemas Lineares

$$
A = \begin{bmatrix}
-\alpha & 0 & 0 & 1 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\alpha & 0 & -1 & 0 & -\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\alpha & -1 & 0 & 0 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \alpha & 0 & 1 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -\alpha & 0 & 0 & 1 & \alpha & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha & 0 & -1 & 0 & -\alpha & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & \alpha & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -\alpha & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha & -1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha & -1
\end{bmatrix}
$$

Resolução de Sistemas Lineares

$$A = \begin{bmatrix}
-\alpha & 0 & 0 & 1 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\alpha & 0 & -1 & 0 & -\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\alpha & -1 & 0 & 0 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \alpha & 0 & 1 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -\alpha & 0 & 0 & 1 & \alpha & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha & 0 & -1 & 0 & -\alpha & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & \alpha & 0 \\
 & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -\alpha & 0 \\
 & & & & & & & 0 & 0 & 0 & 0 & 0 & -\alpha & -1 & 0 & 0 & 1 \\
 & & & & & & & 0 & 0 & 0 & 0 & 0 & \alpha & 0 & 1 & 0 & 0 \\
 & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha & -1
\end{bmatrix}$$

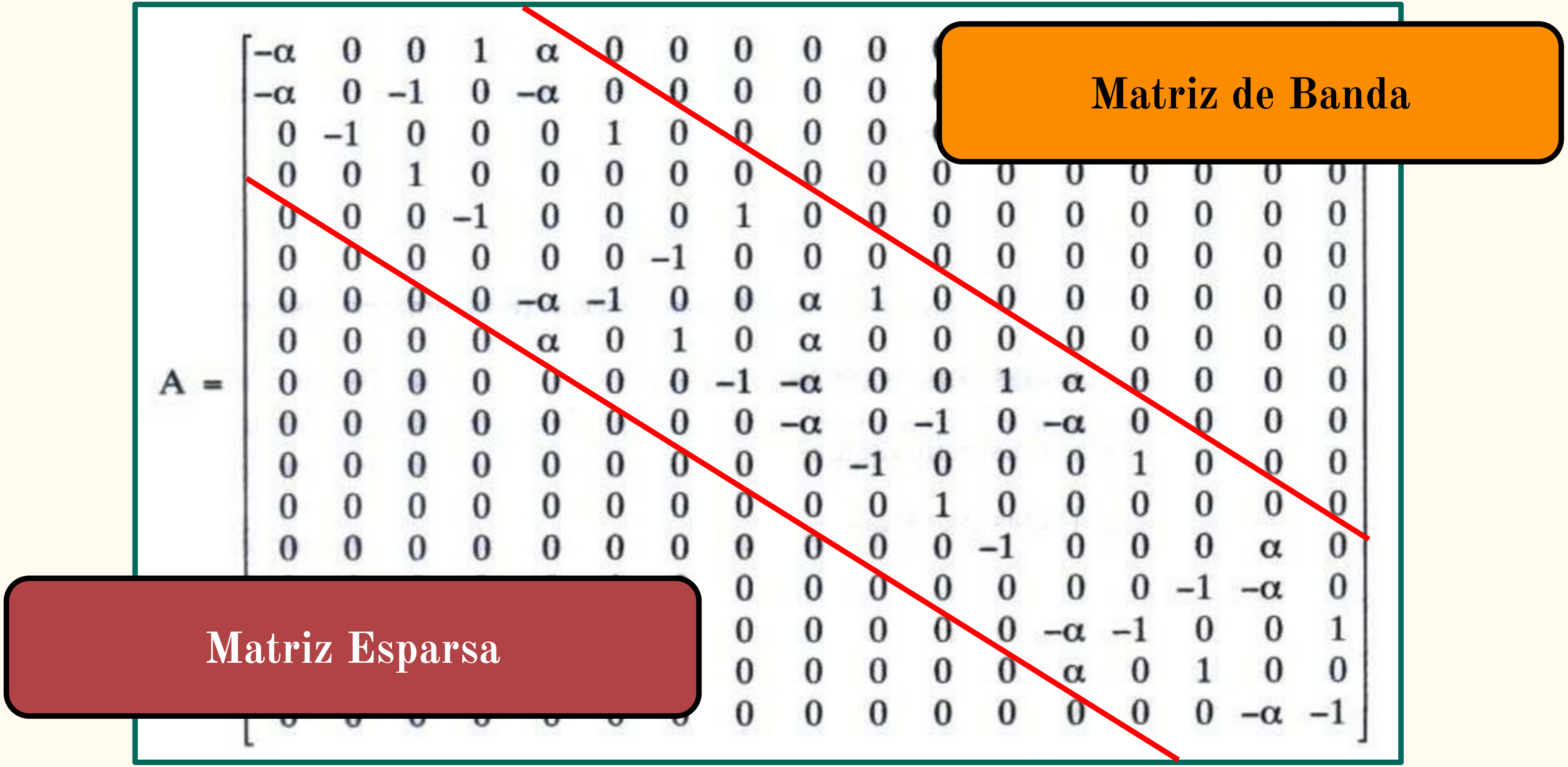
Matriz Esparsa

Resolução de Sistemas Lineares

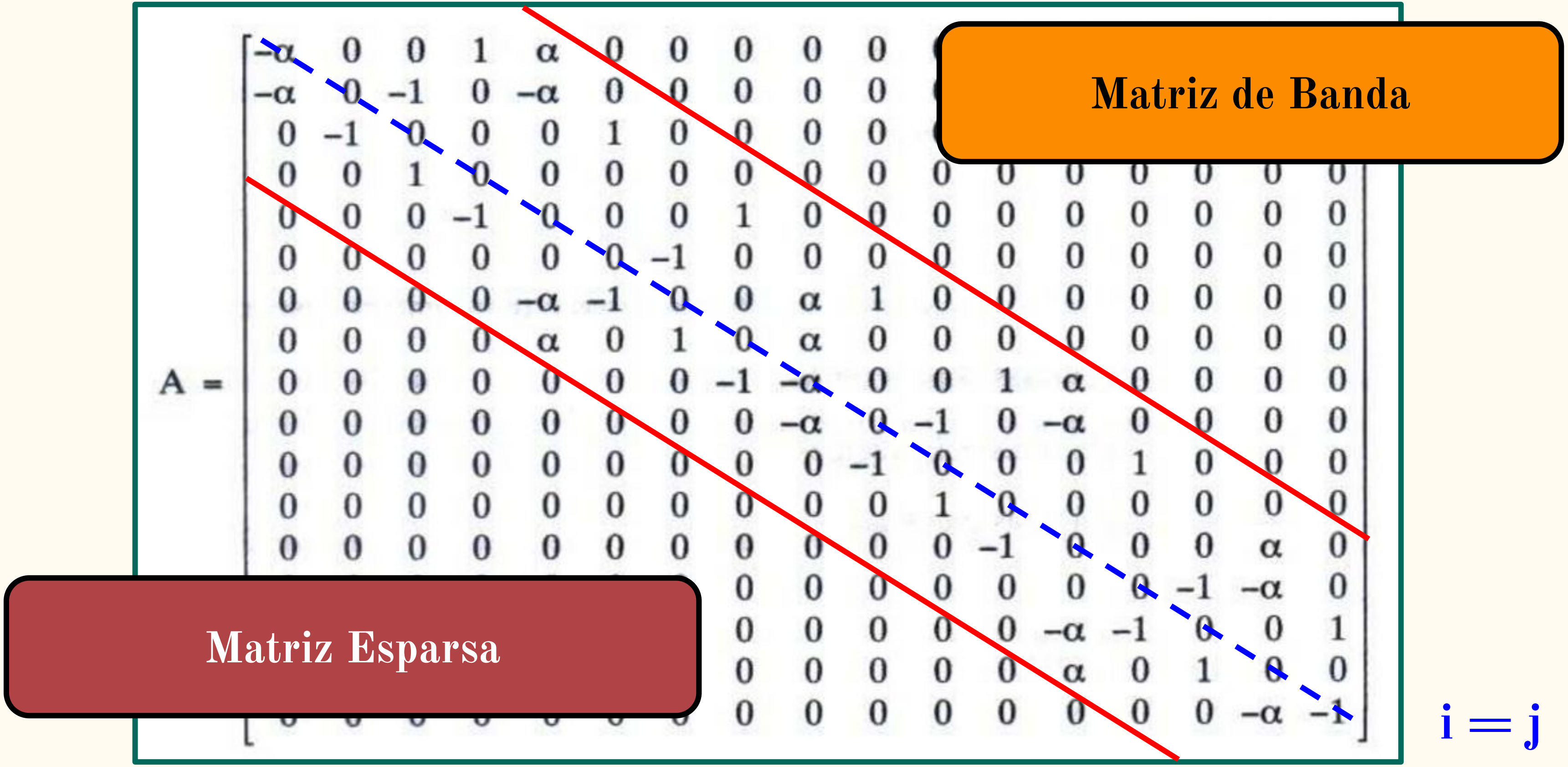
$$
A = \begin{bmatrix}
-\alpha & 0 & 0 & 1 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\alpha & 0 & -1 & 0 & -\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\alpha & -1 & 0 & 0 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \alpha & 0 & 1 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -\alpha & 0 & 0 & 1 & \alpha & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha & 0 & -1 & 0 & -\alpha & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & \alpha & 0 \\
 & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -\alpha & 0 \\
 & & & & & & & 0 & 0 & 0 & 0 & 0 & -\alpha & -1 & 0 & 0 & 1 \\
 & & & & & & & 0 & 0 & 0 & 0 & 0 & \alpha & 0 & 1 & 0 & 0 \\
 & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha & -1
\end{bmatrix}
$$

