HMMT February 2019

February 16, 2019

Combinatorics

1. How many distinct permutations of the letters of the word REDDER are there that do not contain a palindromic substring of length at least two? (A *substring* is a contiguous block of letters that is part of the string. A string is *palindromic* if it is the same when read backwards.)

   Proposed by: Yuan Yao

   Answer: $\boxed{6}$

   If two identical letters are adjacent or have a single letter in between, there is clearly a palindromic substring of length (respectively) two or three. So there cannot be any such substrings.

   Say we have a permutation of the word REDDER without any palindromic substrings. Let us call the first letter X. The second letter has to be different, let us call it Y. The third letter cannot be X or Y, let it be Z. Again, the fourth letter cannot be Y or Z, and we only have 3 letters to choose from, so it has to be X. Continuing analogously, the fifth letter has to be Y, and the sixth letter has to be Z. So any word satisfying the problem statement has to be of the form XYZXYZ. It is easy to check that such a word indeed does not have any palindromic substrings. X, Y, Z can be any permutation of R, E, D, giving a total of 6 possibilities.

2. Your math friend Steven rolls five fair icosahedral dice (each of which is labelled $1, 2, \ldots, 20$ on its sides). He conceals the results but tells you that at least half of the rolls are 20. Suspicious, you examine the first two dice and find that they show 20 and 19 in that order. Assuming that Steven is truthful, what is the probability that all three remaining concealed dice show 20?

   Proposed by: Evan Chen

   Answer: $\boxed{\frac{1}{58}}$

   The given information is equivalent to the first two dice being 20 and 19 and there being at least two 20's among the last three dice. Thus, we need to find the probability that given at least two of the last three dice are 20's, all three are. Since there is only one way to get all three 20's and $3 \cdot 19 = 57$ ways to get exactly two 20's, the probability is $\frac{1}{1+57} = \frac{1}{58}$.

3. Reimu and Sanae play a game using 4 fair coins. Initially both sides of each coin are white. Starting with Reimu, they take turns to color one of the white sides either red or green. After all sides are colored, the 4 coins are tossed. If there are more red sides showing up, then Reimu wins, and if there are more green sides showing up, then Sanae wins. However, if there is an equal number of red sides and green sides, then *neither* of them wins. Given that both of them play optimally to maximize the probability of winning, what is the probability that Reimu wins?

   Proposed by: Yuan Yao

   Answer: $\boxed{\frac{5}{16}}$

   Clearly Reimu will always color a side red and Sanae will always color a side green, because their situation is never worse off when a side of a coin changes to their own color. Since the number of red-only coins is always equal to the number of green-only coins, no matter how Reimu and Sanae color the coins, they will have an equal probability of winning by symmetry, so instead they will cooperate to make sure that the probability of a tie is minimized, which is when all 4 coins have different colors on both sides (which can easily be achieved by Reimu coloring one side of a new coin red and Sanae immediately coloring the opposite side green). Therefore, the probability of Reimu winning is $\frac{\binom{4}{3}+\binom{4}{4}}{2^4} = \frac{5}{16}$.

4. Yannick is playing a game with 100 rounds, starting with 1 coin. During each round, there is a $n\%$ chance that he gains an extra coin, where n is the number of coins he has at the beginning of the round. What is the expected number of coins he will have at the end of the game?

Proposed by: Yuan Yao

Answer: $\boxed{1.01^{100}}$

Let X_i be the random variable which is the number of coins at the end of round i. Say that $X_0 = 1$ for convenience. Fix $i > 0$ and some positive integer x. Conditioning on the event $X_{i-1} = x$, there are only two cases with positive probability. In particular,

$$\Pr[X_i = x + 1 \mid X_{i-1} = x] = \frac{x}{100}$$

and

$$\Pr[X_i = x \mid X_{i-1} = x] = 1 - \frac{x}{100}.$$

Therefore

$$\begin{aligned}
\mathbb{E}[X_i] &= \sum_{x>0} x \cdot \Pr[X_i = x] \\
&= \sum_{x>0} x \cdot \left(\left(1 - \frac{x}{100}\right) \Pr[X_{i-1} = x] + \frac{x-1}{100} \Pr[X_{i-1} = x-1] \right) \\
&= \sum_{x>0} x \Pr[X_{i-1} = x] - \frac{1}{100} \sum_{x>0} x \Pr[X_{i-1} = x - 1] \\
&\quad + \frac{1}{100} \sum_{x>0} x^2 \Pr[X_{i-1} = x - 1] - \frac{1}{100} \sum_{x>0} x^2 \Pr[X_i = x] \\
&= \frac{99}{100} \mathbb{E}[X_{i-1}] - \frac{1}{100} + \frac{1}{50} \mathbb{E}[X_{i-1}] + \frac{1}{100} \\
&= \frac{101}{100} \mathbb{E}[X_{i-1}].
\end{aligned}$$

(A different way to understand this is that no matter how many coins Yannick has currently (as long as he does not have more than 100 coins, which is guaranteed in this problem), the expected number of coins after one round is always 1.01 times the current number of coins, so the expected value is multiplied by 1.01 each round.)

