本征值问题：函数情形

$$\begin{cases} y''(x) + \lambda y(x) = 0, \quad (0 < x < l) \\ y(0) = 0,\ y(l) = 0. \end{cases}$$

一、线性相关要求：

$$y = \alpha y_1 + \beta y_2$$

$$= \begin{cases} \alpha e^{kx} + \beta e^{-kx},\ (\lambda < 0,\ k = \sqrt{-\lambda}) \\ \alpha + \beta x,\ (\lambda = 0) \\ \alpha \cos(kx) + \beta \sin(kx),\ (\lambda > 0,\ k = \sqrt{\lambda}) \end{cases}$$

二、边界条件要求 $\begin{bmatrix} y_1(0) & y_2(0) \\ y_1(l) & y_2(l) \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$；非零解要求 $\det P = 0$，其中 $P = \begin{bmatrix} y_1(0) & y_2(0) \\ y_1(l) & y_2(l) \end{bmatrix}$。

- $\lambda < 0$, $\det P = e^{-kl} - e^{kl}$

  $\det P \neq 0 \Rightarrow \alpha = \beta = 0 \Rightarrow y(x) = 0$

- $\lambda = 0$, $\det P = l$

  $\det P \neq 0 \Rightarrow \alpha = \beta = 0 \Rightarrow y(x) = 0$

- $\lambda > 0$, $\det P = \sin(kl)$

  $\det P = 0 \xrightarrow[\alpha=0]{\lambda=\left(\frac{n\pi}{l}\right)^2} y(x) = \beta\sin(n\pi x/l)$

结论

一、本征子空间（$\lambda_n = (\frac{n\pi}{l})^2, n = 1, 2, \cdots$）:

$$V_n = \left\{\beta_n \sin(\sqrt{\lambda_n} x) \mid \forall \beta_n \in R\right\}, \ \dim V_n = 1$$

二、正交性关系：设 $y_m(x) = \sin(\frac{m\pi}{l}x)$, $y_n(x) = \sin(\frac{n\pi}{l}x)$，当 $m \neq n$ 时，有

$$(y_m, y_n) = \int_0^l y_m(x) y_n(x) dx = 0$$

三、完备正交基：$\{y_n(x)\}_{n=1}^{\infty}$

例：

$$\begin{cases} \phi''(\theta) + \lambda\phi(\theta) = 0, \quad (0 < \theta < 2\pi) \\ \phi(0) = \phi(2\pi), \phi'(0) = \phi'(2\pi). \end{cases}$$

一、线性相关要求：

$$\phi = \alpha\phi_1 + \beta\phi_2$$

$$= \begin{cases} \alpha e^{k\theta} + \beta e^{-k\theta}, (\lambda < 0, k = \sqrt{-\lambda}) \\ \alpha + \beta\theta, (\lambda = 0) \\ \alpha\cos(k\theta) + \beta\sin(k\theta), (\lambda > 0, k = \sqrt{\lambda}) \end{cases}$$

二、边界条件要求

$$\underbrace{\begin{bmatrix} \phi_1(0)-\phi_1(2\pi) & \phi_2(0)-\phi_2(2\pi) \\ \phi_1'(0)-\phi_1'(2\pi) & \phi_2'(0)-\phi_2'(2\pi) \end{bmatrix}}_{P} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

非零解要求 $\det P = 0$。

- $\lambda < 0$:

$$\det P = \begin{vmatrix} 1-e^{2k\pi} & 1-e^{-2k\pi} \\ k(1-e^{2k\pi}) & -k(1-e^{-2k\pi}) \end{vmatrix} \neq 0$$

$$\Rightarrow \quad \phi(\theta) = 0$$

- $\lambda = 0$:

$$\det P = \begin{vmatrix} 0 & -2\pi \\ 0 & 0 \end{vmatrix} = 0 \implies \beta = 0,\ \phi(\theta) = \alpha$$

