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These groups are—

I. Methane.
II. The three methyl compounds.
III. Ethane and its derivatives.
IV. Propane and its derivatives.

If the members of a group have the same ratio of the specific heats, we know, from a well-known equation in the kinetic theory of gases, that the ratio of the internal energy absorbed by the molecule to the total energy absorbed, per degree rise of temperature, is the same for all. Hence we have the result that, with the single exception of marsh gas, the compounds with similar formulæ have the same energy-absorbing power, a result which supplies a link of a kind much needed to connect the graphic formula of a gas with the dynamical properties of its molecules.

From the conclusion we have reached, it follows with a high degree of probability that the atoms which can be interchanged without effect on the ratio of the specific heats have themselves the same energy-absorbing power, their mass and other special peculiarities being of no consequence. Further, the anomalous behaviour of methane confirms what was clear from previous determinations, namely, that the number of atoms in the molecule is not in itself sufficient to fix the distribution of energy, and suggests that perhaps the configuration is the sole determining cause.

If this is so, it follows that ethane and propane have the same configuration as their monohalogen derivatives, but that methane differs from the methyl compounds, a conclusion that in no way conflicts with the symmetry of the graphic formulæ of methane and its derivatives, for this is a symmetry of reactions, not of form.

## VIII. "On Operators in Physical Mathematics. Part II." By OLIVER HEAVISIDE, F.R.S. Received June 8, 1893.

*Algebraical Harmonization of the Forms of the Fundamental Bessel Function in Ascending and Descending Series by means of the Generalized Exponential.*

**27.** As promised in § 22, Part I ('Roy. Soc. Proc.,' vol. 52, p. 504), I will now first show how the formulæ for the Fourier-Bessel function in rising and descending powers of the variable may be algebraically harmonized, without analytical operations. The algebraical conversion is to be effected by means of the generalized exponential theorem, § 20. It was, indeed, used in § 22 to generalize the ascending form of the function in question; but that use was analytical. At present it is to be algebraical only. Thus, let

$$\mathrm{A} = 1+\frac{x^2}{2^2}+\frac{x^4}{2^2 4^2}+\frac{x^6}{2^2 4^2 6^2}+\ldots\ldots, \tag{1}$$

$$\mathrm{B} = \frac{2}{\pi}\left(x+\frac{x^3}{3^2}+\frac{x^5}{3^2 5^2}+\ldots\ldots\right)+\frac{2}{\pi}\left(\frac{1}{x}+\frac{1^2}{x^3}+\frac{1^2 3^2}{x^5}+\ldots\ldots\right), \tag{2}$$

$$\mathrm{C} = \frac{\epsilon^x}{(2\pi x)^{\frac{1}{2}}}\left(1+\frac{1^2}{8x}+\frac{1^2 3^2}{\underline{|2}(8x)^2}+\frac{1^2 3^2 5^2}{\underline{|3}(8x)^3}+\ldots\ldots\right). \tag{3}$$

Here A is the usual form of the Fourier-Bessel function (or, rather, the function $\mathrm{I}_0(x)$ instead of the oscillating function $\mathrm{J}_0(x)$, whose theory is less easy), or the first solution in rising powers of $x^2$ of the differential equation

$$(\nabla^2+x^{-1}\nabla)u = u, \tag{4}$$

as in (71), (72), Part I. Also, B is a particular case, viz., (78), Part I, of the generalization of the same series, (77), Part I, using the odd powers of $x$, and going both ways, in order to complete the series. And C is an equivalent form of the same function in a descending series, (31), Part I, obtained analytically, before the subject of generalized differentiation was introduced. The analytical transformation from A to C was considered in § 14. The present question is, what relation does C bear to A and B algebraically? It cannot be algebraically identical with either of them alone, on account of the radical in C. We may, however, eliminate the radical by employing the particular case of the generalized exponential that will introduce the radical anew. Thus, (63), (64), Part I,

$$\epsilon^x = \ldots\ldots+\frac{x^{-\frac{3}{2}}}{\underline{|-\tfrac{3}{2}}}+\frac{x^{-\frac{1}{2}}}{\underline{|-\tfrac{1}{2}}}+\frac{x^{\frac{1}{2}}}{\underline{|\tfrac{1}{2}}}+\frac{x^{\frac{3}{2}}}{\underline{|\tfrac{3}{2}}}+\ldots\ldots. \tag{5}$$

If we use this in (3), and carry out the multiplications, we obtain a series in integral powers of $x$, positive and negative; thus,

$$\begin{aligned}\mathrm{C} = \frac{1}{(2\pi)^{\frac{1}{2}}}\Bigg[&\frac{1}{\underline{|\tfrac{1}{2}}}+\frac{(\frac{1}{2})^2}{2\underline{|\tfrac{3}{2}}}+\frac{(\frac{1}{2}\cdot\frac{3}{2})^2}{2^2\underline{|2}\,\underline{|\tfrac{5}{2}}}+\frac{(\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2})^2}{2^3\underline{|3}\,\underline{|\tfrac{7}{2}}}+\ldots\ldots\\ &+x\left(\frac{1}{\underline{|\tfrac{3}{2}}}+\frac{(\frac{1}{2})^2}{2\underline{|\tfrac{5}{2}}}+\frac{(\frac{1}{2}\cdot\frac{3}{2})^2}{2^2\underline{|2}\,\underline{|\tfrac{7}{2}}}+\ldots\ldots\right)\\ &+x^2\left(\frac{1}{\underline{|\tfrac{5}{2}}}+\frac{(\frac{1}{2})^2}{2\underline{|\tfrac{7}{2}}}+\frac{(\frac{1}{2}\cdot\frac{3}{2})^2}{2^2\underline{|2}\,\underline{|\tfrac{9}{2}}}+\ldots\ldots\right)+\ldots\ldots\\ &+x^{-1}\left(\frac{1}{\underline{|-\tfrac{1}{2}}}+\frac{(\frac{1}{2})^2}{2\underline{|\tfrac{1}{2}}}+\frac{(\frac{1}{2}\cdot\frac{3}{2})^2}{2^2\underline{|2}\,\underline{|\tfrac{3}{2}}}+\ldots\ldots\right)+\ldots\ldots\Bigg].\end{aligned} \tag{6}$$

**28.** Now B involves all the odd powers of $x$, whilst A involves only the even positive powers. But the terms involving even negative powers in A are zero, if we follow the law of the coefficients. So A is also complete, and C must be some combination of the series A and B. In fact, if we assume that

$$u = a_0 + a_1x + a_2x^2 + \dots + b_1x^{-1} + b_2x^{-2} + \dots \tag{7}$$

is a solution of the characteristic (4), and insert it therein, to find the law of the coefficients in the usual manner, we find that the even $b$'s are zero, whilst the even $a$'s are connected in one way, and the odd $a$'s and even $b$'s are independently connected in another way. This makes

$$u = a\mathrm{A} + b\mathrm{B}, \tag{8}$$

where $a$ and $b$ are independent multipliers. Now, judging from common experience with this rule-of-thumb method of constructing solutions of differential equations, we might hastily conclude that A and B represented the two independent solutions of the characteristic. Here, however, we know (analytically) that they are not independent, but are equivalent. Therefore

$$\mathrm{C} = a\mathrm{A} + b\mathrm{B}, \tag{9}$$

where the sum of $a$ and $b$ is unity. It only remains to find the value of $a$. This is easily obtainable, because the separate series in (6) are rapidly convergent. But we need only employ the first series, viz., to find the coefficient of $x^0$. Thus, the first line of (6) gives

$$a = \frac{\sqrt{2}}{\pi}\left\{1 + \tfrac{1}{12}(1 + \tfrac{9}{40}(1 + \tfrac{25}{84}(1 + \tfrac{49}{144}(1 + \dots\right\}$$

$$= \frac{1{\cdot}1106}{2{\cdot}2214} = 0{\cdot}5. \tag{10}$$

We see, therefore, that the series C is algebraically identical with half the sum of the series A and B.

To further verify, we see that the coefficient of $x$ in (6) should be $2/\pi$ times that of $x^0$. This requires

$$\frac{2}{\pi} \times 1{\cdot}1106 = \tfrac{2}{3}(1 + \tfrac{1}{20}(1 + \tfrac{9}{56}(1 + \tfrac{25}{108}(1 + \dots,$$

or

$$0{\cdot}7068 = 0{\cdot}7067,$$

which is also close. Similarly, from the $x^2$ series we require

$$\frac{1{\cdot}1106}{4} = \tfrac{4}{15}(1 + \tfrac{1}{28}(1 + \tfrac{1}{8}(1 + \tfrac{25}{132}(1 + \dots,$$

or

$$0{\cdot}277 = 0{\cdot}277.$$

The numerical tests in this example are perfectly satisfactory; and if the numerical meaning of a divergent series could be always as easily fixed, it would considerably facilitate the investigation of the subject.

*Condensed Generalized Notation. Generalization of the Descending Series for the Bessel Function through the Generalized Binomial Theorem.*

**29.** Since the series A and B are particular cases of the general formula (77), Part I, or of

$$u = \Sigma \frac{\left(\frac{x^2}{4}\right)^r}{(\underline{|r})^2}, \tag{11}$$

by taking $r = 0$ and $r = \frac{1}{2}$ respectively, it may be desirable to find the general formula of which the series C is a particular case. Notice, in passing, the shorter notation employed in (11). It is certainly easier to see the meaning of a series by inspecting the written-out formula containing several terms, when one is not familiar with the kind of series concerned. As soon, however, as one gets used to the kind of formula, the writing out of several terms becomes first needless, and then tiresome. The short form (11) is then sufficient. One term only is written, with the summation sign before it. The other terms are got by changing $r$ with unit step always, and both ways. The value of $r$ is arbitrary, though of course it should have the same value in every term so far as the fractional part is concerned, so that, in (11), $r$ may be changed to any other number without affecting its truth. Similarly, the exponential formula may be written

$$\epsilon^x = \Sigma \frac{x^r}{\underline{|r}}, \tag{12}$$

with $r$ arbitrary and unit step.

Now, to find the generalized formula wanted, we have, by (25), Part I,

$$\mathrm{I}_0(x) = \epsilon^x(1+2\nabla^{-1})^{-\frac{1}{2}}. \tag{13}$$

Expand this according to the particular form of the binomial theorem got by taking $n = -\frac{1}{2}$ in (84), Part I, leaving $m$ arbitrary. Or writing that general formula thus:—

$$\frac{(1+x)^n}{\underline{|n}} = \Sigma \frac{x^r}{\underline{|r}\,\underline{|n-r}}, \tag{14}$$

which is compact and intelligible, according to the above explanation, take $n = -\frac{1}{2}$, and write $2\nabla^{-1}$ in place of $x$. This makes

$$\frac{(1+2\nabla^{-1})^{-\frac{1}{2}}}{\underline{|-\tfrac{1}{2}}} = \Sigma \frac{(2\nabla^{-1})^r}{\underline{|r}\,\underline{|-\tfrac{1}{2}-r}}. \tag{15}$$

Effect the integration, and we obtain immediately

$$= \Sigma \frac{(2x)^r}{(\underline{|r})^2\,\underline{|-\tfrac{1}{2}-r}}; \tag{16}$$

and therefore, by (13),

$$\mathrm{I}_0(x) = \epsilon^x \underline{|-\tfrac{1}{2}}\,\Sigma \frac{(2x)^r}{(\underline{|r})^2\,\underline{|-\tfrac{1}{2}-r}}. \tag{17}$$

Here we see the great convenience in actual work of the condensed notation. At the same time, it is desirable to expand sometimes and see what the developed formula looks like. We then take the written term as a central basis, making it a factor of all the rest. Thus,

$$\mathrm{I}_0(x) = \frac{(2x)^r \epsilon^x \underline{|-\tfrac{1}{2}}}{\underline{|-\tfrac{1}{2}-r}\,(\underline{|r})^2} \Bigg\{ 1 + \frac{2x(-\frac{1}{2}-r)}{(r+1)^2}\Bigg(1+\frac{2x(-\frac{3}{2}-r)}{(r+2)^2}\Bigg(1+\ldots\ldots$$

$$+\frac{(2x)^{-1}r^2}{(\frac{1}{2}-r)}\Bigg(1+\frac{(2x)^{-1}(r-1)^2}{(\frac{3}{2}-r)}\Bigg(1+\ldots\ldots\Bigg\}. \tag{18}$$

**30.** Take $r = 0$ in this, and we have

$$\mathrm{I}_0(x) = \epsilon^x\Bigg(1-x\Bigg(1-\frac{3x}{2^2}\Bigg(1-\frac{5x}{3^2}\Bigg(1-\ldots\ldots, \tag{19}$$

which is the same as (20), Part I, noting that $\frac{1}{2}at$ there is $x$ here. But of course the exponential factor is now of no service, the ordinary series A, equation (1) above, being the practical formula when $x$ is small.

Take $r = -\frac{1}{2}$ in (18), and we obtain

$$\mathrm{I}_0(x) = \frac{\epsilon^x}{(2\pi x)^{\frac{1}{2}}}\Bigg\{1+\frac{1^2}{8x}\Bigg(1+\frac{3^2}{2\,.\,8x}\Bigg(1+\frac{5^2}{3\,.\,8x}\Bigg(1+\ldots\ldots, \tag{20}$$

which is the formula C, equation (3) above, the practical formula when $x$ is bigger than is suitable for rapid calculation by A. Observe that these are the extreme cases, for the whole of the second line in (18) goes out to make (19), and the whole of the first line, excepting the first term, goes out to make (20). On the other hand, it frequently happens that extreme cases of a generalized formula are numerically uninterpretable.

