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STURM-LIOUVILLE THEORY

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Contents

1. Examples of separation of variables leading to Sturm-Liouville eigenvalue problems

Many partial differential equations which appear in physics can be solved by separation of variables. Two examples are illustrated here.

1.1. **Heat conduction in dimension one.**

Consider a thin rod of length L, perfectly insulated. The temperature $u(x,t)$ at time t and position $x \in [0,L]$ satisfies the heat equation

$$u_t = \alpha\, u_{xx} \tag{1}$$

where α is a parameter (which depends on the rod).

Of course, the heat conduction depends on what happens at the two endpoints of the rod $x = 0$ and $x = L$ (boundary condition), and on the initial temperature distribution the rod (initial condition), which need to be specified.

1.1.1. *Separation of variables.* Looking for solutions in the form $u(x,t) = y(x)T(t)$ (with separated variables) and substituting in (1) we obtain

$$y(x)T'(t) = \alpha y''(x)T(t) \quad \text{therefore} \quad \frac{T'(t)}{\alpha T(t)} = \frac{y''(x)}{y(x)} = \text{constant} = -\lambda$$

and we obtain two ordinary differential equation which can be solved:

$$T'(t) = -\alpha\lambda T(t) \tag{2}$$

and

$$y'' + \lambda y = 0 \tag{3}$$

1.1.2. *Boundary conditions.* 1) Suppose both ends of the rod are kept at constant temperature zero: say $u(0,t) = 0$ and $u(L,t) = 0$ for all t. It follows that $y(0) = 0$ and $y(L) = 0$. Equation (3) with these boundary conditions is a Sturm-Liouville eigenvalue problem.

We saw that the eigenvalues of this problem are $\lambda_n = n^2\pi^2/L^2$ ($n = 1, 2, \ldots$) and the eigenfunctions are $y_n(x) = \sin(n\pi/L\, x)$.

2) Other constant boundary temperatures can be imposed, but these can be reduced to the case of zero boundary conditions. If, say, $u(0,t) = 0$ and $u(L,t) = T_0$ for all t, then substituting

$$u(x,t) = \frac{T_0}{L}\, x + v(x,t)$$

into (1) we find that $v(x,t)$ satisfies the heat equation and has zero boundary conditions: $v(0,t) = 0$ and $v(L,t) = 0$ for all t.

3) If one end radiates (say, $x = L$) then the boundary condition at $x = L$ is $u_x(L,t) = -hu(L,t)$ ($h > 0$ means heat loss due to radiation, $h = 0$ means there is no radiation) and $u(0,t) = 0$. This gives: $y(0) = 0$, $y'(L) + hy(L) = 0$ which together with equation (3) form another Sturm-Liouville eigenvalue problem.

1.1.3. *Initial condition.* The initial distribution of the temperature needs to be specified as well: $u(x,0)=u_0(x)$.

After finding the eigenvalues λ_n and eigenfunctions y_n of the appropriate Sturm-Liouville eigenvalue problem, equation (2) is solved yielding $T_n(t)=c_ne^{-\alpha\lambda_n t}$.

Since the heat equation is linear, then a superposition of solutions with separated variables: $u(x,t)=\sum_n c_ne^{-\alpha\lambda_n t}y_n(x)$ is again a solution.

Now it is the time to require that the initial condition be satisfied: $u(x,0)=u_0(x)=\sum_n c_ny_n(x)$.

If the eigenfunctions y_n are complete in $L^2[0,L]$ then indeed c_n exist, and are uniquely determined for any $u_0\in L^2[0,L]$: since the eigenfunctions will be shown to be orthogonal, and assuming they have been normalized to have norm one, then $c_n=\langle y_n,u_0\rangle=\int_0^L y_n(x)u_0(x)dx$.

1.2. **The vibrating string.** Consider a vibrating string with space-dependent tension $T(x)$ and variable linear density $\rho(x)$, assumed to vibrate only due to the restoring tension.

1.2.1. *Derivation of the equation.* Denote by $y(x,t)$ the displacement at time t. (For each t, the graph of the function $x\mapsto y(x,t)$ represents the string.)

To deduce the equation of the motion we apply Newton's law on each small piece $[x,x+\Delta x]$. The force at each point x is the vertical component of $T(x)$: $T_V(x)=T(x)\sin\alpha$ where α is the angle between $T(x)$ and the x-axis. Assuming the oscillations are small then $\sin\alpha\approx\alpha\approx\tan\alpha=\frac{\partial y}{\partial x}$. Thus $T_V(x)\approx T(x)\frac{\partial y}{\partial x}$.

The force of the piece $[x,x+\Delta x]$ is

$$T_V(x+\Delta x)-T_V(x)\approx\frac{\partial T_V}{\partial x}\,\Delta x=\frac{\partial}{\partial x}\left(T(x)\frac{\partial y}{\partial x}\right)\,\Delta x$$

On the other hand, mass times acceleration is $\rho(x)\Delta x\,\frac{\partial^2 y}{\partial t^2}$, therefore

$$\rho(x)\,\frac{\partial^2 y}{\partial t^2}=\frac{\partial}{\partial x}\left(T(x)\frac{\partial y}{\partial x}\right)\tag{4}$$

The motion depends on what happens at the endpoints of the string (the boundary conditions) and on its initial state (initial condition) which need to be specified for the equation (4).

1.2.2. *Separation of variables.* Solve the partial differential equation (4) by separation of variables: looking for solutions of the form $y(x,t)=u(x)f(t)$ and plugging it into (4) it follows that

$$\rho(x)u(x)f''(t)=\frac{d}{dx}\left(T(x)\frac{du}{dx}\right)\,f(t)$$

therefore

$$\frac{f''(t)}{f(t)}=\frac{1}{\rho(x)u(x)}\,\big(T(x)u'(x)\big)'=\text{constant}=-\lambda$$

so

$$f'' = -\lambda f \tag{5}$$

and

$$\frac{1}{\rho u}(Tu')' = -\lambda \tag{6}$$

We can rewrite (6) as

$$(Tu')' + \lambda\rho u = 0 \tag{7}$$

1.2.3. *Boundary conditions.* Suppose the endpoints of the string are kept fixed: $y(0,t) = 0, \;\; y(L,t) = 0$. Then this implies

$$u(0) = 0, \; u(L) = 0 \tag{8}$$

The problem (7), (8) is a Sturm-Liouville eigenvalue problem. As noted before, this is an eigenvalue/eigenfunction problem for the operator

$$-\frac{1}{\rho(x)}\frac{d}{dx}T(x)\frac{d}{dx}, \quad \text{in } H = L^2([0,L],\rho(x)dx)$$

which is selfadjoint on the domain

$$\{u \in H \,|\, u', u'' \in H, \; u(0) = 0, \; u(L) = 0\}$$

2) Another boundary conditions could be: while $x = 0$ is fixed, the endpoint $x = L$ is attached to a vertical rod, without friction. Then, it is easy to see that $u'(L) = 0$. Equation (7) together with the boundary conditions $u(0) = 0$, $u'(L) = 0$ is another example of a Sturm-Liouville eigenvalue problem.

3) Or, $x = 0$ is fixed, but the endpoint $x = L$ is tied to vertical rod, without friction and to a vertical spring that vibrates: $-Tu'(L) - ku(L) = 0$ (T is the string tension, k is the spring constant, the tail is accelerated up and down but there is no transversal force). In this case, the Sturm-Liouville eigenvalue problem consists of equation (7) together with the boundary conditions $u(0) = 0$, $Tu'(L) + ku(L) = 0$.

The appropriate Sturm-Liouville problem is solved, finding the eigenvalues λ_n and the corresponding eigenfunctions $u_n(x)$.

Remark. The eigenfunctions $u_n(x)$ are the *normal modes* of the string.

Then $f_n(t)$ can be found by solving (5): $f_n(t) = c_n \sin(\sqrt{\lambda_n}t) + d_n \cos(\sqrt{\lambda_n}t)$

1.2.4. *Initial condition.* To determine a unique solution the initial position of the string must be given: $u(x,0) = g(x)$ and the initial velocity $u_t(x,0) = v(x)$ (the equation is of order two in t!). It is then required that a superposition of the solutions $\sum_n f_n(t)u_n(x)$ satisfy the initial condition: $\sum_n f_n(0)u_n(x) = g(x)$ and $\sum_n f_n'(0)u_n(x) = v(x)$. If $u_n(x)$ form a complete set then c_n and d_n can be determined.

1.3. **The wave equation.** In particular, if $T(x)$ and $\rho(x)$ are constant then equation (4) becomes

$$y_{tt} = c^2\, y_{xx} \qquad (\text{where } c^2 = T/\rho)$$

which is the wave equation.

The eigenfunctions u_n satisfty $u_n''+\lambda_n u = 0$ and the appropriate boundary conditions. If these are $u(0) = 0,\ u(L) = 0$ then we showed that $u_n(x) = \sin(n\pi x/L)$ which are the normal modes of the string.

2. Second order linear ordinary differential equations

2.1. **Recall some basic results.**

A second order linear ordinary differential equation (ODE) has the form

$$P(x)u'' + Q(x)u' + R(x)u = 0 \tag{9}$$

Because the equation is linear, any linear combination of solutions is again a solution: if u_1, u_2 are solutions of (9) and c_1, c_2 are constants then $c_1u_1(x) + c_2u_2(x)$ is also a solution of (9).

Assumptions.

1) It is assumed assume that the coefficients $P(x), Q(x), R(x)$ are continuous on an interval $[a, b]$. (However, jump discontinuities do appear in applications, and can also be accommodated; we will discuss this later.)

2) It is assumed that $P(x)$ does not vanish in $[a, b]$. Though, we will sometimes let P vanish at a or b. (Points where $P(x)$ is zero are singular, solutions are usually very special at such points, and care is needed.)

3) It can be assumed without loss of generality that $P(x) > 0$ on (a, b). (Since $P(x)$ is never zero on (a, b), and it is continuous, then $P(x)$ is either positive on (a, b) or negative on (a, b). If $P < 0$ we multiply the equation by -1.)

Existence and uniqueness of solution to the initial value problem: given $x_0 \in [a, b]$, so that $P(x_0) \neq 0$ and given the numbers u_0, u_0' then there exists a unique solution $u(x)$ of (9) so that $u(x_0) = u_0$ and $u'(x_0) = u_0'$. This solution $u(x)$ is twice differentiable (moreover, u'' is continuous, as it is seen from (9)), and it depends continuously on the initial conditions.

