This is Volume 5 of the five-volume book Mathematical Inequalities, which introduces and develops the main types of elementary inequalities. The first three volumes are a great opportunity to look into many old and new inequalities, as well as elementary procedures for solving them: Volume 1 - Symmetric Polynomial Inequalities, Volume 2 - Symmetric Rational and Nonrational Inequalities, Volume 3 - Cyclic and Noncyclic Inequalities. As a rule, the inequalities in these volumes are increasingly ordered according to the number of variables: two, three, four, ... , n-variables. The last two volumes (Volume 4 – Extensions and Refinements of Jensen's Inequality, Volume 5 – Other Recent Methods for Creating and Solving Inequalities) present beautiful and original methods for solving inequalities, such as Half/Partial convex function method, Equal variables method, Arithmetic compensation method, Highest coefficient cancellation method, pqr method etc. The book is intended for a wide audience: advanced middle school students, high school students, college and university students, and teachers. Many problems and methods can be used as group projects for advanced high school students.

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Vasile Cirtoaje

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Vasile Cîrtoaje

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Contents

Chapter 1

Arithmetic Mean Method and Arithmetic Compensation Method

1.1 Theoretical Basis

The Arithmetic Mean - Theorem (AM-Theorem) is useful to prove some symmetric inequalities of n real variables $a_1, a_2, \ldots, a_n$, where the equality occurs when n or $n-1$ variables are equal.

AM-THEOREM. *Let*

$$F(a_1, a_2, \ldots, a_n) : \mathbb{A} \to \mathbb{R}, \qquad \mathbb{A} \in \mathbb{R}^n,$$

be a symmetric continuous function satisfying

$$F(a_1, a_2, \ldots, a_{n-1}, a_n) \geq F\left(\frac{a_1 + a_n}{2}, a_2, \ldots, a_{n-1}, \frac{a_1 + a_n}{2}\right)$$

for all $a_1, a_2, \ldots, a_n \in \mathbb{A}$ such that $a_1 \leq a_2 \leq \cdots \leq a_n$ or $a_1 \geq a_2 \geq \cdots \geq a_n$. Then, for all $a_1, a_2, \ldots, a_n \in \mathbb{A}$, the following inequality holds:

$$F(a_1, a_2, \ldots, a_n) \geq F(a, a, \ldots, a), \qquad a = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Proof. Let

$$A_j = (a_{j1}, a_{j2}, \ldots, a_{jn}), \qquad j = 0, 1, 2, \ldots,$$

where each A_j is constructed from the preceding A_{j-1} by replacing its smallest and largest elements with their arithmetic mean, and

$$A_0 = (a_{01}, a_{02}, \ldots, a_{0n}) = (a_1, a_2, \ldots, a_n).$$

By hypothesis, we have

$$F(a_1, a_2, \ldots, a_n) \geq F(a_{11}, a_{12}, \ldots, a_{1n}) \geq F(a_{21}, a_{22}, \ldots, a_{2n}) \geq \cdots,$$

and by Lemma below it follows that

$$F(a_1,a_2,\dots,a_n) \ge \lim_{j\to\infty} F(a_{j1},a_{j2},\dots,a_{jn}) = F(a,a,\dots,a).$$

Lemma. *Let*

$$A_0,\ A_1,\ A_2,\ \dots$$

be an infinite sequence of n-tuples $A_j = (a_{j1},a_{j2},\dots,a_{jn}) \in \mathbb{R}^n$, where each A_j is constructed from the preceding A_{j-1} by replacing its smallest and largest elements with their arithmetic mean. Then

$$a_{j1}+a_{j2}+\cdots+a_{jn} = a_{01}+a_{02}+\cdots+a_{0n}$$

and

$$\lim_{j\to\infty} A_j = (a,a,\dots,a), \qquad a = \frac{a_{01}+a_{02}+\cdots+a_{0n}}{n}.$$

Proof. For any n-tuple A_j, define the closed interval $\mathbb{J}_j = [a_j, b_j]$, where

$$a_j = \min\{a_{j1},a_{j2},\dots,a_{jn}\}, \qquad b_j = \max\{a_{j1},a_{j2},\dots,a_{in}\},$$

and denote by $|\mathbb{J}_j|$ the length of the closed interval $\mathbb{J}_j$:

$$|\mathbb{J}_j| = b_j - a_j.$$

Clearly, we have

$$\mathbb{J}_0 \supseteq \mathbb{J}_1 \supseteq \mathbb{J}_2 \supseteq \cdots\ , \qquad |\mathbb{J}_0| \ge |\mathbb{J}_1| \ge |\mathbb{J}_2| \ge \cdots\ .$$

We infer that for any integer j there exists an integer k_j such that

$$k_j \le \frac{n}{2}\ , \qquad |\mathbb{J}_{j+k_j}| \le \frac{2}{3}|\mathbb{J}_j|.$$

Under this assumption, we have

$$\lim_{j\to\infty} |\mathbb{J}_j| = 0,$$

therefore

$$\lim_{j\to\infty} A_j = (a,a,\dots,a), \qquad a = \frac{a_{01}+a_{02}+\cdots+a_{0n}}{n}.$$

To end the proof, let

$$\mathbb{B}_j = \left[a_j, \frac{2a_j+b_j}{3}\right), \qquad \mathbb{C}_j = \left[\frac{2a_j+b_j}{3}, \frac{a_j+2b_j}{3}\right], \qquad \mathbb{D}_j = \left(\frac{a_j+2b_j}{3}, b_j\right].$$

Consider that A_j has k_{j1} elements in $\mathbb{B}_j$, k_{j2} elements in $\mathbb{D}_j$ and $n-k_{j1}-k_{j2}$ elements in $\mathbb{C}_j$. Let

$$k_j = \min\{k_{j1},k_{j2}\}.$$

If $c_1 \in \mathbb{B}_j$ and $c_2 \in \mathbb{D}_j$, then

$$\frac{c_1 + c_2}{2} > \frac{a_j + (a_j + 2b_j)/3}{2} = \frac{2a_j + b_j}{3},$$
$$\frac{c_1 + c_2}{2} < \frac{(2a_j + b_j)/3 + b_j}{2} = \frac{a_j + 2b_j}{3},$$

therefore

$$\frac{c_1 + c_2}{2} \in \mathbb{C}_j.$$

As a consequence, A_{j+k_j} has $k_{j1} - k_j$ elements in $\mathbb{B}_j$, $k_{j2} - k_j$ elements in $\mathbb{D}_j$, and the other elements in $\mathbb{C}_j$. Since

$$(k_{j1} - k_j)(k_{j2} - k_j) = 0,$$

all elements of A_{j+k_j} belong to

$$\mathbb{B}_j \cup \mathbb{C}_j = \left[a_j, \frac{a_j + 2b_j}{3} \right]$$

or

$$\mathbb{C}_j \cup \mathbb{D}_j = \left[\frac{2a_j + b_j}{3}, b_j \right].$$

Because

$$|\mathbb{B}_j \cup \mathbb{C}_j| = |\mathbb{C}_j \cup \mathbb{D}_j| = \frac{2(b_j - a_j)}{3} = \frac{2}{3}|\mathbb{J}_j|,$$

it follows that

$$|\mathbb{J}_{j+k_j}| \le \frac{2}{3}|\mathbb{J}_j|.$$

This Lemma is a known result (see, for example, problem 2389 in Crux Mathematicorum, 1999, page 171 and page 520).

Assuming that $a_n = \min\{a_1, a_2, \dots, a_n\}$ or $a_n = \max\{a_1, a_2, \dots, a_n\}$ and fixing it, we get

AM-Corollary (*Vasile Cîrtoaje*, 2005). *Let*

$$F(a_1, a_2, \dots, a_n) : \mathbb{A} \to \mathbb{R}, \qquad \mathbb{A} \in \mathbb{R}^n,$$

be a symmetric continuous function satisfying

$$F(a_1, a_2, \dots, a_{n-2}, a_{n-1}, a_n) \ge F\left(\frac{a_1 + a_{n-1}}{2}, a_2, \dots, a_{n-2}, \frac{a_1 + a_{n-1}}{2}, a_n \right)$$

for all $a_1, a_2, \dots, a_n \in \mathbb{A}$ such that $a_1 \le a_2 \le \cdots \le a_n$ (or $a_1 \ge a_2 \ge \cdots \ge a_n$). Then, for all $a_1, a_2, \dots, a_n \in \mathbb{A}$ such that $a_1 \le a_2 \le \cdots \le a_n$ (or $a_1 \ge a_2 \ge \cdots \ge a_n$), the following inequality holds:

$$F(a_1, a_2, \dots, a_{n-1}, a_n) \ge F(t, t, \dots, t, a_n), \qquad t = \frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1}.$$

From the AM-Theorem and the AM-Corollary, we get the SM-Theorem and the SM-Corollary, respectively.

SM-THEOREM (*Vasile Cîrtoaje*, 2005). *Let*

$$G(a_1, a_2, \ldots, a_n) : \mathbb{A} \to \mathbb{R}, \qquad \mathbb{A} \in \mathbb{R}_+^n,$$

be a symmetric continuous function satisfying

$$G(a_1, a_2, \ldots, a_{n-1}, a_n) \ge G\left(\sqrt{\frac{a_1^2 + a_n^2}{2}}, a_2, \ldots, a_{n-1}, \sqrt{\frac{a_1^2 + a_n^2}{2}}\right)$$

for $a_1, a_2, \ldots, a_n \in \mathbb{A}$ such that $a_1 \le a_2 \le \cdots \le a_n$ or $a_1 \ge a_2 \ge \cdots \ge a_n$. Then, for $a_1, a_2, \ldots, a_n \in \mathbb{A}$, the following inequality holds:

$$G(a_1, a_2, \ldots, a_n) \ge G(t, t, \ldots, t), \qquad t = \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}}.$$

Proof. Let

$$b_1 = a_1^2, \; b_2 = a_2^2, \; \ldots \; , \; b_n = a_n^2$$

and

$$F(b_1, b_2, \ldots, b_n) = G\left(\sqrt{b_1}, \sqrt{b_2}, \ldots, \sqrt{b_n}\right).$$

Since

$$a_1^2 + a_2^2 + \cdots + a_n^2 = b_1 + b_2 + \cdots + b_n,$$

$$G(a_1, a_2, \ldots, a_n) = G\left(\sqrt{b_1}, \sqrt{b_2}, \ldots, \sqrt{b_n}\right) = F(b_1, b_2, \ldots, b_n),$$

$$\begin{aligned} G\left(\sqrt{\frac{a_1^2 + a_n^2}{2}}, a_2, \ldots, a_{n-1}, \sqrt{\frac{a_1^2 + a_n^2}{2}}\right) &= G\left(\sqrt{\frac{b_1 + b_n}{2}}, \sqrt{b_2}, \ldots, \sqrt{b_{n-1}}, \sqrt{\frac{b_1 + b_n}{2}}\right) \\ &= F\left(\frac{b_1 + b_n}{2}, b_2, \ldots, b_{n-1}, \frac{b_1 + b_n}{n}\right), \end{aligned}$$

the SM-Theorem follows immediately from the AM-Theorem.

SM-Corollary (*Vasile Cîrtoaje*, 2005). *Let*

$$G(a_1, a_2, \ldots, a_n) : \mathbb{A} \to \mathbb{R}, \qquad \mathbb{A} \in \mathbb{R}_+^n,$$

be a symmetric continuous function satisfying

$$G(a_1, a_2, \ldots, a_{n-2}, a_{n-1}, a_n) \ge G\left(\sqrt{\frac{a_1^2 + a_{n-1}^2}{2}}, a_2, \ldots, a_{n-2}, \sqrt{\frac{a_1^2 + a_{n-1}^2}{2}}, a_n\right)$$

for all $a_1, a_2, \ldots, a_n \in \mathbb{A}$ such that $a_1 \le a_2 \le \cdots \le a_n$ (or $a_1 \ge a_2 \ge \cdots \ge a_n$). For all $a_1, a_2, \ldots, a_n \in \mathbb{A}$ such that $a_1 \le a_2 \le \cdots \le a_n$ (or $a_1 \ge a_2 \ge \cdots \ge a_n$), the following inequality holds:

$$G(a_1, a_2, \ldots, a_{n-1}, a_n) \ge G(t, t, \ldots, t, a_n), \qquad t = \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_{n-1}^2}{n-1}}.$$

In addition, the GM-Theorem and the GM-Corollary are valid.

GM-THEOREM (*Vasile Cîrtoaje*, 2005). *Let*

$$G(a_1, a_2, \ldots, a_n) : \mathbb{A} \to \mathbb{R}, \qquad \mathbb{A} \in \mathbb{R}_+^n,$$

be a symmetric continuous function satisfying

$$G(a_1, a_2, \ldots, a_{n-1}, a_n) \ge G\left(\sqrt{a_1 a_n}, a_2, \ldots, a_{n-1}, \sqrt{a_1 a_n}\right)$$

for $a_1, a_2, \ldots, a_n \in \mathbb{A}$ such that $a_1 \le a_2 \le \cdots \le a_n$ or $a_1 \ge a_2 \ge \cdots \ge a_n$. Then, for $a_1, a_2, \ldots, a_n \in \mathbb{A}$, the following inequality holds:

$$G(a_1, a_2, \ldots, a_n) \ge G(t, t, \ldots, t), \qquad t = \sqrt[n]{a_1 a_2 \cdots a_n}.$$

GM-Corollary (*Vasile Cîrtoaje*, 2005). *Let*

$$G(a_1, a_2, \ldots, a_n) : \mathbb{A} \to \mathbb{R}, \qquad \mathbb{A} \in \mathbb{R}_+^n,$$

be a symmetric continuous function satisfying

$$G(a_1, a_2, \ldots, a_{n-2}, a_{n-1}, a_n) \ge G\left(\sqrt{a_1 a_{n-1}}, a_2, \ldots, a_{n-2}, \sqrt{a_1 a_{n-1}}, a_n\right)$$

for all $a_1, a_2, \ldots, a_n \in \mathbb{A}$ such that $a_1 \le a_2 \le \cdots \le a_n$ (or $a_1 \ge a_2 \ge \cdots \ge a_n$). For all $a_1, a_2, \ldots, a_n \in \mathbb{A}$ such that $a_1 \le a_2 \le \cdots \le a_n$ (or $a_1 \ge a_2 \ge \cdots \ge a_n$), the following inequality holds:

$$G(a_1, a_2, \ldots, a_{n-1}, a_n) \ge G(t, t, \ldots, t, a_n), \qquad t = \sqrt[n-1]{a_1 a_2 \cdots a_{n-1}}.$$

***

The Arithmetic Compensation - Theorem (AC-Theorem) is a power tool for solving some symmetric inequalities of the form $F(a_1, a_2, \ldots, a_n) \ge 0$, where $a_1, a_2, \ldots, a_n$ are nonnegative real variables satisfying $a_1 + a_2 + \cdots + a_n = s$, $s > 0$. Notice that the AC-method can be applied especially to those inequalities where the equality occurs when $n - k$ variables are zero and the other k are equal to s/k, where $k \in \{1, 2, \ldots, n\}$.

AC-THEOREM (*Vasile Cîrtoaje*, 2005). *Let $s > 0$ and let F be a symmetric continuous function on the compact set in $\mathbb{R}^n$*

$$S = \{(a_1, a_2, \ldots, a_n) : a_1 + a_2 + \cdots + a_n = s,\ a_i \ge 0,\ i = 1, 2, \ldots, n\}.$$

If

$$F(a_1, a_2, a_3, \ldots, a_n) \ge$$
$$\ge \min\left\{F\left(\frac{a_1+a_2}{2}, \frac{a_1+a_2}{2}, a_3, \ldots, a_n\right), F(0, a_1+a_2, a_3, \ldots, a_n)\right\} \qquad (*)$$

for all $(a_1, a_2, \ldots, a_n) \in S$, then

$$F(a_1, \ldots, a_{n-k}, a_{n-k+1}, \ldots, a_n) \ge \min_{1\le k\le n} F\left(0, \ldots, 0, \frac{s}{k}, \ldots, \frac{s}{k}\right)$$

for all $(a_1, a_2, \ldots, a_n) \in S$.

Proof The AC-Theorem is clearly true for $n = 2$. Consider further $n \ge 3$. Since the function F is continuous on the compact set S, F achieves its minimum at one or more points of the set. We need to show that among these global minimum points there is one having $n-k$ coordinates equal to zero and k coordinates equal to s/k, where $k \in \{1, 2, \ldots, n\}$. Using the mathematical induction, it is easy to prove that this is true if among the global minimum points there is one having either a coordinate equal to zero or all coordinates equal to s/n. Let

$$B_0 = (b_1, b_2, \ldots, b_n), \qquad b_1 \le b_2 \le \cdots \le b_n,$$

be a global minimum point of F over the set S. If $b_1 = 0$ or $b_1 = b_2 = \cdots = b_n$, then the proof is completed. Consider further that

$$0 < b_1 < b_n.$$

From the hypothesis (*), it follows that at least one of

$$A_1 = (0, b_2, \ldots, b_{n-1}, b_1 + b_n)$$

and

$$B_1 = \left(\frac{b_1+b_n}{2}, b_2, \ldots, b_{n-1}, \frac{b_1+b_n}{2}\right)$$

is also a global minimum point of F over the set S. In the first case, the proof is completed. In the second case, starting from B_1 as a global minimum point of F, we repeat the process infinitely to get A_∞ with a coordinate equal to zero and, by Lemma above,

$$B_\infty = \left(\frac{s}{n}, \frac{s}{n}, \ldots, \frac{s}{n}\right).$$

Since at least one of A_∞ and B_∞ is a global minimum point of F, the conclusion follows.

AC-Corollary. *Let $s > 0$ and let F be a symmetric continuous function on the compact set in $\mathbb{R}^n$*

$$S = \{(a_1, a_2, \ldots, a_n) : a_1 + a_2 + \cdots + a_n = s,\ a_i \ge 0,\ i = 1, 2, \ldots, n\}.$$

If

$$F(a_1, a_2, a_3, \ldots, a_n) \ge F(0, a_1 + a_2, a_3, \ldots, a_n)$$

for all $(a_1, a_2, \ldots, a_n) \in S$ satisfying

$$F(a_1, a_2, a_3, \ldots, a_n) < F\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n\right), \qquad a_1 \ne a_2,$$

then $F(a_1, a_2, \ldots, a_n)$ is minimal when $n - k$ of the variables $a_1, a_2, \ldots, a_n$ are zero and the other k variables are equal to $\frac{s}{k}$, where $k \in \{1, 2, \ldots, n\}$.

Proof. Consider the following two possible cases:

$$F(a_1, a_2, a_3, \ldots, a_n) \ge F\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n\right)$$

and

$$F(a_1, a_2, a_3, \ldots, a_n) < F\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n\right).$$

If the first case, the condition (*) in AC-Theorem is obviously satisfied. In the second case, which implies $a_1 \ne a_2$, the hypothesis in the AC-Corollary gives

$$F(a_1, a_2, a_3, \ldots, a_n) \ge F(0, a_1 + a_2, a_3, \ldots, a_n),$$

and the condition (*) in AC-Theorem is also satisfied.

AC1-Corollary. *Let $s > 0$ and let G be a symmetric continuous function on the compact set in $\mathbb{R}^n$*

$$S = \{(a_1, a_2, \ldots, a_n) : a_1^2 + a_2^2 + \cdots + a_n^2 = s,\ a_i \ge 0,\ i = 1, 2, \ldots, n\}.$$

If

$$G(a_1, a_2, a_3, \ldots, a_n) \ge G\left(0, \sqrt{a_1^2 + a_2^2}, a_3, \ldots, a_n\right)$$

for all $(a_1, a_2, \ldots, a_n) \in S$ satisfying

$$G(a_1, a_2, a_3, \ldots, a_n) < G\left(\sqrt{\frac{a_1^2 + a_2^2}{2}}, \sqrt{\frac{a_1^2 + a_2^2}{2}}, a_3 \ldots, a_n\right), \qquad a_1 \ne a_2,$$

then $G(a_1, a_2, \ldots, a_n)$ is minimal when $n - k$ of the variables $a_1, a_2, \ldots, a_n$ are zero and the other k variables are equal to $\sqrt{\frac{s}{k}}$, where $k \in \{1, 2, \ldots, n\}$.

AC2-Corollary. *Let $s > 0$ and let G be a symmetric continuous function on the compact set in $\mathbb{R}^n$*

$$S = \{(a_1, a_2, \ldots, a_n) : a_1 a_2 \cdots a_n = p,\ a_i > 0,\ i = 1, 2, \ldots, n\}.$$

If

$$G(a_1, a_2, a_3, \ldots, a_n) \geq G(1, a_1 a_2, a_3, \ldots, a_n)$$

for all $(a_1, a_2, \ldots, a_n) \in S$ satisfying

$$G(a_1, a_2, a_3, \ldots, a_n) < G\left(\sqrt{a_1 a_2}, \sqrt{a_1 a_2}, a_3 \ldots, a_n\right), \qquad a_1 \neq a_2,$$

then $G(a_1, a_2, \ldots, a_n)$ is minimal when $n-k$ of the variables $a_1, a_2, \ldots, a_n$ are 1 and the other k variables are equal to $\sqrt[k]{p}$, where $k \in \{1, 2, \ldots, n\}$.

1.2 Applications

1.1. If $a_1, a_2, \ldots, a_n$ are real numbers such that $a_1 + a_2 + \cdots + a_n = 2$, then

$$(1+a_1^2)(1+a_2^2)\cdots(1+a_n^2) \ge \left(1+\frac{4}{n^2}\right)^n.$$

1.2. If $a_1, a_2, \ldots, a_n$ $(n \ge 5)$ are real numbers such that $a_1 + a_2 + \cdots + a_n = 4$, then

$$(1+a_1^2)(1+a_2^2)\cdots(1+a_n^2) \ge \left(1+\frac{16}{n^2}\right)^n.$$

1.3. If a, b, c, d are nonnegative real numbers such that $a + b + c + d = 4$, then

$$\sqrt{9(a^3+b^3+c^3+d^3)+28} \le 2(a^2+b^2+c^2+d^2).$$

1.4. If $a, b, c, d, e, f \ge 0$ such that $a + b + c + d + e + f = 6$, then

$$5(a^3+b^3+c^3+d^3+e^3+f^3)+36 \ge 11(a^2+b^2+c^2+d^2+e^2+f^2).$$

1.5. If $a, b, c, d \ge 0$ such that $a + b + c + d = 4$, then

$$abc + bcd + cda + dab + a^2b^2c^2 + b^2c^2d^2 + c^2d^2a^2 + d^2a^2b^2 \le 8.$$

1.6. If $a, b, c, d \ge 0$ such that $a + b + c + d = 4$, then

$$\frac{1}{5-abc} + \frac{1}{5-bcd} + \frac{1}{5-cda} + \frac{1}{5-dab} \le 1.$$

1.7. Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be nonnegative real numbers such that

$$a_1 + a_2 + \cdots + a_n = n,$$

and let

$$e_{n-1} = \left(1+\frac{1}{n-1}\right)^{n-1}, \qquad k \ge \frac{(n-1)e_{n-1}}{n-e_{n-1}}.$$

Then,

$$\frac{1}{k-a_1a_2\cdots a_{n-1}} + \frac{1}{k-a_2a_3\cdots a_n} + \cdots + \frac{1}{k-a_na_1\cdots a_{n-2}} \le \frac{n}{k-1}.$$

1.8. If $a,b,c,d \ge 0$ such that $a+b+c+d=4$, then

$$\frac{1}{4-abc}+\frac{1}{4-bcd}+\frac{1}{4-cda}+\frac{1}{4-dab}\le\frac{15}{11}.$$

1.9. Let $a_1,a_2,\ldots,a_n$ $(n\ge 3)$ be nonnegative real numbers such that

$$a_1+a_2+\cdots+a_n=n,$$

and let

$$e_{n-1}=\left(1+\frac{1}{n-1}\right)^{n-1},\qquad e_{n-1}<k\le\frac{(n-1)e_{n-1}}{n-e_{n-1}}\ .$$

Then,

$$\frac{1}{k-a_1a_2\cdots a_{n-1}}+\frac{1}{k-a_2a_3\cdots a_n}+\cdots+\frac{1}{k-a_na_1\cdots a_{n-2}}\le\frac{n-1}{k}+\frac{1}{k-e_{n-1}}\ .$$

1.10. If $a,b,c,d \ge 0$ such that $a+b+c+d=4$, then

$$\frac{1}{10-ab}+\frac{1}{10-bc}+\frac{1}{10-cd}+\frac{1}{10-da}+\frac{1}{10-ac}+\frac{1}{10-bd}\le\frac{2}{3}.$$

1.11. If $a,b,c,d \ge 0$ such that $a+b+c+d=4$, then

$$3(a^3+b^3+c^3+d^3-4)\ge(a^2+b^2+c^2+d^2-4)(a^2+b^2+c^2+d^2-1).$$

1.12. If $a,b,c,d \ge 0$ such that $a+b+c+d=4$, then

$$7(a^3+b^3+c^3+d^3-4)\ge(a^2+b^2+c^2+d^2-4)(a^2+b^2+c^2+d^2+11).$$

1.13. Let $a_1,a_2,\ldots,a_n$ $(n\ge 3)$ be nonnegative real numbers such that

$$a_1+a_2+\cdots+a_n=n.$$

Then

$$\frac{2}{3}\left(a_1^3+a_2^3+\cdots+a_n^3\right)+\sum_{1\le i_1<i_2<i_3\le n}a_{i_1}a_{i_2}a_{i_3}\ge\frac{n}{3}\left(a_1^2+a_2^2+\cdots+a_n^2\right).$$

1.14. Let $a_1, a_2, \ldots, a_n$ $(n \geq 4)$ be nonnegative real numbers such that

$$a_1 + a_2 + \cdots + a_n = n.$$

Then

$$a_1^3 + a_2^3 + \cdots + a_n^3 + \frac{6}{n(n-3)} \sum_{1 \leq i_1 < i_2 < i_3 \leq n} a_{i_1} a_{i_2} a_{i_3} \geq 2\left(a_1^2 + a_2^2 + \cdots + a_n^2\right).$$

1.15. Let $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ be nonnegative real numbers such that

$$a_1 + a_2 + \cdots + a_n = n.$$

Then

$$(n+1)\left(a_1^3 + a_2^3 + \cdots + a_n^3\right) + \frac{6}{n-2} \sum_{1 \leq i_1 < i_2 < i_3 \leq n} a_{i_1} a_{i_2} a_{i_3} \geq 2n\left(a_1^2 + a_2^2 + \cdots + a_n^2\right).$$

1.16. Let $a_1, a_2, ..., a_n$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = 1$. If

$$m \in \{1, 2, \ldots, n+1\},$$

then

$$m(m-1)\left(a_1^3 + a_2^3 + \cdots + a_n^3\right) + 1 \geq (2m-1)\left(a_1^2 + a_2^2 + \cdots + a_n^2\right).$$

1.17. If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$, then

$$(n+1)\left(a_1^2 + a_2^2 + \cdots + a_n^2\right) \geq n^2 + a_1^3 + a_2^3 + \cdots + a_n^3.$$

1.18. If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$, then

$$(n^2 + n + 1)\left(a_1^2 + a_2^2 + \cdots + a_n^2 - n\right) \geq a_1^4 + a_2^4 + \cdots + a_n^4 - n.$$

1.19. Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$. If $m \geq 3$ is an integer, then

$$\frac{n^{m-1} - 1}{n - 1}\left(a_1^2 + a_2^2 + \cdots + a_n^2 - n\right) \geq a_1^m + a_2^m + \cdots + a_n^m - n.$$

1.20. If a,b,c,d,e are nonnegative real numbers such that $a+b+c+d+e=5$, then

$$a^2+b^2+c^2+d^2+e^2-5 \ge \frac{5}{4}(1-abcde).$$

1.21. If $a_1,a_2,\dots,a_n$ are nonnegative real numbers such that $a_1+a_2+\cdots+a_n=n$, then

$$a_1^2+a_2^2+\cdots+a_n^2-n \ge \frac{n}{n-1}\,(1-a_1a_2\cdots a_n).$$

1.22. If a,b,c,d,e are nonnegative real numbers such that $a+b+c+d+e=5$, then

$$a^3+b^3+c^3+d^3+e^3-5 \ge \frac{45}{16}(1-abcde).$$

1.23. If $a_1,a_2,\dots,a_n$ are nonnegative real numbers such that $a_1+a_2+\cdots+a_n=n$, then

$$a_1^3+a_2^3+\cdots+a_n^3-n \ge \frac{n(2n-1)}{(n-1)^2}\,(1-a_1a_2\cdots a_n).$$

1.24. If $a_1,a_2,\dots,a_n$ are nonnegative real numbers such that $a_1+a_2+\cdots+a_n=n$, then

$$\frac{1}{n}\left(a_1^3+a_2^3+\cdots+a_n^3-n\right) \ge a_1^2+a_2^2+\cdots+a_n^2-n+\left(\frac{n-2}{2}\right)^2(a_1a_2\cdots a_n-1).$$

1.25. If $a_1,a_2,\dots,a_n$ are nonnegative real numbers such that $a_1+a_2+\cdots+a_n=n$, then

$$\sqrt{\frac{n(n-1)}{n+1}\,\left(a_1^3+a_2^3+\cdots+a_n^3-n\right)} \ge a_1^2+a_2^2+\cdots+a_n^2-n.$$

1.26. If a,b,c,d are nonnegative real numbers such that $a+b+c+d=4$, then

$$\sqrt{2\,(a^4+b^4+c^4+d^4)+\frac{313}{81}}\;+\;\frac{5}{9} \ge a^2+b^2+c^2+d^2.$$

1.27. If a,b,c,d are nonnegative real numbers such that $a+b+c+d=4$, then

$$2\sqrt{\frac{a^4+b^4+c^4+d^4-4}{7}} \ge a^2+b^2+c^2+d^2-4.$$

1.28. If a,b,c,d are nonnegative real numbers such that $a+b+c+d=4$, then

$$\sqrt{a^4+b^4+c^4+d^4}+\frac{16}{\sqrt{3}}-8\ge\left(\frac{4}{\sqrt{3}}-\frac{3}{2}\right)\left(a^2+b^2+c^2+d^2\right).$$

1.29. If a,b,c,d are nonnegative real numbers such that $a+b+c+d=4$, then

$$\sqrt{a^4+b^4+c^4+d^4}+8(2-\sqrt{2})\ge\left(2-\frac{1}{\sqrt{2}}\right)\left(a^2+b^2+c^2+d^2\right).$$

1.30. If a,b,c,d are nonnegative real numbers such that $a+b+c+d=4$, then

$$\sqrt{a^4+b^4+c^4+d^4}+16\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}\right)\ge\left(\frac{3}{\sqrt{2}}-\frac{2}{\sqrt{3}}\right)\left(a^2+b^2+c^2+d^2\right).$$

1.31. If a,b,c,d are nonnegative real numbers such that $a^2+b^2+c^2+d^2=1$, then

$$a^3+b^3+c^3+d^3+abc+bcd+cda+dab\le 1.$$

1.32. If a,b,c,d are nonnegative real numbers such that $a^2+b^2+c^2+d^2=2$, then

$$a^3+b^3+c^3+d^3+abc+bcd+cda+dab\ge 2.$$

1.33. If a,b,c,d are nonnegative real numbers such that $a^2+b^2+c^2+d^2=3$, then

$$3(a^3+b^3+c^3+d^3)+2(abc+bcd+cda+dab)\ge 11.$$

1.34. If a,b,c,d are nonnegative real numbers such that $a^2+b^2+c^2+d^2=4$, then

$$4(a^3+b^3+c^3+d^3)+abc+bcd+cda+dab\ge 20.$$

1.35. If a,b,c,d are nonnegative real numbers such that $a^2+b^2+c^2+d^2=4$, then

$$a^3+b^3+c^3+d^3+3(a+b+c+d)\le\frac{28}{\sqrt{3}}.$$

1.36. If a,b,c,d are nonnegative real numbers such that $a^2+b^2+c^2+d^2=4$, then

$$a^3+b^3+c^3+d^3+4(a+b+c+d)\le 20.$$

1.37. If a,b,c,d are nonnegative real numbers such that $a^2+b^2+c^2+d^2=4$, then

$$a^3+b^3+c^3+d^3+2\sqrt{2}(a+b+c+d)\ge 4(2+\sqrt{2}\ ).$$

1.38. If a,b,c,d are nonnegative real numbers such that $a^2+b^2+c^2+d^2=4$, then

$$a^3+b^3+c^3+d^3+2\sqrt{\frac{2}{3}}\,(a+b+c+d)\ge 4\left(\sqrt{2}+\frac{2}{\sqrt{3}}\right).$$

1.39. If a,b,c,d are nonnegative real numbers such that $a^2+b^2+c^2+d^2=4$, then

$$a^3+b^3+c^3+d^3-4+\frac{2}{\sqrt{3}}(a+b+c+d-4)\ge 0.$$

1.40. If $a_1,a_2,\ldots,a_n$ are real numbers such that

$$a_1+a_2+\cdots+a_n=n,$$

then

$$\left(a_1^2+a_2^2+\cdots+a_n^2\right)^2-n^2\ge\frac{n(n-1)}{n^2-n+1}\left(a_1^4+a_2^4+\cdots+a_n^4-n\right).$$

1.41. If a,b,c,d are real numbers such that $a+b+c+d=4$, then

$$\left(a^2+b^2+c^2+d^2-4\right)\left(a^2+b^2+c^2+d^2+\frac{26}{5}\right)\ge a^4+b^4+c^4+d^4-4.$$

1.42. If a,b,c,d are nonnegative real numbers such that $a+b+c+d=4$, then

$$\left(a^2+b^2+c^2+d^2-4\right)\left(a^2+b^2+c^2+d^2+\frac{11}{6}\right)\ge\frac{3}{4}\left(a^4+b^4+c^4+d^4-4\right).$$

1.43. If a,b,c,d are nonnegative real numbers such that $a+b+c+d=4$, then

$$\left(a^2+b^2+c^2+d^2-4\right)\left(2a^2+2b^2+2c^2+2d^2-1\right)\ge a^4+b^4+c^4+d^4-4.$$

1.44. If a,b,c,d are nonnegative real numbers such that $a+b+c+d=4$, then

$$\left(a^2+b^2+c^2+d^2-4\right)\left(a^2+b^2+c^2+d^2+12\right)\geq\frac{4}{3}\left(a^4+b^4+c^4+d^4-4\right).$$

1.45. If a,b,c,d are nonnegative real numbers such that $a+b+c+d=4$, then

$$\left(a^2+b^2+c^2+d^2-4\right)\left(a^2+b^2+c^2+d^2+\frac{76}{11}\right)\geq\frac{12}{11}\left(a^4+b^4+c^4+d^4-4\right).$$

1.46. If $a_1,a_2,\ldots,a_n$ are nonnegative real numbers such that

$$a_1+a_2+\cdots+a_n=m,\qquad m\in\{1,2,\ldots,n\},$$

then

$$\frac{1}{1+a_1^2}+\frac{1}{1+a_2^2}+\cdots+\frac{1}{1+a_n^2}\geq n-\frac{m}{2}.$$

1.47. If a,b,c,d are nonnegative real numbers such that $a+b+c+d=2$, then

$$\frac{1}{1+3a^2}+\frac{1}{1+3b^2}+\frac{1}{1+3c^2}+\frac{1}{1+3d^2}\geq\frac{16}{7}.$$

1.48. If a,b,c,d are nonnegative real numbers such that $a+b+c+d=4$, then

$$(1+a^2)(1+b^2)(1+c^2)(1+d^2)\leq 25.$$

1.49. If a,b,c,d are nonnegative real numbers such that $a+b+c+d=1$, then

$$\frac{(1+2a)(1+2b)(1+2c)(1+2d)}{(1-a)(1-b)(1-c)(1-d)}\geq\frac{125}{8}.$$

1.50. If $a_1,a_2,\ldots,a_n$ are nonnegative real numbers such that $a_1+a_2+\cdots+a_n=\frac{2}{3}$, then

$$\sum_{1\leq i\leq j\leq n}\frac{a_ia_j}{(1-a_i)(1-a_j)}\leq\frac{1}{4}.$$

1.51. If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = 1$ and no one of which is 1, then

$$\sum_{1 \le i \le j \le n} \frac{a_i a_j}{(1-a_i)(1-a_j)} \ge \frac{n}{2(n-1)}.$$

1.52. If a, b, c, d are nonnegative real numbers such that $a + b + c + d = 4$, then

$$(1+3a)(1+3b)(1+3c)(1+3d) \le 125 + 131abcd.$$

1.53. If a, b, c, d are nonnegative real numbers such that $a + b + c + d = 4$, then

$$(1+3a^2)(1+3b^2)(1+3c^2)(1+3d^2) \le 255 + a^2b^2c^2d^2.$$

1.54. If $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ are nonnegative real numbers, then

$$\sum a_1^2 + 2\sum_{sym} a_1a_2a_3 + \frac{4n(n-2)}{3(n-1)^2} \ge 2\sum_{sym} a_1a_2.$$

1.55. If a, b, c, d are nonnegative real numbers such that $a + b + c + d = \sqrt{3}$, then

$$ab(a+2b+3c) + bc(b+2c+3d) + cd(c+2d+3a) + da(d+2a+3b) \le 2.$$

1.56. If $a, b, c, d > 0$ such that $abcd = 1$, then

$$\frac{a+b+c+d}{16} + \frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} + \frac{1}{d+1} \ge \frac{9}{4}.$$

1.57. Let

$$F(a,b,c,d) = 4(a^2+b^2+c^2+d^2) - (a+b+c+d)^2,$$

where a, b, c, d, e are positive real numbers such that $a \le b \le c \le d$ and

$$a(b+c+d) \ge 3.$$

Then,

$$F(a,b,c,d) \ge F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}\right).$$

1.58. Let

$$F(a,b,c,d,e) = \sqrt[5]{abcde} - \frac{5}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}}\,,$$

where a,b,c,d,e are positive real numbers such that

$$a = \max\{a,b,c,d,e\}, \qquad bcde \geq 1.$$

Then,

$$F(a,b,c,d,e) \geq F\left(\frac{1}{a},\frac{1}{b},\frac{1}{c},\frac{1}{d},\frac{1}{e}\right).$$

1.59. Let

$$F(a_1,a_2,\ldots,a_n) = a_1+a_2+\cdots+a_n - n\sqrt[n]{a_1a_2\cdots a_n}\,,$$

where $a_1,a_2,\ldots,a_n$ are positive real numbers such that $a_1 \geq a_2 \geq \cdots \geq a_n$ and

$$a_1a_2\cdots a_{n-1}a_n^{n-1} \geq 1.$$

Then,

$$F(a_1,a_2,\ldots,a_n) \geq F\left(\frac{1}{a_1},\frac{1}{a_2},\ldots,\frac{1}{a_n}\right).$$

1.60. Let

$$F(a,b,c,d) = \sqrt{\frac{a^2+b^2+c^2+d^2}{4}} - \frac{a+b+c+d}{4}\,,$$

where a,b,c,d are positive real numbers such that $a \leq b \leq c \leq d$ and

$$a^3(b+c+d) \geq 1.$$

Then,

$$F(a,b,c,d) \geq F\left(\frac{1}{a},\frac{1}{b},\frac{1}{c},\frac{1}{d}\right).$$

1.61. If a,b,c,d are positive real numbers such that

$$a+b+c+d = 4, \qquad d = \max\{a,b,c,d\},$$

then

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1 \geq a^2+b^2+c^2+d^2.$$

1.3 Solutions

P 1.1. *If $a_1, a_2, \ldots, a_n$ are real numbers such that $a_1 + a_2 + \cdots + a_n = 2$, then*

$$(1 + a_1^2)(1 + a_2^2)\cdots(1 + a_n^2) \geq \left(1 + \frac{4}{n^2}\right)^n.$$

Solution. Clearly, it suffices to consider that $a_1, a_2, \ldots, a_n$ are nonnegative numbers. Assume that $0 \leq a_1 \leq a_2 \leq \cdots \leq a_n$ and write the inequality as

$$F(a_1, a_2, \ldots, a_n) \geq 0,$$

where

$$F(a_1, a_2, \ldots, a_n) = (1 + a_1^2)(1 + a_2^2)\cdots(1 + a_n^2) - \left(1 + \frac{4}{n^2}\right)^n.$$

If

$$F(a_1, a_2, \ldots, a_n) \geq F(t, a_2, \ldots, a_{n-1}, t)$$

for

$$t = \frac{a_1 + a_n}{2},$$

then, by the AM-Theorem, we have

$$F(a_1, a_2, \ldots, a_n) \geq F\left(\frac{2}{n}, \frac{2}{n}, \ldots, \frac{2}{n}\right) = 0.$$

The inequality $F(a_1, a_2, \ldots, a_n) \geq F(t, a_2, \ldots, a_{n-1}, t)$ is equivalent to

$$(1 + a_1^2)(1 + a_n^2) \geq (1 + t^2)^2,$$

$$(a_1 - a_n)^2(2 - t^2 - a_1 a_n) \geq 0.$$

Since $a_1 a_n \leq t^2$, it suffices to show that $t \leq 1$. We have

$$t = \frac{a_1 + a_n}{2} \leq \frac{a_1 + a_2 + \cdots + a_n}{2} = 1.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = \frac{2}{n}$.

□

P 1.2. *If $a_1, a_2, \ldots, a_n$ $(n \geq 5)$ are real numbers such that $a_1 + a_2 + \cdots + a_n = 4$, then*

$$(1 + a_1^2)(1 + a_2^2)\cdots(1 + a_n^2) \geq \left(1 + \frac{16}{n^2}\right)^n.$$

Solution. It suffices to consider that $a_1, a_2, \ldots, a_n$ are nonnegative numbers. Assume that $0 \le a_1 \le a_2 \le \cdots \le a_n$ and write the inequality as $F(a_1, a_2, \ldots, a_n) \ge 0$, where

$$F(a_1, a_2, \ldots, a_n) = (1 + a_1^2)(1 + a_2^2)\cdots(1 + a_n^2) - \left(1 + \frac{16}{n^2}\right)^n.$$

If

$$F(a_1, a_2, \ldots, a_n) \ge F(x, a_2, \ldots, a_{n-2}, x, a_n)$$

for

$$x = \frac{a_1 + a_{n-1}}{2},$$

then, by the AM-Corollary, we have

$$F(a_1, a_2, \ldots, a_{n-1}, a_n) \ge F(t, t, \ldots, t, a_n)$$

for $t = \dfrac{a_1 + a_2 + \cdots + a_{n-1}}{n-1}$. The inequality $F(a_1, a_2, \ldots, a_n) \ge F(x, a_2, \ldots, a_{n-2}, x, a_n)$ is equivalent to

$$(1 + a_1^2)(1 + a_{n-1}^2) \ge (1 + x^2)^2,$$
$$(a_1 - a_{n-1})^2(2 - x^2 - a_1 a_{n-1}) \ge 0.$$

Since $a_1 a_{n-1} \le x^2$, it suffices to show that $x \le 1$. We have

$$x = \frac{a_1 + a_{n-1}}{2} \le \frac{a_1 + a_2 + \cdots + a_n}{4} = 1.$$

Thus, we only need to prove the original inequality for $a_1 = a_2 = \cdots = a_{n-1}$, that is $f(x) \ge 0$, where

$$f(x) = (1 + x^2)^{n-1}(1 + y^2) - \left(1 + \frac{16}{n^2}\right)^n,$$

where

$$y = 4 - (n-1)x, \quad 0 \le x \le \frac{4}{n}.$$

Since $y' = -n + 1$, we have

$$\begin{aligned} f'(x) &= 2(n-1)x(1 + x^2)^{n-2}(1 + y^2) + 2(1 + x^2)^{n-1} y' y \\ &= 2(n-1)(1 + x^2)^{n-2}[x(1 + y^2) - (1 + x^2)y] \\ &= 2(n-1)(1 + x^2)^{n-2}(x - y)(1 - xy) \le 0. \end{aligned}$$

Because $x - y \le 0$ and

$$1 - xy = 1 - x[4 - (n-1)x] = 1 - 4x + (n-1)x^2 = (1 - 2x)^2 + (n-5)x^2 \ge 0,$$

we get $f'(x) \le 0$, f is decreasing, hence f is minimal for $x = \dfrac{4}{n} = y$. Therefore

$$f(x) \ge f\left(\frac{4}{n}\right) = 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = \dfrac{4}{n}$.

□

P 1.3. *If a,b,c,d are nonnegative real numbers such that $a+b+c+d=4$, then*

$$\sqrt{9(a^3+b^3+c^3+d^3)+28} \le 2(a^2+b^2+c^2+d^2).$$

(Vasile C., 2006)

Solution. Assume that

$$a \le b \le c \le d,$$

and write the inequality as $F(a,b,c,d) \ge 0$, where

$$F(a,b,c,d) = 4(a^2+b^2+c^2+d^2)^2 - 9(a^3+b^3+c^3+d^3) - 28.$$

First Solution. If

$$F(a,b,c,d) \ge F(t,b,c,t)$$

for

$$t = \frac{a+d}{2},$$

then, by the AM-Theorem, we have

$$F(a,b,c,d) \ge F(1,1,1,1) = 0.$$

Using the identities

$$a^2+d^2-2t^2 = 2(t^2-ad), \qquad a^3+d^3-2t^3 = 3(t^2-ad)(a+d),$$

we may write the inequality $F(a,b,c,d) \ge F(t,b,c,t)$ as follows:

$$4\left(a^2+d^2-2t^2\right)\left(2t^2+a^2+2b^2+2c^2+d^2\right) - 9\left(a^3+d^3-2t^3\right) \ge 0,$$

$$8(t^2-ad)\left(2t^2+a^2+2b^2+2c^2+d^2\right) - 27(t^2-ad)(a+d) \ge 0.$$

Since

$$t^2 - ad = \frac{1}{4}(a-d)^2 \ge 0,$$

this inequality is true if

$$8(2t^2+a^2+2b^2+2c^2+d^2) - 28(a+d) \ge 0,$$

which is equivalent to the homogeneous inequalities

$$12(a^2+d^2) + 8ad + 16(b^2+c^2) - 7(a+d)(a+b+c+d) \ge 0,$$

$$5(a^2+d^2) - 6ad + 16(b^2+c^2) - 7(a+d)(b+c) \ge 0.$$

Since

$$2(b^2+c^2) \ge (b+c)^2,$$

it suffices to show that

$$5(a^2+d^2)-6ad+8(b+c)^2-7(a+d)(b+c)\geq 0.$$

Using the substitution

$$x=\frac{b+c}{2}, \qquad a\leq x\leq d,$$

we may write the inequality as $E(a,x,d)\geq 0$, where

$$E(a,x,d)=5(a^2+d^2)-6ad+32x^2-14x(a+d).$$

We will show that

$$E(a,x,d)\geq E(x,x,d)\geq 0.$$

We have

$$E(a,x,d)-E(x,x,d)=(x-a)(9x+6d-5a)\geq 0$$

and

$$E(x,x,d)=5d^2-20dx+23x^2=5(d-2x)^2+3x^2>0.$$

The equality holds for $a=b=c=d=1$.

Second Solution. We will apply the AM-Corollary. Thus, we need to show that

$$F(a,b,c,d)\geq F(t,b,t,d), \qquad t=\frac{a+c}{2}\leq d.$$

As shown at the first solution, this inequality is true if

$$5(a^2+c^2)-6ac+16(b^2+d^2)-7(a+c)(b+d)\geq 0,$$

which can be written as

$$5(a^2+c^2)-6ac+16d^2+\left(4b-\frac{7a+7c}{8}\right)^2-\frac{49(a+c)^2}{64}-7(a+c)d\geq 0.$$

It suffices to show that

$$5(a^2+c^2)-6ac+16d^2-\frac{3(a+c)^2}{2}-7(a+c)d\geq 0.$$

Since

$$5(a^2+c^2)-6ac\geq(a+c)^2=4t^2,$$

it suffices to show that

$$4t^2+16d^2-6t^2-14td\geq 0,$$

which is equivalent to the obvious inequality

$$(d-t)(8d+t)\geq 0.$$

By the AM-Corollary, it suffices to prove the original inequality for $a = b = c$. That is, to show that $3a + d = 4$ involves

$$4\left[(3a^2 + d^2)^2 - 16\right] - 9(3a^3 + d^3 - 4) \geq 0,$$

which is equivalent to

$$4(3a^2 + d^2 - 4)(3a^2 + d^2 + 4) - 9(3a^3 + d^3 - 4) \geq 0.$$

Since

$$3a^2 + d^2 - 4 = 12(a-1)^2, \quad 3a^2 + d^2 + 4 = 4(3a^2 - 6a + 5),$$
$$3a^3 + d^3 - 4 = 12(a-1)^2(5 - 2a),$$

the inequality reduces to

$$(a-1)^2(48a^2 - 78a + 35) \geq 0,$$

which is true because

$$48a^2 - 78a + 35 > 3(16a^2 - 26a + 11)$$

and

$$16a^2 - 26a + 11 = \left(4a - \frac{13}{4}\right)^2 + \frac{7}{16} > 0.$$

□

P 1.4. *If $a, b, c, d, e, f \geq 0$ such that $a + b + c + d + e + f = 6$, then*

$$5(a^3 + b^3 + c^3 + d^3 + e^3 + f^3) + 36 \geq 11(a^2 + b^2 + c^2 + d^2 + e^2 + f^2).$$

(Vasile C., 2006)

Solution. Write the inequality as $F(a, b, c, d, e, f) \geq 0$, where

$$F(a,b,c,d,e,f) = 5(a^3+b^3+c^3+d^3+e^3+f^3)+36-11(a^2+b^2+c^2+d^2+e^2+f^2).$$

First Solution. Apply the AM-Corollary. We will show first that

$$F(a, b, c, d, e, f) \geq F(t, b, c, d, t, f)$$

for

$$a \geq b \geq c \geq d \geq e \geq f, \qquad t = \frac{a+e}{2}.$$

Using the identities

$$a^2 + e^2 - 2t^2 = 2(t^2 - ae), \qquad a^3 + e^3 - 2t^3 = 3(a+e)(t^2 - ae),$$

we write the desired inequality $F(a,b,c,d,e,f) \geq F(t,b,c,d,t,f)$ as

$$5(a^3+e^3-2t^3)-11(a^2+e^2-2t^2) \geq 0,$$

$$15(a+e)(t^2-ae)-22(t^2-ae) \geq 0.$$

Since

$$t^2-ae = \frac{1}{4}(a-e)^2 \geq 0,$$

we only need to show that

$$15(a+e)-22 \geq 0.$$

This inequality is equivalent to

$$45(a+e)-11(a+b+c+d+e+f) \geq 0,$$

$$34a-11(b+c+d)+34e-11f \geq 0,$$

$$a+11(3a-b-c-d)+23e+11(e-f) \geq 0.$$

Clearly, the last inequality is true. By the AM-Corollary, it suffices to prove the original inequality for $a=b=c=d=e$. Thus, we need to show that $5a+f=6$ involves

$$5(5a^3+f^3)+36 \geq 11(5a^2+f^2),$$

which is equivalent to

$$6-17a+16a^2-5a^3 \geq 0,$$

$$(1-a)^2(6-5a) \leq 0,$$

$$(1-a)^2 b \geq 0.$$

The equality holds for $a=b=c=d=e=f=1$, and also for

$$a=b=c=d=e=\frac{6}{5}, \qquad f=0$$

(or any cyclic permutation).

Second Solution. Apply the AC-Corollary. We will show that

$$F(a,b,c,d,e,f) < F(t,t,c,d,e,f)$$

involves

$$F(a,b,c,d,e,f) \geq F(0,2t,c,d,e,f),$$

where

$$t = \frac{a+b}{2}, \qquad a \neq b.$$

As shown at the first solution, the hypothesis $F(a,b,c,d,e,f) < F(t,t,c,d,e,f)$ involves

$$15(a+b)-22 < 0. \tag{*}$$

Write now the required inequality $F(a,b,c,d,e,f) \geq F(0,a+b,c,d,e,f)$ as follows:

$$5[a^3+b^3-(a+b)^3]-11[a^2+b^2-(a+b)^2] \geq 0,$$

$$-15ab(a+b)+22ab \geq 0,$$

$$ab[15(a+b)-22] \leq 0.$$

Clearly, this inequality follows immediately from (*).

By the AC-Corollary, it suffices to prove that $F(a,b,c,d,e,f) \geq 0$ when $6-k$ of the variables a,b,c,d,e,f are zero and the other k variables are equal to $\frac{6}{k}$, where $k \in \{1,2,3,4,5,6\}$; that is,

$$5k\left(\frac{6}{k}\right)^3+36-11k\left(\frac{6}{k}\right)^2 \geq 0,$$

which is equivalent to the obvious inequality

$$(k-5)(k-6) \geq 0.$$

□

P 1.5. *If* $a,b,c,d \geq 0$ *such that* $a+b+c+d=4$ *, then*

$$abc+bcd+cda+dab+a^2b^2c^2+b^2c^2d^2+c^2d^2a^2+d^2a^2b^2 \leq 8.$$

(Vasile C., 2006)

Solution. Write the desired inequality as $F(a,b,c,d)+8 \geq 0$, where

$$F(a,b,c,d) = -(abc+bcd+cda+dab+a^2b^2c^2+b^2c^2d^2+c^2d^2a^2+d^2a^2b^2).$$

First Solution. According to the AM-Corollary, we need to show that

$$F(a,b,c,d) \geq F(t,b,t,d)$$

for

$$a \geq b \geq c \geq d, \qquad t = \frac{a+c}{2}.$$

Taking into account that

$$F(a,b,c,d) = -ac(b+d)-(a+c)bd-a^2c^2(b^2+d^2)-(a^2+c^2)b^2d^2,$$

we write the desired inequality $F(a,b,c,d) \geq F(t,b,t,d)$ as

$$(t^2-ac)(b+d)+(t^4-a^2c^2)(b^2+d^2)-(a^2+c^2-2t^2)b^2d^2 \geq 0,$$

$$(t^2-ac)\left[b+d+(t^2+ac)(b^2+d^2)-2b^2d^2\right] \geq 0.$$

Since
$$t^2 - ac = \frac{1}{4}(a-c)^2 \ge 0,$$
this inequality is true if
$$ac(b^2+d^2) - 2b^2d^2 \ge 0.$$
We have
$$ac(b^2+d^2) - 2b^2d^2 \ge bd(b^2+d^2) - 2b^2d^2 = bd(b-d)^2 \ge 0.$$

By the AM-Corollary, it suffices to prove the original inequality for $a = b = c$; that is, to show that $3a + d = 4$ involves
$$a^3 + 3a^2d + a^6 + 3a^4d^2 \le 8,$$
which is equivalent to
$$7a^6 - 18a^5 + 12a^4 - 2a^3 + 3a^2 - 2 \le 0,$$
$$(a-1)^2(7a^4 - 4a^3 - 3a^2 - 4a - 2) \le 0.$$
It suffices to show that
$$7a^4 - 4a^3 - 3a^2 - 4a - 2 + \frac{2}{81} \le 0,$$
which is equivalent to
$$567a^4 - 324a^3 - 243a^2 - 324a - 160 \le 0,$$
$$(3a-4)(189a^3 + 144a^2 + 111a + 40) \le 0,$$
$$d(189a^3 + 144a^2 + 111a + 40) \ge 0.$$
The equality holds for $a = b = c = d = 1$.

Second Solution. Apply the AC-Corollary. First, we write $F(a,b,c,d)$ in the form
$$F(a,b,c,d) = -ab(c+d) - (a+b)cd - a^2b^2(c^2+d^2) - (a^2+b^2)c^2d^2.$$
We will show that
$$F(a,b,c,d) < F(t,t,c,d)$$
involves
$$F(a,b,c,d) \ge F(0,2t,c,d),$$
where
$$t = \frac{a+b}{2}, \qquad a \ne b.$$
Write the hypothesis $F(a,b,c,d) < F(t,t,c,d)$ as follows:
$$(t^2 - ab)(c+d) + (t^4 - a^2b^2)(c^2+d^2) - (a^2+b^2-2t^2)c^2d^2 < 0,$$

$$(t^2-ab)\left[c+d+(t^2+ab)(c^2+d^2)-2c^2d^2\right]<0.$$

Dividing by the positive factor t^2-ab, the inequality becomes

$$c+d+(t^2+ab)(c^2+d^2)-2c^2d^2<0. \tag{*}$$

Write now the required inequality $F(a,b,c,d)\ge F(0,2t,c,d)$ as follows:

$$-ab(c+d)-a^2b^2(c^2+d^2)+\left[(a+b)^2-a^2-b^2\right]c^2d^2\ge 0,$$

$$-ab\left[c+d+ab(c^2+d^2)-2c^2d^2\right]\ge 0,$$

$$ab\left[c+d+ab(c^2+d^2)-2c^2d^2\right]\le 0.$$

Clearly, this inequality follows immediately from (*).

By the AC-Corollary, it suffices to prove the original inequality for $a=b=c=d=1$, and for $a=0$ and $b+c+d=4$. In the first case, the equality occurs. In the second case, the inequality becomes

$$bcd+b^2c^2d^2\le 8.$$

By the AM-GM inequality, we have

$$bcd\le\left(\frac{b+c+d}{3}\right)^3=\frac{64}{27},$$

therefore

$$bcd+b^2c^2d^2\le\frac{64}{27}\left(1+\frac{64}{27}\right)=8-\frac{8}{729}<8.$$

□

P 1.6. *If $a,b,c,d\ge 0$ such that $a+b+c+d=4$, then*

$$\frac{1}{5-abc}+\frac{1}{5-bcd}+\frac{1}{5-cda}+\frac{1}{5-dab}\le 1.$$

Solution. By the AM-GM inequality, we have

$$5-abc\ge 5-\left(\frac{a+b+c}{3}\right)^3\ge 5-\left(\frac{4}{3}\right)^3=\frac{71}{27}>0.$$

Write the desired inequality as $F(a,b,c,d)\ge 0$, where

$$F(a,b,c,d)=1-\frac{1}{5-abc}-\frac{1}{5-bcd}-\frac{1}{5-cda}-\frac{1}{5-dab},$$

and apply the AC-Corollary. First, we will show that

$$F(a,b,c,d)<F(t,t,c,d)$$

involves
$$F(a,b,c,d) \ge F(0,2t,c,d),$$
where $t = \dfrac{a+b}{2}$, $a \ne b$. Write the hypothesis $F(a,b,c,d) < F(t,t,c,d)$ as
$$\left(\frac{1}{5-ct^2} - \frac{1}{5-abc}\right) + \left(\frac{1}{5-dt^2} - \frac{1}{5-dab}\right) < \frac{2(5-cdt)}{(5-bcd)(5-cda)} - \frac{2}{5-cdt}.$$
Dividing by the positive factor $t^2 - ab$, the inequality becomes
$$\frac{c}{(5-abc)(5-ct^2)} + \frac{d}{(5-dab)(5-dt^2)} < \frac{2c^2d^2}{(5-bcd)(5-cda)(5-cdt)}.$$
Since
$$\frac{c}{(5-abc)(5-ct^2)} + \frac{d}{(5-dab)(5-dt^2)} \ge \frac{c}{5(5-abc)} + \frac{d}{5(5-dab)},$$
we get
$$\frac{c}{5-abc} + \frac{d}{5-dab} < \frac{10c^2d^2}{(5-bcd)(5-cda)(5-cdt)}. \tag{*}$$
Similarly, write the required inequality $F(a,b,c,d) \ge F(0,2t,c,d)$ as follows:
$$\frac{1}{5} + \frac{1}{5-2cdt} - \left(\frac{1}{5-bcd} + \frac{1}{5-cda}\right) \ge \left(\frac{1}{5-abc} - \frac{1}{5}\right) + \left(\frac{1}{5-dab} - \frac{1}{5}\right),$$
$$\frac{2(5-cdt)}{5-2cdt} - \frac{10(5-cdt)}{(5-bcd)(5-cda)} \ge \frac{abc}{5-abc} + \frac{dab}{5-dab},$$
$$\frac{2c^2d^2(5-cdt)}{(5-bcd)(5-cda)(5-2cdt)} \ge \frac{c}{5-abc} + \frac{d}{5-dab}.$$
Since
$$\frac{5-cdt}{5-2cdt} \ge \frac{5}{5-cdt},$$
it suffices to show that
$$\frac{10c^2d^2}{(5-bcd)(5-cda)(5-cdt)} \ge \frac{c}{5-abc} + \frac{d}{5-dab}.$$
Taking into account (*), the conclusion follows.

By the AC-Corollary, it suffices to prove the original inequality for $a = b = c = d = 1$, and for $a = 0$ and $b + c + d = 4$. In the first case, the equality occurs. In the second case, since
$$5 - bcd \ge 5 - \left(\frac{b+c+d}{3}\right)^3 = 5 - \left(\frac{4}{3}\right)^3 = \frac{71}{27},$$

we have

$$\frac{1}{5-abc}+\frac{1}{5-bcd}+\frac{1}{5-cda}+\frac{1}{5-dab}=\frac{1}{5-bcd}+\frac{3}{5}\leq\frac{27}{71}+\frac{3}{5}<1.$$

The equality holds for $a=b=c=d=1$.

Remark. In the same manner, we can prove the following generalization:

- *If $a_1,a_2,\dots,a_n$ $(n\geq 4)$ are nonnegative real numbers such that*

$$a_1+a_2+\cdots+a_n=n,$$

then

$$\frac{1}{n+1-a_1a_2\cdots a_{n-1}}+\frac{1}{n+1-a_2a_3\cdots a_n}+\cdots+\frac{1}{n+1-a_na_1\cdots a_{n-2}}\leq 1,$$

with equality for $a_1=a_2=\cdots=a_n=1$.

□

P 1.7. *Let $a_1,a_2,\dots,a_n$ $(n\geq 3)$ be nonnegative real numbers such that*

$$a_1+a_2+\cdots+a_n=n,$$

and let

$$e_{n-1}=\left(1+\frac{1}{n-1}\right)^{n-1},\qquad k\geq\frac{(n-1)e_{n-1}}{n-e_{n-1}}.$$

Then,

$$\frac{1}{k-a_1a_2\cdots a_{n-1}}+\frac{1}{k-a_2a_3\cdots a_n}+\cdots+\frac{1}{k-a_na_1\cdots a_{n-2}}\leq\frac{n}{k-1}.$$

(Vasile C., 2005)

Solution. By the AM-GM inequality, we have

$$a_1a_2\cdots a_{n-1}\leq\left(\frac{a_1+a_2+\cdots+a_{n-1}}{n-1}\right)^{n-1}\leq\left(\frac{n}{n-1}\right)^{n-1}=e_{n-1}<k.$$

Write the desired inequality as

$$F(a_1,a_2,\dots,a_n)+\frac{n}{k-1}\geq 0,$$

where

$$F(a_1,a_2,\dots,a_n)=-\left(\frac{1}{k-a_2a_3\cdots a_n}+\frac{1}{k-a_1a_3\cdots a_n}+\cdots+\frac{1}{k-a_1a_2\cdots a_{n-1}}\right).$$

We assert that

$$F(a_1,a_2,a_3,\ldots,a_n) < F(t,t,a_3,\ldots,a_n) \tag{1}$$

involves

$$F(a_1,a_2,a_3,\ldots,a_n) \ge F(0,2t,a_3,\ldots,a_n) \tag{2}$$

for any $k > e_{n-1}$, where

$$t = \frac{a_1+a_2}{2}, \qquad a_1 \ne a_2.$$

Let us denote

$$b = a_3a_4\cdots a_n.$$

Since

$$-F(a_1,a_2,a_3,\ldots,a_n) = \frac{1}{k-a_1b} + \frac{1}{k-a_2b} + \sum_{i=3}^{n} \frac{a_i}{ka_i - a_1a_2b},$$

and

$$-F(t,t,a_3,\ldots,a_n) = \frac{2}{k-tb} + \sum_{i=3}^{n} \frac{a_i}{ka_i - t^2b},$$

the hypothesis (1) is equivalent to

$$b(t^2-a_1a_2)\left[\frac{2b}{(k-tb)(k-a_1b)(k-a_2b)} - \sum_{i=3}^{n} \frac{a_i}{(ka_i-t^2b)(ka_i-a_1a_2b)}\right] > 0,$$

which involves

$$\frac{2b}{(k-tb)(k-a_1b)(k-a_2b)} > \sum_{i=3}^{n} \frac{a_i}{(ka_i-t^2b)(ka_i-a_1a_2b)}.$$

Because

$$\frac{a_i}{ka_i-t^2b} \ge \frac{1}{k},$$

this inequality involves

$$\frac{2kb}{(k-tb)(k-a_1b)(k-a_2b)} > \sum_{i=3}^{n} \frac{1}{ka_i-a_1a_2b}. \tag{3}$$

On the other hand, since

$$-F(a_1,a_2,a_3,\ldots,a_n) = \frac{1}{k-a_1b} + \frac{1}{k-a_2b} + \sum_{i=3}^{n} \frac{a_i}{ka_i - a_1a_2b},$$

and

$$-F(0,2t,a_3,\ldots,a_n) = \frac{n-1}{k} + \frac{1}{k-2tb},$$

the required inequality (2) can be written as follows:

$$\left(\frac{1}{k-2tb}-\frac{1}{k}\right)-\left(\frac{1}{k-a_1b}-\frac{1}{k}\right)-\left(\frac{1}{k-a_2b}-\frac{1}{k}\right)\geq\sum_{i=3}^{n}\left(\frac{a_i}{ka_i-a_1a_2b}-\frac{1}{k}\right),$$

$$\frac{2tb}{k-2tb}-\frac{2(kt-a_1a_2b)b}{(k-a_1b)(k-a_2b)}\geq\sum_{i=3}^{n}\frac{a_1a_2b}{ka_i-a_1a_2b},$$

$$a_1a_2b\left[\frac{2b(k-tb)}{(k-2tb)(k-a_1b)(k-a_2b)}-\sum_{i=3}^{n}\frac{1}{ka_i-a_1a_2b}\right]\geq 0.$$

Because

$$\frac{k-tb}{k-2tb}\geq\frac{k}{k-tb},$$

this inequality is true if

$$\frac{2kb}{(k-tb)(k-a_1b)(k-a_2b)}>\sum_{i=3}^{n}\frac{1}{ka_i-a_1a_2b},$$

which is exactly (3).

According to the AC-Corollary, it suffices to prove the original inequality for $a_1=a_2=\cdots=a_n=1$, and for $a_1=0$ and $a_2+a_3+\cdots+a_n=n$. For the first case, the original inequality is an equality. For the second case, the original inequality becomes

$$\frac{1}{k-a_2a_3\cdots a_n}+\frac{n-1}{k}\leq\frac{n}{k-1},$$

which is equivalent to

$$a_2a_3\cdots a_n\leq\frac{kn}{k+n-1}.$$

Since $a_2a_3\cdots a_n\leq e_{n-1}$, it suffices to show that

$$e_{n-1}\leq\frac{kn}{k+n-1},$$

which is equivalent to the hypothesis

$$k\geq\frac{(n-1)e_{n-1}}{n-e_{n-1}}.$$

The equality holds for $a_1=a_2=\cdots=a_n=1$. In addition, if $k=\dfrac{(n-1)e_{n-1}}{n-e_{n-1}}$, then the equality holds also for

$$a_1=0,\qquad a_2=a_3=\cdots=a_n=\frac{n}{n-1}$$

(or any cyclic permutation).

Remark. For $n=4$, we get the following statement:

- *If a,b,c,d are nonnegative real numbers such that $a+b+c+d=4$, then*

$$\frac{1}{48-11abc}+\frac{1}{48-11bcd}+\frac{1}{48-11cda}+\frac{1}{48-11dab}\le\frac{4}{37},$$

with equality for $a=b=c=d=1$, and also for $a=0$ and $b=c=d=\frac{4}{3}$ (or any cyclic permutation).

□

P 1.8. *If $a,b,c,d\ge 0$ such that $a+b+c+d=4$, then*

$$\frac{1}{4-abc}+\frac{1}{4-bcd}+\frac{1}{4-cda}+\frac{1}{4-dab}\le\frac{15}{11}.$$

Solution. The proof is similar to the one of the preceding P 1.7. Finally, it suffices to prove the original inequality for $a=b=c=d=1$, and for $a=0$ and $b+c+d=4$. The first case is trivial, while the second case leads to the inequality

$$\frac{1}{4-bcd}+\frac{3}{4}\le\frac{15}{11},$$

which is equivalent to

$$bcd\le\frac{64}{27}.$$

By the AM-GM inequality, we have

$$bcd\le\left(\frac{b+c+d}{3}\right)^3=\frac{64}{27}.$$

The equality holds for

$$a=0,\qquad b=c=d=\frac{4}{3}$$

(or any cyclic permutation).

Remark. In the same manner, we can prove the following generalization:

- *Let $a_1,a_2,\ldots,a_n$ $(n\ge 4)$ be nonnegative real numbers such that*

$$a_1+a_2+\cdots+a_n=n.$$

and let

$$e_{n-1}=\left(1+\frac{1}{n-1}\right)^{n-1}.$$

Then,

$$\frac{1}{n-a_1a_2\cdots a_{n-1}}+\frac{1}{n-a_2a_3\cdots a_n}+\cdots+\frac{1}{n-a_na_1\cdots a_{n-2}}\le 1+\frac{e_{n-1}}{n(n-e_{n-1})},$$

with equality for $a_1=0$ and $a_2=\cdots=a_n=\frac{n}{n-1}$ (or any cyclic permutation).

□

P 1.9. *Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be nonnegative real numbers such that*

$$a_1 + a_2 + \cdots + a_n = n,$$

and let

$$e_{n-1} = \left(1 + \frac{1}{n-1}\right)^{n-1}, \qquad e_{n-1} < k \le \frac{(n-1)e_{n-1}}{n - e_{n-1}}.$$

Then,

$$\frac{1}{k - a_1 a_2 \cdots a_{n-1}} + \frac{1}{k - a_2 a_3 \cdots a_n} + \cdots + \frac{1}{k - a_n a_1 \cdots a_{n-2}} \le \frac{n-1}{k} + \frac{1}{k - e_{n-1}}.$$

(Vasile C., 2005)

Solution. Write the desired inequality as

$$F(a_1, a_2, \ldots, a_n) + \frac{n-1}{k} + \frac{1}{k - e_{n-1}} \ge 0,$$

where

$$F(a_1, a_2, \ldots, a_n) = -\left(\frac{1}{k - a_1 a_2 \cdots a_{n-1}} + \frac{1}{k - a_2 a_3 \cdots a_n} + \cdots + \frac{1}{k - a_n a_1 \cdots a_{n-2}}\right).$$

As shown in the proof of P 1.7, the inequality

$$F(a_1, a_2, a_3, \ldots, a_n) < F(t, t, a_3, \ldots, a_n)$$

involves

$$F(a_1, a_2, a_3, \ldots, a_n) \ge F(0, 2t, a_3, \ldots, a_n)$$

for any $k > e_{n-1}$, where

$$t = \frac{a_1 + a_2}{2}, \qquad a_1 \ne a_2.$$

According to the AC-Corollary, it suffices to prove the original inequality for $a_1 = a_2 = \cdots = a_n = 1$, and for $a_1 = 0$ and $a_2 + a_3 + \cdots + a_n = n$. In the first case, the original inequality becomes

$$\frac{n}{k-1} \le \frac{n-1}{k} + \frac{1}{k - e_{n-1}},$$

which is equivalent to the hypothesis

$$k \le \frac{(n-1)e_{n-1}}{n - e_{n-1}}.$$

In the second case, the original inequality becomes

$$\frac{1}{k - a_2 a_3 \cdots a_n} \le \frac{1}{k - e_{n-1}},$$

which is equivalent to

$$a_2a_3\cdots a_n \le e_{n-1}.$$

By the AM-GM inequality, we have

$$a_1a_2\cdots a_{n-1} \le \left(\frac{a_1+a_2+\cdots+a_{n-1}}{n-1}\right)^{n-1} \le \left(\frac{n}{n-1}\right)^{n-1} = e_{n-1}.$$

The equality holds for

$$a_1 = 0, \qquad a_2 = a_3 = \cdots = a_n = \frac{n}{n-1}$$

(or any cyclic permutation). In addition, if $k = \dfrac{(n-1)e_{n-1}}{n-e_{n-1}}$, then the equality holds also for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 1.10. *If $a,b,c,d \ge 0$ such that $a+b+c+d=4$, then*

$$\frac{1}{10-ab}+\frac{1}{10-bc}+\frac{1}{10-cd}+\frac{1}{10-da}+\frac{1}{10-ac}+\frac{1}{10-bd} \le \frac{2}{3}.$$

Solution. By the AM-GM inequality, we have

$$10-ab \ge 10-\left(\frac{a+b}{2}\right)^2 \ge 10-4 > 0.$$

Write the desired inequality as

$$F(a,b,c,d)+\frac{2}{3} \ge 0,$$

where

$$F(a,b,c,d) = -\left(\frac{1}{10-ab}+\frac{1}{10-ac}+\frac{1}{10-ad}+\frac{1}{10-bc}+\frac{1}{10-bd}+\frac{1}{10-cd}\right),$$

and apply the AC-Corollary. First, we will show that

$$F(a,b,c,d) < F(t,t,c,d)$$

involves

$$F(a,b,c,d) \ge F(0,2t,c,d),$$

where $t = \dfrac{a+b}{2}$, $a \ne b$. Write the hypothesis $F(a,b,c,d) < F(t,t,c,d)$ as

$$\left(\frac{1}{10-ac}+\frac{1}{10-bc} - \frac{2}{10-tc}\right)+\left(\frac{1}{10-ad}+\frac{1}{10-bd} - \frac{2}{10-td}\right) >$$

$$> \frac{1}{10-t^2} - \frac{1}{10-ab}.$$

Dividing by the positive factor $t^2 - ab$, the inequality becomes

$$\frac{2c^2}{(10-ac)(10-bc)(10-tc)} + \frac{2d^2}{(10-ad)(10-bd)(10-td)} > \frac{1}{(10-ab)(10-t^2)}.$$

Since

$$\frac{1}{(10-ab)(10-t^2)} \geq \frac{1}{10(10-ab)},$$

we get

$$\frac{20c^2}{(10-ac)(10-bc)(10-tc)} + \frac{20d^2}{(10-ad)(10-bd)(10-td)} > \frac{1}{10-ab}. \qquad (*)$$

Similarly, write the required inequality $F(a,b,c,d) \geq F(0,2t,c,d)$ as follows:

$$\left(\frac{1}{10-ab} - \frac{1}{10}\right) + \left[\left(\frac{1}{10-ac} + \frac{1}{10-bc}\right) - \left(\frac{1}{10} + \frac{1}{10-2tc}\right)\right]$$
$$+ \left[\left(\frac{1}{10-ad} + \frac{1}{10-bd}\right) - \left(\frac{1}{10} + \frac{1}{10-2td}\right)\right] \leq 0,$$

$$\frac{ab}{10(10-ab)} + \left[\frac{2(10-tc)}{(10-ac)(10-bc)} - \frac{2(10-tc)}{10(10-2tc)}\right]$$
$$+ \left[\frac{2(10-td)}{(10-ad)(10-bd)} - \frac{2(10-td)}{10(10-2td)}\right] \leq 0,$$

$$\frac{abc^2(10-tc)}{(10-ac)(10-bc)(5-tc)} + \frac{abd^2(10-td)}{(10-ad)(10-bd)(5-td)} \geq \frac{ab}{10-ab}.$$

This is true if

$$\frac{c^2(10-tc)}{(10-ac)(10-bc)(5-tc)} + \frac{d^2(10-td)}{(10-ad)(10-bd)(5-td)} \geq \frac{1}{10-ab}.$$

Since

$$\frac{10-tc}{5-tc} \geq \frac{20}{10-tc}, \qquad \frac{10-td}{5-td} \geq \frac{20}{10-td},$$

it suffices to show that

$$\frac{20c^2}{(10-ac)(10-bc)(10-tc)} + \frac{20d^2}{(10-ad)(10-bd)(10-td)} \geq \frac{1}{10-ab}.$$

According to (*), the conclusion follows.

By the AC-Corollary, it suffices to prove the original inequality for $a = b = c = d = 1$, for $a = 0$ and $b = c = d = 4/3$, and for $a = b = 0$ and $c + d = 4$. In the first case, the equality holds. In the second case, the inequality becomes

$$\frac{3}{10} + \frac{27}{74} \le \frac{2}{3}.$$

We have

$$\frac{2}{3} - \frac{3}{10} - \frac{27}{74} = \frac{11}{30} - \frac{27}{74} = \frac{4}{2220} > 0.$$

In the third case, the inequality becomes

$$\frac{1}{10 - cd} \le \frac{1}{6},$$

$$cd \le 4.$$

We have

$$cd \le \left(\frac{c + d}{2}\right)^2 = 4.$$

The equality holds for $a = b = c = d = 1$, and for $a = b = 2$ and $c = d = 0$ (or any permutation).

Remark. In the same manner, we can prove the following generalizations (*Vasile Cîrtoaje*, 2005):

- *Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be nonnegative numbers such that $a_1 + a_2 + \cdots + a_n = n$.*

  (a) If $k \ge \dfrac{n(n+1)}{2}$, then

$$\sum_{1 \le i < j \le n} \frac{1}{k - a_i a_j} \le \frac{n(n-1)}{2(k-1)},$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$;

  (b) If $\dfrac{n^2}{4} < k \le \dfrac{n(n+1)}{2}$, then

$$\sum_{1 \le i < j \le n} \frac{1}{k - a_i a_j} \le \frac{(n-2)(n+1)}{2k} + \frac{4}{4k - n^2},$$

with equality for

$$a_1 = a_2 = \frac{n}{2}, \qquad a_3 = \cdots = a_n = 0$$

(or any permutation).

- *Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be nonnegative numbers such that $a_1 + a_2 + \cdots + a_n = n$, and let*

$$G(a_1, a_2, \ldots, a_n) = \sum_{1 \le i_1 < \cdots < i_m \le n} \frac{1}{k - a_{i_1} a_{i_2} \cdots a_{i_m}}.$$

(a) *If* $k \ge \dfrac{\binom{n}{m} - 1}{\binom{n}{m}\left(\frac{m}{n}\right)^m - 1}$, *then*

$$G(a_1, a_2, \ldots, a_n) \le G(1, 1, \ldots, 1)\ ;$$

(b) *If* $\left(\frac{m}{n}\right)^m < k \le \dfrac{\binom{n}{m} - 1}{\binom{n}{m}\left(\frac{m}{n}\right)^m - 1}$, *then*

$$G(a_1, a_2, \ldots, a_n) \le G(\frac{n}{m}, \ldots, \frac{n}{m}, 0 \ldots, 0)\ .$$

□

P 1.11. *If* $a, b, c, d \ge 0$ *such that* $a + b + c + d = 4$ *, then*

$$3(a^3 + b^3 + c^3 + d^3 - 4) \ge (a^2 + b^2 + c^2 + d^2 - 4)(a^2 + b^2 + c^2 + d^2 - 1).$$

(Vasile C., 2005)

Solution. Write the inequality as

$$F(a, b, c, d) \ge 0,$$

where

$$F(a, b, c, d) = 3(a^3 + b^3 + c^3 + d^3 - 4) - \left(a^2 + b^2 + c^2 + d^2 - \frac{5}{2}\right)^2 + \frac{9}{4},$$

and apply the AC-Corollary. We will show that

$$F(a, b, c, d) < F(t, t, c, d) \tag{*}$$

involves

$$F(a, b, c, d) \ge F(0, 2t, c, d), \tag{**}$$

where

$$t = \frac{a+b}{2}, \qquad a \ne b.$$

Using the identities

$$a^2 + b^2 - 2t^2 = 2(t^2 - ab), \qquad a^3 + b^3 - 2t^3 = 3(t^2 - ab)(a + b),$$

we may write (*) as

$$3(a^3 + b^3 - 2t^3) - (a^2 + b^2 - 2t^2)(2t^2 + a^2 + b^2 + 2c^2 + 2d^2 - 5) < 0,$$

$$9(t^2-ab)(a+b)-2(t^2-ab)(2t^2+a^2+b^2+2c^2+2d^2-5)<0,$$

$$9(a+b)-2(2t^2+a^2+b^2+2c^2+2d^2-5)<0. \tag{A}$$

Since

$$a^2+b^2-(a+b)^2=-2ab, \qquad a^3+b^3-(a+b)^3=-3ab(a+b),$$

we may write (**) as

$$3\left[a^3+b^3-(a+b)^3\right]-\left[a^2+b^2-(a+b)^2\right]\left[(a+b)^2+a^2+b^2+2c^2+2d^2-5\right]\ge 0,$$

$$-9ab(a+b)+2ab(4t^2+a^2+b^2+2c^2+2d^2-5)\ge 0,$$

which is true if

$$9(a+b)-2(4t^2+a^2+b^2+2c^2+2d^2-5)\le 0.$$

Clearly, this inequality follows immediately from (A). As a consequence, (*) implies (**). Thus, by the AC-Corollary, we have

$$F(a,b,c,d)\ge \min_{1\le k\le 4} f(k),$$

where

$$f(k)=\frac{16(k-1)(4-k)}{k^2}$$

is the value of F for the case where $4-k$ of the variables a,b,c,d are zero and the other k variables are equal to $4/k$. We have

$$f(1)=f(4)=0, \quad f(2)=8, \quad f(3)=\frac{32}{9},$$

therefore $F(a,b,c,d)\ge 0$.

The equality holds for $a=b=c=0$ and $d=4$ (or any cyclic permutation), and also for $a=b=c=d=1$.

□

P 1.12. *If $a,b,c,d\ge 0$ such that $a+b+c+d=4$, then*

$$7(a^3+b^3+c^3+d^3-4)\ge (a^2+b^2+c^2+d^2-4)(a^2+b^2+c^2+d^2+11).$$

(Vasile C., 2005)

Solution. Write the inequality as $F(a,b,c,d)\ge 0$, where

$$F(a,b,c,d)=7(a^3+b^3+c^3+d^3-4)-\left(a^2+b^2+c^2+d^2+\frac{7}{2}\right)^2+\frac{225}{4},$$

First Solution. Apply the AM-Corollary. We will show first that

$$F(a,b,c,d) \ge F(t,b,t,d)$$

for

$$a \ge b \ge c \ge d, \qquad t = \frac{a+c}{2}.$$

Using the identities

$$a^2 + c^2 - 2t^2 = 2(t^2 - ac), \qquad a^3 + c^3 - 2t^3 = 3(a+c)(t^2 - ac),$$

we write the desired inequality $F(a,b,c,d) \ge F(t,b,t,d)$ as

$$7(a^3 + c^3 - 2t^3) - (a^2 + c^2 - 2t^2)(2t^2 + a^2 + c^2 + 2b^2 + 2d^2 + 7) \ge 0,$$

$$21(t^2 - ac)(a+c) - 2(t^2 - ac)(2t^2 + a^2 + c^2 + 2b^2 + 2d^2 + 7) \ge 0.$$

This is true if

$$21(a+c) - (4t^2 + 2a^2 + 2c^2 + 4b^2 + 4d^2 + 14) \ge 0.$$

Since

$$4t^2 + 2a^2 + 2c^2 + 4b^2 + 4d^2 = (a+c)^2 + 2(a^2 + c^2) + 4(b^2 + d^2) \le 3(a+c)^2 + 4(b+d)^2,$$

it suffices to show that

$$21(a+c) - 3(a+c)^2 - 4(b+d)^2 - 14 \ge 0.$$

Substituting

$$x = a + c, \qquad y = b + d,$$

we need to prove that

$$21x - 3x^2 - 4y^2 - 14 \ge 0$$

for $x + y = 4$, $x \ge y$. We have

$$\begin{aligned} 21x - 3x^2 - 4y^2 - 14 &= \frac{21x(x+y)}{4} - 3x^2 - 4y^2 - \frac{7(x+y)^2}{8} \\ &= \frac{11x^2 + 28xy - 39y^2}{8} \\ &= \frac{(x-y)(11x + 39y)}{8} \ge 0. \end{aligned}$$

By the AM-Corollary, it suffices to prove the original inequality for $a = b = c$; that is, to show that $3a + d = 4$ involves

$$7(3a^3 + d^3 - 4) \ge (3a^2 + d^2 - 4)(3a^2 + d^2 + 11),$$

which is equivalent to

$$84(1-a)^2(5-2a) \ge 36(1-a)^2(4a^2-8a+9),$$

$$(1-a)^2(1+2a)(4-3a) \ge 0,$$

$$(1-a)^2(1+2a)d \ge 0,$$

The equality holds for $a = b = c = d = 1$, and also for $a = b = c = 4/3$ and $d = 0$ (or any cyclic permutation).

Second Solution. Apply the AC-Corollary. We will show that

$$F(a,b,c,d) < F(t,t,c,d) \tag{*}$$

involves

$$F(a,b,c,d) \ge F(0,2t,c,d), \tag{**}$$

where

$$t = \frac{a+b}{2}, \qquad a \ne b.$$

As shown at the first solution, (*) involves

$$21(a+b) - 2(2t^2 + a^2 + b^2 + 2c^2 + 2d^2 + 7) < 0. \tag{A}$$

Since

$$a^2 + b^2 - (a+b)^2 = -2ab, \qquad a^3 + b^3 - (a+b)^3 = -3ab(a+b),$$

we may write (**) as

$$7\left[a^3 + b^3 - (a+b)^3\right] - \left[a^2 + b^2 - (a+b)^2\right]\left[(a+b)^2 + a^2 + b^2 + 2c^2 + 2d^2 + 7\right] \ge 0,$$

$$-21ab(a+b) + 2ab(4t^2 + a^2 + b^2 + 2c^2 + 2d^2 + 7) \ge 0,$$

which is true if

$$21(a+b) - 2(4t^2 + a^2 + b^2 + 2c^2 + 2d^2 + 7) \le 0.$$

Clearly, this inequality follows immediately from (A). As a consequence, (*) implies (**). Thus, by the AC-Corollary, we have

$$F(a,b,c,d) \ge \min_{1\le k\le n} f(k),$$

where

$$f(k) = \frac{16(k-3)(k-4)}{k^2}$$

is the value of F for the case where $4-k$ of the variables a,b,c,d are zero and the other k variables are equal to $4/k$. We have

$$f(1) = 96, \quad f(2) = 8, \quad f(3) = f(4) = 0,$$

therefore $F(a,b,c,d) \ge 0$.

□

P 1.13. *Let $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ be nonnegative real numbers such that*

$$a_1 + a_2 + \cdots + a_n = n.$$

Then

$$\frac{2}{3}\left(a_1^3 + a_2^3 + \cdots + a_n^3\right) + \sum_{1 \leq i_1 < i_2 < i_3 \leq n} a_{i_1} a_{i_2} a_{i_3} \geq \frac{n}{3}\left(a_1^2 + a_2^2 + \cdots + a_n^2\right).$$

(Vasile C., 2005)

Solution. Let

$$b = a_3 + \cdots + a_n$$

and

$$F(a_1, a_2, \ldots, a_n) = k_1\left(a_1^3 + \cdots + a_n^3\right) + \sum_{1 \leq i_1 < i_2 < i_3 \leq n} a_{i_1} a_{i_2} a_{i_3} - k_2 n(a_1^2 + \cdots + a_n^2),$$

where k_1 and k_2 are fixed real numbers. We claim that

$$F(a_1, a_2, a_3, \ldots, a_n) < F(t, t, a_3, \ldots, a_n)$$

involves

$$F(a_1, a_2, a_3, \ldots, a_n) \geq F(0, 2t, a_3, \ldots, a_n)$$

where

$$t = \frac{a_1 + a_2}{2}, \qquad a_1 \neq a_2.$$

Since

$$\sum_{1 \leq i_1 < i_2 < i_3 \leq n} a_{i_1} a_{i_2} a_{i_3} = a_1 a_2 b + (a_1 + a_2) \sum_{3 \leq i_1 < i_2 \leq n} a_{i_1} a_{i_2} + \sum_{3 \leq i_1 < i_2 < i_3 \leq n} a_{i_1} a_{i_2} a_{i_3},$$

$$a_1^2 + a_2^2 - 2t^2 = 2(t^2 - a_1 a_2), \qquad a_1^3 + a_2^3 - 2t^3 = 3(a_1 + a_2)(t^2 - a_1 a_2),$$

the inequality $F(a_1, a_2, a_3, \ldots, a_n) < F(t, t, a_3, \ldots, a_n)$ is equivalent to

$$k_1(a_1^3 + a_2^3 - 2t^3) + (a_1 a_2 - t^2)b - k_2 n(a_1^2 + a_2^2 - 2t^2) < 0,$$

$$(t^2 - a_1 a_2)[3k_1(a_1 + a_2) - b - 2k_2 n] < 0,$$

$$3k_1(a_1 + a_2) - b - 2k_2 n < 0. \tag{*}$$

On the other hand, since

$$a_1^2 + a_2^2 - (a_1 + a_2)^2 = -2a_1 a_2, \qquad a_1^3 + a_2^3 - (a_1 + a_2)^3 = -3a_1 a_2 (a_1 + a_2),$$

the required inequality $F(a_1, a_2, a_3, \ldots, a_n) \geq F(0, 2t, a_3, \ldots, a_n)$ is equivalent to

$$k_1\left[a_1^3 + a_2^3 - (a_1 + a_2)^3\right] + a_1 a_2 b - k_2 n\left[a_1^2 + a_2^2 - (a_1 + a_2)^2\right] \geq 0,$$

$$a_1a_2\left[3k_1(a_1+a_2)-b-2k_2n\right]\le 0.$$

Clearly, this inequality follows immediately from (*).

According to the AC-Corollary, $F(a_1,a_2,\dots,a_n)$ is minimal when $n-k$ of the variables $a_1,a_2,\dots,a_n$ are zero and the other k variables are equal to n/k, where $k\in\{1,2,\dots,n\}$. Thus,

$$F(a_1,\dots,a_{n-k},a_{n-k+1},\dots,a_n)\ge \min_{1\le k\le n} F\left(0,\dots,0,\frac{n}{k},\dots,\frac{n}{k}\right)=\frac{n^3}{6}\min_{1\le k\le n} f(k),$$

where

$$\frac{n^3}{6}f(k)=\binom{n}{3}\left(\frac{n}{k}\right)^3+\frac{k_1n^3}{k^2}-\frac{k_2n^3}{k},$$

$$f(k)=1-\frac{3(2k_2+1)}{k}+\frac{2(3k_1+1)}{k^2}.$$

We only need to show that $f(k)\ge 0$ for all $k\in\{1,2,\dots,n\}$.

For the original inequality, we have

$$k_1=\frac{2}{3},\qquad k_2=\frac{1}{3},$$

hence

$$f(k)=1-\frac{5}{k}+\frac{6}{k^2}=\frac{(k-2)(k-3)}{k^2}.$$

Clearly, for $k\in\{1,2,\dots,n\}$, we have

$$f(k)\ge 0,$$

with equality when $k=2$ and $k=3$. The original inequality is an equality for

$$a_1=a_2=\frac{n}{2},\qquad a_3=\cdots=a_n=0$$

(or any permutation), and also for

$$a_1=a_2=a_3=\frac{n}{3},\qquad a_4=\cdots=a_n=0$$

(or any permutation).

Remark. We can rewrite the inequality in the following homogeneous forms:

$$2\left(a_1^3+a_2^3+\cdots+a_n^3\right)+3\sum_{1\le i_1<i_2<i_3\le n} a_{i_1}a_{i_2}a_{i_3}\ge (a_1+a_2+\cdots+a_n)(a_1^2+a_2^2+\cdots+a_n^2)$$

and

$$a_1^3+a_2^3+\cdots+a_n^3+3\sum_{1\le i_1<i_2<i_3\le n} a_{i_1}a_{i_2}a_{i_3}\ge \sum_{1\le i_1<i_2\le n} a_{i_1}a_{i_2}(a_{i_1}+a_{i_2}).$$

For $n = 3$, we get the third degree Schur's inequality

$$a_1^3 + a_2^3 + a_3^3 + 3a_1a_2a_3 \geq a_1a_2(a_1 + a_2) + a_2a_3(a_2 + a_3) + a_3a_1(a_3 + a_1),$$

with equality for $a_1 = a_2$ and $a_3 = 0$ (or any cyclic permutation), and also for $a_1 = a_2 = a_3$.

□

P 1.14. *Let $a_1, a_2, \ldots, a_n$ ($n \geq 4$) be nonnegative real numbers such that*

$$a_1 + a_2 + \cdots + a_n = n.$$

Then

$$a_1^3 + a_2^3 + \cdots + a_n^3 + \frac{6}{n(n-3)} \sum_{1 \leq i_1 < i_2 < i_3 \leq n} a_{i_1}a_{i_2}a_{i_3} \geq 2\left(a_1^2 + a_2^2 + \cdots + a_n^2\right).$$

(Vasile C., 2005)

Solution. As shown at the preceding P 1.13, it suffices to show that $f(k) \geq 0$ for all $k \in \{1, 2, \ldots, n\}$, where

$$f(k) = 1 - \frac{3(2k_2 + 1)}{k} + \frac{2(3k_1 + 1)}{k^2},$$

with

$$k_1 = \frac{n(n-3)}{6}, \qquad k_2 = \frac{n-3}{3}.$$

We get

$$f(k) = 1 - \frac{2n-3}{k} + \frac{(n-1)(n-2)}{k^2} = \frac{(k-n+1)(k-n+2)}{k^2}.$$

Clearly, for $k \in \{1, 2, \ldots, n\}$, we have

$$f(k) \geq 0,$$

with equality when $k = n-2$ and $k = n-1$. The original inequality is an equality for

$$a_1 = \cdots = a_{n-2} = \frac{n}{n-2}, \qquad a_{n-1} = a_n = 0$$

(or any permutation), and also for

$$a_1 = \cdots = a_{n-1} = \frac{n}{n-1}, \qquad a_n = 0$$

(or any cyclic permutation).

□

P 1.15. *Let $a_1, a_2, \dots, a_n$ $(n \ge 3)$ be nonnegative real numbers such that*

$$a_1 + a_2 + \cdots + a_n = n.$$

Then

$$(n+1)\left(a_1^3 + a_2^3 + \cdots + a_n^3\right) + \frac{6}{n-2} \sum_{1 \le i_1 < i_2 < i_3 \le n} a_{i_1} a_{i_2} a_{i_3} \ge 2n\left(a_1^2 + a_2^2 + \cdots + a_n^2\right).$$

(Vasile C., 2005)

Solution. As shown at P 1.13, it suffices to show that $f(k) \ge 0$ for all $k \in \{1, 2, \dots, n\}$, where

$$f(k) = 1 - \frac{3(2k_2 + 1)}{k} + \frac{2(3k_1 + 1)}{k^2},$$

with

$$k_1 = \frac{(n+1)(n-2)}{6}, \qquad k_2 = \frac{n-2}{3}.$$

We get

$$f(k) = 1 - \frac{2n-1}{k} + \frac{n(n-1)}{k^2} = \frac{(k-n)(k-n+1)}{k^2}.$$

Clearly, for $k \in \{1, 2, \dots, n\}$, we have

$$f(k) \ge 0,$$

with equality when $k = n - 1$ and $k = n$. The original inequality is an equality for

$$a_1 = \cdots = a_{n-1} = \frac{n}{n-1}, \qquad a_n = 0$$

(or any cyclic permutation), and also for

$$a_1 = a_2 = \cdots = a_n = 1.$$

Remark. For $n = 4$, we get the homogeneous inequality

$$a^3 + b^3 + c^3 + d^3 + abc + bcd + cda + dab \ge \frac{2}{3} \sum a^2(b + c + d),$$

where a, b, c, d are nonnegative real numbers. The equality holds for $a = b = c$ and $d = 0$ (or any cyclic permutation), and also for $a = b = c = d$.

□

P 1.16. *Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = 1$. If*

$$m \in \{1, 2, \ldots, n+1\},$$

then

$$m(m-1)\left(a_1^3 + a_2^3 + \cdots + a_n^3\right) + 1 \geq (2m-1)\left(a_1^2 + a_2^2 + \cdots + a_n^2\right).$$

(Vasile C., 2005)

Solution. We need to prove that $F(a_1, a_2, \ldots, a_n) \geq 0$, where

$$F(a_1, a_2, \ldots, a_n) = m(m-1)\left(a_1^3 + a_2^3 + \cdots + a_n^3\right) + 1 - (2m-1)\left(a_1^2 + a_2^2 + \cdots + a_n^2\right).$$

Assume that

$$F(a_1, a_2, a_3, \ldots, a_n) < F(t, t, a_3, \ldots, a_n) \tag{*}$$

involves

$$F(a_1, a_2, a_3, \ldots, a_n) \geq F(0, 2t, a_3, \ldots, a_n), \tag{**}$$

where

$$t = \frac{a_1 + a_2}{2}, \qquad a_1 \neq a_2.$$

Then, by AC-Corollary, we have

$$F(a_1, a_2, \ldots, a_n) \geq \min_{1 \leq k \leq n} f(k),$$

where

$$f(k) = \frac{(k-m)(k-m+1)}{k^2}$$

is the value of F for $a_1 = \cdots = a_{n-k} = 0$ and $a_{n-k+1} = \cdots = a_n = 1/k$. Obviously, for $k \in \{1, 2, \ldots, n\}$, we have

$$f(k) \geq 0,$$

with equality when $k = m-1$ ($m \geq 2$) and $k = m$ ($m \leq n$). From $f(k) \geq 0$ for $k \in \{1, 2, \ldots, n\}$, it follows that $F(a_1, a_2, \ldots, a_n) \geq 0$.

To prove that (*) implies (**), we write (*) as

$$m(m-1)(a_1^3 + a_2^3 - 2t^3) - (2m-1)(a_1^2 + a_2^2 - 2t^2) < 0,$$

$$(t^2 - a_1a_2)[3m(m-1)(a_1 + a_2) - 2(2m-1)] < 0,$$

$$3m(m-1)(a_1 + a_2) - 2(2m-1) < 0, \tag{A}$$

and (**) as

$$m(m-1)(a_1^3 + a_2^3 - 8t^3) - (2m-1)(a_1^2 + a_2^2 - 4t^2) \geq 0,$$

$$a_1a_2[3m(m-1)(a_1 + a_2) - 2(2m-1)] \leq 0. \tag{B}$$

Clearly, (A) implies (B), hence (*) implies (**). This completes the proof.

For $m = 1$, the equality holds when $n-1$ of the numbers a_i are zero. For $m \in \{2,3,\ldots,n\}$, the equality holds when m or $m-1$ of the numbers a_i are equal and the other numbers are zero. For $m = n+1$, the equality holds when all numbers $a_1, a_2, ..., a_n$ are equal.

Remark. Actually, the inequality holds for all $m \in [0,1] \cup \{2,3,\ldots,n\} \cup [n+1,\infty)$, but it does not hold for $m \in (1,2) \cup (2,3) \cup \cdots \cup (n,n+1)$.

□

P 1.17. *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$, then*

$$(n+1)\left(a_1^2 + a_2^2 + \cdots + a_n^2\right) \geq n^2 + a_1^3 + a_2^3 + \cdots + a_n^3.$$

(Vasile C., 2002)

Solution. Write the desired inequality as $F(a_1, a_2, \ldots, a_n) \geq 0$, where

$$F(a_1, a_2, \ldots, a_n) = (n+1)\left(a_1^2 + a_2^2 + \cdots + a_n^2\right) - n^2 - a_1^3 - a_2^3 - \cdots - a_n^3.$$

First Solution. Apply the AM-Corollary. First, we will show that

$$F(a_1, a_2, \ldots, a_{n-2}, a_{n-1}, a_n) \geq F(t, a_2, \ldots, a_{n-2}, t, a_n)$$

for

$$a_1 \leq a_2 \leq \cdots \leq a_{n-1} \leq a_n, \qquad t = \frac{a_1 + a_{n-1}}{2}.$$

Write this inequality as follows:

$$(n+1)(a_1^2 + a_{n-1}^2 - 2t^2) - (a_1^3 + a_{n-1}^3 - 2t^3) \geq 0,$$

$$(t^2 - a_1 a_{n-1})[2(n+1) - 3(a_1 + a_{n-1})] \geq 0.$$

It is true if

$$2(n+1) - 3(a_1 + a_{n-1}) \geq 0,$$

which is equivalent to the homogeneous inequalities

$$2(n+1)(a_1 + a_2 + \cdots + a_n) - 3n(a_1 + a_{n-1}) \geq 0,$$

$$2(n+1)(a_2 + \cdots + a_{n-2} + a_n) - (n-2)(a_1 + a_{n-1}) \geq 0.$$

It suffices to show that

$$2(n+1)a_n - (n-2)(a_1 + a_{n-1}) \geq 0.$$

We have

$$2(n+1)a_n - (n-2)(a_1 + a_{n-1}) > (n-2)(2a_n - a_1 - a_{n-1}) \geq 0.$$

By the AM-Corollary, it suffices to prove the original inequality for $a_1 = a_2 = \cdots = a_{n-1}$; that is, to show that $(n-1)a_1 + a_n = n$ involves

$$(n+1)\left[(n-1)a_1^2 + a_n^2\right] \geq n^2 + (n-1)a_1^3 + a_n^3.$$

This inequality is equivalent to

$$n(n-1)(n-2)a_1(a_1-1)^2 \geq 0.$$

The equality holds when $a_1 = a_2 = \cdots = a_n = 1$, and also when

$$a_1 = \cdots = a_{n-1} = 0, \qquad a_n = n$$

(or any cyclic permutation).

Second Solution. Apply the AC-Corollary. Assume that

$$F(a_1, a_2, a_3, \ldots, a_n) < F(t, t, a_3, \ldots, a_n) \tag{*}$$

involves

$$F(a_1, a_2, a_3, \ldots, a_n) \geq F(0, 2t, a_3, \ldots, a_n), \tag{**}$$

where

$$t = \frac{a_1 + a_2}{2}, \qquad a_1 \neq a_2.$$

Then, by AC-Corollary, we have

$$F(a_1, a_2, \ldots, a_n) \geq \min_{1 \leq k \leq n} f(k),$$

where

$$f(k) = (n+1)\frac{n^2}{k} - n^2 - \frac{n^3}{k^2} = \frac{n^2(k-1)(n-k)}{k^2}$$

is the value of F for $a_1 = \cdots = a_{n-k} = 0$ and $a_{n-k+1} = \cdots = a_n = n/k$. Obviously,

$$f(k) \geq 0$$

for $k \in \{1, 2, \ldots, n\}$.

To prove that (*) implies (**), we write (*) as

$$(n+1)(a_1^2 + a_2^2 - 2t^2) - (a_1^3 + a_2^3 - 2t^3) < 0,$$

$$(t^2 - a_1 a_2)[2(n+1) - 3(a_1 + a_2)] < 0,$$

$$2(n+1) - 3(a_1 + a_2) < 0, \tag{A}$$

and (**) as

$$(n+1)\left[a_1^2 + a_2^2 - (a_1 + a_2)^2\right] - \left[a_1^3 + a_2^3 - (a_1 + a_2)^3\right] \geq 0,$$

$$a_1 a_2 \left[2(n+1) - 3(a_1 + a_2)\right] \leq 0. \tag{B}$$

Clearly, (A) implies (B), hence (*) implies (**). This completes the proof.

□

P 1.18. *If $a_1, a_2, \dots, a_n$ are nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$, then*

$$(n^2+n+1)\left(a_1^2+a_2^2+\cdots+a_n^2-n\right) \ge a_1^4+a_2^4+\cdots+a_n^4-n.$$

(Vasile C., 2002)

Solution. We need to prove that $F(a_1, a_2, \dots, a_n) \ge 0$, where

$$F(a_1,a_2,\dots,a_n) = (n^2+n+1)\left(a_1^2+a_2^2+\cdots+a_n^2-n\right) - \left(a_1^4+a_2^4+\cdots+a_n^4-n\right).$$

First Solution. Apply the AM-Corollary. Let us show that

$$F(a_1,a_2,\dots,a_{n-2},a_{n-1},a_n) \ge F(t,a_2,\dots,a_{n-2},t,a_n)$$

for

$$a_1 \le a_2 \le \cdots \le a_{n-1} \le a_n, \qquad t = \frac{a_1+a_{n-1}}{2}.$$

Since

$$a_1^2+a_{n-1}^2-2t^2 = 2(t^2-a_1a_{n-1}),$$

$$a_1^4+a_{n-1}^4-2t^4 = \frac{1}{2}(t^2-a_1a_{n-1})(7a_1^2+7a_{n-1}^2+10a_1a_{n-1}),$$

the inequality $F(a_1,a_2,\dots,a_{n-2},a_{n-1},a_n) \ge F(t,a_2,\dots,a_{n-2},t,a_n)$ can be written as follows:

$$(n^2+n+1)(a_1^2+a_{n-1}^2-2t^2) - (a_1^4+a_{n-1}^4-2t^3) \ge 0,$$

$$4(n^2+n+1)(t^2-a_1a_{n-1}) - (t^2-a_1a_{n-1})(7a_1^2+7a_{n-1}^2+10a_1a_{n-1}) \ge 0.$$

Since $t^2 - a_1a_{n-1} \ge 0$, we need to show that

$$4(n^2+n+1) \ge 7a_1^2+7a_{n-1}^2+10a_1a_{n-1}.$$

It suffices to prove that

$$4n^2 \ge 7a_1^2+7a_{n-1}^2+10a_1a_{n-1}$$

for $n \ge 3$, which is equivalent to the homogeneous inequality

$$4(a_1+a_2+\cdots+a_n)^2 \ge 7a_1^2+7a_{n-1}^2+10a_1a_{n-1}.$$

Since

$$a_1+a_2+\cdots+a_n \ge (n-2)a_1+a_{n-1}+a_n \ge (n-2)a_1+2a_{n-1} \ge a_1+2a_{n-1},$$

we only need to prove that

$$4(a_1+2a_{n-1})^2 \ge 7a_1^2+7a_{n-1}^2+10a_1a_{n-1},$$

which reduces to the obvious inequality

$$3a_{n-1}^2 + 2a_1a_{n-1} \geq a_1^2.$$

By the AM-Corollary, it suffices to prove the original inequality for $a_1 = a_2 = \cdots = a_{n-1}$; that is, to show that $(n-1)a_1 + a_n = n$ involves

$$(n^2+n+1)[(n-1)a_1^2 + a_n^2 - n] \geq (n-1)a_1^4 + a_n^4 - n.$$

Since

$$(n-1)a_1^2 + a_n^2 - n = n(n-1)(a_1-1)^2,$$

$$(n-1)a_1^4+a_n^4-n = n(n-1)(a_1-1)^2\left[(n^2-3n+3)a_1^2 - 2(n^2-n-1)a_1 + n^2+n+1\right],$$

the inequality is equivalent to

$$n(n-1)a_1(a_1-1)^2[2(n^2-n-1)-(n^2-3n+3)a_1] \geq 0.$$

It is true because

$$2(n^2-n-1)-(n^2-3n+3)a_1 \geq 2(n^2-n-1)-\frac{(n^2-3n+3)n}{n-1} = \frac{n^3-n^2-3n+2}{n-1} > 0.$$

The equality holds when $a_1 = a_2 = \cdots = a_n = 1$, and also when

$$a_1 = \cdots = a_{n-1} = 0, \qquad a_n = n$$

(or any cyclic permutation).

Second Solution. Apply the AC-Corollary. Assume that

$$F(a_1,a_2,a_3,\ldots,a_n) < F(t,t,a_3,\ldots,a_n) \tag{*}$$

involves

$$F(a_1,a_2,a_3,\ldots,a_n) \geq F(0,2t,a_3,\ldots,a_n), \tag{**}$$

where

$$t = \frac{a_1+a_2}{2}, \qquad a_1 \neq a_2.$$

Then, by AC-Corollary, we have

$$F(a_1,a_2,\ldots,a_n) \geq \min_{1\leq k\leq n} f(k),$$

where

$$f(k) = (n^2+n+1)\left(\frac{n^2}{k}-n\right)-\left(\frac{n^4}{k^3}-n\right) = \frac{n^2(k-1)(n-k)(nk+n+k)}{k^3}$$

is the value of F for $a_1 = \cdots = a_{n-k} = 0$ and $a_{n-k+1} = \cdots = a_n = n/k$. Obviously,

$$f(k) \geq 0$$

for $k \in \{1,2,\dots,n\}$.

Let us show now that (*) involves (**). Since

$$a_1^2+a_2^2-2t^2=2(t^2-a_1a_2), \qquad a_1^4+a_2^4-2t^4=\frac{1}{2}(t^2-a_1a_2)(7a_1^2+7a_2^2+10a_1a_2),$$

we may write (*) as follows:

$$(n^2+n+1)(a_1^2+a_2^2-2t^2)-(a_1^4+a_2^4-2t^4)<0,$$

$$4(n^2+n+1)(t^2-a_1a_2)-(t^2-a_1a_2)(7a_1^2+7a_2^2+10a_1a_2)<0,$$

$$7a_1^2+7a_2^2+10a_1a_2>4(n^2+n+1). \tag{A}$$

Since

$$a_1^2+a_2^2-(a_1+a_2)^2=-2a_1a_2, \qquad a_1^4+a_2^4-(a_1+a_2)^4=-a_1a_2(4a_1^2+4a_2^2+6a_1a_2),$$

we may write (**) as follows:

$$(n^2+n+1)\left[a_1^2+a_2^2-(a_1+a_2)^2\right]-\left[a_1^4+a_2^4-(a_1+a_2)^4\right]\geq 0,$$

$$a_1a_2[-2(n^2+n+1)+(4a_1^2+4a_2^2+6a_1a_2)]\geq 0.$$

This is true if

$$2(4a_1^2+4a_2^2+6a_1a_2)\geq 4(n^2+n+1). \tag{B}$$

Taking intp account (A), it suffices to show that

$$2(4a_1^2+4a_2^2+6a_1a_2)\geq 7a_1^2+7a_2^2+10a_1a_2,$$

which is equivalent to $(a_1-a_2)^2\geq 0$.

□

P 1.19. *Let $a_1,a_2,\dots,a_n$ be nonnegative real numbers such that $a_1+a_2+\cdots+a_n=n$. If $m\geq 3$ is an integer, then*

$$\frac{n^{m-1}-1}{n-1}\left(a_1^2+a_2^2+\cdots+a_n^2-n\right)\geq a_1^m+a_2^m+\cdots+a_n^m-n.$$

(Vasile C., 2002)

Solution. Let

$$r=\frac{n^{m-1}-1}{n-1}.$$

We need to prove that $F(a_1,a_2,\dots,a_n)\geq 0$, where

$$F(a_1,a_2,\dots,a_n)=r\left(a_1^2+a_2^2+\cdots+a_n^2-n\right)-\left(a_1^m+a_2^m+\cdots+a_n^m-n\right).$$

Assume that

$$F(a_1, a_2, a_3, \ldots, a_n) < F(t, t, a_3, \ldots, a_n) \tag{*}$$

involves

$$F(a_1, a_2, a_3, \ldots, a_n) \geq F(0, 2t, a_3, \ldots, a_n), \tag{**}$$

where

$$t = \frac{a_1 + a_2}{2}, \qquad a_1 \neq a_2.$$

Then, by AC-Corollary, we have

$$F(a_1, a_2, \ldots, a_n) \geq \min_{1 \leq k \leq n} f(k),$$

where

$$f(k) = r\left(\frac{n^2}{k} - n\right) - \frac{n^m}{k^{m-1}} + n$$

is the value of F for $a_1 = \cdots = a_{n-k} = 0$ and $a_{n-k+1} = \cdots = a_n = n/k$. We need to show that $f(k) \geq 0$ for $k \in \{1, 2, \ldots, n-1\}$. Write the inequality $f(k) \geq 0$ in the form

$$r \geq \frac{n^{m-1} - k^{m-1}}{k^{m-2}(n-k)}.$$

We have

$$r = \frac{n^{m-1} - 1}{n-1} = n^{m-2} + n^{m-3} + \cdots + 1 \geq \left(\frac{n}{k}\right)^{m-2} + \left(\frac{n}{k}\right)^{m-3} + \cdots + 1 = \frac{n^{m-1} - k^{m-1}}{k^{m-2}(n-k)}.$$

Let us show now that (*) involves (**). We may write (*) as

$$\frac{a_1^m + a_2^m - 2t^m}{a_1^2 + a_2^2 - 2t^2} > r,$$

$$\frac{a_1^m + a_2^m - 2t^m}{t^2 - a_1 a_2} > 2r. \tag{A}$$

For the nontrivial case $a_1 a_2 \neq 0$, the desired inequality (**) is equivalent to

$$\frac{a_1^m + a_2^m - (a_1 + a_2)^m}{a_1^2 + a_2^2 - (a_1 + a_2)^2} \leq r,$$

$$\frac{a_1^m + a_2^m - (2t)^m}{a_1 a_2} \geq 2r.$$

This is true if

$$\frac{a_1^m + a_2^m - (2t)^m}{a_1 a_2} \geq \frac{a_1^m + a_2^m - 2t^m}{t^2 - a_1 a_2},$$

$$a_1^m + a_2^m + (2^m - 2)a_1 a_2 t^{m-2} - (2t)^m \leq 0.$$

For $m = 3$, the inequality is an identity. Consider next that $m \ge 4$. Due to homogeneity, we may set $t = 1$. Using the substitution

$$a_1 = 1 + x, \quad a_2 = 1 - x, \qquad 0 \le x < 1,$$

we need to show that $f(x) \le 0$, where

$$f(x) = (1+x)^m + (1-x)^m + (2^m - 2)(1 - x^2) - 2^m, \qquad 0 \le x \le 1.$$

We have

$$f'(x) = m\left[(1+x)^{m-1} - (1-x)^{m-1}\right] - (2^{m+1} - 4)x,$$

$$f''(t) = m(m-1)\left[(1+x)^{m-2} + (1-x)^{m-2}\right] - 2^{m+1} + 4,$$

$$f'''(x) = m(m-1)(m-2)\left[(1+x)^{m-3} - (1-x)^{m-3}\right].$$

Since $f''' > 0$, f'' is strictly increasing. Because

$$f''(0) = 2(m^2 - m + 2 - 2^m) < 0, \qquad f''(1) = 2^{m-2}(m^2 - m - 8) + 4 > 0,$$

there exists $x_1 \in (0,1)$ such that $f''(x_1) = 0$, $f''(x) < 0$ for $x \in [0, x_1)$ and $f''(x) > 0$ for $x \in (x_1, 1]$. Consequently, f' is strictly decreasing on $[0, x_1]$ and strictly increasing on $[x_1, 1]$. Since

$$f'(0) = 0, \qquad f'(1) = 4 + (m-4)2^{m-1} > 0,$$

there exists $x_2 \in (x_1, 1)$ such that $f'(x_2) = 0$, $f'(x) < 0$ for $x \in (0, x_2)$ and $f''(x) > 0$ for $x \in (x_2, 1]$. Thus, f is strictly decreasing on $[0, x_2]$ and strictly increasing on $[x_2, 1]$. Since $f(0) = f(1) = 0$, it follows that $f(x) \le 0$ for $0 \le x \le 1$.

The equality holds when $a_1 = a_2 = \cdots = a_n = 1$, and also when

$$a_1 = \cdots = a_{n-1} = 0, \qquad a_n = n$$

(or any cyclic permutation).

□

P 1.20. *If a, b, c, d, e are nonnegative real numbers such that $a + b + c + d + e = 5$, then*

$$a^2 + b^2 + c^2 + d^2 + e^2 - 5 \ge \frac{5}{4}(1 - abcde).$$

(Vasile C., 2005)

Solution. We apply the AM-Corollary to the function

$$F(a, b, c, d, e) = a^2 + b^2 + c^2 + d^2 + e^2 - 5 - \frac{5}{4}(1 - abcde).$$

So, we need to show that

$$F(a,b,c,d,e) \geq F(t,b,c,t,e)$$

for

$$a \geq b \geq c \geq d \geq e, \qquad t = \frac{a+d}{2}.$$

We have

$$\begin{aligned} F(a,b,c,d,e) - F(t,b,c,t,e) &= a^2 + d^2 - 2t^2 - \frac{5}{4}bce(t^2 - ad) \\ &= 2(t^2 - ad) - \frac{5}{4}bce(t^2 - ad) \\ &= \frac{1}{4}(t^2 - ad)(8 - 5bce). \end{aligned}$$

It suffices to show that

$$8 - 5bce \geq 0,$$

which is equivalent to the homogeneous inequality

$$8(a+b+c+d+e)^3 \geq 625bce.$$

By the AM-GM inequality, we have

$$a+b+c+d+e \geq 2b+c+2e \geq 3\sqrt[3]{(2b)c(2e)} = 3\sqrt[3]{4bce},$$

hence

$$8(a+b+c+d+e)^3 \geq 8 \cdot 27 \cdot (4bce) = 864bce \geq 625bce.$$

According to AM-Corollary, it suffices to prove the original inequality for $a = b = c = d$. Thus, we need to show that $4a + e = 5$ implies

$$4a^2 + e^2 - 5 \geq \frac{5}{4}(1 - a^4 e),$$

which is equivalent to

$$20(1-a)^2 \geq \frac{5}{4}(1-a)^2(4a^3 + 3a^2 + 2a + 1),$$

$$(1-a)^2(15 - 2a - 3a^2 - 4a^3) \geq 0,$$

$$(1-a)^2(3 + 2a + a^2)(5 - 4a) \geq 0,$$

$$(1-a)^2(3 + 2a + a^2)e \geq 0.$$

The equality holds for $a = b = c = d = e = 1$, and also for $a = b = c = d = \frac{5}{4}$ and $e = 0$ (or any cyclic permutation).

□

P 1.21. *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$, then*

$$a_1^2 + a_2^2 + \cdots + a_n^2 - n \geq \frac{n}{n-1}\,(1 - a_1 a_2 \cdots a_n).$$

Solution. Write the inequality as $F(a_1, a_2, \ldots, a_n) \geq 0$, where

$$F(a_1, a_2, \ldots, a_n) = a_1^2 + a_2^2 + \cdots + a_n^2 - n - \frac{n}{n-1}\,(1 - a_1 a_2 \cdots a_n).$$

Assume that

$$F(a_1, a_2, a_3, \ldots, a_n) < F(t, t, a_3, \ldots, a_n) \tag{*}$$

involves

$$F(a_1, a_2, a_3, \ldots, a_n) \geq F(0, 2t, a_3, \ldots, a_n), \tag{**}$$

where

$$t = \frac{a_1 + a_2}{2}, \qquad a_1 \neq a_2.$$

On this assumption, by the AC-Corollary, we have

$$F(a_1, a_2, \ldots, a_n) \geq \min_{1 \leq k \leq n} f(k),$$

where

$$f(n) = F(1, 1, \ldots, 1) = 0$$

and

$$f(k) = \frac{n^2}{k} - \frac{n^2}{n-1}, \qquad k \in \{1, 2, \ldots, n-1\},$$

is the value of F for $a_1 = \cdots = a_{n-k} = 0$ and $a_{n-k+1} = \cdots = a_n = n/k$. Since $f(k) \geq 0$ for $k \in \{1, 2, \ldots, n-1\}$ (with equality when $k = n-1$), we have

$$F(a_1, a_2, \ldots, a_n) \geq 0.$$

To prove that (*) implies (**), we write these inequalities as

$$(t^2 - a_1 a_2)\left(2 - \frac{n}{n-1} a_3 \cdots a_n\right) < 0 \tag{A}$$

and

$$a_1 a_2 \left(2 - \frac{n}{n-1} a_3 \cdots a_n\right) \leq 0, \tag{B}$$

respectively. Clearly, (A) implies (B), therefore (*) implies (**). This completes the proof.

The equality holds when $a_1 = a_2 = \cdots = a_n = 1$, and also when

$$a_1 = 0, \qquad a_2 = \cdots = a_n = \frac{n}{n-1}$$

(or any cyclic permutation).

□

P 1.22. *If a, b, c, d, e are nonnegative real numbers such that $a+b+c+d+e=5$, then*

$$a^3+b^3+c^3+d^3+e^3-5 \geq \frac{45}{16}(1-abcde).$$

(Vasile C., 2005)

Solution. We apply the AM-Corollary to the function

$$F(a,b,c,d,e)=a^3+b^3+c^3+d^3+e^3-5-\frac{45}{16}(1-abcde).$$

So, we need to show that

$$F(a,b,c,d,e) \geq F(t,b,c,t,e)$$

for

$$a \geq b \geq c \geq d \geq e, \qquad t=\frac{a+d}{2}.$$

We have

$$\begin{aligned} F(a,b,c,d,e)-F(t,b,c,t,e) &= a^3+d^3-2t^3-\frac{45}{16}bce(t^2-ad) \\ &= 3(t^2-ad)(a+d)-\frac{45}{16}bce(t^2-ad) \\ &= \frac{3}{16}(t^2-ad)(16a+16d-15bce). \end{aligned}$$

It suffices to show that

$$a+d \geq bce,$$

which is equivalent to the homogeneous inequality

$$(a+d)(a+b+c+d+e)^2 \geq 16bce.$$

Let

$$x=\sqrt{be}.$$

Since

$$a+d \geq b+e \geq 2x, \qquad a+b+c+d+e \geq 2b+c+2e \geq 4x+c,$$

it suffices to show that

$$2x(4x+c)^2 \geq 16cx^2,$$

which is equivalent to

$$2x(16x^2+c^2) \geq 0.$$

According to AM-Corollary, it suffices to prove the original inequality for $a=b=c=d$. Thus, we need to show that $4a+e=5$ implies

$$4a^3+e^3-5 \geq \frac{45}{16}(1-a^4e),$$

which is equivalent to

$$60(1-a)^2(2-a) \geq \frac{45}{16}(1-a)^2(1+2a+3a^2+4a^3),$$

$$(1-a)^2(125-70a-9a^2-12a^3) \geq 0,$$

$$(1-a)^2(25+6a+3a^2)(5-4a) \geq 0,$$

$$(1-a)^2(25+6a+3a^2)e \geq 0.$$

The equality holds for $a = b = c = d = e = 1$, and also for $a = b = c = d = \frac{5}{4}$ and $e = 0$ (or any cyclic permutation).

□

P 1.23. *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that $a_1+a_2+\cdots+a_n = n$, then*

$$a_1^3+a_2^3+\cdots+a_n^3-n \geq \frac{n(2n-1)}{(n-1)^2}\,(1-a_1a_2\cdots a_n).$$

Solution. Write the inequality as $F(a_1, a_2, \ldots, a_n) \geq 0$, where

$$F(a_1, a_2, \ldots, a_n) = a_1^3+a_2^3+\cdots+a_n^3-n-\frac{n(2n-1)}{(n-1)^2}\,(1-a_1a_2\cdots a_n).$$

We claim that

$$F(a_1, a_2, a_3, \ldots, a_n) < F(t, t, a_3, \ldots, a_n) \tag{*}$$

involves

$$F(a_1, a_2, a_3, \ldots, a_n) \geq F(0, 2t, a_3, \ldots, a_n), \tag{**}$$

where

$$t = \frac{a_1+a_2}{2}, \qquad a_1 \neq a_2.$$

On this assumption, by the AC-Corollary, we have

$$F(a_1, a_2, \ldots, a_n) \geq \min_{1\leq k\leq n} f(k),$$

where

$$f(n) = F(1, 1, \ldots, 1) = 0$$

and

$$f(k) = \frac{n^3}{k^2} - \frac{n^3}{(n-1)^2}, \qquad k \in \{1, 2, \ldots, n-1\},$$

is the value of F for $a_1 = \cdots = a_{n-k} = 0$ and $a_{n-k+1} = \cdots = a_n = n/k$. Since $f(k) \geq 0$ for $k \in \{1, 2, \ldots, n-1\}$ (with equality when $k = n-1$), we have

$$F(a_1, a_2, \ldots, a_n) \geq 0.$$

To prove that (*) implies (**), we write these inequalities as

$$(t^2 - a_1a_2)\left[3(a_1+a_2) - \frac{n(2n-1)}{(n-1)^2}a_3\cdots a_n\right] < 0 \tag{A}$$

and

$$a_1a_2\left[3(a_1+a_2) - \frac{n(2n-1)}{(n-1)^2}a_3\cdots a_n\right] \le 0, \tag{B}$$

respectively. Clearly, (A) implies (B), therefore (*) implies (**). This completes the proof.

The equality holds when $a_1 = a_2 = \cdots = a_n = 1$, and also when

$$a_1 = 0, \qquad a_2 = \cdots = a_n = \frac{n}{n-1}$$

(or any cyclic permutation).

□

P 1.24. *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$, then*

$$\frac{1}{n}\left(a_1^3 + a_2^3 + \cdots + a_n^3 - n\right) \ge a_1^2 + a_2^2 + \cdots + a_n^2 - n + \left(\frac{n-2}{2}\right)^2 (a_1a_2\cdots a_n - 1).$$

(Vasile C., 2005)

Solution. Write the inequality as $F(a_1, a_2, \ldots, a_n) \ge 0$, where

$$F(a_1, a_2, \ldots, a_n) = \frac{1}{n}\left(a_1^3 + a_2^3 + \cdots + a_n^3 - n\right) - \left(a_1^2 + a_2^2 + \cdots + a_n^2 - n\right)$$
$$- \left(\frac{n-2}{2}\right)^2 (a_1a_2\cdots a_n - 1).$$

If

$$F(a_1, a_2, a_3, \ldots, a_n) < F(t, t, a_3, \ldots, a_n) \tag{*}$$

involves

$$F(a_1, a_2, a_3, \ldots, a_n) \ge F(0, 2t, a_3, \ldots, a_n), \tag{**}$$

where

$$t = \frac{a_1 + a_2}{2}, \qquad a_1 \ne a_2,$$

then, by the AC-Corollary, we have

$$F(a_1, a_2, \ldots, a_n) \ge \min_{1 \le k \le n} f(k),$$

where

$$f(n) = F(1, 1, \ldots, 1) = 0$$

and

$$f(k)=\frac{1}{n}\left(\frac{n^3}{k^2}-n\right)-\left(\frac{n^2}{k}-n\right)+\left(\frac{n-2}{2}\right)^2=\frac{(k-2)^2n^2}{4k^2}, \qquad k\in\{1,2,\ldots,n-1\},$$

is the value of F for $a_1=\cdots=a_{n-k}=0$ and $a_{n-k+1}=\cdots=a_n=n/k$. Since $f(k)\geq 0$ (with equality when $k=2$), we have

$$F(a_1,a_2,\ldots,a_n)\geq 0.$$

To prove that (*) implies (**), we write these inequalities as

$$(t^2-a_1a_2)\left[\frac{3}{n}(a_1+a_2)-2+\frac{(n-2)^2}{4}a_3\cdots a_n\right]<0 \tag{A}$$

and

$$a_1a_2\left[\frac{3}{n}(a_1+a_2)-2+\frac{(n-2)^2}{4}a_3\cdots a_n\right]\leq 0, \tag{B}$$

respectively. Clearly, (A) implies (B), therefore (*) implies (**). This completes the proof.

The equality holds when $a_1=a_2=\cdots=a_n=1$, and also when

$$a_1=\cdots=a_{n-2}=0, \qquad a_{n-1}=a_n=n/2$$

(or any permutation).

□

P 1.25. *If $a_1,a_2,\ldots,a_n$ are nonnegative real numbers such that $a_1+a_2+\cdots+a_n=n$, then*

$$\sqrt{\frac{n(n-1)}{n+1}}\,\left(a_1^3+a_2^3+\cdots+a_n^3-n\right)\geq a_1^2+a_2^2+\cdots+a_n^2-n.$$

(Vasile C., 2006)

Solution. Write the inequality as $F(a_1,a_2,\ldots,a_n)\geq 0$, where

$$F(a_1,a_2,\ldots,a_n)=\frac{n(n-1)}{n+1}\,\left(a_1^3+a_2^3+\cdots+a_n^3-n\right)-(a_1^2+a_2^2+\cdots+a_n^2-n)^2.$$

If

$$F(a_1,a_2,a_3,\ldots,a_n)<F(t,t,a_3,\ldots,a_n) \tag{*}$$

involves

$$F(a_1,a_2,a_3,\ldots,a_n)\geq F(0,2t,a_3,\ldots,a_n), \tag{**}$$

where

$$t=\frac{a_1+a_2}{2}, \qquad a_1\neq a_2,$$

then, by the AC-Corollary, we have

$$F(a_1, a_2, \ldots, a_n) \geq \min_{1\leq k\leq n} f(k),$$

where $f(k)$ is the value of F for $a_1 = \cdots = a_{n-k} = 0$ and $a_{n-k+1} = \cdots = a_n = n/k$; that is,

$$f(k) = \frac{n(n-1)}{n+1}\left(\frac{n^3}{k^2} - n\right) - \left(\frac{n^2}{k} - n\right)^2 = \frac{2n^3(k-1)(n-k)}{k^2(n+1)}.$$

Since $f(k) \geq 0$ (with equality when $k = 1$ and $k = n$), we have

$$F(a_1, a_2, \ldots, a_n) \geq 0.$$

Using the identities

$$a_1^2 + a_2^2 - 2t^2 = 2(t^2 - a_1a_2), \qquad a_1^3 + a_2^3 - 2t^3 = 3(a_1 + a_2)(t^2 - a_1a_2),$$

we may write (*) as

$$\frac{n(n-1)}{n+1}(a_1^3 + a_2^3 - 2t^3) - 2(a_1^2 + a_2^2 - 2t^2)\left(t^2 + \frac{a_1^2 + a_2^2}{2} + a_3^2 + \cdots + a_n^2 - n\right) < 0,$$

$$(t^2 - a_1a_2)\left[\frac{3n(n-1)}{n+1}(a_1 + a_2) - 4\left(t^2 + \frac{a_1^2 + a_2^2}{2} + a_3^2 + \cdots + a_n^2 - n\right)\right] < 0,$$

$$\frac{3n(n-1)}{n+1}(a_1 + a_2) - 4\left(t^2 + \frac{a_1^2 + a_2^2}{2} + a_3^2 + \cdots + a_n^2 - n\right) < 0. \tag{A}$$

Since

$$a_1^2 + a_2^2 - (a_1 + a_2)^2 = -2a_1a_2, \qquad a_1^3 + a_2^3 - (a_1 + a_2)^3 = -3a_1a_2(a_1 + a_2),$$

we may write (**) as

$$\frac{n(n-1)}{n+1}[a_1^3 + a_2^3 - (a_1 + a_2)^3] -$$

$$-2[a_1^2 + a_2^2 - (a_1 + a_2)^2]\left(2t^2 + \frac{a_1^2 + a_2^2}{2} + a_3^2 + \cdots + a_n^2 - n\right) \geq 0,$$

$$a_1a_2\left[\frac{3n(n-1)}{n+1}(a_1 + a_2) - 4\left(2t^2 + \frac{a_1^2 + a_2^2}{2} + a_3^2 + \cdots + a_n^2 - n\right)\right] \leq 0. \tag{B}$$

Clearly, (A) implies (B), therefore (*) implies (**). This completes the proof.

The equality holds when $a_1 = a_2 = \cdots = a_n = 1$, and also when

$$a_1 = \cdots = a_{n-1} = 0, \qquad a_n = n$$

(or any cyclic permutation).

□

P 1.26. *If a,b,c,d are nonnegative real numbers such that $a+b+c+d=4$, then*

$$\sqrt{2\left(a^4+b^4+c^4+d^4\right)+\frac{313}{81}}\ +\ \frac{5}{9}\geq a^2+b^2+c^2+d^2.$$

(Vasile C., 2006)

Solution. Write the inequality in the form

$$2\left(a^4+b^4+c^4+d^4\right)+\frac{313}{81}\geq\left(a^2+b^2+c^2+d^2-\frac{5}{9}\right)^2.$$

Consider the more general inequality $F(a,b,c,d)\geq 0$, where

$$F(a,b,c,d)=k_1\left(a^4+b^4+c^4+d^4\right)+k_2-\left[k_3(a^2+b^2+c^2+d^2)-k_4\right]^2,$$

with

$$2k_3^2\geq k_1>0,\qquad 4k_3\geq k_4\quad k_3>0.$$

We will show that

$$F(a,b,c,d)<F(t,t,c,d)\tag{*}$$

involves

$$F(a,b,c,d)\geq F(0,2t,c,d),\tag{**}$$

where

$$t=\frac{a+b}{2},\qquad a\neq b.$$

Using the identities

$$a^2+b^2-2t^2=2(t^2-ab),\qquad a^4+b^4-2t^4=\frac{1}{2}(t^2-ab)(7a^2+7b^2+10ab),$$

we may write (*) as

$$k_1(a^4+b^4-2t^4)-2k_3(a^2+b^2-2t^2)\left(k_3(t^2+\frac{a^2+b^2}{2}+c^2+d^2)-k_4\right)<0,$$

$$(t^2-ab)\left[\frac{k_1(7a^2+7b^2+10ab)}{2}-4k_3\left(k_3(t^2+\frac{a^2+b^2}{2}+c^2+d^2)-k_4\right)\right]<0,$$

$$\frac{k_1(7a^2+7b^2+10ab)}{2}-4k_3\left[k_3(t^2+\frac{a^2+b^2}{2}+c^2+d^2)-k_4\right]<0.\tag{A}$$

Since

$$a^2+b^2-(a+b)^2=-2ab,\qquad a^4+b^4-(a+b)^4=-ab(4a^2+4b^2+6ab),$$

we may write (**) as

$$k_1[a^4+b^4-(a+b)^4]-2k_3\left[a^2+b^2-(a+b)^2\right]\left[k_3(2t^2+\frac{a^2+b^2}{2}+c^2+d^2)-k_4\right]\geq 0,$$

$$abk_1(4a^2+4b^2+6ab)-4k_3ab\left[k_3(2t^2+\frac{a^2+b^2}{2}+c^2+d^2)-k_4\right]\le 0.$$

This inequality is true if

$$k_1(4a^2+4b^2+6ab)-4k_3\left[k_3(2t^2+\frac{a^2+b^2}{2}+c^2+d^2)-k_4\right]\le 0,$$

which is equivalent to

$$\frac{k_1(7a^2+7b^2+10ab)}{2}+2k_1t^2-4k_3\left[k_3(2t^2+\frac{a^2+b^2}{2}+c^2+d^2)-k_4\right]\le 0. \quad \text{(B)}$$

Clearly, (A) implies (B) if

$$2k_1t^2-4k_3^2t^2\le 0,$$

which is true because $k_1\le 2k_3^2$. As a consequence, (*) implies (**). Thus, by the AC-Corollary, we have

$$F(a,b,c,d)\ge \min_{1\le k\le n} f(k),$$

where

$$f(k)=\min_{1\le k\le 4} F\left(0,\dots,0,\frac{4}{k},\dots,\frac{4}{k}\right)=\frac{4^4k_1}{k^3}+k_2-\left(\frac{4^2k_3}{k}-k_4\right)^2$$

is the value of F for the case where $4-k$ of the variables a,b,c,d are zero and the other k variables are equal to $4/k$. Since

$$\frac{4^2k_3}{k}-k_4\ge 4k_3-k_4\ge 0,$$

we only need to show that $g(k)\ge 0$ for $k\in\{1,2,3,4\}$, where

$$g(k)=\sqrt{\frac{256k_1}{k^3}+k_2}-\frac{16k_3}{k}+k_4.$$

For the original inequality, we have

$$k_1=2,\qquad k_2=\frac{313}{81},\qquad k_3=1,\qquad k_4=\frac{5}{9},$$

$$2k_3^2-k_1=0,\qquad 4k_3-k_4=4-\frac{5}{9}>0,\quad k_3>0,$$

$$g(k)=\sqrt{\frac{512}{k^3}+\frac{313}{81}}-\frac{16}{k}+\frac{5}{9},\qquad k\in\{1,2,3,4\}.$$

Since

$$g(1)=\sqrt{512+\frac{313}{81}}-16+\frac{5}{9}>\sqrt{512}-16>0,$$

$$g(2)=\sqrt{64+\frac{313}{81}}-8+\frac{5}{9}>\sqrt{64}-8=0,$$

$$g(3)=\sqrt{\frac{512}{27}+\frac{313}{81}}-\frac{16}{3}+\frac{5}{9}=\frac{43}{9}-\frac{16}{3}+\frac{5}{9}=0,$$

$$g(4)=\sqrt{8+\frac{313}{81}}-4+\frac{5}{9}=\frac{31}{9}-4+\frac{5}{9}=0,$$

the conclusion follows.

The equality holds for $a=0$ and $b=c=d=4/3$ (or any cyclic permutation), and also for $a=b=c=d=1$.

□

P 1.27. *If a,b,c,d are nonnegative real numbers such that $a+b+c+d=4$, then*

$$2\sqrt{\frac{a^4+b^4+c^4+d^4-4}{7}}\ge a^2+b^2+c^2+d^2-4.$$

(Vasile C., 2006)

Solution. Write the inequality as $F(a,b,c,d)\ge 0$, where

$$F(a,b,c,d)=k_1\left(a^4+b^4+c^4+d^4\right)+k_2-\left[k_3(a^2+b^2+c^2+d^2)-k_4\right]^2,$$

with

$$k_1=\frac{4}{7},\qquad k_2=-\frac{16}{7},\qquad k_3=1,\qquad k_4=4,$$

$$2k_3^2-k_1=2-\frac{4}{7}>0,\qquad 4k_3-k_4=0\quad k_3>0.$$

As shown in the proof of the preceding P 1.26, we only need to show that $g(k)\ge 0$ for $k=1,2,3,4$, where

$$g(k)=\sqrt{\frac{256k_1}{k^3}+k_2}-\frac{16k_3}{k}+k_4=\frac{4}{k}\sqrt{\frac{64-k^3}{7k}}-\frac{16}{k}+4.$$

We have

$$g(1)=4\sqrt{\frac{63}{7}}-12=0,$$

$$g(2)=2\sqrt{\frac{56}{14}}-4=0,$$

$$g(3)=\frac{4}{3}\sqrt{\frac{37}{21}}-\frac{16}{3}+4>\frac{4}{3}-\frac{16}{3}+4=0,$$

$$g(4)=0.$$

The equality holds for $a=b=c=0$ and $d=4$ (or any cyclic permutation), for $a=b=0$ and $c=d=2$ (or any permutation), and also for $a=b=c=d=1$.

□

P 1.28. *If a, b, c, d are nonnegative real numbers such that $a+b+c+d=4$, then*

$$\sqrt{a^4+b^4+c^4+d^4}+\frac{16}{\sqrt{3}}-8 \geq \left(\frac{4}{\sqrt{3}}-\frac{3}{2}\right)\left(a^2+b^2+c^2+d^2\right).$$

(Vasile C., 2006)

Solution. Write the inequality as $F(a,b,c,d) \geq 0$, where

$$F(a,b,c,d)=k_1\left(a^4+b^4+c^4+d^4\right)+k_2-\left[k_3(a^2+b^2+c^2+d^2)-k_4\right]^2,$$

with

$$k_1=1, \quad k_2=0, \quad k_3=\frac{4}{\sqrt{3}}-\frac{3}{2}, \quad k_4=\frac{16}{\sqrt{3}}-8,$$

$$2k_3^2-k_1=\frac{91-48\sqrt{3}}{6}>4(3-2\sqrt{3}\,)>0, \quad 4k_3-k_4=2>0.$$

As shown in the proof of P 1.26, we only need to show that $g(k) \geq 0$ for $k = 1,2,3,4$, where

$$g(k)=\sqrt{\frac{256k_1}{k^3}+k_2}-\frac{16k_3}{k}+k_4=8h(k),$$

$$h(k)=\frac{2}{k\sqrt{k}}-\frac{2}{\sqrt{3}}\left(\frac{4}{k}-1\right)+\frac{3}{k}-1.$$

We have

$$h(1)=2(2-\sqrt{3}\,)>0,$$

$$h(2)=\frac{1}{\sqrt{2}}-\frac{2}{\sqrt{3}}+\frac{1}{2}>0,$$

$$h(3)=0,$$

$$h(4)=0.$$

The equality holds when $a=0$ and $b=c=d=4/3$ (or any cyclic permutation), and also when $a=b=c=d=1$.

□

P 1.29. *If a, b, c, d are nonnegative real numbers such that $a+b+c+d=4$, then*

$$\sqrt{a^4+b^4+c^4+d^4}+8(2-\sqrt{2}) \geq \left(2-\frac{1}{\sqrt{2}}\right)\left(a^2+b^2+c^2+d^2\right).$$

(Vasile C., 2006)

Solution. Write the inequality as $F(a,b,c,d) \ge 0$, where

$$F(a,b,c,d) = k_1\left(a^4+b^4+c^4+d^4\right)+k_2-\left[k_3(a^2+b^2+c^2+d^2)-k_4\right]^2,$$

with

$$k_1 = 1, \quad k_2 = 0, \quad k_3 = 2-\frac{1}{\sqrt{2}}, \quad k_4 = 8(2-\sqrt{2}\,),$$

$$2k_3^2-k_1 > 2-1 > 0, \quad 4k_3-k_4 = 2(3\sqrt{2}-4) > 0.$$

As shown in the proof of P 1.26, we only need to show that $g(k) \ge 0$ for $k = 1,2,3,4$, where

$$g(k) = \sqrt{\frac{256k_1}{k^3}+k_2}-\frac{16k_3}{k}+k_4 = 16h(k),$$

$$h(k) = \frac{1}{k\sqrt{k}}+1-\frac{2}{k}-\frac{1}{\sqrt{2}}\left(1-\frac{1}{k}\right).$$

We have

$$h(1) = 0,$$

$$h(2) = 0,$$

$$h(3) = \frac{1}{3}\left(1+\frac{1}{\sqrt{3}}-\sqrt{2}\right) > 0,$$

$$h(4) = \frac{5-3\sqrt{2}}{8} > 0.$$

The equality holds when $a = b = c = 0$ and $d = 4$ (or any cyclic permutation), and also when $a = b = 0$ and $c = d = 2$ (or any permutation).

□

P 1.30. *If a,b,c,d are nonnegative real numbers such that $a+b+c+d = 4$, then*

$$\sqrt{a^4+b^4+c^4+d^4}+16\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}\right) \ge \left(\frac{3}{\sqrt{2}}-\frac{2}{\sqrt{3}}\right)\left(a^2+b^2+c^2+d^2\right).$$

(Vasile C., 2006)

Solution. Write the inequality as $F(a,b,c,d) \ge 0$, where

$$F(a,b,c,d) = k_1\left(a^4+b^4+c^4+d^4\right)+k_2-\left[k_3(a^2+b^2+c^2+d^2)-k_4\right]^2,$$

with

$$k_1 = 1, \quad k_2 = 0, \quad k_3 = \frac{3}{\sqrt{2}}-\frac{2}{\sqrt{3}}, \quad k_4 = 16\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}\right),$$

$$2k_3^2 - k_1 > 0, \qquad 4k_3 - k_4 = 4\left(\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{2}}\right) > 0.$$

As shown in the proof of P 1.26, we only need to show that $g(k) \geq 0$ for $k = 1, 2, 3, 4$, where

$$g(k) = \sqrt{\frac{256k_1}{k^3} + k_2} - \frac{16k_3}{k} + k_4 = 16h(k),$$

$$h(k) = \frac{1}{k\sqrt{k}} + \frac{1}{\sqrt{2}}\left(1 - \frac{3}{k}\right) - \frac{1}{\sqrt{3}}\left(1 - \frac{2}{k}\right).$$

We have

$$h(1) = 1 - \sqrt{2} + \frac{1}{\sqrt{3}} > 1 - \frac{3}{2} + \frac{1}{2} = 0,$$

$$h(2) = 0,$$

$$h(3) = 0,$$

$$h(4) = \frac{1}{2}\left(\frac{1}{4} + \frac{1}{2\sqrt{2}} - \frac{1}{\sqrt{3}}\right) > 0.$$

The equality holds when $a = b = 0$ and $c = d = 2$ (or any permutation), and also when $a = 0$ and $b = c = d = 4/3$ (or any cyclic permutation).

□

P 1.31. *If a, b, c, d are nonnegative real numbers such that $a^2 + b^2 + c^2 + d^2 = 4$, then*

$$a^3 + b^3 + c^3 + d^3 + abc + bcd + cda + dab \leq 8.$$

(Vasile C., 2006)

Solution. Write the inequality as $G(a, b, c, d) \geq 0$, where

$$G(a, b, c, d) = 8 - ac(b + d) - (a + c)bd - (a^3 + b^3 + c^3 + d^3),$$

and apply the SM-Corollary. First, we will show that

$$G(a, b, c, d) \geq G(t, b, t, d)$$

for

$$a \leq b \leq c \leq d, \qquad t = \sqrt{\frac{a^2 + c^2}{2}}.$$

Write this inequality as follows:

$$(t^2 - ac)(b + d) + (2t - a - c)bd - (a^3 + c^3 - 2t^3) \geq 0,$$

$$(t^2 - ac)(b + d) + \frac{4t^2 - (a + c)^2}{2t + a + c}bd - \frac{(a^3 + c^3)^2 - 4t^6}{a^3 + c^3 + 2t^3} \geq 0,$$

$$\frac{(a-c)^2(b+d)}{2}+\frac{(a-c)^2}{2t+a+c}bd-\frac{(a-c)^2[(a^2+ac+c^2)^2-3a^2c^2]}{2(a^3+c^3+2t^3)}\geq 0,$$

which is true if

$$b+d+\frac{2bd}{2t+a+c}\geq\frac{(a^2+ac+c^2)^2}{a^3+c^3+2t^3}.$$

Since

$$b+d\geq a+c,\qquad bd\geq ac,$$

$$2t+a+c\leq 3(a+c)$$

and

$$a^3+c^3+2t^3\geq a^3+c^3+\frac{1}{2}(a+c)(a^2+c^2)\geq a^3+c^3+ac(a+c)=(a+c)(a^2+c^2),$$

it suffices to show that

$$a+c+\frac{2ac}{3(a+c)}\geq\frac{(a^2+ac+c^2)^2}{(a+c)(a^2+c^2)},$$

which is equivalent to

$$3(a^2+c^2)+8ac\geq\frac{3(a^2+ac+c^2)^2}{a^2+c^2},$$

$$2ac(a^2+c^2)\geq 3a^2c^2,$$

$$2ac(a-c)^2+a^2c^2\geq 0.$$

By the SM-Corollary, it suffices to prove the original inequality for $a=b=c$; that is, to show that $3a^2+d^2=4$ involves

$$4a^3+d^3+3a^2d\leq 8.$$

This inequality is equivalent to

$$4a^3+4d\leq 8,$$

$$2-a^3\geq d.$$

Since $3a^2\leq 4$, we have

$$2-a^3\geq 2-\frac{8}{3\sqrt{3}}>0.$$

Thus, we only need to show that

$$(2-a^3)^2\geq d^2,$$

which is equivalent to

$$(2-a^3)^2\geq 4-3a^2,$$

$$a^2(a^4-4a+3)\geq 0,$$

$$a^2(a-1)^2(a^2+2a+3)\geq 0.$$

The equality holds for $a=b=c=d=1$, and also for $a=b=c=0$ and $d=2$ (or any cyclic permutation).

