1.138J/2.062J/18.376J, WAVE PROPAGATION

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CHAPTER FOUR. GRAVITY WAVES IN WATER

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1 Governing equations for waves on the sea surface

In this chapter we shall model the water as an inviscid and incompressible fluid, and consider waves of infinitesimal amplitude so that the linearized approximation suffices.

Recall in the first chapter that when compressibility is included the velocity potential defined by $\mathbf{u} = \nabla\Phi$ is governed by the wave equation:

$$\nabla^2\Phi = \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} \tag{1.1}$$

where $c = \sqrt{dp/d\rho}$ is the speed of sound. Consider the ratio

$$\frac{\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}}{\nabla^2\Phi} \sim \frac{\omega^2/k^2}{c^2}$$

As will be shown later, the phase speed of the fastest wave is $\omega/k = \sqrt{gh}$ where g is the gravitational acceleration and h the sea depth. Now h is at most 4000 m in the ocean, and the sound speed in water is $c = 1400$ m/sec^2, so that the ratio above is at most

$$\frac{40000}{1400^2} = \frac{1}{49} \ll 1$$

We therefore approximate (1.1) by

$$\nabla^2\Phi = 0 \tag{1.2}$$

Let the free surface be $z = \zeta(x, y, t)$. Then for a gently sloping free surface the vertical velocity of the fluid on the free surface must be equal to the vertical velocity of the surface itself. i.e.,

$$\frac{\partial\zeta}{\partial t} = \frac{\partial\Phi}{\partial z}, \quad z = 0. \tag{1.3}$$

Having to do with the velocity only, this is called the *kinematic boundary condition.*

For small amplitude motion, the linearized momentum equation reads

$$\rho\frac{\partial\mathbf{u}}{\partial t} = -\nabla P - \rho g\mathbf{e}_z \tag{1.4}$$

Now let the total pressure be split into static and dynamic parts

$$P = p_o + p \tag{1.5}$$

where p_o is the static pressure

$$p_o = -\rho g z \tag{1.6}$$

which satisfies

$$0 = -\nabla p_o + -\rho g\mathbf{e}_z \tag{1.7}$$

Let us assume that wind is not present, so that the air above the sea surface is essentially stagnant. The static pressure is hydrostatic, which can be taken to be zero at $z = 0$ without loss of generality. It follows that

$$\rho\frac{\partial\mathbf{u}}{\partial t} = \rho\frac{\partial\nabla\Phi}{\partial t} = -\nabla p \tag{1.8}$$

so that

$$p = -\rho\frac{\partial\Phi}{\partial t} \tag{1.9}$$

Because of its very small density the passive air motion due to surface waves has negible effect on wate motion. We assume the dynamic air pressure to be zero on the free surface.

It will be shown shortly that for sufficiently long waves, surface tension can be ignored. Continuity of pressure requires that

$$p = 0, \quad z = \zeta.$$

to the leading order of approximation, we have, therefore

$$\rho g \zeta + \rho \frac{\partial \Phi}{\partial t} = 0, \quad z = 0. \tag{1.10}$$

Being a statement on forces, this is called the *dynamic boundary condition.* The two conditions (1.3) and (1.10) can be combined to give

$$\frac{\partial^2 \Phi}{\partial t^2} + g \frac{\partial \Phi}{\partial z} = 0, \quad z = 0 \tag{1.11}$$

If surface tension is also included then we adopt the model where there is a thin film covering the water surface with tension T per unit length. Consider a horizontal rectangle $dxdy$ on the free surface. The net vertical force on water from four sides of the small rectangle is

$$\left(T \frac{\partial \zeta}{\partial x}\bigg|_{x+dx} - T \frac{\partial \zeta}{\partial x}\bigg|_{x} \right) dy + \left(T \frac{\partial \zeta}{\partial y}\bigg|_{y+dy} - T \frac{\partial \zeta}{\partial x}\bigg|_{y} \right) dx = T \left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} \right) dx\, dy$$

Continuity of vertical force on an unit area of the surface requires

$$p + T \left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} \right) = 0.$$

Hence

$$-\rho g \zeta - \rho \frac{\partial \Phi}{\partial t} + T \left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} \right) = 0, \quad z = 0. \tag{1.12}$$

which can be combined with the kinematic condition (1.3) to give

$$\frac{\partial^2 \Phi}{\partial t^2} + g \frac{\partial \Phi}{\partial z} - \frac{T}{\rho} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \frac{\partial \Phi}{\partial z} = 0, \quad z = 0 \tag{1.13}$$

When viscosity is neglected, the normal fluid velocity vanishes on the rigid seabed,

$$\mathbf{n} \cdot \nabla \Phi = 0 \tag{1.14}$$

Let the sea bed be $z = -h(x, y)$ then the unit normal is

$$\mathbf{n} = \frac{(h_x, .h_y, 1)}{\sqrt{1 + h_x^2 + h_y^2}} \tag{1.15}$$

Hence

$$\frac{\partial \Phi}{\partial z} = -\frac{\partial h}{\partial x} \frac{\partial \Phi}{\partial x} - \frac{\partial h}{\partial y} \frac{\partial \Phi}{\partial y}, \quad z = -h(x, y) \tag{1.16}$$

2 Progressive waves on a sea of constant depth

2.1 The velocity potential

Consider the simplest case of constant depth so that

$$\frac{\partial \Phi}{\partial z} = 0, \quad z = -h. \tag{2.1}$$

and sinusoidal waves with infinitively long crests parallel to the y axis. The motion is in the vertical plane (x, z). Let us seek a solution representing a wavetrain advancing along the x direction with frequency ω and wave number k,

$$\Phi = f(z)e^{ikx - i\omega t} \tag{2.2}$$

In order to satisfy (1.2), (1.13) and (2.1) we need

$$f'' + k^2 f = 0, \quad -h < z < 0 \tag{2.3}$$

$$-\omega^2 f + gf' + \frac{T}{\rho}k^2 f' = 0, \quad z = 0, \tag{2.4}$$

$$f' = 0, \quad z = -h \tag{2.5}$$

Clearly the solution to (2.3) and (2.5) is

$$f(z) = B\cosh k(z+h)$$

implying

$$\Phi = B\cosh k(z+h)e^{ikx - i\omega t} \tag{2.6}$$

In order to satisfy (2.4) we require

$$\omega^2 = \left(gk + \frac{T}{\rho}k^3\right)\tanh kh \tag{2.7}$$

which is the dispersion relation between ω and k. From (1.3) we get

$$\frac{\partial \zeta}{\partial t} = \left.\frac{\partial \Phi}{\partial z}\right|_{z=0} = (Bk\sinh kh)e^{ikx - i\omega t} \tag{2.8}$$

Upon integration,

$$\zeta = Ae^{ikx - i\omega t} = \frac{Bk\sinh kh}{-i\omega}e^{ikx - i\omega t} \tag{2.9}$$

where A denotes the surface wave amplitude, it follows that

$$B = \frac{-i\omega A}{k \sinh kh}$$

and

$$\begin{aligned}
\Phi &= \frac{-i\omega A}{k \sinh kh} \cosh k(z+h) e^{ikx-i\omega t} \\
&= \frac{-igA}{\omega}\left(1+\frac{Tk^2}{g\rho}\right)\frac{\cosh k(z+h)}{\cosh kh} e^{ikx-i\omega t}
\end{aligned} \tag{2.10}$$

2.2 The dispersion relation

Let us first examine the dispersion relation (2.7), where three lengths are present : the depth h, the wavelength $\lambda = 2\pi/k$, and the length $\lambda_m = 2\pi/k_m$ with

$$k_m = \sqrt{\frac{g\rho}{T}}, \quad \lambda_m = \frac{2\pi}{k_m} = 2\pi\sqrt{\frac{T}{g\rho}} \tag{2.11}$$

For reference we note that on the air-water interface, $T/\rho = 74\, cm^3/s^2, g = 980\, cm/s^2$, so that $\lambda_m = 1.73 cm$. The depth of oceanographic interest ranges from O(10cm) to thousand of meters. The wavelength ranges from a few centimeters to hundreds of meters.

Let us introduce

$$\omega_m^2 = 2gk_m = 2g\sqrt{\frac{g\rho}{T}} \tag{2.12}$$

then (2.7) is normalized to

$$\frac{\omega^2}{\omega_m^2} = \frac{1}{2}\frac{k}{k_m}\left(1+\frac{k^2}{k_m^2}\right)\tanh kh \tag{2.13}$$

Consider first waves of length of the order of λ_m. For depths of oceanographic interest, $h \gg \lambda$, or $kh \gg 1$, $\tanh kh \approx 1$. Hence

$$\frac{\omega^2}{\omega_m^2} = \frac{1}{2}\frac{k}{k_m}\left(1+\frac{k^2}{k_m^2}\right) \tag{2.14}$$

or, in dimensional form,

$$\omega^2 = gk + \frac{Tk^3}{\rho} \tag{2.15}$$

The phase velocity is

$$c = \frac{\omega}{k} = \sqrt{\frac{g}{k}\left(1+\frac{Tk^2}{g\rho}\right)} \tag{2.16}$$

Defining

$$c_m = \frac{\omega_m}{k_m} \tag{2.17}$$

the preceding equation takes the normalized form

$$\frac{c}{c_m} = \sqrt{\frac{1}{2}\left(\frac{k_m}{k} + \frac{k}{k_m}\right)} \tag{2.18}$$

Clearly

$$c \approx \sqrt{\frac{Tk}{\rho}}, \quad \text{if } \; k/k_m \gg 1, \quad \text{or} \quad \lambda/\lambda_m \ll 1 \tag{2.19}$$

Thus for wavelengths much shorter than 1.7 cm, capillarity alone is important, These are called the capillary waves. On the other hand

$$c \approx \sqrt{\frac{g}{k}}, \quad \text{if } \; k/k_m \ll 1, \quad \text{or} \quad \lambda/\lambda_m \ll 1 \tag{2.20}$$

Thus for wavelength much longer than 1.73 cm, gravity alone is important; these are called the gravity waves. Since in both limits, c becomes large, there must be a minimum for some intermediate k. From

$$\frac{dc^2}{dk} = -\frac{g}{k^2} + \frac{T}{\rho} = 0$$

the minimum c occurs when

$$k = \sqrt{\frac{g\rho}{T}} = k_m, \quad \text{or } \lambda = \lambda_m \tag{2.21}$$

The smallest value of c is c_m. For the intermediate range where both capillarity and gravity are of comparable importance; the dispersion relation is plotted in figure (1).

Next we consider longer gravity waves where the depth effects are essential.

$$\omega = \sqrt{gk \tanh kh} \tag{2.22}$$

For gravity waves on deep water, $kh \gg 1$, $\tanh kh \to 1$. Hence

$$\omega \approx \sqrt{gk}, \quad c \approx \sqrt{\frac{g}{k}} \tag{2.23}$$

Thus longer waves travel faster. These are also called short gravity waves. If however the waves are very long or the depth very small so that $kh \ll 1$, then $\tanh kh \sim kh$ and

$$\omega \approx k\sqrt{gh}, \quad c \approx \sqrt{gh} \tag{2.24}$$

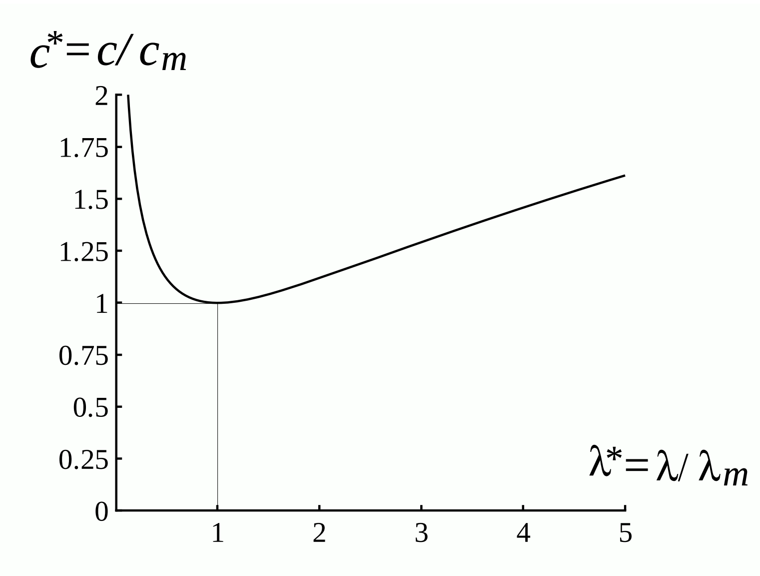

Figure 1: Phase speed of capillary-gravity waves in water much deeper than λ_m.

Form intermediate values of kh, the phase speed decreases monotonically with increasing kh. All long waves with $kh \ll 1$ travel at the same maximum speed limited by the depth, $\sqrt{gh}$, hence there are non-dispersive. The dispersion relation is plotted in figure (**??**).

2.3 The flow field

For arbitrary k/k_m and kh, the velocities and dynamic pressure are easily found from the potential (2.10) as follows

$$u = \frac{\partial \Phi}{\partial x} = \frac{gkA}{\omega}\left(1 + \frac{Tk^2}{g\rho}\right)\frac{\cosh k(z+h)}{\cosh kh} e^{ikx - i\omega t} \tag{2.25}$$

$$w = \frac{\partial \Phi}{\partial z} = \frac{-igkA}{\omega}\left(1 + \frac{Tk^2}{g\rho}\right)\frac{\sinh k(z+h)}{\cosh kh} e^{ikx - i\omega t} \tag{2.26}$$

$$p = -\rho\frac{\partial \Phi}{\partial t} = \rho g A\left(1 + \frac{Tk^2}{g\rho}\right)\frac{\cosh k(z+h)}{\cosh kh} e^{ikx - i\omega t} \tag{2.27}$$

Note that all these quantities decay monotonically in depth.

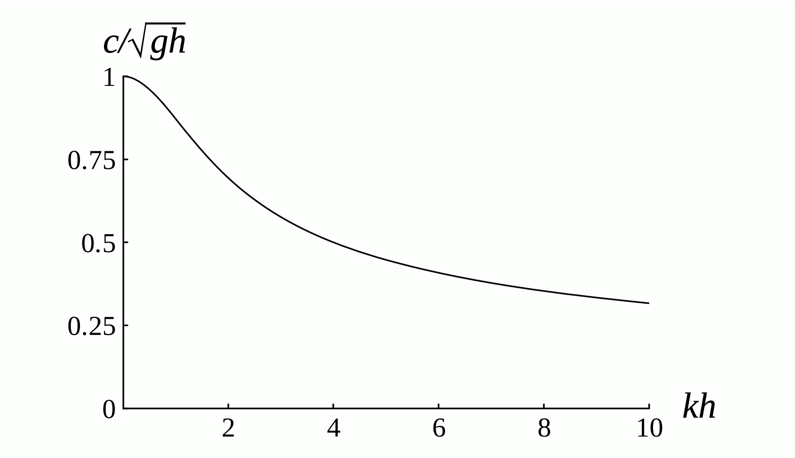

Figure 2: Phase speed of capillary-gravity waves in water of constant depth

In deep water, $kh \gg 1$,

$$u = \frac{gkA}{\omega}\left(1+\frac{Tk^2}{g\rho}\right)e^{kz}e^{ikx-i\omega t} \tag{2.28}$$

$$w = \frac{\partial\Phi}{\partial z} = \frac{-igkA}{\omega}\left(1+\frac{Tk^2}{g\rho}\right)e^{kz}e^{ikx-i\omega t} \tag{2.29}$$

$$p = -\rho\frac{\partial\Phi}{\partial t} = \rho gA\left(1+\frac{Tk^2}{g\rho}\right)e^{kz}e^{ikx-i\omega t} \tag{2.30}$$

All dynamical quantities diminish exponentially to zero as $kz \to -\infty$. Thus the fluid motion is limited to the surface layer of depth $O(\lambda)$. Gravity and capillary-gravity waves are therefore surface waves.

For pure gravity waves in shallow water, $T = 0$ and $kh \ll 1$, we get

$$u = \frac{gkA}{\omega}e^{ikx-i\omega t} \tag{2.31}$$

$$w = 0, \tag{2.32}$$

$$p = -\rho\frac{\partial\Phi}{\partial t} = \rho gAe^{ikx-i\omega t} = \rho g\zeta \tag{2.33}$$

Note that the horizontal velocity is uniform in depth while the vertical velocity is negligible. Thus the fluid motion is essentially horizontal. The total pressure

$$P = p_o + p = \rho g(\zeta - z) \tag{2.34}$$

is hydrostatic and increases linearly with depth from the free surface.

2.4 The particle orbit

In fluid mechanics there are two ways of describing fluid motion. In the Lagrangian scheme, one follows the trajectory x, z of all fluid particles as functions of time. Each fluid particle is identified by its static or initial position x_o, z_o. Therefore the instantaneous position at time t depends parametrically on x_o, z_o. In the Eulerian scheme, the fluid motion at any instant t is described by the velocity field at all fixed positions x, z. As the fluid moves, the point x, z is occupied by different fluid particles at different times. At a particular time t, a fluid particle originally at (x_o, z_o) arrives at x, z, hence its particle velocity must coincide with the fluid velocity there,

$$\frac{dx}{dt} = u(x,z,t), \quad \frac{dz}{dt} = w(x,z,t) \tag{2.35}$$

Once u, w are known for all x, z, t, we can in principle integrate the above equations to get the particle trajectory. This Euler-Lagrange problem is in general very difficult.

In small amplitude waves, the fluid particle oscillates about its mean or initial position by a small distance. Integration of (2.35) is relatively easy. Let

$$x(x_o,z_o,t) = x_o + x'(x_o,z_o,t), \quad \text{and} z(x_o,z_o,t) = z_o + x'(x_o,z_o,t) \tag{2.36}$$

then $x' \ll x, z' \ll z$ in general. Equation (2.35) can be approximated by

$$\frac{dx'}{dt} = u(x_o,z_o,t), \quad \frac{dz'}{dt} = w(x_o,z_o,t) \tag{2.37}$$

From (2.25) and (2.26), we get by integration,

$$\begin{aligned} x' &= \frac{gkA}{-i\omega^2}\left(1+\frac{Tk^2}{g\rho}\right)\frac{\cosh k(z_o+h)}{\cosh kh}e^{ikx_o-i\omega t} \\ &= -\frac{gkA}{\omega^2}\left(1+\frac{Tk^2}{g\rho}\right)\frac{\cosh k(z_o+h)}{\cosh kh}\sin(kx_o-\omega t) \end{aligned} \tag{2.38}$$

(2.39)

$$\begin{aligned} z' &= \frac{gkA}{\omega^2}\left(1+\frac{Tk^2}{g\rho}\right)\frac{\sinh k(z_o+h)}{\cosh kh}e^{ikx_o-i\omega t} \\ &= \frac{gkA}{\omega^2}\left(1+\frac{Tk^2}{g\rho}\right)\frac{\sinh k(z_o+h)}{\cosh kh}\cos(kx_o-\omega t) \end{aligned} \tag{2.40}$$

(2.41)

Letting

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{gkA}{\omega^2 \cosh kh}\left(1+\frac{Tk^2}{g\rho}\right)\begin{pmatrix} \cosh k(z_o+h) \\ \sinh k(z_o+h) \end{pmatrix} \tag{2.42}$$

we get

$$\frac{x'^2}{a^2} + \frac{z'^2}{b^2} = 1 \tag{2.43}$$

The particle trajectory at any depth is an ellipse. Both horizontal (major) and vertical (minor) axes of the ellipse decrease monotonically in depth. The minor axis diminishes to zero at the seabed, hence the ellipse collapses to a horizontal line segment. In deep water, the major and minor axes are equal

$$a = b = \frac{gkA}{\omega^2}\left(1 + \frac{Tk^2}{g\rho}\right) e^{kz_o}, \tag{2.44}$$

therefore the orbits are circles with the radius diminishing exponentially with depth.

