

# GEOMETRIC INEQUALITIES

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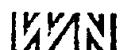
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## Preface

This is a collection of inequalities in elementary plane geometry, the majority of which deals with triangles and the remarkable lines associated with them.

Geometric inequalities are as old as geometry itself. The first book of Euclid's *Elements* contains several theorems on inequalities for the sides and the angles of a triangle, the most important of which is perhaps Proposition XX: the sum of two sides is greater than the third. It seems permissible to state that almost all geometric inequalities are based in some way or another on this theorem.

A multitude of inequalities has been found during the last two centuries. One of the oldest is that with respect to the radii of the circumscribed circle and the inscribed circle:  $R \geqslant 2r$ , given by Euler in 1765. It still maintains its position of high quality in this field, because it shows two properties for mathematical appreciation: it is simple, but by no means trivial. Many others have been given in the course of time, but while the elementary geometry of remarkable points, lines and circles (sometimes disparagingly referred to as the microscopy of the triangle) had its climax in the later part of the nineteenth century, the interest in inequalities has been increasing in, say, the last thirty years. The results are rather disjointed: they are scattered at random over many books, journals, problem sections and examination papers. In this book we tried to collect and present them to the reader, according to a certain classification, although we are fully aware that the latter had to be imperfect in view of the data we had to deal with.

The collection offers a variety of theorems on geometric inequalities. We inserted a few which at first glance may seem

unsympathetic by their complicity or may even be considered as bizarre. But the greater part is in our opinion attractive and interesting and among them are some real gems. The effort involved in collecting, checking and classifying the contents of this book was more than compensated for by the pleasure experienced from so many ingenious theorems and elegant proofs and we have admired the skill and the imagination of that multitude of mathematicians from former and recent years, whose contributions fill the following pages.

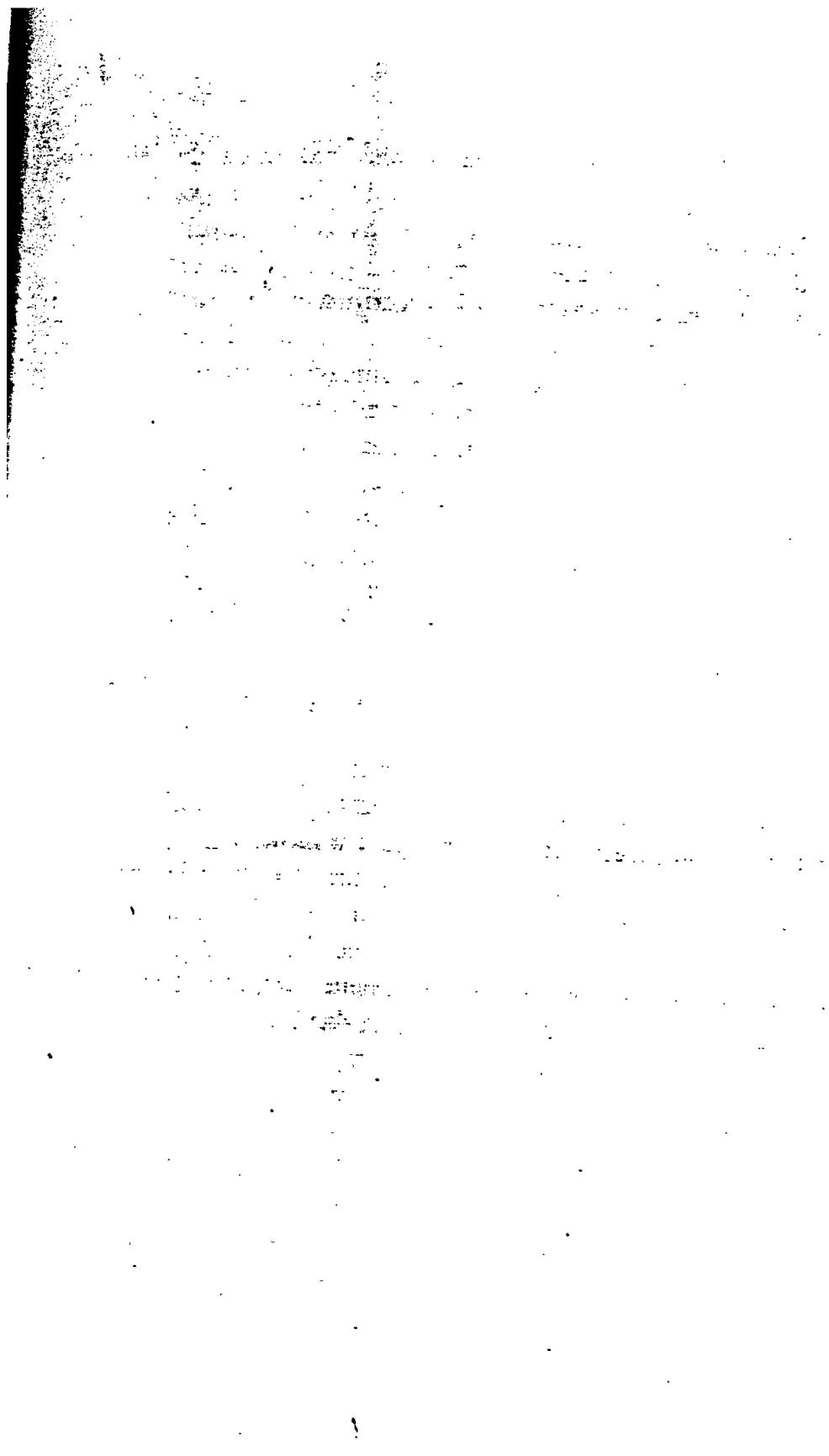
With a few exceptions we were able to add bibliographic references to the theorems. As some identical or similar inequalities have appeared, obviously independent of each other, at different times and places, we are not at all certain that we have always given due honour to priority claims. To many theorems we added a complete proof, sometimes we restricted ourselves to a hint and in many other cases the proof was omitted altogether. We had no strict rules for that; we were guided by the importance of the inequality, the nature of the proof and the accessibility of the reference. For all theorems, but especially for those without an added proof, we liked to think that the inequality concerned could be a challenge to the reader.

This is not a new book. A considerable part of it is a translation of a publication in serbo-croatian *Geometrijske nejednakosti*, by all the authors with the exception of myself. It appeared as number 31 of the *Matematička Biblioteka* at Beograd in 1966. The four members of the Yugoslav group have done the tedious field work; they have consulted innumerable volumes of likely and unlikely mathematical journals. After selecting the material they classified it and checked the proofs. They continued in this manner when preparing the English version and during this second period they unearthed more valuable treasures. It was an honour and a pleasure to be invited to join their group and to be given an advisory task in the making of this edition as well as contributing thereto. Although each of the five authors mentioned in alphabetical order on the title page contributed his own share, the edition is essentially the result of close operation and team work,

A comparison with the original version shows that the collection of inequalities has been extended and classified anew. On the other hand we decided on several grounds to delete all theorems on solid geometry: they are interesting, but rather isolated and we wished to give our work a certain unity and brevity. The character of the book, however, has not been changed. It emphasizes algebraic (including goniometric) methods and with some exaggeration one may state that all our theorems on triangles give inequalities for three real numbers  $a, b, c$ , which are corollaries of the system  $-a+b+c > 0$ ,  $a-b+c > 0$ ,  $a+b-c > 0$ . Consequently this book on *geometric* inequalities does not contain any drawings.

O. B.

The authors intend to keep a systematic check on the further development of geometric inequalities and to make an annual report on their findings in the journal: Univerzitet i Beogradu, Publikacije Elektrotehničkog fakulteta, Serija Matematika i fizika or in some other suitable journal. This report will inform the reader about the state of research in the field. The authors welcome suggestions and critical remarks on the book, so as to make the report more complete. Such material should be sent to: Professor D. S. Mitrinović, Department of Mathematics, Electro-technical Faculty, University of Beograd, Yugoslavia



## Notations

In this book the following notations are used:

$A, B, C$	vertices of a triangle
$a, b, c$	sides $BC, CA, AB$
$\alpha, \beta, \gamma$	its angles
$h_a, h_b, h_c$	altitudes
$m_a, m_b, m_c$	medians
$w_a, w_b, w_c$	angle-bisectors 角平分线
$O$	circumcentre
$R$	radius of circumcircle
$I$	incentre
$r$	radius of incircle
$H$	orthocentre
$G$	centroid
$s$	semi-perimeter
$F$	area of triangle $ABC$
$P$	point in the interior of a triangle
$I_a, I_b, I_c$	excentres
$r_a, r_b, r_c$	radii of excircles
$R_1, R_2, R_3$	distances from $P$ to the vertices of $ABC$
$r_1, r_2, r_3$	distances from $P$ to the sides of $ABC$
$w_1, w_2, w_3$	angle-bisectors of the angles $BPC, CPA, APB$
$r'_1, r'_2, r'_3$	Cevian segments $PD, PE, PF$
$R_i(P) = R_i$	
$r_i(P) = r_i$	
$Q = (b-c)^2 + (c-a)^2 + (a-b)^2$	
$MN^2 = \overline{MN}^2 =  MN ^2$	



$$\begin{aligned}
 M_k(x, y, z) &= (xyz)^{1/3} && \text{for } k = 0 \\
 &= (\frac{1}{3}(x^k + y^k + z^k))^{1/k} && \text{for } k \neq 0, \\
 &= \min(x, y, z) && \text{for } k = - \\
 &= \max(x, y, z) && \text{for } k = +
 \end{aligned}$$

$A, B, C, D$  vertices of a quadrilateral

$a, b, c, d$  sides  $AB, BC, CD, DA$

$\alpha, \beta, \gamma, \delta$  its angles

$p, q$  diagonals  $AC, BD$

$l = 2s$  perimeter of  $ABCD$

$F$  area of quadrilateral  $ABCD$

$P$  intersection of the diagonals

## 1. Inequalities involving only the sides of a triangle

$$1.1 \quad 3(bc+ca+ab) \leq (a+b+c)^2 < 4(bc+ca+ab).$$

Equality holds if and only if the triangle is equilateral.

PROOF. We commence with  $2bc \leq b^2+c^2$ ,  $2ca \leq c^2+a^2$ ,  $2ab \leq a^2+b^2$ . By adding together these inequalities, and then adding  $4(bc+ca+ab)$ , we get

$$3(bc+ca+ab) \leq (a+b+c)^2,$$

where equality holds only if  $a = b = c$ .

Since  $a$ ,  $b$  and  $c$  are the sides of a triangle, we have

$$|b-c| < a, |c-a| < b, |a-b| < c,$$

i.e.

$$(b-c)^2 < a^2, (c-a)^2 < b^2, (a-b)^2 < c^2.$$

By addition, we obtain

$$a^2+b^2+c^2 < 2(bc+ca+ab),$$

i.e.

$$(a+b+c)^2 < 4(bc+ca+ab).$$

F. E. Wood, Problem E 345, Amer. Math. Monthly 45 (1938), 551.

$$1.2 \quad a^2+b^2+c^2 \geq \frac{36}{35} \left( s^2 + \frac{abc}{s} \right).$$

Equality holds if and only if the triangle is equilateral.

PROOF. Using the inequalities

$$a^2+b^2+c^2 \geq \frac{1}{3}(a+b+c)^2$$

and

$$abc \leq [\frac{1}{3}(a+b+c)]^3,$$

where the equalities hold if and only if  $a = b = c$ , we have

$$\begin{aligned} a^2 + b^2 + c^2 &\geq \frac{4}{3}s^2 = \frac{36}{35} \left[ s^2 + (\frac{2}{3}s)^3 \cdot \frac{1}{s} \right] \\ &= \frac{36}{35} \left[ s^2 + \left( \frac{a+b+c}{3} \right)^3 \cdot \frac{1}{s} \right] \\ &\geq \frac{36}{35} \left( s^2 + \frac{abc}{s} \right), \end{aligned}$$

with equality only if  $a = b = c$ .

J. F. Darling-W. Moser, Problem E 1456, Amer. Math. Monthly 68 (1961), 294 and 930.

$$1.3 \quad 8(s-a)(s-b)(s-c) \leq abc. \quad (1)$$

Equality holds if and only if the triangle is equilateral.

PROOF. We have

$$\sqrt{a^2 - (b-c)^2} \leq a, \quad \sqrt{b^2 - (c-a)^2} \leq b, \quad \sqrt{c^2 - (a-b)^2} \leq c,$$

i.e.

$$\sqrt{(a+b-c)^2(b+c-a)^2(c+a-b)^2} \leq abc.$$

Since  $b+c-a$ ,  $c+a-b$ ,  $a+b-c$  are positive numbers, we get

$$(a+b-c)(b+c-a)(c+a-b) \leq abc.$$

Equality in (1) holds if and only if  $a = b = c$ .

A. Padoa, Period. Mat. (4) 5 (1925), 80-85.

$$1.4 \quad 8abc \leq (a+b)(b+c)(c+a),$$

with equality if and only if the triangle is equilateral.

E. Cesàro, Nouvelle Correspondance Math. 6 (1880), 140.

REMARK. This inequality also holds for all non-negative real numbers  $a, b, c$ .

$$1.5 \quad 3(a+b)(b+c)(c+a) \leq 8(a^3 + b^3 + c^3),$$

with equality if and only if  $a = b = c$ .

A. Padoa, Period. Mat. (4) 5 (1925), 80-85.

REMARK. This inequality also holds for all non-negative real numbers  $a, b, c$ .

$$1.6 \quad 2(a+b+c)(a^2+b^2+c^2) \geq 3(a^3+b^3+c^3+3abc),$$

with equality if and only if  $a = b = c$ .

M. Collins, Educational Times 13 (1870), 30–31.

$$1.7 \quad abc < a^2(s-a) + b^2(s-b) + c^2(s-c) \leq \frac{3}{2}abc. \quad (1)$$

PROOF. Since

$$2 \sum a^2(s-a) = \sum a^2(b+c-a) = - \sum a^3 + \sum b^2c$$

and

$$(b+c-a)(c+a-b)(a+b-c) = - \sum a^3 + \sum b^2c - 2abc$$

we have

$$2 \sum a^2(s-a) = (b+c-a)(c+a-b)(a+b-c) + 2abc. \quad (2)$$

From (2) and 1.3 follows (1).

Equality holds if and only if the triangle is equilateral.

$$1.8 \quad bc(b+c) + ca(c+a) + ab(a+b) \geq 48(s-a)(s-b)(s-c). \quad (1)$$

Equality holds if and only if the triangle is equilateral.

PROOF. Since

$$b+c \geq 2\sqrt{bc}, \quad c-a \geq 2\sqrt{ca}, \quad a+b \geq 2\sqrt{ab},$$

we have

$$bc(b+c) + ca(c+a) + ab(a+b) \geq 2[(bc)^{3/2} + (ca)^{3/2} + (ab)^{3/2}].$$

By arithmetic-geometric mean inequality we get

$$(bc)^{3/2} + (ca)^{3/2} + (ab)^{3/2} \geq 3abc,$$

so that

$$bc(b+c) + ca(c+a) + ab(a+b) \geq 6abc. \quad (2)$$

From (2) and 1.3 follows (1).

$$1.9 \quad a^3(s-a) + b^3(s-b) + c^3(s-c) \leq abcs. \quad (1)$$

PROOF. This inequality is equivalent to

$$a^2(a-b)(a-c) + b^2(b-c)(b-a) + c^2(c-a)(c-b) \geq 0. \quad (2)$$

Inequality (2) holds because it is a particular case of Schur's inequality (see: G. H. Hardy, J. E. Littlewood and G. Pólya: Inequalities, Cambridge 1934, p. 64).

Equality in (1) holds if and only if  $a = b = c$ .

J. Andersson, Problem E 1779, Amer. Math. Monthly 59 (1952), 41.

**1.10** For all real  $t$ ,

$$a^t(s-a) + b^t(s-b) + c^t(s-c) \leq \frac{1}{2}abc(a^{t-2} + b^{t-2} + c^{t-2}).$$

Equality holds if and only if  $a = b = c$ .

This result is due to R. Ž. Djordjević.

**1.11**  $a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0$ ,

where equality holds if and only if the triangle is equilateral.

E. Catalan, Educational Times, N.S. 10 (1906), 57.

**1.12**  $64s^3(s-a)(s-b)(s-c) \leq 27a^2b^2c^2$ .

Equality occurs if and only if the triangle is equilateral.

A. Padoa, Period. Mat. (4) 6 (1926), 38–40.

**1.13** If  $q = (Q/2)^{1/2}$ , then

$$2(s-q)(2s+q)^2 \leq 27abc \leq 2(s+q)(2s-q)^2. \quad - (1)$$

The first and second equality in (1) hold respectively for isosceles triangles whose base is the smallest and largest of the three sides; of course, both equality signs apply when the triangle is equilateral, since in that case  $q = 0$ .

R. Frucht, Canad. J. Math. 9 (1957), 227–231.

$$\frac{2s}{abc} \leq \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}. \quad (1)$$

Equality holds if and only if the triangle is equilateral.

PROOF. Since

$$(bc)^2 + (ca)^2 + (ab)^2 \geq (ca)(ab) + (ab)(bc) + (bc)(ca),$$

i.e.,

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq \frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab},$$

and

$$\frac{2s}{abc} = \frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab},$$

we conclude that (1) holds.

Gaz. Mat. B 10 (1959), 162.

$$1.15 \quad \frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} \geq \frac{9}{s}.$$

Equality holds if and only if the triangle is equilateral.

$$1.16 \quad \frac{3}{2} \leq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < 2. \quad (1)$$

Equality occurs if and only if  $a = b = c$ .

PROOF. By means of arithmetic-harmonic mean inequality,

$$\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} \geq \frac{9}{2(a+b+c)}, \quad (2)$$

with equality if and only if  $a = b = c$ .

Since

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = (a+b+c) \left( \frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} \right) - 3,$$

we conclude by (2) that the first inequality in (1) holds.

Since

$$b+c > \frac{1}{2}(a+b+c), \quad c+a > \frac{1}{2}(a+b+c), \quad a+b > \frac{1}{2}(a+b+c),$$

we have

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < \frac{2(a+b+c)}{a+b+c} = 2.$$

This proves the second inequality in (1).

A. M. Nesbitt, Problem 15114, Educational Times (2) 3 (1903), 37-38.

M. Petrović, Računanje sa brojnim razmacima, Beograd 1932, p. 79.

REMARK. The first inequality in (1) also holds for all non-negative real numbers. See, for example, D. S. Mitrinović, Elementary Inequalities, Groningen 1964, 144–147.

$$1.17 \quad \frac{15}{4} \leq \frac{s+a}{b+c} + \frac{s+b}{c+a} + \frac{s+c}{a+b} < \frac{9}{2}.$$

Equality occurs if and only if  $a = b = c$ .

Gaz. Mat. B 7 (1956), 438.

$$1.18 \quad \left(1 - \frac{a}{b}\right)\left(1 - \frac{a}{c}\right) + 1 > \frac{c-a}{b} + \frac{b-a}{c}.$$

Gaz. Mat. B 15 (1964), 422.

$$1.19 \quad \frac{1}{3} \leq \frac{a^2+b^2+c^2}{(a+b+c)^2} < \frac{1}{2}.$$

Equality occurs if and only if  $a = b = c$ .

M. Petrović, Enseign. Math. 18 (1916), 153–163.

$$1.20 \quad \sqrt{s} < \sqrt{s-a} + \sqrt{s-b} + \sqrt{s-c} \leq \sqrt{3s}.$$

Equality holds if and only if the triangle is equilateral.

A. Santalò, Math. Notae 3 (1943), fasc. 2, 65–73.

E. G. Gotman, Matematika v škole 1965, No. 1, 76.

1.21 For all real  $t$ ,

$$\begin{aligned} & 2F \cdot \min[(abc)^{-2/3}, (\max(a, b, c) \min(a, b, c))^{-1}] M_t(a, b, c) \\ & \leq M_t(h_a, h_b, h_c) \\ & \leq 2F \cdot \max[(abc)^{-2/3}, (\max(a, b, c) \min(a, b, c))^{-1}] M_t(a, b, c). \end{aligned}$$

O. Reuter, Elem. Math. 18 (1963), 34–35.

1.22 For all real  $t$ ,

$$\begin{aligned} & (a^2+b^2+c^2)(b^{t-2}c^{t-2}+c^{t-2}a^{t-2}+a^{t-2}b^{t-2}) \\ & < 2(a^t+b^t+c^t)(a^{t-2}+b^{t-2}+c^{t-2}). \end{aligned}$$

PROOF. Consider the point  $P$  which has the barycentric coordinates  $a^t, b^t, c^t$ . Then

$$PA^2 = \frac{(a^t+b^t+c^t)(b^t c^2 + c^t b^2) - a^2 b^2 c^t - a^t b^2 c^t - a^t b^t c^2}{(a^t+b^t+c^t)^2}.$$

whence

$$\frac{PA^2}{b^2c^2} = \frac{(a^t+b^t+c^t)(b^{t-2}+c^{t-2}) - a^t b^{t-2} c^{t-2} + c^{t-2} a^{t-2} + a^{t-2} b^{t-2}}{(a^t+b^t+c^t)^2},$$

with the analogous formulae for  $\frac{PB^2}{c^2a^2}$  and  $\frac{PC^2}{a^2b^2}$ .

Therefore

$$\begin{aligned} & (a^t+b^t+c^t)^2 \left( \frac{PA^2}{b^2c^2} + \frac{PB^2}{c^2a^2} + \frac{PC^2}{a^2b^2} \right) \\ &= 2(a^t+b^t+c^t)(a^{t-2}-b^{t-2}+c^{t-2}) \\ &\quad - (a^2+b^2+c^2)(b^{t-2}c^{t-2}+c^{t-2}a^{t-2}+a^{t-2}b^{t-2}). \end{aligned}$$

Since the first member of this equality is positive, the second member will also be positive.

L. Bénézech, Problem 412, J. Math. Élém. Paris (4) 1 (1892), 234.

$$1.23 \quad (a+b+c)^3 \leq 5[bc(b+c) + ca(c+a) + ab(a+b)] - 3abc, \quad (1)$$

with equality if and only if the triangle is equilateral.

PROOF. We have

$$\begin{aligned} (a+b+c)^3 &= a^3 + b^3 + c^3 + 3abc \\ &\quad + 3[bc(b+c) + ca(c+a) + ab(a+b)] - 3abc. \end{aligned} \quad (2)$$

Since (see: 1.6)

$$a^3 + b^3 + c^3 + 9abc \leq 2[bc(b+c) + ca(c+a) + ab(a+b)],$$

from (2) we obtain (1).

S. Reich, Problem E 1930, Amer. Math. Monthly 73 (1966), 1017-1018.

$$1.24 \quad \left(1 + \frac{a}{b}\right)\left(1 + \frac{a}{c}\right) < 3 + \frac{a+b}{c} + \frac{a+c}{b}.$$

C. Ionescu-Tiu, Gaz. Mat. B 15 (1964), 272.

## 2. Inequalities for the angles of a triangle

$$2.1 \quad 0 < \sin \alpha + \sin \beta + \sin \gamma \leq \frac{3}{2}\sqrt{3}. \quad (1)$$

Equality holds if and only if the triangle is equilateral.

PROOF. Starting from

$$a^2 + b^2 + c^2 = 8R^2(1 + \cos \alpha \cos \beta \cos \gamma)$$

and

$$\frac{(a+b+c)^2}{3} \leq a^2 + b^2 + c^2,$$

we have

$$\frac{(a+b+c)^2}{3} \leq a^2 + b^2 + c^2 = 8R^2(1 + \cos \alpha \cos \beta \cos \gamma) \leq 9R^2, \quad (2)$$

where we used the inequality 2.23.

From (2) and the sine law, we get (1).

A. Padoa, Period. Mat. (4) 5 (1925), 80–85.

T. R. Curry, Problem E 1644, Amer. Math. Monthly 70 (1963), 1099 and 71 (1964), 915–916.

$$2.2 \quad 0 < \sin \alpha + \sin \beta + \sin \gamma \leq \frac{3}{2}\sqrt{3} \quad \text{in each triangle.} \quad (1)$$

$$2 < \sin \alpha + \sin \beta + \sin \gamma \leq \frac{3}{2}\sqrt{3} \quad \text{in acute triangles.} \quad (2)$$

$$0 < \sin \alpha + \sin \beta + \sin \gamma < 1 + \sqrt{2} \quad \text{in obtuse triangles.} \quad (3)$$

O. Bottema, Euclides 30 (1954/55), 114–116.

REMARK. Inequalities (1) and (2) are also proved in:

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

$$2.3 \quad 0 < \sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma \leq \frac{9}{4} \quad \text{in each triangle.}$$

$$2 < \sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma \leq \frac{9}{4} \quad \text{in acute triangles.}$$

$$0 < \sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma < 2 \quad \text{in obtuse triangles.}$$

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

$$2.4 \quad \sin \alpha + \sin \beta + \sin \gamma \geq \sin 2\alpha + \sin 2\beta + \sin 2\gamma.$$

Equality holds if and only if the triangle is equilateral.

PROOF. Applying the sine law we obtain

$$\sin \alpha + \sin \beta + \sin \gamma = \frac{a+b+c}{2R} = \frac{F}{rR}.$$

Also

$$\begin{aligned}\sin 2\alpha + \sin 2\beta + \sin 2\gamma &= 2(\sin \alpha \cos \alpha + \sin \beta \cos \beta + \sin \gamma \cos \gamma) \\ &= \frac{1}{R} (a \cos \alpha + b \cos \beta + c \cos \gamma).\end{aligned}$$

Since

$$a \cdot \cos \alpha + b \cdot \cos \beta + c \cdot \cos \gamma = \frac{2F}{R},$$

we have

$$\sin 2\alpha + \sin 2\beta + \sin 2\gamma = \frac{2F}{R^2}.$$

Therefore

$$\frac{\sin \alpha + \sin \beta + \sin \gamma}{\sin 2\alpha + \sin 2\beta + \sin 2\gamma} = \frac{R}{2r} \geq 1.$$

25.  $\sqrt{\sin \alpha} + \sqrt{\sin \beta} + \sqrt{\sin \gamma} \leq 3\sqrt[4]{\frac{3}{4}}.$

PROOF. Since

$$(x+y+z)^2 \leq 3(x^2+y^2+z^2),$$

putting

$$x = \sqrt{\sin \alpha}, y = \sqrt{\sin \beta}, z = \sqrt{\sin \gamma},$$

we obtain

$$\sqrt{\sin \alpha} + \sqrt{\sin \beta} + \sqrt{\sin \gamma} \leq \sqrt{3(\sin \alpha + \sin \beta + \sin \gamma)}.$$

The given inequality follows from inequality 2.1.

$\square$  Albu, Gaz. Mat. B 14 (1963), 177.

26. If  $k \leq 1$ , then

$$M_k(\sin \alpha, \sin \beta, \sin \gamma) \leq \frac{1}{2}\sqrt{3}.$$

$\square$  Gaz. Mat. B 14 (1963), 681.

27.  $\sin \alpha \sin \beta \sin \gamma \leq \frac{1}{8}\sqrt{3}.$

Equality holds if and only if the triangle is equilateral.

K. P. Wilkins, Amer. Math. Monthly 44 (1937), 579-583.

$$2.8 \quad 0 < \sin \alpha \sin \beta \sin \gamma \leq \frac{3}{8}\sqrt{3} \quad \text{in each triangle.}$$

$$0 < \sin \alpha \sin \beta \sin \gamma \leq \frac{1}{8}\sqrt{3} \quad \text{in acute triangles.}$$

$$0 < \sin \alpha \sin \beta \sin \gamma < \frac{1}{8}\sqrt{3} \quad \text{in obtuse triangles.}$$

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108-115.

$$2.9 \quad 1 < \sin \frac{\alpha}{2} + \sin \frac{\beta}{2} + \sin \frac{\gamma}{2} \leq \frac{3}{2}.$$

Equality occurs if and only if  $\alpha = \beta = \gamma$ .

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108-115.

REMARK. The second inequality has also been proved in:

J. M. Child, Math. Gaz. 23 (1939), 138-143.

$$2.10 \quad \sin \alpha \sin \beta \sin \frac{\gamma}{2} \leq \frac{3}{8}\sqrt{3}.$$

Equality holds if and only if  $\alpha = \beta$ , with  $\cos \alpha = \frac{1}{2}\sqrt{3}$ .

K. P. Wilkins, Amer. Math. Monthly 44 (1937), 579-583.

$$2.11 \quad \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \gamma \leq \frac{1}{54}(2\sqrt{13}-5) \sqrt{2\sqrt{13}+22}$$

Equality holds if and only if  $\alpha = \beta$ , where  $\cos \alpha = \frac{1}{6}(\sqrt{13}-1)$ .

K. P. Wilkins, Amer. Math. Monthly 44 (1937), 579-583.

$$2.12 \quad 0 < \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \leq \frac{1}{8}.$$

Equality holds if and only if the triangle is equilateral.

PROOF. Since

$$\sin \frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

and as we have similar expressions for  $\sin \beta/2$  and  $\sin \gamma/2$ , we

obtain

$$\sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2} = \frac{(s-a)(s-b)(s-c)}{abc} = \frac{r}{4R} \leq \frac{1}{8},$$

where 5.1 has been used.

Equality holds if and only if the triangle is equilateral.

**REMARK.** This inequality was proved in 197. See: T. Radó, Amer. Math. Monthly 39 (1932), 85–90.

V. Krylov, Matematika v škole 1938, No. 5–6, 134.

J. M. Child, Math. Gaz. 23 (1939), 138–142.

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

$$2.13 \quad \sin \frac{\alpha}{4} \sin \frac{\beta}{4} \sin \frac{\gamma}{2} \leq \frac{1}{54}(5\sqrt{10}-14).$$

Equality holds if and only if  $\alpha = \beta$ , when

$$\cos \frac{\alpha}{2} = \frac{1}{6}(\sqrt{10}+2).$$

K. P. Wilkins, Amer. Math. Monthly 44 (1937), 579–583.

$$2.14 \quad \frac{3}{4} \leq \sin^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2} + \sin^2 \frac{\gamma}{2} < 1. \quad (1)$$

Equality holds only if the triangle is equilateral.

**PROOF.** Since

$$\sin^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2} + \sin^2 \frac{\gamma}{2} = \frac{3}{2} - \frac{1}{2}(\cos \alpha - \cos \beta + \cos \gamma),$$

on the basis of 2.16, we conclude that inequality (1) holds.

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

$$2.15 \quad \sin \frac{\beta}{2} \sin \frac{\gamma}{2} + \sin \frac{\gamma}{2} \sin \frac{\alpha}{2} + \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \leq \frac{3}{4}.$$

Equality holds if and only if the triangle is equilateral.

J. M. Child, Math. Gaz. 23 (1939), 138–142.

$$2.16 \quad 1 < \cos \alpha + \cos \beta + \cos \gamma \leq \frac{3}{2}. \quad (1)$$

Equality holds if and only if the triangle is equilateral.

PROOF. Since  $\alpha/2, \beta/2, \gamma/2 < \pi/2$ , from

$$\cos \alpha + \cos \beta + \cos \gamma = 1 + 4 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}, \quad (2)$$

the first inequality in (1) immediately follows.

The second inequality in (1) follows from (2) on the basis of 2.12.

REMARK. For the second inequality in (1) see: J. M. Child, Math. Gaz. 23 (1939), 138–143. Later a weaker inequality was established by A. N. Aheart, Problem E 1398, Amer. Math. Monthly 67(1960), 52.