□

P 1.32. *If a,b,c,d are nonnegative real numbers such that $a^2+b^2+c^2+d^2=2$, then*

$$a^3+b^3+c^3+d^3+abc+bcd+cda+dab\ge 2.$$

(Vasile C., 2006)

Solution. Write the inequality as $G(a,b,c,d)\ge 0$, where

$$G(a,b,c,d)=a^3+b^3+c^3+d^3+kab(c+d)+k(a+b)cd-m,\qquad k=1,\quad m=2,$$

and apply the AC1-Corollary. First, we will show that

$$G(a,b,c,d)<G(t,t,c,d)$$

involves

$$G(a,b,c,d)\ge G(0,\sqrt{2}\,t,c,d)$$

for any $k>0$ and real m, where

$$t=\sqrt{\frac{a^2+b^2}{2}},\qquad a\ne b.$$

Write the hypothesis $G(a,b,c,d)<G(t,t,c,d)$ as

$$(a^3+b^3-2t^3)-k(t^2-ab)(c+d)-k(2t-a-b)cd<0,$$

$$\frac{(a^3+b^3)^2-4t^6}{a^3+b^3+2t^3}-k(t^2-ab)(c+d)-k\cdot\frac{4t^2-(a+b)^2}{2t+a+b}cd<0,$$

$$\frac{(a-b)^2[a^4+b^4+2ab(a^2+b^2)]}{2(a^3+b^3+2t^3)}-\frac{k(a-b)^2(c+d)}{2}-\frac{k(a-b)^2}{2t+a+b}cd<0,$$

$$\frac{a^4+b^4+2ab(a^2+b^2)}{a^3+b^3+2t^3}<k\left(c+d+\frac{2cd}{2t+a+b}\right).\tag{*}$$

Write now the required inequality $G(a,b,c,d)\ge G(0,\sqrt{2}\,t,c,d)$ as follows:

$$kab(c+d)+k(a+b-\sqrt{2}\,t)cd\ge 2\sqrt{2}\,t^3-a^3-b^3,$$

$$kab(c+d)+k\cdot\frac{(a+b)^2-2t^2}{a+b+\sqrt{2}\,t}cd\ge\frac{8t^6-(a^3+b^3)^2}{2\sqrt{2}\,t^3+a^3+b^3},$$

$$kab(c+d)+\frac{2kabcd}{a+b+\sqrt{2}\,t}\ge\frac{a^2b^2(3a^2+3b^2-2ab)}{2\sqrt{2}\,t^3+a^3+b^3},$$

$$k(c+d)+\frac{2kcd}{a+b+\sqrt{2}\,t}\ge\frac{ab(3a^2+3b^2-2ab)}{2\sqrt{2}\,t^3+a^3+b^3}.$$

This inequality is true if

$$k(c+d)+\frac{2kcd}{a+b+2t}\ge\frac{ab(3a^2+3b^2-2ab)}{2t^3+a^3+b^3}.$$

Having in view (*), it suffices to show that

$$ab(3a^2+3b^2-2ab) \le a^4+b^4+2ab(a^2+b^2),$$

which is equivalent to

$$(a^2+b^2)(a^2+b^2-ab) \ge 0.$$

By the AC1-Corollary, it suffices to prove that

$$G\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right) \ge 0, \qquad G\left(0,\sqrt{\frac{2}{3}},\sqrt{\frac{2}{3}},\sqrt{\frac{2}{3}}\right) \ge 0,$$

$$G(0,0,1,1) \ge 0, \qquad G(0,0,0,\sqrt{2}) \ge 0.$$

We have

$$G\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right) = 4\sqrt{2}-2, \qquad G\left(0,\sqrt{\frac{2}{3}},\sqrt{\frac{2}{3}},\sqrt{\frac{2}{3}}\right) = \frac{8\sqrt{6}}{9}-2,$$

$$G(0,0,1,1)=0, \qquad G(0,0,0,\sqrt{2}) = 2\sqrt{2}-2.$$

The equality holds for $a=b=0$ and $c=d=1$ (or any permutation).

□

P 1.33. *If a,b,c,d are nonnegative real numbers such that $a^2+b^2+c^2+d^2=3$, then*

$$3(a^3+b^3+c^3+d^3)+2(abc+bcd+cda+dab) \ge 11.$$

(Vasile C., 2006)

Solution. Write the inequality as $G(a,b,c,d) \ge 0$, where

$$G(a,b,c,d) = 3(a^3+b^3+c^3+d^3)+2ab(c+d)+2(a+b)cd-11,$$

and apply the AC1-Corollary. As shown at the preceding P 1.32, if

$$G(a,b,c,d) < G(t,t,c,d),$$

then

$$G(a,b,c,d) \ge G(0,\sqrt{2}\,t,c,d),$$

where

$$t=\sqrt{\frac{a^2+b^2}{2}}, \qquad a \ne b.$$

By the AC1-Corollary, it suffices to prove that

$$G\left(\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}\right) \ge 0, \qquad G(0,1,1,1) \ge 0,$$

$$G\left(0,0,\sqrt{\frac{3}{2}},\sqrt{\frac{3}{2}}\right)\ge 0, \qquad G(0,0,0,\sqrt{3})\ge 0.$$

We have

$$G\left(\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}\right)=\frac{15\sqrt{3}}{2}-11>0, \qquad G(0,1,1,1)=0,$$

$$G\left(0,0,\sqrt{\frac{3}{2}},\sqrt{\frac{3}{2}}\right)=9\sqrt{\frac{3}{2}}-11>0, \qquad G(0,0,0,\sqrt{3})=9\sqrt{3}-11>0.$$

The equality holds for $a=0$ and $b=c=d=1$ (or any cyclic permutation).

□

P 1.34. *If a,b,c,d are nonnegative real numbers such that $a^2+b^2+c^2+d^2=4$, then*

$$4(a^3+b^3+c^3+d^3)+abc+bcd+cda+dab\ge 20.$$

(Vasile C., 2006)

Solution. Write the inequality as $G(a,b,c,d)\ge 0$, where

$$G(a,b,c,d)=4(a^3+b^3+c^3+d^3)+ac(b+d)+(a+c)bd-20.$$

First Solution. Apply the SM-Corollary. First, we will show that

$$G(a,b,c,d)\ge G(t,b,t,d)$$

for

$$a\ge b\ge c\ge d, \qquad t=\sqrt{\frac{a^2+c^2}{2}}.$$

Write this inequality as follows:

$$4(a^3+c^3-2t^3)-(t^2-ac)(b+d)-(2t-a-c)bd\ge 0,$$

$$\frac{4[(a^3+c^3)^2-4t^6]}{a^3+c^3+2t^3}-(t^2-ac)(b+d)-\frac{4t^2-(a+c)^2}{2t+a+c}bd\ge 0$$

$$\frac{4(a-c)^2[a^4+b^4+2ab(a^2+b^2)]}{2(a^3+c^3+2t^3)}-\frac{(a-c)^2(b+d)}{2}-\frac{(a-c)^2}{2t+a+c}bd\ge 0,$$

which is true if

$$\frac{4a^4+4c^4+8ac(a^2+c^2)}{a^3+c^3+2t^3}\ge b+d+\frac{2bd}{2t+a+c}.$$

Since

$$2t^3\le a^3+c^3, \qquad 2t\ge a+c,$$

it suffices to show that

$$\frac{4a^4+4c^4+8ac(a^2+c^2)}{2(a^3+c^3)} \ge b+d+\frac{2bd}{2(a+c)},$$

which is equivalent to

$$\frac{2a^4+2c^4+4ac(a^2+c^2)}{a^2+c^2-ac} \ge (a+c)(b+d)+bd,$$

Since $b+d \le a+c$ and $bd \le ac$, we only need to prove that

$$\frac{2a^4+2c^4+4ac(a^2+c^2)}{a^2+c^2-ac} \ge (a+c)^2+ac,$$

which is equivalent to

$$2a^4+2c^4+4ac(a^2+c^2) \ge (a^2+c^2-ac)(a^2+c^2+3ac),$$

$$a^4+c^4+2ac(a^2+c^2)+a^2c^2 \ge 0.$$

By the SM-Corollary, it suffices to prove the original inequality for $a=b=c$; that is, to show that $3a^2+d^2=4$ involves

$$4(3a^3+d^3)+a^3+3a^2d \ge 20,$$

which is equivalent to

$$13a^3+4d^3+3a^2d \ge 20,$$

$$13a^3+4d(4-3a^2)+3a^2d \ge 20,$$

$$(16-9a^2)d \ge 20-13a^3. \tag{*}$$

Since $3a^2 \le 4$ involves $16-9a^2>0$, we only need to show that

$$(16-9a^2)^2d^2 \ge (20-13a^3)^2$$

for $a^3 \le \frac{20}{13}$, which is equivalent to

$$312-960a^2+260a^3+594a^4-206a^6 \ge 0,$$

$$a(1-a)^2f(a) \ge 0, \qquad f(a)=\frac{312}{a}+624-24a-412a^2-206a^3.$$

Since f is decreasing, it suffices to show that $f(a) \ge 0$ for $a^3=\frac{20}{13}$. This is true because (*) holds for $a^3=\frac{20}{13}$.

The equality holds for $a=b=c=d=1$.

Second Solution. Apply the AC1-Corollary. As shown at P 1.32, if

$$G(a,b,c,d) < G(t,t,c,d),$$

then

$$G(a,b,c,d) \geq G(0,\sqrt{2}\,t,c,d),$$

where

$$t = \sqrt{\frac{a^2+b^2}{2}}, \qquad a \neq b.$$

Therefore, by the AC1-Corollary, it suffices to prove that

$$G(1,1,1,1) \geq 0, \qquad G\left(0,\frac{2}{\sqrt{3}},\frac{2}{\sqrt{3}},\frac{2}{\sqrt{3}}\right) \geq 0,$$

$$G(0,0,\sqrt{2},\sqrt{2}) \geq 0, \qquad G(0,0,0,2) \geq 0.$$

We have

$$G(1,1,1,1) = 0, \qquad G\left(0,\frac{2}{\sqrt{3}},\frac{2}{\sqrt{3}},\frac{2}{\sqrt{3}}\right) = \frac{104}{3\sqrt{3}} - 20 > 0,$$

$$G(0,0,\sqrt{2},\sqrt{2}\ ) = 16\sqrt{2} - 20 > 0, \qquad G(0,0,0,2) = 12.$$

□

P 1.35. *If a,b,c,d are nonnegative real numbers such that $a^2+b^2+c^2+d^2=4$, then*

$$a^3+b^3+c^3+d^3+3(a+b+c+d) \leq \frac{28}{\sqrt{3}}.$$

(Vasile C., 2006)

Solution. Write the inequality as $G(a,b,c,d) + \dfrac{28}{\sqrt{3}} \geq 0$, where

$$G(a,b,c,d) = -(a^3+b^3+c^3+d^3) - k(a+b+c+d), \qquad k = 3,$$

and apply the SM-Corollary. First, we will show that

$$G(a,b,c,d) \geq G(t,b,t,d)$$

for $k \geq 3$ and

$$a \leq b \leq c \leq d, \qquad t = \sqrt{\frac{a^2+c^2}{2}}.$$

Write this inequality as follows:

$$k(2t-a-c) - (a^3+c^3-2t^3) \geq 0,$$

$$\frac{k[4t^2-(a+c)^2]}{2t+a+c}-\frac{(a^3+c^3)^2-4t^6}{a^3+c^3+2t^3}\geq 0,$$

$$\frac{k(a-c)^2}{2t+a+c}-\frac{(a-c)^2[a^4+c^4+2ac(a^2+c^2)]}{2(a^3+c^3+2t^3)}\geq 0,$$

which is true if

$$\frac{2k}{2t+a+c}\geq\frac{a^4+c^4+2ac(a^2+c^2)}{a^3+c^3+(a^2+c^2)t}.$$

Write this inequality in the homogeneous form

$$\frac{k(a^2+b^2+c^2+d^2)}{2(2t+a+c)}\geq\frac{a^4+c^4+2ac(a^2+c^2)}{a^3+c^3+(a^2+c^2)t},$$

Since

$$k(a^2+b^2+c^2+d^2)\geq 2k(a^2+c^2)\geq 6(a^2+c^2),$$

it suffices to show that

$$\frac{3(a^2+c^2)}{2t+a+c}\geq\frac{a^4+c^4+2ac(a^2+c^2)}{a^3+c^3+(a^2+c^2)t},$$

which is equivalent to

$$(a-c)^4t+(a-c)^2(2a^2+2c^2-ac)\geq 0.$$

By the SM-Corollary, it suffices to prove the original inequality for $a=b=c$; that is, to show that $3a^2+d^2=4$ involves

$$3a^3+d^3+3(3a+d)\leq\frac{28}{\sqrt{3}}.$$

Using the substitution

$$a=\frac{x}{\sqrt{3}},\qquad c=\frac{y}{\sqrt{3}},\qquad x,y\geq 0,$$

we need to prove that

$$3x^2+y^2=12,\qquad x\leq 2$$

involves

$$x^3+\frac{y^3}{3}+9x+3y\leq 28,$$

which is equivalent to

$$x^3+(4-x^2)y+9x+3y\leq 28,$$

$$(7-x^2)y\leq 28-9x-x^3.$$

Since $x\leq 2$ implies

$$28-9x-x^3=2+9(2-x)+(8-x^3)>0,$$

we only need to show that

$$(7-x^2)^2y^2 \le (28-9x-x^3)^2,$$

which is equivalent to

$$(7-x^2)^2(12-3x^2) \le (28-9x-x^3)^2,$$

$$x^6-9x^4-14x^3+99x^2-126x+49 \ge 0,$$

$$(x-1)^2 f(x) \ge 0,$$

where

$$\begin{aligned} f(x) &= x^4+2x^3-6x^2-28x+49 \\ &= (x-2)^2(x^2+6x+14)+4x-7 \\ &= (2-x)(24-2x-4x^2-x^3)+1. \end{aligned}$$

Since $f(x) \ge 0$ for $4x-7 \ge 0$, it suffices to show that $24-2x-4x^2-x^3 \ge 0$ for $0 \le x \le \frac{7}{4}$. Indeed, we have

$$24-2x-4x^2-x^3 = (4-2x)+(13-4x^2)+(7-x^3) \ge \frac{1}{2}+\frac{3}{4}+\frac{105}{64} > 0.$$

The equality holds for $a=b=c=\frac{1}{\sqrt{3}}$ and $d=\sqrt{3}$ (or any cyclic permutation).

□

P 1.36. *If a,b,c,d are nonnegative real numbers such that $a^2+b^2+c^2+d^2=4$, then*

$$a^3+b^3+c^3+d^3+4(a+b+c+d) \le 20.$$

(Vasile C., 2006)

Solution. Write the inequality as $G(a,b,c,d)+20 \ge 0$, where

$$G(a,b,c,d) = -(a^3+b^3+c^3+d^3)-k(a+b+c+d), \qquad k=4,$$

and apply the SM-Corollary. As shown at the preceding P 1.35, we have

$$G(a,b,c,d) \ge G(t,b,t,d)$$

for $k \ge 3$ and

$$a \le b \le c \le d, \qquad t=\sqrt{\frac{a^2+c^2}{2}}.$$

By the SM-Corollary, it suffices to prove the original inequality for $a = b = c$; that is, to show that $3a^2 + d^2 = 4$ involves

$$3a^3 + d^3 + 4(3a + d) \le 20,$$

which is equivalent to

$$3a^3 + (4 - 3a^2)d + 12a + 4d \le 20,$$

$$(8 - 3a^2)d \le 20 - 12a - 3a^3.$$

Since $3a^2 \le 4$ implies

$$20 - 12a - 3a^3 \ge 20 - 12a - 4a = 4(5 - 4a) > 0,$$

we only need to show that

$$(8 - 3a^2)^2 d^2 \le (20 - 12a - 3a^3)^2,$$

which is equivalent to

$$(8 - 3a^2)^2(4 - 3a^2) \le (20 - 12a - 3a^3)^2,$$

$$3a^6 - 9a^4 - 10a^3 + 44a^2 - 40a + 12 \ge 0,$$

$$(a - 1)^2 f(a) \ge 0, \qquad f(a) = 3a^4 + 6a^3 - 16a + 12.$$

It suffices to show that $f(a) \ge 0$. We have

$$f(a) \ge 6a^3 - 18a + 12 = 6(a - 1)^2(a + 2) \ge 0.$$

The equality holds for $a = b = c = d = 1$.

□

P 1.37. *If a, b, c, d are nonnegative real numbers such that $a^2+b^2+c^2+d^2 = 4$, then*

$$a^3 + b^3 + c^3 + d^3 + 2\sqrt{2}\,(a + b + c + d) \ge 4(2 + \sqrt{2}\,).$$

(Vasile C., 2006)

Solution. Write the inequality as $G(a, b, c, d) \ge 0$, where

$$G(a, b, c, d) = a^3 + b^3 + c^3 + d^3 + k(a + b + c + d) - m, \qquad k = 2\sqrt{2}, \quad m = 4(2 + \sqrt{2}\,),$$

and apply the AC1-Corollary. First, we will show that

$$G(a, b, c, d) < G(t, t, c, d)$$

involves

$$G(a,b,c,d) \geq G(0,\sqrt{2}\ t,c,d)$$

for any $k > 0$ and real m, where

$$t = \sqrt{\frac{a^2+b^2}{2}}, \qquad a \neq b.$$

Write the hypothesis $G(a,b,c,d) < G(t,t,c,d)$ as

$$(a^3+b^3-2t^3) - k(2t-a-b) < 0,$$

$$\frac{(a^3+b^3)^2-4t^6}{a^3+b^3+2t^3} - k \cdot \frac{4t^2-(a+b)^2}{2t+a+b}cd < 0,$$

$$\frac{(a-b)^2[a^4+b^4+2ab(a^2+b^2)]}{2(a^3+b^3+2t^3)} - \frac{k(a-b)^2}{2t+a+b} < 0,$$

$$\frac{[a^4+b^4+2ab(a^2+b^2)](2t+a+b)}{a^3+b^3+2t^3} < 2k. \tag{*}$$

Write now the required inequality $G(a,b,c,d) \geq G(0,\sqrt{2}\ t,c,d)$ as follows:

$$k(a+b-\sqrt{2}\ t) \geq 2\sqrt{2}\ t^3 - a^3 - b^3,$$

$$\frac{k[(a+b)^2-2t^2]}{a+b+\sqrt{2}\ t} \geq \frac{8t^6-(a^3+b^3)^2}{2\sqrt{2}\ t^3+a^3+b^3},$$

$$\frac{2kab}{a+b+\sqrt{2}\ t} \geq \frac{a^2b^2(3a^2+3b^2-2ab)}{2\sqrt{2}\ t^3+a^3+b^3},$$

$$2k \geq \frac{ab(a+b+\sqrt{2}\ t)(3a^2+3b^2-2ab)}{2\sqrt{2}\ t^3+a^3+b^3}.$$

Having in view (*), it suffices to show that

$$\frac{[a^4+b^4+2ab(a^2+b^2)](2t+a+b)}{a^3+b^3+2t^3} \geq \frac{(a+b+\sqrt{2}\ t)[ab(3a^2+3b^2-2ab)]}{2\sqrt{2}\ t^3+a^3+b^3}.$$

We can get this inequality by multiplying the inequalities

$$2[a^4+b^4+2ab(a^2+b^2)] \geq 3[ab(3a^2+3b^2-2ab)]$$

and

$$\frac{3(2t+a+b)}{a^3+b^3+2t^3} \geq \frac{2(a+b+\sqrt{2}\ t)}{2\sqrt{2}\ t^3+a^3+b^3}.$$

The first inequality is equivalent to

$$(a-b)^2(2a^2+2b^2-ab) \geq 0,$$

and the second inequality to

$$8\sqrt{2}t^4+2(3\sqrt{2}-2)(a+b)t^3+2(3-\sqrt{2})(a^3+b^3)t+(a+b)(a^3+b^3)\geq 0.$$

By the AC1-Corollary, it suffices to prove that

$$G(1,1,1,1)\geq 0, \qquad G\left(0,\frac{2}{\sqrt{3}},\frac{2}{\sqrt{3}},\frac{2}{\sqrt{3}}\right)\geq 0,$$

$$G(0,0,\sqrt{2},\sqrt{2})\geq 0, \qquad G(0,0,0,2)\geq 0.$$

We have

$$G(1,1,1,1)=4(\sqrt{2}-1)>0,$$

$$G\left(0,\frac{2}{\sqrt{3}},\frac{2}{\sqrt{3}},\frac{2}{\sqrt{3}}\right)=4(\sqrt{3}-1)\left(\sqrt{2}-\frac{2}{\sqrt{3}}\right)>0,$$

$$G(0,0,\sqrt{2},\sqrt{2})=0, \qquad G(0,0,0,2)=0.$$

The equality holds for $a=b=0$ and $c=d=\sqrt{2}$ (or any permutation), and also for $a=b=c=0$ and $d=2$ (or any cyclic permutation).

□

P 1.38. *If a,b,c,d are nonnegative real numbers such that $a^2+b^2+c^2+d^2=4$, then*

$$a^3+b^3+c^3+d^3+2\sqrt{\frac{2}{3}}\,(a+b+c+d)\geq 4\left(\sqrt{2}+\frac{2}{\sqrt{3}}\right).$$

(Vasile C., 2006)

Solution. Write the inequality as $G(a,b,c,d)\geq 0$, where

$$G(a,b,c,d)=a^3+b^3+c^3+d^3+k(a+b+c+d)-m, \quad k=2\sqrt{\frac{2}{3}}, \ \ m=4\left(\sqrt{2}+\frac{2}{\sqrt{3}}\right),$$

and apply the AC1-Corollary. As shown at the preceding P 1.37, if

$$G(a,b,c,d)<G(t,t,c,d),$$

then

$$G(a,b,c,d)\geq G(0,\sqrt{2}\,t,c,d)$$

for any $k>0$ and real m, where

$$t=\sqrt{\frac{a^2+b^2}{2}}, \qquad a\neq b.$$

As a consequence, by the AC1-Corollary, it suffices to prove that

$$G(1,1,1,1) \geq 0, \qquad G\left(0, \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) \geq 0,$$

$$G(0,0,\sqrt{2},\sqrt{2}) \geq 0, \qquad G(0,0,0,2) \geq 0.$$

We have

$$G(1,1,1,1) = 4(\sqrt{2}-1)\left(\frac{2}{\sqrt{3}}-1\right) > 0, \qquad G\left(0, \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) = 0$$

$$G(0,0,\sqrt{2},\sqrt{2}) = 0, \qquad G(0,0,0,2) = 4(2-\sqrt{2})\left(1-\frac{1}{\sqrt{3}}\right) > 0.$$

The equality holds for $a = 0$ and $b = c = d = \dfrac{2}{\sqrt{3}}$ (or any cyclic permutation), and also for $a = b = 0$ and $c = d = \sqrt{2}$ (or any permutation).

□

P 1.39. *If a, b, c, d are nonnegative real numbers such that $a^2+b^2+c^2+d^2 = 4$, then*

$$a^3+b^3+c^3+d^3-4+\frac{2}{\sqrt{3}}(a+b+c+d-4) \geq 0.$$

(Vasile C., 2006)

Solution. Write the inequality as $G(a,b,c,d) \geq 0$, where

$$G(a,b,c,d) = a^3+b^3+c^3+d^3-4+k(a+b+c+d-4), \qquad k = \frac{2}{\sqrt{3}}.$$

First Solution. Apply the SM-Corollary. First, we will show that

$$G(a,b,c,d) \geq G(t,b,t,d)$$

for $k \leq \dfrac{4}{3}$ and

$$a \geq b \geq c \geq d, \qquad t = \sqrt{\frac{a^2+c^2}{2}}.$$

Write this inequality as follows:

$$(a^3+c^3-2t^3) - k(2t-a-c) \geq 0,$$

$$\frac{(a^3+c^3)^2-4t^6}{a^3+c^3+2t^3} - \frac{k[4t^2-(a+c)^2]}{2t+a+c} \geq 0,$$

$$\frac{(a-c)^2[a^4+c^4+2ac(a^2+c^2)]}{2(a^3+c^3+2t^3)} - \frac{k(a-c)^2}{2t+a+c} \geq 0,$$

which is true if

$$\frac{a^4+c^4+2ac(a^2+c^2)}{a^3+c^3+(a^2+c^2)t} \ge \frac{2k}{2t+a+c}.$$

Write this inequality in the homogeneous form

$$\frac{a^4+c^4+2ac(a^2+c^2)}{a^3+c^3+(a^2+c^2)t} \ge \frac{k(a^2+b^2+c^2+d^2)}{2(2t+a+c)},$$

Since

$$k(a^2+b^2+c^2+d^2) \le 2k(a^2+c^2) \le \frac{8(a^2+c^2)}{3},$$

it suffices to show that

$$\frac{a^4+c^4+2ac(a^2+c^2)}{a^3+c^3+(a^2+c^2)t} \ge \frac{4(a^2+c^2)}{3(2t+a+c)}. \tag{*}$$

This inequality can be written in the form $2At+B \ge 0$, where

$$A = a^4+c^4+6ac(a^2+c^2)-4a^2c^2.$$

Since $A>0$ and $t \ge \frac{a+c}{2}$, it suffices to prove the inequality (*) for $t = \frac{a+c}{2}$. That means to show that

$$\frac{a^4+c^4+2ac(a^2+c^2)}{a^3+c^3+(a^2+c^2)(a+c)/2} \ge \frac{4(a^2+c^2)}{6(a+c)},$$

which is true if

$$\frac{a^4+c^4+2ac(a^2+c^2)}{a^2+c^2-ac+(a^2+c^2)/2} \ge \frac{2(a^2+c^2)}{3},$$

which is equivalent to

$$\frac{a^4+c^4+2ac(a^2+c^2)}{3(a^2+c^2)-2ac} \ge \frac{a^2+c^2}{3},$$

$$ac(4a^2+4c^2-3ac) \ge 0.$$

According to the SM-Corollary, it suffices to prove the original inequality for $a = b = c$. Write the original inequality

$$a^3+b^3+c^3+d^3+k(a+b+c+d) \ge 4(1+k), \qquad k = \frac{2}{\sqrt{3}},$$

in the homogeneous inequality

$$\frac{4(a^3+b^3+c^3+d^3)}{a^2+b^2+c^2+d^2} + k(a+b+c+d) \ge 2(1+k)\sqrt{a^2+b^2+c^2+d^2}.$$

We only need to prove this inequality for $a = b = c = 0$ and for $a = b = c = 1$. The first case reduces to $(2-k)d \geq 0$, while the second case to

$$\frac{4(d^3+3)}{d^2+3} + k(d+3) \geq 2(1+k)\sqrt{d^2+3}.$$

By squaring, the inequality becomes

$$Ak^2 + 2Bk + C \geq 0,$$

where

$$A = -\frac{3}{4}(d-1)^2, \qquad B = \frac{3d(d-1)^2}{d^2+3}, \qquad C = \frac{3(d-1)^2(d^4+2d^3+6d+3)}{(d^2+3)^2}.$$

Thus, we need to show that

$$-\frac{3}{4}k^2(d^2+3)^2 + 6kd(d^2+3)^2 + 3(d^4+2d^3+6d+3) \geq 0,$$

which is equivalent to

$$-(d^2+3)^2 + 6kd(d^2+3) + 3(d^4+2d^3+6d+3) \geq 0,$$

$$6kd(d^2+3) + 2d(d^3+3d^2-3d+9) \geq 0.$$

The equality holds for $a = b = c = d = 1$, and also for $a = b = c = \dfrac{2}{\sqrt{3}}$ and $d = 0$ (or any cyclic permutation).

Second Solution. Apply the AC1-Corollary. As shown at P 1.37, if

$$G(a,b,c,d) < G(t,t,c,d),$$

then

$$G(a,b,c,d) \geq G(0,\sqrt{2}\,t,c,d)$$

for any $k > 0$, where

$$t = \sqrt{\frac{a^2+b^2}{2}}, \qquad a \neq b.$$

Therefore, by the AC1-Corollary, it suffices to prove that

$$G(1,1,1,1) \geq 0, \qquad G\left(0,\frac{2}{\sqrt{3}},\frac{2}{\sqrt{3}},\frac{2}{\sqrt{3}}\right) \geq 0,$$

$$G(0,0,\sqrt{2},\sqrt{2}) \geq 0, \qquad G(0,0,0,2) \geq 0.$$

We have

$$G(1,1,1,1) = 0, \qquad G\left(0,\frac{2}{\sqrt{3}},\frac{2}{\sqrt{3}},\frac{2}{\sqrt{3}}\right) = 0$$

$$G(0,0,\sqrt{2},\sqrt{2}\,) = 4(2-\sqrt{2}\,)\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right) > 0, \qquad G(0,0,0,2) = 4\left(1-\frac{1}{\sqrt{3}}\right) > 0.$$

□

P 1.40. *If $a_1, a_2, \ldots, a_n$ are real numbers such that*

$$a_1 + a_2 + \cdots + a_n = n,$$

then

$$\left(a_1^2 + a_2^2 + \cdots + a_n^2\right)^2 - n^2 \ge \frac{n(n-1)}{n^2-n+1}\left(a_1^4 + a_2^4 + \cdots + a_n^4 - n\right).$$

(Vasile C., 2006)

Solution. Write the inequality as $F(a_1, a_2, \ldots, a_n) \ge 0$, where

$$F(a_1, a_2, \ldots, a_n) = \left(a_1^2 + a_2^2 + \cdots + a_n^2\right)^2 - n^2 - k\left(a_1^4 + a_2^4 + \cdots + a_n^4 - n\right),$$

with

$$k = \frac{n(n-1)}{n^2-n+1}.$$

Without loss of generality, assume that

$$a_1 \le a_2 \le \cdots \le a_n$$

or

$$a_1 \ge a_2 \ge \cdots \ge a_n,$$

such that

$$a_n^2 = \max\{a_1^2, a_2^2, \ldots, a_n^2\}.$$

If

$$F(a_1, a_2, \ldots, a_{n-2}, a_{n-1}, a_n) \ge F\left(t, a_2, \ldots, a_{n-2}, t, a_n\right), \qquad t = \frac{a_1 + a_{n-1}}{2},$$

then, by the AM-Corollary, it suffices to prove the original inequality for

$$a_1 = a_2 = \cdots = a_{n-1}.$$

That means to show that

$$(n-1)a + a_n = n$$

involves

$$\left[(n-1)a^2 + a_n^2\right]^2 - n^2 \ge k\left[(n-1)a^4 + a_n^4 - n\right].$$

Since

$$\begin{aligned}\left[(n-1)a^2 + a_n^2\right]^2 - n^2 &= \left[(n-1)a^2 + a_n^2 - n\right]\left[(n-1)a^2 + a_n^2 + n\right]\\ &= n^2(n-1)(a-1)^2\left[(n-1)a^2 - 2(n-1)a + n + 1\right]\end{aligned}$$

and

$$(n-1)a^4 + a_n^4 - n = n(n-1)(a-1)^2\left[(n^2-3n+3)a^2 - 2(n^2-n-1)a + n^2 + n + 1\right],$$

the desired inequality can be written in the obvious form

$$(a-1)^2[(n-1)a-1]^2 \geq 0.$$

Using the identities

$$a_1^2 + a_{n-1}^2 - 2t^2 = 2(t^2 - a_1 a_{n-1}),$$

$$a_1^4 + a_{n-1}^4 - 2t^4 = \frac{1}{2}(t^2 - a_1 a_{n-1})(7a_1^2 + 7a_{n-1}^2 + 10a_1 a_{n-1}),$$

we may write the inequality $F(a_1, a_2, \ldots, a_{n-2}, a_{n-1}, a_n) \geq F(t, a_2, \ldots, a_{n-2}, t, a_n)$ as follows:

$$(a_1^2 + a_{n-1}^2 - 2t^2)(2t^2 + a_1^2 + 2a_2^2 + \cdots + 2a_{n-2}^2 + a_{n-1}^2 + 2a_n^2) \geq k(a_1^4 + a_{n-1}^4 - 2t^4),$$

$$2(t^2 - a_1 a_{n-1})(2t^2 + a_1^2 + 2a_2^2 + \cdots + 2a_{n-2}^2 + a_{n-1}^2 + 2a_n^2) \geq$$
$$\frac{k}{2}(t^2 - a_1 a_{n-1})(7a_1^2 + 7a_{n-1}^2 + 10a_1 a_{n-1}).$$

Since $t^2 - a_1 a_{n-1} \geq 0$ and $2a_2^2 + \cdots + 2a_{n-2}^2 \geq 0$, we only need to prove that

$$4\left(2t^2 + a_1^2 + a_{n-1}^2 + 2a_n^2\right) \geq k(7a_1^2 + 7a_{n-1}^2 + 10a_1 a_{n-1}).$$

Since

$$2a_n^2 \geq a_1^2 + a_{n-1}^2,$$

it suffices to show that

$$8\left(t^2 + a_1^2 + a_{n-1}^2\right) \geq k(7a_1^2 + 7a_{n-1}^2 + 10a_1 a_{n-1}2).$$

This is true because $k < 1$ and

$$8\left(t^2 + a_1^2 + a_{n-1}^2\right) = 10a_1^2 + 10a_{n-1}^2 + 4a_1 a_{n-1} \geq 7a_1^2 + 7a_{n-1}^2 + 10a_1 a_{n-1}.$$

The equality holds when $a_1 = a_2 = \cdots = a_n = 1$, and also when

$$a_1 = \cdots = a_{n-1} = \frac{1}{n-1}, \qquad a_n = n-1$$

(or any cyclic permutation).

□

P 1.41. *If a, b, c, d are real numbers such that $a + b + c + d = 4$, then*

$$\left(a^2 + b^2 + c^2 + d^2 - 4\right)\left(a^2 + b^2 + c^2 + d^2 + \frac{26}{5}\right) \geq a^4 + b^4 + c^4 + d^4 - 4.$$

(Vasile C., 2006)

Solution. Write the inequality as $F(a,b,c,d) \ge 0$, where

$$F(a,b,c,d) = \left(a^2+b^2+c^2+d^2+\frac{3}{5}\right)^2 - \left(\frac{17}{5}\right)^2 - \left(a^4+b^4+c^4+d^4-4\right).$$

Without loss of generality, assume that $a \le b \le c \le d$ or $a \ge b \ge c \ge d$, such that $d^2 = \max\{a^2, b^2, c^2, d^2\}$. We will show that

$$F(a,b,c,d) \ge F(t,b,t,d), \qquad t = \frac{a+c}{2}.$$

Using the identities

$$a^2+c^2-2t^2 = 2(t^2-ac), \qquad a^4+c^4-2t^4 = \frac{1}{2}(t^2-ac)(7a^2+7c^2+10ac),$$

we may write the inequality $F(a,b,c,d) \ge F(t,b,t,d)$ as follows:

$$(a^2+c^2-2t^2)\left(2t^2+a^2+2b^2+c^2+2d^2+\frac{6}{5}\right) - (a^4+c^4-2t^4) \ge 0,$$

$$2(t^2-ac)\left(2t^2+a^2+2b^2+c^2+2d^2+\frac{6}{5}\right) - \frac{1}{2}(t^2-ac)(7a^2+7c^2+10ac) \ge 0.$$

The inequality is true if

$$4\left(2t^2+a^2+2b^2+c^2+2d^2+\frac{6}{5}\right) - (7a^2+7c^2+10ac) \ge 0,$$

which is equivalent to

$$-5(a^2+c^2) - 30ac + 40(b^2+d^2) + 24 \ge 0.$$

It suffices to show that

$$-5(a^2+c^2) - 30ac + 40d^2 \ge 0.$$

Since $2d^2 \ge a^2+c^2$, we have

$$-5(a^2+c^2) - 30ac + 40d^2 \ge -5(a^2+c^2) - 30ac + 20(a^2+c^2) = 15(a-c)^2 \ge 0.$$

By the AM-Corollary, it suffices to prove the original inequality for $a = b = c$. That is, to show that $3a + d = 4$ implies

$$(3a^2+d^2-4)\left(3a^2+d^2+\frac{26}{5}\right) \ge 3a^4+d^4-4.$$

Since

$$3a^2+d^2-4 = 12(a-1)^2,$$

$$3a^2+d^2+\frac{26}{5} = \frac{2(30a^2-60a+53)}{5}$$

and

$$3a^4+d^4-4=12(a-1)^2(7a^2-22a+21),$$

the desired inequality can be written in the obvious form

$$(a-1)^2(5a-1)^2 \geq 0.$$

The equality holds for $a=b=c=d=1$, and also for $a=b=c=\frac{1}{5}$ and $d=\frac{17}{5}$ (or any cyclic permutation).

□

P 1.42. *If a,b,c,d are nonnegative real numbers such that $a+b+c+d=4$, then*

$$\left(a^2+b^2+c^2+d^2-4\right)\left(a^2+b^2+c^2+d^2+\frac{11}{6}\right) \geq \frac{3}{4}\left(a^4+b^4+c^4+d^4-4\right).$$

(Vasile C., 2006)

Solution. Write the inequality as $F(a,b,c,d) \geq 0$, where

$$F(a,b,c,d)=\left(a^2+b^2+c^2+d^2-\frac{13}{12}\right)^2-\left(\frac{35}{12}\right)^2-\frac{3}{4}\left(a^4+b^4+c^4+d^4-4\right),$$

Assume that $a \leq b \leq c \leq d$ and show that

$$F(a,b,c,d) \geq F(t,b,t,d), \qquad t=\frac{a+c}{2}.$$

Using the identities

$$a^2+c^2-2t^2=2(t^2-ac), \qquad a^4+c^4-2t^4=\frac{1}{2}(t^2-ac)(7a^2+7c^2+10ac),$$

we may write the inequality $F(a,b,c,d) \geq F(t,b,t,d)$ as follows:

$$(a^2+c^2-2t^2)\left(2t^2+a^2+2b^2+c^2+2d^2-\frac{13}{6}\right)-\frac{3}{4}(a^4+c^4-2t^4) \geq 0,$$

$$2(t^2-ac)\left(2t^2+a^2+2b^2+c^2+2d^2-\frac{13}{6}\right)-\frac{3}{8}(t^2-ac)(7a^2+7c^2+10ac) \geq 0.$$

The inequality is true if

$$16\left(2t^2+a^2+2b^2+c^2+2d^2-\frac{13}{6}\right)-3(7a^2+7c^2+10ac) \geq 0,$$

which is equivalent to

$$3(a^2+c^2)-14ac+32(b^2+d^2)-\frac{13}{6}(a+b+c+d)^2 \geq 0,$$

$$18(a^2+c^2)-84ac+192(b^2+d^2)-13(a+b+c+d)^2 \ge 0,$$
$$18(a^2+c^2)-84ac+179b^2+192d^2-26b(a+c+d)-13(a+c+d)^2 \ge 0,$$
$$18(a^2+c^2)-84ac+10b^2+192d^2+(13b-a-c-d)^2-14(a+c+d)^2 \ge 0.$$

The inequality is true if

$$18(a^2+c^2)-84ac+192d^2-16(a+c+d)^2 \ge 0,$$

which is equivalent to

$$9(a^2+c^2)-42ac+96d^2-8(a+c+d)^2 \ge 0.$$

Since

$$9(a^2+c^2)-42ac \ge -6(a+c)^2 = -24t^2,$$

it suffices to show that

$$-24t^2+96d^2-8(2t+d)^2 \ge 0,$$

which is equivalent to the obvious inequality

$$(d-t)(11d+7t) \ge 0.$$

By the AM-Corollary, it suffices to prove the original inequality for $a=b=c$. That is, to show that $3a+d=4$ implies

$$(3a^2+d^2-4)\left(3a^2+d^2+\frac{11}{6}\right) \ge \frac{3}{4}\left(3a^4+d^4-4\right).$$

Since

$$3a^2+d^2-4 = 12(a-1)^2,$$
$$3a^2+d^2+\frac{11}{6} = \frac{72a^2-144a+107}{6}$$

and

$$3a^4+d^4-4 = 12(a-1)^2(7a^2-22a+21),$$

the desired inequality can be written in the obvious form

$$(a-1)^2(9a-5)^2 \ge 0.$$

The equality holds for $a=b=c=d=1$, and also for $a=b=c=\frac{5}{9}$ and $d=\frac{7}{3}$ (or any cyclic permutation).

□

P 1.43. *If a,b,c,d are nonnegative real numbers such that $a+b+c+d=4$, then*

$$\left(a^2+b^2+c^2+d^2-4\right)\left(2a^2+2b^2+2c^2+2d^2-1\right)\geq a^4+b^4+c^4+d^4-4.$$

(Vasile C., 2006)

Solution. Write the inequality as $F(a,b,c,d)\geq 0$, where

$$F(a,b,c,d)=2\left(a^2+b^2+c^2+d^2-\frac{9}{4}\right)^2-\frac{49}{8}-\left(a^4+b^4+c^4+d^4-4\right).$$

Assume that $a\leq b\leq c\leq d$ and show that

$$F(a,b,c,d)\geq F(t,b,t,d),\qquad t=\frac{a+c}{2}.$$

Using the identities

$$a^2+c^2-2t^2=2(t^2-ac),\qquad a^4+c^4-2t^4=\frac{1}{2}(t^2-ac)(7a^2+7c^2+10ac),$$

we may write the inequality $F(a,b,c,d)\geq F(t,b,t,d)$ as follows:

$$2(a^2+c^2-2t^2)\left(2t^2+a^2+2b^2+c^2+2d^2-\frac{9}{2}\right)-(a^4+c^4-2t^4)\geq 0,$$

$$8(t^2-ac)\left(2t^2+a^2+2b^2+c^2+2d^2-\frac{9}{2}\right)-(t^2-ac)(7a^2+7c^2+10ac)\geq 0.$$

The inequality is true if

$$8\left(2t^2+a^2+2b^2+c^2+2d^2-\frac{9}{2}\right)-(7a^2+7c^2+10ac)\geq 0,$$

which is equivalent to

$$5(a^2+c^2)-2ac+16(b^2+d^2)-36\geq 0,$$

$$20(a^2+c^2)-8ac+64(b^2+d^2)-9(a+b+c+d)^2\geq 0,$$

$$20(a^2+c^2)-8ac+64d^2+55b^2-18b(a+c+d)-9(a+c+d)^2\geq 0,$$

$$20(a^2+c^2)-8ac+64d^2+6b^2+\left[7b-\frac{9}{7}(a+c+d)\right]^2-\frac{522}{49}(a+c+d)^2\geq 0.$$

The inequality is true if

$$20(a^2+c^2)-8ac+64d^2-\frac{522}{49}(a+c+d)^2\geq 0,$$

which is equivalent to

$$10(a^2+c^2)-4ac+32d^2-\frac{261}{49}(a+c+d)^2\geq 0.$$

Since

$$10(a^2+c^2)-4ac \ge 4(a+c)^2 = 16t^2$$

and

$$\frac{261}{49} < \frac{16}{3},$$

it suffices to show that

$$16t^2+32d^2-\frac{16}{3}(2t+d)^2 \ge 0,$$

which is equivalent to the obvious inequality

$$(d-t)(5d+t) \ge 0.$$

According to the AM-Corollary, it suffices to prove the original inequality for $a = b = c$. That is, to show that $3a+d=4$ implies

$$(3a^2+d^2-4)(6a^2+2d^2-1) \ge 3a^4+d^4-4.$$

Since

$$3a^2+d^2-4 = 12(a-1)^2,$$

$$6a^2+2d^2-1 = 24a^2-48a+31$$

and

$$3a^4+d^4-4 = 12(a-1)^2(7a^2-22a+21),$$

the desired inequality can be written as

$$(a-1)^2(17a^2-26a+10) \ge 0.$$

This is true because

$$17(17a^2-26a+10) = (17a-13)^2+1 > 0.$$

The equality holds for $a=b=c=d=1$.

□

P 1.44. *If a,b,c,d are nonnegative real numbers such that $a+b+c+d=4$, then*

$$\left(a^2+b^2+c^2+d^2-4\right)\left(a^2+b^2+c^2+d^2+12\right) \ge \frac{4}{3}\left(a^4+b^4+c^4+d^4-4\right).$$

(Vasile C., 2006)

Solution. Write the inequality as $F(a,b,c,d) \ge 0$, where

$$F(a,b,c,d) = (a^2+b^2+c^2+d^2+4)^2 - 64 - \frac{4}{3}(a^4+b^4+c^4+d^4-4).$$

Assume that $a \le b \le c \le d$ and show that

$$F(a,b,c,d) \ge F(t,b,t,d), \qquad t = \frac{a+c}{2}.$$

Using the identities

$$a^2+c^2-2t^2 = 2(t^2-ac), \qquad a^4+c^4-2t^4 = \frac{1}{2}(t^2-ac)(7a^2+7c^2+10ac),$$

we may write the inequality $F(a,b,c,d) \ge F(t,b,t,d)$ as

$$(a^2+c^2-2t^2)(2t^2+a^2+2b^2+c^2+2d^2+8) - \frac{4}{3}(a^4+c^4-2t^4) \ge 0,$$

$$2(t^2-ac)(2t^2+a^2+2b^2+c^2+2d^2+8) - \frac{2}{3}(t^2-ac)(7a^2+7c^2+10ac) \ge 0,$$

$$(t^2-ac)(-5a^2-5c^2-14ac+12b^2+12d^2+48) \ge 0.$$

This inequality is true if

$$-5a^2-5c^2-14ac+12b^2+12d^2+48 \ge 0,$$

which is equivalent to

$$-5a^2-5c^2-14ac+12b^2+12d^2+3(a+b+c+d)^2 \ge 0.$$

It suffices to show that

$$-5a^2-5c^2-14ac+12d^2+3(a+c)^2 \ge 0.$$

Since $2d^2 \ge a^2+c^2$, we only need to show that

$$-5a^2-5c^2-14ac+6(a^2+c^2)+3(a+c)^2 \ge 0,$$

which reduces to

$$4(a-c)^2 \ge 0.$$

By the AM-Corollary, it suffices to prove the original inequality for $a=b=c$. That is, to show that $3a+d=4$ involves

$$3(3a^2+d^2-4)(3a^2+d^2+12) \ge 4(3a^4+d^4-4).$$

Since

$$3a^2+d^2-4 = 12(a-1)^2,$$

$$3a^2+d^2+12=4(3a^2-6a+7)$$

and

$$3a^4+d^4-4=12(a-1)^2(7a^2-22a+21),$$

the desired inequality can be written in the obvious form

$$a(a-1)^2(a+2)\ge 0.$$

The equality holds for $a=b=c=d=1$, and also for $a=b=c=0$ and $d=4$ (or any cyclic permutation).

□

P 1.45. *If a,b,c,d are nonnegative real numbers such that $a+b+c+d=4$, then*

$$\left(a^2+b^2+c^2+d^2-4\right)\left(a^2+b^2+c^2+d^2+\frac{76}{11}\right)\ge\frac{12}{11}\left(a^4+b^4+c^4+d^4-4\right).$$

(Vasile C., 2006)

Solution. Write the inequality as $F(a,b,c,d)\ge 0$, where

$$F(a,b,c,d)=\left(a^2+b^2+c^2+d^2+\frac{16}{11}\right)^2-\left(\frac{60}{11}\right)^2-\frac{12}{11}\left(a^4+b^4+c^4+d^4-4\right).$$

Assume that $a\le b\le c\le d$ and show that

$$F(a,b,c,d)\ge F(t,b,t,d),\qquad t=\frac{a+c}{2}.$$

Using the identities

$$a^2+c^2-2t^2=2(t^2-ac),\qquad a^4+c^4-2t^4=\frac{1}{2}(t^2-ac)(7a^2+7c^2+10ac),$$

we may write the inequality $F(a,b,c,d)\ge F(t,b,t,d)$ as follows:

$$\left(a^2+c^2-2t^2\right)\left(2t^2+a^2+2b^2+c^2+2d^2+\frac{32}{11}\right)-\frac{12}{11}\left(a^4+c^4-2t^4\right)\ge 0,$$

$$22(t^2-ac)\left(2t^2+a^2+2b^2+c^2+2d^2+\frac{32}{11}\right)-6(t^2-ac)(7a^2+7c^2+10ac)\ge 0,$$

$$(t^2-ac)(-9a^2-9c^2-38ac+44b^2+44d^2+64)\ge 0.$$

This inequality is true if

$$-9a^2-9c^2-38ac+44b^2+44d^2+4(a+b+c+d)^2\ge 0.$$

It suffices to show that

$$-9a^2-9c^2-38ac+44d^2+4(a+c)^2\geq 0.$$

Since $2d^2\geq a^2+c^2$, we only need to show that

$$-9a^2-9c^2-38ac+22(a^2+c^2)+4(a+c)^2\geq 0.$$

which reduces to

$$2(a^2+c^2)+15(a-c)^2\geq 0.$$

By the AM-Corollary, it suffices to prove the original inequality for $a=b=c$. That is, to show that $3a+d=4$ involves

$$(3a^2+d^2-4)\left(33a^2+11d^2+76\right)\geq 12\left(3a^4+d^4-4\right).$$

Since

$$3a^2+d^2-4=12(a-1)^2,$$

$$33a^2+11d^2+76=12(11a^2-22a+21)$$

and

$$3a^4+d^4-4=12(a-1)^2(7a^2-22a+21),$$

the desired inequality can be written in the obvious form

$$a^2(a-1)^2\geq 0.$$

The equality holds for $a=b=c=d=1$, and also for $a=b=c=0$ and $d=4$ (or any cyclic permutation).

□

P 1.46. *If $a_1,a_2,\ldots,a_n$ are nonnegative real numbers such that*

$$a_1+a_2+\cdots+a_n=m,\qquad m\in\{1,2,\ldots,n\},$$

then

$$\frac{1}{1+a_1^2}+\frac{1}{1+a_2^2}+\cdots+\frac{1}{1+a_n^2}\geq n-\frac{m}{2}.$$

(Vasile C., 2005)

Solution. We need to prove that $F(a_1,a_2,\ldots,a_n)\geq 0$, where

$$F(a_1,a_2,\ldots,a_n)=\frac{1}{1+a_1^2}+\frac{1}{1+a_2^2}+\cdots+\frac{1}{1+a_n^2}-n+\frac{m}{2}.$$

Assume that

$$F(a_1,a_2,a_3,\ldots,a_n)<F(t,t,a_3,\ldots,a_n)\tag{*}$$

involves

$$F(a_1,a_2,a_3,\dots,a_n) \ge F(0,a_1+a_2,a_3,\dots,a_n), \tag{**}$$

where

$$t=\frac{a_1+a_2}{2}, \qquad a_1 \ne a_2.$$

Then, by AC-Corollary, we have

$$F(a_1,a_2,\dots,a_n) \ge \min_{1\le k\le n} f(k),$$

where

$$f(k)=n-k+\frac{k}{1+(m/k)^2}-n+\frac{m}{2}=\frac{m(m-k)^2}{2(m^2+k^2)}$$

is the value of F for $a_1=\cdots=a_{n-k}=0$ and $a_{n-k+1}=\cdots=a_n=\frac{m}{k}$. Obviously, we have $f(k)\ge 0$ (with equality for $k=m$), therefore $F(a_1,a_2,\dots,a_n)\ge 0$.

To prove that (*) implies (**), we write (*) as

$$\frac{1}{1+a_1^2}+\frac{1}{1+a_2^2}-\frac{2}{1+t^2}<0,$$

$$\frac{(a_1-a_2)^2(a_1^2+a_2^2+4a_1a_2-2)}{(1+a_1^2)(1+a_2^2)(1+t^2)}<0,$$

$$a_1^2+a_2^2+4a_1a_2-2<0,$$

$$(a_1+a_2)^2<2-2a_1a_2, \tag{A}$$

and (**) as

$$\frac{1}{1+a_1^2}+\frac{1}{1+a_2^2}-1-\frac{1}{1+(a_1+a_2)^2}\ge 0,$$

$$\frac{a_1a_2[2-2a_1a_2-a_1a_2(a_1+a_2)^2]}{(1+a_1^2)(1+a_2^2)(1+4t^2)}\ge 0.$$

We need to show that

$$2-2a_1a_2-a_1a_2(a_1+a_2)^2\ge 0.$$

Using (A), we have

$$2-2a_1a_2-a_1a_2(a_1+a_2)^2\ge 2-2a_1a_2-a_1a_2(2-2a_1a_2)=2(a_1a_2-1)^2\ge 0.$$

The equality holds for $a_1=\cdots=a_{n-m}=0$ and $a_{n-m+1}=\cdots=a_n=1$ (or any permutation).

$\square$

P 1.47. *If a,b,c,d are nonnegative real numbers such that $a+b+c+d=2$, then*

$$\frac{1}{1+3a^2}+\frac{1}{1+3b^2}+\frac{1}{1+3c^2}+\frac{1}{1+3d^2}\ge\frac{16}{7}.$$

(Vasile C., 2005)

Solution. Write the inequality as $F(a,b,c,d)\ge 0$, where

$$F(a,b,c,d)=\frac{1}{1+3a^2}+\frac{1}{1+3b^2}+\frac{1}{1+3c^2}+\frac{1}{1+3d^2}-\frac{16}{7}.$$

Assume that

$$F(a,b,c,d)<F(t,t,c,d) \tag{*}$$

involves

$$F(a,b,c,d)\ge F(0,a+b,c,d), \tag{**}$$

where

$$t=\frac{a+b}{2},\qquad a\ne b.$$

Then, by AC-Corollary, we have

$$F(a,b,c,d)\ge \min_{1\le k\le 4} f(k),$$

where

$$f(k)=4-k+\frac{k}{1+3(2/k)^2}-\frac{16}{7}=\frac{12(k-3)(k-4)}{7(k^2+12)}$$

is the value of F when $4-k$ of the numbers a,b,c,d are zero, and the other k numbers are equal to $\frac{2}{k}$. Obviously, we have $f(k)\ge 0$ for $k\in\{1,2,3,4\}$ (with equality for $k=3$ and $k=4$), therefore $F(a_1,a_2,\dots,a_n)\ge 0$.

To prove that (*) implies (**), we write (*) as

$$\frac{1}{1+3a^2}+\frac{1}{1+3b^2}-\frac{2}{1+3t^2}<0,$$

$$\frac{(t^2-ab)(6t^2-1+3ab)}{(1+3a^2)(1+3b^2)(1+3t^2)}<0,$$

$$6t^2<1-3ab, \tag{A}$$

and (**) as

$$\frac{1}{1+3a^2}+\frac{1}{1+3b^2}-1-\frac{1}{1+3(a+b)^2}\ge 0,$$

$$\frac{ab(1-3ab-18abt^2)}{(1+3a^2)(1+3b^2)(1+12t^2)}\ge 0.$$

For the nontrivial case $ab\ne 0$, this is true if

$$6t^2\le\frac{1}{3ab}-1.$$

Using (A), we only need to show that

$$1-3ab \le \frac{1}{3ab}-1,$$

which is equivalent to

$$(3ab-1)^2 \ge 0.$$

The equality holds for $a=b=c=d=\frac{1}{2}$, and also for $a=0$ and $b=c=d=\frac{2}{3}$ (or any cyclic permutation).

□

P 1.48. *If a,b,c,d are nonnegative real numbers such that $a+b+c+d=4$, then*

$$(1+a^2)(1+b^2)(1+c^2)(1+d^2) \le 25.$$

(Vasile C., 2005)

Solution. Write the inequality as $F(a,b,c,d) \ge 0$, where

$$F(a,b,c,d) = 25-(1+a^2)(1+b^2)(1+c^2)(1+d^2).$$

Assume that

$$F(a,b,c,d) < F(t,t,c,d) \tag{*}$$

involves

$$F(a,b,c,d) \ge F(0,a+b,c,d), \tag{**}$$

where

$$t=\frac{a+b}{2}, \qquad a \ne b.$$

By the AC-Corollary, it suffices to prove that

$$F(1,1,1,1) \ge 0, \qquad F\left(0,\frac{4}{3},\frac{4}{3},\frac{4}{3}\right) \ge 0,$$

$$F(0,0,2,2) \ge 0, \qquad F(0,0,0,4) \ge 0.$$

We have

$$F(1,1,1,1)=9, \qquad F\left(0,\frac{4}{3},\frac{4}{3},\frac{4}{3}\right)=\frac{2600}{729},$$

$$F(0,0,\sqrt{2},\sqrt{2}\ )=0, \qquad F(0,0,0,4)=8.$$

To prove that (*) implies (**), we write (*) as

$$\left[(1+t^2)^2-(1+a^2)(1+b^2)\right](1+c^2)(1+d^2) < 0,$$

$$(1+t^2)^2-(1+a^2)(1+b^2) < 0,$$

$$(t^2-ab)(t^2+ab-2)<0,$$
$$2-ab>t^2, \tag{A}$$

and (**) as

$$\left[1+(a+b)^2-(1+a^2)(1+b^2)\right](1+c^2)(1+d^2)\geq 0,$$

$$1+(a+b)^2-(1+a^2)(1+b^2)\geq 0,$$
$$ab(2-ab)\geq 0.$$

This inequality is true if $2-ab\geq 0$. Using (A), we have

$$2-ab>t^2\geq 0.$$

The equality holds for $a=b=0$ and $c=d=2$ (or any permutation).

□

P 1.49. *If a,b,c,d are nonnegative real numbers such that $a+b+c+d=1$, then*

$$\frac{(1+2a)(1+2b)(1+2c)(1+2d)}{(1-a)(1-b)(1-c)(1-d)}\geq\frac{125}{8}.$$

Solution. Write the inequality as $F(a,b,c,d)\geq 0$, where

$$F(a,b,c,d)=(1+2a)(1+2b)(1+2c)(1+2d)-k(1-a)(1-b)(1-c)(1-d),\quad k=\frac{125}{8}.$$

If

$$F(a,b,c,d)<F(t,t,c,d) \tag{*}$$

involves

$$F(a,b,c,d)\geq F(0,a+b,c,d), \tag{**}$$

where

$$t=\frac{a+b}{2},\qquad a\neq b.$$

then, by the AC-Corollary, it suffices to prove that

$$F(0,0,0,1)\geq 0,\qquad F\left(0,0,\frac{1}{2},\frac{1}{2}\right)\geq 0,$$

$$F\left(0,\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)\geq 0,\qquad F\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)\geq 0.$$

We have

$$F(0,0,0,1)=3,\qquad F\left(0,0,\frac{1}{2},\frac{1}{2}\right)=\frac{3}{32},$$

$$F\left(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = 0, \qquad F\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) = \frac{243}{2048}.$$

The inequality (*) is equivalent to

$$\left[(1+2a)(1+2b)-(1+2t)^2\right](1+2c)(1+2d)-$$

$$-k\left[(1-a)(1-b)-(1-t)^2\right](1-c)(1-d) < 0,$$

$$-4(t^2-ab)(1+2c)(1+2d)+k(t^2-ab)(1-c)(1-d) < 0,$$

$$4(1+2c)(1+2d) > k(1-c)(1-d), \tag{A}$$

and (**) to

$$[(1+2a)(1+2b)-(1+2a+2b)](1+2c)(1+2d)-$$

$$-k[(1-a)(1-b)-(1-a-b)](1-c)(1-d) \ge 0,$$

$$4ab(1+2c)(1+2d)-kab(1-c)(1-d) \ge 0.$$

This inequality is true if

$$4(1+2c)(1+2d) \ge k(1-c)(1-d) \ge 0,$$

which follows immediately from (A); therefore, (*) implies (**).

The equality holds for $a = 0$ and $b = c = d = \dfrac{1}{3}$ (or any cyclic permutation).

□

P 1.50. *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that $a_1+a_2+\cdots+a_n = \dfrac{2}{3}$, then*

$$\sum_{1\le i\le j\le n} \frac{a_ia_j}{(1-a_i)(1-a_j)} \le \frac{1}{4}.$$

(Vasile C., 2005)

Solution. We need to prove that $F(a_1, a_2, \ldots, a_n) \ge 0$, where

$$F(a_1, a_2, \ldots, a_n) = \frac{1}{4} - \sum_{1\le i\le j\le n} \frac{a_ia_j}{(1-a_i)(1-a_j)}.$$

Assume that

$$F(a_1, a_2, a_3, \ldots, a_n) < F(t, t, a_3, \ldots, a_n) \tag{*}$$

involves

$$F(a_1, a_2, a_3, \ldots, a_n) \ge F(0, a_1+a_2, a_3, \ldots, a_n), \tag{**}$$

where

$$t = \frac{a_1+a_2}{2}, \qquad a_1 \ne a_2.$$

Then, by AC-Corollary, we only need to prove the original inequality for the case when $n-k$ of the variables $a_1, a_2, \ldots, a_n$ are zero and the other k variables are equal to $\frac{2}{3k}$, where $k \in \{1, 2, \cdots, n\}$; that is, to show that

$$\binom{k}{2}\frac{\left(\frac{2}{3k}\right)^2}{\left(1-\frac{2}{3k}\right)^2} \le \frac{1}{4},$$

which is equivalent to

$$\frac{2k(k-1)}{(3k-2)^2} \le \frac{1}{4},$$

$$(k-2)^2 \ge 0.$$

To prove that (*) implies (**), we write the inequality (*) as

$$\frac{t^2}{(1-t)^2} - \frac{a_1a_2}{(1-a_1)(1-a_2)} + \left(\frac{2t}{1-t} - \frac{a_1}{1-a_1} - \frac{a_2}{1-a_2}\right)\sum_{i=3}^{n}\frac{a_i}{1-a_i} < 0,$$

$$\frac{t^2 - a_1a_2}{(1-a_1)(1-a_2)(1-t)^2}\left[1-2t-2(1-t)\sum_{i=3}^{n}\frac{a_i}{1-a_i}\right] < 0,$$

$$1-2t-2(1-t)\sum_{i=3}^{n}\frac{a_i}{1-a_i} < 0, \tag{A}$$

and (**) as

$$\frac{-a_1a_2}{(1-a_1)(1-a_2)} + \left(\frac{2t}{1-2t} - \frac{a_1}{1-a_1} - \frac{a_2}{1-a_2}\right)\sum_{i=3}^{n}\frac{a_i}{1-a_i} \ge 0,$$

$$\frac{a_1a_2}{(1-a_1)(1-a_2)(1-2t)}\left[1-2t-2(1-t)\sum_{i=3}^{n}\frac{a_i}{1-a_i}\right] \le 0.$$

Clearly, this inequality follows immediately from (A), therefore (*) implies (**). This completes the proof.

The equality holds when $a_1 = a_2 = \cdots = a_{n-2} = 0$ and $a_{n-1} = a_n = \frac{1}{3}$ (or any permutation).

□

P 1.51. *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = 1$ and no one of which is 1, then*

$$\sum_{1 \le i \le j \le n}\frac{a_ia_j}{(1-a_i)(1-a_j)} \ge \frac{n}{2(n-1)}.$$

(Gabriel Dospinescu, 2005)

Solution. For $n = 2$, the inequality is an equality. Consider next that $n \geq 3$. We need to show that $F(a_1, a_2, \ldots, a_n) \geq 0$, where

$$F(a_1, a_2, \ldots, a_n = \sum_{1 \leq i \leq j \leq n} \frac{a_i a_j}{(1-a_i)(1-a_j)} - \frac{n}{2(n-1)},$$

Assume that

$$F(a_1, a_2, a_3, \ldots, a_n) < F(t, t, a_3, \ldots, a_n) \tag{*}$$

involves

$$F(a_1, a_2, a_3, \ldots, a_n) \geq F(0, a_1 + a_2, a_3, \ldots, a_n), \tag{**}$$

where

$$t = \frac{a_1 + a_2}{2}, \qquad a_1 \neq a_2.$$

Then, by AC-Corollary, we only need to prove the original inequality for the case when $n-k$ of the variables $a_1, a_2, \ldots, a_n$ are zero and the other k variables are equal to $\frac{1}{k}$, where $k \in \{1, 2, \cdots, n\}$; that is, to show that

$$\binom{k}{2} \frac{\left(\frac{1}{k}\right)^2}{\left(1 - \frac{1}{k}\right)^2} \geq \frac{n}{2(n-1)},$$

which is equivalent to $n - k \geq 0$. As shown at the preceding P 1.50, the inequalities (*) and (**) are equivalent to

$$2t - 1 + 2(1-t) \sum_{i=3}^{n} \frac{a_i}{1-a_i} < 0$$

and

$$\frac{a_1 a_2}{(1-a_1)(1-a_2)(1-2t)} \left[2t - 1 + 2(1-t) \sum_{i=3}^{n} \frac{a_i}{1-a_i} \right] \leq 0,$$

respectively. Thus, (*) involves (**).

The equality holds when $a_1 = a_2 = \cdots = a_n = \dfrac{1}{n}$.

□

P 1.52. *If a, b, c, d are nonnegative real numbers such that $a + b + c + d = 4$, then*

$$(1 + 3a)(1 + 3b)(1 + 3c)(1 + 3d) \leq 125 + 131abcd.$$

(Pham Kim Hung, 2005)

Solution. Write the inequality as $F(a,b,c,d) \ge 0$, where

$$F(a,b,c,d) = 125 - (1+3a)(1+3b)(1+3c)(1+3d) + 131abcd.$$

If

$$F(a,b,c,d) < F(t,t,c,d) \tag{*}$$

involves

$$F(a,b,c,d) \ge F(0,a+b,c,d), \tag{**}$$

where

$$t = \frac{a+b}{2}, \qquad a \ne b.$$

then, by the AC-Corollary, it suffices to prove that

$$F(0,0,0,4) \ge 0, \qquad F(0,0,2,2) \ge 0,$$

$$F\left(0,\frac{4}{3},\frac{4}{3},\frac{4}{3}\right) \ge 0, \qquad F(1,1,1,1) \ge 0.$$

We have

$$F(0,0,0,4) = 112, \qquad F(0,0,2,2) = 76,$$

$$F\left(0,\frac{4}{3},\frac{4}{3},\frac{4}{3}\right) = 0, \qquad F(1,1,1,1) = 0.$$

The inequality (*) is equivalent to

$$\left[(1+3t)^2 - (1+3a)(1+3b)\right](1+3c)(1+3d) - 131(t^2-ab)cd < 0,$$

$$9(t^2-ab)(1+3c)(1+3d) - 131(t^2-ab)cd < 0,$$

$$9(1+3c)(1+3d) - 131cd < 0, \tag{A}$$

and (**) to

$$[(1+3a+3b) - (1+3a)(1+3b)](1+3c)(1+3d) + 131abcd \ge 0,$$

$$-9ab(1+3c)(1+3d) + 131abcd \ge 0,$$

$$ab[9(1+3c)(1+3d) - 131cd] \le 0,$$

This inequality follows immediately from (A); therefore, (*) implies (**).

The equality holds for $a=b=c=d=1$, and also for $a=0$ and $b=c=d=\dfrac{4}{3}$ (or any cyclic permutation).

□

P 1.53. *If a,b,c,d are nonnegative real numbers such that $a+b+c+d=4$, then*

$$(1+3a^2)(1+3b^2)(1+3c^2)(1+3d^2) \le 255 + a^2b^2c^2d^2.$$

(Vasile C., 2005)

Solution. Write the inequality as $F(a,b,c,d) \ge 0$, where

$$F(a,b,c,d) = 255 - (1+3a^2)(1+3b^2)(1+3c^2)(1+3d^2) + a^2b^2c^2d^2.$$

If

$$F(a,b,c,d) < F(t,t,c,d) \tag{*}$$

involves

$$F(a,b,c,d) \ge F(0,a+b,c,d), \tag{**}$$

where

$$t = \frac{a+b}{2}, \qquad a \ne b.$$

then, by the AC-Corollary, it suffices to prove that

$$F(0,0,0,4) \ge 0, \qquad F(0,0,2,2) \ge 0,$$

$$F\left(0, \frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right) \ge 0, \qquad F(1,1,1,1) \ge 0.$$

We have

$$F(0,0,0,4) = 206, \qquad F(0,0,2,2) = 86,$$

$$F\left(0, \frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right) = \frac{404}{3}, \qquad F(1,1,1,1) = 0.$$

The inequality (*) is equivalent to

$$\left[(1+3t^2)^2 - (1+3a^2)(1+3b^2)\right](1+3c^2)(1+3d^2) - (t^4 - a^2b^2)c^2d^2 < 0,$$

$$3(t^2 - ab)(3t^2 + 3ab - 2)(1+3c^2)(1+3d^2) - (t^2 - ab)(t^2 + ab)c^2d^2 < 0,$$

$$3(3t^2 + 3ab - 2)(1+3c^2)(1+3d^2) - (t^2 + ab)c^2d^2 < 0,$$

$$\frac{3t^2 + 3ab - 2}{t^2 + ab} < \frac{c^2d^2}{3(1+3c^2)(1+3d^2)}, \tag{A}$$

and (**) to

$$\left[1 + 3(a+b)^2 - (1+3a^2)(1+3b^2)\right](1+3c^2)(1+3d^2) + a^2b^2c^2d^2 \ge 0,$$

$$-3ab(3ab-2)(1+3c^2)(1+3d^2) + a^2b^2c^2d^2 \ge 0.$$

This inequality is true if

$$-3(3ab-2)(1+3c^2)(1+3d^2) + abc^2d^2 \ge 0,$$

$$\frac{c^2d^2}{3(1+3c^2)(1+3d^2)} \ge \frac{3ab-2}{ab}.$$

Having in view (A), it suffices to show that

$$\frac{3t^2 + 3ab - 2}{t^2 + ab} \ge \frac{3ab-2}{ab},$$

which reduces to $t^2 \ge 0$; therefore, (*) implies (**).

The equality holds for $a = b = c = d = 1$.

□

P 1.54. *If $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ are nonnegative real numbers, then*

$$\sum a_1^2 + 2\sum_{sym} a_1a_2a_3 + \frac{4n(n-2)}{3(n-1)^2} \geq 2\sum_{sym} a_1a_2.$$

(Vasile C., 2005)

Solution. Let us denote

$$s = a_1 + a_2 + \cdots + a_n.$$

Since

$$2\sum_{sym} a_1a_2 = s^2 - \sum a_1^2,$$

we need to show that $F(a_1, a_2, \ldots, a_n) \geq 0$, where

$$F(a_1, a_2, \ldots, a_n) = \sum a_1^2 + \sum_{sym} a_1a_2a_3 + \frac{2n(n-2)}{3(n-1)^2} - \frac{s^2}{2}.$$

Assume that

$$F(a_1, a_2, a_3, \ldots, a_n) < F(t, t, a_3, \ldots, a_n) \tag{*}$$

involves

$$F(a_1, a_2, a_3, \ldots, a_n) \geq F(0, a_1 + a_2, a_3, \ldots, a_n), \tag{**}$$

where

$$t = \frac{a_1 + a_2}{2}, \qquad a_1 \neq a_2.$$

Then, by AC-Corollary, we only need to prove the original inequality for the case when $n-k$ of the variables $a_1, a_2, \ldots, a_n$ are zero and the other k variables are equal to $\frac{s}{k}$, where $k \in \{1, 2, \cdots, n\}$; that is, to show that

$$k\left(\frac{s}{k}\right)^2 + 2\binom{k}{3}\left(\frac{s}{k}\right)^3 + \frac{4n(n-2)}{3(n-1)^2} \geq 2\binom{k}{2}\left(\frac{s}{k}\right)^3,$$

which is equivalent to

$$2 \cdot \frac{(k-1)(k-2)s^3}{k^2} + \frac{8n(n-2)}{(n-1)^2} \geq \frac{6(k-2)s^2}{k}.$$

This inequality is clearly true for $k = 1$ and $k = 2$. For $k \in \{3, \ldots, n\}$, using the AM-GM inequality to three numbers, it suffices to show that

$$3\sqrt[3]{\frac{(k-1)^2(k-2)^2s^6}{k^4} \cdot \frac{8n(n-2)}{(n-1)^2}} \geq \frac{6(k-2)s^2}{k},$$

which is equivalent to

$$\frac{n(n-2)}{(n-1)^2} \geq \frac{k(k-2)}{(k-1)^2},$$

$$\frac{(n-1)^2-1}{(n-1)^2} \ge \frac{(k-1)^2-1}{(k-1)^2},$$

$$\frac{1}{(k-1)^2} \ge \frac{1}{(n-1)^2}.$$

The inequality (*) is equivalent to

$$(a_1^2+a_2^2-2t^2)-(t^2-a_1a_2)(a_3+\cdots+a_n)<0,$$

$$2(t^2-a_1a_2)-(t^2-a_1a_2)(a_3+\cdots+a_n)<0,$$

$$2<a_3+\cdots+a_n, \tag{A}$$

and the inequality (**) to

$$a_1^2+a_2^2-(a_1+a_2)^2+a_1a_2(a_3+\cdots+a_n)\ge 0,$$

$$-2a_1a_2+a_1a_2(a_3+\cdots+a_n)\ge 0,$$

$$a_1a_2(2-a_3-\cdots-a_n)\le 0.$$

This inequality follows from (A), therefore (*) involves (**).

The equality holds when $a_1=a_2=\cdots=a_n=\dfrac{2}{n-1}$.

□

P 1.55. *If a,b,c,d are nonnegative real numbers such that $a+b+c+d=\sqrt{3}$, then*

$$ab(a+2b+3c)+bc(b+2c+3d)+cd(c+2d+3a)+da(d+2a+3b)\le 2.$$

Solution. Write the inequality as $F(a,b,c,d)\le 2$, where

$$F(a,b,c,d)=ab(a+2b+3c)+bc(b+2c+3d)+cd(c+2d+3a)+da(d+2a+3b).$$

From

$$F(a,b,c,d)-f(a+c,b,0,d)=c(b-d)(a+c-b-d)$$

and

$$F(a,b,c,d)-f(0,b,a+c,d)=a(d-b)(c+a-d-b),$$

we get

$$[F(a,b,c,d)-f(a+c,b,0,d)][F(a,b,c,d)-f(0,b,a+c,d)]=$$

$$=-ac(b-d)^2(a+c-b-d)^2\le 0,$$

hence

$$F(a,b,c,d)\le \max\{f(a+c,b,0,d),f(0,b,a+c,d)\}.$$

Similarly, we have

$$F(a,b,c,d) \le \max\{f(a,b+d,c,0), f(a,0,c,b+d)\}.$$

Therefore, $f(a,b,c,d)$ is maximal when one of a and c is zero, and one of b and d is zero. Thus, it suffices to consider one of these cases. For instance, for $c = d = 0$, we need to show that $a + b = \sqrt{3}$ involves

$$ab(a+2b) \le 2.$$

We have

$$\begin{aligned} 2 - ab(a+2b) &= 2 - b(a^2+2ab) = 2 - b[(a+b)^2 - b^2] \\ &= 2 - 3b + b^3 = (1-b)^2(2+b) \ge 0. \end{aligned}$$

The equality holds for $a = \sqrt{3} - 1$, $b = 1$, $c = d = 0$ (or any cyclic permutation).

□

P 1.56. *If $a,b,c,d > 0$ such that $abcd = 1$, then*

$$\frac{a+b+c+d}{16} + \frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} + \frac{1}{d+1} \ge \frac{9}{4}.$$

(Vasile C., 2018)

Solution. Write the inequality as $F(a,b,c,d) \ge 0$ and assume that $a \ge b \ge c \ge d$, which implies $a^2c^2 \ge (ab)(cd) = 1$, therefore $ac \ge 1$. We will show that

$$F(a,b,c,d) \ge F(\sqrt{ac}, b, \sqrt{ac}, d).$$

We have

$$F(a,b,c,d) - F(\sqrt{ac}, b, \sqrt{ac}, d) = \frac{a+c-2\sqrt{ac}}{16} + \frac{1}{a+1} + \frac{1}{c+1} - \frac{2}{\sqrt{ac}+1} \ge 0,$$

because

$$a + c - 2\sqrt{ac} \ge 0$$

and

$$\frac{1}{a+1} + \frac{1}{c+1} - \frac{2}{\sqrt{ac}+1} = \frac{(\sqrt{a}-\sqrt{c})^2(\sqrt{ac}-1)}{(a+1)(c+1)(\sqrt{ac}+1)} \ge 0.$$

According to GM-Corollary, we have

$$F(a,b,c,d) \ge F(t,t,t,d),$$

where $t = \sqrt[3]{abc}$. So, we need to shoe that $t^3d = 1$ implies

$$\frac{3t+d}{16} + \frac{3}{t+1} + \frac{1}{d+1} \ge \frac{9}{4},$$

which is equivalent to

$$\frac{3t^4+1}{16t^3}++\frac{3}{t+1}+\frac{t^3}{t^3+1}\geq\frac{9}{4},$$

$$3t^7-20t^6+48t^5-45t^4+12t^3+1\geq 0,$$

$$(t-1)^2(3t^5-14t^4+17t^3+3t^2+2t+1)\geq 0.$$

The last inequality is true because

$$3t^5-14t^4+17t^3+3t^2+2t+1>t^3(3t^2-14t+17)>0.$$

The equality holds for $a=b=c=d=1$.

□

P 1.57. *Let*

$$F(a,b,c,d)=4(a^2+b^2+c^2+d^2)-(a+b+c+d)^2\ ,$$

where a,b,c,d,e are positive real numbers such that $a\leq b\leq c\leq d$ and

$$a(b+c+d)\geq 3.$$

Then,

$$F(a,b,c,d)\geq F\left(\frac{1}{a},\frac{1}{b},\frac{1}{c},\frac{1}{d}\right).$$

(Vasile C., 2018)

Solution. For fixed a, write the inequality as $E(b,c,d)\geq 0$, where

$$E(b,c,d)=4(a^2+b^2+c^2+d^2)-(a+b+c+d)^2$$

$$-4\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}\right)+\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)^2,$$

and show that

$$E(b,c,d)\geq E(t,t,t)\geq 0,$$

where

$$t=\frac{b+c+d}{3},\qquad at\geq 1.$$

According to AM-Theorem, the left inequality is true if

$$E(b,c,d)\geq E(x,c,x),$$

where

$$x=\frac{b+d}{2}.$$

We have

$$E(b,c,d)-E(x,c,x)=4(b^2+d^2-2x^2)-4\left(\frac{1}{b^2}+\frac{1}{d^2}-\frac{2}{x^2}\right)$$

$$+\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)^2-\left(\frac{1}{a}+\frac{1}{x}+\frac{1}{c}+\frac{1}{x}\right)^2$$

$$=2(b-d)^2-\frac{4(b-d)^2(b^2+d^2+4bd)}{b^2d^2(b+d)^2}+C,$$

where

$$C=\left(\frac{1}{b}+\frac{1}{d}-\frac{2}{x}\right)\left(\frac{2}{a}+\frac{1}{b}+\frac{2}{c}+\frac{1}{d}+\frac{2}{x}\right)$$

$$\geq\frac{(b-d)^2}{bd(b+d)}\left(\frac{3}{b}+\frac{3}{d}+\frac{4}{b+d}\right)=\frac{(b-d)^2(3b^2+3d^2+10bd)}{b^2d^2(b+d)^2}.$$

Thus, we need to show that

$$2-\frac{4(b^2+d^2+4bd)}{b^2d^2(b+d)^2}+\frac{3b^2+3d^2+10bd}{b^2d^2(b+d)^2}\geq 0,$$

that is

$$2\geq\frac{b^2+d^2+6bd}{b^2d^2(b+d)^2}.$$

Since

$$3\leq a(b+c+d)\leq b(b+2d),$$

it suffices to prove the homogeneous inequality

$$\frac{18}{b^2(b+2d)^2}\geq\frac{b^2+d^2+6bd}{b^2d^2(b+d)^2}.$$

Due to homogeneity, we may set $b=1$ (which involves $d\geq 1$), when the inequality becomes

$$\frac{18}{(1+2d)^2}\geq\frac{1+d^2+6d}{d^2(1+d)^2},$$

$$14d^4+8d^3-11d^2-10d-1\geq 0,$$

$$(d-1)(14d^3+22d^2+11d+1)\geq 0.$$

Also, we have

$$E(t,t,t)=4(a^2+3t^2)-(a+3t)^2-4\left(\frac{1}{a^2}+\frac{3}{t^2}\right)+\left(\frac{1}{a}+\frac{3}{t}\right)^2$$

$$=3(a-t)^2-\frac{3(a-t)^2}{a^2t^2}=\frac{3(a^2t^2-1)(a-t)^2}{a^2t^2}\geq 0.$$

The equality occurs for $a=b=c=d\geq 1$ and for $\dfrac{1}{a}=b=c=d\geq 1$.

Remark 1. Similarly, we can prove the following statement:

- *Let*

$$F(a_1,a_2,\dots,a_n)=n(a_1^2+a_2^2+\cdots+a_n^2)-(a_1+a_2+\cdots+a_n)^2\ ,$$

where $a_1,a_2,\dots,a_n$ are positive real numbers such that $a_1\le a_2\le\cdots\le a_n$ and

$$a_1(a_2+a_3+\cdots+a_n)\ge n-1.$$

If $n\le 6$, then

$$F(a_1,a_2,\dots,a_n)\ge F\left(\frac{1}{a_1},\frac{1}{a_2},\dots,\frac{1}{a_n}\right),$$

with equality for $\dfrac{1}{a_1}=a_2=\cdots=a_n$.

Actually, the inequality holds for all $n\ge 2$.

Remark 2. The inequality

$$F(a_1,a_2,\dots,a_n)\ge F\left(\frac{1}{a_1},\frac{1}{a_2},\dots,\frac{1}{a_n}\right)$$

is also valid in the particular case

$$a_1,a_2,\dots,a_n\ge 1.$$

□

P 1.58. *Let*

$$F(a,b,c,d,e)=\sqrt[5]{abcde}-\frac{5}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}}\ ,$$

where a,b,c,d,e are positive real numbers such that

$$a=\max\{a,b,c,d,e\},\qquad bcde\ge 1.$$

Then,

$$F(a,b,c,d,e)\ge F\left(\frac{1}{a},\frac{1}{b},\frac{1}{c},\frac{1}{d},\frac{1}{e}\right).$$

(Vasile C., 2018)

Solution. Assume that

$$a\ge b\ge c\ge d\ge e.$$

For fixed a, write the inequality as $E(b,c,d,e)\ge 0$, where

$$E(b,c,d,e)=\sqrt[5]{abcde}-\frac{5}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}}-\frac{1}{\sqrt[5]{abcde}}+\frac{5}{a+b+c+d+e},$$

and show that

$$E(b,c,d,e) \geq E(t,t,t,t) \geq 0,$$

where

$$t = \sqrt[4]{bcde} \geq 1.$$

Let as denote

$$a+c+d = 3p, \qquad \frac{1}{a}+\frac{1}{c}+\frac{1}{d} = \frac{3}{q}.$$

According to GM-Theorem, the left inequality is true if

$$E(b,c,d,e) \geq E(\sqrt{be},c,d,\sqrt{be}),$$

which is equivalent to

$$\frac{1}{3p+b+e} - \frac{1}{\frac{3}{q}+\frac{1}{b}+\frac{1}{e}} \geq \frac{1}{3p+2\sqrt{be}} - \frac{1}{\frac{3}{q}+\frac{2}{\sqrt{be}}},$$

$$\frac{1}{\frac{3}{q}+\frac{2}{\sqrt{be}}} - \frac{1}{\frac{3}{q}+\frac{1}{b}+\frac{1}{e}} \geq \frac{1}{3p+2\sqrt{be}} - \frac{1}{3p+b+e},$$

$$\frac{(\sqrt{b}-\sqrt{e})^2}{be\left(\frac{3}{q}+\frac{2}{\sqrt{be}}\right)\left(\frac{3}{q}+\frac{1}{b}+\frac{1}{e}\right)} \geq \frac{(\sqrt{b}-\sqrt{e})^2}{(3p+2\sqrt{be})(3p+b+e)}.$$

After dividing by $(\sqrt{b}-\sqrt{e})^2$, we need to show that

$$\left(3p+2\sqrt{be}\right)(3p+b+e) \geq be\left(\frac{3}{q}+\frac{2}{\sqrt{be}}\right)\left(\frac{3}{q}+\frac{b+e}{be}\right), \tag{1}$$

that is

$$A(b+e)+B \geq 0,$$

where

$$A = 3p+2\sqrt{be} - \frac{3}{q} - \frac{2}{\sqrt{be}}.$$

Since

$$a - \frac{1}{a} \geq b - \frac{1}{b},$$

we have $A \geq C$, where

$$C = 3p+2\sqrt{be} - \frac{3}{q} - \frac{2}{\sqrt{be}}.$$

By Lemma 1 below, we have $C \geq 0$, hence $A \geq 0$. Since

$$A(c+d)+B \geq 2A\sqrt{cd}+B,$$

we need to show that $2A\sqrt{cd}+B \geq 0$. This is equivalent to (1) if the sum $b+e$ is replaced by $2\sqrt{be}$:

$$(3p+2\sqrt{be})(3p+2\sqrt{be}) \geq be\left(\frac{3}{q}+\frac{2}{\sqrt{be}}\right)\left(\frac{3}{q}+\frac{2}{\sqrt{be}}\right),$$

that is

$$\left(3p+2\sqrt{be}\right)^2 \ge be\left(\frac{3}{q}+\frac{2}{\sqrt{be}}\right)^2,$$

$$a+c+d+2\sqrt{be} \ge \sqrt{be}\left(\frac{1}{a}+\frac{1}{c}+\frac{1}{d}\right)+2,$$

Since

$$a-\frac{\sqrt{be}}{a} \ge b-\frac{\sqrt{be}}{b},$$

it suffices to show that

$$b+c+d+2\sqrt{be} \ge \sqrt{be}\left(\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)+2.$$

By Lemma 2, this inequality is true.

The right inequality $E(t,t,t,t) \ge 0$ is true for $a \ge t \ge 1$ if

$$\sqrt[5]{at^4}-\frac{5}{\frac{1}{a}+\frac{4}{t}} \ge \frac{1}{\sqrt[5]{at^4}}-\frac{5}{a+4t}.$$

It suffices to prove the homogeneous inequality

$$\sqrt[5]{at^4}-\frac{5at}{4a+t} \ge t^2\left(\frac{1}{\sqrt[5]{at^4}}-\frac{5}{a+4t}\right).$$

Setting $t=1$ and substituting

$$a=x^5, \qquad x \ge 1,$$

the inequality becomes as follows:

$$x-\frac{5x^5}{4x^5+1} \ge \frac{1}{x}-\frac{5}{x^5+4},$$

$$\frac{x^2(4x^5-5x^4+1)}{4x^5+1} \ge \frac{x^5-5x+4}{x^5+4},$$

$$\frac{x^2(x-1)^2(4x^3+3x^2+2x+1)}{4x^5+1} \ge \frac{(x-1)^2(x^3+2x^2+3x+4}{x^5+4},$$

$$(x-1)^5(4x^7+15x^6+31x^5+45x^4+45x^3+31x^2+15x+4) \ge 0.$$

This completes the proof. The equality holds for $a=b=c=d=e \ge 1$.

Lemma 1. *If b,c,d,e are positive real numbers such that*

$$b \ge c \ge d \ge e, \qquad bcde \ge 1,$$

then

$$b+c+d+2\sqrt{be} \ge \frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{2}{\sqrt{be}}.$$

Proof. It suffices to consider the case $bcde = 1$, when the inequality becomes

$$b + c + d + \frac{2}{\sqrt{cd}} \ge \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + 2\sqrt{cd}.$$

Since

$$c + d - 2\sqrt{cd} = (\sqrt{c} - \sqrt{d})^2 \ge 0,$$

it suffices to show that

$$b + \frac{2}{\sqrt{cd}} \ge \frac{1}{b} + \frac{1}{c} + \frac{1}{d}.$$

Write this inequality as

$$\left(b - \frac{1}{d}\right) + \left(\frac{1}{\sqrt{cd}} - \frac{1}{b}\right) + \left(\frac{1}{\sqrt{cd}} - \frac{1}{c}\right) \ge 0.$$

It is true because

$$b - \frac{1}{d} = b - \sqrt{\frac{bce}{d}} = \sqrt{\frac{b}{d}}\,(\sqrt{bd} - \sqrt{ce}) \ge 0,$$

$$\frac{1}{\sqrt{cd}} - \frac{1}{b} = \frac{b - \sqrt{cd}}{b\sqrt{cd}} \ge 0,$$

$$\frac{1}{\sqrt{cd}} - \frac{1}{c} = \frac{1}{\sqrt{cd}}\left(1 - \sqrt{\frac{d}{c}}\right) \ge 0.$$

Lemma 2. *If b, c, d, e are positive real numbers such that*

$$b \ge c \ge d \ge e, \qquad bcde \ge 1,$$

then

$$b + c + d + 2\sqrt{be} \ge \sqrt{be}\left(\frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right) + 2.$$

Proof. Replacing b, c, d, e by b^4, c^4, d^4, e^4, we need to show that

$$b^4 + c^4 + d^4 + 2b^2e^2 \ge b^2e^2\left(\frac{1}{b^4} + \frac{1}{c^4} + \frac{1}{d^4}\right) + 2$$

for

$$b \ge c \ge d \ge e > 0, \qquad bcde \ge 1.$$

It suffices to prove the homogeneous inequality

$$\frac{b^4 + c^4 + d^4 + 2b^2e^2}{bcde} \ge b^2e^2\left(\frac{1}{b^4} + \frac{1}{c^4} + \frac{1}{d^4}\right) + 2,$$

which is equivalent to

$$\frac{b^3}{cde}+\frac{c^3}{bde}+\frac{d^3}{bce}+\frac{2be}{cd}\geq\frac{e^2}{b^2}+\frac{b^2e^2}{c^4}+\frac{b^2e^2}{d^4}+2,$$

or

$$A+B+C\geq 0,$$

where

$$A=\frac{d^3}{bce}-\frac{e^2}{b^2},$$

$$B=\frac{c^3}{bde}+\frac{be}{cd}-2\geq 0,$$

$$C=\frac{b^3}{cde}+\frac{be}{cd}-\frac{b^2e^2}{c^4}-\frac{b^2e^2}{d^4}\geq 0.$$

We will show that $A\geq 0$, $B\geq 0$ and $C\geq 0$. We have

$$A=\frac{bd^3-ce^3}{b^2ce}\geq 0,$$

$$B\geq 2\sqrt{\frac{c^3}{bde}\cdot\frac{be}{cd}}-2=2\left(\frac{c}{d}-1\right)\geq 0,$$

$$C=\frac{b^2e}{cd}D,$$

where

$$D=\frac{b}{e^2}+\frac{1}{b}-\frac{de}{c^3}-\frac{ce}{d^3}.$$

To show that $D\geq 0$, it suffices to consider the case $e=d$. Thus we need to show that

$$\frac{b}{d^2}+\frac{1}{b}-\frac{d^2}{c^3}-\frac{c}{d^2}\geq 0.$$

We will show that

$$\frac{b}{d^2}+\frac{1}{b}\geq\frac{c}{d^2}+\frac{1}{c}\geq\frac{d^2}{c^3}+\frac{c}{d^2}.$$

Indeed, the left inequality is equivalent to

$$\frac{(b-c)(bc-d^2)}{bcd^2}\geq 0,$$

and the right inequality is equivalent to

$$\frac{c^2-d^2}{c^3}\geq 0.$$

Remark 1. The inequality

$$F(a,b,c,d,e)\geq F\left(\frac{1}{a},\frac{1}{b},\frac{1}{c},\frac{1}{d},\frac{1}{e}\right)$$

is also valid in the particular case

$$a, b, c, d, e \geq 1.$$

Note that the property is not valid for six numbers $a, b, c, d, e, f \geq 1$.

Remark 2. We claim that the following generalization is valid:

- *Let*

$$F(a_1, a_2, \ldots, a_n) = \sqrt[n]{a_1 a_2 \cdots a_n} - \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}},$$

where $a_1, a_2, \ldots, a_n$ are positive real numbers such that

$$a_1^k a_2 \cdots a_n \geq 1, \qquad k = \frac{(n-1)(5-n)}{4n-5}.$$

Then,

$$F(a_1, a_2, \ldots, a_n) \geq F\left(\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}\right).$$

□

P 1.59. *Let*

$$F(a_1, a_2, \ldots, a_n) = a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1 a_2 \cdots a_n}\ ,$$

where $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1 \leq a_2 \leq \cdots \leq a_n$ and

$$a_1^{n-1} a_2 a_3 \cdots a_n \geq 1.$$

Then,

$$F(a_1, a_2, \ldots, a_n) \geq F\left(\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}\right).$$

(Vasile C., 2018)

Solution. By the AM-GM inequality, both sides of the inequality are nonnegative. For fixed a_1, write the inequality as $E(a_2, a_3, \ldots, a_n) \geq 0$, where

$$E(a_2, a_3, \ldots, a_n) = a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1 a_2 \cdots a_n} - \frac{1}{a_1} - \frac{1}{a_2} - \cdots - \frac{1}{a_n} + \frac{n}{\sqrt[n]{a_1 a_2 \cdots a_n}}\ ,$$

and show that

$$E(a_2, a_3, \ldots, a_n) \geq E(t, t, \ldots, t) \geq 0\ ,$$

where

$$t = \sqrt[n-1]{a_2 a_3 \cdots a_n}, \qquad a_1 t \geq 1, \quad t \geq a_1, \quad t \geq 1.$$

According to GM-Theorem, the left inequality is true if

$$E(a_2, a_3, \ldots, a_{n-1}, a_n) \geq E\left(\sqrt{a_2 a_n}, a_3, \ldots, a_{n-1}, \sqrt{a_2 a_n}\right),$$

which can be written as

$$(\sqrt{a_2}-\sqrt{a_n})^2\left(1-\frac{1}{a_2a_n}\right)\ge 0.$$

Thus, we need to show that

$$a_2a_n\ge 1,$$

that is true if

$$(a_2a_n)^{n-1}\ge a_1^{n-1}a_2a_3\cdots a_n.$$

This follows by multiplying the inequalities

$$a_2^{n-1}\ge a_1^{n-1},$$

$$a_n^{n-1}\ge a_2a_3\cdots a_n.$$

Write now the right inequality $E(t,t,\ldots,t)\ge 0$ in the form

$$a_1+(n-1)t-n\sqrt[n]{a_1t^{n-1}}\ge \frac{1}{a_1}+\frac{n-1}{t}-\frac{n}{\sqrt[n]{a_1t^{n-1}}}\ .$$

Since $a_1t\ge 1$, it suffices to show the homogeneous inequality

$$a_1+(n-1)t-n\sqrt[n]{a_1t^{n-1}}\ge a_1t\left(\frac{1}{a_1}+\frac{n-1}{t}-\frac{n}{\sqrt[n]{a_1t^{n-1}}}\right)\ ,$$

that is

$$(n-2)(t-a_1)\ge n\left(\sqrt[n]{a_1t^{n-1}}-\sqrt[n]{a_1^{n-1}t}\right)\ .$$

Setting $a_1=1$ and substituting $t=x^n$, $x\ge 1$, we need to show that

$$(n-2)(x^n-1)\ge n(x^{n-1}-x),$$

that is

$$n(x-1)(x^{n-1}+1)-2(x^n-1)\ge 0,$$

$$(x-1)[(n-2)(x^{n-1}+1)-2(x^{n-2}+\cdots+x]\ge 0,$$

$$(x-1)\sum_{i=1}^{n-2}(x^i-1)(x^{n-i-1}-1)\ge 0.$$

The equality holds for $a_1=a_2=\cdots=a_n\ge 1$.

Remark 1. The inequality

$$F(a_1,a_2,\ldots,a_n)\ge F\left(\frac{1}{a_1},\frac{1}{a_2},\ldots,\frac{1}{a_n}\right)$$

is also valid in the particular case

$$a_1,a_2,\ldots,a_n\ge 1.$$

Remark 2. Since $a_1a_2\cdots a_n \ge 1$, from P 1.59 it follows that

$$a_1+a_2+\cdots+a_n \ge \frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}$$

for

$$a_1^{n-1}a_2a_3\cdots a_n \ge 1.$$

□

P 1.60. *Let*

$$F(a,b,c,d)=\sqrt{\frac{a^2+b^2+c^2+d^2}{4}}-\frac{a+b+c+d}{4},$$

where a,b,c,d are positive real numbers such that $a \le b \le c \le d$ and

$$a^3(b+c+d) \ge 1.$$

Then,

$$F(a,b,c,d) \ge F\left(\frac{1}{a},\frac{1}{b},\frac{1}{c},\frac{1}{d}\right).$$

(Vasile C., 2020)

Solution. For fixed a, write the inequality as $E(b,c,d) \ge 0$, where

$$E(b,c,d)=\sqrt{4(a^2+b^2+c^2+d^2)}-(a+b+c+d)$$

$$-\sqrt{4\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}\right)}+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d},$$

and show that

$$E(b,c,d) \ge E(t,t,t) \ge 0,$$

where

$$t=\frac{b+c+d}{3}, \qquad a^3t \ge 1, \quad t \ge a.$$

According to AM-Theorem, the left inequality is true if

$$E(b,c,d) \ge E(x,c,x),$$

where

$$x=\frac{b+d}{2}.$$

Write the inequality $E(b,c,d) \ge E(x,c,x)$ in the form

$$A+B \ge C,$$

where

$$A=\sqrt{4(a^2+b^2+c^2+d^2)}-\sqrt{4(a^2+x^2+c^2+x^2)}$$
$$=\frac{(b-d)^2}{\sqrt{a^2+b^2+c^2+d^2}+\sqrt{a^2+c^2+2x^2}}$$
$$\geq \frac{(b-d)^2}{\sqrt{2b^2+2d^2}+\sqrt{b^2+d^2+2x^2}},$$

$$B=\frac{1}{b}+\frac{1}{d}-\frac{2}{x}=\frac{(b-d)^2}{bd(b+d)},$$

$$C=\sqrt{4\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}\right)}-\sqrt{4\left(\frac{1}{a^2}+\frac{1}{x^2}+\frac{1}{c^2}+\frac{1}{x^2}\right)}$$
$$=\frac{1}{2b^2d^2x^2}\cdot\frac{(b-d)^2(b^2+d^2+4bd)}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}}+\sqrt{\frac{1}{a^2}+\frac{1}{x^2}+\frac{1}{c^2}+\frac{1}{x^2}}}$$
$$\leq \frac{1}{2b^2d^2x^2}\cdot\frac{(b-d)^2(b^2+d^2+4bd)}{\sqrt{\frac{2}{b^2}+\frac{2}{d^2}}+\sqrt{\frac{1}{b^2}+\frac{1}{d^2}+\frac{2}{x^2}}}$$
$$=\frac{1}{2bdx^2}\cdot\frac{(b-d)^2(b^2+d^2+4bd)}{\sqrt{2b^2+2d^2}+\sqrt{b^2+d^2+2b^2d^2/x^2}}.$$

Thus, we need to show that

$$\frac{1}{\sqrt{2b^2+2d^2}+\sqrt{b^2+d^2+2x^2}}+\frac{1}{bd(b+d)}\geq$$
$$\geq \frac{1}{2bdx^2}\cdot\frac{b^2+d^2+4bd}{\sqrt{2b^2+2d^2}+\sqrt{b^2+d^2+2b^2d^2/x^2}}.$$

Since

$$b^2+d^2+4bd=4x^2+2bd,$$

the inequality is true if

$$\frac{1}{bd(b+d)}\geq \frac{1}{2bdx^2}\cdot\frac{4x^2}{\sqrt{2b^2+2d^2}+\sqrt{b^2+d^2+2b^2d^2/x^2}}$$

and

$$\frac{1}{\sqrt{2b^2+2d^2}+\sqrt{b^2+d^2+2x^2}}\geq \frac{1}{2bdx^2}\cdot\frac{2bd}{\sqrt{2b^2+2d^2}+\sqrt{b^2+d^2+2b^2d^2/x^2}}. \tag{*}$$

Rewrite the first inequality in the form

$$\sqrt{2b^2+2d^2}+\sqrt{b^2+d^2+2b^2d^2/x^2}\geq 2(b+d).$$

By squaring, it becomes

$$2\sqrt{2(b^2+d^2)(b^2+d^2+2b^2d^2/x^2)} \ge b^2+d^2+8bd-2b^2d^2/x^2.$$

Since

$$\sqrt{2(b^2+d^2)(b^2+d^2+2b^2d^2/x^2)} \ge \sqrt{(b^2+d^2+2x^2)(b^2+d^2+2b^2d^2/x^2)}$$
$$\ge b^2+d^2+2bd,$$

it suffices to show that

$$2(b^2+d^2+2bd) \ge b^2+d^2+8bd-2b^2d^2/x^2,$$

which is equivalent to

$$b^2+d^2 \ge 4bd-2b^2d^2/x^2,$$
$$b^2+d^2 \ge \frac{4bd(b^2+d^2)}{(b+d)^2},$$
$$\frac{(b^2+d^2)(b-d)^2}{(b+d)^2} \ge 0.$$

From $a^3(b+c+d) \ge 3$, it follows that

$$b^3(b+2d) \ge 3,$$

hence $y \ge 1$, where

$$y = \frac{b^3(b+2d)}{3}.$$

To prove the inequality (*), it suffices to show that the following homogeneous inequality holds:

$$\sqrt{2b^2+2d^2}+\sqrt{b^2+d^2+2b^2d^2/x^2} \ge \frac{\sqrt{y}}{x^2}\left[\sqrt{2b^2+2d^2}+\sqrt{b^2+d^2+2x^2}\right].$$

Due to homogeneity, we may set $b=1$. Thus, we need to show that $d \ge 1$ implies

$$\sqrt{2+2d^2}+\sqrt{1+d^2+2d^2/x^2} \ge \frac{\sqrt{y}}{x^2}\left[\sqrt{2+2d^2}+\sqrt{1+d^2+2x^2}\right],$$

where

$$x = \frac{d+1}{2} \ge 1, \qquad y = \frac{2d+1}{3}.$$

Since

$$\sqrt{y} \le \frac{y+1}{2} = \frac{d+2}{3} \le \frac{d+1}{2} = x,$$

it is enough to show that

$$\sqrt{2+2d^2}+\sqrt{1+d^2+2d^2/x^2} \ge \frac{1}{x}\left[\sqrt{2+2d^2}+\sqrt{1+d^2+2x^2}\right].$$

Since

$$\sqrt{2+2d^2} \ge \frac{1}{x}\sqrt{2+2d^2},$$

it suffices to show that

$$\sqrt{1+d^2+2d^2/x^2} \ge \frac{1}{x}\sqrt{1+d^2+2x^2}.$$

By squaring, we need to show that

$$(d^2+1)(d+1)^2+8d^2 \ge 2(3d^2+2d+3),$$

which is equivalent to

$$d^4+2d^3+4d^2-2d-5 \ge 0,$$

$$(d-1)(d^3+3d^2+7d+5) \ge 0.$$

Write the inequality $E(t,t,t) \ge 0$ in the form

$$2\sqrt{a^2+3t^2}-a-3t \ge 2\sqrt{\frac{1}{a^2}+\frac{3}{t^2}}-\frac{1}{a}-\frac{3}{t}.$$

Since $a^3t \ge 1$, it suffices to show that the following homogeneous inequality holds:

$$2\sqrt{a^2+3t^2}-a-3t \ge \sqrt{a^3t}\left(2\sqrt{\frac{1}{a^2}+\frac{3}{t^2}}-\frac{1}{a}-\frac{3}{t}\right).$$

Due to homogeneity, we may set $a=1$. Thus, we need to show that $t \ge 1$ implies

$$2\sqrt{1+3t^2}-1-3t \ge \sqrt{t}\left(2\sqrt{1+\frac{3}{t^2}}-1-\frac{3}{t}\right),$$

that is

$$2\sqrt{1+3t^2}-3t-1 \ge \frac{2\sqrt{t^2+3}-t-3}{\sqrt{t}},$$

$$\frac{3(t-1)^2}{2\sqrt{1+3t^2}+3t+1} \ge \frac{3(t-1)^2}{\sqrt{t}\,(2\sqrt{t^2+3}+t+3)}.$$

This is true if

$$\sqrt{t}\,(2\sqrt{t^2+3}+t+3) \ge 2\sqrt{1+3t^2}+3t+1.$$

Since

$$\sqrt{t}\,(t+3)-(3t+1) = (\sqrt{t}-1)^3 \ge 0,$$

it is enough to show that

$$2\sqrt{t(t^2+3)} \ge 2\sqrt{1+3t^2}.$$

By squaring, the inequality becomes

$$(t-1)^3 \ge 0.$$

The proof is completed. The equality occurs for $a = b = c = d \ge 1$.
Remark. The inequality

$$F(a,b,c,d) \ge F\left(\frac{1}{a},\frac{1}{b},\frac{1}{c},\frac{1}{d}\right)$$

is also valid in the particular case

$$a,b,c,d \ge 1.$$

□

P 1.61. *If a,b,c,d are positive real numbers such that*

$$a+b+c+d=4, \qquad d=\max\{a,b,c,d\},$$

then

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1 \ge a^2+b^2+c^2+d^2.$$

(Vasile C., 2021)

Solution. Assuming that $a \le b \le c \le d$, we have

$$a+c \le \frac{a+b}{2}+\frac{c+d}{2}=2.$$

Write the inequality as $F(a,b,c) \ge 0$, where

$$F(a,b,c)=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1-(a^2+b^2+c^2+d^2), \qquad d=4-a-b-c,$$

and show that

$$F(a,b,c) \ge F(t,t,t) \ge 0,$$

where

$$t=\frac{a+b+c}{3}, \qquad t \le 1.$$

By the AM-Theorem, the left inequality is true if

$$F(a,b,c) \ge F(x,b,x),$$

where

$$x=\frac{a+c}{2}.$$

Since

$$F(a,b,c)-F(x,b,x)=\frac{1}{a}+\frac{1}{c}-\frac{2}{x}-(a^2+c^2-2x^2)$$

$$=\frac{(a-c)^2}{ac(a+c)}-\frac{(a-c)^2}{2}=\frac{(a-c)^2[2-ac(a+c)]}{2ac(a+c)}$$

and

$$8-4ac(a+c)\geq 8-(a+c)^3\geq 0,$$

we have $F(a,b,c)\geq F(x,b,x)$.

The inequality $F(t,t,t)\geq 0$ is equivalent to

$$\frac{3}{t}+1\geq 3t^2+(4-3t)^2,$$

$$1-5t+8t^2-4t^3\geq 0,$$

$$(1-t)(1-2t)^2\geq 0.$$

The proof is completed. The equality occurs for $a=b=c=d=1$, and also for $a=b=c=\frac{1}{2}$ and $d=\frac{5}{2}$.

Remark 1. Similarly, we can prove the following nice statement:

- *If a,b,c,d,e are positive real numbers such that*

$$a+b+c+d+e=5, \qquad e=\max\{a,b,c,d,e\},$$

then

$$5\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\geq 4(a^2+b^2+c^2+d^2+e^2),$$

with equality for $a=b=c=d=e=1$, and also for $a=b=c=d=\frac{1}{2}$ and $e=3$.

For $a\leq b\leq c\leq d\leq e$, the inequality $F(a,b,c,d)\geq F(x,b,c,x)$ with $x=\frac{a+d}{2}$ is true if $2ad(a+d)\leq 5$ for $3a+2d\leq 5$. So, it suffices to show the homogeneous inequality

$$50ad(a+d)\leq (3a+2d)^3,$$

which is equivalent to

$$27a^3+4a^2d-14ad^2+8d^3\geq 0,$$

$$25a^3+2(a-d)^2(a+4d)\geq 0.$$

The inequality $F(t,t,t,t)\geq 0$ is equivalent to

$$(1-t)(1-2t)^2\geq 0.$$

Remark 2. The following more general statement is valid:

- *If* $a_1, a_2, \ldots, a_n$ $(n \geq 4)$ *are positive real numbers such that*

$$a_1 + a_2 + \cdots + a_n = n, \qquad a_n = \max\{a_1, a_2, \ldots, a_n\},$$

then

$$n\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{n-1}}\right) \geq 4(a_1^2 + a_2^2 + \cdots + a_n^2) + n(n-5),$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$*, and also for* $a_1 = a_2 = \cdots = a_{n-1} = \frac{1}{2}$ *and* $a_n = \frac{n+1}{2}$.

□

Chapter 2

pqr Method

2.1 Theoretical Basis

The pqr method is applicable to prove symmetric and cyclic polynomial inequalities of the form

$$P(a,b,c) \ge 0.$$

If $P(a,b,c)$ is a symmetric polynomial having the degree not very large (practically, less than or equal to eight), then the inequality can be written in the polynomial form

$$P_1(p,q,r) \ge 0,$$

where

$$p = a+b+c, \qquad q = ab+bc+ca, \qquad r = abc.$$

In addition, the pqr method can be applied to prove inequalities where $P_1(p,q,r)$ is not a polynomial function.

The pqr method enables to prove an inequality $P_1(p,q,r) \ge 0$ using the theorems below.

Theorem 1 (see P 2.53 in Volume 1). *If $a \ge b \ge c$ are real numbers such that*

$$a+b+c = p, \qquad ab+bc+ca = q,$$

where p and q are fixed real numbers satisfying $p^2 \ge 3q$, then the product $r = abc$ is minimal only when $a = b$, and maximal only when $b = c$.

Theorem 2 (see P 2.54 in Volume 1). *If a,b,c are real numbers such that*

$$a+b+c = p, \qquad abc = r,$$

where p and r are fixed real numbers, then the sum $q = ab+bc+ca$ is maximal only when two of a,b,c are equal.

Theorem 3 (see P 2.54 in Volume 1). *If $a \ge b \ge c$ are real numbers such that*

$$ab+bc+ca = q, \qquad abc = r \ne 0,$$

where q and r are fixed real numbers, then the product $p_1 = abc(a+b+c)$ is maximal only when two of a, b, c are equal.

Theorem 4 (see P 3.57 in Volume 1). *If $a \ge b \ge c \ge 0$ are nonnegative real numbers such that*

$$a+b+c = p, \qquad ab+bc+ca = q,$$

where p and q are fixed nonnegative real numbers satisfying $p^2 \ge 3q$, then the product $r = abc$ is minimal only when $a = b$ or $c = 0$, and maximal only when $b = c$.

Theorem 5 (see P 3.58 in Volume 1). *If $a \ge b \ge c > 0$ are positive real numbers such that*

$$a+b+c = p, \qquad abc = r,$$

where p and r are fixed positive real numbers satisfying $p^3 \ge 27r$, then $q = ab+bc+ca$ is minimal only when $b = c$, and maximal only when $a = b$.

Theorem 6 (see Remark 2 from P 3.58 in Volume 1). *If $a \ge b \ge c > 0$ are positive real numbers such that*

$$ab+bc+ca = q, \qquad abc = r,$$

where q and r are fixed positive real numbers satisfying $p^3 \ge 27r$, then the sum $p = a+b+c$ is minimal only when $a = b$, and maximal only when $b = c$.

For cyclic polynomial inequalities, the pqr method can be applied by using the formulae

$$\begin{aligned}2C\sum a^2b + 2D\sum ab^2 &= (C+D)\sum ab(a+b) + (C-D)\sum ab(a-b)\\ &= (C+D)(pq-3r) - (C-D)(a-b)(b-c)(c-a),\end{aligned}$$

$$\begin{aligned}2C\sum a^3b + 2D\sum ab^3 &= (C+D)\sum ab(a^2+b^2) + (C-D)\sum ab(a^2-b^2)\\ &= (C+D)(p^2q-2q^2-pr) - (C-D)p(a-b)(b-c)(c-a),\end{aligned}$$

$$\begin{aligned}|(a-b)(b-c)(c-a)| &= \sqrt{(a-b)^2(b-c)^2(c-a)^2}\\ &= \sqrt{\frac{4(p^2-3q)^3-(2p^3-9pq+27r)^2}{27}},\end{aligned} \tag{*}$$

and

Lemma. *If α, β, x, y are real numbers such that*

$$\alpha \ge 0, \quad x \ge 0, \quad x^2 \ge y^2,$$

then

$$\beta y + \alpha\sqrt{x^2-y^2} \le x\sqrt{\alpha^2+\beta^2},$$

with equality for

$$\beta x = y\sqrt{\alpha^2+\beta^2}.$$

Proof. Since

$$x\sqrt{\alpha^2+\beta^2}-\beta y \geq |\beta|x-\beta y \geq |\beta||y|-\beta y \geq 0,$$

we can write the desired inequality as follows:

$$\alpha\sqrt{x^2-y^2} \leq x\sqrt{\alpha^2+\beta^2}-\beta y,$$

$$\alpha^2(x^2-y^2) \leq \left(x\sqrt{\alpha^2+\beta^2}-\beta y\right)^2,$$

$$\left(\beta x - y\sqrt{\alpha^2+\beta^2}\right)^2 \geq 0.$$

Using (*) and Lemma above for

$$\alpha = \frac{1}{\sqrt{27}}, \qquad x = 2(p^2-3q)^{3/2}, \qquad y = 2p^3-9pq+27r,$$

we get the following theorem:

Theorem 4. *Let a, b, c be real numbers. For any real β, the following inequality holds*

$$27\beta r + |(a-b)(b-c)(c-a)| \leq 9\beta pq - 2\beta p^3 + 2\sqrt{\frac{1}{27}+\beta^2}\,\left(p^2-3q\right)^{3/2},$$

with equality for

$$2\beta(p^2-3q)^{3/2} = \sqrt{\frac{1}{27}+\beta^2}\,\left(2p^3-9pq+27r\right).$$

From Theorem 4, we get

Corollary 1. *Let a, b, c be real numbers such that $a+b+c=1$ For any real β, the following inequality holds:*

$$27\beta r + |(a-b)(b-c)(c-a)| \leq 9\beta q - 2\beta + 2\sqrt{\frac{1+27\beta^2}{27}}\,(1-3q)^{3/2}, \qquad (**)$$

with equality for

$$2\beta(1-3q)^{3/2} = \sqrt{\frac{1+27\beta^2}{27}}\,(2-9q+27r). \qquad (***)$$

2.2 Applications

2.1. If a, b, c and k are real numbers, then

$$\sum(a-b)(a-c)(a-kb)(a-kc) \ge 0.$$

2.2. If a, b, c are real numbers such that

$$a+b+c>0, \qquad ab+bc+ca \ge 0,$$

then

(a) $$3(a^3+b^3+c^3) \ge (a+b+c)(a^2+b^2+c^2);$$

(b) $$a^3+b^3+c^3+\frac{15}{4}abc \ge 0;$$

(c) $$4(a^3+b^3+c^3)+15abc \ge 3(a+b+c)(ab+bc+ca).$$

2.3. If a, b, c are real numbers such that

$$a+b+c>0, \qquad a^2+b^2+c^2+3(ab+bc+ca) \ge 0,$$

then

$$2(a^3+b^3+c^3) \ge ab(a+b)+bc(b+c)+ca(c+a).$$

2.4. If a, b, c are real numbers such that

$$a+b+c>0, \qquad 33(ab+bc+ca) \ge 8(a^2+b^2+c^2),$$

then

$$8(a^3+b^3+c^3)+39abc \ge 7(a+b+c)(ab+bc+ca).$$

2.5. If a, b, c are real numbers such that

$$a+b+c>0, \qquad ab+bc+ca \ge 0,$$

then

(a) $$a^4+b^4+c^4+\frac{11}{4}abc(a+b+c) \ge 0;$$

(b) $$4(a^4+b^4+c^4)+abc(a+b+c) \ge 5(a^2b^2+b^2c^2+c^2a^2).$$

2.6. If a,b,c are real numbers such that

$$a+b+c>0, \qquad ab+bc+ca \ge 0,$$

then

$$81(a^3+b^3+c^3)^2 \ge 25(a^2+b^2+c^2)^3.$$

2.7. If a,b,c are real numbers such that

$$a+b+c>0, \qquad ab+bc+ca \ge 0,$$

then

$$a^2(a-2b+c)+b^2(b-2c+a)+c^2(c-2a+b) \ge 0.$$

2.8. If a,b,c are real numbers such that

$$a+b+c>0, \qquad ab+bc+ca \ge 0,$$

then

(a) $$2(a^3+b^3+c^3)+7abc \ge (a-b)(b-c)(c-a);$$

(b) $$a^3+b^3+c^3+2abc \ge ab^2+bc^2+ca^2;$$

(c) $$9(a^3+b^3+c^3)+12abc \ge 2\sum a^2b+11\sum ab^2.$$

2.9. If a,b,c are real numbers such that

$$a+b+c>0, \qquad ab+bc+ca \ge 0,$$

then

(a) $$a^3+b^3+c^3+\frac{11}{4}abc \ge (a-b)(b-c)(c-a);$$

(b) $$4(a^3+b^3+c^3)+5abc+2\sum a^2b \ge 6\sum ab^2;$$

(c) $$36(a^3+b^3+c^3)+30abc+13\sum a^2b \ge 59\sum ab^2.$$

2.10. If a,b,c are real numbers such that

$$a+b+c>0, \qquad ab+bc+ca \ge 0,$$

then

$$a^3+b^3+c^3-\frac{1}{4}abc \ge 2(a-b)(b-c)(c-a).$$

2.11. If a, b, c are real numbers such that

$$a+b+c>0, \qquad ab+bc+ca \geq 0,$$

then

(a) $$2(a^3+b^3+c^3)+3(3\sqrt{3}-2)abc+6\sqrt{3}\,(a^2b+b^2c+c^2a) \geq 0;$$

(b) $$a^3+b^3+c^3-3abc \geq \frac{3\sqrt{3}}{2}(a-b)(b-c)(c-a).$$

2.12. If a, b, c are real numbers such that

$$a+b+c>0, \qquad 2(a^2+b^2+c^2)+7(ab+bc+ca) \geq 0,$$

then

$$a^2(a-b)+b^2(b-c)+c^2(c-a) \geq 0.$$

2.13. If a, b, c are real numbers such that

$$a+b+c>0, \qquad 3(ab+bc+ca) \geq a^2+b^2+c^2,$$

then

$$a^2(a+2b-3c)+b^2(b+2c-3a)+c^2(c+2a-3b) \geq 0.$$

2.14. If a, b, c are real numbers, then

$$(a^2+b^2+c^2)^2 \geq 3(a^3b+b^3c+c^3a).$$

2.15. If a, b, c are real numbers, then

$$a^3(a-2b+c)+b^3(b-2c+a)+c^3(c-2a+b) \geq 0.$$

2.16. If a, b, c are real numbers, then

$$a^4+b^4+c^4-abc(a+b+c) \geq \sqrt{3}\,(a+b+c)(a-b)(b-c)(c-a).$$

2.17. If a, b, c are real numbers, then

$$a^3(a+b)+b^3(b+c)+c^3(c+a) \geq \frac{2}{3}(ab+bc+ca)^2.$$

2.18. If a,b,c are real numbers such that $ab+bc+ca=3$, then

$$a^3(a-2b)+b^3(b-2c)+c^3(c-2a)+3\ge 0.$$

2.19. If a,b,c are real numbers such that $ab+bc+ca\ge 0$, then

$$a^4+b^4+c^4-abc(a+b+c)\ge \frac{\sqrt{15}}{2}\,(a+b+c)(a-b)(b-c)(c-a).$$

2.20. If a,b,c are real numbers such that

$$2(ab+bc+ca)\ge a^2+b^2+c^2,$$

then

$$a^4+b^4+c^4-abc(a+b+c)\ge \frac{\sqrt{39}}{2}\,(a+b+c)(a-b)(b-c)(c-a).$$

2.21. If a,b,c are real numbers, then

$$a^4+b^4+c^4+2abc(a+b+c)\ge ab^3+bc^3+ca^3.$$

2.22. If a,b,c are real numbers, then

$$a^4+b^4+c^4+\sqrt{2}\,(a^3b+b^3c+c^3a)\ge 0.$$

2.23. If a,b,c are real numbers, then

$$(a^2+b^2+c^2)^2+2(a^3b+b^3c+c^3a)\ge 3(ab^3+bc^3+ca^3).$$

2.24. If a,b,c are real numbers, then

$$(a^2+b^2+c^2)^2+\frac{8}{\sqrt{7}}(a^3b+b^3c+c^3a)\ge 0.$$

2.25. If a,b,c are real numbers such that $ab+bc+ca\le 0$, then

$$(a^2+b^2+c^2)^2\ge(2\sqrt{7}-1)(ab^3+bc^3+ca^3).$$

2.26. If a, b, c are real numbers such that $ab + bc + ca \geq 0$, then

$$(a^2 + b^2 + c^2)^2 + (1 + 2\sqrt{7}\,)(a^3b + b^3c + c^3a) \geq 0.$$

2.27. If a, b, c are real numbers such that $ab + bc + ca \leq 0$, then

$$a^4 + b^4 + c^4 \geq (2\sqrt{3} - 1)(ab^3 + bc^3 + ca^3).$$

2.28. If a, b, c are real numbers such that $ab + bc + ca \geq 0$, then

$$a^4 + b^4 + c^4 + (1 + 2\sqrt{3}\,)(a^3b + b^3c + c^3a) \geq 0.$$

2.29. If a, b, c are real numbers such that $ab + bc + ca \geq 0$, then

$$a^4 + b^4 + c^4 + 2\sqrt{2}\,(a^3b + b^3c + c^3a) \geq ab^3 + bc^3 + ca^3.$$

2.30. If a, b, c are real numbers such that $ab + bc + ca \geq 0$, then

$$(a + b + c)(a^3 + b^3 + c^3) + 5(a^3b + b^3c + c^3a) \geq 0.$$

2.31. If a, b, c are real numbers such that

$$k(ab + bc + ca) = a^2 + b^2 + c^2, \qquad k \geq \frac{1 + \sqrt{21 + 8\sqrt{7}}}{2} \approx 3.7468,$$

then

$$a^3b + b^3c + c^3a \geq 0.$$

2.32. If a, b, c are nonnegative real numbers, then

$$3(a^4 + b^4 + c^4) + 4(a^3b + b^3c + c^3a) \geq 7(ab^3 + bc^3 + ca^3).$$

2.33. If a, b, c are nonnegative real numbers, then

$$16(a^4 + b^4 + c^4) + 52(a^3b + b^3c + c^3a) \geq 47(ab^3 + bc^3 + ca^3).$$

2.34. If a, b, c are nonnegative real numbers, then

$$a^4 + b^4 + c^4 + 5(a^3b + b^3c + c^3a) \geq 6(a^2b^2 + b^2c^2 + c^2a^2).$$

2.35. If a, b, c are nonnegative real numbers such that $a + b + c = 4$, then

$$a^3b + b^3c + c^3a + \frac{473}{64}abc \leq 27.$$

2.36. If a, b, c are positive real numbers such that $a + b + c = 3$, then

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} + 5(ab + bc + ca) \geq 18.$$

2.37. If a, b, c are positive real numbers, then

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} - 3 \geq \frac{5(a^2 + b^2 + c^2 - ab - bc - ca)}{a^2 + b^2 + c^2 + ab + bc + ca}.$$

2.38. If a, b, c are positive real numbers, then

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} - 3 \geq \frac{16(a^2 + b^2 + c^2 - ab - bc - ca)}{a^2 + b^2 + c^2 + 6(ab + bc + ca)}.$$

2.39. If a, b, c are real numbers such that $ab + bc + ca \geq 0$, then

$$(a^2 + b^2 + c^2)(a^3 + b^3 + c^3) + 5(a^4b + b^4c + c^4a) \geq 0.$$

2.40. If a, b, c are real numbers such that

$$a + b + c = 3, \qquad ab + bc + ca \geq 0,$$

then

$$a^3b + b^3c + c^3a + 18\sqrt{3} \geq ab^3 + bc^3 + ca^3.$$

2.41. If a, b, c are real numbers such that $a^2 + b^2 + c^2 = 3$, then

$$a^3b + b^3c + c^3a + \frac{81\sqrt{2}}{32} \geq ab^3 + bc^3 + ca^3.$$

2.3 Solutions

P 2.1. *If a, b, c and k are real numbers, then*

$$\sum(a-b)(a-c)(a-kb)(a-kc) \geq 0.$$

(Vasile C., 2005)

***Solution*.** Let

$$p = a+b+c, \qquad q = ab+bc+ca, \qquad r = abc.$$

Write the inequality as

$$\sum(a^2+2bc-q)[a^2+k(k+1)bc-kq] \geq 0,$$

which is equivalent to

$$A_4 - A_2 q + 3kq^2 \geq 0,$$

where

$$A_4 = \sum(a^2+2bc)[a^2+k(k+1)bc], \qquad A_2 = (k+1)\sum a^2 + k(k+3)\sum bc.$$

We have

$$\begin{aligned} a^4+b^4+c^4 &= (a^2+b^2+c^2)^2 - 2(a^2b^2+b^2c^2+c^2a^2) \\ &= (p^2-2q)^2 - 2(q^2-2pr) \\ &= p^4 - 4p^2q + 2q^2 + 4pr, \end{aligned}$$

therefore

$$\begin{aligned} A_4 &= \sum a^4 + (k^2+k+2)abc\sum a + 2k(k+1)\sum b^2c^2 \\ &= p^4 - 4p^2q + 2q^2 + 4pr + (k^2+k+2)pr + 2k(k+1)(q^2-2pr) \\ &= p^4 - 4p^2q + 2(k^2+k+1)q^2 - 3(k^2+k-2)pr, \end{aligned}$$

For given p and q, the left hand side of the desired inequality $A_4 - A_2q + 3kq^2 \geq 0$ is a linear function of r. Therefore, this function is minimal when r is either minimal or maximal. Thus, according to Theorem 1, it suffices to prove the original inequality for $b = c$. In this case, the inequality becomes

$$(a-b)^2(a-kb)^2 \geq 0.$$

The equality holds for $a = b = c$, and also for $\frac{a}{k} = b = c$ (or any cyclic permutation). If $k = 0$, then the equality holds for $a = b = c$, and also for $a = 0$ and $b = c$ (or any cyclic permutation).

□

P 2.2. *If a,b,c are real numbers such that*

$$a+b+c>0, \qquad ab+bc+ca\geq 0,$$

then

(a) $$3(a^3+b^3+c^3)\geq (a+b+c)(a^2+b^2+c^2);$$

(b) $$a^3+b^3+c^3+\frac{15}{4}abc\geq 0;$$

(c) $$4(a^3+b^3+c^3)+15abc\geq 3(a+b+c)(ab+bc+ca).$$

(Vasile C., 2006)

Solution. (a) Write the inequality as

$$3(3r+p^3-3pq)\geq p(p^2-2q),$$

$$2p^3-7pq+9r\geq 0,$$

where

$$p=a+b+c, \qquad q=ab+bc+ca, \qquad r=abc.$$

According to Theorem 1, for fixed p and q, the product r is minimal when

$$a=b\geq c.$$

Thus, it suffices to prove the original inequality for $a=b\geq c$. Since

$$a+b+c=2a+c, \qquad ab+bc+ca=a(a+2c),$$

we need to show that

$$a\geq c, \qquad 2a+c>0, \qquad a+2c\geq 0$$

involve

$$3(2a^3+c^3)\geq (2a+c)(2a^2+c^2),$$

which is equivalent to

$$(a-c)^2(a+c)\geq 0.$$

This inequality is true because

$$3(a+c)=(2a+c)+(a+2c)>0.$$

The equality holds for $a=b=c>0$.

(b) Write the inequality as

$$3r+p^3-3pq+\frac{15}{4}r\geq 0,$$

$$4p^3 - 12pq + 27r \geq 0.$$

According to Theorem 1, for fixed p and q, the product r is minimal when $a = b \geq c$. Thus, it suffices to prove the original inequality for $a = b \geq c$. Since

$$a + b + c = 2a + c, \qquad ab + bc + ca = a(a + 2c),$$

we need to show that

$$a \geq c, \qquad 2a + c > 0, \qquad a + 2c \geq 0$$

involve

$$2a^3 + c^3 + \frac{15}{4}a^2c \geq 0,$$

which is equivalent to

$$(a + 2c)(8a^2 - ac + 2c^2) \geq 0.$$

The equality holds for $a = b = -2c > 0$ (or any cyclic permutation).

(c) Write the inequality as

$$4(3r + p^3 - 3pq) + 15r \geq 3pq,$$

$$4p^3 - 15pq + 27r \geq 0.$$

According to Theorem 1, it suffices to prove the original inequality for $a = b \geq c$. Since

$$a + b + c = 2a + c, \qquad ab + bc + ca = a(a + 2c),$$

we need to show that

$$a \geq c, \qquad 2a + c > 0, \qquad a + 2c \geq 0$$

involve

$$4(2a^3 + c^3) + 15a^2c \geq 3(2a + c)(a^2 + 2ac),$$

which is equivalent to

$$a^3 - 3ac^2 + 2c^3 \geq 0,$$

$$(a + 2c)(a - c)^2 \geq 0.$$

The equality holds for $a = b = c > 0$, and also for $a = b = -2c > 0$ (or any cyclic permutation).

Remark. The inequality in (c) is sharper than the inequalities in (a) and (b) because these last inequalities can be obtained by summing the inequality in (c) to the obvious inequalities

$$2(a^3 + b^3 + c^3 - 3abc) \geq 0$$

and

$$3(a + b + c)(ab + bc + ca) \geq 0,$$

respectively. The inequality $a^3 + b^3 + c^3 - 3abc \geq 0$ is equivalent to

$$(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) \geq 0.$$

□

P 2.3. *If a, b, c are real numbers such that*

$$a+b+c>0, \qquad a^2+b^2+c^2+3(ab+bc+ca)\ge 0,$$

then

$$2(a^3+b^3+c^3)\ge ab(a+b)+bc(b+c)+ca(c+a).$$

(Vasile C., 2006)

First Solution. Write the inequality as

$$2(3r+p^3-3pq)\ge pq-3r,$$

$$2p^3-7pq+9r\ge 0.$$

According to Theorem 1, it suffices to prove the original inequality for $a=b\ge c$. Since

$$a+b+c=2a+c, \qquad a^2+b^2+c^2+3(ab+bc+ca)=(a+c)(5a+c),$$

we need to show that

$$a\ge c, \qquad 2a+c>0, \qquad a+c\ge 0$$

involve

$$2(2a^3+c^3)\ge 2a(a^2+ac+c^2),$$

which is equivalent to

$$a^3-a^2c-ac^2+c^3\ge 0,$$

$$(a+c)(a-c)^2\ge 0.$$

The equality holds for $a=b=c>0$, and also for $a=b=-c>0$ (or any cyclic permutation).

Second Solution. Assume that $a\ge b\ge c$, and write the inequality as follows:

$$\sum(a^3+b^3-a^2b-ab^2)\ge 0,$$

$$\sum(a+b)(a-b)^2\ge 0.$$

$$(a+b)(a-b)^2+(b+c)(b-c)^2+(c+a)(c-a)^2\ge 0,$$

$$(a+b)(a-b)^2+(b+c)(b-c)^2+(a+c)[(a-b)+(b-c)]^2\ge 0,$$

$$(2a+b+c)(a-b)^2+(a+b+2c)(b-c)^2+2(a+c)(a-b)(b-c)\ge 0.$$

Case 1: $a+b+2c\ge 0$. The AM-GM inequality yields

$$(2a+b+c)(a-b)^2+(a+b+2c)(b-c)^2\ge 2\sqrt{(2a+b+c)(a+b+2c)}\,(a-b)(b-c).$$

Thus, it suffices to show that

$$\sqrt{(2a+b+c)(a+b+2c)}+a+c \ge 0,$$

which is true if

$$(2a+b+c)(a+b+2c) \ge (a+c)^2.$$

This inequality is equivalent to the hypothesis $a^2+b^2+c^2+3(ab+bc+ca) \ge 0$.

Case 2: $a+b+2c<0$. This case is not possible because, as shown at the preceding case 1, the hypothesis $a^2+b^2+c^2+3(ab+bc+ca) \ge 0$ involves

$$a+b+2c \ge \frac{(a+c)^2}{2a+b+c} \ge 0.$$

□

P 2.4. *If a,b,c are real numbers such that*

$$a+b+c>0, \qquad 33(ab+bc+ca) \ge 8(a^2+b^2+c^2),$$

then

$$8(a^3+b^3+c^3)+39abc \ge 7(a+b+c)(ab+bc+ca).$$

(Vasile C., 2006)

First Solution. Write the inequality as

$$8(3r+p^3-3pq)+39r \ge 7pq,$$

$$8p^3-31pq+63r \ge 0.$$

According to Theorem 1, it suffices to prove the original inequality for $a=b \ge c$. Since

$$a+b+c=2a+c, \qquad 33(ab+bc+ca)-8(a^2+b^2+c^2)=(17a-2c)(a+4c),$$

we only need to show that

$$a \ge c, \qquad a+4c \ge 0$$

involve

$$8(2a^3+c^3)+39a^2c \ge 7(2a+c)(a^2+2ac),$$

which is equivalent to

$$a^3+2a^2c-7ac^2+4c^3 \ge 0,$$

$$(a+4c)(a-c)^2 \ge 0.$$

The equality holds for $a=b=c>0$, and also for $a=b=-4c>0$ (or any cyclic permutation).

□

P 2.5. *If a,b,c are real numbers such that*

$$a+b+c>0, \qquad ab+bc+ca\ge 0,$$

then

(a) $$a^4+b^4+c^4+\frac{11}{4}abc(a+b+c)\ge 0;$$

(b) $$4(a^4+b^4+c^4)+abc(a+b+c)\ge 5(a^2b^2+b^2c^2+c^2a^2).$$

(Vasile C., 2006)

Solution. We have

$$\begin{aligned}a^4+b^4+c^4 &= (a^2+b^2+c^2)^2-2(a^2b^2+b^2c^2+c^2a^2)\\ &= (p^2-2q)^2-2(q^2-2pr)\\ &= p^4-4p^2q+2q^2+4pr.\end{aligned}$$

(a) Write the inequality as

$$p^4-4p^2q+2q^2+\frac{27}{4}pr\ge 0.$$

According to Theorem 1, it suffices to prove the original inequality for $a=b\ge c$. Clearly, the original inequality is true for $a=b\ge c\ge 0$. Consider further that $a=b>0>c$. Since

$$a+b+c=2a+c, \qquad ab+bc+ca=a(a+2c),$$

we need to show that

$$a>0>c, \qquad 2a+c>0, \qquad a+2c\ge 0$$

involve

$$2a^4+c^4+\frac{11}{4}a^2c(2a+c)\ge 0,$$

which is equivalent to

$$8a^4+22a^3c+11a^2c^2+4c^4\ge 0,$$

$$(a+2c)(8a^3+6a^2c-ac^2+2c^3)\ge 0.$$

The last inequality is true since $a+2c\ge 0$ and

$$\begin{aligned}8a^3+6a^2c-ac^2+2c^3 &> 8a^3+6a^2c-ac^2+38c^3\\ &= (a+2c)(8a^2-10ac+19c^2)\ge 0.\end{aligned}$$

The equality holds for $a=b=-2c>0$ (or any cyclic permutation).

(b) Write the inequality as

$$4(p^4 - 4p^2q + 2q^2 + 4pr) + pr \geq 5(q^2 - 2pr),$$

$$4p^4 - 16p^2q + 3q^2 + 27pr \geq 0.$$

According to Theorem 1, it suffices to prove the original inequality for $a = b \geq c$. Since

$$a + b + c = 2a + c, \qquad ab + bc + ca = a(a + 2c),$$

we need to show that

$$a \geq c, \qquad 2a + c > 0, \qquad a + 2c \geq 0$$

involve

$$4(2a^4 + c^4) + a^2c(2a + c) \geq 5(a^4 + 2a^2c^2),$$

which is equivalent to

$$3a^4 + 2a^3c - 9a^2c^2 + 4c^4 \geq 0,$$

$$(a - c)^2(a + 2c)(3a + 2c) \geq 0.$$

The equality holds for $a = b = c > 0$, and also for $a = b = -2c > 0$ (or any cyclic permutation).

Remark. The inequality in (b) is sharper than the inequalities in (a) because this last inequality can be obtained by summing the inequality in (b) to the obvious inequalities

$$(ab + bc + ca)^2 \geq 0.$$

□

P 2.6. *If a, b, c are real numbers such that*

$$a + b + c > 0, \qquad ab + bc + ca \geq 0,$$

then

$$81(a^3 + b^3 + c^3)^2 \geq 25(a^2 + b^2 + c^2)^3.$$

(Vasile C., 2006)

Solution. Write the inequality as

$$81(3r + p^3 - 3pq)^2 \geq 25(p^2 - 2q)^3,$$

$$p^4 - 4p^2q + 2q^2 + \frac{27}{4}pr \geq 0.$$

According to Theorem 1, it suffices to prove the original inequality for $a = b \geq c$. Since

$$a + b + c = 2a + c, \qquad ab + bc + ca = a(a + 2c),$$

we need to show that

$$a \geq c, \qquad 2a + c > 0, \qquad a + 2c \geq 0$$

involve

$$81(2a^3 + c^3)^2 \geq 25(2a^2 + c^2)^3.$$

Since the inequality is trivial for $c = 0$, consider further the cases $c > 0$ and $c < 0$.

Case 1: $c > 0$. Due to homogeneity, we may set $c = 1$. So, we need to show that $a \geq c = 1$ involves

$$81(2a^3 + 1)^2 \geq 25(2a^2 + 1)^3,$$

which is equivalent to

$$62a^6 - 150a^4 + 162a^3 - 75a^2 + 28 \geq 0.$$

It suffices to show that

$$25(a^6 - 6a^4 + 6a^3 - 3a^2 + 1) \geq 0,$$

which is equivalent to

$$(a-1)^2(2a^4 + 4a^3 + 2a + 1) \geq 0.$$

Case 1: $c < 0$. Due to homogeneity, we may set $c = -1$. So, we need to show that

$$a - 2 \geq 0$$

involves

$$81(2a^3 - 1)^2 \geq 25(2a^2 + 1)^3,$$

which is equivalent to

$$62a^6 - 150a^4 - 162a^3 - 75a^2 + 28 \geq 0.$$

Since

$$62a^6 - 150a^4 - 162a^3 \geq 248a^4 - 150a^4 - 162a^3$$
$$= 98a^4 - 162a^3 \geq 196a^3 - 162a^3 = 34a^3,$$

it suffices to show that

$$34a^3 - 75a^2 + 28 \geq 0,.$$

which is equivalent to

$$(a-2)(34a^2 - 7a - 14) \geq 0.$$

This is true because

$$34a^2 - 7a - 14 > 7a^2 - 7a - 14 = 7(a+1)(a-2) \geq 0.$$

The equality holds for $a = b = -2c > 0$ (or any cyclic permutation).