Matriz Esparsa

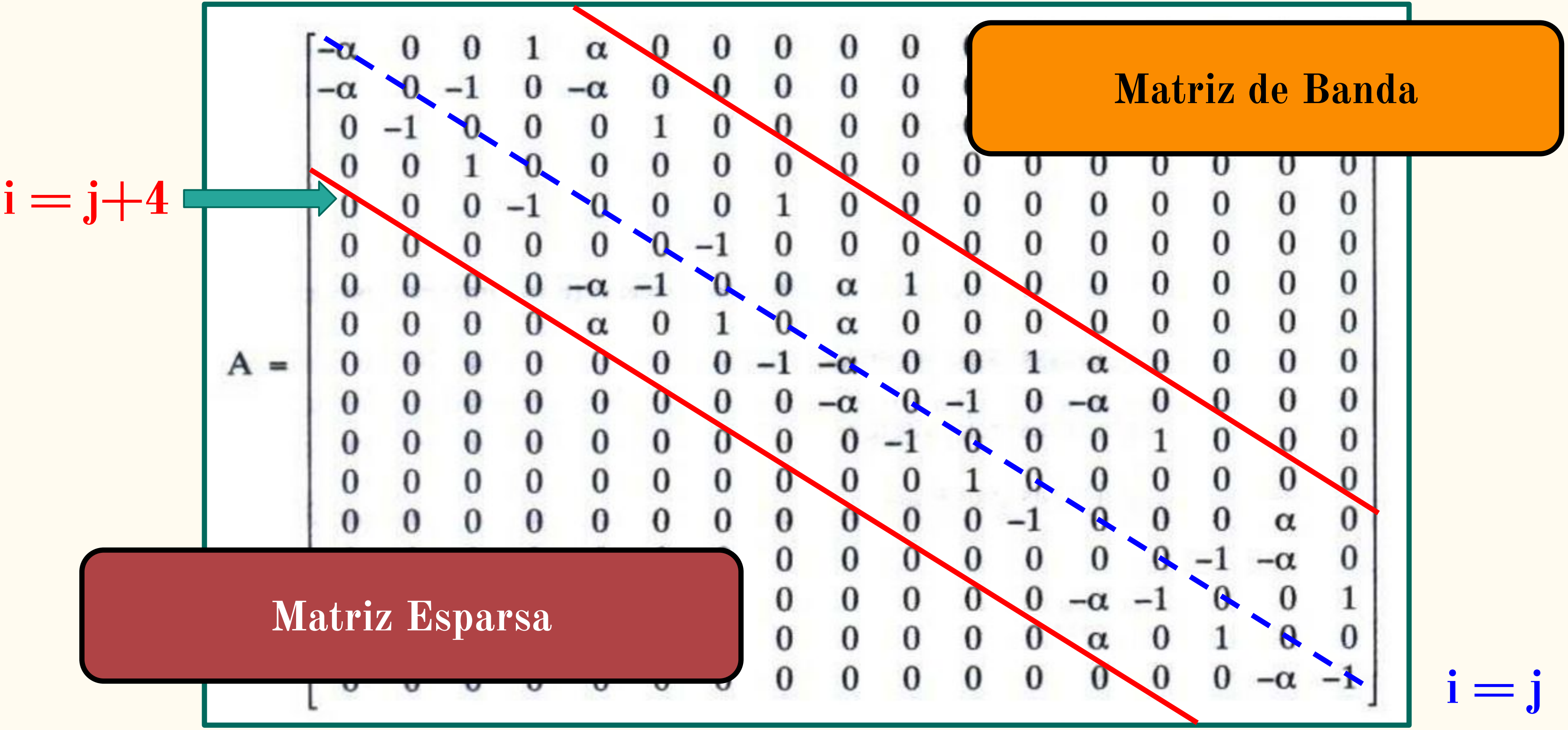
Matrizes k-Diagonais



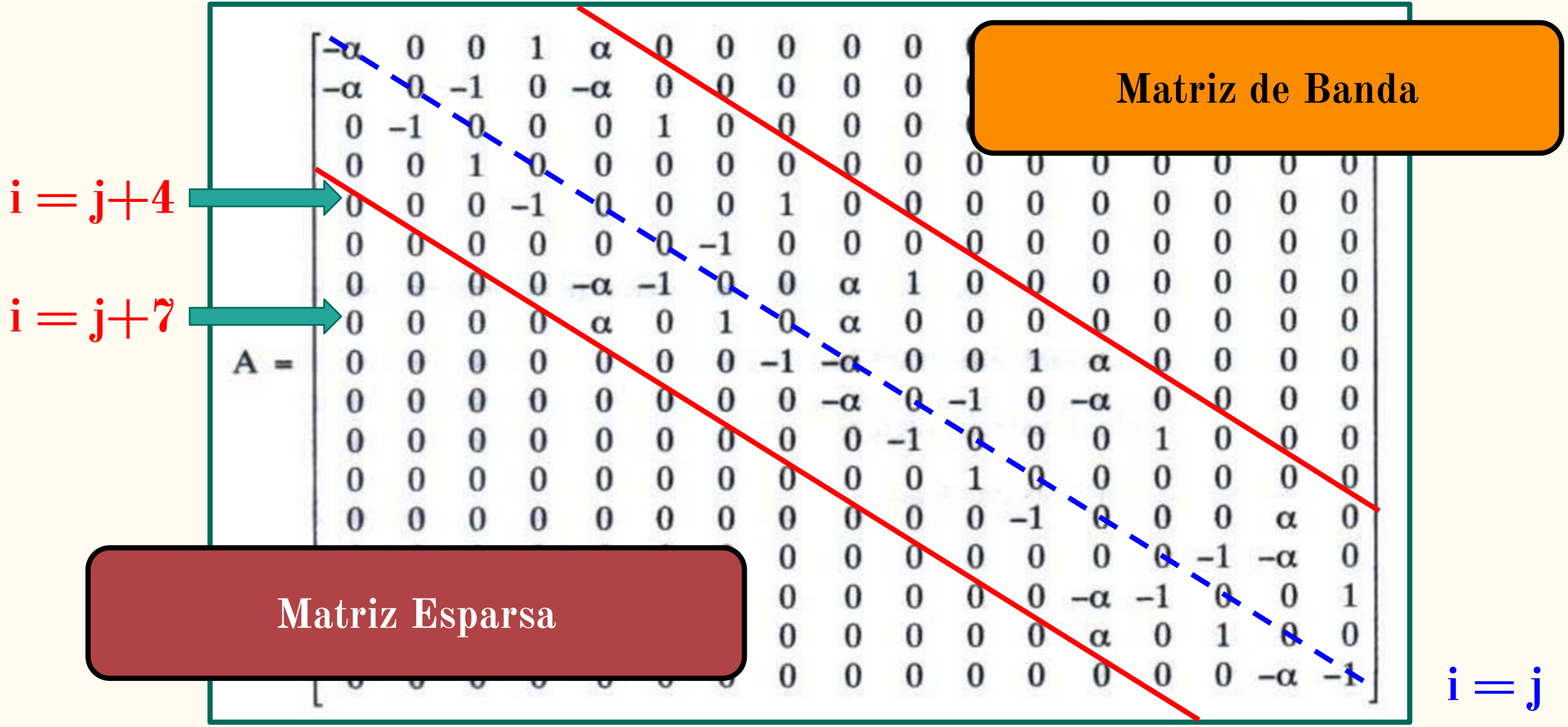
Matrizes k-Diagonais



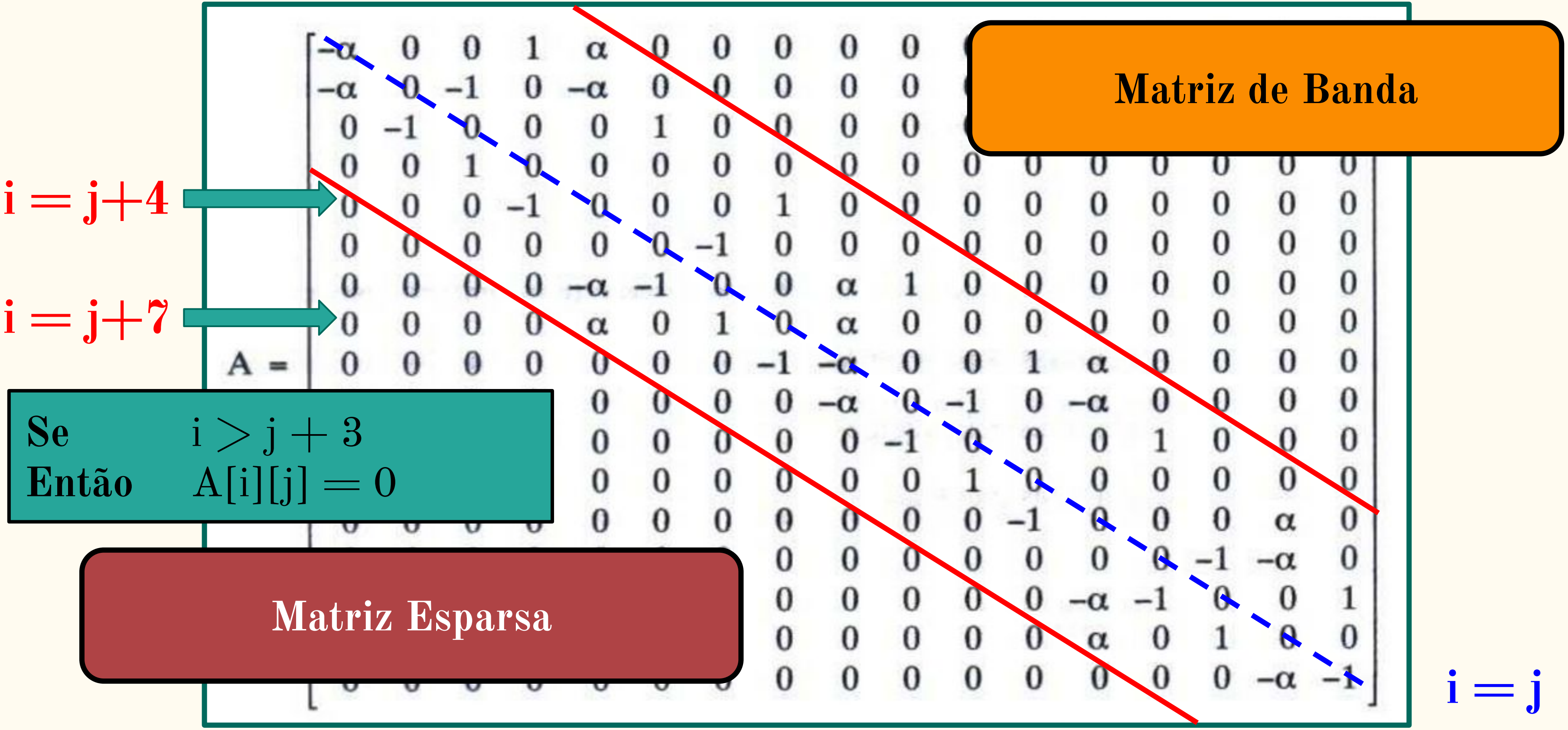
Matrizes k-Diagonais



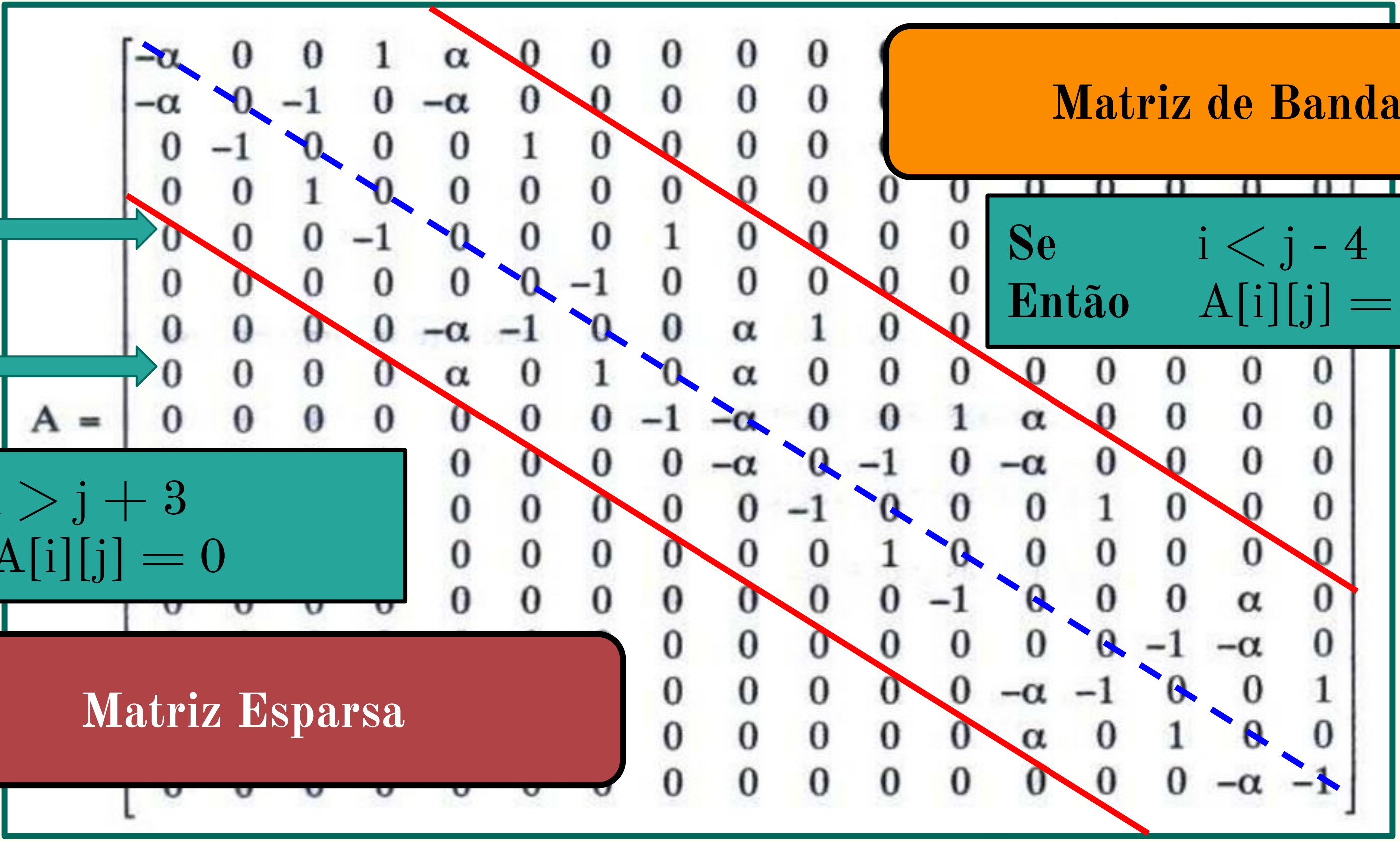
Matrizes k-Diagonais



Matrizes k-Diagonais



Matrizes k-Diagonais



Matriz de Banda

Se $i < j - 4$
Então $A[i][j] = 0$

$i = j+4$

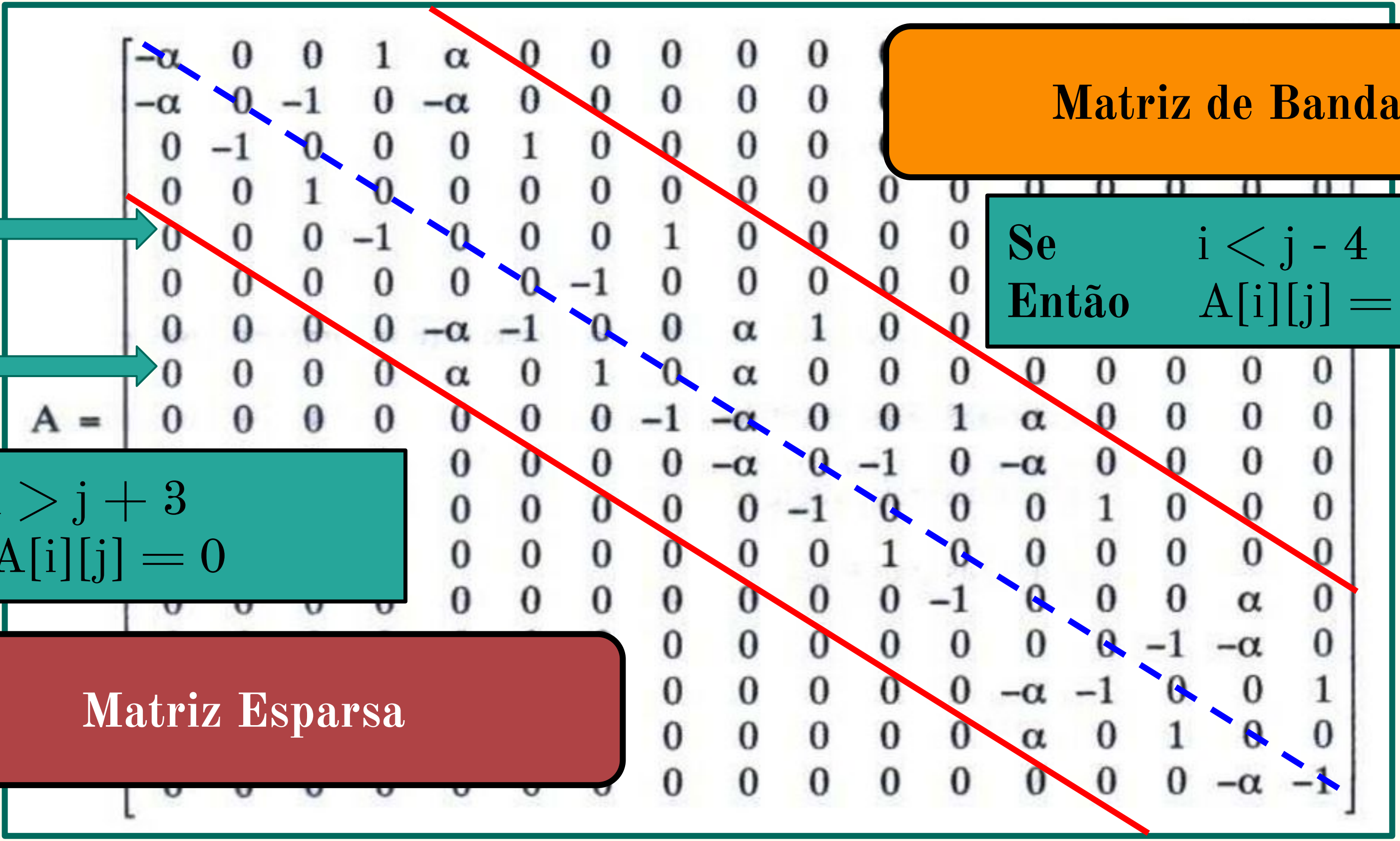
$i = j+7$

Se $i > j + 3$
Então $A[i][j] = 0$

Matriz Esparsa

$i = j$

Matrizes k-Diagonais



Matriz de Banda 8

Se $i < j - 4$
Então $A[i][j] = 0$

$i = j+4$