To convert (18) to the form $a\mathrm{A}+b\mathrm{B}$ algebraically, we may use the exponential expansion in the form (12), but with $r$ negatived, thus,

$$\epsilon^x = \Sigma \frac{x^{-r}}{\underline{|-r}}. \tag{21}$$

Employing this in (18), we can reduce the series to one containing integral powers only. The coefficient of $x^0$ is made to be

$$\Sigma \frac{2^r \underline{|-\frac{1}{2}}}{(\underline{|r})^2 \, \underline{|-\frac{1}{2}-r} \, \underline{|-r}}. \tag{22}$$

That this reduces correctly to a convergent series summing up to $\frac{1}{2}$, when $r = -\frac{1}{2}$, may be anticipated and verified. Also, that when $r = 0$ we obtain unity is sufficiently evident. In these conclusions we merely corroborate the preceding. But I have not been able to reduce (22) to a simple formula showing plainly in what ratio the formulæ A and B are involved when $r$ has any other values than 0 and $\frac{1}{2}$ (or, any integral value, and the same *plus* $\frac{1}{2}$).

*The Extreme Forms of the Binomial Theorem. Obscurities.*

**31.** There are some peculiarities about the extreme forms of the binomial theorem when the exponent is negative unity (or a negative integer) which deserve to be noticed, because they are concerned in failures, or apparent failures, which occur in derived formulæ. These peculiarities are connected with the vanishing of the inverse factorial for any negative integral value of the argument. Thus, in

$$\frac{(1+x)^n}{\underline{|n}} = \Sigma \frac{x^r}{\underline{|r} \, \underline{|n-r}}, \tag{23}$$

take $n = -1$. We obtain

$$\frac{(1+x)^{-1}}{\underline{|-1}} = \frac{x^r}{\underline{|r} \, \underline{|-1-r}} \left\{ (1-x+x^2-x^3+\dots) \right.$$
$$\left. -(x^{-1}-x^{-2}+x^{-3}-x^{-4}+\dots) \right\}. \tag{24}$$

Now, on the left side we have the vanishing factor $(\underline{|-1})^{-1}$. So, on the right side, the quantity in the big brackets should generally vanish. This asserts that

$$1-x+x^2-x^3+\dots = x^{-1}-x^{-2}+x^{-3}-x^{-4}+\dots, \tag{25}$$

where on the left side we have the result of dividing 1 by $1+x$, and, on the right, the result of dividing 1 by $x+1$, or $x^{-1}$ by $1+x^{-1}$. These series are the extreme forms of the expansion of $(1+x)^{-1}$ by the ordinary binomial theorem, and they are asserted to be algebraically equivalent, although the numerical equivalence, which is sometimes recognisable, is often scarcely imaginable.

But observe that if we choose $r = 0$ as well, we have a nullifying factor on the right side also of (24). It is apparently the same as the other, and could be removed from both sides if it were finite. It must not, however, be removed from (24). What is asserted is that $0\times(1+x)^{-1} = 0\times 0$, where the first 0 on the right is $(\underline{|-1})^{-1}$, and also the 0 on the left.

Again, if we put $r = 0$ first in (23), making

$$\frac{(1+x)^n}{\underline{|n}} = \frac{1}{\underline{|n}}+\frac{x}{\underline{|1}\;\underline{|n-1}}+\dots\dots+\frac{x^{-1}}{\underline{|n+1}\;\underline{|-1}}+\frac{x^{-2}}{\underline{|n+2}\;\underline{|-2}}+\dots\dots, \tag{26}$$

and then put $n = -1$, we get $0 = 0$. But if we multiply (26) by $\underline{|n}$, making

$$(1+x)^n = 1+nx+\frac{n(n-1)}{\underline{|2}}x^2+\dots\dots$$
$$+\frac{x^{-1}}{(n+1)\,\underline{|-1}}+\frac{x^{-2}}{(n+1)\,(n+2)\,\underline{|-2}}+\dots\dots, \tag{27}$$

we see that the descending series vanishes when $n$ is any negative integer. That is, it is asserted that

$$(1+x)^n = 1+nx+\frac{n(n-1)}{1\,.\,2}x^2+\dots\dots, \tag{28}$$

unless $n$ is negatively integral. But when it is a negative integer there are additional terms, though always in indeterminate form; for instance, $\infty\times 0$ when $n = -1$ and $x$ is finite. It would appear, however, that the value is zero, because there is every reason to think (28) correct (as a particular form) in the limit.

On the other hand, if we multiply (26) by $\underline{|-1}$, and so make it cancel the $\underline{|-1}$, $\underline{|-2}$, &c., in the denominators, we get, when $n$ is $-1$,

$$(1+x)^{-1} = 1-x+x^2-\dots\dots+x^{-1}-x^{-2}+x^{-3}-\dots\dots, \tag{29}$$

which is quite inadmissible, since the right member is the sum of two series previously found to be equivalent to one another, and to the left member. The right member is therefore twice as great as the left.

### *Improved Statement of the Binomial Theorem with Integral Negative Index.*

**32.** A consideration of the above obscurities suggests the following way of avoiding them. We should recognize that the zeros $(\underline{|n})^{-1}$ and $r$, when we take $n = -1$ and $r = 0$, are independent, and may have any ratio we please. Thus, first put $n = -1+s$ in (23), making

$$\frac{(1+x)^{-1+s}}{\underline{|-1+s}} = \frac{x^r}{\underline{|r}}\,\frac{1}{\underline{|-1+s-r}}\left\{1+\frac{-1+s-r}{r+1}x\left(1+\frac{-2+s-r}{r+2}x\left(1+\ldots\right.\right.\right.$$

$$\left.+\frac{r}{s-r}x^{-1}\left(1+\frac{r-1}{(s-r)(1+s-r)}x^{-1}\left(1+\ldots\right.\right.\right\}. \quad (30)$$

This being general, let $r$ and $s$ be both infinitely small, but without any connexion. We know that the rate of increase of the inverse factorial with $n$ is 1 when $n$ is $-1$. It follows that

$$\frac{1}{\underline{|-1+s}} = s, \qquad \frac{1}{\underline{|-1+s-r}} = s-r. \quad (31)$$

These, used in (30), make it become

$$s\,(1+x)^{-1+s} = \frac{x^r}{\underline{|r}}(s-r)\left\{1+\frac{-1+s-r}{r+1}x+\ldots\right.$$

$$\left.+\frac{r}{s-r}x^{-1}+\frac{r(r-1)}{(s-r)(1+s-r)}x^{-2}+\ldots\right\}. \quad (32)$$

Ultimately, therefore, we obtain in a clear manner

$$(1+x)^{-1}=\left(1-\frac{r}{s}\right)\left(1-x+x^2-x^3+\ldots\right)+\frac{r}{s}\left(x^{-1}-x^{-2}+x^{-3}-\ldots\right). \quad (33)$$

This seems to be the proper limiting form of the binomial theorem when the index is negative unity. It asserts that the two extreme equivalent forms may be combined in any ratio we please, since $r/s$ may have any value. If $r=0$, we have the ascending series only. If $r=s$, then the descending series only. If $s=2r$, we obtain half their sum. The expansion is indeterminate, but the degree of indeterminateness appears to be merely conditioned by the size of the ratio $r/s$.

We may also notice that the suppositions that $s$ is infinitely small and $r$ is finite, so that

$$\frac{1}{\underline{|-1+s}} = s, \qquad \text{and} \qquad \frac{1}{\underline{|-1+s-r}} = \frac{1}{\underline{|-1-r}},$$

used in (30), lead us to

$$(1+x)^{-1} = \frac{x^r}{\underline{|r}}\,\frac{1}{\underline{|-1-r}}\cdot\frac{1}{s}\left\{1-x+x^2-x^3+\ldots\right.$$

$$\left.-x^{-1}+x^{-2}-x^{-3}+\ldots\right\}; \quad (34)$$

that is, the difference of the two extreme equivalent series divided by 0, which is, of course, indeterminate.

*Consideration of a more general Operator,* $(1+\nabla^{-1})^n$. *Suggested Derived Equivalences.*

**33.** Some years since, after noticing first the analytical and then later the numerical equivalence of the different formulæ for the Fourier-Bessel function arising immediately from the operator $(1+\nabla^{-1})^{-\frac{1}{2}}$ by the use of the two extreme forms of the binomial theorem (the only forms then known to me), I endeavoured to extend the results by substituting the operator $(1+\nabla^{-1})^n$, which includes the former, and comparing the extreme forms. Thus, calling $u$ the series in ascending powers of $\nabla^{-1}$, and $v$ the descending series, so that

$$u = 1+n\nabla^{-1}+\frac{n(n-1)}{\underline{|2}}\nabla^{-2}+\dots, \tag{35}$$

$$v = \nabla^{-n}\left(1+n\nabla+\frac{n(n-1)}{\underline{|2}}\nabla^{2}+\dots\right), \tag{36}$$

and integrating (with $x^0$ for operand, as usual when no operand is written), we obtain

$$u = 1+nx+\frac{n(n-1)}{\underline{|2}\,\underline{|2}}x^2+\frac{n(n-1)(n-2)}{\underline{|3}\,\underline{|3}}x^3+\dots, \tag{37}$$

$$v = \frac{x^n}{\underline{|n}}\left(1+\frac{n^2}{x}+\frac{n^2(n-1)^2}{x^2\underline{|2}}+\frac{n^2(n-1)^2(n-2)^2}{x^3\underline{|3}}+\dots\right); \tag{38}$$

and the suggestion is that these are equivalent. If this equivalence is analytical, and we substitute $\nabla^{-1}$ for $x$ and integrate a second time, we obtain

$$1+nx+\frac{n(n-1)}{(\underline{|2})^3}x^2+\frac{n(n-1)(n-2)}{(\underline{|3})^3}x^3+\dots \tag{39}$$

$$= \frac{x^n}{(\underline{|n})^2}\left\{1+\frac{n^3}{x}+\frac{n^3(n-1)^3}{x^2\underline{|2}}+\frac{n^3(n-1)^3(n-2)^3}{x^3\underline{|3}}+\dots\right\}; \tag{40}$$

and obvious repetitions of the same process lead us to

$$1+nx+\frac{n(n-1)}{(\underline{|2})^m}x^2+\frac{n(n-1)(n-2)}{(\underline{|3})^m}x^3+\dots \tag{41}$$

$$= \frac{x^n}{(\underline{|n})^{m-1}}\left\{1+\frac{n^m}{x}+\frac{n^m(n-1)^m}{x^2\underline{|2}}+\frac{n^m(n-1)^m(n-2)^m}{x^3\underline{|3}}+\dots\right\}, \tag{42}$$

which are clearly the cases $r=0$ and $r=n$ of the general expression

$$\Sigma\frac{x^r\underline{|n}}{(\underline{|r})^m\underline{|n-r}}; \tag{43}$$

provided $n$ is not a negative integer, when we know that closer examination is required.

*Apparent Failure of Numerical Equivalence in certain Cases.*

**34.** Now, although the equations following (35), (36) (excepting (43)) are deducible from them by the process used immediately and without trouble, there is considerable difficulty in finding out their meaning. Considering (37) and (38), I knew that in the case $n = -\frac{1}{2}$ the equivalence was satisfactory all round, though not very understandable. When $n$ is 0, or integral, it is also satisfactory, for then we have merely a perversion of terms in passing from $u$ to $v$. But when I tried the case $n = -\frac{1}{4}$, and subjected it to numerical calculation, with the expectation of finding numerical equivalence to the extent permitted by the initial convergence of the divergent series, I found a glaring discrepancy between $u$ and $v$. Furthermore, on taking $n = -1$, we produce

$$u = \epsilon^{-x}, \tag{44}$$

$$v = \frac{x^{-1}}{\underline{|-1}}\left(1+\frac{1}{x}+\frac{\underline{|2}}{x^2}+\frac{\underline{|3}}{x^3}+\dots\right), \tag{45}$$

which show no sort of numerical equivalence whatever. Similarly, $n = -2$ gives

$$u = 1-2x+\frac{3}{\underline{|2}}x^2-\frac{4}{\underline{|3}}x^3+\dots, \tag{46}$$

$$v = \frac{x^{-2}}{\underline{|-2}}\left(1+\frac{2^2}{x}+\frac{2^2 3^2}{\underline{|2}x^2}+\frac{2^2 3^2 4^2}{\underline{|3}x^3}+\dots\right), \tag{47}$$

which also do not show any numerical equivalence. I was therefore led to think that the equivalence in the case of the Fourier-Bessel function was due to some peculiarity of that function, and it is a fact that the function is the meeting-place of many remarkabilities. The matter was therefore put on one side for the time. But, more recently, independent evidence in other directions showed me that there was no particular reason to expect such a complete failure. And, in fact, on returning to the discrepant calculations relating to $n = -\frac{1}{4}$, I found an important numerical error. When corrected, the results for $u$ and $v$ agreed as fairly as could be expected.