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

$$2.17 \quad \cos \alpha + \sqrt{2}(\cos \beta + \cos \gamma) \leq 2.$$

Equality only if  $\alpha = \pi/2, \beta = \gamma$ .

PROOF. Since  $\alpha + \beta + \gamma = \pi$ , we have

$$\begin{aligned} \cos \alpha + \sqrt{2}(\cos \beta + \cos \gamma) &= \cos \alpha + 2\sqrt{2} \cos \frac{\beta + \gamma}{2} \cos \frac{\beta - \gamma}{2} \\ &= \cos \alpha + 2\sqrt{2} \sin \frac{\alpha}{2} \cos \frac{\beta - \gamma}{2} \\ &\leq \cos \alpha + 2\sqrt{2} \sin \frac{\alpha}{2} \\ &= 2 - 2 \left( \sin \frac{\alpha}{2} - \frac{1}{2}\sqrt{2} \right)^2 \leq 2. \end{aligned}$$

This proof is due to R. P. Lučić.

$$2.18 \quad \text{If } \lambda \text{ is real, then}$$

$$\cos z - \lambda(\cos \beta + \cos \gamma) \leq 1 + \frac{\lambda^2}{2}.$$

PROOF. For any three real numbers  $\beta, \gamma, \lambda$ , the following inequality holds:

$$(\cos \beta + \cos \gamma - \lambda)^2 + (\sin \beta - \sin \gamma)^2 \geq 0.$$

After squaring and then dividing into groups, we get

$$-\cos(\beta + \gamma) + \lambda(\cos \beta + \cos \gamma) \leq 1 + \frac{\lambda^2}{2}.$$

Since  $\alpha + \beta + \gamma = \pi$ , we have

$$\cos \alpha + \lambda(\cos \beta + \cos \gamma) \leq 1 + \frac{\lambda^2}{2}.$$

Equality occurs if  $0 < \lambda < 2$ ;  $\cos \alpha = 1 - \lambda^2/2$ ,  $\cos \beta = \cos \gamma = \lambda/2$ .

REMARK. For  $\lambda = \sqrt{2}$ , we obtain 2.17.

Ž. Mitrović, Mat. Vesnik 4 (19) (1967), 341.

**2.19** Let the lengths of the sides of a triangle be  $a, b, c$ , with  $a \geq b \geq c$ , and the opposite angles  $\alpha, \beta, \gamma$ , respectively. Then, for the angles  $\alpha_1, \beta_1, \gamma_1$ , of an arbitrary triangle,

$$bc + ca - ab < bc \cdot \cos \alpha_1 + ca \cdot \cos \beta_1 + ab \cdot \cos \gamma_1 \leq \frac{1}{2}(a^2 + b^2 + c^2). \quad (1)$$

Equality is valid only if  $\alpha_1 = \alpha, \beta_1 = \beta, \gamma_1 = \gamma$ .

REMARK. If  $\alpha = \beta = \gamma$ , inequalities (1) give 2.16.

If  $\alpha = \pi/2$  and  $\beta = \gamma = \pi/4$ , then (1) gives 2.17.

If  $\alpha = 2\pi/3$  and  $\beta = \gamma = \pi/6$ , then (1) reduces to

$$1 < \cos \alpha_1 + \sqrt{3}(\cos \beta_1 + \cos \gamma_1) \leq \frac{5}{2}.$$

Equality holds if and only if  $\alpha_1 = 2\pi/3, \beta_1 = \gamma_1 = \pi/6$ .

P. Szász, Monatsh. Math. 66 (1962), 174–178.

**2.20** If  $x, y, z$  are real numbers such that  $xyz > 0$ , then

$$x \cdot \cos \alpha + y \cdot \cos \beta + z \cdot \cos \gamma \leq \frac{yz}{2x} + \frac{zx}{2y} + \frac{xy}{2z}. \quad (1)$$

If  $xyz < 0$ , then inequality (1) is reversed.

Equality is valid in both cases if and only if

$$\frac{1}{x} : \frac{1}{y} : \frac{1}{z} = \sin \alpha : \sin \beta : \sin \gamma.$$

**PROOF.** If  $x, y, z, \alpha, \beta$  are real numbers, then

$$(xz \cdot \cos \alpha + yz \cdot \cos \beta - xy)^2 + (xz \cdot \sin \alpha - yz \cdot \sin \beta)^2 \geq 0, \quad \square$$

i.e.,

$$y^2z^2 + z^2x^2 + x^2y^2 + 2xyz^2 \cos(\alpha + \beta) - 2xy^2z \cos \beta - 2x^2yz \cos \alpha \geq 0 \quad (3)$$

Since  $\alpha + \beta + \gamma = \pi$ , (3) gives

$$2x^2yz \cos \alpha + 2xy^2z \cos \beta + 2xyz^2 \cos \gamma \leq y^2z^2 + z^2x^2 + x^2y^2.$$

If  $xyz > 0$ , after division by  $2xyz$ , we get (1). If  $xyz < 0$ , we obtain the reversed inequality.

Equality in the considered inequalities holds if and only if

$$xz \cdot \cos \alpha + yz \cdot \cos \beta - xy = 0, \quad zx \cdot \sin \alpha - yz \cdot \sin \beta = 0,$$

i.e.,

$$\frac{1}{x} : \frac{1}{y} : \frac{1}{z} = \sin \alpha : \sin \beta : \sin \gamma.$$

**REMARK.** Inequality (1) in the case where  $x, y, z$  are positive numbers was established by D. F. Barrow. The above generalization and proof are due to R. R. Janić.

D. F. Barrow, Problem 3740, Amer. Math. Monthly 44 (1937), 252–254.

R. R. Janić, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 181–196 (1967), 73–74.

$$2.21 \quad \frac{3}{4} \leq \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma < 3 \quad \text{in each triangle.}$$

$$\frac{3}{4} \leq \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma < 1 \quad \text{in acute triangles.}$$

$$1 < \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma < 3 \quad \text{in obtuse triangles.}$$

Equality holds only for the equilateral triangle.

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

$$2.22 \quad \cos \beta \cos \gamma + \cos \gamma \cos \alpha + \cos \alpha \cos \beta \leq \frac{3}{4}.$$

There is strict inequality except when the triangle is equilateral.

J. M. Child, Math. Gaz. 23 (1939), 138–143.

$$2.23 \quad \cos \alpha \cos \beta \cos \gamma \leq \frac{1}{8}. \quad (1)$$

Equality holds if and only if the triangle is equilateral.

**PROOF.** For acute triangle  $\cos \alpha, \cos \beta, \cos \gamma$  are positive, so that on the basis of arithmetic-geometric mean inequality, we have

$$\cos \alpha \cos \beta \cos \gamma \leq \left( \frac{\cos \alpha + \cos \beta + \cos \gamma}{3} \right)^3.$$

Using 2.16, we conclude that inequality (1) is true.

For right and obtuse triangles inequality (1) is immediately established.

C. C. Popovici, Gaz. Mat. 31 (1925), 132.

J. M. Child, Math. Gaz. 23 (1939), 138–143.

$$2.24 \quad -1 < \cos \alpha \cos \beta \cos \gamma \leq \frac{1}{8} \quad \text{in each triangle.}$$

$$0 \leq \cos \alpha \cos \beta \cos \gamma \leq \frac{1}{8} \quad \text{in acute triangles.}$$

$$-1 < \cos \alpha \cos \beta \cos \gamma < 0 \quad \text{in obtuse triangles.}$$

Equality on the left side holds for a rectangular triangle, and on the right side for an equilateral triangle.

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

$$2.25 \quad \text{If } \alpha < \pi/2 \text{ and } \beta < \pi/2, \text{ then}$$

$$\cos \alpha + \cos \beta > \sin \gamma.$$

If  $\alpha > \pi/2$  or  $\beta > \pi/2$ , then reversed inequality is valid.

A. Pantazi, Gaz. Mat. 23 (1917/18), 144.

V. Cristescu, Gaz. Mat. 26 (1920/21), 88–89.

B. M. Barbalatt, Gaz. Mat. 30 (1924/25), 380–381.

$$2.26 \quad \cos \alpha \cos \beta \cos \gamma \leq \frac{1}{24}(\cos^2(\beta - \gamma) + \cos^2(\gamma - \alpha) + \cos^2(\alpha - \beta)).$$

(1)

**PROOF.** By  $(x+y)^2 \geq 4xy$  we obtain

$$(\cos \alpha + 2 \cos \beta \cos \gamma)^2 \geq 8 \cdot \cos \alpha \cos \beta \cos \gamma. \quad (2)$$

Since

$$\cos \alpha = -\cos(\beta + \gamma),$$

(2) gives

$$(-\cos(\beta + \gamma) + 2 \cos \beta \cos \gamma)^2 \geq 8 \cdot \cos \alpha \cos \beta \cos \gamma,$$

i.e.

$$\cos^2(\beta - \gamma) \geq 8 \cdot \cos \alpha \cos \beta \cos \gamma. \quad (3)$$

Similarly

$$\cos^2(\gamma - \alpha) \geq 8 \cdot \cos \alpha \cos \beta \cos \gamma, \quad \cos^2(\alpha - \beta) \geq 8 \cdot \cos \alpha \cos \beta \cos \gamma. \quad (4)$$

Adding (3) and (4) we get (1).

C. Cosnita and F. Turtoiu, Culegere de probleme de algebra, Bucuresti 1965, pp. 176–177.

$$2.27 \quad 2 < \cos \frac{\alpha}{2} + \cos \frac{\beta}{2} + \cos \frac{\gamma}{2} \leq \frac{3}{2}\sqrt{3}.$$

Equality only if the triangle is equilateral.

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

$$2.28 \quad 0 < \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} \leq \frac{3}{8}\sqrt{3} \quad \text{in each triangle.}$$

$$\frac{1}{2} < \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} \leq \frac{3}{8}\sqrt{3} \quad \text{in acute triangles.}$$

$$0 < \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} < \frac{3}{8}\sqrt{3} \quad \text{in obtuse triangles.}$$

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

$$2.29 \quad 2 < \cos^2 \frac{\alpha}{2} + \cos^2 \frac{\beta}{2} + \cos^2 \frac{\gamma}{2} \leq \frac{9}{4}.$$

Equality holds if and only if the triangle is equilateral.

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

$$2.30 \quad \operatorname{tg} \alpha + \operatorname{tg} \beta + \operatorname{tg} \gamma \geq 3\sqrt{3} \quad \text{in acute triangles.}$$

$$\operatorname{tg} \alpha + \operatorname{tg} \beta + \operatorname{tg} \gamma < 0 \quad \text{in obtuse triangles.}$$

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

2.31  $\operatorname{tg}^2 \alpha + \operatorname{tg}^2 \beta + \operatorname{tg}^2 \gamma > 0$  in each triangle.

$\operatorname{tg}^2 \alpha + \operatorname{tg}^2 \beta + \operatorname{tg}^2 \gamma \geq 9$  in acute triangles.

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

2.32  $\operatorname{tg} \alpha \operatorname{tg} \beta \operatorname{tg} \gamma \geq 3\sqrt{3}$  in acute triangles.

$\operatorname{tg} \alpha \operatorname{tg} \beta \operatorname{tg} \gamma < 0$  in obtuse triangles.

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

2.33  $\operatorname{tg} \frac{\alpha}{2} + \operatorname{tg} \frac{\beta}{2} + \operatorname{tg} \frac{\gamma}{2} \geq \sqrt{3}.$

Equality holds if and only if the triangle is equilateral.

J. Karamata, Problem 119, Glasnik matematičko-fizički i astronomski 3 (1948), 223.

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

2.34  $0 < \operatorname{tg} \frac{\alpha}{2} \operatorname{tg} \frac{\beta}{2} \operatorname{tg} \frac{\gamma}{2} \leq \frac{1}{3}\sqrt{3}.$

Equality holds if and only if the triangle is equilateral.

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

2.35  $\operatorname{tg}^2 \frac{\alpha}{2} + \operatorname{tg}^2 \frac{\beta}{2} + \operatorname{tg}^2 \frac{\gamma}{2} \geq 1.$  (1)

Equality holds if and only if the triangle is equilateral.

PROOF. Since

$$\frac{\gamma}{2} = \frac{\pi}{2} - \frac{\alpha+\beta}{2},$$

we obtain

$$\operatorname{tg} \frac{\gamma}{2} = \operatorname{ctg} \frac{\alpha+\beta}{2} = \frac{1 - \operatorname{tg} \alpha/2 \operatorname{tg} \beta/2}{\operatorname{tg} \alpha/2 + \operatorname{tg} \beta/2},$$

i.e.

$$\operatorname{tg} \frac{\beta}{2} \operatorname{tg} \frac{\gamma}{2} - \operatorname{tg} \frac{\gamma}{2} \operatorname{tg} \frac{\alpha}{2} - \operatorname{tg} \frac{\alpha}{2} \operatorname{tg} \frac{\beta}{2} = 1.$$

By  $\operatorname{tg} \alpha/2 = x$ ,  $\operatorname{tg} \beta/2 = y$ ,  $\operatorname{tg} \gamma/2 = z$  the last equality becomes

$$yz + zx + xy = 1. \quad (2)$$

From

$$2(x^2 + y^2 + z^2) - 2(yz + zx + xy) = (y-z)^2 + (z-x)^2 + (x-y)^2 \geq 0$$

and from (2) follows

$$x^2 + y^2 + z^2 \geq 1,$$

i.e. (1).

C. V. Durell and A. Robson, Advanced Trigonometry, London 1948, p. 277.

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

$$2.36 \quad \operatorname{tg}^6 \frac{\alpha}{2} + \operatorname{tg}^6 \frac{\beta}{2} + \operatorname{tg}^6 \frac{\gamma}{2} \geq \frac{1}{9}.$$

Equality holds if and only if the triangle is equilateral.

$$2.37 \quad \sqrt{\operatorname{tg} \frac{\beta}{2} \operatorname{tg} \frac{\gamma}{2} + 5} + \sqrt{\operatorname{tg} \frac{\gamma}{2} \operatorname{tg} \frac{\alpha}{2} + 5} + \sqrt{\operatorname{tg} \frac{\alpha}{2} \operatorname{tg} \frac{\beta}{2} + 5} \leq 4\sqrt{3}.$$

Ju. A. Izosimov, Matematika v škole 1953, No. 3, 94 and No. 6, 85.

$$2.38 \quad \operatorname{cotg} \alpha + \operatorname{cotg} \beta + \operatorname{cotg} \gamma \geq \sqrt{3},$$

with equality holding if and only if the triangle is equilateral.

PROOF.

$$\begin{aligned} \operatorname{cotg} \alpha + \operatorname{cotg} \beta &= \frac{\sin(\alpha + \beta)}{\sin \alpha \sin \beta} = \frac{2 \sin \gamma}{\cos(\alpha - \beta) + \cos \gamma} \\ &\geq \frac{2 \sin \gamma}{1 + \cos \gamma} = 2 \operatorname{tg} \frac{\gamma}{2}. \end{aligned}$$

Therefore

$$\begin{aligned} \operatorname{cotg} \alpha + \operatorname{cotg} \beta + \operatorname{cotg} \gamma &\geq 2 \cdot \operatorname{tg} \frac{\gamma}{2} + \operatorname{cotg} \gamma \\ &= 2 \cdot \operatorname{tg} \frac{\gamma}{2} + \frac{\operatorname{cotg}^2 \gamma/2 - 1}{2 \operatorname{cotg} \gamma/2} = \frac{\operatorname{cotg}^2 \gamma/2 + 3}{2 \operatorname{cotg} \gamma/2} \\ &= \frac{1}{2} \left( \operatorname{cotg} \frac{\gamma}{2} + 3 \cdot \operatorname{tg} \frac{\gamma}{2} \right) \geq \sqrt{3}. \end{aligned}$$

T. Varopoulos, Bull. Soc. Math. Grèce 15<sub>1</sub> (1934), 17.

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

**2.39**  $\cot^2 \alpha + \cot^2 \beta + \cot^2 \gamma \geq 1$ .

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

**2.40**  $\cot \alpha \cot \beta \cot \gamma \leq \frac{1}{9}\sqrt{3}$       in acute triangles.

$\cot \alpha \cot \beta \cot \gamma < 0$       in obtuse triangles.

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

**2.41**  $\cot \frac{\alpha}{2} + \cot \frac{\beta}{2} + \cot \frac{\gamma}{2} \geq 3\sqrt{3}$ .

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

**2.42**  $\cot \frac{\alpha}{2} \cot \frac{\beta}{2} \cot \frac{\gamma}{2} \geq 3\sqrt{3}$ .

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

**2.43**  $\cot^2 \frac{\alpha}{2} + \cot^2 \frac{\beta}{2} + \cot^2 \frac{\gamma}{2} \geq 9$ .

Equality holds if and only if the triangle is equilateral.

PROOF. By the harmonic-arithmetic mean inequality and by  $\cot \alpha/2, \cot \beta/2, \cot \gamma/2 > 0$ , we have

$$\begin{aligned} & \left( \cot \frac{\beta}{2} \cot \frac{\gamma}{2} + \cot \frac{\gamma}{2} \cot \frac{\alpha}{2} + \cot \frac{\alpha}{2} \cot \frac{\beta}{2} \right) \\ & \times \left( \cot \frac{\beta}{2} \cot \frac{\gamma}{2} + \cot \frac{\gamma}{2} \cot \frac{\alpha}{2} + \cot \frac{\alpha}{2} \cot \frac{\beta}{2} \right) \geq 9. \end{aligned}$$

Since

$$\cot \frac{\beta}{2} \cot \frac{\gamma}{2} + \cot \frac{\gamma}{2} \cot \frac{\alpha}{2} + \cot \frac{\alpha}{2} \cot \frac{\beta}{2} = 1,$$

we get

$$\cot \frac{\beta}{2} \cot \frac{\gamma}{2} + \cot \frac{\gamma}{2} \cot \frac{\alpha}{2} + \cot \frac{\alpha}{2} \cot \frac{\beta}{2} \geq 9. \quad \square$$

By adding the inequality

$$\cot^2 \frac{\alpha}{2} + \cot^2 \frac{\beta}{2} \geq 2 \cot \frac{\alpha}{2} \cot \frac{\beta}{2}$$

to two inequalities obtained by cyclic permutation of  $\alpha, \beta, \gamma$ , we get

$$\begin{aligned} & \cot^2 \frac{\alpha}{2} + \cot^2 \frac{\beta}{2} + \cot^2 \frac{\gamma}{2} \\ & \geq \cot \frac{\alpha}{2} \cot \frac{\beta}{2} + \cot \frac{\beta}{2} \cot \frac{\gamma}{2} + \cot \frac{\gamma}{2} \cot \frac{\alpha}{2} \end{aligned}$$

which together with (2) gives (1).

R. Kooistra, Nieuw Tijdschr. Visk. 45 (1957/58), 108–115.

Ju. I. Gerasimov, Matematika i Škole 1964, No. 3, 75.

$$\begin{aligned} 2.44 \quad & \cot^2 \frac{\alpha}{2} + \cot^2 \frac{\beta}{2} + \cot^2 \frac{\gamma}{2} \\ & \geq \left( \cot \frac{\alpha}{2} + \cot \frac{\beta}{2} + \cot \frac{\gamma}{2} \right) (\cot \alpha + \cot \beta + \cot \gamma). \quad (1) \end{aligned}$$

PROOF. Using the notation  $\cot \alpha/2 = x$ ,  $\cot \beta/2 = y$ ,  $\cot \gamma/2 = z$ , from

$$x+y+z = xyz$$

we have

$$(x+y+z)^2 = xyz(x+y+z)$$

i.e.

$$x^2 + y^2 + z^2 = (x^2 - 1)yz + (y^2 - 1)zx + (z^2 - 1)xy - (yz + zx + xy).$$

Since

$$x^2 + y^2 + z^2 \geq xy + zx + xy$$

we get

$$2(x^2 + y^2 + z^2) \geq (x^2 - 1)yz + (y^2 - 1)zx + (z^2 - 1)xy$$

i.e.,

$$\begin{aligned} x^2 + y^2 + z^2 & \geq xyz \left( \frac{x^2 - 1}{2x} + \frac{y^2 - 1}{2y} + \frac{z^2 - 1}{2z} \right) \\ & = (x+y+z) \left( \frac{x^2 - 1}{2x} + \frac{y^2 - 1}{2y} + \frac{z^2 - 1}{2z} \right). \quad (2) \end{aligned}$$

By  $\cot \alpha/2$ ,  $\cot \beta/2$ ,  $\cot \gamma/2$  substituting for  $x$ ,  $y$ ,  $z$  and using the formula

$$\frac{\cot^2(t/2) - 1}{2 \cot t/2} = \cot t$$

from (2) we obtain (1).

C. Cosnita and F. Turtoiu, Culegere de probleme de algebra, Bucuresti 1965, p. 176.

**2.45**  $\sec \alpha + \sec \beta + \sec \gamma \geq 6$ .

Equality holds if and only if the triangle is equilateral.

J. M. Child, Math. Gaz. 23 (1939), 138–143.

**2.46**  $\sec^2 \alpha + \sec^2 \beta + \sec^2 \gamma \geq 12$  in acute triangles.

$\sec^2 \alpha + \sec^2 \beta + \sec^2 \gamma > 3$  in obtuse triangles.

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

**2.47**  $\sec \beta \sec \gamma + \sec \gamma \sec \alpha + \sec \alpha \sec \beta \geq 12$ .

Equality only applies when the triangle is equilateral.

J. M. Child, Math. Gaz. 23 (1939), 136–143.

**2.48**  $\sec^2 \frac{\alpha}{2} + \sec^2 \frac{\beta}{2} + \sec^2 \frac{\gamma}{2} \geq 4$ .

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1958/57), 108–115.

**2.49**  $\operatorname{cosec} \alpha + \operatorname{cosec} \beta + \operatorname{cosec} \gamma \geq 2\sqrt{3}$ ,

with equality only for equilateral triangles.

T. R. Curry, Problem E 1861, Amer. Math. Monthly 73 (1966), 199.

**2.50**  $\operatorname{cosec}^2 \alpha + \operatorname{cosec}^2 \beta + \operatorname{cosec}^2 \gamma \geq 4$ .

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

**2.51**  $\operatorname{cosec} \frac{\alpha}{2} + \operatorname{cosec} \frac{\beta}{2} + \operatorname{cosec} \frac{\gamma}{2} \geq 6$ ,

with equality holding if and only if the triangle is equilateral.

J. M. Child, Math. Gaz. 23 (1939), 138–143.

M. S. Klamkin, Problem E 1361, Amer. Math. Monthly 66 (1959), 312 and 916–917.

$$2.52 \quad \operatorname{cosec}^2 \frac{\alpha}{2} + \operatorname{cosec}^2 \frac{\beta}{2} + \operatorname{cosec}^2 \frac{\gamma}{2} \geq 12.$$

R. Kooistra, Nieuw Tijdschr. Wisk. 45 (1957/58), 108–115.

$$2.53 \quad \operatorname{cosec} \frac{\beta}{2} \operatorname{cosec} \frac{\gamma}{2} + \operatorname{cosec} \frac{\gamma}{2} \operatorname{cosec} \frac{\alpha}{2} + \operatorname{cosec} \frac{\alpha}{2} \operatorname{cosec} \frac{\beta}{2} \geq 12.$$

Equality holds if and only if the triangle is equilateral.

J. M. Child, Math. Gaz. 23 (1939), 138–143.

$$2.54 \quad \operatorname{cosec} \alpha + \operatorname{cosec} \beta + \operatorname{cosec} \gamma \geq \frac{9}{4} \sec \frac{\alpha}{2} \sec \frac{\beta}{2} \sec \frac{\gamma}{2}. \quad (1)$$

PROOF. Putting  $x = \sin \alpha$ ,  $y = \sin \beta$ ,  $z = \sin \gamma$  in

$$(x+y+z) \left( \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) \geq 9,$$

we obtain

$$\operatorname{cosec} \alpha + \operatorname{cosec} \beta + \operatorname{cosec} \gamma \geq \frac{9}{\sin \alpha + \sin \beta + \sin \gamma}. \quad (2)$$

Since

$$\sin \alpha + \sin \beta + \sin \gamma = 4 \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}$$

from (2), we get (1).

C. Ionescu-Tiu, Gaz. Mat. B 14 (1963), 225.

$$2.55 \quad \operatorname{cosec} 2\alpha + \operatorname{cosec} 2\beta + \operatorname{cosec} 2\gamma \geq \operatorname{cosec} \alpha + \operatorname{cosec} \beta + \operatorname{cosec} \gamma$$

$$\geq \sec \frac{\alpha}{2} + \sec \frac{\beta}{2} + \sec \frac{\gamma}{2}$$

$$\geq 2\sqrt{3}.$$

$$2.56 \quad \left( \sin \frac{\alpha}{2} + \sin \frac{\beta}{2} + \sin \frac{\gamma}{2} \right)^2 \leq \cos^2 \frac{\alpha}{2} + \cos^2 \frac{\beta}{2} + \cos^2 \frac{\gamma}{2}. \quad (1)$$

PROOF. One of the following relations holds for the angles of a triangle

$$\alpha \geq \frac{\pi}{3} \geq \beta \geq \gamma \text{ or } \gamma \geq \beta \geq \alpha \geq \frac{\pi}{3}.$$

In both cases

$$\left( \sin \frac{\beta}{2} - \frac{1}{2} \right) \left( \sin \frac{\gamma}{2} - \frac{1}{2} \right) \geq 0,$$

whence

$$4 \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \geq 2 \left( \sin \frac{\beta}{2} + \sin \frac{\gamma}{2} \right) - 1.$$

Therefore

$$1 + 4 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \geq 2 \cdot \sin \frac{\alpha}{2} \left( \sin \frac{\beta}{2} + \sin \frac{\gamma}{2} \right) + 1 - \sin \frac{\alpha}{2}. \quad (2)$$

Since  $\sin \beta/2 \sin \gamma/2$ , for  $\alpha$  constant, reaches its maximum when  $\beta/2 = \gamma/2 = (\pi - \alpha)/4$ , we have

$$\sin \frac{\beta}{2} \sin \frac{\gamma}{2} \leq \sin^2 \frac{\pi - \alpha}{4} = \frac{1 - \sin \alpha/2}{2}.$$

By the last inequality, from (2) follows

$$\begin{aligned} \cos \alpha + \cos \beta + \cos \gamma &= 1 + 4 \cdot \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \\ &\geq 2 \left( \sin \frac{\beta}{2} \sin \frac{\gamma}{2} + \sin \frac{\gamma}{2} \sin \frac{\alpha}{2} + \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \right). \end{aligned}$$

Therefore inequality (1) holds.

V. Thébault-L. Bankoff, Problem E 1272, Amer. Math. Monthly 67 (1960), 693–694.

$$2.57 \quad \frac{\cos \beta/2 \cos \gamma/2}{\sin \alpha/2} + \frac{\cos \gamma/2 \cos \alpha/2}{\sin \beta/2} + \frac{\cos \alpha/2 \cos \beta/2}{\sin \gamma/2} \geq \frac{9}{2}.$$

C. Ionescu-Tiu, Gaz. Mat. B 14 (1963), 225.

$$2.58 \quad S = \frac{\sin \alpha/2}{\sin \beta/2 \cos \gamma/2} + \frac{\sin \beta/2}{\sin \gamma/2 \cos \alpha/2} + \frac{\sin \gamma/2}{\sin \alpha/2 \cos \beta/2} \geq 2\sqrt{3}. \quad (1)$$

PROOF. By the arithmetic-geometric mean inequality, we have

$$S \geq 3 \left( \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} \right)^{-1/3}. \quad (2)$$

Since (see: 2.28)

$$\cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} \leq \frac{3}{8}\sqrt{3},$$

from (2) we get (1).

D. Nicolae, Gaz. Mat. B 14 (1963), 561.

$$2.59 \quad \frac{1 + \cos \alpha \cos \beta \cos \gamma}{\sin \alpha \sin \beta \sin \gamma} \geq \sqrt{3}.$$

Equality holds if and only if the triangle is equilateral.

H. W. Guggenheimer, Plane Geometry and its Groups, San Francisco, Cambridge, London, Amsterdam 1967, p. 189.

$$2.60 \quad \sqrt{2} \left( \cos \frac{\alpha}{4} + \sin \frac{\alpha}{4} \right) < \cos \frac{\beta}{4} + \cos \frac{\gamma}{4} + \sin \frac{\beta}{4} + \sin \frac{\gamma}{4}.$$

PROOF.

$$\begin{aligned} & \sqrt{2} \left( \cos \frac{\alpha}{4} + \sin \frac{\alpha}{4} \right) \\ &= \sqrt{2} \left( \cos \frac{\pi - \beta - \gamma}{4} + \sin \frac{\pi - \beta - \gamma}{4} \right) \\ &= 2 \cos \frac{\beta + \gamma}{4} < 2 \cos \frac{\beta - \gamma}{4} \\ &= \cos \frac{\beta}{4} \cos \frac{\gamma}{4} + \sin \frac{\beta}{4} \sin \frac{\gamma}{4} + \cos \frac{\gamma}{4} \cos \frac{\beta}{4} + \sin \frac{\gamma}{4} \sin \frac{\beta}{4} \\ &< \cos \frac{\beta}{4} + \cos \frac{\gamma}{4} + \sin \frac{\beta}{4} + \sin \frac{\gamma}{4}. \end{aligned}$$

C. Ionescu-Tiu, Gaz. Mat. B 12 (1961), 304.

2.61 If  $\alpha, \beta, \gamma$  are angles of a triangle measured in radians, then

$$\alpha \cos \beta + \sin \gamma \cos \gamma > 0.$$

L. E. Bush, Amer. Math. Monthly 64 (1957), 24.

2.62  $2(\cot \alpha + \cot \beta + \cot \gamma) \geq \csc \alpha + \csc \beta + \csc \gamma$ .

C. Cosnita and F. Turtoiu, Culegere de probleme de algebra, Bucuresti 1965, p. 176.

2.63  $\operatorname{tg} \beta \operatorname{tg} \gamma + \operatorname{tg} \gamma \operatorname{tg} \alpha + \operatorname{tg} \alpha \operatorname{tg} \beta \geq 3 + \sec \alpha + \sec \beta + \sec \gamma$ .

C. Cosnita and F. Turtoiu, Culegere de probleme de algebra, Bucuresti 1965, p. 176.

$$2.64 \frac{\sqrt{\operatorname{tg} \beta/2 \operatorname{tg} \gamma/2}}{\cos \alpha/2} + \frac{\sqrt{\operatorname{tg} \gamma/2 \operatorname{tg} \alpha/2}}{\cos \beta/2} + \frac{\sqrt{\operatorname{tg} \alpha/2 \operatorname{tg} \beta/2}}{\cos \gamma/2} \leq 2.$$

Ju. A. Izosimov, Matematika v škole 1958, No. 5, 95.

2.65 If  $n$  is a natural number, then

$$\cot^n \alpha + \cot^n \beta + \cot^n \gamma \geq 3 \cdot 3^{-n/2}.$$

Equality holds if and only if the triangle is equilateral.