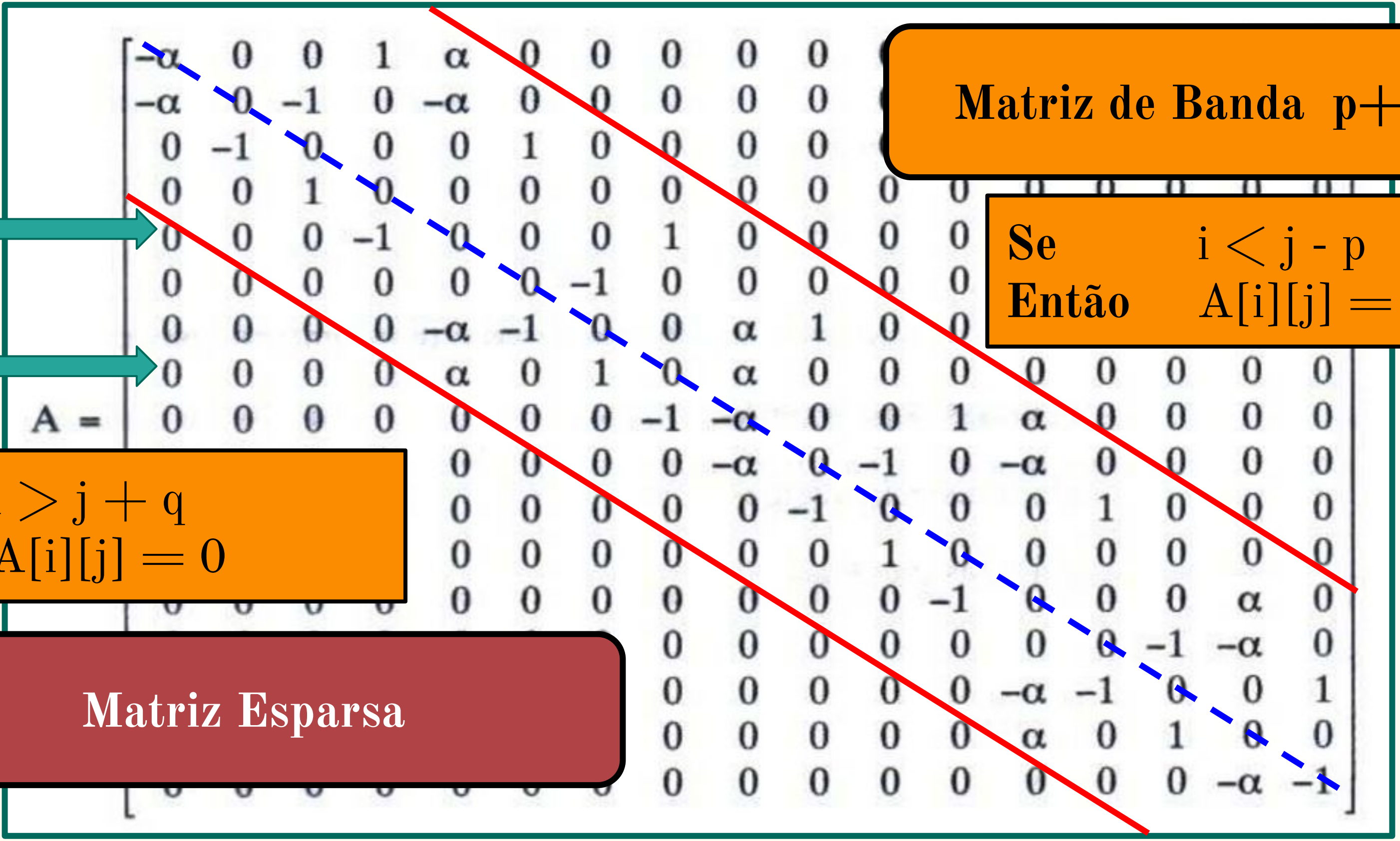
$i = j+7$

Se $i > j + 3$
Então $A[i][j] = 0$

Matriz Esparsa

$i = j$

Matrizes k-Diagonais



Matriz de Banda p+q+1

Se $i < j - p$
Então $A[i][j] = 0$

$i = j+4$

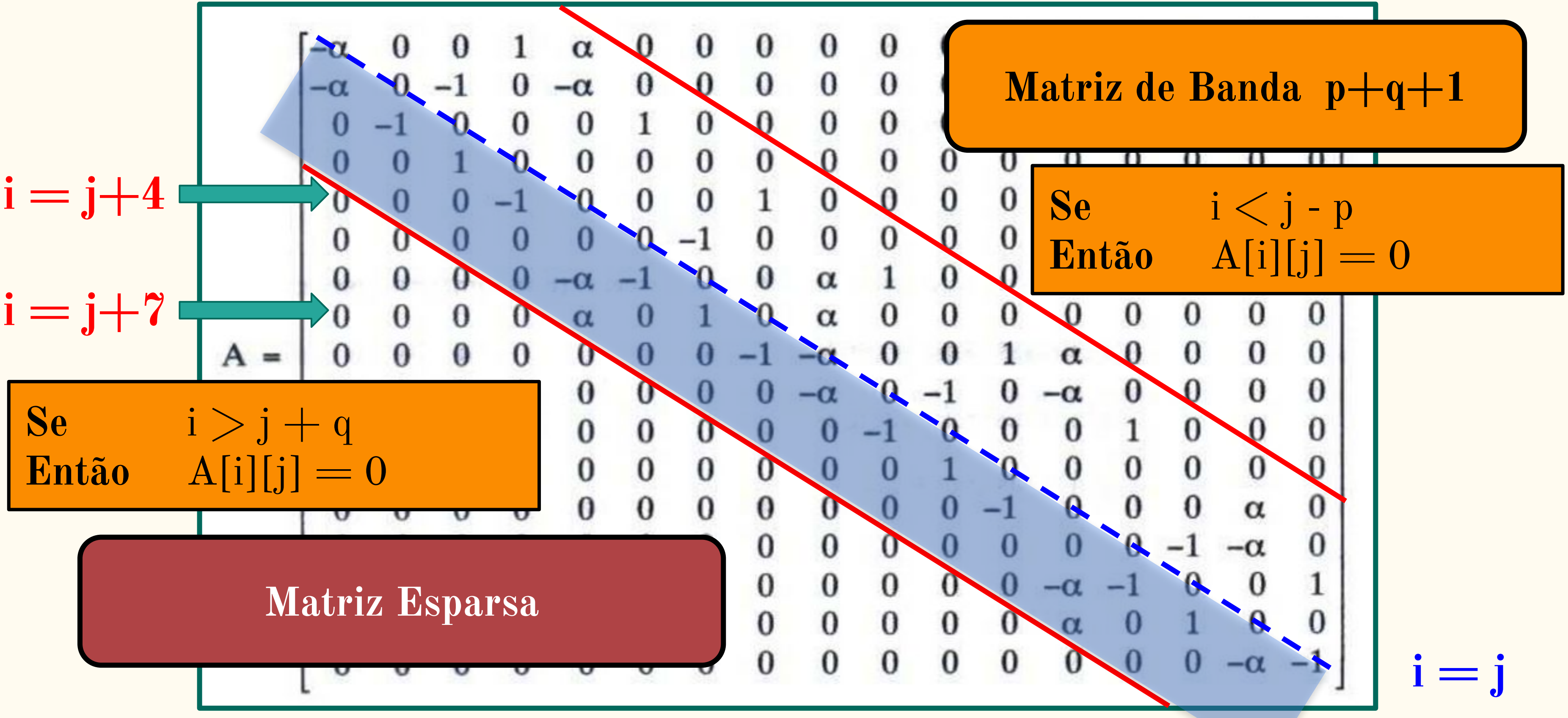
$i = j+7$

Se $i > j + q$
Então $A[i][j] = 0$

Matriz Esparsa

$i = j$

Matrizes k-Diagonais



Matrizes Tridiagonais $(p = q = 1)$

$$\begin{bmatrix} d_1 & c_1 & 0 & 0 & 0 \\ a_1 & d_2 & c_2 & 0 & 0 \\ 0 & a_2 & d_3 & c_3 & 0 \\ 0 & 0 & a_3 & d_4 & c_4 \\ 0 & 0 & 0 & a_4 & d_5 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

