

$$\theta_p = 2 \arccos \left[\frac{\cos \theta_l - \cos \theta_l \cos \theta_s}{\sin \theta_l \sin \theta_s} \right] \cong 111.38^\circ. \quad (7)$$

Using the radian measure of this angle in (2) gives $F = 0.28177$ to five decimal places, which agrees with (5) and gives us additional confidence in this result.

A comparison of (2) and (5) shows that the “planar approximation” used in getting (2) is remarkably good and gives a value just about 1 percent higher than the true value. The makers of soccer balls are evidently well aware of this close convergence, because they put the ball together out of planar pentagonal and hexagonal patches. After the patches are sewn together and the ball is inflated, the patches flex gently to accommodate themselves to the demands of spherical geometry.

Despite the near equality of (2) and (5), it is worth noting that the vertex angles of the spherical pentagons and hexagons on a soccer ball differ appreciably from those of the planar pentagon and hexagon. The vertex angles of the spherical pentagon and hexagon are $\theta_p = 111.38^\circ$ and $\theta_h = 124.31^\circ$ (the latter following from the fact that $\theta_p + 2\theta_h = 360^\circ$), and these differ noticeably from the angles of 108° (for a planar pentagon) and 120° (for a planar hexagon), showing that the differences between spherical and planar geometry are not completely masked in local measurements on a soccer ball.

The truncated icosahedron that underlies a soccer ball also serves as the framework for a molecule of C-60, or “buckyball,” which has a carbon atom at each vertex of this polyhedron. It is interesting to contrast buckyball with diamond and graphite, the other two allotropes of carbon. In diamond, each carbon atom occurs at the center of a tetrahedral cage formed by four other carbon atoms, with the angle between neighboring C-C bonds being $\arccos(-1/3) = 109.47^\circ$. In graphite the carbon atoms are arranged in planar hexagonal sheets, with the angle between neighboring C-C bonds being 120° . Buckyball interpolates neatly between these other two allotropes in having bond angles of 108° and 120° .

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Euler’s Triangle Inequality via Proofs Without Words

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In November 1983, this MAGAZINE published a special issue [7] in tribute to Leonhard Euler (1707–1783) on the occasion of the 200th anniversary of his death. In addition to a number of excellent survey articles, that issue contained a glossary of terms, formulas, equations and theorems that bear Euler’s name, the last one of which was the following:

EULER’S THEOREM FOR A TRIANGLE. *The distance d between the circumcenter and incenter of a triangle is given by $d^2 = R(R - 2r)$, where R, r are the circumradius and inradius, respectively.*

An immediate consequence of this theorem is $R \geq 2r$, which is often referred to as *Euler’s triangle inequality*. In this Note (on the occasion of the 300th anniversary of Euler’s birth) we use “proofs without words” to prove three simple lemmas that can be combined with the arithmetic mean-geometric mean inequality in order to prove Euler’s triangle inequality with only simple algebra (and without reference to the theorem above). The proof is derived from one that appears in [3]. Coxeter [1] notes that although Euler published this inequality in 1767 [2], it had appeared earlier (1746) in a publication by William Chapple.

As we have just noted, the “inequality” in Euler’s triangle inequality is derived from the *arithmetic mean-geometric mean inequality*: For any two positive numbers x and y , the arithmetic mean $(x + y)/2$ is at least as great as the geometric mean \sqrt{xy} . Hence for any three positive numbers x, y , and z , we have $x + y \geq 2\sqrt{xy}$, $y + z \geq 2\sqrt{yz}$, and $z + x \geq 2\sqrt{zx}$. Multiplying these three inequalities yields

$$(x + y)(y + z)(z + x) \geq 8xyz.$$

(1)

Now consider a triangle with side lengths a, b , and c as shown in FIGURE 1(a), and bisect each angle to locate the center of the inscribed circle. Extending an inradius (length r) to each side partitions the triangle into six smaller right triangles with side lengths as indicated in FIGURE 1(b). Noting that $x + y = c, y + z = a$, and $z + x = b$, (1) becomes

$$abc \geq 8xyz.$$

(2)

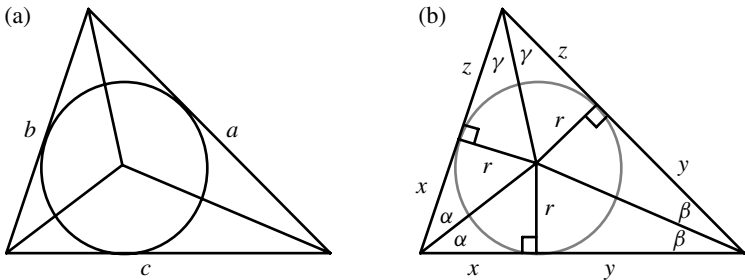


Figure 1

We now show that (2) is equivalent to $R \geq 2r$. To accomplish this, first we prove (wordlessly) three lemmas—which are of interest in their own right—from which Euler’s triangle inequality readily follows. The proofs are elementary, employing nothing more sophisticated than similarity of triangles. The first expresses the area K of the triangle in terms of the three side lengths a, b, c and the circumradius R . The second, whose proof uses a rectangle composed of triangles similar to the right triangles in FIGURE 1(b), expresses the product xyz in terms of the inradius r and the sum $x + y + z$. The third gives the area K in terms of r and $x + y + z$.