M. N. Kritikos, Actes du Congrès interbalkanique de mathématiciens, Athènes 1934, 157–158.

### 3. Inequalities for the angles and other elements of a triangle

3.1 If  $\alpha < \frac{1}{2}(b+c)$ , then  $\alpha < \frac{1}{2}(\beta+\gamma)$ .

PROOF. By virtue of the sine law, the inequality  $\alpha < \frac{1}{2}(b+c)$  gives

$$\sin \alpha < \frac{1}{2}(\sin \beta + \sin \gamma) = \sin \frac{\beta-\gamma}{2} \cos \frac{\beta-\gamma}{2} \leq \sin \frac{\beta+\gamma}{2}.$$

Since  $\alpha$  and  $\frac{1}{2}(\beta+\gamma)$  are acute angles, we infer that

$$\alpha < \frac{1}{2}(\beta+\gamma).$$

G. Pólya-A. Hess, Elem. Math. 13 (1958), 88.

$$3.2 \quad a\alpha + b\beta + c\gamma \geq \frac{1}{2}(b\gamma + c\alpha + a\beta + c\beta + a\gamma + b\alpha). \quad (1)$$

PROOF. Let  $a \geq b \geq c$ . Then  $\alpha \geq \beta \geq \gamma$ . Multiplying  $\alpha \geq \beta$  by  $a-b \geq 0$ ,  $\alpha \geq \gamma$  by  $a-c \geq 0$  and  $\beta \geq \gamma$  by  $b-c \geq 0$ , one obtains

$$a\alpha + b\beta \geq a\beta + b\alpha, \quad a\alpha + c\gamma \geq a\gamma + c\alpha, \quad b\beta + c\gamma \geq b\gamma + c\beta.$$

Adding these inequalities, we have

$$2(a\alpha + b\beta + c\gamma) \geq b\gamma + c\alpha + a\beta + c\beta + a\gamma + b\alpha,$$

i.e. (1).

$$3.3 \quad \frac{\pi}{3} \leq \frac{a\alpha + b\beta + c\gamma}{a+b+c} < \frac{\pi}{2}.$$

Equality holds if and only if  $a = b = c$ .

PROOF. Let  $a \geq b \geq c$ . Then  $\alpha \geq \beta \geq \gamma$  and consequently

$$(a-b)(\alpha-\beta) \geq 0, \quad (b-c)(\beta-\gamma) \geq 0, \quad (c-a)(\gamma-\alpha) \geq 0. \quad (1)$$

Adding these inequalities, we obtain

$$(a-b)(\alpha-\beta) + (b-c)(\beta-\gamma) + (c-a)(\gamma-\alpha) \geq 0,$$

i.e.

$$2(a\alpha + b\beta + c\gamma) \geq (b+c)\alpha + (c+a)\beta + (a+b)\gamma.$$

Adding  $a\alpha + b\beta + c\gamma$  to both sides of the last inequality, we get

$$3(a\alpha + b\beta + c\gamma) \geq (a+b+c)(\alpha + \beta + \gamma). \quad (2)$$

Since  $\alpha + \beta + \gamma = \pi$ , from (2) follows

$$\frac{\pi}{3} \leq \frac{a\alpha + b\beta + c\gamma}{a+b+c},$$

with equality only if  $a = b = c$ , because equality in (1) holds only in this case.

Since  $a, b, c$  are the sides of a triangle, we have

$$a+b+c > 2a, \quad a+b+c > 2b, \quad a+b+c > 2c.$$

Multiplying these inequalities by  $\alpha, \beta, \gamma$  respectively and adding together the inequalities thus obtained, we get

$$(a+b+c)(\alpha + \beta + \gamma) > 2(a\alpha + b\beta + c\gamma),$$

whence

$$\frac{a\alpha+b\beta+c\gamma}{a+b+c} < \frac{\pi}{2}.$$

REMARK. This proof follows the idea of J. Kürschak given in: G. Pólya and G. Szegő, Aufgaben und Lehrsätze aus der Analysis, vol. II, Berlin 1925, p. 166 and 393.

3.4 If  $\alpha \leq \beta \leq \gamma$ , then

$$1^{\circ}. \quad \frac{\pi}{3} \leq \frac{a\alpha+b\beta+c\gamma}{a+b+c} \leq \frac{\pi-\alpha}{2},$$

$$2^{\circ}. \quad \frac{\pi}{3} \leq \frac{a\alpha+b\beta+c\gamma}{a+b+c} \leq \frac{\pi}{2} \left( 1 - \operatorname{tg} \frac{\alpha}{2} \operatorname{tg} \frac{\beta}{2} \right).$$

PROOF. 1°. Since  $\alpha \leq \beta \leq \gamma$  and  $(a-b-c)\alpha + (a-b-c)\gamma = 2a\alpha$ , we have

$$(a-b-c)\beta + (a-b-c)\gamma \geq 2a\alpha,$$

i.e.,

$$2a\alpha+b\beta+c\gamma \leq (a+b-c)\beta + (a+b-c)\gamma.$$

Adding  $b\beta+c\gamma$  to both sides of the last inequality, we get

$$2(a\alpha+b\beta+c\gamma) \leq (a+b-c)(\beta+\gamma)$$

whence

$$\frac{a\alpha+b\beta+c\gamma}{a+b+c} \leq \frac{\pi-\alpha}{2}.$$

2°. If  $a \leq b \leq c$ , then

$$\frac{a\alpha+b\beta+c\gamma}{a+b+c} \leq \frac{c(\alpha+\beta+\gamma)}{a+b+c} = \frac{c\pi}{2s}.$$

Since

$$\operatorname{tg} \frac{\alpha}{2} = \left( \frac{(s-b)(s-c)}{s(s-a)} \right)^{1/2}, \quad \operatorname{tg} \frac{\beta}{2} = \left( \frac{(s-c)(s-a)}{s(s-b)} \right)^{1/2},$$

we obtain

$$\frac{\pi}{2} \left( 1 - \operatorname{tg} \frac{\alpha}{2} \operatorname{tg} \frac{\beta}{2} \right) = \frac{\pi}{2} \left( 1 - \frac{s-c}{s} \right) = \frac{c\pi}{2s}.$$

Therefore

$$\frac{a\alpha + b\beta + c\gamma}{a+b+c} \leq \frac{\pi}{2} \left( 1 - \operatorname{tg} \frac{\alpha}{2} \operatorname{tg} \frac{\beta}{2} \right).$$

This proof is due to G. Kalajdžić.

D. Marković, Bull. Soc. Math. Phys. Serbie 4, No. 3-4 (1952), 71.

$$3.5 \quad \frac{(b\gamma - c\beta)^2 + (c\alpha - a\gamma)^2 + (a\beta - b\alpha)^2}{(a+b+c)^2} < \frac{\pi^2}{4}.$$

This result is the best possible.

D. Blanuša, Problem 175, Glasnik matematičko-fizički i astronomski (2) 8 (1953), 160.

REMARK. A proof of this inequality, due to T. Leko, is very complicated and it would be desirable to have a simpler proof.

$$3.6 \quad \frac{2bc \cos \alpha}{b+c} < b+c-a < \frac{2bc}{a}. \quad (1)$$

PROOF. Let  $a \leq \max(b, c)$ . Since  $\max(b, c) < a + \min(b, c)$ , we have

$$\begin{aligned} b+c-a &= \min(b, c) + \max(b, c) - a < \min(b, c) + \min(b, c) + a - a \\ &= 2\min(b, c) \leq \frac{2\min(b, c)\max(b, c)}{a} = \frac{2bc}{a}. \end{aligned}$$

Let  $a > \max(b, c)$ . Then  $a < 2\max(b, c)$  must hold, for otherwise  $a \geq b+c$ . Further, we have

$$\begin{aligned} b+c-a &= \min(b, c) + \max(b, c) - a < \min(b, c) + a - a \\ &= \min(b, c) < \frac{2\max(b, c)\min(b, c)}{a} = \frac{2bc}{a}. \end{aligned} \quad (2)$$

Therefore we have proved the second inequality in (1). Since

$$(b+c)(b+c-a) - 2bc \cos \alpha = a \left( \frac{2bc}{a} - (b+c-a) \right),$$

using inequality (2), we obtain the first inequality in (1).

J. A. Kalman, Math. Gaz. 47 (1963), 224-225.

$$3.7 \quad \left(1 - \frac{a}{b+c}\right) \left(1 - \frac{b}{c+a}\right) \left(1 - \frac{c}{a+b}\right) \leq \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}.$$

Equality occurs if and only if the triangle is equilateral.

Gaz. Mat. B 8 (1957), 47.

$$3.8 \quad \cot \alpha + \cot \beta + \cot \gamma \geq \frac{\sqrt{3}}{9} \cdot \frac{(a^2 + b^2 + c^2)(a+b+c)}{abc},$$

with equality holding if and only if the triangle is equilateral.

T. R. Curry, Problem E 1861, Amer. Math. Monthly 73 (1966), 199.

3.9 If  $a \leq b \leq c$ , then

$$g \leq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \leq G,$$

where

$$g = \frac{2}{1 + \tan \beta/2 \tan \gamma/2} \geq 2 \cos^2 \frac{\gamma}{2}, \quad G = \frac{2}{1 + \tan \alpha/2 \tan \beta/2} \leq 2 \cos^2 \frac{\alpha}{2}.$$

Equality occurs if and only if  $a = b = c$ .

PROOF. Since  $a \leq b \leq c$  and

$$\tan \frac{\alpha}{2} = \left( \frac{(s-b)(s-c)}{s(s-a)} \right)^{1/2}, \quad \tan \frac{\beta}{2} = \left( \frac{(s-c)(s-a)}{s(s-b)} \right)^{1/2},$$

$$\tan \frac{\gamma}{2} = \left( \frac{(s-a)(s-b)}{s(s-c)} \right)^{1/2}$$

we have

$$g = \frac{2}{1 + \tan \beta/2 \tan \gamma/2} = \frac{2s}{2s-a} = \frac{a+b+c}{b+c} \leq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b};$$

$$G = \frac{2}{1 + \tan \alpha/2 \tan \beta/2} = \frac{2s}{2s-c} = \frac{a+b+c}{a+b} \geq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}.$$

This proof is due to G. Kalajdžić.

D. Marković, Bull. Soc. Math. Phys. Serbie 4, No. 3-4 (1952), 71.

**3.10** If  $k = c^t/(a^t + b^t)$ , then

$$0 < k \leq 2^{t-1} \sin^t \frac{\gamma}{2} \quad \text{for } t < 0,$$

$$2^{t-1} \sin^t \frac{\gamma}{2} \leq k < 1 \quad \text{for } 0 < t \leq 1,$$

$$\min\left(1, 2^{t-1} \sin^t \frac{\gamma}{2}\right) \leq k \leq \max\left(2^{t-1-t/2}, 2^{t-1} \sin^t \frac{\gamma}{2}\right) \quad \text{for } 1 < t \leq 2,$$

$$\min\left(2^{t-1-t/2}, 2^{t-1} \sin^t \frac{\gamma}{2}\right) \leq k \leq \max\left(1, 2^{t-1} \sin^t \frac{\gamma}{2}\right) \text{ for } t \geq 2.$$

**REMARK** These inequalities are due to R. Ballieu, Simon Stevin 26 (1949), 129–134. The upper bound for the case  $t \geq 2$  and  $t$  a positive integer is determined in: F. van der Blij, Simon Stevin 25 (1947), 231–235.

**3.11** If  $\bar{z} = r/R$ , then

$$-1 + 4k - \bar{z}^2 \leq \cos \beta \cos \gamma + \cos \gamma \cos \alpha + \cos \alpha \cos \beta \leq k + k^2$$

$$-1 + 3k - \frac{3}{2}\bar{z}^2 \leq \cos \alpha \cos \beta \cos \gamma \leq \frac{1}{2}k^2.$$

Equality occurs if and only if the triangle is equilateral.

W. J. Eundon, Problem E 1925, Amer. Math. Monthly 73 (1966), 1016.

**3.12** If for a triangle  $\alpha > \beta > \gamma$ , then

$$2R \cos \alpha < R - d < 2R \cos \beta < R + d < 2R \cos \gamma,$$

where  $d$  is the distance from the circumcentre to the incentre.

O. Bottema, Euclides 39 (1963/64), 129–137.

**3.13**  $a \sin x + b \sin \beta + c \sin \gamma \geq \frac{2\sqrt{3}F}{R}$ .

Equality holds if and only if the triangle is equilateral.

**PROOF.** Since

$$\sin x = a/(2R), \sin \beta = b/(2R), \sin \gamma = c/(2R),$$

we have

$$a \sin \alpha + b \sin \beta + c \sin \gamma = \frac{a^2 + b^2 + c^2}{2R} \geq \frac{2\sqrt{3}F}{R},$$

Weizenböck's inequality 4.4 was used here.

$$3.14 \quad 9r \leq a \sin \alpha + b \sin \beta + c \sin \gamma \leq \frac{9}{2}R,$$

with equality holding if and only if the triangle is equilateral.

PROOF. Starting from

$a \sin z = \frac{a^2}{2R} = \frac{2aF}{bc} = \frac{h_b h_c}{h_a}$  and similarly for  $b \sin \beta$  and  $c \sin \gamma$ , we get

$$a \sin \alpha + b \sin \beta + c \sin \gamma = \frac{h_b h_c}{h_a} + \frac{h_c h_a}{h_b} + \frac{h_a h_b}{h_c}.$$

By means of the arithmetic-geometric mean inequality and 6.16, we obtain

$$\frac{h_b h_c}{h_a} + \frac{h_c h_a}{h_b} + \frac{h_a h_b}{h_c} \geq 3\sqrt[3]{h_a h_b h_c} \geq 9r,$$

with equality if and only if  $h_a = h_b = h_c$ , i.e. if and only if the triangle is equilateral.

Therefore

$$a \sin \alpha + b \sin \beta + c \sin \gamma \geq 9r.$$

Using the sine law, we obtain

$$a \sin \alpha + b \sin \beta + c \sin \gamma = \frac{a^2 + b^2 + c^2}{2R}. \quad (1)$$

Since (see: 5.13)

$$a^2 + b^2 + c^2 \leq 9R^2,$$

from (1) we obtain

$$a \sin \alpha + b \sin \beta + c \sin \gamma \leq \frac{9}{2}R.$$

This result is due to R. R. Janić.

$$3.15 \quad \sin \alpha + \sin \beta + \sin \gamma \leq 2 + (3\sqrt{3} - 4) \cdot \frac{r}{R} \leq \frac{3}{2}\sqrt{3}.$$

Equality holds if and only if the triangle is equilateral.

W. J. Blundon, Canad. Math. Bull. 8 (1965), 615-626.

**4. Inequalities for the sides and the area of a triangle**

**4.1**  $4F \leq \min(b^2 + c^2, c^2 + a^2, a^2 + b^2)$ .

PROOF. Let  $a \geq b \geq c$ . From

$$b^2 + c^2 \geq 2bc \text{ and } 2bc \geq 2bc \sin \alpha,$$

follows

$$b^2 + c^2 \geq 2bc \sin \alpha = 4F.$$

This concludes the proof.

**4.2**  $s^2 \geq 3F\sqrt{3}$ .

Equality holds if and only if the triangle is equilateral.

PROOF. Since

$$[(s-a)(s-b)(s-c)]^{1/3} \leq \frac{(s-a) + (s-b) + (s-c)}{3} = \frac{s}{3},$$

i.e.

$$(s-a)(s-b)(s-c) \leq \frac{s^3}{27},$$

we obtain

$$F = [s(s-a)(s-b)(s-c)]^{1/2} \leq \frac{s^2}{3\sqrt{3}}.$$

H. Hadwiger, Jber. Deutsch. Math.-Verein. 49 (1939), 35–39  
kursiv.

L. A. Santaló, Math. Notae 3 (1943), 65–73.

**4.3**  $s^2 \geq 3F\sqrt{3} + \frac{1}{2}Q$ .

J. C. H. Gerretsen, Nieuw Tijdschr. Wisk. 41 (1953), 1–7.

**4.4**  $a^2 + b^2 + c^2 \geq 4F\sqrt{3}$ .

Equality occurs if and only if the triangle is equilateral.

PROOF. By cosine law,

$$a^2 + b^2 + c^2 = 2(b^2 + c^2) - 2bc \cos \alpha.$$

Since

$$F = \frac{1}{2}bc \sin \alpha,$$

one obtains

$$\begin{aligned}
 a^2 + b^2 + c^2 - 4F\sqrt{3} &= 2(b^2 + c^2) - 2bc \cos \alpha - bc\sqrt{3} \sin \alpha \\
 &= 2(b^2 + c^2) - 4bc \left( \frac{1}{2} \cos \alpha - \frac{\sqrt{3}}{2} \sin \alpha \right) \\
 &= 2(b^2 + c^2) - 4bc \cos \left( \frac{\pi}{3} - \alpha \right) \\
 &\geq 2(b^2 + c^2) - 4bc \\
 &= 2(b - c)^2.
 \end{aligned}$$

Equality holds if and only if  $b = c$  and  $\alpha = \pi/3$ , i.e. if and only if the triangle is equilateral.

R. Weitzenböck, Math. Z. 5 (1919), 137–146.

P. Finsler and H. Hadwiger, Comment. Math. Helv. 10 (1937/38), 316–326.

L. A. Santaló, Math. Notae 3 (1943), 65–73.

F. Goldner, Problem 69, Elem. Math. 4 (1945), 120.

R.A., Elementa: matematik, fysik, kemi (Uppsala) 49 (1966), 250.

$$4.5 \quad bc + ca + ab \geq 4F\sqrt{3}. \quad (1)$$

PROOF. Since

$$(\sin \alpha)^{-1} + (\sin \beta)^{-1} + (\sin \gamma)^{-1} \geq 3\sqrt[3]{(\sin \alpha \sin \beta \sin \gamma)^{-1}}$$

and (see: 2.7)

$$\sin \alpha \sin \beta \sin \gamma \leq \frac{3\sqrt{3}}{8},$$

we have

$$(\sin \alpha)^{-1} + (\sin \beta)^{-1} + (\sin \gamma)^{-1} \geq 2\sqrt[3]{1}.$$

Using the last inequality and

$$bc + ca + ab = 2F[(\sin \alpha)^{-1} + (\sin \beta)^{-1} + (\sin \gamma)^{-1}],$$

we get (1).

REMARK. From (1) and

$$(bc+ca+ab)^2 \leq 3(b^2c^2+c^2a^2+a^2b^2) \leq (a^2+b^2+c^2)^2 \leq 3(a^4+b^4+c^4)$$

one obtains the inequalities: 4.12, 4.4, 4.10.

V. O. Gordon, Matematika v škole, 1966, No. 1, 89.

4.6  $bc+ca+ab \geq 4F\sqrt{3} + \frac{1}{2}Q.$

J. C. H. Gerretsen, Nieuw Tijdschr. Wisk. 41 (1953), 1-7.

4.7  $4F\sqrt{3} + Q \leq a^2 + b^2 + c^2 \leq 4F\sqrt{3} + 3Q.$

Equality holds if and only if the triangle is equilateral.

H. Hadwiger, Ber. Deutsch. Math.-Verein. 49 (1939), 35-39  
kursiv.

REMARK. The first inequality was also proved in:

P. Finsler and E. Hadwiger, Comment. Math. Helv. 10 (1937/38), 316-326.

J. Karamata, Problem 119, Glasnik Mat.-Fiz. Astronom. 3 (1948), 223.

O. Kooi, Simon Stevin 32 (1958), 97-101.

J. Steinig, Elem. Math. 18 (1963), 127-131.

S. Zetel', Matematika v škole, 1965, No. 3, 66-69.

4.8  $4F\sqrt{3} + Q \leq a^2 + b^2 + c^2$

$$\leq 4F\left(\sec \frac{\alpha}{2} + \sec \frac{\beta}{2} + \sec \frac{\gamma}{2} - \sqrt{3}\right) + Q.$$

Equality occurs if and only if the triangle is equilateral.

S. T. Berkolaiko, Matematika v škole, 1967, No. 2, 72-74.

4.9  $12F\sqrt{3} + 2Q \leq (a+b+c)^2 \leq 12F\sqrt{3} + 8Q.$

Equality holds if and only if the triangle is equilateral.

H. Hadwiger, Ber. Deutsch. Math.-Verein. 49 (1939), 35-39  
kursiv.

REMARK. The first inequality was also proved in:

P. Finsler and E. Hadwiger, Comment. Math. Helv. 10 (1937/38), 316-326.

$$4.10 \quad a^4 + b^4 + c^4 \geq 16F^2.$$

Equality holds if and only if  $a = b = c$ .

F. Goldner, Problem 69, Elem. Math. 4 (1949), 120.

$$4.11 \quad a^4 + b^4 + c^4 \geq 16F^2 + 4FQ\sqrt{3} + \frac{1}{2}Q^2. \quad (1)$$

Equality holds if and only if  $a = b = c$ .

PROOF. By squaring the Finsler-Hadwiger's inequality 4.7, one obtains

$$a^4 + b^4 + c^4 + 2(b^2c^2 + c^2a^2 + a^2b^2) \geq 48F^2 + 8FQ\sqrt{3} + Q^2.$$

Since

$$2(b^2c^2 + c^2a^2 + a^2b^2) = 16F^2 + a^4 + b^4 + c^4,$$

then

$$2(a^4 + b^4 + c^4) + 16F^2 \geq 48F^2 + 8FQ\sqrt{3} + Q^2,$$

i.e. (1).

Matematika i fizika, Sofia, 6 (1965), 51–52.

$$4.12 \quad b^2c^2 + c^2a^2 + a^2b^2 \geq 16F^2. \quad (1)$$

Equality holds if and only if the triangle is equilateral.

PROOF. On the basis of Heron's formula we have

$$2(b^2c^2 + c^2a^2 + a^2b^2) - (a^4 + b^4 + c^4) = 16F^2.$$

Since

$$(b^2 - c^2)^2 + (c^2 - a^2)^2 + (a^2 - b^2)^2 \geq 0,$$

i.e.,

$$b^2c^2 + c^2a^2 + a^2b^2 \leq a^4 + b^4 + c^4,$$

we get (1).

F. Goldner, Problem 69, Elem. Math. 4 (1949), 120.

$$4.13 \quad 4F\sqrt{3} \leq \frac{9abc}{a+b+c}.$$

Equality occurs if and only if  $a = b = c$ .

T. R. Cutty, Problem E 1861, Amer. Math. Monthly 73 (1966), 199.

$$4.14 \quad (abc)^2 \geq \left( \frac{4F}{\sqrt{3}} \right)^3. \quad (1)$$

Equality holds if and only if  $a = b = c$ .

PROOF. From  $abc = 4RF$  and from  $a+b+c \leq 3R\sqrt{3}$  (see: 5.3), we have

$$\frac{4F}{\sqrt{3}} = \frac{abc}{R\sqrt{3}} \leq \frac{3abc}{2s} = \frac{3abc}{a+b+c} \leq (abc)^{2/3},$$

whence inequality (1) follows.

L. Carlitz-F. Leuenberger, Problem E 1454, Amer. Math. Monthly 68 (1961), 177 and 68 (1961), 805-806.

H. W. Guggenheimer, Problem E 1724, Amer. Math. Monthly 71 (1964), 911 and 72 (1965), 791-793.

4.15 If  $x, y, z$  are real numbers, then

$$a^x b^y c^z + b^x c^y a^z + c^x a^y b^z \geq 3 \cdot \frac{2^{x+y+z} F^{(x+y+z)/2}}{3^{(x+y+z)/4}}. \quad (1)$$

PROOF. By means of the arithmetic-geometric mean inequality, we obtain

$$a^x b^y c^z + b^x c^y a^z + c^x a^y b^z \geq 3(abc)^{(x+y+z)/3}.$$

From this inequality, on the basis of 4.14, we get (1).

This inequality and proof are due to P. M. Vasić.

4.16 If  $x, y, z$  are real numbers, then

$$b^x c^y (b+c)^z + c^x a^y (c+a)^z + a^x b^y (a+b)^z \geq 3 \cdot 2^z \frac{2^{x+y+z} F^{(x+y+z)/2}}{3^{(x+y+z)/4}}. \quad (1)$$

PROOF. Since  $a+b \geq 2\sqrt{ab}$ , etc., we obtain

$$\begin{aligned} b^x c^y (b+c)^z + c^x a^y (c+a)^z + a^x b^y (a+b)^z \\ \geq 2^z (b^{x+(z/2)} c^{y+(z/2)} + c^{x+(z/2)} a^{y+(z/2)} + a^{x+(z/2)} b^{y+(z/2)}) \\ \geq 3 \cdot 2^z \sqrt[3]{(abc)^{x+y+z}}. \end{aligned}$$

From this inequality, on the basis of 4.14, we get (1).

This inequality and proof are due to R. Ž. Djordjević.

4.17 If  $x, y, z$  are real numbers, then

$$\begin{aligned} & a^x(a+b)^y(b+c)^z + b^x(b+c)^y(c+a)^z + c^x(c+a)^y(a+b)^z \\ & \geq 3 \cdot 2^x \frac{2^{2(y+z)} F^{(x+y+z)/2}}{3^{(x+y+z)/4}}. \end{aligned}$$

This inequality is due to R. Đ. Djordjević.

4.18 If  $a^2 + b^2 + c^2 = 2H$  and  $bc + ca + ab = K$ , then

$$\frac{(K-H)(3K-5H)}{12} \leq F^2 \leq \frac{(K-H)^2}{12}. \quad (1)$$

Equality in (1) holds if and only if the triangle is equilateral.  
S. Beatty, Trans. Roy. Soc. Canada, III (3), 48 (1954), 1–5.

4.19 If  $q = (Q/2)^{1/2}$ , then

$$\frac{s(s+q)^2(s-2q)}{27} \leq F^2 \leq \frac{s(s-q)^2(s+2q)}{27}. \quad (1)$$

The first (second) equality sign in (1) holds for an isosceles triangle whose base is the smallest (largest) of the three sides; of course both equality signs apply when the triangle is equilateral, since then  $q = 0$ .

R. Frucht, Canad. J. Math. 9 (1957), 227–231.

4.20  $27(b^2 + c^2 - a^2)^2(c^2 + a^2 - b^2)^2(a^2 + b^2 - c^2)^2 \leq (4F)^6$ .

Equality holds if and only if  $a = b = c$ .

F. Balitrand, Interméd. Math. 23 (1916), 86–87.

4.21 If  $\lambda > 0$ , then

$$F \leq \frac{\sqrt{3}}{4} \left( \frac{a^\lambda + b^\lambda + c^\lambda}{3} \right)^{2/\lambda}.$$

C. N. Mills-O. Dunkel, Problem 3207, Amer. Math. Monthly 34 (1927), 382–384.

## 5. Inequalities for the sides and the radii of a triangle

### 5.1 $2r \leq R$ .

Equality holds if and only if the triangle is equilateral.

PROOF. This inequality is obtained from

$$OI^2 = R(R - 2r) \geq 0.$$

L. Euleri, Novi commentarii academiae scientiarum Petropolitanae 11 (1765), 1767, 103–123.

L. Euleri, Opera Omnia, I 26, 1953, 139–157.

Ramus-E. Rouché, Nouv. Ann. Math. 10 (1851), 353.

### 5.2 If $t$ and $T$ are real numbers such $-\infty < t \leq T \leq 11$ , then

$$2r \leq \frac{4s^2R^{-1}-2tr}{27-t} \leq \frac{4s^2R^{-1}-2Tr}{27-T} \leq R.$$

PROOF. Consider the function  $f$ , defined by

$$f(x) = \frac{4s^2R^{-1}-2rx}{27-x} \text{ for } x < 27.$$

From inequality (see: 1.1 and 5.36)

$$(a+b+c)^2 \geq 3(bc+ca+ab) \geq 54Rr,$$

we have

$$f(x) \geq \frac{54r-2rx}{27-x} = 2r.$$

From inequalities (see: 5.8)

$$4s^2 \leq 4(3r^2 + 4Rr + 4R^2) \text{ and } 2r \leq R,$$

we obtain

$$4s^2 \leq 16R^2 + 22Rr.$$

Since  $f'(x) = \frac{4s^2R^{-1}-54r}{(27-x)^2} \geq 0$ , we infer that  $f$  is a monotone

non-decreasing function so that  $f(11) \leq R$ .

I. Paasche-O. Reutter, Problem 516, Elem. Math. 20 (1965), 140 and 21 (1966), 139.

**5.3**  $a+b+c \leqslant 3R\sqrt{3}$ .

Equality holds if and only if  $a = b = c$ .

S. Nakajima, Tôhoku Math. J. 25 (1925), 115–121.

A. Padoa, Period. Mat. (4) 5 (1925), 80–85.

**REMARK.** This inequality is equivalent to 2.1.

**5.4**  $s \leqslant 2R + (3\sqrt{3} - 4)r$ .

Equality holds if and only if the triangle is equilateral.

W. J. Blundon, Canad. Math. Bull. 8 (1965), 615–626.

W. J. Blundon, Problem E 1935, Amer. Math. Monthly 73 (1966), 1122.

**5.5**  $9r(4R+r) \leqslant 3s^2 \leqslant (4R+r)^2$ . (1)

**PROOF.** The roots of the equation

$$y^3 - 2sy^2 - (s^2 + r(4R+r))y - 4sRr = 0 \quad (2)$$

are  $a, b, c$ .

Taking  $y = \frac{s(x-r)}{x}$ , equation (2) becomes

$$x^3 - (4R+r)x^2 + s^2x - s^2r = 0,$$

with roots  $r_a, r_b, r_c$ .

In view of Rolle's theorem, both equations

$$3y^2 - 4sy + s^2 + r(4R+r) = 0$$

and

$$3x^2 - 2(4R+r)x + s^2 = 0$$

have real roots, which implies

$$3s^2 \geqslant 9r(4R+r) \text{ and } (4R+r)^2 \geqslant 3s^2.$$

This proves (1).

G. Colombier-T. Doucet, Problem 1051, Nouv. Ann. Math. 31 (1872), 467.

**5.6**  $6r(4R+r) \leqslant 2s^2 \leqslant 2(2R+r)^2 + R^2$ .

Equalities hold if and only if the triangle is equilateral.

PROOF. Since  $r = F/s$  and  $R = abc/(4F)$ , we have

$$r(4R+r) = \frac{abc}{s} + \frac{F^2}{s^2} = \frac{1}{3}s^2 - Q \leq \frac{1}{3}s^2,$$

and consequently

$$6r(4R+r) \leq 2s^2.$$

By the last inequality and the inequality  $a^2+b^2+c^2 \leq 9R^2$  (see: 5.13), we obtain

$$(2R+r)^2 + \frac{R^2}{2} = \frac{9R^2}{2} + r(4R+r) \geq \frac{a^2+b^2+c^2}{2} + r(4R+r) = s^2,$$

i.e.

$$2s^2 \leq 2(2R+r)^2 + R^2.$$

F. Leuenberger-L. Carlitz, Problem E 1481, Amer. Math. Monthly 68 (1961), 803 and 69 (1962), 312.

5.7  $2s^2(2R-r) \leq R(4R+r)^2.$

O. Kooi, Simon Stevin, 32 (1958), 97-101.

5.8  $r(16R-5r) \leq s^2 \leq 4R^2+4Rr+3r^2.$

Equalities hold if and only if the triangle is equilateral. —

PROOF. Consider the triangle  $HIO$ . Then

$$OH^2 = 9R^2 - \sum a^2, \quad (1)$$

$$IO^2 = R^2 - 2rR, \quad (2)$$

$$IH^2 = 2r^2 - 4R^2 \cos \alpha \cos \beta \cos \gamma. \quad (3)$$

But

$$4R^2 \cos \alpha \cos \beta \cos \gamma = \frac{1}{4}(\sum a)^2 - (2R+r)^2,$$

and thus

$$IH^2 = 3r^2 + 4rR + 4R^2 - \frac{1}{4}(\sum a)^2.$$

This establishes the inequality

$$(\sum a)^2 \leq 4(3r^2 + 4rR + 4R^2),$$

with equality only in an equilateral triangle.