Also we can rewrite the trajectory as

$$\begin{aligned} x' &= \frac{gkA}{\omega^2}\left(1 + \frac{Tk^2}{g\rho}\right) \frac{\cosh k(z_o + h)}{\cosh kh} \sin(\omega t - kx_o) && (2.45) \\ z' &= \frac{gkA}{\omega^2}\left(1 + \frac{Tk^2}{g\rho}\right) \frac{\sinh k(z_o + h)}{\cosh kh} \sin(\omega t - kx_o - \frac{\pi}{2}) && (2.46) \end{aligned}$$

When $\omega t - kx_o = 0$, $x' = 0$ and $z' = b$. A quarter period later, $\omega t - k_o = \pi/2$, $x' = a$ and $z' = 0$. Hence as time passes, the particle traces the elliptical orbit in the clockwise direction.

2.5 Energy and Energy transport

Beneath a unit length of the free surface, the time-averaged kinetic energy density is

$$\bar{E}_k = \frac{\rho}{2} \int_{-h}^{0} dz \left(\overline{u^2} + \overline{w^2}\right) \tag{2.47}$$

whereas the instantaneous potential energy density is

$$E_p = \frac{1}{2}\rho g \zeta^2 + T\frac{(ds - dx)}{dx} = \frac{1}{2}\rho g \zeta^2 + T\left(\sqrt{1 + \zeta_x^2} - 1\right) = \frac{1}{2}\rho g \zeta^2 + T\zeta_x^2 \tag{2.48}$$

Hence the time-average is

$$\bar{E}_p = \frac{1}{2}\rho g \overline{\zeta^2} + \frac{T}{2}\overline{\zeta_x^2} \tag{2.49}$$

Let us rewrite (2.25) and (2.26) in (2.49):

$$\begin{aligned} u &= \Re\left\{ \frac{gkA}{\omega}\left(1 + \frac{Tk^2}{g\rho}\right) \frac{\cosh k(z + h)}{\cosh kh} e^{ikx} \right\} e^{-i\omega t} && (2.50) \\ w &= \Re\left\{ \frac{-igkA}{\omega}\left(1 + \frac{Tk^2}{g\rho}\right) \frac{\sinh k(z + h)}{\cosh kh} e^{ikx} \right\} e^{-i\omega t} && (2.51) \end{aligned}$$

Then

$$\begin{aligned}
\bar{E}_k &= \frac{\rho}{4}\left(\frac{gkA}{\omega}\right)^2\left(1+\frac{Tk^2}{g\rho}\right)^2\frac{1}{\cosh^2 kh}\int_{-h}^{0}dz\left[\cosh^2 k(z+h)+\sinh^2 k(z+h)\right] \\
&= \frac{\rho}{4}\left(\frac{gkA}{\omega}\right)^2\left(1+\frac{Tk^2}{g\rho}\right)^2\frac{\sinh 2kh}{2k\cosh^2 kh} = \frac{\rho}{4}\left(\frac{gkA}{\omega}\right)^2\left(1+\frac{Tk^2}{g\rho}\right)^2\frac{\sinh kh}{k\cosh kh} \\
&= \frac{\rho g A^2}{4}\left(1+\frac{Tk^2}{g\rho}\right)^2\frac{gk\tanh kh}{\omega^2} = \frac{\rho g A^2}{4}\left(1+\frac{Tk^2}{g\rho}\right) \qquad (2.52)
\end{aligned}$$

after using the dispersion relation. On the other hand,

$$\bar{E}_p = \frac{\rho g A^2}{4}\left(1+\frac{Tk^2}{\rho g}\right) \qquad (2.53)$$

Hence the total energy density is

$$\bar{E} = \bar{E}_k + \bar{E}_p = \frac{\rho g A^2}{2}\left(1+\frac{Tk^2}{\rho g}\right) = \frac{\rho g A^2}{2}\left(1+\frac{k^2}{k_m^2}\right) = \frac{\rho g A^2}{2}\left(1+\frac{\lambda_m^2}{\lambda^2}\right) \qquad (2.54)$$

Note that the total energy is equally divided between kinetic and potential energies; this is called the equipartition of energy.

We leave it as an exercise to show that the power flux (rate of energy flux) across a station x is

$$\frac{d\bar{E}}{dt} = \int_{-h}^{0}\overline{pu}\,dz - T\overline{\zeta_x\zeta_t} = -\rho\int_{-h}^{0}\overline{\Phi_t\Phi_x}\,dz - T\overline{\zeta_x\zeta_t} = \bar{E}c_g \qquad (2.55)$$

where c_g is the speed of energy transport , or the group velocity

$$c_g = \frac{d\omega}{dk} = \frac{c}{2}\left\{\frac{\frac{k_m^2}{k^2}+3}{\frac{k_m^2}{k^2}+1}+\frac{2kh}{\sinh 2kh}\right\} = \frac{c}{2}\left\{\frac{\frac{\lambda^2}{\lambda_m^2}+3}{\frac{\lambda^2}{\lambda_m^2}+1}+\frac{2kh}{\sinh 2kh}\right\} \qquad (2.56)$$

For pure gravity waves, $k/k_m \ll 1$ so that

$$c_g = \frac{c}{2}\left(1+\frac{2kh}{\sinh 2kh}\right) \qquad (2.57)$$

where the phase velocity is

$$c = \sqrt{\frac{g}{k}\tanh kh} \qquad (2.58)$$

In very deep water $kh \gg 1$, we have

$$c_g = \frac{c}{2} = \frac{1}{2}\sqrt{\frac{g}{k}} \qquad (2.59)$$

The shorter the waves the smaller the phase and group velocities. In shallow water $kh \ll 1$,

$$c_g = c = \sqrt{gh} \tag{2.60}$$

Long waves are the fastest and no longer dispersive.

For capillary-gravity waves with $kh \gg 1$, we have

$$c_g = \frac{c}{2}\left\{\frac{\frac{k_m^2}{k^2}+3}{\frac{k_m^2}{k^2}+1}\right\} = \frac{c}{2}\left\{\frac{\frac{\lambda^2}{\lambda_m^2}+3}{\frac{\lambda^2}{\lambda_m^2}+1}\right\}, \quad k_m = \frac{2\pi}{\lambda_m}\sqrt{\frac{\rho g}{T}} \tag{2.61}$$

where

$$c = \sqrt{\frac{g}{k} + \frac{Tk^3}{\rho}} \tag{2.62}$$

Note that $c_g = c$ when $k = k_m$, and

$$c_g \gtrless c, \quad \text{if} \quad k \gtrless k_m \tag{2.63}$$

In the limit of pure capillary waves of $k \gg k_m$, $c_g = 3c/2$. For pure gravity waves $c_g = c/2$ as in (2.59).

3 Wave resistance of a two-dimensional obstacle

Ref: Lecture notes on *Surface Wave Hydrodynamics* Theodore T.Y. WU, Calif. Inst.Tech.

As an application of the information gathered so far, let us examine the wave resistance on a two dimensional body steadily advancing parallel to the free surface. Let the body speed be U from right to left and the sea depth be constant.

Due to two-dimensionality, waves generated must have crests parallel to the axis of the body (y axis). After the steady state is reached, waves that keep up with the ship must have the phase velocity equal to the body speed. In the coordinate system fixed on the body, the waves are stationary. Consider first capillary -gravity waves in deep water $\lambda_* = \lambda/\lambda_m = O(1)$ and $kh \gg 1$. Equating $U = c$ we get from the normalized dispersion relation

$$U_*^2 = c_*^2 = \frac{1}{2}\left(\lambda_* + \frac{1}{\lambda_*}\right) \tag{3.1}$$

where $U_* \equiv U/c_m$. Hence

$$\lambda_*^2 - 2c_*^2\lambda_* + 1 = 0 = (\lambda_* - \lambda_{*1})(\lambda_* - \lambda_{*2})$$

which can be solved to give

$$\begin{bmatrix} \lambda_{*1} \\ \lambda_{*2} \end{bmatrix} = c_*^2 \pm \left(c_*^4 - 1\right)^{1/2} \tag{3.2}$$

and

$$\lambda_{*1} = \frac{1}{\lambda_{*2}} \tag{3.3}$$

Thus, as long as $c_* = U_* > 1$ two wave trains are present: the longer gravity wave with length λ_{*1}, and the shorter capillary wave with length λ_{*2}. Since $c_{g_1} < c = U$ and $c_{g_2} > c = U$, and energy must be sent from the body, the longer gravity waves must follow, while the shorter capillary waves stay ahead of, the body.

Balancing the power supply by the body and the power flux in both wave trains, we get

$$Rc = (c - c_{g_1})\bar{E}_1 + (c_{g_2} - c)\bar{E}_2 \tag{3.4}$$

Recalling that

$$\frac{c_g}{c} = \frac{1}{2}\frac{\lambda_*^2 + 3}{\lambda_*^2 + 1}$$

we find,

$$1 - \frac{c_g}{c} = 1 - \frac{1}{2}\left(1 + \frac{2}{\lambda_*^2 + 1}\right) = \frac{1}{2} - \frac{1/\lambda_*}{\lambda_* + 1/\lambda_*} = \frac{1}{2} - \frac{1/\lambda_*}{2c^2}$$

For the longer wave we replace c_g/c by $c_{g_{*1}}/c_*$ and λ_* by λ_{*1} in the preceding equation, and use (3.2), yielding

$$1 - \frac{c_{g*1}}{c_*} = \left(1 - c_*^{-4}\right)^{1/2} \tag{3.5}$$

Similarly we can show that

$$\frac{c_{g*2}}{c_*} - 1 = \left(1 - c_*^{-4}\right)^{1/2} = 1 - \frac{c_{g*1}}{c_*} \tag{3.6}$$

Since

$$\bar{E}_1 = \frac{\rho g A_1^2}{2}\left(1 + \frac{1}{\lambda_{*1}^2}\right) = \frac{\rho g A_1^2}{2}\frac{1}{\lambda_{*1}}\left(\lambda_{*1} + \frac{1}{\lambda_{*1}}\right) = \rho g A_1^2 \lambda_{*2} c_*^2, \tag{3.7}$$

we get finally

$$R = \frac{1}{2}\rho g \left(\lambda_{*2}A_1^2 + \lambda_{*1}A_2^2\right)\left(c_*^4 - 1\right)^{1/2} = \frac{1}{2}\rho g \left(\lambda_{*2}A_1^2 + \lambda_{*1}A_2^2\right)\left(U_*^4 - 1\right)^{1/2} \tag{3.8}$$

Note that when $U_* = 1$, the two waves become the same; no power input from the body is needed to maintain the single infinite train of waves; the wave resistance vanishes. When $U_* < 1$, no waves are generated; the disturbance is purely local and there is also no wave resistance. To get the magnitude of R one must solve the boundary value problem for the wave amplitudes A_1, A_2 which are affected by the size (relative to the wavelengths), shape and depth of submergence.

When the speed is sufficiently high, pure gravity waves are generated behind the body. Power balance then requires that

$$R = \left(1 - \frac{c_g}{U}\right)\bar{E} = \frac{\rho g A^2}{2}\left(\frac{1}{2} - \frac{kh}{\sinh 2kh}\right) \tag{3.9}$$

The wavelength generated by the moving body is given implicitly by

$$\frac{U}{\sqrt{gh}} = \left(\frac{\tanh kh}{kh}\right)^{1/2} \tag{3.10}$$

When $U \approx \sqrt{gh}$ the waves generated are very long, $kh \ll 1$, $c_g \to c = \sqrt{gh}$, and the wave resistance drops to zero. When $U \ll \sqrt{gh}$, the waves are very short, $kh \gg 1$,

$$R \approx \frac{\rho g A^2}{4} \tag{3.11}$$

For intermediate speeds the dependence of wave resistance on speed is plotted in figure (3).

4 Narrow-banded dispersive waves in general

In this section let us discuss the superposition of progressive sinusoidal waves with the amplitudes spread over a narrow spectrum of wave numbers

$$\zeta(x,t) = \int_0^\infty |\mathcal{A}(k)| \cos(kx - \omega t - \theta_A) dk = \Re \int_0^\infty \mathcal{A}(k) e^{ikx - i\omega t} dk \tag{4.1}$$

where$\mathcal{A}(k)$ is complex denotes the dimensionless amplitude spectrum of dimension (length)^2. The component waves are dispersive with a general nonlinear relation $\omega(k)$. Let $\mathcal{A}(k)$ be different from zero only within a narrow band of wave numbers centered at k_o. Thus the integrand is of significance only in a small neighborhood of k_o. We then

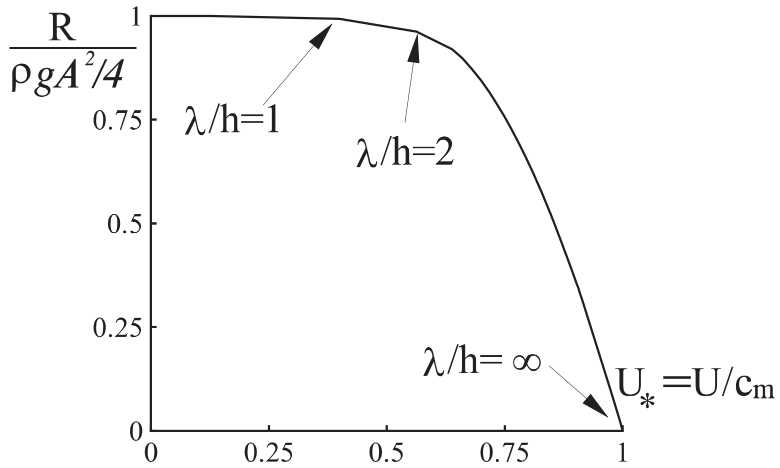

Figure 3: Dependence of wave resistance on speed for pure gravity waves

approximate the integral by expanding for small $\Delta k = k - k_o$ and denote $\omega_o = \omega(k_o)$, $\omega_o' = \omega'(k_o)$, and $\omega_o'' = \omega''(k_o)$,

$$\begin{aligned}
\zeta &= \Re\left\{e^{ik_ox-i\omega_ot}\int_0^\infty \mathcal{A}(k)\, e^{i\Delta kx-i(\omega-\omega_o)t}dk\right\} \\
&= \Re\left\{e^{ik_ox-i\omega_ot}\int_0^\infty dk\,\mathcal{A}(k)\exp\left[i\Delta kx - i\left(\omega_o'\Delta k + \frac{1}{2}\omega_o''(\Delta k)^2\right)t + \cdots\right]\right\} \\
&= \Re\left\{A(x,t)e^{ik_ox-i\omega_ot}\right\} \qquad (4.2)
\end{aligned}$$

where

$$A(x,t) = \int_0^\infty dk\,\mathcal{A}(k)\,\exp\left[i\Delta kx - i\left(\omega_o'\Delta k + \frac{1}{2}\omega_o''(\Delta k)^2\right)t + \cdots\right] \qquad (4.3)$$

Although the integration is formally extends from 0 to ∞, the effective range is only from $k_o - (\Delta k)_m$ to $k_o + (\Delta k)_m$, i,.e., the total range is $O((\Delta k)_m)$, where $(\Delta k)_m$ is the bandwidth. Thus the total wave is almost a sinusoidal wavetrain with frequency ω_o and wave number k_o, and amplitude $A(x,t)$ whose local value is slowly varying in space and time. $A(x,t)$ is also called the envelope. How slow is its variation?

If we ignore terms of $(\Delta k)^2$ in the integrand, (4.3) reduces to

$$A(x,t) = \int_0^\infty dk\,\mathcal{A}(k)\,\exp\left[i\Delta k(x-\omega_o't)\right] \qquad (4.4)$$

Clearly $A = A(x - \omega_o' t)$. Thus the envelope itself is a wave traveling at the speed ω_o'. This speed is called the group velocity,

$$c_g(k_o) = \left.\frac{d\omega}{dk}\right|_{k_o} \tag{4.5}$$

Note that the characteristic length and time scales are $(\Delta k_m)^{-1}$ and $(\omega_o'\Delta k_m)^{-1}$ respectively, therefore much longer than those of the component waves : k_o^{-1} and ω_o^{-1}. In other words, (4.3) is adequate for the slow variation of A_e in the spatial range of $\Delta k_m\, x = O(1)$ and the time range of $\omega_o'\Delta k_m\, t = O(1)$.

As a specific example we let the amplitude spectrum be a real constant within the narrow band of $k_o - \kappa, k_o + \kappa$,

$$\zeta = \mathcal{A}\int_{k_o-\kappa}^{k_o+\kappa} e^{ikx - i\omega(k)t} dk, \quad \kappa \ll k_o \tag{4.6}$$

then

$$\begin{aligned}
\zeta &= k_o \mathcal{A} e^{ik_o x - i\omega_o t}\int_{-\kappa}^{\kappa} d\xi e^{ik_o\xi(x - c_g t)} + \cdots \\
&= \frac{2\mathcal{A}\sin\kappa(x - c_g t)}{x - c_g t} e^{ik_o x - i\omega_o t} = A e^{ik_o x - i\omega_o t}
\end{aligned} \tag{4.7}$$

where $\xi = k - k_o/k_o$ and

$$A = \frac{2\mathcal{A}\sin\kappa(x - c_g t)}{(x - c_g t)} \tag{4.8}$$

as plotted in figure (4).

By differentiation, it can be verified that

$$\frac{\partial A}{\partial t} + c_g \frac{\partial A}{\partial x} = 0, \tag{4.9}$$

Multiplying (4.9) by A^*,

$$A^* \frac{\partial A}{\partial t} + c_g A^* \frac{\partial A}{\partial x} = 0,$$

and adding the result to its complex conjugate,

$$A \frac{\partial A^*}{\partial t} + c_g A \frac{\partial A^*}{\partial x} = 0,$$

we get

$$\frac{\partial |A|^2}{\partial t} + c_g \frac{\partial |A|^2}{\partial x} = 0 \tag{4.10}$$

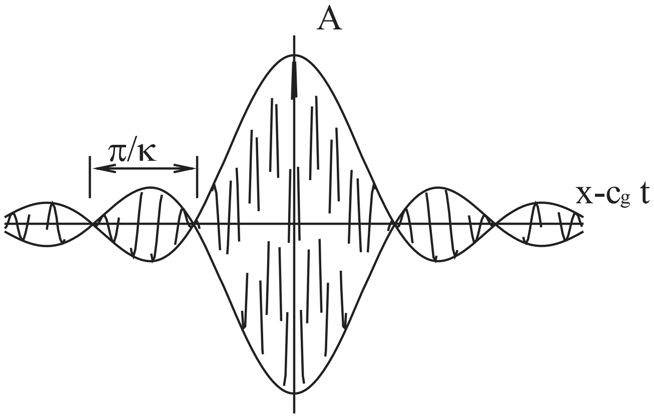

Figure 4: Envelope of waves with a rectangular band of wavenumbers

We have seen that for a monochromatic wave train the energy density is proportional to $|A|^2$. Thus the time rate of change of the *local* energy density is balanced by the net flux of energy by the group velocity.

Now let us examine the more accurate approximation (4.3). By straightforward differentiation, we find

$$\begin{aligned}
\frac{\partial A}{\partial t} &= \int_0^\infty \left[-i\omega'(k_o)\Delta k - \frac{i\omega''(k_o)}{2}(\Delta k)^2\right] \mathcal{A}(k)e^{iS}dk \\
\frac{\partial A}{\partial x} &= \int_0^\infty (i\Delta k)\mathcal{A}(k)e^{iS}dk \\
\frac{\partial^2 A}{\partial x^2} &= \int_0^\infty \left(-(\Delta k)^2\right) \mathcal{A}(k)e^{iS}dk
\end{aligned}$$

where

$$S = \Delta k\, x - \omega_o'\Delta k\, t - \frac{1}{2}\omega_o''(\Delta k)^2\, t \tag{4.11}$$

is the phase function. It can be easily verified that

$$\frac{\partial A}{\partial t} + \omega_o' \frac{\partial A}{\partial x} = \frac{i\omega_o''}{2}\frac{\partial^2 A}{\partial x^2} \tag{4.12}$$

By keeping the quadratic term in the expansion, (4.12) is now valid for a larger spatial range of $(\Delta k)^2 x = O(1)$. In the coordinate system moving at the group velocity, $\xi =$

$x - c_g t, \tau = t$, we easily find

$$\frac{\partial A(\xi,\tau)}{\partial t} = \frac{\partial A}{\partial \tau} - c_g \frac{\partial A}{\partial \xi}, \quad \frac{\partial A(\xi,\tau)}{\partial x} = \frac{\partial A}{\partial x}$$

so that (4.12) simplifies to the Schrödinger equation:

$$\frac{\partial A}{\partial \tau} = \frac{i\omega_o''}{2} \frac{\partial^2 A}{\partial \xi^2} \tag{4.13}$$

By manipulations similar to those leading to (4.10), we get

$$\frac{\partial |A|^2}{\partial \tau} = \frac{i\omega_o''}{2} \frac{\partial}{\partial \xi} \left(A^* \frac{\partial A}{\partial \xi} - A \frac{\partial A^*}{\partial \xi} \right) \tag{4.14}$$

Thus the local energy density is not conserved over a long distance of propagation. Higher order effects of dispersion redistribute energy to other parts of the envelope. For either a wave packet whose envelope has a finite length ($A(\pm\infty) = 0$), or for a periodically modulated envelope ($A(x) = A(x + L)$), we can integrate (4.14) to give

$$\frac{\partial}{\partial \tau} \int |A|^2 d\xi = 0 \tag{4.15}$$

where the integration extends over the entire wave packet or the group period. Thus the total energy in the entire wave packet or in a group period is conserved.