General solution.

There exist two linearly independent solutions of (9): $u_1(x), u_2(x)$ solutions for $x \in (a, b)$ so that the vectors $(u_1(x), u_1'(x))$ and $(u_2(x), u_2'(x))$ are linearly independent at all $x \in (a, b)$.

In fact, $u_1(x), u_2(x)$ are linearly independent at all $x \in (a, b)$ is equivalent to $u_1(x_0), u_2(x_0)$ are linearly independent at some $x_0 \in (a, b)$.

For example, the solutions with the initial conditions $u_1(x_0) = 1, u_1'(x_0) = 0$ and $u_2(x_0) = 0, u_2'(x_0) = 1$ are linearly independent.

Any solution of (9) is a linear combination of two independent solutions:

$$u(x) = C_1u_1(x) + C_2u_2(x) \tag{10}$$

for some constants $C_{1,2}$. In fact, $C_1u_1(x) + C_2u_2(x)$ with $C_{1,2}$ arbitrary parameters is called *the general solution* of (9).

So the set of all the solutions of (9) form a linear space of dimension two (the dimension equals the order of the equation).

Note: The general solution depends on two parameters, so it makes sense that two conditions are required to determine these parameters. However, there is no apriori guarantee that solutions satisfying different types of problems, like boundary conditions, do exist.

An equivalent condition for two solutions to be linearly independent is that their Wronskian

$$W[u_1, u_2] = u_1' u_2 - u_1 u_2'$$

satisfies $W(x) \neq 0$ for all $x \in (a, b)$ (equivalently, at some $x_0 \in (a, b)$).

Recall that the Wronskian satisfies the differential equation

$$W'(x) = -\frac{Q(x)}{P(x)} W(x) \tag{11}$$

and therefore

$$W(x) = C \exp\left[\int -\frac{Q(x)}{P(x)}\, dx\right]$$

2.2. **The selfadjoint form of a linear second order equation.** Consider eigenvalue problems for equations (9):

$$P(x)u'' + Q(x)u' + R(x)u + \lambda u = 0 \tag{12}$$

We will now show that any equation (12) can be written in a self-adjoint form:

$$\frac{1}{w(x)} \left(-\frac{d}{dx} p(x) \frac{du}{dx} + q(x)\right) u = \lambda u \tag{13}$$

or, expanded,

$$\left(pu'\right)' + (-q + \lambda w)u = 0 \tag{14}$$

where $p(x), q(x), w(x)$ are functions which we will determine now. The designation "self-adjoint" comes from the fact that $-\frac{1}{w}\frac{d}{dx}px)\frac{d}{dx}$ is self-adjoint on *suitable domains* of the weighted $L^2_w[a, b]$, the weight being w. We only show formal-self-adjointness (later).

Expanding the left hand-side of (13) we obtain

$$\frac{p}{w}\, u'' + \frac{p'}{w}\, u' + \left(-\frac{q}{w} + \lambda\right) u = 0$$

which must be (12), therefore

$$\frac{p}{w} = P, \quad \frac{p'}{w} = Q, \quad \frac{q}{w} = -R$$

The first two equations imply that $p'/p = Q/P$ therefore

$$p(x) = \exp\left[\int \frac{Q(x)}{P(x)}\, dx\right] \tag{15}$$

Then since $w = p/P$ and $q = wR$ we obtain

$$w(x) = \frac{1}{P(x)} \exp\left[\int \frac{Q(x)}{P(x)}\, dx\right], \quad q(x) = -\frac{R(x)}{P(x)} \exp\left[\int \frac{Q(x)}{P(x)}\, dx\right] \tag{16}$$

2.3. **Homogeneous boundary conditions.** These conditions are usually inherited from the PDEs which produced the ODE (12) by separation of variables. If the values on the boundary are not zero, substitutions can often be made to ensure zero values on the boundary: these are called *homogeneous boundary conditions.*

These could have the form:

Dirichlet conditions: $u(a) = 0,\ u(b) = 0$, or

Neuman conditions: $u'(a) = 0,\ u'(b) = 0$, or

Mixed Dirichlet-Neuman conditions (or Robin conditions):

$$\begin{aligned} B_a[u] &\equiv \alpha u(a) + \alpha' u'(a) = 0 \\ B_b[u] &\equiv \beta u(b) + \beta' u'(b) = 0 \end{aligned} \tag{17}$$

where $\alpha, \alpha', \beta, \beta'$ are constants.

The mixed conditions are the most general, as they have the Dirichlet and the Neuman conditions as particular cases (if $\alpha' = 0 = \beta'$ we obtain Dirichlet conditions, and if $\alpha = 0 = \beta$ we obtain Neuman conditions). Therefore we work with the general mixed Dirichlet-Neuman conditions.

Note: B_a, B_b are linear functionals of u.

It must be assumed that **the boundary conditions are nontrivial**: the linear functionals B_a, B_b are not identically zero; this means that at least one of the numbers α, α' is not zero (note that this condition can be written as $|\alpha| + |\alpha'| \neq 0$), and similarly, at least one of the numbers β, β' is not zero (i.e. $|\beta| + |\beta'| \neq 0$).

2.3.1. *Another way of writing $B_a[u]$, $B_b[u]$.* Clearly if we multiply α and α' by the same constant, we obtain the same boundary condition $B_a[u]$, and similarly for β and β' in $B_b[u]$. It is sometimes convenient (and always possible!) to choose these in the form

$$\begin{aligned} B_a[u] &\equiv \cos(\theta_a)u(a) - \sin(\theta_a)p(a)u'(a) = 0 \\ B_b[u] &\equiv \cos(\theta_b)u(b) - \sin(\theta_b)p(b)u'(b) = 0 \end{aligned} \tag{18}$$

which are very suitable for Prüfer coordinates.

The transformation which brings (17) in the form (18) is the following: dividing α and α' by the quantity $\pm\sqrt{\alpha^2 + (\alpha'/p(a))^2}$ with the sign chosen to be opposite to the sign of α', we obtain $B_a[u] = 0$ in the form $\alpha_1 u(a) - \alpha_2 p(a)u'(a) = 0$ where $\alpha_1^2 + \alpha_2^2 = 1$ and $\alpha_2 \leq 0$ therefore there exists $\theta_a \in [0, \pi)$ so that $\alpha_1 = \cos(\theta_a)$ and $\alpha_2 = -\sin(\theta_a)$ (we choose $\theta_a < \pi/2$ if $\alpha_1 > 0$ and $\theta_a > \pi/2$ if $\alpha_1 < 0$). A similar transformation can be performed on B_b. Note that we can choose θ_b in $[n, (n+1)\pi)$ for any integer n (if n is even, we proceed as for the condition at $x = a$, while if n is odd we choose the opposite sign in front of $\pm\sqrt{\beta^2 + (\beta'/p(b))^2}$, namely the sign of β').

2.3.2. *Singular boundary conditions.* Other type of conditions which appear in applications are:

Periodic conditions: if $p(a) = p(b)$ then it can be required that the solutions be periodic:

$$u(a) = u(b), \quad \text{and } u'(a) = u'(b)$$

More generally:

$$\begin{aligned} \alpha_1 u(a) + \alpha_1' u'(a) + \beta_1 u(b) + \beta_1' u'(b) = 0 \\ \alpha_2 u(a) + \alpha_2' u'(a) + \beta_2 u(b) + \beta_2' u'(b) = 0 \end{aligned}$$

If p vanishes at an endpoint, say $p(a) = 0$: then the boundary condition at $x = a$ is dropped.

2.4. **Formulation of the homogeneous Sturm-Liouville problem.** We will consider real-valued problems: the functions P, Q, R and the numbers $\alpha, \alpha', \beta, \beta'$ are real. In case complex valued functions are needed, then equations can be written and separately solved for the real and imaginary parts of these functions.

Note that with this reality assumption we have $p(x) > 0$ and $w(x) > 0$ (see (15), (16)).

Given the functions p, q, w continuous on $[a, b]$ and $p, w > 0$ on $[a, b]$, and $B_a[u]$, $B_b[u]$ (nontrivial) find the numbers λ so that the following problem has a nontrivial (i.e. nonzero) solution $u(x)$ on $[a, b]$:

$$\left\{ \begin{array}{l} [p(x)u']' + [-q(x) + \lambda w(x)]u = 0 \\ \\ \text{Boundary conditions at } x = a \text{ and } x = b \end{array} \right. \tag{19}$$

The boundary conditions are one of the following:

• *regular conditions:*

$$\begin{aligned} B_a[u] \equiv \alpha u(a) + \alpha' u'(a) = 0 \qquad & (|\alpha| + |\alpha'| \neq 0) \\ B_b[u] \equiv \beta u(b) + \beta' u'(b) = 0 \qquad & (|\beta| + |\beta'| \neq 0) \end{aligned} \tag{20}$$

• *singular conditions:*
if $p(b) = 0$:

$$B_a[u] \equiv \alpha u(a) + \alpha' u'(a) = 0 \qquad (|\alpha| + |\alpha'| \neq 0) \tag{21}$$

or, if $p(a) = 0$:

$$B_b[u] \equiv \beta u(b) + \beta' u'(b) = 0 \qquad (|\beta| + |\beta'| \neq 0) \tag{22}$$

or, if both $p(a) = 0$, $p(b) = 0$, then no boundary conditions are assumed
• *periodic conditions:* (also singular) if $p(a) = p(b)$

$$\begin{aligned} C[u] \equiv u(a) - u(b) = 0 \\ C'[u] \equiv u'(a) - u'(b) = 0 \end{aligned} \tag{23}$$

The numbers λ are called eigenvalues, and the corresponding solutions - eigenfunctions.