According to the theorem of Euler  $OG : GH = 1 : 2$ , we have

$$GI^2 = \frac{2}{3}IO^2 + \frac{1}{3}IH^2 + \frac{2}{9}OH^2. \quad (4)$$

By substituting (1), (2) and (3) in (4), we obtain

$$GI^2 = r^2 - \frac{1}{12}(\sum a)^2 + \frac{2}{9}\sum a^2,$$

which, in view of  $(\sum a)^2 = 2\sum a^2 + 4r(4R+r)$ , is equivalent to

$$36 \cdot GI^2 = (\sum a)^2 + 20r^2 - 64rR,$$

whence

$$(\sum a)^2 \geq 4r(16R - 5r).$$

Equality only when the triangle is equilateral.

J. Steinig, Elem. Math. 18 (1963), 127–131.

REMARK. In view of  $R \geq 2r$  both inequalities strengthen 5.2.

**5.9** Let  $q(R, r)$  and  $Q(R, r)$  be quadratic forms with real coefficients. Then the best possible inequalities of the form

$$q(R, r) \leq s^2 \leq Q(R, r),$$

with equalities only for the equilateral triangle, occur when

$$q(R, r) = 16Rr - 5r^2$$

and

$$Q(R, r) = 4R^2 + 4Rr + 3r^2.$$

W. J. Blundon, Canad. Math. Bull. 8 (1965), 615–626.

**5.10** Let  $f(R, r)$  and  $F(R, r)$  be homogeneous real functions. Then the strongest possible inequalities of the form

$$f(R, r) \leq s^2 \leq F(R, r),$$

with equalities only for the triangle equilateral, occur when

$$f(R, r) = 2R^2 + 10Rr - r^2 - 2(R - 2r)\sqrt{R^2 - 2Rr},$$

and

$$F(R, r) = 2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{R^2 - 2Rr}.$$

W. J. Blundon, Canad. Math. Bull. 8 (1965), 615–626.

$$5.11 \quad s^2 \geq 27r^2.$$

Equality holds if and only if the triangle is equilateral.

PROOF. By the arithmetic-geometric mean inequality, we obtain

$$\frac{s}{3} = \frac{(s-a)+(s-b)+(s-c)}{3} \geq \sqrt[3]{(s-a)(s-b)(s-c)}.$$

Since  $(s-a)(s-b)(s-c) = F^2/s = r^2s$ , we have

$$\frac{s}{3} \geq \sqrt[3]{r^2s},$$

i.e.

$$s^2 \geq 27r^2.$$

N. A. Edwards, Problem 1273. Nouv. Ann. Math. 37 (1878), 475.

J. M. Child, Math. Gaz. 23 (1899), 138-143.

$$5.12 \quad 2s^2 \geq 27Rr. \quad (1)$$

PROOF. Since

$$(a+b+c)^3 \geq 27abc \text{ and } abc = 4RF = 4Rrs,$$

we have

$$8s^3 \geq 27 \cdot 4Rrs,$$

i.e. (1).

C. Cosnita and F. Turtoiu, Culegere de probleme de algebra, Bucuresti 1965, p. 177.

$$5.13 \quad 36r^2 \leq a^2 + b^2 + c^2 \leq 9R^2. \quad (1)$$

Equalities hold if and only if  $a = b = c$ .

PROOF. Since

$$\left( \frac{a^2 + b^2 + c^2}{3} \right)^{1/2} \geq \frac{a+b+c}{3},$$

by virtue of  $6r\sqrt{3} \leq a+b+c$  (see 5.11), one obtains

$$36r^2 \leq a^2 + b^2 + c^2.$$

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The second inequality in (1) follows from

$$9R^2 - (a^2 + b^2 + c^2) = OH^2 \geq 0.$$

J. Neuberg, Ed. Times, News Ser. 9 (1906), 51–52.

T. Kubota, Tôhoku Math. J. 25 (1925), 122–126.

J. Steinig, Elem. Math. 18 (1963), 127–131.

**5.14**  $24Rr - 12r^2 \leq a^2 + b^2 + c^2 \leq 8R^2 + 4r^2.$

Equalities hold if and only if the triangle is equilateral.

J. C. Gerretsen, Nieuw Tijdschr. Wisk. 41 (1953), 1–7.

J. Steinig, Elem. Math. 18 (1963), 127–131.

**5.15** 
$$\begin{aligned} 4R^2 + 16Rr - 3r^2 - 4(R-2r)\sqrt{R^2 - 2Rr} \\ \leq a^2 + b^2 + c^2 \\ \leq 4R^2 + 16Rr - 3r^2 + 4(R-2r)\sqrt{R^2 - 2Rr}. \end{aligned}$$

W. J. Blunden, Canad. Math. Bull. 8 (1965), 615–626.

**5.16**  $36r^2 \leq bc + ca + ab \leq 9R^2.$

Equalities hold if and only if  $a = b = c$ .

PROOF. Multiplying the inequalities (see: 7.12)

$$\frac{9r}{2F} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{9R}{4F},$$

by  $F = abc/(4R^2)$ , we get

$$18Rr \leq bc + ca + ab \leq 9R^2.$$

By using  $R \geq 2r$ , inequalities (1) follow from (2).

F. Leuenberger, Elem. Math. 13 (1958), 121–126.

**5.17**  $4r(5R - r) \leq bc + ca + ab \leq 4(R + r)^2.$

Equalities hold if and only if the triangle is equilateral.

J. Steinig, Elem. Math. 18 (1963), 127–131.

**5.18**  $36r^2 \leq 4r(5R - r) \leq bc + ca + ab \leq 4(R + r)^2 \leq 9R^2.$

Equalities occur if and only if the triangle is equilateral

W. J. Blunden, Canad. Math. Bull. 8 (1965), 615–626.

REMARK. From 5.16, 5.17, and  $R \geq 2r$  follows 5.18.

$$\begin{aligned} 5.19 \quad & 2R^2 + 14Rr - 2(R-2r)\sqrt{R^2 - 2Rr} \\ & \leq bc + ca + ab \\ & \leq 2R^2 + 14Rr + 2(R-2r)\sqrt{R^2 - 2Rr}. \end{aligned}$$

Equalities hold if and only if the triangle is equilateral.

W. J. Blundon, Canad. Math. Bull. 8 (1965), 615-626.

$$5.20 \quad a(s-a) + b(s-b) + c(s-c) \leq 9Rr.$$

S. G. Guba, Matematika v škole, 1966, No. 6, 67.

$$5.21 \quad abc \leq 8R^2r + (12\sqrt{3} - 16)Rr^2.$$

Equality holds if and only if the triangle is equilateral.

W. J. Blundon, Canad. Math. Bull. 8 (1965), 615-626.

$$5.22 \quad \frac{\sqrt{3}}{R} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{\sqrt{3}}{2r}.$$

F. Leuenberger, Elem. Math. 15 (1960), 77-79.

$$5.23 \quad \frac{3\sqrt{3}}{2(R+r)} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{\sqrt{3}}{2r}.$$

Equalities occur if and only if the triangle is equilateral.

J. Steinig, Elem. Math. 18 (1963), 127-131.

REMARK. This inequality improves 5.22.

$$5.24 \quad \frac{1}{R^2} \leq \frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab} \leq \frac{1}{4r^2}. \quad (1)$$

PROOF. Since

$$\frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab} = \frac{2s}{abc} = \frac{1}{2Rr},$$

inequalities (1) are equivalent to

$$\frac{1}{R^2} \leq \frac{1}{2Rr} \leq \frac{1}{4r^2}.$$

These inequalities are true because each of these can be reduced to  $R \geqslant 2r$ .

F. Leuenberger, Elem. Math. 13 (1958), 121–126.

$$5.25 \quad 8r(R-2r) \leqslant Q \leqslant 8R(R-2r).$$

Equalities hold if and only if the triangle is equilateral.

J. C. H. Gerretsen, Nieuw Tijdschr. Wisk. 41 (1953), 1–7.

$$5.26 \quad 4r^2 \leqslant \frac{abc}{a+b+c}.$$

Equality occurs if and only if the triangle is equilateral.

M. S. Klamkin, Math. Teacher 60 (1967), 323–328.

$$5.27 \quad abc \leqslant (R\sqrt{3})^3. \quad (1)$$

Equality holds if and only if  $a = b = c$ .

PROOF. Since

$$\sqrt[3]{abc} \leqslant \frac{a+b+c}{3},$$

on the basis of 5.3, we conclude that inequality (1) holds.

$$5.28 \quad \text{The inequality}$$

$$M_k(a, b, c) \leqslant R\sqrt{3}$$

is valid for every triangle if and only if

$$k \leqslant \frac{\log 9 - \log 4}{\log 4 - \log 3}.$$

A. Makowski-J. Berkes, Elem. Math. 17 (1962), 40–41 and 18, (1963), 31–32.

$$5.29 \quad (a+b+c)\sqrt{3} \leqslant 2(r_a+r_b+r_c).$$

Equality occurs if and only if  $a = b = c$ .

J. C. H. Gerretsen, Nieuw Tijdschr. Wisk. 41 (1953), 1–7.

F. Leuenberger, Elem. Math. 16 (1961), 127–129.

$$5.30 \quad \frac{a^2}{r_b r_c} + \frac{b^2}{r_c r_a} + \frac{c^2}{r_a r_b} \geq 4.$$

Equality holds if and only if the triangle is equilateral.  
This inequality is due to R. R. Janić.

$$5.31 \quad (s-a)\sqrt{3} < 4R - r_a.$$

E. Lemoine, Problem 578, J. Math. Elem. (4) 3 (1894), 263.

$$5.32 \quad (s-a)\left(\sqrt{3} + \frac{a^2 + (b-c)^2}{2F}\right) \leq 4R - r_a. \quad (1)$$

Equality holds if and only if  $a = b = c$ .

PROOF. From

$$4R + r = r_a + r_b + r_c, \quad r_a = \frac{rs}{s-a}, \quad r_b = \frac{rs}{s-b}, \quad r_c = \frac{rs}{s-c},$$

we get

$$\begin{aligned} 4R - r_a &= r_b + r_c - r \\ &= rs\left(\frac{1}{s-b} + \frac{1}{s-c} - \frac{1}{s}\right) \\ &= (s-a)\frac{s^2 - bc}{F} \\ &= (s-a)\frac{a^2 + b^2 + c^2 + 2(-bc + ca + ab)}{4F}. \end{aligned}$$

On the basis of inequality 4.7, we have

$$4R - r_a \geq (s-a) \frac{4F\sqrt{3} + 2(a^2 + b^2 + c^2 - 2bc)}{4F},$$

i.e. (1).

Equality holds if and only if  $a = b = c$ , for in 4.7 we have equality only in that case.

R. R. Janić, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 181-No. 196 (1967), 75-76.

$$5.33 \quad 5R - r \geq s\sqrt{3}. \quad (1)$$

Equalities hold if and only if the triangle is equilateral.

PROOF. Starting from

$$4R+r = r_a + r_b + r_c, \quad r_a = \frac{rs}{s-a}, \quad r_b = \frac{rs}{s-b}, \quad r_c = \frac{rs}{s-c}$$

and from  $R \geq 2r$ , we get

$$\begin{aligned} 5R-r &\geq r_a + r_b + r_c \\ &= rs \left( \frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} \right) \\ &= \frac{s}{F} (bc+ca+ab-s^2) \\ &= \frac{s}{4F} [2(bc+ca+ab)-(a^2+b^2+c^2)]. \end{aligned}$$

On the basis of 4.7, we have (1).

Equality holds only for  $a = b = c$ , because only in this case  $R = 2r$  and equality in 4.7 holds.

R. R. Janić, Univ. Beograd. Pril. Elektrotehn. Fak. Ser. Mat. Fiz. No. 181-No. 196 (1967), 75-76.

$$5.34 \quad s^2 \leq r_a^2 + r_b^2 + r_c^2. \quad (1)$$

Equality occurs if and only if the triangle is equilateral.

PROOF. Since, by 2.35,

$$\operatorname{tg}^2 \frac{\alpha}{2} + \operatorname{tg}^2 \frac{\beta}{2} + \operatorname{tg}^2 \frac{\gamma}{2} \geq 1,$$

and

$$r_a = s \operatorname{tg} \frac{\alpha}{2}, \quad r_b = s \operatorname{tg} \frac{\beta}{2}, \quad r_c = s \operatorname{tg} \frac{\gamma}{2},$$

we obtain (1).

E. Bokov, Matematika v škole, 1956, No. 4, 95 and 1957, No. 1, 93.

$$5.35 \quad 8r_a r_b r_c \leq 3abc\sqrt{3}.$$

Equality holds if and only if the triangle is equilateral.

F. Leuenberger, Elern. Math. 16 (1961), 127-129.

$$5.36 \quad \frac{1}{2}4Rr \leq 3(bc+ca+ab) \leq 4(r_b r_c + r_c r_a + r_a r_b).$$

PROOF. The first of these inequalities was already proved in 5.16. Since

$$4(r_b r_c + r_c r_a + r_a r_b) = (a+b+c)^2 \text{ and } 3(bc+ca+ab) \leq (a+b+c)^2,$$

we conclude that the second inequality also holds.

F. Leinenberger-J. Steinig, Elem. Math. 20 (1965), 89–90.

$$5.37 \quad 32r_a r_b r_c - 14r^2(r_a + r_b + r_c) \leq 9r(r_a + r_b + r_c)^2 + 9r^3$$

holds for acute triangles.

Equality holds if and only if the triangle is equilateral.

S. Reich, Problem E 1930, Amer. Math. Monthly 73 (1966), 1017–1018.

$$5.38 \quad \frac{1}{2}R \leq \max(r_a, r_b, r_c).$$

P. Erdős, Mat. Lapok 1 (1949), 72.

5.39 If  $r_a \leq r_b \leq r_c$ , then

$$r_a \leq \frac{1}{2}R, \quad r_b < 2R, \quad \frac{1}{2}R \leq r_c < 4R.$$

H. J. Baron, Tôhoku Math. J. 48 (1941), 185–192.

$$5.40 \quad \frac{\frac{1}{2}R}{3} < \frac{r_a + r_b + r_c}{3} \leq \frac{1}{2}R \leq \sqrt[4]{\frac{r_a^4 + r_b^4 + r_c^4}{3}}.$$

L. Fejes Tóth, Mat. Lapok 1 (1949), 72.

$$5.41 \quad 9r \leq r_a + r_b + r_c \leq \frac{9}{2}R.$$

M. S. Klamkin, Math. Teacher 60 (1967), 323–328.

5.42 If  $G(x, y, z)$  denotes the geometric mean of  $x, y, z$ , then

$$G\left(r, r, \frac{R}{2}\right) \leq \frac{1}{3}G(r_a, r_b, r_c) \leq G\left(r, \frac{R}{2}, \frac{R}{2}\right).$$

Equalities hold if and only if the triangle is equilateral.

A. Nákowskí, Nieuw Arch. Wisk. (3) 12 (1964), 5–7.

$$5.43 \quad \frac{27}{4}R^2 \leq r_a^2 + r_b^2 + r_c^2.$$

Equality holds if and only if the triangle is equilateral.

V. Thébault, Mat. Lapok, 3 (1952), 59–61.

A. Makowski, Nieuw Arch. Wisk. (3) 12 (1964), 5–7.

$$5.44 \quad \frac{3}{2}R \leq M_k(r_a, r_b, r_c) \quad (k \geq 2).$$

G. Hajós, Math. Lapok 1 (1950), 313.

L. Fejes Tóth, Lagerungen in der Ebene, auf der Kugel und im Raum, Berlin 1953, p. 27.

REMARK. This inequality is a generalization of inequality 5.40.

$$5.45 \quad 7R^2 \leq r^2 + r_a^2 + r_b^2 + r_c^2. \quad (1)$$

PROOF. Since

$$r^2 + r_a^2 + r_b^2 + r_c^2 = 16R^2 - (a^2 + b^2 + c^2),$$

and on the basis of 5.13, we have (1).

Ju. I. Gerasimov, Matematika v škole, 1967, No. 3, 84.

Comment by R. R. Janić. The following stronger inequality holds

$$8R^2 - 4r^2 \leq r^2 + r_a^2 + r_b^2 + r_c^2, \quad (2)$$

because (see: 5.14)  $a^2 + b^2 + c^2 \leq 8R^2 + 4r^2$ .

Inequality (2) is stronger than the inequalities 5.40 and 5.43.

$$5.46 \quad \frac{a^3}{r_a} + \frac{b^3}{r_b} + \frac{c^3}{r_c} \leq \frac{abc}{r}. \quad (1)$$

Equality holds if and only if  $a = b = c$ .

J. Andersson, Problem 1779, Amer. Math. Monthly 59 (1952), 41.

REMARK. This inequality is equivalent to 1.9.

$$5.47 \quad \sqrt{\frac{a}{r_a}} + \sqrt{\frac{b}{r_b}} + \sqrt{\frac{c}{r_c}} \leq \frac{3}{2}\sqrt{\frac{s}{r}}. \quad (1)$$

PROOF. Since  $r_a = rs/(s-a)$ , etc., inequality (1) is equivalent to

$$\sqrt{a(s-a)} + \sqrt{b(s-b)} + \sqrt{c(s-c)} \leq \frac{3}{2}s.$$

By the arithmetic-geometric mean inequality, we get

$$\begin{aligned} & \sqrt{a(s-a)} + \sqrt{b(s-b)} + \sqrt{c(s-c)} \\ & \leq \frac{a+s-a}{2} + \frac{b+s-b}{2} + \frac{c+s-c}{2} = \frac{3s}{2}. \end{aligned}$$

Gaz. Mat. B 15 (1964), 256.

## 6. Inequalities for the sides, the altitudes and the radii of a triangle

6.1  $2(h_a + h_b + h_c) \leq \sqrt{3}(a+b+c)$ .

Equality holds if and only if the triangle is equilateral.

PROOF. First we have

$$(a+b+c)^2 \geq 3(bc+ca+ab) = \frac{3abc}{2R} (h_a + h_b + h_c) = \frac{3abc}{2R} (h_a + h_b + h_c),$$

with equality only in the case of an equilateral triangle.

Since (see: 5.3)  $3R\sqrt{3} \geq a+b+c$ , with equality if and only if the triangle is equilateral, we obtain

$$\sqrt{3}(a+b+c) \geq 2(a+b+c)(h_a + h_b + h_c),$$

so that

$$\sqrt{3}(a+b+c) \geq 2(h_a + h_b + h_c),$$

with equality only in the case of an equilateral triangle.

A. Santaló, Math. Notae 3 (1943), 65–73.

F. Leuenberger-L. Carlitz, Problem E 1427, Amer. Math. Monthly 67 (1960), 692 and 68 (1961), 296–297.

6.2  $h_a + h_b + h_c \leq \sqrt{s}(\sqrt{s-a} + \sqrt{s-b} + \sqrt{s-c}) \leq s\sqrt{3}$ .

Equalities occur if and only if the triangle is equilateral.

L. Santaló, Math. Notae 3 (1943), 65–73.

6.3  $M_2(h_a, h_b, h_c) \leq \frac{\sqrt{3}}{2} M_2(a, b, c)$  and

$$M_{-2}(h_a, h_b, h_c) \leq \frac{\sqrt{3}}{2} M_{-2}(a, b, c).$$

Equality holds if and only if  $a = b = c$ .

PROOF. By means of the obvious inequality

$$h_a^2 + h_b^2 + h_c^2 \leq m_a^2 + m_b^2 + m_c^2$$

and by

$$m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$$

one gets the first inequality.

The first and the second inequality are, however, equivalent (see: 6.6).

L. A. Santaló, Math. Notae 3 (1943), 65–73.

J. Berkes, Elem. Math. 18 (1963), 31–32.

$$6.4 \quad 3(bc + ca + ab) \geq 4(h_b h_c + h_c h_a + h_a h_b).$$

Equality holds if and only if the triangle is equilateral.

Math. Notae 3 (1943), 182.

$$6.5 \quad a^3 + b^3 + c^3 > \frac{8}{7}(h_a^3 + h_b^3 + h_c^3).$$

Gaz. Mat. B 9 (1958), 731.

$$6.6 \quad M_k(h_a, h_b, h_c) \leq \frac{\sqrt{3}}{2} M_k(a, b, c)$$

$$\Leftrightarrow M_{-k}(h_a, h_b, h_c) \leq \frac{\sqrt{3}}{2} M_{-k}(a, b, c).$$

These inequalities hold for every number  $k$  if and only if  $\beta < \pi/3$  and  $\alpha \leq \beta \leq \gamma$ .

A. Makowski, Elem. Math. 17 (1962), 40–41.

J. Steinig, Elem. Math. 20 (1965), 64–65.

$$6.7 \quad \frac{a^2}{h_b^2 + h_c^2} + \frac{b^2}{h_c^2 + h_a^2} + \frac{c^2}{h_a^2 + h_b^2} \geq 1$$

V. O. Gordon, Matematika v škole, 1967, No. 3, 84.

$$6.8 \quad h_a + h_b + h_c \geq 9r. \tag{1}$$

Equality holds if and only if the triangle is equilateral.

By the arithmetic-geometric mean inequality, we get

$$\begin{aligned} & \sqrt{a(s-a)} + \sqrt{b(s-b)} + \sqrt{c(s-c)} \\ & \leq \frac{a+s-a}{2} + \frac{b+s-b}{2} + \frac{c+s-c}{2} = \frac{3s}{2}. \end{aligned}$$

Gaz. Mat. B 15 (1964), 256.

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6.1  $2(h_a + h_b + h_c) \leq \sqrt{3}(a + b + c)$ .

Equality holds if and only if the triangle is equilateral.

PROOF. First we have

$$(a+b+c)^2 \geq 3(bc+ca+ab) = \frac{3abc}{2R} (h_a + h_b + h_c) = R(h_a + h_b + h_c),$$

with equality only in the case of an equilateral triangle.

Since (see: 5.3)  $3R\sqrt{3} \geq a+b+c$ , with equality if and only if the triangle is equilateral, we obtain

$$\sqrt{3}(a+b+c)^2 \geq 2(a+b+c)(h_a + h_b + h_c),$$

so that

$$\sqrt{3}(a+b+c) \geq 2(h_a + h_b + h_c),$$

with equality only in the case of an equilateral triangle.

A. Santaló, Math. Notae 3 (1943), 65-73.

F. Leuenberger-L. Carlitz, Problem E 1427, Amer. Math. Monthly 67 (1960), 692 and 68 (1961), 296-297.

6.2  $h_a + h_b + h_c \leq \sqrt{s}(\sqrt{s-a} + \sqrt{s-b} + \sqrt{s-c}) \leq s\sqrt{3}$ .

Equalities occur if and only if the triangle is equilateral.

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6.3  $M_2(h_a, h_b, h_c) \leq \frac{\sqrt{3}}{2} M_2(a, b, c)$  and

$$M_{-2}(h_a, h_b, h_c) \leq \frac{\sqrt{3}}{2} M_{-2}(a, b, c).$$

Equality holds if and only if  $a = b = c$ .

PROOF. By means of the obvious inequality

$$h_a^2 + h_b^2 + h_c^2 \leq m_a^2 + m_b^2 + m_c^2$$

and by

$$m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$$

one gets the first inequality.

The first and the second inequality are, however, equivalent (see: 6.6).

L. A. Santaló, Math. Notae 3 (1943), 65–71.

J. Berkes, Elem. Math. 18 (1963), 31–32.

$$6.4 \quad 3(bc + ca + ab) \geq 4(h_b h_c + h_c h_a + h_a h_b).$$

Equality holds if and only if the triangle is equilateral.

Math. Notae 3 (1943), 182.

$$6.5 \quad a^3 + b^3 + c^3 > \frac{8}{7}(h_a^3 + h_b^3 + h_c^3).$$

Gaz. Mat. B 9 (1958), 731.

$$6.6 \quad M_k(h_a, h_b, h_c) \leq \frac{\sqrt{3}}{2} M_k(a, b, c)$$

$$\Leftrightarrow M_{-k}(h_a, h_b, h_c) \leq \frac{\sqrt{3}}{2} M_{-k}(a, b, c).$$

These inequalities hold for every number  $k$  if and only if  $\beta < \pi/3$  and  $\alpha \leq \beta \leq \gamma$ .

A. Makowski, Elem. Math. 17 (1962), 40–41.

J. Steinig, Elem. Math. 20 (1965), 64–65.

$$6.7 \quad \frac{a^2}{h_b^2 + h_c^2} + \frac{b^2}{h_c^2 + h_a^2} + \frac{c^2}{h_a^2 + h_b^2} \geq 1.$$

V. O. Gordon, Matematika v škole, 1967, No. 3, 84.

$$6.8 \quad h_a + h_b + h_c \geq 9r. \tag{1}$$

Equality holds if and only if the triangle is equilateral.

PROOF. By

$$\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r}$$

and by

$$\frac{h_a + h_b + h_c}{3} \geq \frac{3}{\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}},$$

we conclude that (1) is true.

From the proof we deduce that equality in (1) holds if and only if  $h_a = h_b = h_c$ , i.e. if and only if the triangle is equilateral.

S. I. Zetel', Zadači na maksimum i minimum, Moskva 1948,  
p. 64.

6.9 If  $a \leq b \leq c$ , then

$$h_a + h_b + h_c \leq \frac{3b(a^2 + ac + c^2)}{4Rs}.$$

Equality holds if and only if the triangle is equilateral.

Z. Živanović, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 181-No. 196 (1967), 69-72.

$$\begin{aligned} 6.10 \quad 9r &\leq 3(h_a h_b h_c)^{1/3} \leq h_a + h_b + h_c \\ &\leq 3r(\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma)^{1/2} \leq \frac{3}{2}R. \end{aligned}$$

J. Berkes, Elem. Math. 14 (1959), 62-64.

$$6.11 \quad h_a + h_b - h_c \leq 3(R+r).$$

Equality holds if and only if the triangle is equilateral.

L. Carlitz, Problem E 1616, Amer. Math. Monthly 70 (1963), 758.

F. Leuenberger, Elem. Math. 19 (1964), 132-133.

$$6.12 \quad h_a + h_b - h_c \leq 2R + 5r.$$

F. Leuenberger, Elem. Math. 19 (1964), 132-133.

L. Bankoff-D. Woods, Math. Mag. 38 (1955), 240 and 36 (1966), 130.

$$6.13 \quad \frac{2r \cdot 5R - r}{R} \leq h_a + h_b + h_c \leq \frac{2(R+r)^2}{R}.$$

Equalities occur if and only if the triangle is equilateral.

J. Steinig, Elem. Math. 18 (1963), 127–131.

$$6.14 \quad h_a + h_b + h_c \leq w_a + w_b + w_c \leq 3(R+r) \leq r_a + r_b + r_c.$$

L. Carlitz, Problem E 1616, Amer. Math. Monthly 70 (1963), 758 and 71 (1964), 558–559.

F. Leuenberger, Elem. Math. 18 (1963), 35–36.

F. Leuenberger, Elem. Math. 19 (1964), 132–133.

$$6.15 \quad 2(h_b h_c + h_c h_a + h_a h_b) \leq 6F\sqrt{3} \leq 27Rr. \quad (1)$$

PROOF. Since

$$F = 4Rr \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2},$$

from

$$abc = 16R^2r \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2},$$

on the basis of 2.28, we get

$$abc \leq 6R^2r\sqrt{3}, \text{ i.e., } 2F \leq 3Rr\sqrt{3}. \quad (2)$$

This proves the second inequality in (1).

If the equality  $h_a h_b h_c = 2F^2/R$  is multiplied by  $1/h_a$ ,  $1/h_b$ ,  $1/h_c$ , respectively, and if the given values are added, we obtain

$$h_b h_c + h_c h_a + h_a h_b = \frac{2F^2}{Rr}.$$

By virtue of the last equality and of inequality (2), we conclude that the first inequality in (1) is true.

F. Leuenberger and J. Steinig, Elem. Math. 20 (1965), 89–90.

$$6.16 \quad h_a h_b h_c \geq 27r^3.$$

PROOF. Since  $\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r}$ , the product  $\frac{1}{h_a} \frac{1}{h_b} \frac{1}{h_c}$  has the maximum value if  $\frac{1}{h_a} = \frac{1}{h_b} = \frac{1}{h_c} = \frac{1}{3r}$ , whence it

follows that the product  $h_a h_b h_c$  attains the minimum value  $27r^3$  if  $h_a = h_b = h_c$ .

Therefore  $h_a h_b h_c \geq 27r^3$ .

S. I. Zetel', Zadači na maksimum i minimum, Moščva 1948, p. 100.

$$6.17 \quad \sqrt{h_a} + \sqrt{h_b} + \sqrt{h_c} \leq \frac{3}{2}\sqrt{6R}.$$

Equality holds if and only if the triangle is equilateral.

E. A. Bokov, Matematika v škole, 1967, No. 1, 83.

$$6.18 \quad h_b h_c + h_c h_a + h_a h_b \leq r_b r_c + r_c r_a + r_a r_b. \quad (1)$$

PROOF. From

$$\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}$$

we get

$$h_b h_c + h_c h_a + h_a h_b = \frac{h_a h_b h_c}{r_a r_b r_c} (r_b r_c + r_c r_a + r_a r_b). \quad (2)$$

Since (see: 6.27)

$$h_a h_b h_c \leq r_a r_b r_c,$$

from (2) follows (1).

A. Makowski, Problem E 1675, Amer. Math. Monthly 71 (1964), 317 and 72 (1965), 187-188.

6.19 If  $-1 < t < 0$ , then

$$h_a^t + h_b^t + h_c^t \geq r_a^t + r_b^t + r_c^t. \quad (1)$$

If  $t > 0$  or  $t < -1$ , then

$$h_a^t + h_b^t + h_c^t \leq r_a^t + r_b^t + r_c^t. \quad (2)$$

PROOF. Since  $h_a = \frac{2F}{a}$ ,  $r_a = \frac{2F}{b+c-a}$ , etc., inequality (1)

is equivalent to

$$a^{-t} + b^{-t} + c^{-t} \geq (b+c-a)^{-t} + (c+a-b)^{-t} + (a+b-c)^{-t} \quad (3)$$

for  $-1 < t < 0$ .

Since the function  $f(x) = -x^{-t}$ , for  $x > 0$  and  $-1 < t < 0$ , is convex, Jensen's inequality can be applied, viz.

$$\frac{f(x_1) + f(x_2)}{2} \geq f\left(\frac{x_1 + x_2}{2}\right) \quad (4)$$

for  $x_1$  and  $x_2 > 0$ .

For  $x_1 = c+a-b$ ,  $x_2 = a+b-c$ , inequality (4) becomes

$$(c+a-b)^{-t} + (a+b-c)^{-t} \leq 2a^{-t}. \quad (5)$$

Permuting  $a, b, c$  in (5) cyclically and adding the inequalities obtained we get inequality (3), i.e., (1).

Inequality (2) is equivalent to

$$a^{-t} + b^{-t} + c^{-t} \leq (b+c-a)^{-t} + (c+a-b)^{-t} + (a+b-c)^{-t},$$

for  $t > 0$  or  $t < -1$ .

The proof of this inequality is analogous to the proof of (3).

A. Makowski, Elem. Math. 16 (1961), 60–61.

**6.20** If  $x, y, z$  are non-negative real numbers, then

$$h_a^x h_b^y h_c^z + h_b^x h_c^y h_a^z + h_c^x h_a^y h_b^z \leq r_a^x r_b^y r_c^z + r_b^x r_c^y r_a^z + r_c^x r_a^y r_b^z.$$

Equality holds if and only if the triangle is equilateral.