2 Two-Dimensional Tsunami

Tsunamis are the water waves generated by submarine earthquakes. If the seafloor displacement is known in the area of the earthquake, the water-wave problem is a purely hydrodynamic one. Unfortunately, direct measurements near the epicenter are too difficult to make, and a good deal of effort has been centered on using water-wave records measured at larger distances from the epicenter to infer roughly the nature of tectonic movement. Hence, there has been considerable theoretical studies on water waves due to a variety of ground movement.

As an introduction, let us consider an open ocean of constant depth and assume that there is no wind and the tectonic disturbance on the bottom is independent of y. The problem is two dimensional in the x, z plane. Thus, the velocity potential $\Phi(x, z, t)$ satisfies

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0\,. \tag{2.1}$$

On the free surface the following conditions hold:

$$\frac{\partial \zeta}{\partial t} = \frac{\partial \Phi}{\partial z}, \qquad z = 0 \tag{2.1.2a}$$

$$\frac{\partial \Phi}{\partial t} + g\zeta = 0, \quad z = 0, \tag{2.1.2b}$$

where the atmospheric pressure is assumed to be absent. the two conditions can be combined to give

$$\frac{\partial^2 \Phi}{\partial t^2} + g\frac{\partial \Phi}{\partial z} = 0, \quad z = 0 \tag{2.1.2c}$$

Let the seafloor be denoted by $z = -h + H(x,t)$. If the ground motion is known, continuity of normal velocity gives

$$\frac{\partial \Phi}{\partial z} = \frac{\partial H}{\partial t} + \frac{\partial \Phi}{\partial x}\frac{\partial H}{\partial x} \quad \text{on } z = -h + H(x,t)\,. \tag{2.3}$$

Within the framework of linearization we assume that the amplitudes of H, $\partial H/\partial t$ and $\partial H/\partial x$ are small so that the quadratic term is negligible; hence

$$\frac{\partial \Phi}{\partial z} = \frac{\partial H}{\partial t} \equiv W(x,t) \quad \text{on } z \cong -h\,. \tag{2.4}$$

From (3.4) two initial conditions must be further prescribed, i.e., $\Phi(x,0,0)$ and $\zeta(x,0)$ on the free surface, but nowhere else, because time derivatives appear only in the free-surface conditions.

What is the physical significance of $\Phi(x,0,0)$? Assume that, before $t = 0$, all is calm, but at $t = 0$ an impulsive pressure $P_a(x,t) = I\delta(t)$ is applied on the free surface. Integrating Bernoulli's equation from $t = 0-$ to $t = 0+$, we obtain

$$\Phi(x,0,0+) - \Phi(x,0,0-) + \int_{0-}^{0+} g\zeta\, dt = -\frac{I}{\rho}\int_{0-}^{0+} \delta(t)\, dt = -\frac{I}{\rho}\,.$$

Since $\Phi(x,0,0-) = 0$ and ζ must be finite, we obtain $\Phi(x,0,0+) = I/\rho$. Thus, the initial value of Φ represents physically an impulsive pressure acting on the free surface at an instant slightly earlier than $t = 0+$.

Equations (2.1), (3.4), and (2.4) now define a boundary-value problem which formally resembles that of a simple harmonic case. For any finite t it is expected that no motion is felt at a great distance from the initial disturbance so that $\Phi(x,t) \to 0$ as $|x| \to \infty$, which implies that $\overline{\Phi} \to 0$ as $|x| \to \infty$. Since the region does not involve any finite bodies, the problem can be readily solved by applying the exponential Fourier transform with respect to x, defined by

$$\tilde{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx\,, \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\tilde{f}(k)\, dk\,. \tag{2.5}$$

By taking Fourier transform of Laplace's equation with respect to x, we get the general solution

$$\tilde{\phi} = A\sinh k(z+h) + B\cosh k(z+h) \tag{2.6}$$

Fourier transforms of the boundary conditions give

$$\tilde{\Phi}_{tt} + g\tilde{\Phi}_z = 0, \quad z = 0 \tag{2.7}$$

on the free surface and

$$\tilde{\phi}_z = \tilde{W}, \quad z = -h \tag{2.8}$$

on the mean sea bed. Let the free surface be undisturbed initially,

$$\tilde{\phi}(k,0,0) = \tilde{\phi}_t(k,0,0) = 0 \tag{2.9}$$

We get from (2.7),

$$B_{tt} + gkA = 0 \tag{2.10}$$

and from (2.8)

$$kA\cosh kh - kB\sinh kh = \tilde{W} \tag{2.11}$$

It follows that

$$B_{tt} + \omega^2 B = \frac{-g\bar{W}}{\cosh kh} \tag{2.12}$$

where $\omega = \sqrt{gk\tanh kh}$, subject to the initial conditions

$$B(k,0) = 0, \quad \text{and} \quad B_t(k,0) = 0 \tag{2.13}$$

For the special case where $W(x,t) = H_0(x)\delta(t)$, integation across the delta function gives the the impulsive vertical displacement

$$\int_{0-}^{0+} W\, dt = H_0(x) \tag{2.14}$$

The initial-value problem can be replaced by

$$B_{tt} + \omega^2 B = 0 \qquad t > 0+ \tag{2.15}$$

$$B(k, 0+) = 0, \quad \text{and} \quad B_t(k, 0+) = \frac{-g\tilde{H}_0}{\cosh kh} \tag{2.16}$$

which can be readily solved fo $\tilde{B}(t)$. Afterwards $\tilde{\Phi}(k, t)$ follows from, from which $\zeta(k, t)$ is found. By invese transform, we finally get $\zeta(x, t)$.

We leave it as an exercise to show that the free surface is given by

$$\zeta = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \frac{\tilde{H}_0(k)}{\cosh kh} \frac{1}{2} [e^{ik(x+\omega t)} + e^{ik(x-\omega t)}] \tag{2.17}$$

which can be written as

$$\zeta = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \frac{\tilde{H}_0(k)}{\cosh kh} e^{ik(x} \cos \omega t = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, \tilde{\zeta}_0(k) e^{ikx} \cos \omega t\,. \tag{2.18}$$

where

$$\tilde{\zeta}_0(k) \equiv \frac{\tilde{H}_0(k)}{\cosh kh} \tag{2.19}$$

Clearly, $\cos \omega t$ is even in k.

In general, we can split $H_0(x)$ into even and odd parts with respect to $x : H_0^e$ and H_0^o. It follows from the definition of Fourier transform that

$$\begin{aligned} \tilde{H}_0(k) &= 2\int_0^{\infty} dx \cos kx H_0^e(x) - 2i \int_0^{\infty} dx \sin kx H_0^o(x)\,, \\ &\equiv \tilde{H}_0^e(k) + \tilde{H}_0^o(k) \end{aligned}$$

where $\tilde{H}_0^e$ is real and even in k and $\tilde{H}_0^o$ is imaginary and odd in k.

For simplicity, let us consider $H_0(x)$ (hence $\zeta_0(x)$) to be even in x. The case of odd $H_0(x)$ can be dealt with similarly. Eq. (2.18) may be written

$$\begin{aligned} \zeta(x, t) &= \frac{1}{\pi} \int_0^{\infty} dk\, \tilde{\zeta}_0^e \cos kx \cos \omega t \\ &= \frac{1}{2\pi} \operatorname{Re} \int_0^{\infty} dk\, \tilde{\zeta}_0^e [e^{i(kx-\omega t)} + e^{i(kx+\omega t)}]\,. \end{aligned} \tag{2.20}$$

The first and second terms in the brackets above represent right- and left-going waves, respectively.

2.1 Asympotic behavior at large time

For a better physical understanding, approximations are necessary. At large t we can employ the *method of stationary phase* devised by Kelvin. Heuristically, the idea is as follows.

Consider the integral

$$I(t) = \int_a^b f e^{itg} \, dk \tag{2.21}$$

where f and g are smooth functions of k. When t is large, the phase tg of the sinusoidal part oscillates rapidly as k varies. If one plots the integrand versus k, there is very little net area under the curve due to cancellation unless there is a point at which the phase is stationary, that is,

$$g'(k) = 0\,, \quad k = k_0\,. \tag{2.22}$$

In the neighborhood of this stationary point the oscillating factor of the integrand of Eq. (2.23) may be written

$$e^{itg(k_0)} \exp\{it[g(k) - g(k_0)]\}\,.$$

The real part of $\exp\{it[g(k) - g(k_0)]\}$ varies slowly, as sketched in Fig. 5, while the imaginary part slowly crosses the k axis at $k = k_0$. Therefore, a significant contribution to th For a better physical understanding, approximations are necessary. At large t we can employ the *method of stationary phase* devised by Kelvin. Heuristically, the idea is as follows.

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$$g'(k) = 0\,, \quad k = k_0\,. \tag{2.24}$$

In the neighborhood of this stationary point the oscillating factor of the integrand of Eq. (2.23) may be written

$$e^{itg(k_0)} \exp\{it[g(k) - g(k_0)]\}\,.$$

The real part of $\exp\{it[g(k) - g(k_0)]\}$ varies slowly, as sketched in Fig. 5, while the imaginary part slowly touches the k axis at $k = k_0$. Therefore, a significant contribution

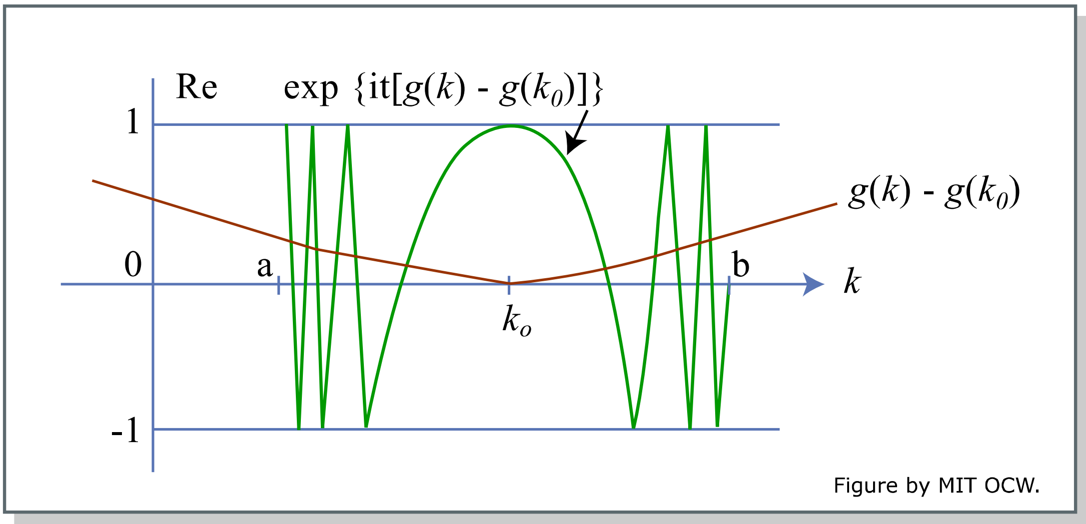

Figure 5: The real part of $\exp\{it[g(k) - g(k_0)]\}$.

to the integral can be expected from this neighborhood. If we approximate $g(k)$ by the first two terms of the Taylor expansion

$$g(k) \cong g(k_0) + \frac{1}{2}(k - k_0)^2 g''(k_0)\,,$$

then the integral may be written

$$I \cong e^{itg(k_0)} f(k_0) \int_{-\infty}^{\infty} dk \exp\left[\frac{1}{2} i (k - k_0)^2 t g''(k_0)\right],$$

where the limits (a, b) have been approximated by $(-\infty, \infty)$. Using the fact that

$$\int_{-\infty}^{\infty} e^{\pm itk^2}\, dk = \left(\frac{\pi}{t}\right)^{1/2} e^{\pm i\pi/4}\,,$$

we finally have

$$I \cong e^{itg(k_0)} f(k_0) \left(\frac{2\pi}{t|g''(k_0)|}\right)^{1/2} e^{\pm i\pi/4}\,, \tag{2.25}$$

where the + sign is to be taken if $g''(k_0) < 0$, and minus ifif $g''(k_0) > 0$, . It can be shown by a more elaborate analysis that the error is of order $O(t^{-1})$. Also if there is no stationary point in the range (a, b), the integral is at most of order $O(t^{-1})$. This and other information can be found in Stoker (1957) or Carrier, Krook, and Pearson (1966).

Returning to Eq. (2.20), we need certain properties of the dispersion curve as sketched in Fig. 6. Consider $x > 0$. For the first integral

$$g(k) = k\frac{x}{t} - \omega\,,$$

it may be seen from Fig. 6(b) that there is a stationary point at

$$\frac{x}{t} = \omega'(k_0) = C_g(k_0) \quad \text{if } \frac{x}{t} < (gh)^{1/2}\,. \tag{2.26}$$

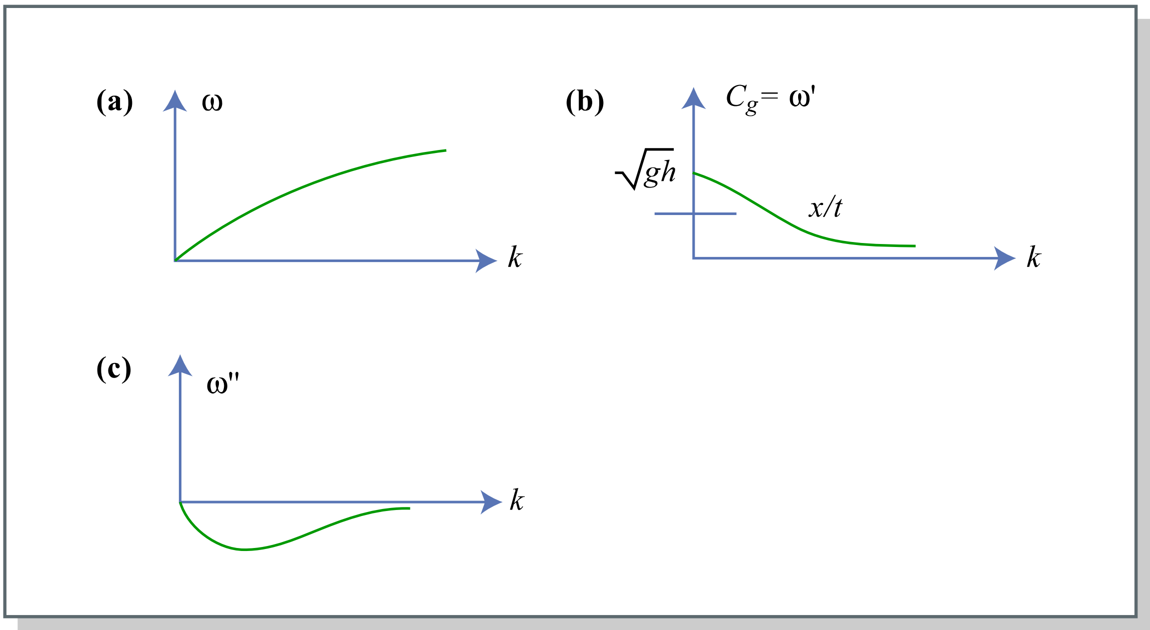

Figure by MIT OCW.

Figure 6: Variations of ω, ω', and ω'' with k.

In the same interval $(0, \infty)$ of k, there is no stationary point for the second integral. It follows from Eq. (2.25) that

$$\begin{aligned}\zeta \quad &\cong \quad \frac{1}{2\pi}\tilde{\zeta}_0^e(k_0)\left[\frac{2\pi}{t|\omega''(k_0)|}\right]^{1/2}\cos\left[k_0 x - \omega(k_0)t + \frac{\pi}{4}\right] \\ &\quad + O(t^{-1})\,, \quad x < (gh)^{1/2}t\,, \end{aligned} \tag{2.27}$$

where use is made of the fact that $\omega''(k) < 0$ [Fig. 6(c)], and

$$\zeta \cong O(t^{-1})\,, \quad x > (gh)^{1/2}t\,. \tag{2.28}$$

Now let us examine the physics represented by Eq. (2.27). An observer moving at a certain speed x/t lower than $(gh)^{1/2}$ sees a train of sinusoidal waves of wavenumber k_0 [and frequency $\omega(k_0)$] whose group velocity equals x/t. The amplitude of the wavetrain decays as $O(t^{-1/2})$. For large x/t we see from Fig. 6(a) that k_0 is small, hence, a faster moving observer sees longer waves which are also of larger amplitude since $(|\omega''(k_0)|)^{1/2}$ is less. The precise shape of $H_0(x)$ affects $\tilde{H}_0(k)$ hence $\tilde{\zeta}_0(k)$ and the amplitude of the dispersed waves.

Summing up the views of many observers for the same t, we obtain a snapshot of the free surface (see Fig. 7). Thus, at a constant t, long waves are found toward the front and short waves toward the rear. Now consider the snapshot at a later time $t_2 > t_1$. Both observers have now moved to the right. The spatial separation, however, has increased. In particular, let $\xi_1 \approx \xi_2$ so that between them $k, \omega \approx$ const. The total extent of a monochromatic wavetrain with k, ω now stretches with increasing t, implying that wave crests are created in the course of propagation.

To follow a particular wave crest at its phase speed, an observer must travel at a varying speed since k_0 and $C(k_0)$ do not remain constant as the crest moves into new territory. However, if one moves at the group velocity of the waves of length $2\pi/k_0$, one only sees sine waves of this length catching up from behind and then running away toward the front, since their phase velocity exceeds the group velocity.

A similar picture exists for the left-going disturbance.

2.2 Leading waves of a tsunami due to symmetric vertical displacement

The fastest waves correspond to $k \simeq 0$ and move at the speed near $(gh)^{1/2}$. In the neighborhood of the wave front, $g'(k) \simeq x/t - (gh)^{1/2}$ is small, and the phase is nearly stationary. Furthermore, $\omega''(k) \simeq -(gh)^{1/2}h^2k$ is also very small and the approximation by the stationary phase method is not valid. A better approximation is needed (Kajiura, 1963).

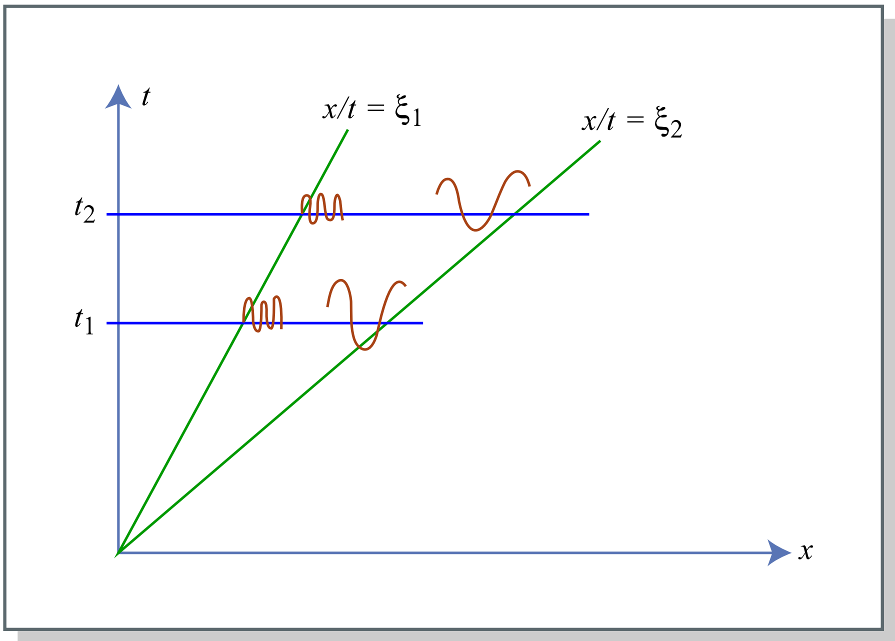

Figure by MIT OCW.