□

P 2.7. *If a, b, c are real numbers such that*

$$a+b+c>0, \qquad ab+bc+ca \ge 0,$$

then

$$a^2(a-2b+c)+b^2(b-2c+a)+c^2(c-2a+b) \ge 0.$$

(Vasile C., 2006)

Solution. Let

$$p=a+b+c, \qquad q=ab+bc+ca, \qquad r=abc.$$

Due to homogeneity, we may set

$$p=1, \qquad 0 \le q \le \frac{p^2}{3}=\frac{1}{3}.$$

Write the inequality as follows:

$$2(a^3+b^3+c^3) \ge \sum ab(a+b)+3\sum ab(a-b),$$

$$2(a^3+b^3+c^3)+3abc \ge (a+b+c)(ab+bc+ca)-3(a-b)(b-c)(c-a),$$

$$2(3r+p^3-3pq)+3r \ge pq-3(a-b)(b-c)(c-a),$$

$$2p^3-7pq+9r \ge -3(a-b)(b-c)(c-a),$$

$$2-7q \ge -9r-3(a-b)(b-c)(c-a).$$

This inequality is true if

$$\frac{2-7q}{3} \ge -3r+|(a-b)(b-c)(c-a)|.$$

Applying Corollary 1 for

$$\beta=\frac{-1}{9},$$

we have

$$-3r+|(a-b)(b-c)(c-a)| \le -q+\frac{2}{9}+\frac{4}{9}(1-3q)^{3/2},$$

with equality for

$$(1-3q)^{3/2}=9q-27r-2. \tag{*}$$

Therefore, it suffices to show that

$$\frac{2-7q}{3} \ge -q+\frac{2}{9}+\frac{4}{9}(1-3q)^{3/2},$$

which is equivalent to

$$1-3q \geq (1-3q)^{3/2},$$
$$(1-3q)(1-\sqrt{1-3q}) \geq 0,$$
$$\frac{3q(1-3q)}{1+\sqrt{1-3q}} \geq 0.$$

The last inequality is true, with equality for $q = 1/3$ and $q = 0$. According to (*), $q = 1/3$ involves $r = 1/27$, and $q = 0$ involves $r = -1/9$. Thus, the original inequality is an equality for $a = b = c > 0$, and also for

$$ab+bc+ca = 0, \quad (a+b+c)^3+9abc = 0, \quad a+b+c > 0, \quad (a-b)(b-c)(c-a) < 0.$$

The last equality conditions are equivalent to the condition that a, b, c satisfy

$$(a-b)(b-c)(c-a) < 0$$

and are proportional to the roots of the equation

$$9w^3 - 9w^2 + 1 = 0.$$

□

P 2.8. *If a, b, c are real numbers such that*

$$a+b+c > 0, \qquad ab+bc+ca \geq 0,$$

then

(a) $$2(a^3+b^3+c^3)+7abc \geq (a-b)(b-c)(c-a);$$

(b) $$a^3+b^3+c^3+2abc \geq ab^2+bc^2+ca^2;$$

(c) $$9(a^3+b^3+c^3)+12abc \geq 2\sum a^2b+11\sum ab^2.$$

(Vasile C., 2006)

Solution. Denote

$$p = a+b+c, \qquad q = ab+bc+ca, \qquad r = abc.$$

Due to homogeneity, we may set

$$p = 1, \qquad 0 \leq q \leq \frac{p^2}{3} = \frac{1}{3}.$$

(a) Write the inequality as

$$2(3r+p^3-3pq)+7r \geq (a-b)(b-c)(c-a),$$

$$2p^3 - 6pq \geq -13r + (a-b)(b-c)(c-a),$$

$$2 - 6q \geq -13r + (a-b)(b-c)(c-a).$$

It suffices to show that

$$2 - 6q \geq -13r + |(a-b)(b-c)(c-a)|.$$

Applying Corollary 1 for $\beta = \dfrac{-13}{27}$, we have

$$-13r + |(a-b)(b-c)(c-a)| \leq \frac{-13q}{3} + \frac{26}{27} + \frac{28}{27}(1-3q)^{3/2},$$

with equality for

$$13(1-3q)^{3/2} = 7(9q - 27r - 2). \qquad (*)$$

Therefore, it suffices to show that

$$2 - 6q \geq \frac{-13q}{3} + \frac{26}{27} + \frac{28}{27}(1-3q)^{3/2},$$

which is equivalent to

$$28 - 45q \geq 28(1-3q)^{3/2}.$$

Using the substitution

$$t = \sqrt{1-3q}, \qquad 0 \leq t \leq 1,$$

the inequality becomes

$$28 - 15(1-t^2) \geq 28t^3,$$

$$13 + 15t^2 - 28t^3 \geq 0.$$

We have

$$13 + 15t^2 - 28t^3 = (1-t)(13 + 13t + 28t^2) \geq 0,$$

with equality for $t = 1$. Notice that $t = 1$ involves $q = 0$, and (*) gives $r = -1/7$. Thus, if $p = 1$, then the original inequality is an equality for $q = 0$, $r = -1/7$ and $(a-b)(b-c)(c-a) > 0$. More general, the equality holds for

$$ab+bc+ca = 0, \quad (a+b+c)^3 + 7abc = 0, \quad a+b+c > 0, \quad (a-b)(b-c)(c-a) > 0.$$

These equality conditions are equivalent to the condition that a, b, c satisfy

$$(a-b)(b-c)(c-a) > 0$$

and are proportional to the roots of the equation

$$7w^3 - 7w^2 + 1 = 0.$$

(b) Write the inequality as

$$2(a^3+b^3+c^3)+4abc \ge \sum ab(a+b) - \sum ab(a-b),$$

$$2(a^3+b^3+c^3)+4abc - \sum ab(a+b) \ge (a-b)(b-c)(c-a),$$

$$2(a^3+b^3+c^3)+7abc-(a+b+c)(ab+bc+ca) \ge (a-b)(b-c)(c-a),$$

$$2(3r+p^3-3pq)+7r-pq \ge (a-b)(b-c)(c-a),$$

$$2p^3-7pq \ge -13r+(a-b)(b-c)(c-a),$$

$$2-7q \ge -13r+(a-b)(b-c)(c-a).$$

It suffices to show that

$$2-7q \ge -13r+|(a-b)(b-c)(c-a)|.$$

As shown at (a), we have

$$-13r+|(a-b)(b-c)(c-a)| \le \frac{-13q}{3}+\frac{26}{27}+\frac{28}{27}(1-3q)^{3/2},$$

Therefore, it suffices to show that

$$2-7q \ge \frac{-13q}{3}+\frac{26}{27}+\frac{28}{27}(1-3q)^{3/2},$$

which is equivalent to

$$7-18q \ge 7(1-3q)^{3/2}.$$

Using the substitution

$$t=\sqrt{1-3q}, \qquad 0 \le t \le 1,$$

the inequality becomes

$$7-6(1-t^2) \ge 7t^3,$$

$$1+6t^2-7t^3 \ge 0.$$

We have

$$1+6t^2-7t^3=(1-t)(1+7t+7t^2) \ge 0,$$

with equality for $t=1$. The original inequality is an equality for

$$ab+bc+ca=0, \quad (a+b+c)^3+7abc=0, \quad a+b+c>0, \quad (a-b)(b-c)(c-a)>0.$$

These equality conditions are equivalent to the condition that a,b,c satisfy

$$(a-b)(b-c)(c-a)>0$$

and are proportional to the roots of the equation

$$7w^3-7w^2+1=0.$$

(c) Write the inequality as

$$18(a^3+b^3+c^3)+24abc \geq 13\sum ab(a+b)-9\sum ab(a-b),$$

$$18(a^3+b^3+c^3)+24abc-13\sum ab(a+b) \geq 9(a-b)(b-c)(c-a),$$

$$18(a^3+b^3+c^3)+63abc-13(a+b+c)(ab+bc+ca) \geq 9(a-b)(b-c)(c-a),$$

$$18(3r+p^3-3pq)+63r-13pq \geq 9(a-b)(b-c)(c-a),$$

$$18p^3-67pq \geq -117r+9(a-b)(b-c)(c-a),$$

$$18-67q \geq -117r+9(a-b)(b-c)(c-a).$$

It suffices to show that

$$2-\frac{67q}{9} \geq -13r+|(a-b)(b-c)(c-a)|.$$

As shown at (a), we have

$$-13r+|(a-b)(b-c)(c-a)| \leq \frac{-13q}{3}+\frac{26}{27}+\frac{28}{27}(1-3q)^{3/2}.$$

Therefore, it suffices to show that

$$2-\frac{67q}{9} \geq \frac{-13q}{3}+\frac{26}{27}+\frac{28}{27}(1-3q)^{3/2},$$

which is equivalent to

$$1-3q \geq (1-3q)^{3/2},$$

$$(1-3q)(1-\sqrt{1-3q}\,) \geq 0,$$

$$\frac{3q(1-3q)}{1-\sqrt{1-3q}} \geq 0.$$

The last inequality is true, with equality for $q=1/3$ and $q=0$. The original inequality is an equality for $a=b=c>0$, and also for

$$ab+bc+ca=0, \quad (a+b+c)^3+7abc=0, \quad a+b+c>0, \quad (a-b)(b-c)(c-a)>0.$$

The last equality conditions are equivalent to the condition that a,b,c satisfy

$$(a-b)(b-c)(c-a)>0$$

and are proportional to the roots of the equation

$$7w^3-71w^2+1=0.$$

Remark. The inequality (c) is sharper than the inequalities (b), and the inequality (b) is sharper than the inequalities (a). Indeed, the inequalities (b) can be obtained by summing the inequality in (c) to the obvious inequality

$$2(a+b+c)(ab+bc+ca) \geq 0,$$

and the inequalities (a) can be obtained by summing the inequality in (b) to the inequality

$$\frac{1}{2}(a+b+c)(ab+bc+ca) \geq 0.$$

□

P 2.9. *If a, b, c are real numbers such that*

$$a+b+c > 0, \qquad ab+bc+ca \geq 0,$$

then

(a) $$a^3+b^3+c^3+\frac{11}{4}abc \geq (a-b)(b-c)(c-a);$$

(b) $$4(a^3+b^3+c^3)+5abc+2\sum a^2b \geq 6\sum ab^2;$$

(c) $$36(a^3+b^3+c^3)+30abc+13\sum a^2b \geq 59\sum ab^2.$$

(Vasile C., 2006)

Solution. Consider

$$p=1, \qquad 0 \leq q \leq \frac{p^2}{3}=\frac{1}{3}.$$

(a) Write the inequality as

$$3r+p^3-3pq+\frac{11}{4}r \geq (a-b)(b-c)(c-a),$$

$$p^3-3pq+\frac{23}{4}r \geq (a-b)(b-c)(c-a),$$

$$1-3q \geq \frac{-23}{4}r+(a-b)(b-c)(c-a).$$

It suffices to show that

$$1-3q \geq \frac{-23r}{4}+|(a-b)(b-c)(c-a)|.$$

Applying Corollary 1 for $\beta=\dfrac{-23}{108}$, we have

$$\frac{-23}{4}r+|(a-b)(b-c)(c-a)| \leq \frac{-23q}{12}+\frac{23}{54}+\frac{31}{54}(1-3q)^{3/2},$$

with equality for

$$46(1-3q)^{3/2} = 31(9q-27r-2). \tag{*}$$

Therefore, it suffices to show that

$$1-3q \ge \frac{-23q}{12} + \frac{23}{54} + \frac{31}{54}(1-3q)^{3/2},$$

which is equivalent to

$$62-117q \ge 62(1-3q)^{3/2}.$$

Using the substitution

$$t = \sqrt{1-3q}, \qquad 0 \le t \le 1,$$

the inequality becomes

$$62-39(1-t^2) \ge 62t^3,$$
$$23+39t^2-62t^3 \ge 0.$$

We have

$$23+39t^2-62t^3 = (1-t)(23+23t+62t^2) \ge 0,$$

with equality for $t=1$. Notice that $t=1$ involves $q=0$, and (*) gives $r=-4/31$. Thus, if $p=1$, then the original inequality is an equality for $q=0$, $r=-4/31$ and $(a-b)(b-c)(c-a)>0$. More general, the equality holds for

$$ab+bc+ca=0, \quad 4(a+b+c)^3+31abc=0, \quad a+b+c>0, \quad (a-b)(b-c)(c-a)>0.$$

These equality conditions are equivalent to the condition that a,b,c satisfy

$$(a-b)(b-c)(c-a)>0$$

and are proportional to the roots of the equation

$$31w^3-31w^2+4=0.$$

(b) Write the inequality as

$$4(a^3+b^3+c^3)+5abc \ge 2\sum ab(a+b) - 4\sum ab(a-b),$$
$$4(a^3+b^3+c^3)+5abc \ge 2\sum ab(a+b) + 4(a-b)(b-c)(c-a),$$
$$4(a^3+b^3+c^3)+11abc \ge 2(a+b+c)(ab+bc+ca) + 4(a-b)(b-c)(c-a),$$
$$4(3r+p^3-3pq)+11r \ge 2pq+4(a-b)(b-c)(c-a),$$
$$p^3-\frac{7}{2}pq \ge \frac{-23}{4}r+(a-b)(b-c)(c-a),$$
$$1-\frac{7q}{2} \ge \frac{-23r}{4}+(a-b)(b-c)(c-a).$$

It suffices to show that

$$1-\frac{7q}{2}\geq\frac{-23}{4}r+|(a-b)(b-c)(c-a)|.$$

As shown at (a), we have

$$\frac{-23}{4}r+|(a-b)(b-c)(c-a)|\leq\frac{-23q}{12}+\frac{23}{54}+\frac{31}{54}(1-3q)^{3/2}.$$

Therefore, it suffices to show that

$$1-\frac{7q}{2}\geq\frac{-23q}{12}+\frac{23}{54}+\frac{31}{54}(1-3q)^{3/2},$$

which is equivalent to

$$62-171q\geq 62(1-3q)^{3/2}.$$

Using the substitution

$$t=\sqrt{1-3q},\qquad 0\leq t\leq 1,$$

the inequality becomes

$$62-57(1-t^2)\geq 62t^3,$$

$$5+57t^2-62t^3\geq 0.$$

We have

$$5+57t^2-62t^3=(1-t)(5+5t+62t^2)\geq 0,$$

with equality for $t=1$. The equality holds for

$$ab+bc+ca=0,\quad 4(a+b+c)^3+31abc=0,\quad a+b+c>0,\quad (a-b)(b-c)(c-a)>0.$$

These equality conditions are equivalent to the condition that a,b,c satisfy

$$(a-b)(b-c)(c-a)>0$$

and are proportional to the roots of the equation

$$31w^3-31w^2+4=0.$$

(c) Write the inequality as

$$36(a^3+b^3+c^3)+30abc\geq 23\sum ab(a+b)-36\sum ab(a-b),$$

$$36(a^3+b^3+c^3)+30abc\geq 23\sum ab(a+b)+36(a-b)(b-c)(c-a),$$

$$36(a^3+b^3+c^3)+99abc-23(a+b+c)(ab+bc+ca)\geq 36(a-b)(b-c)(c-a),$$

$$36(3r+p^3-3pq)+99r-23pq\geq 36(a-b)(b-c)(c-a),$$

$$36p^3-131pq\geq -207r+36(a-b)(b-c)(c-a),$$

$$36 - 131q \geq -207r + 36(a-b)(b-c)(c-a).$$

It suffices to show that

$$\frac{36-131q}{36} \geq \frac{-23r}{4} + |(a-b)(b-c)(c-a)|.$$

As shown at (a), we have

$$\frac{-23}{4}r + |(a-b)(b-c)(c-a)| \leq \frac{-23q}{12} + \frac{23}{54} + \frac{31}{54}(1-3q)^{3/2}.$$

Therefore, it suffices to show that

$$\frac{36-131q}{36} \geq \frac{-23q}{12} + \frac{23}{54} + \frac{31}{54}(1-3q)^{3/2},$$

which is equivalent to

$$1-3q \geq (1-3q)^{3/2},$$

$$(1-3q)(1-\sqrt{1-3q}\,) \geq 0,$$

$$\frac{3q(1-3q)}{1-\sqrt{1-3q}} \geq 0.$$

The last inequality is true, with equality for $q = 1/3$ and $q = 0$. The original inequality is an equality for $a = b = c > 0$, and also for

$$ab+bc+ca = 0, \quad 4(a+b+c)^3+31abc = 0, \quad a+b+c > 0, \quad (a-b)(b-c)(c-a) > 0.$$

The last equality conditions are equivalent to the condition that a, b, c satisfy

$$(a-b)(b-c)(c-a) > 0$$

and are proportional to the roots of the equation

$$31w^3 - 31w^2 + 4 = 0.$$

Remark. The inequality (c) is sharper than the inequalities (b), and the inequality (b) is sharper than the inequalities (a). Indeed, the inequalities (b) can be obtained by summing the inequality in (c) to the obvious inequality

$$5(a+b+c)(ab+bc+ca) \geq 0,$$

and the inequalities (a) can be obtained by summing the inequality in (b) to the inequality

$$2(a+b+c)(ab+bc+ca) \geq 0.$$

□

P 2.10. *If a, b, c are real numbers such that*

$$a+b+c>0, \qquad ab+bc+ca \ge 0,$$

then

$$a^3+b^3+c^3-\frac{1}{4}abc \ge 2(a-b)(b-c)(c-a).$$

(Vasile C., 2006)

Solution. Assume

$$p=1, \qquad 0 \le q \le \frac{p^2}{3} = \frac{1}{3},$$

and write the inequality as

$$3r+p^3-3pq-\frac{1}{4}r \ge 2(a-b)(b-c)(c-a),$$

$$p^3-3pq+\frac{11}{4}r \ge 2(a-b)(b-c)(c-a),$$

$$1-3q \ge \frac{-11}{4}r+2(a-b)(b-c)(c-a).$$

It suffices to show that

$$\frac{1}{2}-\frac{3}{2}q \ge \frac{-11r}{8}+|(a-b)(b-c)(c-a)|.$$

Applying Corollary 1 for $\beta = \dfrac{-11}{216}$, we have

$$\frac{-11}{8}r+|(a-b)(b-c)(c-a)| \le \frac{-11q}{24}+\frac{11}{108}+\frac{43}{108}(1-3q)^{3/2},$$

with equality for

$$22(1-3q)^{3/2} = 43(9q-27r-2). \tag{*}$$

Therefore, it suffices to show that

$$\frac{1}{2}-\frac{3}{2}q \ge \frac{-11q}{24}+\frac{11}{108}+\frac{43}{108}(1-3q)^{3/2},$$

which is equivalent to

$$86-225q \ge 86(1-3q)^{3/2}.$$

Using the substitution

$$t=\sqrt{1-3q}, \qquad 0 \le t \le 1,$$

the inequality becomes

$$86-75(1-t^2) \ge 86t^3,$$

$$11+75t^2-86t^3 \geq 0.$$

We have

$$11+75t^2-86t^3=(1-t)(11+11t+86t^2) \geq 0,$$

with equality for $t=1$. Notice that $t=1$ involves $q=0$, and (*) gives $r=-4/43$. Thus, if $p=1$, then the original inequality is an equality for $q=0$, $r=-4/43$ and $(a-b)(b-c)(c-a)>0$. More general, the equality holds for

$$ab+bc+ca=0, \quad 4(a+b+c)^3+43abc=0, \quad a+b+c>0, \quad (a-b)(b-c)(c-a)>0.$$

These equality conditions are equivalent to the condition that a,b,c satisfy

$$(a-b)(b-c)(c-a)>0$$

and are proportional to the roots of the equation

$$43w^3-43w^2+4=0.$$

□

P 2.11. *If a,b,c are real numbers such that*

$$a+b+c>0, \qquad ab+bc+ca \geq 0,$$

then

(a) $$2(a^3+b^3+c^3)+3(3\sqrt{3}-2)abc+6\sqrt{3}\,(a^2b+b^2c+c^2a) \geq 0;$$

(b) $$a^3+b^3+c^3-3abc \geq \frac{3\sqrt{3}}{2}(a-b)(b-c)(c-a).$$

(Vasile C., 2006)

Solution. Due to homogeneity, we may set

$$p=1, \qquad 0 \leq q \leq \frac{p^2}{3}=\frac{1}{3}.$$

(a) Write the inequality as

$$2(a^3+b^3+c^3)+3(3\sqrt{3}-2)abc+3\sqrt{3}\sum ab(a+b)+3\sqrt{3}\sum ab(a-b) \geq 0,$$

$$2(a^3+b^3+c^3)+3(3\sqrt{3}-2)abc+3\sqrt{3}\sum ab(a+b) \geq 3\sqrt{3}\,(a-b)(b-c)(c-a),$$

$$2(3r+p^3-3pq)-6r+3\sqrt{3}\,pq \geq 3\sqrt{3}\,(a-b)(b-c)(c-a),$$

$$2p^3-3(2-\sqrt{3})pq \geq 3\sqrt{3}\,(a-b)(b-c)(c-a),$$

$$2-3(2-\sqrt{3})q \geq 3\sqrt{3}(a-b)(b-c)(c-a).$$

It suffices to show that

$$2-3(2-\sqrt{3})q \ge 3\sqrt{3}|(a-b)(b-c)(c-a)|.$$

Applying Corollary 1 for $\beta = 0$, we have

$$|(a-b)(b-c)(c-a)| \le \frac{2}{3\sqrt{3}}(1-3q)^{3/2},$$

with equality for

$$27r = 9q-2. \tag{*}$$

Therefore, it suffices to show that

$$2-3(2-\sqrt{3})q \ge 2(1-3q)^{3/2}.$$

Using the substitution

$$t = \sqrt{1-3q}, \qquad 0 \le t \le 1,$$

the inequality becomes

$$2-(2-\sqrt{3})(1-t^2) \ge 2t^3,$$

$$\sqrt{3}+(2-\sqrt{3})t^2-2t^3 \ge 0,$$

$$(1-t)(\sqrt{3}+\sqrt{3}\ t+2t^2) \ge 0,$$

with equality for $t = 1$. Notice that $t = 1$ involves $q = 0$, and (*) gives $r = -2/27$. Thus, if $p = 1$, then the original inequality is an equality for $q = 0$, $r = -2/27$ and $(a-b)(b-c)(c-a) > 0$. More general, the equality holds for

$$ab+bc+ca = 0, \quad 2(a+b+c)^3+27abc = 0, \quad a+b+c > 0, \quad (a-b)(b-c)(c-a) > 0;$$

that is,

$$a = \frac{b}{1+\sqrt{3}} = \frac{c}{1-\sqrt{3}} > 0$$

(or any cyclic permutation).

(b) Write the inequality as

$$p^3-3pq \ge \frac{3\sqrt{3}}{2}(a-b)(b-c)(c-a),$$

$$1-3q \ge \frac{3\sqrt{3}}{2}(a-b)(b-c)(c-a).$$

It suffices to show that

$$1-3q \ge \frac{3\sqrt{3}}{2}|(a-b)(b-c)(c-a)|.$$

As shown at (a), we have

$$|(a-b)(b-c)(c-a)| \le \frac{2}{3\sqrt{3}}(1-3q)^{3/2}.$$

Therefore, it suffices to show that

$$1-3q \ge (1-3q)^{3/2}.$$

which is equivalent to

$$(1-3q)\left(1-\sqrt{1-3q}\right) \ge 0,$$

$$\frac{3q(1-3q)}{1+\sqrt{1-3q}} \ge 0.$$

The equality holds for $q = 1/3$ and $q = 0$. The original inequality is an equality for $a = b = c > 0$, and also for

$$ab+bc+ca = 0, \quad 2(a+b+c)^3+27abc = 0, \quad a+b+c > 0, \quad (a-b)(b-c)(c-a) > 0;$$

this means that

$$a = \frac{b}{1+\sqrt{3}} = \frac{c}{1-\sqrt{3}} > 0$$

(or any cyclic permutation).

Remark. The inequality (b) is sharper than the inequality (a), because the last inequalities can be obtained by summing the inequality in (b) to the obvious inequality

$$\frac{3\sqrt{3}}{2}(a+b+c)(ab+bc+ca) \ge 0.$$

□

P 2.12. *If a, b, c are real numbers such that*

$$a+b+c > 0, \qquad 2(a^2+b^2+c^2)+7(ab+bc+ca) \ge 0,$$

then

$$a^2(a-b)+b^2(b-c)+c^2(c-a) \ge 0.$$

(Vasile C., 2006)

Solution. Due to homogeneity, we may set $p = 1$. From $p^2 \ge 3q$ and

$$2(a^2+b^2+c^2)+7(ab+bc+ca) = 2p^2+3q = 2+3q \ge 0,$$

we get

$$\frac{-2}{3} \le q \le \frac{1}{3}.$$

Write the inequality as

$$2(a^3+b^3+c^3) \geq \sum ab(a+b) + \sum ab(a-b),$$

$$2(a^3+b^3+c^3)+3abc \geq (a+b+c)(ab+bc+ca)-(a-b)(b-c)(c-a),$$

$$2(3r+p^3-3pq)+3r \geq pq-(a-b)(b-c)(c-a),$$

$$2p^3-7pq \geq -9r-(a-b)(b-c)(c-a),$$

$$2-7q \geq -9r-(a-b)(b-c)(c-a).$$

It suffices to show that

$$2-7q \geq -9r+|(a-b)(b-c)(c-a)|.$$

Applying Corollary 1 for $\beta=-1/3$, we have

$$-9r+|(a-b)(b-c)(c-a)| \leq -3q+\frac{2}{3}+\frac{4}{3\sqrt{3}}(1-3q)^{3/2},$$

with equality for

$$\sqrt{3}\,(1-3q)^{3/2}=9q-27r-2. \tag{*}$$

Therefore, it suffices to show that

$$2-7q \geq -3q+\frac{2}{3}+\frac{4}{3\sqrt{3}}(1-3q)^{3/2},$$

which is equivalent to

$$\sqrt{3}(1-3q) \geq (1-3q)^{3/2},$$

$$(1-3q)(\sqrt{3}-\sqrt{1-3q}) \geq 0,$$

$$\frac{(1-3q)(2+3q)}{\sqrt{3}+\sqrt{1-3q}} \geq 0,$$

with equality for $q=1/3$ and $q=-2/3$. According to (*), $q=1/3$ involves $r=1/27$, and $q=-2/3$ involves $r=-17/27$. Thus, the equality holds for $a=b=c>0$, and also for

$$17(a+b+c)^3+27abc=0, \qquad 2(a+b+c)^2+3(ab+bc+ca)=0,$$

$$a+b+c>0, \qquad (a-b)(b-c)(c-a)<0.$$

The last equality conditions are equivalent to the condition that a,b,c satisfy

$$(a-b)(b-c)(c-a)<0$$

and are proportional to the roots of the equation

$$27w^3-27w^2-18w+17=0.$$

$\square$

P 2.13. *If a, b, c are real numbers such that*

$$a+b+c>0, \qquad 3(ab+bc+ca) \ge a^2+b^2+c^2,$$

then

$$a^2(a+2b-3c)+b^2(b+2c-3a)+c^2(c+2a-3b) \ge 0.$$

(Vasile C., 2006)

Solution. Due to homogeneity, we may set $p=1$. From $p^2 \ge 3q$ and

$$3(ab+bc+ca)-(a^2+b^2+c^2) = 3q-(p^2-2q) = 5q-1 \ge 0,$$

we get

$$\frac{1}{5} \le q \le \frac{1}{3}.$$

Write the inequality as follows:

$$2(a^3+b^3+c^3) \ge \sum ab(a+b) - 5\sum ab(a-b),$$

$$2(a^3+b^3+c^3)+3abc \ge (a+b+c)(ab+bc+ca)+5(a-b)(b-c)(c-a),$$

$$2(3r+p^3-3pq)+3r \ge pq+5(a-b)(b-c)(c-a),$$

$$2p^3-7pq \ge -9r+5(a-b)(b-c)(c-a),$$

$$2-7q \ge -9r+5(a-b)(b-c)(c-a).$$

It suffices to show that

$$\frac{2-7q}{5} \ge \frac{-9r}{5} + |(a-b)(b-c)(c-a)|.$$

Applying Corollary 1 for $\beta = -1/15$, we have

$$\frac{-9r}{5} + |(a-b)(b-c)(c-a)| \le \frac{-3q}{5} + \frac{2}{15} + \frac{4}{15}\sqrt{\frac{7}{3}}\,(1-3q)^{3/2},$$

with equality for

$$\sqrt{3}\,(1-3q)^{3/2} = \sqrt{7}\,(9q-27r-2). \tag{*}$$

Therefore, it suffices to show that

$$\frac{2-7q}{5} \ge \frac{-3q}{5} + \frac{2}{15} + \frac{4}{15}\sqrt{\frac{7}{3}}\,(1-3q)^{3/2},$$

which is equivalent to

$$\sqrt{3}\,(1-3q) \ge \sqrt{7}\,(1-3q)^{3/2},$$

$$(1-3q)\left[\sqrt{3}-\sqrt{7(1-3q)}\;\right] \ge 0.$$

This is true if

$$3 \ge 7(1-3q),$$

which is equivalent to $q \ge \frac{4}{21}$. Indeed, we have

$$q \ge \frac{1}{5} > \frac{4}{21}.$$

The equality holds for $a = b = c > 0$.

□

P 2.14. *If a, b, c are real numbers, then*

$$(a^2+b^2+c^2)^2 \ge 3(a^3b+b^3c+c^3a).$$

(Vasile C., 1992)

Solution. Write the inequality as follows:

$$2(a^2+b^2+c^2)^2 \ge 3\sum ab(a^2+b^2)+3\sum ab(a^2-b^2),$$

$$2\left(\sum a^2\right)^2 \ge 3\left(\sum ab\right)\left(\sum a^2\right)-3abc\sum a-3\left(\sum a\right)(a-b)(b-c)(c-a),$$

$$2(p^2-2q)^2-3q(p^2-2q) \ge -3pr-3p(a-b)(b-c)(c-a),$$

$$2p^4-11p^2q+14q^2 \ge -3pr-3p(a-b)(b-c)(c-a).$$

It suffices to show that

$$2p^4-11p^2q+14q^2 \ge 3p\left[-r+|(a-b)(b-c)(c-a)|\right].$$

Since the inequality remains unchanged by replacing a, b, c with $-a, -b, -c$, respectively, we may assume that $p \ge 0$. For the nontrivial case $p \ne 0$, we may set $p = 1$ (due to homogeneity). Thus, we need to show that

$$\frac{2-11q+14q^2}{3} \ge -r+|(a-b)(b-c)(c-a).$$

Applying Corollary 1 for $\beta = -1/27$, we have

$$-r+|(a-b)(b-c)(c-a)| \le \frac{-q}{3}+\frac{2}{27}+\frac{4\sqrt{7}}{27}\,(1-3q)^{3/2},$$

with equality for

$$(1-3q)^{3/2} = \sqrt{7}\,(9q-27r-2). \tag{*}$$

Therefore, it suffices to show that

$$\frac{2-11q+14q^2}{3} \geq \frac{-q}{3} + \frac{2}{27} + \frac{4\sqrt{7}}{27}(1-3q)^{3/2},$$

which is equivalent to

$$8-45q+63q^2 \geq 2\sqrt{7}(1-3q)^{3/2},$$

$$(1-3q)(8-21q) \geq 2\sqrt{7}(1-3q)^{3/2},$$

$$(1-3q)\left[8-21q-2\sqrt{7(1-3q)}\right] \geq 0.$$

Since $1-3q \geq 0$, this is true if

$$(8-21q)^2 \geq 28(1-3q),$$

which is equivalent to

$$(7q-2)^2 \geq 0.$$

According to (*), $q = 1/3$ involves $r = 1/27$, and $q = 2/7$ involves $r = -1/49$. Thus, the original inequality is an equality for $a = b = c$, and also for

$$2p^2 - 7q = 0, \qquad p^3 + 49r = 0, \qquad (a+b+c)(a-b)(b-c)(c-a) \leq 0.$$

The last equality conditions are equivalent to the condition that a, b, c satisfy

$$(a-b)(b-c)(c-a) \leq 0$$

and are proportional to the roots of the equation

$$49w^3 - 49w^2 + 14w + 1 = 0;$$

this means that

$$\frac{a}{\sin^2 \frac{4\pi}{7}} = \frac{b}{\sin^2 \frac{2\pi}{7}} = \frac{c}{\sin^2 \frac{\pi}{7}}$$

(or any cyclic permutation).

□

P 2.15. *If a, b, c are real numbers, then*

$$a^3(a-2b+c) + b^3(b-2c+a) + c^3(c-2a+b) \geq 0.$$

(Vasile C., 1998)

Solution. Write the inequality as follows:

$$2(a^4+b^4+c^4) \ge \sum ab(a^2+b^2)+3\sum ab(a^2-b^2),$$

$$2\sum a^4 \ge \left(\sum ab\right)\left(\sum a^2\right)-abc\sum a-3\left(\sum a\right)(a-b)(b-c)(c-a),$$

$$2(p^4-4p^2q+2q^2+4pr) \ge q(p^2-2q)-pr-3p(a-b)(b-c)(c-a),$$

$$2p^4-9p^2q+6q^2 \ge -9pr-3p(a-b)(b-c)(c-a).$$

Since the inequality remains unchanged by replacing a,b,c with $-a,-b,-c$, respectively, we may assume that $p \ge 0$. For the nontrivial case $p>0$, it suffices to show that

$$2p^4-9p^2q+6q^2 \ge 3p\left[-3r+|(a-b)(b-c)(c-a)|\right].$$

Due to homogeneity, we may set $p=1$. Thus, we need to show that

$$\frac{2-9q+6q^2}{3} \ge -3r+|(a-b)(b-c)(c-a).$$

Applying Corollary 1 for $\beta=-1/9$, we have

$$-3r+|(a-b)(b-c)(c-a)| \le -q+\frac{2}{9}+\frac{4}{9}(1-3q)^{3/2},$$

with equality for

$$(1-3q)^{3/2}=9q-27r-2. \tag{*}$$

Therefore, it suffices to show that

$$\frac{2-9q+6q^2}{3} \ge -q+\frac{2}{9}+\frac{4}{9}(1-3q)^{3/2},$$

which is equivalent to

$$2-9q+9q^2 \ge 2(1-3q)^{3/2},$$

$$(1-3q)(2-3q) \ge 2(1-3q)^{3/2},$$

$$(1-3q)\left[2-3q-2\sqrt{1-3q}\,\right] \ge 0.$$

Since $1-3q \ge 0$, this is true if

$$(2-3q)^2 \ge 4(1-3q),$$

which is equivalent to $q^2 \ge 0$. According to (*), $q=1/3$ involves $r=1/27$, and $q=0$ involves $r=-1/9$. Thus, the original inequality is an equality for $a=b=c$, and also for

$$q=0, \qquad p^3+9r=0, \qquad (a+b+c)(a-b)(b-c)(c-a) \le 0.$$

The last equality conditions are equivalent to the condition that a, b, c satisfy

$$(a-b)(b-c)(c-a) \le 0$$

and are proportional to the roots of the equation

$$9w^3 - 9w^2 + 1 = 0;$$

this means that

$$a \sin\frac{\pi}{9} = b \sin\frac{7\pi}{9} = c \sin\frac{13\pi}{9}$$

(or any cyclic permutation).

□

P 2.16. *If a, b, c are real numbers, then*

$$a^4 + b^4 + c^4 - abc(a+b+c) \ge \sqrt{3}\,(a+b+c)(a-b)(b-c)(c-a).$$

(Vasile C., 2006)

Solution. Since the inequality remains unchanged by replacing a, b, c with $-a, -b, -c$, respectively, we may assume that $p \ge 0$. For the nontrivial case $p > 0$, it suffices to show that

$$a^4 + b^4 + c^4 - abc(a+b+c) \ge \sqrt{3}\,(a+b+c)|(a-b)(b-c)(c-a)|,$$

which is equivalent to

$$p^4 - 4p^2q + 2q^2 + 3pr \ge \sqrt{3}\,p|(a-b)(b-c)(c-a)|.$$

Due to homogeneity, we may set $p = 1$. Thus, we need to show that

$$1 - 4q + 2q^2 \ge \sqrt{3}\left[-\sqrt{3}\,r + |(a-b)(b-c)(c-a)|\right].$$

Applying Corollary 1 for $\beta = -\sqrt{3}/27$, we have

$$-\sqrt{3}\,r + |(a-b)(b-c)(c-a)| \le \frac{-\sqrt{3}\,q}{3} + \frac{2\sqrt{3}}{27} + \frac{2\sqrt{10}}{3\sqrt{3}}\,(1-3q)^{3/2},$$

with equality for

$$2(1-3q)^{3/2} = \sqrt{10}\,(9q - 27r - 2). \tag{*}$$

Therefore, it suffices to show that

$$1 - 4q + 2q^2 \ge -q + \frac{2}{9} + \frac{2\sqrt{10}}{9}\,(1-3q)^{3/2},$$

which is equivalent to

$$7-27q+18q^2 \ge 2\sqrt{10}\,(1-3q)^{3/2},$$

$$(1-3q)(7-6q) \ge 2\sqrt{10}\,(1-3q)^{3/2},$$

$$(1-3q)\left[7-6q-2\sqrt{10(1-3q)}\,\right] \ge 0.$$

Since $1-3q \ge 0$, this is true if

$$(7-6q)^2 \ge 40(1-3q),$$

which is equivalent to

$$(2q+1)^2 \ge 0.$$

According to (*), $q = 1/3$ involves $r = 1/27$, and $q = -1/2$ involves $r = -1/3$. Thus, the original inequality is an equality for $a = b = c$, and also for

$$(a+b+c)^2+2(ab+bc+ca) = 0, \quad (a+b+c)^3+3abc = 0, \quad (a+b+c)(a-b)(b-c)(c-a) \ge 0.$$

The last equality conditions are equivalent to the condition that a, b, c satisfy

$$(a-b)(b-c)(c-a) \ge 0$$

and are proportional to the roots of the equation

$$6w^3-6w^2-3w+2 = 0.$$

□

P 2.17. *If a, b, c are real numbers, then*

$$a^3(a+b)+b^3(b+c)+c^3(c+a) \ge \frac{2}{3}(ab+bc+ca)^2.$$

(Vasile C., 2006)

Solution. Write the inequality as follows:

$$2(a^4+b^4+c^4)+\sum ab(a^2+b^2)+\sum ab(a^2-b^2) \ge \frac{4}{3}(ab+bc+ca)^2,$$

$$2(a^4+b^4+c^4)+\sum ab(a^2+b^2)-(a+b+c)(a-b)(b-c)(c-a) \ge \frac{4}{3}(ab+bc+ca)^2,$$

$$2(p^4-4p^2q+2q^2+4pr)+q(p^2-2q)-pr-p(a-b)(b-c)(c-a) \ge \frac{4}{3}q^2,$$

$$2p^4-7p^2q+\frac{2}{3}q^2+7pr \ge p(a-b)(b-c)(c-a).$$

Since the statement remains unchanged by replacing a, b, c with $-a, -b, -c$, respectively, we may assume $p \geq 0$. For the nontrivial case $p > 0$, it suffices to show that

$$2p^4 - 7p^2q + \frac{2}{3}q^2 + 7pr \geq p|(a-b)(b-c)(c-a)|.$$

Due to homogeneity, we may set $p = 1$. From $p^2 \geq 3q$, we get

$$q \leq \frac{1}{3}.$$

Thus, we need to show that

$$2 - 7q + \frac{2}{3}q^2 \geq -7r + |(a-b)(b-c)(c-a)|.$$

Applying Corollary 1 for $\beta = \dfrac{-7}{27}$, we have

$$-7r + |(a-b)(b-c)(c-a)| \leq \frac{-7q}{3} + \frac{14}{27} + \frac{4\sqrt{19}}{27}(1-3q)^{3/2},$$

with equality for

$$7(1-3q)^{3/2} = \sqrt{19}(9q - 27r - 2). \tag{*}$$

Therefore, it suffices to show that

$$2 - 7q + \frac{2}{3}q^2 \geq \frac{-7q}{3} + \frac{14}{27} + \frac{4\sqrt{19}}{27}(1-3q)^{3/2},$$

which is equivalent to

$$20 - 63q + 9q^2 \geq 2\sqrt{19}(1-3q)^{3/2},$$

$$(1-3q)(20-3q) \geq 2\sqrt{19}(1-3q)^{3/2},$$

$$(1-3q)\left[20 - 3q - 2\sqrt{19(1-3q)}\,\right] \geq 0.$$

Since $1 - 3q \geq 0$, this is true if

$$(20-3q)^2 \geq 76(1-3q),$$

which is equivalent to

$$(q+6)^2 \geq 0,$$

According to (*), $q = 1/3$ involves $r = 1/27$, and $q = -6$ involves $r = -7$. Thus, the original inequality is an equality for $a = b = c$, and also for

$$ab+bc+ca+6(a+b+c)^2 = 0, \quad abc+7(a+b+c)^3 = 0, \quad (a+b+c)(a-b)(b-c)(c-a) \geq 0.$$

The last equality conditions are equivalent to the condition that a,b,c satisfy

$$(a-b)(b-c)(c-a)\geq 0$$

and are proportional to the roots of the equation

$$w^3-w^2-6w+7=0.$$

□

P 2.18. *If a,b,c are real numbers such that $ab+bc+ca=3$, then*

$$a^3(a-2b)+b^3(b-2c)+c^3(c-2a)+3\geq 0.$$

(Vasile C., 2006)

Solution. Write the inequality in the homogeneous form

$$a^4+b^4+c^4-\sum ab(a^2+b^2)-\sum ab(a^2-b^2)+\frac{1}{3}(ab+bc+ca)^2\geq 0,$$

which is equivalent to

$$a^4+b^4+c^4-\sum ab(a^2+b^2)+\frac{1}{3}(ab+bc+ca)^2\geq \sum ab(a^2-b^2),$$

$$(p^4-4p^2q+2q^2+4pr)-q(p^2-2q)+pr+\frac{1}{3}q^2\geq -p(a-b)(b-c)(c-a),$$

$$p^4-5p^2q+\frac{13}{3}q^2+5pr\geq -p(a-b)(b-c)(c-a).$$

Since the inequality remains unchanged by replacing a,b,c with $-a,-b,-c$, respectively, we may assume $p\geq 0$. For the nontrivial case $p>0$, it suffices to show that

$$p^4-5p^2q+\frac{13}{3}q^2+5pr\geq p|(a-b)(b-c)(c-a)|.$$

Due to homogeneity, we may leave out the hypothesis $ab+bc+ca=3$ for $p=1$. From $p^2\geq 3q$, we get

$$q\leq\frac{1}{3}.$$

Thus, we need to show that

$$1-5q+\frac{13}{3}q^2\geq -5r+|(a-b)(b-c)(c-a)|.$$

Applying Corollary 1 for $\beta=\frac{-5}{27}$, we have

$$-5r+|(a-b)(b-c)(c-a)|\leq\frac{-5q}{3}+\frac{10}{27}+\frac{4\sqrt{13}}{27}(1-3q)^{3/2},$$

with equality for

$$5(1-3q)^{3/2} = \sqrt{13}(9q-27r-2). \tag{*}$$

Therefore, it suffices to show that

$$1-5q+\frac{13}{3}q^2 \ge \frac{-5q}{3}+\frac{10}{27}+\frac{4\sqrt{13}}{27}(1-3q)^{3/2},$$

which is equivalent to

$$17-90q+117q^2 \ge 4\sqrt{13}(1-3q)^{3/2},$$

$$(1-3q)(17-39q) \ge 4\sqrt{13}(1-3q)^{3/2},$$

$$(1-3q)\left[17-39q-4\sqrt{13(1-3q)}\right] \ge 0.$$

Since $1-3q \ge 0$, this is true if

$$(17-39q)^2 \ge 208(1-3q),$$

which is equivalent to

$$(13q-3)^2 \ge 0,$$

According to (*), $q = 1/3$ involves $r = 1/27$, and $q = 3/13$ involves $r = -1/169$. Thus, the homogeneous inequality is an equality for $a = b = c$, and also for

$$13(ab+bc+ca) = 3(a+b+c)^2 = 0, \qquad 169abc+(a+b+c)^3 = 0,$$

$$(a+b+c)(a-b)(b-c)(c-a) < 0.$$

Since $ab+bc+ca = 3$, the original inequality is an equality for

$$a+b+c = \sqrt{13}, \qquad abc = \frac{-1}{\sqrt{13}}, \qquad (a-b)(b-c)(c-a) < 0.$$

The last equality conditions are equivalent to the condition that a, b, c satisfy

$$(a-b)(b-c)(c-a) < 0$$

and are the roots of the equation

$$w^3-\sqrt{13}\,w^2+3w+\frac{1}{\sqrt{13}} = 0.$$

Remark 1. The inequalities in P 2.14, ... , P2.18 are particular cases of the following generalization:

- *Let A, B, C, D be real numbers such that*

$$1+A+B+C+D = 0, \qquad 3(1+A) \ge C^2+CD+D^2.$$

Then, for any real a,b,c, the following inequality holds:

$$\sum a^4+A\sum a^2b^2+Babc\sum a+C\sum a^3b+D\sum ab^3\ge 0.$$

To prove this inequality, we write it in the form

$$\sum a^4+A\Big(\sum a^2b^2-abc\sum a\Big)-(1+C+D)abc\sum a+C\sum a^3b+D\sum ab^3\ge 0.$$

Since $\sum a^2b^2-abc\sum a\ge 0$, it suffices to consider the case

$$3(1+A)=C^2+CD+D^2.$$

Proceeding as in the preceding problems (with $p=1$), we need to show that

$$7+2A+C+D-3(2+A-C-D)q\ge$$

$$\ge 2\sqrt{(1-A)^2-(1-A)(C+D)+C^2-CD+D^2}\cdot\sqrt{1-3q}.$$

Since

$$1+A=\frac{C^2+CD+D^2}{3}\ge\frac{(C+D)^2}{4},$$

we have

$$\begin{aligned}2+A-C-D=(1+A)+1-C-D&\ge\frac{(C+D)^2}{4}+1-C-D\\&=\frac{(C+D-2)^2}{4}\ge 0,\end{aligned}$$

hence

$$\begin{aligned}7+2A+C+D-3(2+A-C-D)q&\ge 7+2A+C+D-(2+A-C-D)\\&=(1+A)+4+2(C+D)\\&\ge\frac{(C+D)^2}{4}+4+2(C+D)\\&=\frac{(C+D+4)^2}{4}\ge 0.\end{aligned}$$

Thus, we only need to show that

$$[7+2A+C+D-3(2+A-C-D)q]^2\ge$$

$$4\big[(1-A)^2-(1-A)(C+D)+C^2-CD+D^2\big](1-3q),$$

which is equivalent to

$$\big[(2+A-C-D)^2q-4-5A+(1+A)(C+D)+C^2+D^2\big]^2\ge 0.$$

Remark 2. Replacing A,B,C,D with $A/A_0,B/A_0,C/A_0,D/A_0$, respectively, the statement from Remark 1 becomes as follows:

- *Let A_0,A,B,C,D be real numbers such that*

$$A_0>0, \qquad A_0+A+B+C+D=0, \qquad 3A_0(A_0+A)\geq C^2+CD+D^2.$$

Then, for any real a,b,c, the following inequality holds:

$$A_0\sum a^4+A\sum a^2b^2+Babc\sum a+C\sum a^3b+D\sum ab^3\geq 0.$$

$\square$

P 2.19. *If a,b,c are real numbers such that $ab+bc+ca\geq 0$, then*

$$a^4+b^4+c^4-abc(a+b+c)\geq \frac{\sqrt{15}}{2}(a+b+c)(a-b)(b-c)(c-a).$$

(Vasile C., 2006)

First Solution. Since the statement remains unchanged by replacing a,b,c with $-a,-b,-c$, respectively, we may assume that $p\geq 0$. For the nontrivial case $p>0$, it suffices to show that

$$a^4+b^4+c^4-abc(a+b+c)\geq \frac{\sqrt{15}}{2}(a+b+c)|(a-b)(b-c)(c-a)|,$$

which is equivalent to

$$p^4-4p^2q+2q^2+3pr\geq \frac{\sqrt{15}}{2}p|(a-b)(b-c)(c-a)|.$$

Due to homogeneity, we may set $p=1$. From $p^2\geq 3q$, we get

$$0\leq q\leq \frac{1}{3}.$$

We need to show that

$$1-4q+2q^2\geq \frac{\sqrt{15}}{2}\left[\frac{-6r}{\sqrt{15}}+|(a-b)(b-c)(c-a)|\right].$$

Applying Corollary 1 for $\beta=\dfrac{-2}{9\sqrt{15}}$, we have

$$\frac{-6r}{\sqrt{15}}+|(a-b)(b-c)(c-a)|\leq \frac{-2q}{\sqrt{15}}+\frac{4}{9\sqrt{15}}+\frac{14}{9\sqrt{15}}(1-3q)^{3/2},$$

with equality for

$$4(1-3q)^{3/2} = 7(9q-27r-2). \tag{*}$$

Therefore, it suffices to show that

$$1-4q+2q^2 \geq \frac{\sqrt{15}}{2}\left[\frac{-2q}{\sqrt{15}} + \frac{4}{9\sqrt{15}} + \frac{14}{9\sqrt{15}}\,(1-3q)^{3/2}\right],$$

which is equivalent to

$$7-27q+18q^2 \geq 7(1-3q)^{3/2},$$

$$(1-3q)(7-6q) \geq 7(1-3q)^{3/2},$$

$$(1-3q)\left(7-6q-7\sqrt{1-3q}\ \right) \geq 0.$$

Since $1-3q \geq 0$, this is true if

$$(7-6q)^2 \geq 49(1-3q),$$

which is equivalent to

$$q(4q+7) \geq 0,$$

According to (*), $q = 1/3$ involves $r = 1/27$, and $q = 0$ involves $r = -2/21$. Thus, the original inequality is an equality for $a = b = c$, and also for

$$ab+bc+ca = 0, \quad 21abc+2(a+b+c)^3 = 0, \quad (a+b+c)(a-b)(b-c)(c-a) \geq 0.$$

The last equality conditions are equivalent to the condition that a, b, c satisfy

$$(a-b)(b-c)(c-a) \geq 0$$

and are proportional to the roots of the equation

$$21w^3 - 21w^2 + 2 = 0.$$

Second Solution. We will find a stronger inequality

$$f(a,b,c) \geq 0,$$

where

$$\begin{aligned} f(a,b,c) =& a^4+b^4+c^4 - abc(a+b+c) + \frac{\sqrt{15}}{2}\left(\sum a^3b - \sum ab^3\right) \\ & - k(ab+bc+ca)(a^2+b^2+c^2-ab-bc-ca), \quad k > 0, \end{aligned}$$

satisfies $f(1,1,1) = 0$. Since

$$(ab+bc+ca)(a^2+b^2+c^2-ab-bc-ca) = -\sum a^2b^2 - abc\sum a + \sum a^3b + \sum ab^3,$$

the inequality can be written as

$$\sum a^4 + A\sum a^2b^2 + Babc\sum a + C\sum a^3b + D\sum ab^3 \ge 0,$$

where

$$A = k, \quad B = k-1, \quad C = -k + \frac{\sqrt{15}}{2}, \quad D = -k - \frac{\sqrt{15}}{2}.$$

We see that $1+A+B+C+D=0$. According to the statement in Remark 1 from P 2.18, if

$$3(1+A) = C^2 + CD + D^2,$$

then the inequality holds for all real a, b, c. It is easy to show that this condition is satisfied for $k = 1/2$. Since the inequality $f(a, b, c) \ge 0$ for $k = 1/2$ is stronger than the original inequality, the proof is completed.

□

P 2.20. *If a, b, c are real numbers such that*

$$2(ab + bc + ca) \ge a^2 + b^2 + c^2,$$

then

$$a^4 + b^4 + c^4 - abc(a + b + c) \ge \frac{\sqrt{39}}{2}(a + b + c)(a - b)(b - c)(c - a).$$

(Vasile C., 2006)

Solution. Since the statement remains unchanged by replacing a, b, c with $-a, -b, -c$, respectively, we may assume that $p \ge 0$. For the nontrivial case $p > 0$, it suffices to show that

$$a^4 + b^4 + c^4 - abc(a + b + c) \ge \frac{\sqrt{39}}{2}(a + b + c)|(a - b)(b - c)(c - a)|,$$

which is equivalent to

$$p^4 - 4p^2q + 2q^2 + 3pr \ge \frac{\sqrt{39}}{2}p|(a - b)(b - c)(c - a)|.$$

Due to homogeneity, we may set $p = 1$. From $p^2 \ge 3q$ and $2(ab + bc + ca) \ge a^2 + b^2 + c^2$, we get

$$\frac{1}{4} \le q \le \frac{1}{3}.$$

We need to show that

$$1 - 4q + 2q^2 \ge \frac{\sqrt{39}}{2}\left[\frac{-6r}{\sqrt{39}} + |(a - b)(b - c)(c - a)|\right].$$

Applying Corollary 1 for $\beta = \dfrac{-2}{9\sqrt{39}}$, we have

$$\frac{-6r}{\sqrt{39}} + |(a-b)(b-c)(c-a)| \le \frac{-2q}{\sqrt{39}} + \frac{4}{9\sqrt{39}} + \frac{22}{9\sqrt{39}}\,(1-3q)^{3/2},$$

with equality for

$$4(1-3q)^{3/2} = 11(9q-27r-2). \tag{*}$$

Therefore, it suffices to show that

$$1-4q+2q^2 \ge \frac{\sqrt{39}}{2}\left[\frac{-2q}{\sqrt{39}} + \frac{4}{9\sqrt{39}} + \frac{22}{9\sqrt{39}}\,(1-3q)^{3/2}\right],$$

which is equivalent to

$$7-27q+18q^2 \ge 11(1-3q)^{3/2},$$

$$(1-3q)(7-6q) \ge 11(1-3q)^{3/2},$$

$$(1-3q)\left(7-6q-11\sqrt{1-3q}\,\right) \ge 0.$$

Since $1-3q \ge 0$, this is true if

$$(7-6q)^2 \ge 121(1-3q),$$

which is equivalent to

$$(4q-1)(q+8) \ge 0,$$

According to (*), $q = 1/3$ involves $r = 1/27$, and $q = 1/4$ involves $r = 1/132$. Thus, the original inequality is an equality for $a = b = c$, and also for

$$4(ab+bc+ca) = (ab+c)^2, \quad 132abc+(a+b+c)^3 = 0, \quad (a+b+c)(a-b)(b-c)(c-a) > 0.$$

The last equality conditions are equivalent to the condition that a, b, c satisfy

$$(a-b)(b-c)(c-a) \ge 0$$

and are proportional to the roots of the equation

$$132w^3 - 132w^2 + 33w - 1 = 0.$$

Second Solution. We will find a stronger inequality

$$f(a,b,c) \ge 0,$$

where

$$\begin{aligned} f(a,b,c) =& a^4+b^4+c^4 - abc(a+b+c) + \frac{\sqrt{15}}{2}\left(\sum a^3b - \sum ab^3\right) \\ & - k\left(2\sum ab - \sum a^2\right)\left(\sum a^2 - \sum ab\right), \quad k > 0, \end{aligned}$$

satisfies $f(1,1,1)=0$. Since

$$-\left(2\sum ab-\sum a^2\right)\left(\sum a^2-\sum ab\right)=$$

$$=\left(\sum a^2\right)^2-3\left(\sum ab\right)\left(\sum a^2\right)+2\left(\sum ab\right)^2$$

$$=\sum a^4+4\sum a^2b^2+abc\sum a-3\sum a^3b-3\sum ab^3,$$

the inequality $f(a,b,c)\ge 0$ can be written as

$$A_0\sum a^4+A\sum a^2b^2+Babc\sum a+C\sum a^3b+D\sum ab^3\ge 0,$$

where

$$A_0=k+1,\quad A=4k,\quad B=k-1,\quad C=-3k+\frac{\sqrt{39}}{2},\quad D=-3k-\frac{\sqrt{39}}{2}.$$

We see that $A_0+A+B+C+D=0$. According to the statement in Remark 2 from P 2.18, if $A_0>0$ and

$$3A_0(A_0+A)=C^2+CD+D^2,$$

then the inequality holds for all real a,b,c. It is easy to show that this condition is satisfied for $k=3/4$. Since the inequality $f(a,b,c)\ge 0$ for $k=3/4$ is stronger than the original inequality, the proof is completed.

$\square$

P 2.21. *If a,b,c are real numbers, then*

$$a^4+b^4+c^4+2abc(a+b+c)\ge ab^3+bc^3+ca^3.$$

(Vasile C., 2009)

First Solution. Write the inequality as follows:

$$2(a^4+b^4+c^4)+4abc(a+b+c)\ge\sum ab(a^2+b^2)-\sum ab(a^2-b^2)\ge 0,$$

$$2(p^4-4p^2q+2q^2+4pr)+4pr\ge q(p^2-2q)-pr+p(a-b)(b-c)(c-a)\ge 0,$$

$$2p^4-9p^2q+6q^2\ge -13pr+p(a-b)(b-c)(c-a)\ge 0,$$

Since the statement remains unchanged by replacing a,b,c with $-a,-b,-c$, respectively, we may assume $p\ge 0$. For $p=0$, the inequality is clearly true. For $p>0$, it suffices to show that

$$2p^4-9p^2q+6q^2\ge -13pr+p|(a-b)(b-c)(c-a)|\ge 0.$$

Due to homogeneity, we may set $p = 1$. From $p^2 \ge 3q$, we get

$$q \le \frac{1}{3}.$$

Thus, we need to show that

$$2 - 9q + 6q^2) \ge -13r + |(a-b)(b-c)(c-a)| \ge 0,$$

Applying Corollary 1 for $\beta = \dfrac{-13}{27}$, we have

$$-13r + |(a-b)(b-c)(c-a)| \le \frac{-13q}{3} + \frac{26}{27} + \frac{28}{27}\,(1-3q)^{3/2},$$

with equality for

$$13(1-3q)^{3/2} = 7(9q - 2 - 27r). \tag{*}$$

Therefore, it suffices to show that

$$2 - 9q + 6q^2 \ge \frac{-13q}{3} + \frac{26}{27} + \frac{28}{27}\,(1-3q)^{3/2},$$

which is equivalent to

$$14 - 63q + 81q^2 \ge 14(1-3q)^{3/2}.$$

Using the substitution

$$t = \sqrt{1-3q}, \qquad t \ge 0,$$

the inequality becomes

$$9t^4 - 14t^3 + 3t^2 + 2 \ge 0,$$

which is equivalent to

$$(t-1)^2(9t^2 + 4t + 2) \ge 0.$$

The last inequality is true, with equality for $t = 1$, that is for $q = 0$. From (*), we get $r = -1/7$. Thus, the original inequality is an equality when a, b, c satisfy

$$(a-b)(b-c)(c-a) \ge 0$$

and are proportional to the roots of the equation

$$7w^3 - 7w^2 + 1 = 0.$$

The equality conditions are equivalent to

$$\frac{a}{\sin\frac{2\pi}{7}} = \frac{b}{\sin\frac{4\pi}{7}} = \frac{c}{\sin\frac{8\pi}{7}}$$

(or any cyclic permutation).

Second Solution. We will find a stronger inequality

$$f(a,b,c) \ge 0,$$

where

$$f(a,b,c) = a^4+b^4+c^4+2abc(a+b+c)-(ab^3+bc^3+ca^3)-\frac{2}{27}[p^2+m(p^2-3q)]^2, \quad m \in \mathbb{R},$$

satisfies $f(1,1,1) = 0$. The inequality can be written as

$$\sum a^4 + A\sum a^2b^2 + Babc\sum a + C\sum a^3b + D\sum ab^3 \ge 0,$$

where $1+A+B+C+D = 0$. According to the statement in Remark 1 from P 2.18, if

$$3(1+A) = C^2 + CD + D^2,$$

then the inequality $f(a,b,c) \ge 0$ holds for all real a,b,c. It is easy to show that this condition is satisfied for $m = -1$, when

$$f(a,b,c) = a^4 + b^4 + c^4 + 2abc(a+b+c) - (ab^3 + bc^3 + ca^3) - \frac{2}{3}(ab + bc + ca)^2.$$

□

P 2.22. *If a,b,c are real numbers, then*

$$a^4 + b^4 + c^4 + \sqrt{2}\,(a^3b + b^3c + c^3a) \ge 0.$$

(Vasile C., 2009)

Solution. We will prove a stronger inequality

$$f(a,b,c) \ge 0,$$

where

$$f(a,b,c) = a^4 + b^4 + c^4 + k(a^3b + b^3c + c^3a) - \frac{1+k}{27}(a+b+c)^4, \qquad k = \sqrt{2},$$

satisfies $f(1,1,1) = 0$. Since

$$(a+b+c)^4 = \sum a^4 + 6\sum a^2b^2 + 12abc\sum a + 4\sum a^3b + 4\sum ab^3,$$

the inequality can be written as

$$A_0\sum a^4 + A\sum a^2b^2 + Babc\sum a + C\sum a^3b + D\sum ab^3 \ge 0,$$

where

$$A_0 = 26 - k, \quad A = -6(1+k), \quad B = -12(1+k), \quad C = 23k - 4, \quad D = -4(1+k).$$

We see that $A_0 + A + B + C + D = 0$. According to the statement in Remark 2 from P 2.18, if $A_0 > 0$ and

$$3A_0(A_0 + A) \geq C^2 + CD + D^2,$$

then the inequality holds for all real a, b, c. We have

$$3A_0(A_0 + A) - (C^2 + CD + D^2) = 54(28 - 7k - 8k^2) = 54(12 - 7\sqrt{2}) > 0.$$

Thus, the proof is completed. The equality holds for $a = b = c = 0$.

Remark. From the proof above, it follows that the inequality

$$a^4 + b^4 + c^4 + k(a^3b + b^3c + c^3a) \geq 0$$

holds for all real a, b, c and

$$-1 \leq k \leq \frac{3\sqrt{105} - 7}{16} \approx 1.4838,$$

where $k = \dfrac{3\sqrt{105} - 7}{16}$ is a root of the equation $28 - 7k - 8k^2 = 0$. Actually, the inequality holds for

$$-1 \leq k \leq k_0,$$

where $k_0 \approx 1.4894$ is a root of the equation

$$7k^5 + 17k^4 + 16k^3 + 16k^2 - 64k - 128 = 0.$$

□

P 2.23. *If a, b, c are real numbers, then*

$$(a^2 + b^2 + c^2)^2 + 2(a^3b + b^3c + c^3a) \geq 3(ab^3 + bc^3 + ca^3).$$

(Vasile C., 2009)

First Solution. Write the inequality as follows:

$$2(a^2 + b^2 + c^2)^2 \geq \sum ab(a^2 + b^2) - 5\sum ab(a^2 - b^2) \geq 0,$$

$$2(p^2 - 2q)^2 \geq q(p^2 - 2q) - pr + 5p(a - b)(b - c)(c - a),$$

$$2p^4 - 9p^2q + 10q^2 \geq -pr + 5p(a - b)(b - c)(c - a) \geq 0,$$

Since the statement remains unchanged by replacing a, b, c with $-a, -b, -c$, respectively, we may assume $p \geq 0$. For $p = 0$, the inequality is clearly true. For $p > 0$, it suffices to show that

$$\frac{2p^4 - 9p^2q + 10q^2}{5} \geq -\frac{1}{5}pr + p|(a-b)(b-c)(c-a)| \geq 0.$$

Due to homogeneity, we may set $p = 1$. From $p^2 \geq 3q$, we get

$$q \leq \frac{1}{3}.$$

Thus, we need to show that

$$\frac{2 - 9q + 10q^2}{5} \geq -\frac{1}{5}r + |(a-b)(b-c)(c-a)| \geq 0.$$

Applying Corollary 1 for $\beta = \dfrac{-1}{135}$, we have

$$-\frac{1}{5}pr + |(a-b)(b-c)(c-a)| \leq \frac{-5q}{3} + \frac{10}{27} + \frac{52}{135}(1-3q)^{3/2},$$

with equality for

$$(1-3q)^{3/2} = 13\,(9q - 2 - 27r). \tag{*}$$

Therefore, it suffices to show that

$$\frac{2 - 9q + 10q^2}{5} \geq \frac{-q}{15} + \frac{2}{135} + \frac{52}{135}(1-3q)^{3/2},$$

which is equivalent to

$$26 - 117q + 135q^2 \geq 26(1-3q)^{3/2}.$$

Using the substitution

$$t = \sqrt{1-3q}, \qquad t \geq 0,$$

the inequality becomes

$$15(t^2-1)^2 \geq 13(2t^3 - 3t^2 + 1) \geq 0,$$

which is equivalent to

$$(t-1)^2(15t^2 + 4t + 2) \geq 0.$$

The last inequality is true, with equality for $t = 1$, that is $q = 0$. From (*), we get $r = -1/13$. Thus, the original inequality is an equality when a, b, c satisfy

$$(a-b)(b-c)(c-a) \geq 0$$

and are proportional to the roots of the equation

$$13w^3 - 13w^2 + 1 = 0.$$

Second Solution. We will find a stronger inequality

$$f(a,b,c) \ge 0,$$

where

$$f(a,b,c) = (a^2+b^2+c^2)^2+2(a^3b+b^3c+c^3a)-3(ab^3+bc^3+ca^3)-\frac{2}{27}[p^2+m(p^2-3q)]^2$$

satisfies $f(1,1,1) = 0$. The inequality can be written as

$$\sum a^4 + A\sum a^2b^2 + Babc\sum a + C\sum a^3b + D\sum ab^3 \ge 0,$$

where $1 + A + B + C + D = 0$. According to the statement in Remark 1 from P 2.18, the inequality holds for all real a, b, c if

$$3(1 + A) = C^2 + CD + D^2.$$

It is easy to show that this condition is satisfied for $m = -1$, when

$$f(a,b,c) = (a^2+b^2+c^2)^2+2(a^3b+b^3c+c^3a)-3(ab^3+bc^3+ca^3)-\frac{2}{3}(ab+bc+ca)^2.$$

□

P 2.24. *If a, b, c are real numbers, then*

$$(a^2 + b^2 + c^2)^2 + \frac{8}{\sqrt{7}}(a^3b + b^3c + c^3a) \ge 0.$$

Solution. Write the inequality as follows:

$$\frac{\sqrt{7}}{4}(a^2 + b^2 + c^2)^2 + \sum ab(a^2 + b^2) + \sum ab(a^2 - b^2) \ge 0,$$

$$\frac{\sqrt{7}}{4}(p^2 - 2q)^2 + q(p^2 - 2q) - pr - p(a-b)(b-c)(c-a) \ge 0.$$

Since the statement remains unchanged by replacing a, b, c with $-a, -b, -c$, respectively, we may assume $p \ge 0$. For $p = 0$, the inequality is clearly true. For $p > 0$, it suffices to show that

$$\frac{\sqrt{7}}{4}(p^2 - 2q)^2 + q(p^2 - 2q) \ge pr + p|(a-b)(b-c)(c-a)| \ge 0.$$

Due to homogeneity, we may set $p = 1$. From $p^2 \geq 3q$, we get

$$q \leq \frac{1}{3}.$$

Thus, we need to show that

$$\frac{\sqrt{7}}{4}(1-2q)^2 + q(1-2q) \geq r + |(a-b)(b-c)(c-a)|.$$

Applying Corollary 1 for $\beta = \dfrac{1}{27}$, we have

$$r + |(a-b)(b-c)(c-a)| \leq \frac{q}{3} - \frac{2}{27} + \frac{4\sqrt{7}}{27}\,(1-3q)^{3/2},$$

with equality for

$$(1-3q)^{3/2} = \sqrt{7}(2-9q+27r). \tag{*}$$

Therefore, it suffices to show that

$$\frac{\sqrt{7}}{4}(1-2q)^2 + q(1-2q) \geq \frac{q}{3} - \frac{2}{27} + \frac{4\sqrt{7}}{27}\,(1-3q)^{3/2}.$$

Using the substitution

$$t = \sqrt{1-3q}, \qquad t \geq 0,$$

(*) turns into

$$t^3 = \sqrt{7}\,(3t^2 - 1 + 27r), \tag{**}$$

and the desired inequality becomes

$$12(\sqrt{7}-2)t^4 - 16\sqrt{7}\,t^3 + 12(\sqrt{7}+2)t^2 + 3\sqrt{7} + 8 \geq 0,$$

which is equivalent to

$$\left(t - \frac{3+\sqrt{7}}{2}\right)^2 \left[6(\sqrt{7}-2)t^2 + 2(3-\sqrt{7}\,)t + 1\right] \geq 0.$$

The last inequality is true, with equality for $t = \dfrac{3+\sqrt{7}}{2}$, hence for

$$q = \frac{1-t^2}{3} = -\frac{2+\sqrt{7}}{2}.$$

From (**), we get

$$r = -\frac{3+\sqrt{7}}{4\sqrt{7}}.$$

Thus, the original inequality is an equality when a,b,c satisfy

$$(a-b)(b-c)(c-a)\ge 0$$

and are proportional to the roots of the equation

$$4w^3-4w^2-2(2+\sqrt{7}\,)w+1+\frac{3}{\sqrt{7}}=0.$$

□

P 2.25. *If a,b,c are real numbers such that $ab+bc+ca\le 0$, then*

$$(a^2+b^2+c^2)^2\ge(2\sqrt{7}-1)(ab^3+bc^3+ca^3).$$

(Vasile C., 2009)

First Solution. Write the inequality as follows:

$$\frac{4\sqrt{7}+2}{27}(a^2+b^2+c^2)^2\ge\sum ab(a^2+b^2)-\sum ab(a^2-b^2),$$

$$\frac{4\sqrt{7}+2}{27}(p^2-2q)^2\ge q(p^2-2q)-pr+p(a-b)(b-c)(c-a).$$

Since the statement remains unchanged by replacing a,b,c with $-a,-b,-c$, respectively, we may assume $p\ge 0$. For $p=0$, the inequality is trivial. For $p>0$, it suffices to show that

$$\frac{4\sqrt{7}+2}{27}(p^2-2q)^2-q(p^2-2q)\ge -pr+p|(a-b)(b-c)(c-a)|.$$

Due to homogeneity, we may set $p=1$. Thus, we need to show that

$$\frac{4\sqrt{7}+2}{27}(1-2q)^2-q(1-2q)\ge -r+|(a-b)(b-c)(c-a)|.$$

Applying Corollary 1 for $\beta=\frac{-1}{27}$, we have

$$-r+|(a-b)(b-c)(c-a)|\le\frac{-q}{3}+\frac{2}{27}+\frac{4\sqrt{7}}{27}\,(1-3q)^{3/2},$$

with equality for

$$(1-3q)^{3/2}=\sqrt{7}\,(9q-2-27r). \tag{*}$$

Therefore, it suffices to show that

$$\frac{4\sqrt{7}+2}{27}(1-2q)^2-q(1-2q)\ge\frac{-q}{3}+\frac{2}{27}+\frac{4\sqrt{7}}{27}\,(1-3q)^{3/2},$$

which can be written as

$$\frac{4\sqrt{7}}{27}\left[(1-2q)^2-(1-3q)^{3/2}\right]-\frac{2}{27}q(13-31q)\geq 0.$$

Since $q \leq 0$, this inequality is true if

$$\frac{4\sqrt{7}}{27}\left[(1-2q)^2-(1-3q)^{2}\right]-\frac{2}{27}q(13-31q)\geq 0,$$

which is equivalent to

$$q[-13+4\sqrt{7}+(31-10\sqrt{7})q]\geq 0,$$

$$(-q)[13-4\sqrt{7}+(31-10\sqrt{7})(-q)]\geq 0.$$

The last inequality is true, with equality for $q=0$. From (*), we get

$$r=\frac{-1}{27}\left(2+\frac{1}{\sqrt{7}}\right).$$

Thus, the original inequality is an equality when a,b,c satisfy

$$(a-b)(b-c)(c-a)\geq 0$$

and are proportional to the roots of the equation

$$27w^3-27w^2+2+\frac{1}{\sqrt{7}}=0.$$

Second Solution. We will find a stronger inequality of the form

$$(a^2+b^2+c^2)^2+q[kp^2+m(p^2-3q)]\geq(2\sqrt{7}-1)(ab^3+bc^3+ca^3),$$

where $k>0,\ m>0.$ Since

$$q[kp^2+m(p^2-3q)]=(k+m)\left(\sum ab\right)\left(\sum a^2\right)+(2k-m)\left(\sum ab\right)^2$$

$$=(2k-m)\sum a^2b^2+(5k-m)abc\sum a+(k+m)\left(\sum a^3b+\sum ab^3\right),$$

he inequality can be written as

$$\sum a^4+A\sum a^2b^2+Babc\sum a+C\sum a^3b+D\sum ab^3\geq 0,$$

where

$$A=2k-m+2,\quad B=5k-m,\quad C=k+m,\quad D=k+m+1-2\sqrt{7}.$$

According to the statement in Remark 1 from P 2.18, the inequality holds for all real a,b,c if

$$1+A+B+C+D=0$$

and

$$3(1+A) = C^2 + CD + D^2.$$

These condition are satisfied for $k = \dfrac{2(\sqrt{7}-2)}{9}$, and $m = \dfrac{7\sqrt{7}-5}{9}$. Thus, the proof is completed.

$\square$

P 2.26. *If a,b,c are real numbers such that $ab+bc+ca \ge 0$, then*

$$(a^2+b^2+c^2)^2 + (1+2\sqrt{7}\ )(a^3b+b^3c+c^3a) \ge 0.$$

(Vasile C., 2009)

First Solution. Write the inequality as follows:

$$\frac{4\sqrt{7}-2}{27}(a^2+b^2+c^2)^2 + \sum ab(a^2+b^2) + \sum ab(a^2-b^2) \ge 0,$$

$$\frac{4\sqrt{7}-2}{27}(p^2-2q)^2 + q(p^2-2q) - pr - p(a-b)(b-c)(c-a) \ge 0.$$

Since the statement remains unchanged by replacing a,b,c with $-a,-b,-c$, respectively, we may assume $p \ge 0$. For $p = 0$, the inequality is clearly true. For $p > 0$, it suffices to show that

$$\frac{4\sqrt{7}-2}{27}(p^2-2q)^2 + q(p^2-2q) \ge pr + p|(a-b)(b-c)(c-a)| \ge 0.$$

Due to homogeneity, we may set $p = 1$. From $p^2 \ge 3q$, we get

$$0 \le q \le \frac{1}{3}.$$

Thus, we need to show that

$$\frac{4\sqrt{7}-2}{27}(1-2q)^2 + q(1-2q) \ge r + |(a-b)(b-c)(c-a)|.$$

Applying Corollary 1 for $\beta = \dfrac{1}{27}$, we have

$$r + |(a-b)(b-c)(c-a)| \le \frac{q}{3} - \frac{2}{27} + \frac{4\sqrt{7}}{27}\ (1-3q)^{3/2},$$

with equality for

$$(1-3q)^{3/2} = \sqrt{7}(2-9q+27r). \qquad (*)$$

Therefore, it suffices to show that

$$\frac{4\sqrt{7}-2}{27}(1-2q)^2+q(1-2q)\geq\frac{q}{3}-\frac{2}{27}+\frac{4\sqrt{7}}{27}(1-3q)^{3/2},$$

which can be written as

$$\frac{4\sqrt{7}}{27}\left[(1-2q)^2-(1-3q)^{3/2}\right]+\frac{2}{27}\left[1-(1-2q)^2\right]+\frac{2}{3}q(1-3q)\geq 0.$$

This is true if

$$(1-2q)^2-(1-3q)^{3/2}\geq 0.$$

Using the substitution

$$t=\sqrt{1-3q},\qquad 0\leq t\leq 1,$$

the inequality becomes

$$(1+2t^2)^2\geq 9t^3,$$

which is equivalent to

$$(1-t)(1+t+5t^2-4t^3)\geq 0.$$

Since

$$1+t+5t^2-4t^3\geq 1+t+2t^2-4t^3=(1-t)(1+2t+4t^2)\geq 0,$$

the inequality is true, with equality for $t=1$, hence for $q=0$. From (*), we get

$$r=\frac{1}{27}\left(\frac{1}{\sqrt{7}}-2\right).$$

Thus, the original inequality is an equality when a,b,c satisfy

$$(a-b)(b-c)(c-a)\geq 0$$

and are proportional to the roots of the equation

$$27w^3-27w^2+2-\frac{1}{\sqrt{7}}=0.$$

Second Solution. We will find a stronger inequality of the form

$$(a^2+b^2+c^2)^2+(1+2\sqrt{7}\,)(a^3b+b^3c+c^3a)\geq q[kp^2+m(p^2-3q)],$$

where $k>0,\ m>0.$ Since

$$q[kp^2+m(p^2-3q)]=(k+m)\left(\sum ab\right)\left(\sum a^2\right)+(2k-m)\left(\sum ab\right)^2$$

$$=(2k-m)\sum a^2b^2+(5k-m)abc\sum a+(k+m)\left(\sum a^3b+\sum ab^3\right),$$

the inequality can be written as

$$\sum a^4 + A\sum a^2b^2 + Babc\sum a + C\sum a^3b + D\sum ab^3 \geq 0,$$

where

$$A = -2k + m + 2, \quad B = -5k + m, \quad C = -k - m, \quad D = -k - m + 1 + 2\sqrt{7}.$$

According to the statement in Remark 1 from P 2.18, the inequality holds for all real a, b, c if

$$1 + A + B + C + D = 0$$

and

$$3(1 + A) = C^2 + CD + D^2.$$

These conditions are satisfied for $k = \dfrac{2(2+\sqrt{7})}{9}$ and $m = \dfrac{5+7\sqrt{7}}{9}$. Thus, the proof is completed.

Remark. Because $ab + bc + ca \geq 0$ involves

$$(a + b + c)^2 \geq a^2 + b^2 + c^2,$$

the inequality is sharper than the inequality

$$(a + b + c)^4 + (1 + 2\sqrt{7}\,)(a^3b + b^3c + c^3a) \geq 0. \tag{A}$$

On the other hand, the inequality is weaker than the inequality

$$(a^2 + b^2 + c^2 - ab - bc - ca)^2 + (1 + 2\sqrt{7}\,)(a^3b + b^3c + c^3a) \geq 0. \tag{B}$$

Also, the following inequality holds for $ab + bc + ca \geq 0$:

$$(a^2 + b^2 + c^2 - ab - bc - ca)E + (1 + 2\sqrt{7}\,)(a^3b + b^3c + c^3a) \geq 0, \tag{C}$$

where

$$E = a^2 + b^2 + c^2 - \sqrt{7}\,(ab + bc + ca).$$

We can prove the inequalities (B) and (C) in a similar way. The original inequality and the inequalities (A), (B) and (C) are equalities in the same conditions.

□

P 2.27. *If a, b, c are real numbers such that $ab + bc + ca \leq 0$, then*

$$a^4 + b^4 + c^4 \geq (2\sqrt{3} - 1)(ab^3 + bc^3 + ca^3).$$

(Vasile C., 2009)

First Solution. Let
$$k = \frac{1}{2\sqrt{3}-1} = \frac{1+2\sqrt{3}}{11}.$$
Write the inequality as follows:

$$2k(a^4+b^4+c^4) \geq \sum ab(a^2+b^2) - \sum ab(a^2-b^2),$$

$$2k(p^4-4p^2q+2q^2+4pr) \geq q(p^2-2q) - pr + p(a-b)(b-c)(c-a).$$

Since the statement remains unchanged by replacing a,b,c with $-a,-b,-c$, respectively, we may assume $p \geq 0$. For $p=0$, the inequality is true. For $p>0$, it suffices to show that

$$2k(p^4-4p^2q+2q^2+4pr) \geq q(p^2-2q) - pr + p|(a-b)(b-c)(c-a)|.$$

Due to homogeneity, we may set $p=1$. Thus, we need to show that

$$2k(1-4q+2q^2+4r) \geq q(1-2q) - r + |(a-b)(b-c)(c-a)|,$$

which is equivalent to

$$2k(1-4q+2q^2) - q(1-2q) \geq -(1+8k)r + |(a-b)(b-c)(c-a)|.$$

Applying Corollary 1 for $\beta = \dfrac{-(1+8k)}{27}$, we have

$$-(1+8k)r + |(a-b)(b-c)(c-a)| \leq$$

$$\leq \frac{-(1+8k)q}{3} + \frac{2(1+8k)}{27} + \frac{4(4+19\sqrt{3})}{11\cdot 27}\,(1-3q)^{3/2}$$
$$= \frac{-(1+8k)q}{3} + \frac{2(1+8k)}{27} + \frac{2(19k-1)}{27}\,(1-3q)^{3/2},$$

with equality for

$$2(1+8k)(1-3q)^{3/2} = (19k-1)(9q-2-27r). \tag{*}$$

Therefore, it suffices to show that

$$2k(1-4q+2q^2) - q(1-2q) \geq \frac{-(1+8k)q}{3} + \frac{2(1+8k)}{27} + \frac{2(19k-1)}{27}\,(1-3q)^{3/2},$$

which is equivalent to

$$k(19-72q+54q^2) + 27q^2 - 9q - 1 \geq (19k-1)(1-3q)^{3/2}.$$

Using the substitution
$$t = \sqrt{1-3q}, \qquad t \geq 1,$$

the inequality becomes

$$k(6t^4+12t^2+1)+3t^4-3t^2-1 \ge (19k-1)t^3,$$

$$k(6t^4-19t^3+12t^2+1)+3t^4+t^3-3t^2-1 \ge 0,$$

$$k(t-1)(6t^3-13t^2-t-1)+(t-1)(3t^3+4t^2+t+1) \ge 0.$$

This is true if $E \ge 0$, where

$$E = k(6t^3-13t^2-t-1)+(3t^3+4t^2+t+1).$$

We have

$$E > k(6t^3-13t^2-t-1)+k(3t^3+4t^2+t+1) = 9kt^2(t-1) \ge 0.$$

The equality $E=0$ occurs for $t=1$, that means $q=0$. From (*), we get

$$r = \frac{-2k}{19k-1} = \frac{-(2\sqrt{3}+1)}{19\sqrt{3}+4} = \frac{-(10+\sqrt{3})}{97}.$$

Thus, the original inequality is an equality when a,b,c satisfy

$$(a-b)(b-c)(c-a) \ge 0$$

and are proportional to the roots of the equation

$$97w^3-97w^2+10+\sqrt{3}=0.$$

Second Solution. We will find a stronger inequality of the form

$$a^4+b^4+c^4+q[kp^2+m(p^2-3q)] \ge (2\sqrt{3}-1)(ab^3+bc^3+ca^3),$$

where $k>0,\ m>0$. Since

$$q[kp^2+m(p^2-3q)] = (k+m)\left(\sum ab\right)\left(\sum a^2\right)+(2k-m)\left(\sum ab\right)^2$$

$$=(2k-m)\sum a^2b^2+(5k-m)abc\sum a+(k+m)\left(\sum a^3b+\sum ab^3\right),$$

the inequality can be written as

$$\sum a^4+A\sum a^2b^2+Babc\sum a+C\sum a^3b+D\sum ab^3 \ge 0,$$

where

$$A=2k-m, \quad B=5k-m, \quad C=k+m, \quad D=k+m+1-2\sqrt{3}.$$

According to the statement in Remark 1 from P 2.18, the inequality holds for all real a,b,c if

$$1+A+B+C+D=0$$

and

$$3(1+A)=C^2+CD+D^2.$$

These conditions are satisfied for $k=\dfrac{2(\sqrt{3}-1)}{9}$ and $m=\dfrac{7(\sqrt{3}-1)}{9}$. Thus, the proof is completed.

□

P 2.28. *If a, b, c are real numbers such that $ab + bc + ca \geq 0$, then*

$$a^4 + b^4 + c^4 + (1 + 2\sqrt{3})(a^3b + b^3c + c^3a) \geq 0.$$

(Vasile C., 2009)

Solution. Denoting

$$k = \frac{2\sqrt{3} - 1}{11},$$

we may write the inequality as follows:

$$2k(a^4 + b^4 + c^4) + \sum ab(a^2 + b^2) + \sum ab(a^2 - b^2) \geq 0,$$

$$2k(p^4 - 4p^2q + 2q^2 + 4pr) + q(p^2 - 2q) - pr - p(a-b)(b-c)(c-a) \geq 0.$$

Since the statement remains unchanged by replacing a, b, c with $-a, -b, -c$, respectively, we may assume $p \geq 0$. For $p = 0$, the inequality is true. For $p > 0$, it suffices to show that

$$2k(p^4 - 4p^2q + 2q^2 + 4pr) + q(p^2 - 2q) - pr - p|(a-b)(b-c)(c-a)| \geq 0.$$

Due to homogeneity, we may set

$$p = 1, \quad 0 \leq q \leq \frac{1}{3}.$$

Thus, we need to show that

$$2k(1 - 4q + 2q^2) + q(1 - 2q) \geq (1 - 8k)r + |(a-b)(b-c)(c-a)|.$$

Applying Corollary 1 for $\beta = \dfrac{1 - 8k}{27}$, we have

$$(1 - 8k)r + |(a-b)(b-c)(c-a)| \leq$$

$$\leq \frac{(1-8k)q}{3} - \frac{2(1-8k)}{27} + \frac{4(19\sqrt{3}-4)}{11 \cdot 27}(1-3q)^{3/2}$$

$$= \frac{(1-8k)q}{3} - \frac{2(1-8k)}{27} + \frac{2(19k+1)}{27}(1-3q)^{3/2},$$

with equality for

$$(16\sqrt{3} - 19)(1 - 3q)^{3/2} = (19\sqrt{3} - 4)(9q - 2 - 27r). \tag{*}$$

Therefore, it suffices to show that

$$2k(1 - 4q + 2q^2) + q(1 - 2q) \geq \frac{(1-8k)q}{3} - \frac{2(1-8k)}{27} + \frac{2(19k+1)}{27}(1-3q)^{3/2},$$

which can be written as

$$kA+B\geq 0,$$

where

$$A=19\left[1-(1-3q)^{3/2}\right]-18q(4-3q),$$

$$B=1-(1-3q)^{3/2}+9q(1-3q).$$

Since $B\geq 0$, it suffices to show that $A\geq 0$. This is true if

$$18\left[1-(1-3q)^{3/2}\right]-18q(4-3q)\geq 0,$$

which is equivalent to

$$3\left[1-(1-3q)^{3/2}\right]-3q(4-3q)\geq 0.$$

Using the substitution

$$t=\sqrt{1-3q},\qquad 0\leq t\leq 1,$$

the inequality becomes

$$3(1-t^3)-(1-t^2)(3+t^2)\geq 0,$$

which is equivalent to

$$t^2(1-t)(2-t)\geq 0.$$

This inequality is true, with equality for $t=1$, hence for $q=0$. From (*), we get

$$r=\frac{-(10-\sqrt{3})}{97}.$$

Thus, the original inequality is an equality when a,b,c satisfy

$$(a-b)(b-c)(c-a)\geq 0$$

and are proportional to the roots of the equation

$$97w^3-97w^2+10-\sqrt{3}=0.$$

Second Solution. We will find a stronger inequality of the form

$$a^4+b^4+c^4+(1+2\sqrt{3}\,)(a^3b+b^3c+c^3a)\geq q[kp^2+m(p^2-3q)],$$

where $k>0,\ m>0.$ Since

$$q[kp^2+m(p^2-3q)]=(k+m)\left(\sum ab\right)\left(\sum a^2\right)+(2k-m)\left(\sum ab\right)^2$$

$$=(2k-m)\sum a^2b^2+(5k-m)abc\sum a+(k+m)\left(\sum a^3b+\sum ab^3\right),$$

the inequality can be written as

$$\sum a^4 + A\sum a^2b^2 + Babc\sum a + C\sum a^3b + D\sum ab^3 \ge 0,$$

where

$$A = -2k + m, \quad B = -5k + m, \quad C = -k - m, \quad D = -k - m + 1 + 2\sqrt{3}.$$

According to the statement in Remark 1 from P 2.18, the inequality holds for all real a, b, c if

$$1 + A + B + C + D = 0$$

and

$$3(1 + A) = C^2 + CD + D^2.$$

These conditions are satisfied for $k = \dfrac{2(1+\sqrt{3})}{9}$ and $m = \dfrac{7(1+\sqrt{3})}{9}$. Thus, the proof is completed.

□

P 2.29. *If a, b, c are real numbers such that $ab + bc + ca \ge 0$, then*

$$a^4 + b^4 + c^4 + 2\sqrt{2}\,(a^3b + b^3c + c^3a) \ge ab^3 + bc^3 + ca^3.$$

(Vasile C., 2009)

Solution. For $m = 2\sqrt{2}$, we write the inequality as follows:

$$2(a^4 + b^4 + c^4) + (m-1)\sum ab(a^2 + b^2) + (m+1)\sum ab(a^2 - b^2) \ge 0,$$

$$2(p^4 - 4p^2q + 2q^2 + 4pr) + (m-1)[q(p^2 - 2q) - pr] \ge (m+1)p(a-b)(b-c)(c-a).$$

Since the statement remains unchanged by replacing a, b, c with $-a, -b, -c$, respectively, we may assume $p \ge 0$. For $p = 0$, the inequality is true. For $p > 0$, it suffices to show that

$$2(p^4 - 4p^2q + 2q^2) + (m-1)q(p^2 - 2q) \ge (m-9)r + (m+1)p|(a-b)(b-c)(c-a)|.$$

Due to homogeneity, we may set

$$p = 1, \quad 0 \le q \le \frac{1}{3}.$$

Thus, we need to show that

$$2(1 - 4q + 2q^2) + (m-1)q(1-2q) \ge (m-9)r + (m+1)|(a-b)(b-c)(c-a)|,$$

which is equivalent to

$$\frac{2(1-4q+2q^2)+(m-1)q(1-2q)}{m+1} \ge \frac{m-9}{m+1}r+|(a-b)(b-c)(c-a)|,$$

Applying Corollary 1 for $\beta = \dfrac{m-9}{27(m+1)}$, we have

$$\frac{m-9}{m+1}r+|(a-b)(b-c)(c-a)| \le$$

$$\le \frac{(m-9)q}{3(m+1)}-\frac{2(m-9)}{27(m+1)}+\frac{2(18+m)}{27(m+1)}\,(1-3q)^{3/2},$$

with equality for

$$2(m-9)(1-3q)^{3/2}=(18+m)(2-9q+27r). \tag{*}$$

Therefore, it suffices to show that

$$2(1-4q+2q^2)+(m-1)q(1-2q) \ge \frac{(m-9)q}{3}-\frac{2(m-9)}{27}+\frac{2(18+m)}{27}\,(1-3q)^{3/2},$$

which can be written as

$$mA+2B \ge 0,$$

where

$$A=\frac{2q(1-q)}{3}+\frac{2}{27}\left[1-(1-3q)^{3/2}\right],$$

$$B=-3q(1-q)+\frac{2}{3}\left[1-(1-3q)^{3/2}\right].$$

Since $A \ge 0$, it suffices to show that $B \ge 0$. Using the substitution

$$t=\sqrt{1-3q}, \qquad 0 \le t \le 1,$$

we have

$$B=\frac{-(1-t^2)(2+t^2)}{3}+\frac{2}{3}(1-t^3)=\frac{t^2(1-t)^2}{3} \ge 0.$$

The inequality $mA+2B \ge 0$ is an equality for $q=0$. From (*), we get

$$r=\frac{\sqrt{2}-9}{7}.$$

Thus, the original inequality is an equality when a,b,c satisfy

$$(a-b)(b-c)(c-a) \ge 0$$

and are proportional to the roots of the equation

$$7w^3-7w^2+9-\sqrt{2}=0.$$

Second Solution. We will find a stronger inequality of the form

$$a^4+b^4+c^4+2\sqrt{2}\,(a^3b+b^3c+c^3a)\ge ab^3+bc^3+ca^3+q[kp^2+m(p^2-3q)],$$

where $k>0,\ m>0.$ Since

$$q[kp^2+m(p^2-3q)]=(k+m)\left(\sum ab\right)\left(\sum a^2\right)+(2k-m)\left(\sum ab\right)^2$$

$$=(2k-m)\sum a^2b^2+(5k-m)abc\sum a+(k+m)\left(\sum a^3b+\sum ab^3\right),$$

the inequality can be written as

$$\sum a^4+A\sum a^2b^2+Babc\sum a+C\sum a^3b+D\sum ab^3\ge 0,$$

where

$$A=-2k+m,\qquad B=-5k+m,\qquad C=-k-m+2\sqrt{2},\qquad D=-k-m-1.$$

According to the statement in Remark 1 from P 2.18, the inequality holds for all real a,b,c if

$$1+A+B+C+D=0$$

and

$$3(1+A)=C^2+CD+D^2.$$

These conditions are satisfied for $k=\dfrac{2\sqrt{2}}{9}$ and $m=\dfrac{7\sqrt{2}}{9}$. Thus, the proof is completed.

□

P 2.30. *If a,b,c are real numbers such that $ab+bc+ca\ge 0$, then*

$$(a+b+c)(a^3+b^3+c^3)+5(a^3b+b^3c+c^3a)\ge 0.$$

(Vasile C., 2008)

First Solution. Since the statement remains unchanged by replacing a,b,c with $-a,-b,-c$, respectively, we may assume $p\ge 0$. Since $p=0$ and $q\ge 0$ involve $a=b=c=0$, consider next $p>0$. Due to homogeneity, we may set

$$p=1,\qquad 0\le q\le\frac{p^2}{3}=\frac{1}{3}.$$

The desired inequality becomes as follows:

$$2(a+b+c)(a^3+b^3+c^3)+5\sum ab(a^2+b^2)+5\sum ab(a^2-b^2)\ge 0,$$

$$2p(3r+p^3-3pq)+5q(p^2-2q)-5pr\ge 5p(a-b)(b-c)(c-a),$$

$$2(3r+1-3q)+5q(1-2q)-5r \ge 5(a-b)(b-c)(c-a),$$

$$2-q-10q^2 \ge -r+5(a-b)(b-c)(c-a).$$

Thus, we need to show that

$$\frac{2-q-10q^2}{5} \ge \frac{-1}{5}r+|(a-b)(b-c)(c-a)|.$$

Applying Corollary 1 for $\beta = \dfrac{-1}{135}$, we have

$$\frac{-1}{5}r+|(a-b)(b-c)(c-a)| \le \frac{-9q}{135}+\frac{2}{135}+\frac{52}{135}\,(1-3q)^{3/2},$$

with equality for

$$(1-3q)^{3/2} = 13(9q-2-27r). \tag{*}$$

Therefore, it suffices to show that

$$\frac{2-q-10q^2}{5} \ge \frac{-9q}{135}+\frac{2}{135}+\frac{52}{135}\,(1-3q)^{3/2},$$

which can be rewritten as

$$26-9q-135q^2 \ge 26(1-3q)^{3/2}.$$

Using the substitution

$$t=\sqrt{1-3q}, \qquad 0 \le t \le 1,$$

the inequality becomes

$$26-3(1-t^2)-15(1-t^2)^2 \ge 26t^3,$$

$$8+33t^2-26t^3-15t^4 \ge 0,$$

$$(1-t)(8+8t+41t^2+15t^3) \ge 0.$$

The last inequality is true, with equality for $t=1$. Notice that $t=1$ involves $q=0$, and (*) gives $r=-1/13$. Thus, the original inequality is an equality when a,b,c satisfy

$$(a-b)(b-c)(c-a) \ge 0$$

and are proportional to the roots of the equation

$$13w^3-13w^2+1=0.$$

Second Solution. We will find a stronger inequality of the form

$$(a+b+c)(a^3+b^3+c^3)+5(a^3b+b^3c+c^3a) \ge q[kp^2+m(p^2-3q)],$$

where $k > 0,\ m > 0.$ Since

$$q[kp^2 + m(p^2 - 3q)] = (k+m)\left(\sum ab\right)\left(\sum a^2\right) + (2k-m)\left(\sum ab\right)^2$$

$$= (2k-m)\sum a^2b^2 + (5k-m)abc\sum a + (k+m)\left(\sum a^3b + \sum ab^3\right),$$

the inequality can be written as

$$\sum a^4 + A\sum a^2b^2 + Babc\sum a + C\sum a^3b + D\sum ab^3 \ge 0,$$

where

$$A = -2k+m, \quad B = -5k+m, \quad C = -k-m+6, \quad D = -k-m+1.$$

According to the statement in Remark 1 from P 2.18, the inequality holds for all real a, b, c if

$$1 + A + B + C + D = 0$$

and

$$3(1+A) = C^2 + CD + D^2.$$

These conditions are satisfied for $k = \dfrac{8}{9}$ and $m = \dfrac{28}{9}$. Thus, the proof is completed. $\square$

P 2.31. *If a, b, c are real numbers such that*

$$k(ab+bc+ca) = a^2+b^2+c^2, \qquad k \ge \frac{1+\sqrt{21+8\sqrt{7}}}{2} \approx 3.7468,$$

then

$$a^3b + b^3c + c^3a \ge 0.$$

(Vasile C., 2012)

Solution. Write the inequality as follows:

$$\sum ab(a^2+b^2) + \sum ab(a^2-b^2) \ge 0,$$

$$q(p^2-2q) - pr - p(a-b)(b-c)(c-a) \ge 0.$$

Since the statement remains unchanged by replacing a, b, c with $-a, -b, -c$, respectively, we may assume $p \ge 0$. For $p = 0$, we have

$$2(a^2+b^2+c^2) = 2k(ab+bc+ca) = -k(a^2+b^2+c^2),$$

which implies $a^2+b^2+c^2 = 0$, $a = b = c = 0$. For $p > 0$, it suffices to show that

$$q(p^2-2q) \ge pr + p|(a-b)(b-c)(c-a)|.$$

Due to homogeneity, we may set $p = 1$, which implies

$$q = \frac{1}{k+2}.$$

Thus, we need to show that

$$q(1-2q) \ge r + |(a-b)(b-c)(c-a)|.$$

Applying Corollary 1 for $\beta = \dfrac{1}{27}$, we have

$$r + |(a-b)(b-c)(c-a)| \le \frac{q}{3} - \frac{2}{27} + \frac{4\sqrt{7}}{27}\,(1-3q)^{3/2},$$

with equality for

$$(1-3q)^{3/2} = \sqrt{7}(2-9q+27r). \tag{*}$$

Therefore, it suffices to show that

$$q(1-2q) \ge \frac{q}{3} - \frac{2}{27} + \frac{4\sqrt{7}}{27}\,(1-3q)^{3/2},$$

which can be rewritten as

$$1 + 9q - 27q^2 \ge 2\sqrt{7}\,(1-3q)^{3/2}. \tag{**}$$

Using the substitution

$$t = \sqrt{1-3q} = \sqrt{\frac{k-1}{k+2}},$$

the inequality becomes

$$1 + 3(1-t^2) - 3(1-t^2)^2 \ge 2\sqrt{7}\,t^3,$$

$$1 + 3t^2 - 2\sqrt{7}\,t^3 - 3t^4 \ge 0,$$

$$\left[1 + t + (2+\sqrt{7})t^2\right]\left[1 - t + (2-\sqrt{7})t^2\right] \ge 0.$$

We only need to show that

$$1 - t + (2-\sqrt{7})t^2 \ge 0,$$

which is equivalent to

$$1 - (\sqrt{7}-2)\frac{k-1}{k+2} \ge \sqrt{\frac{k-1}{k+2}},$$

$$(3-\sqrt{7})k + \sqrt{7} \ge \sqrt{(k-1)(k+2)}.$$

By squaring, the inequality becomes

$$(2\sqrt{7}-5)k^2-(2\sqrt{7}-5)k-3 \le 0,$$

$$k^2-k-2\sqrt{7}-5 \le 0.$$

The last inequality is true for $k \ge \dfrac{1+\sqrt{21+8\sqrt{7}}}{2}$, with equality for

$$k=\frac{1+\sqrt{21+8\sqrt{7}}}{2}.$$

From (*) and (**), we get the equality condition

$$1+9q-27q^2=14(2-9q+27r),$$

which is equivalent to

$$14r=-1+5q-q^2=-1+\frac{5}{k+2}-\frac{1}{(k+2)^2}=\frac{5+k-k^2}{(k+2)^2}.$$

The original inequality is an equality when $a=b=c=0$. For

$$k=\frac{1+\sqrt{21+8\sqrt{7}}}{2}$$

(which satisfies $k^2-k-2\sqrt{7}-5=0$), the equality condition above becomes

$$r=\frac{5+k-k^2}{14(k+2)^2}=\frac{-1}{\sqrt{7}(k+2)^2}.$$

Thus, the equality holds also when a,b,c satisfy

$$(a-b)(b-c)(c-a)>0$$

and are proportional to the roots of the equation

$$w^3-w^2+\frac{1}{k+2}w+\frac{1}{\sqrt{7}(k+2)^2}=0.$$

□

P 2.32. *If a,b,c are nonnegative real numbers, then*

$$3(a^4+b^4+c^4)+4(a^3b+b^3c+c^3a) \ge 7(ab^3+bc^3+ca^3).$$

(Vasile C., 2009)

Solution. It suffices to show that there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that the sharper inequality $f(a,b,c) \geq 0$ holds for any real a,b,c and $f(1,1,1)=0$, where

$$f(a,b,c) = 3\sum a^4 + 4\sum a^3 b - 7\sum ab^3 - \sum bc(\alpha a + \beta b + \gamma c)^2,$$

$$f(1,1,1) = -3(\alpha+\beta+\gamma)^2.$$

Since

$$\sum bc(\alpha a + \beta b + \gamma c)^2 = 2\beta\gamma \sum b^2c^2 + \alpha(\alpha + 2\beta + 2\gamma)abc\sum a + \beta^2\sum a^3 b + \gamma^2 \sum ab^3,$$

we have

$$f(a,b,c) = A_0\sum a^4 + A\sum a^2b^2 + Babc\sum a + C\sum a^3b + D\sum ab^3,$$

where

$$A_0 = 3, \quad A = -2\beta\gamma, \quad B = -\alpha(\alpha+2\beta+2\gamma), \quad C = 4-\beta^2, \quad D = -7-\gamma^2.$$

Choosing

$$\gamma = -1,$$

we have

$$A_0 = 3, \quad A = 2\beta, \quad B = -\alpha(\alpha+2\beta-2), \quad C = 4-\beta^2, \quad D = -8.$$

For $\alpha+\beta+\gamma=0$, we have $A_0 + A + B + C + D = 0$. According to the statement in Remark 2 from P 2.18, if $A_0 > 0$, $A_0 + A + B + C + D = 0$ and

$$3A_0(A_0 + A) \geq C^2 + CD + D^2,$$

then the inequality $f(a,b,c) \geq 0$ holds for all real a,b,c. Since

$$3A_0(A_0+A) - (C^2 + CD + D^2) = f(\beta), \quad f(\beta) = 18\beta - 21 - \beta^4,$$

and

$$f\left(\frac{3}{2}\right) = \frac{15}{16} > 0,$$

we choose

$$\beta = \frac{3}{2}$$

and

$$\alpha = -\beta - \gamma = \frac{-1}{2}.$$

As a consequence, the inequality

$$3\sum a^4 + 4\sum a^3 b \geq 7\sum ab^3 + \frac{1}{4}\sum bc(-a+3b-2c)^2$$

holds for any real a,b,c. Thus, the proof is completed. The original inequality is an equality for $a=b=c$.

□

P 2.33. *If a, b, c are nonnegative real numbers, then*

$$16(a^4+b^4+c^4)+52(a^3b+b^3c+c^3a) \geq 47(ab^3+bc^3+ca^3).$$

(Vasile C., 2009)

Solution. It suffices to show that there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that the sharper inequality $f(a,b,c) \geq 0$ holds for any real a, b, c and $f(1,1,1)=0$, where

$$f(a,b,c)=16\sum a^4+52\sum a^3b-47\sum ab^3-\sum bc(\alpha a+\beta b+\gamma c)^2,$$

$$f(1,1,1)=63-3(\alpha+\beta+\gamma)^2.$$

Since

$$\begin{aligned}\sum bc(\alpha a+\beta b+\gamma c)^2 = & 2\beta\gamma\sum b^2c^2+\alpha(\alpha+2\beta+2\gamma)abc\sum a \\ & +\beta^2\sum a^3b+\gamma^2\sum ab^3,\end{aligned}$$

we have

$$f(a,b,c)=A_0\sum a^4+A\sum a^2b^2+Babc\sum a+C\sum a^3b+D\sum ab^3,$$

where

$$A_0=16, \quad A=-2\beta\gamma, \quad B=-\alpha(\alpha+2\beta+2\gamma), \quad C=52-\beta^2, \quad D=-47-\gamma^2.$$

The original inequality is an equality for $a=0$, $b=1$ and $c=2$. Since

$$f(0,1,2)=-2(\beta+2\gamma)^2,$$

we choose

$$\beta=-2\gamma.$$

We have

$$A_0=16, \quad A=4\gamma^2, \quad B=-\alpha(\alpha-2\gamma), \quad C=52-4\gamma^2, \quad D=-47-\gamma^2.$$

According to the statement in Remark 2 from P 2.18, if $A_0>0$, $A_0+A+B+C+D=0$ and

$$3A_0(A_0+A)=C^2+CD+D^2,$$

then the inequality $f(a,b,c) \geq 0$ holds for all real a, b, c. Since

$$3A_0(A_0+A)-(C^2+CD+D^2)=-21(\gamma-3)^2(\gamma+3)^2,$$

we choose

$$\gamma=3$$

and

$$\beta = -2\gamma = -6.$$

In addition, for

$$\alpha = \sqrt{21} - \beta - \gamma = \sqrt{21} + 3$$

we have $3(A_0 + A + B + C + D) = f(1,1,1) = 0$. As a consequence, the inequality

$$16\sum a^4 + 52\sum a^3 b \ge 47\sum ab^3 + \sum bc\left[(\sqrt{21}+3)a - 6b + 3c\right]^2$$

holds for any real a, b, c. The original inequality is an equality for $a = 0$ and $b = c/2$ (or any cyclic permutation).

□

P 2.34. *If a, b, c are nonnegative real numbers, then*

$$a^4 + b^4 + c^4 + 5(a^3 b + b^3 c + c^3 a) \ge 6(a^2 b^2 + b^2 c^2 + c^2 a^2).$$

(Vasile C., 2009)

Solution. It suffices to show that there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that the sharper inequality $f(a, b, c) \ge 0$ holds for any real a, b, c and $f(1,1,1) = 0$, where

$$f(a,b,c) = \sum a^4 + 5\sum a^3 b - 6\sum a^2 b^2 - \sum bc(\alpha a + \beta b + \gamma c)^2,$$

$$f(1,1,1) = -3(\alpha + \beta + \gamma)^2.$$

Since

$$\sum bc(\alpha a + \beta b + \gamma c)^2 = 2\beta\gamma\sum b^2 c^2 + \alpha(\alpha + 2\beta + 2\gamma)abc\sum a + \beta^2\sum a^3 b + \gamma^2\sum ab^3,$$

we have

$$f(a,b,c) = \sum a^4 + A\sum a^2 b^2 + Babc\sum a + C\sum a^3 b + D\sum ab^3,$$

where

$$A = -6 - 2\beta\gamma, \quad B = -\alpha(\alpha + 2\beta + 2\gamma), \quad C = 5 - \beta^2, \quad D = -\gamma^2.$$

Choosing

$$\gamma = -\frac{3}{2},$$

we have

$$A = -6 + 3\beta, \quad B = -\alpha(\alpha + 2\beta - 3), \quad C = 5 - \beta^2, \quad D = -\frac{9}{4}.$$

According to the statement in Remark 1 from P 2.18, if $1+A+B+C+D=0$ and

$$3(1+A) \geq C^2+CD+D^2,$$

then the inequality $f(a,b,c) \geq 0$ holds for all real a,b,c. Since

$$3(1+A)-(C^2+CD+D^2)=f(\beta), \qquad 4f(\beta)=36\beta-\frac{141}{4}-9\beta^2-4(5-\beta^2)^2,$$

and

$$f\left(\frac{9}{4}\right)=\frac{11}{64}>0,$$

we choose

$$\beta=\frac{9}{4}.$$

In addition, for

$$\alpha=-\beta-\gamma=\frac{-3}{4}$$

we have $3(1+A+B+C+D)=f(1,1,1)=0$. As a consequence, the inequality

$$\sum a^4+5\sum a^3b \geq 6\sum a^2b^2+\frac{9}{16}\sum bc(-a+3b-2c)^2$$

holds for any real a,b,c. Thus, the proof is completed. The original inequality is an equality for $a=b=c$.

$\square$

P 2.35. *If a,b,c are nonnegative real numbers such that $a+b+c=4$, then*

$$a^3b+b^3c+c^3a+\frac{473}{64}abc \leq 27.$$

(Vasile C., 2009)

Solution. Write the inequality in the homogeneous form $g(a,b,c) \geq 0$, where

$$g(a,b,c)=27(a+b+c)^4-256(a^3b+b^3c+c^3a)-473abc(a+b+c).$$

It suffices to show that there exist $\alpha,\beta,\gamma \in \mathbb{R}$ such that the sharper inequality $f(a,b,c) \geq 0$ holds for any real a,b,c, where

$$f(a,b,c)=g(a,b,c)-\sum bc(\alpha a+\beta b+\gamma c)^2,$$

$$f(1,1,1)=-3(\alpha+\beta+\gamma)^2.$$

Since

$$(a+b+c)^4=\sum a^4+6\sum a^2b^2+12abc\sum a+4\sum a^3b+4\sum ab^3$$

and

$$\sum bc(\alpha a+\beta b+\gamma c)^2 = 2\beta\gamma\sum b^2c^2+\alpha(\alpha+2\beta+2\gamma)abc\sum a + \beta^2\sum a^3b+\gamma^2\sum ab^3,$$

we have

$$f(a,b,c) = A_0\sum a^4 + A\sum a^2b^2 + Babc\sum a + C\sum a^3b + D\sum ab^3,$$

where

$$A_0 = 27, \quad A = 162-2\beta\gamma, \quad B = -149-\alpha(\alpha+2\beta+2\gamma),$$
$$C = -148-\beta^2, \quad D = 108-\gamma^2.$$

Since

$$g(1,1,1) = g(0,3,1) = 0,$$

we need to have also

$$f(1,1,1) = f(0,3,1) = 0.$$

From

$$f(1,1,1) = -3(\alpha+\beta+\gamma)^2$$

and

$$f(0,3,1) = -3(3\beta+\gamma)^2,$$

we get

$$\gamma = -3\beta, \qquad \alpha = -\beta-\gamma = 2\beta,$$

therefore

$$A_0 = 27, \quad A = 162+6\beta^2, \quad B = -149+4\beta^2, \quad C = -148-\beta^2, \quad D = 108-9\beta^2.$$

We see that $A_0+A+B+C+D = 0$. According to the statement in Remark 2 from P 2.18, if $A_0 > 0$ and

$$3A_0(A_0+A) = C^2+CD+D^2,$$

then the inequality $f(a,b,c) \ge 0$ holds for all real a,b,c. Since

$$3A_0(A_0+A)-(C^2+CD+D^2) = -91(\beta^2-5)^2,$$

we choose

$$\beta = \sqrt{5}, \quad \alpha = 2\beta = 2\sqrt{5}, \quad \gamma = -\alpha-\beta = -3\sqrt{5}.$$

As a consequence, the inequality

$$27(a+b+c)^4-256(a^3b+b^3c+c^3a)-473abc(a+b+c) \ge 5\sum bc(2a+b-3c)^2$$

holds for any real a,b,c. The original inequality is an equality for $a=b=c=4/3$, and also for $a=0$, $b=3$ and $c=1$ (or any cyclic permutation).

□

P 2.36. *If a, b, c are positive real numbers such that $a+b+c=3$, then*

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}+5(ab+bc+ca)\geq 18.$$

(Michael Rozenberg and Vasile C., 2009)

Solution. Write the inequality in the homogeneous forms:

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}+\frac{15(ab+bc+ca)}{a+b+c}\geq 6(a+b+c),$$

$$\sum\left(\frac{a^2}{b}+\frac{b^2}{a}\right)+\sum\left(\frac{a^2}{b}-\frac{b^2}{a}\right)+\frac{30(ab+bc+ca)}{a+b+c}\geq 12(a+b+c),$$

$$\frac{(p^2-2q)q-pr}{r}+\frac{p(a-b)(b-c)(c-a)}{r}+\frac{30q}{p}\geq 12p,$$

$$(p^2-2q)q\geq\left(13p-\frac{30q}{p}\right)r-p(a-b)(b-c)(c-a).$$

It suffices to show that

$$(p^2-2q)q\geq\left(13p-\frac{30q}{p}\right)r+p|(a-b)(b-c)(c-a)|.$$

Due to homogeneity, we may set $p=1$. From $p^2\geq 3q$, we get

$$0\leq q\leq\frac{1}{3}.$$

Thus, we need to show that

$$(1-2q)q\geq(13-30q)r+|(a-b)(b-c)(c-a)|. \tag{*}$$

Case 1: $\frac{1}{9}\leq q\leq\frac{1}{3}$. Applying Corollary 1 for $\beta=\frac{13-30q}{27}$, we have

$$(13-30q)r+|(a-b)(b-c)(c-a)|\leq\frac{(13-30q)q}{3}-\frac{2(13-30q)}{27}+\frac{2A}{27}(1-3q)^{3/2},$$

where

$$A=\sqrt{27+(13-30q)^2}.$$

Therefore, it suffices to show that

$$(1-2q)q\geq\frac{(13-30q)q}{3}-\frac{2(13-30q)}{27}+\frac{2A}{27}(1-3q)^{3/2},$$

which is equivalent to

$$(1-3q)\left(13-36q-A\sqrt{1-3q}\right)\geq 0.$$

This is true if
$$(13-36q)^2 \ge A^2(1-3q),$$
which can be written as
$$27(100q^3-72q^2+16q-1) \ge 0,$$
$$3(10q-3)^2(9q-1)+3q(3-8q) \ge 0.$$
Clearly, the last inequality is true.

Case 2: $0 < q \le \frac{1}{9}$. Since
$$\begin{aligned}|(a-b)(b-c)(c-a)| &= \sqrt{-27r^2-2(2p^3-9pq)r+p^2q^2-4q^3} \\ &= \sqrt{-27r^2-2(2-9q)r+q^2-4q^3},\end{aligned}$$
the inequality (*) becomes
$$(1-2q)q-(13-30q)r \ge \sqrt{-27r^2-2(2-9q)r+q^2-4q^3}.$$
From the known inequality $q^2 \ge 3pr$, we get $r \le q^2/3$, therefore
$$\begin{aligned}(1-2q)q-(13-30q)r &\ge (1-2q)q-\frac{1}{3}(13-30q)q^2 \\ &= \frac{1}{3}q(1-3q)(3-10q) > 0.\end{aligned}$$
Thus, we only need to show that
$$[(1-2q)q-(13-30q)r]^2 \ge -27r^2-2(2-9q)r+q^2-4q^3,$$
which is equivalent to
$$[27+(13-30q)^2]r^2+2[2-9q-q(1-2q)(13-30q)]r+4q^4 \ge 0.$$
It suffices to show that
$$2-9q \ge q(1-2q)(13-30q).$$
We can get this inequality by multiplying the inequalities
$$10(2-9q) > 13-30q$$
and
$$1 > 10q(1-2q).$$
The last inequality is true because
$$1-10q(1-2q) > 1-10q+9q^2 = (1-q)(1-9q) \ge 0.$$

The original inequality is an equality for $a=b=c=1$.

Remark. The following generalization holds:

- *Let a,b,c be positive real numbers such that $a+b+c=3$. If $k \le k_0$, where $k_0 \approx 6.1708$ is the largest positive root of the equation*

$$x^4-11x^3+72x^2-304x+269=0,$$

then
$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}+(k-1)(ab+bc+ca) \ge 3k.$$

□

P 2.37. *If a,b,c are positive real numbers, then*

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3 \ge \frac{5(a^2+b^2+c^2-ab-bc-ca)}{a^2+b^2+c^2+ab+bc+ca}.$$

(Vasile C., 2009)

Solution. Write the inequality as follows:

$$\sum\left(\frac{a}{b}+\frac{b}{a}\right)+\sum\left(\frac{a}{b}-\frac{b}{a}\right)-6 \ge \frac{10(a^2+b^2+c^2-ab-bc-ca)}{a^2+b^2+c^2+ab+bc+ca}.$$

$$\frac{pq-3r}{r}-6+\frac{(a-b)(b-c)(c-a)}{r} \ge \frac{10(p^2-3q)}{p^2-q},$$

$$pq+(a-b)(b-c)(c-a) \ge \frac{19p^2-39q}{p^2-q}r.$$

It suffices to show that

$$pq \ge \frac{19p^2-39q}{p^2-q}r+|(a-b)(b-c)(c-a)|.$$

Due to homogeneity, we may set $p=1$. From $p^2 \ge 3q$, we get

$$0 \le q \le \frac{1}{3}.$$

Thus, we need to show that

$$q \ge \frac{19-39q}{1-q}r+|(a-b)(b-c)(c-a)|. \tag{*}$$

Case 1: $\frac{1}{15} \le q \le \frac{1}{3}$. Applying Corollary 1 for $\beta = \frac{19-39q}{27(1-q)}$, we have

$$\frac{19-39q}{1-q}r+|(a-b)(b-c)(c-a)| \le \frac{(19-39q)q}{3(1-q)}-\frac{2(19-39q)}{27(1-q)}+\frac{2A}{27(1-q)}\,(1-3q)^{3/2},$$

where

$$A = \sqrt{27(1-q)^2+(19-39q)^2}.$$

Therefore, it suffices to show that

$$q \ge \frac{(19-39q)q}{3(1-q)} - \frac{2(19-39q)}{27(1-q)} + \frac{2A}{27}\,(1-3q)^{3/2},$$

which is equivalent to

$$(1-3q)\left(19-54q-A\sqrt{1-3q}\right) \ge 0.$$

This is true if

$$(19-54q)^2 \ge A^2(1-3q),$$

which can be written as

$$172q^3-120q^2+24q-1 \ge 0,$$

$$(3q-1)^2(15q-1)+q(37q^2-21q+3) \ge 0.$$

The last inequality is true because

$$37q^2-21q+3 = 37\left(q-\frac{21}{74}\right)^2+\frac{3}{148} > 0.$$

Case 2: $0 < q \le \frac{1}{15}$. Since

$$\begin{aligned}|(a-b)(b-c)(c-a)| &= \sqrt{-27r^2-2(2p^3-9pq)r+p^2q^2-4q^3}\\ &= \sqrt{-27r^2-2(2-9q)r+q^2-4q^3},\end{aligned}$$

the inequality (*) becomes

$$q-\frac{19-39q}{1-q}r \ge \sqrt{-27r^2-2(2-9q)r+q^2-4q^3}.$$

From the known inequality $q^2 \ge 3pr$, we get $r \le q^2/3$, therefore

$$\begin{aligned}q-\frac{19-39q}{1-q}r &\ge q-\frac{(19-39q)q^2}{3(1-q)}\\ &= \frac{q(1-3q)(3-13q)}{3(1-q)} > 0.\end{aligned}$$

Thus, we only need to show that

$$\left(q-\frac{19-39q}{1-q}r\right)^2 \geq -27r^2-2(2-9q)r+q^2-4q^3,$$

which is equivalent to the obvious inequality

$$Ar^2+4Br+4q^3 \geq 0,$$

where

$$A = 27+\left(\frac{19-39q}{1-q}\right)^2 > 0,$$

$$B = \frac{(1-15q)+24q^2}{1-q} > 0.$$

The original inequality is an equality for $a=b=c$.

□

P 2.38. *If a,b,c are positive real numbers, then*

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3 \geq \frac{16(a^2+b^2+c^2-ab-bc-ca)}{a^2+b^2+c^2+6(ab+bc+ca)}.$$

(Vasile C., 2012)

Solution. Write the inequality as follows:

$$\sum\left(\frac{a}{b}+\frac{b}{a}\right)+\sum\left(\frac{a}{b}-\frac{b}{a}\right)-6 \geq \frac{32(a^2+b^2+c^2-ab-bc-ca)}{a^2+b^2+c^2+6(ab+bc+ca)}.$$

$$\frac{pq-3r}{r}-6+\frac{(a-b)(b-c)(c-a)}{r} \geq \frac{32(p^2-3q)}{p^2+4q},$$

$$pq+(a-b)(b-c)(c-a) \geq \frac{41p^2-60q}{p^2+4q}r.$$

It suffices to show that

$$pq \geq \frac{41p^2-60q}{p^2+4q}r+|(a-b)(b-c)(c-a)|.$$

Due to homogeneity, we may set $p=1$. From $p^2 \geq 3q$, we get

$$0 \leq q \leq \frac{1}{3}.$$

Thus, we need to show that

$$q \geq \frac{41-60q}{1+4q}r+|(a-b)(b-c)(c-a)|. \tag{*}$$

Case 1: $\dfrac{1}{28} \le q \le \dfrac{1}{3}$. Applying Corollary 1 for $\beta = \dfrac{41-60q}{27(1+4q)}$, we have

$$\frac{41-60q}{1+4q}r+|(a-b)(b-c)(c-a)| \le \frac{(41-60q)q}{3(1+4q)}-\frac{2(41-60q)}{27(1+4q)}+\frac{2A}{27(1+4q)}\,(1-3q)^{3/2},$$

where

$$A = \sqrt{27(1+4q)^2+(41-60q)^2},$$

with equality for

$$2(41-60q)(1-3q)^{3/2} = A(2-9q+27r). \tag{**}$$

Therefore, it suffices to show that

$$q \ge \frac{(41-60q)q}{3(1+4q)} - \frac{2(41-60q)}{27(1+4q)} + \frac{2A}{27(1+4q)}\,(1-3q)^{3/2},$$

which is equivalent to

$$(1-3q)\left(41-108q-A\sqrt{1-3q}\right) \ge 0.$$

This is true if

$$(41-108q)^2 \ge A^2(1-3q),$$

which can be written as

$$448q^3-240q^2+36q-1 \ge 0,$$

$$(4q-1)^2(28q-1) \ge 0.$$

The last inequality is true, with equality for $q = 1/4$. From (**), we get $r = 1/56$.

Case 2: $0 < q \le \dfrac{1}{28}$. Since

$$\begin{aligned}|(a-b)(b-c)(c-a)| &= \sqrt{-27r^2-2(2p^3-9pq)r+p^2q^2-4q^3} \\ &= \sqrt{-27r^2-2(2-9q)r+q^2-4q^3},\end{aligned}$$

the inequality (*) becomes

$$q-\frac{41-60q}{1+4q}r \ge \sqrt{-27r^2-2(2-9q)r+q^2-4q^3}.$$

From the known inequality $q^2 \ge 3pr$, we get $r \le q^2/3$, therefore

$$\begin{aligned}q-\frac{41-60q}{1+4q}r &\ge q-\frac{(41-60q)q^2}{3(1+4q)} \\ &= \frac{q(1-3q)(3-20q)}{3(1+4q)} > 0.\end{aligned}$$

Thus, we only need to show that

$$\left(q-\frac{41-60q}{1+4q}r\right)^2 \geq -27r^2-2(2-9q)r+q^2-4q^3,$$

which is equivalent to the obvious inequality

$$Ar^2+4Br+4q^3 \geq 0,$$

where

$$A=27+\left(\frac{41-60q}{1+q}\right)^2>0,$$

$$B=\frac{(1-21q)+12q^2}{1+4q}>0.$$

The original inequality is an equality for $a=b=c$, and also when a,b,c satisfy

$$(a-b)(b-c)(c-a)<0$$

and are proportional to the roots of the equation

$$56w^3-56w^2+14w-1=0.$$

The last equality conditions are equivalent to

$$\frac{a}{\sqrt{7}-\tan\frac{\pi}{7}}=\frac{b}{\sqrt{7}-\tan\frac{2\pi}{7}}=\frac{c}{\sqrt{7}-\tan\frac{4\pi}{7}}$$

(or any cyclic permutation).

Remark. This inequality is stronger than the inequality

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+2 \geq \frac{14(a^2+b^2+c^2)}{(a+b+c)^2}, \tag{A}$$

which is stronger than the inequality

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{7(ab+bc+ca)}{a^2+b^2+c^2} \geq \frac{17}{2}, \tag{B}$$

because

$$\frac{16(a^2+b^2+c^2-ab-bc-ca)}{a^2+b^2+c^2+6(ab+bc+ca)} \geq \frac{14(a^2+b^2+c^2)}{(a+b+c)^2}-2$$

and

$$\frac{14(a^2+b^2+c^2)}{(a+b+c)^2}-2 \geq \frac{17}{2}-\frac{7(ab+bc+ca)}{a^2+b^2+c^2},$$

respectively. Notice that the last inequalities are equivalent to

$$(a^2+b^2+c^2-2ab-2bc-2ca)^2 \ge 0.$$

The inequalities (A) and (B) are equalities for

$$\frac{a}{\sqrt{7}-\tan\frac{\pi}{7}} = \frac{b}{\sqrt{7}-\tan\frac{2\pi}{7}} = \frac{c}{\sqrt{7}-\tan\frac{4\pi}{7}}$$

(or any cyclic permutation).

□

P 2.39. *If a,b,c are real numbers such that $ab+bc+ca \ge 0$, then*

$$(a^2+b^2+c^2)(a^3+b^3+c^3)+5(a^4b+b^4c+c^4a) \ge 0.$$

(Vasile C., 2008)

Solution. Since the statement remains unchanged by replacing a,b,c with $-a,-b,-c$, respectively, we may assume $p \ge 0$. Since $p=0$ and $q \ge 0$ involve $a=b=c=0$, consider next $p>0$. Due to homogeneity, we may set

$$p=1, \qquad 0 \le q \le \frac{p^2}{3} = \frac{1}{3}.$$

The desired inequality becomes as follows:

$$2\left(\sum a^2\right)\left(\sum a^3\right)+5\sum ab(a^3+b^3)+5\sum ab(a^3-b^3) \ge 0,$$

$$2\left(\sum a^2\right)\left(\sum a^3\right)+5\left(\sum ab\right)\left(\sum a^3\right)-5abc\sum a+5\sum ab(a^3-b^3) \ge 0,$$

$$\left(2\sum a^2+5\sum ab\right)\left(\sum a^3\right)-5abc\sum a^2+5\sum ab(a^3-b^3) \ge 0,$$

$$(2p^2+q)(3r+p^3-3pq)-5(p^2-2q)r-5(p^2-q)(a-b)(b-c)(c-a) \ge 0,$$

$$(2+q)(3r+1-3q)-5(12-2q)r-5(1-q)(a-b)(b-c)(c-a) \ge 0,$$

$$2-5q-3q^2 \ge -(1+13q)r+5(1-q)(a-b)(b-c)(c-a).$$

Thus, we need to show that

$$\frac{2-5q-3q^2}{5(1-q)} \ge \frac{-(1+13q)r}{5(1-q)}+|(a-b)(b-c)(c-a)|.$$

Applying Corollary 1 for $\beta = \dfrac{-(1+13q)}{135(1-q)}$, we have

$$\frac{-(1+13q)r}{5(1-q)}+|(a-b)(b-c)(c-a)| \le$$

$$\leq \frac{-9(1+13q)q}{135(1-q)} + \frac{2(1+13q)}{135(1-q)} + \frac{2A}{135(1-q)}\,(1-3q)^{3/2},$$

where

$$A = \sqrt{675(1-q)^2 + (1+13q)^2},$$

with equality for

$$-2(1+13q)(1-3q)^{3/2} = A(2-9q+27r). \tag{*}$$

Therefore, it suffices to show that

$$\frac{2-5q-3q^2}{5(1-q)} \geq \frac{-9(1+13q)q}{135(1-q)} + \frac{2(1+13q)}{135(1-q)} + \frac{2A}{135(1-q)}\,(1-3q)^{3/2},$$

which can be rewritten as

$$26-75q+18q^2 \geq A(1-3q)^{3/2}.$$

Since

$$A = \sqrt{26^2 - 4q(331-211q)} \leq 26$$

and

$$\sqrt{1-3q} \leq 1,$$

it suffices to show that

$$26-75q+18q^2 \geq 26(1-3q),$$

which is equivalent to

$$3q(1+6q) \geq 0.$$

For $q = 0$, (*) gives $r = -1/13$. Thus, the original inequality is an equality when a, b, c satisfy

$$(a-b)(b-c)(c-a) \geq 0$$

and are proportional to the roots of the equation

$$13w^3 - 13w^2 + 1 = 0.$$

□

P 2.40. *If a, b, c are real numbers such that*

$$a+b+c = 3, \qquad ab+bc+ca \geq 0,$$

then

$$a^3b + b^3c + c^3a + 18\sqrt{3} \geq ab^3 + bc^3 + ca^3.$$

(Vasile C., 2008)

Solution. From $0 \le 3q \le p^2$, we get

$$0 \le q \le 3.$$

Write the inequality as

$$18\sqrt{3} \ge (a+b+c)(a-b)(b-c)(c-a),$$

$$6\sqrt{3} \ge (a-b)(b-c)(c-a).$$

It suffices to show that

$$6\sqrt{3} \ge |(a-b)(b-c)(c-a)|.$$

Since

$$|(a-b)(b-c)(c-a)| = \sqrt{\frac{4(p^2-3q)^3-(2p^3-9pq+27r)^2}{27}}$$

$$\le 2\sqrt{\frac{(p^2-3q)^3}{27}} = 2(3-q)\sqrt{3-q},$$

with equality for $27r = 9pq - 2p^3$, i.e. for

$$r = q-2,$$

we only need to show that

$$6\sqrt{3} \ge 2(3-q)\sqrt{3-q}.$$

Since $0 \le 3-q \le 3$, the inequality is clearly true, with equality for $q=0$. From $r = q-2 = -2$, it follows that the original inequality is an equality when a,b,c are the roots of the equation

$$w^3 - 3w^2 + 2 = 0$$

and $(a-b)(b-c)(c-a) \ge 0$. Equivalently, the original inequality is an equality for

$$a = -1, \qquad b = 2-\sqrt{2}, \qquad c = 2+\sqrt{2}$$

(or any cyclic permutation)

□

P 2.41. *If a,b,c are real numbers such that $a^2+b^2+c^2=3$, then*

$$a^3b + b^3c + c^3a + \frac{81\sqrt{2}}{32} \ge ab^3 + bc^3 + ca^3.$$

(Vasile C., 2008)

Solution. Write the inequality in the homogeneous form:

$$\frac{9\sqrt{2}}{32}(a^2+b^2+c^2)^2 \geq (a+b+c)(a-b)(b-c)(c-a). \tag{*}$$

Since the statement remains unchanged by replacing a,b,c with $-a,-b,-c$, respectively, we may assume $p \geq 0$. Thus, it suffices to show that

$$\frac{9\sqrt{2}}{32}(a^2+b^2+c^2)^2 \geq (a+b+c)|(a-b)(b-c)(c-a)|,$$

which can be written as

$$\frac{9\sqrt{2}}{32}(p^2-2q)^2 \geq p|(a-b)(b-c)(c-a)|,$$

Since

$$\begin{aligned}|(a-b)(b-c)(c-a)| &= \sqrt{\frac{4(p^2-3q)^3-(2p^3-9pq+27r)^2}{27}}\\ &\leq 2\sqrt{\frac{(p^2-3q)^3}{27}},\end{aligned}$$

with equality for $27r = 9pq - 2p^3$, we only need to show that

$$\frac{9\sqrt{2}}{32}(p^2-2q)^2 \geq 2p\sqrt{\frac{(p^2-3q)^3}{27}}.$$

This inequality is equivalent to

$$3^7(p^2-2q)^4 \geq 2^{11}p^2(p^2-3q)^3,$$

$$\left(\frac{3p^2-6q}{4}\right)^4 \geq p^2\left(\frac{2p^2-6q}{3}\right)^3.$$

Using the substitution

$$x = \frac{2p^2-6q}{3}, \quad x \geq 0,$$

the inequality becomes

$$\left(\frac{p^2+3x}{4}\right)^4 \geq p^2x^3.$$

This is the AM-GM inequality applied to four nonnegative numbers, with equality for $p^2 = x$. Thus, the homogeneous inequality (*) is an equality for

$$(a+b+c)(a-b)(b-c)(c-a) \geq 0,$$

$$6q = -p^2, \qquad 27r = 9pq - 2p^3 = -\frac{7}{2}p^3.$$

Since $p^2-2q=3$, we get the equality conditions

$$p=\frac{3}{2},\qquad q=\frac{-3}{8},\qquad r=\frac{-7}{16},$$

and

$$p=\frac{-3}{2},\qquad q=\frac{-3}{8},\qquad r=\frac{7}{16}.$$

In the first case, a,b,c are the roots of the equation

$$16w^3-24w^2-6w+7=0,$$

which is equivalent to

$$(2w-1)(8w^2-8w-7)=0.$$

In the second case, a,b,c are the roots of the equation

$$16w^3+24w^2-6w-7=0,$$

which is equivalent to

$$(2w+1)(8w^2+8w-7)=0.$$

Thus,the original inequality is an equality for

$$a=\frac{1}{2},\qquad b=\frac{2+3\sqrt{2}}{4},\qquad c=\frac{2-3\sqrt{2}}{4}$$

(or any cyclic permutation), and for

$$a=\frac{-1}{2},\qquad b=\frac{-2-3\sqrt{2}}{4},\qquad c=\frac{-2+3\sqrt{2}}{4}$$

(or any cyclic permutation)

□

Chapter 3

Highest Coefficient Cancellation Method for Symmetric Homogeneous Inequalities in Real Variables

3.1 Theoretical Basis

The Highest Coefficient Cancellation Method (HCC-Method) is especially applicable to symmetric homogeneous polynomial inequalities of six and eight degree. The main results in this section are based on the following Lemma (see P 2.53 in Volume 1):

Lemma. *If x,y,z are real numbers such that*

$$x+y+z=p, \qquad xy+yz+zx=q,$$

where p and q are given real numbers satisfying $p^2 \ge 3q$, then the product $r=xyz$ is minimal and maximal when two of x,y,z are equal.

3.1.1. Inequalities of degree six

A symmetric and homogeneous polynomial of degree six can be written in the form

$$f_6(x,y,z)=A_1\sum x^6+A_2\sum xy(x^4+y^4)+A_3\sum x^2y^2(x^2+y^2)+A_4\sum x^3y^3 \\ +A_5xyz\sum x^3+A_6xyz\sum xy(x+y)+3A_7x^2y^2z^2,$$

where $A_1,\cdots,A_7$ are real coefficients. In terms of

$$p=x+y+z, \quad q=xy+yz+zx, \quad r=xyz,$$

it can be rewritten as

$$f_6(x,y,z)=Ar^2+g_1(p,q)r+g_2(p,q),$$

where A is the *highest coefficient* of $f_6(x,y,z)$, and $g_1(p,q)$ and $g_2(p,q)$ are polynomial functions of the form

$$g_1(p,q) = Bp^3 + Cpq, \quad g_2(p,q) = Dp^6 + Ep^4q + Fp^2q^2 + Gq^3,$$

where B, C, D, E, F, G are real coefficients.

The highest coefficients of the polynomials

$$\sum x^6, \quad \sum xy(x^4+y^4), \quad \sum x^2y^2(x^2+y^2), \quad \sum x^3y^3,$$
$$xyz\sum x^3, \quad xyz\sum xy(x+y)$$

are, respectively,

$$3, \quad -3, \quad -3, \quad 3, \quad 3, \quad -3.$$

Therefore, the highest coefficient of $f_6(x,y,z)$ is

$$A = 3(A_1 - A_2 - A_3 + A_4 + A_5 - A_6 + A_7).$$

The polynomial

$$P_1(x,y,z) = \sum(A_1x^2 + A_2yz)(B_1x^2 + B_2yz)(C_1x^2 + C_2yz)$$

has the highest coefficient

$$A = 3(A_1 + A_2)(B_1 + B_2)(C_1 + C_2) = P_1(1,1,1). \tag{3.1}$$

Indeed, since

$$\begin{aligned} P_1(x,y,z) = & A_1B_1C_1\sum x^6 + A_2B_2C_2\sum x^3y^3 \\ & + (B_1C_1A_2 + C_1A_1B_2 + A_1B_1C_2)xyz\sum x^3 \\ & + 3(A_1B_2C_2 + B_1C_2A_2 + C_1A_2B_2)x^2y^2z^2, \end{aligned}$$

we have

$$\begin{aligned} A = & 3A_1B_1C_1 + 3A_2B_2C_2 + 3(B_1C_1A_2 + C_1A_1B_2 + A_1B_1C_2) \\ & + 3(A_1B_2C_2 + B_1C_2A_2 + C_1A_2B_2) \\ = & 3(A_1 + A_2)(B_1 + B_2)(C_1 + C_2). \end{aligned}$$

Similarly, we can show that the polynomial

$$P_2(x,y,z) = \sum(A_1x^2 + A_2yz)(B_1y^2 + B_2zx)(C_1z^2 + C_2xy)$$

has the highest coefficient

$$A = 3(A_1 + A_2)(B_1 + B_2)(C_1 + C_2) = P_2(1,1,1), \tag{3.2}$$

and the polynomial

$$P_3(x,y,z) = (A_1x^2 + A_2yz)(A_1y^2 + A_2zx)(A_1z^2 + A_2xy)$$

has the highest coefficient

$$A = (A_1 + A_2)^3 = P_3(1,1,1). \tag{3.3}$$

With regard to

$$P_4(x,y,z) = (x-y)^2(y-z)^2(z-x)^2,$$

from

$$P_4(x,y,z) = (p^2 - 2q - z^2 - 2xy)(p^2 - 2q - x^2 - 2yz)(p^2 - 2q - y^2 - 2zx),$$

it follows that P_4 has the same highest coefficient as

$$P_3(x,y,z) = (-z^2 - 2xy)(-x^2 - 2yz)(-y^2 - 2zx),$$

that is, according to (3.3),

$$A = P_3(1,1,1) = (-1-2)^3 = -27.$$

* * *

Based on Lemma above, Theorem 1 bellow gives for $A \le 0$ the necessary and sufficient conditions to have $f_6(x,y,z) \ge 0$ for all real numbers x,y,z which satisfy

$$k_1p^2 + k_2q \ge 0, \tag{3.4}$$

where k_1 and k_2 are given real numbers (see Remark 3 from P 2.75, Volume 1).

Theorem 1 (*Vasile Cirtoaje*, 2008). *Let $f_6(x,y,z)$ be a symmetric homogeneous polynomial of degree six which has the highest coefficient $A \le 0$. The inequality $f_6(x,y,z) \ge 0$ holds for all real numbers x,y,z satisfying (3.4) if and only if*

$$f_6(x,1,1) \ge 0$$

for all real x satisfying $k_1(x+2)^2 + k_2(2x+1) \ge 0$.

For $k_1 = k_2 = 0$, we get

Corollary 1. *Let $f_6(x,y,z)$ be a symmetric homogeneous polynomial of degree six which has the highest coefficient $A \le 0$. The inequality $f_6(x,y,z) \ge 0$ holds for all real numbers x,y,z if and only if*

$$f_6(x,1,1) \ge 0$$

for all real x.

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Further, consider the inequality

$$f_6(x,y,z) \ge 0,$$

where x,y,z are real numbers and $f_6(x,y,z)$ is a symmetric homogeneous polynomial of degree six with the highest coefficient $A > 0$. The highest coefficient cancellation method for proving such an inequality uses the above Theorem 1 and the following three ideas:

1) finding a nonnegative symmetric homogeneous function $\bar{f}_6(x,y,z)$ of the form

$$\bar{f}_6(x,y,z) = \left(r + A_1pq + A_2p^3 + A_3\frac{q^2}{p}\right)^2, \tag{3.5}$$

where A_1, A_2, A_3 are real numbers chosen such that

$$f_6(x,y,z) \ge A\bar{f}_6(x,y,z) \ge 0$$

for all real numbers x,y,z;

2) seeing that the difference $f_6(x,y,z) - A\bar{f}_6(x,y,z)$ has the highest coefficient equal to zero, therefore the inequality

$$f_6(x,y,z) \ge A\bar{f}_6(x,y,z)$$

holds if and only if it holds for $y = z = 1$ (Theorem 1);

3) choosing a suitable real number

$$\xi \in (-\infty, 0) \cup (3, \infty)$$

and treating successively the cases $p^2 < \xi q$ and $p^2 \ge \xi q$.

• In this chapter, we consider that the function $\bar{f}_6(x,y,z)$ depends on only two parameters α and β. Let us define the following nonnegative functions:

$$f_{\alpha,\beta}(x) = \frac{4(x-1)^4(x-\alpha)^2(x-\beta)^2}{9(4-\alpha-\beta-2\alpha\beta)^2(x+2)^2}, \tag{3.6}$$

$$\begin{aligned} g_{\alpha,\beta}(x) &= (x-\alpha)^2\bar{g}_{\alpha,\beta}(x) \\ &= (x-\alpha)^2\left[\frac{\alpha x^2 + \alpha(\alpha+6)x - 8}{(\alpha+2)^3} + \frac{\beta(x+2)(2\alpha x + x + \alpha - 4)}{(\alpha+2)^2}\right]^2, \end{aligned} \tag{3.7}$$

$$h_{\alpha,\beta}(x) = \left[x + \alpha(x+2)(2x+1) + \beta(x+2)^3\right]^2, \tag{3.8}$$

where α and β are real numbers. Note that the parameters α and β of the function $f_{\alpha,\beta}$ may be also conjugate complex numbers.

- The function $f_{\alpha,\beta}(x)$ has been derived by setting $y = z = 1$ in the associated function

$$\bar{f}_6(x,y,z) = \left(r - \frac{b_1}{a_1}pq + \frac{2}{a_1}p^3 + \frac{c_1}{a_1} \cdot \frac{q^2}{p} \right)^2 \tag{3.9}$$

with

$$a_1 = 3(4-\alpha-\beta-2\alpha\beta), \quad b_1 = 10+\alpha+\beta, \quad c_1 = 2(2+\alpha)(2+\beta),$$

which satisfies

$$\bar{f}_6(1,1,1) = 0, \quad \bar{f}_6(\alpha,1,1) = 0, \quad \bar{f}_6(\beta,1,1) = 0;$$

therefore,

$$f_{\alpha,\beta}(x) = \bar{f}_6(x,1,1) = \left[x - \frac{b_1}{a_1}(x+2)(2x+1) + \frac{2}{a_1}(x+2)^3 + \frac{c_1}{a_1} \cdot \frac{(2x+1)^2}{x+2} \right]^2.$$

Setting

$$\beta = -2, \quad \beta = -1, \quad \beta = 0, \quad \beta = 1, \quad \beta \to \infty$$

in (3.6), we get in succession:

$$f_{\alpha,-2}(x) = \frac{4(x-1)^4(x-\alpha)^2}{81(2+\alpha)^2},$$

$$f_{\alpha,-1}(x) = \frac{4(x+1)^2(x-1)^4(x-\alpha)^2}{9(5+\alpha)^2(x+2)^2},$$

$$f_{\alpha,0}(x) = \frac{4x^2(x-1)^4(x-\alpha)^2}{9(4-\alpha)^2(x+2)^2},$$

$$f_{\alpha,1}(x) = \frac{4(x-1)^6(x-\alpha)^2}{81(1-\alpha)^2(x+2)^2},$$

$$f_{\alpha,\infty}(x) = \frac{4(x-1)^4(x-\alpha)^2}{9(1+2\alpha)^2(x+2)^2}.$$

In particular,

$$f_{\infty,-2}(x) = \frac{4(x-1)^4}{81},$$

$$f_{-1,\infty}(x) = \frac{4(x-1)^4(x+1)^2}{9(x+2)^2},$$

$$f_{0,\infty}(x) = \frac{4x^2(x-1)^4}{9(x+2)^2},$$

$$f_{1,\infty}(x) = \frac{4(x-1)^6}{81(x+2)^2},$$

$$f_{\infty,\infty}(x) = \frac{(x-1)^4}{9(x+2)^2}.$$

Notice that the relative degree of the rational functions $f_{\alpha,\beta}(x)$, $f_{\alpha,\infty}(x)$ and $f_{\infty,\infty}(x)$ are six, four and two, respectively.

- The function $g_{\alpha,\beta}(x)$ is derived by setting $y = z = 1$ in the associated function

$$\bar{f}_6(x, y, z) = \left[r + \beta pq - \frac{\alpha + (\alpha+2)(2\alpha+1)\beta}{(\alpha+2)^3}p^3\right]^2$$

which satisfies

$$\bar{f}_6(\alpha, 1, 1) = 0.$$

Therefore,

$$g_{\alpha,\beta}(x) = \bar{f}_6(x, 1, 1) = \left[x + \beta(x+2)(2x+1) - \frac{\alpha + (\alpha+2)(2\alpha+1)\beta}{(\alpha+2)^3}(x+2)^3\right]^2.$$

Setting

$$\alpha = 0, \quad \alpha = -1, \quad \alpha = 1, \quad \beta = 0$$

in (3.7), we get:

$$\bar{g}_{0,\beta}(x) = \left(\frac{\beta}{4}(x+2)(x-4) - 1\right)^2,$$

$$\bar{g}_{-1,\beta}(x) = \left[x^2 + 5x + 8 + \beta(x+2)(x+5)\right]^2,$$

$$\bar{g}_{1,\beta}(x) = \frac{1}{9}(x-1)^2\left[\frac{x+8}{3} + \beta(x+2)\right]^2,$$

$$\bar{g}_{\alpha,0}(x) = \frac{[\alpha x^2 + \alpha(\alpha+6)x - 8]^2}{(\alpha+2)^6},$$

In particular,

$$\bar{g}_{0,0}(x) = 1,$$

$$\bar{g}_{-1,0}(x) = (x^2 + 5x + 8)^2,$$

$$\bar{g}_{1,0}(x) = \frac{(x-1)^2(x+8)^2}{729}.$$

- The function $h_{\alpha,\beta}(x)$ is derived by setting $y = z = 1$ in the associated function

$$\bar{f}_6(x, y, z) = \left(r + \alpha pq + \beta p^3\right)^2.$$

Therefore,

$$h_{\alpha,\beta}(x) = \bar{f}_6(x,1,1) = \left[x + \alpha(x+2)(2x+1) + \beta(x+2)^3\right]^2.$$

- Let
$$\xi \in (-\infty, 0) \cup (3, \infty).$$

For $y = z = 1$, the condition
$$p^2 < \xi q$$
involves
$$x \in \mathbb{I}, \qquad \mathbb{I} = \left(\xi - 2 - \sqrt{\xi^2 - 3\xi}\ ,\ \xi - 2 + \sqrt{\xi^2 - 3\xi}\right), \tag{3.10}$$
while the condition
$$p^2 \ge \xi q$$
involves
$$x \in \mathbb{R} \setminus \mathbb{I}.$$
Using the substitution
$$\xi = \frac{(2+\eta)^2}{1+2\eta},$$
we have
$$\mathbb{I} = \left(\eta, \frac{4-\eta}{1+2\eta}\right), \qquad \eta \in (-\infty, -2) \cup \left(\frac{-1}{2}, 1\right),$$
$$\mathbb{I} = \left(\frac{4-\eta}{1+2\eta}, \eta\right), \qquad \eta \in \left(-2, \frac{-1}{2}\right) \cup (1, \infty).$$

Thus, we can check that

$$\begin{array}{rclll}
\eta \to -\infty & \Rightarrow & \xi \to -\infty, & \mathbb{I} = (-\infty, -1/2), \\
& \cdots & & \\
\eta = -5 & \Rightarrow & \xi = -1, & \mathbb{I} = (-5, -1), \\
\eta = -4 & \Rightarrow & \xi = -4/7, & \mathbb{I} = (-4, -8/7), \\
\eta = -3 & \Rightarrow & \xi = -1/5, & \mathbb{I} = (-3, -7/5), \\
\eta = -2 & \Rightarrow & \xi = 0, & \mathbb{I} = (-2, -2) = \emptyset,
\end{array}$$

and

$$\begin{array}{rclll}
\eta = 1 & \Rightarrow & \xi = 3, & \mathbb{I} = (1, 1) = \emptyset, \\
\eta = 2 & \Rightarrow & \xi = 16/5, & \mathbb{I} = (2/5, 2), \\
\eta = 3 & \Rightarrow & \xi = 25/7, & \mathbb{I} = (1/7, 3), \\
\eta = 4 & \Rightarrow & \xi = 4, & \mathbb{I} = (0, 4), \\
& \cdots & & \\
\eta \to \infty & \Rightarrow & \xi \to \infty, & \mathbb{I} = (-1/2, \infty).
\end{array}$$

Theorem 2. *Let $f_6(x,y,z)$ be a symmetric homogeneous polynomial of degree six with the highest coefficient $A>0$. The inequality $f_6(x,y,z)\ge 0$ holds for any real numbers x,y,z if there exist five real numbers α, β, γ, δ and ξ, and*

$$E_{\alpha,\beta}\in\{f_{\alpha,\beta},g_{\alpha,\beta},h_{\alpha,\beta}\},$$

$$F_{\gamma,\delta}\in\{f_{\gamma,\delta},g_{\gamma,\delta},h_{\gamma,\delta}\},$$

such that the following two conditions are satisfied:

(a) $f_6(x,1,1)\ge AE_{\alpha,\beta}(x)$ *for* $x\in\mathbb{I}$;

(b) $f_6(x,1,1)\ge AF_{\gamma,\delta}(x)$ *for* $x\in\mathbb{R}\setminus\mathbb{I}$.

Proof. Let

$$E_1(x,y,z)=\left(r+A_1pq+A_2p^3+A_3\frac{q^2}{p}\right)^2$$

and

$$F_1(x,y,z)=\left(r+B_1pq+B_2p^3+B_3\frac{q^2}{p}\right)^2$$

be the functions associated to $E_{\alpha,\beta}(x)$ and $F_{\gamma,\delta}(x)$, respectively; this means that

$$E_1(x,1,1)=E_{\alpha,\beta}(x),$$

$$F_1(x,1,1)=F_{\gamma,\delta}(x).$$

Let us denote

$$E_2(x,y,z)=f_6(x,y,z)-AE_1(x,y,z)$$

and

$$F_2(x,y,z)=f_6(x,y,z)-AF_1(x,y,z).$$

Since $AE_1(x,y,z)\ge 0$ and $AF_1(x,y,z)\ge 0$, the inequality $f_6(x,y,z)\ge 0$ holds for all real x,y,z if

(a) $E_2(x,y,z)\ge 0$ for $p^2<\xi q$;

(b) $F_2(x,y,z)\ge 0$ for $p^2\ge \xi q$.

According to Theorem 1, because $E_2(x,y,z)$ and $F_2(x,y,z)$ has the highest coefficient zero, these conditions are satisfied if and only if

(a) $E_2(x,1,1)\ge 0$ for $(x+2)^2<\xi(2x+1)$;

(b) $F_2(x,1,1)\ge 0$ for $(x+2)^2\ge\xi(2x+1)$.

Since these conditions are equivalent to

(a) $f_6(x,1,1)-AE_{\alpha,\beta}(x)\ge 0$ for $x\in\mathbb{I}$;

(b) $f_6(x,1,1)-AF_{\gamma,\delta}(x)\ge 0$ for $x\in\mathbb{R}\setminus\mathbb{I}$,

the proof is completed.

For $\xi = 0$ or $\xi = 3$, when $\mathbb{I} = \emptyset$, we get

Corollary 2. *Let $f_6(x, y, z)$ be a symmetric homogeneous polynomial of degree six with the highest coefficient $A > 0$. The inequality $f_6(x, y, z) \geq 0$ holds for any real numbers x, y, z if there exist two real numbers γ and δ such that $f_6(x, 1, 1) \geq AF_{\gamma,\delta}(x)$ for all $x \in \mathbb{R}$.*

The following Proposition 1 is useful in many applications based on Theorem 2 or Corollary 2.

Proposition 1. *Let $f_6(x, y, z)$ be a symmetric homogeneous polynomial of degree six, and let*

$$f(x) = f_6(x, 1, 1) - AE_{\alpha,\beta}(x),$$

where A is the highest coefficient of $f_6(x, y, z)$ and

$$E_{\alpha,\beta} \in \{f_{\alpha,-2}, g_{\alpha,\beta}, h_{\alpha,\beta}\}.$$

If

$$f_6(0, 1, -1) = 0,$$

then

$$f(-2) = 0.$$

Proof. Write $f_6(x, y, z)$ in the form

$$f_6(x, y, z) = Ar^2 + g_1(p, q)r + g_2(p, q).$$

where

$$g_1(p, q) = Bp^3 + Cpq,$$

$$g_2(p, q) = Dp^6 + Ep^4q + Fp^2q^2 + Gq^3,$$

For $(x, y, z) = (0, 1, -1)$, we have $r = 0$, $p = 0$ and $q = -1$, therefore

$$f_6(0, 1, -1) = -G.$$

For $(x, y, z) = (-2, 1, 1)$, we have $r = -2$, $p = 0$ and $q = -3$, therefore

$$f_6(-2, 1, 1) = 4A - 27G = 4A + 27f_6(0, 1, -1).$$

Thus, the hypothesis $f_6(0, 1, -1) = 0$ involves

$$f_6(-2, 1, 1) = 4A.$$

Since

$$f_{\alpha,-2}(-2) = 4, \qquad g_{\alpha,\beta}(-2) = 4, \qquad h_{\alpha,\beta}(-2) = 4,$$

we get

$$f(-2) = f_6(-2, 1, 1) - AE_{\alpha,\beta}(-2) = 4A - 4A = 0.$$

Remark 1. The function $\bar{f}_6(x,y,z)$ associated to $f_{\alpha,\beta}(x)$ (given by (3.9)) is zero for

$$(x,y,z)=(1,1,1), \quad (x,y,z)=(\alpha,1,1), \quad (x,y,z)=(\beta,1,1).$$

If $\beta=-2$, then the function associated to $f_{\alpha,-2}(x)$ has the expression

$$\bar{f}_6(x,y,z)=\left(r-\frac{\alpha+8}{9\alpha+18}pq+\frac{2}{9\alpha+18}p^3\right)^2, \tag{3.11}$$

and is zero for

$$(x,y,z)=(1,1,1), \quad (x,y,z)=(\alpha,1,1), \quad (x,y,z)=(0,1,-1).$$

In addition, if $\alpha\to\infty$, then

$$\bar{f}_6(x,y,z)=\left(r-\frac{1}{9}pq\right)^2$$

is zero for

$$(x,y,z)=(1,1,1), \quad (x,y,z)=(1,0,0), \quad (x,y,z)=(0,1,-1).$$

If $\beta\to\infty$, then

$$\bar{f}_6(x,y,z)=\left(r+\frac{1}{6\alpha+3}pq-\frac{2\alpha+4}{6\alpha+3}\cdot\frac{q^2}{p}\right)^2$$

is zero for

$$(x,y,z)=(1,1,1), \quad (x,y,z)=(\alpha,1,1), \quad (x,y,z)=(1,0,0).$$

If, in addition, $\alpha\to\infty$, then

$$\bar{f}_6(x,y,z)=\left(r-\frac{1}{3}\cdot\frac{q^2}{p}\right)^2$$

is zero for

$$(x,y,z)=(1,1,1), \quad (x,y,z)=(1,0,0).$$

Remark 2. If

$$f_6(x,1,1)=Af_{\gamma,-2}(x)$$

for all $x\in\mathbb{R}$, then there is $k\geq 0$ such that the following identity holds:

$$f_6(x,y,z)=A\bar{f}_6(x,y,z)+k(x-y)^2(y-z)^2(z-x)^2,$$

where, according to (3.11),

$$\bar{f}_6(x,y,z)=\left(r-\frac{\gamma+8}{9\gamma+18}pq+\frac{2}{9\gamma+18}p^3\right)^2.$$

In addition, if the coefficient of the product

$$(x-y)^2(y-z)^2(z-x)^2$$

is the best possible in the inequality $f_6(x,y,z) \geq 0$, then $k=0$ and the following identity holds:

$$f_6(x,y,z) = A\left(r - \frac{\gamma+8}{9\gamma+18}pq + \frac{2}{9\gamma+18}p^3\right)^2. \tag{3.12}$$

Remark 3. Theorem 2 is also valid for the case where the parameters α and β of the function $f_{\alpha,\beta}(x)$ are conjugate complex numbers. For example, if $k>0$ and

$$\alpha = \sqrt{-k}, \qquad \beta = -\sqrt{-k},$$

then, according to (3.6), we have

$$f_{\sqrt{-k},-\sqrt{-k}} = \frac{(x-1)^4(x^2+k)^2}{9(k-2)^2(x+2)^2}. \tag{3.13}$$

Remark 4. Consider the inequality $f_6(x,y,z) \geq 0$, where

$$f_6(0,1,-1) = 0.$$

- If

$$f_6(x,1,1) = (x-\alpha)^2 g(x), \quad \alpha \neq -2,$$

where g is a polynomial function, then the condition (a) in Theorem 2 applied for

$$E_{\alpha,\beta} = g_{\alpha,\beta}, \qquad -2 \in \mathbb{I},$$

is satisfied if and only if

$$\bar{g}(x) \geq 0, \qquad x \in \mathbb{I},$$

where

$$\bar{g}(x) = g(x) - A\bar{g}_{\alpha,\beta}(x),$$

$$\bar{g}_{\alpha,\beta}(x) = \left[\frac{\alpha x^2 + \alpha(\alpha+6)x - 8}{(\alpha+2)^3} + \frac{\beta(x+2)(2\alpha x + x + \alpha - 4)}{(\alpha+2)^2}\right]^2. \tag{3.14}$$

Denote

$$f(x) = f_6(x,1,1) - Ag_{\alpha,\beta}(x).$$

From

$$f(x) = (x-\alpha)^2 g(x) - A(x-\alpha)^2 \bar{g}_{\alpha,\beta}(x) = (x-\alpha)^2 \bar{g}(x)$$

and $f(-2) = 0$ (see Proposition 1), it follows that

$$\bar{g}(-2) = 0.$$

To have $\bar{g}(x) \ge 0$ in the vicinity of $x = -2 \in \mathbb{I}$, the condition $\bar{g}'(-2) = 0$ is necessary. This condition involves

$$\beta = \frac{\alpha}{3(\alpha+2)} + \frac{(\alpha+2)^2 g'(-2)}{12A}. \tag{3.15}$$

In conclusion, in the case $x = -2 \in \mathbb{I}$, the condition (a) in Theorem 2 is satisfied if and only if

$$g(x) \ge A\bar{g}_{\alpha,\beta}(x), \qquad x \in \mathbb{I},$$

with β given by (3.15).