2.5. **Green's Identity and self-adjointness of the Sturm-Liouville operator.** We show here that the problem (19) is indeed *selfadjoint*. Let us first show a general formula:

Lemma 1. *Green's identity:*

$$\int_a^b \left[(pu')'v - u(pv')'\right] = p(u'v - uv')\Big|_a^b \tag{24}$$

Relation (24) follows easily using integration by parts:

$$\int_a^b \left[(pu')'v - u(pv')'\right] = \int_a^b (pu')'v \; - \int_a^b u(pv')'$$

$$= pu'v\Big|_a^b - \int_a^b (pu')v' \; - \; upv'\Big|_a^b + \int_a^b u'(pv') = p(u'v - uv')\Big|_a^b$$

□

Theorem 2. *The operator*

$$L = \frac{1}{w(x)} \left(-\frac{d}{dx} p(x) \frac{d}{dx} \, + q(x) \right) \tag{25}$$

is self-adjoint in the weighted (real) Hilbert space $H = L^2([a,b], w(x)dx)$ *on the domain*

$$D = \{u \in H \mid u', u'' \in H, \; B_a[u] = 0, \; B_b[u] = 0\} \tag{26}$$

where $B_a[u]$, $B_b[u]$ *are any of the boundary conditions listed above.*

Proof. To show that $\langle Lu, v\rangle - \langle u, Lv\rangle = 0$ for all $u, v \in D$ we use, as usually integration by parts. This work was done by the Green's identity, which gives (noting that the terms containing q cancel each other):

$$\langle Lu, v\rangle - \langle u, Lv\rangle = \int_a^b \frac{1}{w} \left[-\left(pu'\right)' + qu \right] v \, w \, dx - \int_a^b u \, \frac{1}{w} \left[-\left(pv'\right)' + qv \right] w \, dx$$

$$= -p(u'v - uv')\Big|_a^b = p(a)\,(u'v - uv')\Big|_{x=a} - \, p(b)\,(u'v - uv')\Big|_{x=b} \tag{27}$$

We have that

$$p(a)(u'v - uv')\Big|_{x=a} = 0, \; p(b)(u'v - uv')\Big|_{x=b} = 0 \tag{28}$$

because $B_a[u] = 0 = B_a[v]$ and $B_b[u] = 0 = B_b[v]$.

Indeed, if $\alpha' = 0$ then $u(a) = 0 = v(a)$ therefore $(u'v - uv')\big|_{x=a} = 0$, and if $\alpha' \neq 0$ then $u'(a) = -\alpha/\alpha' u(a)$, $v'(a) = -\alpha/\alpha' v(a)$ which substituted into the first relation of (28) gives again zero. The second relation of (28) follows in a similar way. □

2.6. **Conclusions.** Since L is selfadjoint on D, this implies that its eigenvalues (if any!) are real, and that eigenfunctions corresponding to different eigenvalues are orthogonal.

We will show that indeed, there exist infinitely many eigenvalues λ_n, and that the eigenfunctions u_n form a complete set in the Hilbert space $L^2([a,b], w(x)dx)$.

We will accomplish this program by studying the solutions of the differential equation.

It turns out that, in addition, λ_n can be ordered increasingly, and $\lambda_n \to \infty$, and that eigenfunctions u_n oscillate, and the larger n, the more rapid the oscillations.

3. Eigenfunctions associated to one eigenvalue

3.1. **Regular problems.**

Lemma 3. *For regular problems (19), (20) the eigenspaces are one-dimensional: there is a unique (up to a scalar multiple) eigenfunction associated to each eigenvalue.*

Proof. We show that the eigenspace associated to one eigenvalue of (19) is one dimensional: any two (nonzero) solutions $u_1(x)$, $u_2(x)$ of (19) (for the same λ) are linearly dependent.

Assume, to get a contradiction, that u_1 and u_2 are linearly independent. Then the general solution of the differential equation in (19) is $u = C_1u_1 + C_2u_2$ with C_1, C_2 arbitrary constants. Since the boundary conditions are linear, it follows that $B_a[u] = C_1B_a[u_1] + C_2B_a[u_2] = 0$, $B_b[u] = C_1B_b[u_1] + C_2B_b[u_2] = 0$ therefore the boundary conditions are satisfied by any solution of the differential equation. For u the solution with $u(a) = 1, u'(a) = 0$ we find that $\alpha = 0$ and u the solution with $u(a) = 0, u'(a) = 1$ it follows that $\alpha' = 0$, which contradicts the nontriviality assumption on B_a. (A similiar argument can be made at $x = b$.)

Therefore, u_1 and u_2 must be linearly dependent, hence scalar multiples of each other. □

3.2. **When $p(x)$ vanishes at one endpoint.**

Suppose that $p(b) = 0$. Consider the the Sturm-Liouville problem (19), (22).

The Sturm-Liouville operator (25) is selfadjoint on the domain

$$D = \{u \in H \mid u', u'' \in H,\ B_a[u] = 0\} \tag{29}$$

Indeed, relations (28) hold: at $x = b$ because $p(b) = 0$ and at $x = a$ with the same proof as for Theorem 25.

The eigenspaces are still one-dimensionsl, as the proof of Lemma 3 works.

The singular case with $p(a) = 0$ is similar.

3.3. **Periodic problems.**

If $p(a) = p(b)$ the Sturm-Liouville eigenvalue problem (19), (23) is also selfadjoint: the operator (25) is selfadjoint on

$$D = \{u \in H \mid u', u'' \in H,\ C[u] = 0,\ C'[u] = 0\} \tag{30}$$

since the last quantity in (27) is clearly zero for u, v in the domain (30).

However, for periodic boundary conditions the eigenspaces may be one, or two-dimensional.

Indeed, repeating the argument of §3.1 we find that if for an eigenvalue λ, if there are two independent eigenfunctions then all the solutions of the ODE are periodic (but the space of solutions of a second order ODE is a two-dimensional vector space). This means that the eigenspace is either one-dimensional or two-dimensional, and in the latter case we can choose two orthogonal eigenfunctions.

Here are sufficient conditions which ensure completeness of eigenfunctions:

Theorem 4. Completeness of the eigenfunctions

Suppose that L is a self-adjoint operator in a Hilbert space H, defined on a domain D dense in H.

Assume that L has the eigenvalues $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$ with $\lambda_n \to \infty$, and that each eigenspace is finite dimensional, spanned by a set of orthgonal eigenfunctions u_n.

Then the set of eigenfunctions is complete: they form a basis for the Hilbert space H.

The proof of the theorem is postponed until §5. We will put it to good use first.

4. Fourier series

4.1. **Basic facts.**

It is easy to solve the following periodic Sturm-Liouville problem:

$$u'' + \lambda u = 0, \qquad u(-\pi) = u(\pi),\ u'(-\pi) = u'(\pi) \tag{31}$$

It has the eigenvalues $\lambda_n = n^2,\ n = 0, 1, 2, \ldots$. For $n > 0$ there are two orthogonal eigenfunctions corresponding to $\lambda_n = n^2$: e^{inx} and e^{-inx}, For $n = 0$ the eigenfunction corresponding to $\lambda_0 = 0$ is the constant function, say 1.

Applying Theorem 4 to $L = -\frac{d^2}{dx^2}$ in $H = L^2[-\pi, \pi]$, which is self-adjoint on

$$D = \{u \in H \,|\, u', u'' \in H,\ u(\pi) - u(-\pi) = 0,\ u'(\pi) - u'(-\pi) = 0\}$$

having the eigenvalues $0, 1, 1, 2^2, 2^2, \ldots$ it follows that its eigenfunctions e^{inx}, $n \in \mathbb{Z}$ form an orthogonal basis for $L^2[-\pi, \pi]$.

Sometimes it is preferable to work with real-valued functions. In this case, in each eigenspace $Sp(e^{inx}, e^{-inx})$ corresponding to $\lambda_n = n^2 > 0$ we choose a basis consisting of real function: $\sin(nx)$ and $\cos(nx)$ (which are orthogonal). Therefore also its eigenfunctions 1, $\sin(nx)$, $\cos(nx)$ for $n = 1, 2, \ldots$ form an orthogonal basis for $L^2[-\pi, \pi]$. Therefore:

Theorem 5. *Any function $f \in L^2[-\pi, \pi]$ can be expanded in a Fourier series:*

$$f = \sum_{n=-\infty}^{\infty} \hat{f}_n e^{inx} \tag{32}$$

where

$$\hat{f}_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} f(x)\, dx$$

Also

$$f = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx)\right] \tag{33}$$

where the Fourier coefficients a_n and b_n are given by the formulas

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx\ ,\ \ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx,$$

The series (32) *and* (33) *converge in the L^2-norm of $L^2[-\pi,\pi]$.*

Please note again that convergence is in the sense of tye L^2 norm of the difference going to zero, and not in the sense of convergence at each point!

Exercise. Write $\hat{f}_n$ in terms of a_n, b_n. Write a_n, b_n in terms of $\hat{f}_n$.

4.2. **Fourier series, sine series and cosine series.** Recall that the eigenfunctions of the Sturm-Liouville problem

$$u'' + \lambda u = 0,\ u(0) = 0,\ u(\pi) = 0 \quad \text{on } [0,\pi] \tag{34}$$

are $\sin(nx)$, for $n = 1, 2, \ldots$ and they correspond to the eigenvalues $\lambda_n = n^2$. Theorem 4 does apply and we obtain that any function in $L^2[0,\pi]$ can be expanded in a sine-series.

On the other hand, the eigenfunctions of the Sturm-Liouville problem with Neuman conditions

$$v'' + \lambda v = 0,\ v'(0) = 0,\ v'(\pi) = 0 \quad \text{on } [0,\pi] \tag{35}$$

are $\cos(nx)$, for $n = 0, 1, 2, \ldots$ and they correspond to the eigenvalues $\lambda_n = n^2$. (Indeed, denoting $v' = u$ we obtain the problem (34)). By Theorem 4 it follows that $\cos(nx)$, for $n = 0, 1, 2, \ldots$ also form an orthogonal basis for $L^2[0,\pi]$.

Therefore, we can expand any *function on* $[0,\pi]$ in a sine-series

$$f = \sum_{n=1}^{\infty} b_n \sin(nx), \quad \text{where } b_n = \frac{2}{\pi}\int_0^{\pi} \sin(nx) f(x)\,dx \tag{36}$$

or as a cosine series

$$f = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx), \quad \text{where } a_n = \frac{2}{\pi}\int_0^{\pi} \cos(nx) f(x)\,dx \tag{37}$$

where the convergence of (36) and (37) is in the L^2-norm on $[0,\pi]$ (i.e. in square average).

How do the series (33), (36), (37) reconcile each other?

The most important difference to keep in mind between (33) on one hand, and (36), (37) on the other hand, is that in the first case, of a bone-fide Fourier series, the function f is considered on an interval equal to the period of the functions sin and cos used, while in the case of sine, or cosine series, $f(x)$ is considered the interval of half-period length.

The connection between (33) and (36), (37) can be understood easily if we think in terms of even and odd functions. (This is one reason why it is helpful to choose an interval symmetric with respect to the origin.)