Figure 7: Space-time plot of dispersive waves between two moving observers.

Since $k \simeq 0$, we expand the phase function for small k as follows:

$$\begin{aligned} g(k) &= k\frac{x}{t} - \omega(k) \simeq k\left(\frac{x}{t}\right) - (gh)^{1/2}\left(k - \frac{k^3h^2}{6} + \cdots\right) \\ &= k\left[\frac{x}{t} - (gh)^{1/2}\right] + \frac{(gh)^{1/2}}{6}h^2k^3 + \cdots . \end{aligned} \tag{2.29}$$

Near the leading wave, $x/t - (gh)^{1/2}$ can be zero; we must retain the term proportional to k^3. Again, only the first integral in Eq. (2.20) matters so that

$$\begin{aligned} \zeta &= \frac{1}{2\pi}\int_0^\infty dk\,\tilde{\zeta}_0^e(k)\cos(kx - \omega t) + O\left(\frac{1}{t}\right) \\ &\simeq \frac{1}{2\pi}\tilde{\zeta}_0^e(0)\int_0^\infty \cos\left\{k[x-(gh)^{1/2}t] + \left[\frac{(gh)^{1/2}h^2t}{6}\right]k^3\right\}dk \end{aligned}$$

where use is made of the fact that $\tilde{\zeta}_0^e$ is real. With the change of variables

$$Z^3 = \frac{2[x-(gh)^{1/2}t]^3}{(gh)^{1/2}h^2t} \quad \text{and} \quad k[x-(gh)^{1/2}t] = Z\alpha\,,$$

the integral above becomes

$$\zeta \sim \frac{(2)^{1/3}\tilde{\zeta}_0^e(0)}{2\pi((gh)^{1/2}h^2t)^{1/3}}\int_0^\infty d\alpha\cos\left(Z\alpha + \frac{\alpha^3}{3}\right), \tag{2.30}$$

where

$$\tilde{\zeta}_0^e(0) = \tilde{H}_0^e(0) = \int_{-\infty}^\infty H_0^e(x)dx \tag{2.31}$$

is proportional to the total area of the seafloor displacement. The integral in (2.30) above can be expressed in terms of Airy's function of Z:

$$\mathrm{Ai}(Z) \equiv \frac{1}{\pi}\int_0^\infty d\alpha\cos\left(Z\alpha + \frac{\alpha^3}{3}\right). \tag{2.32}$$

Thus, we have

$$\zeta \sim \left[\frac{2}{(gh)^{1/2}h^2t}\right]^{1/3}\frac{1}{2}\tilde{H}_0^e(0)\,\mathrm{Ai}\left\{\left[\frac{2}{(gh)^{1/2}h^2t}\right]^{1/3}[x-(gh)^{1/2}t]\right\}. \tag{2.33}$$

$\mathrm{Ai}(Z)$ is oscillatory for $Z < 0$ and decays exponentially for $Z > 0$. Its variation is shown in Fig. 8.

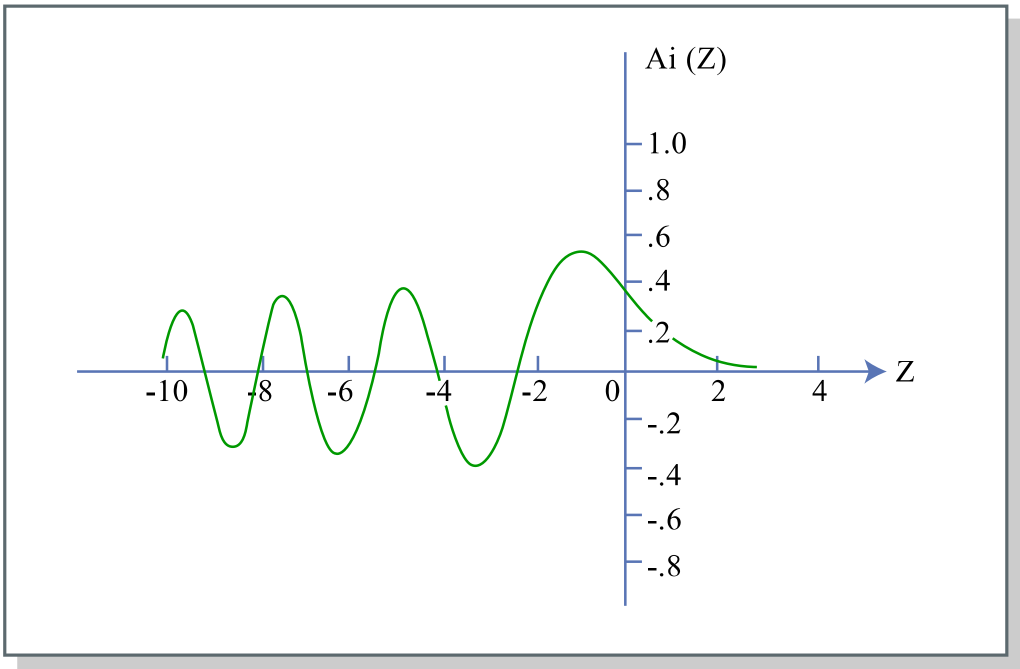

Figure by MIT OCW.

Figure 8: Leading wave due to a symmetrical surface hump or trough. The ordinate is $\zeta[(gh)^{1/2}h^2t/2]^{1/3}[\tilde{\zeta}_0^e(0)]^{-1}$, see Eq. (2.33).

The physical picture is as follows: For a fixed t, Z is proportional to $x-(gh)^{1/2}t$ which is the distance from the wave front $x=(gh)^{1/2}t$. At a fixed instant the amplitude is small ahead of the front, and the highest peak is at some distance behind. Toward the rear, the amplitude and the wavelength decrease. Since Z is proportional to $t^{-1/3}$, the snapshots at different times are of the same form except that the spatial scale is proportional to the factor $t^{1/3}$, meaning that the same wave form is being stretched out in time. During the evolution the amplitude decays as $t^{-1/3}$ while the rest of the wavetrain decays as $t^{-1/2}$. Thus, the head lives longer than the rest of the body. Note that the amplitude of the leading wave is proportional to $\tilde{H}_0^e(0)$ which is equal to the total area of the initial displacement $H_0^e(x)$.

2.3 Tsunami Due to Tilting of the Bottom

Among the many features of tsunamis as recorded near a coast, two have been frequently (but not always) reported (Shepard, 1963). One feature is that the arrival of a tsunami is often preceded by the withdrawal of water from the beaches, and the other is that the first crest may not be the largest. In this section we shall show an idealized model which reproduces these features qualitatively.

Again any $H_0(x)$ can be thought of as the sum of $H_0^o(x)$ and $H_0^e(x)$ which are odd and even in x, respectively. By linearity, the two parts may be treated separately first and their results superimposed later. It is easily shown that the even part $H_0^e(x)$ has effects very similar to the previous example of symmetrical initial displacement on the free surface, the only difference being the factor $(\cosh kh)^{-1}$ which cuts down the influence of the short waves. We shall, therefore, only focus our attention to the odd part.

Let us introduce

$$H_0^o(x) = \frac{dB}{dx} \tag{2.34}$$

so that $\tilde{H}_0^o(k) = ik\tilde{B}(k)$. Since $\tilde{H}_0^o(k)$ is odd, $\tilde{B}$ must be real and even in k; hence,

$$\begin{aligned}
\zeta &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \frac{e^{ikx}}{\cosh kh} ik\tilde{B}(k)\frac{1}{2}(e^{i\omega t} + e^{-i\omega t}) \\
&= \frac{1}{2\pi}\frac{d}{dx}\int_{-\infty}^{\infty} dk \frac{e^{ikx}}{\cosh kh}\tilde{B}(k)\frac{1}{2}(e^{i\omega t} + e^{-i\omega t}) \\
&= \frac{1}{2\pi}\frac{d}{dx}\,\mathrm{Re}\int_{0}^{\infty} dk \frac{e^{ikx}}{\cosh kh}\tilde{B}(k)(e^{i\omega t} + e^{-i\omega t})\,.
\end{aligned} \tag{2.35}$$

For large t and away from the leading waves, the integrals can be dealt with by the stationary phase method just as before, and many of the same qualitative features should be expected. Let us only look at the neighborhood of the *leading waves* propagating to $x > 0$. Again, the second integral dominates and the important contribution comes from the neighborhood of $k \simeq 0$. Hence

$$\begin{aligned}
&\mathrm{Re}\int_0^{\infty} dk \frac{e^{i(kx-\omega t)}}{\cosh kh}\tilde{B}(k) \\
&\cong \mathrm{Re}\,\tilde{B}(0)\int_0^{\infty} dk\, e^{ikx}e^{-i\omega t} \\
&\cong \mathrm{Re}\,\tilde{B}(0)\int_0^{\infty} dk\, \exp\left(i\left\{k[x-(gh)^{1/2}t] + \frac{1}{6}(gh)^{1/2}h^2k^3t\right\}\right) \\
&= \pi\tilde{B}(0)\left[\frac{2}{(gh)^{1/2}h^2t}\right]^{1/3}\mathrm{Ai}\left\{\left[\frac{2}{(gh)^{1/2}h^2t}\right]^{1/3}[x-(gh)^{1/2}t]\right\},
\end{aligned}$$

as discussed earlier. Differentiating with respect to x, we have

$$
\begin{aligned}
\zeta &\simeq \frac{\tilde{B}(0)}{2}\left[\frac{2}{(gh)^{1/2}h^2t}\right]^{1/3}\frac{d}{dx}\,\mathrm{Ai}\left\{\left[\frac{2}{(gh)^{1/2}h^2t}\right]^{1/3}[x-(gh)^{1/2}t]\right\} \\
&= \frac{\tilde{B}(0)}{2}\left[\frac{2}{(gh)^{1/2}h^2t}\right]^{2/3}\mathrm{Ai}'\left\{\left[\frac{2}{(gh)^{1/2}h^2t}\right]^{1/3}[x-(gh)^{1/2}t]\right\},
\end{aligned}
\tag{2.36}
$$

where

$$\mathrm{Ai}'(Z) \equiv \frac{d}{dZ}\,\mathrm{Ai}(Z)\,.$$

The leading wave attenuates with time as $t^{-2/3}$ which is much faster than the case of a pure rise or fall (where $\zeta \sim t^{-1/3}$). This result is due to the fact that the ground movement is half positive and half negative, thereby reducing the net effect. The function $\mathrm{Ai}'(Z)$ behaves as shown in Fig. 9. Note that

$$\tilde{B}(0) = \int_{-\infty}^{\infty} B(x)\,dx = \int_{-\infty}^{\infty} dx \int_{-\infty}^{x} H_0^o(x')\,dx' = -\int_{-\infty}^{\infty} xH_0^o(x)\,dx\,.$$

Thus, if the ground tilts down on the right and up on the left, $\tilde{B}(0) > 0$ and the wave front propagating to the right is led by depression of water surface (hence withdrawal from a beach). The subsequent crests increase in amplitude. On the left side, $x < 0$, the wave front has the opposite phase and is led by a crest. If, however, the ground tilt is opposite in direction, that is, down on the left and up on the right, then the right-going wave front should be led by an elevation.

Kajiura pointed out that retaining the terms gk^3h^2 in $\omega(k)$ implies keeping dispersion to the lowest order, and the same results, Eqs. (2.33) and (2.36), may be obtained alternatively by invoking the long-wave approximation at the outset, which is clearly appropriate far away from the source. It will be shown in Chapter Twelve that such an approximation is given by the linearized Boussinesq equations which are, in one dimension, equivalent to

$$\frac{\partial^2\zeta}{\partial t^2} = gh\left(\frac{\partial^2\zeta}{\partial x^2} + \frac{h^2}{3}\frac{\partial^4\zeta}{\partial x^4}\right). \tag{2.37}$$

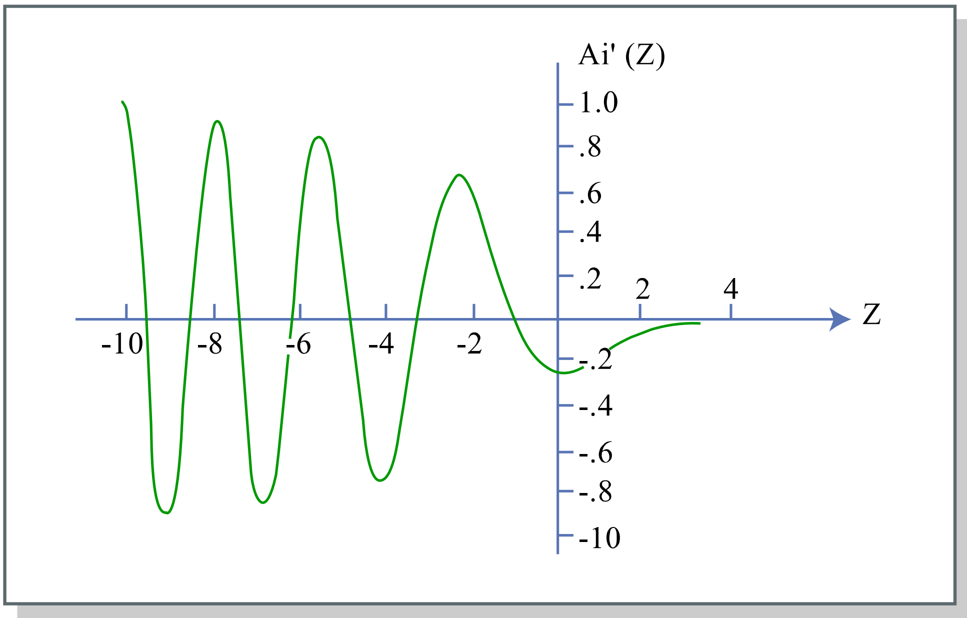

Figure by MIT OCW.

Figure 9: Leading wave due to antisymmetric ground tilt $\zeta[\tilde{B}(0)]^{-1}[(gh)^{1/2}h^2t/2]^{2/3}$, see Eq. (2.36).

3 Radiation of surface waves forced by an oscillating pressure

We demonstrate the reasoning which is typical in many similar radiation problems.

The governing equations are

$$\nabla^2\phi = \phi_{xx} + \phi_{yy} = 0, \qquad -\infty < z < 0. \tag{3.1}$$

with the kinematic boundary condition

$$\phi_z = \zeta_t, \quad z = 0 \tag{3.2}$$

and the dynamic boundary condition

$$\frac{p_a}{\rho} + \phi_t + g\zeta = 0 \tag{3.3}$$

where p_a is the prescribed air pressure. Eliminating the free surface displacement we get

$$\phi_{tt} + g\phi_z = -\frac{(p_a)_t}{\rho}, \quad z = 0. \tag{3.4}$$

Let us consider only sinusoidal time dependence:

$$p_a = P(x)e^{-i\omega t} \tag{3.5}$$

and assume

$$\phi(x,z,t) = \Phi(x,z)e^{-i\omega t}, \quad \zeta(x,t) = \eta(x)e^{-i\omega t} \tag{3.6}$$

then the governing equations become

$$\nabla^2\Phi = \Phi_{xx} + \Phi_{yy} = 0, \qquad -\infty < z < 0. \tag{3.7}$$

$$\Phi_z = -i\omega\eta, \quad z = 0 \tag{3.8}$$

and

$$\Phi_z - \frac{\omega^2}{g}\Phi = \frac{i\omega}{\rho g}P(x), \qquad z = 0. \tag{3.9}$$

Define the Fourier transform and its inverse by

$$\bar{f}(\alpha) = \int_{-\infty}^{\infty} dx\, e^{-i\alpha x} f(x), \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\alpha\, e^{i\alpha x}\bar{f}(\alpha), \tag{3.10}$$

We then get the transforms of (3.1) and (3.4)

$$\bar{\Phi}_{zz} - \alpha^2\bar{\Phi} = 0, \quad z < 0 \tag{3.11}$$

subject to

$$\bar{\Phi}_z - \frac{\omega^2}{g}\bar{\Phi} = \frac{i\omega}{\rho g}\bar{P}(\alpha), \qquad z = 0. \tag{3.12}$$

The solution finite at $z \sim -\infty$ for all α is

$$\bar{\Phi} = Ae^{|\alpha|z}$$

To satisfy the free surface condition

$$|\alpha|A - \frac{\omega^2}{g}A = \frac{i\omega\bar{P}(\alpha)P(\alpha)}{\rho g}$$

hence

$$A = \frac{\frac{i\omega\bar{P}(\alpha)}{\rho g}}{|\alpha| - \omega^2/g}$$

or

$$\begin{aligned}
\Phi &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\alpha e^{i\alpha x}e^{|\alpha|z}\frac{\frac{i\omega\bar{P}(\alpha)}{\rho g}}{|\alpha| - \omega^2/g} \\
&= \frac{i\omega}{\rho g}\frac{1}{2\pi}\int_{-\infty}^{\infty} d\alpha e^{i\alpha x}e^{|\alpha|z}\int_{-\infty}^{\infty} dx'\, e^{-i\alpha x'}P(x')\frac{1}{|\alpha| - \omega^2/g}, \\
&= \frac{i\omega}{\rho g}\int_{-\infty}^{\infty} dx'\, P(x')\frac{1}{2\pi}\int_{-\infty}^{\infty} d\alpha\, e^{i\alpha(x-x')}e^{|\alpha|z}\frac{1}{|\alpha| - \omega^2/g}
\end{aligned} \tag{3.13}$$

Let

$$k = \frac{\omega^2}{g} \tag{3.14}$$

we can rewrite (3.13) as

$$\Phi = \frac{i\omega}{\rho g}\int_{-\infty}^{\infty} dx'\, P(x')\frac{1}{\pi}\int_0^{\infty} d\alpha\, e^{\alpha z}\frac{\cos(\alpha(x-x'))}{\alpha - k} \tag{3.15}$$

The final formal solution is

$$\phi = \frac{i\omega}{\rho g}e^{-i\omega t}\int_{-\infty}^{\infty} dx'\, P(x')\frac{1}{\pi}\int_0^{\infty} d\alpha\, e^{\alpha z}\frac{\cos(\alpha(x-x'))}{\alpha - k} \tag{3.16}$$

If we chose

$$P(x') = P_o\delta(x') \tag{3.17}$$

then

$$\Phi \to \mathcal{G}(x,z) = \frac{i\omega P_o}{\rho g}\frac{1}{\pi}\int_0^{\infty} d\alpha\, e^{\alpha z}\frac{\cos(\alpha x)}{\alpha - k} \tag{3.18}$$

is clearly the response to a concentrated surface pressure and the response to a pressure distribution (3.16) can be written as a superposition of concentrated loads over the free surface,

$$\phi = \int_{-\infty}^{\infty} dx'\, P(x')\mathcal{G}(x-x',z). \tag{3.19}$$

where

$$\mathcal{G}(x,z,t) = \frac{i\omega P_o}{\rho g}e^{-i\omega t}\frac{1}{\pi}\int_0^{\infty} d\alpha\, e^{\alpha z}\frac{\cos(\alpha x)}{\alpha - k} \tag{3.20}$$

In these results, e.g., (3.20), the Fourier integral is so far undefined since the integrand has a real pole at $\alpha = k$ which is on the path of integration. To make it mathematically defined we can chose the principal value, deform the contour from below or from above the pole as shown in figure (3). This indefiniteness is due to the assumption of quasi

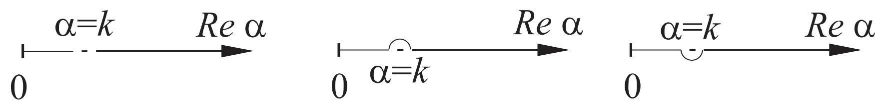

Figure 10: Possible paths of integration

steady state where the influence of the initial condition is no longer traceable. We must now impose the radiation condition that waves must be outgoing as $x \to \infty$. This

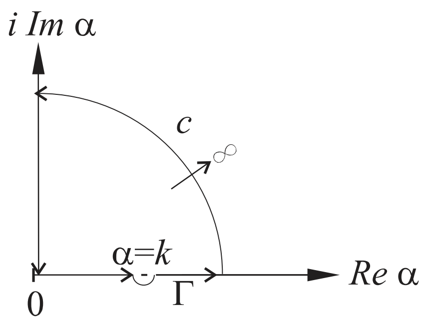

Figure 11: Closed contour in the upper half plane

condition can only be satisfied if we deform the contour from below. Denoting this contour by Γ, we now manipulate the integral to exhibit the behavior at infinity, and to verify the choice of path. For simplicity we focus attention on $\mathcal{G}$. Due to symmetry, it suffices to consider $x > 0$. Rewriting,

$$\begin{aligned}\mathcal{G}(x,z,t) &= \frac{i\omega P_o}{\rho g}e^{-i\omega t}\frac{1}{2\pi}(I_1 + I_2) \\ &= \frac{i\omega P_o}{\rho g}e^{-i\omega t}\frac{1}{2\pi}\int_\Gamma d\alpha\, e^{\alpha z}\left[\frac{e^{i\alpha x}}{\alpha - k} + \frac{e^{-i\alpha x}}{\alpha - k}\right]\end{aligned} \tag{3.21}$$

Consider the first integral in (3.21). In order that the first integral converges for large $|\alpha|$, we close the contour by a large circular arc in the upper half plane, as shown in figure (11), where $\Im\alpha > 0$ along the arc. The term

$$e^{i\alpha x} = e^{i\Re\alpha x}e^{-\Im\alpha x}$$

is exponentially small for positive x. Similarly, for the second integral we must chose the contour by a large circular arc in the lower half plane as shown in figure (12).