Matrizes Tridiagonais $(p = q = 1)$

$$
\begin{bmatrix}
d_1 & c_1 & 0 & 0 & 0 \\
a_1 & d_2 & c_2 & 0 & 0 \\
0 & a_2 & d_3 & c_3 & 0 \\
0 & 0 & a_3 & d_4 & c_4 \\
0 & 0 & 0 & a_4 & d_5
\end{bmatrix}
\begin{bmatrix}
b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5
\end{bmatrix}
$$

```
/* Seja um S.L. de ordem 'n'
*/
void eliminacaoGauss( double **A, double *b, u
   /* para cada linha a partir da primeira */
   for (int i=0; i < n; ++i) {
      for(int k=i+1; k < n; ++k) {
         double m = A[k][i] / A[i][i];
         A[k][i] = 0.0;
         for(int j=i+1; j < n; ++j)
            A[k][j] -= A[i][j] * m;
         b[k] -= b[i] * m;
      }
   }
}
```

Matrizes Tridiagonais $(p = q = 1)$

$$
\begin{bmatrix}
d_1 & c_1 & 0 & 0 & 0 \\
a_1 & d_2 & c_2 & 0 & 0 \\
0 & a_2 & d_3 & c_3 & 0 \\
0 & 0 & a_3 & d_4 & c_4 \\
0 & 0 & 0 & a_4 & d_5
\end{bmatrix}
\begin{bmatrix}
b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5
\end{bmatrix}
$$

```
/* Seja um S.L. de ordem 'n'
*/
void eliminacaoGauss( double **A, double *b, u
   /* para cada linha a partir da primeira */
   for (int i=0; i < n; ++i) {
      for(int k=i+1; k < n; ++k) {
         double m = A[k][i] / A[i][i];
         A[k][i] = 0.0;
         for(int j=i+1; j < n; ++j)
            A[k][j] -= A[i][j] * m;
         b[k] -= b[i] * m;
      }
   }
}
```

Matrizes Tridiagonais $(p = q = 1)$

$$\begin{bmatrix} d_1 & c_1 & 0 & 0 & 0 \\ a_1 & d_2 & c_2 & 0 & 0 \\ 0 & a_2 & d_3 & c_3 & 0 \\ 0 & 0 & a_3 & d_4 & c_4 \\ 0 & 0 & 0 & a_4 & d_5 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

```
/* Seja um S.L. de ordem 'n'
*/
void eliminacaoGauss( double **A, double *b, u
   /* para cada linha a partir da primeira */
   for (int i=0; i < n; ++i) {
      for(int k=i+1; k < n; ++k) {
         double m = A[k][i] / A[i][i];
         A[k][i] = 0.0;
         for(int j=i+1; j < n; ++j)
            A[k][j] -= A[i][j] * m;
         b[k] -= b[i] * m;
      }
   }
}
```

Matrizes Tridiagonais $(p = q = 1)$

$$\begin{bmatrix} d_1 & c_1 & 0 & 0 & 0 \\ a_1 & d_2 & c_2 & 0 & 0 \\ 0 & a_2 & d_3 & c_3 & 0 \\ 0 & 0 & a_3 & d_4 & c_4 \\ 0 & 0 & 0 & a_4 & d_5 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

```
/* Seja um S.L. de ordem 'n'
*/
void eliminacaoGauss( double **A, double *b, u
   /* para cada linha a partir da primeira */
   for (int i=0; i < n; ++i) {
      for(int k=i+1; k < n; ++k) {
         double m = A[k][i] / A[i][i];
         A[k][i] = 0.0;
         for(int j=i+1; j < n; ++j)
            A[k][j] -= A[i][j] * m;
         b[k] -= b[i] * m;
      }
   }
}
```

$k = i+1$

Matrizes Tridiagonais $(p = q = 1)$

$$\begin{bmatrix} d_1 & c_1 & 0 & 0 & 0 \\ a_1 & d_2 & c_2 & 0 & 0 \\ 0 & a_2 & d_3 & c_3 & 0 \\ 0 & 0 & a_3 & d_4 & c_4 \\ 0 & 0 & 0 & a_4 & d_5 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

```
/* Seja um S.L. de ordem 'n'
*/
void eliminacaoGauss( double **A, double *b, u
   /* para cada linha a partir da primeira */
   for (int i=0; i < n; ++i) {
     for(int k=i+1; k < n; ++k) {
        double m = A[k][i] / A[i][i];
        A[k][i] = 0.0;
        for(int j=i+1; j < n; ++j)
           A[k][j] -= A[i][j] * m;
        b[k] -= b[i] * m;
     }
   }
}
```

$k = i+1$

Matrizes Tridiagonais $(p = q = 1)$

$$\begin{bmatrix} d_1 & c_1 & 0 & 0 & 0 \\ a_1 & d_2 & c_2 & 0 & 0 \\ 0 & a_2 & d_3 & c_3 & 0 \\ 0 & 0 & a_3 & d_4 & c_4 \\ 0 & 0 & 0 & a_4 & d_5 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

```
/* Seja um S.L. de ordem 'n'
*/
void eliminacaoGauss( double **A, double *b, u
   /* para cada linha a partir da primeira */
   for (int i=0; i < n; ++i) {
      for(int k=i+1; k < n; ++k) {
         double m = A[k][i] / A[i][i];
         A[k][i] = 0.0;
         for(int j=i+1; j < n; ++j)
            A[k][j] -= A[i][j] * m;
         b[k] -= b[i] * m;
      }
   }
}
```

$k = i+1$

$j = i+1$

Matrizes Tridiagonais $(p = q = 1)$

$$\begin{bmatrix} d_1 & c_1 & 0 & 0 & 0 \\ a_1 & d_2 & c_2 & 0 & 0 \\ 0 & a_2 & d_3 & c_3 & 0 \\ 0 & 0 & a_3 & d_4 & c_4 \\ 0 & 0 & 0 & a_4 & d_5 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

```
double b[5];
double d[5];
double a[4];
double c[4];
```

```
/* Seja um S.L. de ordem 'n'
*/
void eliminacaoGauss( double **A, double *b, u
   /* para cada linha a partir da primeira */
   for (int i=0; i < n; ++i) {
      for(int k=i+1; k < n; ++k) {
         double m = A[k][i] / A[i][i];
         A[k][i] = 0.0;
         for(int j=i+1; j < n; ++j)
            A[k][j] -= A[i][j] * m;
         b[k] -= b[i] * m;
      }
   }
}
```

$k = i+1$

$j = i+1$

Matrizes Tridiagonais (p = q = 1)

$$\begin{bmatrix} d_1 & c_1 & 0 & 0 & 0 \\ a_1 & d_2 & c_2 & 0 & 0 \\ 0 & a_2 & d_3 & c_3 & 0 \\ 0 & 0 & a_3 & d_4 & c_4 \\ 0 & 0 & 0 & a_4 & d_5 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

```
double b[5];
double d[5];
double a[4];
double c[4];
```

```
double *d, double *a, double *c
```

```
/* Seja um S.L. de ordem
*/
void eliminacaoGauss( double **A, double *b, u
   /* para cada linha a partir da primeira */
   for (int i=0; i < n; ++i) {
      for(int k=i+1; k < n; ++k) {
         double m = A[k][i] / A[i][i];
         A[k][i] = 0.0;
         for(int j=i+1; j < n; ++j)
            A[k][j] -= A[i][j] * m;
         b[k] -= b[i] * m;
      }
   }
}
```

Matrizes Tridiagonais (p = q = 1)

$$
\begin{bmatrix}
d_1 & c_1 & 0 & 0 & 0 \\
a_1 & d_2 & c_2 & 0 & 0 \\
0 & a_2 & d_3 & c_3 & 0 \\
0 & 0 & a_3 & d_4 & c_4 \\
0 & 0 & 0 & a_4 & d_5
\end{bmatrix}
\begin{bmatrix}
b_1 \\ \\ b_3 \\ b_4 \\ b_5
\end{bmatrix}
$$

```
double b[5];
double d[5];
double a[4];
double c[4];
```

```
double *d, double *a, double *c
```

```
double m = a[i] / d[i];
```

```
                       ordem
/
void elim          Gauss( double **A, double *b, u
   /* para c     linha a partir da primeira */
   for (int i=0    i < n; ++i) {
      for(int k=i+1; k < n; ++k) {
         double m = A[k][i] / A[i][i];
         A[k][i] = 0.0;
         for(int j=i+1; j < n; ++j)
            A[k][j] -= A[i][j] * m;
         b[k] -= b[i] * m;
      }
   }
}
```

Matrizes Tridiagonais ($p = q = 1$)

$$\begin{bmatrix} d_1 & c_1 & 0 & 0 & 0 \\ a_1 & d_2 & c_2 & 0 & 0 \\ 0 & a_2 & d_3 & c_3 & 0 \\ 0 & 0 & a_3 & d_4 & c_4 \\ 0 & 0 & 0 & a_4 & d_5 \end{bmatrix} \begin{bmatrix} b_1 \\ \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

```
double *d, double *a, double *c
```

```
double m = a[i] / d[i];
```

```
a[i] = 0.0;
```

```
ordem
void elim        Gauss( double **A, double *b, u
  /* para c      lin              a primeira */
  for (int i=0    i
    for(int k=i+1;    < n; ++k) {
      double m = A[k][i] / A[i][i];
      A[k][i] = 0.0;
      for(int j=i+1; j < n; ++j)
        A[k][j] -= A[i][j] * m;
      b[k] -= b[i] * m;
    }
  }
}
```

double b[**5**];
double d[**5**];
double a[**4**];
double c[**4**];

Matrizes Tridiagonais $(p = q = 1)$

$$\begin{bmatrix} d_1 & c_1 & 0 & 0 & 0 \\ a_1 & d_2 & c_2 & 0 & 0 \\ 0 & a_2 & d_3 & c_3 & 0 \\ 0 & 0 & a_3 & d_4 & c_4 \\ 0 & 0 & 0 & a_4 & d_5 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

```
double b[5];
double d[5];
double a[4];
double c[4];
```

```
double *d, double *a, double *c
```

```
double m = a[i] / d[i];
```

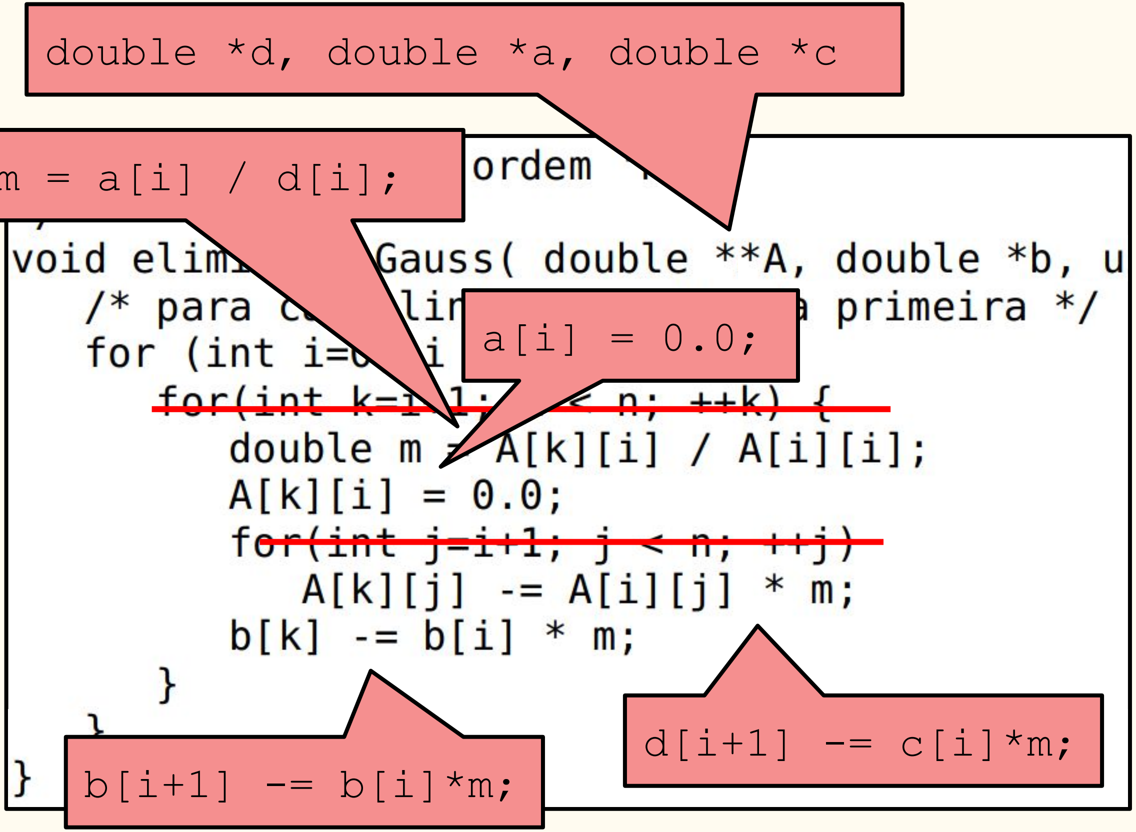
```
a[i] = 0.0;
```

```
d[i+1] -= c[i]*m;
```

```
 ordem *
 */
void elim   Gauss( double **A, double *b, u
   /* para c   lin          a primeira */
   for (int i=0   i
      for(int k=i+1;   < n; ++k) {
         double m = A[k][i] / A[i][i];
         A[k][i] = 0.0;
         for(int j=i+1; j < n; ++j)
            A[k][j] -= A[i][j] * m;
         b[k] -= b[i] * m;
      }
   }
}
```