Recall that a function is called *even* if $f(-x) = f(x)$ for all x. For example the functions $1, x^2, x^4, x^8 - 3x^4, |x|, \cos(x)$ are even.

Recall that a function is called *odd* if $f(-x) = -f(x)$ for all x. For example the functions $x, x^3, x^5, x^9 - 2x, \sin(x), \tan x$ are odd.

4.2.1. *Functions on* $[-\pi, \pi]$.

Note: *If $f(x)$ is an even function then all $b_n = 0$ and therefore f has a cosine-series.*

If $f(x)$ is an odd function then all $a_n = 0$ and therefore f has a sine-series.

Indeed, first of all this is intuitive, since sines are odd, while cosines are even. On the other hand, the equality (33) is on average, so we should better check using formulas. And it is true, since: if $f(x)$ is even, then all $f(x)\sin(nx)$ are odd, hence their integral on a symmetric interval is zero, hence $b_n = 0$; if $f(x)$ is odd, then all $f(x)\cos(nx)$ are odd, hence all $a_n = 0$.

Now if $f(x)$ is an arbitrary function on $[-\pi, \pi]$ (or on any symmetric interval), it can be written as a sum of an even function, $f_e(x) = \frac{1}{2}[f(x) + f(-x)]$ (its *even part*) and an odd function $f_o(x) = \frac{1}{2}[f(x) - f(-x)]$ (its *odd part*): $f = f_e + f_o$. Hence, if $f(x)$ is defined on $[-\pi, \pi]$ its Fourier series (33) equals the sine-series of its odd part f_o plus the cosine-series of its even part f_e.

4.2.2. *Functions on* $[0, \pi]$. Given any function $g(x)$ for $x \in [0, \pi]$, then g can be continued to $x \in [-\pi, \pi]$ by

1) requiring that the function on $[-\pi, \pi]$ be odd, that is continued as

$$g_{odd}(x) = \begin{cases} g(x) & \text{if } 0 < x < \pi \\ -g(-x) & \text{if } -\pi < x < 0 \end{cases}$$

in which case $g(x)$ has a sine-series, or by

2) requiring that the function on $[-\pi, \pi]$ be even, that is continued as

$$g_{even}(x) = \begin{cases} g(x) & \text{if } 0 < x < \pi \\ g(-x) & \text{if } -\pi < x < 0 \end{cases}$$

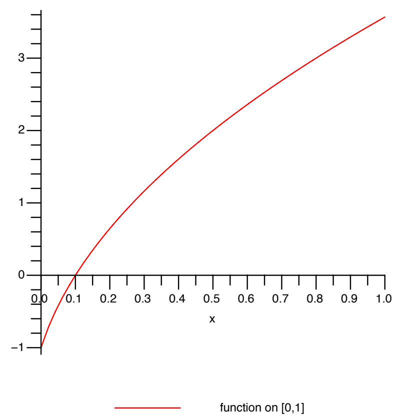
in which case $g(x)$ has a cosine-series.

(Recall that functions that differ by the value at one point, such as $x = 0$, are considered equal in L^2.)

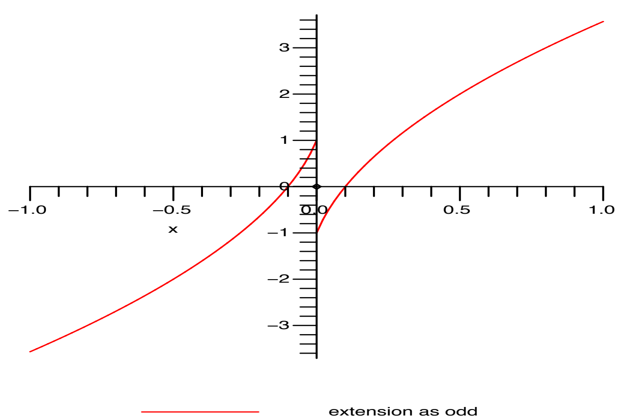
Note that the odd part of g_{odd} is g, hence its Fourier series contains only sines (and equals the sine series of g); the even part of g_{even} is also g, hence its Fourier series contains only cosines (and equals the cosine series of g).

We have now fully reconciled the Fourier series with the sine and cosine series.

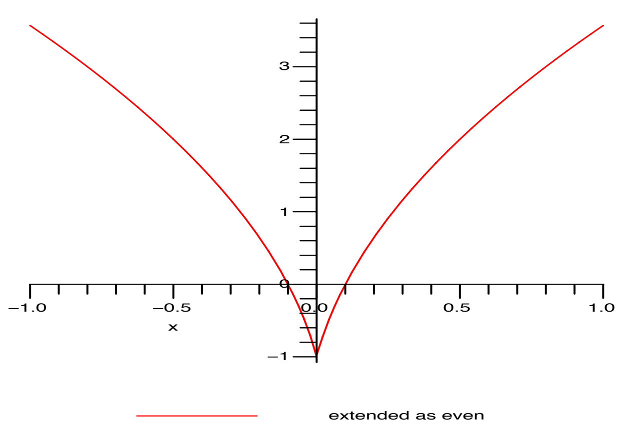
The figure below shows a function on $[0, 1]$



and its extension to $[-1, 1]$ as an odd function



and its extension to $[-1, 1]$ as an even function



4.3. **General intervals.** First note that instead of the interval $[-\pi, \pi]$ we can use any interval of length 2π (but we loose the odd-even symmetry); similarly, instead of the interval $[0, \pi]$ for the sine and cosine series, we can use any interval of length π. For a function $f(x)$ on a general interval $[a, b]$ a linear change of the coordinate x shows that we can write an expansion

$$f = \sum_{n\in\mathbb{Z}} \hat{f}_n e^{2\pi i n x/(b-a)} \quad \text{ on } [a, b]$$

or, we can replace x by any translation $x - c$. Also sine series and cosine series ca be written, with argument $n\pi x/(b-a)$. The general Fourier series is more appropriate if f is periodic on $[a, b]$ (i.e. $f(a) = f(b)$).

4.4. **pointwise convergence of Fourier series.** While the Fourierseries (32) and (33) converges to f in the L^2-norm (that is, in squared average), it is useful to know when the series converges pointwise (that is, for a fixed x, as a series of numbers):

Theorem 6 (Pointwise convergence of a Fourier series)**.**
A. *If f and f' are piecewise continuous on $[-\pi, \pi]$ then the series (33) converges for every x.*
B. *Let $c \in (-\pi, \pi)$.*
(i) If f is continuous at $x = c$ then the Fourier series (33) at $x = c$ converges to $f(c)$.
(ii) If f has a jump discontinuity at $x = c$ then the Fourier series (33) at $x = c$ converges to $[f(c-) + f(c+)]/2$.
C. *The behavior of the Fourier series at the points $x = \pi$ and $x = -\pi$ is seen in the following way. Continue $f(x)$ outside $[-\pi, \pi]$ by 2π-periodicity. If $f(\pi-) = f(-\pi+)$ then $x = \pm\pi$ are points of continuity, and the series (33) converges to $f(\pm\pi)$ for $x = \pm\pi$. Otherwise, $x = \pm\pi$ are points where there is a jump discountinuity and the series (33) converges to $[f(\pi-) + f(-\pi+)]/2$ for $x = \pm\pi$.*

We will give a proof of Theorem 6 shortly.

Remark. *Smoothness of a function is related to the rate of decay of its Fourier coefficients.* Namely, the more derivatives a function has, the faster its Fourier coefficients decay.

Recall that if $f \in \mathcal{H}$ then sequence of its generalized Fourier coefficients with respect to an orthonormal basis of $\mathcal{H}$ belongs to ℓ^2. In particular, the sequence $a_0, a_1, b_1, a_2, b_2, \ldots$ in (33) is in ℓ^2, so $a_n, b_n \to 0$, faster than $1/\sqrt{n}$ *in the weak sense that* $\liminf \sqrt{n} c_n = 0$. It can be shown that if the coefficients decay faster, say like $1/n^2$, then f is continuous and piecewise differentiable. And so on: the faster the decay, the more derivatives f has.

Precise statements can be found in books dedicated to Fourier analysis.

4.5. **An example.** Let us find the sine-series of function $f(x) = 1$ for $x \in [0, \pi]$.

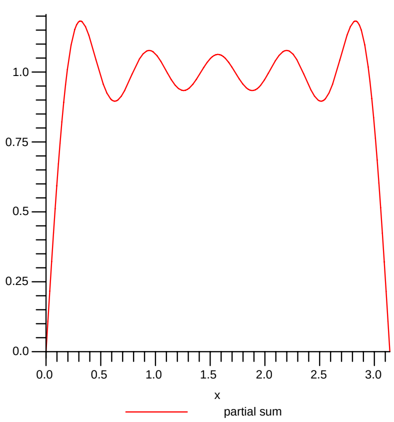
It can be checked that $b_n = 0$ for n even, and its sine-series is

$$\frac{4}{\pi}\sin(x) + \frac{4}{3\pi}\sin(3\,x) + \frac{4}{5\pi}\sin(5\,x) + \frac{4}{7\pi}\sin(7\,x) + \frac{4}{9\pi}\sin(9\,x) + \dots$$

For each $x \in (0, \pi)$ the sine-series converges to 1 by Theorem 6 **B.**(i). To understand the value at which the series converges at the end points $x = 0$ and $x = \pi$ we first continue the function to an odd function on $[-\pi, \pi]$, which is $f_{odd}(x) = 1$ for $x \in (0, \pi]$ and $f_{odd}(x) = -1$ for $x \in [-\pi, 0)$. Using Theorem 6 **B.**(i) the series at $x = 0$ converges to $[f_{odd}(0-) + f_{odd}(0+)]/2 = 0$ and by Theorem 6 **C.** at $x = \pi$ the series converges to $[f_{odd}(-\pi+) + f_{odd}(\pi-)]/2 = 0$.

Of course, by substituting directly $x = 0$ or $x = \pi$ in the series we see that all its terms are zero.

The picture shows the plot of the sum of the first five nonzero terms.



Note the large overshoot of the partial sum at the at the jump discontinuities at $x = 0$ and $x = \pi$: this is called Gibbs phenomenon. This behavior of truncates of Fourier series gives rise to artifacts in signal processing.

4.6. **The Dirichlet Kernel and the pointwise convergence theorem.** We will prove the theorem in the case f is periodic in $C^1[-\pi, \pi]$. Dealing with discontinuities is relatively easy, and could, in principle, be left as an exercise. How would you go about solving such an exercise?