*Probable Satisfaction of Numerical Equivalence by Initial Convergence within a certain Range for n, viz., $n = -\frac{1}{2}$ to $+1$.*

**35.** Thus, $n = -\frac{1}{4}$ in (37), (38) produces

$$u = 1-\frac{x}{4}+\frac{1.5}{\underline{|2}\,\underline{|2}}\left(\frac{x}{4}\right)^2-\frac{1.5.9}{\underline{|3}\,\underline{|3}}\left(\frac{x}{4}\right)^3+\ldots., \tag{48}$$

$$v = \frac{x^{-\frac{1}{4}}}{\underline{|-\frac{1}{4}}}\left(1+\frac{1}{16x}+\frac{1^2 5^2}{\underline{|2}(16x)^2}+\frac{1^2 5^2 9^2}{\underline{|3}(16x)^3}+\ldots.\right). \tag{49}$$

Here take $x = 4$. Then

$$u = 1-1(1-\tfrac{5}{4}(1-1(1-\tfrac{13}{16}(1-\tfrac{17}{25}(1-\ldots., \tag{50}$$

$$v = \frac{1}{4^{\frac{1}{4}}\underline{|-\frac{1}{4}}}\left\{1+\frac{1}{64}\left(1+\frac{25}{2.64}\left(1+\frac{81}{3.64}\left(1+\frac{169}{4.64}\left(1+\ldots.\right.\right.\right.\right.\right. . \tag{51}$$

This was the test case which failed, the error arising from the numerical equality of two consecutive terms, and then, a little later, of another two consecutive terms, which caused a skipping. I now make

$$u = 0{\cdot}5880, \qquad v = \frac{1{\cdot}0216}{4^{\frac{1}{4}}\underline{|-\frac{1}{4}}}.$$

Their equivalence requires that

$$\frac{1}{\underline{|-\frac{1}{4}}} = \frac{0{\cdot}5880\times 1{\cdot}4142}{1{\cdot}0216} = 0{\cdot}814,$$

which is about right.

When $x = 2$, we have

$$u = 1-\tfrac{1}{2}+\tfrac{5}{16}(1-\tfrac{9}{18}(1-\tfrac{13}{32}(1-\tfrac{17}{50}(1-\tfrac{21}{72}(1-\ldots.,$$

$$v = \frac{1}{2^{\frac{1}{4}}\underline{|-\frac{1}{4}}}\left\{1+\tfrac{1}{32}(1+\tfrac{25}{64}(1+\tfrac{81}{96}(1+\ldots.\right\},$$

giving $\qquad u = 0{\cdot}706, \qquad v = \dfrac{1{\cdot}043}{2^{\frac{1}{4}}\underline{|-\frac{1}{4}}};$

which requires $\qquad \dfrac{1}{\underline{|-\frac{1}{4}}} = 0{\cdot}805.$

And when $x = 1$ we have

$$u = 1-\tfrac{1}{4}+\tfrac{5}{64}(1-\tfrac{9}{36}(1-\tfrac{13}{64}(1-\tfrac{17}{100}(1-\ldots.,$$

$$v = \frac{1}{\underline{|-\frac{1}{4}}}(1+\tfrac{1}{16}(1+\tfrac{25}{32}(1+\tfrac{81}{48}(1+\ldots.;$$

giving $$u = 0{\cdot}8123, \qquad v = \frac{1{\cdot}0625}{\lfloor\underline{-\tfrac{1}{4}}},$$

requiring $$\frac{1}{\lfloor\underline{-\tfrac{1}{4}}} = 0{\cdot}76.$$

Of course with such a small value of $x$, we cannot expect more than a very rough agreement, because the convergence of the $v$ series is confined to the first and second terms, and we may expect an error of magnitude of the ratio of the second to the first term.

**36.** Now take $n = \frac{1}{4}$. We have

$$u = 1+\frac{x}{4}-\tfrac{3}{4}\left(\frac{x}{4}\right)^2+\frac{1\,.\,3\,.\,7}{\lfloor\underline{3}\,\lfloor\underline{3}}\left(\frac{x}{4}\right)^3-\frac{1\,.\,3\,.\,7\,.\,11}{\lfloor\underline{4}\,\lfloor\underline{4}}\left(\frac{x}{4}\right)^4+\ldots\ldots, \tag{52}$$

$$v = \frac{x^{\frac{1}{4}}}{\lfloor\underline{\tfrac{1}{4}}}\left\{1+\frac{1}{16x}+\frac{9}{\lfloor\underline{2}\,(16x)^2}+\frac{3^2\,.\,7^2}{\lfloor\underline{3}\,(16x)^3}+\ldots\ldots\right\}; \tag{53}$$

and in case of $x = 1$ we have

$$u = 1+\tfrac{1}{4}(1-\tfrac{3}{16}(1-\tfrac{7}{36}(1-\tfrac{11}{64}(1-\tfrac{15}{100}(1-\ldots\ldots, \tag{54}$$

$$v = \frac{1}{\lfloor\underline{\tfrac{1}{4}}}\left\{1+\tfrac{1}{16}(1+\tfrac{9}{32}(1+\tfrac{49}{48}(1+\ldots\ldots\right\}; \tag{55}$$

which make $$u = 1{\cdot}2109, \qquad v = \frac{1{\cdot}18}{\lfloor\underline{\tfrac{1}{4}}};$$

and therefore $$\frac{1}{\lfloor\underline{\tfrac{1}{4}}} = 1{\cdot}024.$$

Now this shows a large error, for the value is about 1·11. This excess in $v$ is, however, made a deficit by not counting the smallest term in the $v$ series (the third term). Omitting it, we make

$$v = \frac{1{\cdot}0625}{\lfloor\underline{\tfrac{1}{4}}} \quad\text{and}\quad \frac{1}{\lfloor\underline{\tfrac{1}{4}}} = 1{\cdot}14.$$

Again, with $x = 2$, we have

$$u = 1+\tfrac{1}{2}-\tfrac{3}{16}(1-\tfrac{7}{18}(1-\tfrac{11}{32}(1-\tfrac{15}{50}(1-\tfrac{19}{72}(1-\ldots\ldots,$$

$$v = \frac{2^{\frac{1}{4}}}{\lfloor\underline{\tfrac{1}{4}}}\left\{1+\tfrac{1}{32}(1+\tfrac{9}{64}(1+\tfrac{49}{96}(1+\tfrac{121}{128}(1+\ldots\ldots\right\},$$

making $$u = 1{\cdot}365, \qquad v = \frac{1{\cdot}0399}{\lfloor\underline{\tfrac{1}{4}}}\times 2^{\frac{1}{4}}.$$

This makes $$\frac{1}{\lfloor\underline{\tfrac{1}{4}}} = \frac{1{\cdot}365}{1{\cdot}04 \times 1{\cdot}18} = 1{\cdot}11,$$

which is very good.

**37.** Now passing to the case of a bigger $n$, viz., $\frac{1}{2}$, we may remark that this differs from the known good case $n = -\frac{1}{2}$ by an integral differentiation, so we may expect good results again. We have

$$u = 1+\frac{x}{2}\left(1-\frac{x}{8}\left(1-\frac{x}{6}\left(1-\frac{5x}{32}\left(1-\frac{7x}{50}\left(1-\ldots\right.\right.\right.\right.\right., \tag{56}$$

$$v = \frac{2x^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}\left(1+\frac{1}{4x}\left(1+\frac{1}{8x}\left(1+\frac{9}{12x}\left(1+\frac{25}{16x}\left(1+\ldots\right.\right.\right.\right.\right. . \tag{57}$$

Taking $x = 1$ first, giving

$$u = 1+\tfrac{1}{2}(1-\tfrac{1}{8}(1-\tfrac{1}{6}(1-\tfrac{5}{32}(1-\ldots , \tag{58}$$

$$v = \frac{2}{\pi^{\frac{1}{2}}}\left\{1+\tfrac{1}{4}(1+\tfrac{1}{8}(1+\tfrac{3}{4}(1+\tfrac{25}{16}(1+\ldots\right. , \tag{59}$$

we find $\qquad u = 1{\cdot}4464, \qquad v = \dfrac{2{\cdot}5625}{1{\cdot}772} = 1{\cdot}4462,$

by *not* counting the last convergent, that is, the smallest term in the $v$ series. Its inclusion makes $v$ appreciably too big, viz. $1{\cdot}46$.

Next take $x = 2$. Then

$$u = 1+1-\tfrac{1}{4}(1-\tfrac{1}{3}(1-\tfrac{5}{16}(1-\tfrac{7}{25}(1-\tfrac{9}{36}(1-\tfrac{11}{49}(1-\ldots , \tag{60}$$

$$v = \frac{2\sqrt{2}}{\sqrt{\pi}}\left\{1+\tfrac{1}{8}(1+\tfrac{1}{16}(1+\tfrac{9}{24}(1+\tfrac{25}{32}(1+\tfrac{49}{40}(1+\ldots\right. ; \tag{61}$$

giving $\qquad u = 1{\cdot}81275, \qquad v = \dfrac{3{\cdot}2124}{1{\cdot}772} = 1{\cdot}812,$

again not counting the smallest term.

Lastly, with $x = 3$, we have

$$u = 1+\tfrac{3}{2}-\tfrac{9}{16}(1-\tfrac{1}{2}(1-\tfrac{15}{32}(1-\tfrac{21}{50}(1-\tfrac{27}{72}(1-\tfrac{33}{98}(1-\ldots , \tag{62}$$

$$v = \frac{2\sqrt{3}}{\sqrt{\pi}}\left\{1+\tfrac{1}{12}(1+\tfrac{1}{24}(1+\tfrac{1}{4}(1+\tfrac{25}{48}(1+\tfrac{49}{60}(1+\tfrac{81}{72}(1+\ldots\right. ; \tag{63}$$

giving $\qquad u = 2{\cdot}1260, \qquad v = 2{\cdot}1256,$

again neglecting the smallest term in $v$, though it is of little moment in this example. The tendency for $v$ to be too big when the smallest term is fully counted should be noted.

**38.** A further increase of $n$ to $\frac{3}{4}$ gives good results, and likewise $\frac{9}{10}$. Thus, for $\frac{9}{10}$ we have

$$u = 1+\tfrac{9}{10}x(1-\tfrac{1}{40}x(1-\tfrac{11}{90}x(1-\tfrac{21}{160}x(1-\tfrac{31}{250}x(1-\ldots , \tag{64}$$

$$v = \frac{x^{\frac{9}{10}}}{\lfloor\frac{9}{10}}\left\{1+\frac{9^2}{100x}\left(1+\frac{1^2}{200x}\left(1+\frac{11^2}{300x}\left(1+\frac{21^2}{400x}(1+\ldots\right.\right.\right.\right. ; \tag{65}$$

giving in the case of $x = 1$,

$$u = 1{\cdot}880, \qquad v = \frac{1{\cdot}815}{\underline{\vert \tfrac{9}{10}}}; \qquad \therefore \frac{1}{\underline{\vert \tfrac{9}{10}}} = \frac{1{\cdot}880}{1{\cdot}815} = 1{\cdot}035.$$

Thus we have practically gone over the ground from $n = -\frac{1}{2}$ to $= 1$ with good results, so far as the limited examples are concerned, and there can be, so far, scarcely a doubt of the existence of numerical equivalence, in the same sense as before with respect to the ascending and descending series for the Fourier-Bessel function. It remains to examine cases between $n = -\frac{1}{2}$ and $-1$. This is important on account of the complete failure in the latter case of the numerical equivalence when estimated in the above manner. From the already shown indeterminateness of the binomial expansion when $n = -1$, we have the suggestion of a partial explanation, because we should arrive at the form $au + bv$, where $a + b = 1$. But there remains the fact indicated that the extreme forms of the binomial expansion are equivalent, so that we should expect $u$ and $v$ to be equivalent. Since, however, the numerical equivalence of the different forms of $(1 + x)^n$ becomes very unsatisfactory when $n$ is or is near $-1$, so we should not be surprised to find that the unsatisfactoriness becomes emphasized in the case of $u$ and $v$. Such is, in fact, the case.

*Failure of Numerical Equivalence of Derived Series reckoned by Initial Convergence, at first slight, and later complete, when n approaches a Negative Integer.*

**39.** Take $n = -\frac{3}{4}$ in (37), (38). Then

$$u = 1 - \frac{3x}{4}\left(1 - \frac{7x}{16}\left(1 - \frac{11x}{36}\left(1 - \frac{15x}{64}\left(1 - \frac{19x}{100}\right.\right.\right.\right., \tag{66}$$

$$v = \frac{1}{4x^{\frac{3}{4}}\underline{\vert \tfrac{1}{4}}}\left\{1 + \frac{9}{16x}\left(1 + \frac{49}{32x}\left(1 + \frac{121}{48x}\left(1 + \dots\right.\right.\right.\right\}. \tag{67}$$

When $x = 1$, we find that

$$u = 0{\cdot}497, \qquad v = \frac{0{\cdot}25 \text{ or } 0{\cdot}39}{\underline{\vert \tfrac{1}{4}}},$$

according as we do not, or do, count the smallest term in $v$. That is,

$$\frac{1}{\underline{\vert \tfrac{1}{4}}} = \frac{0{\cdot}497}{0{\cdot}25 \text{ or } 0{\cdot}39}.$$

Now the first gives far too great a result, whilst the other, though not so bad, is still too great. That is, the $v$ series gives too small a result, when the smallest term is fully included. A part of the next term is needed, to come to $u$.

When $x = 2$ we deduce that

$$u = 0{\cdot}28, \qquad \frac{1}{\lfloor\frac{1}{4}} = \frac{4 \times 2 \times 0{\cdot}28}{1{\cdot}28 \text{ or } 1{\cdot}49} = \frac{1{\cdot}88}{1{\cdot}28 \text{ or } 1{\cdot}49};$$

the first case being without, and the second with, the smallest term in $v$. Both results are too great, though the error is less than the last term counted. But this rule breaks down when we pass to $x = 3$, when we conclude that

$$u = 0{\cdot}175, \qquad \frac{1}{\lfloor\frac{1}{4}} = \frac{0{\cdot}175 \times 4 \times 3^{\frac{3}{4}}}{1{\cdot}1875 \text{ or } 1{\cdot}2813} = 1{\cdot}3437 \text{ or } 1{\cdot}2454;$$

the former case being without, and the latter with, the smallest term in $v$. But the result is too big, and the error rule just mentioned fails. For if we add on the smallest term a second time, we obtain $1{\cdot}1604$, which is still too big.