Similarly, in the case $-2 \in \mathbb{R} \setminus \mathbb{I}$ $(\mathbb{I} = \emptyset$ in Corollary 2), the condition (b) in Theorem 2 or the condition in Corollary 2 are satisfied if and only if

$$g(x) \ge A\bar{g}_{\gamma,\delta}(x), \qquad x \in \mathbb{R} \setminus \mathbb{I},$$

with δ given by

$$\delta = \frac{\gamma}{3(\gamma+2)} + \frac{(\gamma+2)^2 g'(-2)}{12A} \tag{3.16}$$

and

$$\bar{g}_{\gamma,\delta}(x) = \left[\frac{\gamma x^2 + \gamma(\gamma+6)x - 8}{(\gamma+2)^3} + \frac{\delta(x+2)(2\gamma x + x + \gamma - 4)}{(\gamma+2)^2}\right]^2. \tag{3.17}$$

- The condition (a) in Theorem 2 applied for

$$E_{\alpha,\beta} = h_{\alpha,\beta}, \qquad -2 \in \mathbb{I},$$

has the form

$$f(x) \ge 0, \qquad x \in \mathbb{I},$$

where

$$f(x) = f_6(x,1,1) - Ah_{\alpha,\beta}(x).$$

According to Proposition 1, we have

$$f(-2) = 0.$$

In order to have $f(x) \ge 0$ in the vicinity of $x = -2 \in \mathbb{I}$, the condition $f'(-2) = 0$ is necessary. This condition involves

$$\alpha = \frac{1}{3} + \frac{h'(-2)}{12A}, \tag{3.18}$$

where

$$h(x) = f_6(x,1,1).$$

In conclusion, in the case $x = -2 \in \mathbb{I}$, the condition (a) in Theorem 2 is satisfied if and only if

$$f_6(x,1,1) \geq Ah_{\alpha,\beta}(x), \qquad x \in \mathbb{I},$$

with α given by (3.18).

Similarly, in the case $-2 \in \mathbb{R} \setminus \mathbb{I}$ $\quad$ ($\mathbb{I} = \emptyset$ in Corollary 2), the condition (b) in Theorem 2 or the condition in Corollary 2 are satisfied if and only if

$$f_6(x,1,1) \geq Ah_{\gamma,\delta}(x), \qquad x \in \mathbb{R} \setminus \mathbb{I},$$

with

$$\gamma = \frac{1}{3} + \frac{h'(-2)}{12A} \tag{3.19}$$

and

$$h_{\gamma,\delta}(x) = \left[x + \gamma(x+2)(2x+1) + \delta(x+2)^3\right]^2. \tag{3.20}$$

Remark 5. By Theorem 1, it follows that Theorem 2 and Corollary 2 can be extended to the case where the real numbers x, y, z satisfy the condition

$$k_1(x+y+z)^2 + k_2(xy+yz+zx) \geq 0,$$

where k_1 and k_2 are two fixed real numbers. More precisely, all conditions concerning $f_6(x,1,1)$ in Theorem 2 and Corollary 2 need to be satisfied only for

$$k_1(x+2)^2 + k_2(2x+1) \geq 0.$$

3.1.2. Inequalities of degree eight

A symmetric and homogeneous polynomial of degree eight $f_8(x,y,z)$ can be written in term of

$$p = x+y+z, \quad q = xy+yz+zx, \quad r = xyz,$$

in the form

$$g(r) = A(p,q)r^2 + g_3(p,q)r + g_4(p,q),$$

where $A(p,q)$, $g_3(p,q)$ and $g_4(p,q)$ are polynomial functions. The *highest polynomial* $A(p,q)$ of $f_8(x,y,z)$ has the form

$$A(p,q) = \mu_1 p^2 + \mu_2 q, \tag{3.21}$$

where μ_1 and μ_2 are real constants.

***

The following theorem is an extension of Theorem 1 to symmetric and homogeneous polynomial inequalities of degree eight.

Theorem 3. *Let $f_8(x,y,z)$ be an eighth degree symmetric homogeneous polynomial which has the highest polynomial $A(p,q)$. The inequality $f_8(x,y,z) \ge 0$ holds for all real x,y,z satisfying $A(p,q) \le 0$ if and only if*

$$f_8(x,1,1) \ge 0$$

for all real x such that $A(x+2,2x+1) \le 0$.

Proof. For fixed p and q, the inequality $f_8(x,y,z) \ge 0$ can be written as $g(r) \ge 0$, where

$$g(r) = A(p,q)r^2 + g_3(p,q)r + g_4(p,q)$$

is a quadratic function. Since $g(r)$ is concave for $A(p,q) \le 0$, it is minimal when r is minimal or maximal, that is, when two of x,y,z are equal (see Lemma above). Due to the homogeneity (of even degree), we may take $y = z = 1$ and $y = z = 0$. As shown in Volume 1 (Remark 3 from P 2.75), the case $y = z = 0$ is not necessary.

Corollary 3. *Let $f_8(x,y,z)$ be an eighth degree symmetric homogeneous polynomial having the highest polynomial $A(p,q)$. The inequality $f_8(x,y,z) \ge 0$ holds for all real x,y,z satisfying $A(p,q) \le 0$ if $f_8(x,1,1) \ge 0$ for all real x.*

* * *

The following theorem is an extension of Theorem 2 to symmetric and homogeneous polynomial inequalities of degree eight.

Theorem 4. *Let $f_8(x,y,z)$ be a symmetric homogeneous polynomial of degree eight having the highest coefficient $A(p,q)$. The inequality $f_8(x,y,z) \ge 0$ holds for any real numbers x,y,z satisfying $A(p,q) \ge 0$ if there exist five real numbers α, β, γ, δ and ξ, and*

$$E_{\alpha,\beta} \in \{f_{\alpha,\beta}, g_{\alpha,\beta}, h_{\alpha,\beta}\},$$

$$F_{\gamma,\delta} \in \{f_{\gamma,\delta}, g_{\gamma,\delta}, h_{\gamma,\delta}\},$$

such that the following two conditions are satisfied:

(a) $f_8(x,1,1) \ge AE_{\alpha,\beta}(x)$ *for* $x \in \mathbb{I}$ *and* $A(x+2,2x+1) \ge 0$;

(b) $f_8(x,1,1) \ge AF_{\gamma,\delta}(x)$ *for* $x \in \mathbb{R} \setminus \mathbb{I}$ *and* $A(x+2,2x+1) \ge 0$.

For $\xi = 0$ or $\xi = 3$, when $\mathbb{I} = \emptyset$, we get

Corollary 4. *Let $f_8(x,y,z)$ be a symmetric homogeneous polynomial of degree eight having the highest coefficient $A(p,q)$. The inequality $f_8(x,y,z) \ge 0$ holds for any real numbers x,y,z satisfying $A(p,q) \ge 0$ if there exist two real numbers γ and δ such that*

$$f_8(x,1,1) \ge A(x+2,2x+1)E_{\gamma,\delta}(x)$$

for all real x satisfying $A(x+2,2x+1) \ge 0$, where

$$E_{\gamma,\delta} \in \{f_{\gamma,\delta}, g_{\gamma,\delta}, h_{\gamma,\delta}\}.$$

Remark 6. Theorem 3, Theorem 4, Corollary 3 and Corollary 4 can be extended to the case where the real numbers x, y, z satisfy the condition

$$k_1(x+y+z)^2 + k_2(xy+yz+zx) \geq 0,$$

where k_1 and k_2 are two fixed real numbers. More precisely, all conditions concerning $f_8(x,1,1)$ need to be satisfied for

$$k_1(x+2)^2 + k_2(2x+1) \geq 0.$$

3.2 Applications

3.1. If x, y, z are real numbers, then

$$\sum(x+2y)(x+2z)(2x+y)(2x+z)(x-y)(x-z)+3(x-y)^2(y-z)^2(z-x)^2 \geq 0.$$

3.2. If x, y, z are real numbers, then

$$\sum(x^2-4yz)^2(x-y)(x-z)+3(x-y)^2(y-z)^2(z-x)^2 \geq 0.$$

3.3. If x, y, z are real numbers, then

$$\sum(x^2+yz)(x+2y)(x+2z)(x-y)(x-z)+2(x-y)^2(y-z)^2(z-x)^2 \geq 0.$$

3.4. If x, y, z are real numbers, then

$$\frac{2x^2+3yz}{4x^2+y^2+z^2}+\frac{2y^2+3zx}{4y^2+z^2+x^2}+\frac{2z^2+3xy}{4z^2+x^2+y^2} \leq \frac{5}{2}.$$

3.5. If x, y, z are real numbers, then

$$\frac{9x^2-4yz}{3x^2+2y^2+2z^2}+\frac{9y^2-4zx}{3y^2+2z^2+2x^2}+\frac{9z^2-4xy}{3z^2+2x^2+2y^2} \geq \frac{15}{7}.$$

3.6. Let x, y, z be real numbers, no two of which are zero. If

$$x^2+y^2+z^2 \geq 2(xy+yz+zx),$$

then

$$\frac{x^2-6yz}{y^2+yz+z^2}+\frac{y^2-6zx}{z^2+zx+x^2}+\frac{z^2-6xy}{x^2+xy+y^2} \geq 0.$$

3.7. If x, y, z be real numbers, no two of which are zero, then

$$\frac{8x^2+3yz}{y^2+yz+z^2}+\frac{8y^2+3zx}{z^2+zx+x^2}+\frac{8z^2+3xy}{x^2+xy+y^2} \geq 11.$$

3.8. If x, y, z are real numbers, no two of which are zero, then

$$\frac{8x^2-5yz}{y^2-yz+z^2}+\frac{8y^2-5zx}{z^2-zx+x^2}+\frac{8z^2-5xy}{x^2-xy+y^2} \geq 9.$$

3.9. If x, y, z are real numbers, no two of which are zero, then

$$\frac{5x^2+2yz}{2y^2+3yz+2z^2}+\frac{5y^2+2zx}{2z^2+3zx+2x^2}+\frac{5z^2+2xy}{2x^2+3xy+2y^2} \geq 3.$$

3.10. If x, y, z are real numbers, then

$$\frac{(x+y)(x+z)}{7x^2+y^2+z^2}+\frac{(y+z)(y+x)}{7y^2+z^2+x^2}+\frac{(z+x)(z+y)}{7z^2+x^2+y^2} \leq \frac{4}{3}.$$

3.11. If x, y, z are real numbers, then

$$\frac{6x(y+z)-yz}{12x^2+y^2+z^2}+\frac{6y(z+x)-zx}{12y^2+z^2+x^2}+\frac{6z(x+y)-xy}{12z^2+x^2+y^2} \leq \frac{33}{4}.$$

3.12. If x, y, z are real numbers, then

$$\frac{x(y+z)-yz}{x^2+3y^2+3z^2}+\frac{y(z+x)-zx}{y^2+3z^2+3x^2}+\frac{z(x+y)-xy}{z^2+3x^2+3y^2} \leq \frac{3}{7}.$$

3.13. If x, y, z are real numbers, then

$$\sum yz(2x^2+yz)(x-y)(x-z)+\frac{1}{2}(x-y)^2(y-z)^2(z-x)^2 \geq 0.$$

3.14. If x, y, z are real numbers, then

$$\sum(x^2-yz)^2(x-y)(x-z) \geq \frac{3}{4}(x-y)^2(y-z)^2(z-x)^2.$$

3.15. If x, y, z are real numbers, then

$$\sum(x^2+8yz)^2(x-y)(x-z)+15(x-y)^2(y-z)^2(z-x)^2 \geq 0.$$

3.16. Let x, y, z be real numbers. If $k \in \mathbb{R}$, then

$$\sum(x^2+yz)(x-y)(x-z)(x-ky)(x-kz)+\frac{7k^2-20k-20}{24}(x-y)^2(y-z)^2(z-x)^2 \geq 0.$$

3.17. If x, y, z are distinct real numbers, then

$$\frac{yz}{(y-z)^2}+\frac{zx}{(z-x)^2}+\frac{xy}{(x-y)^2}+\frac{1}{4} \geq 0.$$

3.18. Let x, y, z be distict real numbers. If $k \in \mathbb{R}$, then

$$\frac{(x-ky)(x-kz)}{(y-z)^2}+\frac{(y-kz)(y-kx)}{(z-x)^2}+\frac{(z-kx)(z-ky)}{(x-y)^2} \geq 2+2k-\frac{k^2}{4}.$$

3.19. If x, y, z are real numbers, then

$$\sum(x^2+2yz)(x^2-y^2)(x^2-z^2)+\frac{1}{2}(x-y)^2(y-z)^2(z-x)^2 \geq 0.$$

3.20. Let x, y, z be real numbers. If $k \in \mathbb{R}$, then

$$\sum(x-y)(x-z)(x-ky)(x-kz) \geq \frac{3(k+2)^2(x-y)^2(y-z)^2(z-x)^2}{4(x^2+y^2+z^2-xy-yz-zx)}.$$

3.21. Let x, y, z be real numbers such that $xy+yz+zx \geq 0$. If $k \in \mathbb{R}$, then

$$\sum(x^2+yz)(x-y)(x-z)(x-ky)(x-kz) \geq \left(1+k-\frac{k^2}{4}\right)(x-y)^2(y-z)^2(z-x)^2.$$

3.22. If x, y, z are real numbers, then

$$\sum(x^2+2yz)^2(x-y)(x-z) \geq 0.$$

3.23. If x, y, z are real numbers, then

$$\sum x^2(x^2+yz)(x-y)(x-z) \geq (x-y)^2(y-z)^2(z-x)^2.$$

3.24. Let x, y, z be real numbers. If $0 \le k \le 27$, then

$$\sum(x^2+yz)(x-y)(x-z)(x-ky)(x-kz) \ge \left(1+k-\frac{k^2}{4}\right)(x-y)^2(y-z)^2(z-x)^2.$$

3.25. If x, y, z are real numbers, then

$$\sum(x^2+yz)(x-y)(x-z)(x-28y)(x-28z)+167(x-y)^2(y-z)^2(z-x)^2 \ge 0.$$

3.26. If x, y, z are real numbers, then

$$\sum(x^2+yz)(x^2-y^2)(x^2-z^2)+\frac{1}{4}(x-y)^2(y-z)^2(z-x)^2 \ge 0.$$

3.27. Let x, y, z be real numbers. If $-2 \le k \le \frac{-3}{2}$, then

$$\sum(x^2+yz)(x-y)(x-z)(x-ky)(x-kz) \ge \left(1+k-\frac{k^2}{4}\right)(x-y)^2(y-z)^2(z-x)^2,$$

3.28. Let x, y, z be real numbers. If $-5 \le k \le -2$ and

$$\delta_k = \frac{k^4-8k^3-7k^2-20k-20}{4(k-1)^2},$$

then

$$\sum(x^2+yz)(x-y)(x-z)(x-ky)(x-kz)+\delta_k(x-y)^2(y-z)^2(z-x)^2 \ge 0.$$

3.29. If x, y, z are real numbers, then

$$\sum(x^2+yz)(x+y)(x+z) \ge \frac{15(x-y)^2(y-z)^2(z-x)^2}{32(x^2+y^2+z^2)}.$$

3.30. If x, y, z are real numbers, then

$$\sum(x-y)(x-z)(x^2+y^2)(x^2+z^2) \ge \frac{7}{4}(x-y)^2(y-z)^2(z-x)^2.$$

3.31. If x, y, z are real numbers such that $xy + yz + zx \geq 0$, then

$$\sum (x-y)(x-z)(x^2+y^2)(x^2+z^2) \geq \frac{15}{4}(x-y)^2(y-z)^2(z-x)^2.$$

3.32. If x, y, z are real numbers, then

$$\sum (x-y)(x-z)(x^2+xy+y^2)(x^2+xz+z^2) \geq \frac{3}{4}(x-y)^2(y-z)^2(z-x)^2.$$

3.33. If x, y, z are real numbers such that $xy + yz + zx \geq 0$, then

$$\sum (x-y)(x-z)(x^2+xy+y^2)(x^2+xz+z^2) \geq 3(x-y)^2(y-z)^2(z-x)^2.$$

3.34. If x, y, z are real numbers, then

$$(x^2+y^2)(y^2+z^2)(z^2+x^2) \geq 8x^2y^2z^2 + \frac{3}{8}(x-y)^2(y-z)^2(z-x)^2.$$

3.35. If x, y, z are real numbers, then

$$(x^2+2y^2+2z^2)(y^2+2z^2+2x^2)(z^2+2x^2+2y^2) \geq 125x^2y^2z^2 + \frac{15}{2}(x-y)^2(y-z)^2(z-x)^2.$$

3.36. If x, y, z are real numbers, then

$$(2x^2+y^2+z^2)(2y^2+z^2+x^2)(2z^2+x^2+y^2) \geq 64x^2y^2z^2 + \frac{15}{4}(x-y)^2(y-z)^2(z-x)^2.$$

3.37. If x, y, z are real numbers, then

$$8(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2) \geq 3(x^2+y^2)(y^2+z^2)(z^2+x^2).$$

3.38. If x, y, z are nonnegative real numbers, then

$$\sum (16x^2+3yz)(x-y)(x-z)(x-4y)(x-4z) + 52(x-y)^2(y-z)^2(z-x)^2 \geq 0.$$

3.39. If x,y,z are real numbers, then

$$\sum x^4(x-y)(x-z) \ge (x-y)^2(y-z)^2(z-x)^2.$$

3.40. Let x,y,z be real numbers. If $\dfrac{13-3\sqrt{17}}{2} \le k \le \dfrac{13+3\sqrt{17}}{2}$, then

$$\sum (x^2+kyz)^2(x-y)(x-z) + \left(\frac{k^2}{4}-1\right)(x-y)^2(y-z)^2(z-x)^2 \ge 0.$$

3.41. Let x,y,z be real numbers. If $k \in (-\infty,-4] \cup [-1,0]$, then

$$\sum (x^2+kyz)^2(x-y)(x-z) + \left(\frac{k^2}{4}-1\right)(x-y)^2(y-z)^2(z-x)^2 \ge 0.$$

3.42. Let x,y,z be real numbers. If $k \ge 0$, then

$$\sum (x^2+kyz)^2(x-y)(x-z) + \left(\frac{k^2}{4}-1\right)(x-y)^2(y-z)^2(z-x)^2 \ge 0.$$

3.43. If x,y,z are nonnegative real numbers, then

$$\sum x^2(x^2+8yz)(x-y)(x-z) \ge (x-y)^2(y-z)^2(z-x)^2.$$

3.44. If x,y,z are real numbers, then

$$\sum (x-y)(x-z)(x-2y)(x-2z)(2x-y)(2x-z) + 15(x-y)^2(y-z)^2(z-x)^2 \ge 0.$$

3.45. If x,y,z are real numbers, then

$$\sum (x-y)(x-z)(x-2y)(x-2z)(x-3y)(x-3z) \ge 3(x-y)^2(y-z)^2(z-x)^2.$$

3.46. If x,y,z are real numbers, then

$$\sum (x-y)(x-z)(2x+3y)(2x+3z)(3x+2y)(3x+2z) + 15(x-y)^2(y-z)^2(z-x)^2 \ge 0.$$

3.47. If x, y, z are real numbers, then

$$\sum(x+y)(x+z)(x^2-y^2)(x^2-z^2)+\frac{1}{4}(x-y)^2(y-z)^2(z-x)^2 \geq 0.$$

3.48. If x, y, z are real numbers, then

$$\sum(x-y)(x-z)(x-4y)^2(x-4z)^2+39(x-y)^2(y-z)^2(z-x)^2 \geq 0.$$

3.49. Let x, y, z be real numbers, and let

$$\alpha_k = \begin{cases} \dfrac{20+12k-4k^2-k^4}{4(1-k)^2}, & k \in (-\infty,-2] \cup [4,\infty) \\ 1+k, & k \in [-2,1] \\ 5-3k, & k \in [1,4] \end{cases}.$$

Then,

$$\sum x^2(x-y)(x-z)(x-ky)(x-kz) \geq \alpha_k(x-y)^2(y-z)^2(z-x)^2.$$

3.50. Let x, y, z be real numbers, and let

$$\beta_k = \begin{cases} \dfrac{k^2}{4}, & k \in (-\infty,-2] \cup [1,\infty) \\ \dfrac{-k(8+11k+8k^2)}{4(1-k)^2}, & k \in \left[-2, \dfrac{-1}{2}\right] \\ \dfrac{1}{4}, & k \in \left[\dfrac{-1}{2}, 1\right] \end{cases}.$$

Then,

$$\sum yz(x-y)(x-z)(x-ky)(x-kz)+\beta_k(x-y)^2(y-z)^2(z-x)^2 \geq 0.$$

3.51. Let x, y, z be real numbers, and let

$$\gamma_k = \begin{cases} \dfrac{(k+1)(5k-3)}{16}, & k \in (-\infty,-5] \cup [1,\infty) \\ \dfrac{(k+1)(k^3-7k^2-16k-32)}{4(k-1)^2}, & k \in [-5,-2] \\ \dfrac{k^2}{4}, & k \in [-2,1] \end{cases}.$$

Then,

$$\sum(x^2-y^2)(x^2-z^2)(x-ky)(x-kz)+\gamma_k(x-y)^2(y-z)^2(z-x)^2 \geq 0.$$

3.52. If x, y, z are real numbers, then

$$\sum yz(4x^2+3yz)(x-y)(x-z)+(x-y)^2(y-z)^2(z-x)^2 \ge 0.$$

3.53. If x, y, z are real numbers, then

$$\sum x^4(x+2y)(x+2z)+5x^2y^2z^2+\frac{1}{2}(x-y)^2(y-z)^2(z-x)^2 \ge 0.$$

3.54. If x, y, z are real numbers, no two of which are zero, then

$$\sum \frac{1}{2y^2-3yz+2z^2} \ge \frac{9}{4(x^2+y^2+z^2)-3(xy+yz+zx)}.$$

3.55. Let x, y, z be real numbers. If $k > 1$, then

$$\frac{kx^2+2yz}{kx^2+y^2+z^2}+\frac{ky^2+2zx}{ky^2+z^2+x^2}+\frac{kz^2+2xy}{kz^2+x^2+y^2} \ge \frac{k-1}{k+1}.$$

3.56. If x, y, z are real numbers such that $xy+yz+zx < 0$, then

$$\frac{1}{3x^2+y^2+z^2}+\frac{1}{3y^2+z^2+x^2}+\frac{1}{3z^2+x^2+y^2}+\frac{1}{xy+yz+zx} \le 0.$$

3.57. If x, y, z are real numbers, then

$$\frac{x^2-yz}{3x^2+y^2+z^2}+\frac{y^2-zx}{3y^2+z^2+x^2}+\frac{z^2-xy}{3z^2+x^2+y^2} \le 1.$$

3.58. Let x, y, z be real numbers. If $k > 1$, then

$$\frac{yz}{kx^2+y^2+z^2}+\frac{zx}{ky^2+z^2+x^2}+\frac{xy}{kz^2+x^2+y^2}+\frac{1}{2} \ge 0.$$

3.59. If x, y, z are real numbers, then

$$\frac{yz}{x^2+4y^2+4z^2}+\frac{zx}{y^2+4z^2+4x^2}+\frac{xy}{z^2+4x^2+4y^2}+\frac{1}{8} \ge 0.$$

3.60. If x, y, z are real numbers, then

$$\sum \frac{1}{x^2+4y^2+4z^2} \le \frac{7}{4(x^2+y^2+z^2)+3(xy+yz+zx)}.$$

3.61. If x, y, z are real numbers such that $xy+yz+zx \ge 0$, then

$$\sum \frac{2}{4x^2+y^2+z^2} \ge \frac{45}{14(x^2+y^2+z^2)+xy+yz+zx}.$$

3.62. If x, y, z are real numbers such that $xy+yz+zx \ge 0$, then

$$\sum \frac{1}{x^2+4y^2+4z^2} \ge \frac{45}{44(x^2+y^2+z^2)+xy+yz+zx}.$$

3.63. If x, y, z are real numbers, then

$$\frac{x(-x+4y+4z)}{y^2+z^2} + \frac{y(-y+4z+4x)}{z^2+x^2} + \frac{z(-z+4x+4y)}{x^2+y^2} \le \frac{21}{2}.$$

3.64. If x, y, z are real numbers, no two of which are zero, then

$$\frac{x^2+3yz}{y^2-yz+z^2} + \frac{y^2+3zx}{z^2-zx+x^2} + \frac{z^2+3xy}{x^2-xy+y^2} \ge 1.$$

3.65. If x, y, z are real numbers, then

$$\frac{(4x-y-z)^2}{2y^2-3yz+2z^2} + \frac{(4y-z-x)^2}{2z^2-3zx+2x^2} + \frac{(4z-x-y)^2}{2x^2-3xy+y^2} \ge 12.$$

3.66. If x, y, z are real numbers, then

$$\frac{(3y+3z-4x)^2}{2y^2-3yz+2z^2} + \frac{(3z+3x-4y)^2}{2z^2-3zx+2x^2} + \frac{(3x+3y-4z)^2}{2x^2-3xy+y^2} \ge 12.$$

3.67. Let x, y, z be real numbers. If $k > -2$, then

$$4(x^2+kxy+y^2)(y^2+kyz+z^2)(z^2+kzx+x^2) \ge (2-k)(x-y)^2(y-z)^2(z-x)^2.$$

3.68. If x, y, z are real numbers, then

$$(x^2+y^2)(y^2+z^2)(z^2+x^2)+2(x^2y+y^2z+z^2x)(xy^2+yz^2+zx^2) \geq x^2y^2z^2.$$

3.69. Let x, y, z be real numbers. If $k \in (-\infty, -2] \cup (0, \infty)$, then

$$x^6+y^6+z^6-3x^2y^2z^2+\frac{2}{k}(x^2+kyz)(y^2+kzx)(z^2+kxy) \geq 0.$$

3.70. If x, y, z are real numbers, then

$$2(2x^2+y^2+z^2)(2y^2+z^2+x^2)(2z^2+x^2+y^2) \geq 89x^2y^2z^2+9(x-y)^2(y-z)^2(z-x)^2.$$

3.71. If x, y, z are real numbers such that $x+y+z=3$, then

$$\frac{13x-1}{x^2+23}+\frac{13y-1}{y^2+23}+\frac{13z-1}{z^2+23} \leq \frac{3}{2}.$$

3.72. If x, y, z are real numbers, then

$$5(x^2+y^2+z^2)^3 \geq 108x^2y^2z^2+10(x-y)^2(y-z)^2(z-x)^2.$$

3.73. If x, y, z are real numbers, then

$$(x^2+y^2+z^2)^3+2(2x^2+yz)(2y^2+zx)(2z^2+xy) \geq 27x^2y^2z^2.$$

3.74. If x, y, z are real numbers, no two of which are zero, then

$$\frac{x^2+2yz}{y^2+yz+z^2}+\frac{y^2+2zx}{z^2+zx+x^2}+\frac{z^2+2xy}{x^2+xy+y^2} \geq \frac{3(xy+yz+zx)}{x^2+y^2+z^2}.$$

3.75. If x, y, z are real numbers, no two of which are zero, then

$$\frac{x^2-2yz}{y^2-yz+z^2}+\frac{y^2-2zx}{z^2-zx+x^2}+\frac{z^2-2xy}{x^2-xy+y^2}+\frac{3(xy+yz+zx)}{x^2+y^2+z^2} \geq 0.$$

3.76. If x, y, z are real numbers, no two of which are zero, then

$$\frac{x^2}{y^2-yz+z^2}+\frac{y^2}{z^2-zx+x^2}+\frac{z^2}{x^2-xy+y^2}\geq\frac{(x+y+z)^2}{x^2+y^2+z^2}.$$

3.77. If x, y, z are real numbers such that $xyz \neq 0$, then

$$\frac{(y+z)^2}{x^2}+\frac{(z+x)^2}{y^2}+\frac{(x+y)^2}{z^2}\geq 2+\frac{10(x+y+z)^2}{3(x^2+y^2+z^2)}.$$

3.78. If x, y, z are real numbers, no two of which are zero, then

$$\frac{32x^2+49yz}{y^2+z^2}+\frac{32y^2+49zx}{z^2+x^2}+\frac{32z^2+49xy}{x^2+y^2}\geq\frac{81(x+y+z)^2}{2(x^2+y^2+z^2)}.$$

3.79. If x, y, z are real numbers, no two of which are zero, then

$$(a)\qquad \frac{x^2+4yz}{y^2+z^2}+\frac{y^2+4zx}{z^2+x^2}+\frac{z^2+4xy}{x^2+y^2}\geq\frac{15(xy+yz+zx)}{2(x^2+y^2+z^2)};$$

$$(b)\qquad \frac{2x^2+9yz}{y^2+z^2}+\frac{2y^2+9zx}{z^2+x^2}+\frac{2z^2+9xy}{x^2+y^2}\geq\frac{33(xy+yz+zx)}{2(x^2+y^2+z^2)}.$$

3.80. If x, y, z are distinct real numbers, then

$$\frac{x^2}{(y-z)^2}+\frac{y^2}{(z-x)^2}+\frac{z^2}{(x-y)^2}\geq\frac{4(xy+yz+zx)}{x^2+y^2+z^2}.$$

3.81. If x, y, z are distinct real numbers, then

$$\frac{x^2}{(y-z)^2}+\frac{y^2}{(z-x)^2}+\frac{z^2}{(x-y)^2}\geq\frac{(x+y+z)^2}{x^2+y^2+z^2}.$$

3.82. If x, y, z are real numbers, then

$$\frac{2xy}{x^2+y^2}+\frac{2yz}{y^2+z^2}+\frac{2zx}{z^2+x^2}+1\geq\frac{4(xy+yz+zx)}{x^2+y^2+z^2}.$$

3.3 Solutions

P 3.1. *If x, y, z are real numbers, then*

$$\sum(x+2y)(x+2z)(2x+y)(2x+z)(x-y)(x-z)+3(x-y)^2(y-z)^2(z-x)^2 \geq 0.$$

(Vasile C., 2012)

Solution. Write the inequality as $f_6(x, y, z) \geq 0$, where

$$f_6(x, y, z) = \sum(x+2y)(x+2z)(2x+y)(2x+z)(x-y)(x-z)+3\prod(y-z)^2.$$

Since

$$(x+2y)(x+2z) = 2q + x^2 + 2yz,$$
$$(2x+y)(2x+z) = 2q + 4x^2 - yz,$$
$$(x-y)(x-z) = x^2 + 2yz - q,$$
$$(y-z)^2 = -x^2 - 2yz + p^2 - 2q,$$

$f_6(x, y, z)$ has the same highest coefficient A as

$$P_1(x, y, z) + 3P_3(x, y, z),$$

where

$$P_1(x, y, z) = \sum(x^2+2yz)(4x^2-yz)(x^2+2yz),$$
$$P_3(x, y, z) = \prod(-x^2-2yz).$$

According to (3.1) and (3.3), we get

$$\begin{aligned} A &= P_1(1, 1, 1) + 3P_3(1, 1, 1) \\ &= 3(1+2)(4-1)(1+2) + 3(-1-2)^3 = 81 - 81 = 0. \end{aligned}$$

By Corollary 1, we only need to show that $f_6(x, 1, 1) \geq 0$ for $x \in \mathbb{R}$. We have

$$f(x, 1, 1) = (x+2)^2(2x+1)^2(x-1)^2 \geq 0.$$

The equality holds for $x = y = z$, for $-x/2 = y = z$ (or any cyclic permutation), and for $-2x = y = z$ (or any cyclic permutation).

□

P 3.2. *If x, y, z are real numbers, then*

$$\sum(x^2-4yz)^2(x-y)(x-z)+3(x-y)^2(y-z)^2(z-x)^2 \geq 0.$$

(Vasile C., 2012)

Solution. Write the inequality as $f_6(x,y,z) \ge 0$, where

$$f_6(x,y,z) = f(x,y,z) + 3(x-y)^2(y-z)^2(z-x)^2,$$

$$f(x,y,z) = \sum (x^2-4yz)^2(x-y)(x-z).$$

Since

$$(x-y)(x-z) = x^2 + 2yz - q,$$

$f(x,y,z)$ has the same highest coefficient A_1 as

$$P_1(x,y,z) = \sum (x^2-4yz)^2(x^2+2yz),$$

that is, according to (3.1),

$$A_1 = P_1(1,1,1) = 81.$$

Since the product $(x-y)^2(y-z)^2(z-x)^2$ has the highest coefficient equal to -27, $f_6(x,y,z)$ has the highest coefficient

$$A = A_1 + 3(-27) = 0.$$

By Corollary 1, we only need to show that $f_6(x,1,1) \ge 0$ for all real x. This is true because

$$f(x,1,1) = (x^2-4)^2(x-1)^2 \ge 0.$$

The equality holds for $x=y=z$, for $x+y+z=0$, and for $x/2=y=z$ (or any cyclic permutation).

Observation. The inequality is equivalent to

$$p^2(p^4 - 7p^2q + 16q^2 - 12pr) \ge 0.$$

□

P 3.3. *If x,y,z are real numbers, then*

$$\sum (x^2+yz)(x+2y)(x+2z)(x-y)(x-z) + 2(x-y)^2(y-z)^2(z-x)^2 \ge 0.$$

(Vasile C., 2012)

Solution. Write the inequality as $f_6(x,y,z) \ge 0$, where

$$f_6(x,y,z) = f(x,y,z) + 2(x-y)^2(y-z)^2(z-x)^2,$$

$$f(x,y,z) = (x^2+yz)(x+2y)(x+2z)(x-y)(x-z).$$

Since

$$(x+2y)(x+2z) = x^2 + 2yz + 2q,$$

$$(x-y)(x-z) = x^2 + 2yz - q,$$

$f(x,y,z)$ has the same highest coefficient A_1 as

$$P_1(x,y,z) = \sum (x^2 + yz)(x^2 + 2yz)^2,$$

that is, according to (3.1),

$$A_1 = P_1(1,1,1) = 54.$$

Since the product $(x-y)^2(y-z)^2(z-x)^2$ has the highest coefficient equal to -27, $f_6(x,y,z)$ has the highest coefficient

$$A = A_1 + 2(-27) = 0.$$

According to Corollary 1, we only need to show that $f_6(x,1,1) \geq 0$ for all real x. Indeed,

$$f(x,1,1) = (x^2+1)(x-1)^2(x+2)^2 \geq 0.$$

The equality holds for $x = y = z$ and for $x + y + z = 0$.

Observation. The inequality is equivalent to

$$(x+y+z)^2\left[(x^2+y^2)(x-y)^2 + (y^2+z^2)(y-z)^2 + (z^2+x^2)(z-x)^2\right] \geq 0.$$

□

P 3.4. *If x, y, z are real numbers, then*

$$\frac{2x^2+3yz}{4x^2+y^2+z^2} + \frac{2y^2+3zx}{4y^2+z^2+x^2} + \frac{2z^2+3xy}{4z^2+x^2+y^2} \leq \frac{5}{2}.$$

(Vasile C., 2012)

Solution. Write the inequality as $f_6(x,y,z) \geq 0$, where

$$f_6(x,y,z) = 5\prod(4x^2+y^2+z^2) - 2\sum(2x^2+3yz)(4y^2+z^2+x^2)(4z^2+x^2+y^2).$$

Since

$$4x^2 + y^2 + z^2 = 3x^2 + p^2 - 2q,$$

$f_6(x,y,z)$ has the same highest coefficient A as $f(x,y,z)$, where

$$\begin{aligned} f(x,y,z) &= 5\prod(3x^2) - 2\sum(2x^2+3yz)(3y^2)(3z^2) \\ &= 135x^2y^2z^2 - 108x^2y^2z^2 - 54\sum y^3z^3 \\ &= 27x^2y^2z^2 - 54\sum y^3z^3, \end{aligned}$$

that is

$$A = 27 - 162 = -135.$$

According to Corollary 1, we only need to prove the original inequality for

$$y = z = 1.$$

Thus, we need to show that

$$\frac{2x^2+3}{2(2x^2+1)} + \frac{2(3x+2)}{x^2+5} \le \frac{5}{2},$$

which is equivalent to

$$(x-1)^2(2x-1)^2 \ge 0.$$

The equality holds for $x = y = z$ and for $2x = y = z$ (or any cyclic permutation).

□

P 3.5. *If x, y, z are real numbers, then*

$$\frac{9x^2-4yz}{3x^2+2y^2+2z^2} + \frac{9y^2-4zx}{3y^2+2z^2+2x^2} + \frac{9z^2-4xy}{3z^2+2x^2+2y^2} \ge \frac{15}{7}.$$

(Vasile C., 2012)

Solution. Write the inequality as $f_6(x,y,z) \ge 0$, where

$$\begin{aligned} f_6(x,y,z) =& 7\sum(9x^2-4yz)(3y^2+2z^2+2x^2)(3z^2+2x^2+2y^2) \\ & -15(3x^2+2y^2+2z^2)(3y^2+2z^2+2x^2)(3z^2+2x^2+2y^2). \end{aligned}$$

Since

$$3x^2+2y^2+2z^2 = x^2+2(p^2-2q),$$

$f_6(x,y,z)$ has the same highest coefficient A as $f(x,y,z)$, where

$$\begin{aligned} f(x,y,z) &= 7\sum(9x^2-4yz)(y^2)(z^2) - 15\prod(x^2) \\ &= 174x^2y^2z^2 - 28\sum y^3z^3, \end{aligned}$$

that is

$$A = 174 - 84 = 90.$$

Since $A > 0$ and

$$\begin{aligned} f_6(x,1,1) =& 7(9x^2-4)(2x^2+5)^2 + 14(9-4x)(2x^2+5)(3x^2+4) \\ & -15(2x^2+5)^2(3x^2+4) \\ =& 4(2x^2+5)(x-1)^2(3x-4)^2, \end{aligned}$$

we apply Corollary 2 for

$$F_{\gamma,\delta}(x) = f_{4/3,-2}(x) = \frac{(x-1)^4(3x-4)^2}{81 \cdot 25}.$$

We have

$$Af_{4/3,-2}(x) = \frac{2(x-1)^4(3x-4)^2}{45},$$

$$f_6(x,1,1) - Af_{4/3,-2}(x) = \frac{2(x-1)^2(3x-4)^2 f(x)}{45},$$

where

$$f(x) = 90(2x^2+5) - (x-1)^2 > (2x^2+5) - (x-1)^2 = (x+1)^2 + 3 > 0.$$

The equality holds for $x = y = z$, and also for $\frac{3x}{4} = y = z$ (or any cyclic permutation).

Observation. Similarly, we can prove the following generalization:

- *Let x, y, z be real numbers. If $k > 0,\ k \neq 1,$ then*

$$\sum \frac{\frac{k(k-3)}{k-1}x^2 + 2yz}{kx^2+y^2+z^2} \le \frac{3(k+1)(k-2)}{(k-1)(k+2)},$$

with equality for $x = y = z$, and for $\frac{kx}{2} = y = z$ (or any cyclic permutation).

For

$$f_6(x,y,z) = m\prod(kx^2+y^2+z^2) - \sum(nx^2+2yz)(ky^2+z^2+x^2)(kz^2+x^2+y^2),$$

where

$$m = \frac{3(k+1)(k-2)}{(k-1)(k+2)}, \qquad n = \frac{k(k-3)}{k-1},$$

we have

$$A = \frac{9(k+1)(k-1)(2-k)}{k+2}$$

and

$$f_6(x,1,1) = \frac{2}{k+2}(x^2+k+1)(x-1)^2(kx-2)^2.$$

For $k \in (0,1) \cup [2,\infty)$, we have $A \le 0$. According to Corollary 1, we only need to prove that $f_6(x,1,1) \ge 0$ for $x \in \mathbb{R}$, which is clearly true.

For $k \in (1,2)$, we have $A > 0$. According to Corollary 2, it suffices to prove that

$$f_6(x,1,1) \ge Af_{2/k,-2}(x)$$

for $x \in \mathbb{R}$. We have

$$f_{2/k,-2}(x) = \frac{(x-1)^4(kx-2)^2}{81(k+1)^2},$$

$$Af_{2/k,-2}(x) = \frac{(k-1)(2-k)(x-1)^4(kx-2)^2}{9(k+1)(k+2)},$$

$$f_6(x,1,1) - Af_{2/k,-2}(x) = \frac{(x-1)^2(kx-2)^2 f(x)}{9(k+1)(k+2)},$$

where

$$f(x) = 18(k+1)(x^2+k+1) - (k-1)(2-k)(x-1)^2.$$

Since

$$2(x^2+k+1) - (x-1)^2 = (x+1)^2 + 2k > 0,$$

we have

$$f(x) > 9(k+1)(x-1)^2 - (k-1)(2-k)(x-1)^2 = (k^2+6k+11)(x-1)^2 \geq 0.$$

□

P 3.6. *Let x,y,z be real numbers, no two of which are zero. If*

$$x^2+y^2+z^2 \geq 2(xy+yz+zx),$$

then

$$\frac{x^2-6yz}{y^2+yz+z^2} + \frac{y^2-6zx}{z^2+zx+x^2} + \frac{z^2-6xy}{x^2+xy+y^2} \geq 0.$$

(Vasile C., 2012)

Solution. Write the inequality as $f_6(x,y,z) \geq 0$, where

$$f_6(x,y,z) = \sum (x^2-6yz)(x^2+xy+y^2)(x^2+xz+z^2) \geq 0.$$

Since

$$x^2+xy+y^2 = p^2-2q-z^2+xy, \qquad x^2+xz+z^2 = p^2-2q-y^2+xz,$$

$f_6(x,y,z)$ has the same highest coefficient A as

$$P_2(x,y,z) = \sum (x^2-6yz)(-z^2+xy)(-y^2+zx),$$

that is, according to (3.2),

$$A = P_2(1,1,1) = 0.$$

According to Theorem 1, we only need to show that $f_6(x,1,1) \geq 0$ for $x^2+2 \geq 2(2x+1)$, that is for

$$x \in (-\infty,0] \cup [4,\infty).$$

We have

$$f_6(x,1,1) = (x^2-6)(x^2+x+1)^2 + 6(1-6x)(x^2+x+1)$$
$$= (x^2+x+1)x(x^3+x^2-5x-42).$$

Case 1: $x \le 0$. We need to show that $x^3+x^2-5x-42 \le 0$. Indeed,

$$x^3+x^2-5x-42 = x(x+1)^2-(x+3)^2-33 < 0.$$

Case 2: $x \ge 4$. We need to show that $x^3+x^2-5x-42 \ge 0$. We have

$$x^3+x^2-5x-42 \ge 5x^2-5x-42 \ge 15x-42 > 0.$$

The equality holds for $x=0$ and $y=z$ (or any cyclic permutation).

□

P 3.7. *If x,y,z be real numbers, no two of which are zero, then*

$$\frac{8x^2+3yz}{y^2+yz+z^2} + \frac{8y^2+3zx}{z^2+zx+x^2} + \frac{8z^2+3xy}{x^2+xy+y^2} \ge 11.$$

(Vasile C., 2012)

Solution. Write the inequality as $f_6(x,y,z) \ge 0$, where

$$f_6(x,y,z) = \sum(8x^2+3yz)(x^2+xy+y^2)(x^2+xz+z^2) - 11\prod(y^2+yz+z^2).$$

Since

$$y^2+yz+z^2 = yz-x^2+p^2-2q,$$

$f_6(x,y,z)$ has the same highest coefficient A as

$$P_2(x,y,z) - 11P_3(x,y,z),$$

where

$$P_2(x,y,z) = \sum(8x^2+3yz)(xy-z^2)(xz-y^2), \qquad P_3(x,y,z) = \prod(yz-x^2).$$

Therefore,

$$A = P_2(1,1,1) - 11P_3(1,1,1) = 0.$$

According to Corollary 1, we only need to prove the original inequality for $y=z=1$. Thus, we need to show that

$$\frac{8x^2+3}{3} + \frac{2(8+3x)}{x^2+x+1} \ge 11,$$

which is equivalent to

$$4x^4+4x^3-11x^2-6x+9 \geq 0,$$

$$(x-1)^2(2x+3)^2 \geq 0.$$

The equality holds for $x=y=z$, and also for $\dfrac{-2x}{3}=y=z$ (or any cyclic permutation).

□

P 3.8. *If x,y,z are real numbers, no two of which are zero, then*

$$\frac{8x^2-5yz}{y^2-yz+z^2}+\frac{8y^2-5zx}{z^2-zx+x^2}+\frac{8z^2-5xy}{x^2-xy+y^2} \geq 9.$$

(Vasile C., 2012)

Solution. Write the inequality as $f_6(x,y,z) \geq 0$, where

$$f_6(x,y,z)=\sum(8x^2-5yz)(x^2-xy+y^2)(x^2-xz+z^2)-9\prod(y^2-yz+z^2).$$

From

$$\begin{aligned} f_6(x,y,z) &= \sum(8x^2-5yz)(p^2-2q-z^2-xy)(p^2-2q-y^2-xz) \\ &\quad -9\prod(p^2-2q-x^2-yz), \end{aligned}$$

it follows that $f_6(x,y,z)$ has the same highest coefficient A as

$$P_2(x,y,z)+9P_3(x,y,z),$$

where

$$P_2(x,y,z)=\sum(8x^2-5yz)(z^2+xy)(y^2+xz), \qquad P_3(x,y,z)=\prod(x^2+yz),$$

that is, according to (3.2) and (3.3),

$$A=P_2(1,1,1)+9P_3(1,1,1)=3\cdot 3\cdot 2\cdot 2+9\cdot 8=108.$$

Since

$$\begin{aligned} f_6(x,1,1) &= (8x^2-5)(x^2-x+1)^2+2(8-5x)(x^2-x+1)-9(x^2-x+1)^2 \\ &= 2(x^2-x+1)(x-1)^2(2x+1)^2, \end{aligned}$$

we apply Corollary 2 for

$$F_{\gamma,\delta}(x)=f_{-1/2,-2}(x)=\frac{4(x-1)^4(2x+1)^2}{729}.$$

We need to show that $f_6(x,1,1) \ge Af_{-1/2,-2}(x)$ for $x \in \mathbb{R}$. Indeed,

$$f_6(x,1,1) - Af_{-1/2,-2}(x) = \frac{2(x-1)^2(2x+1)^2(19x^2-11x+19)}{27} \ge 0.$$

The equality holds for $x = y = z$, and for $-2x = y = z$ (or any cyclic permutation).

Observation. Similarly, we can prove the following generalization:

- *Let x, y, z be real numbers, no two of which are zero. If $-2 < k \le 1$, then*

$$\frac{8x^2 + k(4-k)yz}{y^2 + kyz + z^2} + \frac{8y^2 + k(4-k)zx}{z^2 + kzx + x^2} + \frac{8z^2 + k(4-k)xy}{x^2 + kxy + y^2} \ge \frac{3(8 + 4k - k^2)}{k+2},$$

with equality for $x = y = z$, and for $\frac{-2x}{k+2} = y = z$ (or any cyclic permutation).

□

P 3.9. *If x, y, z are real numbers, no two of which are zero, then*

$$\frac{5x^2 + 2yz}{2y^2 + 3yz + 2z^2} + \frac{5y^2 + 2zx}{2z^2 + 3zx + 2x^2} + \frac{5z^2 + 2xy}{2x^2 + 3xy + 2y^2} \ge 3.$$

(Vasile C., 2012)

Solution. Write the inequality as $f_6(x,y,z) \ge 0$, where

$$f_6(x,y,z) = \sum(5x^2+2yz)(2x^2+3xy+2y^2)(2x^2+3xz+2z^2) - 3\prod(2y^2+3yz+2z^2).$$

From

$$\begin{aligned} f_6(x,y,z) = &\sum(5x^2 + 2yz)(3xy - 2z^2 + 2p^2 - 4q)(3xz - 2y^2 + 2p^2 - 4q) \\ &- 3\prod(3yz - 2x^2 + 2p^2 - 4q), \end{aligned}$$

it follows that $f_6(x,y,z)$ has the same highest coefficient A as

$$P_2(x,y,z) - 3P_3(x,y,z),$$

where

$$P_2(x,y,z) = \sum(5x^2+2yz)(3xy-2z^2)(3xz-2y^2), \quad P_3(x,y,z) = \prod(3yz-2x^2),$$

that is,

$$A = P_2(1,1,1) - 3P_3(1,1,1) = 21 - 3 = 18.$$

Since

$$\begin{aligned}\frac{f_6(x,1,1)}{2x^2+3x+2} &= (5x^2+2)(2x^2+3x+2)+14(5+2x)-21(2x^2+3x+2)\\ &= 10x^4+15x^3-28x^2-29x+32\\ &= (x-1)^2(10x^2+35x+32).\end{aligned}$$

we apply Corollary 2 for

$$F_{\gamma,\delta}(x)=f_{\gamma,-2}(x)=\frac{4(x-1)^4(x-\gamma)^2}{81(2+\gamma)^2}.$$

We need to show that

$$f_6(x,1,1)\ge Af_{\gamma,-2}(x), \qquad x\in\mathbb{R},$$

which is equivalent to $f(x)\ge 0$, where

$$f(x)=(2x^2+3x+2)(x-1)^2(10x^2+35x+32)-\frac{8(x-1)^2(x-\gamma)^2}{9(2+\gamma)^2}.$$

Since the original inequality is an equality for $(x,y,z)=(0,1,-1)$, that is

$$f_6(0,1,-1)=0,$$

we have $f(-2)=0$ for all real γ (see Proposition 1). To have $f(x)\ge 0$ in the vicinity of $x=-2$, the condition $f'(-2)=0$ is necessary. This condition involves $\gamma=-50/37$ and

$$f(x)=(x+2)^2(11591x^2+17474x+9743)\ge 0.$$

The equality holds for $x=y=z$, and for $x=0$ and $y+z=0$ (or any cyclic permutation).

Observation. Similarly, we can prove the following generalization:

- *Let x,y,z be real numbers, no two of which are zero. If $\frac{3}{2}\le k<2$, then*

$$\sum\frac{2(4-k)x^2+(2-k)(1+2k)yz}{y^2+kyz+z^2}\ge 3(5-2k),$$

with equality for $x=y=z$, and also for $x=0$ and $y+z=0$ (or any cyclic permutation).

□

P 3.10. *If x, y, z are real numbers, then*

$$\frac{(x+y)(x+z)}{7x^2+y^2+z^2}+\frac{(y+z)(y+x)}{7y^2+z^2+x^2}+\frac{(z+x)(z+y)}{7z^2+x^2+y^2}\le\frac{4}{3}.$$

(Vasile C., 2009)

Solution. Write the inequality as $f_6(x,y,z)\ge 0$, where

$$f_6(x,y,z)=4\prod(7x^2+y^2+z^2)-3\sum(x+y)(x+z)(7y^2+z^2+x^2)(7z^2+x^2+y^2).$$

Since

$$7x^2+y^2+z^2=6x^2+p^2-2q,\qquad (x+y)(x+z)=x^2+q,$$

$f_6(x,y,z)$ has the same highest coefficient A as

$$4\prod(6x^2)-3\sum x^2(6y^2)(6z^2),$$

that is

$$A=4\cdot 6^3-9\cdot 6^2=540.$$

Since

$$\begin{aligned}\frac{f_6(x,1,1)}{x^2+8}&=4(7x^2+2)(x^2+8)-3(x+1)[(x+1)(x^2+8)+4(7x^2+2)]\\&=25x^4-90x^3+121x^2-72x+16=(x-1)^2(5x-4)^2,\end{aligned}$$

we apply Corollary 2 for

$$F_{\gamma,\delta}(x)=f_{4/5,-2}(x)=\frac{(x-1)^4(5x-4)^2}{81\cdot 49}.$$

We need to show that

$$f_6(x,1,1)\ge Af_{4/5,-2}(x)$$

for $x\in\mathbb{R}$. Since

$$Af_{4/5,-2}(x)=\frac{20(x-1)^4(5x-4)^2}{147},$$

we get

$$f_6(x,1,1)-Af_{4/5,-2}(x)=\frac{(x-1)^2(5x-4)^2f(x)}{147},$$

where

$$f(x)=147(x^2+8)-20(x-1)^2>40(x^2+8)-20(x-1)^2=20[(x+1)^2+14]>0.$$

Thus, the proof is completed. The equality holds for $x=y=z$, and also for $\frac{x}{4}=\frac{y}{5}=\frac{z}{5}$ (or any cyclic permutation).

□

P 3.11. *If x, y, z are real numbers, then*

$$\frac{6x(y+z)-yz}{12x^2+y^2+z^2}+\frac{6y(z+x)-zx}{12y^2+z^2+x^2}+\frac{6z(x+y)-xy}{12z^2+x^2+y^2}\le\frac{33}{14}.$$

(Vasile C., 2009)

Solution. Write the inequality as $f_6(x,y,z)\ge 0$, where $f_6(x,y,z)$ is

$$33\prod(12x^2+y^2+z^2)-14\sum(6xy+6xz-yz)(12y^2+z^2+x^2)(12z^2+x^2+y^2).$$

Since

$$12x^2+y^2+z^2=11x^2+p^2-2q,\qquad 6xy+6xz-yz=6q-7yz,$$

$f_6(x,y,z)$ has the same highest coefficient A as $f(x,y,z)$, where

$$\begin{aligned}f(x,y,z)&=33\prod(11x^2)-14\sum(-7yz)(11y^2)(11z^2)\\&=33\cdot 11^3x^2y^2z^2+98\cdot 11^2\sum y^3z^3,\end{aligned}$$

that is

$$A=33\cdot 11^3+3\cdot 98\cdot 11^2=33^2\cdot 73.$$

Since

$$\begin{aligned}\frac{f_6(x,1,1)}{x^2+13}&=33(12x^2+2)(x^2+13)-14[(12x-1)(x^2+13)+2(5x+6)(12x^2+2)]\\&=44(9x^4-42x^3+73x^2-56x+16)=44(x-1)^2(3x-4)^2,\end{aligned}$$

we apply Corollary 2 for

$$F_{\gamma,\delta}(x)=f_{4/3,-2}(x)=\frac{(x-1)^4(3x-4)^2}{2025}.$$

We need to show that

$$f_6(x,1,1)\ge Af_{4/3,-2}(x)$$

for $x\in\mathbb{R}$. Since

$$Af_{4/3,-2}(x)=\frac{8833(x-1)^4(3x-4)^2}{225},$$

we get

$$f_6(x,1,1)-Af_{4/3,-2}(x)=\frac{11(x-1)^2(3x-4)^2f(x)}{225},$$

where

$$\begin{aligned}f(x)&=900(x^2+13)-803(x-1)^2\\&>900(x^2+13)-810(x-1)^2=90[(x+9)^2+40]>0.\end{aligned}$$

Thus, the proof is completed. The equality holds for $x=y=z$, and also for $\frac{x}{4}=\frac{y}{3}=\frac{z}{3}$ (or any cyclic permutation).

□

P 3.12. *If x, y, z are real numbers, then*

$$\frac{x(y+z)-yz}{x^2+3y^2+3z^2}+\frac{y(z+x)-zx}{y^2+3z^2+3x^2}+\frac{z(x+y)-xy}{z^2+3x^2+3y^2}\le\frac{3}{7}.$$

(Vasile C., 2009)

Solution. Write the inequality as $f_6(x,y,z)\ge 0$, where $f_6(x,y,z)$ is

$$3\prod(x^2+3y^2+3z^2)-7\sum(xy+xz-yz)(y^2+3z^2+3x^2)(z^2+3x^2+3y^2).$$

Since

$$x^2+3y^2+3z^2=-2x^2+3(p^2-2q),\qquad xy+xz-yz=q-2yz,$$

$f_6(x,y,z)$ has the same highest coefficient A as $f(x,y,z)$, where

$$\begin{aligned}f(x,y,z)&=3\prod(-2x^2)-7\sum(-2yz)(-2y^2)(-2z^2)\\&=-24x^2y^2z^2+56\sum y^3z^3,\end{aligned}$$

that is

$$A=-24+168=144.$$

Since

$$\begin{aligned}\frac{f_6(x,1,1)}{3x^2+4}&=3(x^2+6)(3x^2+4)-7[(2x-1)(3x^2+4)+2(x^2+6)]\\&=9x^4-42x^3+73x^2-56x+16=(x-1)^2(3x-4)^2,\end{aligned}$$

we apply Corollary 2 for

$$F_{\gamma,\delta}(x)=f_{4/3,-2}(x)=\frac{(x-1)^4(3x-4)^2}{2025}.$$

We need to show that

$$f_6(x,1,1)\ge Af_{4/3,-2}(x)$$

for $x\in\mathbb{R}$. Since

$$Af_{4/3,-2}(x)=\frac{16(x-1)^4(3x-4)^2}{225},$$

we get

$$f_6(x,1,1)-Af_{4/3,-2}(x)=\frac{(x-1)^2(3x-4)^2f(x)}{225},$$

where

$$f(x)=225(3x^2+4)-16(x-1)^2>16(3x^2+4)-16(x-1)^2=16[x^2+2+(x+1)^2]>0.$$

Thus, the proof is completed. The equality holds for $x=y=z$, and also for $\frac{x}{4}=\frac{y}{3}=\frac{z}{3}$ (or any cyclic permutation).

□

P 3.13. *If x, y, z are real numbers, then*

$$\sum yz(2x^2 + yz)(x-y)(x-z) + \frac{1}{2}(x-y)^2(y-z)^2(z-x)^2 \ge 0.$$

(Vasile C., 2012)

Solution. Write the inequality as $f_6(x, y, z) \ge 0$, where

$$f_6(x, y, z) = f(x, y, z) + (x-y)^2(y-z)^2(z-x)^2,$$

$$f(x, y, z) = 2\sum yz(2x^2 + yz)(x-y)(x-z).$$

Since $(x-y)(x-z) = x^2 + 2yz - q$, $f(x, y, z)$ has the same highest coefficient A_1 as

$$P_1(x, y, z) = 2\sum yz(2x^2 + yz)(x^2 + 2yz),$$

that is, according to (3.1),

$$A_1 = P_1(1, 1, 1) = 54.$$

Therefore, $f_6(x, y, z)$ has the highest coefficient

$$A = A_1 + (-27) = 27.$$

Since

$$f_6(x, 1, 1) = f(x, 1, 1) = 2(2x^2 + 1)(x-1)^2,$$

we apply Corollary 2 for

$$F_{\gamma,\delta}(x) = f_{\infty,-2}(x) = \frac{4(x-1)^4}{81}.$$

We need to show that

$$f_6(x, 1, 1) \ge A f_{\infty,-2}(x)$$

for $x \in \mathbb{R}$. Indeed, we get

$$f_6(x, 1, 1) - A f_{\infty,-2}(x) = \frac{2(x-1)^2(2x+1)^2}{3} \ge 0.$$

Thus, the proof is completed. The equality holds for $x = y = z$, and also for $y = z = 0$ (or any cyclic permutation).

□

P 3.14. *If x, y, z are real numbers, then*

$$\sum (x^2 - yz)^2(x-y)(x-z) \ge \frac{3}{4}(x-y)^2(y-z)^2(z-x)^2.$$

(Vasile C., 2012)

Solution. Write the inequality as $f_6(x,y,z) \geq 0$, where

$$f_6(x,y,z) = f(x,y,z) - \frac{3}{4}(x-y)^2(y-z)^2(z-x)^2,$$

$$f(x,y,z) = \sum (x^2 - yz)^2(x-y)(x-z).$$

Since

$$(x-y)(x-z) = x^2 + 2yz - q,$$

$f(x,y,z)$ has the same highest coefficient A_1 as

$$P_1(x,y,z) = \sum (x^2 - yz)^2(x^2 + 2yz),$$

that is, according to (3.1),

$$A_1 = P_1(1,1,1) = 0.$$

Therefore, $f_6(x,y,z)$ has the highest coefficient

$$A = A_1 - \frac{3}{4}(-27) = \frac{81}{4}.$$

Since

$$f_6(x,1,1) = (x^2-1)^2(x-1)^2,$$

we apply Corollary 2 for

$$F_{\gamma,\delta}(x) = f_{-1,-2}(x) = \frac{4(x-1)^4(x+1)^2}{81}.$$

We need to show that

$$f_6(x,1,1) \geq A f_{-1,-2}(x)$$

for $x \in \mathbb{R}$, which is an identity. Thus, the proof is completed. The equality holds for

$$9xyz + 2(x+y+z)^3 = 7(x+y+z)(xy+yz+zx).$$

Observation. The coefficient of the product $(x-y)^2(y-z)^2(z-x)^2$ is the best possible. Setting $x=0$, $y=1$ and $z=-1$ in the inequality

$$\sum (x^2 - yz)^2(x-y)(x-z) \geq k(x-y)^2(y-z)^2(z-x)^2,$$

we get $k \leq \dfrac{3}{4}$. According to (3.12) from Remark 2, the identity holds:

$$f_6(x,y,z) = \frac{1}{4}(9r - 7pq + 2p^3)^2.$$

Therefore, the original inequality is equivalent to

$$(9r - 7pq + 2p^3)^2 \geq 0.$$

□

P 3.15. *If x, y, z are real numbers, then*

$$\sum(x^2+8yz)^2(x-y)(x-z)+15(x-y)^2(y-z)^2(z-x)^2 \ge 0.$$

(Vasile C., 2012)

Solution. Write the inequality as $f_6(x,y,z) \ge 0$, where

$$f_6(x,y,z) = f(x,y,z)+15(x-y)^2(y-z)^2(z-x)^2,$$

$$f(x,y,z) = \sum(x^2+8yz)^2(x-y)(x-z).$$

Since

$$(x-y)(x-z) = x^2+2yz-q,$$

$f(x,y,z)$ has the same highest coefficient A_1 as

$$P_1(x,y,z) = \sum(x^2+8yz)^2(x^2+2yz),$$

that is, according to (3.1),

$$A_1 = P_1(1,1,1) = 729.$$

Therefore, $f_6(x,y,z)$ has the highest coefficient

$$A = A_1+15(-27) = 324.$$

Since

$$f_6(x,1,1) = (x^2+8)^2(x-1)^2,$$

we apply Corollary 2 for

$$F_{\gamma,\delta}(x) = f_{\infty,-2}(x) = \frac{4(x-1)^4}{81}.$$

We need to show that

$$f_6(x,1,1) \ge Af_{\infty,-2}(x)$$

for $x \in \mathbb{R}$. Indeed,

$$f_6(x,1,1)-Af_{\infty,-2}(x) = (x-1)^2g(x),$$

where

$$g(x) = (x^2+8)^2-16(x-1)^2 = (x+2)^2(x^2-4x+12) \ge 0.$$

Thus, the proof is completed. The equality holds for $x = y = z$, and also for $x = 0$ and $y+z = 0$.

Observation. The coefficient of the product $(x-y)^2(y-z)^2(z-x)^2$ is the best possible. Setting $x = 0$, $y = 1$ and $z = -1$ in the inequality

$$\sum(x^2+8yz)^2(x-y)(x-z)+k(x-y)^2(y-z)^2(z-x)^2 \ge 0,$$

we get $k \ge 15$.

□

P 3.16. *Let x, y, z be real numbers. If $k \in \mathbb{R}$, then*

$$\sum(x^2+yz)(x-y)(x-z)(x-ky)(x-kz)+\frac{7k^2-20k-20}{24}(x-y)^2(y-z)^2(z-x)^2 \geq 0.$$

(Vasile C., 2010)

Solution. Write the inequality as $f_6(x, y, z) \geq 0$, where

$$f_6(x, y, z) = f(x, y, z) + \frac{7k^2-20k-20}{24}(x-y)^2(y-z)^2(z-x)^2,$$

$$f(x, y, z) = \sum(x^2+yz)(x-y)(x-z)(x-ky)(x-kz).$$

Since

$$(x-y)(x-z) = x^2+2yz-q,$$

$$(x-ky)(x-kz) = x^2+(k+k^2)yz-kq,$$

$f(x, y, z)$ has the same highest coefficient A_1 as

$$P_1(x, y, z) = \sum(x^2+yz)(x^2+2yz)[x^2+(k+k^2)yz];$$

that is, according to (3.1),

$$A_1 = P_1(1, 1, 1) = 18(1+k+k^2).$$

Therefore, $f_6(x, y, z)$ has the highest coefficient

$$A = A_1 + \frac{7k^2-20k-20}{24}(-27) = \frac{81(k+2)^2}{8}.$$

On the other hand

$$f_6(x, 1, 1) = (x^2+1)(x-1)^2(x-k)^2.$$

For $k = -2$, we have $A = 0$. Since $f_6(x, 1, 1) \geq 0$ for any real x, the conclusion follows by Corollary 1.

For $k \neq -2$, we apply Corollary 2 for

$$F_{\gamma,\delta}(x) = f_{k,-2}(x) = \frac{4(x-1)^4(x-k)^2}{81(2+k)^2}.$$

We have

$$f_6(x, 1, 1) - Af_{k,-2}(x) = \frac{(x^2-1)^2(x-k)^2}{2} \geq 0.$$

Thus, the proof is completed. The equality holds for $x = y = z$, and for $x/k = y = z$ (or any cyclic permutation) if $k \neq 0$. If $k = 0$, then the equality holds also for $x = 0$ and $y = z$ (or any cyclic permutation).

$\square$

P 3.17. *If x, y, z are distinct real numbers, then*

$$\frac{yz}{(y-z)^2}+\frac{zx}{(z-x)^2}+\frac{xy}{(x-y)^2}+\frac{1}{4}\ge 0.$$

Solution. Write the inequality as $f_6(x,y,z)\ge 0$, where

$$f_6(x,y,z)=f(x,y,z)+\frac{1}{4}(x-y)^2(y-z)^2(z-x)^2,$$

$$f(x,y,z)=\sum yz(x-y)^2(x-z)^2.$$

Since

$$(x-y)(x-z)=x^2+2yz-q,$$

$f(x,y,z)$ has the same highest coefficient A_1 as

$$P_1(x,y,z)=\sum yz(x^2+2yz)^2;$$

that is, according to (3.1),

$$A_1=P_1(1,1,1)=27.$$

Therefore, $f_6(x,y,z)$ has the highest coefficient

$$A=A_1-\frac{27}{4}=\frac{81}{4}.$$

Since

$$f_6(x,1,1)=(x-1)^4,$$

we apply Corollary 2 for

$$F_{\gamma,\delta}(x)=f_{\infty,-2}(x)=\frac{4(x-1)^4}{81}.$$

We have

$$f_6(x,1,1)-Af_{\infty,-2}(x)=0.$$

Thus, the proof is completed. The inequality is an equality for all distinct real x,y,z which satisfy

$$9xyz=(x+y+z)(xy+yz+zx).$$

Observation. The coefficient of the product $(x-y)^2(y-z)^2(z-x)^2$ in the inequality $f_6(x,y,z)\ge 0$ is the best possible. Thus, setting $x=0$, $y=1$ and $z=-1$ in the inequality

$$\sum yz(x-y)^2(x-z)^2+k(x-y)^2(y-z)^2(z-x)^2\ge 0,$$

we get $k \geq \frac{1}{4}$. According to (3.12) from Remark 2, the identity holds:

$$f_6(x,y,z) = \frac{1}{4}(9r - pq)^2.$$

Therefore, for distinct x, y, z, the original inequality is equivalent to

$$(9r - pq)^2 \geq 0.$$

□

P 3.18. *Let x, y, z be distinct real numbers. If $k \in \mathbb{R}$, then*

$$\frac{(x-ky)(x-kz)}{(y-z)^2} + \frac{(y-kz)(y-kx)}{(z-x)^2} + \frac{(z-kx)(z-ky)}{(x-y)^2} \geq 2 + 2k - \frac{k^2}{4}.$$

(Vasile C., 2010)

Solution. Write the inequality as $f_6(x,y,z) \geq 0$, where

$$f_6(x,y,z) = f(x,y,z) - \left(2 + 2k - \frac{k^2}{4}\right)(x-y)^2(y-z)^2(z-x)^2,$$

$$f(x,y,z) = \sum (x-y)^2(x-z)^2(x-ky)(x-kz).$$

Since

$$(x-y)(x-z) = x^2 + 2yz - q, \qquad (x-ky)(x-kz) = x^2 + (k+k^2)yz - kq$$

$f(x,y,z)$ has the same highest coefficient A_1 as

$$P_1(x,y,z) = \sum (x^2 + 2yz)^2[x^2 + (k+k^2)yz];$$

that is, according to (3.1),

$$A_1 = P_1(1,1,1) = 27(1 + k + k^2).$$

Therefore, $f_6(x,y,z)$ has the highest coefficient

$$A = A_1 - (-27)\left(2 + 2k - \frac{k^2}{4}\right) = \frac{81(k+2)^2}{4}.$$

In addition,

$$f_6(x,1,1) = (x-1)^4(x-k)^2 \geq 0.$$

For $k = -2$, we have $A = 0$. Since $f_6(x,1,1) \geq 0$ for any real x, the conclusion follows (by Corollary 1).

For $k \neq -2$, we apply Corollary 2 for

$$F_{\gamma,\delta}(x) = f_{k,-2}(x) = \frac{4(x-1)^4(x-k)^2}{81(2+k)^2}.$$

Since

$$f_6(x,1,1) - Af_{k,-2}(x) = 0,$$

the proof is completed. The equality occurs for all distinct real x, y, z which satisfy

$$(9k+18)xyz + 2(x+y+z)^3 = (k+8)(x+y+z)(xy+yz+zx).$$

Observation 1. For distinct x, y, z, the original inequality is equivalent to

$$\left[(9k+18)r - (k+8)pq + 2p^3\right]^2 \geq 0.$$

Observation 2. For $k \to \infty$, we get the inequality in P 3.17. For $k = 0$, $k = 1$ and $k = -1$, we get respectively the inequalities:

$$\frac{x^2}{(y-z)^2} + \frac{y^2}{(z-x)^2} + \frac{z^2}{(x-y)^2} \geq 2,$$

$$\frac{(x-y)(x-z)}{(y-z)^2} + \frac{(y-z)(y-x)}{(z-x)^2} + \frac{(z-x)(z-y)}{(x-y)^2} \geq \frac{15}{4},$$

$$\frac{(x+y)(x+z)}{(y-z)^2} + \frac{(y+z)(y+x)}{(z-x)^2} + \frac{(z+x)(z+y)}{(x-y)^2} + \frac{1}{4} \geq 0.$$

□

P 3.19. *If x, y, z are real numbers, then*

$$\sum (x^2+2yz)(x^2-y^2)(x^2-z^2) + \frac{1}{2}(x-y)^2(y-z)^2(z-x)^2 \geq 0.$$

(Vasile C., 2012)

Solution. Let

$$f(x,y,z) = \sum (x^2+2yz)(x^2-y^2)(x^2-z^2)$$

and

$$f_6(x,y,z) = f(x,y,z) + \frac{1}{2}(x-y)^2(y-z)^2(z-x)^2.$$

Since

$$(x-y)(x-z) = x^2 + 2yz - q, \qquad (x+y)(x+z) = x^2 + q,$$

$f(x,y,z)$ has the same highest coefficient A_1 as

$$P_1(x,y,z) = \sum (x^2+2yz)^2 x^2,$$

that is, according to (3.1),

$$A_1 = P_1(1,1,1) = 3(1+2)^2 = 27.$$

Therefore $f_6(x,y,z)$ has the highest coefficient

$$A = A_1 - \frac{27}{2} = \frac{27}{2}.$$

Since

$$f_6(x,1,1) = (x^2+2)(x^2-1)^2,$$

we apply Corollary 2 for

$$F_{\gamma,\delta}(x) = f_{-1,-2}(x) = \frac{4(x-1)^4(x+1)^2}{81}.$$

Since

$$f_6(x,1,1) - Af_{-1,-2}(x) = \frac{(x^2-1)^2(x+2)^2}{3} \ge 0,$$

the proof is completed. The equality holds for $x = y = z$, for $-x = y = z$ (or any cyclic permutation), and also for $x = 0$ and $y + z = 0$ (or any cyclic permutation).

Observation. Similarly, we can prove the following generalization:

- *Let x, y, z be real numbers. If $k \in \mathbb{R}$, then*

$$\sum (x^2+2yz)(x-y)(x-z)(x-ky)(x-kz) \ge$$
$$\ge \left(1+k-\frac{k^2}{2}\right)(x-y)^2(y-z)^2(z-x)^2 \ge 0,$$

with equality for $x = y = z$, for $x/k = y = z$ (or any cyclic permutation) if $k \ne 0$, and for $x = 0$ and $y + z = 0$ (or any cyclic permutation). If $k = 0$, then the equality holds also for $x = 0$ and $y = z$ (or any cyclic permutation).

We have

$$f_6(x,1,1) = (x^2+2)(x-1)^2(x-k)^2,$$
$$A = \frac{27(k+2)^2}{2},$$
$$f_6(x,1,1) - Af_{k,-2}(x) = \frac{(x-1)^2(x-k)^2(x+2)^2}{3} \ge 0.$$

□

P 3.20. *Let x, y, z be real numbers. If $k \in \mathbb{R}$, then*

$$\sum (x-y)(x-z)(x-ky)(x-kz) \ge \frac{3(k+2)^2(x-y)^2(y-z)^2(z-x)^2}{4(x^2+y^2+z^2-xy-yz-zx)}.$$

(Vasile C., 2012)

Solution. Write the inequality as $f_6(x,y,z) \ge 0$, where

$$\begin{aligned} f_6(x,y,z) =& 4(x^2+y^2+z^2-xy-yz-zx)\sum(x-y)(x-z)(x-ky)(x-kz) \\ & -3(k+2)^2(x-y)^2(y-z)^2(z-x)^2. \end{aligned}$$

Since

$$x^2+y^2+z^2-xy-yz-zx = p^2-3q,$$

$f_6(x,y,z)$ has the highest coefficient

$$A = 81(k+2)^2.$$

On the other hand,

$$f_6(x,1,1) = 4(x-1)^4(x-k)^2.$$

For $k=-2$, we have $A=0$. Since $f_6(x,1,1) \ge 0$ for any real x, the conclusion follows by Corollary 1.

For $k \ne -2$, we apply Corollary 2 for

$$F_{\gamma,\delta}(x) = f_{k,-2}(x) = \frac{4(x-1)^4(x-k)^2}{81(k+2)^2}.$$

We need to show that

$$f_6(x,1,1) \ge A f_{k,-2}(x), \qquad x \in \mathbb{R},$$

which is an identity.

The equality holds for all real x,y,z which satisfy

$$9(k+2)xyz + 2(x+y+z)^3 = (k+8)(x+y+z)(xy+yz+zx)$$

and are not all equal.

Observation 1. The coefficient of the product $(x-y)^2(y-z)^2(z-x)^2$ is the best possible. Setting $x=0$, $y=1$ and $z=-1$ in the inequality

$$\sum(x-y)(x-z)(x-ky)(x-kz) \ge \alpha_k \frac{(x-y)^2(y-z)^2(z-x)^2}{x^2+y^2+z^2-xy-yz-zx},$$

we get $\alpha_k \le \dfrac{3(k+2)^2}{4}$. According to (3.12) from Remark 2, the identity holds:

$$f_6(x,y,z) = \left[9(k+2)r-(k+8)pq+2p^3\right]^2.$$

Therefore, if x,y,z are not all equal, then the original inequality is equivalent to

$$\left[9(k+2)r-(k+8)pq+2p^3\right]^2 \ge 0.$$

Observation 2. For $k = -2$, the original inequality has the form

$$\sum (x-y)(x-z)(x+2y)(x+2z) \geq 0,$$

which is equivalent to

$$(x+y+z)^2(x^2+y^2+z^2-xy-yz-zx) \geq 0.$$

The equality holds for $x = y = z$, and also for $x+y+z = 0$.

Observation 3. Since

$$\frac{1}{x^2+y^2+z^2-xy-yz-zx} \geq \frac{2}{3(x^2+y^2+z^2)},$$

with equality for $x+y+z=0$, the following weaker inequality holds

$$\sum (x-y)(x-z)(x-ky)(x-kz) \geq \frac{(k+2)^2(x-y)^2(y-z)^2(z-x)^2}{2(x^2+y^2+z^2)},$$

with equality for $x = y = z$, for $x/k = y = z$ (or any cyclic permutation) if $k \neq 0$, and for $x = 0$ and $y+z = 0$ (or any cyclic permutation). If $k = 0$, then the equality holds also for $x = 0$ and $y = z$ (or any cyclic permutation).

Observation 4. Adding the inequality from Observation 3 written in the form

$$\sum (x^2+y^2+z^2)(x-y)(x-z)(x-ky)(x-kz) - \frac{(k+2)^2}{2}(x-y)^2(y-z)^2(z-x)^2$$

and the identity

$$\sum (2yz-y^2-z^2)(x-y)(x-z)(x-ky)(x-kz) +$$
$$+(k^2+k+1)(x-y)^2(y-z)^2(z-x)^2 = 0,$$

we get the inequality in Observation from the preceding P 3.19.

Observation 5. Substituting $k-1$ for k in P 3.20, and using then the identity

$$\sum (x-y)(x-z)[x-(k-1)y][x-(k-1)z] = \sum (x-y)(x-z)(x-ky+z)(x-kz+y),$$

we get the following statement:

- *Let x, y, z be real numbers. If $k \in \mathbb{R}$, then*

$$\sum (x-y)(x-z)(x-ky+z)(x-kz+y) \geq \frac{3(k+1)^2(x-y)^2(y-z)^2(z-x)^2}{4(x^2+y^2+z^2-xy-yz-zx)},$$

with equality for all real x, y, z which satisfy

$$9(k+1)xyz + 2(x+y+z)^3 = (k+7)(x+y+z)(xy+yz+zx)$$

and are not all equal.

□

P 3.21. *Let x, y, z be real numbers such that $xy + yz + zx \ge 0$. If $k \in \mathbb{R}$, then*

$$\sum(x^2+yz)(x-y)(x-z)(x-ky)(x-kz) \ge \left(1+k-\frac{k^2}{4}\right)(x-y)^2(y-z)^2(z-x)^2.$$

(Vasile C., 2012)

Solution. Let

$$f(x,y,z) = \sum(x^2+yz)(x-y)(x-z)(x-ky)(x-kz)$$

and

$$f_6(x,y,z) = f(x,y,z) - \left(1+k-\frac{k^2}{4}\right)(x-y)^2(y-z)^2(z-x)^2.$$

Since

$$(x-y)(x-z) = x^2+2yz-q, \qquad (x-ky)(x-kz) = x^2+k(1+k)yz-kq$$

$f(x,y,z)$ has the same highest coefficient A_1 as

$$P_1(x,y,z) = \sum(x^2+yz)(x^2+2yz)[x^2+k(1+k)yz],$$

that is, according to (3.1),

$$A_1 = P_1(1,1,1) = 3(1+1)(1+2)(1+k+k^2) = 18(1+k+k^2).$$

Therefore, $f_6(x,y,z)$ has the highest coefficient

$$A = 18(1+k+k^2) - (-27)\left(1+k-\frac{k^2}{4}\right) = \frac{45(k+2)^2}{4}.$$

For $k=-2$, we only need to show that $f_6(x,1,1) \ge 0$ for $2x+1 \ge 0$ (see Theorem 1). Indeed,

$$f_6(x,1,1) = (x^2+1)(x-1)^2(x+2)^2 \ge 0.$$

Consider next $k \ne -2$. Since $A > 0$ and

$$f_6(x,1,1) = (x^2+1)(x-1)^2(x-k)^2,$$

we will apply Corollary 2 for

$$F_{\gamma,\delta}(x) = f_{k,-2}(x) = \frac{4(x-1)^4(x-k)^2}{81(2+k)^2}.$$

Thus, according to Remark 5, we need to show that

$$f_6(x,1,1) - Af_{k,-2}(x) \ge 0$$

for $2x+1 \ge 0$. Indeed, we have

$$f_6(x,1,1)-Af_{k,-2}(x) = (x^2+1)(x-1)^2(x-k)^2 - \frac{5(x-1)^4(x-k)^2}{9}$$
$$= \frac{2(x-1)^2(x-k)^2(x+2)(2x+1)}{9} \ge 0.$$

The equality holds for $x=y=z$, and for $x/k=y=z$ (or any cyclic permutation) if $k \ne 0$ and $2k+1 \ge 0$. If $k=0$, then the equality holds also for $x=0$ and $y=z$ (or any cyclic permutation).

□

P 3.22. *If x,y,z are real numbers, then*

$$\sum (x^2+2yz)^2(x-y)(x-z) \ge 0.$$

(Vasile C., 2012)

Solution. Let

$$f_6(x,y,z) = \sum (x^2+2yz)^2(x-y)(x-z).$$

Since $(x-y)(x-z) = x^2+2yz-q$, $f_6(x,y,z)$ has the same highest coefficient as

$$P_1(x,y,z) = \sum (x^2+2yz)^3,$$

that is, according to (3.1),

$$A = P_1(1,1,1) = 81.$$

Since

$$f_6(x,1,1) = (x^2+2)^2(x-1)^2,$$

we apply Corollary 2. There are two methods to do this.

First method. By selecting

$$F_{\gamma,\delta}(x) = f_{\gamma,-2}(x) = \frac{4(x-1)^4(x-\gamma)^2}{81(\gamma+2)^2},$$

we need to show that there exists a real γ such that

$$f_6(x,1,1) \ge Af_{\gamma,-2}(x)$$

for all real x. We have

$$Af_{\gamma,-2}(x) = \frac{4(x-1)^4(x-\gamma)^2}{(\gamma+2)^2},$$

$$f_6(x,1,1)-Af_{\gamma,-2}(x) = \frac{(x-1)^2 g_1(x) g_2(x)}{(\gamma+2)^2},$$

where

$$\begin{aligned} g_1(x) &= (\gamma+2)(x^2+2) - 2(x-1)(x-\gamma) \\ &= (x+2)(\gamma x+2), \end{aligned}$$

$$\begin{aligned} g_2(x) &= (\gamma+2)(x^2+2) + 2(x-1)(x-\gamma) \\ &= (\gamma+4)x^2 - 2(\gamma+1)x + 4(\gamma+1). \end{aligned}$$

Choosing $\gamma = 1$, we get

$$g_1(x) = (x+2)^2 \ge 0,$$

$$g_2(x) = 5x^2 - 4x + 8 = (2x-1)^2 + x^2 + 7 > 0.$$

The equality holds for $x = y = z$, and for $x = 0$ and $y + z = 0$ (or any cyclic permutation).

Second method. Since

$$f_6(x,1,1) = (x^2+2)^2(x-1)^2$$

and

$$f_6(0,y,z) = 4y^3z^3 + (y-z)(y^5-z^5), \qquad f_6(0,1,-1) = -4+4 = 0,$$

we select

$$F_{\gamma,\delta} = g_{1,\delta}$$

with δ given by (3.16). We have

$$f_6(x,1,1) = (x-1)^2 g(x), \qquad g(x) = (x^2+2)^2,$$

$$g'(x) = 4x(x^2+2), \qquad g'(-2) = -48,$$

$$\delta = \frac{\gamma}{3(\gamma+2)} + \frac{(\gamma+2)^2 g'(-2)}{12A} = \frac{-1}{3}.$$

According to Remark 4, we need to show that

$$g(x) \ge A\bar{g}_{1,\delta}(x),$$

where

$$\bar{g}_{1,\delta}(x) = \frac{1}{9}(x-1)^2\left[\frac{x+8}{9} + \delta(x+2)\right]^2 = \frac{4}{729}(x-1)^4.$$

We have

$$\begin{aligned} g(x) - A\bar{g}_{\gamma,\delta}(x) &= (x^2+2)^2 - \frac{4}{9}(x-1)^4 \\ &= \frac{1}{9}(x+2)^2(5x^2-4x+8) \ge 0. \end{aligned}$$

□

P 3.23. *If x, y, z are real numbers, then*

$$\sum x^2(x^2+yz)(x-y)(x-z) \ge (x-y)^2(y-z)^2(z-x)^2.$$

(Vasile C., 2012)

Solution. Let

$$f(x,y,z) = \sum x^2(x^2+yz)(x-y)(x-z)$$

and

$$f_6(x,y,z) = f(x,y,z) - (x-y)^2(y-z)^2(z-x)^2.$$

Since

$$(x-y)(x-z) = x^2 + 2yz - q,$$

$f(x,y,z)$ has the same highest coefficient A_1 as

$$P_1(x,y,z) = \sum x^2(x^2+yz)(x^2+2yz),$$

that is, according to (3.1),

$$A_1 = P_1(1,1,1) = 3(1+1)(1+2) = 18.$$

Therefore, $f_6(x,y,z)$ has the highest coefficient

$$A = 18 - (-27) = 45.$$

Since

$$f_6(x,1,1) = x^2(x^2+1)(x-1)^2,$$

$$f_{0,-2}(x) = \frac{x^2(x-1)^4}{81},$$

$$\begin{aligned} f_6(x,1,1) - Af_{0,-2}(x) &= x^2(x^2+1)(x-1)^2 - \frac{5x^2(x-1)^4}{9} \\ &= \frac{2x^2(x-1)^2(x+2)(2x+1)}{9}, \end{aligned}$$

we will apply Theorem 2 for

$$\xi \to \infty, \qquad E_{\alpha,\beta} = f_{0,-2}, \qquad F_{\gamma,\delta} = f_{\gamma,-2}.$$

Notice that $\xi \to \infty$ involves

$$\mathbb{I} = \left(\frac{-1}{2}, \infty\right), \qquad \mathbb{R} \setminus \mathbb{I} = \left(-\infty, \frac{-1}{2}\right].$$

The condition (a), namely $f_6(x,1,1) \ge Af_{0,-2}(x)$ for $x \in \mathbb{I}$, is satisfied.

To prove the condition (b), we see that $-2 \in \mathbb{R} \setminus \mathbb{I}$ and

$$f_6(0,y,z) = (y-z)^2(y^4+y^3z+yz^3+z^4), \qquad f_6(0,1,-1) = 0.$$

According to Proposition 1, the difference $f_6(x,1,1) - Af_{\gamma,-2}(x)$ is zero for $x=-2$. Therefore, we will use the Cauchy-Schwarz inequality

$$[(-2)^2+1](x^2+1) \ge (-2x+1)^2,$$

which is equivalent to $(x+2)^2 \ge 0$, to get

$$f_6(x,1,1) \ge F(x),$$

where the polynomial $F(x)$ is a perfect square:

$$F(x) = \frac{1}{5}x^2(2x-1)^2(x-1)^2.$$

Thus, the condition (b) in Theorem 2 is satisfied if

$$F(x) \ge Af_{\gamma,-2}(x)$$

for $x \le -1/2$. We have

$$f_{\gamma,-2}(x) = \frac{4(x-1)^4(x-\gamma)^2}{81(2+\gamma)^2},$$

$$\begin{aligned} F(x) - Af_{\gamma,-2}(x) &= \frac{1}{5}x^2(2x-1)^2(x-1)^2 - \frac{20(x-1)^4(x-\gamma)^2}{9(2+\gamma)^2} \\ &= \frac{(x-1)^2 g_1(x) g_2(x)}{45(2+\gamma)^2}, \end{aligned}$$

where

$$g_1(x) = 3(2+\gamma)x(2x-1) - 10(x-1)(x-\gamma) = (x+2)[\gamma(6x-5)+2x],$$

$$g_2(x) = 3(2+\gamma)x(2x-1) + 10(x-1)(x-\gamma).$$

Choosing

$$\gamma = \frac{-4}{17},$$

$g_1(x)$ is a perfect square,

$$g_1(x) = \frac{10(x+2)^2}{17} \ge 0,$$

and

$$g_2(x) = \frac{10(35x^2-22x-4)}{17} > \frac{10(-8x-4)}{17} = \frac{-40(2x+1)}{17} \ge 0.$$

The equality holds for $x = y = z$, for $x = 0$ and $y = z$ (or any cyclic permutation), and for $x = 0$ and $y + z = 0$ (or any cyclic permutation).

Observation 1. Similarly, applying Theorem 2 for

$$\xi \to \infty, \qquad E_{\alpha,\beta} = f_{k,-2}, \qquad F_{\gamma,\delta} = f_{\gamma,-2}, \qquad \gamma = \frac{13k-4}{k+17},$$

we can prove the following generalization:

- *Let x, y, z be real numbers. If $\frac{-7}{17} \le k \le 1$, then*

$$\sum (x^2+yz)(x-y)(x-z)(x-ky)(x-kz) \ge \left(1+k-\frac{k^2}{4}\right)(x-y)^2(y-z)^2(z-x)^2,$$

with equality for $x = y = z$, for $x/k = y = z$ (or any cyclic permutation) if $k \ne 0$, and for $x = 0$ and $y + z = 0$ (or any cyclic permutation). If $k = 0$, then the equality holds also for $x = 0$ and $y = z$ (or any cyclic permutation).

We have

$$f_6(x,y,z) = \sum (x^2+yz)(x-y)(x-z)(x-ky)(x-kz) - \left(1+k-\frac{k^2}{4}\right)(x-y)^2(y-z)^2(z-x)^2,$$

$$A = \frac{45(k+2)^2}{4},$$

$$\mathbb{I} = \left(\frac{-1}{2}, \infty\right), \quad \mathbb{R} \setminus \mathbb{I} = \left(-\infty, \frac{-1}{2}\right],$$

$$f(x,1,1) = (x^2+1)(x-1)^2(x-k)^2,$$

$$f_{k,-2}(x) = \frac{4(x-1)^4(x-k)^2}{81(2+k)^2}, \qquad Af_{k,-2}(x) = \frac{5(x-1)^4(x-k)^2}{9},$$

$$f_6(x,1,1) - Af_{k,-2}(x) = \frac{2(x-1)^2(x-k)^2(x+2)(2x+1)}{9},$$

$$F(x) = \frac{1}{5}(2x-1)^2(x-1)^2(x-k)^2,$$

$$f_{\gamma,-2}(x) = \frac{4(x-1)^4[(k+17)x-13k+4]^2}{81 \cdot 225(k+2)^2},$$

$$Af_{\gamma,-2}(x) = \frac{(x-1)^4[(k+17)x-13k+4]^2}{405},$$

$$F(x) - Af_{\gamma,-2}(x) = \frac{(x-1)^2 g_1(x) g_2(x)}{405},$$

$$g_1(x) = (1-k)(x+2)^2 \ge 0,$$

$$g_2(x) = (k+35)x^2 - 2(16k+11)x + 2(11k-2),$$

$$g_2(x) \geq g_2\left(\frac{-1}{2}\right) = \frac{9(17k+7)}{4} \geq 0.$$

Observation 2. By leaving out the inequality

$$x^2 + 1 \geq \frac{1}{5}(2x-1)^2,$$

we can prove the inequality from Observation 1 for the extended range

$$\frac{-8}{19} \leq k \leq 9\sqrt{5} - 17 \approx 3.1246.$$

We have

$$f_6(x,1,1) - Af_{\gamma,-2}(x) = \frac{(x-1)^2 g_1(x)}{405},$$

$$\begin{aligned} g_1(x) &= 405(x^2+1)(x^2-2kx+k^2) - (x-1)^2[(k+17)x - 13k+4]^2 \\ &= (x+2)^2 g_2(x), \end{aligned}$$

$$g_2(x) = (116 - 34k - k^2)x^2 - 2(11 + 86k - 16k^2)x + 59k^2 + 26k - 4,$$

$$116 - 34k - k^2 \geq 0,$$

$$g_2(x) \geq g_2\left(\frac{-1}{2}\right) = \frac{9(k+2)(19k+8)}{4} \geq 0.$$

Observation 3. Actually, the inequality from Observation 1 holds for $k \in [-2, 28]$.

□

P 3.24. *Let x, y, z be real numbers. If $0 \leq k \leq 27$, then*

$$\sum (x^2+yz)(x-y)(x-z)(x-ky)(x-kz) \geq \left(1 + k - \frac{k^2}{4}\right)(x-y)^2(y-z)^2(z-x)^2.$$

(Vasile C., 2014)

Solution. Write the inequality as $f_6(x,y,z) \geq 0$. As shown in the proof of P 3.21, $f_6(x,y,z$ has the highest coefficient

$$A = \frac{45(k+2)^2}{4}.$$

Since

$$f(x,1,1) = (x^2+1)(x-1)^2(x-k)^2,$$

$$f_{k,-2}(x) = \frac{4(x-1)^4(x-k)^2}{81(2+k)^2}$$

and

$$\begin{aligned} f_6(x,1,1) - Af_{k,-2}(x) &= (x^2+1)(x-1)^2(x-k)^2 - \frac{5(x-1)^4(x-k)^2}{9} \\ &= \frac{2(x-1)^2(x-k)^2(x+2)(2x+1)}{9}, \end{aligned}$$

we will apply Theorem 2 for

$$\xi \to \infty, \qquad E_{\alpha,\beta} = f_{k,-2}, \qquad F_{\gamma,\delta} = h_{\gamma,\delta}.$$

Notice that $\xi \to \infty$ involves

$$\mathbb{I} = \left(\frac{-1}{2}, \infty\right), \qquad \mathbb{R} \setminus \mathbb{I} = \left(-\infty, \frac{-1}{2}\right].$$

The condition (a) in Theorem 2, namely $f_6(x,1,1) - Af_{k,-2}(x) \geq 0$ for $x > -1/2$, is satisfied.

To prove the condition (b), namely $f_6(x,1,1) \geq Ah_{\gamma,\delta}(x)$ for $x \leq -1/2$, we see that $-2 \in \left(-\infty, \frac{-1}{2}\right]$ and

$$f_6(0,y,z) = k^2y^3z^3 + (y-z)\left[(y^5-z^5) - kyz(y^3-z^3)\right] - \left(1+k-\frac{k^2}{4}\right)y^2z^2(y-z)^2,$$

$$f_6(0,1,-1) = -k^2 + 2(2+2k) - (4+4k-k^2) = 0.$$

Therefore, according to Remark 4, we select γ given by (3.19),

$$\gamma = \frac{1}{3} + \frac{h'(-2)}{12A},$$

where

$$h(x) = f_6(x,1,1) = (x^2+1)(x-1)^2(x-k)^2.$$

We have

$$h'(x) = 2x(x-1)^2(x-k)^2 + 2(x^2+1)(x-1)(x-k)^2 + 2(x^2+1)(x-1)^2(x-k),$$

$$h'(-2) = -6(k+2)(11k+37),$$

$$\gamma = \frac{1}{3} + \frac{h'(-2)}{12A} = \frac{-7k-44}{45(k+2)},$$

$$h_{\gamma,\delta}(x) = \left[x - \frac{7k+44}{45(k+2)}\,(x+2)(2x+1) + \delta(x+2)^3\right]^2.$$

Write the inequality $f_6(x,1,1) \ge Ah_{\gamma,\delta}(x)$ in the form

$$f(x) \ge 0,$$

where

$$f(x) = (x^2+1)(x-1)^2(x-k)^2 - \frac{45(k+2)^2}{4}h_{\gamma,\delta}(x).$$

We choose

$$\delta = \frac{2}{9(k+2)}$$

to have $f(-1/2) = 0$. Next, we get

$$\begin{aligned} h_{\gamma,\delta}(x) &= \left[x - \frac{7k+44}{45(k+2)}\,(x+2)(2x+1) + \frac{2}{9(k+2)}(x+2)^3\right]^2 \\ &= \frac{4[5x^3-(7k+14)x^2+5(k-1)x-7k-4]^2}{2025(k+2)^2} \end{aligned}$$

and

$$\begin{aligned} 45f(x) &= 45(x^2+1)(x-1)^2(x-k)^2 - [5x^3-(7k+14)x^2+5(k-1)x-7k-4]^2 \\ &= (2x+1)(x+2)^2 f_1(x), \end{aligned}$$

where

$$f_1(x) = 10x^3 - 10(k+2)x^2 - 2(k^2-6k-1)x - k^2 - 14k - 4.$$

We need to show that $f_1(x) \le 0$ for $x \le -1/2$. Since $10x^3 \le -5x^2$, we have

$$\begin{aligned} f_1(x) &\le -5x^2 - 10(k+2)x^2 - 2(k^2-6k-1)x - k^2 - 14k - 4 \\ &= -5(2k+5)x^2 - 2(k^2-6k-1)x - k^2 - 14k - 4. \end{aligned}$$

So, we need to show that $f_2(x) \ge 0$, where

$$f_2(x) = 5(2k+5)x^2 + 2(k^2-6k-1)x + k^2 + 14k + 4.$$

We have

$$f_2(x) = 5(2k+5)\left(x + \frac{k^2-6k-1}{10k+25}\right)^2 + \frac{f_3(k)}{5(2k+5)},$$

where

$$\begin{aligned} f_3(k) &= 99 + 378k + 131k^2 + 22k^3 - k^4 \\ &> 108k + 131k^2 + 22k^3 - k^4 \\ &= k(27-k)(4+5k+k^2) \ge 0. \end{aligned}$$

Thus, the proof is completed.

The equality holds for $x = y = z$, for $x/k = y = z$ (or any cyclic permutation) if $k \ne 0$, and for $x = 0$ and $y + z = 0$ (or any cyclic permutation). If $k = 0$, then the equality holds also for $x = 0$ and $y = z$ (or any cyclic permutation).

□

P 3.25. *If x, y, z are real numbers, then*

$$\sum (x^2 + yz)(x - y)(x - z)(x - 28y)(x - 28z) + 167(x - y)^2(y - z)^2(z - x)^2 \geq 0.$$

(Vasile C., 2014)

Solution. This is the inequality of the preceding P 3.24 for $k = 28$. Having in view the proof of P 3.24, we only need to show that there exists a real δ such that $f(x) \geq 0$ for $x \leq -1/2$, where

$$\begin{aligned} f(x) &= (x^2 + 1)(x - 1)^2(x - k)^2 - \frac{45(k + 2)^2}{4} h_{\gamma,\delta}(x), \\ &= (x^2 + 1)(x - 1)^2(x - 28)^2 - 10125 h_{\gamma,\delta}(x), \end{aligned}$$

$$\gamma = \frac{-7k - 44}{45(k + 2)} = \frac{-8}{45},$$

$$\begin{aligned} h_{\gamma,\delta}(x) &= \left[x + \gamma(x + 2)(2x + 1) + \delta(x + 2)^3\right]^2 \\ &= \left[x - \frac{8}{45}\,(x + 2)(2x + 1) + \delta(x + 2)^3\right]^2. \end{aligned}$$

Actually, there is a unique δ such that $f(x) \geq 0$ for all $x \leq -1/2$, namely

$$\delta = \frac{17}{2250}.$$

For this value of δ, we have

$$f(x) = \frac{(x + 2)^4(211x^2 - 6956x - 3056)}{500},$$

where

$$211x^2 - 6956x - 3056 > -6956x - 3056 > 0.$$

The equality holds for $x = y = z$, for $x/28 = y = z$ (or any cyclic permutation), and for $x = 0$ and $y + z = 0$ (or any cyclic permutation).

Observation 1. Similarly, using

$$\gamma = \frac{-7k - 44}{45(k + 2)}$$

and

$$\delta = \frac{17}{75(k + 2)},$$

therefore

$$h_{\gamma,\delta}(x) = \left[x - \frac{7k + 44}{45(k + 2)}\,(x + 2)(2x + 1) + \frac{17}{75(k + 2)}\,(x + 2)^3\right]^2,$$

we can prove the following statement:

- *Let x,y,z be real numbers. If $-97/200 \le k \le 28$, then*

$$\sum (x^2+yz)(x-y)(x-z)(x-ky)(x-kz) \ge \left(1+k-\frac{k^2}{4}\right)(x-y)^2(y-z)^2(z-x)^2,$$

with equality for $x=y=z$, for $x/k=y=z$ (or any cyclic permutation) if $k \ne 0$, and for $x=0$ and $y+z=0$ (or any cyclic permutation). If $k=0$, then the equality holds also for $x=0$ and $y=z$ (or any cyclic permutation).

□

P 3.26. *If x,y,z are real numbers, then*

$$\sum (x^2+yz)(x^2-y^2)(x^2-z^2)+\frac{1}{4}(x-y)^2(y-z)^2(z-x)^2 \ge 0.$$

(Vasile C., 2014)

Solution. Let

$$f(x,y,z)=\sum (x^2+yz)(x^2-y^2)(x^2-z^2)$$

and

$$f_6(x,y,z)=f(x,y,z)+\frac{1}{4}(x-y)^2(y-z)^2(z-x)^2.$$

Since

$$(x-y)(x-z)=x^2+2yz-q, \qquad (x+y)(x+z)=x^2+q,$$

$f(x,y,z)$ has the same highest coefficient A_1 as

$$P_1(x,y,z)=\sum (x^2+yz)(x^2+2yz)x^2,$$

that is, according to (3.1),

$$A_1=P_1(1,1,1)=3(1+1)(1+2)=18.$$

Therefore, $f_6(x,y,z)$ has the highest coefficient

$$A=18+\frac{1}{4}(-27)=\frac{45}{4}.$$

Since

$$f_6(x,1,1)=(x^2+1)(x^2-1)^2,$$

$$f_{-1,-2}(x)=\frac{4(x-1)^4(x+1)^2}{81},$$

$$\begin{aligned} f_6(x,1,1) - Af_{-1,-2}(x) &= (x^2+1)(x^2-1)^2 - \frac{5(x-1)^4(x+1)^2}{9} \\ &= \frac{2(x^2-1)^2(x+2)(2x+1)}{9}, \end{aligned}$$

we will apply Theorem 2 for

$$\xi \to \infty, \qquad E_{\alpha,\beta} = f_{-1,-2}, \qquad F_{\gamma,\delta} = g_{-1,\delta}.$$

Notice that $\xi \to \infty$ involves

$$\mathbb{I} = \left(\frac{-1}{2}, \infty\right), \qquad \mathbb{R} \setminus \mathbb{I} = \left(-\infty, \frac{-1}{2}\right].$$

The condition (a) in Theorem 2, namely $f_6(x,1,1) - Af_{-1,-2}(x) \geq 0$ for $x > -1/2$, is satisfied.

To prove the condition (b), namely $f_6(x,1,1) \geq Ag_{-1,\delta}(x)$ for $x \leq -1/2$, we see that $-2 \in \left(-\infty, \frac{-1}{2}\right]$ and

$$f_6(0,y,z) = y^3z^3 + (y^2-z^2)(y^4-z^4) + \frac{1}{4}y^2z^2(y-z)^2,$$

$$f_6(0,1,-1) = -1 + 0 + 1 = 0.$$

According to Proposition 1, the difference $f_6(x,1,1) - Ag_{-1,\delta}(x)$ is zero for $x = -2$. Therefore, we will use the inequality

$$x^2 + 1 \geq \frac{1}{5}(2x-1)^2,$$

which is an equality for $x = -2$, to get

$$f_6(x,1,1) \geq F(x),$$

where the polynomial $F(x)$ is a perfect square:

$$F(x) = \frac{1}{5}(2x-1)^2(x^2-1)^2.$$

Thus, the condition (b) in Theorem 2 is satisfied if

$$F(x) \geq Ag_{-1,\delta}(x)$$

for $x \leq -1/2$. We have

$$g_{-1,\delta}(x) = (x+1)^2[x^2+5x+8+\delta(x+2)(x+5)]^2,$$

$$F(x) - Ag_{-1,\delta}(x) = (x+1)^2 g(x),$$

where

$$g(x) = \frac{(2x-1)^2(x-1)^2}{5} - \frac{45}{4}[x^2+5x+8+\delta(x+2)(x+5)]^2$$
$$= \frac{g_1(x)g_2(x)}{20},$$

$$g_1(x) = 2(2x-1)(x-1) - 15[x^2+5x+8+\delta(x+2)(x+5)]$$
$$= -(x+2)[(15\delta+11)x+75\delta+59],$$

$$g_2(x) = 2(2x-1)(x-1) + 15[x^2+5x+8+\delta(x+2)(x+5)].$$

Choosing

$$\delta = \frac{-37}{45},$$

$g_1(x)$ is a perfect square,

$$g_1(x) = \frac{4(x+2)^2}{3} \ge 0,$$

and

$$g_2(x) = \frac{4(5x^2-13x-1)}{3} > 0.$$

The equality holds for $x = y = z$, for $-x = y = z$ (or any cyclic permutation), and for $x = 0$ and $y + z = 0$ (or any cyclic permutation).

Observation 1. Similarly, applying Theorem 2 for

$$\xi \to \infty, \qquad E_{\alpha,\beta} = f_{k,-2}, \qquad F_{\gamma,\delta} = g_{k,\delta}, \qquad \delta = \frac{-7k-44}{45(k+2)},$$

we can prove the following generalization:

- *Let x, y, z be real numbers. If $\frac{-5}{3} \le k \le 1$, then*

$$\sum (x^2+yz)(x-y)(x-z)(x-ky)(x-kz) \ge \left(1+k-\frac{k^2}{4}\right)(x-y)^2(y-z)^2(z-x)^2,$$

with equality for $x = y = z$, for $x/k = y = z$ (or any cyclic permutation) if $k \ne 0$, and for $x = 0$ and $y + z = 0$ (or any cyclic permutation). If $k = 0$, then the equality holds also for $x = 0$ and $y = z$ (or any cyclic permutation).

We have

$$f_6(x,y,z) = \sum (x^2+yz)(x-y)(x-z)(x-ky)(x-kz)$$
$$-\left(1+k-\frac{k^2}{4}\right)(x-y)^2(y-z)^2(z-x)^2,$$

$$A = \frac{45(k+2)^2}{4},$$

$$\mathbb{I} = \left(\frac{-1}{2}, \infty\right), \quad \mathbb{R} \setminus \mathbb{I} = \left(-\infty, \frac{-1}{2}\right],$$

$$f(x,1,1) = (x^2+1)(x-1)^2(x-k)^2,$$

$$f_{k,-2}(x) = \frac{4(x-1)^4(x-k)^2}{81(2+k)^2}, \qquad Af_{k,-2}(x) = \frac{5(x-1)^4(x-k)^2}{9},$$

$$f_6(x,1,1) - Af_{k,-2}(x) = \frac{2(x-1)^2(x-k)^2(x+2)(2x+1)}{9},$$

$$F(x) = \frac{1}{5}(2x-1)^2(x-1)^2(x-k)^2,$$

$$g_{k,\delta}(x) = (x-k)^2\left[\frac{kx^2+k(k+6)x-8}{(k+2)^3} + \frac{\delta(x+2)(2kx+x+k-4)}{(k+2)^2}\right]^2,$$

$$Ag_{k,\delta}(x) = \frac{(x-k)^2 g^2(x)}{180(k+2)^4},$$

$$\begin{aligned} g(x) &= 45kx^2 + 45k(k+6)x - 360 - (7k+44)(x+2)(2kx+x+k-4) \\ &= -2(k+2)[(7k+11)x^2 - (5k+22)x + 7k+2], \end{aligned}$$

$$Ag_{k,\delta}(x) = \frac{(x-k)^2[(7k+11)x^2-(5k+22)x+7k+2]^2}{45(k+2)^2},$$

$$F(x) - Ag_{k,\delta}(x) = \frac{(x-k)^2 g_1(x) g_2(x)}{45(k+2)^2},$$

$$g_1(x) = (1-k)(x+2)^2 \geq 0,$$

$$g_2(x) = (13k+23)x^2 - 2(7k+20)x + 10k+8,$$

$$g_2(x) \geq g_2\left(\frac{-1}{2}\right) = \frac{27(3k+5)}{4} \geq 0.$$

Observation 2. By leaving out the simplifying inequality

$$x^2+1 \geq \frac{1}{5}(2x-1)^2,$$

we can prove the inequality from Observation 1 for the extended range

$$\frac{-17}{10} \leq k \leq \frac{13+9\sqrt{5}}{4} \approx 8.281.$$

We have

$$f_6(x,1,1) - Ag_{k,\gamma}(x) = \frac{(x-k)^2 g_1(x)}{45(k+2)^2},$$

$$\begin{aligned} g_1(x) &= 45(k+2)^2(x^2+1)(x-1)^2 - [(7k+11)x^2-(5k+22)x+7k+2]^2 \\ &= (x+2)^2 g_2(x), \end{aligned}$$

$$g_2(x) = (59+26k-4k^2)x^2 - 2(8+k)(7+2k)x + 44 + 38k - k^2,$$

$$59+26k-4k^2 \ge 0,$$

$$g_2(x) \ge g_2\left(\frac{-1}{2}\right) = \frac{27(10k+17)}{4} \ge 0.$$

□

P 3.27. *Let x,y,z be real numbers. If $-2 \le k \le \frac{-3}{2}$, then*

$$\sum (x^2+yz)(x-y)(x-z)(x-ky)(x-kz) \ge \left(1+k-\frac{k^2}{4}\right)(x-y)^2(y-z)^2(z-x)^2,$$

(Vasile C., 2014)

Solution. Let

$$\begin{aligned} f_6(x,y,z) &= \sum (x^2+yz)(x-y)(x-z)(x-ky)(x-kz) \\ &\quad - \left(1+k-\frac{k^2}{4}\right)(x-y)^2(y-z)^2(z-x)^2. \end{aligned}$$

As shown in the proof of P 3.21, $f_6(x,y,z)$ has the highest coefficient

$$A = \frac{45(k+2)^2}{4}.$$

In addition,

$$f_6(x,1,1) = (x^2+1)(x-1)^2(x-k)^2.$$

For $k=-2$, we have $A=0$. Since $f_6(x,1,1) \ge 0$ for all real x, the conclusion follows from Corollary 1. Consider further that

$$-2 < k \le \frac{-3}{2},$$

and apply Theorem 2 for

$$E_{\alpha,\beta} = g_{k,\beta}, \qquad \beta = \frac{-7k-44}{45(k+2)}, \qquad F_{\gamma,\delta} = f_{k,\infty},$$

and $\eta = 2k+2 \in (-2,-1]$, which involves $\xi = \frac{4(k+2)^2}{4k+5}$ and

$$\mathbb{I} = \left(\frac{2-2k}{5+4k}, 2k+2\right), \qquad \mathbb{R} \setminus \mathbb{I} = \left(-\infty, \frac{2-2k}{5+4k}\right] \cup [2k+2, \infty).$$

Condition (a). We need to show that

$$f(x,1,1) \ge Ag_{k,\beta}(x), \qquad \beta = \frac{-7k-44}{45(k+2)}$$

for $x \in \mathbb{I}$. Since

$$f(x,1,1) \ge F(x) = \frac{1}{5}(2x-1)^2(x-1)^2(x-k)^2,$$

it suffices to show that

$$F(x) \ge Ag_{k,\beta}(x).$$

As shown at Observation 1 of the preceding P 3.26, we have

$$F(x) - Ag_{k,\beta}(x) = \frac{(x-k)^2 g_1(x) g_2(x)}{45(k+2)^2},$$

$$g_1(x) = (1-k)(x+2)^2 \ge 0,$$

$$g_2(x) = (13k+23)x^2 - 2(7k+20)x + 10k + 8.$$

We claim that

$$g_2(x) \ge g_2(2k+2) \ge 0.$$

Since

$$g_2(x) - g_2(2k+2) = (x-2k-2)g_3(x), \qquad g_3(x) = (13k+23)x + 26k^2 + 58k + 6,$$

we need to show that $g_3(x) \le 0$. Since

$$g_3(x) = 13(k+2)(2k+3) + g_4(x) \le g_4(x), \qquad g_4(x) = (13k+23)x - 3(11k+24),$$

it suffices to show that $g_4(x) \le 0$. Because

$$-1 \le \frac{-x}{2k+2}, \qquad -3(11k+24) \le \frac{-3(11k+24)x}{2k+2},$$

we get

$$g_4(x) \le (13k+23)x - \frac{3(11k+24)x}{2k+2} = \frac{13(k+2)(2k-1)x}{2(k+1)} \le 0.$$

Also, we have

$$g_2(2k+2) = 2(k+2)(26k^2+32k+5),$$

$$26k^2 + 32k + 5 = 13k(2k+3) + 7(-k) + 5 > 0,$$

therefore $g_2(2k+2) \ge 0$.

Condition (b). We need to show that

$$f(x,1,1) \ge Af_{k,\infty}(x)$$

for
$$x \in \mathbb{R} \setminus \mathbb{I} = \left(-\infty, \frac{2-2k}{5+4k}\right] \cup [2k+2, \infty).$$
Since
$$f(x,1,1) \ge F(x) = \frac{1}{2}(x-1)^4(x-k)^2,$$
it suffices to show that
$$F(x) \ge Af_{k,\infty}(x),$$
where
$$f_{k,\infty}(x) = \frac{4(x-1)^4(x-k)^2}{9(1+2k)^2(x+2)^2}.$$
We have
$$F(x) - Af_{k,\infty}(x) = \frac{(x-1)^4(x-k)^2 g_1(x)}{2(1+2k)^2(x+2)^2},$$
$$g_1(x) = (1+2k)^2(x+2)^2 - 10(k+2)^2.$$
It suffices to show that $g_2(x) \ge 0$, where
$$g_2(x) = (1+2k)^2(x+2)^2 - 16(k+2)^2.$$
Case 1: $x \le \dfrac{2-2k}{5+4k}$. Since
$$x+2 \le \frac{2-2k}{5+4k} + 2 = \frac{6(k+2)}{5+4k} < 0,$$
the inequality $g_2(x) \ge 0$ holds if
$$(1+2k)(x+2) \ge 4(k+2).$$
Indeed,
$$\begin{aligned}(1+2k)(x+2) - 4(k+2) &\ge (1+2k)\,\frac{6(k+2)}{5+4k} - 4(k+2)\\ &= \frac{-2(k+2)(2k+7)}{5+4k} > 0.\end{aligned}$$
Case 2: $x \ge 2k+2$. Since
$$x+2 \ge 2(k+2) > 0,$$
the inequality $g_2(x) \ge 0$ holds if
$$-(1+2k)(x+2) \ge 4(k+2).$$
Indeed,
$$\begin{aligned}-(1+2k)(x+2) - 4(k+2) &\ge -2(1+2k)(k+2) - 4(k+2)\\ &= -2(k+2)(2k+3) \ge 0.\end{aligned}$$

The equality holds for $x = y = z$, for $x/k = y = z$ (or any cyclic permutation) if $k \neq 0$, and for $x = 0$ and $y + z = 0$ (or any cyclic permutation). If $k = 0$, then the equality holds also for $x = 0$ and $y = z$ (or any cyclic permutation).

Observation 1. From P 3.23-3.27 and the observations attached to them, the following generalization follows:

- *Let x, y, z be real numbers. If $-2 \leq k \leq 28$, then*

$$\sum (x^2+yz)(x-y)(x-z)(x-ky)(x-kz) \geq \left(1+k-\frac{k^2}{4}\right)(x-y)^2(y-z)^2(z-x)^2,$$

with equality for $x = y = z$, for $x/k = y = z$ (or any cyclic permutation) if $k \neq 0$, and for $x = 0$ and $y + z = 0$ (or any cyclic permutation). If $k = 0$, then the equality holds also for $x = 0$ and $y = z$ (or any cyclic permutation).

Note that the coefficient of the product $(x-y)^2(y-z)^2(z-x)^2$ is the best possible. Setting $x = k$, $y = 1 + t$ and $z = 1 - t$, the inequality

$$\sum (x^2 + yz)(x-y)(x-z)(x-ky)(x-kz) \geq \delta_k (x-y)^2(y-z)^2(z-x)^2$$

turns into

$$A(k, \delta_k)t^6 + B(k, \delta_k)t^4 + C(k, \delta_k)t^2 \geq 0,$$

where

$$A(k, \delta_k) = -4\delta_k + 4 + 4k - k^2.$$

From the necessary condition $A(k, \delta_k) \geq 0$, we get

$$\delta_k \leq 1 + k - \frac{k^2}{4}.$$

Observation 2. Substituting $k - 1$ for k, and using then the identity

$$\sum (x^2 + yz)(x-y)(x-z)[x-(k-1)y][x-(k-1)z] =$$

$$= \sum (x^2 + yz)(x-y)(x-z)(x-ky+z)(x-kz+y) + 2k(x-y)^2(y-z)^2(z-x)^2,$$

the statement from Observation 1 becomes as follows:

- *Let x, y, z be real numbers. If $-1 \leq k \leq 29$, then*

$$\sum (x^2+yz)(x-y)(x-z)(x-ky+z)(x-kz+y) + \frac{(k+1)^2}{4}(x-y)^2(y-z)^2(z-x)^2 \geq 0,$$

with equality for $x = y = z$, for $x/(k-1) = y = z$ (or any cyclic permutation) if $k \neq 1$, and for $x = 0$ and $y + z = 0$ (or any cyclic permutation). If $k = 1$, then the equality holds also for $x = 0$ and $y = z$ (or any cyclic permutation).

$\square$

P 3.28. *Let x,y,z be real numbers. If $-5 \le k \le -2$ and*

$$\delta_k = \frac{k^4-8k^3-7k^2-20k-20}{4(k-1)^2},$$

then

$$\sum (x^2+yz)(x-y)(x-z)(x-ky)(x-kz)+\delta_k(x-y)^2(y-z)^2(z-x)^2 \ge 0.$$

(Vasile C., 2013)

Solution. Denote

$$f(x,y,z)=\sum (x^2+yz)(x-y)(x-z)(x-ky)(x-kz),$$

and write the inequality as $f_6(x,y,z)\ge 0$, where

$$f_6(x,y,z)=f(x,y,z)+\delta_k(x-y)^2(y-z)^2(z-x)^2.$$

As shown at P 3.21, $f(x,y,z)$ has the highest coefficient $A_1=18(k^2+k+1)$. As a consequence, $f_6(x,y,z)$ has the highest coefficient

$$A=A_1-27\delta_k=\frac{9(k+2)^2(5k^2-4k+17)}{4(k-1)^2}.$$

In addition,

$$f_6(x,1,1)=(x^2+1)(x-1)^2(x-k)^2.$$

For $k=-2$, we have $A=0$. Since $f_6(x,1,1)\ge 0$ for any real x, the conclusion follows from Corollary 1.

Consider further that $-5\le k<-2$. Since

$$f_{k,-2}(x)=\frac{4(x-1)^4(x-k)^2}{81(k+2)^2}$$

and

$$f_6(x,1,1)-Af_{k,-2}(x)=\frac{(x-1)^2(x-k)^2[(k-4)x+(2k+1)][(2k+1)x+k-4]}{9(k-1)^2},$$

we apply Theorem 2 for $\eta=k$, which involves $\xi=\dfrac{(k+2)^2}{2k+1}$,

$$\mathbb{I}=\left(k,\frac{4-k}{2k+1}\right),\qquad \mathbb{R}\setminus\mathbb{I}=(-\infty,k]\cup\left[\frac{4-k}{2k+1},\infty\right),$$

and for

$$E_{\alpha,\beta}=f_{k,-2},\qquad E_{\gamma,\delta}=f_{k,\infty}.$$

The condition (a), namely $f_6(x,1,1) \ge Af_{k,-2}(x)$ for $x \in \mathbb{I}$, is satisfied since

$$(k-4)x+(2k+1) > (k-4)\cdot\frac{4-k}{2k+1}+(2k+1) = \frac{3(k-1)(k+5)}{2k+1} \ge 0,$$

$$(2k+1)x+k-4 > (2k+1)\cdot\frac{4-k}{2k+1}+k-4 = 0.$$

The condition (b) is satisfied if $f_6(x,1,1) \ge Af_{k,\infty}(x)$ for $x \in \mathbb{R}\setminus\mathbb{I}$, where

$$f_{k,\infty}(x) = \frac{4(x-1)^4(x-k)^2}{9(2k+1)^2(x+2)^2}.$$

We have

$$f_6(x,1,1)-Af_{k,\infty}(x) = \frac{(x-1)^2(x-k)^2 f(x)}{(k-1)^2(2k+1)^2(x+2)^2},$$

where

$$f(x) = (k-1)^2(2k+1)^2(x^2+1)(x+2)^2-(k+2)^2(5k^2-4k+17)(x-1)^2.$$

Thus, we need to show that $f(x) \ge 0$ for

$$x \in (-\infty,k]\cup\left[\frac{4-k}{2k+1},\infty\right), \qquad -5 \le k < -2.$$

The inequality $f(x) \ge 0$ is equivalent to

$$h(x) \le h\left(\frac{4-k}{2k+1}\right),$$

where

$$h(x) = \frac{(x-1)^2}{(x^2+1)(x+2)^2}.$$

From

$$h'(x) = \frac{-2(x-1)(x-3)(x^2+x+1)}{(x^2+1)^2(x+2)^3},$$

it follows that h is increasing on $(-\infty,-2)\cup[1,3]$ and decreasing on $(-2,1]\cup[3,\infty)$.

Case 1: $x \le k$. Since h is increasing on $[-5,-2)$ and $k \in [-5,-2)$, we have $h(x) \le h(k)$. Thus, it suffices to show that

$$h(k) \le h\left(\frac{4-k}{2k+1}\right).$$

We have

$$\begin{aligned} h(k)-\left(\frac{4-k}{2k+1}\right) &= \frac{(k-1)^2}{(k^2+1)(k+2)^2}-\frac{(k-1)^2(2k+1)^2}{(k+2)^2(5k^2-4k+17)} \\ &= \frac{4(1-k)(k^2+2)}{(k^2+1)(k+2)(5k^2-4k+17)} \le 0. \end{aligned}$$

Case 2: $x \ge \dfrac{4-k}{2k+1}$. Since

$$\frac{4-k}{2k+1} \in (-2,-1],$$

h is decreasing on $\left(\dfrac{4-k}{2k+1},1\right] \cup [3,\infty)$ and increasing on $[1,3]$. Therefore, we have $h(x) \le h\left(\dfrac{4-k}{2k+1}\right)$ if and only if $h(3) \le h\left(\dfrac{4-k}{2k+1}\right)$. On the other hand, since h is decreasing on $\left(\dfrac{4-k}{2k+1},-1\right]$, we have

$$h\left(\frac{4-k}{2k+1}\right) \ge h(-1).$$

Therefore, it suffices to show that $h(3) \le h(-1)$. Since

$$h(-1) = 2, \qquad h(3) = \frac{2}{125},$$

the conclusion follows.

The equality holds for $x = y = z$, and for $x/k = y = z$ (or any cyclic permutation).

Observation 1. The coefficient δ_k of the product $(x-y)^2(y-z)^2(z-x)^2$ in the original inequality is the best possible. Setting $x = k$, $y = 1+t$ and $z = 1-t$, the original inequality turns into

$$A(k,\delta_k)t^6 + B(k,\delta_k)t^4 + C(k,\delta_k)t^2 \ge 0,$$

where

$$C(k,\delta_k) = (k-1)^2[4(k-1)^2\delta_k - k^4 + 8k^3 + 7k^2 + 20k + 20].$$

From the necessary conditions $C(k,\delta_k) \ge 0$, we get

$$\delta_k \ge \frac{k^4 - 8k^3 - 7k^2 - 20k - 20}{4(k-1)^2}.$$

Observation 2. Substituting $k-1$ for k and using then the identity

$$\sum(x^2+yz)(x-y)(x-z)[x-(k-1)y][x-(k-1)z] =$$

$$= \sum(x^2+yz)(x-y)(x-z)(x-ky+z)(x-kz+y) + 2k(x-y)^2(y-z)^2(z-x)^2,$$

we get the following statement:

- *Let x,y,z be real numbers, and let*

$$\delta_k^* = \frac{(k+1)^2(k^2-6k+2)}{4(k-2)^2}.$$

If $-4 \le k \le -1$, *then*

$$\sum(x^2+yz)(x-y)(x-z)(x-ky+z)(x-kz+y)+\delta_k^*(x-y)^2(y-z)^2(z-x)^2 \ge 0,$$

with equality for $x=y=z$, *and for* $x/(k-1)=y=z$ (*or any cyclic permutation*). □

P 3.29. *If* x,y,z *are real numbers, then*

$$\sum(x^2+yz)(x+y)(x+z) \ge \frac{15(x-y)^2(y-z)^2(z-x)^2}{32(x^2+y^2+z^2)}.$$

(Vasile C., 2013)

Solution. Write the inequality as $f_6(x,y,z) \ge 0$, where

$$f_6(x,y,z)=(x^2+y^2+z^2)\sum(x^2+yz)(x+y)(x+z)-\frac{15}{32}(x-y)^2(y-z)^2(z-x)^2.$$

The function $f_6(x,y,z)$ has the highest coefficient

$$A=-\frac{15}{32}\,(-27)=\frac{405}{32}.$$

Since

$$f_6(x,1,1)=(x^2+2)(x+1)^2(x^2+5),$$

we apply Corollary 2 for

$$F_{\gamma,\delta}(x)=g_{-1,\delta}(x)=(x+1)^2[x^2+5x+8+\delta(x+2)(x+5)]^2.$$

So, we only need to show that there exists a real number δ such that

$$f_6(x,1,1)-Ag_{-1,\delta}(x) \ge 0$$

for all real x. We have

$$f_6(x,1,1)-Ag_{-1,\delta}(x)=\frac{(x+1)^2f(x)}{32},$$

where

$$f(x)=32(x^2+2)(x^2+5)-405[x^2+5x+8+\delta(x+2)(x+5)]^2.$$

Since

$$f(-5)=0,$$

the condition $f'(-5)=0$ is necessary to have $f(x) \ge 0$ in the vicinity of $x=-5$. This condition involves

$$\delta=\frac{-59}{81}$$

and
$$f(x)=\frac{4g(x)}{81},$$
where
$$\begin{aligned}g(x)&=648(x^2+2)(x^2+5)-5(11x^2-4x+29)^2\\&=43x^4+440x^3+1266x^2+1160x+2275\\&=(x+5)^2(43x^2+10x+91)\geq 0.\end{aligned}$$

The equality holds for $-x=y=z$ (or any cyclic permutation).

□

P 3.30. *If x,y,z are real numbers, then*
$$\sum(x-y)(x-z)(x^2+y^2)(x^2+z^2)\geq\frac{7}{4}(x-y)^2(y-z)^2(z-x)^2.$$

(Vasile C., 2012)

Solution. Let
$$f(x,y,z)=4\sum(x-y)(x-z)(x^2+y^2)(x^2+z^2)$$
and
$$f_6(x,y,z)=f(x,y,z)-7(x-y)^2(y-z)^2(z-x)^2.$$
Since $(x-y)(x-z)=x^2+2yz-q$ and
$$x^2+y^2=p^2-2q-z^2,\qquad x^2+z^2=p^2-2q-y^2,$$
$f(x,y,z)$ has the same highest coefficient A_1 as
$$P_2(x,y,z)=4\sum(x^2+2yz)(-z^2)(-y^2),$$
that is, according to (3.2),
$$A_1=P_2(1,1,1)=36.$$
Therefore, $f_6(x,y,z)$ has the highest coefficient
$$A=36-7(-27)=225.$$
Since
$$f_6(x,1,1)=(x-1)^2(x^2+1)^2,$$
we will apply Corollary 2 for
$$F_{\gamma,\delta}(x)=f_{\gamma,-2}(x)=\frac{4(x-1)^4(x-\gamma)^2}{81(2+\gamma)^2}.$$

Thus, we need to show that there exists a real γ such that

$$f_6(x,1,1)-Af_{\gamma,-2}(x)\geq 0$$

for $x\in\mathbb{R}$. We have

$$f_6(x,1,1)-Af_{\gamma,-2}(x)=\frac{(x-1)^2g(x)}{9(2+\gamma)^2},$$

where

$$\begin{aligned}g(x)&=9(\gamma+2)^2(x^2+1)^2-25(x-1)^2(x-\gamma)^2\\&=(x+2)[(3\gamma+1)x+3-\gamma][(3\gamma+11)x^2-5(\gamma+1)x+8\gamma+6].\end{aligned}$$

Choosing $\gamma=1/7$, we get

$$g(x)=\frac{100(x+2)^2(8x^2-4x+5)}{49}\geq 0.$$

The equality holds for $x=y=z$, and also for $x=0$ and $y+z=0$ (or any cyclic permutation).

□

P 3.31. *If x,y,z are real numbers such that $xy+yz+zx\geq 0$, then*

$$\sum(x-y)(x-z)(x^2+y^2)(x^2+z^2)\geq\frac{15}{4}(x-y)^2(y-z)^2(z-x)^2.$$

(Vasile C., 2012)

Solution. Let

$$f(x,y,z)=4\sum(x-y)(x-z)(x^2+y^2)(x^2+z^2)$$

and

$$f_6(x,y,z)=f(x,y,z)-15(x-y)^2(y-z)^2(z-x)^2.$$

As shown at the preceding P 3.30, $f(x,y,z)$ has the highest coefficient $A_1=36$, therefore $f_6(x,y,z)$ has the highest coefficient

$$A=A_1-15(-27)=441.$$

Since

$$f_6(x,1,1)=4(x-1)^2(x^2+1)^2,$$

we will apply Corollary 2 for

$$F_{\gamma,\delta}(x)=f_{\gamma,-2}(x)=\frac{4(x-1)^4(x-\gamma)^2}{81(2+\gamma)^2}.$$

Having in view Remark 5, we need to show that there exists a real γ such that

$$f_6(x,1,1)-Af_{\gamma,-2}(x)\geq 0$$

for $2x+1\geq 0$. We have

$$f_6(x,1,1)-Af_{\gamma,-2}(x)=\frac{4(x-1)^2g(x)}{9(2+\gamma)^2},$$

where

$$g(x)=9(2+\gamma)^2(x^2+1)^2-49(x-1)^2(x-\gamma)^2.$$

Choosing $\gamma=1/3$, which involves $g(-1/2)=0$, we get

$$\begin{aligned}g(x)&=\frac{49}{9}\left[9(x^2+1)^2-(x-1)^2(3x-1)^2\right]\\&=\frac{196(2x+1)(3x^2-2x+2)}{9}\geq 0.\end{aligned}$$

The equality holds for $x=y=z$, and also when x,y,z are proportional to the roots of the equation

$$7t^3-21t^2+18=0.$$

Observation. The last equality condition follows from the necessary condition $\bar{f}_6(x,y,z)=0$, where, according to (3.11),

$$\bar{f}_6(x,y,z)=\left(r-\frac{\gamma+8}{9\gamma+18}pq+\frac{2}{9\gamma+18}p^3\right)^2=\left(r-\frac{25}{63}pq+\frac{2}{21}p^3\right)^2.$$

Moreover, if $p=3$ and $q=0$, then the condition $\bar{f}_6(x,y,z)=0$ involves $r=-18/7$.

□

P 3.32. *If x,y,z are real numbers, then*

$$\sum(x-y)(x-z)(x^2+xy+y^2)(x^2+xz+z^2)\geq\frac{3}{4}(x-y)^2(y-z)^2(z-x)^2.$$

(Vasile C., 2012)

Solution. Let

$$f(x,y,z)=\sum(x-y)(x-z)(x^2+xy+y^2)(x^2+xz+z^2)$$

and

$$f_6(x,y,z)=f(x,y,z)-\frac{3}{4}(x-y)^2(y-z)^2(z-x)^2.$$

Since $(x-y)(x-z)=x^2+2yz-q$ and

$$x^2+xy+y^2=p^2-2q-z^2+xy, \qquad x^2+xz+z^2=p^2-2q-y^2+xz,$$

$f(x,y,z)$ has the same highest coefficient A_1 as

$$P_2(x,y,z)=\sum(x^2+2yz)(z^2-xy)(y^2-xz),$$

that is, according to (3.2),

$$A_1=P_2(1,1,1)=0.$$

Therefore, $f_6(x,y,z)$ has the highest coefficient

$$A=-\frac{3}{4}(-27)=\frac{81}{4}.$$

Since

$$f_6(x,1,1)=(x-1)^2(x^2+x+1)^2,$$

we will apply Corollary 2 for

$$F_{\gamma,\delta}(x)=f_{\gamma,-2}(x)=\frac{4(x-1)^4(x-\gamma)^2}{81(2+\gamma)^2}.$$

Thus, we need to show that there exists a real γ such that

$$f_6(x,1,1)-Af_{\gamma,-2}(x)\ge 0$$

for $x\in\mathbb{R}$. We have

$$f_6(x,1,1)-Af_{\gamma,-2}(x)=\frac{(x-1)^2g(x)}{(2+\gamma)^2},$$

where

$$\begin{aligned}g(x)&=(\gamma+2)^2(x^2+x+1)^2-(x-1)^2(x-\gamma)^2\\&=(x+2)[(\gamma+1)x+1][(\gamma+3)x^2+x+2\gamma+2].\end{aligned}$$

Choosing $\gamma=-1/2$, we get

$$g(x)=\frac{(x+2)^2(5x^2+2x+2)}{4}\ge 0.$$

The equality holds for $x=y=z$, and also for $x=0$ and $y+z=0$ (or any cyclic permutation).

Observation 1. Similarly, applying Corollary 2 for

$$F_{\gamma,\delta}=f_{\gamma,-2}, \qquad \gamma=\frac{1-4k}{7-k},$$

we can prove the following generalization:

- *Let x,y,z be real numbers. If $k \in \left[-2, \frac{8}{5}\right]$, then*

$$\sum(x-y)(x-z)(x^2+kxy+y^2)(x^2+kxz+z^2) \ge \left(\frac{7}{4}-k\right)(x-y)^2(y-z)^2(z-x)^2,$$

with equality for $x = y = z$, and also for $x = 0$ and $y + z = 0$ (or any cyclic permutation).

For

$$\begin{aligned} f_6(x,y,z) = &\sum(x-y)(x-z)(x^2+kxy+y^2)(x^2+kxz+z^2) \\ &-\left(\frac{7}{4}-k\right)(x-y)^2(y-z)^2(z-x)^2, \end{aligned}$$

we have

$$A = \frac{9}{4}(2k-5)^2,$$

$$f_6(x,1,1) = (x-1)^2(x^2+kx+1)^2,$$

$$f_6(x,1,1) - Af_{\gamma,-2}(x) = \frac{(k+2)(x-1)^2(x+2)^2h(x)}{81},$$

where

$$\begin{aligned} h(x) &= (16-k)x^2 + 2(7k-4)x + 10 - 4k \\ &= (16-k)\left(x + \frac{7k-4}{16-k}\right)^2 + \frac{9(2+k)(8-5k)}{16-k} \ge 0. \end{aligned}$$

Observation 2. Actually, the inequality from Observation 1 holds for $k \in [-2, 5/2]$. For $k = 5/2$ and $k = 2$, we get the inequalities in P 3.1 and P 3.47, respectively.

□

P 3.33. *If x,y,z are real numbers such that $xy + yz + zx \ge 0$, then*

$$\sum(x-y)(x-z)(x^2+xy+y^2)(x^2+xz+z^2) \ge 3(x-y)^2(y-z)^2(z-x)^2.$$

(Vasile C., 2012)

Solution. Let

$$f(x,y,z) = \sum(x-y)(x-z)(x^2+xy+y^2)(x^2+xz+z^2)$$

and

$$f_6(x,y,z) = f(x,y,z) - 3(x-y)^2(y-z)^2(z-x)^2.$$

As shown at the preceding P 3.32, $f(x,y,z)$ has the highest coefficient $A_1 = 0$. Therefore, $f_6(x,y,z)$ has the highest coefficient

$$A = 0 - 3(-27) = 81.$$

On the other hand,

$$f_6(x,1,1) = (x-1)^2(x^2+x+1)^2.$$

We will apply Corollary 2 for

$$F_{\gamma,\delta}(x) = f_{0,-2}(x) = \frac{(x-1)^4x^2}{81}.$$

Having in view Remark 5, we need to show that

$$f_6(x,1,1) - Af_{0,-2}(x) \geq 0$$

for $2x+1 \geq 0$. Indeed, we have

$$\begin{aligned} f_6(x,1,1) - Af_{0,-2}(x) &= (x-1)^2(x^2+x+1)^2 - (x-1)^4x^2 \\ &= (x-1)^2(2x+1)(2x^2+1) \geq 0. \end{aligned}$$

The equality holds for $x = y = z$, and also when x,y,z are proportional to the roots of the equation

$$t^3 - 3t^2 + 3 = 0.$$

Observation 1. The last equality condition follows from the necessary condition $\bar{f}_6(x,y,z) = 0$, where

$$\bar{f}_6(x,y,z) = \left(r - \frac{\gamma+8}{9\gamma+18}pq + \frac{2}{9\gamma+18}p^3\right)^2 = \left(r - \frac{4}{9}pq + \frac{1}{9}p^3\right)^2$$

(see (3.11) from Remark 1). Moreover, if $p = 3$ and $q = 0$, then the condition $\bar{f}_6(x,y,z) = 0$ involves $r = -3$.

Observation 2. Similarly, applying Corollary 2 for

$$F_{\gamma,\delta} = f_{\gamma,-2}, \qquad \gamma = \frac{1-k}{3},$$

we can prove the following generalization:

- *Let x,y,z be real numbers such that $xy+yz+zx \geq 0$. If $k \in [-2, 5/2]$, then*

$$\sum (x-y)(x-z)(x^2+kxy+y^2)(x^2+kxz+z^2) \geq \frac{(3-k)(5+k)}{4}(x-y)^2(y-z)^2(z-x)^2,$$

with equality for $x = y = z$, and also when x,y,z are proportional to the roots of the equation

$$(7-k)t^3 - 3(7-k)t^2 + 18 = 0.$$

For

$$f_6(x,y,z)=\sum(x-y)(x-z)(x^2+kxy+y^2)(x^2+kxz+z^2)$$
$$-\frac{(3-k)(5+k)}{4}(x-y)^2(y-z)^2(z-x)^2,$$

we have

$$A=\frac{9}{4}(k-7)^2,$$
$$f_6(x,1,1)=(x-1)^2(x^2+kx+1)^2,$$

and, for $2x+1\geq 0$,

$$f_6(x,1,1)-Af_{\gamma,-2}(x)=\frac{1}{9}(k+2)(x-1)^2(2x+1)[6x^2+4(k-1)x+4-k]\geq 0.$$

□

P 3.34. *If x,y,z are real numbers, then*

$$(x^2+y^2)(y^2+z^2)(z^2+x^2)\geq 8x^2y^2z^2+\frac{3}{8}(x-y)^2(y-z)^2(z-x)^2.$$

(Vasile C., 2011)

Solution. Write the inequality as $f_6(x,y,z)\geq 0$, where

$$f_6(x,y,z)=\prod(y^2+z^2)-8x^2y^2z^2-\frac{3}{8}(x-y)^2(y-z)^2(z-x)^2.$$

Since

$$\prod(y^2+z^2)=\prod(-x^2+p^2-2q),$$

the polynomial $f_6(x,y,z)$ has the same highest coefficient as

$$-x^2y^2z^2-8x^2y^2z^2-\frac{3}{8}(x-y)^2(y-z)^2(z-x)^2,$$

that is

$$A=-9+\frac{81}{8}=\frac{9}{8}.$$

Since

$$f_6(x,1,1)=2(x^2+1)^2-8x^2=2(x^2-1)^2,$$
$$f_{-1,\infty}(x)=\frac{4(x-1)^4(x+1)^2}{9(x+2)^2},$$
$$f_6(x,1,1)-Af_{-1,\infty}(x)=\frac{3(x^2-1)^2(x+1)(x+5)}{2(x+2)^2},$$

we apply Theorem 2 for

$$\eta = -5, \qquad \xi = -1,$$

$$\mathbb{I} = (-5, -1), \qquad \mathbb{R} \setminus \mathbb{I} = (-\infty, -5] \cup [-1, \infty),$$

and for

$$E_{\alpha,\beta} = f_{-1,-2}, \qquad F_{\gamma,\delta} = f_{-1,\infty}.$$

The condition (b), namely $f_6(x,1,1) - Af_{-1,\infty}(x) \ge 0$ for $x \in (-\infty, -5] \cup [-1, \infty)$, is satisfied.

The condition (a) is satisfied if $f_6(x,1,1) - Af_{-1,-2}(x) \ge 0$ for $x \in (-5, -1)$. We have

$$f_{-1,-2}(x) = \frac{4(x-1)^4(x+1)^2}{81},$$

$$f_6(x,1,1) - Af_{-1,-2}(x) = \frac{(x^2-1)^2(x+5)(7-x)}{18} \ge 0.$$

The equality holds for $x = y = z$, and for $-x = y = z$ (or any cyclic permutation).

□

P 3.35. *If x, y, z are real numbers, then*

$$(x^2+2y^2+2z^2)(y^2+2z^2+2x^2)(z^2+2x^2+2y^2) \ge 125x^2y^2z^2 + \frac{15}{2}(x-y)^2(y-z)^2(z-x)^2.$$

(Vasile C., 2011)

Solution. Write the inequality as $f_6(x, y, z) \ge 0$, where

$$f_6(x,y,z) = \prod(x^2+2y^2+2z^2) - 125x^2y^2z^2 - \frac{15}{2}(x-y)^2(y-z)^2(z-x)^2.$$

Since

$$\prod(x^2+2y^2+2z^2) = \prod(-x^2+2p^2-4q),$$

the polynomial $f_6(x, y, z)$ has the same highest coefficient as

$$-x^2y^2z^2 - 125x^2y^2z^2 - \frac{15}{2}(x-y)^2(y-z)^2(z-x)^2,$$

that is

$$A = -126 + \frac{15 \cdot 27}{2} = \frac{153}{2}.$$

Since

$$f_6(x,1,1) = (x^2+4)(2x^2+3)^2 - 125x^2 = 4(x^2-1)^2(x^2+9),$$

$$f_{-1,-2}(x) = \frac{4(x-1)^4(x+1)^2}{81},$$

$$f_6(x,1,1) - Af_{-1,-2}(x) = \frac{2(x^2-1)^2(x+5)(x+29)}{9},$$

we apply Theorem 2 for

$$\eta = -5, \qquad \xi = -1,$$

$$\mathbb{I} = (-5,-1), \qquad \mathbb{R} \setminus \mathbb{I} = (-\infty,-5] \cup [-1,\infty),$$

and for

$$E_{\alpha,\beta} = f_{-1,-2}, \qquad F_{\gamma,\delta} = f_{-1,-1}.$$

The condition (a), namely $f_6(x,1,1) - Af_{-1,-2}(x) \geq 0$ for $x \in (-5,-1)$, is satisfied.

The condition (b) is satisfied if $f_6(x,1,1) - Af_{-1,-1}(x) \geq 0$ for $x \in (-\infty,-5] \cup [-1,\infty)$. We have

$$f_{-1,-1}(x) = \frac{(x-1)^4(x+1)^4}{36(x+2)^2},$$

$$f_6(x,1,1) - Af_{-1,-1}(x) = \frac{(x^2-1)^2 g(x)}{8(x+2)^2},$$

where

$$\begin{aligned} g(x) &= 32(x^2+9)(x+2)^2 - 17(x^2-1)^2 \\ &= 15x^4 + 128x^3 + 450x^2 + 1152x + 1135 \\ &= (x+5)g_1(x), \qquad g_1(x) = 15x^3 + 53x^2 + 185x + 227. \end{aligned}$$

We need to show that $g_1(x) \leq 0$ for $x \leq -5$, and $g_1(x) \geq 0$ for $x \geq -1$. Indeed, if $x \leq -5$, then

$$g_1(x) < 5(3x^3 + 11x^2 + 37x + 54) = 5(x+2)(3x^2 + 5x + 27) < 0,$$

and if $x \geq -1$, then

$$g_1(x) > 15x^2(x+1) + 185(x+1) \geq 0.$$

The equality holds for $x = y = z$, and for $-x = y = z$ (or any cyclic permutation).

□

P 3.36. *If x, y, z are real numbers, then*

$$(2x^2+y^2+z^2)(2y^2+z^2+x^2)(2z^2+x^2+y^2) \geq 64x^2y^2z^2 + \frac{15}{4}(x-y)^2(y-z)^2(z-x)^2.$$

(Vasile C., 2011)

Solution. Write the inequality as $f_6(x,y,z) \ge 0$, where

$$f_6(x,y,z) = \prod(2x^2+y^2+z^2) - 64x^2y^2z^2 - \frac{15}{4}(x-y)^2(y-z)^2(z-x)^2.$$

Since

$$\prod(2x^2+y^2+z^2) = \prod(x^2+p^2-2q),$$

the polynomial $f_6(x,y,z)$ has the same highest coefficient as

$$x^2y^2z^2 - 64x^2y^2z^2 - \frac{15}{4}(x-y)^2(y-z)^2(z-x)^2,$$

that is

$$A = -63 + \frac{15\cdot 27}{4} = \frac{153}{4}.$$

Since

$$f_6(x,1,1) = 2(x^2+1)(x^2+3)^2 - 64x^2 = 2(x^2-1)^2(x^2+9),$$

$$f_{-1,-2}(x) = \frac{4(x-1)^4(x+1)^2}{81},$$

$$f_6(x,1,1) - Af_{-1,-2}(x) = \frac{(x^2-1)^2(x+5)(x+29)}{9},$$

we apply Theorem 2 for

$$\eta = -5, \qquad \xi = -1,$$

$$\mathbb{I} = (-5,-1), \qquad \mathbb{R}\setminus\mathbb{I} = (-\infty,-5]\cup[-1,\infty),$$

and for

$$E_{\alpha,\beta} = f_{-1,-2}, \qquad F_{\gamma,\delta} = f_{-1,-1}.$$

The condition (a), namely $f_6(x,1,1) - Af_{-1,-2}(x) \ge 0$ for $x \in (-5,-1)$, is satisfied.

The condition (b) is satisfied if $f_6(x,1,1) - Af_{-1,-1}(x) \ge 0$ for $x \in (-\infty,-5]\cup[-1,\infty)$. We have

$$f_{-1,-1}(x) = \frac{(x-1)^4(x+1)^4}{36(x+2)^2},$$

$$f_6(x,1,1) - Af_{-1,-1}(x) = \frac{(x^2-1)^2(x+5)g_1(x)}{16(x+2)^2} \ge 0,$$

where

$$g_1(x) = 15x^3 + 53x^2 + 185x + 227.$$

As shown at the preceding P 3.35, we have $(x+5)g_1(x) \ge 0$.

The equality holds for $x = y = z$, and for $-x = y = z$ (or any cyclic permutation).

Observation 1. The inequalities in P 3.34, P 3.35 and P 3.36 are particular cases of the following statement:

- *Let x, y, z be real numbers. If $k \ge 0$, then*

$$\prod(kx^2+y^2+z^2) \ge (k+2)^3x^2y^2z^2 + \frac{3(k+2)(2k+1)}{16}(x-y)^2(y-z)^2(z-x)^2,$$

with equality for $x = y = z$, and for $-x = y = z$ (or any cyclic permutation).