- $\lambda > 0$:

$$\begin{aligned} \det P &= \begin{vmatrix} 1 - \cos(2k\pi) & -\sin(2k\pi) \\ k\sin(2k\pi) & k(1 - \cos(2k\pi)) \end{vmatrix} \\ &= 2k(1 - \cos(2k\pi)) = 0 \\ &\implies \lambda = n^2 \\ &\implies \phi(\theta) = \alpha\cos(n\theta) + \beta\sin(n\theta) \end{aligned}$$

结论

一、本征子空间：

$$V_0 = \{\alpha_0 \mid \forall\, \alpha_0 \in R\},\ \lambda_0 = 0,\ \dim V_0 = 1$$

$$V_n = \{\alpha_n \cos(n\theta) + \beta_n \sin(n\theta) \mid \forall\, \alpha_n, \beta_n \in R\}$$

$$\lambda_n = n^2,\ \dim V_n = 2 \quad (n = 1, 2, \cdots)$$

二、正交性关系：设 y_m 和 y_n 为本征函数，本征值分别为 λ_m 和 λ_n，当 $\lambda_m \neq \lambda_n$ 时，有 $(y_m, y_n) = \int_0^{2\pi} y_m(\theta) y_n(\theta) d\theta = 0$。

三、完备正交基：

$$\{1, \cos(\theta), \sin(\theta), \cdots, \cos(n\theta), \sin(n\theta), \cdots\}$$

Sturm-Liouville 理论

一般型方程：$\hat{L}y + \lambda py = 0$

$$\left[p_0\frac{d^2}{dx^2} + p_1\frac{d}{dx} + p_2\right]y + \lambda py = 0$$

S-L 型方程：$\hat{L}_s y + \lambda\rho y = 0$

$$\left[\frac{d}{dx}\left(k\frac{d}{dx}\right) - q\right]y + \lambda\rho y = 0$$

思考：二者的异同？如何转换？为何要作转换？

转换函数 $f(x)$ 及权函数 $\rho(x)$：

$$f(x) = p_0^{-1}(x)\exp\left(\int^x \frac{p_1(t)}{p_0(t)}dt\right), \ \rho = fp$$

Green 恒等式： 自伴边界条件将使得表面项恒为零。

$$(f, \hat{L}_s g) - (g, \hat{L}_s f) = [k\,(fg' - gf')]\,|_a^b$$

正交性关系：

$$\underline{(y_n, \hat{L}_s y_m) - (y_m, \hat{L}_s y_n) = (\lambda_n - \lambda_m)(y_m, \rho y_n) = 0}$$

$$\Downarrow \lambda_m \neq \lambda_n$$

$$\underline{(y_m, \rho y_n) = 0}$$

证明 Green 恒等式：

$$\boxed{(f, \hat{L}_s g) - (g, \hat{L}_s f) = [k(fg' - gf')]|_a^b}$$

证明：

$$(f, \hat{L}_s g) - (g, \hat{L}_s f) = \int_a^b (f[kg']' - g[kf']')dx$$

$$= \int_a^b [fkg']' - \cancel{f'[kg']} - [gkf']' + \cancel{g'[kf']})dx$$

$$= \int_a^b [k(fg' - gf')]'dx = [k(fg' - gf')]|_a^b$$

例：正交性关系

$$\begin{cases} y'' + 2y' + \lambda y = 0, & (0 < x < 1) \\ y(0) = 0, y(1) = 0. \end{cases}$$

一、将本征方程转换为 S-L 型：

$$f'(x) = 2f(x) \quad \Rightarrow \quad f(x) = ce^{2x}$$

$$\times\, e^{2x} \quad \Rightarrow \quad e^{2x}y'' + 2e^{2x}y' + \lambda e^{2x}y = 0$$

$$\hat{L} = \frac{d}{dx}e^{2x}\frac{d}{dx} \quad \Rightarrow \quad \hat{L}y + \lambda e^{2x}y = 0$$