**40.** Since the case $n = -\frac{3}{4}$ is bad, we may expect $n = -\frac{9}{10}$ to be worse. We have

$$u = 1 - \frac{9x}{10} + \frac{9 \,.\, 19}{\lfloor 2 \lfloor 2}\left(\frac{x}{10}\right)^2 - \frac{9 \,.\, 19 \,.\, 29}{\lfloor 3 \lfloor 3}\left(\frac{x}{10}\right)^3 + \dots, \tag{68}$$

$$v = \frac{1}{x^{\frac{9}{10}} \lfloor -\frac{9}{10}} \left\{ 1 + \frac{81}{100x} + \frac{81 \,.\, 361}{\lfloor 2\, (100x)^2} + \dots \right\}. \tag{69}$$

Here take $x = 1$, then we conclude that

$$u = 0{\cdot}42, \qquad \frac{1}{\lfloor\frac{1}{10}} = \frac{4{\cdot}2}{1{\cdot}0 \text{ or } 1{\cdot}81}.$$

But it cannot lie between these limits, being only a little over unity. So add on to $v$ the next term, the third in the $v$ series. This will give

$$\frac{1}{\lfloor\frac{1}{10}} = \frac{4{\cdot}2}{3{\cdot}2},$$

which is still too great, and, of course, the error rule is wrong, as we suspected just now.

Whilst there does not appear to be any departure from numerical equivalence of $u$ and $v$ in the sense used between $n = -\frac{1}{2}$ and $n = +1$, it appears that when $n$ is below $-\frac{1}{2}$, there is a tendency for the $v$ series (convergent part) to give too small a result. This tendency, which is at first small, becomes pronounced when $n$ is down to $-\frac{9}{10}$, at least for small values of $x$. It is likely that for large values, the rule in question might still hold good. But sinking below $-\frac{9}{10}$ towards $-1$ makes the tendency become a marked characteristic, and in the end the rule wholly fails except for an infinite value of $x$.

*Success of Alternative Method of Representation by Harmonic Analysis.*

**41.** We may then adopt another method. Thus, (44) and (45) arise from

$$u = 1-\nabla^{-1}+\nabla^{-2}-\nabla^{-3}+\ldots, \tag{70}$$

$$v = \nabla-\nabla^{2}+\nabla^{3}-\nabla^{4}+\ldots. \tag{71}$$

With unit operand, the $u$ series is immediately integrable without any obscurity, giving $\epsilon^{-x}$. The $v$ series leads to an unintelligible result. But let the unit operand be replaced by its simple harmonic equivalent. Then

$$\begin{aligned} v = (1-\nabla+\nabla^2-\ldots)\nabla &= \frac{1-\nabla}{1-\nabla^2}\,\frac{1}{\pi}\int_0^\infty \cos mx\, dm \\ &= (1-\nabla)\,\frac{1}{\pi}\int_0^\infty \frac{\cos mx}{1+m^2}\, dm \\ &= (1-\nabla)\,\tfrac{1}{2}\,\epsilon^{-\sqrt{x^2}} = \epsilon^{-x}, \end{aligned} \tag{72}$$

when $x$ is positive, which is the required result. We are only concerned with positive $x$, but it is worth noting that when $x$ is negative, this method makes $v$ zero. This is also in accordance with the analytical method, or (70) directly integrated, for we suppose the operand to start when $x = 0$, and to be zero for negative $x$, which makes $u$ also zero then.

**42.** As regards the derived formulæ (39) to (42), although I have not examined them thoroughly to ascertain limits within which the suspected numerical equivalence may obtain, I find there is a rough agreement between (41) and (42) when $n = \frac{1}{2}$ and $m = 3$, even with $x = 1$, and the convergency confined to the first three terms of $v$, the results being

$$u = 1+\frac{x}{2}\left(1-\frac{x}{16}\left(1-\frac{3x}{54}\left(1-\frac{5x}{128}\left(1-\ldots\right.\right.\right.\right., \tag{73}$$

$$v = \frac{x^{\frac{1}{2}}}{(\underline{|\tfrac{1}{2}})^2}\left\{1+\frac{1}{8x}\left(1-\frac{1}{16x}\left(1-\frac{27}{24x}\left(1-\ldots\right.\right.\right.\right.; \tag{74}$$

which, when $x = 1$, give

$$u = 1{\cdot}47, \qquad v = 1{\cdot}41.$$

Again, with the much larger value $x = 9$, we have

$$u = 3{\cdot}88, \qquad v = 3{\cdot}87,$$

which is a very close agreement.

This is promising as regards further numerical agreement when $m$

is made larger, but the promise is not fulfilled when $m$ is as big as 10. Take $n = \frac{1}{2}$ and $m = 10$ in the series (41), (42), so that

$$u = 1+\frac{x}{2}\left(1-\frac{x}{2\,.\,2^{10}}\left(1-\frac{3\,x}{2\,.\,3^{10}}\left(1-\ldots\ldots\right.\right.\right., \tag{75}$$

$$v = \frac{x^{\frac{1}{2}}}{(\underline{|\tfrac{1}{2}})^9}\left\{1+\frac{1}{2^{10}x}\left(1+\frac{1}{2\,.\,2^{10}x}\left(1+\frac{3^{10}}{3\,.\,2^{10}x}\left(1+\ldots\ldots\right.\right.\right.\right. . \tag{76}$$

Here $x = 1$ makes $u$ a little less than $1\frac{1}{2}$, while the first term of $v$ is 2·965, which is very little changed by the next two. But observe a fresh peculiarity in the $v$ series. The change from convergency to divergency at the fourth term is so immensely rapid that this fact alone might render the series quite unsuitable for approximate numerical calculation. A portion of the term following the least term might be required (though not in the last example), but when this term is a large multiple of the least term, no definite information is obtainable.

### *What is the Meaning of Equivalence? Sketch of Gradual Development of Ideas concerning Equivalence and Divergent Series (up to § 49).*

**43.** In the preceding, I have purposely avoided giving any definition of "equivalence." Believing in example rather than precept, I have preferred to let the formulæ, and the method of obtaining them, speak for themselves. Besides that, I could not give a satisfactory definition which I could feel sure would not require subsequent revision. Mathematics is an experimental science, and definitions do not come first, but later on. They make themselves, when the nature of the subject has developed itself. It would be absurd to lay down the law beforehand. Perhaps, therefore, the best thing I can do is to describe briefly several successive stages of knowledge relating to equivalent and divergent series, being approximately representative of personal experience.

($a$). Complete ignorance.

($b$). A convergent series has a limit, and therefore a definite value. A divergent series, on the contrary, is of infinite value, of course. So all solutions of physical problems must be in finite terms or in convergent series. Otherwise nonsense is made.

### *The Use of Alternating Divergent Series. Boole's Rejection of Continuous Divergent Series.*

**44.** ($c$). Eye-opening. But in some physical problems divergent series are actually used for calculation. A notable example is Stokes's divergent formula for the oscillating function $J_n(x)$. He showed that the error was less than the last term included. Now series of this kind have the terms alternately positive and negative. This seems to give a clue to the numerical meaning. The terms get bigger and bigger, but the alternation of sign prevents the assumption of an infinite value, either positive or negative. It is possible to imagine a *finite* quantity split up into parts alternately positive and negative, and of successively increasing magnitude (after a certain point, for example). It is a bad arrangement of parts, certainly, but understandable roughly by the initial convergence. So the use of alternating divergent series may be justified by numerical convenience in an approximate calculation of the value of the function.

But, by the same reasoning, a direct divergent series, with all terms of one sign, is of infinite value, and therefore out of court. It cannot have a finite value, and cannot be the solution of a physical problem involving finite values. This seems to be what Boole meant in his remark on p. 475 of his 'Differential Equations' (3rd edition) :—"It is known that in the employment of divergent series an important distinction exists between the cases in which the terms of the series are ultimately all positive, and alternately positive and negative. In the latter case we are, according to a known law, permitted to employ that portion of the series which is convergent for the calculation of the entire value." He proceeded to exemplify this by Petzval's integrals. The argument is equivalent to this. Change the sign of $x$ in the Series C, equation (3) above. Let the result be C′. Then we must use the Series C′ when $x$ is positive, and C when $x$ is negative. This amounts to excluding the direct divergent series altogether, and using only the alternating. That is, we have one solution, not two. Professor Boole did not say what the "known law" was. His above authoritative rejection of direct divergent series led me away from the truth for many years. The plausibility of the argument is evident, as evident as that the value of a direct divergent series is infinity.

*Divergent Series as Differentiating Operators.*

**45.** (*d*). Later on, divergent series presented themselves in an entirely different manner. In the solution of physical problems by means of differentiating or analytical operators, the operators themselves may be either convergent, or alternatingly divergent, or directly divergent. That is, they are so when regarded algebraically, with a differentiator regarded as a quantity. When the operations indicated by the operator are carried out upon a function of the variable, the solution of the problem arises, and in a convergent form. Here, then, we have the secret of the direct divergent series at last. It is numerically meaningless, when considered algebraically, with a quantity and its powers involved. But analytically considered, the question of divergency does not arise. The proper use of divergent series is as analytical operators to obtain convergent algebraical solutions. The series C and C′ above referred to are then truly the two independent solutions of a certain differential equation, and neither should be rejected, for they are natural companions.

*Disappearance of the Distinction between Direct and Alternating Divergent Series.*

**46.** (*e*). But, still pursuing the subject along the same lines, this view is soon found to be imperfect. For a given operator leading to a convergent solution one way may lead to a divergent solution by another. Or it may lead to the same algebraical function by diverse ways. These and other considerations show that divergent series, even when continuously divergent, must be considered numerically as well as algebraically and analytically. But in the analytical use of a direct or continuously divergent series every term must be used, if the result is a convergent series. Yet it is plain that we cannot count the whole divergent series numerically, because it has no limit. And on examination we find that the initial convergent part of the continuously divergent series gives the value of the function in the same sense as an alternatingly divergent series. In the latter case we come nearest to the value by stopping at the smallest term, where the oscillation is least. If we now make all terms positive, so that the series is continuously divergent, and treat it in the same way, and stop when the addition made by a fresh term is the smallest, we come near the true value.

We now seem to have something like a distinct theory of divergent series. The supposed distinction between the alternating and the continuous divergent series has disappeared. Analytical equivalence of two series, one convergent, the other divergent, may require all terms in the divergent one to be counted. Numerical equivalence exists also, but is governed by the initial convergency.

*Broader and Deeper Views obtained by the Generalized Calculus. Analytical, Numerical, and Algebraical Equivalences. Equivalence not necessarily Identity.*

**47.** (*f*). The last view is a distinct advance, and it is certainly true in the case of many equivalences, including some which are of importance in mathematical physics. But, again, further examination shows that the last word has not been said. For on seeking to explain the meaning and origin of equivalent series, we are led to a theory of generalized differentiation, involving the inverse factorial as a completely continuous function both ways, and to methods of multiplying equivalent forms to any extent, and in a generalized manner, all previous examples being merely special extreme cases of the general results. We also come to confirm the idea we have recognized that equivalence may be understood in three distinct senses. viz., analytical, algebraical, and numerical. The first use made by me of equivalent series, one of which is continuously divergent, was analytical only. The second use was numerical. The third is algebraical, through the generalized algebraical theorems. We also see that equivalence does not necessarily or usually mean identity, Thus the series A, B, C are analytically, algebraically, and numerically equivalent with $x$ positive. But they are not algebraically identical. The identity is given by $C = \frac{1}{2}(A+B)$. This point is rather important in some transformations, and explains some previously inexplicable peculiarities. Thus, the series A is real whether $x$ be real or a pure imaginary. In the latter case, we get the oscillating function $J_0(x)$, the original Fourier cylinder function. But the equivalent series C becomes complex by the same transformation. The above-mentioned identity explains it. The second solution of the oscillating kind is brought in, as will appear a little later (§ 70).

*Partial Failure of Interpretation of Numerical Value of Divergent Series by Initial Convergence. Further Explanation yet required.*

**48.** (*g*). But whilst we thus greatly extend our views concerning divergent series, the question of numerical equivalence, which just now in (*f*) seemed to be about settled, becomes again obscured. The property that the value of a divergent series, including the continuously divergent, may be estimated by the initially convergent part, is a very valuable one. But the property is not generally true, and, in fact, sometimes fails in a very marked manner. We must, therefore, reserve for the present the question of numerical equivalence in general, and let the explanation evolve itself in course of time. If definitely understandable numerical equivalence of series were imperative under all circumstances, then I am afraid that the study of the subject would be of doubtful value. But the matter has not this limited range, a very important application of divergent series being their analytical use, which is free from the numerical difficulty. For example, the extreme forms of the binomial theorem may, when considered numerically equivalent, be utterly useless. Yet they may be employed to lead to other series, either convergent, or it may be divergent, but with a satisfactory initial convergence contrasting with the original. Note that the series may sometimes take the form of definite integrals, apparently of infinite or of indefinite value. In any case we should not be misled by apparent unintelligibility to ignore the subject. That is not the way to get on. We have seen the error fallen into by Boole and others on the subject of divergent series. It is not so long ago, either, since mathematicians of the highest repute could not see the validity of investigations based upon the use of the algebraic imaginary. The results reached were, according to them, to be regarded as suggestive merely, and required proof by methods not involving the imaginary. But familiarity has bred contempt, and at the present day the imaginary is a generally used powerful engine, which I should think most mathematicians consider can be trusted (if well treated) to give valid proofs, though it certainly does need cautious treatment sometimes, and perhaps auxiliary aid.*

*Application of Generalized Binomial Theorem to obtain a Generalized Formula for* $\log x$.

