Schur and related inequalities

MathLink Members

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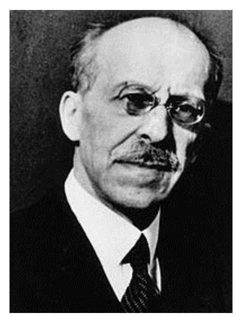
Sommario

This is a study of the classical Schur's Inequality (not Vornicu-Schur) and it's various forms and implications.

1 Schur's Inequality



Issai Schur (January 10, 1875 in Mogilev - January 10, 1941 in Tel Aviv) was a mathematician who worked in Germany for most of his life. He studied at Berlin. He obtained his doctorate in 1901, became lecturer in 1903 and, after a stay at Bonn, professor in 1919.

He considered himself German rather than Jewish, even though he had been born in the Russian Empire in what is now Belarus, and brought up partly in Latvia. For this reason he declined invitations to leave Germany for the United States and Britain in 1934.

Nevertheless he was dismissed from his chair in 1935 and, at the instigation of Ludwig Bieberbach (who had previously sympathised with Schur regarding his treatment at the hands of the Nazis), he was forced to resign from the Prussian Academy in 1938. Schur eventually emigrated to Palestine in 1939, and lived his final years in poverty. He died in Tel Aviv on his 66th birthday.

As a student of Frobenius, he worked on group representations (the subject with which he is most closely associated), but also in combinatorics and number theory and even theoretical physics. He is perhaps best known today for his result on the existence of the Schur decomposition and for his work on group representations (Schur's lemma).

Schur had a number of students, including Richard Brauer, B. H. Neumann, Heinz Prüfer, and Richard Rado. His lectures were very popular with students. He was a foreign member of the Russian Academy of Sciences from 1929.

Here a proof of Schur's inequality with three variables $a, b, c > 0$.

Theorem 1. (**Schur**) *For any $n \in \mathbb{R}$, and for any $a, b, c \in \mathbb{R}_0^+$ we have*

$$a^n(a-b)(a-c) + b^n(b-c)(b-a) + c^n(c-a)(c-b) \geq 0$$

with equality iff $a = b = c$ or $a = 0, b = c$ or $b = 0, a = c$ or $c = 0, a = b$.
Note that in the case $abc = 0$, there is the restriction $n \neq 0$.

Proof. Since the inequality is symmetric, it can be assumed without loss of generality that $a \geq b \geq c$. Rewrite

$$\sum a^n(a-b)(a-c) = [a^n(a-c) - b^n(b-c)] + c^n(c-a)(c-b)$$

Using the ordering assumption, $a \geq b$ we have $a^n \geq b^n$ and $a - c \geq b - c$, so $a^n(a-c) - b^n(b-c) \geq 0$. Clearly $a \geq c$, $b \geq b$ implies $c^n(c-a)(c-b) \geq 0$. □

2 Alternative forms of Schur's Inequality

Assuming $a, b, c > 0$ we have the following results:

Result 1. $n = 0$ is equivalent to the well-known

$$a^2 + b^2 + c^2 \geq ab + bc + ca$$

which actually holds $\forall a, b, c \in \mathbb{R}$.

Result 2. $n = 1$ expanded is

$$a^3 + b^3 + c^3 + 3abc \geq ab(a+b) + bc(b+c) + ca(c+a).$$

Result 3. $n = 1$ is equivalent to

$$abc \geq (a+b-c)(b+c-a)(c+a-b).$$

Result 4. $n = 1$ is equivalent to

$$(a+b+c)^2 + \frac{9abc}{a+b+c} \geq 4(ab+bc+ca).$$

Result 5. Euler's Triangle Inequality states that if the circumradius of a trangle is R and the inradius is r, then $R \geq 2r$. If this is expressed in terms of the sides a, b, c, we get The distance *d* between the circumcenter and incenter is $d^2 = R(R - 2r)$

$$\frac{abc}{\sqrt{(a+b+c)(a+b-c)(b+c-a)(a+c-b)}} \geq \sqrt{\frac{(a+b-c)(b+c-a)(a+c-b)}{a+b+c}}$$

which is equivalent to

$$abc \geq (a+b-c)(b+c-a)(c+a-b).$$

Result 6. Gerretsen's Inequality states that if s, r, R denotes the semiperimeter, inradius and circumradius of a triangle, then

$$16Rr - 5r^2 \leq s^2 \leq 4R^2 + 4Rr + r^2.$$

By **Ravi Transformation**, we have

$$s^2 - 16Rr + 5r^2 \geq 0 \iff \sum x(x-y)(x-z) \geq 0$$

and

$$4R^2 + 4Rr + 3r^2 - s^2 \geq 0 \iff \sum x^4(y-z)^2 + 2 \cdot \sum p(p-q)(p-r) \geq 0$$

where

$$\begin{cases} a = x+y \\ b = y+z \\ c = z+x \end{cases} \iff \begin{cases} 2x = a-b+c \\ 2y = a+b-c \\ 2z = -a+b+c \end{cases}$$

and p=xy, q=yz, r=zx.

Result 7. $n = 2$ expanded is $\frac{2}{3}(a^2+b^2+c^2)^2 \geq a^3(b+c)+b^3(c+a)+c^3(a+b)$

$$a^4 + b^4 + c^4 + abc(a+b+c) \geq a^3b + b^3c + c^3a + ab^3 + bc^3 + ca^3.$$

Result 8. Schur's inequality of fourth degree can be rewritten into

$$(a+b+c)(a^3+b^3+c^3+3abc) \geq 2(a^2+b^2+c^2)(ab+bc+ca).$$

Result 9. Schur's inequality of third degree is equivalent with

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + \frac{4abc}{(a+b)(b+c)(c+a)} \geq 2.$$

Result 10. For $a, b, c > 0$ and $abc = 1$ we have

$$(a-1)\left(\frac{1}{b}-1\right) + (b-1)\left(\frac{1}{c}-1\right) + (c-1)\left(\frac{1}{a}-1\right) \geq 0.$$

(Let $a = \frac{x}{y}$ etc, then this is Schur's inequality of 0-th degree.)

Result 11. Third degree:

$$(a^2+b^2+c^2)(a+b+c) + 9abc \geq 2(a+b+c)(ab+bc+ca).$$

Result 12. (Darij Grinberg) (weaker form of Result 4)

$$a^2 + b^2 + c^2 + 2abc + 1 \geq 2(ab+bc+ca).$$

Result 13. (Stronger than 3rd degree, equivalent with 4th degree)

$$a^3+b^3+c^3+3abc \geq \sum_{cyc} bc(b+c) + \frac{bc(b-c)^2+ca(c-a)^2+ab(a-b)^2}{a+b+c}.$$

Result 14. (4th degree)

$$a^2+b^2+c^2+\frac{6abc(a+b+c)}{a^2+b^2+c^2+ab+bc+ca} \geq 2(ab+bc+ca).$$

Result 15. (D.Duc Lam)

$$\left(\frac{a}{b+c}\right)^2+\left(\frac{b}{c+a}\right)^2+\left(\frac{c}{a+b}\right)^2+\frac{10abc}{(a+b)(b+c)(c+a)} \geq 2.$$

Result 16. (Stronger than Schur of third degree, but weaker than 5th degree)

$$\frac{a^2+b^2+c^2}{ab+bc+ca}+\frac{8abc}{(a+b)(b+c)(c+a)} \geq 2$$

Result 17. (Schur of 5th degree)

$$a^2+b^2+c^2+\frac{6abc}{a+b+c}+\frac{(a+b+c)abc}{a^2+b^2+c^2} \geq 2(ab+bc+ca)$$