Therefore

$$\mathbb{E}[X_{100}] = \left(\frac{101}{100}\right)^{100} \mathbb{E}[X_0] = 1.01^{100}.$$

5. Contessa is taking a random lattice walk in the plane, starting at $(1, 1)$. (In a random lattice walk, one moves up, down, left, or right 1 unit with equal probability at each step.) If she lands on a point of the form $(6m, 6n)$ for $m, n \in \mathbb{Z}$, she ascends to heaven, but if she lands on a point of the form $(6m+3, 6n+3)$ for $m, n \in \mathbb{Z}$, she descends to hell. What is the probability that she ascends to heaven?

   Proposed by: John Michael Wu

   Answer: $\boxed{\frac{13}{22}}$

   Let $P(m, n)$ be the probability that she ascends to heaven from point (m, n). Then $P(6m, 6n) = 1$ and $P(6m + 3, 6n + 3) = 0$ for all integers $m, n \in \mathbb{Z}$. At all other points,

   $$4P(m, n) = P(m - 1, n) + P(m + 1, n) + P(m, n - 1) + P(m, n + 1) \tag{1}$$

   This gives an infinite system of equations. However, we can apply symmetry arguments to cut down the number of variables to something more manageable. We have $P(m, n) = P(m + 6a, n + 6b)$ for $a, b \in \mathbb{Z}$, and $P(m, n) = P(n, m)$, and $P(m, n) = P(-m, n)$, and $P(m, n) = 1 - P(m + 3, n + 3)$ (since any path from the latter point to heaven corresponds with a path from the former point to hell, and vice versa).

Thus for example we have

$$P(1,2) = P(-1,-2) = 1 - P(2,1) = 1 - P(1,2),$$

so $P(1,2) = 1/2$.

Applying Equation (1) to points $(1,1)$, $(0,1)$, and $(0,2)$, and using the above symmetries, we get the equations

$$\begin{aligned} 4P(1,1) &= 2P(0,1) + 1, \\ 4P(0,1) &= P(0,2) + 2P(1,1) + 1, \\ 4P(0,2) &= P(0,1) + 3/2. \end{aligned}$$

Solving yields $P(1,1) = 13/22$.

6. A point P lies at the center of square $ABCD$. A sequence of points $\{P_n\}$ is determined by $P_0 = P$, and given point P_i, point P_{i+1} is obtained by reflecting P_i over one of the four lines AB, BC, CD, DA, chosen uniformly at random and independently for each i. What is the probability that $P_8 = P$?

   Proposed by: Yuan Yao

   Answer: $\boxed{\frac{1225}{16384}}$

   Solution 1. WLOG, AB and CD are horizontal line segments and BC and DA are vertical. Then observe that we can consider the reflections over vertical lines separately from those over horizontal lines, as each reflection over a vertical line moves P_i horizontally to point P_{i+1}, and vice versa. Now consider only the reflections over horizontal segments AB and CD. Note that it is impossible for P_8 to be in the same location vertical location as P if there are an odd number of these reflections. Then we consider the reflections in pairs: let w denote reflecting twice over AB, let x denote reflecting over AB and then CD, let y denote reflecting over CD and then AB, and let z denote reflecting twice over CD. Note that both w and z preserve the position of our point. Also note that in order to end at the same vertical location as P, we must have an equal number of x's and y's. Now we count the number of sequences of length at most 4 with this property:

   - Case 1: Length 0

     There is just the empty sequence here, so 1.
   - Case 2: Length 1

     There are just the sequences w and z, so 2.
   - Case 3: Length 2

     We may either have an x and a y or two characters that are either w or z. There are 2 sequences of the former type and 4 of the latter, for 6 total.
   - Case 4: Length 3

     There are 12 sequences with an x, a y, and either a w or a z, and 8 sequences of only w's and z's, for 12 total.
   - Case 5: Length 4

     There are 6 sequences of 2 x's and 2 y's, 48 with one of each and two terms that are either w or z, and 16 of just w's and z's, for a total of 70.