二、Green 恒等式：设 y_i 为本征函数，本征值为 $\lambda_i\,(i=1,2)$，并且 $\lambda_1 \neq \lambda_2$. 于是有

$$(y_1, \hat{L}y_2) - (y_2, \hat{L}y_1) = [e^{2x}(y_1y_2' - y_2y_1')]\big|_0^1$$

三、边界条件和本征方程：

$$[e^{2x}(y_1y_2' - y_2y_1')]\big|_0^1 \xlongequal{y_i(0)=0,\, y_i(1)=0} 0$$

$$(y_1, \hat{L}y_2) - (y_2, \hat{L}y_1) = 0 \xRightarrow{\hat{L}y_i=-\lambda_i e^{2x} y_i}$$

$$(\lambda_1 - \lambda_2)(y_1, e^{2x}y_2) = 0 \xRightarrow{\lambda_1\neq\lambda_2} (y_1, e^{2x}y_2) = 0$$

分离变量法

例：弦的自由振动

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2}, \quad (0 < x < l, t > 0) \\ u(0,t) = 0,\ u(l,t) = 0, \\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x). \end{cases}$$

(1) PDE：将 $u(x,t) = T(t)X(x)$ 代入方程

$$\frac{T''(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda \ \Rightarrow \begin{cases} T'' + \lambda a^2 T = 0 \\ X'' + \lambda X = 0 \end{cases}$$

(2) BC：将 u 代入边界条件，可得

$$\xRightarrow[T(t)=0]{X(0)=0,\,X(l)=0} \begin{cases} X''(x)+\lambda X(x)=0, \\ X(0)=0,\ X(l)=0. \end{cases}$$

$$\xRightarrow[\omega_n=\frac{an\pi}{l}]{\lambda_n=\left(\frac{n\pi}{l}\right)^2} \begin{cases} X_n(x)=\beta_n \sin(n\pi x/l) \\ T_n(t)=a_n\cos(\omega_n t)+b_n\sin(\omega_n t) \end{cases}$$

由线性叠加原理可得通解：

$$\boxed{u(x,t)=\sum_{n=1}^{\infty}\left(a_n\cos(\omega_n t)+b_n\sin(\omega_n t)\right)y_n(x)}$$

(3) IC：将通解 u 代入初始条件，并由正交性关系 $(y_m, y_n) = 0\,(m \neq n)$，可得

$$\sum_{n=1}^{\infty} a_n y_n(x) = \varphi(x) \quad \Rightarrow \quad a_n = \frac{(y_n, \varphi)}{\|y_n\|^2}$$

$$\sum_{n=1}^{\infty} b_n \omega_n y_n(x) = \psi(x) \quad \Rightarrow \quad b_n = \frac{(y_n, \psi)}{\omega_n \|y_n\|^2}$$

注：此处权函数 $\rho(x) = 1$，$\|y_n\| = \sqrt{(y_n, y_n)}$.

分离变量法

例：圆域内的拉普拉斯问题

$$\begin{cases} \dfrac{\partial^2 u}{\partial \rho^2} + \dfrac{1}{\rho}\dfrac{\partial u}{\partial \rho} + \dfrac{1}{\rho^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0, \quad \xleftarrow[0<\rho<\rho_0]{0<\theta<2\pi} \\ u(\rho, 0) = u(\rho, 2\pi),\ u_\theta(\rho, 0) = u_\theta(\rho, 2\pi), \\ |u(0, \theta)| < \infty,\ u(\rho_0, \theta) = f(\theta). \end{cases}$$

(1) PDE：将 $u(\rho, \theta) = R(\rho)\phi(\theta)$ 代入方程

$$\frac{\rho^2 R'' + \rho R'}{R} = -\frac{\phi''}{\phi} = \lambda \Rightarrow \begin{cases} \rho^2 R'' + \rho R' - \lambda R = 0 \\ \phi'' + \lambda\phi = 0 \end{cases}$$

(2) BC：将 u 代入边界条件，可得

$$\xRightarrow[R(\rho)=0]{\phi(0)=\phi(2\pi),\,\phi'(0)=\phi'(2\pi)} \begin{cases} \phi'' + \lambda\phi = 0, & (0 < \theta < 2\pi) \\ \phi(0) = \phi(2\pi), \ \phi'(0) = \phi'(2\pi). \end{cases}$$

$$\lambda_0 = 0$$

$$\phi_0(\theta) = a_0, \ R_0(\rho) = c_0 + d_0 \ln\rho$$

$$\lambda_n = n^2, \ n = 1, 2, \cdots$$

$$\begin{cases} \phi_n(\theta) = a_n \cos(n\theta) + b_n \sin(n\theta) \\ R_n(\rho) = c_n\rho^n + d_n\rho^{-n} \end{cases}$$