J. I. Nassar, Problem E 1847, Amer. Math. Monthly 73 (1966), 82.

$$6.21 \quad \frac{1}{h_a - 2r} + \frac{1}{h_b - 2r} + \frac{1}{h_c - 2r} \geq \frac{3}{r}. \quad (1)$$

PROOF. By

$$\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r}$$

we obtain

$$\frac{h_a - 2r}{h_a} + \frac{h_b - 2r}{h_b} + \frac{h_c - 2r}{h_c} = 1.$$

From the last equality and from

$$(x+y+z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \geq 9,$$

we have

$$\frac{h_a}{h_a - 2r} + \frac{h_b}{h_b - 2r} + \frac{h_c}{h_c - 2r} \geq 9.$$

Since

$$\begin{aligned} & \frac{2r}{h_a - 2r} + \frac{2r}{h_b - 2r} + \frac{2r}{h_c - 2r} \\ &= \frac{h_a - (h_a - 2r)}{h_a - 2r} + \frac{h_b - (h_b - 2r)}{h_b - 2r} + \frac{h_c - (h_c - 2r)}{h_c - 2r} \geq 9 - 3 = 6, \end{aligned}$$

we obtain (1).

E. A. Bokov, Matematika v škole, 1966, No. 4, 77.

$$6.22 \quad \frac{h_a + r}{h_a - r} + \frac{h_b + r}{h_b - r} + \frac{h_c + r}{h_c - r} \geq 6.$$

C. Cosnita and F. Turtoiu, Culegere de probleme de algebra, Bucuresti 1965, p. 17.

6.23 If  $k > 0$ , then

$$M_k\left(\frac{h_a - r}{h_a + r_a}, \frac{h_b - r}{h_b + r_b}, \frac{h_c - r}{h_c + r_c}\right) \geq \sqrt[3]{\frac{4r^2}{27R^2}}.$$

If  $k < 0$ , then

$$M_k\left(\frac{h_a - r}{h_a + r_a}, \frac{h_b - r}{h_b + r_b}, \frac{h_c - r}{h_c + r_c}\right) \leq \sqrt[3]{\frac{2r}{27R}}.$$

If  $k = 0$ , then

$$\sqrt[3]{\frac{4r^2}{27R^2}} \leq M_k\left(\frac{h_a - r}{h_a + r_a}, \frac{h_b - r}{h_b + r_b}, \frac{h_c - r}{h_c + r_c}\right) \leq \sqrt[3]{\frac{2r}{27R}}.$$

Equalities hold if and only if the triangle is equilateral.  
This result is due to R. Ž. Djordjević.

$$6.24 \quad \frac{h_a - r_a}{h_a + r_a} + \frac{h_b - r_b}{h_b + r_b} + \frac{h_c - r_c}{h_c + r_c} \leq 0.$$

C. Cosnita and F. Turtoiu, Culegere de probleme de algebra, Bucuresti 1965, p. 17.

6.25 If  $a \neq b \neq c \neq a$ , then

$$M_k \left( \frac{h_a - r_b}{r_c - h_a}, \frac{h_b - r_c}{r_a - h_b}, \frac{h_c - r_a}{r_b - h_c} \right) > 1,$$

for  $k > 0$ ,

$$M_k \left( \frac{h_a - r_b}{r_c - h_a}, \frac{h_b - r_c}{r_a - h_b}, \frac{h_c - r_a}{r_b - h_c} \right) < 1,$$

for  $k < 0$ ,

$$M_k \left( \frac{h_a - r_b}{r_c - h_a}, \frac{h_b - r_c}{r_a - h_b}, \frac{h_c - r_a}{r_b - h_c} \right) = 1,$$

for  $k = 0$ .

This result is due to R. Ž. Djordjević.

$$6.26 \quad \frac{r_a}{h_b + h_c} - \frac{r_b}{h_c + h_a} + \frac{r_c}{h_a + h_b} \geq \frac{3}{2}.$$

This inequality is due to R. R. Janić.

$$6.27 \quad (h_a h_b h_c)^{1/3} \leq 3^{1/4} F^{1/2} \leq (r_a r_b r_c)^{1/3}. \quad (1)$$

Equalities hold if and only if the triangle is equilateral.

PROOF. On the basis of 7.9,

$$2R^{-1} \leq 3^{3/4} F^{-1/2} \leq r^{-1}. \quad (2)$$

Commencing with

$$8F^3 = h_a h_b h_c abc = h_a h_b h_c \cdot 4RF, \quad r_a r_b r_c = F^2 r^{-1},$$

we obtain (1) from (2).

A. Makowski, Elem. Math. 16 (1961), 134.

$$6.28 \quad \frac{r_a}{h_a} + \frac{r_b}{h_b} + \frac{r_c}{h_c} \geq 3.$$

Equality occurs if and only if the triangle is equilateral.

PROOF. From

$$\frac{2}{h_a} = \frac{1}{r_b} + \frac{1}{r_c},$$

we get

$$\frac{r_a}{h_a} = \frac{1}{2} \left( \frac{r_a}{r_b} + \frac{r_a}{r_c} \right).$$

Similarly

$$\frac{r_b}{h_b} = \frac{1}{2} \left( \frac{r_b}{r_c} + \frac{r_b}{r_a} \right) \quad \text{and} \quad \frac{r_c}{h_c} = \frac{1}{2} \left( \frac{r_c}{r_a} + \frac{r_c}{r_b} \right).$$

Therefore

$$\frac{r_a}{h_a} + \frac{r_b}{h_b} + \frac{r_c}{h_c} = \frac{1}{2} \left( \frac{r_a}{r_b} + \frac{r_b}{r_a} + \frac{r_b}{r_c} + \frac{r_c}{r_b} + \frac{r_c}{r_a} + \frac{r_a}{r_c} \right) \geq \frac{1}{2} \cdot 6 = 3.$$

Equality holds if and only if  $r_a = r_b = r_c$ , i.e. if and only if the triangle is equilateral.

H. Demir-D. C. B. Marsh, Problem E 1779, Amer. Math. Monthly 72 (1965), 420 and 73 (1956), 668.

**REMARK 1.** H. Guggenheimer (see: Amer. Math. Monthly 73 (1966), 668) proved the generalized inequality

$$\left( \frac{r_a}{h_a} \right)^n + \left( \frac{r_b}{h_b} \right)^n + \left( \frac{r_c}{h_c} \right)^n \geq 3,$$

for  $n \geq 1$ .

**REMARK 2.** T. Figiel (see: Amer. Math. Monthly 73 (1966), 668) observed that the inequality required is equivalent to the fact that the area of an orthic triangle (i.e. the pedal triangle for the orthocentre) is not greater than one-quarter of the area of a given acute-angled triangle.

$$6.29 \quad \frac{3}{h_a^{-1} + h_b^{-1} + h_c^{-1}} = \frac{3}{r_a^{-1} + r_b^{-1} + r_c^{-1}} \leq \sqrt[4]{3F^2}. \quad (\square)$$

Equality holds if and only if the triangle is equilateral.

**PROOF.** Starting from

$$h_a^{-1} + h_b^{-1} + h_c^{-1} = r_a^{-1} + r_b^{-1} + r_c^{-1} = r^{-1}$$

and using the second of inequalities in 7.9, we obtain (1).

A. Makowski, Elem. Math. 16 (1961), 134.

**7. Inequalities for the sides, the area and the radii of a triangle**

$$7.1 \quad F\sqrt{3} \leq (R+r)^2.$$

Equality holds if and only if the triangle is equilateral.

PROOF. Since  $a+b+c \leq 3R\sqrt{3}$  (see: 5.3) by means of the arithmetic-geometric inequality, we get

$$R+r \geq \frac{1}{3} \left( \frac{s}{\sqrt{3}} + \frac{s}{\sqrt{3}} + 3r \right) \geq \sqrt[3]{s^2r} = \sqrt[3]{sF}. \quad (1)$$

By means of  $s^2 \geq 3F\sqrt{3}$ , following 4.2, from (1) follows

$$(R+r)^2 \geq \sqrt[3]{s^2F^2} \geq \sqrt[3]{3F^3\sqrt{3}} = F\sqrt{3}.$$

L. Carlitz-F. Leuenberger, Problem E 1454, Amer. Math. Monthly 68 (1961), 177 and 68 (1961), 805-806.

H. W. Guggenheimer, Amer. Math. Monthly 72 (1965), 791-793.

$$7.2 \quad F\sqrt{3} \leq r(4R+r). \quad (1)$$

Equality occurs if and only if the triangle is equilateral.

PROOF. Since (see: 5.8)

$$F^2 \leq r^2(4R^2+4Rr+3r^2) = r^2[(4R^2+\frac{8}{3}Rr+\frac{1}{3}r^2)+\frac{1}{3}(4Rr+5r^2)],$$

using inequalities  $4r^2 \leq 2Rr \leq R^2$ , we obtain

$$F^2 \leq r^2[(4R^2+\frac{8}{3}Rr+\frac{1}{3}r^2)+\frac{4}{3}R^2] = \frac{1}{3}r^2(4R+r)^2,$$

i.e. (1).

J. C. H. Gerretsen, Nieuw Tijdschr. Wisk. 41 (1953), 1-7.

J. Steinig, Elem. Math. 18 (1963), 127-131.

$$7.3 \quad s^2 \leq 4R^2 - \frac{11}{3\sqrt{3}} F.$$

Equality holds if and only if the triangle is equilateral.

S. Nakajima, Tôhoku Math. J. 25 (1925), 115-121.

**PROOF.** The sides  $a, b, c$  of a triangle are the roots of the equation

$$z^3 - \frac{2F}{r} z^2 + \left( \frac{F^2}{r^2} + 4Rr + r^2 \right) z - 4FR = 0, \quad (2)$$

where  $F$  is the area of given triangle.

On the other hand,  $b+c-a$ ,  $c+a-b$  and  $a+b-c$  are the roots of

$$u^3 - \frac{2F}{r} u^2 + 4r(4R+r)u - 8Fr = 0. \quad (3)$$

The condition that equations (2) and (3) have only real roots, leads to  $R \geq 2r$  and inequality (1).

Ramus-É. Rouché, Nouv. Ann. Math. 10 (1851), 353.

**REMARK 1.** See: A. Laisant. Géométrie du triangle, Paris 1896, p. 112.

**REMARK 2.** See also 5.10.

$$7.12 \quad \frac{9r}{2F} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{9R}{4F}. \quad - (1).$$

Equalities occur if and only if the triangle is equilateral.

**PROOF.** From

$$h_a + h_b + h_c = 2F \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right)$$

and

$$9r \leq h_a + h_b + h_c \leq \frac{9R}{2},$$

on the basis of 6.8 and 6.10, we can immediately conclude that inequalities (1) hold.

Equalities in (1) hold if and only if  $a = b = c$ .

F. Leuenberger, Elem. Math. 13 (1958), 121-126.

$$7.13 \quad F\sqrt{3} \leq 4Rr + r^2 \leq F\sqrt{3} - \frac{1}{2}Q.$$

Equalities occur if and only if the triangle is equilateral.

J. C. H. Gerretsen, Nieuw Tijdschr. Wisk. 41 (1953), 1-7.

**8. Inequalities for the medians, the angle-bisectors and other elements of a triangle**

8.1  $\frac{3}{2}s < m_a + m_b + m_c < \frac{5}{2}s.$

PROOF. Let  $M$  denote the midpoint of  $BC$  and let  $A'$  be the point symmetrical to  $A$  with respect to  $M$ . Then

$$AA' = 2m_a < AB + BA' = b + c,$$

i.e.

$$2m_a < b + c.$$

Similarly,

$$2m_b < c - a, \quad 2m_c < a + b.$$

By addition one obtains

$$m_a + m_b + m_c < 2s. \quad (1)$$

Let  $C'$  be the intersection of  $BC$  and of the straight line passing through  $A$  and parallel to the median  $BN$ .

The sides of the triangle  $AA'C'$  are  $2m_a$ ,  $2m_b$ ,  $2m_c$ , and its medians are  $\frac{3}{2}c$ ,  $\frac{3}{2}b$ ,  $\frac{3}{2}a$ .

By means of (1) we get

$$m_a + m_b + m_c > \frac{3}{2}s.$$

C. Sebastiano, Boll. Mat. (Firenze), 1938, 59.

L. A. Santaló, Math. Notae 3 (1943), 65-73.

8.2  $m_a + m_b + m_c \leq 4R + r. \quad (1)$

PROOF. Let  $p_a$ ,  $p_b$ ,  $p_c$  be the distances of the circumcentre from the sides  $BC$ ,  $CA$ ,  $AB$  respectively. Then

$$m_a \leq R + p_a, \quad m_b \leq R + p_b, \quad m_c \leq R + p_c.$$

By adding these inequalities, we get

$$m_a + m_b + m_c \leq 3R + p_a + p_b + p_c. \quad (2)$$

Since

$$ap_a = R^2 \sin 2\alpha, \quad bp_b = R^2 \sin 2\beta, \quad cp_c = R^2 \sin 2\gamma,$$

we have

$$p_a + p_b + p_c = R(\cos \alpha + \cos \beta + \cos \gamma).$$

Since

$$\cos \alpha + \cos \beta + \cos \gamma = 1 + \frac{r}{R},$$

inequality (2) becomes (1).

F. Leuenberger, Elem. Math. 13 (1958), 121–126.

$$E3 \quad 9r \leq m_a + m_b + m_c \leq \frac{9}{2}R.$$

E. G. Gotman, Matematika v škole, 1966, No. 5, 76.

REMARK. Second inequality is improved by 8.2.

$$E4 \quad \frac{\sqrt{3}}{27} (m_a + m_b + m_c)^2 \geq F.$$

Equality holds if and only if the triangle is equilateral.

L. A. Santaló, Math. Notae 3 (1943), 65–73.

$$E5 \quad bc - \frac{a^2}{4} \leq m_a^2 \leq bc + \frac{a^2}{4}, \quad ca - \frac{b^2}{4} \leq m_b^2 \leq ca + \frac{b^2}{4},$$

$$ab - \frac{c^2}{4} \leq m_c^2 \leq ab + \frac{c^2}{4}.$$

PROOF. Since

$$m_a^2 = \frac{b^2 + c^2}{2} - \frac{a^2}{4},$$

and by  $b^2 + c^2 \geq 2bc$ , we get

$$m_a^2 \geq bc - \frac{a^2}{4}.$$

Equality (2) can be put in the form

$$m_a^2 = \frac{b^2 + c^2 - a^2}{2} + \frac{a^2}{4},$$

whence, by

$$bc \geq \frac{b^2 + c^2 - a^2}{2}$$

we conclude that

$$m_a^2 \leq bc + \frac{a^2}{4}.$$

Analogously, one also proves the other inequalities of (1).  
Gaz. Mat. B 7 (1956), 431.

8.6  $m_a^2 + m_b^2 + m_c^2 \geq 3F\sqrt{3}$ .

Equality holds if and only if the triangle is equilateral.  
L. A. Santaló, Math. Notae 3 (1943), 65–73.

REMARK. As

$$m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2),$$

the inequality follows from 4.4.

8.7 If  $\alpha < \gamma$ , then  $w_a > w_c$ .

PROOF. One has

$$w_a^2 = \frac{4bcs(s-a)}{(b+c)^2}, \quad w_c^2 = \frac{4abs(s-c)}{(a+b)^2}$$

and therefore

$$\begin{aligned} \frac{(b+c)^2(a+b)^2}{2bs} (w_a^2 - w_c^2) &= c(b+c-a)(a+b)^2 - a(a+b-c)(b+c)^2 \\ &= b^3(c-a) + b^2(c^2-a^2) + 3abc(c-a) + ca(c^2-a^2). \end{aligned}$$

Hence  $c-a > 0$  implies  $w_a > w_c$ .

This inequality is due to S. Moiseev and the proof to O. Bottema.

S. Moiseev, Matematika v škole, 1952, No. 5, 92 and 1953, No. 2, 89–90.

8.8  $w_a \leq \sqrt{s(s-a)} < \frac{2bc}{b+c} \leq \sqrt{bc}$ .

H. W. Guggenheimer, Plane Geometry and its Groups, San Francisco, Cambridge, London, Amsterdam 1967, p. 178.

8.9  $s < w_a + w_b + w_c \leq \sqrt{s}(\sqrt{s-a} + \sqrt{s-b} + \sqrt{s-c}) \leq s\sqrt{3}$ .

L. A. Santaló, Math. Notae 3 (1943), 65–73.

$$8.10 \quad w_a^2 + w_b^2 + w_c^2 \geq 3F\sqrt{3}. \quad (1)$$

PROOF. Since

$$w_a^2 = bc - \frac{a^2bc}{(b+c)^2} \quad \text{and} \quad \frac{bc}{(b+c)^2} \leq \frac{1}{4},$$

we have

$$\sum w_a^2 \geq \sum ab - \frac{1}{4} \sum a^2. \quad (2)$$

On the other hand, by means of 4.7

$$\frac{1}{4} \sum a^2 \leq \frac{1}{2} \sum ab - F\sqrt{3},$$

so that (2) becomes

$$\sum w_a^2 \geq \frac{1}{2} \sum ab + F\sqrt{3}, \quad (3)$$

and as  $\sum ab \geq 4F\sqrt{3}$  (see: 4.5) from (3) we get (1).

V. O. Gordon, Matematika v škole, 1962, No. 5, 76.

$$8.11 \quad 16(w_a^4 + w_b^4 + w_c^4) \leq 9(a^4 + b^4 + c^4).$$

J. Berkes, Elem. Math. 19 (1964), 138-139.

8.12 If  $\lambda$  is a real number, then

$$a^\lambda w_a + b^\lambda w_b + c^\lambda w_c \leq \sqrt{\frac{3}{2}abc s(a^{2(\lambda-1)} + b^{2(\lambda-1)} + c^{2(\lambda-1)})}.$$

Equality holds if and only if the triangle is equilateral.

This result is due to R. Ž. Djordjević.

8.13 If  $\lambda$  is a real number, then

$$a^\lambda w_a^2 + b^\lambda w_b^2 + c^\lambda w_c^2 \leq \frac{1}{2} abc s(a^{\lambda-2} + b^{\lambda-2} + c^{\lambda-2}).$$

Equality holds if and only if the triangle is equilateral.

This inequality is due to R. Ž. Djordjević.

$$8.14 \quad w_a w_b w_c \leq rs^2.$$

Equality holds if and only if the triangle is equilateral.

L. Carlitz, Problem E 1628, Amer. Math. Monthly 70 (1963), 891 and 71 (1964), 687.

$$8.15 \quad (w_b w_c)^2 + (w_c w_a)^2 + (w_a w_b)^2 \leq rs^2(4R - r).$$

Equality holds if and only if the triangle is equilateral.

L. Carlitz, Problem E 1628, Amer. Math. Monthly 70 (1963), 891 and 71 (1964), 687.

$$8.16 \quad w_b w_c + w_c w_a + w_a w_b \leq r_b r_c + r_c r_a + r_a r_b.$$

PROOF. Since (see: 8.8),

$$w_a \leq \sqrt{s(s-a)}, \quad w_b \leq \sqrt{s(s-b)},$$

we have

$$w_a w_b \leq s \sqrt{(s-a)(s-b)} \leq s \cdot \frac{c}{2}.$$

Adding  $w_a w_b \leq \frac{1}{2}cs$  and the two other inequalities for  $w_b w_c$  and  $w_c w_a$ , we obtain

$$w_b w_c + w_c w_a + w_a w_b \leq s^2. \quad (1)$$

On the other hand, from  $r_a = F/(s-a)$  etc., we get

$$r_b r_c + r_c r_a + r_a r_b = s^2. \quad (2)$$

From (1) and (2) follows the inequality which was to be proved.

A. Makowski and J. Schopp, Problem E 1675, Amer. Math. Monthly 72 (1965), 187-188.

REMARK. The following stronger inequality exists

$$w_a^2 + w_b^2 + w_c^2 \leq r_b r_c + r_c r_a + r_a r_b.$$

$$8.17 \quad r_a w_a + r_b w_b + r_c w_c$$

$$\leq s(\sqrt{(s-b)(s-c)} + \sqrt{(s-c)(s-a)} + \sqrt{(s-a)(s-b)}) \\ \leq s^2$$

$$\leq F \sqrt{1 - \frac{38}{27} \left( \frac{s}{r} \right)^2 + \left( \frac{\tilde{a}}{27} \right)^2 \left( \frac{s}{r} \right)^4}.$$

Equalities hold if and only if the triangle is equilateral.

PROOF. From the formulas

$$r_a = \sqrt{\frac{s(s-b)(s-c)}{s-a}} \quad \text{and} \quad w_a = \frac{2\sqrt{bc}}{b+c} \sqrt{s(s-a)}, \text{ etc.,}$$

by applying the arithmetic-geometric mean inequality twice we obtain

$$r_a w_a + r_b w_b + r_c w_c$$

$$\leq s[\sqrt{(s-b)(s-c)} + \sqrt{(s-c)(s-a)} + \sqrt{(s-a)(s-b)}] \leq s^2.$$

Equalities hold if and only if the triangle is equilateral.

It remains to be proved that

$$s^2 \leq F \sqrt{1 - \frac{38}{27} \left(\frac{s}{r}\right)^2 + \left(\frac{8}{27}\right)^2 \left(\frac{s}{r}\right)^4},$$

or equivalently  $(x-27)(64x-27) \geq 0$ , with  $x = (s/r)^2$ . This follows immediately from the inequality  $(s/r)^2 \geq 27$  (see: 5.6), in which equality holds if and only if the triangle is equilateral.

H. Guggenheimer-J. Steinig, Problem 520, Elem. Math. 21 (1966), 20.

$$8.18 \quad \left(\frac{r_a}{w_a}\right)^\lambda + \left(\frac{r_b}{w_b}\right)^\lambda + \left(\frac{r_c}{w_c}\right)^\lambda \geq 3,$$

for  $\lambda > 0$ .

Equality holds if and only if the triangle is equilateral.

PROOF. Since

$$r_a = \sqrt{\frac{s(s-b)(s-c)}{s-a}} \text{ and } w_a \leq \sqrt{s(s-a)}, \text{ etc.,}$$

we have

$$\begin{aligned} \left(\frac{r_a}{w_a}\right)^\lambda + \left(\frac{r_b}{w_b}\right)^\lambda + \left(\frac{r_c}{w_c}\right)^\lambda \\ = \frac{\sqrt{(s-b)^\lambda (s-c)^\lambda}}{(s-a)^\lambda} + \frac{\sqrt{(s-c)^\lambda (s-a)^\lambda}}{(s-b)^\lambda} + \frac{\sqrt{(s-a)^\lambda (s-b)^\lambda}}{(s-c)^\lambda}. \end{aligned}$$

The result desired now follows from the geometric-arithmetic mean inequality.

H. Guggenheimer, Amer. Math. Monthly 73 (1966), 668.

$$8.19 \quad w_a^2 + w_b^2 + w_c^2 \leq s^2 \leq m_a^2 + m_b^2 + m_c^2.$$

Equalities occur if and only if the triangle is equilateral.

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PROOF. Starting with

$$w_a^2 \leq s(s-a), \quad w_b^2 \leq s(s-b), \quad w_c^2 \leq s(s-c),$$

we have

$$w_a^2 + w_b^2 + w_c^2 \leq s(3s-a-b-c) = s^2,$$

with equality if and only if the triangle is equilateral.

The inequality

$$(b-c)^2 + (c-a)^2 + (a-b)^2 \geq 0,$$

yields

$$s^2 \leq \frac{3}{4}(a^2 + b^2 + c^2).$$

Since

$$\frac{b^2+c^2}{2} - \frac{a^2}{4} = m_a^2, \text{ etc.,}$$

inequality (1) becomes

$$s^2 \leq m_a^2 + m_b^2 + m_c^2.$$

F. Ryžkov, Matematika v škole, 1939, No. 1, 80; No. 4,

Z. A. Skopec, Matematika v škole, 1963, No. 3, 89.

L. Carlitz-S. Philipp, Problem E 1628, Amer. Math. Mont  
70 (1963), 891 and 71 (1964), 687.

F. Leuenberger, Elem. Math. 18 (1963), 35-36.

$$8.20 \quad w_a + w_b + w_c \leq m_a + m_b + m_c \leq r_a + r_b + r_c.$$

Equalities hold if and only if the triangle is equilateral.

F. Leuenberger, Elem. Math. 16 (1961), 127-129.

$$8.21 \quad w_a w_b w_c \leq r_a r_b r_c \leq m_a m_b m_c.$$

Equalities occur if and only if the triangle is equilateral.

F. Leuenberger, Elem. Math. 16 (1961), 127-129.

**9. Inequalities related to two triangles inscribed one in the other**

**9.1** Consider a triangle  $ABC$  divided into four smaller triangles, a central one  $DEF$  inscribed in  $ABC$  and three others on the three sides of  $DEF$ . Then

$$\text{area } DEF \geq \min(\text{area } AEF, \text{area } BDF, \text{area } CED). \quad (1)$$

Equality holds if and only if the points  $D, E, F$  are the midpoints of the sides of  $ABC$ .

**PROOF.** Let  $F_1, F_2, F_3$  ( $0 < F_1 \leq F_2 \leq F_3$ ) be the areas of the corner triangles and  $F_0$  the area of the central triangle. Then we will prove that

$$F_0 \geq \sqrt{F_1 F_2},$$

with equality if and only if  $D, E, F$  are the midpoints of the sides of the triangle  $ABC$ . This result is slightly stronger than inequality (1).

Let  $BC, CA, AB$  be divided at  $D, E, F$  respectively in the ratio  $x:x', y:y', z:z'$  with  $x+x' = y+y' = z+z' = 1$ . Let  $ABC$  also be of unit area. Then the corner triangles are of areas  $y'z$ ,  $z'x$ ,  $x'y$ . Also

$$F_0 = 1 - y'z - z'x - x'y = xyz + x'y'z'.$$

If  $F_3 < \frac{1}{4}$  then  $F_1 \leq F_2 \leq F_3$  and  $F_1 + F_2 + F_3 + F_0 = 1$  imply

$$F_0 \geq \sqrt{F_1 F_2}.$$

If  $F_3 \geq \frac{1}{4}$  then

$$F_0 = xyz + x'y'z' \geq 2\sqrt{xyz'y'z'} = 2\sqrt{F_1 F_2 F_3} \geq \sqrt{F_1 F_2}.$$

Equality is obtained if and only if  $xyz = x'y'z'$  and  $F_3 = \frac{1}{4}$ , which occurs if and only if  $F_1 = F_2 = F_3 = F_0 = \frac{1}{4}$  and  $x = y = z = \frac{1}{2}$ .

This proof is due to P. H. Diananda.

**REMARK.** Inequality (1) was proposed by H. Debrunner in 1956 (see: Problem 260, Elem. Math. 11 (1956), 20). According to J. Rainwater this inequality may be attributed to P. Erdős (see: J. Rainwater, Problem 4908, Amer. Math. Monthly 67 (1960), 479).

Proof of inequality (1) was first given by A. Bager in Elem. Math. 12 (1957), 43. In the Amer. Math. Monthly 68 (1961), 368, a proof of P. H. Diananda was published.

Inequality (1) was also investigated by:

E. Morgantini, Rend. Sem. Mat. Univ. Padova 30 (1960), 245–247.

G. C. Citterio, Period. Mat. (4) 40 (1962), 41–50.

B. I. Baidaff, Bol. Mat. 35, №. 17–20 (1962), 65–66.

F. E. G.-Rodeja, Gac. Mat. 15 (1963), 23–24.

N. J. Torres, Gac. Mat. 15 (1963), 127–128.

H. T. Croft, Math. Gaz. 49 (1965), 45–49.

E. Szekeres, Elem. Math. 22 (1967), 17–18.

**9.2** Let  $DEF$  be defined as in 9.1, then

$$\text{per } DEF \geq \min(\text{per } AEF, \text{per } BDF, \text{per } CED), \quad (1)$$

where per denotes the perimeter.

Equality holds if and only if the points  $D, E, F$  are the midpoints of the sides of  $ABC$ .

**PROOF.** To prove this it will convenient to regard  $DEF$  as given, and to define angular coordinates  $\varphi, \psi, \theta$  as the angles which the sides of  $ABC$  make with the corresponding sides of  $DEF$ . More precisely, a line through  $D$  and parallel to  $FE$  would have to be rotated anticlockwise i.e. in the sense  $DEF$ , through an angle  $\varphi$  in order to become coincident with  $BC$ ; for a clockwise rotation we would assign a negative value to  $\varphi$ . Similarly  $\psi$  and  $\theta$  are defined by means of anticlockwise rotations about  $E$  and  $F$ .

Let  $D, E, F$  be the angles of triangle  $DEF$ . Then it follows that angle  $A FE = E + \theta$  and angle  $A EF = \bar{E} - \psi$ . The only restrictions on the range of values for  $\varphi, \psi, \theta$  are that  $DEF$  must remain inside  $ABC$ , so that all angles like  $E - \varphi, F - \psi$  must be

positive, and the sum of any pair of angles in the same triangle must be less than two right angles.

Let  $s_0, s_1, s_2, s_3$  denote the perimeters of the triangles  $DEF, AEF, BDF, CED$  respectively. We prove that  $s_0 \geq \min(s_1, s_2, s_3)$ . The sine rule, applied to triangle  $DEF$ , gives

$$\frac{DF+DE}{FE} = \frac{\sin E + \sin F}{\sin(E+F)} = \frac{\cos \frac{1}{2}(E-F)}{\cos \frac{1}{2}(E+F)},$$

and similarly, in triangle  $AEF$ , we have

$$\frac{AF-AE}{FE} = \frac{\cos \frac{1}{2}(E-F+\theta+\psi)}{\cos \frac{1}{2}(E+F+\theta-\psi)}.$$

Since all the denominators are positive,  $s_0 - s_1$  has the same sign as

$$\begin{aligned} & \cos \frac{1}{2}(E-F) \cos \frac{1}{2}(E-F+\theta-\psi) - \cos \frac{1}{2}(E+F) \cos \frac{1}{2}(E-F+\theta+\psi) \\ &= \frac{1}{2} \cos \left( E + \frac{\theta-\psi}{2} \right) + \frac{1}{2} \cos \left( F + \frac{\theta-\psi}{2} \right) \\ &\quad - \frac{1}{2} \cos \left( E + \frac{\theta+\psi}{2} \right) - \frac{1}{2} \cos \left( F - \frac{\theta+\psi}{2} \right) \\ &= \sin \frac{\psi}{2} \sin \left( E + \frac{\theta}{2} \right) - \sin \frac{\theta}{2} \sin \left( F - \frac{\psi}{2} \right). \end{aligned} \quad (2)$$

There are now three possibilities:

1°.  $\varphi, \psi, \theta$  are all zero. Then the sides of  $ABC$  are parallel to those of  $DEF$ ,  $D$  is the midpoint of  $BC$ , etc., and  $s_0 = s_1 = s_2 = s_3$ .

2°.  $\varphi, \psi, \theta$  do not all have the same sign, or not more than two are zero. Then there must be a pair for which, in the cyclic order  $\varphi\psi\theta\varphi$ , negative (or zero) follows positive: if, for instance,  $\psi \geq 0, \theta < 0$  (or  $\psi > 0, \theta \leq 0$ ), then  $s_0 > s_1$ .

3°.  $\varphi, \psi, \theta$  are all of the same sign and different from zero. Then the product  $\sin \psi/2 \sin \theta/2$  is positive, and the expression (2), having the sign of  $s_0 - s_1$ , can be divided by this product to give

$$\sin E \cotg \frac{\varphi}{2} + \cos E - \sin F \cotg \frac{\psi}{2} + \cos F,$$

and dividing further by  $\sin E \sin F$  and bearing in mind that  $\cos E + \cos F > 0$ , we see that there is some number  $k_1 > 0$  such that

$$k_1(s_0 - s_1) > \operatorname{cosec} F \cotg \frac{\theta}{2} - \operatorname{cosec} E \cotg \frac{\psi}{2}.$$

Adding similar results for  $s_0 - s_2$  and  $s_0 - s_3$ , we have

$$k_1(s_0 - s_1) + k_2(s_0 - s_2) + k_3(s_0 - s_3) > 0,$$

so that  $s_0$  is greater than at least one of the  $s_i$ .

