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Nesbitt's inequality

In mathematics, **Nesbitt's inequality** states that for positive real numbers a, b and c,

$$\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\geq\frac{3}{2}.$$

It is an elementary special case (N = 3) of the difficult and much studied Shapiro inequality, and was published at least 50 years earlier.

There is no corresponding upper bound as any of the 3 fractions in the inequality can be made arbitrarily large.

Contents

Proof

First proof: AM-HM inequality

By the AM-HM inequality on $(a+b),(b+c),(c+a)$,

$$\frac{(a+b)+(a+c)+(b+c)}{3}\geq\frac{3}{\frac{1}{a+b}+\frac{1}{a+c}+\frac{1}{b+c}}.$$

Clearing denominators yields

$$((a+b)+(a+c)+(b+c))\left(\frac{1}{a+b}+\frac{1}{a+c}+\frac{1}{b+c}\right)\geq 9,$$

from which we obtain

$$2\frac{a+b+c}{b+c}+2\frac{a+b+c}{a+c}+2\frac{a+b+c}{a+b}\geq 9$$

by expanding the product and collecting like denominators. This then simplifies directly to the final result.

Second proof: Rearrangement

Suppose $a \geq b \geq c$, we have that

$$\frac{1}{b+c}\geq\frac{1}{a+c}\geq\frac{1}{a+b}$$

define

$$\vec{x}=(a,b,c)$$
$$\vec{y}=\left(\frac{1}{b+c},\frac{1}{a+c},\frac{1}{a+b}\right)$$

The scalar product of the two sequences is maximum because of the rearrangement inequality if they are arranged the same way, call $\vec{y}_1$ and $\vec{y}_2$ the vector $\vec{y}$ shifted by one and by two, we have:

$$\vec{x}\cdot\vec{y}\geq\vec{x}\cdot\vec{y}_1$$
$$\vec{x}\cdot\vec{y}\geq\vec{x}\cdot\vec{y}_2$$

Addition yields our desired Nesbitt's inequality.

Third proof: Sum of Squares

The following identity is true for all a, b, c :

$$\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}=\frac{3}{2}+\frac{1}{2}\left(\frac{(a-b)^2}{(a+c)(b+c)}+\frac{(a-c)^2}{(a+b)(b+c)}+\frac{(b-c)^2}{(a+b)(a+c)}\right)$$

This clearly proves that the left side is no less than $\frac{3}{2}$ for positive a, b and c.

Note: every rational inequality can be demonstrated by transforming it to the appropriate sum-of-squares identity, see Hilbert's seventeenth problem.

Fourth proof: Cauchy–Schwarz

Invoking the Cauchy–Schwarz inequality on the vectors $\left\langle\sqrt{a+b}, \sqrt{b+c}, \sqrt{c+a}\right\rangle, \left\langle\frac{1}{\sqrt{a+b}}, \frac{1}{\sqrt{b+c}}, \frac{1}{\sqrt{c+a}}\right\rangle$ yields

$$((b+c)+(a+c)+(a+b))\left(\frac{1}{b+c}+\frac{1}{a+c}+\frac{1}{a+b}\right) \geq 9,$$

which can be transformed into the final result as we did in the AM-HM proof.

Fifth proof: AM-GM

Let $x=a+b, y=b+c, z=c+a$. We then apply the AM-GM inequality to obtain the following

$$\frac{x+z}{y}+\frac{y+z}{x}+\frac{x+y}{z} \geq 6.$$

because $\frac{x}{y}+\frac{z}{y}+\frac{y}{x}+\frac{z}{x}+\frac{x}{z}+\frac{y}{z} \geq 6 \sqrt[6]{\frac{x}{y} \cdot \frac{z}{y} \cdot \frac{y}{x} \cdot \frac{z}{x} \cdot \frac{x}{z} \cdot \frac{y}{z}}=6.$

Substituting out the x, y, z in favor of a, b, c yields

$$\frac{2a+b+c}{b+c}+\frac{a+b+2c}{a+b}+\frac{a+2b+c}{c+a} \geq 6$$
$$\frac{2a}{b+c}+\frac{2c}{a+b}+\frac{2b}{a+c}+3 \geq 6$$

which then simplifies to the final result.

Sixth proof: Titu's lemma

Titu's lemma, a direct consequence of the Cauchy–Schwarz inequality, states that for any sequence of n real numbers (x_k) and any sequence of n positive numbers (a_k), $\sum_{k=1}^{n} \frac{x_k^2}{a_k} \geq \frac{\left(\sum_{k=1}^{n} x_k\right)^2}{\sum_{k=1}^{n} a_k}$.

We use the lemma on $(x_k)=(1,1,1)$ and $(a_k)=(b+c, a+c, a+b)$. This gives,

$$\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b} \geq \frac{3^2}{2(a+b+c)}$$

This results in,

$$\frac{a+b+c}{b+c}+\frac{a+b+c}{c+a}+\frac{a+b+c}{a+b} \geq \frac{9}{2} \text{ i.e.,}$$
$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{9}{2}-3=\frac{3}{2}$$

Seventh proof: Using homogeneity

As the left side of the inequality is homogeneous, we may assume $a+b+c=1$. Now define $x=a+b$, $y=b+c$, and $z=c+a$. The desired inequality turns into $\frac{1-x}{x}+\frac{1-y}{y}+\frac{1-z}{z}\geq\frac{3}{2}$, or, equivalently, $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq 9/2$. This is clearly true by Titu's Lemma.

Eighth proof: Jensen inequality

Define $S=a+b+c$ and consider the function $f(x)=\frac{x}{S-x}$. This function can be shown to be convex in $[0,S]$ and, invoking Jensen inequality, we get

$$\frac{\frac{a}{S-a}+\frac{b}{S-b}+\frac{c}{S-c}}{3}\geq\frac{S/3}{S-S/3}.$$

A straightforward computation yields

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}.$$

Ninth proof: Reduction to a two-variable inequality

By clearing denominators,

$$\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\geq\frac{3}{2}\iff 2(a^3+b^3+c^3)\geq ab^2+a^2b+ac^2+a^2c+bc^2+b^2c.$$

It now suffices to prove that $x^3+y^3\geq xy^2+x^2y$ for $(x,y)\in\mathbb{R}^2_+$, as summing this three times for $(x,y)=(a,b),\ (a,c)$, and (b,c) completes the proof.

As $x^3+y^3\geq xy^2+x^2y\iff(x-y)(x^2-y^2)\geq 0$ we are done.

References

- Nesbitt, A.M., Problem 15114, Educational Times, 55, 1902.
- Ion Ionescu, Romanian Mathematical Gazette, Volume XXXII (September 15, 1926 - August 15, 1927), page 120
- Arthur Lohwater (1982). "Introduction to Inequalities" (http://www.mediafire.com/?1mw1tkgozzu). Online e-book in PDF format.

External links

- See AoPS (http://www.mathlinks.ro/viewtopic.php?t=207221) for more proofs of this inequality.