Matrizes Tridiagonais (p = q = 1)

$$
\begin{bmatrix}
d_1 & c_1 & 0 & 0 & 0 \\
a_1 & d_2 & c_2 & 0 & 0 \\
0 & a_2 & d_3 & c_3 & 0 \\
0 & 0 & a_3 & d_4 & c_4 \\
0 & 0 & 0 & a_4 & d_5
\end{bmatrix}
\begin{bmatrix}
b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5
\end{bmatrix}
$$




```
double b[5];
double d[5];
double a[4];
double c[4];
```

Fatoração LU

Problema: calcular a inversa de uma matriz.

$$\underset{A}{\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}} \underset{A^{-1}}{\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}} = \underset{I}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

Fatoração LU

$a_{11}x_{11} + a_{12}x_{21} + a_{13}x_{31} + a_{14}x_{41} = 1$

<u>Problema</u>: calcular a inversa de uma matriz.

$$\underset{A}{\begin{bmatrix} \boxed{a_{11} \; a_{12} \; a_{13} \; a_{14}} \\ a_{21} \; a_{22} \; a_{23} \; a_{24} \\ a_{31} \; a_{32} \; a_{33} \; a_{34} \\ a_{41} \; a_{42} \; a_{43} \; a_{44} \end{bmatrix}} \underset{A^{-1}}{\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}} = \underset{I}{\begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

Fatoração LU

$a_{11}x_{11} + a_{12}x_{21} + a_{13}x_{31} + a_{14}x_{41} = 1$

$a_{21}x_{11} + a_{22}x_{21} + a_{23}x_{31} + a_{24}x_{41} = 0$

<u>Problema</u>: calcular a inversa de uma matriz.

$$\overset{A}{\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}} \overset{A^{-1}}{\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}} = \overset{I}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ \boxed{0} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

Fatoração LU

$$a_{11}x_{11} + a_{12}x_{21} + a_{13}x_{31} + a_{14}x_{41} = 1$$
$$a_{21}x_{11} + a_{22}x_{21} + a_{23}x_{31} + a_{24}x_{41} = 0$$
$$a_{31}x_{11} + a_{32}x_{21} + a_{33}x_{31} + a_{34}x_{41} = 0$$

Problema: calcular a inversa de uma matriz.

$$\begin{array}{ccc} A & A^{-1} & I \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \boxed{a_{31} \; a_{32} \; a_{33} \; a_{34}} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} & \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix} = & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \boxed{0} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Fatoração LU

<u>Problema</u>: calcular a inversa de uma matriz.

$$\begin{cases} a_{11}x_{11} + a_{12}x_{21} + a_{13}x_{31} + a_{14}x_{41} = 1 \\ a_{21}x_{11} + a_{22}x_{21} + a_{23}x_{31} + a_{24}x_{41} = 0 \\ a_{31}x_{11} + a_{32}x_{21} + a_{33}x_{31} + a_{34}x_{41} = 0 \\ a_{41}x_{11} + a_{42}x_{21} + a_{43}x_{31} + a_{44}x_{41} = 0 \end{cases}$$

$$\underset{A}{\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ \boxed{a_{41} \; a_{42} \; a_{43} \; a_{44}} \end{bmatrix}} \underset{A^{-1}}{\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}} = \underset{I}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \boxed{0} & 0 & 0 & 1 \end{bmatrix}}$$

Fatoração LU

<u>Problema</u>: calcular a inversa de uma matriz.

$$\begin{cases} a_{11}x_{11} + a_{12}x_{21} + a_{13}x_{31} + a_{14}x_{41} = 1 \\ a_{21}x_{11} + a_{22}x_{21} + a_{23}x_{31} + a_{24}x_{41} = 0 \\ a_{31}x_{11} + a_{32}x_{21} + a_{33}x_{31} + a_{34}x_{41} = 0 \\ a_{41}x_{11} + a_{42}x_{21} + a_{43}x_{31} + a_{44}x_{41} = 0 \end{cases}$$

$$Ax_1 = i_1$$

$$\underset{A}{\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}} \underset{A^{-1}}{\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}} = \underset{I}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

Fatoração LU

<u>Problema</u>: calcular a inversa de uma matriz.

$$\begin{cases} a_{11}x_{11} + a_{12}x_{21} + a_{13}x_{31} + a_{14}x_{41} = 1 \\ a_{21}x_{11} + a_{22}x_{21} + a_{23}x_{31} + a_{24}x_{41} = 0 \\ a_{31}x_{11} + a_{32}x_{21} + a_{33}x_{31} + a_{34}x_{41} = 0 \\ a_{41}x_{11} + a_{42}x_{21} + a_{43}x_{31} + a_{44}x_{41} = 0 \end{cases}$$

$$\underset{\mathbf{A}}{\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}} \underset{\mathbf{A}^{-1}}{\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}} = \underset{\mathbf{I}}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

$Ax_1 = i_1$

$Ax_2 = i_2$

$Ax_3 = i_3$

$Ax_4 = i_4$

Fatoração LU

Refinamento

Algoritmo:

Basta aplicar em $\mathbf{r}$ todas as transformações que foram aplicadas em $\mathbf{b}$

1 - Obter solução inicial $\mathbf{x}^{(0)}$ resolvendo $\mathbf{Ax} = \mathbf{b}$ e inicializar $\mathbf{i} = \mathbf{0}$;
2 - Calcular o resíduo $\mathbf{r} = \mathbf{b} - \mathbf{Ax}^{(i)}$ e testar critério de parada (a);
3 - Obter $\mathbf{w}$ resolvendo $\mathbf{Aw} = \mathbf{r}$;
4 - Obter nova solução $\mathbf{x}^{(i+1)} = \mathbf{x}^{(i)} + \mathbf{w}$ e testar critério de parada (b);
5 - Incrementar $\mathbf{i}$ e voltar ao passo 2;

$Ax_4 = i_4$

Fatoração LU

Decompor $\mathbf{A}$ em duas matrizes triangulares $\mathbf{A} = \mathbf{LU}$.

Fatoração LU

Decompor $\mathbf{A}$ em duas matrizes triangulares $\mathbf{A} = \mathbf{LU}$.

$$\mathrm{Ax} = \mathrm{b}$$

Fatoração LU

Decompor $\mathbf{A}$ em duas matrizes triangulares $\mathbf{A} = \mathbf{LU}$.

$$\mathrm{Ax} = \mathrm{b}$$

$$\mathrm{LUx} = \mathrm{b}$$

Fatoração LU

Decompor $\mathbf{A}$ em duas matrizes triangulares $\mathbf{A} = \mathbf{LU}$.

$$\mathrm{Ax} = \mathrm{b}$$

$$\mathrm{L}\underbrace{\mathrm{Ux}}_{\mathrm{y}} = \mathrm{b}$$

Fatoração LU

Decompor $\mathbf{A}$ em duas matrizes triangulares $\mathbf{A} = \mathbf{LU}$.

$$Ax = b$$

$$L\underbrace{Ux}_{y} = b \quad \Rightarrow \quad \begin{array}{l} Ly = b \\ Ux = y \end{array}$$

Fatoração LU

Decompor $\mathbf{A}$ em duas matrizes triangulares $\mathbf{A} = \mathbf{LU}$.

$$Ax = b$$

$$L\underbrace{Ux}_{y} = b \quad \Rightarrow \quad \begin{array}{l} Ly = b \\ Ux = y \end{array}$$

$Ly = b$	n^2 FLOPs
$Ux = y$	n^2 FLOPs

Fatoração LU

Decompor $\mathbf{A}$ em duas matrizes triangulares $\mathbf{A} = \mathbf{LU}$.

$$\mathrm{Ax} = \mathrm{b}$$

?

$$\underbrace{\mathrm{LUx}}_{\mathrm{y}} = \mathrm{b}$$

$\mathrm{Ly} = \mathrm{b}$	n^2 FLOPs
$\mathrm{Ux} = \mathrm{y}$	n^2 FLOPs

Fatoração LU

Decompor $\mathbf{A}$ em duas matrizes triangulares $\mathbf{A} = \mathbf{LU}$.

$Ax_1 = i_1$

$Ax_2 = i_2$

$Ax_3 = i_3$

$\vdots$

$Ax_k = i_k$

?

$$Ax = b$$

$$L\underbrace{Ux}_{y} = b \Rightarrow \begin{array}{l} Ly = b \\ Ux = y \end{array}$$

$Ly = b$	n^2 FLOPs
$Ux = y$	n^2 FLOPs

$$\boxed{\text{custo(decomposição)} + 2kn^2 \text{ FLOPs}}$$

Fatoração LU

Decompor $\mathbf{A}$ em duas matrizes triangulares $\mathbf{A} = \mathbf{LU}$.

$$\mathrm{A}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Fatoração LU

m_{21}

m_{31}

m_{41}

Decompor $\mathbf{A}$ em duas matrizes triangulares $\mathbf{A = LU}$.

$$A \qquad\qquad A_1$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{array}{c} \\ m_{21} \\ m_{31} \\ m_{41} \end{array} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & b_{22} & b_{23} & b_{24} \\ 0 & b_{32} & b_{33} & b_{34} \\ 0 & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

Fatoração LU

m_{21}

$m_{31} \quad m_{32}$

$m_{41} \quad m_{42}$

Decompor **A** em duas matrizes triangulares $\mathbf{A} = \mathbf{LU}$.

$$A_1 \qquad\qquad\qquad\qquad A_2$$

$$\begin{array}{cc} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & b_{22} & b_{23} & b_{24} \\ 0 & b_{32} & b_{33} & b_{34} \\ 0 & b_{42} & b_{43} & b_{44} \end{bmatrix} & \begin{array}{c} \\ \\ m_{32} \\ m_{42} \end{array} \end{array} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & b_{22} & b_{23} & b_{24} \\ 0 & 0 & c_{33} & c_{34} \\ 0 & 0 & c_{43} & c_{44} \end{bmatrix}$$

Fatoração LU

Decompor $\mathbf{A}$ em duas matrizes triangulares $\mathbf{A} = \mathbf{LU}$.

m_{21}
m_{31} m_{32}
m_{41} m_{42} m_{43}

$$A_2 \qquad A_3$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & b_{22} & b_{23} & b_{24} \\ 0 & 0 & c_{33} & c_{34} \\ 0 & 0 & c_{43} & c_{44} \end{bmatrix} m_{43} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & b_{22} & b_{23} & b_{24} \\ 0 & 0 & c_{33} & c_{34} \\ 0 & 0 & 0 & d_{44} \end{bmatrix}$$

Fatoração LU

Decompor $\mathbf{A}$ em duas matrizes triangulares $\mathbf{A} = \mathbf{LU}$.

$$
\begin{array}{lll}
m_{21} & & \\
m_{31} & m_{32} & \\
m_{41} & m_{42} & m_{43}
\end{array}
$$

$$
\begin{bmatrix} 1 & 0 & 0 & 0 \\ m_{21} & 1 & 0 & 0 \\ m_{31} & m_{32} & 1 & 0 \\ m_{41} & m_{42} & m_{43} & 1 \end{bmatrix}
\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & b_{22} & b_{23} & b_{24} \\ 0 & 0 & c_{33} & c_{34} \\ 0 & 0 & 0 & d_{44} \end{bmatrix}}_{\mathrm{A}_3}
=
\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}}_{\mathrm{A}}
$$

Fatoração LU

Decompor $\mathbf{A}$ em duas matrizes triangulares $\mathbf{A} = \mathbf{LU}$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \boxed{m_{21} \quad 1 \quad 0 \quad 0} & & & \\ m_{31} & m_{32} & 1 & 0 \\ m_{41} & m_{42} & m_{43} & 1 \end{bmatrix} \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & b_{22} & b_{23} & b_{24} \\ 0 & 0 & c_{33} & c_{34} \\ 0 & 0 & 0 & d_{44} \end{bmatrix}}_{\mathbf{A_3}} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & \boxed{a_{22}} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}}_{\mathbf{A}}$$

$$\begin{aligned} m_{21}a_{12} + b_{22} &= (a_{21}/a_{11})a_{12} + (a_{22} - a_{12}m_{21}) \\ &= (a_{12}a_{21})/a_{11} + a_{22} - (a_{12}a_{21})/a_{11} \end{aligned}$$