**49.** Let us now pass on to view the logarithm in its generalized aspect. One way of generalizing $\log x$ is to regard it as the limit of $(d/dn)x^n$ when $n = 0$. Now, using the generalized binomial theorem

$$(1+x)^n = \Sigma \frac{x^r \underline{|n}}{\underline{|r}\,\underline{|n-r}}, \tag{77}$$

where $r$ has any value and the step is unity, we obtain by this process

$$\log(1+x) = \Sigma \frac{x^r}{\underline{|r}} \frac{d}{dn}\left(\frac{\underline{|n}}{\underline{|n-r}}\right)_{(n=0)}$$

$$= \Sigma \frac{x^r}{\underline{|r}\,\underline{|-r}}\left(\frac{(\underline{|0})'}{\underline{|0}} - \frac{(\underline{|-r})'}{\underline{|-r}}\right) \tag{78}$$

where the accent means differentiation to $n$, after which the special values are given to the argument. Or, since

$$\frac{1}{\underline{|r}}\frac{1}{\underline{|-r}} = \frac{\sin r\pi}{r\pi}, \qquad \text{and} \qquad \frac{(\underline{|n})'}{\underline{|n}} = -\frac{f'(n)}{f(n)}, \tag{79}$$

if $f(n)$ is the inverse factorial, therefore

$$\log(1+x) = \Sigma\, x^r \frac{\sin r\pi}{r\pi}\left(\frac{f'(-r)}{f(-r)} - \frac{f'(0)}{f(0)}\right). \tag{80}$$

But also

$$f(0) = 1, \qquad f'(0) = C = 0{\cdot}5772, \qquad \Sigma\, x^r \frac{\sin r\pi}{r\pi} = 1,$$

by § 17 and equation (94), Part I. So we reduce to

$$\log(1+x) = -C + \Sigma\, x^r f(r) f'(-r). \tag{81}$$

* Perhaps we may fairly regard the theory of generalized analysis as being now in the same stage of development as the theory of the imaginary was before the development of the modern theory of functions. Not that I know much about the latter; the big book lately turned out by Forsyth reveals to me quite unexpected developments.

**50.** To obtain the common formula for the logarithm, take $r = 0$. Then, since

$$f'(-1) = 1, \quad f'(-2) = -1, \quad f'(-3) = \underline{|2}, \quad f'(-4) = -\underline{|3}, \quad \&c.,$$

we reduce (81) to

$$\log(1+x) = -C + f'(0) + xf'(-1) + \frac{x^2}{\underline{|2}} f'(-2) + \frac{x^3}{\underline{|3}} f'(-3) + \ldots.$$

$$= x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \ldots. \qquad (82)$$

When $r = \frac{1}{2}$ in (81), we have

$$\log(1+x) = -C + \frac{2}{\pi}\left\{ x^{\frac{1}{2}} \frac{f'(-\frac{1}{2})}{f(-\frac{1}{2})} - \tfrac{1}{3}x^{\frac{3}{2}} \frac{f'(-\frac{3}{2})}{f(-\frac{3}{2})} + \tfrac{1}{5}x^{\frac{5}{2}} \frac{f'(-\frac{5}{2})}{f(-\frac{5}{2})} - \ldots \right.$$

$$\left. + x^{-\frac{1}{2}} \frac{f'(\frac{1}{2})}{f(\frac{1}{2})} - \tfrac{1}{3}x^{-\frac{3}{2}} \frac{f'(\frac{3}{2})}{f(\frac{3}{2})} + \tfrac{1}{5}x^{-\frac{5}{2}} \frac{f'(\frac{5}{2})}{f(\frac{5}{2})} - \ldots \right\}. \qquad (83)$$

Now here all the differential coefficients of the inverse factorials may be put in terms of $f'(-\frac{1}{2})$ by means of the formula

$$f(n) + nf'(n) = f'(n-1), \qquad (84)$$

which follows from

$$nf(n) = f(n-1); \qquad (85)$$

but since the resulting formula does not seem to be useful, and is complicated, it need not be given here.

*Deduction of Formula for* $(1+x)^{-1}$.

**51.** If we differentiate (81) with respect to $x$, we obtain

$$\frac{1}{1+x} = \Sigma\, x^{r-1} f(r-1) f'(-r)$$

$$= \Sigma\, x^r f(r) f'(-r-1), \qquad (86)$$

where the second form of the series is got by increasing $r$ by unity in the first. Here note that we have a definite expansion, whereas in § 32 we found the binomial expansion to be indeterminate. When $r = 0$ in (86) we have, of course, the special form $1 - x + x^2 - \ldots.$ It is also right when $r = \frac{1}{2}$.

*Deduction of Formula for* $\epsilon^{-x}$.

**52.** Now regard (86) as true analytically, and we can obtain a formula for $\epsilon^{-x}$. For, first put $\nabla^{-1}$ for $x$, giving

$$\frac{1}{1+\nabla^{-1}} = \Sigma\, \nabla^{-r} f(r)\, f'(-r-1). \tag{87}$$

Integrating, we obtain

$$\epsilon^{-x} = \Sigma\, x^r [f(r)]^2 f'(-r-1). \tag{88}$$

This is quite correct when $r = 0$, when we obtain the ordinary formula $1-x+\ldots$. Another form of (88) is

$$\epsilon^{-x} = -\Sigma\, x^r (\underline{|-r-1})' \frac{\sin^2 (r+1)\pi}{\pi^2}. \tag{89}$$

Now when $r=\frac{1}{2}$, the square of the sine equals unity throughout, giving

$$\epsilon^{-x} = -\frac{1}{\pi^2}\Big\{ (\underline{|-\tfrac{1}{2}})' x^{-\frac{1}{2}} + (\underline{|-\tfrac{3}{2}})' x^{\frac{1}{2}} + (\underline{|-\tfrac{5}{2}})' x^{\frac{3}{2}} + (\underline{|-\tfrac{7}{2}})' x^{\frac{5}{2}} + \ldots$$
$$+ (\underline{|\tfrac{1}{2}})' x^{-\frac{3}{2}} + (\underline{|\tfrac{3}{2}})' x^{-\frac{5}{2}} + (\underline{|\tfrac{5}{2}})' x^{-\frac{7}{2}} + \ldots \Big\}. \tag{90}$$

Since we also have

$$\epsilon^{x} = \frac{x^{\frac{1}{2}}}{\underline{|\tfrac{1}{2}}} + \frac{x^{\frac{3}{2}}}{\underline{|\tfrac{3}{2}}} + \ldots + \frac{x^{-\frac{1}{2}}}{\underline{|-\tfrac{1}{2}}} + \ldots, \tag{91}$$

the product of (90) and (91) should be unity. That is,

$$-\pi^2 = \Big\{ \frac{(\underline{|-\tfrac{1}{2}})'}{\underline{|\tfrac{1}{2}}} + \frac{(\underline{|-\tfrac{3}{2}})'}{\underline{|-\tfrac{1}{2}}} + \frac{(\underline{|-2\tfrac{1}{2}})'}{\underline{|-1\tfrac{1}{2}}} + \frac{(\underline{|-3\tfrac{1}{2}})'}{\underline{|-2\tfrac{1}{2}}} + \ldots$$
$$+ \frac{(\underline{|\tfrac{1}{2}})'}{\underline{|1\tfrac{1}{2}}} + \frac{(\underline{|1\tfrac{1}{2}})'}{\underline{|2\tfrac{1}{2}}} + \frac{(\underline{|2\tfrac{1}{2}})'}{\underline{|3\tfrac{1}{2}}} + \ldots \Big\}$$
$$+ x\Big\{ \frac{(\underline{|-\tfrac{1}{2}})'}{\underline{|1\tfrac{1}{2}}} + \frac{\underline{|-1\tfrac{1}{2}})'}{\underline{|\tfrac{1}{2}}} + \frac{(\underline{|-2\tfrac{1}{2}})'}{\underline{|-\tfrac{1}{2}}} + \ldots + \frac{(\underline{|\tfrac{1}{2}})'}{\underline{|2\tfrac{1}{2}}} + \frac{(\underline{|1\tfrac{1}{2}})'}{\underline{|3\tfrac{1}{2}}} \ldots \Big\}$$
$$+ x^2\Big\{ \frac{(\underline{|-\tfrac{1}{2}})'}{\underline{|2\tfrac{1}{2}}} + \frac{(\underline{|-1\tfrac{1}{2}})'}{\underline{|1\tfrac{1}{2}}} + \frac{(\underline{|-2\tfrac{1}{2}})'}{\underline{|\tfrac{1}{2}}} + \ldots + \frac{(\underline{|\tfrac{1}{2}})'}{\underline{|3\tfrac{1}{2}}} + \frac{(\underline{|1\tfrac{1}{2}})'}{\underline{|4\tfrac{1}{2}}} + \ldots \Big\}$$
$$+ \ldots. \tag{92}$$

Going by the ordinary principles of the algebra of convergent series, we should conclude that the coefficient of $x^0$ was $-\pi^2$, and that the coefficients of the other powers of $x$ were zero. But this rule is not generally true in series of the present kind, as we have already exemplified. Therefore, to see how it goes in the immediate case, I have calculated the value of the coefficient of $x^0$. By (84) we have

$$\frac{1}{n}+\frac{f'(n)}{f(n)}=\frac{f'(n-1)}{f(n-1)}, \tag{93}$$

and from this we may derive, when $r$ is a positive integer,

$$-\frac{f'(r+\frac{1}{2})}{f(r+\frac{1}{2})}=2+\tfrac{2}{3}+\tfrac{2}{5}+\tfrac{2}{7}+\ldots.+\frac{1}{r+\frac{1}{2}}-\frac{f'(-\frac{1}{2})}{f(-\frac{1}{2})}, \tag{94}$$

and also

$$-\frac{f'(-r-1\frac{1}{2})}{f(-r-1\frac{1}{2})}=\frac{1}{r+\frac{1}{2}}+\frac{1}{r-\frac{1}{2}}+\frac{1}{r-\frac{3}{2}}+\ldots.+\frac{1}{1\frac{1}{2}}+2-\frac{f'(-\frac{1}{2})}{f(-\frac{1}{2})}. \tag{95}$$

Therefore, when $r$ is positively integral, we have

$$\frac{f'(r+\frac{1}{2})}{f(r+\frac{1}{2})}=\frac{f'(-r-1\frac{1}{2})}{f(-r-1\frac{1}{2})}, \tag{96}$$

which makes the coefficient of $x^0$ in (92) become

$$-2\mathrm{F}+(\tfrac{2}{3}-2)(2-\mathrm{F})+(\tfrac{2}{5}-\tfrac{2}{3})(\tfrac{2}{3}+2-\mathrm{F})+(\tfrac{2}{7}-\tfrac{2}{5})(\tfrac{2}{5}+\tfrac{2}{3}+2-\mathrm{F})+\ldots.,$$

where, for brevity, F stands for $f'(-\frac{1}{2})/f(-\frac{1}{2})$. It is readily seen that the complete coefficient of F vanishes, and the remainder reduces to

$$-\left(4+\frac{4}{3^2}+\frac{4}{5^2}+\frac{4}{7^2}+\ldots.\right)=-\frac{\pi^2}{2}. \tag{97}$$

Therefore the coefficient of $x^0$ in (92) contributes one-half of the total, and the other half must be given by (or rather, be equivalent to) the sum of the terms involving $x$. Although I have not thoroughly investigated this, there did not appear to be any inconsistency.