LEMMA 1. $4KR = abc$.

Proof.

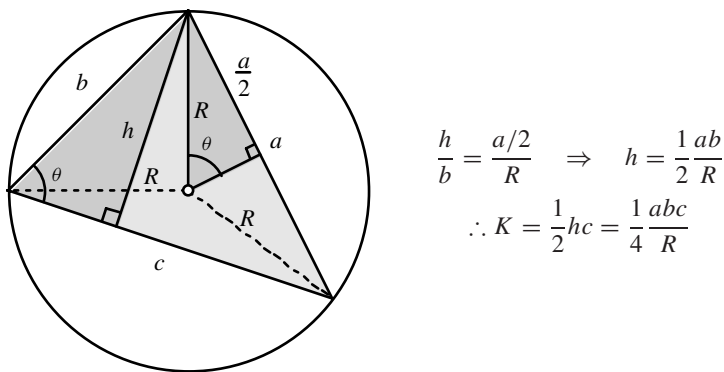


Figure 2 $4KR = abc$

LEMMA 2. $xyz = r^2(x + y + z)$.

Proof. Letting w denote $\sqrt{r^2 + x^2}$, we have

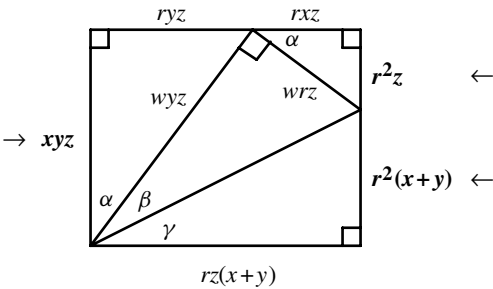


Figure 3 $xyz = r^2(x + y + z)$

LEMMA 3. $K = r(x + y + z)$.

Proof.

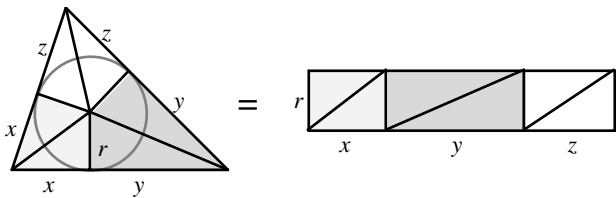


Figure 4 $K = r(x + y + z)$

We now prove

EULER'S TRIANGLE INEQUALITY. In any triangle, the circumradius R and the inradius r satisfy $R \geq 2r$.

Proof. Applying Lemma 1 to (2) yields $4KR \geq 8xyz$; invoking Lemma 2 then gives $4KR \geq 8r^2(x + y + z)$; and with Lemma 3 we have $4KR \geq 8Kr$; from which $R \geq 2r$ follows. ■

We conclude with a few comments about several results related to Euler's triangle inequality and the three lemmas used in our proof.

1. Euler's triangle inequality cannot be improved for general triangles, since $R = 2r$ if and only if the triangle is equilateral. However, for the class of right triangles, we have $R \geq (1 + \sqrt{2})r$ with equality for isosceles right triangles. In fact, if one fixes one of the angles of the triangle, say α , then $R \sin \alpha \geq r(\tan(\alpha/2) + \sec(\alpha/2))$. We leave the proofs of these inequalities as an exercise.
2. Since $2x = b + c - a$, $2y = c + a - b$, and $2z = a + b - c$, (1) can be written entirely in terms of a , b , and c as

$$abc \geq (a + b - c)(c + a - b)(b + c - a).$$

This is known both as the *Lehmus inequality* [1] and *Padoa's inequality* [5], [6].

3. Lemmas 2 and 3 can be employed to produce a proof of *Heron's formula* for the area of the triangle: $K = \sqrt{s(s-a)(s-b)(s-c)}$, where s denotes the semiperimeter, and is given by $s = (a + b + c)/2 = x + y + z$. Since $s - a = x$, $s - b = y$, and $s - c = z$, the result in Lemma 2 can be written as $r^2s = (s-a)(s-b)(s-c)$, or $(rs)^2 = s(s-a)(s-b)(s-c)$. But from Lemma 3 we have $rs = K$, from which Heron's formula for K now follows (for a wordless version of this proof and additional references, see [4]).
4. Dividing both sides of the result in Lemma 2 by r^3 yields $\frac{x}{r} \cdot \frac{y}{r} \cdot \frac{z}{r} = \frac{x}{r} + \frac{y}{r} + \frac{z}{r}$, which proves the following: If α , β , and γ are any three positive angles whose sum is $\pi/2$, then

$$\cot \alpha \cot \beta \cot \gamma = \cot \alpha + \cot \beta + \cot \gamma.$$

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