Back to the first integral in (3.21)

$$I_1 = \int_\Gamma d\alpha \frac{e^{i\alpha x}e^{\alpha z}}{\alpha - k} \tag{3.22}$$

The contour integral is

$$\begin{aligned}\oint d\alpha\frac{e^{i\alpha x}e^{\alpha z}}{\alpha - k} &= \int_\Gamma d\alpha\frac{e^{i\alpha x}e^{\alpha z}}{\alpha - k} + \int_C d\alpha\frac{e^{i\alpha x}e^{\alpha z}}{\alpha - k} + \int_{i\infty}^0 d\alpha\frac{e^{i\alpha x}e^{\alpha z}}{\alpha - k} \\ &= I_1 + 0 + \int_{i\infty}^0 d\alpha\frac{e^{i\alpha x}e^{\alpha z}}{\alpha - k}\end{aligned}$$

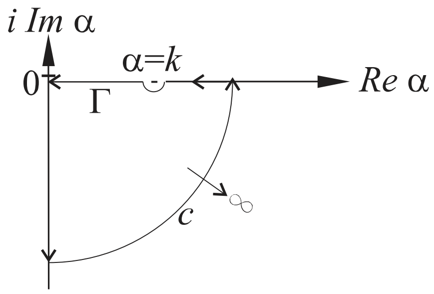

Figure 12: Closed contour in the lower half plane

The contribution by the circular arc C vanishes by Jordan's lemma. The left hand side is

$$LHS = 2\pi i e^{ikx} e^{kz} \tag{3.23}$$

by Cauchy's residue theorem. By the change of variable $\alpha = i\beta$, the right hand side becomes

$$RHS = I_1 + i\int_{\infty}^{0} d\beta \frac{e^{-\beta x} e^{i\beta z}}{i\beta - k}$$

Hence

$$I_1 = 2\pi i e^{ikx} e^{kz} + i\int_{0}^{\infty} d\beta \frac{e^{-\beta x} e^{i\beta z}}{i\beta - k} \tag{3.24}$$

Now consider I_2

$$I_2 = \int_{\Gamma} d\alpha \frac{e^{-i\alpha x} e^{\alpha z}}{\alpha - k} \tag{3.25}$$

and the contour integral along the contour closed in the lower half plane,

$$-\oint d\alpha \frac{e^{-i\alpha x} e^{\alpha z}}{\alpha - k} = I_2 + 0 + \int_{0}^{\infty} d\alpha \frac{e^{-i\alpha x} e^{\alpha z}}{\alpha - k}$$

Again no contribution comes from the circular arc C. Now the pole is outside the contour hence $LHS = 0$. Let $\alpha = -i\beta$ in the last integral we get

$$I_2 = -i\int_{0}^{\infty} d\beta \frac{e^{-\beta x} e^{-i\beta y}}{-i\beta - k} \tag{3.26}$$

Adding the results (3.24) and (3.26).,

$$\begin{aligned} I_1 + I_2 &= 2\pi i e^{ikx} e^{kz} + \int_{0}^{\infty} d\beta \left(\frac{i e^{-\beta x} e^{i\beta z}}{i\beta - k} - \frac{i e^{-\beta x} e^{-i\beta z}}{-i\beta - k} \right) \\ &= 2\pi i e^{ikx} e^{kz} + 2\int_{0}^{\infty} d\beta \frac{e^{-\beta x}}{\beta^2 + k^2} \left(\beta \cos \beta y + k \sin \beta y \right) \end{aligned} \tag{3.27}$$

Finally, the total potential is, on the side of $x > 0$,

$$
\begin{aligned}
\mathcal{G}(x,z,t) &= -\frac{\omega}{\rho g} e^{-i\omega t}\left(\frac{1}{2\pi i}(I_1 + I_2)\right) e^{-i\omega t} \\
&= -\frac{\omega}{\rho g} e^{-i\omega t}\left\{ e^{ikx}e^{kz} + \frac{1}{\pi}\int_0^\infty d\beta \frac{e^{-\beta x}}{\beta^2 + k^2}\left(\beta\cos\beta z + k\sin\beta z\right)\right\}
\end{aligned}
\tag{3.28}
$$

The first term gives an outgoing waves. For a concentrated load with amplitude P_o, the displacement amplitude is $P_o/\rho g$. The integral above represent local effects important only near the applied pressure. If the concentrated load is at $x = x'$, one simply replaces x by $x - x'$ everywhere.

Note on Rayleigh's fictitious damping: If frictional dissipation is accounted for, there should be no wave at infinity. A way to derive the solution for dissipationless theory is to allow a small damping at first, obtain the solution subject to the condition that there is no radiation at infinity, and then take the limit of zero damping. In some problem it is easy to include the real damping mechancism. in some others a fictitious damping can be used.

For linearized water waves we can introduce a fictitious damping term in the momentum equation:

$$
\frac{\partial \mathbf{u}}{\partial t} = -\frac{1}{\rho}\nabla p - g\mathbf{e}_z - \epsilon\mathbf{u} \tag{3.29}
$$

where ϵ is the Rayleigh damping coefficient. Introducing the velocity potential $\mathbf{u} = \nabla\Phi$ we get after integration with respect to space the fictitious Bernoulli equation

$$
\frac{\partial \Phi}{\partial t} + \frac{p}{\rho} + gz - \epsilon\Phi = 0 \tag{3.30}
$$

On the free surface it gives

$$
\frac{\partial \Phi}{\partial t} + \frac{p_a}{\rho} + g\zeta - \epsilon\Phi = 0 \tag{3.31}
$$

Combined with the kinematic boundary coondition we get on the free surface

$$
\frac{\partial^2 \Phi}{\partial t^2} + g\frac{\partial \Phi}{\partial z} - \epsilon\frac{\partial \Phi}{\partial t} = -\frac{1}{\rho}\frac{\partial p_a}{\partial t} \tag{3.32}
$$

With the new term the Fourier integral has two simple poles slighlty off the real axis. One is in the first quadrant and the other in the third. When the limit of zero damping is taken a the end. they give the same residues as before, by Cauchy' theorem.

This devise is of course artificial since in real fluid dissipation by viscosity is of a different form.

We illustrate the use of Rayleigh's damping for two ship-wave problems.

4 Linearized equations in the moving coordinate system

For simplicity let us assume that the disturbance travels at constant speed U from right ($x' \sim \infty$) to left ($x' \sim -\infty$) in the stationary coordinate system (x', y, z, t) where water at infinity is at rest. Let (x, y, z, t) be the coordinate system fixed on the disturbance, then the two systems are related by

$$x = x' + Ut', \quad y = y', \quad z = z', \quad t = t'. \tag{4.1}$$

Using the chain rule

$$\frac{\partial F(x(x',t), y, z, t)}{\partial t'} = \frac{\partial F}{\partial t}\frac{\partial t}{\partial t'} + \frac{\partial F}{\partial x}\frac{\partial x}{\partial t'} = \frac{\partial F}{\partial t} + U\frac{\partial F}{\partial x},$$

$$\frac{\partial F(x(x',t), y, z, t)}{\partial x'} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial x'} = \frac{\partial F}{\partial x}.$$

$$\frac{\partial F}{\partial y'} = \frac{\partial F}{\partial y}\frac{\partial y}{\partial y'} = \frac{\partial F}{\partial y}, \quad \frac{\partial F}{\partial z'} = \frac{\partial F}{\partial y}\frac{\partial z}{\partial z'} = \frac{\partial F}{\partial z}$$

The dynamic and kinematic boundary conditions on the free surface $z = 0$ become, respectively:

$$\Phi_t + U\Phi_x + g\zeta + \epsilon\Phi = -\frac{p_a(x)}{\rho}, \tag{4.2}$$

where ϵ is the Rayleigh damping factor, and

$$\zeta_t + U\zeta_x = \Phi_z \tag{4.3}$$

Inside the fluid we still have

$$\Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 0, \quad z < 0. \tag{4.4}$$

Assuming steady state, (4.2) and (4.3) reduce to

$$U\Phi_x + g\zeta + \epsilon\phi = -\frac{p_a(x)}{\rho}, \quad z = 0, \tag{4.5}$$

and

$$U\zeta_x = \Phi_z, \quad z = 0, \tag{4.6}$$

which can be combined to

$$\Phi_z + \frac{U^2}{g}\Phi_{xx} + \epsilon\frac{U}{g}\Phi_x = -\frac{U}{\rho g}\frac{\partial p_a}{\partial x}, \quad z = 0 \tag{4.7}$$

In the following examples we restrict to deep water so that $\Phi \to 0$ as $z \to -\infty$.

5 Two-dimensional waves due to a traveling surface pressure

5.1 Solution by Fourier transform

The governing equations for $\Phi(x,z)$ are

$$\Phi_{xx} + \Phi_{zz} = 0, \qquad -h < z < 0 \tag{5.1}$$

$$\Phi_{xx} + \frac{g}{U^2}\Phi_z + \frac{\epsilon}{U}\Phi_x = -\frac{1}{\rho U}\frac{\partial p_a}{\partial x}, \qquad z = 0; \tag{5.2}$$

$$\Phi \to 0, \qquad z \to -\infty, \tag{5.3}$$

Applying the exponential Fourier transform with respect to x, defined with its inversion by

$$\bar{\Phi}(k,z) = \int_{-\infty}^{\infty} e^{-ikx}\Phi(x,z)dx, \qquad \Phi(x,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\bar{\Phi}(k,z)dk \tag{5.4}$$

we get a two-point boundary value problem for $\bar{\Phi}$,

$$\frac{d^2\bar{\Phi}}{dz^2} - k^2\bar{\Phi} = 0, \qquad -h < z < 0 \tag{5.5}$$

$$\frac{g}{U^2}\bar{\Phi}_z - k^2\bar{\Phi} + \frac{ik\epsilon}{U}\bar{\Phi} = -\frac{ik}{\rho U}\bar{p}_a(k), \qquad z = 0; \tag{5.6}$$

As an example we take

$$p_a(x) = P_o e^{x^2/4L^2}, \quad \text{so that} \quad \bar{p}_a(k) = \frac{P_o L}{\sqrt{\pi}} e^{-k^2L^2} \tag{5.7}$$

The solution to (5.5) is

$$\bar{\Phi} = Ae^{|k|z} \tag{5.8}$$

which vanishes at great depth. To satisfy the surface boundary condition we must have,

$$\left[\frac{g}{U^2}|k| - k^2 + \frac{i\epsilon k}{U}\right] A = \frac{-ik\bar{p}_a}{\rho U} \tag{5.9}$$

hence

$$A = \frac{-\frac{ik}{\rho U}\bar{P}_a(k)}{\frac{g}{U^2}|k| - k^2 + \frac{i\epsilon k}{U}} \tag{5.10}$$

and

$$\bar{\Phi}(k,y,z) = \frac{-\frac{ik}{\rho U}\bar{P}_a(k)e^{|k|z}}{\frac{g}{U^2}|k| - k^2 + \frac{i\epsilon k}{U}} \tag{5.11}$$

By inverse transform the potential is

$$\Phi(x,y,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\; e^{ikx}\frac{-\frac{ik}{\rho U}\bar{p}_a(k)e^{|k|z}}{\frac{g}{U^2}|k| - k^2 + \frac{i\epsilon k}{U}} \tag{5.12}$$

The Fourier integral can be split into two:

$$\begin{aligned} \Phi(x,y,z) &= \frac{1}{2\pi}\int_{0}^{\infty} dk\; e^{ikx}\frac{-\frac{ik}{\rho U}\bar{p}_a(k)e^{|k|z}}{\frac{g}{U^2}|k| - k^2 + \frac{i\epsilon k}{U}} \\ &+ \frac{1}{2\pi}\int_{-\infty}^{0} dk\; e^{ikx}\frac{-\frac{ik}{\rho U}\bar{p}_a(k)e^{|k|z}}{\frac{g}{U^2}|k| - k^2 + \frac{i\epsilon k}{U}} \end{aligned} \tag{5.13}$$

In the first integral $|k| = k$. In the second $|k| = -k$, hence

$$\begin{aligned} \Phi(x,y,z) &= \frac{1}{2\pi}\int_{0}^{\infty} dk\; e^{ikx}\frac{-\frac{ik}{\rho U}\bar{p}_a(k)e^{kz}}{\frac{g}{U^2}k - k^2 + \frac{i\epsilon k}{U}} \\ &+ \frac{1}{2\pi}\int_{-\infty}^{0} dk\; e^{ikx}\frac{-\frac{ik}{\rho U}\bar{p}_a(k)e^{-kz}}{-\frac{g}{U^2}k - k^2 + \frac{i\epsilon k}{U}} \end{aligned} \tag{5.14}$$

Let us replace k by -k in the second integral so that

$$\begin{aligned} \Phi(x,y,z) &= \frac{1}{2\pi}\int_{0}^{\infty} dk\; e^{ikx}\frac{-\frac{ik}{\rho U}\bar{p}_a(k)e^{kz}}{\frac{g}{U^2}k - k^2 + \frac{i\epsilon k}{U}} \\ &+ \frac{1}{2\pi}\int_{0}^{\infty} dk\; e^{-ikx}\frac{\frac{ik}{\rho U}\bar{p}_a(k)e^{kz}}{\frac{g}{U^2}k - k^2 - \frac{i\epsilon k}{U}} \\ &= \frac{i}{2\pi\rho U}\int_{0}^{\infty} dk\; e^{ikx}\frac{\bar{p}_a(k)e^{kz}}{k - \frac{g}{U^2} - \frac{i\epsilon}{U}} \\ &- \frac{i}{2\pi\rho U}\int_{0}^{\infty} dk\; e^{-ikx}\frac{\bar{p}_a(k)e^{kz}}{k - \frac{g}{U^2} + \frac{i\epsilon}{U}} \end{aligned} \tag{5.15}$$

Use has been made of the fact $\bar{p}_a(k)$ given by (5.7) is even in k.

5.2 Asymptotic solution in the far wake

For brevity we shall write (5.15) as

$$\Phi(x,y,z) = \Phi_1 + \Phi_2 \tag{5.16}$$

where Φ_1 and Φ_2 represent the first and second integrals respectively. Let us examine each integral in turn in the complex k plane. The first integrand has a pole in the

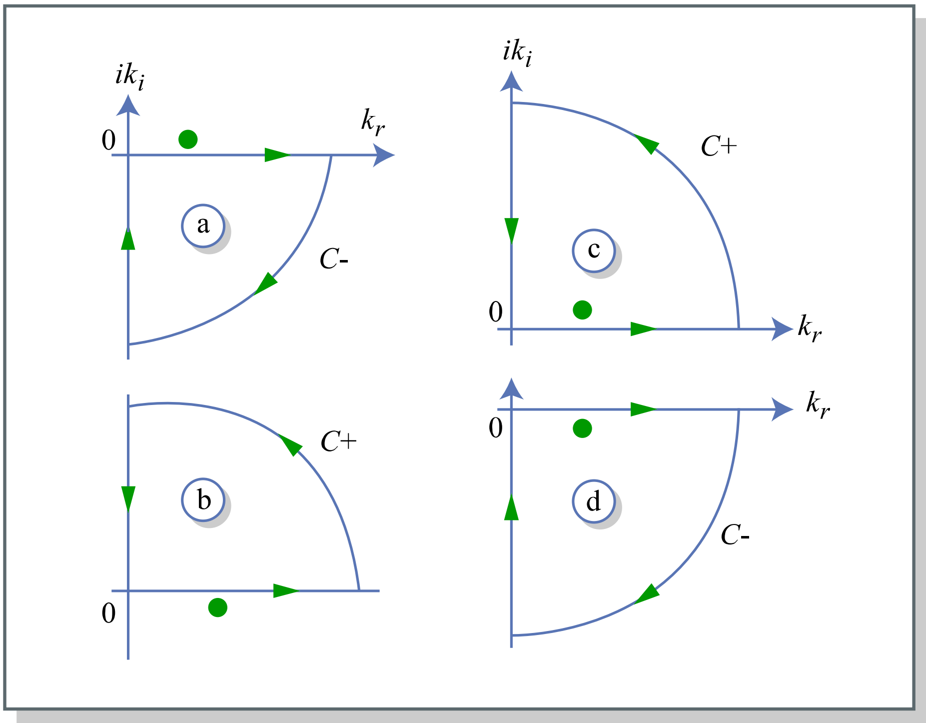

Figure by MIT OCW.

Figure 13: Contours in the complex k plane: (a) for Φ_1 and $x < 0$.,(b) for Φ_2 and $x < 0$, (c) for Φ_1 and $x > 0$, (d) for Φ_2 and $x > 0$.

first quadrant; the second has one in the fourth quadrant. If no Rayleigh damping were assumed, both poles would be on the path of integration, rendering the integrals undefined, unless the radiation condition is added, see §3.

Consider the first $x < 0$ i.e., ahead of the ship. We replace the integral path for Φ_1 by the closed clockwise contour in the fourth quadrant, as shown in Figure 5.2-a. The contour consists of the orginal path, hence Φ_1, an integral along a circular arc C_- with very large radius, and the positive imaginary axis, all with the same integrand,

$$\frac{i}{2\pi\rho U}\int_0^\infty dk\ e^{ikx}\frac{\bar{p}_a(k)e^{kz}}{k-\frac{g}{U^2}-\frac{i\epsilon}{U}} = \Phi_1 + \frac{i}{2\pi\rho U}\left\{\int_{C_-}+\int_{-i\infty}^0\right\}dke^{ikx}\frac{\bar{p}_a(k)e^{kz}}{k-\frac{g}{U^2}-\frac{i\epsilon}{U}} \quad (5.17)$$

Note that the contour is closed in the fourth quadrant where $k_i < 0$ to ensure convergence for $x < 0$, since

$$e^{ikx} = e^{ik_r x}e^{-k_i x} \quad (5.18)$$

Let us show first that the integral along the imaginary axis has only local effects,

$$I = \int_{-i\infty}^0 dke^{ikx}\frac{\bar{p}_a(k)e^{kz}}{k-\frac{g}{U^2}-\frac{i\epsilon}{U}} \quad (5.19)$$

Let $k = -iK$ where Kb is real and positive, we get

$$I = i \int_{\infty}^{0} dK \frac{e^{-Kx} e^{-iKz} \bar{p}_a(-iK)}{iK + \frac{g}{U^2}} \tag{5.20}$$

This integral clearly diminishes to zero as $x \to \infty$. Next, the integral along the arc C_- vanishes by Jordan's lemma. Since no pole is inside the contour, the contour integral is zero by Cauchy's residue theorem. Hence Φ_1 has only local effect and diminishes to zero as far ahead of the moving pressure.