Fatoração LU

Decompor $\mathbf{A}$ em duas matrizes triangulares $\mathbf{A} = \mathbf{LU}$.

$$\underset{\mathbf{L}}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ m_{21} & 1 & 0 & 0 \\ m_{31} & m_{32} & 1 & 0 \\ m_{41} & m_{42} & m_{43} & 1 \end{bmatrix}} \underset{\mathbf{U}}{\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & b_{22} & b_{23} & b_{24} \\ 0 & 0 & c_{33} & c_{34} \\ 0 & 0 & 0 & d_{44} \end{bmatrix}} = \underset{\mathbf{A}}{\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}}$$

Fatoração LU

Exemplo: encontrar a inversa da matriz **A**.

$$\mathbf{A} = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Fatoração LU

<u>Exemplo</u>: encontrar a inversa da matriz **A**.

$$\underset{\textstyle A}{\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}} \underset{\textstyle X}{\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}} = \underset{\textstyle B}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

Fatoração LU

<u>Exemplo</u>: encontrar a inversa da matriz **A**.

$$A = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

$m_{21} = 2.56$

$m_{31} = 5.76$

Fatoração LU

Exemplo: encontrar a inversa da matriz $\mathbf{A}$.

$$\begin{array}{c} A \\ \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \end{array} \begin{array}{l} m_{21} = 2.56 \\ m_{31} = 5.76 \end{array} \Rightarrow \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

Fatoração LU

<u>Exemplo</u>: encontrar a inversa da matriz **A**.

$$\overset{\textstyle A}{\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}} \begin{matrix} \\ m_{21} = 2.56 \\ m_{31} = 5.76 \end{matrix} \Rightarrow \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix} m_{32} = 3.5$$

Fatoração LU

Exemplo: encontrar a inversa da matriz $\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \quad \begin{array}{l} m_{21} = 2.56 \\ m_{31} = 5.76 \end{array} \Rightarrow \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix} \quad m_{32} = 3.5$$

$$\Rightarrow \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Fatoração LU

<u>Exemplo</u>: encontrar a inversa da matriz **A**.

$$\mathbf{A} = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \quad \begin{matrix} m_{21} = 2.56 \\ m_{31} = 5.76 \end{matrix} \Rightarrow \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix} m_{32} = 3.5$$

$$\Rightarrow \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} = \mathbf{U}$$

Fatoração LU

<u>Exemplo</u>: encontrar a inversa da matriz **A**.

$$A = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \quad \begin{matrix} m_{21} = 2.56 \\ m_{31} = 5.76 \end{matrix} \Rightarrow \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix} m_{32} = 3.5$$

$$\Rightarrow U = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix}$$

Fatoração LU

$$Ax = b$$

$$\underbrace{LUx}_{y} = b \quad \Rightarrow \quad Ly = b$$

$$Ux = y$$

<u>Exemplo</u>: encontrar a inversa da matriz **A**.

$$A = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{matrix} m_{21} = 2.56 \\ m_{31} = 5.76 \end{matrix} \Rightarrow \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix} m_{32} = 3.5$$

$$\Rightarrow \underset{U}{\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}} \quad \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix}}$$

Fatoração LU

<u>Exemplo</u>: encontrar a inversa da matriz **A**.

$$Ax = b$$

$$\underbrace{LUx}_{y} = b \quad \Rightarrow \quad Ly = b$$

$$Ux = y$$

L

$$\begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2.56 \\ 3.2 \end{bmatrix}$$

Aplica o mesmo escalonamento no vetor b

U

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2.56 \\ 3.2 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.04762 \\ -0.9524 \\ 4.571 \end{bmatrix}$$

Fatoração LU

<u>Exemplo</u>: encontrar a inversa da matriz **A**.

$$Ax = b$$

$$\underbrace{LUx}_{y} = b \Rightarrow \begin{array}{l} Ly = b \\ Ux = y \end{array}$$

$$\overset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix}} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -3.5 \end{bmatrix}$$

Aplica o mesmo escalonamento no vetor b

$$\overset{U}{\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -3.5 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -0.08333 \\ 1.417 \\ -5.000 \end{bmatrix}$$

Fatoração LU

Exemplo: encontrar a inversa da matriz **A**.

$$Ax = b$$

$$\underbrace{LUx}_{y} = b \Rightarrow Ly = b$$

$$Ux = y$$

L

$$\begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Aplica o mesmo escalonamento no vetor b

U

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.03571 \\ -0.4643 \\ 1.429 \end{bmatrix}$$

Fatoração LU

Exemplo: encontrar a inversa da matriz **A**.

$$A^{-1}$$

$$\begin{bmatrix} 0.04762 & -0.08333 & 0.03571 \\ -0.9524 & 1.417 & -0.4643 \\ 4.571 & -5.000 & 1.429 \end{bmatrix}$$

$$Ax = b$$

$$\underbrace{LUx}_{y} = b \Rightarrow \begin{array}{l} Ly = b \\ Ux = y \end{array}$$

Fatoração LU

<u>Exemplo</u>: encontrar a inversa da matriz $\mathbf{A}$.

Para pivoteamento parcial, efetuar as mesmas trocas de linhas no vetor $\mathbf{b}$.

$$Ax = b$$

$$L\underbrace{Ux}_{y} = b \quad \Rightarrow \quad \begin{array}{l} Ly = b \\ Ux = y \end{array}$$

Fatoração LU

$$Ax = b$$

$$\underbrace{LUx}_{y} = b \Rightarrow Ly = b$$

$$Ux = y$$

Exemplo: encontrar a inversa da matriz **A**.

Para pivoteamento parcial, efetuar as mesmas trocas de linhas no vetor **b**.

Exemplo:

$$\overset{A}{\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}} \Rightarrow \overset{A'}{\begin{bmatrix} 144 & 12 & 1 \\ 64 & 8 & 1 \\ 25 & 5 & 1 \end{bmatrix}} \Rightarrow A' = LU$$

Fatoração LU

Exemplo: encontrar a inversa da matriz **A**.

$$Ax = b$$

Para pivoteamento parcial, efetuar as m
de linhas no vetor **b**.

$$LU\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Exemplo:

$$\mathbf{A} \qquad \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \Rightarrow \mathbf{A'} \qquad \begin{bmatrix} 144 & 12 & 1 \\ 64 & 8 & 1 \\ 25 & 5 & 1 \end{bmatrix} \Rightarrow A' = LU$$

Fatoração LU

Exemplo: encontrar a inversa da matriz **A**.

Para pivoteamento parcial, efetuar as m
de linhas no vetor **b**.

$$Ax = b$$

$$LU \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Atenção: além do vetor **b**, as mesmas trocas de linhas devem ser feitas a cada iteração na matriz **L**.

Exemplo:

$$\mathbf{A} \quad \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \Rightarrow \mathbf{A'} \quad \begin{bmatrix} 144 & 12 & 1 \\ 64 & 8 & 1 \\ 25 & 5 & 1 \end{bmatrix} \Rightarrow A' = LU$$

Fatoração LU

Exemplo:

$$\begin{array}{ccc} \mathbf{A} & \mathbf{x} & \mathbf{b} \\ \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} & \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = & \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \end{array}$$

Fatoração LU

Exemplo:

A

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

L

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

b

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Fatoração LU

<u>Exemplo</u>:

$$A = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Escolhendo o primeiro pivô:
trocar as linhas 1 e 3.

$$L = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

$$b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Fatoração LU

<u>Exemplo</u>:

$$A = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

$$b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

A mesma troca deve ser feita no vetor **b.**

Fatoração LU

Exemplo:

$$A = \begin{bmatrix} 144 & 12 & 1 \\ 64 & 8 & 1 \\ 25 & 5 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

A mesma troca deve ser feita no vetor **b**.

Fatoração LU

Exemplo:

$$A = \begin{bmatrix} 144 & 12 & 1 \\ 64 & 8 & 1 \\ 25 & 5 & 1 \end{bmatrix} \quad \begin{matrix} m_{21} = 0.4444 \\ m_{31} = 0.1736 \end{matrix}$$

Definindo multiplicadores da primeira iteração.

$$L = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Fatoração LU

Exemplo:

Armazenando na matriz **L**.

$$A = \begin{bmatrix} 144 & 12 & 1 \\ 64 & 8 & 1 \\ 25 & 5 & 1 \end{bmatrix} \quad \begin{array}{l} m_{21} = 0.4444 \\ m_{31} = 0.1736 \end{array}$$

$$L = \begin{bmatrix} & & \\ 0.4444 & & \\ 0.1736 & & \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Fatoração LU

Exemplo:

A

$$\begin{bmatrix} 144 & 12 & 1 \\ 64 & 8 & 1 \\ 25 & 5 & 1 \end{bmatrix} \quad \begin{matrix} m_{21} = 0.4444 \\ m_{31} = 0.1736 \end{matrix}$$

Aplicando operações sobre as equações.

$$\begin{bmatrix} 144 & 12 & 1 \\ 0 & 2.6672 & 0.5556 \\ 0 & 2.9168 & 0.8264 \end{bmatrix}$$

L

$$\begin{bmatrix} & & \\ 0.4444 & & \\ 0.1736 & & \end{bmatrix}$$

b

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Fatoração LU

Exemplo:

A

$$\begin{bmatrix} 144 & 12 & 1 \\ 0 & 2.6672 & 0.5556 \\ 0 & 2.9168 & 0.8264 \end{bmatrix}$$

Na segunda iteração é necessário trocar as linhas 2 e 3.

L

$$\begin{bmatrix} & & \\ 0.4444 & & \\ 0.1736 & & \\ & & \end{bmatrix}$$

b

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Fatoração LU

<u>Exemplo</u>:

L

$$\begin{bmatrix} & & \\ 0.4444 & & \\ 0.1736 & & \end{bmatrix}$$

A

A mesma troca deve ser feita no vetor **b** e nos elementos já preenchidos da matriz **L**.

b

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 144 & 12 & 1 \\ 0 & 2.6672 & 0.5556 \\ 0 & 2.9168 & 0.8264 \end{bmatrix}$$

Fatoração LU

Exemplo:

$$\mathbf{L} = \begin{bmatrix} & & \\ 0.1736 & & \\ 0.4444 & & \\ & & \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 144 & 12 & 1 \\ 0 & 2.9168 & 0.8264 \\ 0 & 2.6672 & 0.5556 \end{bmatrix}$$

A mesma troca deve ser feita no vetor **b** e nos elementos já preenchidos da matriz **L**.

$$\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Fatoração LU

Exemplo:

A

L

$$\begin{bmatrix} & & \\ 0.1736 & & \\ 0.4444 & & \\ & & \end{bmatrix}$$

b

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Definindo multiplicadores da segunda iteração.

$$\begin{bmatrix} 144 & 12 & 1 \\ 0 & 2.9168 & 0.8264 \\ 0 & 2.6672 & 0.5556 \end{bmatrix} \quad m_{32} = 0.9144$$

Fatoração LU

Armazenando na matriz **L**.

<u>Exemplo</u>:

A

$$\begin{bmatrix} 144 & 12 & 1 \\ 0 & 2.9168 & 0.8264 \\ 0 & 2.6672 & 0.5556 \end{bmatrix} \quad m_{32} = 0.9144$$

L

$$\begin{bmatrix} & & \\ 0.1736 & & \\ 0.4444 & 0.9144 & \\ & & \end{bmatrix}$$

b

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Fatoração LU

Exemplo:

A

$$\begin{bmatrix} 144 & 12 & 1 \\ 0 & 2.9168 & 0.8264 \\ 0 & 0 & -0.2001 \end{bmatrix}$$

Aplicando operações sobre as equações.

$$\begin{bmatrix} 144 & 12 & 1 \\ 0 & 2.9168 & 0.8264 \\ 0 & 2.6672 & 0.5556 \end{bmatrix} \quad m_{32} = 0.9144$$

L

$$\begin{bmatrix} & & \\ 0.1736 & & \\ 0.4444 & 0.9144 & \\ & & \end{bmatrix}$$

b

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Fatoração LU

Completando matriz **L**.