In assessing pointwise convergence of Fourier series, we want to estimate sums of the form

$$f(x) - \sum_{-N}^{M} \hat{f}_n e^{inx} \tag{38}$$

The sum

$$\sum_{k=-n}^{n} e^{ikx} = \frac{\sin((n+1/2)x)}{\sin(x/2)} \tag{39}$$

is called the Dirichlet kernel. To show (39), one way is to use the geometric series,

$$\sum_{k=-n}^{n} r^k = r^{-n}\,\frac{1-r^{2n+1}}{1-r} = \frac{r^{-n-1/2}}{r^{-1/2}}\,\frac{1-r^{2n+1}}{1-r} = \frac{r^{-n-1/2}-r^{n+1/2}}{r^{-1/2}-r^{1/2}} \tag{40}$$

and thus

$$\sum_{k=-n}^{n} e^{ikx} = \frac{e^{-(n+1/2)ix}-e^{(n+1/2)ix}}{e^{-ix/2}-e^{ix/2}} = \frac{-2i\sin((n+1/2)x)}{-2i\sin(x/2)} = \frac{\sin((n+1/2)x)}{\sin(x/2)} \tag{41}$$

Noting that we can write the Fourier series of a function on $[-\pi,\pi]$ in the form

$$a_0 + \sum_{k=0}^{\infty}(a_k\cos kx + b_k\sin kx) \tag{42}$$

where we remember that we got the a_n, b_n by using Euler's formula in

$$f = \sum_{-\infty}^{\infty} \hat{f}_n e^{inx}, \quad \text{equality in the sense of } L^2 \tag{43}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt,\ a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)dt\ b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)dt \tag{44}$$

we have

$$\begin{aligned}
f(x) - \left[a_0 + \sum_{k=0}^{n}(a_k\cos kx + b_k\sin kx)\right] &= f(x) - \sum_{k=-n}^{n}\hat{f}_n e^{inx} \\
= f(x) - \frac{1}{2\pi}\sum_{k=-n}^{n}\int_{-\pi}^{\pi} f(t)e^{-int}e^{inx}dt &= f(x) - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\sum_{k=-n}^{n} e^{-in(t-x)} \\
&= f(x) - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\frac{\sin((n+1/2)(x-t))}{\sin((x-t)/2)} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)-f(t)]\frac{\sin((n+1/2)(x-t))}{\sin((x-t)/2)}
\end{aligned} \tag{45}$$

Here we used the fact that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sin((n+1/2)(x-t))}{\sin((x-t)/2)} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sin((n+1/2)u)}{\sin(u/2)}du = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-n}^{n} e^{iku}du = 1 \tag{46}$$

Now,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)-f(t)]\frac{\sin((n+1/2)(x-t))}{\sin((x-t)/2)} \tag{47}$$
$$=\int_{-\pi}^{\pi}\left[\frac{f(x)-f(t)}{x-t}\frac{x-t}{\sin((x-t)/2)}\right]\sin((n+1/2)u)du$$
$$=\int_{x-\pi}^{x+\pi}\left[\frac{f(x)-f(x-u)}{u}\frac{u}{\sin(u/2)}\right]\sin(nu+u/2)du$$
$$=\int_{x-\pi}^{x+\pi}G_x(u)\sin(nu+u/2)du$$

where, because f is continuously differentiable, G_x is continuous (why?). Thus

$$\int_{x-\pi}^{x+\pi}G_x(u)\sin(nu+u/2)du \text{ as } n\to\infty \text{ (why?)} \tag{48}$$

5. Completeness of the eigenfunctions: Proof of Theorem 4

Consider a self-adjoint operator L on a domain D dense in a Hilbert space $\mathcal{H}$. We found many problems where there exists a sequence of eigenvalues $\lambda_1 \le \lambda_2 \le \lambda_3 \le \ldots$ with $\lambda_n \to \infty$ and that the corresponding eigenfunctions u_n are orthogonal. The hypotheses of Theorem 4 are thus satisfied. It only remains to prove Theorem 4.

5.1. **Proof of Theorem 4.** As in finite dimensions, the eigenvalues of L can be calculated using the maximin principle. (We cannot speak about a minimax since there is no maximum eigenvalue).

Recall that, using the Rayleigh quotient

$$R[u]=\frac{\langle u,Lu\rangle}{\langle u,u\rangle},\quad u\in D,\ u\neq 0$$

we have

$$\lambda_1=\min R[u]$$

then

$$\lambda_2=\min\{R[u]\,|\,\langle u_1,u\rangle=0\},\quad \lambda_3=\min\{R[u]\,|\,\langle u_1,u\rangle=0,\langle u_2,u\rangle=0\}\ \ldots$$

or, with the notation $W_1 = Sp(u_1)$, $\ldots W_n = Sp(u_1,u_2,\ldots,u_n)\ldots$ and $V_n = W_1^{\perp}\cap D$ then

$$\lambda_2=\min_{u\in V_1}R[u],\ \lambda_2=\min_{u\in V_2}R[u],\ldots,\lambda_{n+1}=\min_{u\in V_n}R[u]\ \ldots$$

Let u_n be eigenfunctions of L: $u_n \in D$ so that $Lu_n = \lambda_n u_n$ which by assumption satisfy $u_n \perp u_k$ if $n \neq k$ (note that this is automatic if $\lambda_n \neq \lambda_k$ since L is selfadjoint). We can assume $\|u_n\| = 1$.

To prove completeness of the eigenfunctions u_n we first show that any f in the domain D of the operator L can be expanded in terms of u_n, in other words that the space $S = Sp(u_1,u_2,u_2,\ldots)$ is dense in D. Then since

D was assumed to be dense in H it follows that S is dense in H, therefore $u_1, u_2, u_3, \ldots$ form a basis for the Hilbert space (see details in §5.2 below).

To show that S is dense in D for any arbitrary $f \in D$, form the series

$$\sum_{n=1}^{\infty} \hat{f}_n u_n, \quad \text{with } \hat{f}_n = \langle u_n, f\rangle \tag{49}$$

and show that this series converges to f. Note that the partial sums of the series (49) belong to S:

$$f^{[N]} \equiv \sum_{n=1}^{N} \langle u_n, f\rangle u_n \in S$$

We show that the error when approximating f by partial sums

$$h^{[N]} = f - f^{[N]} \tag{50}$$

goes to zero as $N \to \infty$, that is

$$\|h^{[N]}\| \to 0 \ \text{ as } N \to \infty$$

Since $h^{[N]} \in V_N$ (why?) then

$$\lambda_{N+1} = \min_{u \in V_N} R[u] \le R[h^{[N]}] = \frac{\langle h^{[N]}, Lh^{[N]}\rangle}{\|h^{[N]}\|^2}$$

and therefore

$$\|h^{[N]}\|^2 \le \frac{1}{\lambda_{N+1}} \langle h^{[N]}, Lh^{[N]}\rangle \tag{51}$$

Now expand, using (50) (recall that $\lambda_n \in \mathbb{R}$),

$$\langle h^{[N]}, Lh^{[N]}\rangle = \langle f, Lf\rangle - \sum_{n=1}^{N} \overline{\hat{f}_n} \langle u_n, Lf\rangle - \sum_{n=1}^{N} \hat{f}_n \langle f, Lu_n\rangle + \sum_{n,m=1}^{N} \overline{\hat{f}_n} \hat{f}_m \langle u_n, Lu_m\rangle$$

$$= \langle f, Lf\rangle - \sum_{n=1}^{N} 2\overline{\hat{f}_n} \lambda_n \langle u_n, f\rangle + \sum_{n=1}^{N} |\hat{f}_n|^2 \lambda_n = \langle f, Lf\rangle - \sum_{n=1}^{N} |\hat{f}_n|^2 \lambda_n$$

Since $\lim_n \lambda_n = +\infty$ then the eigenvalues are positive starting with a certain rank N_0: suppose $\lambda_n \ge 0$ for $n > N_0$. Then (for $N > N_0$)

$$\langle f, Lf\rangle - \sum_{n=1}^{N} |\hat{f}_n|^2 \lambda_n = \langle f, Lf\rangle - \sum_{n=1}^{N_0} |\hat{f}_n|^2 \lambda_n - \sum_{n=N_0+1}^{N} |\hat{f}_n|^2 \lambda_n \le \langle f, Lf\rangle - \sum_{n=1}^{N_0} |\hat{f}_n|^2 \lambda_n \tag{52}$$

Using (52) in (51) we obtain

$$\|h^{[N]}\|^2 \le \frac{1}{\lambda_{N+1}} \left(\langle f, Lf\rangle - \sum_{n=1}^{N_0} |\hat{f}_n|^2 \lambda_n \right) \to 0 \ \text{ as } N \to \infty$$

which completes the proof of the convergence and of completeness. $\Box$

5.2. **If S is dense in D and D is dense in H then S is dense in H.** Intuitively: the statement that D is dense in H means that for any $f \in H$ we can find an $f_D \in D$ as close to f as we wish. Similarly, if S is dense in D then we can find $f_S \in S$ as close to f_D as we wish. By the triangle's inequality: if f_D is close to f, and f_S is close to f_D, then f_S is close to f.

So, let $f \in H$. We want (for an arbitrary $\epsilon > 0$) to find $f_S \in S$ so that $d(f, f_S) < \epsilon$. But we can certainly find $f_D \in D$ so that $d(f, f_D) < \epsilon/2$ and for that f_D we can certainly find $f_S \in S$ so that $d(f_D, f_D) < \epsilon/2$.

By the triangle's inequality then

$$d(f, f_S) < d(f, f_D) + d(f_D, f_S) < \epsilon/2 + \epsilon/2 = \epsilon$$

which proves the claim.

5.3. **Problems on infinite intervals.** If we have eigenvalue problems on infinite intervals (like $[a, +\infty)$ or $\mathbb{R} = (-\infty, +\infty)$) a similar theory can be developed, only series need to be replaced by integrals.

For example, consider the differentiation operator in $L^2(\mathbb{R})$: $L = -i\frac{d}{dx}$. Since

$$-i\frac{d}{dx}e^{ikx} = ke^{ikx}$$

then the function e^{ikx} is a *generalized* eigenfunction of L (it must be called "generalized" because it does not belong to the Hilbert space $L^2(\mathbb{R})$).