   Now let the number of such sequences of length k be a_k (so $a_3 = 20$). Note that these counts work also if we consider only reflections over vertical line segments BC and AD. Now to finish, we only need to count the number of ways to combine 2 valid sequences of total length 4. This is

   $$\sum_{i=0}^{4} a_i a_{4-i} \binom{8}{2i},$$

   as there are a_i sequences of reflections over AB and CD, a_{4-i} sequences of reflections over BC and AD such that there are 8 total reflections, and $\binom{8}{2i}$ ways to choose which of the 8 reflections will be over AB or CD. We compute that this sum is $1 \cdot 70 \cdot 1 + 2 \cdot 20 \cdot 28 + 6 \cdot 6 \cdot 70 + 20 \cdot 2 \cdot 28 + 70 \cdot 1 \cdot 1 = 4900$ total sequences of reflections that place P_8 at P. There are of course $4^8 = 65536$ total sequences of 8 reflections, each chosen uniformly at random, so our answer is $\frac{4900}{65536} = \frac{1225}{16384}$.

Solution 2. Suppose that P_0 is the origin and the four lines are $x = \pm 0.5$ and $y = \pm 0.5$. We consider a permutation of the lattice points on the coordinate plane, where all points with even x-coordinates are reflected across the y-axis and all points with even y-coordinates are reflected across the x-axis, so that the x- and y-coordinates are both rearranged in the following order:

$$\ldots, 4, -3, 2, -1, 0, 1, -2, 3, -4, \ldots.$$

It is not difficult to see that a reflection across one of the lines corresponds to changing one of the coordinates from one number to either the previous number of the next number. Therefore, after the permutation, the question is equivalent to asking for the number of lattice walks of length 8 that returns to the origin. For such a lattice walk to return to origin, there needs to be the same number of up and down moves, and the same number of left and right moves. This condition is equivalent to having four moves that are left or up (LU), and four moves that are right or up (RU). Moreover, knowing whether a move is LU and whether it is RU uniquely determines what the move is, so it suffices to designate four LU moves and four RU moves, giving $\binom{8}{4}^2 = 4900$ possible walks. Hence the probability is $\frac{4900}{4^8} = \frac{1225}{16384}$.

7. In an election for the Peer Pressure High School student council president, there are 2019 voters and two candidates Alice and Celia (who are voters themselves). At the beginning, Alice and Celia both vote for themselves, and Alice's boyfriend Bob votes for Alice as well. Then one by one, each of the remaining 2016 voters votes for a candidate randomly, with probabilities proportional to the current number of the respective candidate's votes. For example, the first undecided voter David has a $\frac{2}{3}$ probability of voting for Alice and a $\frac{1}{3}$ probability of voting for Celia.

   What is the probability that Alice wins the election (by having more votes than Celia)?

   Proposed by: Yuan Yao

   Answer: $\boxed{\frac{1513}{2017}}$

   Let $P_n(m)$ be the probability that after n voters have voted, Alice gets m votes. We show by induction that for $n \geq 3$, the ratio $P_n(2) : P_n(3) : \cdots : P_n(n-1)$ is equal to $1 : 2 : \cdots : (n-2)$. We take a base case of $n = 3$, for which the claim is obvious. Then suppose the claim holds for $n = k$. Then $P_k(m) = \frac{2m-2}{(k-1)(k-2)}$. Then

$$P_{k+1}(i) = \frac{k-i}{k} P_k(i) + \frac{i-1}{k} P_k(i-1) = \frac{(k-i)(2i-2) + (i-1)(2i-4)}{k(k-1)(k-2)} = \frac{2i-2}{k(k-1)}.$$

   Also, we can check $P_{k+1}(2) = \frac{2}{k(k-1)}$ and $P_{k+1}(k) = \frac{2}{k}$, so indeed the claim holds for $n = k+1$, and thus by induction our claim holds for all $n \geq 3$. The probability that Ceila wins the election is then

$$\frac{\sum_{m=2}^{1009} P_{2019}(m)}{\sum_{m=2}^{2018} P_{2019}(m)} = \frac{1008 \cdot (1 + 1008)/2}{2017 \cdot (1 + 2017)/2} = \frac{504}{2017},$$

   and thus the probability that Alice wins is $\frac{1513}{2017}$.

8. For a positive integer N, we color the positive divisors of N (including 1 and N) with four colors. A coloring is called *multichromatic* if whenever a, b and $\gcd(a, b)$ are pairwise distinct divisors of N, then they have pairwise distinct colors. What is the maximum possible number of multichromatic colorings a positive integer can have if it is not the power of any prime?

   Proposed by: Evan Chen

   Answer: $\boxed{192}$

First, we show that N cannot have three distinct prime divisors. For the sake of contradiction, suppose $pqr|N$ for three distinct primes p, q, r. Then by the problem statement, $(p, q, 1)$, $(p, r, 1)$, and $(q, r, 1)$ have three distinct colors, so $(p, q, r, 1)$ has four distinct colors. In addition, $(pq, r, 1)$, (pq, pr, p), and (pq, qr, q) have three distinct colors, so $(pq, p, q, r, 1)$ has five distinct colors, contradicting the fact that there are only four possible colors.