We have

$$A = \frac{9(2k^2+29k+2)}{16},$$

$$f_6(x,1,1) = (x^2-1)^2[kx^2+2(k+1)^2],$$

$$f_6(x,1,1) - Af_{-1,-2}(x) = \frac{(x^2-1)^2(x+5)g(x)}{36},$$

$$g(x) = (-2k^2+7k-2)x + 14k^2 + 23k + 14.$$

Observation 2. The coefficient of the product $(x-y)^2(y-z)^2(z-x)^2$ in the inequality from Observation 1 is the best possible. Setting $x = 1$, $y = 1+t$ and $z = 1-t$, the inequality

$$\prod(kx^2+y^2+z^2) \ge (k+2)^3x^2y^2z^2 + \delta_k(x-y)^2(y-z)^2(z-x)^2$$

turns into

$$A(k,\delta_k)t^6 + B(k,\delta_k)t^4 + 4C(k,\delta_k)t^2 \ge 0,$$

where

$$C(k,\delta_k) = 3(k+2)(2k+1) - 16\delta_k.$$

From the necessary condition $C(k,\delta_k) \ge 0$, we get

$$\delta_k \le \frac{3(k+2)(2k+1)}{16}.$$

□

P 3.37. *If x, y, z are real numbers, then*

$$8(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2) \ge 3(x^2+y^2)(y^2+z^2)(z^2+x^2).$$

(Vasile C., 2013)

Solution. Write the inequality as $f_6(x,y,z) \ge 0$, where

$$f_6(x,y,z) = 8\prod(y^2+yz+z^2) - 3\prod(y^2+z^2).$$

Since

$$f_6(x,y,z) = 8\prod(p^2-2q+yz-x^2) - 3\prod(p^2-2q-x^2),$$

$f_6(x,y,z)$ has the same highest coefficient as

$$f(x,y,z) = 8\prod(yz - x^2) - 3\prod(-x^2),$$

that is, according to (3.3),

$$A = f(1,1,1) = 3.$$

Since

$$f_6(x,1,1) = 24(x^2+x+1)^2 - 6(x^2+1)^2 = 6(x+1)^2(3x^2+2x+3),$$

we apply Corollary 2 for

$$F_{\gamma,\delta}(x) = g_{-1,\delta}(x) = (x+1)^2[x^2+5x+8+\delta(x+2)(x+5)]^2.$$

So, we only need to show that there exists a real number δ such that

$$f_6(x,1,1) - Ag_{-1,\delta}(x) \geq 0$$

for all real x. We have

$$f_6(x,1,1) - Ag_{-1,\delta}(x) = 3(x+1)^2 g(x),$$

where

$$g(x) = 2(3x^2+2x+3) - [x^2+5x+8+\delta(x+2)(x+5]^2.$$

Choosing $\delta = -1$, we get

$$h(x) = 2(3x^2+2x+3) - 4(x+1)^2 = 2(x-1)^2 \geq 0.$$

The equality holds for $-x = y = z$ (or any cyclic permutation).

□

P 3.38. *If x,y,z are nonnegative real numbers, then*

$$\sum(16x^2+3yz)(x-y)(x-z)(x-4y)(x-4z) + 52(x-y)^2(y-z)^2(z-x)^2 \geq 0.$$

(Vasile C., 2013)

Solution. Write the inequality as $f_6(x,y,z) \geq 0$, where

$$f_6(x,y,z) = \sum(16x^2+3yz)(x-y)(x-z)(x-4y)(x-4z)+52(x-y)^2(y-z)^2(z-x)^2.$$

Since

$$(x-y)(x-z) = x^2+2yz-q, \qquad (x-4y)(x-4z) = x^2+20yz-4q,$$

$f_6(x,y,z)$ has the same highest coefficient A as

$$\sum(16x^2+3yz)(x^2+2yz)(x^2+20yz)+52(x-y)^2(y-z)^2(z-x)^2,$$

that is, according to (3.1),

$$A=3(16+3)(1+2)(1+20)-27\cdot 52=3\cdot 729.$$

On the other hand,

$$f_6(x,1,1)=(16x^2+3)(x-1)^2(x-4)^2,$$

$$f_{4,\beta}=\frac{4(x-1)^4(x-4)^2(x-\beta)^2}{729\beta^2(x+2)^2},$$

$$f_6(x,1,1)-Af_{4,\beta}(x)=\frac{(x-1)^2(x-4)^2f(x)}{\beta^2(x+2)^2},$$

$$f(x)=\beta^2(16x^2+3)(x+2)^2-12(x-1)^2(x-\beta)^2.$$

Since $f(0)=0$, we chose $\beta=-2/3$ to have $f'(0)=0$ and

$$f(x)=\frac{4x^2(100+82x-11x^2)}{9}\geq\frac{4x^2(48+82x-11x^2)}{9}=\frac{4x^2(8-x)(6+11x)}{9}.$$

Thus, we apply Theorem 2 for $\eta=8$, which involves $\xi=100/17$,

$$\mathbb{I}=\left(\frac{-4}{17},8\right),\qquad \mathbb{R}\setminus\mathbb{I}=\left(-\infty,\frac{-4}{17}\right]\cup[8,\infty),$$

and for

$$E_{\alpha,\beta}=f_{4,-2/3},\qquad F_{\gamma,\delta}=f_{2,-2}.$$

The condition (a), namely $f_6(x,1,1)\geq Af_{4,-2/3}(x)$ for $x\in(-4/17,8)$, is satisfied because $f(x)\geq 0$.

The condition (b), namely $f_6(x,1,1)\geq Af_{2,-2}(x)$ for $x\in(-\infty,-4/17]\cup[8,\infty)$, where

$$f_{2,-2}(x)=\frac{(x-1)^4(x-2)^2}{324},$$

is equivalent to $(x-1)^2f(x)\geq 0$, where

$$f(x)=(16x^2+3)(x-4)^2-\frac{27}{4}(x-1)^2(x-2)^2.$$

To show that $f(x)\geq 0$, we use the Cauchy-Schwarz inequality

$$(4+3)(16x^2+3)\geq(-8x+3)^2.$$

Thus, it suffices to show that

$$(8x-3)^2(x-4)^2\geq\frac{27}{28}(x-1)^2(x-2)^2.$$

This inequality is true if

$$(8x-3)(x-4) \geq (x-1)(x-2).$$

Indeed,

$$\begin{aligned}(8x-3)(x-4)-(x-1)(x-2) &= 7x^2-32x+10 > 6x^2-32x+10\\ &= 2(x-5)(3x-1) > 0.\end{aligned}$$

The equality holds for $x=y=z$, and for $x/4=y=z$ (or any cyclic permutation).

Observation. The coefficient of the product $(x-y)^2(y-z)^2(z-x)^2$ is the best possible. Setting $x=4$, $y=1+t$ and $z=1-t$, the inequality

$$\sum(16x^2+3yz)(x-y)(x-z)(x-4y)(x-4z)+\alpha(x-y)^2(y-z)^2(z-x)^2 \geq 0$$

turns into

$$At^6+Bt^4+Ct^2 \geq 0,$$

where $C=324(\alpha-52)$. The necessary condition $C \geq 0$ involves $\alpha \geq 52$.

□

P 3.39. *If x,y,z are real numbers, then*

$$\sum x^4(x-y)(x-z) \geq (x-y)^2(y-z)^2(z-x)^2.$$

(Vasile C., 2009)

Solution. Write the inequality as $f_6(x,y,z) \geq 0$, where

$$f_6(x,y,z)=\sum x^4(x-y)(x-z)-(x-y)^2(y-z)^2(z-x)^2.$$

Since $(x-y)(x-z)=x^2+2yz-q$, $f_6(x,y,z)$ has the same highest coefficient A as

$$\sum x^4(x^2+2yz)-(x-y)^2(y-z)^2(z-x)^2,$$

that is, according to (3.1),

$$A=3(1+2)+27=36.$$

Because

$$f_6(x,1,1)=x^4(x-1)^2,$$

$$f_{0,0}(x)=\frac{x^4(x-1)^4}{36(x+2)^2},$$

$$f_6(x,1,1)-Af_{0,0}(x)=\frac{3x^4(x-1)^2(2x+1)}{(x+2)^2},$$

we apply Theorem 2 for

$$E_{\alpha,\beta} = f_{0,0}, \qquad F_{\gamma,\delta} = f_{\gamma,-2},$$

and for $\xi \to \infty$, which involves

$$\mathbb{I} = \left(\frac{-1}{2}, \infty\right), \qquad \mathbb{R} \setminus \mathbb{I} = \left(-\infty, \frac{-1}{2}\right].$$

The condition (a), namely $f_6(x,1,1) \ge Af_{0,0}(x)$ for $x > -1/2$, is satisfied.

The condition (b) is satisfied if there is a real number γ such that

$$f_6(x,1,1) \ge Af_{\gamma,-2}(x)$$

for $x \le -1/2$. We have

$$f_{\gamma,-2}(x) = \frac{4(x-1)^4(x-\gamma)^2}{81(2+\gamma)^2},$$

$$f_6(x,1,1) - Af_{\gamma,-2}(x) = \frac{(x-1)^2 g_1(x) g_2(x)}{9(2+\gamma)^2},$$

where

$$\begin{aligned} g_1(x) &= 3(2+\gamma)x^2 - 4(x-1)(x-\gamma) \\ &= (x+2)[(3\gamma+2)x - 2\gamma], \end{aligned}$$

$$\begin{aligned} g_2(x) &= 3(2+\gamma)x^2 + 4(x-1)(x-\gamma) \\ &= (3\gamma+10)x^2 - 4(\gamma+1)x + 4\gamma. \end{aligned}$$

Choosing $\gamma = \dfrac{-1}{2}$, we have

$$g_1(x) = \frac{1}{2}(x+2)^2 \ge 0,$$

$$g_2(x) = \frac{1}{2}(17x^2 - 4x - 4) > 0.$$

The equality holds for $x = y = z$, for $x = 0$ and $y = z$ (or any cyclic permutation), and for $x = 0$ and $y + z = 0$ (or any cyclic permutation).

□

P 3.40. *Let x, y, z be real numbers. If* $\dfrac{13 - 3\sqrt{17}}{2} \le k \le \dfrac{13 + 3\sqrt{17}}{2}$, *then*

$$\sum (x^2 + kyz)^2 (x-y)(x-z) + \left(\frac{k^2}{4} - 1\right)(x-y)^2(y-z)^2(z-x)^2 \ge 0.$$

(Vasile C., 2012)

Solution. Write the inequality as $f_6(x,y,z) \ge 0$, where

$$f_6(x,y,z) = \sum (x^2+kyz)^2(x-y)(x-z) + \left(\frac{k^2}{4}-1\right)(x-y)^2(y-z)^2(z-x)^2.$$

Since $(x-y)(x-z) = x^2+2yz-q$, $f_6(x,y,z)$ has the same highest coefficient A as

$$\sum (x^2+kyz)^2(x^2+2yz) + \left(\frac{k^2}{4}-1\right)(x-y)^2(y-z)^2(z-x)^2,$$

that is, according to (3.1),

$$A = 9(1+k)^2 + \left(\frac{k^2}{4}-1\right)(-27) = \frac{9}{4}(k+4)^2.$$

Since

$$f_6(x,1,1) = (x^2+1)^2(x-1)^2,$$

we apply Corollary 2 for

$$F_{\gamma,\delta}(x) = g_{1,\delta}(x) = \frac{1}{9}(x-1)^4\left[\frac{x+8}{9}+\delta(x+2)\right]^2.$$

Thus, we only need to show that there exists a real number δ such that

$$f_6(x,1,1) \ge A g_{1,\delta}(x)$$

for all $x \in \mathbb{R}$. We have

$$f_6(x,1,1) - A g_{1,\delta}(x) = \frac{(x-1)^2 g_1(x) g_2(x)}{324},$$

where

$$g_1(x) = 18(x^2+k) + (k+4)(x-1)[x+8+3\delta(x+2)],$$
$$g_2(x) = 18(x^2+k) - (k+4)(x-1)[x+8+3\delta(x+2)].$$

Since

$$g_1(x) = (x+2)[(k+22)x+5k-16+3(k+4)\delta(x-1)],$$

we choose

$$\delta = \frac{k-20}{3(k+4)}$$

to get

$$\begin{aligned} g_1(x) &= (x+2)[(k+22)x+5k-16+(k-20)(x-1)] \\ &= 2(k+1)(x+2)^2 \ge 0, \end{aligned}$$

$$\begin{aligned} g_2(x) &= 18(x^2+k) - (x-1)[(k+4)(x+8)+(k-20)(x+2)] \\ &= (17-k)x^2 - 4(k+1)x + 2(7k-2) \\ &= (17-k)\left(x-\frac{2k+2}{17-k}\right)^2 - \frac{18(k^2-13k+4)}{17-k} \ge 0. \end{aligned}$$

Thus, the proof is completed.

The equality holds for $x = y = z$, and for $x = 0$ and $y + z = 0$ (or any cyclic permutation).

Observation. We may give a similar solution by applying Corollary 2 for

$$F_{\gamma,\delta} = f_{\gamma,-2}.$$

Since

$$f_{\gamma,-2}(x) = \frac{4(x-1)^4(x-\gamma)^2}{81(2+\gamma)^2},$$

we have

$$f_6(x,1,1) - Af_{\gamma,-2}(x) = \frac{(x-1)^2 f_1(x) f_2(x)}{9(2+\gamma)^2},$$

where

$$\begin{aligned} f_1(x) &= 3(2+\gamma)(x^2+k) - (k+4)(x-1)(x-\gamma) \\ &= (x+2)[(3\gamma+2-k)x + (k-2)\gamma + 3k], \end{aligned}$$

$$f_2(x) = 3(2+\gamma)(x^2+k) + (k+4)(x-1)(x-\gamma).$$

By choosing

$$\gamma = \frac{5k-4}{8-k}, \qquad k \neq 8,$$

we get

$$f_1(x) = \frac{(k+1)(k+4)(x+2)^2}{8-k},$$

$$f_2(x) = \frac{(k+4)[(17-k)x^2 - 4(k+1)x + 2(7k-2)]}{8-k},$$

therefore

$$f_1(x) f_2(x) = \frac{(k+1)(k+4)^2(x+2)^2[(17-k)x^2 - 4(k+1)x + 2(7k-2)]}{(8-k)^2} \geq 0.$$

For $k = 8$, by choosing $\gamma \to \infty$ (see P 3.15), we have

$$f_{\infty,-2}(x) = \frac{4(x-1)^4}{81},$$

$$f_6(x,1,1) - Af_{\infty,-2}(x) = (x+2)^2(x^2-4x+12) \geq 0.$$

$\square$

P 3.41. *Let x, y, z be real numbers. If $k \in (-\infty, -4] \cup [-1, 0]$, then*

$$\sum (x^2 + kyz)^2 (x-y)(x-z) + \left(\frac{k^2}{4} - 1\right)(x-y)^2(y-z)^2(z-x)^2 \ge 0.$$

(Vasile C., 2012)

Solution. Denote

$$m = \sqrt{-k}, \qquad m \ge 0.$$

Write the inequality as $f_6(x, y, z) \ge 0$, where

$$f_6(x, y, z) = \sum (x^2 + kyz)^2 (x-y)(x-z) + \left(\frac{k^2}{4} - 1\right)(x-y)^2(y-z)^2(z-x)^2.$$

As shown at the preceding P 3.40, $f_6(x, y, z)$ has the highest coefficient

$$A = \frac{9}{4}(k+4)^2.$$

Also,

$$f_6(x, 1, 1) = (x^2 + k)^2 (x-1)^2.$$

Case 1: $k \le -4$. The case $k = -4$ is treated in P 3.2. Further, consider $k < -4$. Since

$$f_{-m,m}(x) = \frac{4(x-1)^4(x-m)^2(x+m)^2}{9(4+2m^2)^2(x+2)^2} = \frac{(x^2+k)^2(x-1)^2}{9(2-k)^2(x+2)^2},$$

$$f_6(x, 1, 1) - Af_{-m,m}(x) = \frac{3(x^2+k)^2(x-1)^2[(8-k)x + 4 - 5k][(-k)x + 4 - k]}{4(2-k)^2(x+2)^2},$$

we apply Theorem 2 and Remark 3 for

$$E_{\alpha,\gamma} = g_{1,\beta}, \qquad F_{\gamma,\delta} = f_{-m,m},$$

and $\eta = \dfrac{5k-4}{8-k} < -2$, which involves $\xi = \dfrac{(4-k)^2}{k(8-k)}$ and

$$\mathbb{I} = \left(\frac{5k-4}{8-k}, \frac{4-k}{k}\right), \qquad \mathbb{R} \setminus \mathbb{I} = \left(-\infty, \frac{5k-4}{8-k}\right] \cup \left[\frac{4-k}{k}, \infty\right).$$

The condition (b), namely $f_6(x, 1, 1) - Af_{-m,m}(x) \ge 0$ for $x \in \mathbb{R} \setminus \mathbb{I}$, is satisfied.

The condition (a) is satisfied if $f_6(x, 1, 1) - Ag_{1,\beta}(x) \ge 0$ for $x \in \mathbb{I}$. By choosing

$$\beta = \frac{k-20}{3(k+4)},$$

we have (see the preceding P 3.40)

$$f_6(x, 1, 1) - Ag_{1,\beta}(x) = \frac{(k+1)(x-1)^2(x+2)^2 g_2(x)}{162},$$

where
$$g_2(x) = (17-k)x^2 - 4(1+k)x + 2(7k-2).$$
Since $k+1<0$, we must show that $g_2(x) \le 0$ for $x \in \mathbb{I}$. Since $17-k>0$, this is true if $g_2\left(\frac{5k-4}{8-k}\right) \le 0$ and $g_2\left(\frac{4-k}{k}\right) \le 0$. We have
$$g_2\left(\frac{5k-4}{8-k}\right) = \frac{9(k+1)(k+4)^2}{(8-k)^2} < 0,$$
$$g_2\left(\frac{4-k}{k}\right) = \frac{(k+4)(17k^2-59k+68)}{k^2} < 0.$$

Case 2: $k \in [-1,0]$. Since the case $k=-1$ is treated in P 3.14, and the case $k=0$ in P 3.39, consider further $k \in (-1,0)$. The proof is similar to the one of the case 1. We set
$$\eta = \frac{5k-4}{8-k} \in \left(-1, \frac{-1}{2}\right),$$
we involves
$$\mathbb{I} = \left(\frac{4-k}{k}, \frac{5k-4}{8-k}\right), \qquad \mathbb{R} \setminus \mathbb{I} = \left(-\infty, \frac{4-k}{k}\right] \cup \left[\frac{5k-4}{8-k}, \infty\right).$$
The condition (b) is clearly satisfied. Since $k+1>0$, the condition (a) is satisfied if $g_2(x) \ge 0$ for
$$x \in \mathbb{I} = \left(\frac{4-k}{k}, \frac{5k-4}{8-k}\right).$$
Since g_2 is decreasing, we have
$$g_2(x) \ge g_3\left(\frac{5k-4}{8-k}\right) = \frac{9(k+1)(k+4)^2}{(8-k)^2} > 0.$$
Thus, the proof is completed.

The equality holds for $x=y=z$, for $x/\sqrt{-k} = y = z$ (or any cyclic permutation) if $k \ne 0$, for $-x/\sqrt{-k} = y = z$ (or any cyclic permutation) if $k \ne 0$, and for $x=0$ and $y+z=0$ (or any cyclic permutation). If $k=0$, then the equality holds also for $x=0$ and $y=z$ (or any cyclic permutation).

□

P 3.42. *Let x,y,z be real numbers. If $k \ge 0$, then*
$$\sum (x^2+kyz)^2(x-y)(x-z) + \left(\frac{k^2}{4}-1\right)(x-y)^2(y-z)^2(z-x)^2 \ge 0.$$
(Vasile C., 2012)

Solution. Write the inequality as $f_6(x,y,z) \ge 0$, where $f_6(x,y,z)$ has the highest coefficient (see P 3.40)

$$A = \frac{9}{4}(k+4)^2.$$

We have

$$f_6(x,1,1) = (x^2+k)^2(x-1)^2,$$

and

$$f_{\sqrt{-k},-\sqrt{-k}}(x) = \frac{(x^2+k)^2(x-1)^2}{9(2-k)^2(x+2)^2},$$

$$f_6(x,1,1) - Af_{\sqrt{-k},-\sqrt{-k}}(x) = \frac{3(x^2+k)^2(x-1)^2[(8-k)x+4-5k][(-k)x+4-k]}{4(2-k)^2(x+2)^2}.$$

Case 1: $0 \le k \le 2$. The cases $k=2$ and $k=0$ are treated in P 3.22 and P 3.39, respectively. Consider further $0<k<2$. We choose

$$\eta = \frac{5k-4}{8-k} \in \left(\frac{-1}{2}, 1\right),$$

which involves

$$\mathbb{I} = \left(\frac{5k-4}{8-k}, \frac{4-k}{k}\right), \qquad \mathbb{R}\setminus\mathbb{I} = \left(-\infty, \frac{5k-4}{8-k}\right] \cup \left[\frac{4-k}{k}, \infty\right).$$

According to Theorem 2 and Remark 3, it suffices to prove that

(a) $f_6(x,1,1) - Af_{\sqrt{-k},-\sqrt{-k}}(x) \ge 0$ for $x \in \mathbb{I}$;

(b) $f_6(x,1,1) - Ag_{1,\delta}(x) \ge 0$ for $x \in \mathbb{R}\setminus\mathbb{I}$.

The condition (a) is satisfied. With regard to the condition (b), by choosing

$$\delta = \frac{k-20}{3(k+4)},$$

we have (see P 3.40)

$$f_6(x,1,1) - Ag_{1,\delta}(x) = \frac{2(k+1)(x-1)^2(x+2)^2 g_2(x)}{162},$$

where

$$g_2(x) = (17-k)x^2 - 4(1+k)x + 2(7k-2).$$

Since $k+1>0$, we have to show that $g_2(x) \ge 0$ for

$$x \in \mathbb{R}\setminus\mathbb{I} = \left(-\infty, \frac{5k-4}{8-k}\right] \cup \left[\frac{4-k}{k}, \infty\right).$$

If

$$\frac{13-3\sqrt{17}}{2} \le k \le \frac{13+3\sqrt{17}}{2},$$

then the inequality $g_2(x) \ge 0$ holds for all $x \in \mathbb{R}$. So, we only need to show that $g_2(x) \ge 0$ for

$$0 < k < \frac{13 - 3\sqrt{17}}{2} \approx 0.315.$$

Since g_2 is decreasing on $\left(-\infty, \frac{5k-4}{8-k}\right]$ and increasing on $\left[\frac{4-k}{k}, \infty\right)$, we only need to show that $g_2\left(\frac{5k-4}{8-k}\right) \ge 0$ and $g_2\left(\frac{4-k}{k}\right) \ge 0$. Indeed, we have

$$g_2\left(\frac{5k-4}{8-k}\right) = \frac{9(k+1)(k+4)^2}{(8-k)^2} > 0$$

and

$$g_2\left(\frac{4-k}{k}\right) = \frac{(k+4)(17k^2 - 59k + 68)}{k^2} > 0.$$

Thus, the proof is completed.

Case 2: $2 < k \le 8$. The inequality is treated in P 3.40.

Case 3: $k > 8$. We choose

$$\eta = \frac{5k-4}{8-k} \in (-\infty, -5),$$

which involves

$$\mathbb{I} = \left(\frac{5k-4}{8-k}, \frac{4-k}{k}\right), \qquad \mathbb{R} \setminus \mathbb{I} = \left(-\infty, \frac{5k-4}{8-k}\right] \cup \left[\frac{4-k}{k}, \infty\right).$$

According to Theorem 2 and Remark 3, it suffices to prove that

(a) $f_6(x,1,1) - Ag_{1,\delta}(x) \ge 0$ for $x \in \mathbb{I}$;

(b) $f_6(x,1,1) - Af_{-m,m}(x) \ge 0$ for $x \in \mathbb{R} \setminus \mathbb{I}$.

The condition (b) is satisfied because

$$[(8-k)x + 4 - 5k][(-k)x + 4 - k] \ge 0.$$

With regard to the condition (a), by choosing

$$\delta = \frac{k-20}{3(k+4)},$$

we have (see P 3.40)

$$f_6(x,1,1) - Ag_{1,\delta}(x) = \frac{2(k+1)(x-1)^2(x+2)^2 g_2(x)}{162},$$

where

$$g_2(x) = (17-k)x^2 - 4(1+k)x + 2(7k-2).$$

We have to show that $g_2(x) \ge 0$ for $x \in \mathbb{I}$. For $8 < k \le 17$, g_2 is decreasing on $\mathbb{I}$, therefore

$$g_2(x) \ge g_2\left(\frac{4-k}{k}\right) = \frac{(k+4)(17k^2-59k+68)}{k^2} > 0.$$

For $k > 17$, since $17-k < 0$, it suffices to show that $g_2\left(\frac{5k-4}{8-k}\right) \ge 0$ and $g_2\left(\frac{4-k}{k}\right) \ge 0$. Indeed, we have

$$g_2\left(\frac{5k-4}{8-k}\right) = \frac{9(k+1)(k+4)^2}{(8-k)^2} > 0$$

and

$$g_2\left(\frac{4-k}{k}\right) = \frac{(k+4)(17k^2-59k+68)}{k^2} > 0.$$

Thus, the proof is completed.

For $k > 0$, the equality holds for $x = y = z$, and for $x = 0$ and $y + z = 0$ (or any cyclic permutation).

Observation. From P 3.40, P 3.41 and P 3.42, we get the following generalization:

- *Let x, y, z be real numbers. If $k \in (-\infty, -4] \cup [-1, \infty)$, then*

$$\sum (x^2 + kyz)^2(x-y)(x-z) + \left(\frac{k^2}{4} - 1\right)(x-y)^2(y-z)^2(z-x)^2 \ge 0.$$

The inequalities in P 3.2, P 3.14, P 3.22 and P 3.39 are particular cases of this general statement.

The coefficient of the product $(x-y)^2(y-z)^2(z-x)^2$ is the best possible. Setting $x = 1$, $y = 1+t$ and $z = 1-t$, the inequality

$$\sum (x^2 + kyz)^2(x-y)(x-z) + \alpha_k(x-y)^2(y-z)^2(z-x)^2 \ge 0.$$

turns into

$$At^6 + Bt^4 + Ct^2 \ge 0,$$

where $A = 4\alpha_k + 4 - k^2$. The necessary condition $A \ge 0$ involves $\alpha_k \ge \frac{k^2}{4} - 1$.

□

P 3.43. *If x, y, z are real numbers, then*

$$\sum x^2(x^2 + 8yz)(x-y)(x-z) \ge (x-y)^2(y-z)^2(z-x)^2.$$

(Vasile C., 2013)

Solution. Write the inequality as $f_6(x,y,z) \ge 0$, where

$$f_6(x,y,z) = \sum x^2(x^2+8yz)(x-y)(x-z) - (x-y)^2(y-z)^2(z-x)^2.$$

Since $(x-y)(x-z) = x^2+2yz-q$, $f_6(x,y,z)$ has the same highest coefficient A as

$$\sum x^2(x^2+8yz)(x^2+2yz) - (x-y)^2(y-z)^2(z-x)^2,$$

that is, according to (3.1),

$$A = 3(1+8)(1+2)+27 = 108.$$

On the other hand,

$$f_6(x,1,1) = x^2(x^2+8)(x-1)^2.$$

Since

$$f_{0,-2}(x) = \frac{x^2(x-1)^4}{81}$$

and

$$\begin{aligned} f_6(x,1,1) - Af_{0,-2}(x) &= x^2(x^2+8)(x-1)^2 - \frac{4x^2(x-1)^4}{3} \\ &= \frac{x^2(x-1)^2(x+2)(10-x)}{3}, \end{aligned}$$

we apply Theorem 2 for

$$E_{\alpha,\gamma} = f_{0,-2}, \qquad F_{\gamma,\delta} = h_{\gamma,\delta},$$

and for $\eta = 10$, which implies $\xi = 48/7$ and

$$\mathbb{I} = \left(\frac{-2}{7}, 10\right), \qquad \mathbb{R} \setminus \mathbb{I} = \left(-\infty, \frac{-2}{7}\right] \cup [10, \infty).$$

Condition (a): $x \in \mathbb{I} = \left(\frac{-2}{7}, 10\right)$. The condition $f_6(x,1,1) \ge Af_{0,-2}(x)$ is satisfied.

Condition (b): $x \in \mathbb{R} \setminus \mathbb{I} = \left(-\infty, \frac{-2}{7}\right] \cup [10, \infty)$. We need to show that there are γ and δ such that $f_6(x,1,1) \ge Ah_{\gamma,\delta}$. Since

$$f_6(0,y,z) = (y-z)(y^5-z^5) - y^2z^2(y-z)^2,$$

$$f_6(0,1,-1) = 2 \cdot 2 - 4 = 0,$$

according to (3.19), we will choose

$$\gamma = \frac{1}{3} + \frac{h'(-2)}{12A},$$

where

$$h(x) = f_6(x,1,1) = x^2(x^2+8)(x-1)^2.$$

We get

$$h'(-2) = -864, \qquad \frac{h'(-2)}{12A} = \frac{-2}{3}, \qquad \gamma = \frac{-1}{3},$$

therefore

$$h_{-1/3,\delta} = \left[x - \frac{1}{3}(x+2)(2x+1) + \delta(x+2)^3\right]^2,$$

$$f_6(x,1,1) - Ah_{-1/3,\delta}(x) = x^2(x^2+8)(x-1)^2 - 12[-2x^2-2x-2+3\delta(x+2)^3]^2.$$

The inequality $f_6(x,1,1) - Ah_{-1/3,\delta}(x) \geq 0$ holds in the vicinity of -2 only for $\delta = 5/54$. Setting this value for δ, we get

$$\begin{aligned}
f_6(x,1,1) - Ah_{-1/3,\delta}(x) &= x^2(x^2+8)(x-1)^2 - 12\left[-2x^2-2x-2+\frac{5}{18}(x+2)^3\right]^2 \\
&= x^2(x^2+8)(x-1)^2 - \frac{1}{27}(5x^3-6x^2+24x+4)^2 \\
&= \frac{1}{27}(2x^6+6x^5-33x^4-184x^3-312x^2-192x-16) \\
&= \frac{1}{27}(x+2)^4(2x^2-10x-1) \geq 0.
\end{aligned}$$

The equality holds for $x = y = z$, for $x = 0$ and $y = z$ (or any cyclic permutation), and for $x = 0$ and $y + z = 0$ (or any cyclic permutation).

Observation 1. From the inequalities in P 3.39 and P 3.43, we get the following inequality:

$$\sum x^2(x^2+kyz)(x-y)(x-z) \geq (x-y)^2(y-z)^2(z-x)^2, \qquad 0 \leq k \leq 8.$$

Indeed, since the left hand side is linear in k, it suffices to prove this inequality only for $k = 0$ (see P 3.39) and $k = 8$ (see P 3.43).

Observation 2. Notice that the coefficient of the product $(x-y)^2(y-z)^2(z-x)^2$ in the inequality from Observation 1 is the best possible. Indeed, setting $x = 0$, $y = 1+t$ and $z = 1-t$, the inequality

$$\sum x^2(x^2+kyz)(x-y)(x-z) \geq \alpha(x-y)^2(y-z)^2(z-x)^2$$

turns into

$$At^4 + Bt^2 + C \geq 0,$$

where $A = 1-\alpha$. The necessary condition $A \geq 0$ involves $\alpha \leq 1$.

□

P 3.44. *If x, y, z are real numbers, then*

$$\sum (x-y)(x-z)(x-2y)(x-2z)(2x-y)(2x-z)+15(x-y)^2(y-z)^2(z-x)^2 \ge 0.$$

(Vasile C., 2012)

Solution. Let

$$f(x,y,z)=\sum (x-y)(x-z)(x-2y)(x-2z)(2x-y)(2x-z)$$

and

$$f_6(x,y,z)=f(x,y,z)+15(x-y)^2(y-z)^2(z-x)^2.$$

Using the identities

$$(x-y)(x-z)=x^2+2yz-q,$$
$$(x-2y)(x-2z)=x^2+6yz-2q,$$
$$(2x-y)(2x-z)=4x^2+3yz-2q,$$

it follows that $f(x,y,z)$ has the same highest coefficient A_1 as

$$P_1(x,y,z)=\sum (x^2+2yz)(x^2+6yz)(4x^2+3yz),$$

that is, according to (3.1),

$$A_1=P_1(1,1,1)=3(1+2)(1+6)(4+3)=441.$$

Therefore, $f_6(x,y,z)$ has the highest coefficient

$$A=441+15(-27)=36.$$

Since

$$f_6(x,1,1)=(x-1)^2(x-2)^2(2x-1)^2,$$
$$f_{2,1/2}(x)=\frac{4(x-1)^4(x-2)^2(2x-1)^2}{9(x+2)^2},$$
$$f_6(x,1,1)-Af_{2,1/2}(x)=\frac{3(x-1)^2(x-2)^2(2x-1)^2(5x-2)(2-x)}{(x+2)^2},$$

apply Theorem 2 for

$$E_{\alpha,\gamma}=f_{2,1/2}, \qquad F_{\gamma,\delta}=f_{2,-2},$$

and for $\eta=2$, which implies $\xi=16/5$ and

$$\mathbb{I}=\left(\frac{2}{5},2\right), \qquad \mathbb{R}\setminus\mathbb{I}=\left(-\infty,\frac{2}{5}\right]\cup[2,\infty).$$

The condition (a), namely $f_6(x,1,1)\ge Af_{2,1/2}(x)$ for $x\in\left(\frac{2}{5},2\right)$, is satisfied.

The condition (b) is satisfied if $f_6(x,1,1) \ge Af_{2,-2}(x)$ for $x \in \left(-\infty, \frac{2}{5}\right] \cup [2,\infty)$. From

$$f_{2,-2}(x) = \frac{(x-1)^4(x-2)^2}{324},$$

it follows that this inequality is equivalent to

$$(x-1)^2(x-2)^2(5x-2)(7x-4) \ge 0,$$

which is true.

The equality holds for $x = y = z$, for $x/2 = y = z$ (or any cyclic permutation), and for $2x = y = z$ (or any cyclic permutation).

Observation. The coefficient of the product $(x-y)^2(y-z)^2(z-x)^2$ is the best possible. Setting $x = y + z$ in

$$\sum(x-y)(x-z)(x-2y)(x-2z)(2x-y)(2x-z) + \alpha(x-y)^2(y-z)^2(z-x)^2 \ge 0,$$

we get

$$(y-z)^2[(\alpha-15)y^2z^2 + 4(y^2+z^2)(y-z)^2] \ge 0,$$

which holds for all real numbers y and z only if $\alpha \ge 15$.

□

P 3.45. *If x, y, z are real numbers, then*

$$\sum(x-y)(x-z)(x-2y)(x-2z)(x-3y)(x-3z) \ge 3(x-y)^2(y-z)^2(z-x)^2.$$

(Vasile C., 2012)

Solution. Let

$$f(x,y,z) = \sum(x-y)(x-z)(x-2y)(x-2z)(x-3y)(x-3z),$$

and

$$f_6(x,y,z) = f(x,y,z) - 3(x-y)^2(y-z)^2(z-x)^2.$$

From

$$(x-y)(x-z) = x^2 + 2yz - q,$$
$$(x-2y)(x-2z) = x^2 + 6yz - 2q,$$
$$(x-3y)(x-3z) = x^2 + 12yz - 3q,$$

it follows that $f(x,y,z)$ has the same highest coefficient A_1 as

$$P_1(x,y,z) = \sum(x^2+2yz)(x^2+6yz)(x^2+12yz),$$

that is, according to (3.1),

$$A_1 = P_1(1,1,1) = 3(1+2)(1+6)(1+12) = 819.$$

Thus, $f_6(x,y,z)$ has the highest coefficient

$$A = 819 - 3(-27) = 900.$$

Since

$$f_6(x,1,1) = (x-1)^2(x-2)^2(x-3)^2,$$

$$f_{2,3}(x) = \frac{4(x-1)^4(x-2)^2(x-3)^2}{1521(x+2)^2},$$

$$f_6(x,1,1) - Af_{2,3}(x) = \frac{3(x-1)^2(x-2)^2(x-3)^2(2+11x)(46-7x)}{169(x+2)^2},$$

apply Theorem 2 for

$$E_{\alpha,\gamma} = f_{2,3}, \qquad F_{\gamma,\delta} = f_{\gamma,-2},$$

and for $\eta = 46/7$, which implies $\xi = 400/77$ and

$$\mathbb{I} = \left(\frac{-2}{11}, \frac{46}{7}\right), \qquad \mathbb{R} \setminus \mathbb{I} = \left(-\infty, \frac{-2}{11}\right] \cup \left[\frac{46}{7}, \infty\right).$$

The condition (a), namely $f_6(x,1,1) \ge Af_{2,3}(x)$ for $x \in \left(\frac{-2}{11}, \frac{46}{7}\right)$, is satisfied.

The condition (b) is satisfied if there exists a real γ such that $f_6(x,1,1) \ge Af_{\gamma,-2}(x)$ for $x \in \left(-\infty, \frac{-2}{11}\right] \cup \left[\frac{46}{7}, \infty\right)$. We have

$$f_{\gamma,-2}(x) = \frac{4(x-1^4(x-\gamma)^2}{81(2+\gamma)^2},$$

$$f_6(x,1,1) - Af_{\gamma,-2}(x) = \frac{(x-1)^2 g(x)}{9(2+\gamma)^2},$$

where

$$g(x) = 9(2+\gamma)^2(x-2)^2(x-3)^2 - 400(x-1)^2(x-\gamma)^2 = g_1(x)g_2(x),$$

$$g_1(x) = 3(2+\gamma)(x-2)(x-3) - 20(x-1)(x-\gamma),$$

$$g_2(x) = 3(2+\gamma)(x-2)(x-3) + 20(x-1)(x-\gamma).$$

Since

$$g_1(x) = (x+2)[(3\gamma-14)x + 18 - \gamma],$$

we choose

$$\gamma = \frac{46}{7}$$

to get

$$g_1(x) = \frac{40}{7}(x+2)^2 \geq 0,$$

$$g_2(x) = \frac{40}{7}(8x^2 - 49x + 50) > 0.$$

The equality holds for $x = y = z$, for $x/2 = y = z$ (or any cyclic permutation), for $x/3 = y = z$ (or any cyclic permutation), and for $x = 0$ and $y + z = 0$ (or any cyclic permutation).

Observation. The coefficient of the product $(x-y)^2(y-z)^2(z-x)^2$ is the best possible. Setting $x = 0$, $y = 1$ and $z = -1$ in the inequality

$$\sum (x-y)(x-z)(x-2y)(x-2z)(x-3y)(x-3z) \geq \alpha(x-y)^2(y-z)^2(z-x)^2$$

involves $\alpha \leq 3$.

□

P 3.46. *If x, y, z are real numbers, then*

$$\sum (x-y)(x-z)(2x+3y)(2x+3z)(3x+2y)(3x+2z) + 15(x-y)^2(y-z)^2(z-x)^2 \geq 0.$$

(Vasile C., 2012)

Solution. Let

$$f(x,y,z) = \sum (x-y)(x-z)(2x+3y)(2x+3z)(3x+2y)(3x+2z)$$

and

$$f_6(x,y,z) = f(x,y,z) + 15(x-y)^2(y-z)^2(z-x)^2.$$

From

$$(x-y)(x-z) = x^2 + 2yz - q,$$
$$(2x+3y)(2x+3z) = 4x^2 + 3yz + 6q,$$
$$(3x+2y)(3x+2z) = 9x^2 - 2yz + 6q,$$

it follows that $f(x,y,z)$ has the same highest coefficient A_1 as

$$P_1(x,y,z) = \sum (x^2 + 2yz)(4x^2 + 3yz)(9x^2 - 2yz),$$

that is, according to (3.1),

$$A_1 = P_1(1,1,1) = 3(1+2)(4+3)(9-2) = 441.$$

Thus, $f_6(x,y,z)$ has the highest coefficient

$$A = 441 + 15(-27) = 36.$$

Since

$$f_6(x,1,1) = (x-1)^2(2x+3)^2(3x+2)^2,$$

$$f_{-3/2,-2/3}(x) = \frac{4(x-1)^4(2x+3)^2(3x+2)^2}{9 \cdot 625(x+2)^2},$$

$$f_6(x,1,1) - Af_{-3/2,-2/3}(x) = \frac{3(x-1)^2(2x+3)^2(3x+2)^2(7x+18)(29x+46)}{625(x+2)^2},$$

apply Theorem 2 for $\eta = -46/29$, which implies $\xi = -16/203$ and

$$\mathbb{I} = \left(\frac{-18}{7}, \frac{-46}{29}\right), \qquad \mathbb{R} \setminus \mathbb{I} = \left(-\infty, \frac{-18}{7}\right] \cup \left[\frac{-46}{29}, \infty\right),$$

and for

$$E_{\alpha,\gamma} = f_{\alpha,-2}, \qquad F_{\gamma,\delta} = f_{-3/2,-2/3}.$$

The condition (a) is satisfied if there exists a real α such that $f_6(x,1,1) \geq Af_{\alpha,-2}(x)$ for $x \in \left(\frac{-18}{7}, \frac{-46}{29}\right)$. We have

$$f_{\alpha,-2}(x) = \frac{4(x-1^4(x-\alpha)^2}{81(2+\alpha)^2},$$

$$f_6(x,1,1) - Af_{\alpha,-2}(x) = \frac{(x-1)^2 g(x)}{9(2+\alpha)^2},$$

where

$$g(x) = 9(2+\alpha)^2(2x+3)^2(3x+2)^2 - 16(x-1)^2(x-\alpha)^2.$$

It is easy to check that $g(-2) = 0$. Choosing $\alpha = -46/29$, the inequality $g(x) \geq 0$ is equivalent to

$$(x+2)^2(83x^2 + 134x + 8) \geq 0,$$

which is true because $83x^2 + 134x + 8 > 0$ for $x \leq -46/29$.

The condition (b), namely $f_6(x,1,1) \geq Af_{-3/2,-2/3}(x)$ for $x \in \left(-\infty, \frac{-18}{7}\right] \cup \left[\frac{-46}{29}, \infty\right)$, is satisfied.

The equality holds for $x = y = z$, for $-3x/2 = y = z$ (or any cyclic permutation), for $-2x/3 = y = z$ (or any cyclic permutation), and for $x = 0$ and $y + z = 0$ (or any cyclic permutation).

Observation. The coefficient of the product $(x-y)^2(y-z)^2(z-x)^2$ is the best possible. Setting $x = 0$, $y = 1$ and $z = -1$ in

$$\sum (x-y)(x-z)(2x+3y)(2x+3z)(3x+2y)(3x+2z) + \alpha(x-y)^2(y-z)^2(z-x)^2 \geq 0,$$

yields $\alpha \geq 15$.

□

P 3.47. *If x, y, z are real numbers, then*

$$\sum(x+y)(x+z)(x^2-y^2)(x^2-z^2)+\frac{1}{4}(x-y)^2(y-z)^2(z-x)^2 \ge 0.$$

(Vasile C., 2012)

Solution. Let

$$f(x,y,z)=\sum(x+y)^2(x+z)^2(x-y)(x-z)$$

and

$$f_6(x,y,z)=f(x,y,z)+\frac{1}{4}(x-y)^2(y-z)^2(z-x)^2.$$

Since

$$(x+y)(x+z)=x^2+q, \qquad (x-y)(x-z)=x^2+2yz-q,$$

$f(x,y,z)$ has the same highest coefficient A_1 as

$$P_1(x,y,z)=\sum x^4(x^2+2yz),$$

that is, according to (3.1),

$$A_1=P_1(1,1,1)=9.$$

Therefore, $f_6(x,y,z)$ has the highest coefficient

$$A=9+\frac{1}{4}(-27)=\frac{9}{4}.$$

On the other hand, since

$$f_6(x,1,1)=(x+1)^4(x-1)^2,$$

$$f_{-1,-1}(x)=\frac{(x+1)^4(x-1)^4}{36(x+2)^2},$$

$$Af_{-1,-1}(x)=\frac{(x+1)^4(x-1)^4}{16(x+2)^2},$$

$$f_6(x,1,1)-Af_{-1,-1}(x)=\frac{3(x+1)^4(x-1)^2(x+3)(5x+7)}{16(x+2)^2},$$

we will apply Theorem 2 for $\eta=-3$, which involves $\xi=-1/55$,

$$\mathbb{I}=\left(-3,\frac{-7}{5}\right), \qquad \mathbb{R}\setminus\mathbb{I}=(-\infty,-3]\cup\left[\frac{-7}{5},\infty\right),$$

and for

$$E_{\alpha,\beta}=f_{\alpha,-2}, \qquad F_{\gamma,\delta}=f_{-1,-1}.$$

Condition (a): $x \in \mathbb{I} = \left(-3, \frac{-7}{5}\right)$. We need to show that there exists a real α such that

$$f_6(x,1,1) \ge Af_{\alpha,-2}(x).$$

Since

$$f_{\alpha,-2}(x) = \frac{4(x-1)^4(x-\alpha)^2}{81(\alpha+2)^2},$$

$$Af_{\alpha,-2}(x) = \frac{(x-1)^4(x-\alpha)^2}{9(\alpha+2)^2},$$

we have

$$f_6(x,1,1) - Af_{\alpha,-2}(x) = \frac{g_1(x)g_2(x)}{9(\alpha+2)^2},$$

where

$$g_1(x) = 3(\alpha+2)(x+1)^2 - (x-1)(x-\alpha) = (x+2)[(3\alpha+5)x+\alpha+3],$$

$$g_2(x) = 3(\alpha+2)(x+1)^2 + (x-1)(x-\alpha).$$

Since $-2 \in \mathbb{I}$, we choose

$$\alpha = \frac{-7}{5}$$

to get

$$g_1(x) = \frac{4}{5}(x+2)^2 \ge 0$$

and

$$g_2(x) = \frac{2}{5}(7x^2+10x+1).$$

For $x \le \frac{-7}{5}$, we have $g_2(x) > 0$ because

$$7x^2+10x+1 > 7x^2+10x+\frac{7}{25} = \frac{(5x+7)(35x+1)}{25} \ge 0.$$

Condition (b): $x \in \mathbb{R} \setminus \mathbb{I} = (-\infty,-3] \cup \left[\frac{-7}{5}, \infty\right)$. As shown above, we have

$$f_6(x,1,1) - Af_{-1,-1}(x) = \frac{3(x+1)^4(x-1)^2(x+3)(5x+7)}{16(x+2)^2} \ge 0.$$

The equality holds for $x = y = z$, for $-x = y = z$ (or any cyclic permutation), and for $x = 0$ and $y + z = 0$ (or any cyclic permutation).

Observation 1. Similarly, we can prove the following generalization:

- *Let x, y, z be real numbers. If*

$$\alpha_k = \frac{1}{4}(k^2-2k-2)(k^2+2k+2), \qquad -2 \le k \le 0,$$

then

$$\sum(x-y)(x-z)(x-ky)^2(x-kz)^2+\alpha_k(x-y)^2(y-z)^2(z-x)^2 \ge 0,$$

with equality for $x=y=z$, for $x/k=y=z$ (or any cyclic permutation) if $k\neq 0$, and for $x=0$ and $y+z=0$ (or any cyclic permutation). If $k=0$, then the equality holds also for $x=0$ and $y=z$ (or any cyclic permutation).

For $k=0$, we get the inequality from P 3.39. Further, consider that

$$-2\le k<0.$$

We have

$$A=\frac{9}{4}(k+2)^4,$$

$$f_6(x,y,z)=(x-1)^2(x-k)^2,$$

$$f_{k,k}(x)=\frac{(x-1)^4(x-k)^4}{9(k-1)^2(k+2)^2},$$

$$f_6(x,1,1)-Af_{k,k}(x)=\frac{(x-1)^2(x-k)^4}{4(k-1)^2(x+2)^2}h(x),$$

$$\begin{aligned}h(x)&=4(k-1)^2(x+2)^2-(k+2)^2(x-1)^2\\&=(-kx+2-k)[(4-k)x+2-5k].\end{aligned}$$

Therefore, we apply Theorem 2 for

$$\eta=\frac{2-k}{k}\le -2,\qquad \xi=\frac{(k+2)^2}{k(4-k)},$$

$$\mathbb{I}=\left(\frac{2-k}{k},\frac{5k-2}{4-k}\right),\qquad \mathbb{R}\setminus\mathbb{I}=\left(-\infty,\frac{2-k}{k}\right]\cup\left[\frac{5k-2}{4-k},\infty\right),$$

and for

$$E_{\alpha,\beta}=f_{\alpha,-2},\qquad F_{\gamma,\delta}=f_{k,k}.$$

Condition (b): $x\in\mathbb{R}\setminus\mathbb{I}=\left(-\infty,\frac{2-k}{k}\right]\cup\left[\frac{5k-2}{4-k},\infty\right)$. Since $h(x)\ge 0$, we have $f_6(x,1,1)-Af_{k,k}(x)\ge 0$.

Condition (a): $x\in\mathbb{I}=\left(\frac{2-k}{k},\frac{5k-2}{4-k}\right)$. By choosing

$$\alpha=\frac{5k-2}{4-k},$$

we get

$$f_6(x,1,1)-Af_{\alpha,-2}(x)=\frac{(x-1)^2}{9}g_1(x)g_2(x),$$

$$\begin{aligned} g_1(x) &= 3(\alpha+2)(x-k)^2-(k+2)^2(x-1)(x-\alpha) \\ &= (x+2)[(3\alpha-k^2-4k+2)x+(k^2-2k-2)\alpha+3k^2] \\ &= \frac{(1-k)^2(2+k)}{4-k}\,(x+2)^2 \geq 0, \end{aligned}$$

$$g_2(x) = 3(\alpha+2)(x-k)^2+(k+2)^2(x-1)(x-\alpha) = \frac{k+2}{4-k}\,g_3(x),$$

$$g_3(x) = (17+2k-k^2)x^2-4(1+7k+k^2)x-2(2-4k-7k^2).$$

Since

$$g_3(x) \geq g_3\left(\frac{5k-2}{4-k}\right) > 0$$

for $x \in \mathbb{I}$ and $-2 \leq k < 0$, the condition (a) is satisfied, too.

Observation 2. Similarly, we can prove that the inequality from Observation 1 is also valid for $k \in (-\infty,-2) \cup (4,\infty)$. To prove this, we choose

$$\eta = \frac{2-k}{k} \in \left(-2, \frac{-1}{2}\right), \qquad \xi = \frac{(k+2)^2}{k(4-k)},$$

$$\mathbb{I} = \left(\frac{5k-2}{4-k}, \frac{2-k}{k}\right), \qquad \mathbb{R} \setminus \mathbb{I} = \left(-\infty, \frac{5k-2}{4-k}\right] \cup \left[\frac{2-k}{k}, \infty\right).$$

If $x \in \mathbb{I}$, we have

$$g_3(x) \geq g_3\left(\frac{5k-2}{4-k}\right) > 0, \qquad k \in (-\infty,-2),$$

$$g_3(x) \geq g_3\left(\frac{2-k}{k}\right) > 0, \qquad k \in (4,14].$$

In addition, we have $g_3(x) > 0$ for $k \in [6,\infty)$ because

$$g_3\left(\frac{5k-2}{4-k}\right) > 0, \quad g_3\left(\frac{2-k}{k}\right) > 0.$$

□

P 3.48. *If x, y, z are real numbers, then*

$$\sum (x-y)(x-z)(x-4y)^2(x-4z)^2 + 39(x-y)^2(y-z)^2(z-x)^2 \geq 0.$$

(Vasile C., 2015)

Solution. Let

$$f(x,y,z)=\sum(x-y)(x-z)(x-4y)^2(x-4z)^2$$

and

$$f_6(x,y,z)=f(x,y,z)-39(x-y)^2(y-z)^2(z-x)^2.$$

Since

$$(x-y)(x-z)=x^2+2yz-q, \qquad (x-4y)(x-4z)=x^2+20yz-4q,$$

$f(x,y,z)$ has the same highest coefficient A_1 as

$$P_1(x,y,z)=\sum(x^2+2yz)(x^2+20yz)^2,$$

that is, according to (3.1),

$$A_1=P_1(1,1,1)=3969$$

Therefore, $f_6(x,y,z)$ has the highest coefficient

$$A=3969+39(-27)=2916.$$

On the other hand, since

$$f_6(x,1,1)=(x-1)^2(x-4)^4,$$

$$f_{4,4}(x)=\frac{(x-1)^4(x-4)^4}{2916(x+2)^2},$$

$$Af_{4,4}(x)=\frac{(x-1)^4(x-4)^4}{(x+2)^2},$$

$$f_6(x,1,1)-Af_{4,4}(x)=\frac{3(x-1)^2(x-4)^4(2x+1)}{(x+2)^2},$$

we will apply Theorem 2 for $\eta\to\infty$, which involves $\xi\to\infty$,

$$\mathbb{I}=\left(\frac{-1}{2},\infty\right), \qquad \mathbb{R}\setminus\mathbb{I}=\left(-\infty,\frac{-1}{2}\right],$$

and for

$$E_{\alpha,\beta}=f_{4,4} \qquad F_{\gamma,\delta}=f_{\infty,-2}.$$

Condition (a): $x\in\mathbb{I}=\left(\frac{-1}{2},\infty\right)$. Because

$$f_6(x,1,1)-Af_{4,4}(x)\ge 0,$$

this condition is satisfied.

Condition (b): $x \in \mathbb{R} \setminus \mathbb{I} = \left(-\infty, \frac{-1}{2}\right]$. We have

$$f_{\infty,-2}(x) = \frac{4(x-1)^4}{81},$$

$$Af_{\infty,-2}(x) = 144(x-1)^4,$$

$$f_5(x,1,1) - Af_{\infty,-2}(x) = (x-1)^2(x+2)^2(x^2-20x+29) \geq 0.$$

The equality holds for $x = y = z$, for $x/4 = y = z$ (or any cyclic permutation), and for $x = 0$ and $y + z = 0$ (or any cyclic permutation).

Observation 1. Similarly, we can prove the following generalization:

- *Let x, y, z be real numbers. If*

$$\alpha_k = \frac{1}{4}(k^2-2k-2)(k^2+2k+2), \qquad 1 \leq k \leq 4,$$

then

$$\sum (x-y)(x-z)(x-ky)^2(x-kz)^2 + \alpha_k(x-y)^2(y-z)^2(z-x)^2 \geq 0,$$

with equality for $x = y = z$, for $x/k = y = z$ (or any cyclic permutation) if $k \neq 0$, and for $x = 0$ and $y + z = 0$ (or any cyclic permutation).

The cases $k = 1$ and $k = 4$ are treated in P 3.18 and P 3.48, respectively. Further, assume that

$$1 < k < 4.$$

We have

$$A = \frac{9}{4}(k+2)^4,$$

$$f_6(x,y,z) = (x-1)^2(x-k)^2,$$

$$f_{k,k}(x) = \frac{(x-1)^4(x-k)^4}{9(k-1)^2(k+2)^2},$$

$$f_6(x,1,1) - Af_{k,k}(x) = \frac{(x-1)^2(x-k)^4}{4(k-1)^2(x+2)^2}(-kx+2-k)[(4-k)x+2-5k].$$

Therefore, we apply Theorem 2 for

$$\eta = \frac{2-k}{k} \in \left(\frac{-1}{2}, 1\right), \qquad \xi = \frac{(k+2)^2}{k(4-k)},$$

$$\mathbb{I} = \left(\frac{2-k}{k}, \frac{5k-2}{4-k}\right), \qquad \mathbb{R} \setminus \mathbb{I} = \left(-\infty, \frac{2-k}{k}\right] \cup \left[\frac{5k-2}{4-k}, \infty\right),$$

and for

$$E_{\alpha,\beta} = f_{k,k}, \qquad F_{\gamma,\delta} = f_{\gamma,-2}.$$

Condition (a): $x \in \mathbb{I} = \left(\dfrac{2-k}{k}, \dfrac{5k-2}{4-k}\right)$. Since $f_6(x,1,1) - Af_{k,k}(x) \geq 0$, this condition is satisfied.

Condition (b): $x \in \mathbb{R} \setminus \mathbb{I} = \left(-\infty, \dfrac{2-k}{k}\right] \cup \left[\dfrac{5k-2}{4-k}, \infty\right)$. By choosing

$$\alpha = \frac{5k-2}{4-k},$$

we get

$$f_6(x,1,1) - Af_{\alpha,-2}(x) = \frac{(x-1)^2}{9} g_1(x) g_2(x),$$

$$\begin{aligned} g_1(x) &= (x+2)[(3\alpha - k^2 - 4k + 2)x + (k^2 - 2k - 2)\alpha + 3k^2] \\ &= \frac{(1-k)^2(2+k)}{4-k}\,(x+2)^2 \geq 0, \end{aligned}$$

$$g_2(x) = 3(\alpha+2)(x-k)^2 + (k+2)^2(x-1)(x-\alpha) = \frac{k+2}{4-k}\, g_3(x),$$

$$g_3(x) = (17 + 2k - k^2)x^2 - 4(1 + 7k + k^2)x - 2(2 - 4k - 7k^2).$$

We have $g_3(x) > 0$ for $x \in \mathbb{R} \setminus \mathbb{I}$ and $1 < k < 4$, because

$$\frac{2-k}{k} < \frac{2(1+7k+k^2)}{17+2k-k^2} < \frac{5k-2}{4-k}$$

and

$$g_3\left(\frac{2-k}{k}\right) > 0, \qquad g_3\left(\frac{5k-2}{4-k}\right) > 0.$$

Observation 2. The inequality from Observation 1 is also valid for

$$k \in (0,1).$$

The proof is similar to the one from Observation 1, but we choose

$$\eta = \frac{2-k}{k} \in (1,\infty), \qquad \xi = \frac{(k+2)^2}{k(4-k)},$$

$$\mathbb{I} = \left(\frac{5k-2}{4-k}, \frac{2-k}{k}\right), \qquad \mathbb{R} \setminus \mathbb{I} = \left(-\infty, \frac{5k-2}{4-k}\right] \cup \left[\frac{2-k}{k}, \infty\right).$$

We have $g_3(x) > 0$ for $x \in \mathbb{R} \setminus \mathbb{I}$ and $0 < k < 1$, because

$$\frac{5k-2}{4-k} < \frac{2(1+7k+k^2)}{17+2k-k^2} < \frac{2-k}{k}$$

and

$$g_3\left(\frac{5k-2}{4-k}\right) > 0, \qquad g_3\left(\frac{2-k}{k}\right) > 0.$$

Observation 3. From P 3.47, P 3.48 and the observations attached to them, the following generalization follows:

- *Let x, y, z be real numbers. If*

$$\alpha_k = \frac{1}{4}(k^2-2k-2)(k^2+2k+2), \qquad k \in \mathbb{R},$$

then

$$\sum (x-y)(x-z)(x-ky)^2(x-kz)^2 + \alpha_k (x-y)^2(y-z)^2(z-x)^2 \geq 0,$$

with equality for $x = y = z$, for $x/k = y = z$ (or any cyclic permutation) if $k \neq 0$, and for $x = 0$ and $y + z = 0$ (or any cyclic permutation). If $k = 0$, then the equality holds also for $x = 0$ and $y = z$ (or any cyclic permutation).

□

P 3.49. *Let x, y, z be real numbers, and let*

$$\alpha_k = \begin{cases} \dfrac{20+12k-4k^2-k^4}{4(1-k)^2}, & k \in (-\infty, -2] \cup [4, \infty) \\ 1+k, & k \in [-2, 1] \\ 5-3k, & k \in [1, 4] \end{cases}.$$

Then,

$$\sum x^2(x-y)(x-z)(x-ky)(x-kz) \geq \alpha_k (x-y)^2(y-z)^2(z-x)^2.$$

(Vasile C., 2009)

Solution. Denote

$$f(x,y,z) = \sum x^2(x-y)(x-z)(x-ky)(x-kz),$$

and write the inequality as $f_6(x, y, z) \geq 0$, where

$$f_6(x,y,z) = f(x,y,z) - \alpha_k (x-y)^2(y-z)^2(z-x)^2,$$

Since

$$(x-y)(x-z) = x^2 + 2yz - q, \qquad (x-ky)(x-kz) = x^2 + (k+k^2)yz - kq,$$

$f(x, y, z)$ has the same highest coefficient A_1 as

$$P_1(x,y,z) = \sum x^2(x^2+2yz)[x^2+(k+k^2)yz],$$

that is, according to (3.1),

$$A_1 = P_1(1,1,1) = 3(1+2)(1+k+k^2) = 9(k^2+k+1).$$

Therefore, since the product $(x-y)^2(y-z)^2(z-x)^2$ has the highest coefficient equal to -27, $f_6(x,y,z)$ has the highest coefficient

$$A = A_1 + 27\alpha_k = \begin{cases} \dfrac{9(2+k)^2(4-k)^2}{4(1-k)^2}, & k \in (-\infty,-2] \cup [4,\infty) \\ 9(2+k)^2, & k \in [-2,1] \\ 9(4-k)^2, & k \in [1,4] \end{cases}.$$

In addition, we have

$$f_6(x,1,1) = x^2(x-1)^2(x-k)^2,$$

$$f_{k,0}(x) = \frac{4x^2(x-1)^4(x-k)^2}{9(k-4)^2(x+2)^2}, \qquad f_{k,-2}(x) = \frac{4(x-1)^4(x-k)^2}{81(k+2)^2}.$$

We will consider the following cases:

$$k \in \{-2,4\}, \qquad k = 0,$$

$$k \in (-\infty,-2), \qquad k \in (4,\infty), \qquad k \in (-2,0), \qquad k \in (0,1], \qquad k \in [1,4).$$

Case 1: $k \in \{-2,4\}$. Since $A = 0$ and $f_6(x,1,1) \ge 0$ for any real x, the conclusion follows by Corollary 1.

Case 2: $k = 0$. This case is treated in P 3.39.

Case 3: $k \in (-\infty,-2)$. Since

$$A = \frac{9(2+k)^2(4-k)^2}{4(1-k)^2},$$

$$f_6(x,1,1) - Af_{k,0}(x) = \frac{3x^2(x-1)^2(x-k)^3[-(2k+1)x+4-k]}{(1-k)^2(x+2)^2},$$

we apply Theorem 2 for

$$\eta = \frac{4-k}{2k+1} \in \left(-2, \frac{-1}{2}\right), \qquad \xi = \frac{(k+2)^2}{2k+1},$$

$$\mathbb{I} = \left(k, \frac{4-k}{2k+1}\right), \qquad \mathbb{R} \setminus \mathbb{I} = (-\infty,k] \cup \left[\frac{4-k}{2k+1}, \infty\right),$$

and for

$$E_{\alpha,\beta} = f_{k,-2}, \qquad F_{\gamma,\delta} = f_{k,0}.$$

The condition (b), namely $f_6(x,1,1) \ge Af_{k,0}(x)$ for $x \in (-\infty,k] \cup \left[\dfrac{4-k}{2k+1},\infty\right)$, is satisfied.

The condition (a) is satisfied if $f_6(x,1,1) \ge Af_{k,-2}(x)$ for $x \in \left(k,\dfrac{4-k}{2k+1}\right)$. From

$$f_6(x,1,1) - Af_{k,-2}(x) = \frac{(x-1)^2(x-k)^2[-(2k+1)x+4-k][(7-4k)x+k-4]}{9(1-k)^2},$$

it follows that $f_6(x,1,1) - Af_{k,-2}(x) \ge 0$ for $x \in \left(-\infty,\dfrac{4-k}{2k+1}\right] \cup \left[\dfrac{4-k}{7-4k},\infty\right)$, therefore for $x \in \left(k,\dfrac{4-k}{2k+1}\right)$.

Case 4: $k \in (4,\infty)$. Since

$$A = \frac{9(2+k)^2(4-k)^2}{4(1-k)^2},$$

$$f_6(x,1,1) - Af_{k,0}(x) = \frac{3x^2(x-1)^2(x-k)^3[4-k-(2k+1)x]}{(1-k)^2(x+2)^2},$$

we apply Theorem 2 for

$$\eta = k, \qquad \xi = \frac{(k+2)^2}{2k+1},$$

$$\mathbb{I} = \left(\frac{4-k}{2k+1},k\right), \quad \mathbb{R}\setminus\mathbb{I} = \left(-\infty,\frac{4-k}{2k+1}\right] \cup [k,\infty),$$

and for

$$E_{\alpha,\beta} = f_{k,0}, \qquad F_{\gamma,\delta} = f_{k,-2}.$$

The condition (a), namely $f_6(x,1,1) \ge Af_{k,0}(x)$ for $x \in \left(\dfrac{4-k}{2k+1},k\right)$, is satisfied.

The condition (b) is satisfied if $f_6(x,1,1) \ge Af_{k,-2}(x)$ for $x \in \left(-\infty,\dfrac{4-k}{2k+1}\right] \cup [k,\infty)$. From

$$f_6(x,1,1) - Af_{k,-2}(x) = \frac{(x-1)^2(x-k)^2[(2k+1)x+k-4][(4k-7)x+4-k]}{9(1-k)^2},$$

it follows that $f_6(x,1,1) - Af_{k,-2}(x) \ge 0$ for $x \in \left(-\infty,\dfrac{4-k}{2k+1}\right] \cup \left[\dfrac{k-4}{4k-7},\infty\right)$, therefore for $x \in \left(-\infty,\dfrac{4-k}{2k+1}\right] \cup [k,\infty)$.

Case 5: $k \in (-2,0)$. Since

$$A = 9(2+k)^2,$$

$$f_6(x,1,1)-Af_{k,0}(x)=\frac{3x^2(x-1)^2(x-k)^2(-kx+4)[(k+8)x+4-4k]}{(4-k)^2(x+2)^2},$$

we apply Theorem 2 for

$$\eta=\frac{4k-4}{k+8}\in\left(-2,\frac{-1}{2}\right),\qquad \xi=\frac{4(k+2)^2}{k(k+8)},$$

$$\mathbb{I}=\left(\frac{4}{k},\frac{4k-4}{k+8}\right),\qquad \mathbb{R}\setminus\mathbb{I}=\left(-\infty,\frac{4}{k}\right]\cup\left[\frac{4k-4}{k+8},\infty\right),$$

and for

$$E_{\alpha,\beta}=f_{\alpha,-2},\qquad F_{\gamma,\delta}=f_{k,0}.$$

The condition (b), namely $f_6(x,1,1)\geq Af_{k,0}(x)$ for $x\in\left(-\infty,\frac{4}{k}\right]\cup\left[\frac{4k-4}{k+8},\infty\right)$, is satisfied.

The condition (a), namely $f_6(x,1,1)\geq Af_{\alpha,-2}(x)$ for $x\in\mathbb{I}=\left(\frac{4}{k},\frac{4k-4}{k+8}\right)$, is true if $g_1(x)g_2(x)\geq 0$, where

$$g_1(x)=3(2+\alpha)x(x-k)-2(2+k)(x-1)(x-\alpha),$$

$$g_2(x)=3(2+\alpha)x(x-k)+2(2+k)(x-1)(x-\alpha).$$

Since

$$g_1(x)=(x+2)[(3\alpha+2-2k)x-(2+k)\alpha],$$

we choose

$$\alpha=\frac{4(k-1)}{k+8}$$

to get

$$g_1(x)=\frac{2(1-k)(k+2)}{k+8}(x+2)^2\geq 0,$$

$$g_2(x)=\frac{2(k+2)}{k+8}g_3(x),\qquad g_3(x)=(k+17)x^2-2(7k+2)x+4k-4.$$

Since g_3 is strictly decreasing on $\mathbb{I}$, we have

$$g_3(x)\geq g_3\left(\frac{4k-4}{k+8}\right)=\frac{36(1-k)(k+2)^2}{(k+8)^2}>0.$$

Case 6: $k\in(0,1]$. Since

$$A=9(2+k)^2,$$

$$f_6(x,1,1)-Af_{k,0}(x)=\frac{3x^2(x-1)^2(x-k)^2(-kx+4)[(k+8)x+4-4k]}{(4-k)^2(x+2)^2},$$

we apply Theorem 2 for

$$\eta = \frac{4}{k} \in [4, \infty), \qquad \xi = \frac{4(k+2)^2}{k(k+8)},$$

$$\mathbb{I} = \left(\frac{4k-4}{k+8}, \frac{4}{k}\right), \qquad \mathbb{R} \setminus \mathbb{I} = \left(-\infty, \frac{4k-4}{k+8}\right] \cup \left[\frac{4}{k}, \infty\right),$$

and for

$$E_{\alpha,\beta} = f_{k,0}, \qquad F_{\gamma,\delta} = f_{\gamma,-2}.$$

The condition (a), namely $f_6(x,1,1) \ge Af_{k,0}(x)$ for $x \in \left(\frac{4k-4}{k+8}, \frac{4}{k}\right)$, is satisfied.

The condition (b) is satisfied if $f_6(x,1,1) \ge Af_{\gamma,-2}(x)$ for $x \in \left(-\infty, \frac{4k-4}{k+8}\right] \cup \left[\frac{4}{k}, \infty\right)$. This is true if $g_1(x)g_2(x) \ge 0$, where

$$g_1(x) = 3(2+\alpha)x(x-k) - 2(2+k)(x-1)(x-\alpha),$$

$$g_2(x) = 3(2+\alpha)x(x-k) + 2(2+k)(x-1)(x-\alpha).$$

Since

$$g_1(x) = (x+2)[(3\alpha+2-2k)x - (2+k)\alpha],$$

we choose

$$\alpha = \frac{4(k-1)}{k+8}$$

to get

$$g_1(x) = \frac{2(1-k)(k+2)}{k+8}(x+2)^2 \ge 0,$$

$$g_2(x) = \frac{2(k+2)}{k+8} g_3(x), \qquad g_3(x) = (k+17)x^2 - 2(7k+2)x + 4k - 4.$$

Since g_3 is strictly decreasing on $\left(-\infty, \frac{4k-4}{k+8}\right]$ and strictly increasing on $\left[\frac{4}{k}, \infty\right)$, we get

$$g_3(x) \ge g_3\left(\frac{4k-4}{k+8}\right) = \frac{36(1-k)(k+2)^2}{(k+8)^2} \ge 0$$

for $x \le \frac{4k-4}{k+8}$, and

$$g_3(x) \ge g_3\left(\frac{4}{k}\right) = \frac{4(k^3 - 15k^2 + 68)}{k^2} > 0$$

for $x \ge \frac{4}{k}$.

Case 7: $k \in [1,4)$. Since

$$A = 9(4-k)^2,$$

$$f_6(x,1,1) - Af_{k,0}(x) = \frac{3x^3(x-1)^2(x-k)^2(4-x)}{(x+2)^2},$$

we apply Theorem 2 for

$$\eta = 4, \qquad \xi = 4,$$

$$\mathbb{I} = (0,4), \qquad \mathbb{R} \setminus \mathbb{I} = (-\infty, 0] \cup [4, \infty),$$

and for

$$E_{\alpha,\beta} = f_{k,0}, \qquad F_{\gamma,\delta} = f_{0,-2}.$$

The condition (a), namely $f_6(x,1,1) \geq Af_{k,0}(x)$ for $x \in (0,4)$, is satisfied.

The condition (b) is satisfied if $f_6(x,1,1) \geq Af_{0,-2}(x)$ for $x \in (-\infty, 0] \cup [4, \infty)$. We have

$$f_6(x,1,1) - Af_{0,-2}(x) = \frac{(k-1)x^2(x-1)^2(x-4)[(7-k)x - 2k - 4]}{9} \geq 0,$$

since

$$(x-4)[(7-k)x - 2k - 4] \geq 0$$

for $x \leq 0$, and

$$(7-k)x - 2k - 4 \geq 4(7-k) - 2k - 4 = 6(4-k) > 0$$

for $x \geq 4$.

The proof is completed. The equality hold for $x = y = z$, for $x = 0$ and $y = z$ (or any cyclic permutation), for $x/k = y = z$ (or any cyclic permutation) if $k \neq 0$, and for $x = 0$ and $y + z = 0$ (or any cyclic permutation) if $k \in [-2, 1]$.

Observation 1. The coefficient α_k of the product $(x-y)^2(y-z)^2(z-x)^2$ is the best possible. Setting $x = k$, $y = 1+t$ and $z = 1-t$, the inequality in P 3.49 becomes

$$A(k,\alpha_k)t^6 + B(k,\alpha_k)t^4 + C(k,\alpha_k)t^2 \geq 0,$$

where

$$\begin{aligned} A(k,\alpha_k) &= 4(1+k-\alpha_k), \\ C(k,\alpha_k) &= (1-k)^2[20 + 12k - 4k^2 - k^4 - 4(1-k)^2\alpha_k]. \end{aligned}$$

From the necessary conditions $A(k,\alpha_k) \geq 0$ and $C(k,\alpha_k) \geq 0$, we get

$$\alpha_k \leq 1 + k, \quad \alpha_k \leq \frac{20 + 12k - 4k^2 - k^4}{4(1-k)^2},$$

respectively. In addition, setting $x = 0$, the inequality in P 3.49 becomes

$$(y-z)^2[y^4+z^4-(k-1)yz(y^2+z^2)+(1-k-\alpha_k)y^2z^2] \ge 0.$$

For $y = z = 1$, from the necessary inequality

$$y^4+z^4-(k-1)yz(y^2+z^2)+(1-k-\alpha_k)y^2z^2 \ge 0,$$

we get

$$\alpha_k \le 5-3k.$$

Observation 2. There are some relevant particular cases of the inequality in P 3.49.

- For $k = -2$, the inequality turns into

$$\sum x^2(x-y)(x-z)(x+2y)(x+2z)+(x-y)^2(y-z)^2(z-x)^2 \ge 0,$$

which is equivalent to

$$(x+y+z)^2\left[x^4+y^4+z^4+xyz(x+y+z)-\sum xy(x^2+y^2)\right] \ge 0.$$

The equality occurs for $x = y = z$, for $x + y + z = 0$, and for $x = 0$ and $y = z$ (or any cyclic permutation).

- For $k = 1$, the inequality turns into

$$\sum x^2(x-y)^2(x-z)^2 \ge 2(x-y)^2(y-z)^2(z-x)^2,$$

which is equivalent to

$$\left[x^3+y^3+z^3+3xyz-\sum xy(x+y)\right]^2 \ge 0.$$

The equality occurs for $x^3+y^3+z^3+3xyz = \sum xy(x+y)$.

- For $k = 4$, the inequality turns into

$$\sum x^2(x-y)(x-z)(x-4y)(x-4z)+7(x-y)^2(y-z)^2(z-x)^2 \ge 0,$$

which is equivalent to

$$(x^2+y^2+z^2-xy-yz-zx)(x^2+y^2+z^2-2xy-2yz-2zx)^2 \ge 0.$$

The equality occurs for $x = y = z$, and for $\sqrt{x} = \sqrt{y} + \sqrt{z}$ (or any cyclic permutation).

Observation 3. Substituting $k-1$ for k and using then the identity

$$\sum x^2(x-y)(x-z)[x-(k-1)y][x-(k-1)z] =$$

$$= \sum x^2(x-y)(x-z)(x-ky+z)(x-kz+y) + k(x-y)^2(y-z)^2(z-x)^2,$$

we get the following statement:

- *Let x, y, z be real numbers, and let*

$$\alpha_k^* = \begin{cases} \dfrac{(3-k)(1+k)^3}{4(2-k)^2}, & k \in (-\infty, -1] \cup [5, \infty) \\ 0, & k \in [-1, 2] \\ 4(2-k), & k \in [2, 5] \end{cases}.$$

Then,

$$\sum x^2(x-y)(x-z)(x-ky+z)(x-kz+y) \ge \alpha_k^*(x-y)^2(y-z)^2(z-x)^2,$$

with equality for $x = y = z$, for $x = 0$ and $y = z$ (or any cyclic permutation), for $x/(k-1) = y = z$ (or any cyclic permutation) if $k \ne 1$, and for $x = 0$ and $y + z = 0$ (or any cyclic permutation) if $k \in [-1, 2]$.

□

P 3.50. *Let x, y, z be real numbers, and let*

$$\beta_k = \begin{cases} \dfrac{k^2}{4}, & k \in (-\infty, -2] \cup [1, \infty) \\ \dfrac{-k(8+11k+8k^2)}{4(1-k)^2}, & k \in \left[-2, \dfrac{-1}{2}\right] \\ \dfrac{1}{4}, & k \in \left[\dfrac{-1}{2}, 1\right] \end{cases}.$$

Then,

$$\sum yz(x-y)(x-z)(x-ky)(x-kz) + \beta_k(x-y)^2(y-z)^2(z-x)^2 \ge 0.$$

(Vasile C., 2009)

Solution. Denote

$$f(x, y, z) = \sum yz(x-y)(x-z)(x-ky)(x-kz),$$

and write the inequality as $f_6(x, y, z) \ge 0$, where

$$f_6(x, y, z) = f(x, y, z) + \beta_k(x-y)^2(y-z)^2(z-x)^2.$$

Since $(x-y)(x-z) = x^2 + 2yz - q$ and $(x-ky)(x-kz) = x^2 + (k+k^2)yz - kq$, $f(x, y, z)$ has the same highest coefficient A_1 as

$$P_1(x, y, z) = \sum yz(x^2 + 2yz)[x^2 + (k+k^2)yz],$$

that is, according to (3.1),

$$A_1 = P_1(1,1,1) = 3(1+2)(1+k+k^2) = 9(k^2+k+1).$$

Therefore, $f_6(x,y,z)$ has the highest coefficient

$$A = A_1 - 27\beta_k = \begin{cases} \dfrac{9(k+2)^2}{4}, & k \in (-\infty,-2] \cup [1,\infty) \\ \dfrac{9(k+2)^2(2k+1)^2}{4(k-1)^2}, & k \in \left[-2, \dfrac{-1}{2}\right] \\ \dfrac{9(2k+1)^2}{4}, & k \in \left[\dfrac{-1}{2}, 1\right] \end{cases}.$$

Also, we have

$$f_6(x,1,1) = (x-1)^2(x-k)^2,$$

$$f_{k,-2}(x) = \frac{4(x-1)^4(x-k)^2}{81(k+2)^2}, \qquad f_{k,\infty}(x) = \frac{4(x-1)^4(x-k)^2}{9(1+2k)^2(x+2)^2}.$$

We will consider the following cases:

$$k \in \left\{-2, \frac{-1}{2}\right\}, \qquad k = 1,$$

$$k \in \left(-2, \frac{-1}{2}\right), \qquad k \in \left(\frac{-1}{2}, 1\right), \qquad k \in (-\infty,-2) \cup (1,\infty).$$

Case 1: $k \in \left\{-2, \dfrac{-1}{2}\right\}$. Since $A = 0$ and $f_6(x,1,1) \ge 0$ for any real x, the conclusion follows by Corollary 1.

Case 2: $k = 1$. Since $A = 81/4$, $f_{\infty,-2} = \dfrac{4(x-1)^4}{81}$ and

$$f_6(x,1,1) - Af_{\infty,-2}(x) = (x-1)^4 - (x-1)^4 = 0,$$

the inequality follows from Corollary 1. Notice that this case is treated in P 3.17.

Case 3: $k \in \left(-2, \dfrac{-1}{2}\right)$. Since

$$A = \frac{9(k+2)^2(2k+1)^2}{4(k-1)^2}, \qquad Af_{k,\infty}(x) = \frac{(k+2)^2(x-1)^4(x-k)^2}{(k-1)^2(x+2)^2},$$

$$f_6(x,1,1) - Af_{k,\infty}(x) = \frac{3(x-1)^2(x-k)^3[-(2k+1)x+4-k]}{(k-1)^2(x+2)^2},$$

we apply Theorem 2 for

$$\eta = k, \qquad \xi = \frac{(k+2)^2}{2k+1},$$

$$\mathbb{I}=\left(\frac{4-k}{2k+1},k\right),\qquad \mathbb{R}\setminus\mathbb{I}=\left(-\infty,\frac{4-k}{2k+1}\right]\cup[k,\infty),$$

and for

$$E_{\alpha,\beta}=f_{k,-2},\qquad F_{\gamma,\delta}=f_{k,\infty}.$$

The condition (b), namely $f_6(x,1,1)-Af_{k,\infty}(x)\geq 0$ for $x\in\left(-\infty,\frac{4-k}{2k+1}\right]\cup[k,\infty)$, is satisfied.

The condition (a) is satisfied if $f_6(x,1,1)\geq Af_{k,-2}(x)$ for $x\in\left(\frac{4-k}{2k+1},k\right)$. Indeed, we have

$$Af_{k,-2}(x)=\frac{(2k+1)(x-1)^4(x-k)^2}{9(k-1)^2}$$

and

$$f_6(x,1,1)-Af_{k,-2}(x)=$$
$$=\frac{(x-1)^2(x-k)^2[4-k-(2k+1)x][2-5k+(2k+1)x]}{9(k-1)^2}\geq 0,$$

since

$$4-k-(2k+1)x>0,$$
$$2-5k+(2k+1)x>2-5k+(2k+1)k=2(k-1)^2>0.$$

Case 4: $k\in\left(\frac{-1}{2},1\right)$. Since

$$A=\frac{9(2k+1)^2}{4},\qquad Af_{k,\infty}(x)=\frac{(x-1)^4(x-k)^2}{(x+2)^2},$$

$$f_6(x,1,1)-Af_{k,\infty}(x)=\frac{3(x-1)^2(x-k)^2(2x+1)}{(x+2)^2},$$

we apply Theorem 2 for

$$\eta\to\infty,\qquad \xi\to\infty,$$
$$\mathbb{I}=\left(\frac{-1}{2},\infty\right),\qquad \mathbb{R}\setminus\mathbb{I}=\left(-\infty,\frac{-1}{2}\right],$$

and for

$$E_{\alpha,\beta}=f_{k,\infty},\qquad F_{\gamma,\delta}=f_{\infty,-2}.$$

The condition (a), namely $f_6(x,1,1)-Af_{k,\infty}(x)\geq 0$ for $x>\frac{-1}{2}$, is satisfied.

The condition (b) is satisfied if $f_6(x,1,1)\geq Af_{\infty,-2}(x)$ for $x\leq\frac{-1}{2}$. Indeed, we have

$$Af_{\infty,-2}(x)=\frac{(2k+1)^2(x-1)^4}{9},$$

$$f_6(x,1,1) - Af_{\infty,-2}(x) = \frac{(1-k)(x-1)^2(2x+1)[2(k+2)x-5k-1]}{9} \geq 0,$$

since $1-k>0$, $2x+1 \leq 0$ and

$$2(k+2)x-5k-1 \leq -(k+2)-5k-1 = -3(2k+1) < 0.$$

Case 5: $k \in (-\infty,-2) \cup (1,\infty)$. This case reduces to the case $k \in \left(\frac{-1}{2},1\right)$ by substituting $\frac{1}{x}$, $\frac{1}{y}$, $\frac{1}{z}$, $\frac{1}{k}$ for x, y, z, k, respectively.

The proof is completed. The equality hold for $x=y=z$, for $y=z=0$ (or any cyclic permutation), for $x/k=y=z$ (or any cyclic permutation) if $k \neq 0$, and for $x=0$ and $y+z=0$ (or any cyclic permutation) if $k \in (-\infty,-2]\cup[1,\infty)$. If $k=0$, then the equality holds also for $x=0$ and $y=z$ (or any cyclic permutation).

Observation 1. The coefficient β_k of the product $(x-y)^2(y-z)^2(z-x)^2$ is the best possible. Setting $x=k$, $y=1+t$ and $z=1-t$, the inequality in P 3.50 becomes

$$A(k,\beta_k)t^6 + B(k,\beta_k)t^4 + C(k,\beta_k)t^2 \geq 0,$$

where

$$A(k,\beta_k) = 4\beta_k - k^2,$$
$$C(k,\beta_k) = (1-k)^2[4(1-k)^2\beta_k + k(8+11k+8k^2)].$$

From the necessary conditions $A(k,\beta_k) \geq 0$ and $C(k,\beta_k) \geq 0$, we get

$$\beta_k \geq \frac{k^2}{4}, \qquad \beta_k \geq \frac{-k(8+11k+8k^2)}{4(1-k)^2},$$

respectively. In addition, for $x \to \infty$, the inequality in P 3.50 becomes

$$yz + \beta_k(y-z)^2 \geq 0.$$

Setting $y=1$ and $z=-1$, we get $\beta_k \geq \frac{1}{4}$.

Observation 2. There are some relevant particular cases of the inequality in P 3.50.

- For $k=-2$, the inequality turns into

$$\sum yz(x-y)(x-z)(x+2y)(x+2z) + (x-y)^2(y-z)^2(z-x)^2 \geq 0,$$

which is equivalent to

$$(x+y+z)^2\left[x^2(y-z)^2 + y^2(z-x)^2 + z^2(x-y)^2\right] \geq 0.$$

The equality holds for $x=y=z$, for $x+y+z=0$, and for $y=z=0$ (or any cyclic permutation).

- For $k = -1/2$, the inequality turns into

$$\sum yz(x-y)(x-z)(2x+y)(2x+z)+(x-y)^2(y-z)^2(z-x)^2 \ge 0,$$

which is equivalent to

$$(xy+yz+zx)^2(x^2+y^2+z^2-xy-yz-zx) \ge 0.$$

The equality holds for $x = y = z$, and for $xy + yz + zx = 0$.

- For $k = 1$, the inequality turns into

$$\sum yz(x-y)^2(x-z)^2+\frac{1}{4}(x-y)^2(y-z)^2(z-x)^2 \ge 0,$$

which is equivalent to

$$\left[x(y-z)^2+y(z-x)^2+z(x-y)^2\right]^2 \ge 0.$$

The equality holds for $x(y-z)^2 + y(z-x)^2 + z(x-y)^2 = 0$.

Observation 3. Adding the inequality in P 3.50 multiplied by 2 and the identity

$$\sum(y^2+z^2-2yz)(x-y)(x-z)(x-ky)(x-kz) = (k^2+k+1)(x-y)^2(y-z)^2(z-x)^2,$$

we get the following statement:

- *Let x, y, z be real numbers, and let*

$$A_k = \begin{cases} \dfrac{k^2}{2}+k+1, & k \in (-\infty,-2] \cup [1,\infty) \\ \dfrac{2k^4+6k^3+11k^2+6k+2}{2(k-1)^2}, & k \in \left[-2, \dfrac{-1}{2}\right] \\ k^2+k+\dfrac{1}{2}, & k \in \left[\dfrac{-1}{2}, 1\right] \end{cases}.$$

Then,

$$\sum(y^2+z^2)(x-y)(x-z)(x-ky)(x-kz) \ge A_k(x-y)^2(y-z)^2(z-x)^2,$$

with equality for $x = y = z$, for $y = z = 0$ (or any cyclic permutation), for $x/k = y = z$ (or any cyclic permutation) if $k \ne 0$, and for $x = 0$ and $y + z = 0$ (or any cyclic permutation) if $k \in (-\infty,-2] \cup [1,\infty)$. If $k = 0$, then the equality holds also for $x = 0$ and $y = z$ (or any cyclic permutation).

Observation 4. Adding the inequality in P 3.50 multiplied by 4 and the identity

$$\sum(y^2+z^2-2yz)(x-y)(x-z)(x-ky)(x-kz) = (k^2+k+1)(x-y)^2(y-z)^2(z-x)^2,$$

we get the following statement:

- *Let x, y, z be real numbers, and let*

$$B_k = \begin{cases} k+1, & k \in (-\infty, -2] \cup [1, \infty) \\ \dfrac{k^4+7k^3+11k^2+7k+1}{(k-1)^2}, & k \in \left[-2, \dfrac{-1}{2}\right] \\ k^2+k, & k \in \left[\dfrac{-1}{2}, 1\right] \end{cases}.$$

Then,

$$\sum (y+z)^2(x-y)(x-z)(x-ky)(x-kz) \geq B_k(x-y)^2(y-z)^2(z-x)^2,$$

with equality for $x=y=z$, for $y=z=0$ (or any cyclic permutation), for $x/k = y = z$ (or any cyclic permutation) if $k \neq 0$, and for $x=0$ and $y+z=0$ (or any cyclic permutation) if $k \in (-\infty, -2] \cup [1, \infty)$. If $k=0$, then the equality holds also for $x=0$ and $y=z$ (or any cyclic permutation).

Observation 5. Substituting $k-1$ for k and using then the identity

$$\sum yz(x-y)(x-z)[x-(k-1)y][x-(k-1)z] =$$

$$= \sum yz(x-y)(x-z)(x-ky+z)(x-kz+y) + k(x-y)^2(y-z)^2(z-x)^2,$$

we get the following statement:

- *Let x, y, z be real numbers, and let*

$$\beta_k^* = \begin{cases} \dfrac{(1+k)^2}{4}, & k \in (-\infty, -1] \cup [2, \infty) \\ \dfrac{(5-4k)(1+k)^2}{4(2-k)^2}, & k \in \left[-1, \dfrac{1}{2}\right] \\ \dfrac{1+4k}{4}, & k \in \left[\dfrac{1}{2}, 2\right] \end{cases}.$$

Then,

$$\sum yz(x-y)(x-z)(x-ky+z)(x-kz+y) + \beta_k^*(x-y)^2(y-z)^2(z-x)^2 \geq 0,$$

with equality for $x=y=z$, for $y=z=0$ (or any cyclic permutation), for $x/(k-1) = y = z$ (or any cyclic permutation) if $k \neq 1$, and for $x=0$ and $y+z=0$ (or any cyclic permutation) if $k \in (-\infty, -1] \cup [2, \infty)$. If $k=1$, then the equality holds also for $x=0$ and $y=z$ (or any cyclic permutation).

$\square$

P 3.51. *Let x, y, z be real numbers, and let*

$$\gamma_k = \begin{cases} \dfrac{(k+1)(5k-3)}{16}, & k \in (-\infty, -5] \cup [1, \infty) \\ \dfrac{(k+1)(k^3-7k^2-16k-32)}{4(k-1)^2}, & k \in [-5, -2] \\ \dfrac{k^2}{4}, & k \in [-2, 1] \end{cases}.$$

Then,

$$\sum (x^2-y^2)(x^2-z^2)(x-ky)(x-kz) + \gamma_k (x-y)^2(y-z)^2(z-x)^2 \ge 0.$$

(Vasile C., 2009)

Solution. Denote

$$f(x,y,z) = \sum (x^2-y^2)(x^2-z^2)(x-ky)(x-kz),$$

and write the desired inequality as $f_6(x,y,z) \ge 0$, where

$$f_6(x,y,z) = f(x,y,z) + \gamma_k (x-y)^2(y-z)^2(z-x)^2.$$

Since

$$(x+y)(x+z) = x^2+q, \qquad (x-y)(x-z) = x^2+2yz-q,$$
$$(x-ky)(x-kz) = x^2+(k+k^2)yz-kq,$$

$f(x,y,z)$ has the same highest coefficient A_1 as

$$P_1(x,y,z) = \sum x^2(x^2+2yz)[x^2+(k+k^2)yz],$$

that is, according to (3.1),

$$A_1 = P_1(1,1,1) = 9(k^2+k+1).$$

Therefore, $f_6(x,y,z)$ has the highest coefficient

$$A = A_1 - 27\gamma_k = \begin{cases} \dfrac{9(k+5)^2}{16}, & k \in (-\infty, -5] \cup [1, \infty) \\ \dfrac{9(k+2)^2(k+5)^2}{4(k-1)^2}, & k \in [-5, 2] \\ \dfrac{9(k+2)^2}{4}, & k \in [-2, 1] \end{cases}.$$

In addition, we have

$$f_6(x,1,1) = (x^2-1)^2(x-k)^2,$$

$$f_{-1,k}(x) = \frac{4(x-1)^4(x+1)^2(x-k)^2}{9(k+5)^2(x+2)^2},$$

$$f_{-1,-2}(x) = \frac{4(x-1)^4(x+1)^2}{81}, \qquad f_{k,-2}(x) = \frac{4(x-1)^4(x-k)^2}{81(k+2)^2}.$$

We will consider the following cases:

$$k \in \{-5,-2\}, \qquad k \in (-\infty,-5) \cup [1,\infty), \qquad k \in (-5,-2), \qquad k \in (-2,1).$$

Case 1: $k \in \{-5,-2\}$. Since $A = 0$ and $f_6(x,1,1) \geq 0$ for any real x, the conclusion follows by Corollary 1.

Case 2: $k \in (-\infty,-5) \cup [1,\infty)$. Since

$$A = \frac{9(k+5)^2}{16}, \qquad Af_{-1,k}(x) = \frac{(x-1)^4(x+1)^2(x-k)^2}{4(x+2)^2},$$

$$f_6(x,1,1) - Af_{-1,k}(x) = \frac{3(x^2-1)^2(x-k)^2(x+5)(x+1)}{4(x+2)^2},$$

we apply Theorem 2 for

$$\eta = -1, \qquad \xi = -1,$$

$$\mathbb{I} = (-5,-1), \qquad \mathbb{R} \setminus \mathbb{I} = (-\infty,-5] \cup [-1,\infty),$$

and for

$$E_{\alpha,\beta} = f_{-1,-2}, \qquad F_{\gamma,\delta} = f_{-1,k}.$$

The condition (b), namely $f_6(x,1,1) - Af_{-1,k}(x) \geq 0$ for $x \in (-\infty,-5] \cup [-1,\infty)$, is satisfied.

The condition (a) is satisfied if $f_6(x,1,1) \geq Af_{-1,-2}(x)$ for $x \in (-5,-1)$. We have

$$Af_{-1,-2}(x) = \frac{(k+5)^2(x-1)^4(x+1)^2}{36},$$

$$f_6(x,1,1) - Af_{-1,-2}(x) = \frac{(k-1)(x+5)(x^2-1)^2[7k+5-(k+11)x]}{36}.$$

Thus, it suffices to show that $7k+5-(k+11)x \leq 0$ for $k \in (-\infty,-5)$, and $7k+5-(k+11)x \geq 0$ for $k \in [1,\infty)$. Indeed, for $k \in (-\infty,-5)$, we have

$$7k+5-(k+11)x = k(7-x)+5-11x < (-5)(7-x)+5-11x = -6(x+5) < 0,$$

and for $k \in [1,\infty)$, we have

$$7k+5-(k+11)x > 7k+5-(k+11)(-1) = 8(k+2) > 0.$$

Case 3: $k \in (-5,-2)$. Since

$$A = \frac{9(k+2)^2(k+5)^2}{4(k-1)^2}, \qquad Af_{-1,k}(x) = \frac{(k+2)^2(x-1)^4(x+1)^2(x-k)^2}{(k-1)^2(x+2)^2},$$

$$f_6(x,1,1) - Af_{-1,k}(x) = \frac{3(x^2-1)^2(x-k)^3[-(2k+1)x+4-k]}{(k-1)^2(x+2)^2},$$

we apply Theorem 2 for

$$\eta = \frac{4-k}{2k+1}, \qquad \xi = \frac{(k+2)^2}{2k+1},$$

$$\mathbb{I} = \left(k, \frac{4-k}{2k+1}\right), \qquad \mathbb{R} \setminus \mathbb{I} = (-\infty, k] \cup \left[\frac{4-k}{2k+1}, \infty\right),$$

and for

$$E_{\alpha,\beta} = f_{k,-2}, \qquad F_{\gamma,\delta} = f_{-1,k}.$$

The condition (b), namely $f_6(x,1,1) - Af_{-1,k}(x) \geq 0$ for $x \in (-\infty, k] \cup \left[\frac{4-k}{2k+1}, \infty\right)$, is satisfied.

The condition (a) is satisfied if $f_6(x,1,1) \geq Af_{k,-2}(x)$ for $x \in \left(k, \frac{4-k}{2k+1}\right)$. We have

$$Af_{k,-2}(x) = \frac{(k+5)^2(x-1)^4(x-k)^2}{9(k-1)^2}$$

and

$$f_6(x,1,1) - Af_{k,-2}(x) =$$
$$= \frac{4(x-1)^2(x-k)^2[2k+1+(k-4)x][k-4+(2k+1)x]}{9(k-1)^2} \geq 0$$

because

$$2k+1+(k-4)x > 2k+1+(k-4)\cdot\frac{4-k}{2k+1} = \frac{3(k-1)(k+5)}{2k+1} > 0,$$

$$k-4+(2k+1)x > k-4+(2k+1)\cdot\frac{4-k}{2k+1} = 0.$$

Case 4: $k \in (-2,1)$. Since

$$A = \frac{9(k+2)^2}{4}, \qquad Af_{-1,k}(x) = \frac{(k+2)^2(x-1)^4(x+1)^2(x-k)^2}{(k+5)^2(x+2)^2},$$

$$f_6(x,1,1) - Af_{-1,k}(x) = \frac{(x-1)^2(x+1)^2(x-k)^2h(x)}{(k+5)^2(x+2)^2},$$

where

$$h(x) = (k+5)^2(x+2)^2 - (k+2)^2(x-1)^2$$
$$= 3(x+k+4)[(2k+7)x+k+8],$$

we apply Theorem 2 for

$$\eta = -k-4, \qquad \xi = \frac{-(k+2)^2}{2k+7},$$

$$\mathbb{I} = \left(-k-4,\ \frac{-k-8}{2k+7}\right), \qquad \mathbb{R} \setminus \mathbb{I} = (-\infty, -k-4] \cup \left[\frac{-k-8}{2k+7}, \infty\right).$$

and for

$$E_{\alpha,\beta} = f_{\alpha,-2}, \qquad F_{\gamma,\delta} = f_{-1,k}.$$

The condition (b), namely $f_6(x,1,1) - Af_{-1,k}(x) \ge 0$ for $x \in (-\infty, -k-4] \cup \left[\frac{-k-8}{2k+7}, \infty\right)$, is satisfied.

The condition (a) is satisfied if there is a real α such that $f_6(x,1,1) \ge Af_{\alpha,-2}(x)$ for $x \in \left(-k-4,\ \frac{-k-8}{2k+7}\right)$. Since

$$Af_{\alpha,-2} = \frac{(k+2)^2(x-1)^4(x-\alpha)^2}{9(2+\alpha)^2}$$

the inequality is equivalent to

$$(x-1)^2 g_1(x) g_2(x) \ge 0,$$

where

$$g_1(x) = 3(\alpha+2)(x+1)(x-k) - (k+2)(x-1)(x-\alpha)$$
$$= (x+2)[(3\alpha-k+4)x - (2k+1)\alpha - 3k],$$

$$g_2(x) = 3(\alpha+2)(x+1)(x-k) + (k+2)(x-1)(x-\alpha).$$

Since $-2 \in \left(-k-4,\ \frac{-k-8}{2k+7}\right)$, we choose

$$\alpha = \frac{-k-8}{2k+7}$$

to get

$$g_1(x) = \frac{2(1-k)(k+2)(x+2)^2}{2k+7} \ge 0, \qquad g_2(x = \frac{2(k+2)g_3(x)}{2k+7},$$

where

$$g_3(x) = (k+8)x^2 + 5(1-k)x - 5k - 4.$$

Since

$$\begin{aligned} g_3'(x) &= 2(k+8)x + 5(1-k) < 2(k+8) \cdot \frac{-k-8}{2k+7} + 5(1-k) \\ &= \frac{-3(4k^2+19k+31)}{2k+7} < \frac{-3(2k^2+19k+30)}{2k+7} \\ &= \frac{-3(k+2)(2k+15)}{2k+7} < 0, \end{aligned}$$

g_3 is strictly decreasing on $\mathbb{I}$, therefore

$$g_3(x) > g_3\left(\frac{-k-8}{2k+7}\right) = \frac{9(1-k)(k+2)^2}{(2k+7)^2} > 0.$$

The proof is completed. The equality holds for $x = y = z$, for $x/k = y = z$ (or any cyclic permutation) if $k \neq 0$, for $-x = y = z$ (or any cyclic permutation), and for $x = 0$ and $y + z = 0$ (or any cyclic permutation) if $k \in [-2, 1]$. If $k = 0$, then the equality holds also for $x = 0$ and $y = z$ (or any cyclic permutation).

Observation 1. The coefficient γ_k of the product $(x-y)^2(y-z)^2(z-x)^2$ is the best possible. Setting $x = k$, $y = 1+t$ and $z = 1-t$, the inequality in P 3.51 turns into

$$A(k,\gamma_k)t^6 + B(k,\gamma_k)t^4 + C(k,\gamma_k)t^2 \geq 0,$$

where

$$A(k,\gamma_k) = 4\gamma_k - k^2,$$

$$C(k,\gamma_k) = (k-1)^2[4(k-1)^2\gamma_k - (k+1)(k^3 - 7k^2 - 16k - 32)].$$

From the necessary conditions $A(k,\gamma_k) \geq 0$ and $C(k,\gamma_k) \geq 0$, we get

$$\gamma_k \geq \frac{k^2}{4}, \qquad \gamma_k \geq \frac{(k+1)(k^3 - 7k^2 - 16k - 32)}{4(k-1)^2}.$$

Also, for $x = -1$, $y = 1 + t$ and $z = 1 - t$, the inequality in P 3.51 becomes

$$A(k,\gamma_k)t^6 + B(k,\gamma_k)t^4 + C(k,\gamma_k)t^2 \geq 0,$$

where

$$A(k,\gamma_k) = 4\gamma_k - k^2, \qquad C(k,\gamma_k) = 4[16\gamma_k - (k+1)(5k-3)].$$

From the necessary conditions $A(k,\gamma_k) \geq 0$ and $C(k,\gamma_k) \geq 0$, we get

$$\gamma_k \geq \frac{k^2}{4}, \qquad \gamma_k \geq \frac{(k+1)(5k-3)}{16}.$$

Observation 2. There are some relevant particular cases of the inequality in P 3.51.

- For $k=1$, the inequality turns into

$$\sum(x+y)(x+z)(x-y)^2(x-z)^2+\frac{1}{4}(x-y)^2(y-z)^2(z-x)^2 \ge 0,$$

which is equivalent to

$$\left[x^2(2x-y-z)+y^2(2y-z-x)+z^2(2z-x-y)\right]^2 \ge 0.$$

The equality holds for $x^2(2x-y-z)+y^2(2y-z-x)+z^2(2z-x-y)=0$.

- For $k=-2$, the inequality turns into

$$\sum(x^2-y^2)(x^2-z^2)(x+2y)(x+2z)+(x-y)^2(y-z)^2(z-x)^2 \ge 0,$$

which is equivalent to

$$(x+y+z)^2(x^4+y^4+z^4-x^2y^2-y^2z^2-z^2x^2) \ge 0.$$

The equality holds for $x=y=z$, for $x+y+z=0$, and for $-x=y=z$ (or any cyclic permutation).

- For $k=-5$, the inequality turns into

$$\sum(x^2-y^2)(x^2-z^2)(x+5y)(x+5z)+7(x-y)^2(y-z)^2(z-x)^2 \ge 0,$$

which is equivalent to

$$(x^2+y^2+z^2+3xy+3yz+3zx)^2(x^2+y^2+z^2-xy-yz-zx) \ge 0.$$

The equality holds for $x=y=z$, and for $x^2+y^2+z^2+3xy+3yz+3zx=0$.

Observation 3. Substituting $k-1$ for k and using then the identity

$$\sum(x^2-y^2)(x^2-z^2)[x-(k-1)y][x-(k-1)z]=$$

$$=\sum(x^2-y^2)(x^2-z^2)(x-ky+z)(x-kz+y)+k(x-y)^2(y-z)^2(z-x)^2,$$

we get the following statement:

- *Let x,y,z be real numbers, and let*

$$\gamma_k^* = \begin{cases} \dfrac{k(5k+8)}{16}, & k\in(-\infty,-4]\cup[2,\infty) \\ \dfrac{k(k+1)^2(k-8)}{4(k-2)^2}, & k\in[-4,-1] \\ \dfrac{(k+1)^2}{4}, & k\in[-1,2] \end{cases}.$$

Then,

$$\sum(x^2-y^2)(x^2-z^2)(x-ky+z)(x-kz+y)+\gamma_k^*(x-y)^2(y-z)^2(z-x)^2 \ge 0,$$

with equality for $x=y=z$, for $x/(k-1)=y=z$ (or any cyclic permutation) if $k \neq 1$, for $-x=y=z$ (or any cyclic permutation), and for $x=0$ and $y+z=0$ (or any cyclic permutation) if $k \in [-1,2]$. If $k=1$, then the equality holds also for $x=0$ and $y=z$ (or any cyclic permutation).

□

P 3.52. *If x,y,z are real numbers, then*

$$\sum yz(4x^2+3yz)(x-y)(x-z)+(x-y)^2(y-z)^2(z-x)^2 \ge 0.$$

(Vasile C., 2014)

Solution. We write the inequality as $f_6(x,y,z)\ge 0$, where

$$f_6(x,y,z)=\sum yz(4x^2+3yz)(x-y)(x-z)+(x-y)^2(y-z)^2(z-x)^2.$$

Since $(x-y)(x-z)=x^2+2yz-q$, $f_6(x,y,z)$ has the same highest coefficient A as

$$\sum yz(4x^2+3yz)(x^2+2yz)+(x-y)^2(y-z)^2(z-x)^2,$$

that is, according to (3.1),

$$A=63-27=36.$$

Since

$$f_6(x,1,1)=(4x^2+3)(x-1)^2,$$

$$f_{\infty,-2}(x)=\frac{4(x-1)^4}{81},$$

$$f_6(x,1,1)-Af_{\infty,-2}(x)=\frac{(x-1)^2(10x+11)(2x+1)}{9},$$

we apply Theorem 2 for $\xi\to\infty$, which involves

$$\mathbb{I}=\left(\frac{-1}{2},\infty\right), \qquad \mathbb{R}\setminus\mathbb{I}=\left(-\infty,\frac{-1}{2}\right],$$

and for

$$E_{\alpha,\beta}=f_{\infty,-2}, \qquad E_{\gamma,\delta}=h_{0,\delta},$$

The condition (a), namely $f_6(x,1,1)-Af_{\infty,-2}(x)\ge 0$ for $x>-1/2$, is satisfied.

The condition (b) is satisfied if $f_6(x,1,1)\ge Ah_{0,\delta}(x)$ for $x\le -1/2$. We have

$$Ah_{0,\delta}(x)=36[x+\delta(x+2)^3]^2,$$

$$f_6(x,1,1)-Ah_{0,\delta}(x)=f(x),$$
$$f(x)=(4x^2+3)(x-1)^2-36[x+\delta(x+2)(2x+1)]^2.$$

Since $f(-1/2)=0$, a necessary condition to have $f(x)\ge 0$ in a vicinity of $x=-1/2$ is $f'(-1/2)=0$. This implies $\delta=-5/36$, when

$$\begin{aligned}36f(x)&=36(4x^2+3)(x-1)^2-(10x^2-11x+10)^2\\&=44x^4-68x^3-69x^2+4x+8\\&=(2x+1)^2(11x^2-28x+8)\ge 0.\end{aligned}$$

The equality holds for $x=y=z$, and for $y=z=0$ (or any cyclic permutation).

Observation 1. Similarly, we can prove the following generalization:

- *Let x,y,z be real numbers. If*

$$k\in\left[\frac{1}{2},k_1\right],\qquad k_1=\frac{13+3\sqrt{33}}{32}\approx 0.9448,$$

then

$$\sum yz(x^2+kyz)(x-y)(x-z)+\frac{1}{4}(x-y)^2(y-z)^2(z-x)^2\ge 0,$$

with equality for $x=y=z$, and also for $y=z=0$ (or any cyclic permutation).

For

$$f_6(x,y,z)=\sum yz(x^2+kyz)(x-y)(x-z)+\frac{1}{4}(x-y)^2(y-z)^2(z-x)^2,$$

we have

$$A=\frac{9}{4}(4k+1),$$
$$f_6(x,1,1)=(x^2+k)(x-1)^2,$$

Condition (a). For $x>-1/2$, we have

$$f_6(x,1,1)-Af_{\infty,-2}(x)=\frac{(x-1)^2[2(2-k)x+5k-1](2x+1)}{9}\ge 0.$$

Condition (b). For $x\le -1/2$, choosing

$$\delta=\frac{1-8k}{9(1+4k)},$$

we get

$$\begin{aligned}f_6(x,1,1)-Ah_{0,\delta}(x)&=\frac{9(1+4k)f_6(x,1,1)-[(1-8k)x^2+(7-2k)x+1-8k]^2}{9(1+4k)}\\&=\frac{(2x+1)^2g(x)}{9(1+4k)},\end{aligned}$$

where

$$g(x) = (2+13k-16k^2)x^2 - 2(5+k-4k^2)x - 1 + 25k - 28k^2$$
$$= (2+13k-16k^2)(x^2+2) - (5+k-4k^2)(2x+1) \geq 0.$$

Observation 2. Actually, the following more general statement holds:

- *Let x, y, z be real numbers. If*

$$k \in [0, k_1], \qquad k_1 = \frac{13+3\sqrt{33}}{32} \approx 0.9448,$$

then

$$\sum yz(x^2+kyz)(x-y)(x-z) + \frac{1}{4}(x-y)^2(y-z)^2(z-x)^2 \geq 0,$$

with equality for $x = y = z$, and also for $y = z = 0$ (or any cyclic permutation).

Since the left hand side of the inequality is linear in k, the inequality holds for $k \in [0, k_1]$ if and only if it holds for $k = 0$ and $k = k_1$. These cases are treated in P 3.50 and Observation 1, respectively.

Observation 3. Replacing x, y, z with $1/x, 1/y, 1/z$, respectively, the statement from Observation 2 becomes as follows:

- *Let x, y, z be real numbers. If*

$$k \in [0, k_1], \qquad k_1 = \frac{13+3\sqrt{33}}{32} \approx 0.9448,$$

then

$$\sum yz(kx^2+yz)(x-y)(x-z) + \frac{1}{4}(x-y)^2(y-z)^2(z-x)^2 \geq 0,$$

with equality for $x = y = z$, and also for $y = z = 0$ (or any cyclic permutation).

□

P 3.53. *If x, y, z are real numbers, then*

$$\sum x^4(x+2y)(x+2z) + 5x^2y^2z^2 + \frac{1}{2}(x-y)^2(y-z)^2(z-x)^2 \geq 0.$$

(Vasile C., 2011)

Solution. Write the inequality as $f_6(x, y, z) \geq 0$, where

$$f_6(x, y, z) = \sum x^4(x+2y)(x+2z) + 5x^2y^2z^2 + \frac{1}{2}(x-y)^2(y-z)^2(z-x)^2.$$

Since

$$(x+2y)(x+2z) = x^2 + 2yz + 2q,$$

$f_6(x,y,z)$ has the same highest coefficient as

$$\sum x^4(x^2+2yz)+5x^2y^2z^2+\frac{1}{2}(x-y)^2(y-z)^2(z-x)^2,$$

that is, according to (3.1),

$$A=9+5-\frac{27}{2}=\frac{1}{2}.$$

We have

$$f_6(0,y,z)=y^6+z^6+2yz(y^4+z^4)+\frac{1}{2}y^2z^2(y-z)^2,$$

$$f_6(0,1,-1)=2-4+2=0,$$

$$\begin{aligned}f_6(x,1,1)&=x^4(x+2)^2+6(2x+1)+5x^2\\&=x^6+4x^5+4x^4+5x^2+12x+6\\&=(x+1)^2g(x),\end{aligned}$$

where

$$g(x)=x^4+2x^3-x^2+6.$$

Since

$$f_6(-1,1,1)=0, \qquad f_6(0,1,-1)=0,$$

we apply Corollary 2 for $F_{\gamma,\delta}=g_{-1,\delta}$, where δ is given by (3.16):

$$\delta=\frac{\gamma}{3(\gamma+2)}+\frac{(\gamma+2)^2g'(-2)}{12A}=\frac{-1}{3}+\frac{g'(-2)}{6}.$$

According to Remark 4, it suffices to show that

$$g(x)\ge A\bar{g}_{-1,\delta}(x)$$

for $x\in\mathbb{R}$, where

$$\bar{g}_{-1,\delta}(x)=[x^2+5x+8+\delta(x+2)(x+5)]^2.$$

We have

$$g'(x)=4x^3+6x^2-2x, \qquad g'(-2)=-4, \qquad \delta=-1,$$

$$\bar{g}_{-1,-1}(x)=[x^2+5x+8-(x+2)(x+5)]^2=4(x+1)^2,$$

$$g(x)-A\bar{g}_{-1,-1}(x)=x^4+2x^3-3x^2-4x+4=(x+2)^2(x-1)^2\ge 0.$$

The equality holds for $-x=y=z$ (or any cyclic permutation), and for $x=0$ and $y+z=0$ (or any cyclic permutation).

□

P 3.54. *If x, y, z are real numbers, no two of which are zero, then*

$$\sum \frac{1}{2y^2-3yz+2z^2} \geq \frac{9}{4(x^2+y^2+z^2)-3(xy+yz+zx)}.$$

(Vasile C., 2009)

Solution. Write the inequality as $f_6(x,y,z) \geq 0$, where

$$f_6(x,y,z) = P(x,y,z)\sum(2x^2-3xy+2y^2)(2x^2-3xz+2z^2)-9\prod(2y^2-3yz+2z^2),$$

$$P(x,y,z) = 4(x^2+y^2+z^2)-3(xy+yz+zx).$$

Since

$$2y^2-3yz+2z^2 = -2x^2-3yz+2(p^2-2q),$$

$f_6(x,y,z)$ has the same highest coefficient A as

$$P_3(x,y,z) = -9\prod(-2x^2-3yz),$$

that is

$$A = P_3(1,1,1) = -9(-2-3)^3 = 1125.$$

Since

$$P(x,1,1) = 4x^2-6x+5,$$

$$\begin{aligned}\frac{f_6(x,1,1)}{2x^2-3x+2} &= (4x^2-6x+5)[(2x^2-3x+2)+2]-9(2x^2-3x+2)\\ &= 2(4x^4-12x^3+13x^2-6x+1) = 2(x-1)^2(2x-1)^2,\end{aligned}$$

we apply Corollary 2 for

$$F_{\gamma,\delta}(x) = f_{1/2,-2}(x) = \frac{4(x-1)^4(2x-1)^2}{2025}.$$

We need to show that

$$f_6(x,1,1) \geq Af_{1/2,-2}(x)$$

for $x \in \mathbb{R}$. We have

$$Af_{1/2,-2}(x) = \frac{20(x-1)^4(2x-1)^2}{9},$$

$$f_6(x,1,1) - Af_{1/2,-2}(x) = \frac{2(x-1)^2(2x-1)^2 f(x)}{9},$$

where

$$f(x) = 9(2x^2-3x+2)-10(x-1)^2 = 8x^2-7x+8 > 0.$$

The equality holds for $x = y = z$, and for $2x = y = z$ (or any cyclic permutation).

Observation. Similarly, we can prove the following generalization:

- *Let x, y, z be real numbers, no two of which are zero. If $-2 \le k \le 2$, then*

$$\sum \frac{1}{y^2+kyz+z^2} \ge \frac{9}{2(x^2+y^2+z^2)+k(xy+yz+zx)},$$

with equality for $x = y = z$, and for $-x/(k+1) = y = z$ (or any cyclic permutation) if $k \ne -1$. If $k = -1$, then the equality holds also for $x = 0$ and $y = z$ (or any cyclic permutation).

For

$$f_6(x,y,z) = P(x,y,z)\sum(x^2+kxy+y^2)(x^2+kxz+z^2) - 9\prod(y^2+kyz+z^2),$$

$$P(x,y,z) = 2(x^2+y^2+z^2)+k(xy+yz+zx),$$

we have

$$A = -9(k-1)^3,$$

$$P(x,1,1) = 2x^2+2kx+k+4,$$

$$\begin{aligned}\frac{f_6(x,1,1)}{x^2+kx+1} =& P(x,1,1)[x^2+kx+1+2(k+2)] - 9(k+2)(x^2+kx+1)\\ &= 2x^4+4kx^3+(2k^2-4k-4)x^2-4k(k+1)x+2(k+1)^2\\ &= 2(x-1)^2(x+k+1)^2.\end{aligned}$$

Case 1: $1 \le k \le 2$. Since $A \le 0$, it suffices to show that $f_6(x,1,1) \ge 0$ for $x \in \mathbb{R}$ (see Corollary 1), which is true.

Case 2: $-2 \le k < 1$. We apply Corollary 2 for

$$F_{\gamma,\delta}(x) = f_{-k-1,-2}(x) = \frac{4(x-1)^4(x+k+1)^2}{81(1-k)^2}.$$

We need to show that

$$f_6(x,1,1) \ge Af_{-k-1,-2}(x)$$

for $x \in \mathbb{R}$. We have

$$Af_{-k-1,-2}(x) = \frac{4(1-k)(x-1)^4(x+k+1)^2}{9},$$

$$f_6(x,1,1) - Af_{-k-1,-2}(x) = \frac{2(x-1)^2(x-k-1)^2 f(x)}{9},$$

where

$$\begin{aligned}f(x) &= 9(x^2+kx+1) - 2(1-k)(x-1)^2\\ &= (2k+7)x^2+(5k+4)x+2k+7\\ &= (2k+7)\left(x+\frac{5k+4}{4k+14}\right)^2 + \frac{9(k+2)(10-k)}{4(2k+7)} \ge 0.\end{aligned}$$

$\square$

P 3.55. *Let x,y,z be real numbers. If $k>1$, then*

$$\frac{kx^2+2yz}{kx^2+y^2+z^2}+\frac{ky^2+2zx}{ky^2+z^2+x^2}+\frac{kz^2+2xy}{kz^2+x^2+y^2}\ge\frac{k-1}{k+1}.$$

(Vasile C., 2011)

Solution. We write the inequality as $f_6(x,y,z)\ge 0$, where

$$\begin{aligned}f_6(x,y,z)=&(k+1)\sum(kx^2+2yz)(ky^2+z^2+x^2)(kz^2+x^2+y^2)\\&-(k-1)\prod(kx^2+y^2+z^2).\end{aligned}$$

From

$$\begin{aligned}f_6(x,y,z)=&(k+1)\sum(kx^2+2yz)[(k-1)y^2+p^2-2q][(k-1)z^2+p^2-2q]\\&-(k-1)\prod[(k-1)x^2+p^2-2q],\end{aligned}$$

it follows that $f_6(x,y,z)$ has the same highest coefficient A as

$$\begin{aligned}f(x,y,z)=&(k+1)(k-1)^2\sum y^2z^2(kx^2+2yz)-(k-1)^4x^2y^2z^2\\=&(k+1)(k-1)^2\left(3kr^2+2\sum y^3z^3\right)-(k-1)^4r^2,\end{aligned}$$

that is

$$A=(k+1)(k-1)^2(3k+6)-(k-1)^4=(k-1)^2(k+5)(2k+1)>0.$$

We have

$$\begin{aligned}f_6(0,y,z)=&(k+1)[2yz(ky^2+z^2)(kz^2+y^2)+k(y^2+z^2)(2ky^2z^2+y^4+z^4]\\&-(k-1)(y^2+z^2)(ky^2+z^2)(kz^2+y^2),\end{aligned}$$

$$f_6(0,1,-1)=(k+1)[-2(k+1)^2+4k(k+1)]-2(k-1)(k+1)^2=0,$$

$$\begin{aligned}f_6(x,1,1)=&(kx^2+2)(x^2+k+1)[(k+1)(x^2+k+1)+2(k+1)(2x+k)\\&-(k-1)(x^2+k+1)]=(x+k+1)^2g(x),\end{aligned}$$

where

$$g(x)=2(kx^2+2)(x^2+k+1).$$

Since

$$f_6(-k-1,1,1)=0,\qquad f_6(0,1,-1)=0,$$

we apply Corollary 2 for $F_{\gamma,\delta}=g_{-k-1,\delta}$, where δ is given by (3.16):

$$\begin{aligned}\delta&=\frac{\gamma}{3(\gamma+2)}+\frac{(\gamma+2)^2g'(-2)}{12A}\\&=\frac{k+1}{3(k-1)}+\frac{g'(-2)}{12(k+5)(2k+1)}.\end{aligned}$$

According to Remark 4, it suffices to show that

$$g(x) \ge A\bar{g}_{-k-1,\delta}(x)$$

for $x \in \mathbb{R}$, where

$$\begin{aligned}\bar{g}_{-k-1,\delta}(x) &= \left[\frac{\gamma x^2+\gamma(\gamma+6)x-8}{(\gamma+2)^3}+\frac{\delta(x+2)(2\gamma x+x+\gamma-4)}{(\gamma+2)^2}\right]^2\\ &= \left[\frac{(k+1)x^2-(k+1)(k-5)x+8}{(k-1)^3}-\frac{\delta(x+2)(2kx+x+k+5)}{(k-1)^2}\right].\end{aligned}$$

We have

$$g'(x) = 4x(2kx^2+k^2+k+2), \qquad g'(-2) = -8(k^2+9k+2),$$

$$\delta = \frac{-k^2+10k+3}{(k-1)(k+5)(2k+1)},$$

$$\bar{g}_{-k-1,\delta}(x) = \frac{4[(2k+1)x^2-(k^2+k-2)x+k+5]^2}{(k-1)^2(k+5)^2(2k+1)^2},$$

$$A\bar{g}_{-k-1,\delta}(x) = \frac{4[(2k+1)x^2-(k^2+k-2)x+k+5]^2}{(k+5)(2k+1)},$$

$$g(x)-A\bar{g}_{-k-1,\delta}(x) = \frac{2g_1(x)}{(k+5)(2k+1)},$$

$$\begin{aligned}g_1(x) &= (k+5)(2k+1)(kx^2+2)(x^2+k+1)-2[(2k+1)x^2-(k^2+k-2)x+k+5]^2\\ &= (k-1)(k+2)(x+2)^2[(2k+1)x^2+k+5] \ge 0.\end{aligned}$$

The equality holds for $-x/(k+1) = y = z$ (or any cyclic permutation), and for $x = 0$ and $y+z = 0$ (or any cyclic permutation).

□

P 3.56. *If x,y,z are real numbers such that $xy+yz+zx<0$, then*

$$\frac{1}{3x^2+y^2+z^2}+\frac{1}{3y^2+z^2+x^2}+\frac{1}{3z^2+x^2+y^2}+\frac{1}{xy+yz+zx} \le 0.$$

(Vasile C., 2011)

Solution. Since $xy+yz+zx<0$, we may write the inequality as $f_6(x,y,z) \ge 0$, where

$$f_6(x,y,z) = (xy+yz+zx)\sum(3y^2+z^2+x^2)(3z^2+x^2+y^2)+\prod(3x^2+y^2+z^2).$$

Since
$$\prod(3x^2+y^2+z^2)=\prod(2x^2+p^2-2q),$$
$f_6(x,y,z)$ has the same highest coefficient as $\prod(2x^2)$, that is
$$A=8.$$
We have
$$f_6(0,y,z)=yz[(3y^2+z^2)(3z^2+y^2)+4(y^2+z^2)^2]+(y^2+z^2)(3y^2+z^2)(3z^2+y^2),$$
$$f_6(0,1,-1)=-(16+16)+32=0,$$
$$\begin{aligned}f_6(x,1,1)&=(2x+1)[(x^2+4)^2+2(x^2+4)(3x^2+2)]+(x^2+4)^2(3x^2+2)\\&=(x^2+4)(3x^4+14x^3+21x^2+16x+16).\end{aligned}$$
Since
$$f_6(0,1,-1)=0,$$
we apply Corollary 2 for $F_{\gamma,\delta}=h_{\gamma,\delta}$, where γ is given by (3.19):
$$\gamma=\frac{1}{3}+\frac{h'(-2)}{12A}=\frac{1}{3}+\frac{h'(-2)}{96},$$
with
$$\begin{aligned}h(x)&=(x^2+4)(3x^4+14x^3+21x^2+16x+16)\\&=3x^6+14x^5+33x^4+72x^3+100x^2+64x+64.\end{aligned}$$
We have
$$h'(x)=2(9x^5+35x^4+66x^3+108x^2+100x+32),$$
$$h'(-2)=16,\qquad \gamma=\frac{1}{3}+\frac{1}{6}=\frac{1}{2},$$
$$h_{1/2,\delta}(x)=[x+\frac{1}{2}(x+2)(2x+1)+\delta(x+2)^3]^2.$$
Choosing
$$\delta=\frac{-1}{4},$$
we get
$$\begin{aligned}h_{1/2,-1/4}(x)&=\left[x+\frac{1}{2}(x+2)(2x+1)-\frac{1}{4}(x+2)^3\right]^2\\&=\frac{1}{16}(x^3+2x^2-2x+4)^2,\end{aligned}$$
$$f_6(x,1,1)-Ah_{1/2,-1/4}(x)=\frac{1}{2}(x+2)^2(5x^4+4x^3+30x^2+8x+28).$$

For $x \geq 0$, the condition $f_6(x,1,1) - Ah_{1/2,-1/4}(x) \geq 0$ is clearly true. It is also true for $x < 0$, since

$$\begin{aligned} 5x^4 + 4x^3 + 30x^2 + 8x + 28 &> 5(x^4 + x^3 + 6x^2 + 2x + 5) \\ &= 5x^2(x^2 + x + 1) + 5(5x^2 + 2x + 5)] > 0. \end{aligned}$$

The equality holds for $x = 0$ and $y + z = 0$ (or any cyclic permutation).

□

P 3.57. *If x, y, z are real numbers, then*

$$\frac{x^2 - yz}{3x^2 + y^2 + z^2} + \frac{y^2 - zx}{3y^2 + z^2 + x^2} + \frac{z^2 - xy}{3z^2 + x^2 + y^2} \leq 1.$$

(Vasile C., 2011)

Solution. Write the inequality as $f_6(x,y,z) \geq 0$, where

$$f_6(x,y,z) = \prod(3x^2 + y^2 + z^2) - \sum(x^2 - yz)(3y^2 + z^2 + x^2)(3z^2 + x^2 + y^2).$$

From

$$f_6(x,y,z) = \prod(2x^2 + p^2 - 2q) - \sum(x^2 - yz)(2y^2 + p^2 - 2q)(2z^2 + p^2 - 2q),$$

it follows that $f_6(x,y,z)$ has the same highest coefficient as

$$g(x,y,z) = 8x^2y^2z^2 - 4\sum y^2z^2(x^2 - yz) = -4x^2y^2z^2 + 4\sum y^3z^3.$$

Therefore, $f_6(x,y,z)$ has the highest coefficient

$$A = -4 + 12 = 8.$$

We have

$$f_6(0,y,z) = (y^2 + z^2 + yz)(ky^2 + z^2)(kz^2 + y^2) - (y^2 + z^2)(y^4 + z^4 + 6y^2z^2),$$

$$f_6(0,1,-1) = 16 - 16 = 0,$$

$$\begin{aligned} f_6(x,1,1) &= (3x^2 + 2)(x^2 + 4)^2 - (x^2 - 1)(x^2 + 4)^2 - 2(1 - x)(3x^2 + 2)(x^2 + 4) \\ &= (x^2 + 4)(2x^4 + 6x^3 + 5x^2 + 4x + 8). \end{aligned}$$

Since

$$f_6(0,1,-1) = 0,$$

we apply Corollary 2 for $F_{\gamma,\delta} = h_{\gamma,\delta}$, where γ is given by (3.19):

$$\gamma = \frac{1}{3} + \frac{h'(-2)}{12A} = \frac{1}{3} + \frac{h'(-2)}{96},$$

with

$$\begin{aligned} h(x) &= (x^2+4)(2x^4+6x^3+5x^2+4x+8) \\ &= 2x^6+6x^5+13x^4+28x^3+28x^2+16x+32. \end{aligned}$$

We have

$$h'(x) = 2(6x^5+15x^4+26x^3+42x^2+28x+8),$$

$$h'(-2) = -80, \qquad \gamma = \frac{1}{3} - \frac{5}{6} = \frac{-1}{2},$$

$$h_{-1/2,\delta}(x) = [x - \frac{1}{2}(x+2)(2x+1) + \delta(x+2)^3]^2.$$

Choosing

$$\delta = 0,$$

we get

$$\begin{aligned} h_{-1/2,0}(x) &= \left[x - \frac{1}{2}(x+2)(2x+1)\right]^2 \\ &= \frac{1}{4}(2x^2+3x+2)^2, \end{aligned}$$

$$\begin{aligned} f_6(x,1,1) - Ah_{-1/2,0}(x) &= (x+2)^2(2x^4-2x^3+5x^2-8x+6) \\ &\ge (x+2)^2(x^4-2x^3+5x^2-8x+4) \\ &= (x+2)^2(x-1)^2(x^2+4) \ge 0. \end{aligned}$$

The equality holds for $x = 0$ and $y + z = 0$ (or any cyclic permutation).

□

P 3.58. *Let x, y, z be real numbers. If $k > 1$, then*

$$\frac{yz}{kx^2+y^2+z^2} + \frac{zx}{ky^2+z^2+x^2} + \frac{xy}{kz^2+x^2+y^2} + \frac{1}{2} \ge 0.$$

(Vasile C., 2011)

Solution. Write the inequality as $f_6(x,y,z) \ge 0$, where

$$f_6(x,y,z) = 2\sum yz(ky^2+z^2+x^2)(kz^2+x^2+y^2) + \prod(kx^2+y^2+z^2).$$

From

$$f_6(x,y,z) = 2\sum yz[(k-1)y^2+p^2-2q][(k-1)z^2+p^2-2q] + \prod[(k-1)x^2+p^2-2q],$$

it follows that $f_6(x,y,z)$ has the same highest coefficient as

$$g(x,y,z) = 2(k-1)^2\sum y^3z^3 + (k-1)^3x^2y^2z^2,$$

that is
$$A = 6(k-1)^2 + (k-1)^3 = (k-1)^2(k+5) \geq 0.$$
We have
$$f_6(0,y,z) = 2yz(ky^2+z^2)(kz^2+y^2) + (y^2+z^2)(ky^2+z^2)(kz^2+y^2),$$
$$f_6(0,1,-1) = -2(k+1)^2 + 2(k+1)^2 = 0,$$
$$\begin{aligned} f_6(x,1,1) &= 2(x^2+k+1)^2 + 4x(x^2+k+1)(kx^2+2) + (kx^2+2)(x^2+k+1)^2 \\ &= (x^2+k+1)[kx^4+4kx^3+(k^2+k+4)x^2+8x+4k+4]. \end{aligned}$$
Since
$$f_6(0,1,-1) = 0,$$
we apply Corollary 2 for $F_{\gamma,\delta} = h_{\gamma,0}$, where γ is given by (3.19):
$$\gamma = \frac{1}{3} + \frac{h'(-2)}{12A} = \frac{1}{3} + \frac{h'(-2)}{12(k-1)^2(k+5)},$$
with
$$\begin{aligned} h(x) =& (x^2+k+1)[kx^4+4kx^3+(k^2+k+4)x^2+8x+4k+4] \\ =& kx^6+4kx^5+2(k^2+k+2)(x^4+2x^3)+(k+1)(k^2+k+8)x^2 \\ &+8(k+1)x+4(k+1)^2. \end{aligned}$$
We have
$$h'(x) = 2[3kx^5+10kx^4+2(k^2+k+2)(2x^3+3x^2)+(k+1)(k^2+k+8)x+4k+4],$$
$$h'(-2) = -4(k-1)(k^2+7k-14),$$
$$\gamma = \frac{1}{3} - \frac{k^2+7k-14}{3(k-1)(k+5)} = \frac{3-k}{(k-1)(k+5)},$$
$$h_{\gamma,0}(x) = \left[x - \frac{k-3}{(k-1)(k+5)}(x+2)(2x+1)\right]^2,$$
$$Ah_{\gamma,0}(x) = \frac{1}{k+5}\left[(6-2k)x^2+(k^2-k+10)x+6-2k\right]^2,$$
$$f_6(x,1,1) - Ah_{\gamma2,0}(x) = \frac{1}{k+5}(x+2)^2 g(x),$$
where
$$g(x) = k(k+5)x^4 + 2(k^3+2k^2+9k-8)x^2 - 8(k^2-k+2)x + k^3+6k^2+17k-4.$$
Since
$$k^3+2k^2+9k-8 \geq 3k^2+9k-8 > 2(k^2-k+2)$$

and

$$k^3+6k^2+17k-4 \ge 7k^2+17k-4 > 4(k^2-k+2),$$

we have

$$\begin{aligned} g(x) &> 4(k^2-k+2)x^2-8(k^2-k+2)x+4(k^2-k+2)\\ &= 4(k^2-k+2)(x-1)^2 \ge 0. \end{aligned}$$

The equality holds for $x=0$ and $y+z=0$ (or any cyclic permutation).

□

P 3.59. *If x,y,z are real numbers, then*

$$\frac{yz}{x^2+4y^2+4z^2}+\frac{zx}{y^2+4z^2+4x^2}+\frac{xy}{z^2+4x^2+4y^2}+\frac{1}{8}\ge 0.$$

(Vasile C., 2011)

Solution. We will apply Corollary 2 for $F_{\gamma,\delta}=h_{\gamma,\delta}$, where

$$\delta=\frac{1}{2(k+5)}=\frac{2}{21},$$

$$\gamma=\frac{3-k}{(k-1)(k+5)}=\frac{-44}{63}.$$

We have

$$f_6(x,y,z)=2\sum yz(ky^2+z^2+x^2)(kz^2+x^2+y^2)+\prod(kx^2+y^2+z^2),$$

$$A=(k-1)^2(k+5),$$

$$\begin{aligned} f_6(x,1,1)=&(x^2+k+1)[kx^4+4kx^3+(k^2+k+4)x^2+8x+4k+4]\\ =&kx^6+4kx^5+2(k^2+k+2)(x^4+2x^3)+(k+1)(k^2+k+8)x^2\\ &+8(k+1)x+4(k+1)^2, \end{aligned}$$

$$h_{\gamma,\delta}(x)=\left[x-\frac{k-3}{(k-1)(k+5)}(x+2)(2x+1)+\frac{1}{2(k+5)}(x+2)^3\right]^2,$$

$$f_6(x,1,1)-Ah_{\gamma,\delta}(x)=\frac{1}{4(k+5)}(x+2)^2(A_1x^4+B_1x^3+C_1x^2+D_1x+E_1),$$

where

$$A_1=3k^2+22k-1,\quad B_1=16(1-k),\quad C_1=4k(k+1)(k+3),$$

$$D_1=8(k+1)(1-4k-k^2),\quad E_1=4(k+1)^2(k+4).$$

For $k = 1/4$, the inequality $f_6(x,1,1) \geq Ah_{\gamma,\delta}(x)$ reduces to $(x+2)^2 g(x) \geq 0$, where

$$g(x) = 75x^4 + 192x^3 + 65x^2 - 10x + 425.$$

Clearly, $g(x) > 0$ for $x \geq 0$. Also, for $x < 0$, we have

$$g(x) > 64x^4 + 192x^3 + 64x^2 + 256 = 64(x+2)^2(x^2 - x + 1) \geq 0.$$

The equality holds for $x = 0$ and $y + z = 0$ (or any cyclic permutation).

□

P 3.60. *If x, y, z are real numbers, then*

$$\sum \frac{1}{x^2 + 4y^2 + 4z^2} \leq \frac{7}{4(x^2 + y^2 + z^2) + 3(xy + yz + zx)}.$$

(Vasile C., 2011)

Solution. Write the inequality as $f_6(x, y, z) \geq 0$, where

$$f_6(x, y, z) = 7\prod(x^2 + 4y^2 + 4z^2) - P\sum(y^2 + 4z^2 + 4x^2)(z^2 + 4x^2 + 4y^2),$$

$$P = 4(x^2 + y^2 + z^2) + 3(xy + yz + zx).$$

The highest coefficient A of $f_6(x, y, z)$ is equal with the highest coefficient of the product $7\prod(x^2 + 4y^2 + 4z^2)$. Since

$$x^2 + 4y^2 + 4z^2 = -3x^2 + 4(p^2 - 2q),$$

we have

$$A = 7(-3)^3 = -189.$$

By Corollary 1, we only need to prove the original inequality for $y = z = 1$. Thus, we need to show that

$$\frac{1}{x^2 + 9} + \frac{2}{4x^2 + 5} \leq \frac{7}{4x^2 + 6x + 11},$$

which is equivalent to

$$(x-1)^2(2x-7)^2 \geq 0.$$

The equality holds for $x = y = z$, and also for $2x = 7y = 7z$ (or any cyclic permutation).

Observation. For $xy + yz + zx > 0$, using the Cauchy-Schwartz inequality

$$\frac{4}{x^2 + y^2 + z^2} + \frac{3}{xy + yz + zx} \geq \frac{(4+3)^2}{4(x^2 + y^2 + z^2) + 3(xy + yz + zx)},$$

we get the following inequality

$$\sum \frac{7}{x^2+4y^2+4z^2} \le \frac{4}{x^2+y^2+z^2} + \frac{3}{xy+yz+zx},$$

with equality for $x = y = z$.

□

P 3.61. *If x, y, z are real numbers such that $xy + yz + zx \ge 0$, then*

$$\sum \frac{2}{4x^2+y^2+z^2} \ge \frac{45}{14(x^2+y^2+z^2)+xy+yz+zx}.$$

(Vasile C., 2011)

Solution. Write the inequality as $f_6(x, y, z) \ge 0$, where

$$f_6(x,y,z) = 2P\sum(4y^2+z^2+x^2)(4z^2+x^2+y^2) - 45\prod(4x^2+y^2+z^2),$$

$$P = 14(x^2+y^2+z^2)+xy+yz+zx.$$

The highest coefficient A of $f_6(x, y, z)$ is equal with the highest coefficient of the product $-45\prod(4x^2+y^2+z^2)$. Since

$$4x^2+y^2+z^2 = 3x^2+4(p^2-2q),$$

we have

$$A = -45(3)^3 < 0.$$

By Theorem 1, we only need to prove the original inequality for $y = z = 1$ and $2x + 1 \ge 0$. Thus, we need to prove that

$$\frac{1}{2x^2+1} + \frac{4}{x^2+5} \ge \frac{45}{14x^2+2x+29},$$

which is equivalent to

$$(x-1)^2(2x+1)(x+2) \ge 0.$$

The equality holds for $x = y = z$, and also for $-2x = y = z$ (or any cyclic permutation).

Observation. Similarly, we can prove the following generalization:

- *Let x, y, z be real numbers such that $xy + yz + zx \ge 0$. If $k > 1$, then*

$$\sum \frac{1}{kx^2+y^2+z^2} \ge \frac{27(2k+7)}{(k+8)(4k+5)(x^2+y^2+z^2)+2(k-1)^2(xy+yz+zx)},$$

with equality for $x = y = z$, and also for $-2x = y = z$ (or any cyclic permutation).

For

$$f_6(x,y,z) = P(x,y,z)\sum(ky^2+z^2+x^2)(kz^2+x^2+y^2)-27(2k+7)\prod(kx^2+y^2+z^2),$$

$$P(x,y,z) = (k+8)(4k+5)(x^2+y^2+z^2)+2(k-1)^2(xy+yz+zx),$$

we have

$$A = -27(2k+7)(k-1)^3 < 0,$$

$$P(x,1,1) = (k+8)(4k+5)x^2+4(k-1)^2x+2(5k^2+35k+41),$$

$$\begin{aligned}\frac{f_6(x,1,1)}{x^2+k+1} &= P(x,1,1)[x^2+k+1+2(kx^2+2)]-27(2k+7)(x^2+k+1)(kx^2+2)\\ &= 2(k-1)^2(x-1)^2(2x+1)[2(k+5)x+5k+16] \ge 0\end{aligned}$$

for $2x+1 \ge 0$.

□

P 3.62. *If x,y,z are real numbers such that $xy+yz+zx \ge 0$, then*

$$\sum \frac{1}{x^2+4y^2+4z^2} \ge \frac{45}{44(x^2+y^2+z^2)+xy+yz+zx}.$$

(Vasile C., 2011)

Solution. Write the inequality as $f_6(x,y,z) \ge 0$, where

$$f_6(x,y,z) = P\sum(y^2+4z^2+4x^2)(z^2+4x^2+4y^2)-45\prod(x^2+4y^2+4z^2),$$

$$P = 44(x^2+y^2+z^2)+xy+yz+zx.$$

The highest coefficient A of $f_6(x,y,z)$ is equal with the highest coefficient of the product $-45\prod(x^2+4y^2+4z^2)$. Since

$$x^2+4y^2+4z^2 = -3x^2+4(p^2-2q),$$

we have

$$A = -45(-3)^3 = 1215.$$

Since

$$\begin{aligned}\frac{f_6(x,y,z)}{4x^2+5} &= (44x^2+2x+89)[4x^2+5+2(x^2+8)]-45(4x^2+5)(x^2+8)\\ &= 3(44x^2+2x+89)(2x^2+7)-45(4x^2+5)(x^2+8)\\ &= 3(x-1)^2(2x+1)(14x+23),\end{aligned}$$

we apply Corollary 2 for

$$F_{\gamma,\delta}(x) = f_{-1/2,-2}(x) = \frac{4(x-1)^4(2x+1)^2}{729}.$$

Thus, according to Remark 2, it suffices to show that

$$f_6(x,1,1) \geq Af_{-1/2,-2}(x)$$

for $2x+1 \geq 0$. We have

$$Af_{-1/2,-2}(x) = \frac{20(x-1)^4(2x+1)^2}{3},$$

$$f_6(x,1,1) - Af_{-1/2,-2}(x) = \frac{(x-1)^2(2x+1)f(x)}{3},$$

where

$$f(x) = 9(4x^2+5)(14x+23) - 20(x-1)^2(2x+1).$$

Since $4x^2+5 > (x-1)^2$, we have

$$f(x) > 9(x-1)^2(14x+23) - 20(x-1)^2(2x+1) = (x-1)^2(86x+187) \geq 0.$$

The equality holds for $x = y = z$, and also for $-2x = y = z$ (or any cyclic permutation).

Observation 1. Similarly, we can prove the following generalization:

- *Let x, y, z be real numbers such that $xy + yz + zx \geq 0$. If $0 \leq k < 1$, then*

$$\sum \frac{1}{kx^2+y^2+z^2} \geq \frac{27(2k+7)}{(k+8)(5k+5)(x^2+y^2+z^2)+2(k-1)^2(xy+yz+zx)}.$$

with equality for $x = y = z$, and also for $-2x = y = z$ (or any cyclic permutation).

For

$$f_6(x,y,z) = P\sum(ky^2+z^2+x^2)(kz^2+x^2+y^2) - 27(2k+7)\prod(kx^2+y^2+z^2),$$

$$P = (k+8)(5k+5)(x^2+y^2+z^2) + 2(k-1)^2(xy+yz+zx),$$

we have

$$A = -27(2k+7)(k-1)^3 > 0,$$

$$Af_{-1/2,-2}(x) = \frac{4(2k+7)(1-k)^3(x-1)^4(2x+1)^2}{27},$$

$$f_6(x,1,1) = 2(k-1)^2(x^2+k+1)(x-1)^2(2x+1)[2(k+5)x+5k+16],$$

$$f_6(x,1,1) - Af_{-1/2,-2}(x) = \frac{2(1-k)^2(x-1)^2(2x+1)f(x)}{27},$$

where

$$f(x)=27(x^2+k+1)[2(k+5)x+5k+16]-2(2k+7)(1-k)(x-1)^2(2x+1).$$

Since

$$2(x^2+k+1)-(1-k)(x-1)^2\ge 2(x^2+1)-(x-1)^2=(x+1)^2>0,$$

it suffices to show that

$$27[2(k+5)x+5k+16]\ge 4(2k+7)(2x+1)$$

for $2x+1\ge 0$. This is true if

$$2[2(k+5)x+5k+16]\ge (2k+7)(2x+1),$$

which is equivalent to

$$6x+8k+25\ge 0.$$

Observation 2. Having in view Observation 1 above and Observation from the preceding P 3.61, it follows that the concerned inequality holds for all $k\ge 0$.

For $k=0$, the following inequality holds under the condition $xy+yz+zx\ge 0$:

$$\frac{2}{x^2+y^2}+\frac{2}{y^2+z^2}+\frac{2}{z^2+x^2}\ge\frac{189}{20(x^2+y^2+z^2)+xy+yz+zx}.$$

□

P 3.63. *If x,y,z are real numbers, then*

$$\frac{x(-x+4y+4z)}{y^2+z^2}+\frac{y(-y+4z+4x)}{z^2+x^2}+\frac{z(-z+4x+4y)}{x^2+y^2}\le\frac{21}{2}.$$

(Vasile C., 2011)

Solution. Write the inequality as $f_6(x,y,z)\ge 0$, where

$$f_6(x,y,z)=21\prod(y^2+z^2)-2\sum x(-x+4y+4z)(x^2+z^2)(x^2+y^2).$$

Since

$$y^2+z^2=-x^2+p^2-2q,\qquad -x+4y+4z=-5x+4p,$$

$f_6(x,y,z)$ has the same highest coefficient A as

$$21(-x^2)(-y^2)(-z^2)-2\sum x(-5x)(-y^2)(-z^2),$$

that is

$$A=9.$$

Since

$$f_6(x,1,1) = 42(x^2+1)^2 - 2x(-x+8)(x^2+1)^2 - 8(4x+3)(x^2+1),$$

$$\begin{aligned}\frac{f_6(x,1,1)}{2(x^2+1)} &= 21(x^2+1) + x(x-8)(x^2+1) - 4(4x+3)\\ &= x^4 - 8x^3 + 22x^2 - 24x + 9 = (x-1)^2(x-3)^2,\end{aligned}$$

$$f_6(x,1,1) = 2(x^2+1)(x-1)^2(x-3)^2,$$

we apply Corollary 2 for

$$F_{\gamma,\delta}(x) = f_{3,-2}(x) = \frac{4(x-1)^4(x-3)^2}{2025}.$$

Thus, we need to show that $f_6(x,1,1) \ge Af_{3,-2}(x)$ for $x \in \mathbb{R}$. We have

$$Af_{3,-2}(x) = \frac{4(x-1)^4(x-3)^2}{225},$$

$$f_6(x,1,1) - Af_{3,-2}(x) = \frac{2(x-1)^2(x-3)^2 f(x)}{225},$$

where

$$f(x) = 225(x^2+1) - 2(x-1)^2 > 4(x^2+1) - 2(x-1)^2 = 2(x+1)^2 \ge 0.$$

The equality holds for $x = y = z$, and also for $x/3 = y = z$ (or any cyclic permutation).

□

P 3.64. *If x, y, z are real numbers, no two of which are zero, then*

$$\frac{x^2+3yz}{y^2-yz+z^2} + \frac{y^2+3zx}{z^2-zx+x^2} + \frac{z^2+3xy}{x^2-xy+y^2} \ge 1.$$

(Vasile C., 2011)

Solution. Write the inequality as $f_6(x,y,z) \ge 0$, where

$$f_6(x,y,z) = \sum (x^2+3yz)(x^2-xy+y^2)(x^2-xz+z^2) - \prod (y^2-yz+z^2).$$

Since

$$y^2 - yz + z^2 = -x^2 - yz + p^2 - 2q,$$

$f_6(x,y,z)$ has the same highest coefficient A as

$$P_2(x,y,z) - P_3(x,y,z),$$

where

$$P_2(x,y,z)=\sum(x^2+3yz)(-z^2-xy)(-y^2-xz), \qquad P_3(x,y,z)=\prod(-x^2-yz).$$

According to (3.2) and (3.3), $f_6(x,y,z)$ has the highest coefficient

$$A=P_2(1,1,1)-P_3(1,1,1)=48-(-8)=56.$$

On the other hand,

$$\begin{aligned} f_6(x,1,1)&=(x^2+3)(x^2-x+1)^2+2(3x+1)(x^2-x+1)-(x^2-x+1)^2\\ &=(x^2-x+1)(x^4-x^3+3x^2+4x+4). \end{aligned}$$

Since the original inequality is an equality for $(x,y,z)=(0,1,-1)$, that means

$$f_6(0,1,-1)=0,$$

we apply Corollary 2 for $F_{\gamma,\delta}=h_{\gamma,\delta}$, where γ is given by (3.19):

$$\gamma=\frac{1}{3}+\frac{h'(-2)}{12A},$$

$$h(x)=f_6(x,1,1)=(x^2-x+1)(x^4-x^3+3x^2+4x+4)$$

We have

$$h'(-2)=-524, \qquad \gamma=\frac{1}{3}+\frac{h'(-2)}{12A}=\frac{-25}{56}.$$

We need to show that there is a real δ such that $f_6(x,1,1)\ge Ah_{\gamma,\delta}(x)$ for $x\in\mathbb{R}$. Choosing $\delta=\dfrac{1}{8}$, we have

$$\begin{aligned} h_{\gamma,\delta}(x)&=\left[x-\frac{25}{56}(x+2)(2x+1)+\frac{1}{8}(x+2)^3\right]^2\\ &=\frac{1}{56^2}(7x^3-8x^2+15x+6)^2, \end{aligned}$$

$$\begin{aligned} f_6(x,1,1)-Ah_{\gamma,\delta}(x)&=\frac{1}{56}(7x^6+6x^4+156x^3+39x^2-180x+188)\\ &=\frac{1}{56}(x+2)^2(7x^4-28x^3+90x^2-92x+47)\\ &\ge\frac{1}{56}(x+2)^2(7x^4-28x^3+74x^2-92x+46)\\ &=\frac{1}{56}(x+2)^2[7x^2(x-2)^2+46(x-1)^2]\ge 0. \end{aligned}$$

The equality holds for $x=0$ and $y+z=0$ (or any cyclic permutation).