However, any $f \in L^2(\mathbb{R})$ can be developed in terms of these generalized eigenfunctions:

$$f(x) = \int_{-\infty}^{\infty} e^{ikx}\hat{F}(k)\,dk$$

(the inverse Fourier transform, up to a scalar multiple... this normalization issue again...) to be compared to the Fourier series for $f \in L^2[-\pi, \pi]$

$$f = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{ikx}$$

We will discuss Fourier integrals in detail later; the main point is that on finite intervals we have series, while on infinite intervals we have integrals.

6. Abel's Theorem

The following results is (11) in disguise:

Theorem 7. Abel's Theorem

Let u, v be two solutions of the second order linear differential equation

$$[p(x)u']' + [-q(x) + \lambda w(x)]u = 0 \tag{53}$$

Then

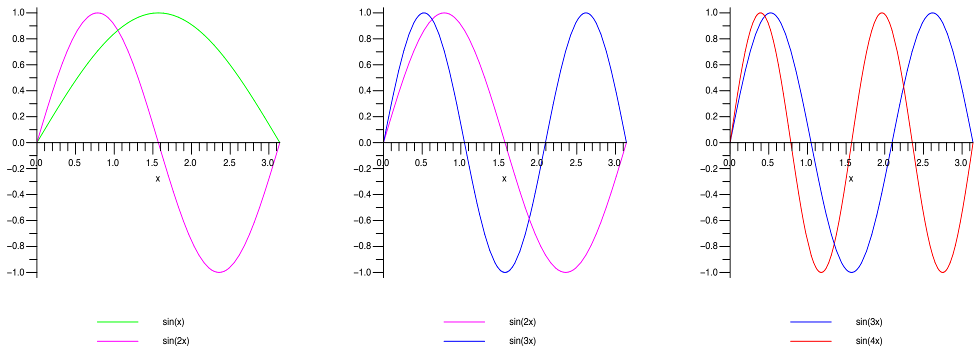
$$p(u'v - uv') = \text{constant}$$

Proof. Equation (53) expanded is $pu'' + p'u' + (-q + \lambda w)u = 0$. For $W = W[u, v] = u'v - uv'$ relation (11) implies that $W' = -p'/p\,W$ so $\ln W = -\ln p + const$ which implies $pW = const$. □

Note that once a solution u of (53) another independent solution v can be found by solving $p(u'v - uv') = c$, which is a first order differential equation, linear nonhomogeneous for v.

7. Sturm's Oscillation Theorems

7.1. **A simple example.** It is always useful to consider simple examples (or, "toy models") for the more complicated system studied. A simple example (and exactly solvable!) of a Sturm-Liouville problem is obtained for for $w(x) = 1,\ p(x) = 1,\ q(x) = 0$, and $\alpha' = 0, \beta' = 0$, and $[a, b] = [0, \pi]$, namely the problem (34). We studied this equation and we found the eigenvalues $\lambda_n = n^2$ $(n = 1, 2, \ldots)$ and the eigenfunctions $u_n = \sin(nx)$.



The figures show, on the same plot, u_n and u_{n+1} for $n = 1, 2, 3$. Please note how their zeroes interlace: between two consecutive zeroes on u_n there is a zero of u_{n+1}. Sturm's Comparison Theorem shows that this is a general feature. (Please note that the numbers of zeros is linked to the number of oscillations.)

7.2. **Sturm's Comparison Theorem.**

Theorem 8. *Consider two solutions $u(x)$ and $v(x)$ of two equations*

$$[p(x)u']' + g(x)u = 0 \tag{54}$$

and

$$[p(x)v']' + h(x)v = 0 \tag{55}$$

where $p(x) > 0$ on $[a, b]$.

(i) If $g(x) < h(x)$ on $[a, b]$ then $v(x)$ oscillates more rapidly than $u(x)$: between any two zeroes of $u(x)$ there is a zero of $v(x)$.

(ii) If $g(x) < 0$ on $[a, b]$ then $u(x)$ does not oscillate: it vanishes at most once on $[a, b]$.

Sturms's Comparison Theorem applied to $g(x) = -q(x) + \lambda w(x)$ with $w > 0$ shows that if λ is negative enough there cannot be solutions of (54) which vanish at both endpoints a, b. However, once there is a solution vanishing at $x = a$ and $x = b$, then the higher λ the more zeroes solutions have (i.e. more oscillations) and that zeroes of solutions for different λs do interlace.

Proof of Sturm's Comparison Theorem

(i) Multiplying (54) by v, (55) by u and subtracting it is found that

$$\frac{d}{dx}\left[p(u'v - uv')\right] = (h - g)uv$$

therefore by integration

$$p(u'v - uv')\Big|_{x_1}^{x_2} = \int_{x_1}^{x_2} (h - g)uv \tag{56}$$

Choose $x_1 < x_2$ be two consecutive zeroes of u. Assume, to get a contradiction, that v has no zero on (x_1, x_2). Then v has a constant sign on

(x_1, x_2), say $v > 0$ (otherwise replace v by $-v$). We can also assume $u > 0$ on (x_1, x_2). Then the right-hand side of (56) is positive, while the left-hane side is negative since it equals

$$\underbrace{p(x_2)}_{>0}\underbrace{u'(x_2)}_{\leq 0}\underbrace{v(x_2)}_{>0} - \underbrace{p(x_1)}_{>0}\underbrace{u'(x_1)}_{\geq 0}\underbrace{v(x_1)}_{>0} \leq 0$$

which is a contradiction.

(ii) For $h(x) = 0$: since $(pv')' = 0$ then the general solution of (55) is $c_1 + c_2 \int_a^x (1/p)$. Choosing $v = 1 + \int_a^x (1/p)$ we have $v > 0$ hence v has no zero. Therefore by (i) u has at most one zero. □.

8. Existence of eigenvalues: The Prüfer transformation

The differential equation (in selfadjoint form) is

$$(p(x)u')' + g(x)u = 0 \qquad \text{with } g(x) = -q(x) + \lambda w(x) \tag{57}$$

Assume that $p(x) > 0$ and that p', g' are continuous.

Sturm's Comparison Theorem shows the oscillatory character of the solutions of (57). Polar coordinates in the phase space are then natural.

8.1. **The Prüfer system.**

The Prüfer transformation consists in writing in polar coordinates the (phase-space like) quantities u and $v = pu'$:

$$u(x) = r(x)\,\sin\theta(x), \quad u'(x) = \frac{r(x)}{p(x)}\,\cos\theta(x) \tag{58}$$

The equation (57) is $v' + gu = 0$, therefore

$$\frac{d}{dx}\,(r\cos\theta) + gr\sin\theta = 0 \tag{59}$$

The second equation is obtained from $u' = r/p\cos\theta$ which expanded gives

$$r'\sin\theta + r\theta'\cos\theta = r/p\cos\theta \tag{60}$$

On the other hand, expanding (59) we obtain

$$r'\cos\theta - r\theta'\sin\theta + gr\sin\theta = 0 \tag{61}$$

Solving (60), (61) for r', θ' we obtain

$$r' = \frac{1}{2}\left(\frac{1}{p} - g\right)\,r\sin 2\theta \tag{62}$$

and

$$\theta' = g\sin^2\theta + \frac{1}{p}\cos^2\theta \equiv F(x,\theta) \tag{63}$$

(multiplying (60) by $\sin\theta$, multiplying (61) by $\cos\theta$ and adding them up we obtain (62), while multiplying (60) by $\cos\theta$, multiplying (61) by $-\sin\theta$ and adding them up we obtain (63)).

Remarkably, equation (63) is a first order equation for $\theta(x)$, independent of $r(x)$! With $\theta(x)$ determined by (63) we can integrate (62) to obtain $r(x)$:

$$r(x) = K \exp \int_a^x \frac{1}{2}\left(\frac{1}{p(s)} - g(s)\right) \sin 2\theta(s)\, ds$$

Note that $r(x) \neq 0$. Note that we can take the constant $K = 1$ (otherwise we divide $u(x)$ by K).

It is easy to see (from (58)) that the boundary conditions (18) become

$$\theta(a) = \theta_a,\ \ \theta(b) = \theta_b \tag{64}$$

8.2. **Behavior of $\theta(x)$ and the zeroes of $u(x)$.**

Choosing λ large enough assume that $g(x) > 0$ on $[a, b]$, so that oscillations are possible.

(A) Note that (from (58))

$$u(x) = 0 \quad \text{if and only if } \sin\theta = 0 \text{ so if and only if } \theta = k\pi,\ k \in \mathbb{Z}$$

(B) On the other hand $\theta'(x) = F(x, \theta) > 0$, so $\theta(x)$ is an increasing function. In fact, the larger g, the larger θ' (the rate of increase of θ).

(C) There can be no accumulation of zeroes of $u(x)$ as the distance between two successive zeroes is no smaller that a positive number d.

Why: Denoting $M_F = \max\{F(x,\theta)\,|x \in [a,b],\ \theta \in [0,2\pi]\}$ we have $\theta'(x) < M_F$. Therefore, if $x_1 < x_2$ are two successive zeroes of $u(x)$ then we have $\theta(x_1) = k\pi$ and since $\theta(x_2) = (k+1)\pi$ and $\theta(x_2) - \theta(x_1) = \theta'(c)(x_2 - x_1)$ for some $c \in (a, b)$ it follows that

$$x_2 - x_1 = \frac{1}{\theta'(c)}(\theta(x_2) - \theta(x_1)) \geq \frac{\pi}{M_F} \equiv d \tag{65}$$

8.3. **Boundary conditions and existence of eigenvalues.**

We showed that the Sturm-Liouville eigenvalue problem (19) becomes, in Prüfer coordinates: determine the values λ so that the following boundary value problem has a nontrivial solution:

$$\theta' = (-q(x) + \lambda w(x)) \sin^2\theta + \frac{1}{p(x)} \cos^2\theta \tag{66}$$

$$\theta(a) = \theta_a \in [0, \pi) \tag{67}$$

$$\theta(b) = \theta_b \in [n\pi, (n+1)\pi) \tag{68}$$

To solve the problem, first solve the equation (66) with the initial condition (67). We obtain $\theta(x; \lambda)$ depending on the parameter λ, chosen large enough so that $-q(x) + \lambda w(x) > 0$ on $[a, b]$.

The solution $\theta(x; \lambda)$ has the following properties:

1) is continuous in λ (since g depends continuously on the parameter λ) and

is increasing in λ (since θ' is increasing in λ);
2) for x fixed $\lim_{\lambda\to\infty}\theta(x;\lambda) = +\infty$ (since θ' is increasing in λ, without any bound);
3) for x fixed $\lim_{\lambda\to-\infty}\theta(x;\lambda) = 0$ (because for large λ equation (63) is approximately $\theta' \approx \lambda w(x)\sin^2\theta$ with the solution $\cot\theta \approx -\lambda\int_c^x w$. In the limit $\lambda \to -\infty$ then $\cot\theta \to -\infty \Rightarrow \theta \to 0$).