Similarly for Φ_2 we choose a closed contour in the first quadrant where $k_i > 0$ as in Figure 5.2-b, since

$$e^{-ikx} = e^{-ik_r x} e^{k_i x}. \tag{5.21}$$

Again there is no pole inside the contour, hence the contour integral vanishes. Since nothing comes from the circular arc, and the integral along the imaginary axis has only local effect, Φ_2 has only local effects also, In summary, far ahead of the moving disturance, wave motion ($\Phi = \Phi_1 + \Phi_2$) is negligible.

Let us now consider $x > 0$, i.e., the wake. For Φ_1 the closed contour must now be in the first quadrant where there is the pole at

$$k = \frac{g}{U^2} + \frac{i\epsilon}{U} \tag{5.22}$$

see Figure 5.2-c. By Cauchy's residue theorem the (counter-clockwise) contour integral is

$$\frac{i}{2\pi\rho U} \oint dk \; e^{ikx} \frac{\bar{p}_a(k) e^{kz}}{k - \frac{g}{U^2} - \frac{i\epsilon}{U}} = -\frac{1}{\rho U} e^{igx/U^2} \bar{P}_a \left(\frac{gz}{U^2}\right) e^{gz/U^2} \tag{5.23}$$

By repeating earlier reasoning, integrals along the arc and along the positive imarginary axis do not matter in the far field, hence

$$\Phi_1 \approx = -\frac{1}{\rho U} \frac{P_0}{\sqrt{\pi}} e^{igx/U^2} e^{-(gL/U^2)^2} e^{gz/U^2} \tag{5.24}$$

On the other hand, for Φ_1 the closed contour must now be in the fourth quadrant where there is the pole at

$$k = \frac{g}{U^2} - \frac{i\epsilon}{U} \tag{5.25}$$

see Figure 5.2-d. By Cauchy's residue theorem the contour integral in the clockwise direction is

$$-\frac{i}{2\pi\rho U} \oint dk \; e^{-ikx} \frac{\bar{p}_a(k) e^{kz}}{k - \frac{g}{U^2} + \frac{i\epsilon}{U}} = -\frac{1}{\rho U} e^{-igx/U^2} \bar{P}_a \left(\frac{gz}{U^2}\right) e^{gz/U^2} \tag{5.26}$$

Again, integrals along the arc and along the positive imarginary axis do not matter in the far field, hence

$$\Phi_2 \approx -\frac{1}{\rho U}\frac{P_0}{\sqrt{\pi}}e^{-igx/U^2}e^{-(gL/U^2)^2}e^{gz/U^2} \tag{5.27}$$

Finally in the far wake,

$$\Phi = \Phi_1 + \Phi_2 \approx -\frac{2}{\rho U}\frac{P_0}{\sqrt{\pi}}\cos(gx/U^2)e^{-(gL/U^2)^2}e^{gz/U^2} \tag{5.28}$$

Thus one sees a train of sinusoidal surface waves of length U^2/g following the disturbance. The wave amplitude is

$$P_0 e^{-(gL/U^2)^2} \tag{5.29}$$

which is large if the pressure variation is sharp, and small if it is flat.

We leave it as an exercise to derive the wave drag.

6 The physics of Kelvin's ship-wave pattern

Refs: Explansions are due to LIGHTHILL, First ONR SYMPOSIUM on Naval Hydrodynamics (1959?).

Material here is borrowed from the lecture notes by T. Y. Wu, Caltech.

Mathematical details can be found in Stoker: Water Waves 1957.

– – – – – – – – – – –

The action of the ship's propeller
Has a thrust pattern
To which the ship reacts by moving forward,
Which also results secondarily,
In the ship's bow elevated waves,
And its depressed transverse stern wave,
Which wave disturbances of the water
Are separate from the propeller's thrust waves.

– –

R.Buckminster Fuller, *Intuition- Metaphysical Mosaic.* 1972.

Anyone flying over a moving ship must be intrigued by the beautiful pattern in the ship's wake. The theory behind it was first completed by Lord Kelvin, who invented the method of stationary phase for the task. Here we shall give a physical/geometrical derivation of the key results

Consider first two coordinate systems. The first $\mathbf{r} = (x, y, z)$ moves with ship at the uniform horizontal velocity $\mathbf{U}$. The second $\mathbf{r'} = (x', y', z)$ is fixed on earth so that water is stationary while the ship passes by at the velocity $\mathbf{U}$. The two systems are related by the Galilean transformation,

$$\mathbf{r'} = \mathbf{r} + \mathbf{U}t \tag{6.1}$$

A train of simple harmonic progressive wave

$$\zeta = \Re\{A \exp[i(\mathbf{k} \cdot \mathbf{r'} - \omega t)]\} \tag{6.2}$$

in the moving coordinates should be expressed as

$$\begin{aligned} \zeta &= \Re\{A \exp[i\mathbf{k} \cdot (\mathbf{r} - \mathbf{U}t) - i\omega t]\} = \Re\{A \exp[i\mathbf{k} \cdot \mathbf{r} - i(\omega - \mathbf{k}.\mathbf{U})t]\} \\ &= \Re\{A \exp[i\mathbf{k} \cdot \mathbf{r} - i\sigma t]\} \end{aligned} \tag{6.3}$$

in the stationary coordinates. Therefore the apparent frequency in the moving coordinates is

$$\sigma = \omega - \mathbf{k} \cdot \mathbf{U} \tag{6.4}$$

The last result is essentially the famous Doppler's effect. To a stationary observer, the whistle from an approaching train has an increasingly high pitch, while that from a leaving train has a decreasing pitch.

If a ship moves in very deep water at the constant speed $-\mathbf{U}$ in stationary water, then relative to the ship, water appears to be washed downstream at the velocity $\mathbf{U}$. A stationary wave pattern is formed in the wake. Once disturbed by the passing ship, a fluid parcel on the ship's path radiates waves in all directions and at all frequencies. Wave of frequency ω spreads out radially at the phase speed of $c = g/\omega$ according to the dispersion relation. Only those parts of the waves that are stationary relative to the ship will form the ship wake, and they must satisfy the condition

$$\sigma = 0, \tag{6.5}$$

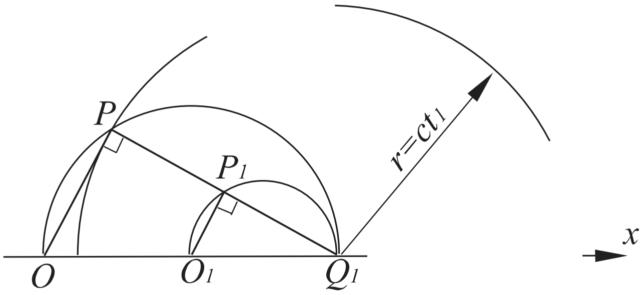

Figure 14: Waves radiated from disturbed fluid parcel

i.e.,

$$\omega = \mathbf{k} \cdot \mathbf{U}, \text{or} \quad c = \frac{\omega}{k} = \frac{\mathbf{k}}{k} \cdot \mathbf{U} \tag{6.6}$$

Referring to figure 14, let $O, (x = 0)$ represents the point ship in the ship-bound coordinates. The current is in the positive x direction. Any point x_1 is occupied by a fluid parcel Q_1 which was disturbed directly by the passing ship at time $t_1 = x_1/U$ earlier. This disurbed parcel radiates waves of all frequencies radially. The phase of wave at the frequency ω reaches the circle of radius ct_1 where c=g/ω by the deep water dispersion relation. Along the entire circle however only the point that satisfies (6.6) can contribute to the stationary wave pattern, as marked by P. Since $OQ_1 = x_1 = Ut_1$, $Q_1P = ct_1$ and $OP = \mathbf{Ut_1} \cdot \mathbf{k}/k$, where $\mathbf{k}$ is in the direction of $\vec{Q_1P}$. It follows that $\triangle OPQ_1$ is a right triangle, and P lies on a semi circle with diameter OQ_1. Accounting for the radiated waves of all frequencies, hence all c, every point on the semi circle can be a part of the stationary wave phase formed by signals emitted from Q_1. Now this argument must be rectified because wave energy only travels at the group velocity which is just half of the phase velocity in deep water. Therefore stationary crests due to signals from Q_1 can only lie on the semi-circle with the diameter $O_1Q_1 = OQ_1/2$. Thus P_1 instead of P is one of the points forming a stationary crest in the ship's wake, as shown in figure 14.

Any other fluid parcel Q_2 at x_2 must have been disturbed by the passing ship at time $t_2 = x_2/U$ earlier. Its radiated signals contribute to the stationary wave pattern only along the semi circle with diameter $O_2Q_2 = OQ_2/2$. Combining the effects of all fluid parcels along the $+x$ axis, stationary wave pattern must be confined inside the wedge

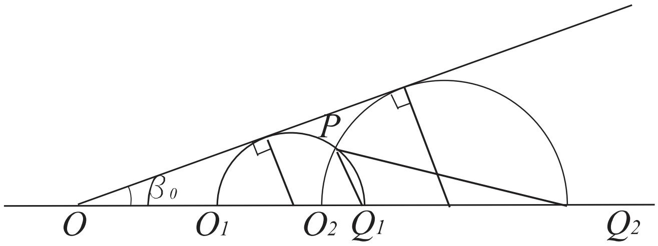

Figure 15: Wedge angle of the ship wake

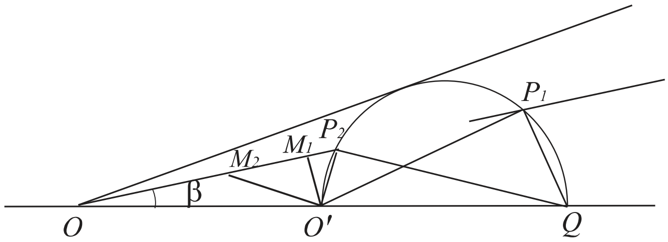

Figure 16: Geometrical relation to find Points of dependence

which envelopes all these semi circles. The half apex angle β_o of the wedge, which defines the wake, is given by

$$\sin \beta_o = \frac{Ut/4}{3Ut/4} = 1/3, \tag{6.7}$$

hence $\beta_o = \sin^{-1} 1/3 = 19.5°$, see figure 15.

Now any point P inside the wedge is on two semicircles tangent to the boundary of the wedge, i.e., there are two segments of the wave crests intersecting at P: one perpendicular to PQ_1 and one to PQ_2, as shown in figure 15.

Another way of picturing this is to examine an interior ray from the ship. In figure (16), draw a semi circle with the diameter $O'Q = OQ/2$, then at the two intersections P_1 and P_2 with the ray are the two segments of the stationary wave crests. In other words, signals originated from Q contribute to the stationary wave pattern only at the two points P_1 and P_2, as shown in figure 16. Point Q can be called the point of dependence for points P_1 and P_2 on the crests.

For any interior point P there is a graphical way of finding the two points of dependence Q_1 and Q_2. Referring to figure 16, $\triangle O'QP_1$ and $\triangle O'QP_2$ are both right triangles. Draw $O_1M_1 \parallel QP_1$ and $O_2M_2 \parallel QP_2$ where M_1 and M_2 lie on the ray inclined at the an-

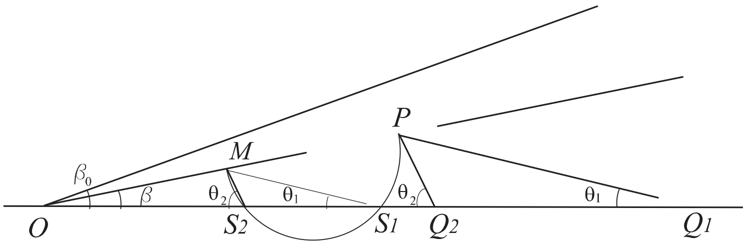

Figure 17: Points of dependence

gle β. it is clear that $OM_1 = OP_1/2$ and $OM_2 = OP_2/2$, and $\triangle M_1O'P_1$ and $\triangle M_2O'P_2$ are both right triangles. Hence O' lies on two semi circles with diameters M_1P_1 and M_2P_2.

We now reverse the process, as shown in figure 17. For any point P on an interior ray, let us mark the mid point M of OP and draw a semi circle with diameter MP. The semi circle intersects the x axis at two points S_1 and S_2. We then draw from P two lines parallel to MS_1 and MS_2, the two points of intersection Q_1 an Q_2 on the x axis are just the two points of dependence.

Let $\angle PQ_1O = \angle MS_1O = \theta_1$ and $\angle PQ_2O = \angle MS_2O = \theta_2$. then

$$\tan(\theta_i + \beta) = \frac{PS_i}{MS_i} = \frac{PS_i}{PQ_i/2} = 2\tan\theta_i \quad i = 1, 2.$$

hence

$$2\tan\theta_i = \frac{\tan\theta_i + \tan\beta}{1 - \tan\theta_i \tan\beta}$$

which is a quadratic equation for θ_i, with two solutions:

$$\left\{ \begin{array}{c} \tan\theta_1 \\ \tan\theta_2 \end{array} \right\} = \frac{1 \pm \sqrt{1 - 8\tan^2\beta}}{4\tan\beta} \tag{6.8}$$

They are real and distinct if

$$1 - 8\tan^2\beta > 0 \tag{6.9}$$

These two angles define the local stationary wave crests crossing P, and they must be perpendicular to PQ_1 and PQ_2. There are no solutions if $1 - 8\tan^2\beta < 0$, which corresponds to $\sin\beta > 1/3$ or $\beta > 19.5°$, i.e., outside the wake. At the boundary of the wake, $\beta = 19,5°$ and $\tan\beta = \sqrt{1/8}$, the two angles are equal

$$\theta_1 = \theta_2 = \tan^{-1}\frac{\sqrt{2}}{2} = 35°16'. \tag{6.10}$$

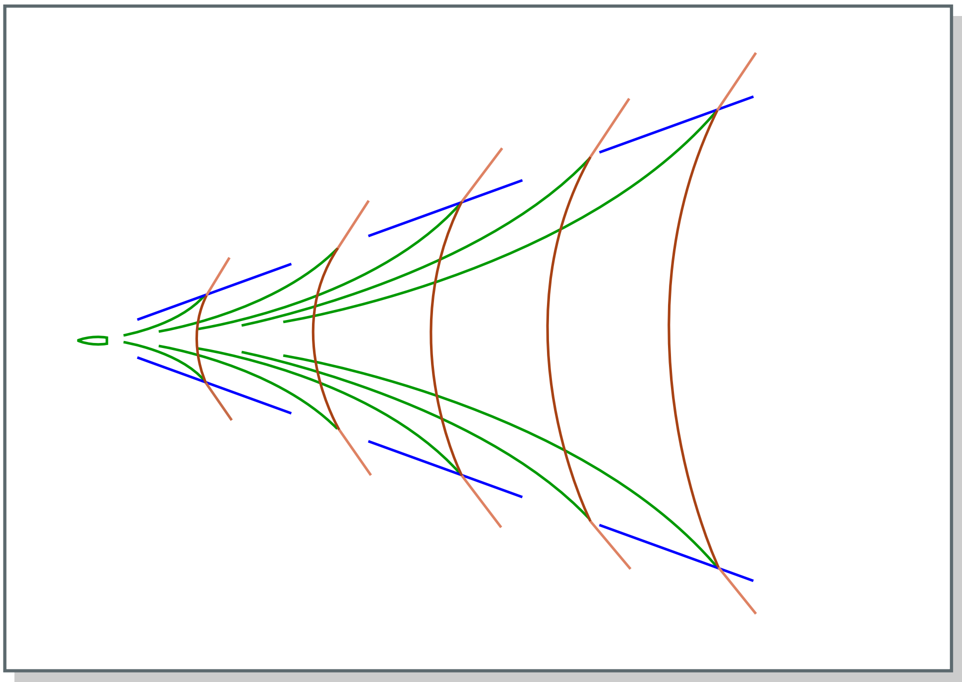

Figure by MIT OCW.

Figure 18: Diverging and transverse waves in a ship wake

By connecting these segments at all points in the wedge, one finds two systems of wave crests, the *diverging waves* and the *transverse waves*, as shown in figure div-trans.

A beautiful photograph is shown in Figure 19

Knowing that waves are confined in a wedge, we can estimate the behavior of the wave amplitude by balancing in order of magnitude work done by the wave drag R and the steady rate of energy flux

$$RU = (\bar{E}c_g)r \sim (|A|^2 c_g)r \tag{6.11}$$

hence

$$A \sim r^{1/2} \tag{6.12}$$

This estimate is valid throughout the wedge except near the outer boundaries, where

$$A \sim r^{-1/3} \tag{6.13}$$

by a more refined analysis (Stoker, 1957, or Wehausen & Laitone, 1960).

"Photograph removed due to copyright restrictions."

Figure 19: Ships in a straight course. From Stoker, 1957.p. 280.

7 Three-dimensional analysis of Kelvin's ship wave

References: J. N. Newman: *Marine Hydrodynamics*

J. J. Stoker: *Water Waves*,

J. V. Wehausen and E. V. Laitone *Surface Waves* in *Handbuch der Physik*: Band IX. Springer.

T Y. Wu *Lecture Notes on Water Waves*, Calif. Inst. Tech.