$\square$

P 3.65. *If x, y, z are real numbers, then*

$$\frac{(4x-y-z)^2}{2y^2-3yz+2z^2}+\frac{(4y-z-x)^2}{2z^2-3zx+2x^2}+\frac{(4z-x-y)^2}{2x^2-3xy+y^2}\ge 12.$$

(Vasile C., 2011)

Solution. Write the inequality as $f_6(x,y,z)\ge 0$, where

$$f_6=\sum(4x-y-z)^2(2z^2-3zx+2x^2)(2x^2-3xy+y^2)-12\prod(2y^2-3yz+2z^2).$$

Since

$$4x-y-z=5x-p, \qquad 2y^2-3yz+2z^2=-2x^2-3yz+2(p^2-2q),$$

$f_6(x,y,z)$ has the same highest coefficient A as

$$25P_2(x,y,z)-12P_3(x,y,z),$$

where

$$P_2(x,y,z=\sum x^2(-2y^2-3zx)(-2z^2-3xy), \qquad P_3(x,y,z)=\prod(-2x^2-3yz),$$

that is

$$A=25P_2(1,1,1)-12P_3(1,1,1)=25\cdot 75-12(-125)=3375.$$

Since

$$\begin{aligned}\frac{f_6(x,1,1)}{2(2x^2-3x+2)}&=2(2x-1)^2(2x^2-3x+2)+(x-3)^2-6(2x^2-3x+2)\\&=16x^4-40x^3+33x^2-10x+1=(x-1)^2(4x-1)^2,\end{aligned}$$

$$f_6(x,1,1)=2(2x^2-3x+2)(x-1)^2(4x-1)^2,$$

we apply Corollary 2 for

$$F_{\gamma,\delta}(x)=f_{1/4,-2}(x)=\frac{4(x-1)^4(4x-1)^2}{81\cdot 81}.$$

Thus, we need to show that $f_6(x,1,1)\ge Af_{1/4,-2}(x)$ for $x\in\mathbb{R}$. We have

$$Af_{1/4,-2}(x)=\frac{500(x-1)^4(4x-1)^2}{243}.$$

$$f_6(x,1,1)-Af_{1/4,-2}(x)=\frac{2(x-1)^2(4x-1)^2f(x)}{243},$$

where

$$\begin{aligned} f(x) &= 243(2x^2-3x+2)-250(x-1)^2 \\ &> 150(2x^2-3x+2)-250(x-1)^2 = 50(x^2+x+1) > 0. \end{aligned}$$

The equality holds for $x=y=z$, and also for $4x=y=z$ (or any cyclic permutation).

Observation. Similarly, we can prove the following generalization:

- *Let x,y,z be real numbers. If $-2<k\le 1-\sqrt{5}$ and $m=\dfrac{2-\sqrt{4+2k}}{4}$, then*

$$\sum \frac{(my+mz-x)^2}{y^2+kyz+z^2} \ge \frac{3(2m-1)^2}{k+2}.$$

with equality for $x=y=z$, and also for $x/\alpha=y=z$ (or any cyclic permutation), where $\alpha=\dfrac{-k-\sqrt{2k+4}}{2}$.

□

P 3.66. *If x,y,z are real numbers, then*

$$\frac{(3y+3z-4x)^2}{2y^2-3yz+2z^2}+\frac{(3z+3x-4y)^2}{2z^2-3zx+2x^2}+\frac{(3x+3y-4z)^2}{2x^2-3xy+y^2} \ge 12.$$

(Vasile C., 2011)

Solution. Write the inequality as $f_6(x,y,z)\ge 0$, where

$$f_6=\sum(3y+3z-4x)^2(2z^2-3zx+2x^2)(2x^2-3xy+y^2)-12\prod(2y^2-3yz+2z^2).$$

Since

$$3y+3z-4x=-7x+3p, \qquad 2y^2-3yz+2z^2=-2x^2-3yz+2(p^2-2q),$$

$f_6(x,y,z)$ has the same highest coefficient A as

$$49P_2(x,y,z)-12P_3(x,y,z),$$

where

$$P_2(x,y,z=\sum x^2(-2z^2-3xy)(-2y^2-3zx), \qquad P_3(x,y,z)=\prod(-2x^2-3yz),$$

that is

$$A=49P_2(1,1,1)-12P_3(1,1,1)=49\cdot 75-12(-125)=5175.$$

Since

$$\frac{f_6(x,1,1)}{2(2x^2-3x+2)} = 2(2x-3)^2(2x^2-3x+2)+(3x-1)^2-6(2x^2-3x+2)$$
$$= 16x^4-72x^3+121x^2-90x+25=(x-1)^2(4x-5)^2,$$

$$f_6(x,1,1) = 2(2x^2-3x+2)(x-1)^2(4x-5)^2,$$

we apply Corollary 2 for

$$F_{\gamma,\delta}(x) = f_{5/4,-2}(x) = \frac{4(x-1)^4(4x-5)^2}{81\cdot 169}.$$

Thus, we need to show that $f_6(x,1,1) \geq Af_{5/4,-2}(x)$ for $x \in \mathbb{R}$. We have

$$Af_{5/4,-2}(x) = \frac{2300(x-1)^4(4x-1)^2}{1521}.$$

$$f_6(x,1,1) - Af_{5/4,-2}(x) = \frac{(x-1)^2(4x-1)^2 f(x)}{1521},$$

where

$$f(x) = 3042(2x^2-3x+2) - 2300(x-1)^2$$
$$> 1500(2x^2-3x+2) - 2500(x-1)^2 = 500(x^2+x+1) > 0.$$

The equality holds for $x = y = z$, and also for $\frac{4x}{5} = y = z$ (or any cyclic permutation).

Observation. Similarly, we can prove the following generalization:

- *Let x,y,z be real numbers. If $-2 < k \leq 2$ and $m = \dfrac{2+\sqrt{4+2k}}{4}$, then*

$$\sum \frac{(my+mz-x)^2}{y^2+kyz+z^2} \geq \frac{3(2m-1)^2}{k+2}.$$

with equality for $x = y = z$, and also for $x/\alpha = y = z$ (or any cyclic permutation), where $\alpha = \dfrac{\sqrt{2k+4}-k}{2}$.

□

P 3.67. *Let x,y,z be real numbers. If $k > -2$, then*

$$4(x^2+kxy+y^2)(y^2+kyz+z^2)(z^2+kzx+x^2) \geq (2-k)(x-y)^2(y-z)^2(z-x)^2.$$

(Vasile C., 2011)

Solution. Write the inequality as $f_6(x,y,z) \ge 0$, where

$$f_6(x,y,z) = 4\prod(x^2+kxy+y^2)-(2-k)(x-y)^2(y-z)^2(z-x)^2.$$

From

$$\prod(x^2+kxy+y^2) = \prod(p^2-2q+kxy-z^2),$$

it follows that $f_6(x,y,z)$ has the same highest coefficient as

$$4\prod(kxy-z^2)-(2-k)(x-y)^2(y-z)^2(z-x)^2,$$

that is, according to (3.3),

$$A = 4(k-1)^3+27(2-k) = (k+2)(2k-5)^2.$$

For $k = 5/2$, we have $A = 0$. Then, by Corollary 1, it suffices to show that $f_6(x,1,1) \ge 0$ for any real x. Indeed,

$$f_6(x,1,1) = 4(k+2)(x^2+kx+1)^2 > 0.$$

Further, consider $k > -2$, $k \neq \dfrac{5}{2}$. Since $f_6(x,1,1)$ is a polynomial function of degree four and

$$f_6(0,y,z) = 4y^2z^2(y^2+kyz+z^2)-(2-k)y^2z^2(y-z)^2,$$

$$f_6(0,1,-1) = 4(2-k)-4(2-k) = 0,$$

we apply Corollary 2 for $F_{\gamma,\delta} = h_{\gamma,0}$, where γ is given by (3.19):

$$\gamma = \frac{1}{3}+\frac{h'(-2)}{12A},$$

with

$$h(x) = f_6(x,1,1) = 4(k+2)(x^2+kx+1)^2.$$

We only need to show that $f_6(x,1,1) \ge Ah_{\gamma,0}(x)$ for $x \in \mathbb{R}$. We have

$$h'(x) = 8(k+2)(2x+k)(x^2+kx+1),$$

$$h'(-2) = -8(k+2)(k-4)(2k-5), \qquad \gamma = \frac{1}{3}-\frac{2k-8}{3(2k-5)} = \frac{1}{2k-5},$$

$$h_{\gamma,0}(x) = \left[x+\frac{1}{2k-5}(x+2)(2x+1)\right]^2 = \frac{4(x^2+kx+1)^2}{(2k-5)^2},$$

therefore

$$f_6(x,1,1) = Ah_{\gamma,0}(x).$$

Actually, the inequality is equivalent to

$$(k+2)[xy(x+y)+yz(y+z)+zx(z+x)+2(k-1)xyz]^2 \ge 0.$$

Therefore, the equality holds for

$$xy(x+y)+yz(y+z)+zx(z+x)+2(k-1)xyz = 0.$$

□

P 3.68. *If x, y, z are real numbers, then*

$$(x^2+y^2)(y^2+z^2)(z^2+x^2)+2(x^2y+y^2z+z^2x)(xy^2+yz^2+zx^2) \ge x^2y^2z^2.$$

(Vasile C., 2011)

Solution. Write the inequality as $f_6(x,y,z) \ge 0$. From

$$f_6(x,y,z) = \prod(p^2-2q-x^2)+2(3x^2y^2z^2+\sum x^3y^3+xyz\sum x^3)-x^2y^2z^2,$$

it follows that $f_6(x,y,z)$ has the highest coefficient

$$A = -1+2(3+3+3)-1 = 16.$$

Since $f_6(x,1,1)$ is a polynomial function of degree four and

$$f_6(0,y,z) = y^2z^2(y^2+z^2)+2y^3z^3, \qquad f_6(0,1,-1) = 2-2 = 0,$$

we apply Corollary 2 for $F_{\gamma,\delta} = h_{\gamma,0}$, where γ is given by (3.19):

$$\gamma = \frac{1}{3}+\frac{h'(-2)}{12A},$$

with

$$h(x) = f_6(x,1,1) = 4x^4+4x^3+9x^2+4x+4 = (2x^2+x+2)^2.$$

We only need to show that $f_6(x,1,1) \ge Ah_{\gamma,0}(x)$ for $x \in \mathbb{R}$. We have

$$h'(x) = 2(4x+1)(2x^2+x+2), \qquad h'(-2) = -112,$$

$$\gamma = \frac{1}{3}-\frac{7}{12} = \frac{-1}{4},$$

$$h_{\gamma,0}(x) = \left[x-\frac{1}{4}(x+2)(2x+1)\right]^2 = \frac{1}{16}(2x^2+x+2)^2,$$

therefore

$$f_6(x,1,1) = Ah_{\gamma,0}(x).$$

Actually, the inequality is equivalent to

$$[xy(x+y)+yz(y+z)+zx(z+x)-xyz]^2 \ge 0.$$

Thus, the equality holds for

$$xy(x+y)+yz(y+z)+zx(z+x) = xyz.$$

□

P 3.69. *Let x, y, z be real numbers. If $k \in (-\infty, -2] \cup (0, \infty)$, then*

$$x^6 + y^6 + z^6 - 3x^2y^2z^2 + \frac{2}{k}(x^2 + kyz)(y^2 + kzx)(z^2 + kxy) \ge 0.$$

(Vasile C., 2011)

Solution. Denote the left side of the inequality by $f_6(x, y, z)$. According to (3.3), the polynomial $f_6(x, y, z)$ has the highest coefficient

$$A = 3 - 3 + \frac{2}{k}(1 + k)^3 = \frac{2(k + 1)^3}{k} > 0.$$

Since

$$f_6(0, y, z) = y^6 + z^6 + 2y^3z^3, \qquad f_6(0, 1, -1) = 1 + 1 - 2 = 0,$$

we apply Corollary 2 for $F_{\gamma,\delta} = h_{\gamma,\delta}$, where γ is given by (3.19):

$$\gamma = \frac{1}{3} + \frac{h'(-2)}{12A},$$

with

$$h(x) = f_6(x, 1, 1) = x^6 - 3x^2 + 2 + \frac{2}{k}(x^2 + k)(kx + 1)^2.$$

We have

$$h'(x) = 6x^5 - 6x + \frac{4}{k}x(kx + 1)^2 + 4(x^2 + k)(kx + 1),$$

$$h'(-2) = \frac{-4(2k^3 + 15k^2 + 33k + 2)}{k},$$

$$\gamma = \frac{1}{3} - \frac{2k^3 + 15k^2 + 33k + 2}{6(k + 1)^3} = \frac{-3k(k + 3)}{2(k + 1)^3},$$

Thus, the condition $f_6(x, 1, 1) \ge Ah_{\gamma,\delta}(x)$ is equivalent to

$$x^6 - 3x^2 + 2 + \frac{2}{k}(x^2 + k)(kx + 1)^2 \ge$$

$$\ge \frac{2(k + 1)^3}{k}\left[x - \frac{3k(k + 3)}{2(k + 1)^3}(x + 2)(2x + 1) + \delta(x + 2)^3\right]^2.$$

Setting $x = 1$, this inequality becomes an equality for $\delta = \dfrac{k(k + 3)}{2(k + 1)^3}$. Choosing this δ, the inequality turns into

$$k(k - 1)^2(k + 2)(x + 2)^2(x - 1)^4 \ge 0,$$

which is true for $x \in \mathbb{R}$.

The equality holds for $x = 0$ and $y + z = 0$ (or any cyclic permutation).

Observation. For $k=-2$ and $k=1$, the following identities hold:

$$x^6+y^6+z^6-3x^2y^2z^2-(x^2-2yz)(y^2-2zx)(z^2-2xy)=(x^3+y^3+z^3-2xyz)^2,$$

$$x^6+y^6+z^6-3x^2y^2z^2+2(x^2+yz)(y^2+zx)(z^2+xy)=(x^3+y^3+z^3+xyz)^2.$$

□

P 3.70. *If x,y,z are real numbers, then*

$$2(2x^2+y^2+z^2)(2y^2+z^2+x^2)(2z^2+x^2+y^2)\geq 89x^2y^2z^2+9(x-y)^2(y-z)^2(z-x)^2.$$

(Vasile C., 2011)

Solution. Write the inequality as $f_6(x,y,z)\geq 0$. Since

$$\prod(2x^2+y^2+z^2)=\prod(x^2+p^2-2q),$$

the polynomial $f_6(x,y,z)$ has the same highest coefficient as

$$2x^2y^2z^2-89x^2y^2z^2-9(x-y)^2(y-z)^2(z-x)^2,$$

that is

$$A=2-89+9\cdot 27=156.$$

Since

$$f_6(0,y,z)=2(y^2+z^2)(2y^2+z^2)(2z^2+y^2)-9y^2z^2(y-z)^2,$$

$$f_6(0,1,-1)=36-36=0,$$

we apply Corollary 2 for $F_{\gamma,\delta}=h_{\gamma,\delta}$, where γ is given by (3.19):

$$\gamma=\frac{1}{3}+\frac{h'(-2)}{12A},$$

with

$$h(x)=f_6(x,1,1)=4(x^2+1)(x^2+3)^2-89x^2.$$

We have

$$h'(x)=8x(x^2+3)^3+16x(x^2+1)(x^2+3)-178x,$$

$$h'(-2)=-1548,\qquad \gamma=\frac{1}{3}-\frac{43}{52}=\frac{-77}{156},$$

Thus, the condition $f_6(x,1,1)\geq Ah_{\gamma,\delta}(x)$ is equivalent to

$$4(x^2+1)(x^2+3)^2-89x^2\geq 156\left[x-\frac{77}{156}(x+2)(2x+1)+\delta(x+2)^3\right]^2.$$

Setting $x=1$, we can check that this inequality becomes an equality for $\delta=\frac{17}{156}$. Choosing $\delta=\frac{17}{156}$, the inequality becomes

$$(x+2)^2(x-1)^2(335x^2-1098x+1323)\geq 0,$$

which is true for $x\in\mathbb{R}$.

The equality holds for $x=0$ and $y+z=0$ (or any cyclic permutation).

Observation. The following related statement holds, too.

- *Let x,y,z be real numbers. If $k\geq 0$, then*

$$2(kx^2+y^2+z^2)(ky^2+z^2+x^2)(kz^2+x^2+y^2)\geq (k+1)^2(x-y)^2(y-z)^2(z-x)^2,$$

with equality for $x=0$ and $y+z=0$ (or any cyclic permutation).

□

P 3.71. *If x,y,z are real numbers such that $x+y+z=3$, then*

$$\frac{13x-1}{x^2+23}+\frac{13y-1}{y^2+23}+\frac{13z-1}{z^2+23}\leq\frac{3}{2}.$$

(Vasile C., 2011)

Solution. Write the inequality in the homogeneous form $f_6(x,y,z)\geq 0$, where

$$f_6(x,y,z)=3\prod(9x^2+23p^2)-2p\sum(39x-p)(9y^2+23p^2)(9z^2+23p^2).$$

Clearly, the polynomial $f_6(x,y,z)$ has the highest coefficient

$$A=3\cdot 729.$$

For $p=0$, we have $f_6(x,y,z)=3\cdot 729\ x^2y^2z^2$, therefore

$$f_6(0,-1,1)=0.$$

Also, we have

$$f_6(x,1,1)=12(23x^2+92x+101)(x-1)^2(7x+11)^2.$$

$$f_{-11/7,-2}(x)=\frac{4(x-1)^4(7x+11)^2}{729},$$

$$Af_{-11/7,-2}(x)=12(x-1)^4(7x+11)^2,$$

$$f_6(x,1,1)-Af_{-11/7,-2}(x)=24(x-1)^2(7x+11)^2(x+2)(11x+25).$$

As a consequence, we will apply Theorem 2 for

$$\eta = \frac{-25}{11}, \qquad \xi = \frac{-3}{143},$$

$$\mathbb{I} = \left(\frac{-25}{11}, \frac{-23}{13}\right), \qquad \mathbb{R} \setminus \mathbb{I} = \left(-\infty, \frac{-25}{11}\right] \cup \left[\frac{-23}{13}, \infty\right),$$

and for

$$E_{\alpha,\beta} = h_{\alpha,\beta}, \qquad E_{\gamma,\delta} = f_{-11/7,-2},$$

where α is given by (3.18):

$$\alpha = \frac{1}{3} + \frac{h'(-2)}{12A},$$

with

$$h(x) = f_6(x,1,1) = 12(23x^2 + 92x + 101)(x-1)^2(7x+11)^2.$$

The condition (b), namely $f_6(x,1,1) \ge Af_{-11/7,-2}(x)$, is satisfied for $x \in \left(-\infty, \frac{-25}{11}\right] \cup$ $[-2, \infty)$, hence for $x \in \mathbb{R} \setminus \mathbb{I} = \left(-\infty, \frac{-25}{11}\right] \cup \left[\frac{-23}{13}, \infty\right)$.

The condition (a) is satisfied if there is a real β such that $f_6(x,1,1) \ge Af_{\alpha,\beta}(x)$ for $x \in \mathbb{I} = \left(\frac{-25}{11}, \frac{-23}{13}\right)$, where

$$\alpha = \frac{1}{3} + \frac{h'(-2)}{12A}.$$

From

$$h'(-2) = -729 \cdot 64,$$

we get

$$\alpha = \frac{1}{3} + + \frac{-16}{9} = \frac{-13}{9},$$

$$h_{\alpha,\beta}(x) = \left[x - \frac{13}{9}(x+2)(2x+1) + \beta(x+2)^3\right]^2.$$

We choose

$$\beta = \frac{28}{9},$$

to have $h_{\alpha,\beta}(-11/7) = 0$. Therefore,

$$\begin{aligned} h_{\alpha,\beta}(x) &= \left[x - \frac{13}{9}(x+2)(2x+1) + \frac{28}{9}(x+2)^3\right]^2 \\ &= \frac{4}{81}(7x+11)^2(2x^2+7x+9)^2, \\ Ah_{\alpha,\beta}(x) &= 108(7x+11)^2(2x^2+7x+9)^2, \end{aligned}$$

$$f_6(x,1,1)-Ah_{\alpha,\beta}(x)=-12(7x+11)^2(x+2)^2(13x^2+154x+167)$$

and

$$13x^2+154x+167<14(2x^2+11x+12)=14(x+4)(2x+3)<0.$$

The original inequality is an equality for $x=y=z=1$, and also for $x=-11$ and $y=z=7$ (or any cyclic permutation).

□

P 3.72. *If x,y,z are real numbers, then*

$$5(x^2+y^2+z^2)^3\geq 108x^2y^2z^2+10(x-y)^2(y-z)^2(z-x)^2.$$

(Vo Quoc Ba Can and Vasile Cirtoaje, 2011)

Solution. Write the inequality as $f_6(x,y,z)\geq 0$, where

$$f_6(x,y,z)=5(x^2+y^2+z^2)^3-108x^2y^2z^2-10(x-y)^2(y-z)^2(z-x)^2.$$

The polynomial $f_6(x,y,z)$ has the highest coefficient

$$A=-108-10(-27)=162.$$

We have

$$f_6(0,y,z)=5(y^2+z^2)^3-10y^2z^2(y-z)^2,\qquad f_6(0,1,-1)=40-40=0,$$

$$h(x)=f_6(x,1,1)=5x^6+30x^4-48x^2+40,$$

$$h'(-2)=-1728,\qquad \frac{1}{3}+\frac{h'(-2)}{12A}=\frac{-5}{9},$$

$$h_{-5/9,\beta}(x)=\left[x-\frac{5}{9}(x+2)(2x+1)+\beta(x+2)^3\right]^2,$$

$$Ah_{-5/9,\beta}(x)=2\left[9x-5(x+2)(2x+1)+9\beta(x+2)^3\right]^2,$$

$$f_6(x,1,1)-Ah_{-5/9,\beta}(x)=(x+2)^3H_\beta(x),$$

where

$$H_\beta(x)=5(x^3-6x^2-10x-4)+72\beta(5x^2+8x+5)+162\beta^2(x+2)^3.$$

First Solution. We will apply Theorem 2 for $\xi\to-\infty$, which involves

$$\mathbb{I}=\left(-\infty,\frac{-1}{2}\right),\qquad \mathbb{R}\setminus\mathbb{I}=\left[\frac{-1}{2},\infty\right),$$

and for

$$E_{\alpha,\beta} = h_{-5/9,\beta}, \qquad E_{\gamma,\delta} = h_{-5/9,\delta}.$$

The condition (a), namely $f_6(x,1,1) \ge Ah_{-5/9,\beta}(x)$ for $x < -1/2$, can be satisfied only if $H_\beta(-2) = 0$. This yields $\beta = \dfrac{10}{81}$ and

$$f_6(x,1,1) - Ah_{-5/9,10/81}(x) = \frac{5}{81}(x+2)^4(41x^2 - 88x + 38) \ge 0.$$

The condition (b) is satisfied if there is a real δ such that $f_6(x,1,1) \ge Ah_{-5/9,\delta}(x)$ for $x \ge -1/2$, that is $H_\delta(x) \ge 0$ for $x \ge -1/2$. Choosing $\delta = \dfrac{4}{27}$, the condition $H_\delta(x) \ge 0$ is equivalent to

$$13x^3 + 18x^2 - 66x + 44 \ge 0,$$

which is true because

$$13x^3 + 18x^2 - 66x + 44 = 13(x-1)^2(x+1) + (31x^2 - 53x + 31) > 0.$$

The equality holds for $x = 0$ and $y + z = 0$ (or any cyclic permutation).

Second Solution. Apply Theorem 2 for

$$\eta = \frac{2}{5}, \qquad \xi = \frac{16}{5},$$

$$\mathbb{I} = \left(\frac{2}{5}, 2\right), \quad \mathbb{R} \setminus \mathbb{I} = \left(-\infty, \frac{2}{5}\right] \cup [2, \infty),$$

and for

$$E_{\alpha,\beta} = h_{-5/9,4/27}, \qquad E_{\gamma,\delta} = h_{-5/9,10/81}.$$

The condition (a) is satisfied if $f_6(x,1,1) \ge Ah_{-5/9,4/27}(x)$ for $2/5 < x < 2$. As shown in the first proof, this inequality can be written as

$$(x+2)^3(13x^3 + 18x^2 - 66x + 44) \ge 0,$$

which is true since

$$13x^3 + 18x^2 - 66x + 44 = 13(x-1)^2(x+1) + (31x^2 - 53x + 31) > 0.$$

The condition (b) is satisfied if $f_6(x,1,1) \ge Ah_{-5/9,10/81}$ for $x \in (-\infty, 2/5] \cup [2, \infty)$. As shown in the first proof, this inequality is equivalent to

$$(x+2)^4(41x^2 - 88x + 38) \ge 0,$$

which is true because

$$41x^2-88x+38>36x^2-88x+32=4(9x-4)(x-2)\geq 0.$$

Observation. The coefficient of the product $(x-y)^2(y-z)^2(z-x)^2$ in P 3.72 has the best possible value. Indeed, setting $x=0$, $y=1$, $z=-1$ in

$$5(x^2+y^2+z^2)^3\geq \alpha x^2y^2z^2+\beta(x-y)^2(y-z)^2(z-x)^2,$$

we get $\beta\leq 10$. In addition, for $\beta=10$, the best value of the coefficient α of the product $x^2y^2z^2$ is 108. Setting $x=5t$, $y=2t+1$, $z=2t-1$, the inequality

$$5(x^2+y^2+z^2)^3\geq \alpha x^2y^2z^2+10(x-y)^2(y-z)^2(z-x)^2$$

becomes

$$A(\alpha)t^6+B(\alpha)t^4+C(\alpha)t^2\geq 0,$$

where $C(\alpha)=108-\alpha$. The necessary condition $C(\alpha)\geq 0$ involves $\alpha\leq 108$.

□

P 3.73. *If x,y,z are real numbers, then*

$$(x^2+y^2+z^2)^3+2(2x^2+yz)(2y^2+zx)(2z^2+xy)\geq 27x^2y^2z^2.$$

(Vasile C., 2011)

Solution. Write the inequality as $f_6(x,y,z)\geq 0$, where

$$f_6(x,y,z)=(x^2+y^2+z^2)^3-27x^2y^2z^2+2(2x^2+yz)(2y^2+zx)(2z^2+xy).$$

According to (3.3), $f_6(x,y,z)$ has the highest coefficient

$$A=-27+2(2+1)^3=27.$$

We have

$$f_6(0,y,z)=(y^2+z^2)^3+8y^3z^3,\qquad f_6(0,1,-1)=8-8=0,$$

$$\begin{aligned}h(x)=f_6(x,1,1)&=(x^2+2)^3+2(2x^2+1)(x+2)^2-27x^2\\&=x^6+10x^4+16x^3+3x^2+8x+16,\end{aligned}$$

$$h'(-2)=-324,\qquad \frac{1}{3}+\frac{h'(-2)}{12A}=\frac{-2}{3},$$

$$h_{-2/3,\beta}(x)=\left[x-\frac{2}{3}(x+2)(2x+1)+\beta(x+2)^3\right]^2,$$

$$Ah_{-2/3,\beta}(x)=2\left[9x-5(x+2)(2x+1)+9\beta(x+2)^3\right]^2,$$

$$f_6(x,1,1)-Ah_{-5/9,\beta}(x)=(x+2)^3H_\beta(x),$$

where

$$H_\beta(x)=x^3-6x^2-14x-4+18\beta(4x^2+7x+4)-27\beta^2(x+2)^3.$$

We will apply Theorem 2 for $\xi\to-\infty$, which involves

$$\mathbb{I}=\left(-\infty,\frac{-1}{2}\right),\qquad \mathbb{R}\setminus\mathbb{I}=\left[\frac{-1}{2},\infty\right),$$

and for

$$E_{\alpha,\beta}=h_{-2/3,\beta},\qquad E_{\gamma,\delta}=h_{-2/3,\delta}.$$

The condition (a), namely $f_6(x,1,1)\ge Ah_{-2/3,\beta}(x)$ for $x<-1/2$, can be satisfied only if $H_\beta(-2)=0$. This yields $\beta=\dfrac{2}{27}$ and

$$f_6(x,1,1)-Ah_{-2/3,2/27}(x)=\frac{1}{27}(x+2)^4(23x^2-88x+2)\ge 0$$

for $x<-1/2$.

The condition (b) is satisfied if there is a real δ such that $f_6(x,1,1)\ge Ah_{-2/3,\delta}(x)$ for $x\ge-1/2$, that is $H_\delta(x)\ge 0$ for $x\ge-1/2$. Choosing $\delta=\dfrac{1}{6}$, we get

$$\begin{aligned}4H_\gamma(x)&=x^3+6x^2-8x+8\\&\ge 2x^2-8x+8=2(x-2)^2\ge 0.\end{aligned}$$

The equality holds for $x=0$ and $y+z=0$ (or any cyclic permutation).

$\square$

P 3.74. *If x,y,z are real numbers, no two of which are zero, then*

$$\frac{x^2+2yz}{y^2+yz+z^2}+\frac{y^2+2zx}{z^2+zx+x^2}+\frac{z^2+2xy}{x^2+xy+y^2}\ge\frac{3(xy+yz+zx)}{x^2+y^2+z^2}.$$

(Vasile C., 2014)

Solution. Write the inequality as $f_8(x,y,z)\ge 0$, where

$$\begin{aligned}f_8(x,y,z)=&(x^2+y^2+z^2)\sum(x^2+2yz)(z^2+zx+x^2)(x^2+xy+y^2)\\&-3(xy+yz+zx)\prod(y^2+yz+z^2).\end{aligned}$$

Since

$$y^2+yz+z^2=yz-x^2+p^2-2q,$$

$f_8(x,y,z)$ has the same highest polynomial as

$$(x^2+y^2+z^2)\sum(x^2+2yz)(zx-y^2)(xy-z^2)-3(xy+yz+zx)\prod(yz-x^2),$$

that is, according to (3.2) and (3.3),

$$A(p,q)=(x^2+y^2+z^2)\cdot 0-3(xy+yz+zx)\cdot 0=0.$$

By Theorem 3, it suffices to show that $f_8(x,1,1)\ge 0$ for all real x. We have

$$\begin{aligned}\frac{f_8(x,1,1)}{x^2+x+1}&=(x^2+2)[(x^2+2)(x^2+x+1)+6(2x+1)]-9(2x+1)(x^2+x+1)\\&=x^6+x^5+5x^4-2x^3-13x^2+x+7\\&=(x^2-1)^2(x^2+x+7)\ge 0.\end{aligned}$$

The equality holds for $x=y=z$, and for $-x=y=z$ (or any cyclic permutation).

□

P 3.75. *If x,y,z are real numbers, no two of which are zero, then*

$$\frac{x^2-2yz}{y^2-yz+z^2}+\frac{y^2-2zx}{z^2-zx+x^2}+\frac{z^2-2xy}{x^2-xy+y^2}+\frac{3(xy+yz+zx)}{x^2+y^2+z^2}\ge 0.$$

(Vasile C., 2014)

Solution. Write the inequality as $f_8(x,y,z)\ge 0$, where

$$\begin{aligned}f_8(x,y,z)=&(x^2+y^2+z^2)\sum(x^2-2yz)(z^2-zx+x^2)(x^2-xy+y^2)\\&+3(xy+yz+zx)\prod(y^2-yz+z^2).\end{aligned}$$

Since

$$y^2-yz+z^2=-yz-x^2+p^2-2q,$$

$f_8(x,y,z)$ has the same highest polynomial as

$$(x^2+y^2+z^2)\sum(x^2-2yz)(zx+y^2)(xy+z^2)-3(xy+yz+zx)\prod(yz+x^2),$$

that is, according to (3.2) and (3.3),

$$A(p,q)=(x^2+y^2+z^2)(-12)-3(xy+yz+zx)\cdot 8=-12p^2\le 0.$$

By Theorem 3, it suffices to show that $f_8(x,1,1)\ge 0$ for all real x. We have

$$\begin{aligned}\frac{f_8(x,1,1)}{x^2-x+1}&=(x^2+2)[(x^2-2)(x^2-x+1)+2(1-2x)]+3(2x+1)(x^2-x+1)\\&=x^6-x^5+x^4+2x^3-5x^2-x+3\\&=(x^2-1)^2(x^2-x+3)\ge 0.\end{aligned}$$

The equality holds for $x=y=z$, and for $-x=y=z$ (or any cyclic permutation).

□

P 3.76. *If x, y, z are real numbers, no two of which are zero, then*

$$\frac{x^2}{y^2-yz+z^2}+\frac{y^2}{z^2-zx+x^2}+\frac{z^2}{x^2-xy+y^2} \ge \frac{(x+y+z)^2}{x^2+y^2+z^2}.$$

(Vasile C., 2014)

Solution. Write the inequality as $f_8(x,y,z) \ge 0$, where

$$f_8(x,y,z) =(x^2+y^2+z^2)\sum x^2(z^2-zx+x^2)(x^2-xy+y^2) \\ -(x+y+z)^2\prod(y^2-yz+z^2).$$

Since

$$y^2-yz+z^2=-yz-x^2+p^2-2q,$$

$f_8(x,y,z)$ has the same highest polynomial as

$$(x^2+y^2+z^2)\sum x^2(zx+y^2)(xy+z^2)+(x+y+z)^2\prod(yz+x^2),$$

that is, according to (3.2) and (3.3),

$$A(p,q)=(x^2+y^2+z^2)\cdot 12+(x+y+z)^2\cdot 8=20p^2-24q=12p^2+8(p^2-3q)\ge 0.$$

We have

$$A(x+2,2x+1)=20(x+2)^2-24(2x+1)=4(5x^2+8x+14),$$

$$\begin{aligned}\frac{f_8(x,1,1)}{x^2-x+1} &= (x^2+2)[x^2(x^2-x+1)+2]-(x+2)^2(x^2-x+1)\\ &= x^6-x^5+2x^4-5x^3+3x^2\\ &= x^2(x-1)^2(x^2+x+3)\ge 0.\end{aligned}$$

Since $A(p,q) \ge 0$ for all real x,y,z, $f_8(1,1,1)=0$ and $f_8(0,1,1)=0$, we apply Corollary 4 for

$$E_{0,-2}(x)=f_{0,-2}(x)=\frac{x^2(x-1)^4}{81}.$$

We have

$$A(x+2,2x+1)f_{0,-2}(x)=\frac{4x^2(x-1)^4(5x^2+8x+14)}{81},$$

$$f_8(x,y,z)-A(x+2,2x+1)f_{0,-2}(x)=\frac{x^2(x-1)^2g(x)}{81},$$

where

$$\begin{aligned}g(x) &= 81(x^2-x+1)(x^2+x+3)-4(x-1)^2(5x^2+8x+14)\\ &= 61x^4+8x^3+231x^2-82x+187\\ &= 57x^4+4x^2(x+1)^2+227x^2-82x+187>0.\end{aligned}$$

The equality holds for $x=y=z$, and for $x=0$ and $y=z$ (or any cyclic permutation).

□

P 3.77. *If x,y,z are real numbers such that $xyz \neq 0$, then*

$$\frac{(y+z)^2}{x^2}+\frac{(z+x)^2}{y^2}+\frac{(x+y)^2}{z^2} \geq 2+\frac{10(x+y+z)^2}{3(x^2+y^2+z^2)}.$$

(Vasile C., 2014)

Solution. Write the inequality as $f_8(x,y,z) \geq 0$, where

$$f_8(x,y,z)=3(x^2+y^2+z^2)\left[\sum y^2z^2(y+z)^2-2x^2y^2z^2\right]-10x^2y^2z^2(x+y+z)^2.$$

Since

$$\sum y^2z^2(y+z)^2=\sum y^2z^2(p-x)^2=p^2\sum y^2z^2-2pqr+3r^2,$$

$f_8(x,y,z)$ has the highest polynomial

$$A(p,q)=3(x^2+y^2+z^2)(3-2)-10(x+y+z)^2=3(p^2-2q)-10p^2=-7p^2-6q.$$

We have

$$\begin{aligned}
f_8(x,1,1)&=3(x^2+2)[4+2x^2(x+1)^2-2x^2]-10x^2(x+2)^2\\
&=2(3x^6+6x^5+x^4-8x^3-14x^2+12)\\
&=2(x-1)^2(3x^4+12x^3+22x^2+24x+12)\\
&=2(x-1)^2[3(x+1)^4+(2x+3)^2]\geq 0,
\end{aligned}$$

$$A(x+2,2x+1)=-7(x+2)^2-6(2x+1)=-7x^2-40x-34.$$

Case 1: $-7p^2-6q \leq 0$. Apply Theorem 3. Since $f_8(x,1,1) \geq 0$ for all real x, the conclusion follows.

Case 2: $-7p^2-6q \geq 0$. Apply Corollary 4 for

$$E_{\alpha,\beta}=h_{\alpha,\beta}.$$

Thus, we need to show that there exist two real numbers α and β such that

$$f_8(x,1,1) \geq A(x+2,2x+1)h_{\alpha,\beta}(x)$$

for $x \in \mathbb{R}$. Write the required inequality as $h(x) \geq 0$, where

$$h(x)=f_8(x,1,1)+(7x^2+40x+34)\left[x+\alpha(x+2)(2x+1)+\beta(x+2)^3\right]^2,$$

Since

$$h(-2)=72-18(-2)^2=0,$$

the condition $h'(-2)=0$ is necessary to have $h(x) \geq 0$ in the vicinity of -2. This condition involves $\alpha=-1$. For this α, we can check that $h''(-2)=0$. Thus, to have $h(x) \geq 0$ in the vicinity of -2, the condition $h'''(-2) = 0$ is necessary. This condition involves $\beta = 2/3$. For these values of α and β, we need to show that

$$f_8(x,1,1) + (7x^2 + 40x + 34)\left[x - (x+2)(2x+1) + \frac{2}{3}(x+2)^3\right]^2 \geq 0$$

for $x \in \mathbb{R}$. This inequality is equivalent to

$$(x+2)^4(14x^4 + 52x^3 + 117x^2 + 154x + 113) \geq 0,$$

which is true because

$$14x^4+52x^3+117x^2+154x+113 = (x^2+1)^2+13x^2(x+2)^2+7(9x^2+22x+16) > 0.$$

The equality holds for $x = y = z$.

□

P 3.78. *If x, y, z are real numbers, no two of which are zero, then*

$$\frac{32x^2 + 49yz}{y^2 + z^2} + \frac{32y^2 + 49zx}{z^2 + x^2} + \frac{32z^2 + 49xy}{x^2 + y^2} \geq \frac{81(x+y+z)^2}{2(x^2+y^2+z^2)}.$$

(Vasile C., 2014)

Solution. Consider the more general inequality

$$\sum \frac{x^2 + kyz}{y^2 + z^2} \geq \frac{(1+k)(x+y+z)^2}{2(x^2+y^2+z^2)}, \quad k > -1,$$

which can be written as $f_8(x,y,z) \geq 0$, where

$$\begin{aligned} f_8(x,y,z) =& 2(x^2+y^2+z^2)\sum(x^2+kyz)(z^2+x^2)(x^2+y^2) \\ &- (1+k)(x+y+z)^2\prod(y^2+z^2). \end{aligned}$$

Since $y^2 + z^2 = -x^2 + p^2 - 2q$, $f_8(x,y,z)$ has the same highest polynomial as

$$2(p^2 - 2q)\sum(x^2 + kyz)y^2z^2 - (1+k)p^2(-x^2y^2z^2),$$

that is

$$A(p,q) = 2(p^2 - 2q)(3+3k) + (1+k)p^2 = (1+k)(7p^2 - 12q).$$

We have

$$A(p,q) = (1+k)[3p^2 + 4(p^2 - 3q)] \geq 0,$$

$$A(x+2, 2x+1) = (1+k)[7(x+2)^2 - 12(2x+1)] = (1+k)(7x^2 + 4x + 16),$$

$$\frac{f_8(x,1,1)}{x^2+1} = 2(x^2+2)[(x^2+k)(x^2+1)+4(kx+1)] - 2(1+k)(x+2)^2(x^2+1),$$

$$f_8(x,1,1) = 2(x^2+1)(x-1)^2(x^4+2x^3+5x^2+4x+4-2k).$$

For $k = \frac{49}{32}$, we get

$$A(x+2,2x+1) = \frac{81}{32}(7x^2+4x+16),$$

$$f_8(x,1,1) = \frac{1}{8}(x^2+1)(x-1)^2(2x+1)^2(4x^2+4x+15).$$

Since $A(p,q) \geq 0$ for all real x,y,z, $f_8(1,1,1) = 0$ and $f_8(-1/2,1,1) = 0$, we apply Corollary 4 for

$$E_{-1/2,-2}(x) = f_{-1/2,-2}(x) = \frac{4}{729}(x-1)^4(2x+1)^2.$$

We only need to show that

$$f_8(x,1,1) - A(x+2,2x+1)f_{-1/2,-2}(x) \geq 0$$

for $x \in \mathbb{R}$. Indeed, we have

$$A(x+2,2x+1)f_{-1/2,-2}(x) = \frac{9}{8}(x-1)^4(2x+1)^2(7x^2+4x+16),$$

$$f_8(x,1,1) - A(x+2,2x+1)f_{-1/2,-2}(x) = \frac{1}{72}(x-1)^2(2x+1)^2 g(x),$$

where

$$\begin{aligned} g(x) &= 29x^4+46x^3+156x^2+64x+119 \\ &= (23x^2+32)(x+1)^2+6x^4+101x^2+87 > 0. \end{aligned}$$

The equality holds for $x = y = z$, and also for $-2x = y = z$ (or any cyclic permutation).

□

P 3.79. *If x,y,z are real numbers, no two of which are zero, then*

$$(a) \qquad \frac{x^2+4yz}{y^2+z^2} + \frac{y^2+4zx}{z^2+x^2} + \frac{z^2+4xy}{x^2+y^2} \geq \frac{15(xy+yz+zx)}{2(x^2+y^2+z^2)};$$

$$(b) \qquad \frac{2x^2+9yz}{y^2+z^2} + \frac{2y^2+9zx}{z^2+x^2} + \frac{2z^2+9xy}{x^2+y^2} \geq \frac{33(xy+yz+zx)}{2(x^2+y^2+z^2)}.$$

(Vasile C., 2014)

Solution. Consider the more general inequality

$$\sum \frac{x^2+kyz}{y^2+z^2} \geq \frac{3(1+k)(xy+yz+zx)}{2(x^2+y^2+z^2)}, \quad k > -1,$$

which can be written as $f_8(x,y,z) \geq 0$, where

$$\begin{aligned} f_8(x,y,z) =& 2(x^2+y^2+z^2)\sum(x^2+kyz)(z^2+x^2)(x^2+y^2) \\ & -3(1+k)(xy+yz+zx)\prod(y^2+z^2). \end{aligned}$$

Since $y^2+z^2 = -x^2+p^2-2q$, $f_8(x,y,z)$ has the same highest polynomial as

$$2(p^2-2q)\sum(x^2+kyz)y^2z^2 - 3(1+k)q(-x^2y^2z^2),$$

that is

$$A(p,q) = 2(p^2-2q)(3+3k) + 3(1+k)q = 3(1+k)(2p^2-3q).$$

We have

$$A(p,q) = 3(1+k)[p^2+(p^2-3q)] \geq 0,$$

$$A(x+2,2x+1) = 3(1+k)[2(x+2)^2 - 3(2x+1)] = 3(1+k)(2x^2+2x+5),$$

$$\frac{f_8(x,1,1)}{x^2+1} = 2(x^2+2)[(x^2+k)(x^2+1)+4(kx+1)] - 6(1+k)(2x+1)(x^2+1),$$

$$f_8(x,1,1) = 2(x^2+1)(x-1)^2[x^4+2x^3+(k+6)x^2+4x+5-k].$$

Since $A(p,q) \geq 0$ for all real x,y,z, we apply Corollary 4 for

$$E_{0,-2}(x) = f_{0,-2}(x) = \frac{1}{81}x^2(x-1)^4.$$

Thus, we need to show that

$$f_8(x,1,1) - A(x+2,2x+1)f_{0,-2}(x) \geq 0$$

for $x \in \mathbb{R}$. Since

$$A(x+2,2x+1)f_{0,-2}(x) = \frac{(1+k)(2x^2+2x+5)x^2(x-1)^4}{27},$$

the inequality $f_8(x,1,1) - A(x+2,2x+1)f_{0,-2}(x) \geq 0$ is true if

$$54(x^2+1)[x^4+2x^3+(k+6)x^2+4x+5-k] \geq (1+k)x^2(2x^2+2x+5)(x-1)^2.$$

Since $2(x^2+1) \geq (x-1)^2$, it suffices to show that

$$27[x^4+2x^3+(k+6)x^2+4x+5-k] \geq (1+k)x^2(2x^2+2x+5),$$

which is equivalent to $g(x) \ge 0$, where

$$g(x) = (25-2k)x^4 + (52-2k)x^3 + (22k+157)x^2 + 108x + 27(5-k).$$

(a) For $k = 4$, we have

$$\begin{aligned} g(x) &= 17x^4 + 44x^3 + 245x^2 + 108x + 27 \\ &= x^4 + (4x+1)^2\left(x^2 + \frac{9}{4}x + 2\right) + \left(194x^2 + \frac{359}{4}x + 25\right) > 0. \end{aligned}$$

The equality holds for $x = y = z$.

(b) For $k = 9/2$, we have

$$\begin{aligned} 2g(x) &= 32x^4 + 86x^3 + 512x^2 + 216x + 27 \\ &= (4x+1)^2\left(2x^2 + \frac{35}{8}x + \frac{5}{2}\right) + \left(435x^2 + \frac{1533}{8}x + \frac{49}{2}\right) > 0. \end{aligned}$$

The equality holds for $x = y = z$.

□

P 3.80. *If x, y, z are distinct real numbers, then*

$$\frac{x^2}{(y-z)^2} + \frac{y^2}{(z-x)^2} + \frac{z^2}{(x-y)^2} \ge \frac{4(xy+yz+zx)}{x^2+y^2+z^2}.$$

(Vasile C., 2014)

Solution. Write the inequality as $f_8(x,y,z) \ge 0$, where

$$f_8(x,y,z) = (p^2-2q)\sum x^2(x-y)^2(x-z)^2 - 4q(x-y)^2(y-z)^2(z-x)^2.$$

Since

$$(x-y)(x-z) = x^2 + 2yz - q,$$

$f_8(x,y,z)$ has the same highest polynomial as

$$(p^2-2q)\sum x^2(x^2+2yz)^2 - 4q(x-y)^2(y-z)^2(z-x)^2,$$

that is, according to (3.1),

$$A(p,q) = (p^2-2q)\cdot 27 - 4q(-27) = 27(p^2+2q).$$

We have

$$f_8(x,1,1) = (x^2+2)x^2(x-1)^4 \ge 0.$$

Case 1: $p^2 + 2q \le 0$. Since $f_8(x,1,1) \ge 0$ for all real x, the conclusion follows from Theorem 3.

Case 2: $p^2+2q \ge 0$. Apply Corollary 4 for

$$E_{\gamma,\delta}(x) = f_{\gamma,0}(x) = \frac{4x^2(x-1)^4(x-\gamma)^2}{9(4-\gamma)^2(x+2)^2}.$$

Thus, we need to show that there exist a real number α such that

$$f_8(x,1,1) \ge A(x+2,2x+1)f_{\gamma,0}(x)$$

for $x \in \mathbb{R}$. Since

$$A(x+2,2x+1) = 27(x^2+8x+6),$$

$$A(x+2,2x+1)f_{\gamma,0}(x) = \frac{12x^2(x-1)^4(x^2+8x+6)(x-\gamma)^2}{(4-\gamma)^2(x+2)^2},$$

$$f_8(x,1,1) - A(x+2,2x+1)f_{\gamma,0}(x) = \frac{x^2(x-1)^4h(x)}{(4-\gamma)^2(x+2)^2},$$

$$h(x) = (4-\gamma)^2(x+2)^2(x^2+2) - 12(x^2+8x+6)(x-\gamma)^2,$$

we need to show that $h(x) \ge 0$ for $x \in \mathbb{R}$. Since $h(4) = 0$, the condition $h'(4) = 0$ is necessary to have $h(x) \ge 0$ in the vicinity of 4. This implies $\alpha = -2/13$ and

$$\begin{aligned} h(x) &= \frac{2(37x^4 - 216x^3 + 12x^2 + 800x + 960)}{169} \\ &= \frac{2(x-4)^2(37x^2+80x+60)}{169} \ge 0. \end{aligned}$$

The equality holds for $x = 0$ and $y = z$ (or any cyclic permutation).

□

P 3.81. *If x, y, z are distinct real numbers, then*

$$\frac{x^2}{(y-z)^2} + \frac{y^2}{(z-x)^2} + \frac{z^2}{(x-y)^2} \ge \frac{(x+y+z)^2}{x^2+y^2+z^2}.$$

(Vasile C., 2014)

Solution. By the Cauchy-Schwarz inequality, we have

$$\sum \frac{x^2}{(y-z)^2} \ge \frac{(x+y+z)^2}{\sum (y-z)^2}.$$

Thus, it suffices to show that

$$\frac{1}{\sum (y-z)^2} \ge \frac{1}{x^2+y^2+z^2},$$

which is equivalent to

$$x^2 + y^2 + z^2 \le 2(xy + yz + zx),$$

$$p^2 \le 4q.$$

Consider next that

$$p^2 \ge 4q,$$

and write the inequality as $f_8(x, y, z) \ge 0$, where

$$f_8(x,y,z) = (x^2+y^2+z^2)\sum x^2(x-y)^2(x-z)^2-(x+y+z)^2(x-y)^2(y-z)^2(z-x)^2.$$

Since

$$(x-y)(x-z) = x^2 + 2yz - q,$$

$f_8(x, y, z)$ has the same highest polynomial as

$$(x^2 + y^2 + z^2)\sum x^2(x^2 + 2yz)^2 - (x + y + z)^2(x - y)^2(y - z)^2(z - x)^2,$$

that is, according to (3.1),

$$A(p,q) = (x^2 + y^2 + z^2)\cdot 27 - (x + y + z)^2(-27) = 54(p^2 - q).$$

Since

$$A(p,q) = 18[2p^2 + (p^2 - 3q)] \ge 0,$$

$$A(x + 2, 2x + 1) = 54(x^2 + 2x + 3),$$

$$f_8(x, 1, 1) = (x^2 + 2)x^2(x - 1)^4,$$

we apply Corollary 4 for $p^2 \ge 4q$ (see Remark 6) and

$$E_{\gamma,\delta}(x) = f_{0,-2}(x) = \frac{1}{81}x^2(x-1)^4.$$

Thus, we need to show that

$$f_8(x, 1, 1) \ge A(x + 2, 2x + 1)f_{0,-2}(x)$$

for $(x + 2)^2 \ge 4(2x + 1)$, that is for

$$x \in (-\infty, 0] \cup [4, \infty).$$

Since

$$A(x + 2, 2x + 1)f_{0,-2}(x) = \frac{2(x^2 + 2x + 3)x^2(x - 1)^4}{3},$$

we have

$$f_8(x,1,1)-A(x + 2, 2x + 1)f_{0,-2}(x) = \frac{x^2(x - 1)^4x(x - 4)}{3} \ge 0.$$

The equality holds for $x = 0$ and $y = z$ (or any cyclic permutation).

□

P 3.82. *If x, y, z are real numbers, then*

$$\frac{2xy}{x^2+y^2}+\frac{2yz}{y^2+z^2}+\frac{2zx}{z^2+x^2}+1 \ge \frac{4(xy+yz+zx)}{x^2+y^2+z^2}.$$

(Vasile C., 2014)

Solution. Write the inequality as $f_8(x,y,z) \ge 0$, where

$$f_8(x,y,z) = 2(p^2-2q)\sum yz(x^2+y^2)(x^2+z^2)+(p^2-6q)(x^2+y^2)(y^2+z^2)(z^2+x^2).$$

Since

$$y^2+z^2 = -x^2+p^2-2q,$$

the polynomial of degree eight $f_8(x,y,z)$ has the same highest polynomial $A(p,q)$ as

$$2(p^2-2q)\sum yz(-z^2)(-y^2)+(p^2-6q)(-z^2)(-x^2)(-y^2),$$

that is

$$A(p,q) = 6(p^2-2q)-(p^2-6q) = 5p^2-6q = 2(p^2-3q)+3p^2 > 0,$$

$$A(x+2,2x+1) = 5x^2+8x+14 > 0.$$

Since

$$\begin{aligned} f_8(x,1,1) &= 2(x^2+2)\left[(x^2+1)^2+4x(x^2+1)\right]+2(x^2-8x-2)(x^2+1)^2 \\ &= 4x^2(x^2+1)(x-1)^2, \end{aligned}$$

$$f_{0,\infty}(x) = \frac{4x^2(x-1)^4}{9(x+2)^2}, \qquad A(x+2,2x+1)f_{0,\infty}(x) = \frac{4x^2(x-1)^4(5x^2+8x+14)}{9(x+2)^2},$$

$$f_8(x,1,1)-A(x+2,2x+1)f_{0,\infty}(x) = \frac{4x^2(x-1)^2 f(x)}{9(x+2)^2},$$

where

$$\begin{aligned} f(x) &= 9(x^2+1)(x+2)^2-(x-1)^2(5x^2+8x+14) \\ &= (2x+1)(x^3+9x^2+6x+11), \end{aligned}$$

and

$$f_{\infty,-2}(x) = \frac{4(x-1)^4}{81}, \qquad A(x+2,2x+1)f_{\infty,-2}(x) = \frac{4(x-1)^4(5x^2+8x+14)}{81},$$

$$f_8(x,1,1)-A(x+2,2x+1)f_{\infty,-2}(x) = \frac{4(x-1)^2 g(x)}{81},$$

where

$$\begin{aligned} g(x) &= 81x^2(x^2+1)-(x-1)^2(5x^2+8x+14) \\ &= (2x+1)(19x^3-9x^2+24x-7), \end{aligned}$$

we apply Theorem 4 for

$$\eta \to \infty, \qquad \xi \to \infty,$$

$$\mathbb{I} = \left(\frac{-1}{2}, \infty\right), \qquad \mathbb{R} \setminus \mathbb{I} = \left(-\infty, \frac{-1}{2}\right],$$

and for

$$E_{\alpha,\beta} = f_{0,\infty}, \qquad F_{\gamma,\delta} = f_{\infty,-2}.$$

The condition (a), namely $f_8(x,1,1) - A(x+2, 2x+1) f_{0,\infty}(x) \geq 0$ for $x > \frac{-1}{2}$, is satisfied because

$$f(x) = (2x+1)[x^2(x+9) + (6x+11)] > 0.$$

The condition (b), namely $f_8(x,1,1) \geq A(x+2, 2x+1) f_{\infty,-2}(x)$ for $x \leq \frac{-1}{2}$, is satisfied because

$$g(x) = (2x+1)[19x^3 + (-9x^2) + 24x - 7] \geq 0.$$

The equality holds for $x = y = z$, and also for $x = 0$ and $y = z$ (or any cyclic permutation).

$\square$

Chapter 4

Highest Coefficient Cancellation Method for Symmetric Homogeneous Inequalities in Nonnegative Variables

4.1 Theoretical Basis

The Highest Coefficient Cancellation Method (HCC-Method) is especially applicable to symmetric homogeneous polynomial inequalities of six and eight degree. The main results in this section are based on the following Lemma (see P 3.57 in Volume 1):

Lemma. *If $x \le y \le z$ are nonnegative real numbers such that*

$$x + y + z = p, \qquad xy + yz + zx = q,$$

where p and q are given nonnegative real numbers satisfying $p^2 \ge 3q$, then the product $r = xyz$ is maximal when $x = y$, and is minimal when $y = z$ (for $p^2 \le 4q$) or $x = 0$ (for $p^2 \ge 4q$).

4.1.1. Inequalities of degree six

A symmetric and homogeneous polynomial of degree six can be written in the form

$$f_6(x,y,z) = A_1 \sum x^6 + A_2 \sum xy(x^4 + y^4) + A_3 \sum x^2y^2(x^2 + y^2) + A_4 \sum x^3y^3$$
$$+ A_5 xyz \sum x^3 + A_6 xyz \sum xy(x + y) + 3A_7 x^2y^2z^2,$$

where $A_1, \cdots, A_7$ are real coefficients. In terms of

$$p = x + y + z, \quad q = xy + yz + zx, \quad r = xyz,$$

it can be rewritten as

$$f_6(x,y,z) = Ar^2 + g_1(p,q)r + g_2(p,q),$$

where A is the *highest coefficient* of $f_6(x,y,z)$ (see section Theoretical Basis from Chapter 3), and $g_1(p,q)$ and $g_2(p,q)$ are polynomial functions of the form

$$g_1(p,q) = Bp^3 + Cpq, \quad g_2(p,q) = Dp^6 + Ep^4q + Fp^2q^2 + Gq^3,$$

where B,C,D,E,F,G are real coefficients.

The highest coefficients of the polynomials

$$\sum x^6, \quad \sum xy(x^4+y^4), \quad \sum x^2y^2(x^2+y^2), \quad \sum x^3y^3,$$
$$xyz\sum x^3, \quad xyz\sum xy(x+y)$$

are, respectively,

$$3, \quad -3, \quad -3, \quad 3, \quad 3, \quad -3.$$

As shown in Chapter 3, the polynomials

$$P_1(x,y,z) = \sum (A_1x^2 + A_2yz)(B_1x^2 + B_2yz)(C_1x^2 + C_2yz),$$

$$P_2(x,y,z) = \sum (A_1x^2 + A_2yz)(B_1y^2 + B_2zx)(C_1z^2 + C_2xy),$$

$$P_3(x,y,z) = (A_1x^2 + A_2yz)(A_1y^2 + A_2zx)(A_1z^2 + A_2xy)$$

and

$$P_4(x,y,z) = (x-y)^2(y-z)^2(z-x)^2$$

has the highest coefficients $P_1(1,1,1)$, $P_2(1,1,1)$, $P_3(1,1,1)$ and -27, respectively.

Based on Lemma above, Theorem 1 bellow gives for $A \le 0$ the necessary and sufficient conditions to have $f_6(x,y,z) \ge 0$ for all nonnegative real numbers x,y,z.

Theorem 1. *Let $f_6(x,y,z)$ be a sixth degree symmetric homogeneous polynomial having the highest coefficient $A \le 0$. The inequality $f_6(x,y,z) \ge 0$ holds for all nonnegative real numbers x,y,z if and only if $f_6(x,1,1) \ge 0$ and $f_6(0,y,z) \ge 0$ for all nonnegative real numbers x,y,z.*

Theorem 1 can be extended in the following form (see P 3.76 in Volume 1, page 173):

Theorem $1'$. *Let $f_6(x,y,z)$ be a sixth degree symmetric homogeneous polynomial having the highest coefficient $A \le 0$.*

(a) The inequality $f_6(x,y,z) \ge 0$ holds for all nonnegative real numbers x,y,z satisfying $p^2 \le 4q$ if and only if $f_6(x,1,1) \ge 0$ for $x \in [0,4]$;

(b) The inequality $f_6(x,y,z) \ge 0$ holds for all nonnegative real numbers x,y,z satisfying $p^2 \ge 4q$ if and only if $f_6(x,1,1) \ge 0$ for $x \ge 4$, and $f_6(0,y,z) \ge 0$ for $y,z \ge 0$.

***

Further, consider the inequality

$$f_6(x,y,z) \geq 0,$$

where x,y,z are nonnegative real numbers and $f_6(x,y,z)$ is a symmetric homogeneous polynomial of degree six with the highest coefficient $A > 0$. The highest coefficient cancellation method for proving such an inequality uses Theorem 1 and the following three ideas:

1) finding a nonnegative symmetric homogeneous function $\bar{f}_6(x,y,z)$ of the form

$$\bar{f}_6(x,y,z) = \left(r + A_1pq + A_2p^3 + A_3\frac{q^2}{p}\right)^2, \tag{4.1}$$

where A_1, A_2, A_3 are real numbers chosen such that

$$f_6(x,y,z) \geq A\bar{f}_6(x,y,z) \geq 0$$

for all nonnegative real numbers x,y,z;

2) seeing that the the difference $f_6(x,y,z) - A\bar{f}_6(x,y,z)$ has the highest coefficient equal to zero, therefore the inequality

$$f_6(x,y,z) \geq A\bar{f}_6(x,y,z)$$

holds for all nonnegative real numbers x,y,z if and only if it holds for $y = z = 1$ and for $x = 0$ (see Theorem 1);

3) treating successively the cases $p^2 < 4q$ and $p^2 \geq 4q$.

***

Let us define the following nonnegative functions:

$$f_{\alpha,\beta}(x) = \frac{4(x-1)^4(x-\alpha)^2(x-\beta)^2}{9(4-\alpha-\beta-2\alpha\beta)^2(x+2)^2}, \tag{4.2}$$

$$\hat{f}_{\alpha,\beta}(y,z) = \frac{[2(y+z)^4 - (10+\alpha+\beta)yz(y+z)^2 + 2(2+\alpha)(2+\beta)y^2z^2]^2}{9(4-\alpha-\beta-2\alpha\beta)^2(y+z)^2}, \tag{4.3}$$

$$g_{\alpha,\beta}(x) = \left[x + \alpha(x+2)(2x+1) + \beta\frac{(2x+1)^2}{x+2}\right]^2, \tag{4.4}$$

$$\hat{g}_{\alpha,\beta}(y,z) = \frac{y^2z^2}{(y+z)^2}\left[\alpha(y+z)^2 + \beta yz\right]^2, \tag{4.5}$$

$$h_{\alpha,\beta}(x) = \left[x + \alpha(x+2)(2x+1) + \beta(x+2)^3\right]^2, \tag{4.6}$$

$$\hat{h}_{\alpha,\beta}(y,z) = (y+z)^2\left[\alpha yz + \beta(y+z)^2\right]^2. \tag{4.7}$$

The functions $f_{\alpha,\beta}(x)$ and $\hat{f}_{\alpha,\beta}(y,z)$ have been derived by setting respectively $y = z = 1$ and $x = 0$ in the associated function

$$\bar{f}_6(x,y,z) = \left(r - \frac{b_1}{a_1}pq + \frac{2}{a_1}p^3 + \frac{c_1}{a_1}\cdot\frac{q^2}{p}\right)^2 \tag{4.8}$$

with

$$a_1 = 3(4-\alpha-\beta-2\alpha\beta), \quad b_1 = 10+\alpha+\beta, \quad c_1 = 2(2+\alpha)(2+\beta),$$

which satisfies $\bar{f}_6(1,1,1) = 0$, $\bar{f}_6(\alpha,1,1) = 0$, $\bar{f}_6(\beta,1,1) = 0$:

$$f_{\alpha,\beta}(x) = \bar{f}_6(x,1,1) = \left[x - \frac{b_1}{a_1}(x+2)(2x+1) + \frac{2}{a_1}(x+2)^3 + \frac{c_1}{a_1}\cdot\frac{(2x+1)^2}{x+2}\right]^2,$$

$$\hat{f}_{\alpha,\beta}(y,z) = \bar{f}_6(0,y,z) = \left[-\frac{b_1}{a_1}yz(y+z) + \frac{2}{a_1}(y+z)^3 + \frac{c_1}{a_1}\cdot\frac{y^2z^2}{y+z}\right)^2.$$

The functions $g_{\alpha,\beta}(x)$ and $\hat{g}_{\alpha,\beta}(y,z)$ have been derived by setting respectively $y = z = 1$ and $x = 0$ in the associated function

$$\bar{f}_6(x,y,z) = \left(r + \alpha pq + \beta\frac{q^2}{p}\right)^2.$$

The functions $h_{\alpha,\beta}(x)$ and $\hat{h}_{\alpha,\beta}(y,z)$ have been derived by setting respectively $y = z = 1$ and $x = 0$ in the associated function

$$\bar{f}_6(x,y,z) = \left(r + \alpha pq + \beta p^3\right)^2.$$

* * *

With regard to the functions $f_{\alpha,\beta}(x)$ and $\hat{f}_{\alpha,\beta}(x)$, we get the following expressions for $\beta = -2$, $\beta = -1$, $\beta = 0$, $\beta = 1$ and $\beta \to \infty$:

$$f_{\alpha,-2}(x) = \frac{4(x-1)^4(x-\alpha)^2}{81(2+\alpha)^2},$$

$$f_{\alpha,-1}(x) = \frac{4(x+1)^2(x-1)^4(x-\alpha)^2}{9(5+\alpha)^2(x+2)^2},$$

$$f_{\alpha,0}(x) = \frac{4x^2(x-1)^4(x-\alpha)^2}{9(4-\alpha)^2(x+2)^2},$$

$$f_{\alpha,1}(x) = \frac{4(x-1)^6(x-\alpha)^2}{81(1-\alpha)^2(x+2)^2},$$

$$f_{\alpha,\infty}(x) = \frac{4(x-1)^4(x-\alpha)^2}{9(1+2\alpha)^2(x+2)^2}$$

and

$$\hat{f}_{\alpha,-2}(y,z) = \frac{(y+z)^2[2(y+z)^2-(8+\alpha)yz]^2}{81(2+\alpha)^2},$$

$$\hat{f}_{\alpha,-1}(y,z) = \frac{[2(y+z)^4-(9+\alpha)yz(y+z)^2+2(2+\alpha)y^2z^2]^2}{9(5+\alpha)^2(y+z)^2},$$

$$\hat{f}_{\alpha,0}(y,z) = \frac{(y-z)^4[2(y+z)^2-(2+\alpha)yz]^2}{9(4-\alpha)^2(y+z)^2},$$

$$\hat{f}_{\alpha,1}(y,z) = \frac{[2(y+z)^4-(11+\alpha)yz(y+z)^2+6(2+\alpha)y^2z^2]^2}{81(1-\alpha)^2(y+z)^2},$$

$$\hat{f}_{\alpha,\infty}(y,z) = \frac{y^2z^2[(y+z)^2-2(2+\alpha)yz]^2}{9(1+2\alpha)^2(y+z)^2}.$$

In particular,

$$f_{\infty,-2}(x) = \frac{4(x-1)^4}{81}, \qquad \hat{f}_{\infty,-2}(y,z) = \frac{y^2z^2(y+z)^2}{81},$$

$$f_{-1,\infty}(x) = \frac{4(x-1)^4(x+1)^2}{9(x+2)^2}, \qquad \hat{f}_{-1,\infty}(y,z) = \frac{y^2z^2(y^2+z^2)^2}{9(y+z)^2},$$

$$f_{0,\infty}(x) = \frac{4x^2(x-1)^4}{9(x+2)^2}, \qquad \hat{f}_{0,\infty}(y,z) = \frac{y^2z^2(y-z)^4}{9(y+z)^2},$$

$$f_{1,\infty}(x) = \frac{4(x-1)^6}{81(x+2)^2}, \qquad \hat{f}_{1,\infty}(y,z) = \frac{y^2z^2(y^2+z^2-4yz)^2}{81(y+z)^2},$$

$$f_{\infty,\infty}(x) = \frac{(x-1)^4}{9(x+2)^2}, \qquad \hat{f}_{\infty,\infty}(y,z) = \frac{y^4z^4}{9(y+z)^2}.$$

With regard to the functions $g_{\alpha,\beta}(x)$ and $\hat{g}_{\alpha,\beta}(x)$, we get the following particular expressions:

$$g_{\alpha,0}(x) = [x+\alpha(x+2)(2x+1)]^2, \qquad \hat{g}_{\alpha,0}(y,z) = \alpha^2y^2z^2(y+z)^2,$$

$$g_{0,\beta}(x) = \left[x+\beta\frac{(2x+1)^2}{x+2}\right]^2, \qquad \hat{g}_{0,\beta}(y,z) = \frac{\beta^2y^4z^4}{(y+z)^2},$$

$$g_{0,0}(x) = x^2, \qquad \hat{g}_{0,0}(y,z) = 0.$$

With regard to the functions $h_{\alpha,\beta}(x)$ and $\hat{h}_{\alpha,\beta}(x)$, we have

$$h_{\alpha,0}(x) = [x + \alpha(x+2)(2x+1)]^2, \qquad \hat{h}_{\alpha,0}(y,z) = \alpha^2 y^2 z^2 (y+z)^2,$$

$$h_{0,\beta}(x) = \left[x + \beta(x+2)^3\right]^2, \qquad \hat{h}_{0,\beta}(y,z) = \beta^2 (y+z)^6,$$

$$h_{0,0}(x) = x^2, \qquad \hat{h}_{0,0}(y,z) = 0.$$

Notice that the relative degree of the rational functions $f_{\alpha,\beta}(x)$, $f_{\alpha,\infty}(x)$ and $f_{\infty,\infty}(x)$ are six, four and two, respectively. Also, $g_{\alpha,\beta}(x)$ and $g_{0,0}(x)$ have the relative degree equal to four and two, respectively, while $h_{\alpha,\beta}(x)$, $h_{\alpha,0}(x)$ and $h_{0,0}(x)$ have the relative degree equal to six, four and two, respectively.

* * *

The following theorem is useful to prove symmetric homogeneous polynomial inequalities of sixth degree in nonnegative real variables x, y, z and having the highest coefficient $A \ge 0$.

Theorem 2. *Let $f_6(x,y,z)$ be a symmetric homogeneous polynomial of degree six having the highest coefficient $A \ge 0$. The inequality $f_6(x,y,z) \ge 0$ holds for all $x,y,z \ge 0$ if there exist four real numbers α, β, γ and δ, and*

$$E_{\alpha,\beta} \in \{f_{\alpha,\beta}, g_{\alpha,\beta}, h_{\alpha,\beta}\},$$

$$F_{\gamma,\delta} \in \{f_{\gamma,\delta}, g_{\gamma,\delta}, h_{\gamma,\delta}\},$$

such that the following three conditions are satisfied:

(a) $f_6(x,1,1) \ge AE_{\alpha,\beta}(x)$ *for* $0 \le x \le 4$;

(b) $f_6(x,1,1) \ge AF_{\gamma,\delta}(x)$ *for* $x \ge 4$;

(c) $f_6(0,y,z) \ge A\hat{F}_{\gamma,\delta}(y,z)$ *for* $y,z \ge 0$.

Proof. Let

$$E_1(x,y,z) = \left(r + A_1 pq + A_2 p^3 + A_3 \frac{q^2}{p}\right)^2$$

and

$$F_1(x,y,z) = \left(r + B_1 pq + B_2 p^3 + B_3 \frac{q^2}{p}\right)^2$$

be the functions associated to $E_{\alpha,\beta}(x)$ and $F_{\gamma,\delta}(x)$, respectively; this means that

$$E_1(x,1,1) = E_{\alpha,\beta}(x),$$

$$F_1(x,1,1) = F_{\gamma,\delta}(x).$$

Let us denote

$$E_2(x,y,z) = f_6(x,y,z) - AE_1(x,y,z)$$

and

$$F_2(x,y,z) = f_6(x,y,z) - AF_1(x,y,z).$$

Since $AE_1(x,y,z) \geq 0$ and $AF_1(x,y,z) \geq 0$, the inequality $f_6(x,y,z) \geq 0$ holds for all nonnegative real x,y,z if

(a) $E_2(x,y,z) \geq 0$ for $p^2 \leq 4q$;

(b) $F_2(x,y,z) \geq 0$ for $p^2 \geq 4q$.

According to Theorem $1'$, since $E_2(x,y,z)$ and $F_2(x,y,z)$ has the highest coefficient zero, these conditions are satisfied if and only if

(a) $E_2(x,1,1) \geq 0$ for $(x+2)^2 \leq 4(2x+1)$;

(b) $F_2(x,1,1) \geq 0$ for $(x+2)^2 \geq 4(2x+1)$, and $F_2(0,y,z) \geq 0$ for $(y+z)^2 \geq 4yz$.

Since these conditions are equivalent to the condition (a), (b) and (c) in Theorem 2, the proof is completed.

For $F_{\gamma,\delta} = g_{0,0}$, when $\hat{F}_{\gamma,\delta} = 0$, we get

Corollary 1. *Let $f_6(x,y,z)$ be a symmetric homogeneous polynomial of degree six having the highest coefficient $A \geq 0$. The inequality $f_6(x,y,z) \geq 0$ holds for all $x,y,z \geq 0$ if there exist two real numbers α and β, and*

$$E_{\alpha,\beta} \in \{f_{\alpha,\beta}, g_{\alpha,\beta}, h_{\alpha,\beta}\},$$

such that the following three conditions are satisfied:

(a) $f_6(x,1,1) \geq AE_{\alpha,\beta}(x)$ *for* $0 \leq x \leq 4$;

(b) $f_6(x,1,1) \geq Ax^2$ *for* $x \geq 4$;

(c) $f_6(0,y,z) \geq 0$ *for* $y,z \geq 0$.

Remark 1. The function $\bar{f}_6(x,y,z)$ associated to $f_{\alpha,\beta}(x)$ (given by (4.8)) is zero for

$$(x,y,z) = (1,1,1), \quad (x,y,z) = (\alpha,1,1), \quad (x,y,z) = (\beta,1,1),$$

and also for $x = 0$ and

$$\frac{y}{z} + \frac{z}{y} = \frac{\alpha+\beta+2 \pm \sqrt{(\alpha+\beta+10)^2 - 16(\alpha+2)(\beta+2)}}{4}.$$

If $\beta = -2$, then the function associated to $f_{\alpha,-2}(x)$ has the expression

$$\bar{f}_6(x,y,z) = \left(r - \frac{\alpha+8}{9\alpha+18}pq + \frac{2}{9\alpha+18}p^3\right)^2, \tag{4.9}$$

and is zero for

$$(x,y,z) = (1,1,1), \quad (x,y,z) = (\alpha,1,1).$$

In addition, if $\alpha \to \infty$, then

$$\bar{f}_6(x,y,z) = \left(r - \frac{1}{9}pq\right)^2$$

is zero for

$$(x,y,z) = (1,1,1), \quad (x,y,z) = (1,0,0).$$

If $\beta \to \infty$, then

$$\bar{f}_6(x,y,z) = \left(r + \frac{1}{6\alpha+3}pq - \frac{2\alpha+4}{6\alpha+3} \cdot \frac{q^2}{p}\right)^2$$

is zero for

$$(x,y,z) = (1,1,1), \quad (x,y,z) = (\alpha,1,1), \quad (x,y,z) = (1,0,0).$$

If, in addition, $\alpha \to \infty$, then

$$\bar{f}_6(x,y,z) = \left(r - \frac{1}{3} \cdot \frac{q^2}{p}\right)^2$$

is zero for

$$(x,y,z) = (1,1,1), \quad (x,y,z) = (1,0,0).$$

Remark 2. If

$$f_6(x,1,1) = Af_{\gamma,-2}(x)$$

for all $x \in \mathbb{R}$, then there is $k \geq 0$ such that the following identity holds:

$$f_6(x,y,z) = A\bar{f}_6(x,y,z) + k(x-y)^2(y-z)^2(z-x)^2,$$

where, according to (4.9),

$$\bar{f}_6(x,y,z) = \left(r - \frac{\gamma+8}{9\gamma+18}pq + \frac{2}{9\gamma+18}p^3\right)^2.$$

In addition, if the coefficient of the product

$$(x-y)^2(y-z)^2(z-x)^2$$

in the inequality $f_6(x,y,z) \geq 0$ is the best possible, then

$$k = 0,$$

and the following identity holds:

$$f_6(x,y,z) = A\left(r - \frac{\gamma+8}{9\gamma+18}pq + \frac{2}{9\gamma+18}p^3\right)^2. \tag{4.10}$$

Remark 3. Theorem 2 is also valid for the case where the parameters α and β of the function $f_{\alpha,\beta}(x)$ are conjugate complex numbers. For example, if $k>0$ and

$$\alpha=\sqrt{-k}, \qquad \beta=-\sqrt{-k},$$

then, according to (4.2), we have

$$f_{\sqrt{-k},-\sqrt{-k}}=\frac{(x-1)^4(x^2+k)^2}{9(k-2)^2(x+2)^2}. \tag{4.11}$$

* * *

The following theorem is also useful to prove some inequalities $f_6(x,y,z)\ge 0$, where $f_6(x,y,z)$ is a symmetric homogeneous polynomial of degree six having the highest coefficient $A\ge 0$ and satisfying $f_6(1,1,1)=0$ and/or $f_6(0,1,1)=0$.

Theorem 3. *Let $f_6(x,y,z)$ be a symmetric homogeneous polynomial of degree six having the highest coefficient $A\ge 0$. The inequality $f_6(x,y,z)\ge 0$ holds for all $x,y,z\ge 0$ if*

(a) $f_6(x,1,1)\ge 0$ *for* $0\le x\le 1$;

(b) $f_6(x,1,1)\ge \dfrac{4Ax(x-1)^3}{27}$ *for* $x\ge 1$;

(c) $f_6(0,y,z)\ge 0$ *for* $y,z\ge 0$.

Proof. From

$$-27r^2+2(9pq-2p^3)r+p^2q^2-4q^3=(a-b)^2(b-c)^2(c-a)^2\ge 0,$$

we get

$$r\ge\frac{9pq-2p^3-2(p^2-3q)\sqrt{p^2-3q}}{27}.$$

Define the nonnegative function

$$E(x,y,z)=r\left[r-\frac{9pq-2p^3-2(p^2-3q)\sqrt{p^2-3q}}{27}\right],$$

which has the highest coefficient equal to 1. We have

$$E(0,y,z)=0,$$

$$E(x,1,1)=\begin{cases}0, & 0\le x\le 1\\[2mm] \dfrac{4x(x-1)^3}{27}, & x\ge 1\end{cases}.$$

Let us denote

$$E_1(x,y,z)=f_6(x,y,z)-AE(x,y,z).$$

Since $AE(x,y,z) \geq 0$, the inequality $f_6(x,y,z) \geq 0$ holds for all nonnegative real x,y,z if $E_1(x,y,z) \geq 0$ for $x,y,z \geq 0$. According to Theorem 1, because $E_1(x,y,z)$ has the highest coefficient zero, the inequality $E_1(x,y,z) \geq 0$ holds for all $x,y,z \geq 0$ if and only if $E_1(x,1,1) \geq 0$ and $E_1(0,y,z) \geq 0$ for all nonnegative numbers x,y,z. These conditions are equivalent to the condition (a), (b) and (c) in Theorem 3. Thus, the proof is completed.

Remark 4. Theorem 2 and Corollary 1 are also valid by replacing the condition (a) with the following two conditions:

(a_1) $f_6(x,1,1) \geq 0$ *for* $0 \leq x \leq 1$;

(a_2) $f_6(x,1,1) \geq \dfrac{4Ax(x-1)^3}{27}$ *for* $1 \leq x \leq 4$.

4.1.2. Inequalities of degree seven and eight

A symmetric and homogeneous polynomial of degree seven has the highest polynomial of the form

$$A(p,q) = \mu_1 p, \qquad \mu_1 \in \mathbb{R},$$

while a symmetric and homogeneous polynomial of degree eight has the highest polynomial of the form

$$A(p,q) = \mu_1 p^2 + \mu_2 q, \qquad \mu_1, \mu_2 \in \mathbb{R}.$$

Theorems 1, $1'$, 2 and 3 can be extended to these polynomials as follows:

Theorem 4. *Let* $f(x,y,z)$ *be a symmetric homogeneous polynomial of degree seven or eight which has the highest polynomial* $A(p,q)$*. The inequality* $f(x,y,z) \geq 0$ *holds for all nonnegative real numbers* x,y,z *satisfying* $A(p,q) \leq 0$ *if and only if* $f(x,1,1) \geq 0$ *and* $f(0,y,z) \geq 0$ *for all nonnegative real numbers* x,y,z *such that* $A(x+2,2x+1) \leq 0$ *and* $A(y+z,yz) \leq 0$.

Corollary 2. *Let* $f(x,y,z)$ *be a symmetric homogeneous polynomial of degree seven or eight having the highest polynomial* $A(p,q)$*. The inequality* $f(x,y,z) \geq 0$ *holds for all nonnegative real numbers* x,y,z *satisfying* $A(p,q) \leq 0$ *if* $f(x,1,1) \geq 0$ *and* $f(0,y,z) \geq 0$ *for all nonnegative real numbers* x,y,z.

Theorem 5. *Let* $f(x,y,z)$ *be a symmetric homogeneous polynomial of degree seven or eight having the highest polynomial* $A(p,q)$*. The inequality* $f_6(x,y,z) \geq 0$ *holds for all nonnegative* x,y,z *satisfying* $A(p,q) \geq 0$ *if there exist four real numbers* α*,* β*,* γ *and* δ*, and*

$$E_{\alpha,\beta} \in \{f_{\alpha,\beta}, g_{\alpha,\beta}, h_{\alpha,\beta}\},$$

$$F_{\gamma,\delta} \in \{f_{\gamma,\delta}, g_{\gamma,\delta}, h_{\gamma,\delta}\},$$

such that the following three conditions are satisfied:

(a) $f(x,1,1) \geq A(x+2,2x+1)E_{\alpha,\beta}(x)$ *for* $0 \leq x \leq 4$ *and* $A(x+2,2x+1) \geq 0$;

(b) $f(x,1,1) \geq A(x+2,2x+1)F_{\gamma,\delta}(x)$ *for* $x \geq 4$ *and* $A(x+2,2x+1) \geq 0$;

(c) $f(0,y,z) \geq A(y+z,yz)\hat{F}_{\gamma,\delta}(y,z)$ *for* $y,z \geq 0$ *and* $A(y+z,yz) \geq 0$.

For $F_{\gamma,\delta} = g_{0,0}$, we get the following corollaries:

Corollary 3. *Let* $f(x,y,z)$ *be a symmetric homogeneous polynomial of degree degree seven or eight having the highest polynomial* $A(p,q)$ *. The inequality* $f_6(x,y,z) \geq 0$ *holds for all nonnegative* x,y,z *satisfying* $A(p,q) \geq 0$ *if there exist two real numbers* α *and* β*, and*

$$E_{\alpha,\beta} \in \{f_{\alpha,\beta}, g_{\alpha,\beta}, h_{\alpha,\beta}\},$$

such that the following three conditions are satisfied:

(a) $f(x,1,1) \geq A(x+2,2x+1)E_{\alpha,\beta}(x)$ *for* $0 \leq x \leq 4$ *and* $A(x+2,2x+1) \geq 0$;

(b) $f(x,1,1) \geq A(x+2,2x+1)x^2$ *for* $x \geq 4$ *and* $A(x+2,2x+1) \geq 0$;

(c) $f(0,y,z) \geq 0$ *for* $y,z \geq 0$ *and* $A(y+z,yz) \geq 0$.

Theorem 6. *Let* $f(x,y,z)$ *be a symmetric homogeneous polynomial of degree seven or eight having the highest polynomial* $A(p,q)$*. The inequality* $f_6(x,y,z) \geq 0$ *holds for all nonnegative* x,y,z *satisfying* $A(p,q) \geq 0$ *if*

(a) $f(x,1,1) \geq 0$ *for* $0 \leq x \leq 1$ *and* $A(x+2,2x+1) \geq 0$;

(b) $f(x,1,1) \geq \dfrac{4A(x+2,2x+1)x(x-1)^3}{27}$ *for* $x \geq 1$ *and* $A(x+2,2x+1) \geq 0$;

(c) $f(0,y,z) \geq 0$ *for* $y,z \geq 0$ *and* $A(y+z,yz) \geq 0$.

4.2 Applications

4.1. If x, y, z are nonnegative real numbers, then

$$\sum x(y+z)(x-y)(x-z)(x-3y)(x-3z) \geq 0.$$

4.2. If x, y, z are nonnegative real numbers, then

$$\sum x(2x+y+z)(x-y)(x-z)(2x-11y)(2x-11z)+102(x-y)^2(y-z)^2(z-x)^2 \geq 0.$$

4.3. If x, y, z are nonnegative real numbers, then

$$\sum x(2x+y+z)(x-y)(x-z)(x-3y)(x-3z)+8(x-y)^2(y-z)^2(z-x)^2 \geq 0.$$

4.4. If x, y, z are nonnegative real numbers, no two of which are zero, then

$$\frac{1}{4x^2+yz}+\frac{1}{4y^2+zx}+\frac{1}{4z^2+xy} \geq \frac{9}{(x^2+4yz)+(y^2+4zx)+(z^2+4xy)}.$$

4.5. If x, y, z are nonnegative real numbers, then

$$\frac{2x^2-yz}{4x^2+y^2+z^2}+\frac{2y^2-zx}{4y^2+z^2+x^2}+\frac{2z^2-xy}{4z^2+x^2+y^2} \leq \frac{1}{2}.$$

4.6. If x, y, z are real numbers, then

$$\frac{21x^2+4yz}{x^2+2y^2+2z^2}+\frac{21y^2+4zx}{y^2+2z^2+2x^2}+\frac{21z^2+4xy}{z^2+2x^2+2y^2} \geq 15.$$

4.7. If x, y, z are real numbers, then

$$\frac{xy-yz+zx}{x^2+3y^2+3z^2}+\frac{yz-zx+xy}{y^2+3z^2+3x^2}+\frac{zx-xy+yz}{z^2+3x^2+3y^2} \leq \frac{3}{7}.$$

4.8. If x, y, z are nonnegative real numbers, then

$$\sum x^3(2x+y+z)(x-y)(x-z) \geq 13(x-y)^2(y-z)^2(z-x)^2.$$

4.9. Let x,y,z be nonnegative real numbers. If $k \le \frac{11}{2}$, then

$$\sum x(2x+y+z)(x-y)(x-z)(x-ky)(x-kz)+(7k-13)(x-y)^2(y-z)^2(z-x)^2 \ge 0.$$

4.10. If x,y,z are nonnegative real numbers, then

$$\sum x^2(x-y)(x-z) \ge \frac{12(x-y)^2(y-z)^2(z-x)^2}{(x+y+z)^2}.$$

4.11. Let x,y,z be nonnegative real numbers. If k is a real number, then

$$\sum x(y+z)(x-y)(x-z)(x-ky)(x-kz)+(k-3)(x-y)^2(y-z)^2(z-x)^2 \ge 0.$$

4.12. Let x,y,z be nonnegative real numbers. If $k \le 4$, then

$$\sum x^2(x-y)(x-z)(x-ky)(x-kz)+(3k-5)(x-y)^2(y-z)^2(z-x)^2 \ge 0.$$

4.13. Let x,y,z be nonnegative real numbers. If k is a real numbers, then

$$\sum yz(x-y)(x-z)(x-ky)(x-kz) \ge 0.$$

4.14. If x,y,z are nonnegative real numbers, then

$$\sum x^2(x-y)(x-z) \ge \frac{3(x-y)^2(y-z)^2(z-x)^2}{xy+yz+zx}.$$

4.15. If x,y,z are nonnegative real numbers, then

$$\sum (x-y)(x-z)(x+2y)(x+2z) \ge \frac{9(x-y)^2(y-z)^2(z-x)^2}{xy+yz+zx}.$$

4.16. If x,y,z are nonnegative real numbers, then

$$\sum (x-y)(x-z)(x-3y)(x-3z) \ge \frac{6(x-y)^2(y-z)^2(z-x)^2}{xy+yz+zx}.$$

4.17. Let x, y, z be nonnegative real numbers, and let

$$\alpha_k = \begin{cases} 3(1-k), & k \le 0 \\ 3+k, & k \ge 0 \end{cases}.$$

Then,

$$\sum (x-y)(x-z)(x-ky)(x-kz) \ge \frac{\alpha_k (x-y)^2(y-z)^2(z-x)^2}{xy+yz+zx}.$$

4.18. If x, y, z are nonnegative real numbers, then

$$\sum x^2(y+z)(x-y)(x-z) \ge \frac{4(x-y)^2(y-z)^2(z-x)^2}{x+y+z}.$$

4.19. Let x, y, z be nonnegative real numbers. If k is a real numbers, then

$$\sum (y+z)(x-y)(x-z)(x-ky)(x-kz) \ge \frac{(2+|k|)^2(x-y)^2(y-z)^2(z-x)^2}{x+y+z}.$$

4.20. If x, y, z are nonnegative real numbers, then

$$\sum x(x^2-y^2)(x^2-z^2) \ge \frac{12(x-y)^2(y-z)^2(z-x)^2}{x+y+z}.$$

4.21. Let x, y, z be nonnegative real numbers, and let

$$\alpha_k = \begin{cases} 4(k-2), & k \le 6 \\ \dfrac{(k+2)^2}{4}, & k \ge 6 \end{cases}.$$

Then,

$$\sum x(x-y)(x-z)(x-ky)(x-kz) + \frac{\alpha_k (x-y)^2(y-z)^2(z-x)^2}{x+y+z} \ge 0.$$

4.22. If x, y, z are nonnegative real numbers, then

$$\sum (x^2+yz)(x-y)(x-z) \ge \frac{5(x-y)^2(y-z)^2(z-x)^2}{xy+yz+zx}.$$

4.23. If x, y, z are nonnegative real numbers, then

$$\sum (4x^2 + yz)(x - y)(x - z) \geq \frac{16(x - y)^2(y - z)^2(z - x)^2}{xy + yz + zx}.$$

4.24. Let x, y, z be nonnegative real numbers. If $k \geq 0$, then

$$\sum (x^2 + kyz)(x - y)(x - z) \geq \frac{(3 + 2\sqrt{k}\,)(x - y)^2(y - z)^2(z - x)^2}{xy + yz + zx}.$$

4.25. If x, y, z are nonnegative real numbers, then

$$\sum (x^2 - yz)^2(x - y)(x - z) \geq 4(\sqrt{2} + 1)(x - y)^2(y - z)^2(z - x)^2.$$

4.26. If x, y, z are nonnegative real numbers, then

$$\sum \frac{1}{4x^2 + y^2 + z^2} \geq \frac{9}{4(x^2 + y^2 + z^2) + 2(xy + yz + zx)}.$$

4.27. If x, y, z are nonnegative real numbers, no two of which are zero, then

$$\frac{2}{x^2 + y^2} + \frac{2}{y^2 + z^2} + \frac{2}{z^2 + x^2} \geq \frac{45}{4(x^2 + y^2 + z^2) + xy + yz + zx}.$$

4.28. If x, y, z are nonnegative real numbers, no two of which are zero, then

$$\sum \frac{1}{2y^2 + yz + 2z^2} \geq \frac{18}{5(x^2 + y^2 + z^2 + xy + yz + zx)}.$$

4.29. If x, y, z are nonnegative real numbers, no two of which are zero, then

$$\sum \frac{x - (2 + \sqrt{2})(y + z)}{(y + z)^2} + \frac{9(3 + 2\sqrt{2})}{4(x + y + z)} \geq 0.$$

4.30. If x, y, z are nonnegative real numbers, no two of which are zero, then

$$\frac{6x - y - z}{y^2 + z^2} + \frac{6y - z - x}{z^2 + x^2} + \frac{6z - x - y}{x^2 + y^2} \geq \frac{18}{x + y + z}.$$

4.31. If x, y, z are nonnegative real numbers, no two of which are zero, then

$$\sum \frac{2x-3y-3z}{y^2+4yz+z^2} + \frac{6}{x+y+z} \geq 0.$$

4.32. If x, y, z are nonnegative real numbers, no two of which are zero, then

$$\sum \frac{7x+4y+4z}{4x^2+yz} \geq \frac{27}{x+y+z}.$$

4.33. If x, y, z are nonnegative real numbers, then

$$\sum \frac{9x-2y-2z}{7x^2+8yz} \leq \frac{3}{x+y+z}.$$

4.34. If x, y, z are nonnegative real numbers, then

$$\sum \frac{y+z}{7x^2+y^2+z^2} \geq \frac{2}{x+y+z}.$$

4.35. If x, y, z are nonnegative real numbers, then

$$\sum \frac{7x-2y-2z}{x^2+4y^2+4z^2} \geq \frac{3}{x+y+z}.$$

4.36. If x, y, z are nonnegative real numbers, no two of which are zero, then

$$\sum \frac{2x^2+yz}{y^2+z^2} \geq \frac{9}{2} + \frac{31(x-y)^2(y-z)^2(z-x)^2}{2(x^2+y^2)(y^2+z^2)(z^2+x^2)}.$$

4.37. If x, y, z are nonnegative real numbers, no two of which are zero, then

$$\sum \frac{2x^2-yz}{y^2-yz+z^2} \geq 3 + \frac{9(x-y)^2(y-z)^2(z-x)^2}{(x^2-xy+y^2)(y^2-yz+z^2)(z^2-zx+x^2)}.$$

4.38. If x, y, z are nonnegative real numbers, no two of which are zero, then

$$\sum \frac{xy-yz+zx}{y^2+z^2} \geq \frac{3}{2} + \frac{5(x-y)^2(y-z)^2(z-x)^2}{2(x^2+y^2)(y^2+z^2)(z^2+x^2)}.$$

4.39. If x, y, z are nonnegative real numbers, no two of which are zero, then

$$\sum \frac{xy - 2yz + zx}{y^2 - yz + z^2} \geq \frac{3(x-y)^2(y-z)^2(z-x)^2}{(x^2 - xy + y^2)(y^2 - yz + z^2)(z^2 - zx + x^2)}.$$

4.40. If x, y, z are nonnegative real numbers, no two of which are zero, then

$$\sum \frac{x + 3y + 3z}{(y + 2z)(2y + z)} \geq \frac{7(x + y + z)}{3(xy + yz + zx)}.$$

4.41. If x, y, z are nonnegative real numbers, no two of which are zero, then

$$\sum \frac{9x - 5y - 5z}{2y^2 - 3yz + 2z^2} + \frac{3(x + y + z)}{xy + yz + zx} \geq 0.$$

4.42. If x, y, z are nonnegative real numbers, no two of which are zero, then

$$\sum \frac{3x - y - z}{y^2 + z^2} \geq \frac{3(x + y + z)}{2(xy + yz + zx)}.$$

4.43. Let x, y, z be nonnegative real numbers, no two of which are zero. If

$$k \in [a, b], \qquad a = \frac{1 - \sqrt{17}}{2} \approx -1.56155, \qquad b = \frac{1 + \sqrt{17}}{2} \approx 2.56155\ ,$$

then

$$\sum \frac{(3-k)x + (k-1)(y+z)}{y^2 + kyz + z^2} \geq \frac{3(k+1)}{k+2} \cdot \frac{x + y + z}{xy + yz + zx}.$$

4.44. If x, y, z are nonnegative real numbers, no two of which are zero, then

$$\sum \frac{x + 13y + 13z}{y^2 + 4yz + z^2} \geq \frac{27(x + y + z)}{2(xy + yz + zx)}.$$

4.45. If x, y, z are nonnegative real numbers, no two of which are zero, then

$$\sum \frac{-x + y + z}{2x^2 + yz} \geq \frac{x + y + z}{xy + yz + zx}.$$

4.46. If x, y, z are nonnegative real numbers, no two of which are zero, then

$$\sum \frac{11x - 3y - 3z}{2x^2 + 3yz} \leq \frac{3(x + y + z)}{xy + yz + zx}.$$

4.47. If x, y, z are nonnegative real numbers, no two of which are zero, then

$$\frac{1}{2x^2 + yz} + \frac{1}{2y^2 + zx} + \frac{1}{2z^2 + xy} \geq \frac{1}{xy + yz + zx} + \frac{2}{x^2 + y^2 + z^2}.$$

4.48. If x, y, z are nonnegative real numbers, no two of which are zero, then

$$\frac{x(y+z)}{x^2 + 5yz} + \frac{y(z+x)}{y^2 + 5zx} + \frac{z(x+y)}{z^2 + 5xy} \leq \frac{x^2 + y^2 + z^2}{xy + yz + zx}.$$

4.49. If x, y, z are nonnegative real numbers, no two of which are zero, then

$$\frac{x(y+z)}{x^2 + yz} + \frac{y(z+x)}{y^2 + zx} + \frac{z(x+y)}{z^2 + xy} + 2 \geq \frac{15(xy + yz + zx)}{(x + y + z)^2}.$$

4.50. If x, y, z are nonnegative real numbers, no two of which are zero, then

$$\frac{x(y+z)}{x^2 + 2yz} + \frac{y(z+x)}{y^2 + 2zx} + \frac{z(x+y)}{z^2 + 2xy} \geq 1 + \frac{xy + yz + zx}{x^2 + y^2 + z^2}.$$

4.51. If x, y, z are nonnegative real numbers such that $xy + yz + zx = 3$, then

$$18\left(\frac{1}{x^2 + y^2} + \frac{1}{y^2 + z^2} + \frac{1}{z^2 + x^2}\right) + 5(x^2 + y^2 + z^2) \geq 42.$$

4.52. If x, y, z are nonnegative real numbers, then

$$\frac{2xy}{x^2 + y^2} + \frac{2yz}{y^2 + z^2} + \frac{2zx}{z^2 + x^2} + 7 \geq \frac{30(xy + yz + zx)}{(x + y + z)^2}.$$

4.53. If x, y, z are nonnegative real numbers, then

$$\frac{2xy}{(x+y)^2} + \frac{2yz}{(y+z)^2} + \frac{2zx}{(z+x)^2} + \frac{x^2 + y^2 + z^2}{xy + yz + zx} \geq \frac{5}{2}.$$

4.54. If x, y, z are nonnegative real numbers, no two of which are zero, then

$$\frac{2}{x^2 + y^2} + \frac{2}{y^2 + z^2} + \frac{2}{z^2 + x^2} \geq \frac{8}{x^2 + y^2 + z^2} + \frac{1}{xy + yz + zx}.$$

4.55. If x, y, z are nonnegative real numbers, then

$$\frac{2xy}{x^2 + y^2} + \frac{2yz}{y^2 + z^2} + \frac{2zx}{z^2 + x^2} + 1 \geq \frac{4(xy + yz + zx)}{x^2 + y^2 + z^2}.$$

4.3 Solutions

P 4.1. *If x, y, z are nonnegative real numbers, then*

$$\sum x(y+z)(x-y)(x-z)(x-3y)(x-3z) \geq 0.$$

(Vasile C., 2008)

Solution. Write the inequality as $f_6(x, y, z) \geq 0$, where

$$f_6(x, y, z) = \sum x(x+y)(x-y)(x-z)(x-3y)(x-3z).$$

Since

$$x(y+z) = x(p-x),$$
$$(x-y)(x-z) = x^2 + 2yz - q,$$
$$(x-3y)(x-3z) = x^2 + 12yz - 3q,$$

$f_6(x, y, z)$ has the same highest coefficient A as

$$P_1(x, y, z) = \sum -x^2(x^2+2yz)(x^2+12yz),$$

that is

$$A = P_1(1, 1, 1) = -3(1+2)(1+12) < 0.$$

By Theorem 1, we only need to show that $f_6(x, 1, 1) \geq 0$ and $f_6(0, y, z) \geq 0$ for $x, y, z \geq 0$. We have

$$f_6(x, 1, 1) = 2x(x-1)^2(x-3)^2 \geq 0,$$

$$f_6(0, y, z) = yz(y-z)^4 \geq 0.$$

The equality holds for $x = y = z$, for $x/3 = y = z$ (or any cyclic permutation), for $x = 0$ and $y = z$ (or any cyclic permutation), and for $y = z = 0$ (or any cyclic permutation).

Observation. Similarly, we can prove the following generalization:

- *Let x, y, z be nonnegative real numbers. If*

$$k \in (-\infty, -2-2\sqrt{3}] \cup [-2+2\sqrt{3}, \infty),$$

then

$$\sum x(y+z)(x-y)(x-z)(x-ky)(x-kz) + (k-3)(x-y)^2(y-z)^2(z-x)^2 \geq 0,$$

with equality for $x = y = z$, for $x/k = y = z$ (or any cyclic permutation) if $k \geq -2+2\sqrt{3}$, for $x = 0$ and $y = z$ (or any cyclic permutation), and for $y = z = 0$ (or any cyclic permutation).

We have

$$A = -3(1+2)(1+k+k^2) + (k-3)(-27) = 9(8-4k-k^2) \le 0,$$

$$f_6(x,1,1) = 2x(x-1)^2(x-k)^2 \ge 0,$$

$$f_6(0,y,z) = yz(y-z)^4 \ge 0.$$

□

P 4.2. *If x,y,z are nonnegative real numbers, then*

$$\sum x(2x+y+z)(x-y)(x-z)(2x-11y)(2x-11z)+102(x-y)^2(y-z)^2(z-x)^2 \ge 0.$$

(Vasile C., 2011)

Solution. Write the inequality as $f_6(x,y,z) \ge 0$, where

$$f_6(x,y,z) = f(x) + 102(x-y)^2(y-z)^2(z-x)^2,$$

$$f(x) = \sum x(2x+y+z)(x-y)(x-z)(2x-11y)(2x-11z).$$

Since

$$x(2x+y+z) = x(x+p),$$

$$(x-y)(x-z) = x^2 + 2yz - q,$$

$$(2x-11y)(2x-11z) = 4x^2 + 143yz - 22q,$$

$f(x,y,z)$ has the same highest coefficient A_1 as

$$P_1(x,y,z) = \sum x^2(x^2+2yz)(4x^2+143yz),$$

that is

$$A_1 = P_1(1,1,1) = 3(1+2)(147) = 1323.$$

Therefore, $f_6(x,y,z)$ has the highest coefficient

$$A = 1323 + 102(-27) = -1431.$$

By Theorem 1, we only need to show that $f_6(x,1,1) \ge 0$ and $f_6(0,y,z) \ge 0$ for $x,y,z \ge 0$. We have

$$f_6(x,1,1) = 2x(x+1)(x-1)^2(2x-11)^2 \ge 0,$$

$$\begin{aligned} f_6(0,y,z) &= (y-z)[8(y^5-z^5) - 40yz(y^3-z^3) - 22y^2z^2(y-z)] + 102y^2z^2(y-z)^2 \\ &= 8(y-z)^6 \ge 0. \end{aligned}$$

The equality holds for $x = y = z$, for $2x = 11y = 11z$ (or any cyclic permutation), and for $x = 0$ and $y = z$ (or any cyclic permutation).

Observation. Similarly, we can prove the following generalization:

- *Let x, y, z be nonnegative real numbers. If*

$$\alpha_k = \frac{4k^2 + 12k + 17}{8}, \qquad k \geq \frac{11}{2},$$

then

$$\sum x(2x + y + z)(x - y)(x - z)(x - ky)(x - kz) + \alpha_k (x - y)^2 (y - z)^2 (z - x)^2 \geq 0,$$

with equality for $x = y = z$, for $x/k = y = z$ (or any cyclic permutation), for $x = 0$ and $y = z$ (or any cyclic permutation), and for $x = 0$ and $\frac{y}{z} + \frac{z}{y} = \frac{2k-3}{4}$ (or any cyclic permutation).

Denoting the left side of the inequality by $f_6(x, y, z)$, we have

$$\begin{aligned}
A &= 3(1+2)(1+k+k^2) + \frac{4k^2 + 12k + 17}{8}(-27) \\
&= \frac{-9}{8}(4k^2 + 28k + 387) \leq 0,
\end{aligned}$$

$$f_6(x, 1, 1) = 2x(x+1)(x-1)^2(x-k)^2 \geq 0,$$

$$f_6(0, y, z) = \frac{1}{8}(y - z)^2[4y^2 + 4z^2 - (2k-3)yz]^2 \geq 0.$$

□

P 4.3. *If x, y, z are nonnegative real numbers, then*

$$\sum x(2x + y + z)(x - y)(x - z)(x - 3y)(x - 3z) + 8(x - y)^2 (y - z)^2 (z - x)^2 \geq 0.$$

(Vasile C., 2011)

Solution. Write the inequality as $f_6(x, y, z) \geq 0$, where

$$f_6(x, y, z) = f(x) + 8(x - y)^2(y - z)^2(z - x)^2,$$

$$f(x) = \sum x(2x + y + z)(x - y)(x - z)(x - 3y)(x - 3z).$$

Since

$$x(2x + y + z) = x(x + p),$$

$$(x - y)(x - z) = x^2 + 2yz - q,$$

$$(x - 3y)(x - 3z) = x^2 + 12yz - 3q,$$

$f(x,y,z)$ has the same highest coefficient A_1 as

$$P_1(x,y,z)=\sum x^2(x^2+2yz)(x^2+12yz),$$

that is

$$A_1=P_1(1,1,1)=3(1+2)(1+12)=117.$$

Therefore, $f_6(x,y,z)$ has the highest coefficient

$$A=117+8(-27)=-99.$$

By Theorem 1, we only need to show that $f_6(x,1,1)\geq 0$ and $f_6(0,y,z)\geq 0$ for $x,y,z\geq 0$. We have

$$f_6(x,1,1)=2x(x+1)(x-1)^2(x-3)^2\geq 0,$$

$$\begin{aligned}f_6(0,y,z)&=(y-z)[2(y^5-z^5)-5yz(y^3-z^3)-3y^2z^2(y-z)]+8y^2z^2(y-z)^2\\&=(y-z)^4(2y^2+2y^2+yz)\geq 0.\end{aligned}$$

The equality holds for $x=y=z$, for $x/3=y=z$ (or any cyclic permutation), and for $x=0$ and $y=z$ (or any cyclic permutation).

Observation. Similarly, we can prove the following generalization:

- *Let x,y,z be nonnegative real numbers. If*

$$10-2\sqrt{15}\leq k\leq\frac{11}{2},$$

then

$$\sum x(2x+y+z)(x-y)(x-z)(x-ky)(x-kz)+(7k-13)(x-y)^2(y-z)^2(z-x)^2\geq 0,$$

with equality for $x=y=z$, for $x/k=y=z$ (or any cyclic permutation), and for $x=0$ and $y=z$ (or any cyclic permutation).

We have

$$A=3(1+2)(1+k+k^2)+(7k-13)(-27)=9(k^2-20k+40)\leq 0,$$

$$f_6(x,1,1)=2x(x+1)(x-1)^2(x-k)^2\geq 0,$$

$$f_6(0,y,z)=(y-z)^4[2y^2+2z^2+(7-2k)yz]\geq 0.$$

□

P 4.4. *If x,y,z are nonnegative real numbers, no two of which are zero, then*

$$\frac{1}{4x^2+yz}+\frac{1}{4y^2+zx}+\frac{1}{4z^2+xy}\geq\frac{9}{(x^2+4yz)+(y^2+4zx)+(z^2+4xy)}.$$

(Vasile C., 2008)

Solution. Write the inequality as $f_6(x,y,z) \ge 0$, where

$$f_6(x,y,z) = [x^2+y^2+z^2+4(xy+yz+zx)]\sum(4y^2+zx)(4z^2+xy)-9\prod(4x^2+yz).$$

The highest coefficient A of $f_6(x,y,z)$ is equal to the highest coefficient of the product

$$P_3(x,y,z) = -9\prod(4x^2+yz),$$

that is

$$A = P_3(1,1,1) = -1125.$$

By Theorem 1, we only need to prove the original inequality for $y = z = 1$, and for $x = 0$. Thus, we need to show that

$$\frac{1}{4x^2+1} + \frac{2}{x+4} \ge \frac{9}{x^2+8x+6}$$

and

$$\frac{1}{yz} + \frac{1}{4y^2} + \frac{1}{4z^2} \ge \frac{9}{y^2+z^2+4yz}.$$

These inequalities are respectively equivalent to

$$x(x-1)^2(8x+45) \ge 0$$

and

$$(y-z)^2(y^2+z^2+10yz) \ge 0.$$

The equality holds for $x = y = z$, and for $x = 0$ and $y = z$ (or any cyclic permutation).

Observation. Similarly, we can prove the following generalization:

- *Let x,y,z be nonnegative real numbers, no two of which are zero. If*

$$1 \le k \le 3+\sqrt{5},$$

then

$$\sum\frac{1}{kx^2+yz} \ge \frac{9(k+2)}{(-k^2+6k-2)(x^2+y^2+z^2)+(2k^2-3k+4)(xy+yz+zx)},$$

with equality for $x = y = z$, and for $x = 0$ and $y = z$ (or any cyclic permutation).

□

P 4.5. *If x,y,z are nonnegative real numbers, then*

$$\frac{2x^2-yz}{4x^2+y^2+z^2} + \frac{2y^2-zx}{4y^2+z^2+x^2} + \frac{2z^2-xy}{4z^2+x^2+y^2} \le \frac{1}{2}.$$

(Vasile C., 2008)

Solution. Write the inequality as $f_6(x,y,z) \ge 0$, where

$$f_6(x,y,z) = \prod(4x^2+y^2+z^2) - 2\sum(2x^2-yz)(4y^2+z^2+x^2)(4z^2+x^2+y^2).$$

Since

$$4x^2+y^2+z^2 = 3x^2+p^2-2q,$$

the highest coefficient A of $f_6(x,y,z)$ is equal to the highest coefficient of

$$\prod(3x^2) - 2P_2(x,y,z),$$

where

$$P_2(x,y,z = \sum(2x^2-yz)(3y^2)(3z^2),$$

that is

$$A = 27 - 2P_2(1,1,1) = 27 - 54 = -27.$$

By Theorem 1, we only need to prove the original inequality for $y = z = 1$, and for $x = 0$. Thus, we need to show that

$$\frac{2x^2-1}{2(2x^2+1)} + \frac{2(2-x)}{x^2+5} \le \frac{1}{2}$$

and

$$\frac{-yz}{y^2+z^2} + \frac{2y^2}{4y^2+z^2} + \frac{2z^2}{4z^2+y^2} \le \frac{1}{2}.$$

These inequalities are respectively equivalent to

$$(x-1)^2(4x+1) \ge 0$$

and

$$\frac{-yz}{y^2+z^2} + 2\frac{(y^2+z^2)^2+6y^2z^2}{4(y^2+z^2)^2+9y^2z^2} \le \frac{1}{2}.$$

For $yz = 0$, the last inequality is an equality. For $yz \ne 0$, using the substitution

$$t = \frac{y^2+z^2}{yz}, \qquad t \ge 2,$$

the inequality becomes

$$\frac{-1}{t} + \frac{2(t^2+6)}{4t^2+9} \le \frac{1}{2},$$

$$8t^2 - 15t + 18 \ge 0,$$

$$t(8t-15) + 18 > 0.$$

The equality holds for $x = y = z$, and for $y = z = 0$ (or any cyclic permutation).

Observation. Similarly, we can prove the following generalization:

- *Let x, y, z be nonnegative real numbers. If $k \geq 2$, then*

$$\sum \frac{3kx^2 - 2(k-1)yz}{kx^2 + y^2 + z^2} \leq 3,$$

with equality for $x = y = z$, and for $y = z = 0$ (or any cyclic permutation). If $k = 2$, then the equality holds also for $x = 0$ and $y = z$ (or any cyclic permutation).

□

P 4.6. *If x, y, z are real numbers, then*

$$\frac{21x^2 + 4yz}{x^2 + 2y^2 + 2z^2} + \frac{21y^2 + 4zx}{y^2 + 2z^2 + 2x^2} + \frac{21z^2 + 4xy}{z^2 + 2x^2 + 2y^2} \geq 15.$$

(Vasile C., 2012)

Solution. Write the inequality as $f_6(x, y, z) \geq 0$, where

$$\begin{aligned} f_6(x, y, z) = &\sum (21x^2 + 4yz)(y^2 + 2z^2 + 2x^2)(z^2 + 2x^2 + 2y^2) \\ &- 15(x^2 + 2y^2 + 2z^2)(y^2 + 2z^2 + 2x^2)(z^2 + 2x^2 + 2y^2). \end{aligned}$$

Since

$$x^2 + 2y^2 + 2z^2 = -x^2 + 2(p^2 - 2q),$$

$f_6(x, y, z)$ has the same highest coefficient A as

$$P_2(x, y, z) - 15 \prod (-x^2),$$

where

$$P_2(x, y, z) = \sum (21x^2 + 4yz)(-y^2)(-z^2),$$

that is

$$A = P_2(1, 1, 1) + 15 = 75 + 15 = 90.$$

Since

$$\begin{aligned} f_6(x, 1, 1) = &(21x^2 + 4)(2x^2 + 3)^2 + 2(4x + 21)(2x^2 + 3)(x^2 + 4) \\ &- 15(2x^2 + 3)^2(x^2 + 4) \\ = &4x(2x^2 + 3)(x - 1)^2(3x + 8), \end{aligned}$$

we apply Theorem 3. The condition (a) in Theorem 3 is clearly satisfied.

The condition (b) in Theorem 3 is satisfied if $f_6(x, 1, 1) \geq \dfrac{4Ax(x-1)^3}{27}$ for $x \geq 1$. We have

$$\frac{4Ax(x-1)^3}{27} = \frac{40x(x-1)^3}{3},$$

$$f_6(x,1,1)-\frac{4Ax(x-1)^3}{27}=\frac{4x(x-1)^2f(x)}{3},$$

where

$$f(x)=3(2x^2+3)(3x+8)-10(x-1)\geq 15(3x+8)-10(x-1)=5(7x+26)>0.$$

The condition (c) in Theorem 3 is satisfied if the original inequality holds for $x=0$. Thus, we need to show that

$$\frac{2yz}{y^2+z^2}+42\frac{(y^2+z^2)^2-y^2z^2}{2(y^2+z^2)^2+y^2z^2}\geq 15.$$

Using the substitution

$$t=\frac{y^2+z^2}{yz},\qquad t\geq 2,$$

we may write the inequality as follows:

$$\frac{2}{t}+\frac{42(t^2-1)}{2t^2+1}\geq 15,$$

$$12t^3+4t^2-57t+2\geq 0,$$

$$(t-2)(12t^2+28t-1)\geq 0.$$

The equality holds for $x=y=z$, and for $x=0$ and $y=z$ (or any cyclic permutation).

Observation. Similarly, we can prove the following generalization:

- *Let x,y,z be nonnegative real numbers. If $0\leq k\leq 2$, $k\neq 1$, then*

$$\sum\frac{\frac{(1+k)(4-k)}{1-k}x^2+2yz}{kx^2+y^2+z^2}\geq\frac{3(3-k)}{1-k},$$

with equality for $x=y=z$, and for $x=0$ and $y=z$ (or any cyclic permutation). If $k=2$, then the equality holds also for $y=z=0$ (or any cyclic permutation).

□

P 4.7. *If x,y,z are real numbers, then*

$$\frac{xy-yz+zx}{x^2+3y^2+3z^2}+\frac{yz-zx+xy}{y^2+3z^2+3x^2}+\frac{zx-xy+yz}{z^2+3x^2+3y^2}\leq\frac{3}{7}.$$

(Vasile C., 2012)

Solution. Write the inequality as $f_6(x,y,z) \geq 0$, where

$$\begin{aligned} f_6(x,y,z) =& 3(x^2+3y^2+3z^2)(y^2+3z^2+3x^2)(z^2+3x^2+3y^2) \\ & -7\sum(xy-yz+zx)(y^2+3z^2+3x^2)(z^2+3x^2+3y^2). \end{aligned}$$

Since

$$x^2+3y^2+3z^2 = -2x^2+3(p^2-2q), \qquad xy-yz+zx = -2yz+q$$

$f_6(x,y,z)$ has the same highest coefficient A as

$$3(-2x^2)(-2y^2)(-2z^2) - 7\sum(-2yz)(-2y^2)(-2z^2),$$

that is

$$A = -24+168 = 144.$$

Since

$$\begin{aligned} f_6(x,1,1) =& 3(x^2+6)(3x^2+4)^2 - 7(2x-1)(3x^2+4)^2 - 14(x^2+6)(3x^2+4) \\ =& (3x^2+4)(9x^4-42x^3+73x^2-56x+16) \\ =& (3x^2+4)(x-1)^2(3x-4)^2, \end{aligned}$$

we apply Corollary 1 for

$$E_{\alpha,\beta}(x) = f_{4/3,-2}(x) = \frac{(x-1)^4(3x-4)^2}{81\cdot 25}.$$

The condition (a) of Corollary 1 is satisfied if $f_6(x,1,1) \geq Af_{4/3,-2}(x)$ for $x \in [0,4]$. We have

$$Af_{4/3,-2}(x) = \frac{16(x-1)^4(3x-4)^2}{225},$$

$$f_6(x,1,1) - Af_{4/3,-2}(x) = \frac{(x-1)^2(3x-4)^2 f(x)}{225},$$

where

$$f(x) = 225(3x^2+4) - 16(x-1)^2 > 16(3x^2+4) - 16(x-1)^2 = 16(2x^2+2x+3) > 0.$$

The condition (b) of Corollary 1 is satisfied if $f_6(x,1,1) \geq Ax^2$ for $x \geq 4$. We have

$$\begin{aligned} f_6(x,1,1) - Ax^2 &= (3x^2+4)(x-1)^2(3x-4)^2 - 144x^2 \\ &> 3x^2[(x-1)^2(3x-4)^2 - 48] > 3x^2(9\cdot 64 - 48) > 0. \end{aligned}$$

The condition (c) of Corollary 1 is satisfied if the original inequality holds for $x = 0$. Thus, we need to show that

$$\frac{-yz}{3(y^2+z^2)} + \frac{yz}{y^2+3z^2} + \frac{yz}{z^2+3y^2} \leq \frac{3}{7},$$

which can be rewritten as

$$\frac{-yz}{3(y^2+z^2)}+\frac{4yz(y^2+z^2)}{3(y^2+z^2)^2+4y^2z^2}\le\frac{3}{7}.$$

For the nontrivial case $yz\neq 0$, using the substitution

$$t=\frac{y^2+z^2}{yz},\qquad t\ge 2,$$

we may write the inequality as follows:

$$\frac{-1}{3t}+\frac{4t}{3t^2+4}\le\frac{3}{7},$$

$$\frac{9t^2-4}{3t(3t^2+4}\le\frac{3}{7}.$$

It suffices to show that

$$\frac{9t^2}{3t(3t^2+4)}\le\frac{3}{8},$$

which reduces to

$$(t-2)(3t-2)\ge 0.$$

The equality holds for $x=y=z$, and for $\frac{3x}{4}=y=z$ (or any cyclic permutation).

Observation. Similarly, we can prove the following generalization:

- *Let x,y,z be nonnegative real numbers, and let*

$$\alpha=\frac{3k(k-1)}{3k^2-1},\qquad \beta=\frac{3k^2-6k+1}{3k^2-1},$$

where $k>0$, $\quad k\neq\frac{1}{\sqrt{3}}$. Then,

$$\sum\frac{\alpha x(y+z)-\beta yz}{3k^2x^2+y^2+z^2}\le\frac{3(2\alpha-\beta)}{3k^2+2},$$

with equality for $x=y=z$, and for $x/\gamma=y=z$ (or any cyclic permutation), where

$$\gamma=k+\frac{2}{3k}-1.$$

□

P 4.8. *If x, y, z are nonnegative real numbers, then*

$$\sum x^3(2x+y+z)(x-y)(x-z) \ge 13(x-y)^2(y-z)^2(z-x)^2.$$

(Vasile C., 2011)

Solution. Write the inequality as $f_6(x,y,z) \ge 0$, where

$$f_6(x,y,z) = f(x) - 13(x-y)^2(y-z)^2(z-x)^2,$$

$$f(x) = \sum x^3(2x+y+z)(x-y)(x-z).$$

Since

$$2x+y+z) = x+p, \qquad (x-y)(x-z) = x^2+2yz-q,$$

$f(x,y,z)$ has the same highest coefficient A_1 as

$$P_1(x,y,z) = \sum x^4(x^2+2yz),$$

that is

$$A_1 = P_1(1,1,1) = 3(1+2) = 9.$$

Therefore, $f_6(x,y,z)$ has the highest coefficient

$$A = 9 - 13(-27) = 360.$$

Since

$$f_6(x,1,1) = 2x^3(x+1)(x-1)^2,$$

we apply Corollary 1 for

$$E_{\alpha,\beta}(x) = f_{0,0}(x) = \frac{x^4(x-1)^4}{36(x+2)^2}.$$

The condition (a) of Corollary 1 is satisfied if $f_6(x,1,1) \ge Af_{0,0}(x)$ for $x \in [0,4]$. We have

$$\begin{aligned} f_6(x,1,1) - Af_{0,0}(x) &= 2x^3(x+1)(x-1)^2 - \frac{10x^4(x-1)^4}{(x+2)^2} \\ &= \frac{2x^3(x-1)^2(4-x)(4x^2+x+1)}{(x+2)^2} \ge 0. \end{aligned}$$

The condition (b) of Corollary 1 is satisfied if $f_6(x,1,1) \ge Ax^2$ for $x \ge 4$. We have

$$\begin{aligned} f_6(x,1,1) - Ax^2 &= 2x^2[x(x+1)(x-1)^2 - 180] \\ &\ge 2x^2[4 \cdot 5 \cdot 9 - 180] = 0. \end{aligned}$$

The condition (c) of Corollary 1 is satisfied if $f_6(0,y,z) \ge 0$ for $y,z \ge 0$. We have

$$\begin{aligned} f_6(0,y,z) &= (y-z)[2(y^5-z^5)+yz(y^3-z^3)]-13y^2z^2(y-z)^2 \\ &= (y-z)^2[2(y^2+z^2)^2+3yz(y^2+z^2)-14y^2z^2] \\ &= (y-z)^4(2y^2+2z^2+7yz) \ge 0. \end{aligned}$$

The equality holds for $x=y=z$, and for $x=0$ and $y=z$ (or any cyclic permutation).

□

P 4.9. *Let x,y,z be nonnegative real numbers. If $k \le \frac{11}{2}$, then*

$$\sum x(2x+y+z)(x-y)(x-z)(x-ky)(x-kz)+(7k-13)(x-y)^2(y-z)^2(z-x)^2 \ge 0.$$

(Vasile C., 2011)

Solution. Write the inequality as $f_6(x,y,z) \ge 0$, where

$$f_6(x,y,z) = f(x)+(7k-13)(x-y)^2(y-z)^2(z-x)^2,$$

$$f(x) = \sum x(2x+y+z)(x-y)(x-z)(x-ky)(x-kz).$$

Since $2x+y+z = x+p$,

$$(x-y)(x-z) = x^2+2yz-q, \qquad (x-ky)(x-kz) = x^2+(k+k^2)yz-kq,$$

$f(x,y,z)$ has the same highest coefficient A_1 as

$$P_1(x,y,z) = \sum x^2(x^2+2yz)[x^2+(k+k^2)yz],$$

that is

$$A_1 = P_1(1,1,1) = 9(k^2+k+1).$$

Therefore, $f_6(x,y,z)$ has the highest coefficient

$$A = 9(k^2+k+1)+(7k-13)(-27) = 9(k^2-20k+40).$$

We have

$$f_6(x,1,1) = 2x(x+1)(x-1)^2(x-k)^2 \ge 0,$$

$$\begin{aligned} f_6(0,y,z) =& (y-z)[2(y^5-z^5)-(2k-1)yz(y^3-z^3)-ky^2z^2(y-z)] \\ &+(7k-13)y^2z^2(y-z)^2 \\ =& (y-z)^4[2y^2+2y^2+(7-2k)yz] \\ =& (y-z)^4[2(y-z)^2+(11-2k)yz] \ge 0 \end{aligned}$$

Case 1: $k \in [10-2\sqrt{15}, 11/2]$. Since $A \le 0$, we only need to show that $f_6(x,1,1) \ge 0$ and $f_6(0,y,z) \ge 0$ for $x,y,z \ge 0$ (Theorem 1). Both conditions are satisfied.

Case 2: $k < 10-2\sqrt{15}$. We apply Corollary 1 for

$$E_{\alpha,\beta}(x) = f_{k,0}(x) = \frac{4x^2(x-1)^4(x-k)^2}{9(4-k)^2(x+2)^2}.$$

The condition (a) in Corollary 1 is satisfied if $f_6(x,1,1) \ge Af_{k,0}(x)$ for $x \in [0,4]$. We have

$$Af_{k,0}(x) = \frac{4(k^2-20k+40)x^2(x-1)^4(x-k)^2}{(4-k)^2(x+2)^2},$$

$$f_6(x,1,1) - Af_{k,0}(x) = \frac{2x(x-1)^2(x-k)^2 g(x)}{(4-k)^2(x+2)^2},$$

where

$$g(x) = (4-k)^2(x+1)(x+2)^2 - 2(k^2-20k+40)x(x-1)^2.$$

Since

$$(x+1)(x+2)^2 - 5x(x-1)^2 = (4-x)(1+x+4x^2) \ge 0,$$

we get

$$g(x) \ge 5(4-k)^2x(x-1)^2 - 2(k^2-20k+40)x(x-1)^2 = 3k^2x(x-1)^2 \ge 0.$$

The condition (b) in Corollary 1 is satisfied if $f_6(x,1,1) \ge Ax^2$ for $x \ge 4$. We have

$$f_6(x,1,1) - Ax^2 = xg(x),$$

where

$$g(x) = 2(x+1)(x-1)^2(x-k)^2 - 9(k^2-20k+40)x.$$

Since

$$4(x-1)^2 - 9x = (x-4)(4x-1) \ge 0,$$

we get

$$\begin{aligned}
2g(x) &\ge 9x(x+1)(x-k)^2 - 18(k^2-20k+40)x \\
&= 9x[(x+1)(x-k)^2 - 2(k^2-20k+40)] \\
&\ge 9x[5(4-k)^2 - 2(k^2-20k+40)] = 18k^2x \ge 0.
\end{aligned}$$

The condition (c) in Corollary 1 is satisfied because $f_6(0,y,z) \ge 0$ for $y,z \ge 0$.

The equality holds for $x = y = z$, and for $x/k = y = z$ (or any cyclic permutation) if $k \ne 0$. If $k = 0$, then the equality holds also for $x = 0$ and $y = z$.

Observation 1. Having in view the inequality in Observation from P 4.2 and the inequality in P 4.9, we can formulate the following statement:

- *Let x,y,z be nonnegative real numbers. If*

$$\alpha_k = \begin{cases} 7k-13, & k \le \dfrac{11}{2} \\ \dfrac{4k^2+12k+17}{8}, & k \ge \dfrac{11}{2} \end{cases},$$

then

$$\sum x(2x+y+z)(x-y)(x-z)(x-ky)(x-kz) + \alpha_k (x-y)^2(y-z)^2(z-x)^2 \ge 0.$$

Observation 2. The coefficient α_k in the inequality from Observation 1 is the best possible.

Setting $x=0$, the inequality becomes

$$(y-z)[2(y^5-z^5)-(2k-1)yz(y^3-z^3)-ky^2z^2(y-z)] + \alpha_k y^2z^2(y-z)^2,$$

$$(y-z)^2 f(y,z) \ge 0,$$

where

$$f(y,z) = 2(y^2+z^2)^2 - (2k-3)yz(y^2+z^2) + (\alpha_k - 3k-1)y^2z^2.$$

From the necessary condition $f(1,1) \ge 0$, we get $\alpha_k \ge 7k-13$.

Assume now that $k \ge 11/2$. Since $(2k-3)/4 \ge 2$, there exist $y>0$ and $z>0$ such that

$$y^2+z^2 = \frac{2k-3}{4}yz.$$

For this case, we have

$$\begin{aligned} \frac{f(y,z)}{y^2z^2} &= \frac{1}{8}(2k-3)^2 - \frac{1}{4}(2k-3)^2 + (\alpha_k - 3k-1) \\ &= \alpha_k - \frac{4k^2+12k+17}{8}. \end{aligned}$$

From the necessary condition $f(y,z) \ge 0$, we get $\alpha_k \ge \dfrac{4k^2+12k+17}{8}$.

□

P 4.10. *If x,y,z are nonnegative real numbers, then*

$$\sum x^2(x-y)(x-z) \ge \frac{12(x-y)^2(y-z)^2(z-x)^2}{(x+y+z)^2}.$$

(Vasile C., 2010)

Solution. Write the inequality as $f_6(x,y,z) \ge 0$, where

$$f_6(x,y,z) = (x+y+z)^2 \sum x^2(x-y)(x-z) - 12(x-y)^2(y-z)^2(z-x)^2$$

has the highest coefficient

$$A = -12(-27) = 324.$$

Since

$$f_6(x,1,1) = x^2(x+2)^2(x-1)^2,$$

we apply Corollary 1 for

$$E_{\alpha,\beta}(x) = f_{0,-2}(x) = \frac{x^2(x-1)^4}{81}.$$

Since

$$Af_{0,-2}(x) = 4x^2(x-1)^4,$$

$$f_6(x,1,1) - Af_{0,-2}(x) = 3x^3(x-1)^2(4-x) \ge 0,$$

the condition (a) from Corollary 1 is satisfied.

The condition (b) from Corollary 1 is satisfied if $f_6(x,1,1) \ge Ax^2$ for $x \ge 4$. This is true since

$$f_6(x,1,1) - Ax^2 = x^2\left[(x+2)^2(x-1)^2 - 324\right] \ge x^2(36 \cdot 9 - 324) = 0.$$

The condition (c) from Corollary 1 is also satisfied because

$$\begin{aligned} f_6(0,y,z) &= (y+z)^2(y-z)^2(y^2+z^2+yz) - 12y^2z^2(y-z)^2 \\ &= (y-z)^4(y^2+z^2+5yz) \ge 0. \end{aligned}$$

The equality occurs for $x = y = z$, and for $x = 0$ and $y = z$ (or any cyclic permutation).

□

P 4.11. *Let x,y,z be nonnegative real numbers. If k is a real number, then*

$$\sum x(y+z)(x-y)(x-z)(x-ky)(x-kz) + (k-3)(x-y)^2(y-z)^2(z-x)^2 \ge 0.$$

(Vasile C., 2011)

Solution. Write the inequality as $f_6(x,y,z) \ge 0$, where

$$f_6(x,y,z) = f(x) + (k-3)(x-y)^2(y-z)^2(z-x)^2,$$

$$f(x) = \sum x(y+z)(x-y)(x-z)(x-ky)(x-kz).$$

Since $y+z = -x+p$,

$$(x-y)(x-z) = x^2 + 2yz - q, \qquad (x-ky)(x-kz) = x^2 + (k+k^2)yz - kq,$$

$f(x,y,z)$ has the same highest coefficient A_1 as

$$P_1(x,y,z) = -\sum x^2(x^2+2yz)[x^2+(k+k^2)yz],$$

that is

$$A_1 = P_1(1,1,1) = -9(k^2+k+1).$$

Therefore, $f_6(x,y,z)$ has the highest coefficient

$$A = -9(k^2+k+1) + (k-3)(-27) = 9(8-4k-k^2).$$

We have

$$f_6(x,1,1) = 2x(x-1)^2(x-k)^2 \ge 0,$$

$$\begin{aligned} f_6(0,y,z) =& yz(y-z)[y^3-z^3-kyz(y-z)] + (k-3)y^2z^2(y-z)^2] \\ =& yz(y-z)^4 \ge 0 \end{aligned}$$

Case 1: $k \in (-\infty, -2-2\sqrt{3}] \cup [-2+2\sqrt{3}, \infty)$. Since $A \le 0$, we only need to show that $f_6(x,1,1) \ge 0$ and $f_6(0,y,z) \ge 0$ for $x,y,z \ge 0$ (Theorem 1). Both conditions are satisfied.

Case 2: $k \in (-2-2\sqrt{3}, -2+2\sqrt{3})$. We apply Corollary 1 for

$$E_{\alpha,\beta}(x) = f_{k,0}(x) = \frac{4x^2(x-1)^4(x-k)^2}{9(4-k)^2(x+2)^2}.$$

The condition (a) of Corollary 1 is satisfied if $f_6(x,1,1) \ge Af_{k,0}(x)$ for $x \in [0,4]$. We have

$$Af_{k,0}(x) = \frac{4(8-4k-k^2)x^2(x-1)^4(x-k)^2}{(4-k)^2(x+2)^2},$$

$$f_6(x,1,1) - Af_{k,0}(x) = \frac{2x(x-1)^2(x-k)^2 g(x)}{(4-k)^2(x+2)^2},$$

where

$$g(x) = (4-k)^2(x+2)^2 - 2(8-4k-k^2)x(x-1)^2.$$

Since

$$(x+2)^2 - x(x-1)^2 = (4-x)(1+x+x^2) \ge 0,$$

we get

$$g(x) \ge (4-k)^2 x(x-1)^2 - 2(8-4k-k^2)x(x-1)^2 = 3k^2 x(x-1)^2 \ge 0.$$

The condition (b) of Corollary 1 is satisfied if $f_6(x,1,1) \ge Ax^2$ for $x \ge 4$. We have

$$f_6(x,1,1) - Ax^2) = xg(x),$$

where

$$g(x) = 2(x-1)^2(x-k)^2 - 9(8-4k-k^2)x.$$

Since

$$4(x-1)^2 - 9x = (x-4)(4x-1) \ge 0,$$

we get

$$\begin{aligned} 2g(x) &\ge 9x(x-k)^2 - 18(8-4k-k^2)x \\ &= 9x[(x-k)^2 - 2(8-4k-k^2)] \\ &\ge 9x[(4-k)^2 - 2(8-4k-k^2)] = 27k^2 x \ge 0. \end{aligned}$$

The condition (c) of Corollary 1 is satisfied since $f_6(0,y,z) \ge 0$.

The equality holds for $x = y = z$, and for $x/k = y = z$ (or any cyclic permutation) if $k \ne 0$, for $x = 0$ and $y = z$ (or any cyclic permutation, and for $y = z$ (or any cyclic permutation.

Observation. The coefficient of the product $(x-y)^2(y-z)^2(z-x)^2$ is the best possible.

Setting $x = 0$, the inequality

$$\sum x(y+z)(x-y)(x-z)(x-ky)(x-kz) + \alpha_k (x-y)^2(y-z)^2(z-x)^2 \ge 0$$

becomes

$$yz(y-z)[y^3 - z^3 - kyz(y-z)] + \alpha_k y^2 z^2 (y-z)^2] \ge 0,$$
$$yz(y-z)^2[y^2 + z^2 + (1-k+\alpha_k)yz] \ge 0.$$

The necessary condition

$$y^2 + z^2 + (1-k+\alpha_k)yz \ge 0$$

leads to $\alpha_k \ge k-3$ for $y = z = 1$.

□

P 4.12. *Let x, y, z be nonnegative real numbers. If $k \le 4$, then*

$$\sum x^2(x-y)(x-z)(x-ky)(x-kz) + (3k-5)(x-y)^2(y-z)^2(z-x)^2 \ge 0.$$

(Vasile C., 2011)

Solution. Denote

$$f(x,y,z)=\sum x^2(x-y)(x-z)(x-ky)(x-kz),$$

and write the desired inequality as $f_6(x,y,z)\ge 0$, where

$$f_6(x,y,z)=f(x,y,z)+(3k-5)(x-y)^2(y-z)^2(z-x)^2.$$

From

$$(x-y)(x-z)=x^2+2yz-q, \qquad (x-ky)(x-kz)=x^2+(k+k^2)yz-kq,$$

it follows that $f(x,y,z)$ has the same highest coefficient A_1 as

$$P_1(x,y,z)=\sum x^2(x^2+2yz)[x^2+(k+k^2)yz],$$

that is,

$$A_1=P_1(1,1,1)=9(k^2+k+1).$$

Since the highest coefficient of the product $(x-y)^2(y-z)^2(z-x)^2$ is equal to -27, $f_6(x,y,z)$ has the highest coefficient

$$A=A_1+(3k-5)(-27)=9(4-k)^2.$$

On the other hand, we have

$$f_6(x,1,1)=x^2(x-1)^2(x-k)^2,$$

$$\begin{aligned}
f_6(0,y,z)&=(y-z)[y^5-z^5-kyz(y^3-z^3)]+(3k-5)y^2z^2(y-z)^2\\
&=(y-z)^2[(y^2+z^2)^2-(k-1)yz(y^2+z^2)+2(k-3)y^2z^2]\\
&=(y-z)^4[y^2+z^2-(k-3)yz]\\
&=(y-z)^2[(y-z)^2+(5-k)yz]\ge 0.
\end{aligned}$$

For $k=4$, we have $A=0$. According to Theorem 1, the desired inequality is true since $f_6(x,1,1)\ge 0$ and $f_6(0,y,z)\ge 0$ for all $x,y,z\ge 0$.

For $k<4$, we have $A>0$. To prove the desired inequality, we will apply Corollary 1 for

$$E_{\alpha,\beta}(x)=f_{k,0}(x)=\frac{4x^2(x-1)^4(x-k)^2}{9(4-k)^2(x+2)^2}.$$

The condition (a) in Corollary 1 is satisfied if $f_6(x,1,1)\ge Af_{k,0}(x)$ for $x\in[0,4]$. We have

$$Af_{k,0}(x)=\frac{4x^2(x-1)^4(x-k)^2}{(x+2)^2},$$

$$f_6(x,1,1)-Af_{k,0}(x)=\frac{3x^3(x-1)^2(x-k)^2(4-x)}{(x+2)^2}\ge 0.$$

The condition (b) in Corollary 1 is satisfied if $f_6(x,1,1) \ge Ax^2$ for $x \ge 4$. We have

$$f_6(x,1,1) - Ax^2 = x^2 g(x),$$

where

$$g(x) = (x-1)^2(x-k)^2 - 9(4-k)^2.$$

Since

$$(x-1)^2 \ge 9,$$

we get

$$g(x) \ge 9[(x-k)^2 - (4-k)^2] \ge 9[(4-k)^2 - (4-k)^2] = 0.$$

The condition (c) in Corollary 1 is satisfied because $f_6(0,y,z) \ge 0$ for $y,z \ge 0$.

The equality occurs for $x = y = z$, for $x = 0$ and $y = z$ (or any cyclic permutation), and for $x/k = y = z$ (or any cyclic permutation) if $k \ne 0$.

Observation 1. The coefficient of the product $(x-y)^2(y-z)^2(z-x)^2$ is the best possible.

Setting $x = 0$, the inequality

$$\sum x^2(x-y)(x-z)(x-ky)(x-kz) + \alpha_k(x-y)^2(y-z)^2(z-x)^2 \ge 0$$

becomes as follows:

$$(y-z)[y^5 - z^5 - kyz(y^3 - z^3)] + \alpha_k y^2 z^2 (y-z)^2 \ge 0,$$

$$(y-z)^2 f(y,z) \ge 0,$$

where

$$f(y,z) = y^4 + z^4 - (k-1)yz(y^2+z^2) + (\alpha_k - k + 1)y^2z^2 \ge 0.$$

For $y = z = 1$, the necessary condition $f(1,1) \ge 0$ involves $\alpha_k \ge 3k - 5$.

Observation 2. For $k = 4$, the inequality turns into

$$\sum x^2(x-y)(x-z)(x-4y)(x-4z) + 7(x-y)^2(y-z)^2(z-x)^2 \ge 0,$$

which is equivalent to

$$(x^2+y^2+z^2-xy-yz-zx)(x^2+y^2+z^2-2xy-2yz-2zx)^2 \ge 0.$$

The equality occurs when $x = y = z$, and when $\sqrt{x} = \sqrt{y} + \sqrt{z}$ (or any cyclic permutation).

Observation 3. The inequality can be extended for $k \ge 4$, as follows:

$$\sum x^2(x-y)(x-z)(x-ky)(x-kz) \ge \frac{20+12k-4k^2-k^4}{4(k-1)^2}(x-y)^2(y-z)^2(z-x)^2.$$

Actually, this inequality is valid for all real x, y, z (see P 3.49 from chapter 3).

Observation 4. Substituting $k-1$ for k in P 4.12, and using then the identity

$$\sum x^2(x-y)(x-z)[x-(k-1)y](x-(k-1)z] =$$
$$= \sum x^2(x-y)(x-z)(x-ky+z)(x-kz+y) + k(x-y)^2(y-z)^2(z-x)^2,$$

we get the following equivalent statement:

- *Let x, y, z be nonnegative real numbers. If $k \le 5$, then*

$$\sum x^2(x-y)(x-z)(x-ky+z)(x-kz+y) \ge 4(2-k)(x-y)^2(y-z)^2(z-x)^2,$$

with equality for $x = y = z$, for $x = 0$ and $y = z$ (or any cyclic permutation), and for $x/(k-1) = y = z$ (or any cyclic permutation) if $k \ne 1$.

□

P 4.13. *Let x, y, z be nonnegative real numbers. If k is a real numbers, then*

$$\sum yz(x-y)(x-z)(x-ky)(x-kz) \ge 0.$$

(Vasile C., 2010)

Solution. If one of x, y, z is zero, the inequality is trivial. On the other hand, the inequality remains unchanged by replacing x, y, z and k with $1/x, 1/y, 1/z$ and $1/k$, respectively. Therefore, it suffices to consider that

$$k \in (-\infty, -1] \cup [0, 1].$$

Write the inequality as $f_6(x, y, z) \ge 0$, where

$$f_6(x, y, z) = \sum yz(x-y)(x-z)(x-ky)(x-kz).$$

Since

$$(x-y)(x-z) = x^2 + 2yz - q, \qquad (x-ky)(x-kz) = x^2 + (k^2+k)yz - kq,$$

$f_6(x, y, z)$ has the same highest coefficient as

$$P_1(x, y, z) = \sum yz(x^2+2yz)[x^2+(k^2+k)yz],$$

that is,

$$A = P_1(1, 1, 1) = 9(k^2+k+1).$$

On the other hand,

$$f_6(x, 1, 1) = (x-1)^2(x-k)^2,$$

$$f_6(0,y,z) = k^2y^3z^3.$$

Thus, we apply Theorem 2 for

$$E_{\alpha,\beta}(x) = f_{k,\infty}(x) = \frac{4(x-1)^4(x-k)^2}{9(2k+1)^2(x+2)^2}.$$

Since

$$f_6(x,1,1) – Af_{k,\infty}(x) = \frac{(x-1)^2(x-k)^2[(2k+1)^2(x+2)^2-4(k^2+k+1)(x-1)^2]}{(2k+1)^2(x+2)^2},$$

the condition (a) of Theorem 2 is satisfied if

$$(2k+1)^2(x+2)^2 \geq 4(k^2+k+1)(x-1)^2$$

for $0 \leq x \leq 4$. This is true because

$$(2k+1)^2 \geq k^2+k+1, \qquad (x+2)^2 \geq 4(x-1)^2.$$

In order to prove the conditions (b) and (c) of Theorem 2, consider two cases: $0 \leq k \leq 1$ and $k \leq -1$.

Case 1: $0 \leq k \leq 1$. Apply Theorem 2 for

$$F_{\gamma,\delta}(x) = g_{0,\delta}(x) = \left[x+\delta\frac{(2x+1)^2}{x+2}\right]^2.$$

We have

$$\hat{g}_{0,\delta}(y,z) = \frac{\delta^2y^4z^4}{(y+z)^2},$$

$$A\hat{g}_{0,\delta}(y,z) = \frac{9(k^2+k+1)\delta^2y^4z^4}{(y+z)^2} \leq \frac{9(k^2+k+1)\delta^2y^3z^3}{4},$$

$$f_6(0,y,z) - A\hat{g}_{0,\delta}(y,z) \geq f(k)y^3z^3, \qquad f(k) = k^2 - \frac{9(k^2+k+1)\delta^2}{4}.$$

Choosing

$$\delta = \frac{-2k}{3(k+1)},$$

we have

$$f(k) = k^2 - \frac{k^2(k^2+k+1)\delta^2}{(k+1)^2} = \frac{k^3}{(k+1)^2} \geq 0,$$

therefore the condition (c) of Theorem 2 is satisfied.

The condition (b) of Theorem 2 is satisfied if $f_6(x,1,1) \geq Ag_{0,\delta}(x)$ for $x \geq 4$. We have

$$\begin{aligned}
Ag_{0,\delta}(x) &\leq 9(k+1)^2\left[x+\delta\frac{(2x+1)^2}{x+2}\right]^2 \\
&= 9(k+1)^2\left[x-\frac{2k(2x+1)^2}{3(k+1)(x+2)}\right]^2 \\
&= \left[3(k+1)x+\frac{2k(2x+1)^2}{x+2}\right] \\
&= \frac{[(3-5k)x^2+(6-2k)x-2k]^2}{(x+2)^2} \\
&\leq \frac{[(3-5k)x^2+(6-2k)x-2k]^2}{36},
\end{aligned}$$

$$f_6(x,1,1)-Ag_{0,\delta}(x) \geq \frac{g_1(x)g_2(x)}{36},$$

where

$$\begin{aligned}
g_1(x) &= 6(x-1)(x-k)-(3-5k)x^2-(6-2k)x+2k \\
&= (3+5k)x^2-(12+4k)x+8k \\
&\geq 4(3+5k)x-(12+4k)x+8k \\
&= 11kx+8k \geq 0,
\end{aligned}$$

$$\begin{aligned}
g_2(x) &= 6(x-1)(x-k)+(3-5k)x^2+(6-2k)x-2k \\
&= (9-5k)x^2-8kx+4k \\
&\geq 4(9-5k)x-8kx+4k \\
&= (36-28k)x+4k > 0.
\end{aligned}$$

Case 2: $k \leq -1$. We apply Theorem 2 for

$$F_{\gamma,\delta}(x) = f_{\infty,\infty}(x) = \frac{(x-1)^4}{9(x+2)^2}.$$

The condition (b) of Theorem 2 is satisfied if $f_6(x,1,1) \geq Af_{\infty,\infty}(x)$ for $x \geq 4$. We have

$$Af_{\infty,\infty}(x) = \frac{(k^2+k+1)(x-1)^4}{(x+2)^2} \leq \frac{k^2(x-1)^4}{(x+2)^2},$$

therefore

$$f_6(x,1,1)-Af_{\infty,\infty}(x) \geq \frac{(x-1)^2[(x+2)^2(x-k)^2-k^2(x-1)^2]}{(x+2)^2}.$$

It suffices to show that

$$(x+2)(x-k) \geq (-k)(x-1).$$

This is true because

$$x+2>x-1>0, \qquad x-k>-k>0.$$

The condition (c) of Theorem 2 is satisfied if $f_6(0,y,z) \geq A\hat{f}_{\infty,\infty}(y,z)$ for $y,z \geq 0$. Since

$$\hat{f}_{\infty,\infty}(y,z) = \frac{y^4z^4}{9(y+z)^2} \leq \frac{y^3z^3}{36},$$

we have

$$f_6(0,y,z) - A\hat{f}_{\infty,\infty}(y,z) \geq \left(k^2 - \frac{9(k^2+k+1)}{36}\right) y^3z^3 \geq 0.$$

The equality occurs for $x=y=z$, for $y=z=0$ (or any cyclic permutation), and for $x/k=y=z$ (or any cyclic permutation) if $k \neq 0$. If $k=0$, then the equality holds also for $x=0$ and $y=z$.

Observation 1. The coefficient (zero) of the product $(x-y)^2(y-z)^2(z-x)^2$ is the best possible.