Therefore, for some $\lambda = \lambda_n$, condition (68) is satisfied.

We thus find an increasing sequence of eigenvalues λ_n, with $\lambda_n \to \infty$.

8.4. **The distance between two consecutive zeroes of the eigenfunction u_n.** Using the formula (65) for $g(x) = -q(x) + \lambda_n w(x)$ we obtain that the distance between two consecutive zeroes of the eigenfunction u_n is no smaller that $d_n = \pi/M_n$ where

$$M_n = \max\{[-q(x) + \lambda_n w(x)]\sin^2\theta + \frac{1}{p(x)}\cos^2\theta \,\big|\, x \in [a,b],\, \theta \in [0,2\pi]\}$$

(Note that this estimate can be used only when p does not vanish at the endpoints of the interval $[a,b]$.)

9. More examples of separation of variables

9.1. **The wave equation.** Vibrations of a string and propagation of waves is modeled by the wave equation

$$u_{tt} = c^2\, u_{xx} \tag{69}$$

It can be easily checked that if $h_{1,2}$ are two arbitrary twice differentiable functions then

$$u(x,t) = h_1(x-ct) + h_2(x+ct) \tag{70}$$

satisfies the wave equation and therefore (70) is the general solution of the wave equation. Note that c represents the speed of propagation of the wave.

Let us solve (69) for $x \in [0,L]$ and $t > 0$. For this we need the boundary conditions at $x = 0$ and at $x = L$, which we take for simplicity to be the homogeneous Dirichlet problem:

$$u(0,t) = 0, \quad u(L,t) = 0 \tag{71}$$

and we also need initial conditions, at $t = 0$. Since (69) is second order in t we need the initial positions and the initial velocity. Therefore the conditions are

$$u(x,0) = f(x), \quad u_t(x,0) = g(x) \tag{72}$$

9.1.1. *Separation of variables.* Looking for solutions of (69) in the form $u(x,t) = X(x)T(t)$ the PDE becomes

$$T''(t)X(x) = c^2T(t)X''(x) \quad \text{therefore} \quad \frac{T''(t)}{c^2T(t)} = \frac{X''(x)}{X(x)} = \text{constant} = -\lambda$$

(we include c^2 with T just for the convenience of solving the equation for X). The boundary conditions (71) imply that

$$X(0) = 0, \quad X(L) = 0 \tag{73}$$

We solved the Sturm-Liouville problem

$$X''(x) + \lambda X(x) = 0$$

with (73); the solutions are

$$\lambda = \lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2 \ldots, \qquad X_n(x) = \sin\frac{n\pi x}{L} \tag{74}$$

Next solve

$$T''(t) + c^2\lambda_n T(t) = 0$$

which gives

$$T(t) = T_n(t) = a_n \cos\frac{cn\pi t}{L} + b_n \sin\frac{cn\pi t}{L} \tag{75}$$

(note the undetermined constants a_n, b_n). We obtained the solutions

$$u(x,t) = u_n(x,t) = \left[a_n \cos\frac{cn\pi t}{L} + b_n \sin\frac{cn\pi t}{L}\right] \sin\frac{n\pi x}{L} \tag{76}$$

which represent the modes of vibration: the mode u_n has frequency $\omega_n = \frac{1}{2\pi}\frac{cn\pi}{L} = \frac{c}{2L}$ cycles per unit time, and it is called *the nth harmonic.* The harmonic with $n = 1$ is called *the fundamental frequency, or first harmonic.* In the case of the vibrating sting all the harmonic frequency are multiples of the fundamental one.

9.1.2. *Superposition.* Since the PDE (69) is linear, then any sum of solutions (76) is again a solution, so let

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos\frac{cn\pi t}{L} + b_n \sin\frac{cn\pi t}{L}\right] \sin\frac{n\pi x}{L} \tag{77}$$

9.1.3. *Initial conditions.* We now require that the solution (77) satisfies (72):

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} = f(x) \tag{78}$$

and

$$u_t(x,0) = \sum_{n=1}^{\infty} b_n \frac{cn\pi}{L} \sin\frac{n\pi x}{L} = g(x) \tag{79}$$

Assuming that $f, g \in L^2[0,L]$ we can expand them as a sine-series (recall that $\sin\frac{n\pi x}{L}$ are eigenfunctions of a selfadjoint operator satisfying the hypothesis of the Completeness Theorem 4), and

$$a_n = \frac{\langle \sin\frac{n\pi x}{L}, f\rangle}{\|\sin\frac{n\pi x}{L}\|^2}, \quad b_n = \frac{L}{cn\pi}\,\frac{\langle \sin\frac{n\pi x}{L}, g\rangle}{\|\sin\frac{n\pi x}{L}\|^2}$$

Since

$$\|\sin\frac{n\pi x}{L}\|^2 = \int_0^L \sin^2\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_0^L\left(1-\cos\frac{2n\pi x}{L}\right)dx = \frac{L}{2}$$

we obtain

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx, \qquad b_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx \tag{80}$$

9.1.4. *Connection with the general solution.* To see how the solution (77), (80) is related to the general solution (70) use the trigonometric formulas $2\sin A\cos B = \sin(A+B)+\sin(A-B)$ and $2\sin A\sin B = \cos(A-B) - \cos(A+B)$ to rewrite (77) as

$$\begin{aligned} u(x,t) = \sum_{n=1}^{\infty}\Bigg[\frac{a_n}{2}\left(\sin\frac{(x+ct)n\pi}{L} + \sin\frac{(x-ct)n\pi}{L}\right) \\ +\frac{b_n}{2}\left(\cos\frac{(x-ct)n\pi}{L} - \cos\frac{(x+ct)n\pi}{L}\right)\Bigg] \\ \equiv \frac{1}{2}\left[F(x+ct)+G(x-ct)\right] \end{aligned}$$

where

$$F(x) = \sum_{n=1}^{\infty}\left(a_n\,\sin\frac{nx\pi x}{L} - b_n\,\cos\frac{nx\pi x}{L}\right)$$

$$G(x) = \sum_{n=1}^{\infty}\left(a_n\,\sin\frac{nx\pi x}{L} + b_n\,\cos\frac{nx\pi x}{L}\right)$$

Note that the graph of $F(x-ct)$ (for any fixed t) is the same as the graph of $F(x)$ (the initial shape), only shifted to the right by ct: it represents the initial "wave" traveling to the right with speed c. Similarly, $F(x+ct)$ represents the initial "wave" traveling to the left with speed c.

9.2. **Laplace's equation.** The two-dimensional heat equation (modeling the temperature distribution in a lamina) is

$$u_t = \alpha^2\left(u_{xx}+u_{yy}\right)$$

The stationary solutions (for which $u(x,y,t)\equiv u(x,y)$) satisfy Laplace's equation

$$u_{xx}+u_{yy} = 0 \tag{81}$$

It is clear that in order to solve (81) we need information about the temperature on the boundary of the lamina.

Consider as example a lamina in the shape of a seminfinite vertical strip: $0 \leq x \leq L$ and $y \geq 0$ and the Dirichlet problem: we need the temperature along the segment $(x,0)$ with $0 \leq x \leq L$, along the half lines $(0,y),\ y \geq 0$ and $(L,y),\ y \geq 0$ and for $y \to +\infty$:

$$\begin{array}{ll} u(x,0) = f(x) & \text{for } 0 \leq x \leq L \\ u(0,y) = 0 & \text{for } y \geq 0 \\ u(L,y) = 0 & \text{for } y \geq 0 \\ \lim_{y\to+\infty} u(x,y) = 0 & \text{for } 0 \leq x \leq L \end{array} \tag{82}$$

9.2.1. *Separation of variables.* Looking for solutions of (81) in the form $u(x,Y) = X(x)Y(y)$ the PDE becomes

$$X''(x)Y(y) + X(x)Y''(y) = 0 \quad \text{therefore } \frac{X''(x)}{X(x)} = \frac{-Y''(y)}{Y(y)} = -\lambda$$

Solving for $X(x)$ we obtain (74) then solving for $Y(y)$:

$$Y(y) = Y_n(y) = a_n \exp(-n\pi y/L) + b_n \exp(n\pi y/L)$$

The last condition in (82) implies that $b_n = 0$.

9.2.2. *Solving the problem by superposition.* Since the equation (81) is linear, then a sum of the solutions with separated variables is again a solution:

$$u(x,y) = \sum_{n=1}^{\infty} X_n(x)Y_n(y) = \sum_{n=1}^{\infty} a_n \exp(-n\pi y/L)\, \sin\frac{n\pi x}{L}$$

Requiring that the first condition in (82) be satisfied we obtain

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} = f(x)$$

If $f \in L^2[0,L]$ then a_n are found by (80).

9.3. **The vibrating rod.** Transverse vibrations of a homogeneous rod are described by the equation

$$u_{xxxx} + u_{tt} = 0 \tag{83}$$

Assume that the rest position of the rod is for $0 \leq x \leq \pi$.

The problem needs two initial conditions in t and four conditions in x (it has order four in x).

The boundary conditions come from the system studied. For example, they could be

1) free ends: $X'' = X''' = 0$ for $x = 0$ and $x = \pi$, or
2) supported ends: $X = X'' = 0$ for $x = 0$ and $x = \pi$, or
3) clamped ends: $X = X' = 0$ for $x = 0$ and $x = \pi$, or
4) $X' = X''' = 0$ for $x = 0$ and $x = \pi$, or
5) periodicity: $X(0) = X(\pi),\ X'(0) = X'(\pi),\ X''(0) = X''(\pi),\ X'''(0) = X'''(\pi)$.

Let us solve the problem with free ends:

$$X'' = X''' = 0 \text{ for } x = 0,\ x = \pi$$

As before, we solve by separation of variable: $u(x,t) = X(x)T(t)$ which gives

$$\frac{X^{(IV)}(x)}{X(x)} = \frac{-T''(t)}{T(t)} = \lambda$$

Can we expect a complete set of eigenfunctions?