*Remarks on Equivalences in Factorial Formulæ. Verifications.*

**53.** If it is given that

$$\mathrm{F}(x)=\Sigma\, x^r\phi(r), \tag{98}$$

it does not, as already remarked in effect, follow that $\phi(r)$ is a definitely unique function of $r$. But it is sometimes true, and then the equation

$$0=\Sigma\, x^r\phi(r) \tag{99}$$

may require the vanishing of every coefficient. For example, using (88) above, if we differentiate it to $x$ we obtain

$$-\epsilon^{-x}=\Sigma\, x^{r-1}f(r-1)f(r)f'(-r-1)$$

$$=\Sigma\, x^r[f(r)]^2\frac{f'(-r-2)}{r+1}. \tag{100}$$

Therefore, by adding this equation to (88), we obtain

$$0 = \Sigma\, x^r [f(r)]^2 \left\{ f'(-r-1) + \frac{f'(-r-2)}{r+1} \right\}. \tag{101}$$

Now it is a fact that this is true, term by term, when $r = 0, 1, 2, 3$, &c. But (101) is not true in the same manner generally. Only when $f(n) = 0$, that is, when $n$ is a negative integer, do we have

$$nf'(n) = f'(n-1), \tag{102}$$

by (93), which is general. Put $n = -r-1$ to suit (101). But

$$f'(-r-1) - \frac{f'(-r-2)}{-r-1} = \frac{f(-r-1)}{r+1}, \tag{103}$$

by (93). Therefore (101) is the same as

$$0 = \Sigma\, x^r [f(r)]^2 \frac{f(-r-1)}{r+1} = \Sigma \frac{x^r}{\underline{|r}} \frac{\sin (r+1)\pi}{(r+1)\pi}, \tag{104}$$

which does not vanish term by term, except for the special values of $r$ indicated. Integrating (104), we obtain

$$\text{constant} = \Sigma \frac{x^r}{\underline{|r}} \frac{\sin r\pi}{r\pi}. \tag{105}$$

**54.** The case $r = 0$ we have already had, when the constant is 1, so it should be 1 generally. The case $r = \frac{1}{2}$ is represented by

$$1 = \frac{2}{\pi} \left\{ \frac{x^{\frac{1}{2}}}{\underline{|\tfrac{1}{2}}} + \frac{x^{-\frac{1}{2}}}{\underline{|-\tfrac{1}{2}}} - \tfrac{1}{3}\left( \frac{x^{\frac{3}{2}}}{\underline{|\tfrac{3}{2}}} + \frac{x^{-\frac{3}{2}}}{\underline{|-\tfrac{3}{2}}} \right) + \tfrac{1}{5}\left( \frac{x^{\frac{5}{2}}}{\underline{|\tfrac{5}{2}}} + \frac{x^{-\frac{5}{2}}}{\underline{|-\tfrac{5}{2}}} \right) - \dots \right\}, \tag{106}$$

and the following is a verification :—The right member is

$$\frac{2}{\pi}\left( \nabla^{-\frac{1}{2}} - \tfrac{1}{3}\nabla^{-\frac{3}{2}} + \tfrac{1}{5}\nabla^{-\frac{5}{2}} - \dots + \nabla^{\frac{1}{2}} - \tfrac{1}{3}\nabla^{\frac{3}{2}} + \tfrac{1}{5}\nabla^{\frac{5}{2}} - \dots \right)$$

$$= \frac{2}{\pi}(\tan^{-1}\nabla^{-\frac{1}{2}} + \tan^{-1}\nabla^{\frac{1}{2}}) = \frac{2}{\pi}\tan^{-1}\frac{\nabla^{-\frac{1}{2}} + \nabla^{\frac{1}{2}}}{1 - \nabla^{\frac{1}{2}}\nabla^{-\frac{1}{2}}}$$

$$= \frac{2}{\pi}\tan^{-1}\infty = 1. \tag{107}$$

Although the validity of this process of evaluation may be doubted, there is no inconsistency exhibited.

**55.** The other formula of a similar kind, viz.,

$$1 = \Sigma \frac{x^r}{\underline{|r}\ \underline{|-r}} = \Sigma\, x^r \frac{\sin r\pi}{r\pi}, \tag{108}$$

when similarly treated, gives

$$1 = \Sigma\, \nabla^{-r} f(-r) = \Sigma\, \nabla^{r} f(r) = \epsilon^{\nabla} \tag{109}$$

That is, $\epsilon^{\nabla} 1 = 1$, which is a case of Taylor's theorem, if we do not go too close to the boundary where the operand begins. That is, regarding the operand as $F(x)$, it is turned to $F(x+1)$.

*Application of Generalized Exponential to obtain other Generalized Formulæ involving the Logarithm.*

**56.** Now return to the fundamental exponential formula

$$\epsilon^{x} = \Sigma\, x^{r} f(r), \tag{110}$$

and derive from it some other logarithmic formulæ. Differentiate to $r$, then

$$0 = \epsilon^{x} \log x + \Sigma\, x^{r} f'(r). \tag{111}$$

A second differentiation to $r$ gives

$$0 = -\epsilon^{x} (\log x)^{2} + \Sigma\, x^{r} f''(r). \tag{112}$$

A third differentiation gives

$$0 = \epsilon^{x} (\log x)^{3} + \Sigma\, x^{r} f'''(r), \tag{113}$$

and so on. Or, all together,

$$\log x = -\frac{\Sigma\, x^{r} f'(r)}{\Sigma\, x^{r} f(r)} = -\frac{\Sigma\, x^{r} f''(r)}{\Sigma\, x^{r} f'(r)} = -\frac{\Sigma\, x^{r} f'''(r)}{\Sigma\, x^{r} f''(r)} = \dots\dots \tag{114}$$

Now combine them to see if they fit. Thus, we have the elementary formula

$$x\epsilon^{x} = \epsilon^{x} \left\{ 1 + \log x + \frac{(\log x)^{2}}{\lfloor 2} + \dots\dots \right\}, \tag{115}$$

and this, by the use of (114), becomes

$$\Sigma\, x^{r} \left( f - f' + \frac{f''}{\lfloor 2} - \frac{f'''}{\lfloor 3} + \dots\dots \right)(r), \tag{116}$$

which, by Taylor's theorem, is the same as

$$\Sigma\, x^{r} f(r-1) = x\, \Sigma\, x^{r} f(r) = x\epsilon^{x}, \tag{117}$$

as required.

**57.** Again, differentiate (111) to $x$. We obtain

$$\begin{aligned} 0 &= \epsilon^{x} \log x + \epsilon^{x} x^{-1} + \Sigma\, r x^{r-1} f'(r) \\ &= -\Sigma\, x^{r} f'(r) + \epsilon^{x} x^{-1} + \Sigma\, x^{r} (r+1) f'(r+1), \end{aligned} \tag{118}$$

by using (111) again, and (110). So

$$\epsilon^x = \Sigma\, x^{r+1}f'(r) - \Sigma\, x^{r+1}(r+1)f'(r+1)$$
$$= \Sigma\, x^r\{f'(r-1) - rf'(r)\}. \tag{119}$$

Here the factor of $x^r$ is identical with $f(r)$, by (84), which corroborates.

**58.** Returning to (111), if we try to make a series for $\log x$ in powers of $x$ we obtain

$$-\log x = \Sigma\, x^r\left(1 - x + \frac{x^2}{\underline{|2}} - \dots\right)f'(r)$$
$$= \Sigma\, x^r\left\{f'(r) - f'(r-1) + \frac{1}{\underline{|2}}f'(r-2) - \frac{1}{\underline{|3}}f'(r-3) + \dots\right\}. \tag{120}$$

This is done by making $x^r$ be the representative power throughout, by reducing the value of $r$ by unity in the second term in the first series, by two in the third term, and so on. Or

$$= \Sigma\, x^r \frac{d}{dr}\left\{f(r) - f(r-1) + \frac{1}{\underline{|2}}f(r-2) - \frac{1}{\underline{|3}}f(r-3) + \dots\right\} \tag{121}$$

$$= \Sigma\, x^r \frac{d}{dr}\,\frac{1 - r + \dfrac{r(r-1)}{\underline{|2}} - \dots}{\underline{|r}} = \Sigma\, x^r \frac{d}{dr}\frac{(1-1)^r}{\underline{|r}}. \tag{122}$$

This is striking, but not usable.

Also, if we try to get a series for $x^{-1}$ we fail. The property (84) comes in, and brings us to $x^{-1} = x^{-1}$ in the end. This failure is not obvious *a priori* in factorial mathematics.

### *Deduction of a Special Logarithmic Formula.*

**59.** Now let the formula (111) be specialized by taking $r = 0$. We then have

$$-\log x = \epsilon^{-x}[f'(0) + xf'(1) + x^2f'(2) + \dots$$
$$+ x^{-1}f'(-1) + x^{-2}f'(-2) + \dots]. \tag{123}$$

Here, for the negative values of $n$ we have

$$f'(-1) = 1,\quad f'(-2) = -1,\quad f'(-3) = \underline{|2},\quad f'(-4) = -\underline{|3}, \tag{124}$$

and so on, whilst for the positive we have

$$f'(0) = \mathrm{C},\quad f'(1) = \mathrm{C} - 1,\quad f'(2) = \tfrac{1}{2}(\mathrm{C} - 1 - \tfrac{1}{2}),$$
$$f'(n) = \frac{1}{\underline{|n}}\left\{\mathrm{C} - \left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \dots + \frac{1}{n}\right)\right\}, \tag{125}$$

by (84). Using these in (123) we obtain

$$-(\log x+\mathrm{C}) = \epsilon^{-x}\left(\frac{\lfloor 0}{x}-\frac{\lfloor 1}{x^2}+\frac{\lfloor 2}{x^3}-\frac{\lfloor 3}{x^4}+\dots\right)$$

$$-\epsilon^{-x}\left\{x+\frac{x^2}{\lfloor 2}(1+\tfrac{1}{2})+\frac{x^3}{\lfloor 3}(1+\tfrac{1}{2}+\tfrac{1}{3})+\dots\right\}. \qquad (126)$$

The first series is the ordinary expression for $\epsilon^{-x}x^{-1}$ with the terms inverted, whilst the latter contains a reminiscence of the companion to the Fourier cylinder function.

**60.** To see whether there is a notable convergency for calculation, take $x = 2$. Then

$$\epsilon^{-2} = 0{\cdot}1353,$$

$$1-\frac{1}{x}+\frac{\lfloor 2}{x^2}-\dots = 1-\tfrac{1}{2}+\tfrac{1}{2}-\tfrac{6}{8}+\dots.$$

This is evidently about $\frac{3}{4}$ by the look of it, especially when diagrammatically represented. Also

$$x+\frac{x^2}{\lfloor 2}(1+\tfrac{1}{2})+\dots = 9{\cdot}7479.$$

So (126) gives

$$\log 2 = 0{\cdot}1353\times 9{\cdot}7479-0{\cdot}5772-0{\cdot}375\times 0{\cdot}1353$$
$$= 0{\cdot}6909.$$

By common logarithmic tables we find log 2 = 0·6923. The difference is 0·0014. Doing it another way, we may prove by multiplication that

$$\epsilon^{-x}\left\{x+\frac{x^2}{\lfloor 2}(1+\tfrac{1}{2})+\frac{x^3}{\lfloor 3}(1+\tfrac{1}{2}+\tfrac{1}{3})+\dots\right\}$$

$$= x-\tfrac{1}{2}\frac{x^2}{\lfloor 2}+\tfrac{1}{3}\frac{x^3}{\lfloor 3}-\tfrac{1}{4}\frac{x^4}{\lfloor 4}+\dots, \qquad (127)$$

which is an interesting transformation. This, with $x = 2$, gives 1·3203, and produces a much closer agreement. It is probably fortuitous.

*Independent Establishment of the Last.*

**61.** We can establish (126) independently thus:—We have

$$\frac{1}{x} = \frac{\epsilon^{-x}}{x}+\frac{1-\epsilon^{-x}}{x} \qquad (128)$$

$$= \frac{\epsilon^{-x}}{x}+1-\frac{x}{\lfloor 2}+\frac{x^2}{\lfloor 3}-\frac{x^3}{\lfloor 4}+\dots. \qquad (129)$$

Integrate to $x$. Then

$$\log x + C = -\epsilon^{-x}\left(\frac{1}{x} - \frac{\underline{|1}}{x^2} + \frac{\underline{|2}}{x^3} - \dots\right) + x - \frac{1}{2}\frac{x^2}{\underline{|2}} + \frac{1}{3}\frac{x^3}{\underline{|3}} - \dots, \qquad (130)$$

when C is some constant introduced by the integration. To find it, note that the series with the exponential factor vanishes when $x$ is infinite; so (130) gives

$$C = x - \frac{1}{2}\frac{x^2}{\underline{|2}} + \frac{1}{3}\frac{x^3}{\underline{|3}} - \dots - \log x, \quad \text{with } x = \infty. \qquad (131)$$

It is not immediately obvious that the function preceding the logarithm in (131) increases infinitely with $x$. But by (127) we may regard it as the ratio

$$\frac{x + \frac{x^2}{\underline{|2}}(1+\frac{1}{2}) + \frac{x^3}{\underline{|3}}(1+\frac{1}{2}+\frac{1}{3}) + \dots}{1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \dots}, \qquad (132)$$

and we see that the terms in the numerator become infinitely greater than those to correspond in the denominator.

### *A Formula for Euler's Constant.*

**62.** Next examine whether (131) gives a rapid approximation to the value of C. When $x = 1$ we get

$$1 - \tfrac{1}{4} + \tfrac{1}{18} - \tfrac{1}{96} + \dots - 0 = 0{\cdot}77, \text{ say.}$$

When $x = 2$ we get $\quad 1{\cdot}3203 - 0{\cdot}6903 = 0{\cdot}6300.$

When $x = 3$ we get $\quad 1{\cdot}6888 - 1{\cdot}1098 = 0{\cdot}5790.$

So with $x = 3$ the error is about $\frac{1}{500}$ only. The usual formula

$$C = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \dots + \frac{1}{r} - \log r, \qquad \text{with } r = \infty,$$

is very slow. Ten terms make 0·62. Twenty make about 0·602, which is still far wrong. We see that (131) will give C pretty quickly with a moderate value of $x$.

**63.** In passing, we may note that the function

$$x + \frac{x^2}{\underline{|2}}(1+\tfrac{1}{2}) + \frac{x^3}{\underline{|3}}(1+\tfrac{1}{2}+\tfrac{1}{3}) + \dots \qquad (133)$$

is represented by

$$\nabla^{-1} + \nabla^{-2}(1+\tfrac{1}{2}) + \nabla^{-3}(1+\tfrac{1}{2}+\tfrac{1}{3}) + \dots, \qquad (134)$$

and also by

$$\epsilon^x\left(x-\frac{1}{2}\frac{x^2}{\lfloor 2}+\frac{1}{3}\frac{x^3}{\lfloor 3}-\ldots\right)=\epsilon^x(\nabla^{-1}-\tfrac{1}{2}\nabla^{-2}+\tfrac{1}{3}\nabla^{-3}-\ldots) \tag{135}$$

$$=\epsilon^x\log(1+\nabla^{-1})=\log\left(\frac{\nabla}{\nabla-1}\right)\epsilon^x=\frac{\nabla}{\nabla-1}\log\frac{\nabla}{\nabla-1}, \tag{136}$$

which may be useful later.