由线性叠加原理可得通解：

$$u(\rho,\theta)=\sum_{n=0}^{\infty}A_n u_n(\rho,\theta)=\sum_{n=0}^{\infty}A_n R_n(\rho)\phi_n(\theta)$$

$$=A_0\times a_0\times(c_0+d_0\ln\rho)+$$

$$\sum_{n=1}^{\infty}A_n(c_n\rho^n+d_n\rho^{-n})(a_n\cos(n\theta)+b_n\sin(n\theta))$$

$$=(a_0+b_0\ln\rho)1+\sum_{n=1}^{\infty}[(a_n\rho^n+b_n\rho^{-n})\cos(n\theta)$$

$$+(c_n\rho^n+d_n\rho^{-n})\sin(n\theta)]$$

(3) IC：将通解 u 代入初始条件，利用本征函数的正交性关系可得系数：

$$|u(0, \theta)| < \infty \quad \Rightarrow \quad b_0 = b_n = d_n = 0$$

于是，有通解

$$u(\rho, \theta) = a_0 1 + \sum_{n=1}^{\infty}[a_n\rho^n \cos(n\theta) + c_n\rho^n \sin(n\theta)]$$

$$= a_0 C_0(\theta) + \sum_{n=1}^{\infty}[a_n\rho^n C_n(\theta) + b_n\rho^n S_n(\theta)]$$

由

$$a_0 C_0 + \sum_{n=1}^{\infty} [a_n \rho_0^n C_n + b_n \rho_0^n S_n] = f$$

$$\Rightarrow \begin{cases} a_0 = \dfrac{(C_0, f)}{\|C_0\|^2} = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} f(\theta) d\theta \\ a_n = \dfrac{(C_n, f)}{\rho_0^n \|C_n\|^2} = \dfrac{1}{\pi \rho_0^n} \displaystyle\int_0^{2\pi} \cos(n\theta) f(\theta) d\theta \\ b_n = \dfrac{(S_n, f)}{\rho_0^n \|S_n\|^2} = \dfrac{1}{\pi \rho_0^n} \displaystyle\int_0^{2\pi} \sin(n\theta) f(\theta) d\theta \end{cases}$$

几何意义

齐次和非齐次常微分方程的解集

(1) 以下集合哪一个构成线性空间？

$$\begin{cases} S_1 = \{(x_1, x_2, 0) \mid \forall x_{1,2} \in \mathbb{R}\} \\ S_2 = \{(x_1, x_2, 1) \mid \forall x_{1,2} \in \mathbb{R}\} \end{cases}$$

(2) 以下函数集合哪一个构成线性空间？

$$\begin{cases} S_1 = \{y \mid y'' + \omega^2 y = 0\} \\ S_2 = \{y \mid y'' + \omega^2 y = f \; (f \neq 0)\} \end{cases}$$

常数变易法

或参数变易法

非齐次方程：$y''(x)+\omega^2 y(x) = f(x), \omega > 0$

$$\begin{aligned} y(x) &= y_h(x) + y_p(x) \\ &= \alpha_1 y_1(x) + \alpha_2 y_2(x) + y_p(x) \\ &= \alpha_1 \cos(\omega x) + \alpha_2 \sin(\omega x) + y_p(x) \end{aligned}$$