Consider the operator $L = \frac{d^4}{dx^4}$ in $H = L^2[0,\pi]$ on the domain

$$D = \{f \in H \,|\, f', f'', f''', f^{(IV)} \in H,\ f''(0) = f'''(0) = 0,\ f''(\pi) = f'''(\pi) = 0\}$$

Is the operator (formally) selfadjoint on D?

For $f, g \in D$ calculate

$$\langle Lf, g\rangle = \int_0^\pi f^{(IV)}(x)g(x)\,dx = f'''g\big|_0^\pi - \int_0^\pi f^{'''}(x)g'(x)\,dx$$

$$= -\int_0^\pi f^{'''}(x)g'(x)\,dx = f''g'\big|_0^\pi + \int_0^\pi f^{''}(x)g''(x)\,dx$$

$$= \int_0^\pi f^{''}(x)g''(x)\,dx = f'g''\big|_0^\pi - \int_0^\pi f^{'}(x)g'''(x)\,dx \tag{84}$$

$$-\int_0^\pi f^{'}(x)g'''(x)\,dx = fg'''\big|_0^\pi + \int_0^\pi f(x)g^{(IV)}(x)\,dx = \langle f, Lg\rangle$$

therefore L is indeed (formally) selfadjoint on D.

Note also that L is positive semidefinite, so the eigenvalues λ are nonnegative. Indeed we have from (84) that

$$\langle Lf, f\rangle = \int_0^\pi |f^{''}(x)|^2\,dx \geq 0$$

and $\langle Lf, f\rangle = 0$ implies $f^{''}(x) = 0$ hence $f(x) = a + bx \in D$.

To calculate the eigenfunctions we find the general solution of the ODE.

For $\lambda \neq 0$ denote $\nu = \lambda^{1/4}$ and then

$$X(x) = c_1\cos(\nu x) + c_2\sin(\nu x) + c_3e^{\nu x} + c_4e^{-\nu x} \tag{85}$$

and for $\lambda = 0$

$$X(x) = d_1 + d_2x + d_3x^2 + d_4x^4$$

which belongs to D if $d_3 = d_4 = 0$. Hence the eigenspace corresponding to $\lambda = 0$ is two-dimansional, consisting of linear functions $X_0(x) = d_1 + d_2x$.

Imposing the boundary condition in (85) we obtain a linear system of four equations for the four constants $c_1, \ldots, c_4$. The condition that this system has a nontrivial solution is that its determinant be zero. Calculation of this determinant yields the condition

$$\cosh\nu\pi\,\cos\nu\pi = 1 \tag{86}$$

which determines λ. However, (86) is a transcendental equation for ν (we cannot solve it explicitly). We can deduct that there is a sequence of solutions ν tending to $+\infty$ in the following way. Rewrite (86) as

$$\cosh x = \frac{1}{\cos x}, \qquad (x = \nu\pi) \tag{87}$$

The function $\cosh x$ is increasing and greater than 1 for $x > 0$. The graph of the right-hand side of (87) has vertical asymptotes. On intervals $(-\pi/2 + 2n\pi, 2n\pi]$ (n integer) it decreases from $+\infty$ to 1, and on intervals $[2n\pi, \pi/2 + 2n\pi)$ it increases from 1 to $+\infty$. Therefore, in each of these intervals there is a solution of equation (87). On the other intervals $1/\cos x$ is negative and equation (87) has no solutions.

A sequence of nonnegative eigenvalues λ_n do exist and $\lambda_n \to \infty$.The Completeness Theorem applies and the eigenfunctions are complete in $L^2[0\pi]$.

The equation is then solved using specified initial conditions.

10. An introduction to the Fourier transform

10.1. **The space $L^2(\mathbb{R})$.** Consider any function in the vector space of continuous functions which are zero outside a closed interval:

$$C_0(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} \,|\, f \text{continuous}, \ f(x) = 0 \text{ for all } x \notin [a, b] \text{ for some } a < b\}$$

(they are called *contiuous functions with compact support*). If $f \in C_0$ vanished outside some $[a, b]$ then clearly it has finite $L^2(\mathbb{R})$-norm since

$$\|f\|^2 = \int_{-\infty}^{+\infty} |f(x)|^2 \, dx = \int_a^b |f(x)|^2 \, dx < \infty$$

The space $L^2(\mathbb{R})$ is defined as the completion of C_0 with respect to the $L^2(\mathbb{R})$-norm

$$\|f\| = \left(\int_{-\infty}^{+\infty} |f(x)|^2 \, dx \right)^{1/2}$$

and it is a Hilbert space with respect to the inner product

$$\langle f, g \rangle = \int_{-\infty}^{+\infty} \overline{f(x)} g(x) \, dx$$

(which is finite by the Cauchy-Schwarz inequality).

The Hilbert space $L^2(\mathbb{R})$ has many features common to $L^2[a, b]$; for example, if two functions differ at only a number of points (finitely many, or countably many) are considered equal in L^2.

One novel feature is that, for a function, even continuous on $\mathbb{R}$, to belong to $L^2(\mathbb{R})$, this function needs *to decay to zero fast enough for $x \to \pm\infty$* (so that the improper integral converges).

For example, consider functions decaying like a power, say $f(x) = 1/(1 + |x|^a)$. For which a is such a function in $L^2(\mathbb{R})$?

For large x, $f(x) \sim x^{-a}$ and $|f(x)|^2 \sim x^{-2a}$ which integrated gives a multiple of x^{1-2a}. The improper integral converges if $1-2a<0$, therefore for $a>1/2$.

Other examples of functions belonging to $L^2(\mathbb{R})$ are $e^{-|x|}$, e^{-x^2}.

10.2. **The Fourier Transform.** For $f \in C_0$ its Fourier transform, $\mathcal{F}f$, is defined as

$$\hat{f}(\xi) = (\mathcal{F}f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} f(x)\, dx \tag{88}$$

It can be shown that $\|\hat{f}\| = \|f\|$ (Parseval's identity) and that the Fourier transform can be extended as a unitary operator from $L^2(\mathbb{R})$ to itself, and that its inverse is:

$$f(x) = (\mathcal{F}^{-1}\hat{f})(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} \hat{f}(\xi)\, d\xi \tag{89}$$

Note the similarity with Fourier series, only we have an integral instead of a series: recall $f = \sum_n e^{inx} f_n$.

Note also the similarity with matrix multiplication: if $U_{\xi,x} = e^{-i\xi x}$ and $f = f_x$ then (88) is like $\sum_x U_{\xi,x} f_x$ while the inverse of U, which equals its adjoint, is $U^*_{x,\xi} = \overline{U_{\xi,x}} = e^{i\xi x}$.

Remark. Some books define the Fourier transform (88) without the prefactor $\frac{1}{\sqrt{2\pi}}$. In this case, the transform is no longer a unitary operator, and the inverse (89) must have the prefactor $\frac{1}{2\pi}$.

10.3. **The Fourier transform diagonalizes the diffferentiation operator.** Indeed, consider the linear operator $\frac{d}{dx}$ in $L^2(\mathbb{R})$, defined on the domain

$$D = C_0^1 \equiv \{f \in C_0 \,|\, f' \in C_0\}$$

Just like for finite intervals, the space C_0^1 is dense in $L^2(\mathbb{R})$.

For $f, g \in C_0^1$ we have

$$\langle \frac{d}{dx} f, g\rangle = \int_{-\infty}^{\infty} \overline{f'(x)} g(x)\, dx = \overline{f(x)} g(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \overline{f(x)} g'(x)\, dx = -\langle f, \frac{d}{dx} g\rangle$$

and therefore $\frac{d}{dx}$ is skew-symmetric (hence its eigenvalues - if any!- are purely imaginary).

We can see that $e^{ix\xi}$ (for $\xi \in \mathbb{R}$) are eigenfunctions of $\frac{d}{dx}$. However they are *generalized eigenfunctions* , since they do not belong to the Hilbert space $L^2(\mathbb{R})$, and we have expansions as integrals (88) rather than series. Indeed,

$$\frac{d}{dx} e^{ix\xi} = i\xi e^{ix\xi}$$

Moreover, integrating by parts we find that

$$(\mathcal{F}\frac{d}{dx}f)(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ix\xi}f'(x)\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} i\xi e^{-ix\xi}f(x)\,dx = i\xi(\mathcal{F}f)(\xi)$$

or

$$\widehat{\frac{df}{dx}} = i\xi\hat{f} \tag{90}$$

or $\mathcal{F}\frac{d}{dx}\mathcal{F}^{-1}$ is the operator of multiplication by $i\xi$ - hence it is diagonal!.

10.4. **An example.** Problem: solve the initial value problem for the heat equation on the line

$$u_t = \alpha u_{xx}, \quad \text{for } x\in\mathbb{R},\ t>0 \tag{91}$$

with

$$u(x,0) = u_0(x) \in L^2(\mathbb{R}) \tag{92}$$

Take the Fourier transform in x in the heat equation: denoting

$$\hat{u}(\xi,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ix\xi}u(x,t)\,dx$$

we obtain

$$\hat{u}_t = \alpha\widehat{u_{xx}}$$

which gives using (90)

$$\hat{u}_t = -\xi^2\alpha\hat{u}$$

which is an ODE in t whose general solution is

$$\hat{u}(\xi,t) = F(\xi)e^{-\xi^2\alpha t} \tag{93}$$

Using the initial condition (92) we see that we must have $F(\xi) = \widehat{u_0}(\xi)$ and taking the inverse Fourier transform in (93) we obtain the solution

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ix\xi-\xi^2\alpha t}\widehat{u_0}(\xi)\,d\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\xi-\xi^2\alpha t}\int_{-\infty}^{\infty} e^{-iy\xi}u_0(y)\,dy\,d\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} K(x,y,t)\,u_0(y)\,dy \end{aligned}$$

where

$$K(x,y,t) = \int_{-\infty}^{\infty} e^{ix\xi-\xi^2\alpha t-iy\xi}\,d\xi$$

which is easily calculated by completing the squares:

$$\xi^2\alpha t - i(x-y)\xi = \left(\xi\sqrt{\alpha}\sqrt{t} - i\frac{x-y}{2\sqrt{\alpha}\sqrt{t}}\right)^2 + \frac{(x-y)^2}{4\alpha t}$$

therefore

$$K(x,y,t) = \frac{1}{\sqrt{4\pi\alpha t}}\,e^{-\frac{(x-y)^2}{4\alpha t}}$$

and therefore the solution to $(91), (92)$ is

$$u(x,t) = \frac{1}{\sqrt{4\pi\alpha t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\alpha t}} u_0(y)\, dy$$