*Deduction of Second Kind of Bessel Function,* $K_0(x)$, *from the Generalized Formula of the First Kind.*

**64.** A similar treatment of the generalized formula for the Fourier-Bessel function leads to the companion function. Thus, take

$$I_0(x)=\Sigma\, y^r[f(r)]^2, \tag{137}$$

as in (76) Part I, the value of $y$ being $\frac{1}{4}x^2$. Differentiate to $r$. Then

$$0=I_0(x)\log y+2\Sigma\, y^r f(r)f'(r). \tag{138}$$

Here take the special case $r=0$. Then we have

$$\begin{aligned}0&=I_0(x)\log y\\&+2\left\{C+y(C-1)+\frac{y^2}{(\lfloor 2)^2}(C-1-\tfrac{1}{2})+\frac{y^3}{(\lfloor 3)^2}(C-1-\tfrac{1}{2}-\tfrac{1}{3})+\ldots\right\}\\&+2\left\{\frac{y^{-1}}{\lfloor -1}-\frac{y^{-2}}{\lfloor -2}\lfloor 1+\frac{y^{-3}}{\lfloor -3}\lfloor 2-\ldots\right\}.\end{aligned} \tag{139}$$

The third line is apparently zero. But it must, as we shall see, be retained, though in a changed form. Or

$$\begin{aligned}\frac{y^{-1}}{\lfloor -1}-\frac{y^{-2}}{\lfloor -2}\lfloor 1+\frac{y^{-3}\lfloor 2}{\lfloor -3}-\ldots&=y+\frac{y^2}{(\lfloor 2)^2}(1+\tfrac{1}{2})+\frac{y^3}{(\lfloor 3)^2}(1+\tfrac{1}{2}+\tfrac{1}{3})+\ldots\\&-I_0(x)(\log y^{\frac{1}{2}}+C).\end{aligned} \tag{140}$$

*Another way. Automatic Standardization.*

**65.** Now the right member is certainly not zero, for it represents the companion of $I_0(x)$, as may be proved in various ways, classical and unclassical. One way is from the formula for $I_n(x)$, thus,

$$I_n(x)=\frac{y^{\frac{n}{2}}}{\lfloor n}\left(1+\frac{y}{\lfloor 1\,(1+n)}+\frac{y^2}{\lfloor 2\,(1+n)(2+n)}+\ldots\right). \tag{141}$$

When $n$ is not an integer, $\mathrm{I}_n(x)$ and $\mathrm{I}_{-n}(x)$ are different, and represent two independent solutions of the characteristic differential equation. But when $n$ is any integer, positive or negative, they become identical, so only one solution is got. Then another is (when $n = 0$) represented by the rate of variation of $\mathrm{I}_n(x)$ with $n$ when $n = 0$. Thus,

$$-\frac{d\mathrm{I}_n(x)}{dn} = -\mathrm{I}_n(x)\left\{\tfrac{1}{2}\log y + \frac{f'(n)}{f(n)}\right\}$$

$$+\frac{y^{\frac{n}{2}}}{\underline{|n}}\left\{\frac{y}{\underline{|1}\,(1+n)^2} + \frac{y^2\left(\frac{1}{1+n}+\frac{1}{2+n}\right)}{\underline{|2}\,(1+n)(2+n)} + \dots\right\}, \qquad (142)$$

which, when $n = 0$, is by inspection the function on the right side of (140). Notice that this method of obtaining the second solution, like the just preceding method, gives it immediately in the form properly standardized so as to vanish at infinity. The constant C comes in automatically, and requires no separate evaluation.

### *The Operator producing* $\mathrm{K}_0(x)$.

**66.** But our immediate object of attention should be the function on the left side of (140). How it can be equivalent to the right member is a mystery. It is certainly an extreme form, if correct. We may write it in the form

$$\Delta - \Delta^2\underline{|1} + \Delta^3\underline{|2} - \Delta^4\underline{|3} + \dots, \qquad (143)$$

where $\Delta$ is $d/dy$. Now the other function $\mathrm{I}_0(x)$ is

$$1 + \frac{\Delta^{-1}}{\underline{|1}} + \frac{\Delta^{-2}}{\underline{|2}} + \frac{\Delta^{-3}}{\underline{|3}} + \dots, \qquad (144)$$

without any mystery, and we see at once that these forms are analogous to

$$\epsilon^{-x} = \Delta - \Delta^2 + \Delta^3 - \Delta^4 + \dots, \qquad (145)$$

$$\epsilon^{x} = 1 + \Delta^{-1} + \Delta^{-2} + \Delta^{-3} + \dots, \qquad (146)$$

the latter, corresponding to (144), being obvious, whilst the former, analogous to (143), is an extreme form already considered and explained; see equations (71), (72). The unintelligibility of (143) is no evidence of its inaccuracy. More puzzling things than it have been cleared up.

**67.** We may also employ the special formula (126), of which we had separate verifications. Multiply it by $\epsilon^x$ and then write $\Delta^{-1}$ for $x$. Thus,

$$0=(C+\log\Delta^{-1})\epsilon^{\Delta^{-1}}+(\Delta-\Delta^2\lfloor 1+\Delta^3\lfloor 2-\Delta^4\lfloor 3+\ldots)$$
$$-\left(\Delta^{-1}+\frac{\Delta^{-2}}{\lfloor 2}(1+\tfrac{1}{2})+\ldots\right), \quad (147)$$

where we see that the operator (143) appears. Integrating, we have

$$\Delta-\Delta^2\lfloor 1+\ldots = y+\frac{y^2}{(\lfloor 2)^2}(1+\tfrac{1}{2})+\ldots-(C+\log\Delta^{-1})\,I_0(x), \quad (148)$$

comparing which with (140), we see that

$$(\log\Delta^{-1})\,I_0(x)=I_0(x)\log\frac{x}{2}; \quad (149)$$

for which a verification would be desirable.

*Companion Formulæ, $H_0(qx)$ and $K_0(qx)$, derived from Companion Operators, expressed in Descending Series. Also in Ascending Series.*

68. Passing, however, at present to more manageable operators involving the two solutions and different forms thereof, it will be convenient to introduce a notation and standardization which shall exhibit the symmetry of relations most clearly. Thus, let

$$H_0(qx)=\frac{2\nabla}{(\nabla^2-q^2)^{\frac{1}{2}}}; \quad (150)$$

$$K_0(qx)=\frac{2\nabla}{(q^2-\nabla^2)^{\frac{1}{2}}}. \quad (151)$$

Here $q$ is a constant and $\nabla$ is $d/dx$. Superficially considered, these functions only differ in one being $i$ times the other. But the common theory of the imaginary does not hold good here, or in operators generally. In a descending series we have

$$H_0(qx)=\epsilon^{qx}\left(\frac{2}{\pi qx}\right)^{\frac{1}{2}}\left\{1+\frac{1}{8qx}+\frac{1^2 3^2}{\lfloor 2(8qx)^2}+\frac{1^2 3^2 5^2}{\lfloor 3(8qx)^3}+\ldots\right\}, \quad (152)$$

as already shown. It is twice the function C, equation (3). Similarly, we may integrate (151). Introduce the factor $\epsilon^{-qx}$, thus,

$$K_0(qx)=\epsilon^{-qx}\,\epsilon^{qx}\frac{2\nabla}{(q^2-\nabla^2)^{\frac{1}{2}}}=2\epsilon^{-qx}\frac{\nabla-q}{(2q\nabla-\nabla^2)^{\frac{1}{2}}}\,\frac{\nabla}{\nabla-q}$$
$$=2\epsilon^{-qx}\left(\frac{\nabla}{2q-\nabla}\right)^{\frac{1}{2}}. \quad (153)$$

Expand in ascending powers of $\nabla$, and then integrate; then

$$K_0(qx) = \epsilon^{-qx}\left\{1+\frac{\nabla}{4q}+\frac{1.3}{\underline{|2}}\left(\frac{\nabla}{4q}\right)^2+\frac{1.3.5}{\underline{|3}}\left(\frac{\nabla}{4q}\right)^3+\dots\right\}\left(\frac{2}{\pi qx}\right)^{\frac{1}{2}}. \quad (154)$$

$$= \epsilon^{-qx}\left(\frac{2}{\pi qx}\right)^{\frac{1}{2}}\left\{1-\frac{1}{8qx}+\frac{1^2 3^2}{\underline{|2}(8qx)^2}-\frac{1^2 3^2 5^2}{\underline{|3}(8qx)^3}+\dots\right\} . \quad (155)$$

Thus the function $K_0(qx)$ only differs from $H_0(qx)$ in the changed sign of $qx$, except under the radical. These are the most primitive solutions of the characteristic equation, and are useful as operators relating to inward and outward going cylindrical waves, as well as for numerical purposes. The function $K_0(qx)$ is also expressed by

$$K_0(qx) = \frac{2}{\pi}\left\{\frac{q^2x^2}{2^2}+\frac{q^4x^4}{2^2 4^2}(1+\tfrac{1}{2})+\frac{q^6x^6}{2^2 4^2 6^2}(1+\tfrac{1}{2}+\tfrac{1}{3})+\dots\right.$$
$$\left.-I_0(qx)\left(\log\frac{qx}{2}+C\right)\right\}. \quad (156)$$

By $I_0(qx)$ here and later should be understood merely the ascending series

$$I_0(qx) = 1+\frac{q^2x^2}{2^2}+\frac{q^4x^4}{2^2 4^2}+\frac{q^6x^6}{2^2 4^2 6^2}+\dots . \quad (157)$$

*Transformation from $K_0(qx)$ to the Companion Oscillating Functions $J_0(sx)$ and $G_0(sx)$, both in Ascending and Descending Series.*

**69.** The connection between these functions $H_0$ and $K_0$ and the oscillatory functions is very important, but was in one respect exceedingly obscure to me until lately. Thus (157) and (156) are usually reckoned to be companion solutions (unless as regards the numerical factor). But if we take $q = si$ in (157), the function remains real, and becomes the oscillatory function, the original cylinder function of Fourier. Thus

$$I_0(qx) = J_0(sx) = 1-\frac{s^2x^2}{2^2}+\frac{s^4x^4}{2^2 4^2}-\frac{s^6x^6}{2^2 4^2 6^2}+\dots . \quad (158)$$

On the other hand, the same transformation in (156) makes it complex, on account of the logarithm. Thus, using

$$\log qx = \log six = \log sx + \log i = \log sx + \tfrac{1}{2}i\pi, \quad (159)$$

by the well-known formula for $\epsilon^{i\pi/2}$, we convert (156) to

$$K_0(qx) = G_0(sx) - iJ_0(sx), \quad (160)$$

where $J_0(sx)$ is the same as in (158), and $G_0(sx)$ is its oscillatory companion given by*

$$G_0(sx) = \frac{2}{\pi}\left\{ -\frac{s^2x^2}{2^2}+\frac{s^4x^4}{2^24^2}(1+\tfrac{1}{2})-\frac{s^6x^6}{2^24^26^2}(1+\tfrac{1}{2}+\tfrac{1}{3})+\dots \right.$$
$$\left. -J_0(sx)\left(\log\frac{sx}{2}+C\right)\right\}. \qquad (161)$$

What is obscure here is the getting of only one oscillating function from $I_0(qx)$, and of two from $K_0(qx)$. In corresponding forms of the first and second solutions we should expect both oscillating solutions to arise in both cases. However this be, the transformation (160) is in agreement with the other form (155). For, if we make the change $q = si$ in it, we obtain the same formula (160), provided $J_0$ and $G_0$ are given by

$$J_0(sx) = \left(\frac{1}{\pi sx}\right)^{\frac{1}{2}}\left[R(\cos+\sin)\,sx+Si\,(\sin-\cos)\,sx\right], \qquad (162)$$

$$G_0(sx) = \left(\frac{1}{\pi sx}\right)^{\frac{1}{2}}\left[R(\cos-\sin)\,sx+Si\,(\cos+\sin)\,sx\right], \qquad (163)$$

where R and $Si$ are the real functions of $sx$ given by

$$R = 1+\frac{1^23^2}{\underline{|2}\,(8qx)^2}+\frac{1^23^25^27^2}{\underline{|4}\,(8qx)^4}+\dots = 1-\frac{1^23^2}{\underline{|2}\,(8sx)^2}+\frac{1^23^25^2}{\underline{|3}\,(8sx)^4}-\dots, \qquad (164)$$

$$S = \frac{1}{8qx}+\frac{1^23^25^2}{\underline{|3}\,(8qx)^3}+\dots = \frac{1}{i}\left(\frac{1}{8sx}-\frac{1^23^25^2}{\underline{|3}\,(8sx)^3}+\dots\right). \qquad (165)$$

Now here (162) is Stokes's formula for $J_0(sx)$, known to be equivalent to (158). And (163) shows that this kind of formula for the oscillating functions allows us to obtain the second solution from the first by the change of sin to cos and cos to −sin. The function $G_0(sx)$ of (163) may be shown to be equivalent to the $G_0(sx)$ of (161) by other means, and certainly verifications are desirable, because transformations involving the square root of the imaginary are sometimes treacherous.

*Transformation from $H_0(qx)$ to the same $J_0(sx)$ and $G_0(sx)$. Explanation of Apparent Discrepancies.*

**70.** Now as regards the changed form of the $H_0(qx)$ function of (152), there is a real and once apparently insurmountable difficulty.

* I have changed the sign of $K_0$ and $G_0$ from that used in my 'Electrical Papers' (in particular, vol. 2, p. 445), in order to make them positive at the origin.