Setting $x=0$, the inequality

$$\sum yz(x-y)(x-z)(x-ky)(x-kz) + \alpha_k(x-y)^2(y-z)^2(z-x)^2 \geq 0$$

becomes as follows:

$$k^2y^3z^3 + \alpha_k y^2z^2(y-z)^2 \geq 0.$$

Setting $y=1$ and $z=0$ in the necessary condition

$$k^2yz + \alpha_k(y-z)^2 \geq 0,$$

we get $\alpha_k \geq 0$.

Observation 2. Substituting $k-1$ for k in P 4.13, and using then the identity

$$\sum yz(x-y)(x-z)[x-(k-1)y][x-(k-1)z] =$$
$$= \sum yz(x-y)(x-z)(x-ky+z)(x-kz+y) + k(x-y)^2(y-z)^2(z-x)^2,$$

we get the following equivalent statement:

- *Let x,y,z be nonnegative real numbers. If k is a real number, then*

$$\sum yz(x-y)(x-z)(x-ky+z)(x-kz+y) + k(x-y)^2(y-z)^2(z-x)^2 \geq 0,$$

with equality for $x=y=z$, for $y=z=0$ (or any cyclic permutation), and for $x/(k-1)=y=z$ (or any cyclic permutation) if $k \neq 1$.

□

P 4.14. *If x, y, z are nonnegative real numbers, then*

$$\sum x^2(x-y)(x-z) \ge \frac{3(x-y)^2(y-z)^2(z-x)^2}{xy+yz+zx}.$$

Solution. Write the inequality as $f_6(x,y,z) \ge 0$, where

$$f_6(x,y,z) = (xy+yz+zx)\sum x^2(x-y)(x-z) - 3(x-y)^2(y-z)^2(z-x)^2.$$

We have

$$A = -3(-27) = 81.$$

Since

$$f_6(x,1,1) = (2x+1)x^2(x-1)^2,$$

we apply Corollary 1 for

$$E_{\alpha,\beta}(x) = f_{0,-2}(x) = \frac{x^2(x-1)^4}{81}.$$

Condition (a). We need to show that $f_6(x,1,1) \ge Af_{0,-2}(x)$ for $x \in [0,4]$. We have

$$Af_{0,-2}(x) = x^2(x-1)^4,$$

$$f_6(x,1,1) - Af_{0,-2}(x) = (2x+1)x^2(x-1)^2 - x^2(x-1)^4 = x^3(x-1)^2(4-x) \ge 0.$$

Condition (b). This condition is satisfied if $f_6(x,1,1) \ge Ax^2$ for $x \ge 4$. We have

$$Ax^2 = 81x^2,$$

$$f_6(x,1,1) - Ax^2 = x^2[(2x+1)(x-1)^2 - 81] \ge x^2(81-81) = 0.$$

Condition (c). This condition is satisfied if $f_6(0,y,z) \ge 0$ for $y,z \ge 0$. We have

$$f_6(0,y,z) = yz(y-z)(y^3-z^3) - 3y^2z^2(y-z)^2 = yz(y-z)^4 \ge 0.$$

The equality occurs for $x = y = z$, and for $x = 0$ and $y = z$ (or any cyclic permutation).

$\square$

P 4.15. *If x, y, z are nonnegative real numbers, then*

$$\sum (x-y)(x-z)(x+2y)(x+2z) \ge \frac{9(x-y)^2(y-z)^2(z-x)^2}{xy+yz+zx}.$$

Solution. Write the inequality as $f_6(x,y,z) \ge 0$, where

$$f_6(x,y,z) = (xy+yz+zx)\sum(x-y)(x-z)(x+2y)(x+2z) - 9(x-y)^2(y-z)^2(z-x)^2.$$

We have

$$A = -9(-27) = 243.$$

Since

$$f_6(x,1,1) = (2x+1)(x-1)^2(x+2)^2,$$

$$\begin{aligned} f_6(0,y,z) &= yz\left[4y^2z^2 + (y-z)(y^3-z^3) + 2yz(y-z)^2\right] - 9y^2z^2(y-z)^2 \\ &= yz\left[4y^2z^2 + (y-z)^4 - 8yz(y-z)^2\right] \\ &= yz\left[(y-z)^2 - 2yz\right]^2 = yz(y^2+z^2-4yz)^2, \end{aligned}$$

$$\hat{f}_{1,\infty}(y,z) = \frac{y^2z^2(y^2+z^2-4yz)^2}{81(y+z)^2},$$

we apply Theorem 2 for

$$E_{\alpha,\beta}(x) = f_{2,-2}(x) = \frac{(x-1)^4(x-2)^2}{324},$$

$$F_{\gamma,\delta}(x) = f_{1,\infty}(x) = \frac{4(x-1)^6}{81(x+2)^2}$$

Condition (a). We need to show that $f_6(x,1,1) \ge Af_{2,-2}(x)$ for $x \in [0,4]$. We have

$$Af_{2,-2}(x) = \frac{3(x-1)^4(x-2)^2}{4},$$

$$f_6(x,1,1) - Af_{2,-2}(x) = \frac{(x-1)^2 f(x)}{4}, \quad f(x) = 4(2x+1)(x+2)^2 - 3(x-1)^2(x-2)^2.$$

Since $(x+2)^2 \ge (x-2)^2$ and $2x+1 \ge (x-1)^2$, we have $f(x) > 0$.

Condition (b). This condition is satisfied if $f_6(x,1,1) \ge Af_{1,\infty}(x)$ for $x \ge 4$. We have

$$Af_{1,\infty}(x) = \frac{12(x-1)^6}{(x+2)^2},$$

$$f_6(x,1,1) - Af_{1,\infty}(x) = \frac{(x-1)^2 f(x)}{(x+2)^2},$$

where

$$f(x) = (2x+1)(x+2)^4 - 12(x-1)^4.$$

The necessary inequality $f(x) \ge 0$ can be obtained by multiplying the inequalities

$$(x+2)^2 > (x-1)^2,$$

$$2x+1>2(x-1),$$
$$(x+2)^2>6(x-1).$$

Condition (c). We need to show that $f_6(0,y,z)\ge A\hat{f}_{1,\infty}(y,z)$. We have

$$A\hat{f}_{1,\infty}(y,z)=\frac{3y^2z^2(y^2+z^2-4yz)^2}{(y+z)^2},$$

$$f_6(0,y,z)-A\hat{f}_{1,\infty}(y,z)=yz(y^2+z^2-4yz)^2\left[1-\frac{3yz}{(y+z)^2}\right]\ge 0.$$

The equality occurs for $x=y=z$, and for $x=0$ and $y/x+x/y=4$ (or any cyclic permutation).

□

P 4.16. *If x,y,z are nonnegative real numbers, then*

$$\sum(x-y)(x-z)(x-3y)(x-3z)\ge\frac{6(x-y)^2(y-z)^2(z-x)^2}{xy+yz+zx}.$$

Solution. Write the inequality as $f_6(x,y,z)\ge 0$, where

$$f_6(x,y,z)=(xy+yz+zx)\sum(x-y)(x-z)(x-3y)(x-3z)-6(x-y)^2(y-z)^2(z-x)^2.$$

We have

$$A=-6(-27)=162.$$

Since

$$f_6(x,1,1)=(2x+1)(x-1)^2(x-3)^2,$$

$$\begin{aligned}f_6(0,y,z)&=yz\left[9y^2z^2+(y-z)(y^3-z^3)-3yz(y-z)^2\right]-6y^2z^2(y-z)^2\\&=yz\left[9y^2z^2+(y-z)^4-6yz(y-z)^2\right]\\&=yz\left[(y-z)^2-3yz\right]^2=yz(y^2+z^2-5yz)^2,\end{aligned}$$

$$\hat{f}_{3/2,\infty}(y,z)=\frac{y^2z^2(y^2+z^2-5yz)^2}{144(y+z)^2},$$

we apply Theorem 2 for

$$E_{\alpha,\beta}(x)=f_{3,-2}(x)=\frac{4(x-1)^4(x-3)^2}{2025},$$

$$F_{\gamma,\delta}(x)=f_{3/2,\infty}(x)=\frac{(x-1)^4(2x-3)^2}{144(x+2)^2}$$

Condition (a). We need to show that $f_6(x,1,1) \geq Af_{3,-2}(x)$ for $x \in [0,4]$. We have

$$Af_{3,-2}(x) = \frac{8(x-1)^4(x-3)^2}{25},$$

$$f_6(x,1,1) - Af_{3,-2}(x) = \frac{(x-1)^2(x-3)^2 f(x)}{25},$$

with

$$f(x) = 25(2x+1) - 8(x-1)^2 > 8[2x+1-(x-1)^2] \geq 0.$$

Condition (b). This condition is satisfied if $f_6(x,1,1) \geq Af_{3/2,\infty}(x)$ for $x \geq 4$. We have

$$Af_{3/2,\infty}(x) = \frac{9(x-1)^4(2x-3)^2}{8(x+2)^2},$$

$$f_6(x,1,1) - Af_{3/2,\infty}(x) = \frac{(x-1)^2 f(x)}{8(x+2)^2},$$

where

$$f(x) = 8(2x+1)(x-3)^2(x+2)^2 - 9(x-1)^2(2x-3)^2.$$

Since $8(2x+1) \geq 72 > 64$, we have $f(x) \geq 0$ if

$$8(x-3)(x+2) \geq 3(x-1)(2x-3).$$

Indeed,

$$8(x-3)(x+2) - 3(x-1)(2x-3) = 2x^2 + 7x - 57 \geq 32 + 28 - 57 > 0.$$

Condition (c). We need to show that $f_6(0,y,z) \geq A\hat{f}_{3/2,\infty}(y,z)$. We have

$$A\hat{f}_{3/2,\infty}(y,z) = \frac{9y^2z^2(y^2+z^2-5yz)^2}{8(y+z)^2},$$

$$f_6(0,y,z) - A\hat{f}_{3/2,\infty}(y,z) = yz(y^2+z^2-5yz)^2\left[1 - \frac{9yz}{8(y+z)^2}\right] \geq 0.$$

The equality occurs for $x = y = z$, for $x/3 = y = z$ (or any cyclic permutation), and for $x = 0$ and $y/x + x/y = 5$ (or any cyclic permutation).

□

P 4.17. *Let x, y, z be nonnegative real numbers, and let*

$$\alpha_k = \begin{cases} 3(1-k), & k \leq 0 \\ 3+k, & k \geq 0 \end{cases}.$$

Then,

$$\sum (x-y)(x-z)(x-ky)(x-kz) \geq \frac{\alpha_k (x-y)^2(y-z)^2(z-x)^2}{xy+yz+zx}.$$

(Vasile C., 2010)

Solution. Write the inequality as $f_6(x,y,z) \ge 0$, where

$$f_6(x,y,z) = (xy+yz+zx)\sum(x-y)(x-z)(x-ky)(x-kz) - \alpha_k(x-y)^2(y-z)^2(z-x)^2.$$

We have

$$A = 27\alpha_k > 0,$$
$$f_6(x,1,1) = (2x+1)(x-1)^2(x-k)^2,$$

$$\begin{aligned} f_6(0,y,z) &= yz\left[k^2y^2z^2 + (y-z)(y^3-z^3) - kyz(y-z)^2\right] - \alpha_k y^2z^2(y-z)^2 \\ &= yz\left[k^2y^2z^2 + (y-z)^4 - 2|k|yz(y-z)^2\right] = yz\left[(y-z)^2 - |k|yz\right]^2 \\ &= yz\left[(y+z)^2 - (4+|k|)yz\right]^2, \end{aligned}$$

$$\hat{g}_{\gamma,\delta}(y,z) = \frac{y^2z^2}{(y+z)^2}\left[\gamma(y+z)^2 + \delta yz\right]^2.$$

We will apply Theorem 2 for

$$E_{\alpha,\beta}(x) = f_{|k|,-2}(x) = \frac{4(x-1)^4(x-|k|)^2}{81(2+|k|)^2}.$$

Condition (a). Since

$$\frac{f_6(x,1,1) - Af_{|k|,-2}(x)}{(x-1)^2} = \frac{3(2+|k|)^2(2x+1)(x-k)^2 - 4\alpha_k(x-1)^2(x-|k|)^2}{3(2+|k|)^2},$$

the condition (a) of Theorem 2 is satisfied if

$$3(2+|k|)^2(2x+1)(x-k)^2 \ge 4\alpha_k(x-1)^2(x-|k|)^2$$

for $0 \le x \le 4$. Since $(x-k)^2 \ge (x-|k|)^2$ and $2x+1 \ge (x-1)^2$, it suffices to show that

$$3(2+|k|)^2 \ge 4\alpha_k.$$

If $k \le 0$, then

$$3(2+|k|)^2 - 4\alpha_k = 3k^2 \ge 0.$$

Also, if $k \ge 0$, then

$$3(2+|k|)^2 - 4\alpha_k = k(3k+8) \ge 0.$$

Conditions (b) *and* (c). To prove the conditions (b) and (c) of Theorem 2, we consider four cases:

$$k \le 0, \qquad k \ge 2, \qquad 0 \le k \le 1, \qquad 1 \le k \le \frac{56}{25}.$$

In the first three cases, we choose

$$F_{\gamma,\delta}(x) = g_{\gamma,\delta}(x) = \left[x + \gamma(x+2)(2x+1) + \delta\frac{(2x+1)^2}{x+2}\right]^2.$$

Having in view the expression of $f_6(0,y,z)$, we need to choose

$$\delta = -(4+|k|)\gamma,$$

to have

$$\hat{g}_{\gamma,\delta}(y,z) = \frac{\gamma^2 y^2 z^2}{(y+z)^2}[(y+z)^2 - (4+|k|)yz]^2.$$

In addition,

$$g_{\gamma,\delta}(x) = \left[x + \gamma(x+2)(2x+1) - \frac{(4+|k|)\gamma(2x+1)^2}{x+2}\right]^2.$$

Case 1: $k \le 0$. Choosing

$$\gamma = \frac{2}{27\sqrt{1-k}},$$

$$\delta = -(4-k)\gamma = \frac{-2(4-k)}{27\sqrt{1-k}},$$

we have

$$A = 27\alpha_k = 81(1-k),$$

$$f_6(x,1,1) - Ag_{\gamma,\delta}(x) = (2x+1)(x-1)^2(x-k)^2$$

$$-81(1-k)\left[x + \frac{2}{27\sqrt{1-k}}(x+2)(2x+1) - \frac{2(4-k)}{27\sqrt{1-k}} \cdot \frac{(2x+1)^2}{x+2}\right]^2,$$

$$9[f_6(x,1,1) - Ag_{\gamma,\delta}(x)] = 9(2x+1)(x-1)^2(x-k)^2$$

$$-\left[27\sqrt{1-k}\,x + 2(x+2)(2x+1) - 2(4-k)\frac{(2x+1)^2}{x+2}\right]^2,$$

The condition (b) is satisfied if $f_6(x,1,1) \ge Ag_{\gamma,\delta}(x)$ for $x \ge 4$. Since $x \ge 4$ involves

$$9(2x+1) \ge 81,$$

this condition is true if

$$81(x-1)^2(x-k)^2 - \left[27\sqrt{1-k}\,x + 2(x+2)(2x+1) - 2(4-k)\frac{(2x+1)^2}{x+2}\right]^2 \ge 0,$$

which can be written as

$$f_1(x)f_2(x) \ge 0,$$

where

$$f_1(x) = 9(x-1)(x-k) - 27\sqrt{1-k}\,x - 2(x+2)(2x+1) + \frac{2(4-k)(2x+1)^2}{x+2}$$

$$= 5x^2 - (9k + 27\sqrt{1-k} + 19)x + 9k - 4 + \frac{2(4-k)(2x+1)^2}{x+2},$$

$$f_2(x)=9(x-1)(x-k)+27\sqrt{1-k}\,x+2(x+2)(2x+1)-\frac{2(4-k)(2x+1)^2}{x+2}.$$

Since

$$\sqrt{1-k}\le 1-\frac{k}{2},$$

$$9k+27\sqrt{1-k}+19\le 9k+27\left(1-\frac{k}{2}\right)+19=46-\frac{9k}{2},$$

we have

$$\begin{aligned}f_1(x)&\ge 5x^2-\left(46-\frac{9k}{2}\right)x+9k-4+\frac{2(4-k)(2x+1)^2}{x+2}\\&=\frac{x-4}{2(x+2)}[2x(5x+16)+(-k)(7x+8)]\ge 0.\end{aligned}$$

To show that $f_2(x)\ge 0$ for $x\ge 4$, it suffices to prove that

$$9(x-1)(x-k)\ge\frac{2(4-k)(2x+1)^2}{x+2}.$$

Since $x-k\ge 4-k$, we only need to show that

$$9(x-1)\ge\frac{2(2x+1)^2}{x+2}.$$

We have

$$9(x-1)-\frac{2(2x+1)^2}{x+2}=\frac{(x-4)(x+5)}{x+2}\ge 0.$$

The condition (c) of Theorem 2 is satisfied if $f_6(0,y,z)\ge A\hat{g}_{\gamma,\delta}(y,z)$ for $y,z\ge 0$. We have

$$\begin{aligned}\hat{g}_{\gamma,\delta}(y,z)&=\frac{\gamma^2y^2z^2}{(y+z)^2}[(y+z)^2-(4+|k|)yz]^2\\&=\frac{4y^2z^2}{729(1-k)(y+z)^2}[(y+z)^2-(4-k)yz]^2,\end{aligned}$$

$$A\hat{g}_{\gamma,\delta}(y,z)=\frac{4y^2z^2}{9\sqrt{1-k}(y+z)^2}[(y+z)^2-(4-k)yz]^2\le\frac{yz}{9\sqrt{1-k}}[(y+z)^2-(4-k)yz]^2,$$

$$f_6(0,y,z)-A\hat{g}_{\gamma,\delta}(y,z)\ge\left(1-\frac{1}{9\sqrt{1-k}}\right)yz[(y+z)^2-(4-k)yz]^2\ge 0.$$

Case 2: $k\ge 2$. From the condition $g_{\gamma,\delta}(k)=0$, where

$$g_{\gamma,\delta}(x)=\left[x+\gamma(x+2)(2x+1)-\frac{(4+k)\gamma(2x+1)^2}{x+2}\right]^2,$$

we get

$$\gamma = \frac{k+2}{(2k+1)(k+5)},$$

$$\delta = -(4+k)\gamma = \frac{-(k+2)(k+4)}{(2k+1)(k+5)},$$

therefore

$$\begin{aligned} g_{\gamma,\delta}(x) &= \left[x + \frac{(k+2)(x+2)(2x+1)}{(2k+1)(k+5)} - \frac{(4+k)(k+2)(2x+1)^2}{(2k+1)(k+5)(x+2)}\right]^2 \\ &= \frac{g_1^2(x)}{(2k+1)^2(k+5)^2(x+2)^2}, \end{aligned}$$

where

$$\begin{aligned} g_1(x) &= (2k+1)(k+5)x(x+2) + (k+2)(x+2)^2(2x+1) - (k+2)(k+4)(2x+1)^2 \\ &= (x-k)[2(k+2)x^2 - (5k+9)x + k + 4]. \end{aligned}$$

Since

$$A = 27\alpha_k = 27(k+3),$$

the condition (b), namely $f_6(x,1,1) \ge A g_{\gamma,\delta}(x)$ for $x \ge 4$, is true if

$$(2k+1)^2(k+5)^2(2x+1)(x-1)^2(x+2)^2 \ge 27(k+3)[2(k+2)x^2-(5k+9)x+k+4]^2.$$

Since $2x+1 \ge 9$, it suffices to show that

$$9(2k+1)^2(k+5)^2(x-1)^2(x+2)^2 \ge 27(k+3)[2(k+2)x^2-(5k+9)x+k+4]^2.$$

In addition, because of

$$2(k+2)x^2-(5k+9)x+k+4 \ge 8(k+2)x-(5k+9)x+k+4 = (3k+7)x+k+4 > 0,$$

the inequality is true if

$$(2k+1)(k+5)(x-1)(x+2) \ge \sqrt{3(k+3)}\,[2(k+2)x^2-(5k+9)x+k+4].$$

Having in view that

$$(2k+1)(k+5) > (2k+4)(k+2) > (2k+4)\sqrt{3(k+3)},$$

it suffices to show that

$$(2k+4)(x-1)(x+2) \ge 2(k+2)x^2-(5k+9)x+k+4,$$

which is equivalent to

$$7(k+7)x - 5k - 12 \ge 0.$$

The condition (c) of Theorem 2 is satisfied if $f_6(0,y,z) \ge A\hat{g}_{\gamma,\delta}(y,z)$ for $y,z \ge 0$. We have

$$\hat{g}_{\gamma,\delta}(y,z) = \frac{\gamma^2 y^2 z^2}{(y+z)^2}[(y+z)^2-(4+|k|)yz]^2$$
$$= \frac{(k+2)^2 y^2 z^2}{(2k+1)^2(k+5)^2(y+z)^2}[(y+z)^2-(4+k)yz]^2,$$

$$A\hat{g}_{\gamma,\delta}(y,z) = \frac{27(k+3)(k+2)^2 y^2 z^2}{(2k+1)^2(k+5)^2(y+z)^2}[(y+z)^2-(4+k)yz]^2$$
$$\le \frac{27(k+3)(k+2)^2 yz}{4(2k+1)^2(k+5)^2}[(y+z)^2-(4+k)yz]^2,$$

$$f_6(0,y,z) - A\hat{g}_{\gamma,\delta}(y,z) \ge \left[1 - \frac{27(k+3)(k+2)^2}{4(2k+1)^2(k+5)^2}\right] yz[(y+z)^2-(4+k)yz]^2.$$

It suffices to show that

$$\frac{7(k+3)(k+2)^2}{(2k+1)^2(k+5)^2} \le 1.$$

This is true because

$$\frac{(k+2)^2}{(2k+1)(k+5)} < 1$$

and

$$\frac{7(k+3)}{(2k+1)(k+5)} \le 1.$$

Case 3: $0 \le k \le 1$. Since

$$f_6(0,y,z) = yz[(y+z)^2-(4+k)yz]^2,$$

$$\hat{g}_{\gamma,\delta}(y,z) = \frac{\gamma^2 y^2 z^2}{(y+z)^2}[(y+z)^2-(4+k)yz]^2,$$

$$A\hat{g}_{\gamma,\delta}(y,z) = \frac{27(k+3)\gamma^2 y^2 z^2}{(y+z)^2}[(y+z)^2-(4+k)yz]^2$$
$$\le \frac{27(k+3)\gamma^2 yz}{4}[(y+z)^2-(4+k)yz]^2$$

$$f_6(0,y,z) - A\hat{g}_{\gamma,\delta}(y,z) \ge \left[1 - \frac{27(k+3)\gamma^2}{4}\right] yz[(y+z)^2-(4+k)yz]^2,$$

we choose

$$\gamma = \frac{2}{3\sqrt{3(k+3)}}$$

to have $f_6(0,y,z) - A\hat{g}_{\gamma,\delta}(y,z) \ge 0$ for $y,z \ge 0$. Thus, the condition (c) of Theorem 2 is satisfied.

The condition (b) of Theorem 2 is satisfied if $f_6(x,1,1) \geq Ag_{\gamma,\delta}(x)$ for $x \geq 4$. We have

$$g_{\gamma,\delta}(x) = \left[x + \gamma(x+2)(2x+1) - \frac{(4+k)\gamma(2x+1)^2}{x+2}\right]^2,$$

$$Ag_{\gamma,\delta}(x) = 27(k+3)\left[x + \gamma(x+2)(2x+1) - \frac{(4+k)\gamma(2x+1)^2}{x+2}\right]^2 = g^2(x).$$

where

$$g(x) = 3\sqrt{3(k+3)}\; x + 2(x+2)(2x+1) - \frac{2(4+k)(2x+1)^2}{x+2}.$$

For $k \in [0,1]$, we get

$$\begin{aligned} g(x) &> 9x + 2(x+2)(2x+1) - \frac{10(2x+1)^2}{x+2} \\ &= \frac{4x^3 - 13x^2 + 2x - 2}{x+2} \\ &= \frac{4x^2(x-4) + 3x^2 + 2x - 2}{x+2} > 0. \end{aligned}$$

According to the AM-GM inequality

$$3 + (k+3) \geq 2\sqrt{3(k+3)},$$

it follows that $g(x) \leq g_1(x)$, where

$$g_1(x) = \frac{3(k+6)x}{2} + 2(x+2)(2x+1) - \frac{2(4+k)(2x+1)^2}{x+2}.$$

Thus it suffices to show that $f_6(x,1,1) \geq g_1^2(x)$, which is equivalent to $g_2(k,x) \geq 0$, where

$$g_2(k,x) = \sqrt{f_6(x,1,1)} - g_1(x)$$

$$= (x-k)(x-1)\sqrt{2x+1} - \frac{3(k+6)x}{2} - 2(x+2)(2x+1) + \frac{2(4+k)(2x+1)^2}{x+2}.$$

For $k = 0$, we have

$$\begin{aligned} g_2(0,x) &= x(x-1)\sqrt{2x+1} - 9x - 2(x+2)(2x+1) + \frac{8(2x+1)^2}{x+2} \\ &= x(x-1)\sqrt{2x+1} - \frac{x(4x^2 - 5x + 10)}{x+2}. \end{aligned}$$

The inequality $g_2(0,x) \geq 0$ is equivalent to

$$(x-1)\sqrt{2x+1} \geq \frac{4x^2 - 5x + 10}{x+2}. \qquad (*)$$

It is true if

$$(x-1)^2(2x+1)(x+2)^2 \geq (4x^2-5x+10)^2,$$

which is equivalent to $(x-4)g_3(x) \geq 0$, where

$$g_3(x) = x^3(2x-3) + x(24x-20) + 24 > 0.$$

Having in view (*), to show that $g_2(k,x) \geq 0$, it suffices to prove that

$$\frac{(x-k)(4x^2-5x+10)}{x+2} - \frac{3(k+6)x}{2} - 2(x+2)(2x+1) + \frac{2(4+k)(2x+1)^2}{x+2} \geq 0.$$

This inequality reduces to

$$k(5x^2+20x-16) \geq 0,$$

which is true.

Case 4: $1 \leq k \leq 3$. We choose

$$F_{\gamma,\delta}(x) = f_{\gamma,\infty}(x) = \frac{4(x-1)^4(x-\gamma)^2}{9(1+2\gamma)^2(x+2)^2}.$$

Having in view the expression of $f_6(0,y,z)$, we will choose

$$\gamma = \frac{k}{2}$$

to have

$$\hat{f}_{k/2,\infty}(y,z) = \frac{y^2z^2[(y+z)^2-(4+k)yz]^2}{9(1+k)^2(y+z)^2},$$

$$A\hat{f}_{k/2,\infty}(y,z) = \frac{3(k+3)y^2z^2[(y+z)^2-(4+k)yz]^2}{(1+k)^2(y+z)^2},$$

therefore,

$$f_6(0,y,z) - A\hat{f}_{k/2,\infty}(y,z) = \left[1 - \frac{3(k+3)yz}{(1+k)^2(y+z)^2}\right] yz\left[(y+z)^2-(4+k)yz\right]^2 \geq 0.$$

Thus, the condition (c) of Theorem 2 is satisfied. With regard to the condition (b), we have

$$f_{k/2,\infty}(x) = \frac{(x-1)^4(2x-k)^2}{9(1+k)^2(x+2)^2},$$

$$Af_{k/2,\infty}(x) = \frac{3(k+3)(x-1)^4(2x-k)^2}{(1+k)^2(x+2)^2},$$

$$f_6(x,1,1) - Af_{k/2,\infty}(x) = \frac{(x-1)^2 f(x)}{(1+k)^2(x+2)^2},$$

$$f(x) = (1+k)^2(2x+1)(x+2)^2(x-k)^2 - 3(k+3)(x-1)^2(2x-k)^2.$$

Since

$$(2x+1)(x+2)^2-12(x+1)^2 \geq 6(2x+1)(x+2)-12(x+1)^2 = 6x > 0,$$

we get

$$f(x) > 12(1+k)^2(x+1)^2(x-k)^2-3(k+3)(x-1)^2(2x-k)^2.$$

Therefore, it suffices to show that

$$2(1+k)(x+1)(x-k) \geq \sqrt{k+3}\,(x-1)(2x-k).$$

In addition, since

$$4+(k+3) \geq 4\sqrt{k+3},$$

it suffices to show that

$$8(1+k)(x+1)(x-k) \geq (k+7)(x-1)(2x-k).$$

This inequality can be written as

$$6(k-1)x^2+(22+9k-7k^2)x-7-9k-8k^2 \geq 0.$$

Since

$$6(k-1)x^2 \geq 24(k-1)x,$$

it suffices to show that

$$24(k-1)x+(22+9k-7k^2)x-7-9k-8k^2 \geq 0,$$

which is

$$(-2+33k-7k^2)x-7-9k-8k^2 \geq 0.$$

Since

$$-2+33k-7k^2 > -9+30k-9k^2 = 3(3-k)(3k-1) \geq 0,$$

we have

$$(-2+33k-7k^2)x-7-9k-8k^2 \geq 4(-2+33k-7k^2)-7-9k-8k^2$$

$$= -15+123k-36k^2 > -27+117k-36k^2 = 9(3-k)(4k-1) \geq 0.$$

The equality holds for $x=y=z$, for $x=0$ and $y/z+z/y=2+|k|$ (or any cyclic permutation), and for $x/k=y=z$ (or any cyclic permutation) if $k>0$

Observation. The coefficient α_k of the product $(x-y)^2(y-z)^2(z-x)^2$ is the best possible. Setting $x=0$, the inequality turns into

$$(y^2+z^2)^2-(1+k+\alpha_k)yz(y^2+z^2)+(k^2+2k-2+2\alpha_k)y^2z^2 \geq 0.$$

For $k > 0$, choosing y and z such that $y^2 + z^2 = (2 + k)yz$, we get

$$k(\alpha_k - k - 3)y^2z^2 \le 0,$$

which involves $\alpha_k \le k + 3$.

For $k < 0$, choosing $y^2 + z^2 = (2 - k)yz$, we get

$$(-k)(\alpha_k + 3k - 3)y^2z^2 \le 0,$$

which provides $\alpha_k \le 3(1 - k)$.

For $k = 0$, we get

$$(y - z)^2[(y - z)^2 + (3 - \alpha_0)yz] \ge 0,$$

which yields $\alpha_0 \le 3$.

□

P 4.18. *If x, y, z are nonnegative real numbers, then*

$$\sum x^2(y + z)(x - y)(x - z) \ge \frac{4(x - y)^2(y - z)^2(z - x)^2}{x + y + z}.$$

(Vasile C., 2010)

Solution. Write the inequality as $f_6(x, y, z) \ge 0$, where

$$f_6(x, y, z) = (x + y + z)\sum x^2(y + z)(x - y)(x - z) - 4(x - y)^2(y - z)^2(z - x)^2.$$

We have

$$A = -4(-27) = 108.$$

Since

$$f_6(x, 1, 1) = 2x^2(x + 2)(x - 1)^2,$$

we apply Corollary 1 for

$$E_{\alpha,\beta}(x) = f_{0,-2}(x) = \frac{x^2(x - 1)^4}{81}.$$

Since

$$f_6(x, 1, 1) - Af_{0,-2}(x) = \frac{2x^2(x - 1)^2(2x + 1)(4 - x)}{3} \ge 0$$

for $0 \le x \le 4$, the condition (a) in Corollary 1 is satisfied.

The condition (b) in Corollary 1 is satisfied if $f_6(x, 1, 1) \ge Ax^2$ for $x \ge 4$. This is true since

$$f_6(x, 1, 1) - Ax^2 = x^2\left[2(x + 2)(x - 1)^2 - 108\right] \ge x^2(2 \cdot 6 \cdot 9 - 108) = 0.$$

The condition (c) in Corollary 1 is also satisfied because

$$f_6(0,y,z) = yz(y+z)(y-z)(y^2-z^2) - 4y^2z^2(y-z)^2 = yz(y-z)^4 \geq 0.$$

The equality occurs for $x=y=z$, for $x=0$ and $y=z$ (or any cyclic permutation), and for $y=z=0$ (or any cyclic permutation).

□

P 4.19. *Let x,y,z be nonnegative real numbers. If k is a real numbers, then*

$$\sum (y+z)(x-y)(x-z)(x-ky)(x-kz) \geq \frac{(2+|k|)^2(x-y)^2(y-z)^2(z-x)^2}{x+y+z}.$$

(Vasile C., 2010)

Solution. Write the inequality as $f_6(x,y,z) \geq 0$, where

$$\begin{aligned} f_6(x,y,z) =& (x+y+z)\sum (y+z)(x-y)(x-z)(x-ky)(x-kz) \\ & -(2+|k|)^2(x-y)^2(y-z)^2(z-x)^2. \end{aligned}$$

We have

$$A = 27(2+|k|)^2 > 0,$$

$$f_6(x,1,1) = 2(x+2)(x-1)^2(x-k)^2,$$

$$\begin{aligned} f_6(0,y,z) &= (y+z)\left[k^2y^2z^2(y+z) + yz(y-z)(y^2-z^2)\right] - (2+|k|)^2y^2z^2(y-z)^2 \\ &= k^2y^2z^2(y+z)^2 + yz(y-z)^2\left[(y+z)^2 - (2+|k|)^2yz\right] \\ &= k^2y^2z^2(y+z)^2 + yz\left[(y+z)^2 - 4yz\right]\left[(y+z)^2 - (2+|k|)^2yz\right] \\ &= yz(y+z)^4 - 4(2+|k|)y^2z^2(y+z)^2 + 4(2+|k|)^2y^3z^3 \\ &= yz\left[(y+z)^2 - 2(2+|k|)yz\right]^2, \end{aligned}$$

$$\hat{g}_{\gamma,\delta}(y,z) = \frac{y^2z^2}{(y+z)^2}\left[\gamma(y+z)^2 + \delta yz\right]^2.$$

We will apply Theorem 2 for

$$E_{\alpha,\beta}(x) = f_{|k|,-2}(x) = \frac{4(x-1)^4(x-|k|)^2}{81(2+|k|)^2}.$$

Condition (a). Since

$$Af_{|k|,-2}(x) = \frac{4(x-1)^4(x-|k|)^2}{3},$$

$$\frac{f_6(x,1,1) - Af_{|k|,-2}(x)}{(x-1)^2} = \frac{3(x+2)(x-k)^2 - 2(x-1)^2(x-|k|)^2}{3},$$

the condition (a) of Theorem 2, namely $f_6(x,1,1)-Af_{|k|,-2}(x)\ge 0$ for $0\le x\le 4$, is satisfied if

$$3(x+2)(x-k)^2\ge 2(x-1)^2(x-|k|)^2.$$

This is true since $3(x+2)\ge 2(x-1)^2$ and $(x-k)^2\ge (x-|k|)^2$. Indeed,

$$3(x+2)-2(x-1)^2=(4-x)(1+2x)\ge 0,$$

$$(x-k)^2-(x-|k|)^2=2(|k|-k)x\ge 0.$$

Conditions (b) *and* (c). To prove the conditions (b) and (c) of Theorem 2, we consider two cases:

$$|k|\ge 1,\qquad |k|\le 1.$$

Case 1: $|k|\ge 1$. We choose

$$F_{\gamma,\delta}(x)=f_{\gamma,\infty}(x)=\frac{4(x-1)^4(x-\gamma)^2}{9(1+2\gamma)^2(x+2)^2}.$$

Having in view the expression of $f_6(0,y,z)$, we will choose

$$\gamma=|k|$$

to have

$$\hat{f}_{|k|,\infty}(y,z)=\frac{y^2z^2[(y+z)^2-2(2+|k|)yz]^2}{9(1+2|k|)^2(y+z)^2},$$

$$A\hat{f}_{|k|,\infty}(y,z)=\frac{3(2+|k|)^2y^2z^2[(y+z)^2-2(2+|k|)yz]^2}{(1+2|k|)^2(y+z)^2},$$

therefore,

$$f_6(0,y,z)-A\hat{f}_{|k|,\infty}(y,z)=\left[1-\frac{3(2+|k|)^2yz}{(1+2|k|)^2(y+z)^2}\right]yz\left[(y+z)^2-(2+|k|)^2yz\right]^2.$$

Thus, the condition (c) of Theorem 2 is satisfied if

$$(1+2|k|)^2(y+z)^2\ge 3(2+|k|)^2yz.$$

Since $(y+z)^2\ge 4yz$, it suffices to show that

$$2(1+2|k|)\ge\sqrt{3}(2+|k|).$$

Indeed,

$$2(1+2|k|)-\sqrt{3}(2+|k|)=(4-\sqrt{3})|k|+2(1-\sqrt{3})\ge(4-\sqrt{3})+2(1-\sqrt{3})>0.$$

With regard to the condition (b), we have

$$f_{|k|,\infty}(x)=\frac{4(x-1)^4(x-|k|)^2}{9(1+2|k|)^2(x+2)^2},$$

$$Af_{|k|,\infty}(x) = \frac{12(2+|k|)^2(x-1)^4(x-|k|)^2}{(1+2|k|)^2(x+2)^2},$$

$$f_6(x,1,1) - Af_{|k|,\infty}(x) = \frac{(x-1)^2 f(x)}{(1+2|k|)^2(x+2)^2},$$

$$f(x) = (1+2|k|)^2(x+2)^3(x-k)^2 - 6(2+|k|)^2(x-1)^2(x-|k|)^2.$$

Since $(x-k)^2 \ge (x-|k|)^2$, it suffices to show that

$$(1+2|k|)^2(x+2)^3 \ge 6(2+|k|)^2(x-1)^2.$$

Since $x+2 > 6$, this is true if

$$(1+2|k|)^2(x+2)^2 \ge (2+|k|)^2(x-1)^2,$$

which is equivalent to

$$(1+2|k|)(x+2) \ge (2+|k|)(x-1).$$

This is true for $x \ge 4$ and $|k| \ge 1$ because $1+2|k| \ge 2+|k|$ and $x+2 > x-1$.

Case 2: $|k| \le 1$. We choose

$$F_{\gamma,\delta}(x) = g_{\gamma,\delta}(x) = \left[x + \gamma(x+2)(2x+1) + \delta\frac{(2x+1)^2}{x+2}\right]^2.$$

Having in view the expression of $f_6(0,y,z)$, we need to set

$$\delta = -2(2+|k|)\gamma,$$

to have

$$\hat{g}_{\gamma,\delta}(y,z) = \frac{\gamma^2 y^2 z^2}{(y+z)^2}[(y+z)^2 - 2(2+|k|)yz]^2.$$

In addition, by choosing

$$\gamma = \frac{1}{3(2+|k|)},$$

we have

$$\hat{g}_{\gamma,\delta}(y,z) = \frac{y^2z^2}{9(2+|k|)^2(y+z)^2}[(y+z)^2 - 2(2+|k|)yz]^2,$$

$$A\hat{g}_{\gamma,\delta}(y,z) = \frac{3y^2z^2}{(y+z)^2}[(y+z)^2 - 2(2+|k|)yz]^2,$$

$$f_6(0,y,z) - A\hat{g}_{\gamma,\delta}(y,z) = \left[1 - \frac{3yz}{(y+z)^2}\right] yz\left[(y+z)^2 - 2(2+|k|)^2 yz\right]^2 \ge 0.$$

Thus, the condition (c) of Theorem 2 is satisfied.

With regard to the condition (b), we have

$$g_{\gamma,\delta}(x) = \left[x + \frac{(x+2)(2x+1)}{3(2+|k|)} - \frac{2(2x+1)^2}{3(x+2)}\right]^2$$
$$= \left[\frac{(x+2)(2x+1)}{3(2+|k|)} - \frac{5x^2+2x+2}{3(x+2)}\right]^2,$$
$$Ag_{\gamma,\delta}(x) = 3\left[(x+2)(2x+1) - \frac{(2+|k|)(5x^2+2x+2)}{x+2}\right]^2,$$
$$f_6(x,1,1) = 2(x+2)(x-1)^2(x-k)^2 \ge 12(x-1)^2(x-|k|)^2,$$
$$f_6(x,1,1) - Ag_{\gamma,\delta}(x) \ge 12(x-1)^2(x-|k|)^2 - Ag_{\gamma,\delta}(x) = 3g_1(x)g_2(x),$$

where

$$g_1(x) = 2(x-1)(x-|k|) - (x+2)(2x+1) + \frac{(2+|k|)(5x^2+2x+2)}{x+2}$$
$$= \frac{3[(1+|k|)x^2 - 4x + 2|k|]}{x+2} \ge \frac{3(x^2-4x)}{x+2} \ge 0.$$

$$g_2(x) = 2(x-1)(x-k) + (x+2)(2x+1) - \frac{(2+|k|)(5x^2+2x+2)}{x+2}$$
$$\ge 2(x-1)(x-1) + (x+2)(2x+1) - \frac{3(5x^2+2x+2)}{x+2}$$
$$\ge 2(x-1)^2 + (x+2)(2x+1) - \frac{5x^2+2x+2}{2}$$
$$= 4x^2 + x + 4 - \frac{5x^2+2x+2}{2} = \frac{3(x^2+2)}{2} > 0.$$

The equality occurs for $x = y = z$, for $x/k = y = z$ (or any cyclic permutation) if $k \ne 0$, for $y = z = 0$ (or any cyclic permutation), and for $x = 0$ and $y/z + z/y = 2 + 2|k|$ (or any cyclic permutation).

Observation. The coefficient of the product $(x-y)^2(y-z)^2(z-x)^2$ is the best possible. Setting $x = 0$, the inequality

$$\sum (y+z)(x-y)(x-z)(x-ky)(x-kz) \ge \frac{\alpha_k(x-y)^2(y-z)^2(z-x)^2}{x+y+z}$$

becomes

$$yz\left[(y^2+z^2)^2 - (\alpha_k - k^2)yz(y^2+z^2) + 2(\alpha_k + k^2 - 2)y^2z^2\right] \ge 0.$$

For $y^2 + z^2 = 2(1+|k|)yz$, this inequality leads to

$$|k|[\alpha_k - (2+|k|)^2]y^2z^2 \le 0,$$

which implies the necessary condition $\alpha_k \le (2+|k|)^2$.

□

P 4.20. *If x, y, z are nonnegative real numbers, then*

$$\sum x(x^2-y^2)(x^2-z^2) \ge \frac{12(x-y)^2(y-z)^2(z-x)^2}{x+y+z}.$$

(Vasile C., 2010)

Solution. Write the inequality as $f_6(x,y,z) \ge 0$, where

$$f_6(x,y,z) = (x+y+z)\sum x(x^2-y^2)(x^2-z^2) - 12(x-y)^2(y-z)^2(z-x)^2$$

has the highest coefficient

$$A = -12(-27) = 324.$$

Since

$$f_6(x,1,1) = x(x+2)(x^2-1)^2,$$

we apply Corollary 1 for

$$E_{\alpha,\beta}(x) = f_{0,-1}(x) = \frac{4x^2(x-1)^4(x+1)^2}{225(x+2)^2}.$$

Since

$$f_6(x,1,1) - Af_{0,-1}(x) = \frac{x(x-1)^2(x+1)^2 f(x)}{225(x+2)^2},$$

where

$$f(x) = 225(x+2)^3 - 1296x(x-1)^2 > 216[(x+2)^3 - 6x(x-1)^2],$$

the condition (a) in Corollary 1 is satisfied if $(x+2)^3 \ge 6x(x-1)^2$ for $0 \le x \le 4$. This is true since

$$2(x+2) \ge 3x, \qquad (x+2)^2 \ge 4(x-1)^2.$$

The condition (b) in Corollary 1 is satisfied if $f_6(x,1,1) \ge Ax^2$ for $x \ge 4$. This is true if

$$(x+2)(x^2-1)^2 \ge 324x.$$

It suffices to show that

$$2(x+2)(x^2-1)^2 \ge 675x,$$

which follows by multiplying the inequalities

$$2(x+2)(x-1) \ge 9x, \qquad (x-1)(x+1)^2 \ge 75.$$

Indeed, we have

$$2(x+2)(x-1) - 9x = (x-4)(2x+1) \ge 0, \qquad (x-1)(x+1)^2 - 75 \ge 3\cdot 25 - 75 = 0.$$

The condition (c) in Corollary 1 is also satisfied because

$$f_6(0,y,z) = (y+z)(y^2-z^2)(y^3-z^3)-12y^2z^2(y-z)^2 = (y-z)^4(y^2+z^2+5yz) \geq 0.$$

The equality occurs for $x = y = z$, for $-x = y = z$ (or any cyclic permutation), and for $x = 0$ and $y = z$ (or any cyclic permutation).

□

P 4.21. *Let x, y, z be nonnegative real numbers, and let*

$$\alpha_k = \begin{cases} 4(k-2), & k \leq 6 \\ \dfrac{(k+2)^2}{4}, & k \geq 6 \end{cases}.$$

Then,

$$\sum x(x-y)(x-z)(x-ky)(x-kz) + \frac{\alpha_k(x-y)^2(y-z)^2(z-x)^2}{x+y+z} \geq 0.$$

(Vasile C., 2010)

Solution. Write the inequality as $f_6(x,y,z) \geq 0$, where

$$f_6(x,y,z) = (x+y+z)\sum x(x-y)(x-z)(x-ky)(x-kz)+\alpha_k(x-y)^2(y-z)^2(z-x)^2.$$

Since the product $(x-y)^2(y-z)^2(z-x)^2$ has the highest coefficient equal to -27, $f_6(x,y,z)$ has the highest coefficient

$$A = -27\alpha_k.$$

Also, we have

$$f_6(x,1,1) = x(x+2)(x-1)^2(x-k)^2,$$

$$f_6(0,y,z) = (y-z)^2[(y+z)^4-(k+2)yz(y+z)^2+\alpha_k y^2z^2].$$

There are three cases to consider.

Case 1: $k \geq 6$. Since

$$\alpha_k = (k+2)^2,$$

$$A = -27\alpha_k = \frac{-27(k+2)^2}{4} < 0,$$

the desired inequality is true if $f_6(x,1,1) \geq 0$ and $f_6(0,y,z) \geq 0$ for $x,y,z \geq 0$ (Theorem 1). The first condition is clearly true and

$$f_6(0,y,z) = (y-z)^2[(y+z)^4-(k+2)yz(y+z)^2+\frac{(k+2)^2}{4}y^2z^2]$$

$$= (y-z)^2\left[(y+z)^2-\frac{k+2}{2}yz\right]^2 \geq 0.$$

Case 2: $2 \le k \le 6$. Since

$$\alpha_k = 4(k-2),$$

$$A = -27\alpha_k = -108(k-2) \le 0,$$

the desired inequality is true if $f_6(x,1,1) \ge 0$ and $f_6(0,y,z) \ge 0$ for $x,y,z \ge 0$ (Theorem 1). The first condition is true and

$$\begin{aligned} f_6(0,y,z) &= (y-z)^2[(y+z)^4 - (k+2)yz(y+z)^2 + 4(k-2)y^2z^2] \\ &= (y-z)^4[(y+z)^2 - (k-2)yz] \ge (y-z)^6 \ge 0. \end{aligned}$$

Case 3: $k \le 2$. We have

$$\alpha_k = 4(k-2),$$

$$A = -27\alpha_k = 108(2-k) \ge 0.$$

We will apply Corollary 1 for

$$E_{\alpha,\beta}(x) = f_{k,0}(x) = \frac{4x^2(x-1)^4(x-k)^2}{9(4-k)^2(x+2)^2}.$$

The condition (a) in Corollary 1 is satisfied if $f_6(x,1,1) \ge Af_{k,0}(x)$ for $x \in [0,4]$. We have

$$Af_{k,0}(x) = \frac{48(2-k)x^2(x-1)^4(x-k)^2}{(4-k)^2(x+2)^2},$$

$$f_6(x,1,1) - Af_{k,0}(x) = \frac{x(x-1)^2(x-k)^2[(4-k)^2(x+2)^3 - 48(2-k)x(x-1)^2]}{(4-k)^2(x+2)^2}.$$

The condition (a) is true if

$$(4-k)^2(x+2)^3 \ge 48(2-k)x(x-1)^2$$

for $0 \le x \le 4$. This inequality follows by multiplying the inequalities

$$(4-k)^2 \ge 8(2-k)$$

and

$$(x+2)^3 \ge 6x(x-1)^2,$$

which are equivalent to $k^2 \ge 0$ and $(4-x)(2+2x+5x^2) \ge 0$, respectively.

The condition (b) in Corollary 1 is satisfied if $f_6(x,1,1) \ge Ax^2$ for $x \ge 4$. It suffices to show that

$$(x+2)(x-1)^2(x-k)^2 \ge 108(2-k)x.$$

This inequality follows from

$$4(x-1)^2 \ge 9x$$

and

$$(x+2)(x-k)^2 \ge 48(2-k).$$

Indeed, we have

$$4(x-1)^2-9x=(x-4)(4x-1)\geq 0,$$

$$(x+2)(x-k)^2-48(2-k)\geq 6(4-k)^2-48(2-k)=6k^2\geq 0.$$

The condition (c) in Corollary 1 is satisfied if $f_6(0,y,z)\geq 0$ for $y,z\geq 0$. Indeed,

$$f_6(0,y,z)=(y-z)^4[(y+z)^2+(2-k)yz]\geq 0.$$

The equality occurs for $x=y=z$, for $x=0$ and $y=z$ (or any cyclic permutation), and for $x/k=y=z$ (or any cyclic permutation) if $k\neq 0$, and for $x=0$ and $y/z+z/y=(k-2)/2$ (or any cyclic permutation) if $k>6$

Observation. The coefficient α_k of the product $(x-y)^2(y-z)^2(z-x)^2$ is the best possible. Setting $x=0$, the inequality becomes

$$(y-z)^2[(y^2+z^2)^2-(k-2)yz(y^2+z^2)+(\alpha_k-2k)y^2z^2]\geq 0.$$

For $y=z=1$, the necessary condition

$$(y^2+z^2)^2-(k-2)yz(y^2+z^2)+(\alpha_k-2k)y^2z^2\geq 0$$

involves $\alpha_k\geq 4(k-2)$.

Also, for

$$y^2+z^2=\frac{k-2}{2}yz,\quad k\geq 6,$$

the necessary condition

$$(y^2+z^2)^2-(k-2)yz(y^2+z^2)+(\alpha_k-2k)y^2z^2\geq 0$$

becomes

$$[4\alpha_k-(k+2)^2]y^2z^2\geq 0,$$

which involves $\alpha_k\geq (k+2)^2/4$.

□

P 4.22. *If x,y,z are nonnegative real numbers, then*

$$\sum(x^2+yz)(x-y)(x-z)\geq \frac{5(x-y)^2(y-z)^2(z-x)^2}{xy+yz+zx}.$$

(Vasile C., 2010)

Solution. Write the inequality as $f_6(x,y,z) \ge 0$, where

$$f_6(x,y,z) = (xy+yz+zx)\sum(x^2+yz)(x-y)(x-z) - 5(x-y)^2(y-z)^2(z-x)^2.$$

We have

$$A = -5(-27) = 135,$$

$$f_6(x,1,1) = (2x+1)(x^2+1)(x-1)^2,$$

$$\begin{aligned}\frac{f_6(0,y,z)}{yz} &= y^2z^2 + (y-z)^2(y^2+z^2+yz) - 5yz(y-z)^2 \\ &= y^2z^2 + (y^2+z^2-2yz)\left[y^2+z^2-4yz\right] \\ &= (y^2+z^2)^2 - 6(y^2+z^2)yz + 9y^2z^2 \\ &= (y^2+z^2-3yz)^2,\end{aligned}$$

$$\hat{f}_{1/2,\infty}(y,z) = \frac{y^2z^2(y^2+z^2-3yz)^2}{36(y+z)^2}.$$

Thus, we apply Theorem 2 for

$$E_{\alpha,\beta}(x) = F_{\gamma,\delta} = f_{1/2,\infty}(x).$$

The conditions (a) and (b) in Theorem 2 are satisfied if $f_6(x,1,1) \ge Af_{1/2,\infty}(x)$ for $x \ge 0$. Since

$$Af_{1/2,\infty}(x) = \frac{15(x-1)^4(2x-1)}{4(x+2)^2},$$

$$f_6(x,1,1) - Af_{1/2,\infty}(x) = \frac{(x-1)^2 f(x)}{4(x+2)^2},$$

where

$$f(x) = 4(x+2)^2(2x+1)(x^2+1) - 15(x-1)^2(2x-1)^2,$$

we need to show that $f(x) \ge 0$ for $x \ge 0$. This is true if

$$(x+2)^2(2x+1)(x^2+1) \ge 4(x-1)^2(2x-1)^2.$$

Since $x^2+1 \ge (x-1)^2$, we only need to prove that

$$(x+2)^2(2x+1) \ge 4(2x-1)^2,$$

that is

$$x(2x^2-7x+28) \ge 0.$$

The condition (c) in Theorem 2 is also satisfied because

$$A\hat{f}_{1/2,\infty}(y,z) = \frac{15y^2z^2(y^2+z^2-3yz)^2}{4(y+z)^2},$$

$$f_6(0,y,z)-A\hat{f}_{1/2,\infty}(y,z)=yz(y^2+z^2-3yz)^2\left[1-\frac{15yz}{4(y+z)^2}\right]$$
$$\geq yz(y^2+z^2-3yz)^2\left(1-\frac{15}{16}\right)\geq 0.$$

The equality occurs for $x=y=z$, and for $x=0$ and $y/z+z/y=3$ (or any cyclic permutation).

□

P 4.23. *If x,y,z are nonnegative real numbers, then*

$$\sum(4x^2+yz)(x-y)(x-z)\geq\frac{16(x-y)^2(y-z)^2(z-x)^2}{xy+yz+zx}.$$

(Vasile C., 2010)

Solution. Write the inequality as $f_6(x,y,z)\geq 0$, where

$$f_6(x,y,z)=(xy+yz+zx)\sum(4x^2+yz)(x-y)(x-z)-16(x-y)^2(y-z)^2(z-x)^2.$$

We have

$$A=-16(-27)=432,$$
$$f_6(x,1,1)=(2x+1)(4x^2+1)(x-1)^2,$$
$$\begin{aligned}\frac{f_6(0,y,z)}{yz}&=y^2z^2+4(y-z)^2(y^2+z^2+yz)-16yz(y-z)^2\\&=y^2z^2+4(y^2+z^2-2yz)\left[y^2+z^2-3yz\right]\\&=4(y^2+z^2)^2-20(y^2+z^2)yz+25y^2z^2\\&=(2y^2+2z^2-5yz)^2.\end{aligned}$$

Apply Theorem 2 for

$$E_{\alpha,\beta}(x)=f_{1/2,-2}(x)=\frac{4(x-1)^4(2x-1)^2}{2025},$$

Condition (a). Since

$$Af_{1/2,-2}(x)=\frac{64(x-1)^4(2x-1)^2}{75},$$

$$f_6(x,1,1)-Af_{1/2,-2}(x)=\frac{(x-1)^2f(x)}{75},$$

where

$$f(x)=75(2x+1)(4x^2+1)-64(x-1)^2(2x-1)^2,$$

the condition (a) of Theorem 2 is satisfied if $f(x) \ge 0$ for $0 \le x \le 4$. It suffices to show that

$$(2x+1)(4x^2+1) \ge (x-1)^2(2x-1)^2.$$

This is true because

$$2x+1 \ge (x-1)^2,$$
$$4x^2+1 \ge (2x-1)^2.$$

Indeed,

$$2x+1-(x-1)^2 = x(4-x) \ge 0,$$
$$4x^2+1-(2x-1)^2 = 4x \ge 0.$$

Conditions (b) and (c). Having in view the expression of $f_6(0,y,z)$, we will apply Theorem 2 for

$$F_{\gamma,\delta}(x) = g_{\gamma,\delta}(x), \qquad \delta = -\frac{9\gamma}{2},$$

which leads to

$$\hat{g}_{\gamma,\delta}(y,z) = \frac{\gamma^2 y^2 z^2}{4(y+z)^2}(2y^2+2z^2-5yz)^2.$$

Since

$$A\hat{g}_{\gamma,\delta}(y,z) = \frac{108\gamma^2 y^2 z^2}{(y+z)^2}(2y^2+2z^2-5yz)^2,$$

$$f_6(0,y,z) - A\hat{g}_{\gamma,\delta}(y,z) = yz(2y^2+2z^2-5yz)^2 g(y,z),$$

where

$$g(y,z) = 1 - \frac{108\gamma^2 yz}{(y+z)^2} \ge 1 - 27\gamma^2,$$

we choose

$$\gamma = \frac{1}{3\sqrt{3}}$$

to have $g(y,z) \ge 0$. Thus, the condition (c) in Theorem 2 is satisfied.

The condition (b) is satisfied if $f_6(x,1,1) \ge Ag_{\gamma,\delta}(x)$ for $x \ge 4$, where

$$g_{\gamma,\delta}(x) = \left[x + \gamma(x+2)(2x+1) - \frac{9\gamma(2x+1)^2}{2(x+2)}\right]^2.$$

Since

$$Ag_{\gamma,\delta}(x) = 4\left[6\sqrt{3}x + 6\sqrt{3}\gamma(x+2)(2x+1) - \frac{27\sqrt{3}\gamma(2x+1)^2}{x+2}\right]^2 = 4f^2(x),$$

$$f(x) = 6\sqrt{3}x + 2(x+2)(2x+1) - \frac{9(2x+1)^2}{x+2},$$

we need to show that

$$(2x+1)(4x^2+1)(x-1)^2 \ge 4f^2(x)$$

for $x \ge 4$. Since

$$f(x) > 9x + 2(x+2)(2x+1) - \frac{9(2x+1)^2}{x+2}$$
$$= \frac{x^2(4x-9)+6x-1}{x+2} > 0,$$

it suffices to show that

$$(2x+1)(4x^2+1)(x-1)^2 \ge 4f_1^2(x),$$

where

$$f_1(x) = 11x + 2(x+2)(2x+1) - \frac{9(2x+1)^2}{x+2} > f(x).$$

Since

$$(2x+1)(4x^2+1)(x-1)^2 > (2x+1)(4x^2)(x-1)^2$$

and

$$f_1(x) = \frac{4x^3-7x^2+10x-1}{x+2} < \frac{x(4x^2-7x+10)}{x+2},$$

it suffices to show that

$$(2x+1)(x-1)^2(x+2)^2 \ge (4x^2-7x+10)^2,$$

which can be rewritten as

$$(2x+1)^2(x-1)^2(x+2)^2 \ge (2x+1)(4x^2-7x+10)^2.$$

Since

$$2x+1 \le \frac{(x+5)^2}{9},$$

it suffices to show that

$$9(2x+1)^2(x-1)^2(x+2)^2 \ge (x+5)^2(4x^2-7x+10)^2,$$

which is equivalent to

$$3(2x+1)(x-1)(x+2) \ge (x+5)(4x^2-7x+10),$$

$$x^3-2x^2+8x-28 \ge 0,$$

$$(x-4)(x^2+2x+16)+36 \ge 0.$$

The equality occurs for $x = y = z$, and for $x = 0$ and $y/z + z/y = 5/2$ (or any cyclic permutation).

□

P 4.24. *Let x, y, z be nonnegative real numbers. If $k \ge 0$, then*

$$\sum(x^2+kyz)(x-y)(x-z) \ge \frac{(3+2\sqrt{k}\,)(x-y)^2(y-z)^2(z-x)^2}{xy+yz+zx}.$$

(Vasile C., 2010)

Solution. Write the inequality as $f_6(x,y,z) \ge 0$, where

$$f_6(x,y,z) = q\sum(x^2+kyz)(x-y)(x-z) - (3+2\sqrt{k})(x-y)^2(y-z)^2(z-x)^2.$$

We have

$$A = 27(3+2\sqrt{k}),$$

$$f_6(x,1,1) = (2x+1)(x^2+k)(x-1)^2,$$

$$\begin{aligned}\frac{f_6(0,y,z)}{yz} &= ky^2z^2 + (y-z)^2(y^2+z^2+yz) - (3+2\sqrt{k})yz(y-z)^2 \\ &= ky^2z^2 + (y^2+z^2-2yz)\left[y^2+z^2-2(1+\sqrt{k})yz\right] \\ &= (y^2+z^2)^2 - 2(2+\sqrt{k})(y^2+z^2)yz + (2+\sqrt{k})^2y^2z^2 \\ &= \left[y^2+z^2-(2+\sqrt{k})yz\right]^2,\end{aligned}$$

We will apply Theorem 2 for

$$E_{\alpha,\beta}(x) = f_{\sqrt{k},-2}(x) = \frac{4(x-1)^4\left(x-\sqrt{k}\,\right)^2}{81\left(2+\sqrt{k}\,\right)^2}.$$

Condition (a). Since

$$Af_{\sqrt{k},-2}(x) = \frac{4\left(3+2\sqrt{k}\,\right)(x-1)^4\left(x-\sqrt{k}\,\right)^2}{3\left(2+\sqrt{k}\,\right)^2},$$

$$f_6(x,1,1) - Af_{\sqrt{k},-2}(x) = (x-1)^2 f(x),$$

where

$$f(x) = (2x+1)(x^2+k) - \frac{4\left(3+2\sqrt{k}\,\right)}{3\left(2+\sqrt{k}\,\right)^2}(x-1)^2\left(x-\sqrt{k}\,\right)^2,$$

the condition (a) of Theorem 2 is satisfied if $f(x) \ge 0$ for $0 \le x \le 4$. This is true because

$$1 \ge \frac{4\left(3+2\sqrt{k}\,\right)}{3\left(2+\sqrt{k}\,\right)^2},$$

$$2x+1 \ge (x-1)^2,$$

$$x^2+k \ge \left(x-\sqrt{k}\,\right)^2.$$

Indeed,

$$2x+1-(x-1)^2 = x(4-x) \ge 0,$$

$$x^2+k-\left(x-\sqrt{k}\,\right)^2 = 2\sqrt{k}\,x \ge 0.$$

With regard to the conditions (b) and (c), we consider two cases: $0 \le k \le 1$ and $k \ge 1$.

Case 1: $k \ge 1$. Having in view the expression of $f_6(0,y,z)$, we will apply Theorem 2 for

$$F_{\gamma,\delta}(x) = f_{\sqrt{k}/2,\infty}(x) = \frac{(x-1)^4\left(2x-\sqrt{k}\,\right)^2}{9\left(1+\sqrt{k}\,\right)^2(x+2)^2},$$

$$\hat{f}_{\sqrt{k}/2,\infty}(y,z) = \frac{y^2z^2\left[y^2+z^2-(2+\sqrt{k})yz\right]^2}{9(1+\sqrt{k}\,)^2(y+z)^2}.$$

The condition (c) is satisfied because

$$f_6(0,y,z) - A\hat{f}_{\sqrt{k}/2,\infty}(y,z) = yz\left[y^2+z^2-(2+\sqrt{k})yz\right]^2 f(y,z),$$

where

$$f(y,z) = 1-\frac{3(3+2\sqrt{k}\,)yz}{(1+\sqrt{k}\,)^2(y+z)^2}$$

$$\ge 1-\frac{3(3+2\sqrt{k}\,)}{4(1+\sqrt{k}\,)^2} = \frac{4k+2\sqrt{k}-5}{4(1+\sqrt{k}\,)^2} > 0.$$

The condition (b) is satisfied if $f_6(x,1,1) \ge Af_{\sqrt{k}/2,\infty}(x)$ for $x \ge 4$. Since

$$Af_{\sqrt{k}/2,\infty}(x) = \frac{3(3+2\sqrt{k}\,)(x-1)^4\left(2x-\sqrt{k}\,\right)^2}{\left(1+\sqrt{k}\,\right)^2(x+2)^2},$$

we need to show that

$$(2x+1)(x^2+k) \ge \frac{3(3+2\sqrt{k}\,)(x-1)^2\left(2x-\sqrt{k}\,\right)^2}{\left(1+\sqrt{k}\,\right)^2(x+2)^2}.$$

Since

$$\frac{3(3+2\sqrt{k}\,)}{\left(1+\sqrt{k}\,\right)^2} < 4$$

and

$$\left(2x-\sqrt{k}\,\right)^2 \le 4x^2+k,$$

it suffices to show that

$$(2x+1)(x^2+k) \geq \frac{4(x-1)^2(4x^2+k)}{(x+2)^2}.$$

Since

$$\frac{x^2+k}{4x^2+k} - \frac{x^2+1}{4x^2+1} = \frac{3(k-1)x^2}{(4x^2+k)(4x^2+1)} \geq 0,$$

we only need to prove that

$$(2x+1)(x^2+1) \geq \frac{4(x-1)^2(4x^2+1)}{(x+2)^2}.$$

This is true because $x^2+1 > (x-1)^2$ and

$$(2x+1)(x+2)^2 \geq 4(4x^2+1).$$

The last inequality is equivalent to

$$x(2x^2-7x+12) \geq 0.$$

Case 2: $0 \leq k \leq 1$. Having in view the expression of $f_6(0,y,z)$, we will apply Theorem 2 for

$$F_{\gamma,\delta}(x) = g_{\gamma,\delta}(x), \qquad \delta = -(4+\sqrt{k}\,)\gamma,$$

which leads to

$$\hat{g}_{\gamma,\delta}(y,z) = \frac{\gamma^2 y^2 z^2}{(y+z)^2}\left[y^2+z^2-(2+\sqrt{k}\,)yz\right]^2.$$

Since

$$A\hat{g}_{\gamma,\delta}(y,z) = \frac{27(3+2\sqrt{k}\,)\gamma^2 y^2 z^2}{(y+z)^2}\left[y^2+z^2-(2+\sqrt{k}\,)yz\right]^2,$$

$$f_6(0,y,z) - A\hat{g}_{\gamma,\delta}(y,z) = yz\left[y^2+z^2-(2+\sqrt{k}\,)yz\right]^2 g(y,z),$$

where

$$g(y,z) = 1 - \frac{27(3+2\sqrt{k}\,)\gamma^2 yz}{(y+z)^2} \geq 1 - \frac{27(3+2\sqrt{k})\gamma^2}{4},$$

we choose

$$\gamma = \frac{2}{3\sqrt{3(3+2\sqrt{k}\,)}}$$

to have $g(y,z) \geq 0$. Thus, the condition (c) in Theorem 2 is satisfied.

The condition (b) is satisfied if $f_6(x,1,1) \geq A g_{\gamma,\delta}(x)$ for $x \geq 4$, where

$$g_{\gamma,\delta}(x) = \left[x + \gamma(x+2)(2x+1) - (4+\sqrt{k}\,)\gamma\frac{(2x+1)^2}{x+2}\right]^2.$$

Since

$$Ag_{\gamma,\delta}(x) = \frac{4}{\gamma^2} g_{\gamma,\delta}(x) = f^2(x),$$

$$f(x) = \frac{2x}{\gamma} + 2(x+2)(2x+1) - 2(4+\sqrt{k})\frac{(2x+1)^2}{x+2},$$

we need to show that

$$(2x+1)(x^2+k)(x-1)^2 \ge f^2(x).$$

Since $2/\gamma \ge 9$ and $4+\sqrt{k} \le 5$, we have

$$\begin{aligned} f(x) &\ge 9x + 2(x+2)(2x+1) - \frac{10(2x+1)^2}{x+2} \\ &= \frac{x^2(4x-13)+2(x-1)}{x+2} > 0. \end{aligned}$$

On the other hand, by the AM-GM inequality,

$$\frac{2}{\gamma} = 3\sqrt{3(3+2\sqrt{k})} \le \frac{3}{2}[3+(3+2\sqrt{k})] = 3(3+\sqrt{k}).$$

Therefore,

$$\begin{aligned} f(x) &\le 3(3+\sqrt{k})x + 2(x+2)(2x+1) - 2(4+\sqrt{k})\frac{(2x+1)^2}{x+2} \\ &= \frac{x(4x^2-5x+10) - \sqrt{k}(5x^2+2x+2)}{x+2} \\ &\le \frac{x(4x^2-5x+10)}{x+2}. \end{aligned}$$

In addition,

$$(2x+1)(x^2+k)(x-1)^2 \ge (2x+1)x^2(x-1)^2.$$

Thus, it suffices to prove that

$$(2x+1)(x-1)^2(x+2)^2 \ge (4x^2-5x+10)^2,$$

which may be rewritten as

$$(2x+1)^2(x-1)^2(x+2)^2 \ge (2x+1)(4x^2-5x+10)^2.$$

According to

$$2x+1 \le \frac{(x+5)^2}{9},$$

it suffices to show that

$$(2x+1)(x-1)(x+2) \ge \frac{1}{3}(x+5)(4x^2-5x+10).$$

This is equivalent to the obvious inequality

$$(x-4)(x^2+x+7) \geq 0.$$

The equality occurs for $x = y = z$, and for $x = 0$ and $y/z + z/y = 2 + \sqrt{k}$ (or any cyclic permutation).

Observation. The coefficient of the product $(x-y)^2(y-z)^2(z-x)^2$ is the best possible. Setting $x = 0$, the inequality

$$\sum(x^2+kyz)(x-y)(x-z) \geq \frac{\alpha_k(x-y)^2(y-z)^2(z-x)^2}{xy+yz+zx}$$

reduces to

$$yz\left[y^2+z^2-(2+\sqrt{k})yz\right]^2 + (3+2\sqrt{k}-\alpha_k)y^2z^2(y-z)^2 \geq 0.$$

In addition, for

$$y^2+z^2 = (2+\sqrt{k})yz,$$

we get the necessary condition

$$\left(3+2\sqrt{k}-\alpha_k\right)y^2z^2(y-z)^2 \geq 0,$$

which involves $\alpha_k \leq 3+2\sqrt{k}$.

□

P 4.25. *If x, y, z are nonnegative real numbers, then*

$$\sum(x^2-yz)^2(x-y)(x-z) \geq 4(\sqrt{2}+1)(x-y)^2(y-z)^2(z-x)^2.$$

(Vasile C., 2014)

Solution. Denote

$$C = 4(\sqrt{2}+1)$$

and write the inequality as $f_6(x, y, z) \geq 0$, where

$$f_6(x,y,z) = \sum(x^2-yz)^2(x-y)(x-z) - C(x-y)^2(y-z)^2(z-x)^2.$$

Since $(x-y)(x-z) = x^2+2yz-q$, the sum $\sum(x^2-yz)^2(x-y)(x-z)$ has the same highest coefficient A_1 as

$$P_1(x,y,z) = \sum(x^2-yz)^2(x^2+2yz)^2,$$

that is $A_1 = P_1(1,1,1) = 0$. Therefore, $f_6(x, y, z)$ has the highest coefficient

$$A = A_1 - C(-27) = 108(\sqrt{2}+1).$$

We have

$$f_6(x,1,1)=(x+1)^2(x-1)^4,$$

$$\begin{aligned}
f_6(0,y,z)&=y^3z^3+(y-z)(y^5-z^5)-Cy^2z^2(y-z)^2\\
&=y^3z^3+(y-z)^2\left[y^4+z^4+yz(y^2+z^2)+(1-C)y^2z^2\right]\\
&=y^2z^2+(y^2+z^2-2yz)\left[(y^2+z^2)^2+(y^2+z^2)yz-(1+C)y^2z^2\right]\\
&=(y^2+z^2)^3-(y^2+z^2)^2yz-(3+C)(y^2+z^2)y^2z^2+(3+2C)y^3z^3\\
&=(y^2+z^2)^3-(y^2+z^2)^2yz-(4\sqrt{2}+7)(y^2+z^2)y^2z^2+(8\sqrt{2}+11)y^3z^3\\
&=\left[y^2+z^2+(2\sqrt{2}+1)yz\right]\left[y^2+z^2-(\sqrt{2}+1)yz\right]^2.
\end{aligned}$$

In addition, for

$$\gamma=\frac{\sqrt{2}-1}{2},$$

we have

$$\hat{f}_{\gamma,\infty}(y,z)=\frac{y^2z^2[y^2+z^2-(\sqrt{2}+1)yz]^2}{18(y+z)^2}.$$

We apply Theorem 2 for

$$E_{\alpha,\beta}(x)=f_{2/3,-2},\qquad F_{\gamma,\delta}(x)=f_{\gamma,\infty}(x).$$

Condition (a). We need to show that $f_6(x,1,1)\ge Af_{2/3,-2}(x)$ for $0\le x\le 4$. We have

$$f_{2/3,\infty}(x)=\frac{(x-1)^4(3x-2)^2}{1296},$$

$$Af_{2/3,\infty}(x)=\frac{(\sqrt{2}+1)(x-1)^4(3x-2)^2}{12},$$

$$f_6(x,1,1)-Af_{2/3,-2}(x)=\frac{(x-1)^4f(x)}{12},$$

$$f(x)=12(x+1)^2-(\sqrt{2}+1)(3x-2)^2.$$

Since

$$\sqrt{2}+1<\frac{3}{2}+1=\frac{5}{2},$$

we have

$$\begin{aligned}
f(x)&>12(x+1)^2-\frac{5(3x-2)^2}{2}\\
&>10(x+1)^2-\frac{5(3x-2)^2}{2}\\
&=\frac{25x(4-x)}{2}\ge 0.
\end{aligned}$$

Condition (b). We need to show that $f_6(x,1,1) \ge Af_{\gamma,\infty}(x)$ for $\gamma = \dfrac{\sqrt{2}-1}{2}$ and $x \ge 4$. Since

$$f_{\gamma,\infty}(x) = \frac{4(x-1)^4(x-\gamma)^2}{9(1+2\gamma)^2(x+2)^2} = \frac{(x-1)^4(2x-\sqrt{2}+1)^2}{18(x+2)^2},$$

$$Af_{\gamma,\infty}(x) = \frac{6(\sqrt{2}+1)(x-1)^4(2x-\sqrt{2}+1)^2}{(x+2)^2},$$

$$f_6(x,1,1) - Af_{\gamma,\infty}(x) = \frac{(x-1)^4 f(x)}{(x+2)^2},$$

where

$$f(x) = (x+1)^2(x+2)^2 - 6(\sqrt{2}+1)(2x-\sqrt{2}+1)^2,$$

we need to show that $f(x) \ge 0$ for $x \ge 4$. Since

$$6(\sqrt{2}+1) < 15,$$

it suffices to show that

$$(x+1)(x+2) \ge \sqrt{15}(2x-\sqrt{2}+1),$$

which is equivalent to $g(x) \ge 0$, where

$$g(x) = x^2 - (2\sqrt{15}-3)x + 2 + \sqrt{30} - \sqrt{15}.$$

We will show that

$$g(x) \ge g(4) > 0.$$

We have

$$g(x) - g(4) = (x-4)(x+7-2\sqrt{15}) \ge (x-4)(11-2\sqrt{15}) \ge 0,$$

$$g(4) = \sqrt{15}(\sqrt{60}+\sqrt{2}-9) > 0.$$

Condition (c). We need to show that $f_6(0,y,z) \ge A\hat{f}_{\gamma,\infty}(y,z)$ for $y,z \ge 0$. This condition is satisfied because

$$A\hat{f}_{\gamma,\infty}(y,z) = \frac{6(\sqrt{2}+1)y^2z^2[y^2+z^2-(\sqrt{2}+1)yz]^2}{(y+z)^2},$$

$$f_6(0,y,z) - A\hat{f}_{\gamma,\infty}(y,z) = \left[y^2+z^2-(\sqrt{2}+1)yz\right]^2 f_0(y,z),$$

where

$$\begin{aligned} f_0(y,z) &= y^2+z^2+(2\sqrt{2}+1)yz - \frac{6(\sqrt{2}+1)y^2z^2}{(y+z)^2} \\ &\ge 2yz + (2\sqrt{2}+1)yz - \frac{3(\sqrt{2}+1)yz}{2} = \frac{(\sqrt{2}+3)yz}{2} \ge 0. \end{aligned}$$

The equality occurs for $x = y = z$, and for $x = 0$ and $y/z + z/y = \sqrt{2}+1$ (or any cyclic permutation).

$\square$

P 4.26. *If x, y, z are nonnegative real numbers, then*

$$\sum \frac{1}{4x^2+y^2+z^2} \geq \frac{9}{4(x^2+y^2+z^2)+2(xy+yz+zx)}.$$

(Vasile C., 2011)

Solution. Write the inequality as $f_6(x, y, z) \geq 0$, where

$$f_6(x,y,z) = P\sum(4y^2+z^2+x^2)(4z^2+x^2+y^2) - 9\prod(4x^2+y^2+z^2),$$

$$P = 4(x^2+y^2+z^2)+2(xy+yz+zx).$$

The highest coefficient A of $f_6(x, y, z)$ is equal with the highest coefficient of the product $-9\prod(4x^2+y^2+z^2)$. Since

$$4x^2+y^2+z^2 = 3x^2+p^2-2q,$$

we have

$$A = -9(3)^3 = -243.$$

By Theorem 1, we only need to prove the original inequality for $y = z = 1$ and for $x = 0$.

Case 1: $y = z = 1$. We need to show that

$$\frac{1}{4x^2+2} + \frac{2}{x^2+5} \geq \frac{9}{4x^2+4x+10},$$

which is equivalent to

$$x(x-1)^2 \geq 0.$$

Case 2: $x = 0$. We need to show that

$$\frac{1}{y^2+z^2} + \frac{1}{4y^2+z^2} + \frac{1}{4z^2+y^2} \geq \frac{9}{4(y^2+z^2)+2yz},$$

which is equivalent to

$$\frac{1}{y^2+z^2} + \frac{5(y^2+z^2)}{4(y^4+z^4)+17y^2z^2} \geq \frac{9}{4(y^2+z^2)+2yz},$$

For $yz = 0$, the inequality is an equality. For $yz \neq 0$, using the substitution

$$t = \frac{y^2+z^2}{yz}, \qquad t \geq 2,$$

we may write the inequality as follows:

$$\frac{yz}{y^2+z^2} + \frac{5yz(y^2+z^2}{4(y^2+z^2)^2+9y^2z^2} \geq \frac{9yz}{4(y^2+z^2)+2yz},$$

$$\frac{1}{t} + \frac{5t}{4t^2+9} \geq \frac{9}{4t+2},$$
$$(t-2)(2t-1) \geq 0,$$

The equality holds for $x = y = z$, for $x = 0$ and $y = z$ (or any cyclic permutation), and for $y = z = 0$ (or any cyclic permutation).

Observation 1. Similarly, we can prove the following generalization:

- *Let x, y, z be nonnegative real numbers. If $k \geq 4$, then*

$$\sum \frac{1}{kx^2 + y^2 + z^2} \geq \frac{9(2k+1)}{9k(x^2+y^2+z^2) + 2(k-1)^2(xy+yz+zx)},$$

with equality for $x = y = z$, and for $y = z = 0$ (or any cyclic permutation). If $k = 4$, then the equality holds also for $x = 0$ and $y = z$ (or any cyclic permutation).

For

$$f_6(x,y,z) = P\sum(ky^2+z^2+x^2)(kz^2+x^2+y^2) - 9(2k+1)\prod(kx^2+y^2+z^2),$$
$$P = 9k(x^2+y^2+z^2) + 2(k-1)^2(xy+yz+zx),$$

we have

$$A = -9(2k+1)(k-1)^3.$$

Since $A < 0$, it suffices to show that the original inequality holds for $y = z = 1$ and for $x = 0$. In these cases, the inequality respectively reduces to

$$(k-1)^2(x-1)^2[(2k+1)x + k - 4] \geq 0$$

and $f(t) \geq 0$, where

$$f(t) = 2(2k+1)t^2 - 9(k+1)t + 2(k-1)^2, \qquad t \geq 2.$$

We have

$$f(t) \geq f(2) \geq 0$$

because

$$f(t) - f(2) = (t-2)[2(2k+1)t - k - 5] \geq (t-2)[4(2k+1) - k - 5] \geq 0,$$

$$f(2) = 2(k-4)(k+1) \geq 0.$$

Observation 2. Also, the following generalization is valid:

- *Let x, y, z be nonnegative real numbers. If $1 < k \leq 4$, then*

$$\sum \frac{1}{kx^2 + y^2 + z^2} \geq \frac{9(k+5)}{(8+11k-k^2)(x^2+y^2+z^2) + 2(k-1)^2(xy+yz+zx)},$$

with equality for $x=y=z$, and for $x=0$ and $y=z$ (or any cyclic permutation). If $k=4$, then the equality holds also for $y=z=0$ (or any cyclic permutation).

For

$$f_6(x,y,z)=P\sum(ky^2+z^2+x^2)(kz^2+x^2+y^2)-9(k+5)\prod(kx^2+y^2+z^2),$$

$$P=(8+11k-k^2)(x^2+y^2+z^2)+2(k-1)^2(xy+yz+zx),$$

we have

$$A=-9(k+5)(k-1)^3.$$

Since $A<0$, it suffices to show that the original inequality holds for $y=z=1$ and for $x=0$. In these cases, the inequality respectively reduces to

$$(k-1)^2x(x-1)^2[(4-k)x+2k+10]\ge 0$$

and

$$(k-1)^2(t-2)[2(4-k)t^2+18t-(k-1)^2]\ge 0,$$

where $t\ge 2$.

□

P 4.27. *If x,y,z are nonnegative real numbers, no two of which are zero, then*

$$\frac{2}{x^2+y^2}+\frac{2}{y^2+z^2}+\frac{2}{z^2+x^2}\ge\frac{45}{4(x^2+y^2+z^2)+xy+yz+zx}.$$

(Vasile C., 2011)

Solution. Write the inequality as $f_6(x,y,z)\ge 0$, where

$$f_6(x,y,z)=2P\sum(x^2+y^2)(x^2+z^2)-45\prod(y^2+z^2),$$

$$P=4(x^2+y^2+z^2)+xy+yz+zx.$$

The highest coefficient A of $f_6(x,y,z)$ is equal with the highest coefficient of the product $-45\prod(y^2+z^2)$. Since

$$y^2+z^2=-x^2+p^2-2q,$$

we have

$$A=-45(-1)^3=45.$$

In addition,

$$\begin{aligned}f_6(x,1,1)&=2(4x^2+2x+9)[(x^2+1)^2+4(x^2+1)]-90(x^2+1)^2\\&=4x(x^2+1)(2x^3+x^2-8x+5)=4x(x^2+1)(x-1)^2(2x+5)\ge 0\end{aligned}$$

and

$$f_6(0,y,z) \geq 0.$$

The last inequality is true if the original inequality holds for $x = 0$. Thus, we need to show that

$$\frac{2}{y^2} + \frac{2}{z^2} + \frac{2}{y^2+z^2} \geq \frac{45}{4(y^2+z^2)+yz},$$

which can be written as

$$\frac{2(y^2+z^2)}{y^2z^2} + \frac{2}{y^2+z^2} \geq \frac{45}{4(y^2+z^2)+yz},$$

$$\frac{2(y^2+z^2)}{yz} + \frac{2yz}{y^2+z^2} \geq \frac{45yz}{4(y^2+z^2)+yz}.$$

Using the notation

$$t = \frac{y^2+z^2}{yz}, \qquad t \geq 2,$$

the inequality becomes

$$2t + \frac{2}{t} \geq \frac{45}{4t+1},$$

$$(t-2)(8t+18t-1) \geq 0.$$

First Solution. Since the inequality is an equality for $x = y = z = 1$ and for $x = 0$ and $y = z$, we apply Corollary 1 for

$$E_{\alpha,\beta}(x) = f_{0,-2}(x) = \frac{x^2(x-1)^4}{81}.$$

Since

$$f_6(x,1,1) - Af_{0,-2}(x) = \frac{x(x-1)^2 f(x)}{9},$$

where

$$\begin{aligned} f(x) =& 36(x^2+1)(2x+5) - 5x(x-1)^2 \\ &\geq 36(x-1)^2(2x+5) - 5x(x-1)^2 \\ &= (x-1)^2(67x+180) > 0. \end{aligned}$$

the condition (a) in Corollary 1 is satisfied.

The condition (b) in Corollary 1 is satisfied if $f_6(x,1,1) \geq 45x^2$ for $x \geq 4$. This is true if

$$4(x^2+1)(x-1)^2(2x+5) \geq 45x.$$

Since $x^2+1 \geq 2x$, it suffices to show that

$$8(x-1)^2(2x+5) \geq 45.$$

which is clearly true for $x \ge 4$.

The condition (c) in Corollary 1 is satisfied because $f_6(0,y,z) \ge 0$ for $y,z \ge 0$.

The equality holds for $x = y = z$, and for $x = 0$ and $y = z$ (or any cyclic permutation).

Second Solution. Apply Theorem 3. Since the conditions (a) and (c) are satisfied, we only need to show that

$$f_6(x,1,1) \ge \frac{4Ax(x-1)^3}{27}$$

for $x \ge 1$. We have

$$f_6(x,1,1) - \frac{4Ax(x-1)^3}{27} = \frac{4x(x-1)^2 f(x)}{3},$$

where

$$f(x) = 3(x^2+1)(2x+5) - 5(x-1) \ge 6(2x+5) - 5(x-1) = 7(x+5) > 0.$$

Observation 1. Similarly, we can prove the following generalization:

- *Let x,y,z be nonnegative real numbers. If $0 \le k < 1$, then*

$$\sum \frac{1}{kx^2+y^2+z^2} \ge \frac{9(k+5)}{(8+11k-k^2)(x^2+y^2+z^2)+2(k-1)^2(xy+yz+zx)},$$

with equality for $x = y = z$, and for $x = 0$ and $y = z$ (or any cyclic permutation).

As shown in Observation 2 from the preceding P 4.26, for

$$f_6(x,y,z) = P(x,y,z)\sum(ky^2+z^2+x^2)(kz^2+x^2+y^2) - 9(k+5)\prod(kx^2+y^2+z^2),$$

$$P(x,y,z) = (8+11k-k^2)(x^2+y^2+z^2)+2(k-1)^2(xy+yz+zx),$$

we have

$$A = 9(k+5)(1-k)^3 > 0,$$

$$P(x,1,1) = (8+11k-k^2)x^2+4(k-1)^2x+18(k+1),$$

$$\begin{aligned}\frac{f_6(x,1,1)}{x^2+k+1} &= P(x,1,1)[x^2+k+1+2(kx^2+2)] - 9(k+5)(x^2+k+1)(kx^2+2)\\ &= 2(1-k)^2x(x-1)^2[(4-k)x+2k+10] \ge 0,\end{aligned}$$

and

$$f_6(0,y,z) \ge 0.$$

Since the conditions (a) and (c) in Theorem 3 are satisfied, it suffices to show that

$$f_6(x,1,1) \geq \frac{4Ax(x-1)^3}{27}$$

for $x \geq 1$. We have

$$f_6(x,1,1) - \frac{4Ax(x-1)^3}{27} = \frac{2(1-k)^2x(x-1)^2f(x)}{3},$$

where

$$\begin{aligned} f(x) &= 3(x^2+k+1)[(4-k)x+2k+10] - 2(k+5)(1-k)(x-1) \\ &\geq 3(k+2)[(4-k)x+2k+10] - 2(k+5)(1-k)(x-1) \\ &= (14+14k-k^2)x + 70 + 34k + 4k^2 \\ &\geq (14+14k-k^2) + 70 + 34k + 4k^2 \\ &= 3(28+16k+k^2) > 0. \end{aligned}$$

Observation 2. Having in view Observation 1 above and Observation 2 from the preceding P 4.26, it follows that the concerned inequality holds for $0 \leq k \leq 4$.

□

P 4.28. *If x, y, z are nonnegative real numbers, no two of which are zero, then*

$$\sum \frac{1}{2y^2+yz+2z^2} \geq \frac{18}{5(x^2+y^2+z^2+xy+yz+zx)}.$$

(Vasile C., 2009)

Solution. Write the inequality as $f_6(x,y,z) \geq 0$, where

$$f_6(x,y,z) = 5P(x,y,z)\sum(2x^2+xy+2y^2)(2x^2+xz+2z^2) - 18\prod(2y^2+yz+2z^2),$$

$$P(x,y,z) = x^2+y^2+z^2+xy+yz+zx.$$

Since

$$2y^2+yz+2z^2 = -2x^2+yz+2(p^2-2q),$$

$f_6(x,y,z)$ has the same highest coefficient A as

$$P_3(x,y,z) = -18\prod(-2x^2+yz),$$

that is

$$A = P_3(1,1,1) = -18(-2+1)^3 = 18.$$

We have

$$P(x,1,1) = x^2+2x+3,$$

$$\frac{f_6(x,1,1)}{2x^2+x+2} = 5(x^2+2x+3)[(2x^2+x+2)+10] - 90(2x^2+x+2)$$
$$= 5x(2x^3+5x^2-16x+9) = 5x(x-1)^2(2x+9),$$

$$f_6(x,1,1) = 5x(x-1)^2(2x^2+x+2)(2x+9) \ge 0.$$

Also,
$$f_6(0,y,z) \ge 0.$$

This inequality is true if the original inequality holds for $x=0$. Thus, we need to show that
$$\frac{1}{2y^2+yz+2z^2} + \frac{1}{2y^2} + \frac{1}{2z^2} \ge \frac{18}{5(y^2+z^2+yz)},$$
which can be rewritten as
$$\frac{yz}{2(y^2+z^2)+yz} + \frac{y^2+z^2}{2yz} \ge \frac{18yz}{5(y^2+z^2+yz)}.$$

Using the substitution
$$t = \frac{y^2+z^2}{yz}, \qquad t \ge 2,$$
the inequality becomes
$$\frac{1}{2t+1} + \frac{t}{2} \ge \frac{18}{5(t+1)},$$
$$10t^3+15t^2-57t-26 \ge 0,$$
$$(t-2)(10t^2+35t+13) \ge 0.$$

First Solution. We apply Corollary 1 for
$$E_{\alpha,\beta}(x) = f_{0,-2}(x) = \frac{(x-1)^4x^2}{81}.$$

Condition (a). Since
$$Af_{0,-2}(x) = \frac{2(x-1)^4x^2}{9},$$
$$f_6(x,1,1) - Af_{0,-2}(x) = \frac{x(x-1)^2f(x)}{9}, \quad f(x) = 45(2x^2+x+2)(2x+9) - 2x(x-1)^2,$$
we need to show that $f(x) \ge 0$ for $x \in [0,4]$. This follows immediately from
$$2x^2+x+2 \ge 2(x-1)^2,$$
$$45(2x^2+x+2) > x.$$

Condition (b). Since
$$f_6(x,1,1) - Ax^2 = xg(x), \qquad g(x) = 5(x-1)^2(2x^2+x+2)(2x+9) - 18x,$$

we need to show that $g(x) \ge 0$ for $x \ge 4$, which is trivial.

Condition (c). This condition is satisfied because $f_6(0,y,z) \ge 0$ for $y,z \ge 0$.

The equality holds for $x = y = z$, and for $x = 0$ and $y = z$ (or any cyclic permutation).

Second Solution. Apply Theorem 3. Since the conditions (a) and (c) are satisfied, we only need to show that the condition (b) is satisfied. Thus, we need to prove that

$$f_6(x,1,1) \ge \frac{4Ax(x-1)^3}{27}$$

for $x \ge 1$. We have

$$f_6(x,1,1) - \frac{4Ax(x-1)^3}{27} = \frac{x(x-1)^2 f(x)}{3},$$

where

$$f(x) = 15(2x^2+x+2)(2x+9) - 8(x-1) \ge 75(2x+9) - 8(x-1) = 7(x+5) > 0.$$

Observation. Similarly, we can prove the following generalization:

- *Let x,y,z be nonnegative real numbers, no two of which are zero. If $-1 \le k \le 2$, then*

$$\sum \frac{k+2}{y^2+kyz+z^2} \ge \frac{9(2k+5)}{2(2-k)(x^2+y^2+z^2)+(4k+1)(xy+yz+zx)},$$

with equality for $x = y = z$, and for $x = 0$ and $y = z$ (or any cyclic permutation).

For

$$f_6(x,y,z) = P(x,y,z)\sum(x^2+kxy+y^2)(x^2+kxz+z^2) - 9(2k+5)\prod(y^2+kyz+z^2),$$

$$\frac{P(x,y,z)}{k+2} = 2(2-k)(x^2+y^2+z^2)+(4k+1)(xy+yz+zx),$$

we have

$$A = -9(2k+5)(k-1)^3,$$

$$\frac{P(x,1,1)}{k+2} = 2(2-k)x^2+2(4k+1)x+9,$$

$$f_6(x,1,1) = 2(k+2)(x^2+kx+1)x(x-1)^2[(2-k)x+(k+1)(5-k)].$$

Case 1: $-1 \le k < 1$. Since $A > 0$, we apply Corollary 1 for

$$E_{\alpha,\beta}(x) = f_{0,-2}(x) = \frac{(x-1)^4x^2}{81}.$$

Condition (a). We have

$$Af_{0,-2}(x) = \frac{(2k+5)(1-k)^3(x-1)^4x^2}{9},$$

$$f_6(x,1,1) - Af_{0,-2}(x) = \frac{x(x-1)^2 f(x)}{9},$$

$$f(x) = 18(k+2)(x^2+kx+1)[(2-k)x+(k+1)(5-k)] - (2k+5)(1-k)^3x(x-1)^2.$$

Since

$$(2-k)x + (k+1)(5-k) \ge (2-k)x,$$

it suffices to show that $x \ge 4$ involves

$$18(k+2)(2-k)(x^2+kx+1) \ge (2k+5)(1-k)^3(x-1)^2.$$

This is true because

$$x^2 + kx + 1 \ge x^2 - x + 1 > (x-1)^2,$$

$$3(k+2) \ge 2k+5,$$

$$4 \ge (1-k)^2$$

and

$$3(2-k) > 2(1-k).$$

Condition (b). Since

$$f_6(x,1,1) - Ax^2 \ge x^2 g(x),$$

$$g(x) = 2(k+2)(2-k)(x^2+kx+1)(x-1)^2 - 9(2k+5)(1-k)^3,$$

we need to show that $g(x) \ge 0$ for $x \ge 4$. It suffices to show that $g(4) \ge 0$, which is true.

Condition (c). For $x = 0$, the original inequality becomes

$$\frac{k+2}{y^2+kyz+z^2} + (k+2)\left(\frac{1}{y^2} + \frac{1}{z^2}\right) \ge \frac{9(2k+5)}{2(2-k)(y^2+z^2)+(4k+1)yz},$$

$$\frac{k+2}{t+k} + (k+2)t \ge \frac{9(2k+5)}{2(2-k)t+4k+1},$$

$$(t-2)h(t) \ge 0,$$

where

$$h(t) = 2(4-k^2)t^2 + (k+2)(9+4k-2k^2)t + 7k^2 + 18k - 1,$$

$$h(t) \ge h(2) = 67 + 52k - k^2 - 4k^3 > 0.$$

Case 2: $1 \le k \le 2$. Since $A \le 0$, according to Theorem 1, we only need to show that $f_6(x,1,1) \ge 0$ and $f_6(0,y,z) \ge 0$ for $x,y,z \ge 0$, which are true.

$\square$

P 4.29. *If x, y, z are nonnegative real numbers, no two of which are zero, then*

$$\sum \frac{x-(2+\sqrt{2})(y+z)}{(y+z)^2} + \frac{9(3+2\sqrt{2})}{4(x+y+z)} \geq 0.$$

(Vasile C., 2014)

Solution. Write the inequality as $f_6(x, y, z) \geq 0$, where

$$f_6(x, y, z) = 4(x+y+z)f(x, y, z) + 9(3+2\sqrt{2})\prod(y+z)^2,$$

$$f(x, y, z) = \sum\left[x-(2+\sqrt{2})(y+z)\right](x+y)^2(x+z)^2.$$

Since

$$(y+z)^2 = (p-x)^2 = x^2 - 2px + p^2,$$

the product $\prod(y+z)^2$ has the same highest coefficient A_1 as $x^2y^2z^2$, that is

$$A_1 = 1.$$

Therefore, $f_6(x, y, z)$ has the highest coefficient

$$A = 9(3+2\sqrt{2}) = 9(1+\sqrt{2})^2.$$

We have

$$\begin{aligned}\frac{f(x,1,1)}{(x+1)^2} &= (x-4-2\sqrt{2})(x+1)^2 - 8\left[(2+\sqrt{2})x+1+\sqrt{2}\right]\\ &= x^3 - 2(1+\sqrt{2})x^2 - (23+12\sqrt{2})x - 12 - 10\sqrt{2},\end{aligned}$$

$$\begin{aligned}f_6(x,1,1) &= 4(x+2)f(x,1,1) + 36(3+2\sqrt{2})(x+1)^4\\ &= 4(x+1)^2 g(x),\end{aligned}$$

$$\begin{aligned}g(x) =&(x+2)\left[x^3 - 2(1+\sqrt{2})x^2 - (23+12\sqrt{2})x - 12 - 10\sqrt{2}\right]\\ &+ 9(3+2\sqrt{2})(x+1)^2\\ =&x^4 - 2\sqrt{2}x^3 + 2\sqrt{2}x^2 - 2(2-\sqrt{2})x + 3 - 2\sqrt{2}\\ =&(x-1)^2(x-\sqrt{2}+1)^2,\end{aligned}$$

therefore

$$f_6(x,1,1) = 4(x+1)^2(x-1)^2(x-\sqrt{2}+1)^2.$$

Thus, we apply Theorem 2 for

$$E_{\alpha,\beta}(x) = F_{\gamma,\delta}(x) = f_{\sqrt{2}-1,-2}(x) = \frac{4(x-1)^4(x-\sqrt{2}+1)^2}{81(1+\sqrt{2})^2}.$$

The conditions (a) and (b) in Theorem 2 are satisfied if $f_6(x,1,1)\ge Af_{\sqrt{2}-1,-2}(x)$ for $x\ge 0$. We have

$$Af_{\sqrt{2}-1,-2}(x)=\frac{4(x-1)^4(x-\sqrt{2}+1)^2}{9},$$

$$f_6(x,1,1)-Af_{\sqrt{2}-1,-2}(x)=\frac{16(x-1)^2(x-\sqrt{2}+1)^2(x+2)(2x+1)}{9}\ge 0.$$

The condition (c) in Theorem 2 is satisfied if $f_6(0,y,z)\ge A\hat{f}_{\sqrt{2}-1,-2}(y,z)$ for $y,z\ge 0$. We have

$$f(0,y,z)=-(2+\sqrt{2})(y+z)y^2z^2+(y+z)^2\left[y^3+z^3-(2+\sqrt{2})yz(y+z)\right],$$

$$\begin{aligned}\frac{f(0,y,z)}{y+z}&=-(2+\sqrt{2})y^2z^2+(y^2+z^2+2yz)\left[y^2+z^2-(3+\sqrt{2})yz\right]\\&=(y^2+z^2)^2-(1+\sqrt{2})(y^2+z^2)yz-(8+3\sqrt{2})y^2z^2,\end{aligned}$$

$$f_6(0,y,z)=4(y+z)f(0,y,z)+9(3+2\sqrt{2})y^2z^2(y+z)^2,$$

$$\begin{aligned}\frac{f_6(0,y,z)}{(y+z)^2}=&4\left[(y^2+z^2)^2-(1+\sqrt{2})(y^2+z^2)yz-(8+3\sqrt{2})y^2z^2\right]\\&+9(3+2\sqrt{2})y^2z^2\\=&4(y^2+z^2)^2-4(1+\sqrt{2})(y^2+z^2)yz+(6\sqrt{2}-5)y^2z^2\end{aligned}$$

and

$$\hat{f}_{\sqrt{2}-1,-2}(y,z)=\frac{(y+z)^2\left[2(y^2+z^2)^2-(3+\sqrt{2})yz\right]^2}{81(1+\sqrt{2})^2},$$

$$A\hat{f}_{\sqrt{2}-1,-2}(y,z)=\frac{(y+z)^2\left[2(y^2+z^2)^2-(3+\sqrt{2})yz\right]^2}{9}.$$

Therefore,

$$f_6(0,y,z)-A\hat{f}_{\sqrt{2}-1,-2}(y,z)=\frac{8(y+z)^2g(y,z)}{9},$$

where

$$\begin{aligned}g(y,z)&=4(y^2+z^2)^2-(3+4\sqrt{2})yz(y^2+z^2)+(6\sqrt{2}-7)y^2z^2\\&=(y^2+z^2-2yz)\left[4(y^2+z^2)-(4\sqrt{2}-5)yz\right]+(3-2\sqrt{2})y^2z^2\ge 0.\end{aligned}$$

The equality occurs for $x=y=z$, and for $\dfrac{x}{\sqrt{2}-1}=y=z$ (or any cyclic permutation).

□

P 4.30. *If x,y,z are nonnegative real numbers, no two of which are zero, then*

$$\frac{6x-y-z}{y^2+z^2}+\frac{6y-z-x}{z^2+x^2}+\frac{6z-x-y}{x^2+y^2}\ge\frac{18}{x+y+z}.$$

(Vasile C., 2014)

Solution. Write the inequality as $f_6(x,y,z)\ge 0$, where

$$f_6(x,y,z)=(x+y+z)f(x,y,z)-18\prod(y^2+z^2),$$

$$f(x,y,z)=\sum(6x-y-z)(x^2+y^2)(x^2+z^2).$$

Since

$$y^2+z^2=-x^2+p^2-2q,$$

the product $\prod(y^2+z^2)$ has the highest coefficient A_1 as $(-x^2)(-y^2)(-z^2)$, that is

$$A_1=-1.$$

Therefore, $f_6(x,y,z)$ has the highest coefficient

$$A=-18A_1=18.$$

We have

$$\frac{f(x,1,1)}{x^2+1}=2(3x-1)(x^2+1)+4(5-x)=2(3x^3-x^2+x+9),$$

$$\begin{aligned}f_6(x,1,1)&=(x+2)f(x,1,1)-36(x^2+1)^2\\&=2(x^2+1)\left[(x+2)(3x^3-x^2+x+9)-18(x^2+1)\right]\\&=2(x^2+1)x(3x^3+5x^2-19x+11)\\&=2x(x^2+1)(x-1)^2(3x+11)\ge 0.\end{aligned}$$

Also, we have

$$f_6(0,y,z)\ge 0$$

for $y,z\ge 0$. This is true if the original inequality holds for $x=0$. Thus, we need to show that

$$\frac{-y-z}{y^2+z^2}+\frac{6y-z}{z^2}+\frac{6z-y}{y^2}\ge\frac{18}{y+z},$$

which is equivalent to

$$\frac{-y-z}{y^2+z^2}+\frac{6(y^3+z^3)-yz(y+z)}{y^2z^2}\ge\frac{18}{y+z},$$

$$\frac{-1}{y^2+z^2}+\frac{6(y^2+z^2)-7yz}{y^2z^2}\ge\frac{18}{(y+z)^2},$$

$$\frac{-yz}{y^2+z^2}+\frac{6(y^2+z^2)-7yz}{yz}\geq\frac{18yz}{(y+z)^2}.$$

Using the substitution

$$t=\frac{y^2+z^2}{yz},\qquad t\geq 2,$$

the inequality can be written as follows:

$$\frac{-1}{t}+6t-7\geq\frac{18}{t+2},$$

$$6t^3+5t^2-33t-2\geq 0,$$

$$(t-2)(6t^2+17t+1)\geq 0.$$

First Solution. We apply Theorem 3. The conditions (a) and (c) are clearly satisfied. The condition (b) is satisfied if

$$f_6(x,1,1)\geq\frac{4Ax(x-1)^3}{27}$$

for $x\geq 1$. We have

$$f_6(x,1,1)-\frac{4Ax(x-1)^3}{27}=\frac{2x(x-1)^2f(x)}{3},$$

where

$$f(x)=3(x^2+1)(3x+11)-4(x-1)\geq 6(3x+11)-4(x-1)=14(x+5)>0.$$

The equality occurs for $x=y=z$, and for $x=0$ and $y=z$ (or any cyclic permutation).

Second Solution. We apply Corollary 1 for

$$E_{\alpha,\beta}(x)=f_{0,-2}(x)=\frac{(x-1)^4x^2}{81}.$$

The condition (a) in Corollary 1 are satisfied if $f_6(x,1,1)\geq Af_{0,-2}(x)$ for $x\in[0,4]$. We have

$$Af_{0,-2}(x)=\frac{2(x-1)^4x^2}{9},$$

$$f_6(x,1,1)-Af_{0,-2}(x)=\frac{2x(x-1)^2g(x)}{9}\geq 0,$$

where

$$g(x)=9(x^2+1)(3x+11)-x(x-1)^2.$$

Since $x^2+1\geq(x-1)^2$, it suffices to show that

$$9(3x+11)\geq x,$$

which is clearly true.

The condition (b) in Corollary 1 is satisfied if $f_6(x,1,1) \ge Ax^2$ for $x \ge 4$. We have

$$f_6(x,1,1) - Ax^2 = 2xg(x),$$

where

$$g(x) = (x^2+1)(x-1)^2(3x+11) - 9x \ge x[2(x-1)^2(3x+11) - 9] \ge 0.$$

The condition (c) in Corollary 1, namely $f_6(0,y,z) \ge 0$ for $y,z \ge 0$, is satisfied.

Observation 1. Similarly, we can prove the following generalization:

- *Let x,y,z be nonnegative real numbers, no two of which are zero. If*

$$-2 < k \le \frac{3}{2},$$

then

$$\sum \frac{(6-4k)x + (4k-1)(y+z)}{y^2+kyz+z^2} \ge \frac{36(k+1)}{(k+2)(x+y+z)},$$

with equality for $x = y = z$, and for $x = 0$ and $y = z$ (or any cyclic permutation).

For

$$f_6(x,y,z) = (k+2)(x+y+z)f(x,y,z) - 36(k+1)\prod(y^2+kyz+z^2),$$

where

$$f(x,y,z) = \sum[(6-4k)x + (4k-1)(y+z)](x^2+kxy+y^2)(x^2+kxz+z^2),$$

we get

$$A = 36(k+1)(1-k)^3.$$

We have

$$\begin{aligned}\frac{f(x,1,1)}{2(x^2+kx+1)} &= [(3-2k)x+4k-1](x^2+kx+1) + (k+2)[(4k-1)x+5] \\ &= (3-2k)x^3 - (2k^2-7k+1)x^2 + (8k^2+4k+1)x + 9k + 9,\end{aligned}$$

$$\begin{aligned}f_6(x,1,1) &= (k+2)(x+2)f(x,1,1) - 36(k+1)(k+2)(x^2+kx+1)^2 \\ &= 2(k+2)x(x^2+kx+1)g(x),\end{aligned}$$

$$\begin{aligned}g(x) &= (3-2k)x^3 - (2k^2-3k-5)x^2 + (4k^2-19)x + 11 - k - 2k^2 \\ &= (x-1)^2[(3-2k)x + 11 - k - 2k^2],\end{aligned}$$

therefore

$$f_6(x,1,1) = 2(k+2)x(x^2+kx+1)(x-1)^2[(3-2k)x + 11 - k - 2k^2].$$

Since $3-2k\ge 0$ and $11-k-2k^2>0$ for $-2<k\le 3/2$, we have $f_6(x,1,1)\ge 0$ for all $x\ge 0$. Also, we have

$$f_6(0,y,z)\ge 0$$

for $y,z\ge 0$. To show this, we only need to prove the original inequality for $x=0$. Using the substitution

$$t=\frac{y^2+z^2}{yz},\qquad t\ge 2,$$

the original inequality can be written as

$$(t-2)(B_1t^2+B_2t+B_3)\ge 0,$$

where

$$B_1=(k+2)(6-4k),\quad B_2=(k+2)(17-2k-4k^2),\quad B_3=2+25k+5k^2-8k^3.$$

Since $B_1\ge 0$ and

$$B_3\ge 2+25k-4k^2-8k^3=(k+2)(1+12k-8k^2),$$

we have

$$\begin{aligned}B_1t^2+B_2t+B_3 &\ge (2B_1+B_2)t+B_3=(k+2)(29-10k-4k^2)t+B_3\\&\ge 2(k+2)(29-10k-4k^2)+(k+2)(1+12k-8k^2)\\&=(k+2)(59-8k-16k^2)>8(k+2)(6-k-2k^2)\\&=8(k+2)^2(3-2k)\ge 0.\end{aligned}$$

Case 1: $k\in(-2,-1]\cup[1,3/2]$. Since $A\le 0$, $f_6(x,1,1)\ge 0$ and $f_6(0,y,z)\ge 0$ for all $x,y,z\ge 0$, the conclusion follows by Theorem 1.

Case 2: $k\in[-1,1]$. Since $A\ge 0$, we apply Corollary 1 for

$$E_{\alpha,\beta}(x)=f_{0,-2}(x)=\frac{(x-1)^4x^2}{81}.$$

The condition (a) in Corollary 1, namely $f_6(x,1,1)\ge Af_{0,-2}(x)$ for $x\in[0,4]$, is satisfied if

$$9(k+2)(x^2+kx+1)[(3-2k)x+11-k-2k^2]\ge 2(k+1)(1-k)^3x(x-1)^2.$$

Since

$$9>8\ge 2(1-k)^2,$$

$$k+2>k+1,$$

$$x^2+kx+1\ge x^2-x+1\ge (x-1)^2,$$

$$(3-2k)x+11-k-2k^2>(3-2k)x\ge(1-k)x,$$

the conclusion follows.

The condition (b) in Corollary 1, namely $f_6(x,1,1) \ge Ax^2$ for $x \ge 4$, is satisfied if

$$(k+2)(x^2+kx+1)(x-1)^2[(3-2k)x+11-k-2k^2] \ge 18(k+1)(1-k)^3x.$$

Since

$$k+2 > k+1,$$

$$x^2+kx+1 \ge x^2-x+1 \ge x,$$

$$(x-1)^2[(3-2k)x+11-k-2k^2] \ge 9[4(3-2k)+11-k-2k^2] = 9(23-9k-2k^2),$$

it suffices to show that

$$23-9k-2k^2 \ge 2(1-k)^3.$$

Indeed,

$$23-9k-2k^2-2(1-k)^3 \ge 23-9k-2k^2-8(1-k) = 15-k-2k^2 > 0.$$

The condition (c) in Corollary 1 is satisfied because $f_6(0,y,z) \ge 0$ for $y,z \ge 0$.

Observation 2. For $k=-1$, $k=0$, $k=1/4$, $k=1$ and $k=3/2$, we get the following particular inequalities from the inequality in Observation 1:

$$\sum \frac{2x-y-z}{y^2-yz+z^2} \ge 0,$$

$$\sum \frac{6x-y-z}{y^2+z^2} \ge \frac{18}{x+y+z},$$

$$\sum \frac{x}{4y^2+yz+z^2} \ge \frac{1}{x+y+z},$$

$$\sum \frac{2x+3y+3z}{y^2+yz+z^2} \ge \frac{24}{x+y+z},$$

$$\sum \frac{y+z}{2y^2+3yz+2z^2} \ge \frac{18}{7(x+y+z)}.$$

□

P 4.31. *If x,y,z are nonnegative real numbers, no two of which are zero, then*

$$\sum \frac{2x-3y-3z}{y^2+4yz+z^2} + \frac{6}{x+y+z} \ge 0.$$

(Vasile C., 2014)

Solution. Write the inequality as $f_6(x,y,z) \ge 0$, where

$$f_6(x,y,z) = (x+y+z)f(x,y,z) + 6\prod(y^2+4yz+z^2),$$

$$f(x,y,z) = \sum(2x-3y-3z)(x^2+4xy+y^2)(x^2+4xz+z^2).$$

Since

$$y^2+4yz+z^2 = 4yz - x^2 + p^2 - 2q,$$

the product $\prod(y^2+4yz+z^2)$ has the same highest coefficient as

$$P_3(x,y,z) = \prod(4yz-x^2),$$

that is

$$A_1 = P_3(1,1,1) = 27.$$

Therefore, $f_6(x,y,z)$ has the highest coefficient

$$A = 6A_1 = 162.$$

We have

$$\frac{f(x,1,1)}{x^2+4x+1} = 2(x-3)(x^2+4x+1) - 12(3x+1) = 2(x^3+x^2-29x-9),$$

$$\begin{aligned}
f_6(x,1,1) &= (x+2)f(x,1,1) + 36(x^2+4x+1)^2 \\
&= 2(x^2+4x+1)\left[(x+2)(x^3+x^2-29x-9) + 18(x^2+4x+1)\right] \\
&= 2(x^2+4x+1)x(x^3+3x^2-9x+5) \\
&= 2x(x-1)^2(x+5)(x^2+4x+1) \ge 0.
\end{aligned}$$

Also, we have

$$f_6(0,y,z) \ge 0$$

for $y,z \ge 0$. This is true if the original inequality holds for $x=0$. Thus, we need to show that

$$\frac{-3(y+z)}{y^2+4yz+z^2} + \frac{2y-3z}{z^2} + \frac{2z-3y}{y^2} + \frac{6}{y+z} \ge 0,$$

which is equivalent to

$$\frac{-3(y+z)}{y^2+z^2+4yz} + \frac{2(y^3+z^3)-3yz(y+z)}{y^2z^2} + \frac{6}{y+z} \ge 0,$$

$$\frac{-3}{(y+z)^2+2yz} + \frac{2(y+z)^2-9yz}{y^2z^2} + \frac{6}{(y+z)^2} \ge 0,$$

$$\frac{-3yz}{(y+z)^2+2yz} + \frac{2(y+z)^2-9yz}{yz} + \frac{6yz}{(y+z)^2} \ge 0.$$

Using the substitution

$$t = \frac{(y+z)^2}{yz}, \qquad t \geq 4,$$

the inequality can be written as follows:

$$\frac{-3}{t+2} + 2t - 9 + \frac{6}{t} \geq 0,$$

$$2t^3 - 5t^2 - 15t + 12 \geq 0,$$

$$(t-4)(2t^2 + 3t - 3) \geq 0.$$

First Solution. We apply Theorem 3. The conditions (a) and (c) are satisfied. The condition (b) is satisfied if

$$f_6(x,1,1) \geq \frac{4Ax(x-1)^3}{27}$$

for $x \geq 1$. We have

$$f_6(x,1,1) - \frac{4Ax(x-1)^3}{27} = 2x(x-1)^2 f(x),$$

where

$$f(x) = (x+5)(x^2+4x+1) - 12(x-1) \geq 6(x^2+4x+1) - 12(x-1) = 6(x^2+2x+3) > 0.$$

The equality occurs for $x = y = z$, and for $x = 0$ and $y = z$ (or any cyclic permutation).

Second Solution. We apply Corollary 1 for

$$E_{\alpha,\beta}(x) = f_{0,-2}(x) = \frac{(x-1)^4 x^2}{81}.$$

The condition (a) in Corollary 1 are satisfied if $f_6(x,1,1) \geq Af_{0,-2}(x)$ for $x \in [0,4]$. We have

$$Af_{0,-2}(x) = \frac{2(x-1)^4 x^2}{9},$$

$$f_6(x,1,1) - Af_{0,-2}(x) = 2x(x-1)^2 g(x),$$

where

$$g(x) = (x^2 + 4x + 1)(x+5) - x(x-1)^2.$$

Since $x^2 + 4x + 1 \geq (x-1)^2$ and $x + 5 > x$, it follows that $g(x) > 0$.

The condition (b) in Corollary 1 is satisfied if $f_6(x,1,1) \geq Ax^2$ for $x \geq 4$. We have

$$f_6(x,1,1) - Ax^2 = 2xg(x),$$

where

$$g(x) = (x^2+4x+1)(x-1)^2(x+5) - 81x > x[x(x-1)^2(x+5) - 81]$$
$$\geq x(4\cdot 9\cdot 9 - 81) > 0.$$

The condition (c) in Corollary 1 is satisfied since $f_6(0,y,z) \geq 0$ for $y,z \geq 0$.

Observation. Similarly, we can prove the following generalization:

- *Let x,y,z be nonnegative real numbers, no two of which are zero. If*

$$k \geq \frac{\sqrt{89}-1}{4} \approx 2.1085,$$

then

$$\sum \frac{(4k-6)x + (1-4k)(y+z)}{y^2+kyz+z^2} + \frac{36(k+1)}{(k+2)(x+y+z)} \geq 0,$$

with equality for $x = y = z$, and for $x = 0$ and $y = z$ (or any cyclic permutation).

Write the inequality as $f_6(x,y,z) \geq 0$, where

$$f_6(x,y,z) = (k+2)(x+y+z)f(x,y,z) + 36(k+1)\prod(y^2+kyz+z^2),$$

$$f(x,y,z) = \sum[(4k-6)x + (1-4k)(y+z)](x^2+kxy+y^2)(x^2+kxz+z^2).$$

Since $f_6(x,y,z)$ is the opposite of $f_6(x,y,z)$ in Observation 1 from the preceding P 4.30, we have

$$A = 36(k+1)(k-1)^3,$$

$$f_6(x,1,1) = 2(k+2)x(x^2+kx+1)(x-1)^2[(2k-3)x + 2k^2 + k - 11].$$

Since $A > 0$, we apply Corollary 1 for

$$E_{\alpha,\beta}(x) = f_{0,-2}(x) = \frac{(x-1)^4x^2}{81}.$$

The condition (a) in Corollary 1, namely $f_6(x,1,1) \geq Af_{0,-2}(x)$ for $x \in [0,4]$, is satisfied if

$$9(k+2)(x^2+kx+1)[(2k-3)x + 2k^2 + k - 11] \geq 2(k+1)(k-1)^3x(x-1)^2.$$

Since

$$9 \geq (x-1)^2,$$

$$k+2 > k+1,$$

$$x^2+kx+1 > kx,$$

$$(2k-3)x + 2k^2 + k - 11 \geq 4(2k-3) + 2k^2 + k - 11 = 2k^2 + 9k - 23,$$

it suffices to show that

$$k(2k^2+9k-23) \geq 2(k-1)^3.$$

Indeed,

$$k(2k^2+9k-23)-2(k-1)^3 = 15k^2-29k+2 > 15k^2-30k = 15k(k-2) > 0.$$

The condition (b) in Corollary 1, namely $f_6(x,1,1) \geq Ax^2$ for $x \geq 4$, is satisfied if

$$(k+2)(x^2+kx+1)(x-1)^2[(2k-3)x+2k^2+k-11] \geq 18(k+1)(k-1)^3x.$$

Since

$$k+2 > k+1,$$

$$x^2+kx+1 > kx,$$

$$(x-1)^2[(2k-3)x+2k^2+k-11] \geq 9[4(3-2k)+11-k-2k^2] = 9(2k^2+9k-23),$$

it suffices to show that

$$k(2k^2+9k-23) \geq 2(k-1)^3.$$

Indeed,

$$k(2k^2+9k-23)-2(k-1)^3 = 15k^2-29k+2 > 15k^2-30k = 15k(k-2) > 0.$$

The condition (c) in Corollary 1, namely $f_6(0,y,z) \geq 0$ for $y,z \geq 0$, is satisfied if the original inequality holds for $x=0$. Thus, we need to show that

$$(t-2)(B_1t^2+B_2t+B_3) \geq 0,$$

where

$$t = \frac{y^2+z^2}{yz} \geq 2,$$

$$B_1 = (k+2)(4k-6), \qquad B_2 = (k+2)(4k^2+2k-17), \qquad B_3 = 8k^3-5k^2-25k-2.$$

Since $t \geq 2$, $B_1 > 0$ and $B_2 \geq 0$, we have

$$B_1t^2+B_2t+B_3 \geq 4B_1+B_3 = (8k^3-50)+k(11k-17) > 0.$$

□

P 4.32. *If x,y,z are nonnegative real numbers, no two of which are zero, then*

$$\sum \frac{7x+4y+4z}{4x^2+yz} \geq \frac{27}{x+y+z}.$$

(Vasile C., 2014)

Solution. Write the inequality as $f_6(x,y,z) \ge 0$, where

$$f_6(x,y,z) = (x+y+z)f(x,y,z) + 27\prod(4x^2+yz),$$

$$f(x,y,z) = \sum(7x+4y+4z)(4y^2+zx)(4z^2+xy).$$

The product

$$P_3(x,y,z) = \prod(4x^2+yz)$$

has the highest coefficient

$$A_1 = P_3(1,1,1) = 125.$$

Therefore, $f_6(x,y,z)$ has the highest coefficient

$$A = -27A_1 < 0.$$

Since $A < 0$, we only need to show that $f_6(x,1,1) \ge 0$ and $f_6(0,y,z) \ge 0$ for $x,y,z \ge 0$. The first inequality is true if the original inequality holds for $y = z = 1$. So, we need to show that

$$\frac{7x+8}{4x^2+1} + \frac{2(4x+11)}{x+4} \ge \frac{27}{x+2},$$

which is equivalent to

$$x(32x^3+51x^2-198x+115) \ge 0,$$

$$x(x-1)^2(32x+115) \ge 0.$$

The second inequality is true if the original inequality holds for $x = 0$. We need to show that

$$\frac{4(y+z)}{yz} + \frac{7y+4z}{4y^2} + \frac{7z+4y}{4z^2} \ge \frac{27}{y+z},$$

which can be rewritten as follows:

$$\frac{4(y+z)}{yz} + \frac{4(y^3+z^3)+7yz(y+z)}{4y^2z^2} \ge \frac{27}{y+z},$$

$$\frac{4}{yz} + \frac{4(y^2+z^2)+3yz}{4y^2z^2} \ge \frac{27}{(y+z)^2},$$

$$4 + \frac{4(y+z)^2-5yz}{4yz} \ge \frac{27yz}{(y+z)^2}.$$

Substituting

$$t = \frac{(y+z)^2}{yz}, \qquad t \ge 4,$$

the inequality becomes

$$4+\frac{4t-5}{4}\ge\frac{27}{t},$$
$$(t-4)(4t+27)\ge 0.$$

The equality occurs for $x=y=z$, and for $x=0$ and $y=z$ (or any cyclic permutation).

□

P 4.33. *If x,y,z are nonnegative real numbers, then*

$$\sum\frac{9x-2y-2z}{7x^2+8yz}\le\frac{3}{x+y+z}.$$

(Vasile C., 2014)

Solution. Write the inequality as $f_6(x,y,z)\ge 0$, where

$$f_6(x,y,z)=3\prod(7x^2+8yz)-(x+y+z)f(x,y,z),$$

$$f(x,y,z)=\sum(9x-2y-2z)(7y^2+8zx)(7z^2+8xy).$$

The product

$$P_3(x,y,z)=\prod(7x^2+8yz)$$

has the highest coefficient

$$A_1=P_3(1,1,1)=15^3.$$

Therefore, $f_6(x,y,z)$ has the highest coefficient

$$A=3A_1=3\cdot 15^3.$$

We have

$$\begin{aligned}\frac{f(x,1,1)}{8x+7}&=(9x-4)(8x+7)+2(2x-7)(7x^2+8)\\&=-(28x^3-170x^2+x-84),\end{aligned}$$

$$\begin{aligned}f_6(x,1,1)&=3(7x^2+8)(8x+7)^2-(x+2)f(x,1,1)\\&=(8x+7)\left[3(7x^2+8)(8x+7)+(x+2)(28x^3-170x^2+x-84)\right]\\&=2x(8x+7)(14x^3+27x^2-96x+55)\\&=2x(x-1)^2(8x+7)(14x+55)\ge 0.\end{aligned}$$

Also, we have

$$f_6(0,y,z)\ge 0$$

for $y,z \ge 0$. This is true if the original inequality holds for $x = 0$. Thus, we need to show that

$$\frac{y+z}{4yz} + \frac{2z-9y}{7y^2} + \frac{2y-9z}{7z^2} + \frac{3}{y+z} \ge 0,$$

which is equivalent to

$$\frac{y+z}{4yz} + \frac{2(y^3+z^3)-9yz(y+z)}{7y^2z^2} + \frac{3}{y+z} \ge 0,$$

$$\frac{1}{4yz} + \frac{2(y^2+z^2)-11yz}{7y^2z^2} + \frac{3}{(y+z)^2} \ge 0,$$

$$\frac{1}{4} + \frac{2(y+z)^2-15yz}{7yz} + \frac{3yz}{(y+z)^2} \ge 0,$$

Using the substitution

$$t = \frac{(y+z)^2}{yz}, \qquad t \ge 4,$$

the inequality can be written as follows:

$$\frac{1}{4} + \frac{2t-15}{7} + \frac{3}{t} \ge 0,$$

$$8t^2 - 53t + 84 \ge 0,$$

$$(t-4)(8t-21) \ge 0.$$

First Solution. We apply Theorem 3. The conditions (a) and (c) are satisfied. The condition (b) is satisfied if

$$f_6(x,1,1) \ge \frac{4Ax(x-1)^3}{27}$$

for $x \ge 1$. We have

$$f_6(x,1,1) - \frac{4Ax(x-1)^3}{27} = 2x(x-1)^2 f(x),$$

where

$$\begin{aligned} f(x) &= (8x+7)(14x+55) - 750(x-1) = 112x^2 - 212x + 1135 \\ &> 106x^2 - 212x + 106 = 106(x-1)^2 \ge 0. \end{aligned}$$

The equality occurs for $x = y = z$, and for $x = 0$ and $y = z$ (or any cyclic permutation).

Second Solution. We apply Corollary 1 for

$$E_{\alpha,\beta}(x) = f_{0,-2}(x) = \frac{(x-1)^4x^2}{81}.$$

The condition (a) in Corollary 1 are satisfied if $f_6(x,1,1) \ge Af_{0,-2}(x)$ for $x \in [0,4]$. We have

$$Af_{0,-2}(x) = 125(x-1)^4x^2,$$
$$f_6(x,1,1) - Af_{0,-2}(x) = x(x-1)^2g(x),$$

where

$$\begin{aligned} g(x) &= 2(8x+7)(14x+55) - 125x(x-1)^2 \\ &\ge 2(8x+7)(14x+55) - 500(x-1)^2 \\ &> 2(8x+6)(15x+50) - 500(x-1)^2 \\ &= 20\left[(4x+3)(3x+10) - 25(x-1)^2\right] \\ &= 20[5 + x(99-13x)] > 0. \end{aligned}$$

The condition (b) in Corollary 1 is satisfied if $f_6(x,1,1) \ge Ax^2$ for $x \ge 4$. We have

$$f_6(x,1,1) - Ax^2 = xg(x),$$

where

$$g(x) = 2(8x+7)(x-1)^2(14x+55) - 3 \cdot 15^3 x.$$

Since

$$2(8x+7)(x-1)^2 > 15x(x-1)^2 \ge 135x,$$

we get

$$g(x) > 135x(14x+55) - 3 \cdot 15^3 x = 135x(14x+55-75) > 0.$$

The condition (c) in Corollary 1 is satisfied because $f_6(0,y,z) \ge 0$ for $y,z \ge 0$.

□

P 4.34. *If x,y,z are nonnegative real numbers, then*

$$\sum \frac{y+z}{7x^2+y^2+z^2} \ge \frac{2}{x+y+z}.$$

(Vasile C., 2014)

Solution. Write the inequality as $f_6(x,y,z) \ge 0$, where

$$f_6(x,y,z) = (x+y+z)\sum(y+z)(7y^2+z^2+x^2)(7z^2+x^2+y^2) - 2\prod(7x^2+y^2+z^2).$$

Since

$$7x^2+y^2+z^2 = 6x^2+p^2-2q,$$

the product $\prod(x^2+4y^2+4z^2)$ has the same highest coefficient A_1 as $(6x^2)(6y^2)(6z^2)$, that is $A_1 = 216$. Therefore, $f_6(x,y,z)$ has the highest coefficient

$$A = -2A_1 = -432.$$

According to Theorem 1, we only need to show that $f_6(x,1,1) \ge 0$ and $f_6(0,y,z) \ge 0$ for $x,y,z \ge 0$. The first condition is true if the original inequality holds for $y=z=1$. Thus, we need to show that

$$\frac{2}{7x^2+2}+\frac{2(x+1)}{x^2+8} \ge \frac{2}{x+2},$$

which is equivalent to

$$(x-1)^2(11x+2) \ge 0.$$

The second condition is true if the original inequality holds for $x=0$. Thus, we need to show that

$$\frac{y+z}{y^2+z^2}+\frac{z}{7y^2+z^2}+\frac{y}{7z^2+y^2} \ge \frac{2}{y+z},$$

which is equivalent to

$$\frac{y+z}{y^2+z^2}+\frac{7(y^3+z^3)+yz(y+z)}{7(y^4+z^4)+50y^2z^2} \ge \frac{2}{y+z},$$

$$\frac{1}{y^2+z^2}+\frac{7(y^2+z^2)-6yz}{7(y^4+z^4)+50y^2z^2} \ge \frac{2}{(y+z)^2}.$$

For $yz=0$, the inequality is an equality. For $yz \ne 0$, we write the inequality as

$$\frac{yz}{y^2+z^2}+\frac{7yz(y^2+z^2)-6y^2z^2}{7(y^2+z^2)^2+36y^2z^2} \ge \frac{2yz}{y^2+z^2+2yz}.$$

Substituting

$$t=\frac{y^2+z^2}{yz}, \qquad t \ge 2,$$

the inequality becomes as follows:

$$\frac{1}{t}+\frac{7t-6}{7t^2+36} \ge \frac{2}{t+2},$$

$$11t^2-24t+36 \ge 0.$$

We have

$$11t^2-24t+36 > 6t^2-24t+24 = 6(t-2)^2 \ge 0.$$

The equality occurs for $x=y=z$.

□

P 4.35. *If x,y,z are nonnegative real numbers, then*

$$\sum \frac{7x-2y-2z}{x^2+4y^2+4z^2} \ge \frac{3}{x+y+z}.$$

(Vasile C., 2014)

Solution. Write the inequality as $f_6(x,y,z) \ge 0$, where

$$f_6(x,y,z) = (x+y+z)f(x,y,z) - 3\prod(x^2+4y^2+4z^2),$$

$$f(x,y,z) = \sum(7x-2y-2z)(y^2+4z^2+4x^2)(z^2+4x^2+4y^2).$$

Since

$$x^2+4y^2+4z^2 = -3x^2+4(p^2-2q),$$

the product $\prod(x^2+4y^2+4z^2)$ has the same highest coefficient A_1 as $(-3x^2)(-3y^2)(-3z^2)$, that is $A_1=-27$. Therefore, $f_6(x,y,z)$ has the highest coefficient

$$A = -3A_1 = 81.$$

We have

$$\begin{aligned}\frac{f(x,1,1)}{4x^2+5} &= (7x-4)(4x^2+5)+2(5-2x)(x^2+8)\\ &= 3(8x^3-2x^2+x+20),\end{aligned}$$

$$\begin{aligned}f_6(x,1,1) &= (x+2)f(x,1,1) - 3(x^2+8)(4x^2+5)^2\\ &= 3(4x^2+5)\left[(x+2)(8x^3-2x^2+x+20)-(x^2+8)(4x^2+5)\right]\\ &= 6x(4x^2+5)(2x^3+7x^2-20x+11)\\ &= 6x(4x^2+5)(x-1)^2(2x+11) \ge 0.\end{aligned}$$

Also, we have

$$f_6(0,y,z) \ge 0$$

for $y,z \ge 0$. This is true if the original inequality holds for $x=0$. Thus, we need to show that

$$\frac{-(y+z)}{2(y^2+z^2)} + \frac{7y-2z}{y^2+4z^2} + \frac{7z-2y}{z^2+4y^2} \ge \frac{3}{y+z},$$

which is equivalent to

$$\frac{-(y+z)}{2(y^2+z^2)} + \frac{26(y^3+z^3)-yz(y+z)}{4(y^4+z^4)+17y^2z^2} \ge \frac{3}{y+z},$$

$$\frac{-1}{2(y^2+z^2)} + \frac{26(y^2+z^2)-27yz}{4(y^2+z^2)^2+9y^2z^2} \ge \frac{3}{(y+z)^2}.$$

If $yz=0$, then the inequality is an equality. For $yz \ne 0$, write the inequality as

$$\frac{-yz}{2(y^2+z^2)} + \frac{26yz(y^2+z^2)-27y^2z^2}{4(y^2+z^2)^2+9y^2z^2} \ge \frac{3yz}{y^2+z^2+2yz}.$$

Using the substitution

$$t = \frac{y^2+z^2}{yz}, \qquad t \ge 2,$$

the inequality becomes

$$\frac{-1}{2t}+\frac{26t-27}{4t^2+9}\geq\frac{3}{t+2},$$

$$8t^3+14t^2-57t-6\geq 0,$$

$$(t-2)(8t^2+30t+3)\geq 0.$$

First Solution. We apply Theorem 3. The conditions (a) and (c) are satisfied. The condition (b) is satisfied if

$$f_6(x,1,1)\geq\frac{4Ax(x-1)^3}{27}$$

for $x\geq 1$. We have

$$f_6(x,1,1)-\frac{4Ax(x-1)^3}{27}=6x(x-1)^2f(x),$$

where

$$\begin{aligned}f(x)&=(4x^2+5)(2x+11)-2(x-1)\\&\geq 9(2x+11)-2(x-1)=16x+101>0.\end{aligned}$$

The equality occurs for $x=y=z$, and for $x=0$ and $y=z$ (or any cyclic permutation).

Second Solution. We apply Corollary 1 for

$$E_{\alpha,\beta}(x)=f_{0,-2}(x)=\frac{(x-1)^4x^2}{81}.$$

The condition (a) in Corollary 1 are satisfied if $f_6(x,1,1)\geq Af_{0,-2}(x)$ for $x\in[0,4]$. We have

$$Af_{0,-2}(x)=(x-1)^4x^2,$$

$$f_6(x,1,1)-Af_{0,-2}(x)=x(x-1)^2g(x),$$

where

$$g(x)=6(4x^2+5)(2x+11)-x(x-1)^2.$$

We have $g(x)>0$ since $4x^2+5>(x-1)^2$ and $2x+11>x$.

The condition (b) in Corollary 1 is satisfied if $f_6(x,1,1)\geq Ax^2$ for $x\geq 4$. We have

$$f_6(x,1,1)-Ax^2=3xg(x),$$

where

$$g(x)=2(4x^2+5)(x-1)^2(2x+11)-27x>0.$$

The condition (c) in Corollary 1, namely $f_6(0,y,z)\geq 0$ for $y,z\geq 0$, is satisfied.

$\square$

P 4.36. *If x, y, z are nonnegative real numbers, no two of which are zero, then*

$$\sum \frac{2x^2+yz}{y^2+z^2} \ge \frac{9}{2} + \frac{31(x-y)^2(y-z)^2(z-x)^2}{2(x^2+y^2)(y^2+z^2)(z^2+x^2)}.$$

(Vasile C., 2014)

Solution. Write the inequality as $f_6(x,y,z) \ge 0$, where

$$f_6(x,y,z) = 2\sum(2x^2+yz)(x^2+y^2)(x^2+z^2) - 9\prod(y^2+z^2) - 31\prod(y-z)^2.$$

Since

$$y^2+z^2 = -x^2+p^2-2q,$$

$f_6(x,y,z)$ has the same highest coefficient A as

$$2\sum(2x^2+yz)(-z^2)(-y^2) - 9(-x^2)(-y^2)(-z^2) - 31(x-y)^2(y-z)^2(z-x)^2,$$

that is

$$A = 2(6+3)+9-31(-27) = 32\cdot 27.$$

Also,

$$\begin{aligned} f_6(x,1,1) =& 2[(2x^2+1)(x^2+1)^2+4(2+x)(x^2+1)] - 18(x^2+1)^2 \\ &= 4x(x^2+1)(x^3-3x+2) = 4x(x^2+1)(x-1)^2(x+2). \end{aligned}$$

Thus, we apply Corollary 1 for

$$E_{\alpha,\beta}(x) = f_{0,-2}(x) = \frac{(x-1)^4x^2}{81}.$$

The condition (a) in Corollary 1 are satisfied if $f_6(x,1,1) \ge Af_{0,-2}(x)$ for $x \in [0,4]$. We have

$$Af_{0,-2}(x) = \frac{32(x-1)^4x^2}{3},$$

$$f_6(x,1,1) - Af_{0,-2}(x) = \frac{4x(x-1)^2g(x)}{3},$$

where

$$g(x) = 3(x^2+1)(x+2) - 8x(x-1)^2.$$

For $0 \le x \le 1$, we have $x+2 \ge 3x$, hence

$$g(x) \ge 9x(x^2+1) - 8x(x-1)^2 \ge 8x(x^2+1) - 8x(x-1)^2 = 16x^2 \ge 0.$$

For $1 \le x \le 4$, we have

$$3(x^2+1)(x+2) = 3x^3+6x^2+3x+6 > 3x^3+6x^2,$$

hence

$$g(x) > x[3x^2+6x-8(x-1)^2] = x(5x-2)(4-x) \geq 0.$$

The condition (b) in Corollary 1 is satisfied if $f_6(x,1,1) \geq Ax^2$ for $x \geq 4$. We have

$$f_6(x,1,1) - Ax^2 = 4xg(x),$$

where

$$g(x) = (x^2+1)(x-1)^2(x+2) - 216x.$$

Since $2(x+2) \geq 3x$, we have

$$\begin{aligned}2g(x) &\geq 3x(x^2+1)(x-1)^2 - 432x \\ &= 3x[(x^2+1)(x-1)^2 - 144] \\ &\geq 3x(17 \cdot 9 - 144] = 27x > 0.\end{aligned}$$

The condition (c) in Corollary 1 is satisfied if the original inequality holds for $x = 0$. Thus, we need to show that

$$\frac{yz}{y^2+z^2} + 2\left(\frac{y^2}{z^2} + \frac{z^2}{y^2}\right) \geq \frac{9}{2} + \frac{31(y-z)^2}{2(y^2+z^2)}.$$

Using the substitution

$$t = \frac{y^2+z^2}{yz}, \qquad t \geq 2,$$

we may write the inequality as follows:

$$\frac{1}{t} + 2(t^2-2) \geq \frac{9}{2} + \frac{31(t-2)}{2t},$$

$$(t-2)^2(t+4) \geq 0.$$

The equality occurs for $x = y = z$, and for $x = 0$ and $y = z$ (or any cyclic permutation).

□

P 4.37. *If x, y, z are nonnegative real numbers, no two of which are zero, then*

$$\sum \frac{2x^2-yz}{y^2-yz+z^2} \geq 3 + \frac{9(x-y)^2(y-z)^2(z-x)^2}{(x^2-xy+y^2)(y^2-yz+z^2)(z^2-zx+x^2)}.$$

(Vasile C., 2014)

Solution. Write the inequality as $f_6(x,y,z) \ge 0$, where

$$f_6(x,y,z) = \sum(2x^2-yz)(x^2-xy+y^2)(x^2-xz+z^2)-3\prod(y^2-yz+z^2)-9\prod(y-z)^2.$$

Since

$$y^2-yz+z^2 = -x^2-yz+p^2-2q,$$

$f_6(x,y,z)$ has the same highest coefficient A as

$$P_2(x,y,z)-3P_3(x,y,z)-9(x-y)^2(y-z)^2(z-x)^2,$$

where

$$P_2(x,y,z) = \sum(2x^2-yz)(-z^2-xy)(-y^2-xz), \qquad P_3(x,y,z) = \prod(-x^2-yz),$$

that is

$$A = P_2(1,1,1)-3P_3(1,1,1)-9(-27) = 3(2-1)(-1-1)^2-3(-1-1)^3-9(-27) = 279.$$

Also,

$$\begin{aligned} f_6(x,1,1) =& [(2x^2-1)(x^2-x+1)^2+2(2-x)(x^2-x+1)]-3(x^2-x+1)^2 \\ &= 2x(x^2-x+1)(x^3-x^2-x+1) = 2x(x^2-x+1)(x-1)^2(x+1). \end{aligned}$$

Thus, we apply Corollary 1 for

$$E_{\alpha,\beta}(x) = f_{0,-2}(x) = \frac{(x-1)^4x^2}{81}.$$

The condition (a) in Corollary 1 are satisfied if $f_6(x,1,1) \ge Af_{0,-2}(x)$ for $x \in [0,4]$. We have

$$Af_{0,-2}(x) = \frac{31(x-1)^4x^2}{9},$$

$$f_6(x,1,1)-Af_{0,-2}(x) = \frac{x(x-1)^2g(x)}{9},$$

where

$$g(x) = 18(x^2-x+1)(x+1)-31x(x-1)^2.$$

For $0 \le x \le 1$, we have $x+1 \ge 2x$, hence

$$g(x) \ge 36x(x^2-x+1)-31x(x-1)^2 \ge 31x(x^2-x+1)-31x(x-1)^2 = 31x^2 \ge 0.$$

For $1 \le x \le 4$, we have

$$18(x^2-x+1)(x+1) = 18x^3+18 > 18x^3,$$

hence

$$g(x) > x[18x^2-31(x-1)^2] \ge x[18x^2-32(x-1)^2] = 2x(7x-4)(4-x) \ge 0.$$

The condition (b) in Corollary 1 is satisfied if $f_6(x,1,1) \ge Ax^2$ for $x \ge 4$. We have

$$f_6(x,1,1) - Ax^2 = xg(x),$$

where

$$g(x) = 2(x^2 - x + 1)(x-1)^2(x+1) - 279x.$$

Since

$$4(x^2 - x + 1) - 13x = (x-4)(4x-1) \ge 0,$$

we have

$$\begin{aligned} 2g(x) &\ge 13x(x-1)^2(x+1) - 558x \\ &= x\left[13(x-1)^2(x+1) - 558\right] \\ &\ge x(13 \cdot 9 \cdot 5 - 558) = 27x > 0. \end{aligned}$$

The condition (c) in Corollary 1 is satisfied if the original inequality holds for $x = 0$. Thus, we need to show that

$$\frac{-yz}{y^2 - yz + z^2} + 2\left(\frac{y^2}{z^2} + \frac{z^2}{y^2}\right) \ge 3 + \frac{9(y-z)^2}{y^2 - yz + z^2}.$$

Using the substitution

$$t = \frac{y^2 + z^2}{yz}, \qquad t \ge 2,$$

we may write the inequality as

$$\frac{-1}{t-1} + 2(t^2 - 2) \ge 3 + \frac{9(t-2)}{t-1},$$

$$(t-2)^2(t+3) \ge 0.$$

The equality occurs for $x = y = z$, and for $x = 0$ and $y = z$ (or any cyclic permutation).

Observation 1. Similarly, we can prove the following generalization:

- *Let x, y, z be nonnegative real numbers, no two of which are zero. If*

$$-2 < k \le 2,$$

then

$$\sum \frac{2x^2 + (2k+1)yz}{y^2 + kyz + z^2} \ge \frac{3(2k+3)}{k+2} + \frac{(8k^2 + 30k + 31)(x-y)^2(y-z)^2(z-x)^2}{(k+2)\prod(y^2 + kyz + z^2)}.$$

with equality for $x = y = z$, and for $x = 0$ and $y = z$ (or any cyclic permutation).

For

$$f_6(x,y,z) = (k+2)\sum[2x^2 + (2k+1)yz](x^2 + kxy + y^2)(x^2 + kxz + z^2)$$

$$-3(2k+3)\prod(y^2+kyz+z^2)-(8k^2+30k+31)(x-y)^2(y-z)^2(z-x)^2,$$

we have

$$A=9(k+2)(2k^2+19k+48),$$

$$f_6(x,1,1)=2(k+2)x(x^2+kx+1)(x-1)^2(x+k+2).$$

Observation 2. For $k=0$ and $k=-1$, the inequality in Observation 1 leads to the particular inequalities in P 4.36 and P 4.37. For $k=-1/2$ and $k=1$, we get to the particular inequalities

$$\sum\frac{x^2}{2y^2-yz+2z^2}\geq 1+\frac{24(x-y)^2(y-z)^2(z-x)^2}{\prod(2y^2-yz+2z^2)},$$

$$\sum\frac{2x^2+3yz}{y^2+yz+z^2}\geq 5+\frac{23(x-y)^2(y-z)^2(z-x)^2}{\prod(y^2+yz+z^2)}.$$

□

P 4.38. *If x,y,z are nonnegative real numbers, no two of which are zero, then*

$$\sum\frac{xy-yz+zx}{y^2+z^2}\geq\frac{3}{2}+\frac{5(x-y)^2(y-z)^2(z-x)^2}{2(x^2+y^2)(y^2+z^2)(z^2+x^2)}.$$

(Vasile C., 2014)

Solution. Write the inequality as $f_6(x,y,z)\geq 0$, where

$$f_6(x,y,z)=2\sum(xy-yz+zx)(x^2+y^2)(x^2+z^2)-3\prod(y^2+z^2)-5\prod(y-z)^2.$$

Since

$$xy-yz+zx=-2yz+q,\qquad y^2+z^2=-x^2+p^2-2q,$$

$f_6(x,y,z)$ has the same highest coefficient A as

$$2\sum(-2yz)(-z^2)(-y^2)-3(-x^2)(-y^2)(-z^2)-5(x-y)^2(y-z)^2(z-x)^2,$$

that is

$$A=-12+3-5(-27)=126.$$

Also,

$$\begin{aligned}f_6(x,1,1)&=2[(2x-1)(x^2+1)^2+4(x^2+1)]-6(x^2+1)^2\\&=4x(x^2+1)(x-1)^2.\end{aligned}$$

Thus, we may apply Corollary 1 (for $E_{\alpha,\beta}=f_{0,-2}$) or Theorem 3. Since the last method is more simple, we will apply it.

The condition (a) in Theorem 3, namely $f_6(x,1,1) \geq 0$ for $0 \leq x \leq 1$, is clearly satisfied.

The condition (b) in Theorem 3 is satisfied if $f_6(x,1,1) \geq \dfrac{4Ax(x-1)^3}{27}$ for $x \geq 1$. We have

$$\frac{4Ax(x-1)^3}{27} = \frac{56x(x-1)^3}{3},$$

$$f_6(x,1,1) - \frac{4Ax(x-1)^3}{27} = \frac{4x(x-1)^2 f(x)}{3},$$

where

$$f(x) = 3(x^2+1) - 14(x-1) = 3x^2 - 14x + 17 = 3\left(x - \frac{7}{3}\right)^2 + \frac{2}{3} > 0.$$

The condition (c) in Theorem 3 is satisfied if the original inequality holds for $x = 0$. Thus, we need to show that

$$\frac{-yz}{y^2+z^2} + \frac{y^2+z^2}{yz} \geq \frac{3}{2} + \frac{5(y-z)^2}{2(y^2+z^2)}.$$

Using the substitution

$$t = \frac{y^2+z^2}{yz}, \qquad t \geq 2,$$

we may write the inequality as follows:

$$\frac{-1}{t} + t \geq \frac{3}{2} + \frac{5(t-2)}{2t},$$

$$(t-2)^2 \geq 0.$$

The equality occurs for $x = y = z$, and for $x = 0$ and $y = z$ (or any cyclic permutation).