This proof is due to L. A. G. Dresel.

**REMARK.** Inequality (1) appeared as problem 4964 in the Amer. Math. Monthly 68 (1961), 384 (E. Trost and A. Bager). However, before that Debrunner in 1956 and Oppenheim in 1960 posed the question of validity of inequality (1). A proof of (1) was given by L. A. G. Dresel, Nabla (Bull. of the Malayan Math. Soc.), 8 (1961), 97–99. A proof of R. Breusch was published in the Amer. Math. Monthly 69 (1962), 672.

Inequality (1) was also proved by:

F. H. Croft, Math. Gaz. 49 (1965), 45–49.

W. A. Zalgaller, Matematyka, Warsaw, 19 (1966), 49–53.

E. Szekeres, Elem. Math. 22 (1967), 17–18.

**9.3** Let  $r_0, r_1, r_2, r_3$  denote the inradii of  $DEF$ ,  $AEF$ ,  $BDF$ ,  $CED$  respectively, where  $DEF$  is defined in 9.1. Then

$$r_0 \geq \min(r_1, r_2, r_3).$$

Equality holds if and only if  $D, E, F$  are the midpoints of the sides of  $ABC$ .

**PROOF.** With the notations of 9.2, we have

$$r_0 \left( \cotg \frac{E}{2} - \cotg \frac{F}{2} \right) = EF = r_1 \left( \cotg \frac{E+\theta}{2} + \cotg \frac{F-\psi}{2} \right).$$

Hence  $r_0 - r_1$  has the same sign as

$$\begin{aligned} & \cotg \frac{E+\theta}{2} + \cotg \frac{F-\psi}{2} - \cotg \frac{E}{2} - \cotg \frac{F}{2} \\ &= \sin \frac{\psi}{2} \operatorname{cosec} \frac{F}{2} \cos \frac{F-\psi}{2} - \sin \frac{\theta}{2} \operatorname{cosec} \frac{E}{2} \operatorname{cosec} \frac{E+\theta}{2}. \end{aligned}$$

The argument is now the same as before in cases 1° and 2° (see: 9.2). In case 3°, where  $\varphi, \psi, \theta$  are either all positive or negative, we find that  $r_0 - r_1$  has the same sign as

$$\begin{aligned} & \operatorname{cosec} \frac{F}{2} \operatorname{cosec} \frac{\theta}{2} \sin \frac{E+\theta}{2} - \operatorname{cosec} \frac{E}{2} \operatorname{cosec} \frac{\psi}{2} \sin \frac{F-\psi}{2} \\ &= \left( \sin \frac{E}{2} \cotg \frac{\theta}{2} + \cos \frac{E}{2} \right) \operatorname{cosec} \frac{F}{2} \\ &\quad - \left( \sin \frac{F}{2} \cotg \frac{\psi}{2} - \cos \frac{E}{2} \right) \operatorname{cosec} \frac{E}{2} \\ &> \left( \operatorname{cosec}^2 \frac{F}{2} \cotg \frac{\theta}{2} - \operatorname{cosec}^2 \frac{E}{2} \cotg \frac{\psi}{2} \right) \sin \frac{E}{2} \sin \frac{F}{2}. \end{aligned}$$

Since the terms within brackets, when added to similar terms, cancel each other out, we see that there are positive numbers  $k_i$  such that

$$k_1(r_0 - r_1) + k_2(r_0 - r_2) + k_3(r_0 - r_3) > 0,$$

so that  $r_0$  is greater than at least one of the  $r_i$ .

We note that in case 3° our proof does not tell us whether one and the same  $i$  gives  $s_0 > s_i$  and  $r_0 > r_i$ .

L. A. G. Dresel, Nábla (Bull. of the Malayan Math. Soc.), 2 (1961), 97–99.

**9.4** If  $ABC$  and  $DEF$  are acute triangles, where  $DEF$  is defined in 9.1, then

$$\min(R_1, R_2, R_3) \leq R_0 \leq \max(R_1, R_2, R_3),$$

where  $R_1, R_2, R_3$  are the circumradii of the three corner triangles and  $R_0$  the circumradius of the central triangle  $DEF$ .

Equalities occur if and only if  $DEF$  is similar to  $ABC$ .

**PROOF.** We have

$$FE = 2R_1 \sin \alpha = 2R_0 \sin D.$$

Therefore

$$\frac{R_0}{R_1} = \frac{\sin \alpha}{\sin D}.$$

Similarly

$$\frac{R_0}{R_2} = \frac{\sin \beta}{\sin E}, \quad \frac{R_0}{R_3} = \frac{\sin \gamma}{\sin F}.$$

Since  $\alpha + \beta + \gamma = \pi = D + E + F$ , it follows that at least one of the ratios  $\alpha:D$ ,  $\beta:E$ ,  $\gamma:F$  is  $\geq 1$ , and at least one of these ratios is  $\leq 1$ . If all the angles are acute, the same is true of the ratios of the sides, and hence of the ratios  $R_0:R_1$ ,  $R_0:R_2$ ,  $R_0:R_3$ .

Therefore

$$\min(R_1, R_2, R_3) \leq R_0 \leq \max(R_1, R_2, R_3),$$

and equalities hold if and only if  $ABC$  and  $DEF$  are similar.

A. Oppenheim-L. A. G. Dresel, Nabla 7 (1960), 175 and 8 (1961), 72.

**9.5** Let  $P$  be a point inside a circle concentric with the circumscribed circle and with radius  $< R\sqrt{3}$ . If  $D$ ,  $E$ ,  $F$  denote the orthogonal projections of  $P$  on the sides of this triangle, then

$$4 \cdot \text{area } DEF \leq F. \quad (1)$$

Equality holds if and only if  $P$  is the circumcentre of the triangle or if  $P$  lies on a circle with centre  $O$  and radius  $\pm\sqrt{2}$ .

**PROOF.** If  $PO = p$  then according to a theorem which was given by Gergonne (1823):

$$\text{area } DEF = \frac{1}{2}(R^2 - p^2) \sin \alpha \sin \beta \sin \gamma.$$

For  $p = R$  the area  $DEF$  is zero; for  $p > R$  area is negative. If we consider positive areas only, then

$$\text{area } DEF = \frac{|R^2 - p^2|}{4R^2} F.$$

From this follows (1).

**9.6** Among all the triangles inscribed in a given acute-angled triangle, the triangle that has for its vertices the feet of the altitudes of the given triangle has the least perimeter.

**REMARK.** In 1775 in the Acta Eruditorum, J. F. de Fuschiis a Fagnano proved this theorem using differential calculus. Other

proofs, which were geometric in nature, appeared later, a most ingenious and elegant one being the proof given by H. A. Schwarz. Several proofs of Fagnano's theorem, including Schwarz's proof are given in: N. A. Court, Scripta Math. 17 (1951), 147–150 and 18 (1952), 95–96. In 1930, L. Fejér proved the theorem in a very simple way.

9.7 Let  $D, E, F$  be the feet of the altitudes of an acute-angled triangle  $ABC$ . If  $p$  denotes the perimeter of the triangle  $DEF$ , then

$$s \geq p.$$

Equality holds only for the equilateral triangle.

PROOF. If  $D, E, F$  be the feet of the altitudes of the triangle  $ABC$ , then  $AF = b \cdot \cos \alpha$ ,  $AE = c \cdot \cos \alpha$ , so that

$$EF^2 = AF^2 + AE^2 - 2 \cdot AF \cdot AE \cdot \cos \alpha = a^2 \cos^2 \alpha,$$

i.e.,

$$EF = a \cdot \cos \alpha.$$

Similarly

$$FD = b \cdot \cos \beta \text{ and } DE = c \cdot \cos \gamma.$$

Therefore we have

$$p = EF + FD + DE = a \cdot \cos \alpha + b \cdot \cos \beta + c \cdot \cos \gamma. \quad (1)$$

Since

$$a = 2R \sin \alpha, \quad b = 2R \sin \beta, \quad c = 2R \sin \gamma, \quad (2)$$

(1) becomes

$$\begin{aligned} p &= R(\sin 2\alpha + \sin 2\beta + \sin 2\gamma) \\ &= 4R \sin \alpha \sin \beta \sin \gamma, \end{aligned}$$

or, on the basis of (2)

$$p = 4R \cdot \frac{a}{2R} \cdot \frac{b}{2R} \cdot \frac{c}{2R} = \frac{2}{R} \cdot \frac{abc}{4R} = \frac{2F}{R},$$

i.e.,

$$p = \frac{2rs}{R}. \quad (3)$$

Using  $R \geq 2r$ , from (3), we obtain  $s \geq p$ .

This proof is due to R. R. Janić.

A. Zirakzadeh, Math. Mag. 39 (1966), 96–99.

L. Carlitz, Math. Mag. 39 (1966), 289.

**9.8** Let  $D, E, F$  be the points in which the bisectors of angles  $\alpha, \beta, \gamma$ , meet the sides of a triangle  $ABC$ . Then

$$4 \cdot \text{area } DEF \leq F.$$

Equality holds only for the equilateral triangle.

V. Gridasov, Matematika i fizika, Sofia, 6 (1965), 52–53.

**9.9** Let  $p$  denote the perimeter of the triangle whose vertices are the points of contact with the incircle of the triangle  $ABC$ . Then

$$p \geq 6r \sqrt[3]{\frac{s}{4R}}.$$

PROOF. Let  $D, E, F$  be the points of contact with the incircle of the triangle  $ABC$ . Then

$$EF = 2(s-a) \sin \frac{\alpha}{2}, \quad FD = 2(s-b) \sin \frac{\beta}{2}, \quad DE = 2(s-c) \sin \frac{\gamma}{2}.$$

By means of the arithmetic-geometric mean inequality, we obtain

$$\begin{aligned} EF + FD + DE &= 2 \left[ (s-a) \sin \frac{\alpha}{2} + (s-b) \sin \frac{\beta}{2} + (s-c) \sin \frac{\gamma}{2} \right] \\ &\geq 6 \sqrt[3]{(s-a)(s-b)(s-c) \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}} \\ &= 6r \sqrt[3]{\frac{s}{4R}}. \end{aligned}$$

Ž. Živanović, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 181–No. 196 (1967), 69–72.

**9.10** If  $D, E, F$  are the points on the sides of the triangle  $ABC$ , such that the sides of the triangle  $DEF$  are parallel to the alti-

tudes of the triangle  $ABC$ , then

$$\text{area } DEF \geq \frac{16}{81R^4} F^3.$$

Ž. Živanović, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 181-No. 196 (1967), 69-72.

**9.11** If  $D, E, F$  are the points on the sides  $BC, CA, AB$  respectively of the triangle  $ABC$ , such that  $BD = CE = AF = x$  ( $x \leq \min(a, b, c)$ ), then

$$\text{area } DEF \geq \frac{a^2b(2c-b) + b^2c(2a-c) + c^2a(2b-a)}{32Rs}.$$

Ž. Živanović, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 181-No. 196 (1967), 69-72.

O. Bottema, id. No. 200-No. 209 (1967), 11-12.

**9.12** Let  $DEF$  be defined as in 9.1 and let

$$2a' = b+c, 2b' = c+a, 2c' = a+b,$$

$$p = AE+AF, q = BF+BD, r = CD+CE,$$

$$3t = a+b+c.$$

If  $t \leq p \leq a, t \leq q \leq b$  and  $t \leq r \leq c$ , then

$$\text{per } DEF \geq s,$$

and equality holds if and only if the points  $D, E, F$  are the midpoints of the sides of the triangle  $ABC$ .

B. Bollobás, Can. J. Math. 19 (1967), 523-528.

**9.13**  $P$  is a point inside the triangle  $ABC$ ; the intersection of  $AP, BP$  and  $CP$  and the opposite side is  $D, E$ , and  $F$  respectively. The area of  $DEF$  is  $F_1$ . Then

$$AD \cdot BE \cdot CF \geq 4F_1s.$$

PROOF. If  $x, y, z$  are the barycentric coordinates of  $P$  with respect to  $ABC$  and  $x_1 = x/a, y_1 = y/b, z_1 = z/c$  the distance-

coordinates, then

$$BD = \frac{az}{y+z}, \quad DC = \frac{ay}{y+z}$$

and therefore

$$a \cdot AD^2 = \frac{c^2y^2 + 2bcyz \cos \alpha + b^2z^2}{(y+z)^2}.$$

If  $F_1$  is the area of  $DEF$ , then

$$F_1 = F \left[ 1 - \sum \frac{xy}{(x+z)(y+z)} \right] = \frac{2Fxyz}{(y+z)(z+x)(x+y)}.$$

Hence

$$\begin{aligned} k^2 &= \frac{AD^2 BE^2 CF^2}{F_1^2} = \frac{1}{4F^2 x^2 y^2 z^2} \prod (c^2 y^2 + 2bcyz \cos \alpha + b^2 z^2) \\ &= \frac{a^2 b^2 c^2}{4F^2 x_1^2 y_1^2 z_1^2} \prod (y_1^2 + 2y_1 z_1 \cos \alpha + z_1^2) \\ &= \frac{a^2 b^2 c^2}{4F^2} \prod \left( \frac{y_1}{z_1} + \frac{z_1}{y_1} + 2 \cos \alpha \right). \end{aligned}$$

As  $x_1, y_1, z_1 > 0$ , we have  $y_1/z_1 + z_1/y_1 \geq 2$  and thus

$$k^2 \geq \frac{a^2 b^2 c^2}{4F^2} \prod (2 + 2 \cos \alpha) = \frac{16a^2 b^2 c^2}{F^2} \prod \frac{s(s-a)}{bc} = 16s^2$$

and thus  $k \geq 4s$ .

Equality holds for the incentre.

O. Bottema, Nieuw Arch. Wisk. 14 (1966), 268.

9.14 If  $D, E, F$  the points defined in 9.13, then

$$\frac{AD}{AP} + \frac{BE}{BP} + \frac{CF}{CP} \geq \frac{9}{2}.$$

Ž. Živanović, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 181-No. 196 (1967), 69-72.

9.15 If  $D, E, F$  are the points of contact of the incircle of the triangle  $ABC$  and its sides, then

$$\left( \frac{a}{EF} \right)^2 + \left( \frac{b}{FD} \right)^2 + \left( \frac{c}{DE} \right)^2 \geq 12.$$

Equality holds if and only if the points  $D, E, F$  are the midpoints of the sides, i.e., if the triangle  $ABC$  is equilateral.

E. A. Bokov, Matematika v škole 1954, №. 5, 76.

**9.16** If  $D, E, F$  are the feet of the altitudes of a triangle, then

$$\left(\frac{EF}{a}\right)^2 + \left(\frac{FD}{b}\right)^2 + \left(\frac{DE}{c}\right)^2 \geq \frac{3}{4}. \quad (1)$$

Equality holds if and only if the triangle is equilateral.

PROOF. Starting from (see: 2.21)

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma \geq \frac{3}{4}$$

and putting

$$EF = a |\cos \alpha|, \quad FD = b |\cos \beta|, \quad DE = c |\cos \gamma|,$$

we get (1).

Ju. I. Gerasimov, Matematika v škole 1965, No. 1, 79.

**9.17** If  $F_1$  be the area of the equilateral triangle inscribed in  $ABC$ , then

$$F_1 \geq \frac{2F^2\sqrt{3}}{(a^2+b^2+c^2+4F\sqrt{3})}.$$

Equality holds if and only if  $ABC$  is equilateral.

E. W. Hobson, Problem 4, Treatise on Plane and Advanced Trigonometry, 1957, p. 211.

## 10. Inequalities involving elements of two triangles

**10.1** If  $A_1, B_1, C_1$  are the second points of intersection of the angle-bisectors and the circumcircle of the triangle  $ABC$ , then

$$\text{area } A_1B_1C_1 \geq \text{area } ABC,$$

where equality occurs only for the equilateral triangle.

M. S. Klamkin, Math. Teacher 60 (1967), 323–325.

**10.2** If  $A_1, B_1, C_1$  are the second points of intersection of the medians and the circumcircle of the triangle  $ABC$ , then

$$\text{area } A_1B_1C_1 \geq \text{area } ABC.$$

Equality holds if and only if the triangle is equilateral.  
This result is due to R. R. Janić.

**10.3** If  $F_1$  is the area of a triangle with sides  $\sqrt{a}$ ,  $\sqrt{b}$ ,  $\sqrt{c}$ , then

$$4F_1^2 \geq \sqrt{3} \cdot F.$$

P. Finsler and H. Hadwiger, Comment. Math. Helv. 10 (1937/38), 316–326.

**10.4** Let  $P$  be a point inside the equilateral triangle  $ABC$ , and let  $D, E, F$  be the points situated symmetrically to  $P$  with respect to the sides  $BC, CA, AB$ . Then

$$\text{area } DEF \leq \text{area } ABC.$$

Equality holds if and only if  $P$  is the centre of the triangle  $ABC$ .

PROOF. Since

$$\text{area } ABC = \frac{\sqrt{3}}{3} (r_1 + r_2 + r_3)^2,$$

$$\text{area } DEF = \sqrt{3}(r_2 r_3 + r_3 r_1 + r_1 r_2),$$

we obtain

$$\text{area } ABC - \text{area } DEF = \frac{\sqrt{3}}{3} [(r_2 - r_3)^2 + (r_3 - r_1)^2 + (r_1 - r_2)^2] \geq 0.$$

Z. Živanović, Univ. Beograd. Pušl. Elektrotehn. Fak. Ser. Mat. Fiz. No. 181-No. 195 (1967), 65–72.

**10.5** If  $a, b, c$  and  $a', b', c'$  are the sides of two triangles  $ABC$  and  $A'B'C'$  inscribed in the same circle such that  $AA' \parallel BC$ ,  $BB' \parallel CA$ ,  $CC' \parallel AB$ , then

$$\frac{a' + b' + c'}{a + b + c} \leq 1, \quad \frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c} \leq 3.$$

Equalities hold only if the triangle  $ABC$  is equilateral.

These inequalities are due to H. Demir.

**10.6** Let the triangles  $A_1A_2A_3$  and  $B_1B_2B_3$  be inscribed in a circle of radius  $R$ . Let  $T_1$  and  $T_2$  be the centroids of these triangles respectively. Then

$$T_1T_2 < \frac{1}{3}(4R + \min A_iB_j) \quad (i, j = 1, 2, 3).$$

E. Jucovič, Problem 353, Elem. Math. 15 (1960), 85–87.

**10.7** Let the triangles  $A_1A_2A_3$  and  $B_1B_2B_3$ , with the orthocentres  $H_1$  and  $H_2$  respectively, be inscribed in a circle of radius  $R$ . Then

$$H_1H_2 < 4R + \min A_iB_j \quad (i, j = 1, 2, 3).$$

E. Jucovič, Problem 352, Elem. Math. 15 (1960), 85–87.

**10.8** Let  $A_1B_1C_1$  be a triangle with sides  $a_1, b_1, c_1$  and let  $A_2B_2C_2$  be another triangle with sides  $a_2, b_2, c_2$ . Let  $F_1$  and  $F_2$  be their areas respectively. Then

$$\begin{aligned} a_1^2(-a_2^2 - b_2^2 + c_2^2) + b_1^2(a_2^2 - b_2^2 + c_2^2) + c_1^2(a_2^2 + b_2^2 - c_2^2) \\ \geq 16F_1F_2, \end{aligned}$$

with equality if and only if the triangles  $A_1B_1C_1$  and  $A_2B_2C_2$  are similar.

**PROOF.** Let  $A_3$  be the point such that the triangles  $A_3B_1C_1$  and  $A_2B_2C_2$  are directly similar. By virtue of the equalities

$$\frac{B_1C_1}{a_2} = \frac{C_1A_3}{b_2} = \frac{A_3B_1}{c_2},$$

implied from the assumed similarity, applying the law of cosines to the triangle  $A_3C_1A_1$ , we get

$$\begin{aligned} a_2^2(A_1A_3)^2 &= a_2^2b_1^2 + a_1^2b_2^2 - 2a_1a_2b_1b_2 \cos(C_1 - C_2) \\ &= a_2^2b_1^2 + a_1^2b_2^2 - 2a_1a_2b_1b_2 \cos C_1 \cos C_2 \\ &\quad - 2a_1a_2b_1b_2 \sin C_1 \sin C_2 \geq 0. \end{aligned}$$

Since

$$a_1a_2b_1b_2 \sin C_1 \sin C_2 = 4F_1F_2,$$

we obtain

$$a_2^2b_1^2 + a_1^2b_2^2 - 2a_1a_2b_1b_2 \cos C_1 \cos C_2 \geq 8F_1F_2,$$

i.e.,

$$a_1^2 b_1^2 + a_1^2 b_2^2 - \frac{1}{2}(b_1^2 + a_1^2 - c_1^2)(b_2^2 + a_2^2 - c_2^2) \geq 8F_1 F_2.$$

The last inequality is equivalent to the one we had to prove.

**REMARK.** D. Pedoe has proposed this inequality as a problem E 152 in the Amer. Math. Monthly 70 (1963), 92 and 1012.

D. Pedoe, Proc. Cambridge Philos. Soc. 38 (1942), 357–358.

D. Pedoe, Math. Gaz. 26 (1942), 202–208.

**10.9** Let the triangle  $ABC$  be divided by the straight line  $BD$  in two triangles. If  $r_1, r_2, r$  are the radii of incircles of triangles  $ABD, CBD, ABC$ , then

$$r_1 + r_2 > r.$$

**PROOF.** Since

$$2s = AB + BC + CA > AB + BD + DC = 2s_1,$$

$$2s = AB + BC + CA > CB + BD + DC = 2s_2,$$

we have

$$r_1 + r_2 = \frac{F_1}{s_1} + \frac{F_2}{s_2} > \frac{F_1}{s} + \frac{F_2}{s} = \frac{F}{s} = r,$$

where  $F_1, F_2, F$  are the areas of the triangles  $ABD, BCD, ABC$ , respectively.

**10.10** Let a circle be inscribed in a triangle  $ABC$  and let a triangle  $A_1B_1C_1$  be inscribed in the same circle. If  $s$  is the semi-perimeter of the triangle  $ABC$  and  $s_1$  that of  $A_1B_1C_1$ , then

$$s \geq 2s_1.$$

Equality occurs if and only if both triangles are equilateral.

$$10.11 \quad 4F \leq \text{area } I_a I_b I_c \leq \left(\frac{R}{r}\right)^2 F.$$

Inequalities hold if and only if the triangle  $ABC$  is equilateral.

M. S. Klamkin, Math. Teacher 60 (1967), 323–328.

**10.12** Suppose that  $A_i B_i C_i$  ( $i = 1, 2$ ) are triangles with sides  $a_i, b_i, c_i$ , area  $F_i$  and altitudes  $p_i, q_i, r_i$ . Define numbers  $a_3, b_3, c_3$

by  $a_3 = (a_1^2 + a_2^2)^{1/2}$ , etc. Then

1°.  $a_3, b_3, c_3$  are the sides of a triangle;

2°.  $p_3^2 \geq p_1^2 + p_2^2$ ,  $q_3^2 \geq q_1^2 + q_2^2$ ,  $r_3^2 \geq r_1^2 + r_2^2$ ,

equality occurring in all three if and only if the original two triangles are similar;

3°.  $F_3 \geq F_1 + F_2$ ,

with equality if and only if the triangles are similar;

4°.  $F_3^2 \geq 4 \cdot F_1 F_2$ ,

with equality if and only if the triangles are congruent.

**PROOF.** We give the Nolan's proof without any change. Taking

$$(ux + vy)^2 \leq (u^2 + v^2)(x^2 + y^2), \quad (1)$$

with equality if and only if  $vx = uy$ .

$$\begin{aligned} 1^\circ. \quad a_3^2 &= a_1^2 + a_2^2 < (b_1 + c_1)^2 + (b_2 + c_2)^2 \\ &= b_3^2 + c_3^2 + 2(b_1 c_1 + b_2 c_2). \end{aligned}$$

But  $b_1 c_1 + b_2 c_2 \leq (b_1^2 + b_2^2)^{1/2} (c_1^2 + c_2^2)^{1/2} = b_3 c_3$  by (1).

Therefore  $a_3^2 < (b_3 + c_3)^2$ , i.e.  $a_3 < b_3 + c_3$ , the symmetry of which gives the result.

2°. The cosine formula gives

$$c_3 a_3 \cos \beta_3 = c_1 a_1 \cos \beta_1 + c_2 a_2 \cos \beta_2.$$

Squaring, applying (1), and dividing by  $a_3^2 = a_1^2 + a_2^2$ , we obtain

$$c_3^2 \cos^2 \beta_3 \leq c_1^2 \cos^2 \beta_1 + c_2^2 \cos^2 \beta_2.$$

Subtraction gives

$$c_3^2 \sin^2 \beta_3 \geq c_1^2 \sin^2 \beta_1 + c_2^2 \sin^2 \beta_2,$$

which is  $p_3^2 \geq p_1^2 + p_2^2$  as required. Similarly for  $q_3^2$  and  $r_3^2$ . Equality holds for  $p_3^2$  if and only if  $a_1/a_2 = c_1 \cos \beta_1/c_2 \cos \beta_2$ , similarly for  $r_3^2$  if and only if  $c_1/c_2 = a_1 \cos \beta_1/a_2 \cos \beta_2$ , giving

$$\cos \beta_1 = \cos \beta_2, \quad a_1/a_2 = c_1/c_2.$$

Therefore for equality for one altitude, similarity is sufficient but not necessary (e.g. both triangles isosceles is sufficient). Equality for two or three altitudes holds if and only if the original triangles are similar.

$$\begin{aligned} 3^\circ. \quad & 2(F_1 + F_2) = p_1 a_1 + p_2 a_2 \\ & \leq (p_1^2 + p_2^2)^{1/2}(a_1^2 + a_2^2)^{1/2} \\ & \leq p_3 a_3 = 2 \cdot F_3, \end{aligned}$$

as required, equality occurring if and only if  $a_1 \cos \beta_1 / c_2 \cos \beta_2 = a_1 / a_2$  as in  $2^\circ$ , and  $a_1 / a_2 = p_1 / p_2 = c_1 \sin \beta_1 c_2 \sin \beta_2$ , giving  $\tan \beta_1 = \tan \beta_2$ , etc., i.e. the triangles are similar.

$$4^\circ. \quad F_3^2 \geq (F_1 + F_2)^2 = (F_1 - F_2)^2 + 4F_1 F_2 \geq 4F_1 F_2,$$

with equality if and only if the triangles are similar, as in  $3^\circ$ , and  $F_1 = F_2$ , i.e. the triangles are congruent.

A. Oppenheim-R. P. Nolan, Problem 502 Amer. Math. Monthly 70 (1963), 444 and 71 (1964), 444.

**10.13** For two triangles with sides  $a, b, c$  and  $n_a, m_b, m_c$ , whose angles are  $\alpha, \beta, \gamma$  and  $\alpha_m, \beta_m, \gamma_m$  respectively, the following implications are valid:

$$\begin{aligned} a > b > c & \Rightarrow m_a < m_b < m_c; \\ \alpha > \beta > \gamma & \Rightarrow \alpha_m < \beta_m < \gamma_m; \\ \alpha > \alpha_m & \Rightarrow \gamma < \gamma_m; \\ \alpha > \beta_m & \Rightarrow \beta < \gamma_m; \\ \beta > \alpha_m & \Rightarrow \gamma < \beta_m. \end{aligned}$$

L. Toscano, Archimede 8 (1956), 278-279.

## 11. Special triangles

**11.1** In each acute triangle two angles exist whose difference is less than or equal to  $\pi/6$ .

**11.2** For each acute triangle,

$$\begin{aligned} -\cot 2\alpha - \cot 2\beta - \cot 2\gamma &\geq \cot \alpha + \cot \beta + \cot \gamma \\ &\geq \operatorname{tg} \frac{\alpha}{2} + \operatorname{tg} \frac{\beta}{2} + \operatorname{tg} \frac{\gamma}{2} \\ &\geq \operatorname{tg} \frac{\pi-\alpha}{4} + \operatorname{tg} \frac{\pi-\beta}{4} + \operatorname{tg} \frac{\pi-\gamma}{4} \\ &\geq \sqrt{3}. \end{aligned}$$

Equalities hold if and only if the triangle is equilateral.

C. Ionescu-Tiu, *Găz. Mat. B* 14 (1963), 560.

**11.3** A necessary and sufficient condition for a triangle to be acute-angled is

$$\operatorname{tg} \alpha \cdot \operatorname{tg} \beta > 1.$$

G. Piskarev, *Matematika v škole*, 1952, No. 5, 93 and 1963, No. 3, 88.

**11.4** For each acute triangle,

$$\begin{aligned} \operatorname{tg} \alpha (\cot \beta - \cot \gamma) + \operatorname{tg} \beta (\cot \gamma + \cot \alpha) \\ + \operatorname{tg} \gamma (\cot \alpha + \cot \beta) \geq 6. \end{aligned} \quad (1)$$

**PROOF.** According to our assumption,

$$\operatorname{tg} \alpha > 0, \operatorname{tg} \beta > 0, \operatorname{tg} \gamma > 0.$$

Using

$$x + \frac{1}{x} \geq 2 \quad (x > 0)$$

for  $x = \operatorname{tg} \alpha / \operatorname{tg} \beta$ , etc., and then adding these inequalities, we obtain

$$\frac{\operatorname{tg} \alpha}{\operatorname{tg} \beta} + \frac{\operatorname{tg} \beta}{\operatorname{tg} \alpha} - \frac{\operatorname{tg} \beta}{\operatorname{tg} \gamma} + \frac{\operatorname{tg} \gamma}{\operatorname{tg} \beta} + \frac{\operatorname{tg} \alpha}{\operatorname{tg} \gamma} + \frac{\operatorname{tg} \gamma}{\operatorname{tg} \alpha} \geq 6,$$

whence (1).

**11.5** For each acute triangle

$$\operatorname{tg} \alpha \operatorname{tg} \beta \operatorname{tg} \gamma \geq 3\sqrt{3}. \quad (1)$$

Equality holds if and only if the triangle is equilateral.

PROOF. Starting from the identity

$$\operatorname{tg} \alpha \operatorname{tg} \beta \operatorname{tg} \gamma = \operatorname{tg} \alpha + \operatorname{tg} \beta + \operatorname{tg} \gamma - 3,$$

which holds for any triangle, we obtain

$$\operatorname{tg} \alpha \operatorname{tg} \beta \operatorname{tg} \gamma = \operatorname{tg} \alpha + \operatorname{tg} \beta + \operatorname{tg} \gamma \geq 3\sqrt[3]{\operatorname{tg} \alpha \cdot \operatorname{tg} \beta \cdot \operatorname{tg} \gamma},$$

whence

$$\operatorname{tg}^3 \alpha \operatorname{tg}^3 \beta \operatorname{tg}^3 \gamma \geq 27 \operatorname{tg} \alpha \operatorname{tg} \beta \operatorname{tg} \gamma,$$

i.e. (1).

N. Dzigava, Matematika v škole, 1949, No. 4, 60.

11.6 For each acute triangle,

$$\operatorname{tg} \alpha + \operatorname{tg} \beta + \operatorname{tg} \gamma \geq 3\sqrt{3}.$$

Equality holds if and only if the triangle is equilateral.