其中 $\alpha_{1,2} \in R$，y_h 为相应齐次方程的通解，而 y_p 为非齐次方程的任一特解。

常数变易法：$y_p(x) = A_1(x)y_1(x)+A_2(x)y_2(x)$

$$y_p = A_1 y_1 + A_2 y_2$$

$$y_p' = A_1 y_1' + A_2 y_2' + \boxed{A_1' y_1 + A_2' y_2} = 0$$

$$y_p'' = A_1 y_1'' + A_2 y_2'' + \boxed{A_1' y_1' + A_2' y_2'} = f$$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\begin{bmatrix} A_1' \\ A_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix} \Rightarrow \begin{cases} A_1' = -y_2 f/\omega \\ A_2' = y_1 f/\omega \end{cases}$$

$$y_p(x) = \frac{1}{\omega}\int_0^x (y_1(x')y_2(x) - y_2(x')y_1(x))f(x')dx'$$

$$= \frac{1}{\omega}\int_0^x \sin\omega(x-x')f(x')dx'$$

本征函数法

例：弦的受迫振动

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2} + f, \quad (0 < x < l, t > 0) \\ u(0,t) = 0,\ u(l,t) = 0, \\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x). \end{cases}$$

注：非齐次项 $f(x,t)$ 源于外力的贡献，故称做受迫振动。对于含非齐次方程的定解问题，狭义上的分离变量法不再适用。

(1) 确定基函数：将 $u(x,t) = T(t)X(x)$ 代入相应的齐次方程（$f = 0$）和边界条件，可得本定解问题的相关本征值问题：

$$\begin{cases} X'' + \lambda X = 0, & (0 < x < l) \\ X(0) = 0,\ X(l) = 0. \end{cases}$$

可求得本征值和本征函数：

$$\lambda_n = (n\pi/l)^2, \quad X_n(x) = \beta_n \sin(n\pi x/l)$$

选取基函数：

$$y_n(x) = \sin(n\pi x/l), \quad n = 1, 2, \cdots$$

(2) 计算系数：为此，将 u、f、φ 和 ψ 以 $\{y_n\}_{n=1}^{\infty}$ 为函数基展开：

$$u(x,t)=\sum_{n=1}^{\infty}u_n(t)y_n(x),\quad \varphi(x)=\sum_{n=1}^{\infty}\varphi_n y_n(x)$$

$$f(x,t)=\sum_{n=1}^{\infty}f_n(t)y_n(x),\quad \psi(x)=\sum_{n=1}^{\infty}\psi_n y_n(x)$$

由正交性关系 $(y_m, y_n)=0.5l\delta_{mn}$ 可知

$$f_n(t)=\frac{(y_n,f)}{\|y_n\|^2},\ \varphi_n=\frac{(y_n,\varphi)}{\|y_n\|^2},\ \psi_n=\frac{(y_n,\psi)}{\|y_n\|^2}$$

将 u 和 f、φ 和 ψ 的展开式分别代入PDE和IC，可得到展开系数 u_n 所满足的初值问题：

$$\begin{cases} u_n''(t) + \omega_n^2 u_n(t) = f_n(t), \quad (\omega_n = n\pi a/l) \\ u_n(0) = \varphi_n,\ u_n'(0) = \psi_n. \end{cases}$$

可解得：

$$u_n(t) = \varphi_n \cos(\omega_n t) + \frac{\psi_n}{\omega_n}\sin(\omega_n t) + \frac{1}{\omega_n}\int_0^t \sin(\omega_n(t-\tau)) f_n(\tau)\,d\tau$$

齐次化原理：物理意义

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2} + f, \\ u(0,t) = 0,\ u(l,t) = 0, \\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x). \end{cases}$$

$$\xRightarrow{u=w+v} \begin{cases} \dfrac{\partial^2 w}{\partial t^2} = a^2 \dfrac{\partial^2 w}{\partial x^2}, \\ w(0,t) = 0,\ w(l,t) = 0, \\ w(x,0) = \varphi(x),\ w_t(x,0) = \psi(x). \end{cases}$$

$$+ \begin{cases} \dfrac{\partial^2 v}{\partial t^2} = a^2 \dfrac{\partial^2 v}{\partial x^2} + f, \\ v(0,t) = 0,\ v(l,t) = 0, \\ v(x,0) = 0,\ v_t(x,0) = 0. \end{cases}$$