□

P 4.39. *If x, y, z are nonnegative real numbers, no two of which are zero, then*

$$\sum \frac{xy - 2yz + zx}{y^2 - yz + z^2} \geq \frac{3(x-y)^2(y-z)^2(z-x)^2}{(x^2-xy+y^2)(y^2-yz+z^2)(z^2-zx+x^2)}.$$

(Vasile C., 2014)

Solution. Write the inequality as $f_6(x,y,z) \geq 0$, where

$$f_6(x,y,z) = \sum (xy - 2yz + zxz)(x^2 - xy + y^2)(x^2 - xz + z^2) - 3\prod (y-z)^2.$$

Since

$$xy - 2yz + zxz = -3yz + q, \qquad x^2 - xy + y^2 = -z^2 - xy + p^2 - 2q,$$

$f_6(x, y, z)$ has the same highest coefficient A as

$$\sum(-3yz)(-z^2 - xy)(-y^2 - xz) - 3(x-y)^2(y-z)^2(z-x)^2,$$

that is

$$A = 3(-3)(-1-1)^2 - 3(-27) = 45.$$

Also,

$$\begin{aligned} f_6(x,1,1) =& [(2x-2)(x^2-x+1)^2 + 2(1-x)(x^2-x+1)] \\ =& \, 2x(x^2-x+1)(x-1)^2. \end{aligned}$$

Thus, we apply Corollary 1 for

$$E_{\alpha,\beta}(x) = f_{0,-2}(x) = \frac{(x-1)^4x^2}{81}.$$

The condition (a) in Corollary 1 are satisfied if $f_6(x,1,1) \geq Af_{0,-2}(x)$ for $x \in [0,4]$. We have

$$Af_{0,-2}(x) = \frac{5(x-1)^4x^2}{9},$$

$$f_6(x,1,1) - Af_{0,-2}(x) = \frac{x(x-1)^2g(x)}{9},$$

where

$$g(x) = 18(x^2-x+1) - 5x(x-1)^2.$$

For $0 \leq x \leq 1$, we have $18(x^2-x+1) \geq 18(1-x) \geq 5(1-x)$, hence

$$g(x) \geq 5(1-x) - 5x(1-x) = 5(1-x)^2 \geq 0.$$

For $1 \leq x \leq 4$, we have $18(x^2-x+1) \geq 18x(x-1) \geq 15x(x-1)$, hence

$$g(x) \geq 15x(x-1) - 5x(x-1)^2 = 5x(x-1)(4-x) \geq 0.$$

The condition (b) in Corollary 1 is satisfied if $f_6(x,1,1) \geq Ax^2$ for $x \geq 4$. We have

$$f_6(x,1,1) - Ax^2 = xg(x),$$

where

$$g(x) = 2(x^2-x+1)(x-1)^2 - 45x.$$

We have

$$\begin{aligned} g(x) &\geq 2(x^2-x)(x-1)^2 - 45x = x[2(x-1)^3 - 45] \\ &\geq x(54-45) > 0. \end{aligned}$$

The condition (c) in Corollary 1 is satisfied if the original inequality holds for $x=0$. Thus, we need to show that

$$\frac{-2yz}{y^2-yz+z^2}+\frac{y^2+z^2}{yz}\geq\frac{3(y-z)^2}{y^2-yz+z^2}.$$

Using the substitution

$$t=\frac{y^2+z^2}{yz},\qquad t\geq 2,$$

we may write the inequality as

$$\frac{-2}{t-1}+t\geq\frac{3(t-2)}{t-1},$$

$$(t-2)^2\geq 0.$$

The equality occurs for $x=y=z$, and for $x=0$ and $y=z$ (or any cyclic permutation).

Observation 1. Similarly, we can prove the following generalization:

- *Let x,y,z be nonnegative real numbers, no two of which are zero. If*

$$-2<k\leq 2,$$

then

$$\sum\frac{x(y+z)+(k-1)yz}{y^2+kyz+z^2}\geq\frac{3(k+1)}{k+2}+\frac{(k^2+3k+5)(x-y)^2(y-z)^2(z-x)^2}{(k+2)\prod(y^2+kyz+z^2)}.$$

with equality for $x=y=z$, and for $x=0$ and $y=z$ (or any cyclic permutation).

For

$$\begin{aligned}f_6(x,y,z)=&(k+2)\sum[x(y+z)+(k-1)yz](x^2+kxy+y^2)(x^2+kxz+z^2)\\&-3(k+1)\prod(y^2+kyz+z^2)-(k^2+3k+5)(x-y)^2(y-z)^2(z-x)^2,\end{aligned}$$

we have

$$A=9(k+2)(2k+7),$$

$$f_6(x,1,1)=2(k+2)x(x^2+kx+1)(x-1)^2.$$

Observation 2. For $k=0$ and $k=-1$, the inequality in Observation 1 leads to the particular inequalities in P 4.38 and P 4.39. For $k=1$ and $k=2$, we get to the particular inequalities

$$\sum\frac{x(y+z)}{y^2+yz+z^2}\geq 2+\frac{3(x-y)^2(y-z)^2(z-x)^2}{(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2)},$$

$$(xy+yz+zx)\sum\frac{1}{(x+y)^2}\geq\frac{9}{4}+\frac{15}{4}\left(\frac{x-y}{x+y}\right)^2\left(\frac{y-z}{y+z}\right)^2\left(\frac{x-x}{z+x}\right)^2.$$

□

P 4.40. *If x, y, z are nonnegative real numbers, no two of which are zero, then*

$$\sum \frac{x+3y+3z}{(y+2z)(2y+z)} \geq \frac{7(x+y+z)}{3(xy+yz+zx)}.$$

(Vasile C., 2014)

Solution. Write the inequality as $f_7(x,y,z) \geq 0$, where

$$\begin{aligned} f_7(x,y,z) =& 3(xy+yz+zx)\sum(x+3y+3z)(2x^2+5xy+2y^2)(2x^2+5xz+2z^2) \\ & -7(x+y+z)\prod(2y^2+5yz+2z^2). \end{aligned}$$

Since

$$2y^2+5yz+2z^2 = 5yz-2x^2+2(p^2-2q),$$

the product $\prod(2y^2+5yz+2z^2)$ has the same highest coefficient A_1 as

$$P_3(x,y,z) = \prod(5yz-2x^2),$$

that is

$$A_1 = P_3(1,1,1) = 27.$$

Therefore, $f_7(x,y,z) \geq 0$ has the highest polynomial

$$A(p,q) = -7(x+y+z)(27) = -189p \leq 0.$$

According to Corollary 2, we only need to show that

$$f_7(x,1,1) \geq 0$$

and

$$f_7(0,y,z) \geq 0$$

for $x, y, z \geq 0$. The first inequality is true if the original inequality holds for $y = z = 1$. So, we need to show that

$$\frac{x+6}{9} + \frac{2(3x+4)}{(x+2)(2x+1)} \geq \frac{7(x+2)}{3(2x+1)},$$

which is equivalent to

$$x(x-1)^2 \geq 0.$$

The second inequality is true if the original inequality holds for $x = 0$. Thus, we need to prove that

$$\frac{3(y+z)}{(y+2z)(2y+z)} + \frac{y+3z}{2z^2} + \frac{3y+z}{2y^2} \geq \frac{7(y+z)}{3yz},$$

which is equivalent to

$$\frac{3(y+z)}{(y+2z)(2y+z)}+\frac{(y+z)^3}{2y^2z^2}\geq\frac{7(y+z)}{3yz},$$

$$\frac{3}{(y+2z)(2y+z)}+\frac{(y+z)^2}{2y^2z^2}\geq\frac{7}{3yz},$$

$$\frac{3}{2(y^2+z^2)+5yz}+\frac{3(y^2+z^2)-8yz}{6y^2z^2}\geq 0,$$

$$6(y^2+z^2)^2-(y^2+z^2)yz-22y^2z^2\geq 0,$$

$$(y-z)^2[6(y^2+z^2)+11yz]\geq 0.$$

The equality occurs for $x=y=z$, and for $x=0$ and $y=z$ (or any cyclic permutation).

□

P 4.41. *If x,y,z are nonnegative real numbers, no two of which are zero, then*

$$\sum\frac{9x-5y-5z}{2y^2-3yz+2z^2}+\frac{3(x+y+z)}{xy+yz+zx}\geq 0.$$

(Vasile C., 2014)

Solution. Write the inequality as $f_7(x,y,z)\geq 0$, where

$$\begin{aligned}f_7(x,y,z)=&3(xy+yz+zx)\sum(9x-5y-5z)(2x^2-3xy+2y^2)(2x^2-3xz+2z^2)\\&+3(x+y+z)\prod(2y^2-3yz+2z^2).\end{aligned}$$

Since

$$2y^2-3yz+2z^2=-3yz-2x^2+2(p^2-2q),$$

the product $\prod(2y^2-3yz+2z^2)$ has the same highest coefficient A_1 as

$$P_3(x,y,z)=\prod(-3yz-2x^2),$$

that is

$$A_1=P_3(1,1,1)=-125.$$

Therefore, $f_7(x,y,z)\geq 0$ has the highest polynomial

$$A(p,q)=3(x+y+z)(-125)=-375p<0.$$

According to Corollary 2, we only need to prove the original inequality for $y=z=1$ and for $x=0$.

Case 1: $y = z = 1$. We need to show that

$$9x - 10 + \frac{2(4-5x)}{2x^2-3x+2} + \frac{3(x+2)}{2x+1} \ge 0,$$

which is equivalent to

$$x(x-1)^2 \ge 0.$$

Case 2: $x = 0$. We need to show that

$$\frac{-5(y+z)}{2y^2-3yz+2z^2} + \frac{9y-5z}{2z^2} + \frac{9z-5y}{2y^2} + \frac{3(y+z)}{yz} \ge 0,$$

which is equivalent to

$$\frac{-5(y+z)}{2y^2-3yz+2z^2} + \frac{(y+z)(9y^2+9z^2-14yz)}{2y^2z^2} + \frac{3(y+z)}{yz} \ge 0,$$

$$\frac{-5}{2y^2-3yz+2z^2} + \frac{9y^2+9z^2-14yz}{2y^2z^2} + \frac{3}{yz} \ge 0,$$

$$\frac{-5}{2(y^2+z^2)-3yz} + \frac{9(y^2+z^2)-8yz}{2y^2z^2} \ge 0,$$

$$18(y^2+z^2)^2 - 43(y^2+z^2) + 14y^2z^2 \ge 0,$$

$$(y-z)^2(18y^2+18z^2-7yz) \ge 0.$$

The equality occurs for $x = y = z$, and for $x = 0$ and $y = z$ (or any cyclic permutation).

Observation 1. Similarly, we can prove the following generalization:

- *Let x, y, z be nonnegative real numbers, no two of which are zero. If*

$$k \in [a, -1] \cup [1, b], \qquad a = \frac{1-\sqrt{17}}{2} \approx -1.56155, \qquad b = \frac{1+\sqrt{17}}{2} \approx 2.56155\ ,$$

then

$$\sum \frac{(3-k)x + (k-1)(y+z)}{y^2+kyz+z^2} \ge \frac{3(k+1)}{k+2} \cdot \frac{x+y+z}{xy+yz+zx},$$

with equality for $x = y = z$, and for $x = 0$ and $y = z$ (or any cyclic permutation).

For

$$f_7(x,y,z) = (k+2)(xy+yz+zx)f(x,y,z) - 3(k+1)(x+y+z)\prod(y^2+kyz+z^2),$$

where

$$f(x,y,z) = \sum[(3-k)x + (k-1)(y+z)](x^2+kxy+y^2)(x^2+kxz+z^2),$$

we have

$$A = -3(k+1)(k-1)^3 p \le 0,$$

$$f_7(x,1,1) = 2(k+2)x(x-1)^2(x^2+kx+1)[(3-k)x - k^2 + k + 4] \ge 0,$$

$$f_7(0,y,z) = yz(y+z)(y-z)^2[(k+2)(3-k)(y-z)^2 + (13+7k-k^2-k^3)yz] \ge 0.$$

Observation 2. For $k = 2$, the inequality from Observation 1 turns into the well-known Iran inequality:

$$\sum \frac{1}{(y+z)^2} \ge \frac{9}{4(xy+yz+zx)}.$$

□

P 4.42. *If x, y, z are nonnegative real numbers, no two of which are zero, then*

$$\sum \frac{3x-y-z}{y^2+z^2} \ge \frac{3(x+y+z)}{2(xy+yz+zx)}.$$

(Vasile C., 2014)

Solution. Write the inequality as $f_7(x,y,z) \ge 0$, where

$$f_7(x,y,z) = 2(xy+yz+zx)f(x,y,z) - 3(x+y+z)\prod(y^2+z^2),$$

$$f(x,y,z) = \sum(3x-y-z)(x^2+y^2)(x^2+z^2).$$

Since

$$y^2+z^2 = -x^2+p^2-2q,$$

the product $\prod(y^2+z^2)$ has the same highest coefficient A_1 as $(-x^2)(-y^2)(-z^2)$, that is $A_1 = -1$. Therefore, $f_7(x,y,z) \ge 0$ has the highest polynomial

$$A(p,q) = -3pA_1 = 3p \ge 0, \qquad A(x+2,2x+1) = 3(x+2).$$

We have

$$\begin{aligned} f_7(x,1,1) &= 2(2x+1)\left[(3x-2)(x^2+1)^2 + 4(2-x)(x^2+1)\right] - 6(x+2)(x^2+1)^2 \\ &= 4x(x-1)^2(x^2+1)(3x+4). \end{aligned}$$

On the other hand, for $x = 0$, the original inequality becomes

$$\frac{-y-z}{y^2+z^2} + \frac{3y-z}{z^2} + \frac{3z-y}{y^2} \ge \frac{3(y+z)}{2yz},$$

$$\frac{-yz}{y^2+z^2} + \frac{3(y^2+z^2)-4yz}{yz} \ge \frac{3}{2}.$$

Using the substitution

$$t = \frac{y^2+z^2}{yz}, \qquad t \geq 2,$$

the inequality becomes

$$-\frac{1}{t} + 3t - 4 \geq \frac{3}{2},$$

$$(t-2)(6t+1) \geq 0.$$

First Solution. Apply Theorem 6. The conditions (a) and (c) are satisfied. In what concerns the condition (b), we have

$$\frac{4A(x+2,2x+1)x(x-1)^3}{27} = \frac{4(x+2)x(x-1)^3}{9},$$

$$f_7(x,1,1) - \frac{4A(x+2,2x+1)x(x-1)^3}{27} = \frac{4x(x-1)^2 g(x)}{9},$$

where

$$g(x) = 9(x^2+1)(3x+4) - (x+2)(x-1).$$

For $x \geq 1$, we get

$$\begin{aligned} g(x) &> (x^2+1)(3x+4) - (x+2)(x-1) \\ &\geq 7(x^2+1) - (x+2)(x-1) = 6x^2 - x + 3 > 0. \end{aligned}$$

The equality occurs for $x = y = z$, and for $x = 0$ and $y = z$ (or any cyclic permutation).

Second Solution. Apply Corollary 3 for

$$E_{\alpha,\beta}(x) = f_{0,-2}(x) = \frac{x^2(x-1)^4}{81}.$$

Condition (a). It suffices to show that $f_7(x,1,1) \geq A(x+2,2x+1)f_{0,-2}(x)$ for $0 \leq x \leq 4$. We have

$$A(x+2,2x+1)f_{0,-2}(x) = \frac{x^2(x-1)^4(x+2)}{27},$$

$$f_7(x,1,1) - A(x+2,2x+1)f_{0,-2}(x) = \frac{x(x-1)^2 f(x)}{27},$$

$$f(x) = 108(x^2+1)(3x+4) - x(x-1)^2(x+2).$$

Since $x^2+1 \geq (x-1)^2$ and $3x+4 > 3x$, we get

$$\begin{aligned} f(x) &> 108(x-1)^2(3x) - x(x-1)^2(x+2) \\ &= x(x-1)^2(322-x) \geq 0. \end{aligned}$$

Condition (b). It suffices to show that $f_7(x,1,1) \ge A(x+2,2x+1)x^2$ for $x \ge 4$. Since

$$f_7(x,1,1) \ge 4x(x-1)^2(x^2+1)(3x),$$

we get

$$\begin{aligned} f_7(x,1,1) - A(x+2,2x+1)x^2 &\ge 4x(x-1)^2(x^2+1)(3x) - 3(x+2)x^2 \\ &= 3x^2[4(x-1)^2(x^2+1) - x - 2] > 0. \end{aligned}$$

Condition (c). This condition is satisfied because the original inequality holds for $x = 0$.

□

P 4.43. *Let x,y,z be nonnegative real numbers, no two of which are zero. If*

$$k \in [a,b], \qquad a = \frac{1-\sqrt{17}}{2} \approx -1.56155, \quad b = \frac{1+\sqrt{17}}{2} \approx 2.56155\ ,$$

then

$$\sum \frac{(3-k)x+(k-1)(y+z)}{y^2+kyz+z^2} \ge \frac{3(k+1)}{k+2} \cdot \frac{x+y+z}{xy+yz+zx}.$$

(Vasile C., 2014)

Solution. Write the inequality as $f_7(x,y,z) \ge 0$, where

$$f_7(x,y,z) = (k+2)(xy+yz+zx)f(x,y,z) - 3(k+1)(x+y+z)\prod(y^2+kyz+z^2),$$

$$f(x,y,z) = \sum[(3-k)x+(k-1)(y+z)](x^2+kxy+y^2)(x^2+kxz+z^2).$$

Since

$$y^2+kyz+z^2 = -x^2+kyz+p^2-2q,$$

the product $\prod(y^2+kyz+z^2)$ has the same highest coefficient A_1 as

$$P_3(x,y,z) = \prod(-x^2+kyz),$$

that is

$$A_1 = P_3(x1,1,1) = (k-1)^3.$$

Therefore, $f_7(x,y,z)$ has the highest polynomial

$$A(p,q) = -3(k+1)pA_1 = 3(1+k)(1-k)^3p.$$

We have

$$\begin{aligned} \frac{f(x,1,1)}{x^2+kx+1} &= [(3-k)x+2k-2](x^2+kx+1)+2(k+2)[(k-1)x+2] \\ &= (3-k)x^3-(k^2-5k+2)x^2+(2k^2-3k+3)x+2k-2, \end{aligned}$$

$$\begin{aligned}
f_7(x,1,1) &= (k+2)(2x+1)f(x,1,1) - 3(k+1)(k+2)(x+2)(x^2+kx+1)^2 \\
&= (k+2)(x^2+kx+1)g(x),
\end{aligned}$$

$$\begin{aligned}
g(x) &= (2x+1)\frac{f(x,1,1)}{x^2+kx+1} - 3(k+1)(x+2)(x^2+kx+1) \\
&= 2x\left[(3-k)x^3 - (k^2-3k+2)x^2 + (2k^2-3k-5)x - k^2+k+4\right] \\
&= 2x(x-1)^2\left[(3-k)x - k^2+k+4\right],
\end{aligned}$$

$$f_7(x,1,1) = 2(k+2)x(x-1)^2(x^2+kx+1)\left[(3-k)x-k^2+k+4\right] \ge 0.$$

and

$$f(0,y,z) = (k-1)(y+z)y^2z^2 + (y^2+kyz+z^2)\left[(3-k)(y^3+z^3)+(k-1)yz(y+z)\right],$$

$$\begin{aligned}
\frac{f(0,y,z)}{y+z} &= (k-1)y^2z^2 + (y^2+z^2+kyz)\left[(3-k)(y^2+z^2)+(2k-4)yz\right] \\
&= (3-k)(y^2+z^2)^2 - (k^2-5k+4)(y^2+z^2)yz + (2k^2-3k-1)y^2z^2,
\end{aligned}$$

$$\begin{aligned}
\frac{f_7(0,y,z)}{yz(y+z)} =& (k+2)\frac{f(0,y,z)}{y+z} - 3(k+1)yz(y^2+z^2+kyz) \\
=& (k+2)(3-k)(y^2+z^2)^2 - (k+2)(k^2-5k+4)(y^2+z^2)yz \\
& + (k+2)(2k^2-3k-1)y^2z^2 - 3(k+1)yz(y^2+z^2+kyz) \\
=& (k+2)(3-k)(y^2+z^2)^2 - (k^3-3k^2-3k+11)(y^2+z^2)yz \\
& + 2(k^3-k^2-5k-1)y^2z^2 \\
=& (y^2+z^2-2yz)\left[(k+2)(3-k)(y^2+z^2) - (k^3-k^2-5k-1)y^2z^2\right] \\
=& (y-z)^2\left[(k+2)(3-k)(y-z)^2 + (13+7k-k^2-k^3)yz\right] \ge 0.
\end{aligned}$$

Case 1: $k \in [a,-1] \cup [1,b]$. Since $A(p,q) \le 0$, it suffices to show that $f_7(x,1,1) \ge 0$ and $f_7(0,y,z) \ge 0$ for $x,y,z \ge 0$ (see Corollary 2). These condition are satisfied.

Case 2: $k \in [-1,1]$. Since $A(p,q) \ge 0$, we apply Corollary 3 for

$$E_{\alpha,\beta}(x) = f_{0,-2}(x) = \frac{x^2(x-1)^4}{81}.$$

Condition (a). It suffices to show that $f_7(x,1,1) \ge A(x+2,2x+1)f_{0,-2}(x)$ for $0 \le x \le 4$, where

$$A(x+2,2x+1) = 3(1+k)(1-k)^3(x+2).$$

We have

$$A(x+2,2x+1)f_{0,-2}(x) = \frac{(1+k)(1-k)^3x^2(x-1)^4(x+2)}{27},$$

$$f_7(x,1,1)-A(x+2,2x+1)f_{0,-2}(x)=\frac{x(x-1)^2f(x)}{27},$$

$$f(x)=54(k+2)(x^2+kx+1)[(3-k)x-k^2+k+4]-(1+k)(1-k)^3x(x-1)^2(x+2).$$

Since

$$k+2>k+1,$$

$$x^2+kx+1\geq(x-1)^2,$$

$$(3-k)x-k^2+k+4>(3-k)x,$$

$$6\geq x+2,$$

it suffices to show that

$$9\geq(1-k)^3.$$

Actually, we have $8\geq(1-k)^3$, which is equivalent to $2\geq 1-k$.

Condition (b). It suffices to show that $f_7(x,1,1)\geq A(x+2,2x+1)x^2$ for $x\geq 4$. Since

$$A(x+2,2x+1)x^2=3(1+k)(1-k)^3(x+2)x^2,$$

we need to prove that

$$2(k+2)(x^2+kx+1)(x-1)^2[(3-k)x-k^2+k+4]\geq 3(1+k)(1-k)^3(x+2)x.$$

Since

$$k+2>k+1,$$

$$x^2+kx+1>x+2,$$

$$(x-1)^2\geq 9,$$

$$(3-k)x-k^2+k+4>(3-k)x,$$

it suffices to show that

$$6(3-k)\geq(1-k)^3.$$

Indeed, we have

$$6(3-k)\geq 6(3-1)=12>8\geq(1-k)^3.$$

Condition (c). This condition is satisfied because $f_7(0,y,z)\geq 0$ for all $y,z\geq 0$.

The equality occurs for $x=y=z$, and for $x=0$ and $y=z$ (or any cyclic permutation).

□

P 4.44. *If x, y, z are nonnegative real numbers, no two of which are zero, then*

$$\sum \frac{x+13y+13z}{y^2+4yz+z^2} \ge \frac{27(x+y+z)}{2(xy+yz+zx)}.$$

(Vasile C., 2014)

Solution. Write the inequality as $f_7(x,y,z) \ge 0$, where

$$f_7(x,y,z) = 2(xy+yz+zx)f(x,y,z) - 27(x+y+z)\prod(y^2+4yz+z^2),$$

$$f(x,y,z) = \sum(x+13y+13z)(x^2+4xy+y^2)(x^2+4xz+z^2).$$

Since

$$y^2+4yz+z^2 = -x^2+4yz+p^2-2q,$$

the product $\prod(y^2+4yz+z^2)$ has the same highest coefficient A_1 as

$$P_3(x,y,z) = \prod(-x^2+4yz),$$

that is

$$A_1 = P_3(1,1,1) = 27.$$

Therefore, $f_7(x,y,z)$ has the highest polynomial

$$A(p,q) = -27pA_1 = -729p.$$

Since $A(p,q) \le 0$, according to Corollary 2, we only need to show that the original inequality holds for $y=z=1$, and for $x=0$. For $y=z=1$, the original inequality becomes

$$\frac{x+26}{6} + \frac{2(13x+14)}{x^2+4x+1} \ge \frac{27(x+2)}{2(2x+1)},$$

$$x^4-10x^3+33x^2-40x+16 \ge 0,$$

$$(x-1)^2(x-4)^2 \ge 0.$$

Also, for $x=0$, the original inequality becomes

$$\frac{13(y+z)}{y^2+4yz+z^2} + \frac{y+13z}{z^2} + \frac{z+13y}{y^2} \ge \frac{27(y+z)}{2yz},$$

$$\frac{13(y+z)}{y^2+4yz+z^2} + \frac{y^3+z^3+13yz(y+z)}{y^2z^2} \ge \frac{27(y+z)}{2yz},$$

$$\frac{13yz}{y^2+z^2+4yz} + \frac{y^2+z^2+12yz}{yz} \ge \frac{27}{2},$$

$$\frac{13}{t+4} + t + 12 \ge \frac{27}{2},$$

$$2t^2+5t+14 \ge 0,$$

where
$$t = \frac{y^2+z^2}{yz}.$$

The equality occurs for $x = y = z$, and for $x/4 = y = z$ (or any cyclic permutation).
□

P 4.45. *If x, y, z are nonnegative real numbers, no two of which are zero, then*
$$\sum \frac{-x+y+z}{2x^2+yz} \ge \frac{x+y+z}{xy+yz+zx}.$$

(Vasile C., 2014)

Solution. Write the inequality as $f_7(x,y,z) \ge 0$, where
$$f_7(x,y,z) = q\sum(-x+y+z)(2y^2+zx)(2z^2+xy) - (x+y+z)\prod(2x^2+yz).$$

Since
$$P_3(x,y,z) = \prod(2x^2+yz)$$
has the highest coefficient
$$A_1 = P_3(1,1,1) = 27,$$
$f_7(x,y,z)$ has the highest polynomial
$$A(p,q) = -A_1(x+y+z) = -27p < 0.$$

According to Corollary 2, we only need to show that $f_7(x,1,1) \ge 0$ and $f_7(0,y,z) \ge 0$ for $x,y,z \ge 0$. The first condition is true if the original inequality holds for $y = z = 1$. Thus, we need to show that
$$\frac{-x+2}{2x^2+1} + \frac{2x}{x+2} \ge \frac{x+2}{2x+1},$$
which is equivalent to
$$x(x+1)(x-1)^2) \ge 0.$$

The second condition is true if the original inequality holds for $x = 0$. Thus, we need to show that
$$\frac{y+z}{yz} + \frac{z-y}{2y^2} + \frac{y-z}{2z^2} \ge \frac{y+z}{yz},$$
which is equivalent to
$$(y+z)(y-z)^2 \ge 0.$$

The equality occurs for $x = y = z$, and for $x = 0$ and $y = z$ (or any cyclic permutation).

Observation. Similarly, we can prove the following generalization:

- *Let x, y, z be nonnegative real numbers, no two of which are zero. If $k \ge 1$, then*

$$\sum \frac{(k^2-4k+1)x+(2k-1)(y+z)}{kx^2+yz} \ge \frac{3(k-1)(x+y+z)}{xy+yz+zx},$$

with equality for $x = y = z$, and for $x = 0$ and $y = z$ (or any cyclic permutation).

For $k = 1$, the following particular inequality holds:

$$\frac{2x-y-z}{x^2+yz}+\frac{2y-z-x}{y^2+zx}+\frac{2z-x-y}{z^2+xy} \le 0.$$

□

P 4.46. *If x, y, z are nonnegative real numbers, no two of which are zero, then*

$$\sum \frac{11x-3y-3z}{2x^2+3yz} \le \frac{3(x+y+z)}{xy+yz+zx}.$$

(Vasile C., 2014)

Solution. Write the inequality as $f_7(x, y, z) \ge 0$, where

$$f_7(x,y,z) = 3(x+y+z)\prod(2x^2+3yz)-(xy+yz+zx)f(x,y,z),$$

$$f(x,y,z) = \sum(11x-3y-3z)(2y^2+3zx)(2z^2+3xy).$$

The product

$$P_3(x,y,z) = \prod(2x^2+3yz)$$

has the highest coefficient

$$A_1 = P_3(1,1,1) = 125.$$

Therefore, $f_6(x, y, z)$ has the highest polynomial

$$A(p,q) = 3A_1(x+y+z) = 375p, \qquad A(x+2,2x+1) = 375(x+2).$$

We have

$$\begin{aligned}\frac{f(x,1,1)}{3x+2} &= (11x-6)(3x+2)+2(8-3x)(2x^2+3)\\ &= -(12x^3-65x^2+14x-36),\end{aligned}$$

$$\begin{aligned}f_7(x,1,1) &= 3(x+2)(2x^2+3)(3x+2)^2-(2x+1)f(x,1,1)\\ &= (3x+2)\left[3(x+2)(2x^2+3)(3x+2)+(2x+1)(12x^3-65x^2+14x-36)\right]\\ &= 14x(3x+2)(3x^3-5x^2+x+1)\\ &= 14x(3x+2)(x-1)^2(3x+1).\end{aligned}$$

For $x=0$, the original inequality becomes

$$\frac{-y-z}{yz}+\frac{11y-3z}{2y^2}+\frac{11z-3y}{2z^2}\le\frac{3(y+z)}{yz},$$

which is equivalent to

$$\frac{4(y+z)}{yz}+\frac{3(y^3+z^3)-11yz(y+z)}{2y^2z^2}\ge 0,$$

$$(y+z)(y-z)^2\ge 0.$$

First Solution. Apply Theorem 6. The conditions (a) and (c) are satisfied. In what concerns the condition (b), we have

$$\frac{4A(x+2,2x+1)x(x-1)^3}{27}=\frac{500(x+2)x(x-1)^3}{9},$$

$$f_7(x,1,1)-\frac{4A(x+2,2x+1)x(x-1)^3}{27}=\frac{2x(x-1)^2g(x)}{9},$$

where

$$g(x)=63(3x+2)(3x+1)-250(x+2)(x-1).$$

For $x\ge 1$, we get

$$\begin{aligned}g(x) &> 50(3x+2)(3x+1)-250(x+2)(x-1)\\ &= 50(4x^2+4x+12)>0.\end{aligned}$$

The equality occurs for $x=y=z$, and for $x=0$ and $y=z$ (or any cyclic permutation).

Second Solution. Apply Corollary 3 for

$$E_{\alpha,\beta}(x)=f_{0,-2}(x)=\frac{(x-1)^4x^2}{81}.$$

The condition (a) in Corollary 3 are satisfied if $f_7(x,1,1)\ge A(x+2,2x+1)f_{0,-2}(x)$ for $x\in[0,4]$. We have

$$A(x+2,2x+1)f_{0,-2}(x)=\frac{125(x+2)x^2(x-1)^2}{27}\le 5(x+2)x^2(x-1)^2,$$

$$f_7(x,1,1)-A(x+2,2x+1)f_{0,-2}(x) \le x(x-1)^2 g(x),$$

where

$$g(x) = 14(3x+2)(3x+1) - 5x(x+2)(x-1)^2.$$

Since

$$(3x+2)(3x+1) > 5x(x+2),$$

we get

$$\begin{aligned} g(x) &\ge 70x(x+2) - 5x(x+2)(x-1)^2 = 5x(x+2)[14-(x-1)^2] \\ &\ge 5x(x+2)[9-(x-1)^2] = 5x(x+2)^2(4-x) \ge 0. \end{aligned}$$

The condition (b) in Corollary 3 is satisfied if $f_7(x,1,1) \ge A(x+2,2x+1)x^2$ for $x \ge 4$. We have

$$f_6(x,1,1)-A(x+2,2x+1)x^2 = xg(x),$$

where

$$g(x) = 14(3x+2)(x-1)^2(3x+1) - 375x(x+2).$$

Since

$$3x+2 > x+2, \qquad 3x+1 > 3x,$$

we get

$$g(x) \ge 3x(x+2)\left[14(x-1)^2 - 125\right] \ge 3x(x+2)(14\cdot 9 - 125) > 0.$$

The condition (c) in Corollary 3 is satisfied if $f_7(0,y,z) \ge 0$ for $y,z \ge 0$. This is true because the original inequality holds for $x = 0$.

Observation 1. Similarly, we can prove the following generalization:

- *Let x,y,z be nonnegative real numbers, no two of which are zero. If $\frac{1}{2} \le k \le 1$, then*

$$\sum \frac{(k^2-4k+1)x + (2k-1)(y+z)}{kx^2+yz} \ge \frac{3(k-1)(x+y+z)}{xy+yz+zx},$$

with equality for $x = y = z$, and for $x = 0$ and $y = z$ (or any cyclic permutation).

For $k = 1/2$, the following particular inequality holds (see P 1.101 in Volume 2):

$$\frac{x}{x^2+2yz} + \frac{y}{y^2+2zx} + \frac{z}{z^2+2xy} \le \frac{x+y+z}{xy+yz+zx}.$$

Observation 2. Having in view Observation 1 above and Observation from the preceding P 4.45, it follows that the concerned inequality holds for $k \ge \frac{1}{2}$.

□

P 4.47. *If x, y, z are nonnegative real numbers, no two of which are zero, then*

$$\frac{1}{2x^2+yz}+\frac{1}{2y^2+zx}+\frac{1}{2z^2+xy}\geq \frac{1}{xy+yz+zx}+\frac{2}{x^2+y^2+z^2}.$$

Solution. Write the inequality as $f_8(x,y,z)\geq 0$, where

$$f_8(x,y,z)=q(p^2-2q)\sum(2y^2+zx)(2z^2+xy)-p^2\prod(2x^2+yz).$$

The polynomial of degree eight $f_8(x,y,z)$ has the highest polynomial

$$A(p,q)=-A_1p^2,$$

where A_1 is the highest coefficient of the polynomial of degree six

$$P_3(x,y,z)=\prod(2x^2+yz).$$

Since

$$A_1=P_3(1,1,1)=27,$$

we have

$$A(p,q)=-27p^2<0.$$

According to Corollary 2, we only need to prove the original inequality for $y=z=1$ and for $x=0$. For $y=z=1$, the original inequality becomes

$$\frac{1}{2x^2+1}+\frac{2}{x+2}\geq\frac{1}{2x+1}+\frac{2}{x^2+2},$$

which is equivalent to

$$\frac{2x(x-1)}{(x+2)(x^2+2)}\geq\frac{2x(x-1)}{(2x+1)(2x^2+1)},$$

$$x(x^2+x+1)(x-1)^2\geq 0.$$

For $x=0$, the original inequality becomes

$$\frac{1}{2y^2}+\frac{1}{2z^2}\geq\frac{2}{y^2+z^2},$$

which is equivalent to

$$(y^2-z^2)^2\geq 0.$$

The equality occurs for $x=y=z$, and for $x=0$ and $y=z$ (or any cyclic permutation).

Observation 1. Similarly, we can prove the following generalization:

- *Let x, y, z be nonnegative real numbers, no two of which are zero.*

(a) *If $k \ge 2$, then*

$$k(k+1)\sum \frac{1}{x^2+kyz} \ge \frac{4(k+1)}{xy+yz+zx} + \frac{5k-4}{x^2+y^2+z^2};$$

(b) *If $\frac{1}{8} \le k \le 2$, then*

$$k(k+1)\sum \frac{1}{x^2+kyz} \ge \frac{4k^2-3k+2}{xy+yz+zx} + \frac{2(-2k^2+6k-1)}{x^2+y^2+z^2}.$$

Observation 2. From the inequalities in Observation 1, we get the following particular inequalities:

$$\frac{2}{x^2+8yz} + \frac{2}{y^2+8zx} + \frac{2}{z^2+8xy} \ge \frac{1}{xy+yz+zx} + \frac{1}{x^2+y^2+z^2},$$

$$\frac{1}{x^2+2yz} + \frac{1}{y^2+2zx} + \frac{1}{z^2+2xy} \ge \frac{2}{xy+yz+zx} + \frac{1}{x^2+y^2+z^2},$$

$$\frac{2}{x^2+yz} + \frac{2}{y^2+zx} + \frac{2}{z^2+xy} \ge \frac{3}{xy+yz+zx} + \frac{6}{x^2+y^2+z^2},$$

$$\frac{1}{2x^2+yz} + \frac{1}{2y^2+zx} + \frac{1}{2z^2+xy} \ge \frac{1}{xy+yz+zx} + \frac{2}{x^2+y^2+z^2},$$

$$\frac{5}{4x^2+yz} + \frac{5}{4y^2+zx} + \frac{5}{4z^2+xy} \ge \frac{6}{xy+yz+zx} + \frac{3}{x^2+y^2+z^2}.$$

□

P 4.48. *If x, y, z are nonnegative real numbers, no two of which are zero, then*

$$\frac{x(y+z)}{x^2+5yz} + \frac{y(z+x)}{y^2+5zx} + \frac{z(x+y)}{z^2+5xy} \le \frac{x^2+y^2+z^2}{xy+yz+zx}.$$

(Vasile C., 2012)

Solution. Write the inequality as $f_8(x,y,z) \ge 0$, where

$$f_8(x,y,z) = (x^2+y^2+z^2)\prod(x^2+5yz) - (xy+yz+zx)\sum x(y+z)(y^2+5zx)(z^2+5xy).$$

Since $x(y+z) = -yz+q$, the polynomial of degree eight $f_8(x,y,z)$ has the highest polynomial

$$A(p,q) = A_1(p^2-2q) + A_2 q,$$

where A_1 and A_2 are the highest coefficients of the polynomials of degree six

$$P_3(x,y,z) = \prod(x^2+5yz)$$

and

$$P_2(x,y,z) = \sum yz(y^2+5zx)(z^2+5xy),$$

respectively. Therefore, we have

$$A(p,q) = P_3(1,1,1)(p^2-2q) + P_2(1,1,1)q = 6^3(p^2-2q) + 3\cdot 6^2 q = 108(2p^2-3q).$$

On the other hand,

$$\begin{aligned} f_8(x,1,1) =& (x^2+2)(x^2+5)(5x+1)^2 \\ & -(2x+1)[2x(5x+1)^2 + 2(x+1)(x^2+5)(5x+1)] \\ =& x(5x+1)(5x^4-3x^3+9x^2-29x+18) \\ =& x(5x+1)(x-1)^2(5x^2+7x+18). \end{aligned}$$

Since $A(p,q) > 0$, we apply Corollary 3 for

$$E_{\alpha,\beta}(x) = f_{0,-2}(x) = \frac{x^2(x-1)^4}{81}.$$

Condition (a). It suffices to show that $f_8(x,1,1)(x) \ge A(x+2,2x+1)f_{0,-2}(x)$ for $0 \le x \le 4$, where

$$A(x+2,2x+1) = 108(2x^2+2x+5).$$

We have

$$A(x+2,2x+1)f_{0,-2}(x) = \frac{4x^2(x-1)^4(2x^2+2x+5)}{3},$$

$$f_8(x,1,1)(x) - A(x+2,2x+1)f_{0,-2}(x) = \frac{x(x-1)^2 f(x)}{3},$$

where

$$f(x) = 3(5x+1)(5x^2+7x+18) - 4x(2x^2+2x+5)(x-1)^2.$$

Since $5x+1 > 5x$ and $2(5x^2+7x+18) \ge 5(2x^2+2x+5)$, we get

$$\begin{aligned} 2f(x) &> 75x(2x^2+2x+5) - 8x(2x^2+2x+5)(x-1)^2 \\ &= x(2x^2+2x+5)[75-8(x-1)^2] \ge x(2x^2+2x+5)(75-72) \ge 0. \end{aligned}$$

Condition (b). It suffices to show that $f_8(x,1,1)(x) \ge A(x+2,2x+1)x^2$ for $x \ge 4$. The inequality is equivalent to

$$(5x+1)(x-1)^2(5x^2+7x+18) \ge 108(2x^2+2x+5)x,$$

which can be obtained by multiplying the inequalities

$$5x+1 > 5x,$$

$$5x^2+7x+18 > \frac{5}{2}(2x^2+2x+5),$$

$$25(x-1)^2 \geq 216.$$

Condition (c). It suffices to show that $f_8(0,y,z) \geq 0$ for $y,z \geq 0$. This is true if the original inequality holds for $x=0$. Thus, we need to show that

$$\frac{z}{y}+\frac{y}{z} \leq \frac{y^2+z^2}{yz},$$

which is an identity.

The equality occurs for $x=y=z$, and also for $x=0$ (or any cyclic permutation).

Observation 1. Similarly, we can prove the following generalization:

- *Let x,y,z be nonnegative real numbers, no two of which are zero. If $\frac{1}{4} \leq k \leq 5$, then*

$$\frac{x(y+z)}{x^2+kyz}+\frac{y(z+x)}{y^2+kzx}+\frac{z(x+y)}{z^2+kxy} \leq \frac{5-k}{1+k}+\frac{x^2+y^2+z^2}{xy+yz+zx}.$$

Observation 2. From the inequalities in Observation 1, we get the following particular inequalities:

$$\frac{2x(y+z)}{2x^2+yz}+\frac{2y(z+x)}{2y^2+zx}+\frac{2z(x+y)}{2z^2+xy} \leq 3+\frac{x^2+y^2+z^2}{xy+yz+zx},$$

$$\frac{x(y+z)}{x^2+yz}+\frac{y(z+x)}{y^2+zx}+\frac{z(x+y)}{z^2+xy} \leq \frac{(x+y+z)^2}{xy+yz+zx},$$

$$\frac{x(y+z)}{x^2+2yz}+\frac{y(z+x)}{y^2+2zx}+\frac{z(x+y)}{z^2+2xy} \leq 1+\frac{x^2+y^2+z^2}{xy+yz+zx},$$

$$\frac{x(y+z)}{x^2+4yz}+\frac{y(z+x)}{y^2+4zx}+\frac{z(x+y)}{z^2+4xy} \leq \frac{1}{5}+\frac{x^2+y^2+z^2}{xy+yz+zx},$$

$$\frac{x(y+z)}{x^2+5yz}+\frac{y(z+x)}{y^2+5zx}+\frac{z(x+y)}{z^2+5xy} \leq \frac{x^2+y^2+z^2}{xy+yz+zx}.$$

□

P 4.49. *If x,y,z are nonnegative real numbers, no two of which are zero, then*

$$\frac{x(y+z)}{x^2+yz}+\frac{y(z+x)}{y^2+zx}+\frac{z(x+y)}{z^2+xy}+2 \geq \frac{15(xy+yz+zx)}{(x+y+z)^2}.$$

(Vasile C., 2012)

Solution. Write the inequality as $f_8(x,y,z) \ge 0$, where

$$f_8(x,y,z) = p^2 \sum x(y+z)(y^2+zx)(z^2+xy) - (15q-2p^2)\prod(x^2+yz).$$

Consider further the nontrivial case

$$15q - 2p^2 \ge 0.$$

Since $x(y+z) = -yz+q$, the polynomial of degree eight $f_8(x,y,z)$ has the highest polynomial

$$A(p,q) = -A_1p^2 - A_2(15q-2p^2),$$

where A_1 and A_2 are the highest coefficients of the polynomials of degree six

$$P_2(x,y,z) = \sum yz(y^2+zx)(z^2+xy)$$

and

$$P_3(x,y,z) = \prod(x^2+yz),$$

respectively. Therefore, we have

$$A(p,q) = -P_2(1,1,1)p^2 - P_3(1,1,1)(15q-2p^2) = -12p^2 - 8(15q-2p^2) < 0.$$

According to Theorem 4, it suffices to prove the original inequality for $y = z = 1$ and for $x = 0$.

Case 1: $y = z = 1$. We need to show that

$$\frac{2x}{x^2+1} + 4 \ge \frac{15(2x+1)}{(x+2)^2},$$

which is equivalent to

$$(x-1)^2(2x-1)^2 \ge 0.$$

Case 2: $x = 0$. We need to show that

$$\frac{z}{y} + \frac{y}{z} + 2 \ge \frac{15yz}{(y+z)^2},$$

which is equivalent to

$$(y+z)^4 \ge 15y^2z^2,$$

$$(y+z)^4 - 16y^2z^2 + y^2z^2 \ge 0,$$

$$(y-z)^2[(y+z)^2 + 4yz] + y^2z^2 \ge 0.$$

The equality occurs for $x = y = z$, and for $\frac{x}{2} = y = z$ (or any cyclic permutation).

□

P 4.50. *If x, y, z are nonnegative real numbers, no two of which are zero, then*

$$\frac{x(y+z)}{x^2+2yz}+\frac{y(z+x)}{y^2+2zx}+\frac{z(x+y)}{z^2+2xy}\geq 1+\frac{xy+yz+zx}{x^2+y^2+z^2}.$$

Solution. Write the inequality as $f_8(x,y,z)\geq 0$, where

$$f_8(x,y,z)=(x^2+y^2+z^2)\sum x(y+z)(y^2+2zx)(z^2+2xy)-(p^2-q)\prod(x^2+2yz).$$

Since $x(y+z)=-yz+q$, the polynomial of degree eight $f_8(x,y,z)$ has the highest polynomial

$$A(p,q)=-A_1(p^2-2q)-A_2(p^2-q),$$

where A_1 and A_2 are the highest coefficients of the polynomials of degree six

$$P_2(x,y,z)=\sum yz(y^2+2zx)(z^2+2xy)$$

and

$$P_3(x,y,z)=\prod(x^2+2yz),$$

respectively. Therefore, we have

$$\begin{aligned}A(p,q)&=-P_2(1,1,1)(p^2-2q)-P_3(1,1,1)(p^2-q)\\&=-27(p^2-2q)-27(p^2-q)<0.\end{aligned}$$

According to Theorem 4, we only need to prove the original inequality for $y=z=1$ and for $x=0$.

Case 1: $y=z=1$. We need to show that

$$\frac{2x}{x^2+2}+\frac{2(x+1)}{2x+1}\geq 1+\frac{2x+1}{x^2+2},$$

which is equivalent to

$$(x-1)^2\geq 0.$$

Case 2: $x=0$. We need to show that

$$\frac{z}{y}+\frac{y}{z}\geq 1+\frac{yz}{y^2+z^2},$$

which is equivalent to

$$(y^2+z^2-yz)^2+yz(y-z)^2\geq 0.$$

The equality occurs for $x=y=z$.

Observation 1. Similarly, we can prove the following generalization:

- *Let x, y, z be nonnegative real numbers, no two of which are zero.*

(a) *If* $0 \le k \le 2$, *then*

$$\frac{k}{2}\sum \frac{x(y+z)}{x^2+kyz} \ge \frac{2k-1}{k+1} + \frac{xy+yz+zx}{x^2+y^2+z^2};$$

(b) *If* $k \ge 2$, *then*

$$\frac{k}{2}\sum \frac{x(y+z)}{x^2+kyz} \ge 1 + \frac{2k-1}{k+1} \cdot \frac{xy+yz+zx}{x^2+y^2+z^2}.$$

Observation 2. From the inequalities in Observation 1, we get the following particular inequalities:

$$\frac{x(y+z)}{x^2+4yz} + \frac{y(z+x)}{y^2+4zx} + \frac{z(x+y)}{z^2+4xy} \ge \frac{1}{2} + \frac{7}{10} \cdot \frac{xy+yz+zx}{x^2+y^2+z^2},$$

$$\frac{x(y+z)}{x^2+2yz} + \frac{y(z+x)}{y^2+2zx} + \frac{z(x+y)}{z^2+2xy} \ge 1 + \frac{xy+yz+zx}{x^2+y^2+z^2},$$

$$\frac{x(y+z)}{x^2+yz} + \frac{y(z+x)}{y^2+zx} + \frac{z(x+y)}{z^2+xy} \ge \frac{(x+y+z)^2}{x^2+y^2+z^2},$$

$$\frac{x(y+z)}{2x^2+yz} + \frac{y(z+x)}{2y^2+zx} + \frac{z(x+y)}{2z^2+xy} \ge \frac{2(xy+yz+zx)}{x^2+y^2+z^2}.$$

□

P 4.51. *If* x, y, z *are nonnegative real numbers such that* $xy + yz + zx = 3$, *then*

$$18\left(\frac{1}{x^2+y^2} + \frac{1}{y^2+z^2} + \frac{1}{z^2+x^2}\right) + 5(x^2+y^2+z^2) \ge 42.$$

(Vasile C., 2012)

Solution. Write the inequality in the homogeneous form

$$2(xy+yz+zx)\left(\frac{1}{x^2+y^2} + \frac{1}{y^2+z^2} + \frac{1}{z^2+x^2}\right) + \frac{5(x^2+y^2+z^2)}{xy+yz+zx} \ge 14.$$

which is equivalent to $f_8(x,y,z) \ge 0$, where

$$f_8(x,y,z) = 2q^2\sum(x^2+y^2)(x^2+z^2) + (5p^2-24q)(x^2+y^2)(y^2+z^2)(z^2+x^2).$$

Since $y^2+z^2 = -x^2+p^2-2q$, the polynomial of degree eight $f_8(x,y,z)$ has the same highest polynomial $A(p,q)$ as

$$(5p^2-24q)(-z^2)(-x^2)(-y^2),$$

that is

$$A(p,q) = -5p^2 + 24q.$$

We have

$$\begin{aligned} f_8(x,1,1) &= 2(2x+1)^2[(x^2+1)^2 + 4(x^2+1)] + 2(5x^2 - 28x - 4)(x^2+1)^2 \\ &= 2(x^2+1)(9x^4 - 24x^3 + 22x^2 - 8x + 1) \\ &= 2(x^2+1)(x-1)^2(3x-1)^2, \end{aligned}$$

$$\begin{aligned} f_8(0,y,z) &= 2y^2z^2[y^2z^2 + (y^2+z^2)^2] + [5(y^2+z^2) - 14yz]y^2z^2(y^2+z^2) \\ &= 7y^2z^2(y^2+z^2)(y-z)^2 + 2y^4z^4. \end{aligned}$$

Case 1: $-5p^2 + 24q \le 0$. Since $A(p,q) \le 0$, it suffices to show that $f_8(x,1,1) \ge 0$ and $f_8(0,y,z) \ge 0$ for $x,y,z \ge 0$ (see Corollary 2). These conditions are clearly satisfied.

Case 2: $-5p^2 + 24q \ge 0$. Since $A(p,q) \ge 0$, we apply Corollary 3 for

$$E_{\alpha,\beta}(x) = f_{1/3,-2}(x) = \frac{4(x-1)^4(3x-1)^2}{3969}.$$

The condition (a) in Corollary 3 are satisfied if $f_8(x,1,1) \ge A(x+2,2x+1)f_{1/3,-2}(x)$ for $x \in [0,4]$, where

$$A(x+2,2x+1) = -5x^2 + 28x + 4.$$

We have

$$f_8(x,1,1) - A(x+2,2x+1)f_{1/3,-2}(x) = \frac{2(x-1)^2(3x-1)^2 g(x)}{3969},$$

where

$$\begin{aligned} g(x) &= 3969(x^2+1) - 2(-5x^2 + 28x + 4)(x-1)^2 \\ &\ge 3969(x-1)^2 - 2(-5x^2 + 28x + 4)(x-1)^2 \\ &\ge 98(x-1)^2 - 2(-5x^2 + 30x + 4)(x-1)^2 \\ &= 10(x-1)^2(x-3)^2 \ge 0. \end{aligned}$$

The condition (b) in Corollary 3 is satisfied if $f_8(x,1,1) \ge A(x+2,2x+1)x^2$ for $x \ge 4$. The inequality is equivalent to

$$2(x^2+1)(x-1)^2(3x-1)^2 \ge (-5x^2 + 28x + 4)x^2.$$

For the nontrivial case $-5x^2 + 28x + 4 \ge 0$, since $x^2 + 1 > x^2$ and $2(x-1)^2 > 1$, it suffices to show that

$$(3x-1)^2 \ge -5x^2 + 28x + 4.$$

Indeed,

$$(3x-1)^2-(-5x^2+28x+4)=14x^2-34x-3\ge 56x-34x-3=22x-3>0.$$

The condition (c) in Corollary 3 is satisfied because $f_8(0,y,z)\ge 0$ for $y,z\ge 0$.

The equality occurs for $x=y=z=1$, and for $x=\dfrac{1}{\sqrt{5}}$ and $y=z=\dfrac{3}{\sqrt{5}}$ (or any cyclic permutation).

□

P 4.52. *If x,y,z are nonnegative real numbers, then*

$$\frac{2xy}{x^2+y^2}+\frac{2yz}{y^2+z^2}+\frac{2zx}{z^2+x^2}+7\ge\frac{30(xy+yz+zx)}{(x+y+z)^2}.$$

(Vasile C., 2012)

Solution. Write the inequality as $f_8(x,y,z)\ge 0$, where

$$f_8(x,y,z)=2p^2\sum yz(x^2+y^2)(x^2+z^2)+(7p^2-30q)(x^2+y^2)(y^2+z^2)(z^2+x^2).$$

Since $y^2+z^2=-x^2+p^2-2q$, the polynomial of degree eight $f_8(x,y,z)$ has the same highest polynomial $A(p,q)$ as

$$2p^2\sum yz(-z^2)(-y^2)+(7p^2-30q)(-z^2)(-x^2)(-y^2),$$

that is

$$A(p,q)=6p^2-(7p^2-30q)=30q-p^2.$$

We have

$$\begin{aligned}f_8(x,1,1)&=2(x+2)^2[(x^2+1)^2+4x(x^2+1)]+2(7x^2-32x-2)(x^2+1)^2\\&=4(x^2+1)(4x^4-12x^3+13x^2-6x+1)\\&=4(x^2+1)(x-1)^2(2x-1)^2,\end{aligned}$$

$$\begin{aligned}f_8(0,y,z)&=2(y+z)^2y^3z^3+[7(y^2+z^2)-16yz]y^2z^2(y^2+z^2)\\&=7y^2z^2(y^2+z^2)(y-z)^2+4y^4z^4.\end{aligned}$$

Case 1: $30q-p^2\le 0$. Since $A(p,q)\le 0$, it suffices to show that $f_8(x,1,1)\ge 0$ and $f_8(0,y,z)\ge 0$ for $x,y,z\ge 0$ (see Corollary 2). These conditions are satisfied.

Case 2: $30q-p^2\ge 0$. Since $A(p,q)\ge 0$, we apply Corollary 3 for

$$E_{\alpha,\beta}(x)=f_{1/2,-2}(x)=\frac{4(x-1)^4(2x-1)^2}{2025}.$$

The condition (a) in Corollary 3 are satisfied if $f_8(x,1,1) \ge A(x+2,2x+1)f_{1/2,-2}(x)$ for $x \in [0,4]$, where

$$A(x+2,2x+1) = -x^2+56x+26.$$

We have

$$f_8(x,1,1) - A(x+2,2x+1)f_{1/2,-2}(x) = \frac{4(x-1)^2(2x-1)^2 g(x)}{2025},$$

where

$$\begin{aligned} g(x) &= 2025(x^2+1) - (-x^2+56x+26)(x-1)^2 \\ &\ge 2025(x-1)^2 - (-x^2+56x+26)(x-1)^2 \\ &\ge 234(x-1)^2 - (-x^2+56x+26)(x-1)^2 \\ &= (x-1)^2(4-x)(52-x) \ge 0. \end{aligned}$$

The condition (b) in Corollary 3 is satisfied if $f_8(x,1,1) \ge A(x+2,2x+1)x^2$ for $x \ge 4$. The inequality is equivalent to

$$4(x^2+1)(x-1)^2(2x-1)^2 \ge (-x^2+56x+26)x^2.$$

For the nontrivial case $-x^2+56x+26 \ge 0$, since $x^2+1 > x^2$ and $(x-1)^2 > 2$, it suffices to show that

$$8(2x-1)^2 \ge -x^2+56x+26.$$

Indeed,

$$\begin{aligned} 8(2x-1)^2 - (-x^2+56x+26) &= 33x^2 - 88x - 18 > 32x - 88x - 24 \\ &= 8(x-3)(4x+1) > 0. \end{aligned}$$

The condition (c) in Corollary 3 is satisfied because $f_8(0,y,z) \ge 0$ for $y,z \ge 0$.

The equality occurs for $x = y = z$, and for $2x = y = z$ (or any cyclic permutation).

Observation. Similarly, we can prove the following generalization:

- *Let x,y,z be nonnegative real numbers. If $-2 < k \le 1$, then*

$$\frac{(4-k^2)(k+2)}{8(5-2k)} \sum \left(\frac{xy}{x^2+kxy+y^2} - \frac{3}{k+2} \right) + 1 - \frac{3(xy+yz+zx)}{(x+y+z)^2} \ge 0,$$

with equality for $x = y = z$, and also for $\frac{(4-k)x}{2-2k} = y = z$ (or any cyclic permutation).

For $k = -1$ and $k = 1$, we get the inequalities:

$$\sum \frac{xy}{x^2-xy+y^2} + \frac{47}{3} \ge \frac{56(xy+yz+zx)}{(x+y+z)^2};$$

$$\sum \frac{xy}{x^2+xy+y^2} + 7 \ge \frac{8(xy+yz+zx)}{(x+y+z)^2}.$$

$\square$

P 4.53. *If x, y, z are nonnegative real numbers, then*

$$\frac{2xy}{(x+y)^2} + \frac{2yz}{(y+z)^2} + \frac{2zx}{(z+x)^2} + \frac{x^2+y^2+z^2}{xy+yz+zx} \ge \frac{5}{2}.$$

(Vasile C., 2012)

Solution. Write the inequality as $f_8(x,y,z) \ge 0$, where

$$f_8(x,y,z) = 4q\sum yz(x+y)^2(x+z)^2 + (2p^2-9q)(x+y)^2(y+z)^2(z+x)^2.$$

Since $(y+z)^2 = (p-x)^2 = x^2+p(p-2x)$, the polynomial of degree eight $f_8(x,y,z)$ has the same highest polynomial $A(p,q)$ as

$$4q\sum yz(z^2)(y^2) + (2p^2-9q)(z^2)(x^2)(y^2),$$

that is

$$A(p,q) = 12q - (2p^2-9q) = 21q - 2p^2.$$

We have

$$\begin{aligned} f_8(x,1,1) &= 4(2x+1)\left[(x+1)^4 + 8x(x+1)^2\right] + 4(2x^2-10x-1)(x+1)^4 \\ &= 8x^2(x+1)^2(x-1)^2, \end{aligned}$$

$$\begin{aligned} f_8(0,y,z) &= 4y^4z^4 + \left[2(y+z)^2 - 9yz\right]y^2z^2(y^2+z^2) \\ &= y^2z^2(y-z)^2(2y^2+2z^2+3yz). \end{aligned}$$

Case 1: $21q-2p^2 \le 0$. Since $A(p,q) \le 0$, it suffices to show that $f_8(x,1,1) \ge 0$ and $f_8(0,y,z) \ge 0$ for $x,y,z \ge 0$ (see Corollary 2). These conditions are clearly satisfied.

Case 2: $21q-2p^2 \ge 0$. Since $A(p,q) \ge 0$, we may apply Corollary 3 for

$$E_{\alpha,\beta}(x) = f_{0,-2}(x) = \frac{(x-1)^4x^2}{81}.$$

The condition (a) in Corollary 3 are satisfied if $f_8(x,1,1) \ge A(x+2,2x+1)f_{0,-2}(x)$ for $x \in [0,4]$, where

$$A(x+2,2x+1) = -2x^2+34x+13.$$

We have

$$f_8(x,1,1)-A(x+2,2x+1)f_{0,-2}(x)=\frac{(x-1)^2x^2g(x)}{81},$$

where

$$\begin{aligned}g(x)&=648(x+1)^2-(-2x^2+34x+13)(x-1)^2\\&\geq 648(x-1)^2-(-2x^2+34x+13)(x-1)^2\\&\geq 117(x-1)^2-(-2x^2+34x+13)(x-1)^2\\&=2(x-1)^2(4-x)(13-x)\geq 0.\end{aligned}$$

The condition (b) in Corollary 3 is satisfied if $f_8(x,1,1)\geq A(x+2,2x+1)x^2$ for $x\geq 4$. The inequality is true if

$$8(x+1)^2(x-1)^2\geq -2x^2+34x+13.$$

For the nontrivial case $-2x^2+34x+13\geq 0$, since $(x-1)^2>1$, it suffices to show that

$$8(x+1)^2\geq -2x^2+34x+13.$$

Indeed,

$$\begin{aligned}8(x+1)^2-(-2x^2+34x+13)&=10x^2-18x-5\geq 40x-18x-5\\&=22x-5>0.\end{aligned}$$

The condition (c) in Corollary 3 is satisfied since $f_8(0,y,z)\geq 0$ for all $y,z\geq 0$.

The equality occurs for $x=y=z$, and for $x=0$ and$y=z$ (or any cyclic permutation).

Observation 1. In the case 2, we can give a more simple solution based on Theorem 6. The conditions (a) and (c) are satisfied. Thus, it suffices to show that $f_8(x,1,1)\geq A(x+2,2x+1)\dfrac{4x(x-1)^3}{27}$ for $x\geq 1$. We have

$$f_8(x,1,1)-A(x+2,2x+1)\frac{4x(x-1)^3}{27}=\frac{4x(x-1)^2g(x)}{27},$$

where

$$\begin{aligned}g(x)&=54x(x+1)^2-(-2x^2+34x+13)(x-1)\\&>13(x-1)(x+1)^2-(-2x^2+34x+13)(x-1)\\&=x(x-1)(15x-8)\geq 0.\end{aligned}$$

Observation 2. Similarly, we can prove the following generalization:

- *Let x, y, z be nonnegative real numbers. If $-2 < k \le 2$, then*

$$(k+2)\left(k-1+\sqrt{5-2k}\right)\sum\left(\frac{xy}{x^2+kxy+y^2}-\frac{3}{k+2}\right)+4\left(\frac{x^2+y^2+z^2}{xy+yz+zx}-1\right)\ge 0,$$

with equality for $x = y = z$, and also for $\dfrac{2x}{\sqrt{5-2k}-1} = y = z$ (or any cyclic permutation).

For $k=-1$, $k=0$, $k=1/2$, $k=1$ and $k=2$, we get the inequalities:

$$\sum\frac{xy}{x^2-xy+y^2}-3+\frac{4(\sqrt{7}+2)}{3}\left(\frac{x^2+y^2+z^2}{xy+yz+zx}-1\right)\ge 0;$$

$$\sum\frac{2xy}{x^2+y^2}-3+(\sqrt{5}+1)\left(\frac{x^2+y^2+z^2}{xy+yz+zx}-1\right)\ge 0;$$

$$\sum\frac{5xy}{2x^2+xy+2y^2}-3+\frac{8}{3}\left(\frac{x^2+y^2+z^2}{xy+yz+zx}-1\right)\ge 0;$$

$$\sum\frac{3xy}{x^2+xy+y^2}-3+\frac{4}{\sqrt{3}}\left(\frac{x^2+y^2+z^2}{xy+yz+zx}-1\right)\ge 0;$$

$$\sum\frac{4xy}{(x+y)^2}-3+2\left(\frac{x^2+y^2+z^2}{xy+yz+zx}-1\right)\ge 0.$$

□

P 4.54. *If x, y, z are nonnegative real numbers, no two of which are zero, then*

$$\frac{2}{x^2+y^2}+\frac{2}{y^2+z^2}+\frac{2}{z^2+x^2}\ge\frac{8}{x^2+y^2+z^2}+\frac{1}{xy+yz+zx}.$$

(Vasile C., 2012)

Solution. Write the inequality as $f_8(x,y,z)\ge 0$, where

$$f_8(x,y,z)=2q(p^2-2q)\sum(x^2+y^2)(x^2+z^2)-(p^2+6q)\prod(x^2+y^2).$$

Since $x^2+y^2=-z^2+p^2-2q$, the polynomial of degree eight $f_8(x,y,z)$ has the same highest polynomial $A(p,q)$ as

$$-(p^2+6q)(-z^2)(-x^2)(-y^2),$$

that is

$$A(p,q)=p^2+6q, \qquad A(x+2,2x+1)=x^2+16x+10.$$

We have

$$\begin{aligned}f_8(x,1,1) &= 2(2x+1)x^2+2)\left[(x^2+1)^2+4(x^2+1)\right]-2(x^2+16x+10)(x^2+1)^2\\ &= 4x(x^2+1)(x^4-x^2-2x+2)\\ &= 4x(x^2+1)(x-1)^2(x^2+2x+2).\end{aligned}$$

On the other hand, for $x=0$, the desired inequality becomes

$$\frac{2}{y^2}+\frac{2}{z^2}+\frac{2}{y^2+z^2}\geq\frac{8}{y^2+z^2}+\frac{1}{yz},$$

which is equivalent to

$$(y-z)^2(2y^2+2z^2+3yz)\geq 0.$$

First Solution. Apply Theorem 6. The conditions (a) and (c) are satisfied. In what concerns the condition (b), we have

$$\frac{4A(x+2,2x+1)x(x-1)^3}{27}=\frac{4(x^2+16x+10)x(x-1)^3}{27},$$

$$f_7(x,1,1)-\frac{4A(x+2,2x+1)x(x-1)^3}{27}=\frac{4x(x-1)^2g(x)}{27},$$

where

$$g(x)=27(x^2+1)(x^2+2x+2)-(x^2+16x+10)(x-1).$$

For $x\geq 1$, we get

$$\begin{aligned}g(x) &> 27(x-1)(x^2+2x+2)-(x^2+16x+10)(x-1)\\ &\geq 6(x-1)(x^2+2x+2)-(x^2+16x+10)(x-1)\\ &=(x-1)(5x^2-4x+2)\geq 0.\end{aligned}$$

The equality occurs for $x=y=z$, and for $x=0$ and $y=z$ (or any cyclic permutation).

Second Solution. Apply Corollary 3 for

$$E_{\alpha,\beta}(x)=f_{0,-2}(x)=\frac{(x-1)^4x^2}{81}.$$

The condition (a) in Corollary 3 is satisfied if $f_8(x,1,1)\geq A(x+2,2x+1)f_{0,-2}(x)$ for $x\in[0,4]$. We have

$$f_8(x,1,1)-A(x+2,2x+1)f_{0,-2}(x)=\frac{x(x-1)^2g(x)}{81},$$

where

$$\begin{aligned}
g(x) &= 324(x^2+1)(x^2+2x+2) - x(x^2+16x+10)(x-1)^2 \\
&\ge 324(x-1)^2(x^2+2x+2) - 4(x^2+16x+10)(x-1)^2 \\
&= 4(x-1)^2[81(x^2+2x+2) - (x^2+16x+10)] \\
&\ge 4(x-1)^2[6(x^2+2x+2) - (x^2+16x+10)] \\
&= 4(x-1)^2[3x^2+2(x-1)^2] \ge 0.
\end{aligned}$$

The condition (b) in Corollary 3 is satisfied if $f_8(x,1,1) \ge A(x+2,2x+1)x^2$ for $x \ge 4$. This is true if

$$4(x^2+1)(x-1)^2(x^2+2x+2) \ge x(x^2+16x+10).$$

Since $4 \ge x$ and $(x^2+1)(x-1)^2 > 5$, it suffices to show that

$$5(x^2+2x+2) \ge x^2+16x+10,$$

which is equivalent to

$$2x(2x-3) \ge 0.$$

The condition (c) in Corollary 3 is satisfied since the original inequality holds for $x = 0$.

Observation. Similarly, we can prove the following generalization:

- *Let x,y,z be nonnegative real numbers. If*

$$1-\sqrt{2} \le k \le \frac{10+\sqrt{136}}{9},$$

then

$$\sum \frac{k+2}{x^2+kxy+y^2} \ge \frac{8-4k}{x^2+y^2+z^2} + \frac{4k+1}{xy+yz+zx},$$

with equality for $x=y=z$, and also for $x=0$ and $y=z$ (or any cyclic permutation).

For $k=-1/4$, $k=0$, $k=1$ and $k=2$, we get the inequalities:

$$\sum \frac{1}{4x^2-xy+4y^2} \ge \frac{9}{7(x^2+y^2+z^2)},$$

$$\sum \frac{2}{x^2+y^2} \ge \frac{8}{x^2+y^2+z^2} + \frac{1}{xy+yz+zx},$$

$$\sum \frac{3}{x^2+xy+y^2} \ge \frac{4}{x^2+y^2+z^2} + \frac{5}{xy+yz+zx},$$

$$\sum \frac{4}{(x+y)^2} \ge \frac{9}{xy+yz+zx}.$$

□

P 4.55. *If x,y,z are nonnegative real numbers, then*

$$\frac{2xy}{x^2+y^2}+\frac{2yz}{y^2+z^2}+\frac{2zx}{z^2+x^2}+1\ge\frac{4(xy+yz+zx)}{x^2+y^2+z^2}.$$

(Vasile C., 2014)

Solution. Write the inequality as $f_8(x,y,z)\ge 0$, where

$$f_8(x,y,z)=2(p^2-2q)\sum yz(x^2+y^2)(x^2+z^2)+(p^2-6q)(x^2+y^2)(y^2+z^2)(z^2+x^2).$$

Since

$$y^2+z^2=-x^2+p^2-2q,$$

the polynomial of degree eight $f_8(x,y,z)$ has the same highest polynomial $A(p,q)$ as

$$2(p^2-2q)\sum yz(-z^2)(-y^2)+(p^2-6q)(-z^2)(-x^2)(-y^2),$$

that is

$$A(p,q)=6(p^2-2q)-(p^2-6q)=5p^2-6q=2(p^2-3q)+3p^2>0,$$

$$A(x+2,2x+1)=5x^2+8x+14>0.$$

For $x=0$, the original inequality becomes

$$\frac{2yz}{y^2+z^2}+1\ge\frac{4yz}{y^2+z^2},$$

which is equivalent to

$$(y-z)^2\ge 0.$$

Also, we have

$$\begin{aligned}f_8(x,1,1)&=2(x^2+2)\left[(x^2+1)^2+4x(x^2+1)\right]+2(x^2-8x-2)(x^2+1)^2\\&=4x^2(x^2+1)(x-1)^2.\end{aligned}$$

First Solution. Apply Theorem 6. The conditions (a) and (c) are satisfied. In what concerns the condition (b), we have

$$\frac{4A(x+2,2x+1)x(x-1)^3}{27}=\frac{4x(x-1)^3(5x^2+8x+14)}{27},$$

$$f_8(x,1,1)-\frac{4A(x+2,2x+1)x(x-1)^3}{27}=\frac{4x(x-1)^2g(x)}{9},$$

where

$$g(x)=27x(x^2+1)-(x-1)(5x^2+8x+14).$$

For $x \ge 1$, we get

$$\begin{aligned} g(x) &> 27(x-1)(x^2+1)-(x-1)(5x^2+8x+14) \\ &= (x-1)(27x^2-8x+13) \ge 0. \end{aligned}$$

The equality holds for $x = y = z$, and also for $x = 0$ and $y = z$ (or any cyclic permutation).

Second Solution. Apply Corollary 3 for

$$E_{\alpha,\beta}(x) = f_{0,-2}(x) = \frac{x^2(x-1)^4}{81}.$$

Condition (a). It suffices to show that $f_8(x,1,1) \ge A(x+2,2x+1)f_{0,-2}(x)$ for $0 \le x \le 4$. We have

$$A(x+2,2x+1)f_{0,-2}(x) = \frac{x^2(x-1)^4(5x^2+8x+14)}{81},$$

$$f_8(x,1,1) - A(x+2,2x+1)f_{0,-2}(x) = \frac{x^2(x-1)^2 f(x)}{81},$$

$$f(x) = 324(x^2+1)-(x-1)^2(5x^2+8x+14).$$

Since $x^2+1 \ge (x-1)^2$, we get

$$\begin{aligned} f(x) &\ge 324(x-1)^2-(x-1)^2(5x^2+8x+14) \\ &= (x-1)^2(310-5x^2-8x) \ge 0. \end{aligned}$$

Condition (b). It suffices to show that $f_8(x,1,1) \ge A(x+2,2x+1)x^2$ for $x \ge 4$. We have

$$\begin{aligned} f_8(x,1,1) - A(x+2,2x+1)x^2 &= x^2[4(x^2+1)(x-1)^2-5x^2-8x-14] \\ &\ge x^2[36(x^2+1)-5x^2-8x-14] \\ &= x^2(31x^2-8x+22) > 0. \end{aligned}$$

Condition (c). This condition is satisfied because the original inequality holds for $x = 0$.

Observation. Similarly, we can prove the following generalization:

- *Let x,y,z be real numbers. If $-2 < k \le 0$, then*

$$\sum \frac{(k+2)yz}{y^2+kyz+z^2} - 3 + \frac{8}{3-\sqrt{2k^2+1}}\left(1 - \frac{xy+yz+zx}{x^2+y^2+z^2}\right) \ge 0,$$

with equality for $x = y = z$, and also for $\dfrac{kx}{1-\sqrt{2k^2+1}} = y = z$ (or any cyclic permutation).

□

Chapter 5

On Popoviciu's Inequality

5.1 Theoretical Basis

In 1965, the Romanian mathematician T. Popoviciu proved the following inequality:

$$f(a)+f(b)+f(c)+3f\left(\frac{a+b+c}{3}\right)\geq 2f\left(\frac{a+b}{2}\right)+2f\left(\frac{b+c}{2}\right)+2f\left(\frac{c+a}{2}\right),$$

where f is a convex function on a real interval $\mathbb{I}$, and $a,b,c\in\mathbb{I}$.

In 2002, we gave the following proof for the variant below of Popoviciu's inequality for n variables:

Theorem 1. *If f is a convex function on a real interval $\mathbb{I}$ and $a_1,a_2,\dots,a_n\in\mathbb{I}$, then*

$$f(a_1)+f(a_2)+\cdots+f(a_n)+n(n-2)f(a)\geq(n-1)[f(b_1)+f(b_2)+\cdots+f(b_n)],$$

where $a=\frac{1}{n}\sum_{j=1}^{n}a_j$ and $b_i=\frac{1}{n-1}\sum_{j\neq i}a_j$ for all i.

Proof. Assume that $n\geq 3$ and $a_1\leq a_2\leq\cdots\leq a_n$. There exists an integer m, $1\leq m\leq n-1$, such that

$$a_1\leq\cdots\leq a_m\leq a\leq a_{m+1}\leq\cdots\leq a_n,$$

$$b_1\geq\cdots\geq b_m\geq a\geq b_{m+1}\geq\cdots\geq b_n.$$

We can get the desired inequality by summing the inequalities

$$f(a_1)+\cdots+f(a_m)+n(n-m-1)f(a)\geq(n-1)[f(b_{m+1})+\cdots+f(b_n)],$$

$$f(a_{m+1})+\cdots+f(a_n)+n(m-1)f(a)\geq(n-1)[f(b_1)+\cdots+f(b_m)].$$

Let

$$b=\frac{a_1+\cdots+a_m+(n-m-1)a}{n-1},\qquad c=\frac{a_{m+1}+\cdots+a_n+(m-1)a}{n-1}.$$

Using Jensen's inequalities

$$f(a_1)+\cdots+f(a_m)+(n-m-1)f(a)\ge(n-1)f(b),$$

$$f(a_{m+1})+\cdots+f(a_n)+(m-1)f(a)\ge(n-1)f(c),$$

it suffices to show that

$$(n-m-1)f(a)+f(b)\ge f(b_{m+1})+\cdots+f(b_n), \tag{*}$$

$$f(c)+(m-1)f(a)\ge f(b_1)+\cdots+f(b_m). \tag{**}$$

Since

$$a\ge b_{m+1}\ge\cdots\ge b_n,\qquad (n-m-1)a+b=b_{m+1}+\cdots+b_n,$$

it follows that the decreasingly ordered sequence $A_{n-m}=(a,\cdots,a,b)$ majorizes the decreasingly ordered sequence $B_{n-m}=(b_{m+1},b_{m+2}\cdots,b_n)$. Similarly, since

$$b_1\ge\cdots\ge b_m\ge a,\qquad c+(m-1)a=b_1+\cdots+b_m,$$

the decreasingly ordered sequence $C_m=(c,a,\cdots,a)$ majorizes the decreasingly ordered sequence $D_m=(b_1,b_2\cdots,b_m)$. Therefore, the inequalities $(*)$ and $(**)$ are consequences of Karamata's inequality.

Another variant of Popoviciu's inequality for n variables is the following:

Theorem 2. *If f is a convex function on a real interval $\mathbb{I}$ and $a_1,a_2,\ldots,a_n\in\mathbb{I}$, then*

$$(n-2)[f(a_1)+f(a_2)+\cdots+f(a_n)]+nf\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)\ \ge\ 2\sum_{1\le i<j\le n}f\left(\frac{a_i+a_j}{2}\right).$$

Proof. We use the induction method. For $n=2$, the equality occurs. Suppose that the inequality holds for $n-1$ numbers, $n\ge 3$, and show that it also holds for n numbers. Let

$$a=\frac{a_1+a_2+\cdots+a_n}{n},\qquad b=\frac{a_1+a_2+\cdots+a_{n-1}}{n-1}.$$

By the induction hypothesis,

$$(n-3)[f(a_1)+f(a_2)+\cdots+f(a_{n-1})]+(n-1)f(b)\ \ge\ 2\sum_{1\le i<j\le n-1}f\left(\frac{a_i+a_j}{2}\right).$$

Thus, it suffices to show that

$$f(a_1)+f(a_2)+\cdots+f(a_{n-1})+(n-2)f(a_n)+nf(a)\ge(n-1)f(b)+2\sum_{i=1}^{n-1}f\left(\frac{a_i+a_n}{2}\right).$$

Since

$$f(a_1)+f(a_2)+\cdots+f(a_{n-1})\ge(n-1)f(b)$$

(by Jensen's inequality), it suffices to show that

$$(n-2)f(a_n)+nf(a)\geq 2\sum_{i=1}^{n-1} f\left(\frac{a_i+a_n}{2}\right). \tag{***}$$

Since

$$(n-2)a_n+na=2\sum_{i=1}^{n-1}\frac{a_i+a_n}{2},$$

we will apply Karamata's inequality.

Case 1: $a_1\leq a_2\leq\cdots\leq a_n,\quad 2a\leq a_1+a_n$. Since

$$a_n\geq\max\left\{\frac{a_1+a_n}{2},\frac{a_2+a_n}{2},\ldots,\frac{a_{n-1}+a_n}{2}\right\}=\frac{a_{n-1}+a_n}{2}$$

and

$$a\leq\min\left\{\frac{a_1+a_n}{2},\frac{a_2+a_n}{2},\ldots,\frac{a_{n-1}+a_n}{2}\right\}=\frac{a_1+a_n}{2},$$

the inequality (***) follows from Karamata's inequality.

Case 2: $a_1\geq a_2\geq\cdots\geq a_n,\quad 2a\geq a_1+a_n$. Since

$$a\geq\max\left\{\frac{a_1+a_n}{2},\frac{a_2+a_n}{2},\ldots,\frac{a_{n-1}+a_n}{2}\right\}=\frac{a_1+a_n}{2},$$

$$a_n\leq\min\left\{\frac{a_1+a_n}{2},\frac{a_2+a_n}{2},\ldots,\frac{a_{n-1}+a_n}{2}\right\}=\frac{a_{n-1}+a_n}{2},$$

the inequality (***) follows from Karamata's inequality.

For $n=4$, the inequality in Theorem 2 has the elegant form

$$f(a)+f(b)+f(c)+f(d)+2f\left(\frac{a+b+c+d}{4}\right)\geq\sum_{sym}f\left(\frac{a+b}{2}\right).$$

Actually, the following generalization holds:

Popoviciu's Theorem. *If f is a convex function on a real interval $\mathbb{I}$, $a_1,a_2,\ldots,a_n\in\mathbb{I}$ and $2\leq k\leq n-1$, then*

$$\frac{1}{k}\binom{n-2}{k-2}\left[\frac{n-k}{k-1}\sum_{i=1}^{n}f(x_i)+nf\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)\right]\geq\sum_{1\leq i_1<\cdots<i_k\leq n}f\left(\frac{1}{k}\sum_{j=1}^{k}x_{i_j}\right).$$

Note. We can rewrite Popoviciu's inequality in Theorem 1 as

$$E_n(a_1,a_2,\ldots,a_n)\geq n-1,$$

where

$$E_n(a_1,a_2,\ldots,a_n)=\frac{f(a_1)+f(a_2)+\cdots+f(a_n)-nf(a)}{f(b_1)+f(b_2)+\cdots+f(b_n)-nf(a)}.$$

For some convex functions, the minimum (greatest lower bound) of E_n is just $n-1$, but it is greater for other functions. Thus, for $f(x)=x^2$, we have

$$\frac{a_1^2+a_2^2+\cdots+a_n^2-na^2}{b_1^2+b_2^2+\cdots+b_n^2-na^2}=(n-1)^2,$$

where $a=\frac{1}{n}\sum_{j=1}^{n}a_j$ and $b_i=\frac{1}{n-1}\sum_{j\neq i}a_j$ for all i. Also, for $f(x)=x^3$, $x\geq 0$, if $a_1,a_2,\ldots,a_n$ are nonnegative numbers, then

$$\frac{a_1^3+a_2^3+\cdots+a_n^3-na^3}{b_1^3+b_2^3+\cdots+b_n^3-na^3}\geq\frac{(2n-1)(n-1)^3}{3n^2-5n+1},$$

with equality for $a_1=0$ and $a_2=a_3=\cdots=a_n$ (or any cyclic permutation). On the assumption that $a_1+a_2+\cdots+a_n=n$, this inequality can be rewritten as

$$(n-1)(a_1^3+a_2^3+\cdots+a_n^3)+n^2\geq(2n-1)(a_1^2+a_2^2+\cdots+a_n^2).$$

5.2 Applications

5.1. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that

$$a_1 a_2 \cdots a_n = 1,$$

then

$$a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1} + n(n-2) \geq (n-1)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).$$

5.2. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that

$$a_1 a_2 \cdots a_n = 1,$$

then

$$a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1} + n(n-2) \geq \frac{n-1}{2}\left(a_1 + a_2 + \cdots + a_n + \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).$$

5.3. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that

$$a_1 a_2 \cdots a_n = 1,$$

then

$$(n-1)(a_1^2 + a_2^2 + \cdots + a_n^2) + n \geq (a_1 + a_2 + \cdots + a_n)^2.$$

5.4. If a, b, c, d are positive real numbers such that

$$ab + bc + cd + da = 4,$$

then

$$\left(1 + \frac{a}{b}\right)\left(1 + \frac{b}{c}\right)\left(1 + \frac{c}{d}\right)\left(1 + \frac{d}{a}\right) \geq (a + b + c + d)^2.$$

5.5. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that

$$a_1 + a_2 + \cdots + a_n = n,$$

then

$$\left(1 + \frac{1-a_1}{n-1}\right)\left(1 + \frac{1-a_2}{n-1}\right)\cdots\left(1 + \frac{1-a_n}{n-1}\right) \geq \sqrt[n-1]{a_1 a_2 \cdots a_n}.$$

5.6. If $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ are positive real numbers such that

$$a_1 + a_2 + \cdots + a_n = 1,$$

then

$$\left(a_1 + \frac{1}{a_1} - 2\right)\left(a_2 + \frac{1}{a_2} - 2\right)\cdots\left(a_n + \frac{1}{a_n} - 2\right) \ge \left(n + \frac{1}{n} - 2\right)^n.$$

5.7. If $a_1, a_2, \ldots, a_n$ are positive real numbers, then

$$\frac{b_1}{a_1} + \frac{b_2}{a_2} + \cdots + \frac{b_n}{a_n} \ge \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n},$$

where

$$b_i = \frac{1}{n-1}\sum_{j \ne i} a_j$$

for all i.

5.8. If $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ are positive real numbers such that

$$a_1 + a_2 + \cdots + a_n = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n},$$

then

(a) $$\frac{1}{1+(n-1)a_1} + \frac{1}{1+(n-1)a_2} + \cdots + \frac{1}{1+(n-1)a_n} \ge 1;$$

(b) $$\frac{1}{n-1+a_1} + \frac{1}{n-1+a_2} + \cdots + \frac{1}{n-1+a_n} \le 1.$$

5.9. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that

$$a_1 + a_2 + \cdots + a_n = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = ns,$$

then

$$\frac{1}{a_1+n-1} + \frac{1}{a_2+n-1} + \cdots + \frac{1}{a_n+n-1} \ge \frac{1}{1+ns-a_1} + \frac{1}{1+ns-a_2} + \cdots + \frac{1}{1+ns-a_n}.$$

5.10. If a, b, c are nonnegative real numbers such that $a+b+c=3$, then

$$4\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)+15 \le 9\left(\sqrt{\frac{a+b}{2}}+\sqrt{\frac{b+c}{2}}+\sqrt{\frac{c+a}{2}}\right).$$

5.11. If a, b, c, d, e are positive real numbers such that $abcde = 1$, then

$$\frac{1}{2+\sqrt{4+5a}}+\frac{1}{2+\sqrt{4+5b}}+\frac{1}{2+\sqrt{4+5c}}+\frac{1}{2+\sqrt{4+5d}}+\frac{1}{2+\sqrt{4+5e}} \le 1.$$

5.12. Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be positive real numbers such that $a_1 a_2 \cdots a_n = 1$. If

$$0 < p \le \frac{2n-1}{(n-1)^2},$$

then

$$\frac{1}{\sqrt{1+pa_1}}+\frac{1}{\sqrt{1+pa_2}}+\frac{1}{\sqrt{1+pa_n}} \le \frac{n}{\sqrt{1+p}}.$$

5.13. Let f be a convex function on a real interval $\mathbb{I}$. If $a_1, a_2, \ldots, a_n \in \mathbb{I}$, then

$$2f(a_1)+2f(a_2)+\cdots+2f(a_n)+n(n-2)f(a) \ge n\sum_{i=1}^{n} f\left(a+\frac{a_i-a_{i+1}}{n}\right),$$

where

$$a = \frac{1}{n}(a_1+a_2+\cdots+a_n).$$

5.14. If $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ are positive real numbers such that

$$a_1 a_2 \cdots a_n = 1,$$

then

$$2(a_1^n+a_2^n+\cdots+a_n^n)+n(n-2) \ge n\left(\frac{a_1}{a_2}+\frac{a_2}{a_3}+\cdots+\frac{a_n}{a_1}\right).$$

5.15. If $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ are positive real numbers such that

$$a_1+a_2+\cdots+a_n = n,$$

then

$$\left(1+\frac{a_1-a_2}{n}\right)\left(1+\frac{a_2-a_3}{n}\right)\cdots\left(1+\frac{a_n-a_1}{n}\right) \ge (a_1 a_2 \cdots a_n)^{\frac{2}{n}}.$$

5.16. If $a_1, a_2, \dots, a_n$ $(n \ge 3)$ are positive real numbers such that

$$a_1 + a_2 + \cdots + a_n = n,$$

then

$$\frac{2}{n}\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) + n - 2 \ge \frac{1}{1 + \frac{a_1 - a_2}{n}} + \frac{1}{1 + \frac{a_2 - a_3}{n}} + \cdots + \frac{1}{1 + \frac{a_n - a_1}{n}}.$$

5.17. Let f be a convex function on $(0, \infty)$. If $a_1, a_2, \dots, a_n$ are positive real numbers, then

$$f\left(a_1 + \frac{1}{a_2}\right) + f\left(a_2 + \frac{1}{a_3}\right) + \cdots + f\left(a_n + \frac{1}{a_1}\right) \ge$$

$$\ge f\left(a_1 + \frac{1}{a_1}\right) + f\left(a_2 + \frac{1}{a_2}\right) + \cdots + f\left(a_n + \frac{1}{a_n}\right).$$

5.3 Solutions

P 5.1. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that*

$$a_1 a_2 \cdots a_n = 1,$$

then

$$a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1} + n(n-2) \ge (n-1)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).$$

Solution. Let $x_1, x_2, \ldots, x_n$ be real numbers such that

$$x_1 + x_2 + \cdots + x_n = 0.$$

Applying Popoviciu's inequality from Theorem 1 to the convex function

$$f(x) = e^x, \quad x \in \mathbb{R},$$

we get

$$e^{x_1} + e^{x_2} + \cdots + e^{x_n} + n(n-2) \ge (n-1)\left(e^{\frac{-x_1}{n-1}} + e^{\frac{-x_2}{n-1}} + \cdots + e^{\frac{-x_n}{n-1}}\right).$$

Using the substitution

$$x_1 = (n-1)\ln a_1, \quad x_2 = (n-1)\ln a_2, \quad \ldots, \quad x_n = (n-1)\ln a_n,$$

gives the desired inequality. For $n \ge 3$, the equality holds if and only if

$$a_1 = a_2 = \cdots = a_n = 1.$$

Remark. For $n = 3$, using the substitution

$$a_1 = x^3, \quad a_2 = y^3, \quad a_3 = z^3,$$

we get the known homogeneous inequality

$$x^6 + y^6 + z^6 + 3x^2y^2z^2 \ge 2(x^3y^3 + y^3z^3 + z^3x^3),$$

which holds for any real numbers x, y, z.

□

P 5.2. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that*

$$a_1 a_2 \cdots a_n = 1,$$

then

$$a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1} + n(n-2) \ge \frac{n-1}{2}\left(a_1 + a_2 + \cdots + a_n + \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).$$

(Bin Zhao, 2005)

Solution. We can get this inequality by adding the inequality in P 5.1 and the inequality

$$a_1^{n-1}+a_2^{n-1}+\cdots+a_n^{n-1}+n(n-2)\geq(n-1)(a_1+a_2+\cdots+a_n).$$

This inequality follows by adding the AM-GM inequalities

$$a_i^{n-1}+n-2\geq(n-1)a_i, \qquad i=1,2,\ldots,n.$$

For $n\geq 3$, the equality holds if and only if $a_1=a_2=\cdots=a_n=1$.

□

P 5.3. *If $a_1,a_2,\ldots,a_n$ are positive real numbers such that*

$$a_1a_2\cdots a_n=1,$$

then

$$(n-1)(a_1^2+a_2^2+\cdots+a_n^2)+n\geq(a_1+a_2+\cdots+a_n)^2.$$

(F. Shleifer, 1979)

Solution. Let $x_1,x_2,\ldots,x_n$ be real numbers such that

$$x_1+x_2+\cdots+x_n=0.$$

Applying the inequality from Theorem 2 to the convex function $f(x)=e^x$, we get

$$(n-2)(e^{x_1}+e^{x_2}+\cdots+e^{x_n})+n\geq 2\sum_{1\leq i<j\leq n}e^{\frac{x_i+x_j}{2}}.$$

Using the substitution

$$x_1=2\ln a_1,\ \ x_2=2\ln a_2,\ \ \ldots,\ \ x_n=2\ln a_n,$$

this inequality becomes

$$(n-2)(a_1^2+a_2^2+\cdots+a_n^2)+n\geq 2\sum_{1\leq i<j\leq n}a_ia_j.$$

Since

$$2\sum_{1\leq i<j\leq n}a_ia_j=(a_1+a_2+\cdots+a_n)^2-(a_1^2+a_2^2+\cdots+a_n^2),$$

the conclusion follows. For $n\geq 3$, the equality holds if and only if

$$a_1=a_2=\cdots=a_n=1.$$

□

P 5.4. *If a, b, c, d are positive real numbers such that*

$$ab + bc + cd + da = 4,$$

then

$$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{d}\right)\left(1+\frac{d}{a}\right) \geq (a+b+c+d)^2.$$

Solution. Applying the inequality from Theorem 2 to the convex function

$$f(x) = -\ln x, \qquad x > 0,$$

we get

$$(a+b)(b+c)(c+d)(d+a)(a+c)(b+d) \geq 4abcd(a+b+c+d)^2.$$

Since

$$(a+c)(b+d) = ab + bc + cd + da = 4,$$

the inequality is equivalent to

$$(a+b)(b+c)(c+d)(d+a) \geq 4abcd(a+b+c+d)^2,$$

$$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{d}\right)\left(1+\frac{d}{a}\right) \geq (a+b+c+d)^2.$$

The equality holds for $a = b = c = d = 1$.

□

P 5.5. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that*

$$a_1 + a_2 + \cdots + a_n = n,$$

then

$$\left(1+\frac{1-a_1}{n-1}\right)\left(1+\frac{1-a_2}{n-1}\right)\cdots\left(1+\frac{1-a_n}{n-1}\right) \geq \sqrt[n-1]{a_1 a_2 \cdots a_n}.$$

Solution. Applying Popoviciu's inequality from Theorem 1 to the convex function

$$f(x) = -\ln x, \qquad x > 0,$$

gives

$$(n-1)\left(\ln\frac{n-a_1}{n-1} + \ln\frac{n-a_2}{n-1} + \cdots + \ln\frac{n-a_n}{n-1}\right) \geq \ln a_1 + \ln a_2 + \cdots + \ln a_n,$$

$$\left(\frac{n-a_1}{n-1}\right)\left(\frac{n-a_2}{n-1}\right)\cdots\left(\frac{n-a_1}{n-1}\right) \geq \sqrt[n-1]{a_1 a_2 \cdots a_n},$$

$$\left(1+\frac{1-a_1}{n-1}\right)\left(1+\frac{1-a_2}{n-1}\right)\cdots\left(1+\frac{1-a_n}{n-1}\right) \geq \sqrt[n-1]{a_1 a_2 \cdots a_n}.$$

For $n \geq 3$, the equality holds if and only if $a_1 = a_2 = \cdots = a_n = 1$.

□

P 5.6. *If $a_1, a_2, \ldots, a_n$ ($n \ge 3$) are positive real numbers such that*

$$a_1 + a_2 + \cdots + a_n = 1,$$

then

$$\left(a_1 + \frac{1}{a_1} - 2\right)\left(a_2 + \frac{1}{a_2} - 2\right)\cdots\left(a_n + \frac{1}{a_n} - 2\right) \ge \left(n + \frac{1}{n} - 2\right)^n.$$

Solution. Write the inequality as

$$\frac{(1-a_1)^2(1-a_2)^2\cdots(1-a_n)^2}{a_1a_2\cdots a_n} \ge \frac{(n-1)^{2n}}{n^n},$$

$$(1-a_1)(1-a_2)\cdots(1-a_n) \ge \frac{(n-1)^n}{n^{n/2}}\sqrt{a_1a_2\cdots a_n}\ .$$

Applying Popoviciu's inequality from Theorem 1 to the convex function

$$f(x) = -\ln x, \qquad x > 0,$$

gives

$$(b_1b_2\cdots b_n)^{n-1} \ge (a_1a_2\cdots a_n)\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^{n(n-2)},$$

where

$$b_i = \frac{1}{n-1}\sum_{j\ne i} a_j, \qquad i = 1, 2, \ldots, n.$$

Under the hypothesis $a_1 + a_2 + \cdots + a_n = 1$, this inequality becomes

$$(1-a_1)^{n-1}(1-a_2)^{n-1}\cdots(1-a_n)^{n-1} \ge \frac{(n-1)^{n(n-1)}}{n^{n(n-2)}}a_1a_2\cdots a_n,$$

$$(1-a_1)(1-a_2)\cdots(1-a_n) \ge \frac{(n-1)^n}{n^{\frac{n(n-2)}{n-1}}}\sqrt[n-1]{a_1a_2\cdots a_n}\ .$$

Thus, it suffices to show that

$$\sqrt[n-1]{a_1a_2\cdots a_n} \ge n^{\frac{n(n-3)}{2(n-1)}}\sqrt{a_1a_2\cdots a_n}\ ,$$

which is equivalent to

$$\frac{1}{n^{n(n-3)}} \ge (a_1a_2\cdots a_n)^{n-3}.$$

This inequality is valid if

$$\frac{1}{n^n} \ge a_1a_2\cdots a_n,$$

which is just the AM-GM inequality

$$\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^n \ge a_1a_2\cdots a_n.$$

The equality holds if and only if $a_1 = a_2 = \cdots = a_n = \dfrac{1}{n}$.

□

P 5.7. *If $a_1, a_2, \ldots, a_n$ are positive real numbers, then*

$$\frac{b_1}{a_1} + \frac{b_2}{a_2} + \cdots + \frac{b_n}{a_n} \ge \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n},$$

where

$$b_i = \frac{1}{n-1}\sum_{j\neq i} a_j$$

for all i.

Solution. Let

$$a = a_1 + a_2 + \cdots + a_n.$$

Since

$$(n-1)\frac{b_i}{a_i} = \frac{a}{a_i} - 1, \quad \frac{a_i}{b_i} = \frac{a}{b_i} - n + 1, \qquad i = 1, 2, \ldots, n,$$

the inequality becomes

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} + \frac{n-2}{a} \ge (n-1)\left(\frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_n}\right),$$

which is just Popoviciu's inequality from Theorem 1 applied to the convex function

$$f(x) = \frac{1}{x}, \qquad x > 0.$$

For $n \ge 3$, the equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

Remark. We can also prove this inequality using the Cauchy-Schwarz inequality

$$\frac{n-1}{\sum\limits_{j\neq i} a_j} \le \frac{1}{n-1}\sum_{j\neq i}\frac{1}{a_j}, \qquad i \in \{1, 2, \ldots, n\}.$$

Setting

$$a = a_1 + a_2 + \cdots + a_n, \qquad \frac{1}{A} = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n},$$

we can write this inequalities as

$$\frac{1}{b_i} \le \frac{1}{n-1}\left(\frac{1}{A} - \frac{1}{a_i}\right),$$

$$\frac{a_i}{b_i} \le \frac{1}{n-1}\left(\frac{a_i}{A} - 1\right).$$

Therefore,

$$\sum_{i=1}^{n}\frac{a_i}{b_i} \le \frac{1}{n-1}\left(\frac{a}{A} - n\right) = \frac{1}{n-1}\sum_{i=1}^{n}\left(\frac{a}{a_i} - 1\right) = \sum_{i=1}^{n}\frac{b_i}{a_i}.$$

$\square$

P 5.8. *If $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ are positive real numbers such that*

$$a_1 + a_2 + \cdots + a_n = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n},$$

then

(a) $$\frac{1}{1+(n-1)a_1} + \frac{1}{1+(n-1)a_2} + \cdots + \frac{1}{1+(n-1)a_n} \ge 1;$$

(b) $$\frac{1}{n-1+a_1} + \frac{1}{n-1+a_2} + \cdots + \frac{1}{n-1+a_n} \le 1.$$

(Vasile C., 1996)

Solution. (a) We use the contradiction method. For the sake of contradiction, assume that

$$\frac{1}{1+(n-1)a_1} + \frac{1}{1+(n-1)a_2} + \cdots + \frac{1}{1+(n-1)a_n} < 1.$$

Using the substitution

$$a_i = \frac{1-x_i}{(n-1)x_i}, \quad i = 1, 2, \ldots, n,$$

we get

$$x_1 + x_2 + \cdots + x_n < 1,$$

which is equivalent to

$$1 - x_i > (n-1)y_i,$$

where

$$y_i = \frac{1}{n-1}\sum_{j \ne i} x_j, \quad i = 1, 2, \ldots, n.$$

Therefore, we have

$$a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} \frac{1-x_i}{(n-1)x_i} > \sum_{i=1}^{n} \frac{y_i}{x_i}.$$

Taking account of the inequality from the preceding P 5.7, we get

$$a_1 + a_2 + \cdots + a_n > \sum_{i=1}^{n} \frac{x_i}{y_i}.$$

In addition, since

$$\sum_{i=1}^{n} \frac{x_i}{y_i} > \sum_{i=1}^{n} \frac{(n-1)x_i}{1-x_i} = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n},$$

we have

$$a_1 + a_2 + \cdots + a_n > \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n},$$

which contradicts the hypothesis

$$a_1 + a_2 + \cdots + a_n = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}.$$

For $n \geq 3$, the equality holds if and only if $a_1 = a_2 = \cdots = a_n = 1$.

(b) Replacing $a_1, a_2, \ldots, a_n$ respectively with $1/a_1, 1/a_2, \ldots, 1/a_n$, the inequality in (a) becomes

$$\frac{a_1}{n-1+a_1} + \frac{a_2}{n-1+a_2} + \cdots + \frac{a_n}{n-1+a_n} \geq 1,$$

which is equivalent to the desired inequality.

$\square$

P 5.9. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that*

$$a_1 + a_2 + \cdots + a_n = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = ns,$$

then

$$\frac{1}{a_1+n-1} + \frac{1}{a_2+n-1} + \cdots + \frac{1}{a_n+n-1} \geq \frac{1}{1+ns-a_1} + \frac{1}{1+ns-a_2} + \cdots + \frac{1}{1+ns-a_n}.$$

(Gabriel Dospinescu, 2004)

Solution. By the Cauchy-Schwarz inequality, we have

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \geq n^2,$$

which leads to

$$s \geq 1.$$

Applying Popoviciu's inequality from Theorem 1 to the convex function

$$f(x) = \frac{1}{1+(n-1)x}, \qquad x > 0,$$

we get

$$\sum_{i=1}^{n} \frac{1}{1+(n-1)a_i} + \frac{n(n-2)}{1+(n-1)s} \geq (n-1)\sum_{i=1}^{n} \frac{1}{1+ns-a_i}.$$

Thus, it suffices to show that

$$(n-1)\sum_{i=1}^{n}\frac{1}{a_i+n-1} \ge \sum_{i=1}^{n}\frac{1}{1+(n-1)a_i}+\frac{n(n-2)}{1+(n-1)s},$$

which is equivalent to

$$\sum_{i=1}^{n}\frac{1}{(a_i+n-1)\left(\frac{1}{a_i}+n-1\right)} \ge \frac{1}{1+(n-1)s}.$$

Write this inequality as

$$\frac{1}{A_1}+\frac{1}{A_2}+\cdots+\frac{1}{A_n} \ge \frac{1}{1+(n-1)s},$$

where

$$A_i=(a_i+n-1)\left(\frac{1}{a_i}+n-1\right)=(n-1)\left(a_i+\frac{1}{a_i}\right)+n^2-2n+2.$$

By the AM-HM inequality, we have

$$\frac{1}{A_1}+\frac{1}{A_2}+\cdots+\frac{1}{A_n} \ge \frac{n^2}{A_1+A_2+\cdots+A_n}=\frac{n}{2(n-1)s+n^2-2n+2}.$$

Consequently, it is enough to prove the inequality

$$\frac{n}{2(n-1)s+n^2-2n+2} \ge \frac{1}{1+(n-1)s},$$

which reduces to $s \ge 1$. For $n \ge 3$, the equality holds if and only if $a_1=\cdots=a_n=1$.
□

P 5.10. *If a,b,c are nonnegative real numbers such that $a+b+c=3$, then*

$$4\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\,\right)+15 \le 9\left(\sqrt{\frac{a+b}{2}}+\sqrt{\frac{b+c}{2}}+\sqrt{\frac{c+a}{2}}\right).$$

Solution. Applying Popoviciu's inequality from Theorem 1 to the convex function

$$f(x)=-\sqrt{x}, \quad x \ge 0,$$

we get the inequality

$$\sqrt{a}+\sqrt{b}+\sqrt{c}\;+3 \le 2\left(\sqrt{\frac{a+b}{2}}+\sqrt{\frac{b+c}{2}}+\sqrt{\frac{c+a}{2}}\right),$$

which is weaker that the desired inequality. To prove the desired inequality, consider the nontrivial case where at least two of a, b, c are positive, and write it in the homogeneous form

$$10\left[\sqrt{3(a+b+c)}-\sqrt{a}-\sqrt{b}-\sqrt{c}\right] \le 9\sum\left[\sqrt{2(a+b)}-\sqrt{a}-\sqrt{b}\right],$$

or, equivalently,

$$\frac{10}{\sqrt{3(a+b+c)}+\sqrt{a}+\sqrt{b}+\sqrt{c}}\sum\left(\sqrt{a}-\sqrt{b}\right)^2 \le 9\sum\frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{\sqrt{2(a+b)}+\sqrt{a}+\sqrt{b}}.$$

This is true if

$$\frac{10}{\sqrt{3(a+b+c)}+\sqrt{a}+\sqrt{b}+\sqrt{c}} \le \frac{9}{\sqrt{2(a+b)}+\sqrt{a}+\sqrt{b}},$$

which is equivalent to

$$\left(9\sqrt{\frac{3}{2}}-10\right)\sqrt{2(a+b)} \ge \sqrt{a}+\sqrt{b}.$$

This inequality is true because

$$9\sqrt{\frac{3}{2}}-10>1, \qquad \sqrt{2(a+b)} \ge \sqrt{a}+\sqrt{b}.$$

The equality holds for $a=b=c=1$.

Remark. We can rewrite the inequality as

$$\sqrt{a}+\sqrt{b}+\sqrt{c}-3 \le \frac{9}{4}\left(\sqrt{\frac{a+b}{2}}+\sqrt{\frac{b+c}{2}}+\sqrt{\frac{c+a}{2}}-3\right).$$

As shown in P 1.47 from Volume 4, the best inequality of the form

$$\sqrt{a}+\sqrt{b}+\sqrt{c}-3 \le k\left(\sqrt{\frac{a+b}{2}}+\sqrt{\frac{b+c}{2}}+\sqrt{\frac{c+a}{2}}-3\right), \quad k>0,$$

is for

$$k=(\sqrt{3}-1)(\sqrt{3}+\sqrt{2}) \approx 2.303.$$

□

P 5.11. *If a, b, c, d, e are positive real numbers such that $abcde=1$, then*

$$\frac{1}{2+\sqrt{4+5a}}+\frac{1}{2+\sqrt{4+5b}}+\frac{1}{2+\sqrt{4+5c}}+\frac{1}{2+\sqrt{4+5d}}+\frac{1}{2+\sqrt{4+5e}} \le 1.$$

Solution. Use the contradiction method. Assume that

$$\frac{1}{2+\sqrt{4+5a}}+\frac{1}{2+\sqrt{4+5b}}+\frac{1}{2+\sqrt{4+5c}}+\frac{1}{2+\sqrt{4+5d}}+\frac{1}{2+\sqrt{4+5e}}>1,$$

and show that

$$abcde<1.$$

Using the substitution

$$\frac{1}{2+\sqrt{4+5a}}=\frac{5-x}{30},\quad \frac{1}{2+\sqrt{4+5b}}=\frac{5-y}{30},\quad \frac{1}{2+\sqrt{4+5c}}=\frac{5-z}{30},$$

$$\frac{1}{2+\sqrt{4+5d}}=\frac{5-u}{30},\quad \frac{1}{2+\sqrt{4+5e}}=\frac{5-v}{30},$$

which involves

$$a=\frac{16x}{(5-x)^2},\quad b=\frac{16y}{(5-y)^2},\quad c=\frac{16z}{(5-z)^2},\quad d=\frac{16u}{(5-u)^2},\quad e=\frac{16v}{(5-v)^2},$$

and

$$0<x<5,\quad 0<y<5,\quad 0<z<5,\quad 0<u<5,\quad 0<v<5,$$

we need to show that

$$x+y+z+u+v<5$$

implies

$$xyzuv<\left(\frac{5-x}{4}\right)^2\left(\frac{5-y}{4}\right)^2\left(\frac{5-z}{4}\right)^2\left(\frac{5-u}{4}\right)^2\left(\frac{5-v}{4}\right)^2.$$

It is easy to see that if x increases, then the left side of this inequality increases, while the right side decreases. Therefore, it suffices to show that

$$x+y+z+u+v=5$$

implies

$$\left(\frac{5-x}{4}\right)^2\left(\frac{5-y}{4}\right)^2\left(\frac{5-z}{4}\right)^2\left(\frac{5-u}{4}\right)^2\left(\frac{5-v}{4}\right)^2\ge xyzuv.$$

Popoviciu's inequality from Theorem 1 applied to the convex function

$$f(x)=-\ln x,\qquad x>0,$$

gives

$$\left(\frac{5-x}{4}\right)^4\left(\frac{5-y}{4}\right)^4\left(\frac{5-z}{4}\right)^4\left(\frac{5-u}{4}\right)^4\left(\frac{5-v}{4}\right)^4\ge xyzuv\left(\frac{x+y+z+u+v}{5}\right)^{15},$$

$$\left(\frac{5-x}{4}\right)^4\left(\frac{5-y}{4}\right)^4\left(\frac{5-z}{4}\right)^4\left(\frac{5-u}{4}\right)^4\left(\frac{5-v}{4}\right)^4 \ge xyzuv.$$

Thus, it suffices to show that $xyzuv \le 1$. By the AM-GM inequality, we have

$$xyzuv \le \left(\frac{x+y+z+u+v}{5}\right)^5 = 1.$$

The equality holds for $a = b = c = d = e = 1$.

Remark. In the same manner, we can prove the following generalization:

- *If $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ are positive real numbers such that*

$$a_1 a_2 \cdots a_n = 1,$$

then

$$\sum_{i=1}^{n} \frac{1}{n-1+\sqrt{(n-1)^2+4na_i}} \le \frac{1}{2},$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 5.12. *Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be positive real numbers such that $a_1 a_2 \cdots a_n = 1$. If*

$$0 < p \le \frac{2n-1}{(n-1)^2},$$

then

$$\frac{1}{\sqrt{1+pa_1}} + \frac{1}{\sqrt{1+pa_2}} + \frac{1}{\sqrt{1+pa_n}} \le \frac{n}{\sqrt{1+p}}.$$

(Vasile Cîrtoaje and Gabriel Dospinescu, 2006)

Solution. We will apply the contradiction method. Assume that the reverse inequality holds, namely

$$\frac{1}{\sqrt{1+pa_1}} + \frac{1}{\sqrt{1+pa_2}} + \frac{1}{\sqrt{1+pa_n}} > \frac{n}{\sqrt{1+p}},$$

and show that

$$a_1 a_2 \cdots a_n < 1.$$

Using the substitution

$$\sqrt{1+pa_i} = \frac{\sqrt{1+p}}{x_i}, \quad 0 < x_i < \sqrt{p+1}, \quad i = 1, 2, \ldots, n,$$

we need to show that $x_1+x_2+\cdots+x_n>n$ yields

$$\left(\frac{1+p}{x_1^2}-1\right)\left(\frac{1+p}{x_2^2}-1\right)\cdots\left(\frac{1+p}{x_n^2}-1\right)<p^n.$$

It suffices to prove that

$$x_1+x_2+\cdots+x_n=n$$

involves

$$\left(\frac{1+p}{x_1^2}-1\right)\left(\frac{1+p}{x_2^2}-1\right)\cdots\left(\frac{1+p}{x_n^2}-1\right)\le p^n.$$

Denoting

$$\sqrt{1+p}=q, \quad 1<q\le\frac{n}{n-1},$$

we need to show that

$$(q^2-x_1^2)(q^2-x_2^2)\cdots(q^2-x_n^2)\le(q^2-1)^n(x_1x_2\cdots x_n)^2 \tag{*}$$

for all $x_i\in(0,q)$ satisfying $x_1+x_2+\cdots+x_n=n$. Applying Popoviciu's inequality to the convex function

$$f(x)=-\ln\left(\frac{n}{n-1}-x\right), \quad 0<x<1,$$

we get

$$(x_1x_2\cdots x_n)^{n-1}\ge[n-(n-1)x_1][n-(n-1)x_2]\cdots[n-(n-1)x_n]. \tag{**}$$

On the other hand, Jensen's inequality applied to the convex function

$$f(x)=\ln\frac{n-(n-1)x}{q-x}$$

yields

$$\frac{[n-(n-1)x_1][n-(n-1)x_2]\cdots[n-(n-1)x_n]}{(q-x_1)(q-x_2)\cdots(q-x_n)}\ge\frac{1}{(q-1)^n}.$$

Multiplying this inequality and (**) gives

$$(x_1x_2\cdots x_n)^{n-1}\ge\frac{(q-x_1)(q-x_2)\cdots(q-x_n)}{(q-1)^n}.$$

Therefore, in order to prove (*), we still have to show that

$$(x_1x_2\cdots x_n)^{n-3}(q+x_1)(q+x_2)\cdots(q+x_n)\le(q+1)^n.$$

This is true because, by the AM-GM inequality, we have

$$x_1x_2\cdots x_n\le\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)^n=1$$

and

$$(q+x_1)(q+x_2)\cdots(q+x_n) \le \left(q+\frac{x_1+x_2+\cdots+x_n}{n}\right)^n = (q+1)^n.$$

The equality occurs for $a_1 = a_2 = \cdots = a_n = 1$.

Remark. For $p = \dfrac{2n-1}{(n-1)^2}$, we get the following inequality

$$\sum_{i=1}^{n} \frac{1}{\sqrt{(n-1)^2+(2n-1)a_i}} \le 1,$$

which holds for all positive numbers $a_1, a_2, \ldots, a_n$ ($n \ge 3$) satisfying $a_1a_2\cdots a_n = 1$.

□

P 5.13. *Let f be a convex function on a real interval $\mathbb{I}$. If $a_1, a_2, \ldots, a_n \in \mathbb{I}$, then*

$$2f(a_1)+2f(a_2)+\cdots+2f(a_n)+n(n-2)f(a) \ge n\sum_{i=1}^{n} f\left(a+\frac{a_i-a_{i+1}}{n}\right),$$

where

$$a = \frac{1}{n}(a_1+a_2+\cdots+a_n).$$

(Darij Grinberg and Vasile Cîrtoaje, 2006)

Solution. Let

$$b_i = \frac{1}{n-1}\sum_{j\neq i} a_j, \quad i = 1,2,\ldots,n.$$

By Jensen's inequality, we have

$$f\left(a+\frac{a_i-a_{i+1}}{n}\right) = f\left(\frac{a_i+(n-1)b_{i+1}}{n}\right) \le \frac{1}{n}f(a_i)+\left(1-\frac{1}{n}\right)f(b_{i+1}),$$

hence

$$n\sum_{i=1}^{n} f\left(a+\frac{a_i-a_{i+1}}{n}\right) \le \sum_{i=1}^{n} f(a_i)+(n-1)\sum_{i=1}^{n} f(b_i).$$

Therefore, it suffices to show that

$$2f(a_1)+2f(a_2)+\cdots+2f(a_n)+n(n-2)f(a) \ge \sum_{i=1}^{n} f(a_i)+(n-1)\sum_{i=1}^{n} f(b_i),$$

which is just Popoviciu's inequality from Theorem 1:

$$f(a_1)+f(a_2)+\cdots+f(a_n)+n(n-2)f(a) \ge (n-1)\sum_{i=1}^{n} f(b_i).$$

□

P 5.14. *If* $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ *are positive real numbers such that*

$$a_1 a_2 \cdots a_n = 1,$$

then

$$2(a_1^n + a_2^n + \cdots + a_n^n) + n(n-2) \ge n\left(\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_n}{a_1}\right).$$

Solution. Let $x_1, x_2, \ldots, x_n$ be real numbers such that

$$x_1 + x_2 + \cdots + x_n = 0.$$

Applying the inequality in the preceding P 5.13 to the convex function

$$f(x) = e^x, \qquad x \in \mathbb{R},$$

we get

$$2e^{x_1} + 2e^{x_2} + \cdots + 2e^{x_n} + n(n-2) \ge n\left(e^{\frac{x_1 - x_2}{n}} + e^{\frac{x_2 - x_3}{n}} + \cdots + e^{\frac{x_n - x_1}{n}}\right).$$

Using the substitution

$$x_1 = n \ln a_1, \quad x_2 = n \ln a_2, \quad \ldots, \quad x_n = n \ln a_n,$$

we get the desired inequality. The equality occurs if and only if

$$a_1 = a_2 = \cdots = a_n = 1.$$

□

P 5.15. *If* $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ *are positive real numbers such that*

$$a_1 + a_2 + \cdots + a_n = n,$$

then

$$\left(1 + \frac{a_1 - a_2}{n}\right)\left(1 + \frac{a_2 - a_3}{n}\right)\cdots\left(1 + \frac{a_n - a_1}{n}\right) \ge (a_1 a_2 \cdots a_n)^{\frac{2}{n}}.$$

Solution. Applying the inequality in P 5.13 to the convex function

$$f(x) = -\ln x, \qquad x > 0,$$

we get

$$n\left[\ln\left(1 + \frac{a_1 - a_2}{n}\right) + \ln\left(1 + \frac{a_2 - a_3}{n}\right) + \cdots + \ln\left(1 + \frac{a_n - a_1}{n}\right)\right] \ge$$

$$\ge 2(\ln a_1 + \ln a_2 + \cdots + \ln a_n),$$

which is equivalent to the desired inequality. The equality occurs if and only if $a_1 = a_2 = \cdots = a_n = 1$.

□

P 5.16. *If $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ are positive real numbers such that*

$$a_1 + a_2 + \cdots + a_n = n,$$

then

$$\frac{2}{n}\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) + n - 2 \ge \frac{1}{1 + \dfrac{a_1 - a_2}{n}} + \frac{1}{1 + \dfrac{a_2 - a_3}{n}} + \cdots + \frac{1}{1 + \dfrac{a_n - a_1}{n}}.$$

Solution. Apply the inequality in P 5.13 to the convex function

$$f(x) = \frac{1}{x}, \qquad x > 0.$$

The equality occurs if and only if $a_1 = a_2 = \cdots = a_n = 1$.

□

P 5.17. *Let f be a convex function on $(0, \infty)$. If $a_1, a_2, \ldots, a_n$ are positive real numbers, then*

$$f\left(a_1 + \frac{1}{a_2}\right) + f\left(a_2 + \frac{1}{a_3}\right) + \cdots + f\left(a_n + \frac{1}{a_1}\right) \ge$$

$$\ge f\left(a_1 + \frac{1}{a_1}\right) + f\left(a_2 + \frac{1}{a_2}\right) + \cdots + f\left(a_n + \frac{1}{a_n}\right).$$

(Vasile C., 2009)

Solution. For $n = 2$, the inequality is

$$f\left(a_1 + \frac{1}{a_2}\right) + f\left(a_2 + \frac{1}{a_1}\right) \ge f\left(a_1 + \frac{1}{a_1}\right) + f\left(a_2 + \frac{1}{a_2}\right).$$

Assume that $a_1 \ge a_2$. Since

$$\left(a_1 + \frac{1}{a_2}\right) + \left(a_2 + \frac{1}{a_1}\right) = \left(a_1 + \frac{1}{a_1}\right) + \left(a_2 + \frac{1}{a_2}\right)$$

and

$$a_1 + \frac{1}{a_2} \ge \max\left\{a_1 + \frac{1}{a_1},\ a_2 + \frac{1}{a_2}\right\},$$

the inequality for $n = 2$ follows from Lemma below (which is a consequence of Karamata's inequality). To prove the original inequality, consider that

$$a_{n+1} \le \min\{a_1, a_2, \ldots, a_n\},$$

and use the induction method. Based on the induction hypothesis, we only need to show that

$$f\left(a_n+\frac{1}{a_{n+1}}\right)+f\left(a_{n+1}+\frac{1}{a_1}\right)\geq f\left(a_n+\frac{1}{a_1}\right)+f\left(a_{n+1}+\frac{1}{a_{n+1}}\right).$$

This inequality follows also by Lemma below, since

$$\left(a_n+\frac{1}{a_{n+1}}\right)+\left(a_{n+1}+\frac{1}{a_1}\right)=\left(a_n+\frac{1}{a_1}\right)+\left(a_{n+1}+\frac{1}{a_{n+1}}\right)$$

and

$$a_n+\frac{1}{a_{n+1}}\geq \max\left\{a_n+\frac{1}{a_1},\ a_{n+1}+\frac{1}{a_{n+1}}\right\}.$$

The equality occurs for $a_1=a_2=\cdots=a_n$.

Lemma. *Let f be a convex function on a real interval $\mathbb{I}$. If $a,b,c,d\in\mathbb{I}$ such that*

$$a+b=c+d,\qquad a\geq\max\{c,d\},$$

then

$$f(a)+f(b)\geq f(c)+f(d).$$

Proof. Without loss of generality, assume that $c\geq d$; then,

$$a\geq c\geq d\geq b.$$

If $a=c$, then $b=d$, and the inequality is an equality. Consider now that $a>c$, when

$$a>c\geq d>b.$$

First proof. The desired inequality follows by adding the following Jensen's inequalities

$$(c-b)f(a)+(a-c)f(b)\geq(a-b)f(c),$$
$$(a-c)f(a)+(c-b)f(b)\geq(a-b)f(d).$$

Second Proof. Since $c,d\in(b,a)$, there are $p,q\in(0,1)$ such that

$$c=pa+(1-p)b,\qquad d=qa+(1-q)b.$$

From $a+b=c+d$, we get

$$a+b=(p+q)a+(2-p-q)b,$$
$$(a-b)(p+q-1)=0,$$
$$p+q=1.$$

Using Jensen's inequalities

$$f(c)\leq pf(a)+(1-p)f(b),$$
$$f(d)\leq qf(a)+(1-q)f(b),$$

we get

$$f(c)+f(d)\leq(p+q)f(a)+(2-p-q)f(b)=f(a)+f(b).$$

$\square$

Appendix A

Glosar

1. AM-GM (ARITHMETIC MEAN-GEOMETRIC MEAN) INEQUALITY

If $a_1, a_2, \dots, a_n$ are nonnegative real numbers, then

$$a_1 + a_2 + \cdots + a_n \ge n\sqrt[n]{a_1 a_2 \cdots a_n},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

2. WEIGHTED AM-GM INEQUALITY

Let $p_1, p_2, \dots, p_n$ be positive real numbers satisfying

$$p_1 + p_2 + \cdots + p_n = 1.$$

If $a_1, a_2, \dots, a_n$ are nonnegative real numbers, then

$$p_1 a_1 + p_2 a_2 + \cdots + p_n a_n \ge a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

3. AM-HM (ARITHMETIC MEAN-HARMONIC MEAN) INEQUALITY

If $a_1, a_2, \dots, a_n$ are positive real numbers, then

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \ge n^2,$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

4. POWER MEAN INEQUALITY

The power mean of order k of positive real numbers $a_1, a_2, \ldots, a_n$,

$$M_k = \begin{cases} \left(\frac{a_1^k + a_2^k + \cdots + a_n^k}{n}\right)^{\frac{1}{k}}, & k \neq 0 \\ \sqrt[n]{a_1 a_2 \cdots a_n}, & k = 0 \end{cases},$$

is an increasing function with respect to $k \in \mathbb{R}$. For instant, $M_2 \geq M_1 \geq M_0 \geq M_{-1}$ is equivalent to

$$\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} \geq \frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{n}{\dfrac{1}{a_1} + \dfrac{1}{a_2} + \cdots + \dfrac{1}{a_n}}.$$

5. BERNOULLI'S INEQUALITY

For any real number $x \geq -1$, we have

a) $(1+x)^r \geq 1 + rx$ for $r \geq 1$ and $r \leq 0$;

b) $(1+x)^r \leq 1 + rx$ for $0 \leq r \leq 1$.

If $a_1, a_2, \ldots, a_n$ are real numbers such that either $a_1, a_2, \ldots, a_n \geq 0$ or

$$-1 \leq a_1, a_2, \ldots, a_n \leq 0,$$

then

$$(1+a_1)(1+a_2)\cdots(1+a_n) \geq 1 + a_1 + a_2 + \cdots + a_n.$$

6. SCHUR'S INEQUALITY

For any nonnegative real numbers a, b, c and any positive number k, the inequality holds

$$a^k(a-b)(a-c) + b^k(b-c)(b-a) + c^k(c-a)(c-b) \geq 0,$$

with equality for $a = b = c$, and for $a = 0$ and $b = c$ (or any cyclic permutation). For $k = 1$, we get the third degree Schur's inequality, which can be rewritten as follows

$$a^3 + b^3 + c^3 + 3abc \geq ab(a+b) + bc(b+c) + ca(c+a),$$

$$(a+b+c)^3 + 9abc \geq 4(a+b+c)(ab+bc+ca),$$

$$a^2 + b^2 + c^2 + \frac{9abc}{a+b+c} \geq 2(ab+bc+ca),$$

$$(b-c)^2(b+c-a)+(c-a)^2(c+a-b)+(a-b)^2(a+b-c)\ge 0.$$

For $k=2$, we get the fourth degree Schur's inequality, which holds for any real numbers a,b,c, and can be rewritten as follows

$$a^4+b^4+c^4+abc(a+b+c)\ge ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2),$$

$$a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2\ge(ab+bc+ca)(a^2+b^2+c^2-ab-bc-ca),$$

$$(b-c)^2(b+c-a)^2+(c-a)^2(c+a-b)^2+(a-b)^2(a+b-c)^2\ge 0,$$

$$6abcp\ge(p^2-q)(4q-p^2),\qquad p=a+b+c,\ \ q=ab+bc+ca.$$

A generalization of the fourth degree Schur's inequality, which holds for any real numbers a, b, c and any real number m, is the following (*Vasile Cirtoaje*, 2004)

$$\sum(a-mb)(a-mc)(a-b)(a-c)\ge 0,$$

with equality for $a=b=c$, and also for $a/m=b=c$ (or any cyclic permutation). This inequality is equivalent to

$$\sum a^4+m(m+2)\sum a^2b^2+(1-m^2)abc\sum a\ge(m+1)\sum ab(a^2+b^2),$$

$$\sum(b-c)^2(b+c-a-ma)^2\ge 0.$$

7. CAUCHY-SCHWARZ INEQUALITY

If $a_1,a_2,\dots,a_n$ and $b_1,b_2,\dots,b_n$ are real numbers, then

$$(a_1^2+a_2^2+\cdots+a_n^2)(b_1^2+b_2^2+\cdots+b_n^2)\ge(a_1b_1+a_2b_2+\cdots+a_nb_n)^2,$$

with equality for

$$\frac{a_1}{b_1}=\frac{a_2}{b_2}=\cdots=\frac{a_n}{b_n}.$$

Notice that the equality conditions are also valid for $a_i=b_i=0$, where $1\le i\le n$.

8. HÖLDER'S INEQUALITY

If x_{ij} $(i=1,2,\cdots,m;\, j=1,2,\cdots n)$ are nonnegative real numbers, then

$$\prod_{i=1}^{m}\left(\sum_{j=1}^{n}x_{ij}\right)\ge\left(\sum_{j=1}^{n}\sqrt[m]{\prod_{i=1}^{m}x_{ij}}\right)^m.$$

9. CHEBYSHEV'S INEQUALITY

Let $a_1 \ge a_2 \ge \cdots \ge a_n$ be real numbers.

a) If $b_1 \ge b_2 \ge \cdots b_n$, then

$$n\sum_{i=1}^{n} a_i b_i \ge \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right);$$

b) If $b_1 \le b_2 \le \cdots \le b_n$, then

$$n\sum_{i=1}^{n} a_i b_i \le \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right).$$

10. REARRANGEMENT INEQUALITY

(1) If $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are two increasing (or decreasing) real sequences, and $(i_1, i_2, \cdots, i_n)$ is an arbitrary permutation of $(1, 2, \cdots, n)$, then

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n \ge a_1b_{i_1} + a_2b_{i_2} + \cdots + a_nb_{i_n}$$

and

$$n(a_1b_1 + a_2b_2 + \cdots + a_nb_n) \ge (a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n).$$

(2) If $(a_1, a_2, \ldots, a_n)$ is decreasing and $(b_1, b_2, \ldots, b_n)$ is increasing, then

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n \le a_1b_{i_1} + a_2b_{i_2} + \cdots + a_nb_{i_n}$$

and

$$n(a_1b_1 + a_2b_2 + \cdots + a_nb_n) \le (a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n).$$

(3) Let $b_1, b_2, \ldots, b_n)$ and $(c_1, c_2, \ldots, c_n)$ be two real sequences such that

$$b_1 + \cdots + b_i \ge c_1 + \cdots + c_i, \quad i = 1, 2, \cdots, n.$$

If $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$, then

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n \ge a_1c_1 + a_2c_2 + \cdots + a_nc_n.$$

Notice that all these inequalities follow immediately from the identity

$$\sum_{i=1}^{n} a_i(b_i - c_i) = \sum_{i=1}^{n}(a_i - a_{i+1})\left(\sum_{j=1}^{i} b_j - \sum_{j=1}^{i} c_j\right), \qquad a_{n+1} = 0.$$

11. SQUARE PRODUCT INEQUALITY

Let a, b, c be real numbers, and let

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc,$$

$$s = \sqrt{p^2 - 3q} = \sqrt{a^2 + b^2 + c^2 - ab - bc - ca}.$$

From the identity

$$(a-b)^2(b-c)^2(c-a)^2 = -27r^2 + 2(9pq - 2p^3)r + p^2q^2 - 4q^3,$$

it follows that

$$\frac{-2p^3 + 9pq - 2(p^2 - 3q)\sqrt{p^2 - 3q}}{27} \le r \le \frac{-2p^3 + 9pq + 2(p^2 - 3q)\sqrt{p^2 - 3q}}{27},$$

which is equivalent to

$$\frac{p^3 - 3ps^2 - 2s^3}{27} \le r \le \frac{p^3 - 3ps^2 + 2s^3}{27}.$$

Therefore, for constant p and q, the product r is minimum and maximum when two of a, b, c are equal.

On the other hand, the identity

$$27(a-b)^2(b-c)^2(c-a)^2 = 4(p^2 - 3q)^3 - (2p^3 - 9pq + 27r)^2,$$

leads to the inequality

$$27(a-b)^2(b-c)^2(c-a)^2 \le 4(p^2 - 3q)^3,$$

with equality for $2p^3 - 9pq + 27r = 0$.

12. KARAMATA'S MAJORIZATION INEQUALITY

Let f be a convex function on a real interval $\mathbb{I}$. If a decreasingly ordered sequence

$$A = (a_1, a_2, \ldots, a_n), \quad a_i \in \mathbb{I},$$

majorizes a decreasingly ordered sequence

$$B = (b_1, b_2, \ldots, b_n), \quad b_i \in \mathbb{I},$$

then

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge f(b_1) + f(b_2) + \cdots + f(b_n).$$

We say that a sequence $A = (a_1, a_2, \ldots, a_n)$ with $a_1 \ge a_2 \ge \cdots \ge a_n$ majorizes a sequence $B = (b_1, b_2, \ldots, b_n)$ with $b_1 \ge b_2 \ge \cdots \ge b_n$, and write it as

$$A \succ B,$$

if

$$\begin{gathered} a_1 \geq b_1, \\ a_1 + a_2 \geq b_1 + b_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_1 + a_2 + \cdots + a_{n-1} \geq b_1 + b_2 + \cdots + b_{n-1}, \\ a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n. \end{gathered}$$

13. CONVEX FUNCTIONS

A function f defined on a real interval $\mathbb{I}$ is said to be *convex* if

$$f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y)$$

for all $x,\ y \in \mathbb{I}$ and any $\alpha,\ \beta \geq 0$ with $\alpha + \beta = 1$. If the inequality is reversed, then f is said to be concave.
If f is differentiable on $\mathbb{I}$, then f is (strictly) convex if and only if the derivative f' is (strictly) increasing. If $f'' \geq 0$ on $\mathbb{I}$, then f is convex on $\mathbb{I}$. Also, if $f'' \geq 0$ on $(a,\ b)$ and f is continuous on $[a,\ b]$, then f is convex on $[a,\ b]$.

Jensen's inequality. *Let $p_1, p_2, \ldots, p_n$ be positive real numbers. If f is a convex function on a real interval $\mathbb{I}$, then for any $a_1,\ a_2,\ \ldots,\ a_n \in \mathbb{I}$, the inequality holds*

$$\frac{p_1 f(a_1) + p_2 f(a_2) + \cdots + p_n f(a_n)}{p_1 + p_2 + \cdots + p_n} \geq f\left(\frac{p_1 a_1 + p_2 a_2 + \cdots + p_n a_n}{p_1 + p_2 + \cdots + p_n}\right).$$

For $p_1 = p_2 = \cdots = p_n$, Jensen's inequality becomes

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right).$$

Right Half Convex Function Theorem (*Vasile Cîrtoaje*, 2004). *Let f be a real function defined on an interval $\mathbb{I}$ and convex on $\mathbb{I}_{\geq s}$, where $s \in int(\mathbb{I})$. The inequality*

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ satisfying

$$a_1 + a_2 + \cdots + a_n = ns$$

if and only if

$$f(x) + (n-1)f(y) \geq nf(s)$$

for all $x, y \in \mathbb{I}$ such that $x \leq s \leq y$ and $x + (n-1)y = ns$.

Left Half Convex Function Theorem (*Vasile Cîrtoaje*, 2004). *Let f be a real function defined on an interval $\mathbb{I}$ and convex on $\mathbb{I}_{\leq s}$, where $s \in int(\mathbb{I})$. The inequality*

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ *satisfying*

$$a_1 + a_2 + \cdots + a_n = ns$$

if and only if

$$f(x) + (n-1)f(y) \geq nf(s)$$

for all $x, y \in \mathbb{I}$ *such that* $x \geq s \geq y$ *and* $x + (n-1)y = ns$.

Right Partially Convex Function Theorem (*Vasile Cîrtoaje*, 2012). *Let* f *be a real function defined on an interval* $\mathbb{I}$ *and convex on* $[s, s_0]$, *where* $s, s_0 \in \mathbb{I}$, $s < s_0$. *In addition,* f *is decreasing on* $\mathbb{I}_{\leq s_0}$ *and* $f(u) \geq f(s_0)$ *for* $u \in \mathbb{I}$. *The inequality*

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ *satisfying*

$$a_1 + a_2 + \cdots + a_n = ns$$

if and only if

$$f(x) + (n-1)f(y) \geq nf(s)$$

for all $x, y \in \mathbb{I}$ *such that* $x \leq s \leq y$ *and* $x + (n-1)y = ns$.

Left Partially Convex Function Theorem (*Vasile Cîrtoaje*, 2012). *Let* f *be a real function defined on an interval* $\mathbb{I}$ *and convex on* $[s_0, s]$, *where* $s_0, s \in \mathbb{I}$, $s_0 < s$. *In addition,* f *is increasing on* $\mathbb{I}_{\geq s_0}$ *and* $f(u) \geq f(s_0)$ *for* $u \in \mathbb{I}$. *The inequality*

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ *satisfying*

$$a_1 + a_2 + \cdots + a_n = ns$$

if and only if

$$f(x) + (n-1)f(y) \geq nf(s)$$

for all $x, y \in \mathbb{I}$ *such that* $x \geq s \geq y$ *and* $x + (n-1)y = ns$.

Equal Variables Theorem for Nonnegative Variables (*Vasile Cirtoaje*, 2005). *Let* $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ *be fixed nonnegative real numbers, and let*

$$0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$$

such that

$$x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n, \qquad x_1^k + x_2^k + \cdots + x_n^k = a_1^k + a_2^k + \cdots + a_n^k,$$

where k is a real number ($k \neq 1$); for $k = 0$, assume that

$$x_1 x_2 \cdots x_n = a_1 a_2 \cdots a_n.$$

Let f be a real-valued function, continuous on $[0, \infty)$ and differentiable on $(0, \infty)$, such that the associated function

$$g(x) = f'\left(x^{\frac{1}{k-1}}\right)$$

is strictly convex on $(0, \infty)$. Then, the sum

$$S_n = f(x_1) + f(x_2) + \cdots + f(x_n)$$

is maximum for

$$x_1 = x_2 = \cdots = x_{n-1} \le x_n,$$

and is minimum for

$$0 < x_1 \le x_2 = x_3 = \cdots = x_n$$

or

$$0 = x_1 = \cdots = x_j \le x_{j+1} \le x_{j+2} = \cdots = x_n, \quad j \in \{1, 2, \ldots, n-1\}.$$

Equal Variables Theorem for Real Variables (*Vasile Cirtoaje*, 2010). *Let $a_1, a_2, \ldots, a_n$ ($n \ge 3$) be fixed real numbers, and let*

$$0 \le x_1 \le x_2 \le \cdots \le x_n$$

such that

$$x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n, \qquad x_1^k + x_2^k + \cdots + x_n^k = a_1^k + a_2^k + \cdots + a_n^k,$$

where k is an even positive integer. If f is a differentiable function on $\mathbb{R}$ such that the associated function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = f'\left(\sqrt[k-1]{x}\right)$$

is strictly convex on $\mathbb{R}$, then the sum

$$S_n = f(x_1) + f(x_2) + \cdots + f(x_n)$$

is minimum for $x_2 = x_3 = \cdots = x_n$, and is maximum for $x_1 = x_2 = \cdots = x_{n-1}$.

Best Upper Bound of Jensen's Difference Theorem (*Vasile Cirtoaje*, 1990). *Let $p_1, p_2, \ldots, p_n$ ($n \ge 3$) be fixed positive real numbers, and let f be a convex function on $\mathbb{I} = [a, b]$. If $a_1, a_2, \ldots, a_n \in \mathbb{I}$, then Jensen's difference*

$$\frac{p_1 f(a_1) + p_2 f(a_2) + \cdots + p_n f(a_n)}{p_1 + p_2 + \cdots + p_n} - f\left(\frac{p_1 a_1 + p_2 a_2 + \cdots + p_n a_n}{p_1 + p_2 + \cdots + p_n}\right)$$

is maximum when all $a_i \in \{a, b\}$.

14. ARITHMETIC MEAN METHOD

Arithmetic Mean Theorem. *Let*

$$F(a_1,a_2,\dots,a_n):\mathbb{A}\to\mathbb{R},\qquad \mathbb{A}\in\mathbb{R}^n,$$

be a symmetric continuous function satisfying

$$F(a_1,a_2,\dots,a_{n-1},a_n)\ge F\left(\frac{a_1+a_n}{2},a_2,\dots,a_{n-1},\frac{a_1+a_n}{2}\right)$$

for all $a_1,a_2,\dots,a_n\in\mathbb{A}$ such that $a_1\le a_2\le\cdots\le a_n$ or $a_1\ge a_2\ge\cdots\ge a_n$. Then, for all $a_1,a_2,\dots,a_n\in\mathbb{A}$, the following inequality holds:

$$F(a_1,a_2,\dots,a_n)\ge F(a,a,\dots,a),\qquad a=\frac{a_1+a_2+\cdots+a_n}{n}.$$

Arithmetic Mean Corollary (*Vasile Cîrtoaje*, 2005). *Let*

$$F(a_1,a_2,\dots,a_n):\mathbb{A}\to\mathbb{R},\qquad \mathbb{A}\in\mathbb{R}^n,$$

be a symmetric continuous function satisfying

$$F(a_1,a_2,\dots,a_{n-2},a_{n-1},a_n)\ge F\left(\frac{a_1+a_{n-1}}{2},a_2,\dots,a_{n-2},\frac{a_1+a_{n-1}}{2},a_n\right)$$

for all $a_1,a_2,\dots,a_n\in\mathbb{A}$ such that $a_1\le a_2\le\cdots\le a_n$ (or $a_1\ge a_2\ge\cdots\ge a_n$). Then, for all $a_1,a_2,\dots,a_n\in\mathbb{A}$ such that $a_1\le a_2\le\cdots\le a_n$ (or $a_1\ge a_2\ge\cdots\ge a_n$), the following inequality holds:

$$F(a_1,a_2,\dots,a_{n-1},a_n)\ge F(t,t,\dots,t,a_n),\qquad t=\frac{a_1+a_2+\cdots+a_{n-1}}{n-1}.$$

15. ARITHMETIC COMPENSATION METHOD

Arithmetic Compensation Theorem (*Vasile Cîrtoaje*, 2005). *Let $s>0$ and let F be a symmetric continuous function on the compact set in $\mathbb{R}^n$*

$$S=\{(a_1,a_2,\dots,a_n):a_1+a_2+\cdots+a_n=s,\ a_i\ge 0,\ i=1,2,\dots,n\}.$$

If

$$F(a_1,a_2,a_3,\dots,a_n)\ge$$
$$\ge\min\left\{F\left(\frac{a_1+a_2}{2},\frac{a_1+a_2}{2},a_3,\dots,a_n\right),F(0,a_1+a_2,a_3,\dots,a_n)\right\}\qquad (*)$$

for all $(a_1,a_2,\dots,a_n)\in S$, then

$$F(a_1,\dots,a_{n-k},a_{n-k+1},\dots,a_n)\ge\min_{1\le k\le n}F\left(0,\dots,0,\frac{s}{k},\dots,\frac{s}{k}\right)$$

for all $(a_1,a_2,\dots,a_n)\in S$.

Arithmetic Compensation Corollary (*Vasile Cîrtoaje*, 2005). *Let $s>0$ and let F be a symmetric continuous function on the compact set in $\mathbb{R}^n$*

$$S=\{(a_1,a_2,\ldots,a_n): a_1+a_2+\cdots+a_n=s,\ a_i\ge 0,\ i=1,2,\ldots,n\}.$$

If

$$F(a_1,a_2,a_3,\ldots,a_n)\ge F(0,a_1+a_2,a_3,\ldots,a_n)$$

for all $(a_1,a_2,\ldots,a_n)\in S$ satisfying

$$F(a_1,a_2,a_3,\ldots,a_n)<F\left(\frac{a_1+a_2}{2},\frac{a_1+a_2}{2},a_3,\ldots,a_n\right),\qquad a_1\ne a_2,$$

then $F(a_1,a_2,\ldots,a_n)$ is minimum when $n-k$ of the variables $a_1,a_2,\ldots,a_n$ are zero and the other k variables are equal to $\frac{s}{k}$, where $k\in\{1,2,\ldots,n\}$.

16. pqr METHOD

Theorem 1. *If $a\ge b\ge c$ are real numbers such that*

$$a+b+c=p,\qquad ab+bc+ca=q,$$

where p and q are fixed real numbers satisfying $p^2\ge 3q$, then the product $r=abc$ is minimal only when $a=b$, and maximal only when $b=c$.

Theorem 2. *If a,b,c are real numbers such that*

$$a+b+c=p,\qquad abc=r,$$

where p and r are fixed real numbers, then the sum $q=ab+bc+ca$ is maximal only when two of a,b,c are equal.

Theorem 3. *If $a\ge b\ge c$ are real numbers such that*

$$ab+bc+ca=q,\qquad abc=r\ne 0,$$

where q and r are fixed real numbers, then the product $p_1=abc(a+b+c)$ is maximal only when two of a,b,c are equal.

Theorem 4. *If $a\ge b\ge c\ge 0$ are nonnegative real numbers such that*

$$a+b+c=p,\qquad ab+bc+ca=q,$$

where p and q are fixed nonnegative real numbers satisfying $p^2\ge 3q$, then the product $r=abc$ is minimal only when $a=b$ or $c=0$, and maximal only when $b=c$.

Theorem 5. *If $a\ge b\ge c>0$ are positive real numbers such that*

$$a+b+c=p,\qquad abc=r,$$

where p and r are fixed positive real numbers satisfying $p^3 \geq 27r$, then $q = ab + bc + ca$ is minimal only when $b = c$, and maximal only when $a = b$.

Theorem 6. *If $a \geq b \geq c > 0$ are positive real numbers such that*

$$ab + bc + ca = q, \qquad abc = r,$$

where q and r are fixed positive real numbers satisfying $p^3 \geq 27r$, then the sum $p = a + b + c$ is minimal only when $a = b$, and maximal only when $b = c$.

pqr Theorem. *Let a, b, c be real numbers and*

$$p = a + b + c, \qquad q = ab + bc + ca, \qquad r = abc.$$

For any real β, the following inequality holds

$$27\beta r + |(a-b)(b-c)(c-a)| \leq 9\beta pq - 2\beta p^3 + 2\sqrt{\frac{1}{27} + \beta^2}\,\left(p^2 - 3q\right)^{3/2},$$

with equality for

$$2\beta(p^2 - 3q)^{3/2} = \sqrt{\frac{1}{27} + \beta^2}\,\left(2p^3 - 9pq + 27r\right).$$

17. SYMMETRIC INEQUALITIES OF DEGREE THREE, FOUR, FIVE AND SIX

Theorem 1. *Let $f_n(a, b, c)$ be a symmetric homogeneous polynomial of degree n.*

(a) *The inequality $f_4(a, b, c) \geq 0$ holds for all real numbers a, b, c if and only if $f_4(a, 1, 1) \geq 0$ for all real a;*

(b) *For $n \in \{3, 4, 5\}$, the inequality $f_n(a, b, c) \geq 0$ holds for all $a, b, c \geq 0$ if and only if $f_n(a, 1, 1) \geq 0$ and $f_n(0, b, c) \geq 0$ for all $a, b, c \geq 0$.*

A symmetric and homogeneous polynomial of degree six can be written in the form

$$f_6(a, b, c) = Ar^2 + g_1(p, q)r + g_2(p, q),$$

where

$$p = a + b + c, \qquad q = ab + bc + ca, \qquad r = abc,$$

A is the *highest coefficient* of $f_6(a, b, c)$, and $g_1(p, q)$ and $g_2(p, q)$ are polynomial functions.

Theorem 2 (*Vasile Cîrtoaje*, 2008). *Let $f_6(a, b, c)$ be a symmetric homogeneous polynomial of degree six which has the highest coefficient $A \leq 0$. The inequality $f_6(a, b, c) \geq 0$ holds for all real numbers a, b, c if and only if*

$$f_6(a, 1, 1) \geq 0$$

for all real a.

Theorem 3 (*Vasile Cîrtoaje*, 2008). *Let $f_6(a,b,c)$ be a sixth degree symmetric homogeneous polynomial having the highest coefficient $A \le 0$. The inequality $f_6(a,b,c) \ge 0$ holds for all nonnegative real numbers a,b,c if and only if $f_6(a,1,1) \ge 0$ and $f_6(0,b,c) \ge 0$ for all nonnegative real numbers a,b,c.*

*****

Consider the inequality

$$f_6(a,b,c) \ge 0,$$

where a,b,c are *real numbers* and $f_6(a,b,c)$ is a symmetric homogeneous polynomial of degree six with the highest coefficient $A > 0$. The *highest coefficient cancellation method* for proving such an inequality uses the above Theorem 2 and the following three ideas:

1) finding a nonnegative symmetric homogeneous function $\bar{f}_6(a,b,c)$ of the form

$$\bar{f}_6(a,b,c) = \left(r + A_1 pq + A_2 p^3 + A_3 \frac{q^2}{p} \right)^2, \tag{A.1}$$

where A_1, A_2, A_3 are real numbers chosen such that

$$f_6(a,b,c) \ge A\bar{f}_6(a,b,c) \ge 0$$

for all real numbers a,b,c;

2) seeing that the difference $f_6(a,b,c) - A\bar{f}_6(a,b,c)$ has the highest coefficient equal to zero, therefore the inequality

$$f_6(a,b,c) \ge A\bar{f}_6(a,b,c)$$

holds if and only if it holds for $b = c = 1$ (see Theorem 2);

3) choosing a suitable real number

$$\xi \in (-\infty, 0) \cup (3, \infty)$$

and treating successively the cases $p^2 < \xi q$ and $p^2 \ge \xi q$.

*****

Consider the inequality

$$f_6(a,b,c) \ge 0,$$

where a,b,c are *nonnegative numbers* and $f_6(a,b,c)$ is a symmetric homogeneous polynomial of degree six with the highest coefficient $A > 0$. The *highest coefficient cancellation method* for proving such an inequality uses the above Theorem 3 and the following three ideas:

1) finding a nonnegative symmetric homogeneous function $\bar{f}_6(a,b,c)$ of the form

$$\bar{f}_6(a,b,c) = \left(r + A_1pq + A_2p^3 + A_3\frac{q^2}{p}\right)^2, \tag{A.2}$$

where A_1, A_2, A_3 are real numbers chosen such that

$$f_6(a,b,c) \ge A\bar{f}_6(a,b,c) \ge 0$$

for all nonnegative real numbers a, b, c;

2) seeing that the the difference $f_6(a,b,c) - A\bar{f}_6(a,b,c)$ has the highest coefficient equal to zero, therefore the inequality

$$f_6(a,b,c) \ge A\bar{f}_6(a,b,c)$$

holds for all nonnegative real numbers a, b, c if and only if it holds for $b = c = 1$ and for $a = 0$ (see Theorem 3);

3) treating successively the cases $p^2 < 4q$ and $p^2 \ge 4q$.

18. POPOVICIU'S INEQUALITY

If f is a convex function on a real interval $\mathbb{I}$ and $a_1, a_2, \ldots, a_n \in \mathbb{I}$, then

$$f(a_1) + f(a_2) + \cdots + f(a_n) + n(n-2)f\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) \ge$$

$$\ge (n-1)[f(b_1) + f(b_2) + \cdots + f(b_n)],$$

where

$$b_i = \frac{1}{n-1}\sum_{j \ne i} a_j, \quad i = 1, 2, \cdots, n.$$

In the same conditions, the following similar inequality holds:

$$f(a_1) + f(a_2) + \cdots + f(a_n) + \frac{n}{n-2}f\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) \ge$$

$$\ge \frac{2}{n-2}\sum_{1 \le i < j \le n} f\left(\frac{a_i + a_j}{2}\right).$$

Appendix B

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