7.1 Solution by Fourier transform

Define the double Fourier transform:

$$\bar{\Phi}(\alpha,\beta,z) = \iint_{-\infty}^{\infty} e^{-i\alpha x - i\beta y}\Phi(x,y,z)dxdy \tag{7.14}$$

and its inverse:

$$\Phi(x,y,z) = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} e^{i\alpha x + i\beta y}\bar{\Phi}(\alpha,\beta,z)d\alpha d\beta \tag{7.15}$$

Applying the Fourier Transform to Laplace's equation and the boundary conditions we get

$$\bar{\Phi}_{zz} - k^2\Phi = 0, \quad \text{where } \ k = \sqrt{\alpha^2+\beta^2}, \quad z<0 \tag{7.16}$$

$$\bar{\Phi}_z - \left(\frac{U\alpha^2}{g} - i\epsilon\alpha\right)\Phi = -i\alpha\frac{U}{\rho g}\bar{p}_a \tag{7.17}$$

The solution is easily found to be

$$\bar{\Phi} = \frac{U}{\rho g}\frac{i\alpha\bar{p}_a\ e^{kz}}{\frac{U^2\alpha^2}{g} - i\epsilon\alpha - k} \tag{7.18}$$

Formally the inverse transform is

$$\Phi(x,y,z) = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} d\alpha d\beta\ e^{i\alpha x + i\beta y}\frac{U}{\rho g}\frac{i\alpha\bar{p}_a\ e^{kz}}{\frac{U^2\alpha^2}{g} - i\epsilon\alpha - k} \tag{7.19}$$

The Fourier transform of the kinematic free surface condition is

$$i\alpha U\bar{\zeta}(\alpha,\beta) = \bar{\Phi}_z(\alpha,\beta,0) \tag{7.20}$$

Hence the displacement is

$$\zeta(x,y) = \frac{1}{4\pi^2}\frac{1}{\rho g}\iint_{-\infty}^{\infty} d\alpha d\beta\ e^{i\alpha x + i\beta y}\frac{\bar{p}_a}{\frac{U^2\alpha^2}{g} - i\epsilon\alpha - k} \tag{7.21}$$

We shall take a Gaussian pressure distribution for illustration,

$$p_a = P_0 e^{(x^2+y^2)/4L^2} \tag{7.22}$$

The Fourier transform is

$$\bar{p}_a = \frac{P_0 L^2}{\pi} e^{k^2 L^2} \tag{7.23}$$

From here on we shall discuss the free surface displacment only,

$$\zeta(x,y) = \frac{P_o L^2}{\rho g \pi} \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} d\alpha d\beta \; e^{i\alpha x + i\beta y} \frac{k e^{-k^2 L^2}}{\frac{U^2 \alpha^2}{g} - i\epsilon\alpha - k} \tag{7.24}$$

7.2 Asymptotic analysis of the wake

If we let

$$x = r\cos\theta, \quad y = r\sin\theta; \quad \alpha = k\cos\psi, \beta = k\sin\psi \tag{7.25}$$

then

$$\alpha x + \beta y = kr\cos(\theta - \psi) \tag{7.26}$$

(7.24) becomes

$$\begin{aligned} \zeta(x,y) &= \frac{P_o L^2}{\rho g \pi} \frac{1}{4\pi^2} \int_0^{2\pi} d\psi \int_0^{\infty} dk \; e^{ikr\cos(\psi-\theta)} \frac{k e^{-k^2 L^2}}{\frac{U^2 k \cos^2\psi}{g} - i\epsilon\cos\psi - 1} \\ &= \frac{P_o L^2}{\rho g \pi} \frac{g}{U^2} \frac{1}{4\pi^2} \int_0^{2\pi} \frac{d\psi}{\cos^2\psi} \int_0^{\infty} dk \; e^{ikr\cos(\psi-\theta)} \frac{k e^{-k^2 L^2}}{\left(k - \frac{g}{U^2\cos^2\psi} - \frac{i\epsilon g}{U^2\cos\psi}\right)} \end{aligned} \tag{7.27}$$

We now evaluate the $k-$integral in the complex plane of $k = |k|e^{i\sigma}$. The integrand has a simple pole at

$$k = \frac{g}{U^2\cos^2\psi} + \frac{i\epsilon g}{U^2\cos\psi} \tag{7.28}$$

which is slightly above the real axis. Let us use Cauchy residue theorem by choosing a closed contour in the complex k plane. Since

$$ikr\cos(\psi - \theta) = (i|k|r\cos\sigma - |k|r\sin\sigma)\cos(\psi - \theta) \tag{7.29}$$

For $0 < |\psi - \theta| < \pi/2$ we must choose $0 < \sigma < \pi/2$ for the Fourier integral to converge. Thus we close the counter-wise contour in the first quadrant by adding a large circular arc connecting the ends of the real and imaginary axes $k \sim \infty$ and $k \sim i\infty$, and then the positive imaginary k axis, as shown in Figure 5.2. The contour integral in k is equal to the residue at the simple pole by Cauchy's theorem,

$$\left\{\int_0^\infty + \int_C + \int_{i\infty}^0\right\} dk\,[...] = \frac{2\pi i g}{U^2\cos^4\psi}\exp\left[-\left(\frac{gL}{U^2\cos^2\psi}\right)^2\right]\exp\left[i\frac{gr}{U^2}\frac{\cos(\psi-\theta)}{\cos^2\psi}\right]. \tag{7.30}$$

The limit of $\epsilon = 0$ has been taken after calculating the residue. It can be shown that the line integral along the imaginary axis has only local effects and dies out quickly for large r. The line integral along the circular arc also vanishes in the limit of large $|k|$ by Jordan's lemma. Hence the residue is the only important term representing the line integral along the real k axis. Subsitituting this result in (7.27), we get

$$\zeta(x,z) \approx \frac{i}{2\pi^2}\frac{P_0L^2}{\rho g}\frac{g}{U^2}\int_0^{2\pi}\frac{d\psi}{\cos^4\psi}\exp\left[-\left(\frac{gL}{U^2\cos^2\psi}\right)^2\right]\exp\left[i\frac{gr}{U^2}\frac{\cos(\psi-\theta)}{\cos^2\psi}\right] \tag{7.31}$$

For $\pi/2 < |\psi-\theta| < \pi$ we must choose $-\pi/2 < \sigma < 0$ for the Fourier integral to converge. In other words, we take a clock-wise countour closed by a circular arc in the fourth quadrant. The pole is now outside the contour and the residue is zero. Hence the k integral is negligible.

In the far wake $\frac{gr}{U^2} \gg 1$, we apply the method of stationary phase to (7.31). Let the phase function be

$$F(\psi) = \frac{\cos(\psi-\theta)}{\cos^2\psi} \tag{7.32}$$

Its derivative is

$$\begin{aligned} F_\psi(\psi,\theta) = \frac{dF}{d\psi} &= -\frac{\sin(\psi-\theta)}{\cos^2\psi} + 2\frac{\sin\psi\cos(\psi-\theta)}{\cos^3\psi} \\ &= \frac{\cos(\psi-\theta)}{\cos^2\psi}\left[-\tan(\psi-\theta)+2\tan\psi\right] \\ &= F\left[-\tan(\psi-\theta)+2\tan\psi\right] \end{aligned} \tag{7.33}$$

The points of stationary phase are the roots of

$$-\tan(\psi-\theta)+2\tan\psi = 0 \tag{7.34}$$

or

$$2\tan\theta\tan^2\psi + \tan\psi + \tan\theta = 0 \tag{7.35}$$

Note that $\tan\theta = y/x$ represents the ray where an observer sits. Because of symmetry with respect to the x axis we need to consider $x > 0, y > 0$ only. There are two negative solutions

$$\begin{pmatrix} \tan\psi_1 \\ \tan\psi_2 \end{pmatrix} = \frac{-1 \mp \sqrt{1 - 8\,(y/x)^2}}{4y/x} \tag{7.36}$$

if (x, y) is inside the half-wedge $\tan\theta = y/x < 1/\sqrt{8}$, i.e., $\theta < 19.3°$. Outside the wedge no stationary phase point exists.

By definition (7.26),

$$\tan\psi = \frac{\beta}{\alpha} \tag{7.37}$$

gives the direction of the local wavenumber vector. For a point along the edge of the wake, $y/x = 1/\sqrt{8}$,

$$\tan\psi_1 = \tan\psi_2 = -\frac{1}{\sqrt{2}}, \quad \text{or,} \quad \psi_1 = \psi_2 = -35°16'. \tag{7.38}$$

For a point along the ray $y/x \approx 0$,

$$\tan\psi_1 \approx -\frac{1}{2y/x} \sim -\infty, \tag{7.39}$$

hence $\psi_1 \approx -\frac{\pi}{2}$, and

$$\tan\psi_2 = -\frac{1 - \sqrt{1 - 8(y/x)^2}}{4y/x} \approx -\frac{4\,(y/x)^2}{4y/x} = -\frac{y}{x}, \tag{7.40}$$

hence $\psi_2 \approx -0$. Wave crests corresponding to ψ_1 are locally parallel to the the ship's path (the x axis). These are the diverging waves. On the other hand, wave crests corresponding to ψ_2 are locally perpendicular to the ship's path; they are the transverse waves. The ranges of $\tan\psi$ are

$$\begin{aligned} \text{Diverging wave} \quad &: \quad -\frac{1}{\sqrt{2}} > \tan\psi_1 > -\infty, \\ \text{Transverse wave} \quad &: \quad 0 > \tan\psi_2 < -\frac{1}{\sqrt{2}} \end{aligned} \tag{7.41}$$

To apply the formula of the stationary phase method we calculate $F_{\psi\psi}(\psi)$ from (7.33) at the stationary points:

$$F_{\psi\psi}(\psi_m) = F_\psi[-\tan(\psi_m - \theta) + 2\tan\psi_m] + F[-\sec^2(\psi_m - \theta) + 2\sec^2\psi_m], \quad m = 1, 2.$$

At the stationary points the first term vanishes. We rewrite the second term as

$$F_{\psi\psi}(\psi_m) = F\left[1 - \tan^2(\psi_m - \theta) + 2\tan^2\psi_m\right] = F\left[1 - 2\tan^2\psi_m\right] \tag{7.42}$$

Use has been made of (7.34). It is clear from (7.41) that $F_{\psi\psi}(\psi_1) < 0$ for the diverging wave and $F_{\psi\psi}(\psi_2) > 0$ for the transverse wave. It follows that the free surface elevation in the wake is dominated by

$$\begin{aligned} \zeta(x,y) \quad \approx \quad & \frac{i}{2\pi^2}\frac{P_0L^2}{\rho g}\frac{g}{U^2}\left\{\left(\frac{2\pi U^2}{gr|F_{\psi\psi}(\psi_2)|}\right)^{1/2}\frac{\exp\left[-\left(\frac{gL}{U^2\cos^2\psi_1}\right)^2\right]}{\cos^4\psi_1}\exp\left[i\frac{gr}{U^2}\frac{\cos(\psi_1-\theta)}{\cos^2\psi_1} - \frac{i\pi}{4}\right]\right. \\ + \quad & \left.\left(\frac{2\pi U^2}{gr|F_{\psi\psi}(\psi_2)|}\right)^{1/2}\frac{\exp\left[-\left(\frac{gL}{U^2\cos^2\psi_1}\right)^2\right]}{\cos^4\psi_2}\exp\left[i\frac{gr}{U^2}\frac{\cos(\psi_2-\theta)}{\cos^2\psi_2} + \frac{i\pi}{4}\right]\right\} \end{aligned} \tag{7.43}$$

Thus the two wave systems are out of phase bu $\pi/2$. Each wave decays with distance as $1/\sqrt{r}$. The wave length of each wave system is

$$\lambda_m = \frac{2\pi U^2\cos\psi_m}{g} \tag{7.44}$$

For the diverging waves, $\cos\psi_1 \approx 0$ along the centerline, where the wave length is the shortest and increases outwards.

Note that near the edge of the wake, $F_{\psi\psi} = 0$. The present approximation breaks down. A better approximation can be made (see Lamb, Stoker).

8 Two-dimensional Internal waves in a continuously stratified fluid

[References]:

C.S. Yih, 1965, *Dynamics of Inhomogeneous Fluids*, MacMillan.

O. M. Phillips, 1977, *Dynamics of the Upper Ocean*, Cambridge U. Press.

P. G. Baines, 1995, *Topographical Effects in Stratified Flows* Cambridge U. Press.

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Due to seasonal changes of temperature, the density of water or atmosphere can have significant variations in the vertical direction. Variation of salt content can also lead to density stratification. Freshwater from rivers can rest on top of the sea water. Due to the small diffusivity, the density contrast remains for a long time.

Consider a calm and stratified fluid with a static density distribution $\overline{\rho}_o(z)$ which decreases with height (z). If a fluid parcel is moved from the level z upward to $z+\zeta$, it is surrounded by lighter fluid of density $\overline{\rho}(z+dz)$. The upward buoyancy force per unit volume is

$$g(\overline{\rho}(z+\zeta)-\overline{\rho}(z)) \approx g\frac{d\overline{\rho}}{dz}\zeta$$

and is negative. Applying Newton's law to the fluid parcel of unit volume

$$\overline{\rho}\frac{d^2\zeta}{dt^2} = g\frac{d\overline{\rho}}{dz}\zeta$$

or

$$\frac{d^2\zeta}{dt^2} + N^2\zeta = 0 \tag{8.1}$$

where

$$N = \left(-\frac{g}{\overline{\rho}}\frac{d\overline{\rho}}{dz}\right)^{1/2} \tag{8.2}$$

is called the Brunt-Väisälä frequency. This elementary consideration shows that once a fluid is displaced from its equilibrium position, gravity and density gradient provides restoring force to enable oscillations. In general there must be horizontal nonunifomities, hence waves are possible.

We start from the exact equations for an inviscid and incompressible fluid with variable density.

For an incompressible fluid the density remains constant as the fluid moves,

$$\rho_t + \mathbf{q} \cdot \nabla \rho = 0 \tag{8.3}$$

where $\mathbf{q} = (u, w)$ is the velocity vector in the vertical plane of (x, z). Conservation of mass requires that

$$\nabla \cdot \mathbf{q} = 0 \tag{8.4}$$

The law of momentum conservation reads

$$\rho(\mathbf{q}_t + \mathbf{q} \cdot \nabla \mathbf{q}) = -\nabla p - \rho g \mathbf{e}_z \tag{8.5}$$

and $\mathbf{e}_z$ is the unit vector in the upward vertical direction.

8.1 Linearized equations

Consider small disturbances

$$p = \overline{p} + p', \quad \rho = \overline{\rho}(z) + \rho', \quad \vec{q} = (u', w') \tag{8.6}$$

with

$$\overline{\rho} \gg \rho', \quad \overline{p} \gg p' \tag{8.7}$$

and u', v', w' are small. Linearizing by omitting quadratically small terms associated with the fluid motion, we get

$$\rho'_t + w' \frac{d\overline{\rho}}{dz} = 0. \tag{8.8}$$

$$u'_x + w'_z = 0 \tag{8.9}$$

$$\overline{\rho} u'_t = -p'_x \tag{8.10}$$

$$\overline{\rho} w'_t = -\overline{p}_z - p'_z - g\overline{\rho} - g\rho' \tag{8.11}$$

In the last equation the static part must be in balance

$$0 = -\overline{p}_z - g\overline{\rho}, \tag{8.12}$$

hence

$$\overline{p}(z) = \int_0^z \bar{\rho}(z) dz. \tag{8.13}$$

The remaining dynamically part must satisfy

$$\overline{\rho} w'_t = -p'_z - g\rho' \tag{8.14}$$

Upon eliminating p' from the two momentum equations we get

$$\frac{d\overline{\rho}}{dz} u'_t + \overline{\rho}(u'_z - w'_x)_t = g\rho'_x \tag{8.15}$$

Eliminating ρ' from (8.8) and (8.15) we get

$$\frac{d\overline{\rho}}{dz} u'_{tt} + \overline{\rho}(u'_z - w'_x)_{tt} = g\rho'_{xt} = -g\frac{d\overline{\rho}}{dz} w'_x \tag{8.16}$$

Let us introduce the disturbance stream function ψ:

$$u' = \psi_z, \quad w' = -\psi_x \tag{8.17}$$

It follows from (8.16) that

$$\overline{\rho}\,(\psi_{xx} + \psi_{zz})_{tt} = \frac{d\overline{\rho}}{dz}\,(g\psi_{xx} - \psi_{ztt}) \tag{8.18}$$

by virture of Eqns. (8.8) and (8.17). Note that

$$N = \sqrt{-\frac{g}{\overline{\rho}}\frac{d\overline{\rho}}{dz}} \tag{8.19}$$

is the Brunt-Väisälä frequency. In the ocean, density gradient is usually very small ($N \sim 5 \times 10^{-3}$ rad/sec). Hence $\overline{\rho}$ can be approximated by a constant reference value, say, $\rho_0 = \overline{\rho}(0)$ in (8.10) and (8.14) without much error in the inertia terms. However density variation must be kept in the buoyancy term associated with gravity, which is the only restoring force responsible for wave motion. This is called the *Boussinesq approximation* and amounts to taking $\overline{\rho}$ to be constant in (Eq:17.1) only. With it (8.18) reduces to

$$(\psi_{xx} + \psi_{zz})_{tt} + N^2(z)\,\psi_{xx} = 0. \tag{8.20}$$

Note that because of linearity, u' and w' satisfy Eqn. (8.20) also, i.e.,

$$(w'_{xx} + w'_{zz})_{tt} + N^2\, w'_{xx} = 0 \tag{8.21}$$

etc.

8.2 Linearized Boundary conditions on the sea surface

Dynamic boundary condition : Total pressure is equal to the atmospheric pressure

$$(\overline{p} + p')_{z=\zeta} = 0. \tag{8.22}$$

On the free surface $z = \zeta$, we have

$$\overline{p} \approx -g \int_0^{\zeta} \overline{\rho}(0) dz = -g\overline{\rho}(0)\zeta$$

Therefore,

$$-\overline{\rho} g \zeta + p' = 0, \quad z = 0, \tag{8.23}$$

implying

$$-\overline{\rho} g \zeta_{xxt} = -p'_{xxt}, \quad z = 0. \tag{8.24}$$

Kinematic condition :

$$\zeta_t = w, \quad z = 0. \tag{8.25}$$

The left-hand-side of (8.24) can be written as

$$-\overline{\rho} g \zeta_{xxt} = -\overline{\rho} g \, w'_{xx}$$

Using 8.10, the right-hand-side of 8.24 may be written,

$$-p_{xxt} = \overline{\rho} \, u'_{xtt} = -\overline{\rho} w'_{ztt}$$

hence

$$w'_{ztt} - g w'_{xx} = 0, \quad \text{on} \quad z = 0. \tag{8.26}$$

Since $w' = -\psi_x$, ψ also satisfies the same boundary condition

$$\psi_{ztt} - g\psi_{xx} = 0, \quad \text{on} \quad z = 0. \tag{8.27}$$

On the seabed, $z = -h(x)$ the normal velocity vanishes. For a horizontal bottom we have

$$\psi(x, -h, t) = 0. \tag{8.28}$$

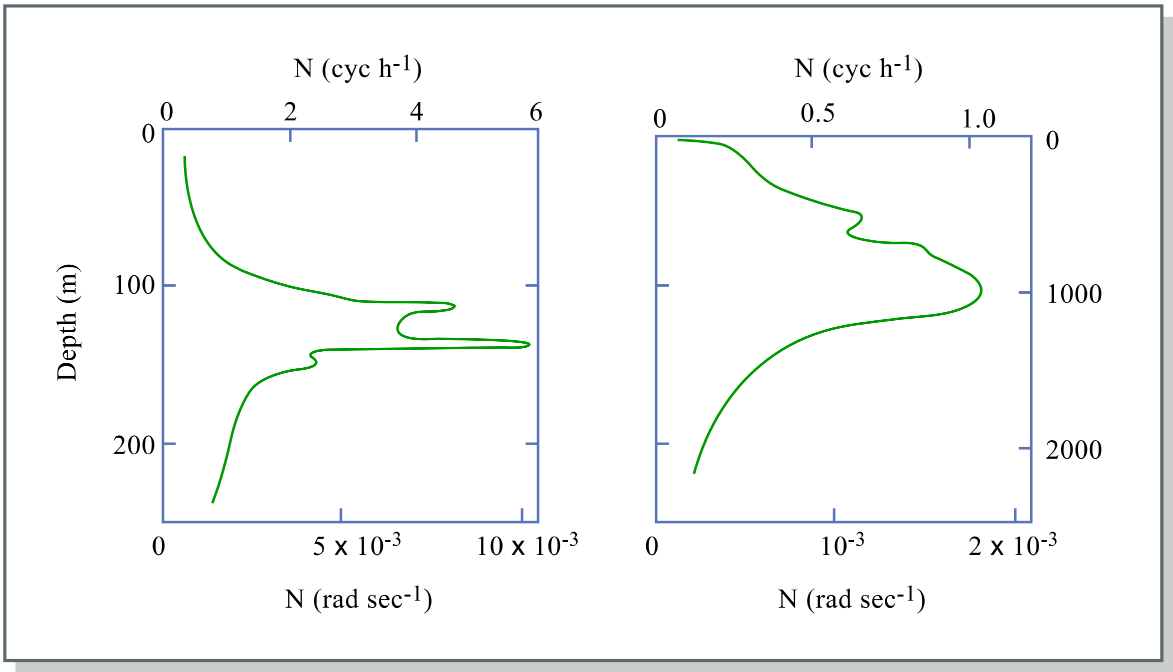

Figure by MIT OCW.

Figure 20: Typical variation of Brunt-Väisälä frequency in the ocean. From O. M. Phillips, 1977

9 Internal waves modes for finite N

Consider a horizontally propagating wave beneath the sea surface. Let

$$\psi = F(z)\, e^{\pm ikx} e^{-i\omega t}. \tag{9.1}$$

From Eqn. (8.21),

$$-\omega^2 \left(\frac{d^2F}{dz^2} - k^2 F \right) + N^2 \left(-k^2 \right) F = 0$$

or,

$$\frac{d^2F}{dz^2} + \frac{N^2 - \omega^2}{\omega^2} k^2 F = 0 \quad z < 0. \tag{9.2}$$

On the (horizontal) sea bottom

$$F = 0 \qquad z = -h. \tag{9.3}$$

From Eqn. (8.27),

$$\frac{dF}{dz} - g \frac{k^2}{\omega^2} F = 0 \qquad z = 0. \tag{9.4}$$

Equations (9.2), (9.3) and (9.4) constitute an eigenvalue condition.