We know that $H_0(qx)$ and $2I_0(qx)$ are equivalent, both analytically and numerically. Why, then, does the first become complex, whilst the second remains real when we take $q = si$? They cannot be both true in changed form. Thus (152) becomes (doing it in detail)

$$\left(\frac{2}{\pi qx}\right)^{\frac{1}{2}} \epsilon^{qx}(R+S) = \left(\frac{2}{\pi sx}\right)^{\frac{1}{2}} \frac{1-i}{\sqrt{2}}(\cos + i\sin)\, sx \,.\, (R - i\,.\,Si)$$

$$= \left(\frac{1}{\pi sx}\right)^{\frac{1}{2}} (\cos + i\sin) sx \,.\, \left\{ (R - Si) - i(R + Si) \right\}$$

$$= \left(\frac{1}{\pi sx}\right)^{\frac{1}{2}} \Big[ R(\cos + \sin)\, sx + Si\,(\sin - \cos)\, sx \Big]$$

$$-i\left(\frac{1}{\pi sx}\right)^{\frac{1}{2}} \Big[ R(\cos - \sin)\, sx + Si\,(\cos + \sin)\, sx \Big]. \qquad (166)$$

That is, using the functions (162), (163) again, we have the transformation

$$H_0(qx) = J_0(sx) - iG_0(sx), \qquad (167)$$

whereas $2I_0(qx)$ becomes $2J_0(sx)$. This was formerly a perfect mystery, indicative of an imperfection in the theory of the Bessel functions. But the reader who has gone through Part I and §§ 27, 28 of Part II will have little trouble in understanding the meaning of (167). The functions $H_0$ and $2I_0$, though equivalent (with positive argument), are not algebraically identical. To have identity we require to use a second equivalent form, so that, as in § 28,

$$H_0(qx) = I_0(qx) + \frac{2}{\pi}\left\{ \frac{1}{qx} + \frac{1^2}{q^3x^3} + \frac{1^2 3^2}{q^5x^5} + \ldots + qx + \frac{q^3x^3}{1^2 3^2} + \ldots \right\}. \ (168)$$

In this form we may take $q = si$, and still have agreement in the changed form. We obtain the relation (167), provided that

$$G_0(sx) = \frac{2}{\pi}\left\{ \frac{1}{sx} - \frac{1}{s^3x^3} + \frac{1^2 3^2}{s^5x^5} - \frac{1^2 3^2 5^2}{s^7x^7} + \ldots - sx + \frac{s^3x^3}{1^2 3^2} - \frac{s^5x^5}{1^2 3^2 5^3} + \ldots \right\}. \qquad (169)$$

As I mentioned before in § 22, this formula for $G_0(sx)$ may be deduced from formulæ in Lord Rayleigh's 'Sound,' derived by a method due to Lipschitz, which investigation, however, I find it rather difficult to follow.

We have, therefore, three principal forms of the first solution with $q$ real and positive, viz., $I_0(qx)$, $\frac{1}{2}H_0(qx)$, and the intermediate form (168). We have also three forms of the oscillatory function $G_0(sx)$, viz., (161), (163), and (169). But we have only employed two forms of $K_0(qx)$, and two of $J_0(sx)$, in obtaining and harmonizing the previous three forms. It would therefore appear probable that there is an additional principal formula for $K_0(qx)$, and another for $J_0(sx)$, not yet investigated.

*Conjugate Property of Companion Functions.*

**71.** The conjugate property of the oscillating functions is

$$J_0(sx)\frac{d}{dx}G_0(sx)-G_0(sx)\frac{d}{dx}J_0(sx) = -\frac{2}{\pi x}, \tag{170}$$

using the pair (162), (163), or the pair (158), (161). And, similarly,

$$H_0(qx)\frac{d}{dx}K_0(qx)-K_0(qx)\frac{d}{dx}H_0(qx) = -\frac{4}{\pi x}. \tag{171}$$

But, in the transition from (171) to (170) by the relation $q = si$, it is indifferent whether we take $H_0(qx) = 2I_0(qx) = 2J_0(sx)$, or else $= J_0(sx) - iG_0(sx)$. This conjugate property is of some importance in the treatment of cylindrical problems by the operators.

*Operators with two Differentiators leading to* $H_0$ *and* $K_0$ *and showing their Mutual Connections compactly in reference to Cylindrical Waves.*

**72.** The fundamental mutual relations of $H_0$ and $K_0$ are exhibited concisely in the following, employing operators containing two differentiators, say $\nabla$ and $q$, viz.,

$$\frac{\nabla q}{(\nabla^2-q^2)^{\frac{1}{2}}} \quad \text{and} \quad \frac{\nabla q}{(q^2-\nabla^2)^{\frac{1}{2}}}. \tag{172}$$

Here it should be understood that either $\nabla$ or $q$ may be passive, when it may be regarded as a constant. But when both are active, there are two independent operands, one for $\nabla$ and the other for $q$. In a cylinder problem relating to elastic waves, we may regard $\nabla$ as being $d/dr$, where $r$ is distance from the axis, and $q$ as $d/d(vt)$, where $t$ is the time, and $v$ the speed of propagation. We have

$$\left.\begin{aligned}
[\mathrm{P}]\ldots\ \frac{\nabla q}{(\nabla^2-q^2)^{\frac{1}{2}}} &= q\mathrm{I}_0(qr), \ldots\ldots [a] \\
= \epsilon^{qr}\left(\frac{\nabla}{2q+\nabla}\right)^{\frac{1}{2}} q &= \tfrac{1}{2}q\mathrm{H}_0(qr), \ldots. [b] \\
= \epsilon^{-vt\nabla}\left(\frac{q}{2\nabla-q}\right)^{\frac{1}{2}}\nabla &= \tfrac{1}{2}\nabla\mathrm{K}_0(vt\nabla), \ldots [c] \\
&= \frac{1}{\pi(r^2-v^2t^2)^{\frac{1}{2}}}, \ldots [d] \\
&= \frac{1}{\pi}\,\mathrm{I}_0(vt\nabla)\,\frac{1}{r}, \ldots [e]
\end{aligned}\right\}, \qquad (173)$$

where the letters in square brackets are for the purpose of concise reference. Similarly, we have this other set,

$$\left.\begin{aligned}
[\mathrm{Q}]\ldots\ \frac{\nabla q}{(q^2-\nabla^2)^{\frac{1}{2}}} &= \nabla\mathrm{I}_0(vt\nabla), \ \ldots [\mathrm{A}] \\
= \epsilon^{vt\nabla}\left(\frac{q}{2\nabla+q}\right)^{\frac{1}{2}}\nabla &= \tfrac{1}{2}\nabla\mathrm{H}_0(vt\nabla), \ldots [\mathrm{B}] \\
= \epsilon^{-qr}q\left(\frac{\nabla}{2q-\nabla}\right)^{\frac{1}{2}} &= \tfrac{1}{2}q\mathrm{K}_0(qr), \ldots. [\mathrm{C}] \\
&= \frac{1}{\pi(v^2t^2-r^2)^{\frac{1}{2}}}, \ldots [\mathrm{D}] \\
&= \frac{1}{\pi}\,\mathrm{I}_0(qr)\,\frac{1}{vt}, \ldots [\mathrm{E}]
\end{aligned}\right\} \qquad (174)$$

The first set is usually, though not essentially, concerned with an inward-going, and the second set with an outward-going wave. The exchange of $r$ and $vt$ and of $\nabla$ and $q$, transforms one set to the other, so that the proof of one set proves the other.

In obtaining $[a]$ from [P] we regard $q$ as a constant, or at any rate, as passive for the time, expand [P] in descending powers of $\nabla$, and integrate directly with the result $[a]$, as in § 13, equations (28), (29).

To obtain $[b]$, introduce the factor $\epsilon^{qr}$ to [P], and expand the transformed operator in descending powers of $q$, as in § 14, equations (30), (31).

To obtain $[c]$, we make $q$ passive, and introduce the factor $\epsilon^{-vt\nabla}$. Then expand the transformed operator in descending powers of $\nabla$, and integrate as in § 68, eqautions (153), (155) (only there the operator is $q$, making the case [C]).

*Details concerning the above Relations.*

**73.** As regards [$d$], it may be obtained from [$a$], [$b$], or [$c$]. These have not yet been done, so a little detail is now given. Thus, from [$b$] to [$d$]:—

$$\tfrac{1}{2}q\mathrm{H}_0(qr) = \frac{\epsilon^{qr}}{(2\pi r)^{\frac{1}{2}}}\left\{1+\frac{1}{8qr}+\frac{1^2 3^2}{\underline{|2}(8qr)^2}+\dots\right\}\frac{1}{(\pi vt)^{\frac{1}{2}}}$$

$$= \epsilon^{qr}\left\{1+\tfrac{1}{2}\frac{vt}{2r}+\frac{1.3}{2^2\underline{|2}}\left(\frac{vt}{2r}\right)^2+\dots\right\}\frac{1}{\pi(2vtr)^{\frac{1}{2}}}$$

$$= \frac{\epsilon^{qr}}{\pi}\frac{1}{(vt)^{\frac{1}{2}}(2r-vt)^{\frac{1}{2}}} = \frac{1}{\pi(r^2-v^2t^2)^{\frac{1}{2}}}. \tag{175}$$

In the first line we expand the function $\mathrm{H}_0$; to get the second line we integrate with unit operand; and, finally, let $\epsilon^{qr}$ operate to get (175).

**74.** Next, from [$c$] to [$d$]:—

$$\tfrac{1}{2}\nabla\mathrm{K}_0(vt\nabla) = \frac{\epsilon^{-vt\nabla}}{(2\pi vt)^{\frac{1}{2}}}\left\{1-\frac{1}{8vt\nabla}+\frac{1^2 3^2}{\underline{|2}(8vt\nabla)^2}-\dots\right\}\nabla^{\frac{1}{2}}$$

$$= \frac{\epsilon^{-vt\nabla}}{\pi(2rvt)^{\frac{1}{2}}}\left\{1-\tfrac{1}{2}\left(\frac{r}{2vt}\right)+\frac{1.3}{2^2\underline{|2}}\left(\frac{r}{2vt}\right)^2-\dots\right\}$$

$$= \frac{\epsilon^{-vt\nabla}}{\pi}\frac{1}{r^{\frac{1}{2}}(2vt+r)^{\frac{1}{2}}} = \frac{1}{\pi(r^2-v^2t^2)^{\frac{1}{2}}}, \tag{176}$$

which needs no explanation, as the course is similar to the previous leading to (175).

**75.** As regards deriving [$d$] from [$a$], this may be done by harmonic decomposition, thus,

$$q\mathrm{I}_0(qr) = \mathrm{I}_0(qr)\frac{1}{\pi}\int_0^\infty \cos svt\,ds = \frac{1}{\pi}\int_0^\infty \mathrm{J}_0(sr)\cos svt\,ds, \tag{177}$$

the value of which is known to be (175). Conversely, we may evaluate the definite integral by turning it to the analytical form $q\mathrm{I}_0(qr)$, which may be done by inspection, and then integrating through the equivalent operator $\mathrm{H}_0(qr)\frac{1}{2}q$. But this definite integral is only one of several that may be immediately derived from the operators in (173), (174) by harmonic decomposition, and it will be more convenient to consider them separately in later sections along with applications and extensions of the preceding.

*Cylindrical Elastic Wave compared with corresponding Diffusive Wave through the Operators.*

**76.** The formulæ [$e$] and [E] are of a somewhat different kind, since the operand is the reciprocal of the independent variable. They are proved at once by carrying out the differentiations. Thus, for [E],

$$\mathrm{I}_0(qr)\frac{1}{vt} = \left(1+\frac{q^2r^2}{2^2}+\frac{q^4r^4}{2^24^2}+\ldots\right)\frac{1}{vt}$$

$$= \left\{1+\frac{\underline{|2}}{2^2}\left(\frac{r}{vt}\right)^2+\frac{\underline{|4}}{2^24^2}\left(\frac{r}{vt}\right)^4+\ldots\right\}\frac{1}{vt}$$

$$= \frac{1}{(v^2t^2-r^2)^{\frac{1}{2}}}. \tag{178}$$

So, by [C] and [E] we have

$$\frac{\pi}{2}\mathrm{K}_0(qr)\,q = \mathrm{I}_0(qr)\frac{1}{vt} = \frac{1}{(v^2t^2-r^2)^{\frac{1}{2}}}. \tag{179}$$

There is an interesting analogue to this transformation from $\mathrm{K}_0$ to $\mathrm{I}_0$ occurring in the theory of pure diffusion. Change the meaning of $q$ from $d/d(vt)$ to $\{d/d(vt)\}^{\frac{1}{2}}$, that is, to its square root. Then we shall have

$$\frac{\pi}{2}\mathrm{K}_0(qr)\,q = \mathrm{I}_0(qr)\frac{1}{2vt} = \frac{\epsilon^{-r^2/4vt}}{2vt}. \tag{180}$$

The quantity $v$ is no longer a velocity, however. In the theory of heat diffusion it is the ratio of the conductivity to the capacity. This example belongs to cylindrical diffusion, and is only put here to compare with the preceding example, which belongs to the corresponding problem with elastic waves without local dissipation.

## IX. "On a Failure of the Law in Photography that when the Products of the Intensity of the Light acting and of the Time of Exposure are Equal, Equal Amounts of Chemical Action will be produced." By Captain W. DE W. ABNEY, C.B., F.R.S. Received June 13, 1893.

It has been generally assumed that when the products of the intensity of light acting on a sensitive surface and the time of exposure are equal similar amounts of chemical action are produced, and with the ordinary exposures and intensities of light employed such, no doubt, is practically the case, and any methods of measurement hitherto practicable have been insufficiently delicate to discover any departure from this law, if such departure existed. In some recent experiments