Math. Notae 6 (1946), 196.

11.7 If  $n$  is a positive integer, then, for each acute triangle,

$$\operatorname{tg}^n \alpha + \operatorname{tg}^n \beta + \operatorname{tg}^n \gamma > 3 + \frac{3}{2}n.$$

Iur. I. Gerasimov, Matematika v škole, 1964, No. 1, 82–83.

11.8 If  $n$  denotes a non-negative real number, then, for each acute triangle

$$\operatorname{tg}^n \alpha + \operatorname{tg}^n \beta + \operatorname{tg}^n \gamma \geq 3 \cdot 3^{n/2}. \quad (1)$$

Equality holds if and only if the triangle is equilateral.

PROOF. On the basis of 11.5, we have

$$\left( \frac{\operatorname{tg}^n \alpha + \operatorname{tg}^n \beta + \operatorname{tg}^n \gamma}{3} \right)^{1/n} \geq (\operatorname{tg} \alpha \operatorname{tg} \beta \operatorname{tg} \gamma)^{1/3} \geq 3^{1/2},$$

whence follows (1).

M. N. Kritikos, Actes du Congrès interbalkanique de mathématiciens, Athènes, 2–9 septembre 1934, 157–158.

REMARK. Inequality (1) in the paper mentioned was proved only for  $n$  an odd, positive integer.

**11.9**  $\sin \alpha + \sin \beta + \sin \gamma + \tan \alpha + \tan \beta + \tan \gamma > 2\pi$  holds for each acute triangle.

Ju. I. Gerasimov, Matematika v škole, 1965, No. 2, 63.

**11.10** For each acute triangle

$$1 + \frac{\sin 2\alpha + \sin 2\beta + \sin 2\gamma}{\sin 4\alpha + \sin 4\beta + \sin 4\gamma} \leq 0. \quad (1)$$

PROOF. By virtue of

$$\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 4 \sin \alpha \sin \beta \sin \gamma$$

and

$$\sin 4\alpha + \sin 4\beta + \sin 4\gamma = -4 \sin 2\alpha \sin 2\beta \sin 2\gamma,$$

inequality (1) becomes

$$1 - \frac{\sin \alpha \sin \beta \sin \gamma}{\sin 2\alpha \sin 2\beta \sin 2\gamma} \leq 0,$$

i.e.

$$\sin 2\alpha \sin 2\beta \sin 2\gamma \leq \sin \alpha \sin \beta \sin \gamma.$$

The last inequality is true by virtue of 2.23.

**11.11**  $\sum r_i(H) \leq \sum r_i(O)$  holds for each acute triangle.

F. Leuenberger, Problem 444, Elem. Math. 18 (1963), 18.

**11.12** For each acute triangle,

$$2 \sum r_i(H) \leq 6r \leq 2 \sum r_i(G) \leq 2 \sum r_i(O) = \sum R_i(H) \leq 3R.$$

L. Bernstein-J. Steinig, Elem. Math. 19 (1964), 870.

**11.13** For each acute triangle

$$6r \leq \sum R_i(I) \leq \sum R_i(H) \leq 3R.$$

L. Bernstein-J. Steinig, Elem. Math. 19 (1964), 870.

**11.14** If  $D, E, F$  are the feet of the altitudes of an acute triangle then

$$AI + BI + CI \geq 2(DH + EH + FH).$$

Equality holds if and only if the triangle is equilateral.

L. Bankoff, Problem E 1564, Amer. Math. Monthly 70 (1963), 210 and 70 (1963), 1101.

11.15 If  $t_a, t_b, t_c$  are the lengths of transversals drawn from the vertices  $A, B, C$  of an acute-angled triangle to the opposite sides, then

$$\frac{1}{2} < \min_{t_a, t_b, t_c} \left\{ \frac{t_a^2 + t_b^2 + t_c^2}{a^2 + b^2 + c^2} \right\} \leq \frac{3}{4},$$

$$1 \leq \max_{t_a, t_b, t_c} \left\{ \frac{t_a^2 + t_b^2 + t_c^2}{a^2 + b^2 + c^2} \right\} < \frac{3}{2}.$$

Equalities hold if and only if the triangle is equilateral.

A. S. B. Holland, Elem. Math. 22 (1967), 49–55.

11.16 If a triangle is non-obtuse, then

$$R + r \leq \max(h_a, h_b, h_c).$$

REMARK. In fact, this is an inequality of P. Erdős. A proof can be found in: Matematika v škole, 1962, No. 6, 87–88.

11.17 If  $c = \min(a, b, c)$ , then, for a non-obtuse triangle, we have

$$\frac{R}{r} \leq \frac{a+b}{c}.$$

Matematika v škole, 1962, No. 6, 87–88.

11.18 For an acute or right-angled triangle

$$4(\cos^2 \beta \cos^2 \gamma + \cos^2 \gamma \cos^2 \alpha + \cos^2 \alpha \cos^2 \beta) \leq \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma,$$

$$4(\cos^2 \beta \cos^2 \gamma + \cos^2 \gamma \cos^2 \alpha + \cos^2 \alpha \cos^2 \beta)$$

$$+ 16 \cos^2 \alpha \cos^2 \beta \cos^2 \gamma \leq 1.$$

Equalities occur when the triangle is equilateral or right-angled isosceles and in no other case.

A. Oppenheim, Problem E 1838, Amer. Math. Monthly 72 (1965), 1129.

11.19 If  $n$  is an integer greater than 2, then for any right triangle,

$$a^n + b^n < c^n.$$

PROOF. Since  $a < c$ ,  $b < c$ , and  $c^2 = a^2 + b^2$ , we o

$$\begin{aligned}c^n &= c^{n-2}c^2 = c^{n-2}(a^2 + b^2) \\&= a^2c^{n-2} + b^2c^{n-2} \\&> a^2a^{n-2} + b^2b^{n-2} \\&= a^n + b^n.\end{aligned}$$

11.20 For any right triangle,

$$a - b \leq c\sqrt{2}.$$

PROOF. Since

$$c^2 = a^2 + b^2 = \frac{(a+b)^2}{2} + \frac{(a-b)^2}{2},$$

we conclude

$$2c^2 \geq (a+b)^2, \text{ i.e. } a+b \leq c\sqrt{2}.$$

Equality in (1) holds only for  $a = b$ .

Matematika v škole, 1965, No. 5, 76.

11.21 For any right triangle,

$$\cos^2 \frac{\alpha - \beta}{2} \geq \frac{2ab}{c^2}.$$

PROOF. Suppose that  $\alpha \neq \beta$ . Then

$$\text{i.e. } \frac{1}{2}(\sin \alpha + \sin \beta) > \sqrt{\sin \alpha \sin \beta},$$

$$\sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} > \sqrt{\sin \alpha \sin \beta}.$$

Since

$$\sin \frac{\alpha + \beta}{2} = \frac{\sqrt{2}}{2}, \quad \sin \alpha = \frac{a}{c}, \quad \sin \beta = \frac{b}{c},$$

we get

$$\frac{\sqrt{2}}{2} \cos \frac{\alpha - \beta}{2} > \sqrt{\frac{ab}{c^2}}.$$

Equality in (1) holds only for  $\alpha = \beta$ .

N. Dzgava, Matematika v škole, 1949, No. 1, 62  
No. 4, 59.

## SPECIAL TRIANGLES

**11.22** For any right triangle

$$\sqrt{2}-1 \leq \frac{r}{h} < \frac{1}{2},$$

where  $h$  is the altitude of the hypotenuse.

Equality holds only for the isosceles triangle.

PROOF. Since

$$F = \frac{1}{2}(a+b+c)r = \frac{1}{2}ch,$$

we get

$$\frac{r}{h} = \frac{c}{a+b+c}.$$

Since  $a+b > c$ , we have

$$\frac{r}{h} < \frac{c}{c+c} = \frac{1}{2}.$$

Furthermore by 11.21,

$$\frac{r}{h} > \frac{c}{c\sqrt{2}+c} = \sqrt{2}-1.$$

**11.23** For any right triangle,

$$R+r \geq \sqrt{2F}.$$

Matematika v škole, 1965, No. 5, 77.

**11.24** If  $\alpha > \pi/2$ , then

$$|\cos \alpha| \leq \frac{a^6}{54b^3c^3}.$$

PROOF. For  $\alpha > \pi/2$  we have  $\cos \alpha < 0$ , so that

$$\cos \alpha = -|\cos \alpha|, \quad a^2 = b^2 + c^2 + 2bc|\cos \alpha|.$$

On the basis of the arithmetic-geometric mean inequality we get

$$\frac{a^2}{3} = \frac{b^2 + c^2 - 2bc|\cos \alpha|}{3} \geq \sqrt[3]{b^2c^2 \cdot 2bc|\cos \alpha|},$$

whence (I) follows.

H. Hamzin, Matematika v škole, 1949, No. 5, 62 and 1950, No. 2, 54.

**11.25** If  $t_a, t_b, t_c$  are the lengths of transversals drawn from the vertices  $A, B, C$  of an obtuse triangle, then

$$\frac{1}{3} < \frac{t_a^2 + t_b^2 + t_c^2}{a^2 + b^2 + c^2} < \frac{3 + \sqrt{3}}{3}.$$

A. S. B. Holland, Elem. Math. 22 (1967), 49–55.

**11.26** A triangle is acute, right or obtuse, depending upon whether the expression

$$(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)(a^2 + b^2 - c^2)$$

is positive, zero or negative.

C. Ciamberlini, Rassegna di Matematica e Fisica (Roma) 5 (1925), 241–244.

**11.27** Depending upon the fact whether a triangle is acute, right or obtuse, one of the following statements hold:

$$1^\circ. s > 2R + r \text{ or } s = 2R + r \text{ or } s < 2R + r,$$

$$2^\circ. \sum \sin^2 \alpha > 2 \text{ or } \sum \sin^2 \alpha = 2 \text{ or } \sum \sin^2 \alpha < 2.$$

$$3^\circ. \sum a^2 > 8R^2 \text{ or } \sum a^2 = 8R^2 \text{ or } \sum a^2 < 8R^2.$$

$$4^\circ. R > 3 \cdot OG \text{ or } R = 3 \cdot OG \text{ or } R < 3 \cdot OG.$$

$$5^\circ. \prod (s - r_a) > 0 \text{ or } \prod (s - r_a) = 0 \text{ or } \prod (s - r_a) < 0.$$

C. Ciamberlini, Boll. Un. Mat. Ital. (2) 5 (1943), 37–41.

**11.28** Whether a triangle is acute, right or obtuse, we have

$$h_a + h_b + h_c \geqslant 2R + 4r + \frac{r^2}{R},$$

respectively.

We were informed of this result by A. Bager.

**12. Inequalities for the distances of a point to the vertices and the sides of a triangle**

**12.1**  $6r \leq AI + BI + CI \leq 3R.$

Equalities hold if and only if the triangle is equilateral.

L. Carlitz, Math. Mag. 36 (1963), 264.

**12.2**  $AI + BI + CI \leq 2(R+r) = AH + BH + CH.$

PROOF. According to 2.56, we have

$$\left( \sum \sin \frac{\alpha}{2} \right)^2 \leq \sum \cos^2 \frac{\alpha}{2}.$$

Hence

$$\begin{aligned} 2 \sum \sin \frac{\alpha}{2} \sin \frac{\beta}{2} &\leq \sum \cos^2 \frac{\alpha}{2} - \sum \sin^2 \frac{\alpha}{2} \\ &= \sum \cos \alpha = 4 \left( \prod \sin \frac{\alpha}{2} \right) + 1. \end{aligned}$$

Then

$$2 \sum \operatorname{cosec} \frac{\alpha}{2} \leq 4 + \prod \operatorname{cosec} \frac{\alpha}{2} = 4 + \frac{4R}{r},$$

or

$$AI + BI + CI \leq 2(R+r).$$

- G. Daniellson, Elementa: matematika, fysika, kemi 36 (1953), 135.

L. Bankoff, Problem E 1397, Amer. Math. Monthly 67 (1960), 82 and 67 (1960), 695.

**12.3** Let  $x, y, z$  be the distances of the circumcentre of a triangle to its sides. Then

$$x_a + x_b + x_c \leq 3(x+y+z).$$

Equality holds if and only if the triangle is equilateral.

F. Leuenberger, Problem E 1573, Amer. Math. Monthly 70 (1963), 331 and 71 (1964), 93.

12.4 If  $ABC$  is an equilateral triangle and  $P$  a point outside its plane,  $PA = p$ ,  $PB = q$ ,  $PC = r$  then

$$(-p+q+r)(p-q+r)(q-q-r) > 0. \quad (1)$$

REMARK. This is a generalization of Pompeiu's theorem. An elegant geometric proof was given by G. R. Veldkamp, Nieuw Tijdschr. Wisk. 44 (1956), 1-4. An analytic one by O. Bottema, ibid. 44 (1956), 183-184, who proved furthermore

$$a^2F^2 \geq 12V^2.$$

Here  $a$  represents the sides of  $ABC$ ,  $F$  the area of the triangle the sides of which are  $PA$ ,  $PB$ ,  $PC$  and  $V$  the volume of the tetrahedron  $PABC$ . In (1) the equality sign holds if  $P$  is on the sphere through  $A$ ,  $B$ ,  $C$  with its centre at the centre of  $ABC$ .

12.5 For any acute triangle

$$9 \frac{AG^2 + BG^2 + CG^2}{AO^2 + BO^2 + CO^2} - \frac{A\bar{E} \cdot BH \cdot CH}{\underline{rI} \cdot BI \cdot CI} \leq 8.$$

Equality holds if and only if the triangle is equilateral.

S. Reich, Problem E 1887, Amer. Math. Monthly 73 (1966), 538.

12.6 Let  $P$  be a point in the interior of the triangle  $ABC$ . If the straight lines  $AP$ ,  $BP$ ,  $CP$  intersect the sides  $BC$ ,  $CA$ ,  $AB$  in  $D$ ,  $E$ ,  $F$  respectively, then

$$PD + PE + PF < \max(a, b, c).$$

PROOF. Assume that  $a > b > c$ . Let  $PX \parallel AB$ ,  $PY \parallel AC$ ,  $XK \parallel PF$ ,  $YL \parallel PE$ , where  $X$  and  $Y$  are points on  $BC$ ,  $K$  a point on  $AB$  and  $L$  a point on  $AC$ . Since  $a$  is greater than  $AD$ ,  $CF$ ,  $BE$  by virtue of the similarity of the triangles  $PXY$  and  $ABC$ ,  $BXK$  and  $BCF$ , and finally  $CYL$  and  $CBE$ , we get, respectively

$$XY > PD, BX > XK = PF, YC > YL = PE.$$

Hence,

$$a = BX + XY + YC > PD + PE + PF.$$

P. Erdős, Problem 3746, Amer. Math. Monthly 42 (1935), 454 and 44 (1937), 400.

**12.7** If the point  $P$  is inside a triangle  $ABC$ , then

$$s < AP + BP + CP < 2s.$$

H. W. Guggenheimer, Plane Geometry and Its Groups, San Francisco, Cambridge, London, Amsterdam 1967, p. 178.

**12.8** If  $a = \max(a, b, c)$ , then

$$R_1 + R_2 - R_3 > b + c.$$

E. G. Gotman, Matematika v škole, 1966, No. 1, 89.

**12.9**  $\min(h_a, h_b, h_c) \leq r_1 + r_2 - r_3 \leq \max(h_a, h_b, h_c)$ .

S. I. Zetel', Zadaci na maksimum i minimum, Moskva 1948, pp. 21-22.

REMARK. A consequence of these inequalities is

$$\min(h_a, h_b, h_c) \leq 3r \leq \max(h_a, h_b, h_c).$$

**12.10**  $(h_a - r_1)(h_b - r_2)(h_c - r_3) \geq 8r_1 r_2 r_3$ .

Equality holds iff and only if the triangle is equilateral and  $P$  is its centre.

L. Carlitz, Amer. Math. Monthly 71 (1964), 881-885.

**12.11**  $h_a h_b h_c \geq 27r_1 r_2 r_3$ .

Equality holds iff and only if the triangle is equilateral and  $P$  is its centre.

L. Carlitz, Amer. Math. Monthly 71 (1964), 881-885.

**12.12**  $\frac{h_a}{r_1} + \frac{h_b}{r_2} + \frac{h_c}{r_3} \geq 9$ .

S. I. Zetel', Zadaci na maksimum i minimum, Moskva 1948, p. 63.

L. Carlitz, Amer. Math. Monthly 71 (1964), 881-885.

**12.13**  $R_1 + R_2 + R_3 \geq 2(r_1 + r_2 + r_3)$ .

Equality holds iff and only if the triangle is equilateral and  $P$  is its centre.

PROOF. Let  $L, M, N$  be the feet of the perpendiculars from  $P$  to  $BC, CA, AB$  respectively. Then

$$MN = (r_2^2 + r_3^2 + 2r_2r_3 \cos \alpha)^{1/2}, \quad MN = R_1 \sin \alpha.$$

We now have, in turn,

$$\begin{aligned} R_1 + R_2 + R_3 &= \sum \frac{(r_2^2 + r_3^2 + 2r_2r_3 \cos \alpha)^{1/2}}{\sin \alpha} \\ &= \sum \frac{[(r_2 \sin \gamma + r_3 \sin \beta)^2 + (r_2 \cos \gamma - r_3 \cos \beta)^2]^{1/2}}{\sin \alpha} \\ &\geq \sum \frac{(r_2 \sin \gamma + r_3 \sin \beta)}{\sin \alpha} \\ &= \sum r_1 \left( \frac{\sin \beta}{\sin \gamma} + \frac{\sin \gamma}{\sin \beta} \right) \\ &\geq 2(r_1 + r_2 + r_3). \end{aligned}$$

P. Erdős-L. J. Mordell, Problem 3740, Amer. Math. Monthly 42 (1935), 396 and 44 (1937), 252-254.

G. R. Veldkamp, Nieuw Tijdschr. Wisk. 45 (1957/58), 193-196.

L. J. Mordell, Math. Gaz. 46 (1962), 213-215.

**12.14** Let  $P$  be any point in the plane of the triangle  $ABC$ . Then

$$R_1 + R_2 + R_3 \geq 6r.$$

M. Schreiber, Aufgabe 196, Jber. Deutsch. Math.-Verein. 45 (1935), 63 kursiv.

J. M. Child, Math. Gaz. 23 (1939), 138-143.

L. Bankoff, Math. Mag. 39 (1966), 69.

**12.15** In any non-equilateral triangle  $T$ , the line through the incentre  $I$  and perpendicular to the line joining  $I$  with the circumcentre  $O$  divides  $T$  and its boundary in two regions. For all points in that region which contains the vertex opposite the triangle's smallest side, we have

$$R_1 + R_2 + R_3 \geq 2(r_1 + r_2 + r_3) \geq 6r.$$

In the second region,

$$R_1 + R_2 + R_3 \geqslant 6r \geqslant 2(r_1 + r_2 + r_3),$$

while on the dividing line,

$$R_1 + R_2 + R_3 \geqslant 2(r_1 + r_2 + r_3) = 6r.$$

J. Steinig, Acta Math. Acad. Sci. Hungar. 16 (1965), 19–22.

$$\begin{aligned} 12.16 \quad R_1 + R_2 + R_3 &\geqslant r_1\left(\frac{b}{c} + \frac{c}{b}\right) + r_2\left(\frac{a}{c} + \frac{c}{a}\right) + r_3\left(\frac{a}{b} + \frac{b}{a}\right) \\ &\geqslant 2(r_1 + r_2 + r_3). \end{aligned}$$

Short and elegant proofs were given by Veldkamp and by Eggleston.

PROOF. Let  $P'$  be the reflection of  $P$  in the internal bisector of angle  $BAC$ . Then  $P'$  is distant  $R_1$  from  $A$ ,  $r_3$  from the side  $AC$ , and  $r_2$  from side  $AB$ . The triangles  $ABP'$ ,  $ACP'$  do not meet except along their common side  $AP'$ . We have

$$\begin{aligned} \frac{1}{2}aR_1 &\geqslant \text{area } ABP'C \\ &= \text{area } ABP' + \text{area } ACP' \\ &= \frac{1}{2}r_2c + \frac{1}{2}r_3b. \end{aligned}$$

Hence

$$R_1 \geqslant r_2\left(\frac{c}{a}\right) + r_3\left(\frac{b}{a}\right),$$

and equality holds if and only if  $AP'$  is perpendicular to  $BC$ , that is, if

$$\angle BAP = \angle ABC + \angle BAC - \frac{\pi}{2}.$$

By reflecting  $P$  in the internal bisectors of angles  $CBA$  and  $BCA$  we obtain

$$R_2 \geqslant r_1\left(\frac{c}{b}\right) + r_3\left(\frac{a}{b}\right), \quad R_3 \geqslant r_1\left(\frac{b}{c}\right) + r_2\left(\frac{a}{c}\right).$$

Equality holds in the latter of these inequalities if and only if

$$\not\propto PBA = \not\propto ABC + \not\propto BAC - \frac{\pi}{2}.$$

Thus if equality occurs in the first and third inequality then  $AP = BP$ , and equality occurs in all three if and only if  $P$  is the circumcentre of a triangle  $ABC$ . Adding these inequalities we obtain

$$R_1 + R_2 + R_3 \geq r_1 \left( \frac{b}{c} + \frac{c}{b} \right) + r_2 \left( \frac{c}{a} + \frac{a}{c} \right) + r_3 \left( \frac{a}{b} + \frac{b}{a} \right).$$

Since  $b/c + c/b \geq 2$ , etc., we have

$$r_1 \left( \frac{b}{c} + \frac{c}{b} \right) + r_2 \left( \frac{c}{a} + \frac{a}{c} \right) + r_3 \left( \frac{a}{b} + \frac{b}{a} \right) \geq 2(r_1 + r_2 + r_3)$$

and equality holds if and only if  $a = b = c$  and  $P$  is the centre of the triangle  $ABC$ .

J. M. Child, Math. Gaz. 23 (1939), 138–143.

D. K. Kazarinoff, Michigan Math. J. 4 (1957), 97–98.

G. R. Veldkamp, Nieuw Tijdschr. Wisk. 45 (1957/58), 193–196.

H. G. Eggleston, Math. Gaz. 42 (1958), 54–55.

Z. A. Skopec, Matematičeskoje prosveščenie 5 (1960), 151–152.

**12.17** Let  $P$  be a point within the triangle  $AEC$  and let  $D, E, F$  be its orthogonal projections on sides  $BC, CA, AB$  respectively, then

$$\begin{aligned} R_1 + R_2 + R_3 &\geq r_1 \left( \frac{FD \cdot PC}{DE \cdot PB} + \frac{DE \cdot PB}{FD \cdot PC} \right) + r_2 \left( \frac{DE \cdot PA}{FE \cdot PC} + \frac{FE \cdot PC}{DE \cdot PA} \right) \\ &\quad + r_3 \left( \frac{FD \cdot PA}{FE \cdot PB} + \frac{FE \cdot PB}{FD \cdot PA} \right) \\ &\geq 2(r_1 + r_2 + r_3). \end{aligned}$$

L. Bankoff, Amer. Math. Monthly 65 (1958), 521.

REMARK. This theorem and 12.16 interpolate an expression 12.13.

$$12.18 \quad R_1 + R_2 + R_3 \geq 2\sqrt{F\sqrt{3}}. \quad (1)$$

Equality occurs if and only if the triangle is equilateral and  $P$  is its centre.

**PROOF.** Let  $BT \parallel CS \parallel PA$  and  $BT = CS = PA$ ,  $CR \parallel AT \parallel PB$  and  $CR = AT = PB$ ,  $AS \parallel BR \parallel PC$  and  $AS = BR = PC$ . Then the hexagon  $ATBRCS$  has the area  $2F$  and the perimeter  $2(R_1 + R_2 + R_3)$ . Between all hexagons of given area the regular hexagon has the smallest perimeter. Therefore

$$R_1 + R_2 + R_3 \geq 2\sqrt{F\sqrt{3}}.$$

The hexagon  $ATBRCS$  will be regular only if  $P$  is the centre of an equilateral triangle.

U. T. Bödewadt, Jber. Deutsch. Math.-Verein. 46 (1936), 7 kursiv.

**REMARK 1.** Inequality also holds when the point lies anywhere in the plane of the triangle considered.

**REMARK 2.** Inequality (1) is stronger than 12.14, because (see: 5.11)

$$6r \leq 2\sqrt{F\sqrt{3}}.$$

$$12.19 \quad aR_1 + bR_2 + cR_3 \geq 2(ar_1 + br_2 + cr_3).$$

G. Steensholt, Amer. Math. Monthly 63 (1956), 571–572.

**REMARK.** J. Schopp has proved a similar result for  $n$ -dimensional simplex. See Amer. Math. Monthly 66 (1959), 896–897.

$$12.20 \quad R_1 \sin \frac{\alpha}{2} + R_2 \sin \frac{\beta}{2} + R_3 \sin \frac{\gamma}{2} \geq r_1 + r_2 + r_3. \quad (1)$$

Equality occurs if and only if  $P$  is the incentre of the triangle.

**PROOF.** Since

$$r_1 = R_3 \sin \angle PCB \text{ and } r_2 = R_2 \sin \angle PCA,$$

we have

$$r_1 + r_2 = R_3(\sin \angle PCB + \sin \angle PCA) \leq 2R_3 \sin \frac{\gamma}{2}. \quad (2)$$

By virtue of two analogous relations valid for  $r_2+r_3$  and  $r_3+r_1$ , we obtain the inequality (1).

Equality in (2) holds if and only if  $\angle PCB = \angle PCA$ , i.e., only if  $P$  is a point of the bisector of the angle  $ACB$ . Therefore in (1) equality holds if and only if  $P$  is the incentre of the triangle.

L. Carlitz, Elem. Math. 21 (1966), 115.

$$12.21 \quad R_2R_3 - R_3R_1 + R_1R_2 \geq 4(r_2r_3 + r_3r_1 + r_1r_2).$$

J. M. Child, Math. Gaz. 23 (1939), 138–143.

A. Oppenheim, Amer. Math. Monthly 68 (1961), 226–230.

$$12.22 \quad R_2R_3 - R_3R_1 + R_1R_2$$

$$\geq (r_1+r_2)(r_3+r_1) + (r_2+r_3)(r_1+r_2) + (r_3+r_1)(r_2+r_3).$$

Equality holds only for the equilateral triangle, the point  $P$  being its centre.

A. Oppenheim, Amer. Math. Monthly 68 (1961), 226–230.

A. Oppenheim, Ann. Math. Pura Appl. (4) 53 (1961), 157–164.

$$12.23 \quad R_1^{-1} - R_2^{-1} + R_3^{-1} \leq 2^{-1}(r_1^{-1} + r_2^{-1} + r_3^{-1}).$$

Equality holds if and only if the triangle is equilateral and if  $P$  is its centre.

J. M. Child, Math. Gaz. 23 (1939), 138–143.

A. Oppenheim, Amer. Math. Monthly 68 (1961), 226–230.

$$12.24 \quad R_1^t + R_2^t + R_3^t < 2(r_1^t + r_2^t + r_3^t),$$

for  $t < -1$ .

$$R_1^t + R_2^t + R_3^t \leq 2^t(r_1^t + r_2^t + r_3^t),$$

for  $-1 \leq t \leq 1$ .

$$R_1^t + R_2^t + R_3^t \geq 2^t(r_1^t + r_2^t + r_3^t),$$

for  $0 \leq t \leq 1$ .

$$R_1^t + R_2^t + R_3^t > 2(r_1^t + r_2^t + r_3^t),$$

for  $t > 1$ .

Equality occurs only for the equilateral triangle, the point  $P$  being its centre.

A. Florian, Elem. Math. 13 (1959), 55–58.

A. Oppenheim, Amer. Math. Monthly 68 (1961), 226–230.

$$12.25 \quad R_1 R_2 R_3 \geq 8r_1 r_2 r_3.$$

J. M. Child, Math. Gaz. 23 (1939), 138–143.

J. Birkes, Elem. Math. 12 (1957), 121–123.

A. Oppenheim, Amer. Math. Monthly 68 (1961), 226–230.

$$12.26 \quad R_1 R_2 R_3 S \geq r_1 r_2 r_3,$$

with  $S = \sin \alpha/2 \sin \beta/2 \sin \gamma/2$ .

Equality holds if and only if  $P$  coincides with  $I$ .

O. Bittema, Nieuw Tijdschr. Wisk. 53 (1965), 50–51.

REMARK. Since  $S \leq \frac{1}{2}$  (see: 2.12), this inequality improves 12.25.

$$12.27 \quad R_1 R_2 R_3 \geq (r_2 + r_3)(r_3 + r_1)(r_1 + r_2).$$

Equality holds if and only if the triangle is equilateral and  $P$  being its centre.

A. Oppenheim, Problem E 1433, Amer. Math. Monthly 67 (1960), 502 and 68 (1961), 380.

A. Oppenheim, Ann. Math. Pura Appl. (4) 53 (1961), 157–163.

$$12.28 \quad \frac{1}{2} S R_1 R_2 R_3 \geq (r_2 + r_3)(r_3 + r_1)(r_1 + r_2), \quad (1)$$

with  $S = \sin \alpha/2 \sin \beta/2 \sin \gamma/2$ .

Equality occurs if and only if  $P$  is incentre of the triangle.

PROOF. Applying the law of sines and the law of cosines to the triangle  $AMN$ , where  $M$  and  $N$  are defined in 12.13, we obtain

$$\begin{aligned} R_1^2 \sin^2 \alpha &= r_2^2 + r_3^2 - 2r_2 r_3 \cos \alpha \\ &= (r_2 + r_3)^2 \cos^2 \frac{\alpha}{2} + (r_2 - r_3)^2 \sin^2 \frac{\alpha}{2} \\ &\geq (r_2 + r_3)^2 \cos^2 \frac{\alpha}{2} \end{aligned}$$

and so

$$2R_1 \sin \frac{\alpha}{2} \geq r_2 + r_3,$$

with equality only when  $r_2 = r_3$ . Hence (1).

In (1) equality holds if and only if  $r_1 = r_2 = r_3$ .

L. J. Mordell, Math. Gaz. 46 (1962), 213–215.

**REMARK.** Since  $S \leq \frac{1}{2}F^2$  (see: 2.12), this inequality improves 12.27.

$$12.29 \quad r_1r_2r_3 \leq \frac{2}{27} \frac{F^2}{R}. \quad (1)$$

Equality holds if and only if point  $P$  is the centroid of the triangle.

**PROOF.** By

$$ar_1 + br_2 + cr_3 = 2F$$

and by arithmetic-geometric mean inequality, we have

$$\frac{ar_1}{2F} \cdot \frac{br_2}{2F} \cdot \frac{cr_3}{2F} \leq \frac{\left(\frac{ar_1}{2F} + \frac{br_2}{2F} + \frac{cr_3}{2F}\right)^3}{27} = \frac{1}{27},$$

where equality holds if and only if  $ar_1 = br_2 = cr_3$ , i.e.,

$$\text{area } PBC = \text{area } PCA = \text{area } PAB,$$

which is equivalent to the condition that  $P$  is the centroid of the triangle.

Since  $abc = 4FR$ , we obtain (1).

L. Carlitz-J. H. Tyrrell, Math. Mag. 37 (1964), 279.

$$12.30 \quad R_2R_3 + R_3R_1 + R_1R_2 \geq 2(r_1R_1 + r_2R_2 + r_3R_3).$$

Equality holds if and only if the triangle is equilateral and the point is its centre.