<u>Exemplo</u>:

A

$$\begin{bmatrix} 144 & 12 & 1 \\ 0 & 2.9168 & 0.8264 \\ 0 & 0 & -0.2001 \end{bmatrix}$$

U

L

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.1736 & 1 & 0 \\ 0.4444 & 0.9144 & 1 \end{bmatrix}$$

b

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

<u>Na prática</u>: efetuar as trocas no(s) vetor(es) **b** posteriormente. No momento da fatoração, apenas armazenar as trocas em uma estrutura de dados.

Métodos de Resolução de Sistemas Lineares

- **<u>Métodos Exatos ou Diretos</u>:** permitiram a solução exata com um número finito de operações, se não fosse por erros numéricos.

- **<u>Métodos Iterativos</u>:** permitem uma solução aproximada através de um processo infinito convergente.

O Método de Gauss-Jacobi

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \dots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & a_{n3}x_3 & + & \dots & + & a_{nn}x_n & = & b_n \end{cases}$$

O Método de Gauss-Jacobi

$$
\begin{cases}
a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \dots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \dots & + & a_{2n}x_n & = & b_2 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
a_{n1}x_1 & + & a_{n2}x_2 & + & a_{n3}x_3 & + & \dots & + & a_{nn}x_n & = & b_n
\end{cases}
$$

O Método de Gauss-Jacobi

$$
\begin{cases}
a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \ldots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \ldots & + & a_{2n}x_n & = & b_2 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
a_{n1}x_1 & + & a_{n2}x_2 & + & a_{n3}x_3 & + & \ldots & + & a_{nn}x_n & = & b_n
\end{cases}
$$

$$
\begin{aligned}
x_1 &= \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3 - \ldots - a_{1n}x_n\right] \\
x_2 &= \frac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3 - \ldots - a_{2n}x_n\right] \\
\vdots & \quad \vdots \\
x_n &= \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1 - a_{n2}x_2 - \ldots - a_{n,n-1}x_{n-1}\right]
\end{aligned}
$$

O Método de Gauss-Jacobi

$$\boxed{x_1^{(0)}, \ldots, x_n^{(0)}}$$

$$\begin{aligned}
x_1 &= \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3 - \ldots - a_{1n}x_n\right] \\
x_2 &= \frac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3 - \ldots - a_{2n}x_n\right] \\
&\vdots \\
x_n &= \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1 - a_{n2}x_2 - \ldots - a_{n,n-1}x_{n-1}\right]
\end{aligned}$$

O Método de Gauss-Jacobi

$$\boxed{x_1^{(0)}, \ldots, x_n^{(0)}}$$

$$\boxed{x_1^{(1)}, \ldots, x_n^{(1)}} \left\{ \begin{array}{rcl} x_1 & = & \dfrac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3 - \ldots - a_{1n}x_n\right] \\ x_2 & = & \dfrac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3 - \ldots - a_{2n}x_n\right] \\ \vdots & & \vdots \\ x_n & = & \dfrac{1}{a_{nn}}\left[b_n - a_{n1}x_1 - a_{n2}x_2 - \ldots - a_{n,n-1}x_{n-1}\right] \end{array} \right.$$

O Método de Gauss-Jacobi

$$\boxed{x_1^{(1)}, \dots, x_n^{(1)}}$$

$$\boxed{x_1^{(1)}, \dots, x_n^{(1)}}$$

$$\begin{aligned}
x_1 &= \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3 - \dots - a_{1n}x_n\right]\\
x_2 &= \frac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3 - \dots - a_{2n}x_n\right]\\
\vdots &\quad \vdots\\
x_n &= \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1 - a_{n2}x_2 - \dots - a_{n,n-1}x_{n-1}\right]
\end{aligned}$$

O Método de Gauss-Jacobi

$$\boxed{x_1^{(1)}, \ldots, x_n^{(1)}}$$

$$\boxed{x_1^{(2)}, \ldots, x_n^{(2)}} \left\{ \begin{array}{lcl} x_1 & = & \dfrac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3 - \ldots - a_{1n}x_n\right] \\ x_2 & = & \dfrac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3 - \ldots - a_{2n}x_n\right] \\ \vdots & & \vdots \\ x_n & = & \dfrac{1}{a_{nn}}\left[b_n - a_{n1}x_1 - a_{n2}x_2 - \ldots - a_{n,n-1}x_{n-1}\right] \end{array} \right.$$

O Método de Gauss-Jacobi

$$\boxed{x_1^{(2)}, \ldots, x_n^{(2)}}$$

$$\boxed{x_1^{(2)}, \ldots, x_n^{(2)}}$$

$$\begin{aligned}
x_1 &= \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3 - \ldots - a_{1n}x_n\right] \\
x_2 &= \frac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3 - \ldots - a_{2n}x_n\right] \\
\vdots & \quad \vdots \\
x_n &= \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1 - a_{n2}x_2 - \ldots - a_{n,n-1}x_{n-1}\right]
\end{aligned}$$

O Método de Gauss-Jacobi

$$\boxed{x_1^{(2)}, \ldots, x_n^{(2)}}$$

$$\boxed{x_1^{(3)}, \ldots, x_n^{(3)}} \left\{ \begin{array}{lcl} x_1 & = & \dfrac{1}{a_{11}} [b_1 - a_{12} x_2 - a_{13} x_3 - \ldots - a_{1n} x_n] \\ x_2 & = & \dfrac{1}{a_{22}} [b_2 - a_{21} x_1 - a_{23} x_3 - \ldots - a_{2n} x_n] \\ \vdots & & \vdots \\ x_n & = & \dfrac{1}{a_{nn}} [b_n - a_{n1} x_1 - a_{n2} x_2 - \ldots - a_{n,n-1} x_{n-1}] \end{array} \right.$$

O Método de Gauss-Jacobi

$$\boxed{Max\ \left\{\left|x_i^{(k+1)} - x_i^{(k)}\right|\right\} \leq \varepsilon \qquad i = 1, 2, 3, \ldots, n}$$

$$\boxed{x_1^{(2)}, \ldots, x_n^{(2)}}$$

$$\boxed{x_1^{(3)}, \ldots, x_n^{(3)}} \left\{ \begin{array}{lcl} x_1 & = & \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3 - \ldots - a_{1n}x_n\right] \\ x_2 & = & \frac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3 - \ldots - a_{2n}x_n\right] \\ \vdots & & \vdots \\ x_n & = & \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1 - a_{n2}x_2 - \ldots - a_{n,n-1}x_{n-1}\right] \end{array} \right.$$

O Método de Gauss-Jacobi

$$Max\left\{\left|\frac{x_i^{(k+1)} - x_i^{(k)}}{x_i^{(k+1)}}\right|\right\} \le \varepsilon \qquad i = 1, 2, 3, \ldots, n$$

$$x_1^{(2)}, \ldots, x_n^{(2)}$$

$$x_1^{(3)}, \ldots, x_n^{(3)} \left\{ \begin{array}{rcl} x_1 & = & \frac{1}{a_{11}}[b_1 - a_{12}x_2 - a_{13}x_3 - \ldots - a_{1n}x_n] \\ x_2 & = & \frac{1}{a_{22}}[b_2 - a_{21}x_1 - a_{23}x_3 - \ldots - a_{2n}x_n] \\ \vdots & & \vdots \\ x_n & = & \frac{1}{a_{nn}}[b_n - a_{n1}x_1 - a_{n2}x_2 - \ldots - a_{n,n-1}x_{n-1}] \end{array} \right.$$

O Método de Gauss-Jacobi

$$Max \left\{\left|r_i^{(k+1)}\right|\right\} \leq \varepsilon \qquad i = 1, 2, 3, \ldots, n$$

$$r_i^{(k+1)} = b_i - \sum_{j=1}^{n} a_{ij} x_j^{(k+1)}$$

$$x_1^{(2)}, \ldots, x_n^{(2)}$$

$$x_1^{(3)}, \ldots, x_n^{(3)} \quad \left\{ \begin{array}{lcl} x_1 & = & \frac{1}{a_{11}} [b_1 - a_{12} x_2 - a_{13} x_3 - \ldots - a_{1n} x_n] \\ x_2 & = & \frac{1}{a_{22}} [b_2 - a_{21} x_1 - a_{23} x_3 - \ldots - a_{2n} x_n] \\ \vdots & & \vdots \\ x_n & = & \frac{1}{a_{nn}} [b_n - a_{n1} x_1 - a_{n2} x_2 - \ldots - a_{n,n-1} x_{n-1}] \end{array} \right.$$

Exemplo

$$\begin{cases} 3x_1 - \ \ x_2 - \ \ x_3 = 1 \\ \ \ x_1 + 3x_2 + \ \ x_3 = 5 \\ \ \ x_1 - x_2 + 2x_3 = 2 \end{cases}$$

Fonte: Cálculo Numérico Computacional (Peters & Szeremeta)

Exemplo

$$\begin{cases} 3x_1 - x_2 - x_3 = 1 \\ x_1 + 3x_2 + x_3 = 5 \\ x_1 - x_2 + 2x_3 = 2 \end{cases} \Rightarrow \begin{cases} x_1^{(k+1)} = (1 + x_2^{(k)} + x_3^{(k)}) / 3 \\ x_2^{(k+1)} = (5 - x_1^{(k)} - x_3^{(k)}) / 3 \\ x_3^{(k+1)} = (2 - x_1^{(k)} + x_2^{(k)}) / 2 \end{cases}$$

Fonte: Cálculo Numérico Computacional (Peters & Szeremeta)

Exemplo

$$\begin{cases} 3x_1 - x_2 - x_3 = 1 \\ x_1 + 3x_2 + x_3 = 5 \\ x_1 - x_2 + 2x_3 = 2 \end{cases} \Rightarrow \begin{cases} x_1^{(k+1)} = (1 + x_2^{(k)} + x_3^{(k)})/3 \\ x_2^{(k+1)} = (5 - x_1^{(k)} - x_3^{(k)})/3 \\ x_3^{(k+1)} = (2 - x_1^{(k)} + x_2^{(k)})/2 \end{cases}$$

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$\lvert x_1^{(k+1)} - x_1^{(k)} \rvert$	$\lvert x_2^{(k+1)} - x_2^{(k)} \rvert$	$\lvert x_3^{(k+1)} - x_3^{(k)} \rvert$
0	0	0	0	-	-	-
1	0.333	1.667	1	0.333	1.667	1
2	1.222	1.222	1.667	0.889	0.555	0.667
3	1.296	0.704	1	0.074	0.518	0.667
4	0.901	0.901	0.704	0.395	0.197	0.296
5	0.868	1.132	1	0.033	0.197	0.296
6	1.044	1.044	1.132	0.176	0.088	0.132

Fonte: Cálculo Numérico Computacional (Peters & Szeremeta)

Exemplo

$$\begin{cases} 3x_1 - x_2 - x_3 = 1 \\ x_1 + 3x_2 + x_3 = 5 \\ x_1 - x_2 + 2x_3 = 2 \end{cases}$$

$$\begin{cases} x_1^{(k+1)} = (1 + x_2^{(k)} + x_3^{(k)})/3 \\ x_2^{(k+1)} = (5 - x_1^{(k)} - x_3^{(k)})/3 \\ x_3^{(k+1)} = (2 - x_1^{(k)} + x_2^{(k)})/2 \end{cases}$$

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$\lvert x_1^{(k+1)} - x_1^{(k)} \rvert$	$\lvert x_2^{(k+1)} - x_2^{(k)} \rvert$	$\lvert x_3^{(k+1)} - x_3^{(k)} \rvert$
0	0	0	0	-	-	-
1	0.333	1.667	1	0.333	1.667	1
2	1.222	1.222	1.667	0.889	0.555	0.667
3	1.296	0.704	1	0.074	0.518	0.667
4	0.901	0.901	0.704	0.395	0.197	0.296
5	0.868	1.132	1	0.033	0.197	0.296
6	1.044	1.044	1.132	$\boxed{0.176}$	0.088	0.132

Fonte: Cálculo Numérico Computacional (Peters & Szeremeta)

O Método de Gauss-Seidel

$$x_1^{(0)}, \ldots, x_n^{(0)}$$

$$\begin{aligned}
x_1 &= \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3 - \ldots - a_{1n}x_n\right] \\
x_2 &= \frac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3 - \ldots - a_{2n}x_n\right] \\
&\vdots \\
x_n &= \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1 - a_{n2}x_2 - \ldots - a_{n,n-1}x_{n-1}\right]
\end{aligned}$$