本征函数法

例：环形域内的泊松方程

$$\begin{cases} \dfrac{\partial^2 u}{\partial \rho^2} + \dfrac{1}{\rho}\dfrac{\partial u}{\partial \rho} + \dfrac{1}{\rho^2}\dfrac{\partial^2 u}{\partial \theta^2} = 12\rho^2 \cos(2\theta), \quad \xleftarrow[a<\rho<b]{0<\theta<2\pi} \\ u(\rho, 0) = u(\rho, 2\pi), \; u_\theta(\rho, 0) = u_\theta(\rho, 2\pi), \\ u(a, \theta) = 0, \; u_\rho(b, \theta) = 0. \end{cases}$$

(1) 确定基函数：将 $u(\rho, \theta) = R(\rho)\phi(\theta)$ 代入相应的齐次方程和边界条件，可得相关本征值问题：

$$\begin{cases} \phi'' + \lambda\phi = 0, \quad (0 < \theta < 2\pi) \\ \phi(0) = \phi(2\pi), \phi'(0) = \phi'(2\pi) \end{cases}$$

可求得本征值和本征函数：

$$\boxed{\begin{aligned} &\lambda_0 = 0, \quad \phi_0(\theta) = \alpha_0 \\ &\lambda_n = n^2, \ \phi_n(\theta) = \alpha_n \cos(n\theta) + \beta_n \sin(n\theta) \end{aligned}}$$

选取基函数：

$$\{1, \cos(\theta), \sin(\theta), \cdots, \cos(n\theta), \sin(n\theta), \cdots\}$$

(2) 计算系数：为此，将展开式

$$u(\rho,\theta)=a_0(\rho)1+\sum_{n=1}^{\infty}[a_n(\rho)\cos(n\theta)+b_n(\rho)\sin(n\theta)]$$

代入方程和边界条件，可得系数满足的边值问题：

$$\begin{cases}\rho^2 a_0''+\rho a_0'=0, & a_0(a)=0,\ a_0'(b)=0\\ \rho^2 a_n''+\rho a_n'-n^2 a_n=0, & a_n(a)=0,\ a_n'(b)=0\\ \rho^2 b_n''+\rho b_n'-n^2 b_n=0, & b_n(a)=0,\ b_n'(b)=0\\ \rho^2 a_2''+\rho a_2'-4a_2=12\rho^4, & a_2(a)=0,\ a_2'(b)=0\end{cases}$$

非齐次边界条件的处理

齐次化函数的引入

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2} + f, \\ u(0,t) = \mu(t), \; u(l,t) = \nu(t), \\ u(x,0) = \varphi(x), \; u_t(x,0) = \psi(x). \end{cases}$$

齐次化函数 $w(x,t)$:

$$w(0,t) = \mu(t), \; w(l,t) = \nu(t)$$

设 $w(x,t) = a(t)x + b(t)$，代入边界条件可得 $w(x,t) = \frac{\nu(t)-\mu(t)}{l} x + \mu(t)$.

令 $u(x,t)=v(x,t)+w(x,t)$，将其代入原定解问题，可得到关于 v 的新定解问题：

$$\begin{cases} \dfrac{\partial^2 v}{\partial t^2}=a^2\dfrac{\partial^2 v}{\partial x^2}+f^*(x,t), \\ v(0,t)=0,\ v(l,t)=0, \\ v(x,0)=\varphi^*(x),\ v_t(x,0)=\psi^*(x). \end{cases}$$

其中 $$\begin{cases} f^*(x,t)=f(x,t)+a^2\dfrac{\partial^2 w}{\partial x^2}-\dfrac{\partial^2 w}{\partial t^2}, \\ \varphi^*(x)=\varphi(x)-w(x,0), \\ \psi^*(x)=\psi(x)-w_t(x,0). \end{cases}$$

注意：

- 齐次化函数必须满足原定解问题的非齐次边界条件；
- 齐次化函数的选取不是唯一的；
- 齐次化函数的选取应遵循：容易求解，同时尽量使方程齐次化。

通常可设 $w(x, t)$ 为以下形式：

$$w(x, t) = a(t)x^2 + b(t)x + c(t)$$