If $\omega^2 < N^2$, then F is oscillatory in z within the thermocline. Away from the thermocline, $\omega^2 > N^2$, W must decay exponentially. Therefore, the thermocline is a waveguide within which waves are trapped. Waves that have the greatest amplitude beneath the free surface is called internal waves.

Since for internal waves, $\omega < N$ while N is very small in oceans, oceanic internal waves have very low natural frequencies. For most wavelengths of practical interests $\omega^2 \ll gk$ so that

$$F \cong 0 \quad \text{on } z = 0. \tag{9.5}$$

This is called the *rigid lid approximation*, which will be adopted in the following.

With the rigid-lid approximation, if N=constant (if the total depth is relataively small compared to the vertical scale of stratification, the solution for F is

$$F = A \sin\left(k(z+h)\frac{\sqrt{N^2-\omega^2}}{\omega}\right) \tag{9.6}$$

where

$$kh\frac{\sqrt{N^2-\omega^2}}{\omega} = n\pi, \quad n = 1, 2, 3... \tag{9.7}$$

This is an eigen-value condition. For a fixed wave number k, it gives the eigen-frequencies,

$$\omega_n = \frac{N}{\sqrt{1+\left(\frac{n\pi}{kh}\right)^2}} \tag{9.8}$$

For a given wavenumber k, this dispersion relation gives the eigen-frequency ω_n. For a given frequency ω, it gives the eigen-wavenumbers k_n,

$$k_n = \frac{n\pi}{h}\frac{\omega}{\sqrt{N^2-\omega^2}} \tag{9.9}$$

For a simple lake with vertical banks and length L, $0 < x < L$, we must impose the conditions :

$$u' = 0, \quad \text{hence} \quad \psi = 0, \quad x = 0, L \tag{9.10}$$

The solution is

$$\psi = A \sin k_m x \, \exp(-i\omega_{nm}t) \sin\left[k_m(z+h)\frac{\sqrt{N^2-\omega_{nm}^2}}{\omega_{nm}}\right]. \tag{9.11}$$

with

$$k_m L = m\pi, \quad m = 1, 2, 3, ... \tag{9.12}$$

The eigen-frequencies are:

$$\omega_{nm} = \frac{N}{\sqrt{1+\left(\frac{nL}{mh}\right)^2}} \tag{9.13}$$

10 Internal waves in a vertically unbounded fluid

Consider $N =$ constant (which is good if attention is limited to a small vertical extent), and denote by (α, β) the (x, z) components of the wave number vector $\vec{k}$ Let the solution be a plane wave in the vertical plane

$$\psi = \psi_0 \, e^{i(\alpha x + \beta z - \omega t)}$$

Then

$$\omega^2 = N^2 \, \frac{\alpha^2}{\alpha^2 + \beta^2}$$

or

$$\omega = \pm N \, \frac{\alpha}{k} \tag{10.1}$$

$$k^2 = \alpha^2 + \beta^2 \tag{10.2}$$

This is the dispersion relation. Note that is

$$\frac{\omega}{N} = \pm \cos\theta' \tag{10.3}$$

where θ' is the inclination of $\vec{k}$ with repect to the x axis. For a given frequency, there are two possible signs for α. Since the above relation is also even in β, there are four possible inclinations for the wave crests and troughs with respect to the horizon; the angle of inclination is

$$|\theta'| = \cos^{-1} \frac{\omega}{N} \tag{10.4}$$

For $\omega > N$, there is no wave.

To under the physics better we note first that the phase velocity is

$$\vec{C} = \frac{\omega}{k} \frac{\vec{k}}{k} = \pm \frac{\omega}{k^2} \, (\alpha, \beta) \tag{10.5}$$

while the group velocity components are

$$C_{gx} = \frac{\partial \omega}{\partial \alpha} = \pm N \left(\frac{1}{k} - \frac{\alpha}{k^2} \frac{\alpha}{k} \right) = \pm \frac{N}{k} \left(1 - \frac{\alpha^2}{k^2} \right) = \pm \frac{N}{k^3} \, \beta^2 \tag{10.6}$$

$$C_{gz} = \frac{\partial \omega}{\partial \beta} = \mp \frac{\alpha \beta}{k^3}. \tag{10.7}$$

Thus

$$\vec{C_g} = \pm N \, \frac{\beta}{k^2} \left(\frac{\beta}{k}, \frac{-\alpha}{k} \right). \tag{10.8}$$

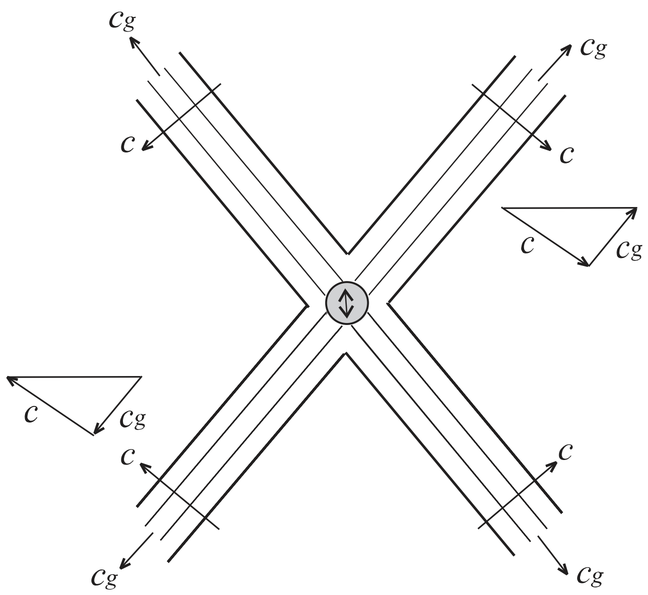

Figure 21: Phase and group velocities

Two conclusions can be drawn. First, the group velocity is perpendicular to the phase velocity,

$$\vec{C}_g \cdot \vec{C} = 0. \tag{10.9}$$

Second,

$$\vec{C} + \vec{C}_g = \pm \frac{N}{k^3}\left(\alpha^2 + \beta^2, 0\right) = \pm \frac{N}{k^2}\left(k, 0\right) \tag{10.10}$$

The vector sum of $\vec{C}$ and $\vec{C}_g$ is a horizontal vector, as shown by any of the sketches in Figure 17. Note that when the phase velocity as an upward component, the group velocity has a downward component, and vice versa. Now let us consider energy transport. from (8.10) we get

$$-p'_x = \overline{\rho}\psi_{zt} = \overline{\rho}\omega\beta\psi_o e^{i(\alpha x + \beta z - \omega t)}$$

hence the dynamic pressure is

$$p' = i\omega\overline{\rho}\frac{\beta}{\alpha}\psi_o e^{i(\alpha x + \beta z - \omega t)} \tag{10.11}$$

The fluid velocity is easily calculated

$$\vec{q}' = (u', v') = (\psi_z, -\psi_x) = i\overline{\rho}(\beta, -\alpha)\psi_o e^{i(\alpha x + \beta z - \omega t)} \tag{10.12}$$

The averaged rate of energy transport is therefore

$$\vec{E} = \frac{1}{2}\bar{\rho}^2|\psi|^2\frac{\beta}{\alpha}(\beta, -\alpha) \tag{10.13}$$

which is in the same direction of the group velocity.

Energy must radiate outward from the oscillating source, hence the group velocity vectors must all be outward. Since there are 4 directions for $\vec{k}$. There are four radial beams parallel to $\vec{c}_g$, in four quadrants, forming *St. Andrews Cross.* The crests (phase lines) in the beam in the first quadrant must be in the south-easterly direction. Similarly the crests in all four beams must be outward and toward the horizontal axis. Let θ be the inclination of a beam (i.e., $(\vec{C}_g)$ with respect to the y axis, then $\theta = \pi/2 - \theta'$ in the first quadrant. The dispersion relation can be written as

$$\frac{\omega}{N} = \pm \sin\theta \tag{10.14}$$

where θ is the inclination of a beam and not of the wavenumber vector.

Movie records indeed confirm these predictions. Within each of the four beams which have widths comparable to the cylinder diameter, only one or two wave lengths can be seen.

This unique property of anisotropy has been verified in dramatic experiments by Mowbray and Stevenson. By oscillating a long cylinder at various frequencies vertically in a stratified fluid, equal phase lines are only found along four beams forming St Andrew's Cross, see Figure (22) for $\omega/N = 0.7$, 0.9 and 1.11. It can be verified that angles are $|\theta| = 45°$ for $\omega/N = 0.7$, and $|\theta| = 64°$ for $\omega/N = 0.9$. In the last photo, $\omega/N = 1.11$. There is no wave. These results are all in accord with the condition (10.4).

Comparison between measured and predicted angles is plotted in Figure (23) for a wide range of ω/N

11 Reflection of internal waves at a plane boundary

For another interesting feature, consider the reflection of an internal wave from a slope.

Recall that $\theta' = \pm\cos^{-1}\frac{\omega}{N}$, i.e., for a fixed frequency there are only two allowable directions with respect to the horizon. Relative to the sloping bottom inclined at θ_o the

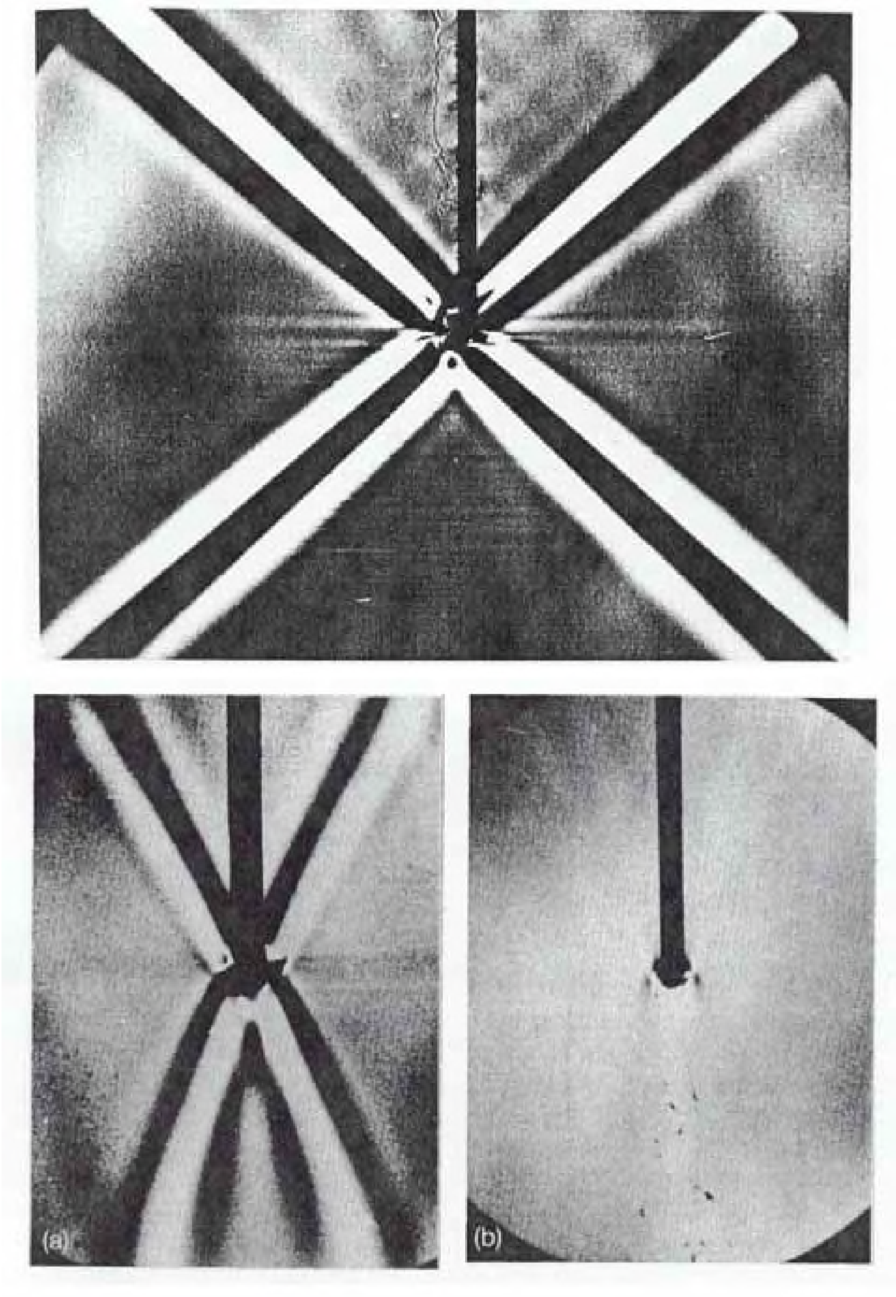

Figure 22: St Andrew's Cross in a stratified fluid. Top: $\omega/N = 0.7$; bottom left $\omega/N = 0.9$; bottom right: $\omega/N = 1.11$. From Mowbray & Rarity, 1965, JFM

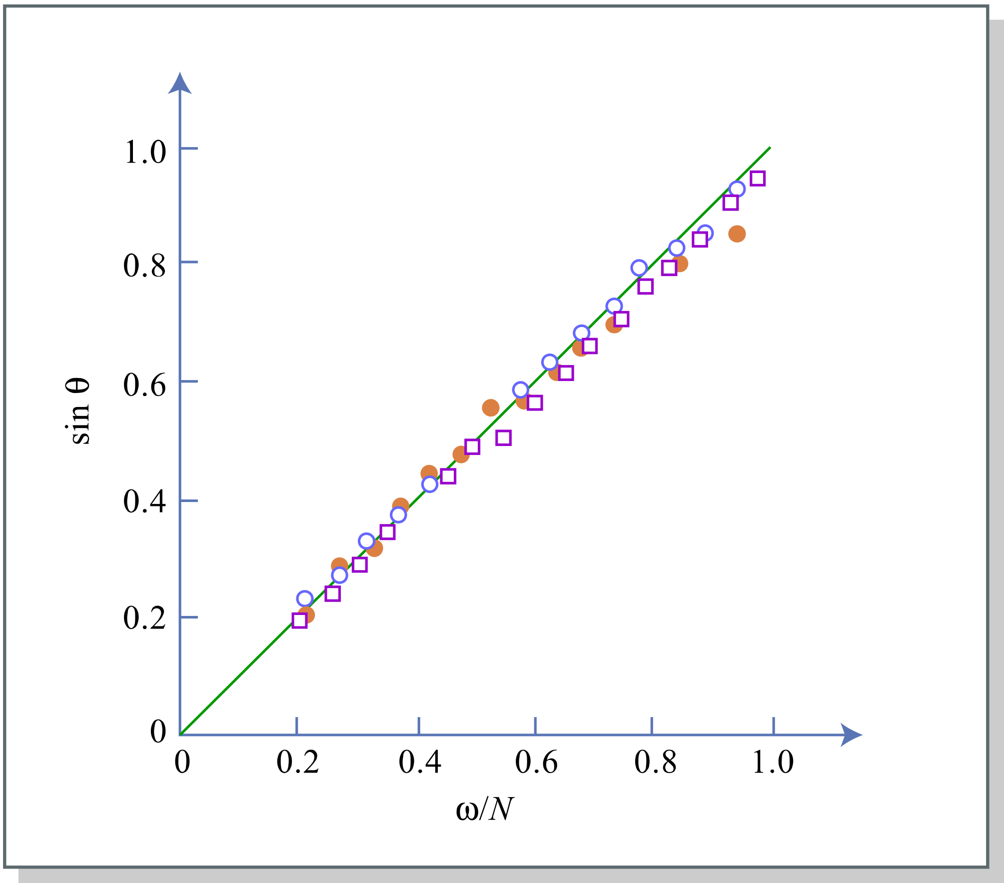

Figure by MIT OCW.

Figure 23: Comparison of measured and predicted angles of internal-wave beams. ω/N vs. $\sin\theta$. From Mowbray & Rarity, 1965, JFM

inclinations of the incident and reflected waves must be different, and are respectively $\theta' + \theta_o$ and $\theta' - \theta_o$, see Figure 24.

Let ξ be along, and η be normal to the slope. Since the slope must be a streamline, $\psi_i + \psi_r$ must vanish along $\eta = 0$ and be proportional to $e^{i(\alpha\xi - \omega t)}$; the total stream function must be of the form

$$\psi_i\, e^{i(k_t^{(i)}\xi - \omega t)} + \psi_r\, e^{i(k_t^{(r)}\xi - \omega t)} \propto \sin \beta\eta e^{i(\alpha\xi - \omega t)}.$$

In particular the wavenumber component along the slope must be equal,

$$k_t^{(i)} = k_t^{(r)} = \alpha$$

Consider first $\theta_o > \theta'$, as shown in the left diagram in Figure(24). We must have

$$k^{(i)}\,\cos(\theta' + \theta_o) = k^{(r)}\cos(\theta' - \theta_o),$$

which implies first that

$$k^{(i)} \neq k^{(r)}. \tag{11.1}$$

The incident wave and the reflected wave have different wavelengths! Furthermore, the reflected wave is directed up-slope.If, however, $\theta_o > \theta'$, as shown in the right diagram in Figure(24), then

$$k^{(i)}\,\cos(\theta' + \theta_o) = k^{(r)}\cos(\pi - (\theta_o - \theta')) = -k^{(r)}\cos(\theta_o - \theta'),$$

The reflected wave is directed down-slope instead (Phillips, 1977).

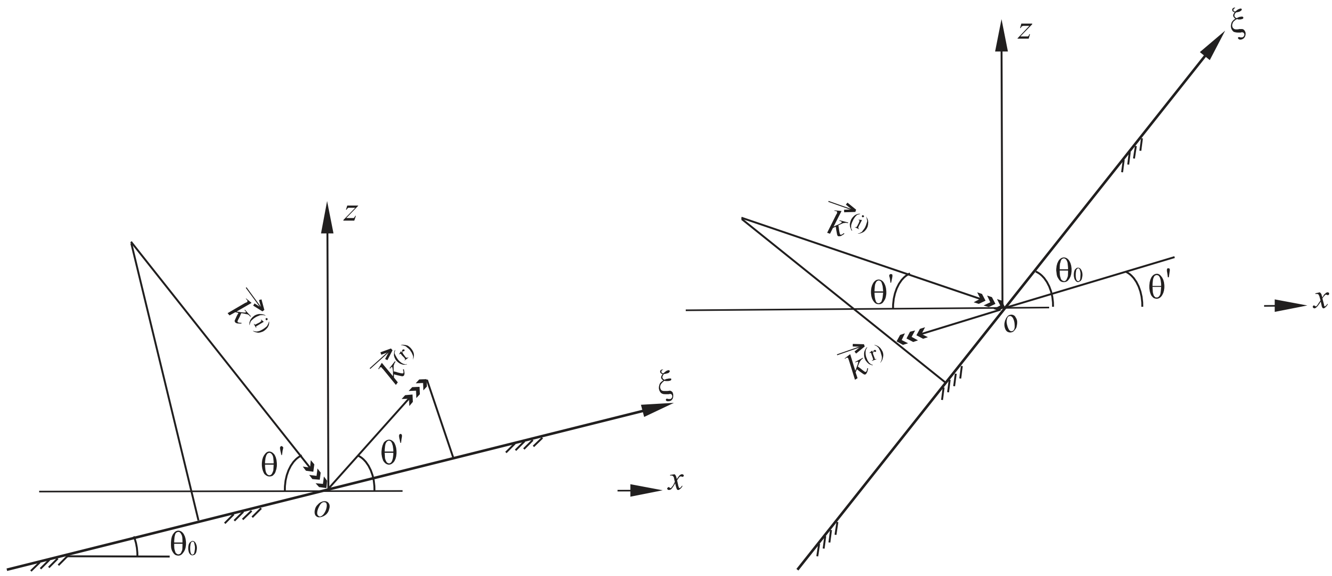

Figure 24: Internal wave reflected by in inclined surface. Top: $\theta_o < \theta'$. Bottom: $\theta_o > \theta'$.