Similarly, if $p^3q|N$ for some distinct primes p and q, then $(p, q, 1)$, $(p^2, q, 1)$, $(p^3, q, 1)$, (p^2, pq, p), (p^3, pq, p), and (p^3, p^2q, p^2) are all triples with distinct colors, so $(1, q, p, p^2, p^3)$ must have five distinct colors, which is again a contradiction. In addition, if $p^2q^2|N$ for some distinct primes p and q, then $(p, q, 1)$, $(p^2, q^2, 1)$, $(p^2, q, 1)$, and $(p, q^2, 1)$ are all triples with pairwise distinct colors, so $(1, p, q, p^2, q^2)$ must have five distinct colors, another contradiction.

We are therefore left with two possibilities:

- Case 1: $N = pq$

  In this case, the only triple of factors that must have pairwise distinct colors is $(p, q, 1)$. We have $4 \cdot 3 \cdot 2 = 24$ choices for these three, and 4 choices for pq itself, giving $4 \cdot 24 = 96$ multichromatic colorings.

- Case 2: $N = p^2q$

  In this case, the triples of pairwise distinctly colored factors are $(p, q, 1)$, $(p^2, q, 1)$, and (p^2, pq, p). From this, we see that $(1, p, q, p^2)$ must have four distinct colors, and the color of pq must be distinct from p and p^2. There are $4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways to assign the four distinct colors, 2 ways to assign the color of pq after that, and 4 ways to color p^2q after that, giving a total of $24 \cdot 2 \cdot 4 = 192$ monochromatic colorings.

Therefore, there can be at most 192 multichromatic colorings.

9. How many ways can one fill a 3×3 square grid with nonnegative integers such that no *nonzero* integer appears more than once in the same row or column and the sum of the numbers in every row and column equals 7?

   Proposed by: Sam Korsky

   Answer: $\boxed{216}$

   In what ways could we potentially fill a single row? The only possibilities are if it contains the numbers $(0, 0, 7)$ or $(0, 1, 6)$ or $(0, 2, 5)$ or $(0, 3, 4)$ or $(1, 2, 4)$. Notice that if we write these numbers in binary, in any choices for how to fill the row, there will be exactly one number with a 1 in its rightmost digit, exactly one number with a 1 in the second digit from the right, and exactly exactly one number with a 1 in the third digit from the right. Thus, consider the following operation: start with every unit square filled with the number 0. Add 1 to three unit squares, no two in the same row or column. Then add 2 to three unit squares, no two in the same row or column. Finally, add 4 to three unit squares, no two in the same row or column. There are clearly $6^3 = 216$ ways to perform this operation and every such operation results in a unique, suitably filled-in 3 by 3 square. Hence the answer is 216.

10. Fred the Four-Dimensional Fluffy Sheep is walking in 4-dimensional space. He starts at the origin. Each minute, he walks from his current position (a_1, a_2, a_3, a_4) to some position (x_1, x_2, x_3, x_4) with integer coordinates satisfying

    $$(x_1-a_1)^2+(x_2-a_2)^2+(x_3-a_3)^2+(x_4-a_4)^2 = 4 \text{ and } |(x_1 + x_2 + x_3 + x_4) - (a_1 + a_2 + a_3 + a_4)| = 2.$$

    In how many ways can Fred reach $(10, 10, 10, 10)$ after exactly 40 minutes, if he is allowed to pass through this point during his walk?

    Proposed by: Brice Huang

    Answer: $\boxed{\binom{40}{10}\binom{40}{20}^3}$

The possible moves correspond to the vectors $\pm\langle 2,0,0,0\rangle$, $\pm\langle 1,1,1,-1\rangle$, and their permutations. It's not hard to see that these vectors form the vertices of a 4-dimensional hypercube, which motivates the change of coordinates

$$(x_1,x_2,x_3,x_4) \Rightarrow \left(\frac{x_1+x_2+x_3+x_4}{2}, \frac{x_1+x_2-x_3-x_4}{2}, \frac{x_1-x_2+x_3-x_4}{2}, \frac{x_1-x_2-x_3+x_4}{2}\right).$$

Under this change of coordinates, Fred must travel from $(0,0,0,0)$ to $(20,0,0,0)$, and the possible moves correspond to the sixteen vectors $\langle \pm 1, \pm 1, \pm 1, \pm 1\rangle$. The new x_1-coordinate must increase 30 times and decrease 10 times during Fred's walk, while the other coordinates must increase 20 times and decrease 20 times. Therefore, there are $\binom{40}{10}\binom{40}{20}^3$ possible walks.