O Método de Gauss-Seidel

$$\boxed{x_1^{(0)}, \ldots, x_n^{(0)}}$$

$$\boxed{x_1^{(1)}} \left\{ \begin{aligned} x_1 &= \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3 - \ldots - a_{1n}x_n\right] \\ x_2 &= \frac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3 - \ldots - a_{2n}x_n\right] \\ \vdots \quad & \quad \vdots \\ x_n &= \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1 - a_{n2}x_2 - \ldots - a_{n,n-1}x_{n-1}\right] \end{aligned} \right.$$

O Método de Gauss-Seidel

$$x_1^{(0)}, \ldots, x_n^{(0)}$$

$$x_1^{(1)}$$

$$\begin{aligned}
x_1 &= \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3 - \ldots - a_{1n}x_n\right] \\
x_2 &= \frac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3 - \ldots - a_{2n}x_n\right] \\
\vdots & \quad \vdots \\
x_n &= \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1 - a_{n2}x_2 - \ldots - a_{n,n-1}x_{n-1}\right]
\end{aligned}$$

O Método de Gauss-Seidel

$$x_1^{(0)}, \ldots, x_n^{(0)}$$

$$\boxed{x_1^{(1)}} \qquad \boxed{x_2^{(1)}}$$

$$\begin{aligned}
x_1 &= \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3 - \ldots - a_{1n}x_n\right] \\
x_2 &= \frac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3 - \ldots - a_{2n}x_n\right] \\
&\vdots \\
x_n &= \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1 - a_{n2}x_2 - \ldots - a_{n,n-1}x_{n-1}\right]
\end{aligned}$$

O Método de Gauss-Seidel

$$\boxed{x_1^{(0)}, \ldots, x_n^{(0)}}$$

$$\boxed{x_1^{(1)}} \quad x_1 = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3 - \ldots - a_{1n}x_n\right]$$

$$\boxed{x_2^{(1)}} \quad x_2 = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3 - \ldots - a_{2n}x_n\right]$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\boxed{x_n^{(1)}} \quad x_n = \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1 - a_{n2}x_2 - \ldots - a_{n,n-1}x_{n-1}\right]$$

O Método de Gauss-Seidel

$$\boxed{x_1^{(1)}, \ldots, x_n^{(1)}}$$

$$\boxed{x_1^{(1)}} \quad \boxed{x_2^{(1)}} \quad \vdots \quad \boxed{x_n^{(1)}}$$

$$
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\end{aligned}
$$

O Método de Gauss-Seidel

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\vdots & \qquad \vdots \\
x_n &= \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1 - a_{n2}x_2 - \ldots - a_{n,n-1}x_{n-1}\right]
\end{aligned}$$

O Método de Gauss-Seidel

$$\boxed{x_1^{(1)}, \ldots, x_n^{(1)}}$$

$$\boxed{x_1^{(2)}}$$

$$\begin{aligned}
x_1 &= \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3 - \ldots - a_{1n}x_n\right] \\
x_2 &= \frac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3 - \ldots - a_{2n}x_n\right] \\
\vdots & \quad \vdots \\
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O Método de Gauss-Seidel

$$\boxed{x_1^{(1)}, \ldots, x_n^{(1)}}$$

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x_1 &= \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3 - \ldots - a_{1n}x_n\right] \\
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\vdots &\quad\ \vdots \\
x_n &= \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1 - a_{n2}x_2 - \ldots - a_{n,n-1}x_{n-1}\right]
\end{aligned}$$

O Método de Gauss-Seidel

$$\boxed{x_1^{(1)}, \ldots, x_n^{(1)}}$$

$$\boxed{x_1^{(2)}} \quad \boxed{x_2^{(2)}}$$

$$\begin{aligned}
x_1 &= \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3 - \ldots - a_{1n}x_n\right] \\
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\vdots & \quad \vdots \\
x_n &= \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1 - a_{n2}x_2 - \ldots - a_{n,n-1}x_{n-1}\right]
\end{aligned}$$

O Método de Gauss-Seidel

$$\boxed{x_1^{(1)}, \dots, x_n^{(1)}}$$

$$
\begin{array}{ll}
\boxed{x_1^{(2)}} & x_1 = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3 - \ldots - a_{1n}x_n\right] \\
\boxed{x_2^{(2)}} & x_2 = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3 - \ldots - a_{2n}x_n\right] \\
\vdots & \qquad \vdots \qquad\qquad \vdots \\
\boxed{x_n^{(2)}} & x_n = \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1 - a_{n2}x_2 - \ldots - a_{n,n-1}x_{n-1}\right]
\end{array}
$$

O Método de Gauss-Seidel

$$\boxed{x_1^{(2)}, \ldots, x_n^{(2)}}$$

$$\boxed{x_1^{(2)}} \quad \boxed{x_2^{(2)}} \quad \vdots \quad \boxed{x_n^{(2)}}$$

$$\begin{aligned}
x_1 &= \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3 - \ldots - a_{1n}x_n\right] \\
x_2 &= \frac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3 - \ldots - a_{2n}x_n\right] \\
\vdots & \quad\ \vdots \\
x_n &= \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1 - a_{n2}x_2 - \ldots - a_{n,n-1}x_{n-1}\right]
\end{aligned}$$

Comparação

Gauss-Jacobi

$$
\begin{aligned}
x_1^{(1)} &= \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)} - \ldots - a_{1n}x_n^{(0)}\right] \\
x_2^{(1)} &= \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(0)} - a_{23}x_3^{(0)} - \ldots - a_{2n}x_n^{(0)}\right] \\
\vdots \quad & \quad \vdots \\
x_n^{(1)} &= \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1^{(0)} - a_{n2}x_2^{(0)} - \ldots - a_{n,n-1}x_{n-1}^{(0)}\right]
\end{aligned}
$$

Gauss-Seidel

$$
\begin{aligned}
x_1^{(1)} &= \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)} - \ldots - a_{1n}x_n^{(0)}\right] \\
x_2^{(1)} &= \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(1)} - a_{23}x_3^{(0)} - \ldots - a_{2n}x_n^{(0)}\right] \\
\vdots \quad & \quad \vdots \\
x_n^{(1)} &= \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1^{(1)} - a_{n2}x_2^{(1)} - \ldots - a_{n,n-1}x_{n-1}^{(1)}\right]
\end{aligned}
$$

Exemplo

$$\begin{cases} 3x_1 - x_2 - x_3 = 1 \\ x_1 + 3x_2 + x_3 = 5 \\ x_1 - x_2 + 2x_3 = 2 \end{cases} \Rightarrow \begin{cases} x_1^{(k+1)} = (1 + x_2^{(k)} + x_3^{(k)}) / 3 \\ x_2^{(k+1)} = (5 - x_1^{(k+1)} - x_3^{(k)}) / 3 \\ x_3^{(k+1)} = (2 - x_1^{(k+1)} + x_2^{(k+1)}) / 2 \end{cases}$$

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$\lvert x_1^{(k+1)} - x_1^{(k)} \rvert$	$\lvert x_2^{(k+1)} - x_2^{(k)} \rvert$	$\lvert x_3^{(k+1)} - x_3^{(k)} \rvert$
0	0	0	0	-	-	-
1	0.333	1.555	1.611	0.333	1.555	1.611
2	1.388	0.666	0.638	1.055	0.888	0.972
3	0.768	1.197	1.214	0.620	0.531	0.575
4	1.137	0.882	0.872	0.368	0.314	0.341
5	0.918	1.069	1.075	0.218	0.186	0.202
6	1.048	0.958	0.955	0.129	0.110	0.120

Fonte: Cálculo Numérico Computacional (Peters & Szeremeta)

Exemplo

$$\begin{cases} 3x_1 - x_2 - x_3 = 1 \\ x_1 + 3x_2 + x_3 = 5 \\ x_1 - x_2 + 2x_3 = 2 \end{cases} \Rightarrow \begin{cases} x_1^{(k+1)} = (1 + x_2^{(k)} + x_3^{(k)}) / 3 \\ x_2^{(k+1)} = (5 - x_1^{(k+1)} - x_3^{(k)}) / 3 \\ x_3^{(k+1)} = (2 - x_1^{(k+1)} + x_2^{(k+1)}) / 2 \end{cases}$$

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$\lvert x_1^{(k+1)} - x_1^{(k)} \rvert$	$\lvert x_2^{(k+1)} - x_2^{(k)} \rvert$	$\lvert x_3^{(k+1)} - x_3^{(k)} \rvert$
0	0	0	0	-	-	-
1	0.333	1.555	1.611	0.333	1.555	1.611
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Fonte: Cálculo Numérico Computacional (Peters & Szeremeta)

Critérios de Convergência

Se o sistema $\mathbf{Ax} = \mathbf{b}$ tiver **diagonal dominante**, os métodos Gauss-Jacobi e Gauss-Seidel convergem para a solução do sistema.

$$\alpha_i = \frac{\sum_{j=1, j \neq i}^{n} |a_{ij}|}{|a_{ii}|}$$

$$\alpha = max(\alpha_i) < 1$$

Critérios de Convergência

Se o sistema $\mathbf{Ax} = \mathbf{b}$ tiver **diagonal dominante**, os métodos Gauss-Jacobi e Gauss-Seidel convergem para a solução do sistema.

Exemplo:

$$\begin{cases} 3x_1 - x_2 - x_3 = 1 & |3| \geq |-1| + |-1| \quad V \\ x_1 + 3x_2 + x_3 = 5 & |3| \geq |+1| + |+1| \quad V \\ x_1 - x_2 + 2x_3 = 2 & |2| \geq |+1| + |-1| \quad V \end{cases}$$

Critérios de Convergência

Para o método de Gauss-Seidel, o critério pode ser relaxado (**Critério de Sassenfeld**).

$$\beta_i = \frac{\sum_{j=1}^{i-1} |a_{ij}| \beta_j + \sum_{j=i+1}^{n} |a_{ij}|}{|a_{ii}|}$$

$$\beta = max(\beta_i) < 1$$

Critérios de Convergência

Para o método de Gauss-Seidel, o critério pode ser relaxado (**Critério de Sassenfeld**).

Exemplo:

$$\begin{cases} 3x_1 - x_2 - x_3 = 1 & |3| \geq |-1| + |-1| \quad V \\ x_1 + 3x_2 + x_3 = 5 & |3| \geq |+1| + |+1| \quad V \\ x_1 - x_2 + 2x_3 = 2 & |2| \geq |+1| + |-1| \quad V \end{cases} \Rightarrow 2/3 = 0{,}6666 = \beta_1$$

Critérios de Convergência

Para o método de Gauss-Seidel, o critério pode ser relaxado (**Critério de Sassenfeld**).

Exemplo:

$$\begin{cases} 3x_1 - x_2 - x_3 = 1 & |3| \geq |-1| + |-1| \quad V \\ x_1 + 3x_2 + x_3 = 5 & |3| \geq |+1| + |+1| \quad V \\ x_1 - x_2 + 2x_3 = 2 & |2| \geq |+1| + |-1| \quad V \end{cases}$$

$2/3 = 0{,}6666 = \beta_1$

$(0{,}6666+1)$

Critérios de Convergência

Para o método de Gauss-Seidel, o critério pode ser relaxado (**Critério de Sassenfeld**).

Exemplo:

$$\begin{cases} 3x_1 - x_2 - x_3 = 1 & |3| \geq |-1| + |-1| \quad V \\ x_1 + 3x_2 + x_3 = 5 & |3| \geq |+1| + |+1| \quad V \\ x_1 - x_2 + 2x_3 = 2 & |2| \geq |+1| + |-1| \quad V \end{cases}$$

$2/3 = 0{,}6666 = \beta_1$

$(0{,}6666 + 1)/3 = 0{,}5555 = \beta_2$

Critérios de Convergência

Para o método de Gauss-Seidel, o critério pode ser relaxado (**Critério de Sassenfeld**).

<u>Exemplo:</u>

$$\begin{cases} 3x_1 - x_2 - x_3 = 1 & |3| \geq |-1| + |-1| \quad V \\ x_1 + 3x_2 + x_3 = 5 & |3| \geq |+1| + |+1| \quad V \\ x_1 - x_2 + 2x_3 = 2 & |2| \geq |+1| + |-1| \quad V \end{cases}$$

$2/3 = 0{,}6666 = \beta_1$

$(0{,}6666+1)/3 = 0{,}5555 = \beta_2$

$(0{,}6666+0{,}5555)/2 = 0{,}61105 = \beta_3$

Critérios de Convergência

Se os critérios de convergência forem satisfeitos, os métodos Gauss-Jacobi e Gauss-Seidel convergem para a solução do sistema para qualquer $\mathbf{X}^{(0)}$.

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