A. Oppenheim, Amer. Math. Monthly 68 (1961), 226–230.

$$12.31 \quad r_1R_1 + r_2R_2 + r_3R_3$$

$$\begin{aligned} &\geq \left(\frac{b}{c} + \frac{c}{b}\right)r_2r_3 + \left(\frac{c}{a} + \frac{a}{c}\right)r_3r_1 + \left(\frac{a}{b} + \frac{b}{a}\right)r_1r_2 \\ &\geq 2(r_2r_3 + r_3r_1 + r_1r_2). \end{aligned}$$

Equality holds if and only if the triangle is equilateral and if  $P$  is its centre or if  $P$  is one of the vertices of the triangle.

A. Oppenheim, Amer. Math. Monthly 68 (1961), 226–230.

$$12.32 \quad \frac{1}{r_2 r_3} + \frac{1}{r_3 r_1} + \frac{1}{r_1 r_2} \geq 2 \left( \frac{1}{r_1 R_1} + \frac{1}{r_2 R_2} + \frac{1}{r_3 R_3} \right).$$

Equality holds if and only if the triangle is equilateral and if  $P$  is its centre.

A. Oppenheim, Amer. Math. Monthly 68 (1961), 226–230.

$$12.33 \quad \text{If } 0 < t < 1, \text{ then}$$

$$(r_1 R_1)^t + (r_2 R_2)^t + (r_3 R_3)^t \geq 2^t [(r_2 r_3)^t + (r_3 r_1)^t - (r_1 r_2)^t].$$

Equality holds if and only if the triangle is equilateral and if  $P$  is its centre or if  $P$  is one of the vertices of the triangle.

A. Oppenheim, Amer. Math. Monthly 68 (1961), 226–230.

$$12.34 \quad \frac{1}{r_1 R_1} + \frac{1}{r_2 R_2} + \frac{1}{r_3 R_3} \geq 2 \left( \frac{1}{R_2 R_3} + \frac{1}{R_3 R_1} + \frac{1}{R_1 R_2} \right).$$

Equality holds if and only if the triangle is equilateral and if  $P$  is its centre.

A. Oppenheim, Amer. Math. Monthly 68 (1961), 226–230.

$$12.35 \quad \frac{1}{r_2 r_3} + \frac{1}{r_3 r_1} + \frac{1}{r_1 r_2} \geq 4 \left( \frac{1}{R_2 R_3} + \frac{1}{R_3 R_1} + \frac{1}{R_1 R_2} \right).$$

Equality holds if and only if the triangle is equilateral and if  $P$  is its centre.

J. M. Child, Math. Gaz. 23 (1939), 138–143.

$$12.36 \quad \frac{R_1}{R_1 + r_1} + \frac{R_2}{R_2 + r_2} + \frac{R_3}{R_3 + r_3} \geq 2.$$

J. Berkes, Elem. Math. 12 (1957), 121–123.

12.37 If  $x, y, z$  are arbitrary positive numbers, then

$$xR_1 + yR_2 + zR_3$$

$$\geq 2 \left( \frac{yz(R_2 + R_3)}{yR_2 + zR_3} w_1 + \frac{zx(R_3 + R_1)}{zR_3 + xR_1} w_2 + \frac{xy(R_1 + R_2)}{xR_1 + yR_2} w_3 \right).$$

Equality occurs if and only if  $xR_1 = yR_2 = zR_3$  and the angles  $BPC, CPA, APB$  are  $2\pi/3$ .

A. Oppenheim, Ann. Math. Pura Appl. (4) 53 (1951), 157–163.

12.38 Let  $P$  be a point inside the triangle  $ABC$ . Let  $D, E, F$  be the points of intersection of the lines  $AP, BP, CP$  with the sides  $BC, CA, AB$  respectively. Then

$$\frac{AP}{PD} + \frac{BP}{PE} + \frac{CP}{PF} \geq 6.$$

Equality holds if and only if the point  $P$  is the centroid of the triangle.

PROOF. Let  $S_1, S_2, S_3$  represent the areas of triangles  $PBC, PCA, PAB$ . Since the triangles  $ABC$  and  $PBC$  have the common side  $BC$ ,

$$\frac{AD}{PD} = \frac{S_1 + S_2 + S_3}{S_1},$$

whence

$$\frac{AD - PD}{PD} = \frac{S_2 + S_3}{S_1}, \text{ i.e. } \frac{PA}{PD} = \frac{S_2}{S_1} + \frac{S_3}{S_1}.$$

Analogously, one obtains

$$\frac{PB}{PE} = \frac{S_3}{S_2} + \frac{S_1}{S_2}, \quad \frac{PC}{PF} = \frac{S_1}{S_3} + \frac{S_2}{S_3}.$$

By adding together these equalities, we get

$$\begin{aligned} & \frac{AP}{PD} + \frac{BP}{PE} + \frac{CP}{PF} \\ &= \left( \frac{S_1}{S_2} + \frac{S_2}{S_1} \right) + \left( \frac{S_2}{S_3} + \frac{S_3}{S_2} \right) + \left( \frac{S_3}{S_1} + \frac{S_1}{S_3} \right) \geq 2 + 2 + 2 = 6. \end{aligned}$$

O. J. Ramler, Problem E 1043, Amer. Math. Monthly 59 (1952), 597 and 60 (1953), 421.

L. Carlitz, Amer. Math. Monthly 71 (1964), 881-885.

12.39 With the notations of 12.38, we have

$$\frac{AP}{PD} \cdot \frac{BP}{PE} \cdot \frac{CP}{PF} \geq 8.$$

Equality holds if and only if the point  $P$  is the centroid of the triangle.

PROOF. By virtue of 12.38, we obtain

$$\frac{AP}{PD} \cdot \frac{BP}{PE} \cdot \frac{C\bar{E}}{P\bar{E}} = \frac{(S_2+S_3)(S_3+S_1)(S_1+S_2)}{S_1S_2S_3} \geq \frac{8S_1S_2S_3}{S_1S_2S_3} = 8.$$

O. J. Ramler, Problem 1043, Amer. Math. Monthly 59 (1952), 697 and 60 (1953), 421.

Ch. W. Trigg, Math. Mag. 36 (1963), 244.

L. Carlitz, Amer. Math. Monthly 71 (1964), 881–885.

**12.40** If  $P$  is a point inside the triangle  $ABC$ , and  $A'$ ,  $B'$ ,  $C'$  are the intersections of  $AP$ ,  $BP$ ,  $CP$  and the sides  $BC$ ,  $CA$ ,  $AB$ , respectively, then

$$\frac{AA'}{AP} + \frac{BB'}{BP} + \frac{CC'}{CP} \geq \frac{9}{2}.$$

Ž. Živanović, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 191–No. 196 (1967), 69–72.

**12.41** Let  $ABC$  be any triangle and let  $P$  be any point inside it; let  $AP$ ,  $BP$ ,  $CP$  produced meet the opposite sides of the triangle at  $A'$ ,  $B'$ ,  $C'$  respectively. Then at least one of the inequalities

$$\frac{AA'}{BC} \geq \frac{\sqrt{3}}{2}, \quad \frac{BB'}{CA} \geq \frac{\sqrt{3}}{2}, \quad \frac{CC'}{AB} \geq \frac{\sqrt{3}}{2}. \quad (1)$$

holds.

This statement of G. N. Watson is included in the following theorem due to L. W. Chaundy:

That one of the above inequalities which involves the shortest of the three sides of the triangle is true.

Or in the following theorem of Watson:

That one of the above inequalities which involves the longest of the three transversals  $AA'$ ,  $BB'$ ,  $CC'$  is true.

$\sqrt{3}/2$  in (1) is the best possible constant.

G. N. Watson, Quart. J. Math. (Oxford Ser.) 11 (1940), 273–276.

A. Rosenblat offers an elementary proof of the theorem due to Watson (see Actas. Acad. Sci. Lima 4 (1941), 155–161 and 213–220).

**REMARK.** In fact, the priority of the theorem is due to J. J. L. Hinrichsen, who proposed in the Amer. Math. Monthly 39 (1932), 549, the following problem:

Given any triangle with three line segments concurrent in a point interior to the triangle and joining each vertex to its opposite side. The length of the longest of these three lines cannot be less than  $\frac{1}{2}\sqrt{3}$  times the length of the opposite side of the triangle.

J. J. L. Hinrichsen, Problem 3576, Amer. Math. Monthly 39 (1932), 549 and 41 (1934), 193–194.

In the same Journal, 41 (1934), 193–196 a solution of this problem was published by F. Underwood and the editors of the Amer. Math. Monthly.

**12.42** If  $a \leq b \leq c$ , then

$$R_1 + R_2 + R_3 > \frac{s}{c} (r'_1 + r'_2 + r'_3).$$

L. Carlitz, Amer. Math. Monthly 71 (1964), 881–885.

**12.43**  $R_1 + R_2 + R_3 \geq 2(\sqrt{r'_2 r'_3} + \sqrt{r'_3 r'_1} + \sqrt{r'_1 r'_2}).$

Equality occurs if and only if the triangle is equilateral and if  $P$  is its centre.

L. Carlitz, Amer. Math. Monthly 71 (1964), 881–885.

**12.44**  $R_1^2 + R_2^2 + R_3^2 \geq 2(r'_2 r'_3 + r'_3 r'_1 + r'_1 r'_2) + 6\sqrt[3]{(r'_1 r'_2 r'_3)^2}.$

Equality holds if and only if the triangle is equilateral and if  $P$  is its centre.

L. Carlitz, Amer. Math. Monthly 71 (1964), 881–885.

**12.45**  $R_2 R_3 + R_3 R_1 + R_1 R_2 \geq r'_2 r'_3 + r'_3 r'_1 + r'_1 r'_2 + 9\sqrt[3]{(r'_1 r'_2 r'_3)^2}.$

Equality holds if and only if the triangle is equilateral and if  $P$  is its centre.

L. Carlitz, Amer. Math. Monthly 71 (1964), 881–885.

**12.46**  $2(\sqrt{R_2 R_3} + \sqrt{R_3 R_1} + \sqrt{R_1 R_2})$

$$\leq R_1 + R_2 + R_3 + 2(r'_1 + r'_2 + r'_3).$$

Equality holds if and only if  $P$  is centroid of the triangle.

L. Carlitz, Amer. Math. Monthly 71 (1964), 881–885.

$$12.47 \quad \frac{R_2 R_3}{r'_2 r'_3} + \frac{R_3 R_1}{r'_3 r'_1} + \frac{R_1 R_2}{r'_1 r'_2} \geq 12.$$

Equality occurs if and only if the triangle is equilateral and  $P$  is its centre.

L. Carlitz, Amer. Math. Monthly 71 (1964), 881–885.

$$12.48 \quad R_1 + R_2 + R_3 \geq 2(w_1 + w_2 + w_3).$$

Equality holds if and only if the triangle is equilateral and if  $P$  is its centre.

D. F. Barrow, Problem 3740, Amer. Math. Monthly 44 (1937), 252–254.

$$12.49 \quad R_2 R_3 + R_3 R_1 + R_1 R_2 \geq 2(R_1 w_1 + R_2 w_2 + R_3 w_3).$$

A. Oppenheim, Ann. Math. Pura Appl. (4) 53 (1961), 157–63.

$$12.50 \quad R_2 R_3 + R_3 R_1 + R_1 R_2 \geq 4(w_2 w_3 + w_3 w_1 + w_1 w_2).$$

Equality holds if and only if the triangle is equilateral and if  $P$  is its centre.

L. Carlitz, Math. Gaz. 47 (1964), 181–182.

$$12.51 \quad R_1 R_2 R_3 \geq 8w_1 w_2 w_3.$$

Equality holds if and only if the triangle is equilateral and if  $P$  is its centre.

L. Carlitz, Math. Gaz. 47 (1964), 181–182.

$$12.52 \quad R_1 R_2 R_3 \geq (w_2 + w_3)(w_3 + w_1)(w_1 + w_2).$$

Equality occurs if and only if the triangle is equilateral and if  $P$  is its centre.

A. Oppenheim, Ann. Math. Pura Appl. (4) 53 (1961), 157–63.

$$12.53 \quad R_1^2 + R_2^2 + R_3^2 \geq \frac{1}{3}(a^2 + b^2 + c^2).$$

Equality holds if and only if  $P$  coincides with  $G$ .

Encyklopädie der Mathematischen Wissenschaften, Bnd. II', Heft 6, p. 1118, and Heft 7, p. 1185.

R. Sturm, Maxima und minima in der elementaren Geometrie, Leipzig, Berlin 1910, p. 71.

T. Lalesco, La géométrie du triangle, Paris 1937, p. 41.

$$12.54 \quad r_1^2 + r_2^2 + r_3^2 \geq \frac{4F^2}{a^2 + b^2 + c^2}.$$

Equality holds if  $P$  coincides with Lemoine's point.

Encyklopädie der Mathematischen Wissenschaften, Bd. III', Heft 6, p. 1118, and Heft 7, p. 1155.

T. Lalesco, La géométrie du triangle, Paris 1937, p. 41.

O. Bottema, Hoofdstukken uit de elementaire meetkunde, Den Haag 1944, p. 91.

12.55 If no angle  $\geq \frac{2}{3}\pi$ , then

$$R_1 + R_2 + R_3 \geq \left( \frac{a^2 + b^2 - c^2}{2} + 2F\sqrt{3} \right)^{1/2}.$$

If  $\alpha \geq \frac{2}{3}\pi$ , then

$$R_1 + R_2 + R_3 \geq b + c.$$

Equality holds if  $P$  coincides with Torricelli's point.

Encyklopädie der Mathematischen Wissenschaften, Bd. III', Heft 6, p. 1118, and Heft 7, p. 1155.

T. Lalesco, La géométrie du triangle, Paris 1937, p. 41.

12.56  $a_1R_1 + b_1R_2 + c_1R_3$

$$\geq \left[ \frac{a^2(-a_1^2 + b_1^2 + c_1^2) + b^2(a_1^2 - b_1^2 + c_1^2) + c^2(a_1^2 + b_1^2 - c_1^2)}{2} + 8FF_1 \right]^{1/2},$$

where  $a_1, b_1, c_1, F_1$  are sides and area of a second triangle  $A_1B_1C_1$ .

The triangles  $ABC$  and  $A_1B_1C_1$  are interchangeable.

O. Bottema, Hoofdstukken uit de elementaire meetkunde, Den Haag 1944, p. 97-99.

**13. Necessary and sufficient conditions for the existence of a triangle**

**13.1** A necessary and sufficient condition for a triangle to exist is

$$\max(a, b, c) < \frac{1}{2}(a+b+c).$$

**13.2** If  $a = \max(a, b, c)$ , then a necessary but insufficient condition for  $a, b, c$  to be the sides of a triangle is

$$\frac{1}{2}a^2 < b^2 + c^2 < 2a^2.$$

**13.3** Let  $f(x)$  be any non-negative, non-decreasing, subadditive function on the domain  $x > 0$ ; for instance  $f(x) = \sqrt[n]{x}$  ( $n$  a natural number).

If  $a, b, c$  form a triangle, then  $f(a), f(b), f(c)$  form a triangle.

**PROOF.** From

$$a \leq b+c,$$

we have

$$f(a) \leq f(b+c) \leq f(b)+f(c).$$

Cyclic permutation of  $a, b, c$  then yields the additional inequalities

$$f(b) \leq f(c)-f(a), \quad f(c) \leq f(a)+f(b),$$

therefore  $f(a), f(b), f(c)$  form a triangle.

This proof is due to J. L. Brown, Jr. Amer. Mathem. Monthly 67 (1960), 82-83.

**REMARK.** The theorem is a generalization of Problem E 1366, Amer. Math. Monthly 66 (1959) 423, and 67 (1960), 82-83, proposed by W. E. Hoggatt, Jr., where the case  $f(x) = \sqrt{x}$  is considered.

**13.4** Let  $p, q$  be real numbers such that  $p+q=1$ . Then a triangle with the sides  $a, b, c$  exists if and only if

$$pa^2 + qb^2 > pqc^2 \text{ for all } p, q. \quad (1)$$

**PROOF.** A triangle with sides  $a, b, c$ , exists if and only if

$$a+b-c > 0, \quad b+c-a > 0, \quad c+a-b > 0. \quad (2)$$

R. Sturm, Maxima und minima in der elementaren Geometrie, Leipzig, Berlin 1910, p. 71.

T. Lalesco, La géométrie du triangle, Paris 1937, p. 41.

$$12.54 \quad r_1^2 + r_2^2 + r_3^2 \geq \frac{4F^2}{a^2 + b^2 + c^2}.$$

Equality holds if  $P$  coincides with Lemoine's point.

Encyklopädie der Mathematischen Wissenschaften, Bd. III', Heft 6, p. 1118, and Heft 7, p. 1125.

T. Lalesco, La géométrie du triangle, Paris 1937, p. 41.

O. Bottema, Hoofdstukken uit de elementaire meetkunde, Den Haag 1944, p. 91.

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where  $a_1, b_1, c_1, F_1$  are sides and area of a second triangle  $A_1B_1C_1$ .

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**13.4** Let  $p, q$  be real numbers such that  $p+q=1$ . Then a triangle with the sides  $a, b, c$  exists if and only if

$$pa^2 + qb^2 > pqc^2 \text{ for all } p, q. \quad (1)$$

**PROOF.** A triangle with sides  $a, b, c$ , exists if and only if

$$a+b-c > 0, \quad b+c-a > 0, \quad c+a-b > 0. \quad (2)$$

We shall prove that (2) is equivalent to (1). Let

$$K = p a^2 + q b^2 - p q c^2.$$

Since  $q = 1-p$ , we have

$$\begin{aligned} K &= p a^2 + (1-p) b^2 - p(1-p)c^2 \\ &= c^2 p^2 + (a^2 - b^2 - c^2)p + b^2, \end{aligned}$$

where  $a, b, c$  are fixed and  $p$  is a variable. Therefore  $K$  is a quadratic trinomial in respect to  $p$ . In order to have  $K > 0$ , it is necessary that

$$D = (a^2 - b^2 - c^2)^2 - 4b^2c^2 < 0,$$

i.e.

$$(a+b+c)(b+c-a)(c+a-b)(a+b-c) > 0. \quad (3)$$

If the triangle exists, then (2) holds, so that  $D < 0$ . Therefore if  $a, b, c$  are the sides of a triangle, then (1) is valid.

Inversely, if  $D < 0$ , then (3) holds, i.e.  $a, b, c$  are sides of a triangle.

V. B. Lidskii, L. V. Ovsjannikov, A. N. Tulaikov and M. I. Sabunin, Zadači po elementarnoi matematike, Moskva 1962, p.21.

**13.5** Let  $\alpha, \beta, \gamma$  be the angles of a triangle. Then a triangle with sides  $\cos^2 \alpha/2, \cos^2 \beta/2, \cos^2 \gamma/2$  exists.

T. P. Čerepanova, Matematika v škole 1966, No. 6, 66.

**13.6** Let  $\alpha, \beta, \gamma$  be the angles of a triangle. Then a triangle whose sides are  $\cos \alpha/2, \cos \beta/2, \cos \gamma/2$  exists.

Matematika v škole 1963, No. 3, 89.

REMARK. 13.6 follows from 13.5 by means of 13.3.

**13.7** G. Petrov has given a series of the necessary and sufficient conditions which must be satisfied by certain elements of a triangle:

$$1^\circ. 0 < m_c < \frac{2ab}{a+b}.$$

$$2^\circ. a - 2b - 2m_a > 0, a - 2b + 2m_a > 0, -a + 2b + 2m_a > 0.$$

3°.  $2r < h_b \leq a$ .

4°.  $8Rw_a \geq a^2 + 4w_a^2$  for  $a \leq 2w_a$ ,

$2R > a$  for  $a > 2w_a$ .

5°.  $a < b+c$ ,  $2m_a < b+c$ ,  $(b+c)^2 \leq a^2 + 4m_a^2$ .

6°.  $2R \geq a$ ,  $2R \geq \frac{a^2 + 4h_a^2}{4h_a}$ .

7°.  $r > 0$ ,  $4r^6 + 4(2a^2 + 2b^2 + 7ab)r^4 + 4(a^4 - a^3b - a^2b^2 - ab^3 + b^4)r^2 - a^2b^2(a-b)^2 \leq 0$ .

G. Petrov, Časopis pro peštování matematiky 77 (1952), 77-92.

**13.8** A necessary and sufficient condition for the existence of a triangle with the elements  $R$ ,  $r$  and  $s$ , is

$$s^4 - 2(2R^2 + 10Rr - r^2)s^2 + r(4R - r)^2 \leq 0.$$

R. Sondat, Nouv. Ann. Math. (3) 10 (1891), 43\*-47\*.

#### 14. Miscellaneous inequalities for the elements of a triangle

**14.1** If  $\lambda_1, \lambda_2, \lambda_3$  are real numbers, then

$$(\lambda_1 + \lambda_2 + \lambda_3)^2 R^2 \geq \lambda_2 \lambda_3 a^2 + \lambda_3 \lambda_1 b^2 + \lambda_1 \lambda_2 c^2. \quad (1)$$

REMARK 1. Taking for  $\lambda_1, \lambda_2, \lambda_3$  special values we deduce from (1) the following inequalities:

4.4 for  $\lambda_1 = a^2, \lambda_2 = b^2, \lambda_3 = c^2$ .

4.7 for  $\lambda_1 = a(4s^2 - a^2 - b^2 - c^2 - 2sa)$ ,

$$\lambda_2 = b(4s^2 - a^2 - b^2 - c^2 - 2sb),$$

$$\lambda_3 = c(4s^2 - a^2 - b^2 - c^2 - 2sc).$$

5.1 for  $\lambda_1 = a, \lambda_2 = b, \lambda_3 = c$ .

5.13 for  $\lambda_1 = 2s - 3a, \lambda_2 = 2s - 3b, \lambda_3 = 2s - 3c$  and for

$$\lambda_1 = \lambda_2 = \lambda_3 = 1.$$

5.25 for  $\lambda_1 = 2s-a$ ,  $\lambda_2 = 2s-b$ ,  $\lambda_3 = 2s-c$ .

5.7 for  $\lambda_1 = a(s-a)$ ,  $\lambda_2 = b(s-b)$ ,  $\lambda_3 = c(s-c)$ .

REMARK 2. O. Bottema has given the following remarks:

1°.  $(\lambda_1 + \lambda_2 + \lambda_3)^2 R^2 - (\lambda_2 \lambda_3 a^2 + \lambda_3 \lambda_1 b^2 + \lambda_1 \lambda_2 c^2)$  has a geometrical meaning: it is equal to  $2\lambda^2 GO^2$  ( $\lambda = \lambda_1 + \lambda_2 + \lambda_3 \neq 0$ ) in which  $G$  is the centroid of the points  $A$ ,  $B$ ,  $C$  if the weights  $\lambda_1$ ,  $\lambda_2$ ,  $\lambda_3$  are given to them. From this the inequality (1) follows.

2°. Equality holds in (1) if and only if  $G$  and  $O$  coincide. If  $\lambda_1$ ,  $\lambda_2$ ,  $\lambda_3$  are given a triangle for which this occurs exists if and only if a triangle exists, the sides of which are  $|\lambda_1|$ ,  $|\lambda_2|$ ,  $|\lambda_3|$ .

O. Kooi, Simon Stevin 32 (1958), 97-101.

O. Bottema, Simon Stevin 33 (1959), 97-100.

14.2  $2F \leq a^2 - ab + b^2$ ,  $4F \leq a^2 + b^2$ ,  $8F \leq (a+b)^2$  hold for each triangle.

Equalities hold only for the right isosceles triangle.

Z. A. Skopec and V. A. Žarov, Zadaci i teoremi po geometriji, Moskva 1962, pp. 81-82.

14.3 If  $s^2 = \lambda ab$ , where  $\lambda > 1$ , then

$$s < \lambda a, s < \lambda b, c < (\lambda - 1)a, c < (\lambda - 1)b.$$

W. O. Pennell and A. L. Epstein, Problem E 1068, Amer. Math. Monthly 61 (1954), 49.

14.4 If  $a \geq b \geq c$ , then

$$1^\circ. \frac{2}{3}s \leq a < s, \frac{1}{2}s < b < s, 0 < c \leq \frac{2}{3}s.$$

$$2^\circ. \pi/3 \leq \alpha < \pi, 0 < \beta < \pi/2, 0 < \gamma \leq \pi/3.$$

$$3^\circ. 0 < h_a \leq \frac{1}{3}s\sqrt{3}, 0 < h_b \leq \frac{1}{2}s\sqrt{3}, \frac{1}{3}s\sqrt{3} \leq h_c < s.$$

$$4^\circ. 0 < w_a \leq \frac{1}{3}s\sqrt{3}, 0 < w_b < \frac{2}{3}s, \frac{1}{3}s\sqrt{3} \leq w_c < s.$$

$$5^\circ. 0 < m_a < \frac{1}{3}s\sqrt{3}, \frac{1}{2}s < m_b < \frac{3}{4}s, \frac{1}{3}s\sqrt{3} \leq m_c \leq s.$$

$$6^\circ. \frac{2}{3}s\sqrt{3} \leq R < +\infty.$$

$$7^\circ. 0 < r \leq \frac{1}{6}s\sqrt{3}.$$

V. M. Černov, Matematika v škole, 1955 No. 3, 91-95.

14.5 If  $a \geq b \geq c$ , then

$$a + h_a \geq b + h_b \geq c + h_c.$$

A. V. Aljaev, Matematika v škole, 1963, No. 3, 88 and 1964, No. 2, 74.

14.6 If  $b = \text{med}(a, b, c)$ , then

$$\frac{s^2}{9} + r^2 \geq 2k,$$

for  $c+a \leq 2b$ .

S. G. Guba, Matematika v škole, 1965, No. 5, 69 and 1966, No. 4, 77–78.

14.7 If  $a$  is the basis of an isosceles triangle and  $w$  the angle-bisector of an angle on the basis, then

$$\frac{2}{3}a < w < a\sqrt{2}$$

Matematika v škole, 1963, No. 2, 86

14.8  $a \geq 2h_a \operatorname{tg} \alpha/2$ .

Matematika v škole, 1965, No. 4, 72.

14.9 Of a triangle is one of the angles  $4\varphi$ , then

$$\frac{F}{s^2} \leq \frac{2 \operatorname{tg} \varphi (1 - \operatorname{tg} \varphi)}{(1 + \operatorname{tg} \varphi)^2}.$$

O. Bottema, Wisk. Opgaven 20 (1955), 4.

14.10 Let  $\rho_1, \rho_2, \rho_3$  be the radii of the circles which touch two sides and the circumcircle of a given triangle. Then

$$4r \leq \rho_1 + \rho_2 + \rho_3 \leq 2R.$$

REMARK. The first inequality is due to K. V. Vetrov and the second to S. T. Berkolaiko. See: K. V. Vetrov, Matematika v škole, 1966, No. 3, 59 and S. T. Berkolaiko, Matematika v škole, 1967, No. 2, 74.

14.11 If  $A_1, B_1, C_1$  are points on the sides  $BC, CA, AB$  of a

triangle, respectively, then

$$\frac{1}{2} < \frac{AA_1 + BB_1 + CC_1}{a+b+c} < \frac{3}{2}.$$

Gaz. Mat. B 8 (1957), 163.

**14.12** Let  $\lambda$  be a positive number. Let  $X$  and  $Y$  denote given points on the sides of a triangle  $ABC$ , such that

$$\frac{\angle XAB}{\angle CAB} = \frac{\angle YBA}{\angle CBA} = \lambda.$$

Then the following equivalences hold:

$$AX > BY \Leftrightarrow AC > BC,$$

for  $0 < \lambda < 1$ ,

$$CY > CX \Leftrightarrow AC > BC,$$

for  $0 < \lambda < 1$ ,

$$AY > BX \Leftrightarrow AC > BC,$$

for  $0 < \lambda < \frac{1}{2}$ .

C. Mat., Problem E 1626 Amer. Math. Monthly 70 (1963), 891 and 71 (1964), 684–686.

**14.13** Let  $P$  be a point inside a triangle  $ABC$ . Let  $A'$ ,  $B'$ ,  $C'$  be the intersections of the lines  $AP$ ,  $BP$ ,  $CP$  with  $BC$ ,  $CA$ ,  $AB$  respectively. Then, if  $AA' \geq BB'$  and  $AA' \geq CC'$ , we have

$$AA' \geq PA' + PB' + PC',$$

with equality only for  $AA' = BB' = CC'$ .

**14.14**  $CH \geq IH\sqrt{2}$ .

Equality holds if and only if the triangle is equilateral.

S. G. Šuba, Matematika v škole, 1966, No. 3, 60.

$$\begin{aligned} 14.15 \quad & \frac{\cos^2 \beta + \cos^2 \gamma}{\cos \beta + \cos \gamma} + \frac{\cos^2 \gamma + \cos^2 \alpha}{\cos \gamma + \cos \alpha} \\ & + \frac{\cos^2 \alpha + \cos^2 \beta}{\cos \alpha + \cos \beta} \geq 1 + \frac{r}{R}. \end{aligned} \quad (1)$$

PROOF. By adding together the relations

$$x^2 + y^2 \geq 2xy \text{ and } x^2 + y^2 = x^2 + y^2,$$

we get

$$2(x^2 + y^2) \geq (x+y)^2.$$

Whence, for  $x+y > 0$ ,

$$\frac{x^2 + y^2}{x+y} \geq \frac{x+y}{2},$$

i.e.

$$\frac{y^2 + z^2}{y+z} + \frac{z^2 + x^2}{z+x} + \frac{x^2 + y^2}{x+y} \geq x + y + z. \quad (2)$$

Since  $\cos \beta + \cos \gamma > 0$ , etc., for  $x = \cos \alpha$ ,  $y = \cos \beta$ ,  $z = \cos \gamma$ , from (2) follows (1), because  $\cos x + \cos \beta + \cos \gamma = 1 + r/R$ .

Gaz. Mat. B 13 (1962), 133–134.

**14.16** If  $m_b$  and  $m_c$  are perpendicular to each other, then

$$\cot \beta + \cot \gamma \geq \frac{2}{3}.$$

S. Reich, Problem E 1839, Amer. Math. Monthly 72 (1965), 1129.

**14.17** Let  $P$  be a point on the median  $m_a$  of a triangle  $ABC$ . Let  $BP$  and  $CP$  meet the sides  $AC$  and  $AB$  in the points  $E$  and  $F$  respectively. Then, if  $AB > AC$ , we have  $BE > CF$ .

K. Tan and N. Harrell, Math. Mag. 38 (1965), 57–58.

**14.18** Let a square with the side  $a$  be inscribed in a triangle. Then

$$r\sqrt{2} < a < 2r.$$

**14.19** If a square lies inside a triangle, the area of the square does not exceed half the area of the triangle.

D. J. Newman, Problem E 1425, Amer. Math. Monthly 67 (1960), 593 and 68 (1961), 180.

**14.20** Let  $P$  be a fixed point in a plane and consider in the plane the set of triangles  $ABC$  with

$$PA^2 + PB^2 + PC^2 = 3\varepsilon^2,$$

where  $e$  is a constant. Then  $BC \cdot CA \cdot AB$  is maximum if and only if  $ABC$  is an equilateral triangle with its vertices on the circle with centre  $P$  and radius  $e$ .

A. C. Aitken, Proc. Edinburgh Math. Soc. (2) :3 (1962/63) 173–174.

**14.21** If  $D, E, F$  are the intersections of  $AI, BI, CI$  with the sides  $BC, CA, AB$  respectively, then

$$AI > ID, BI > IE, CI > IF.$$

PROOF. Since the angle-bisectors are cevians, we have

$$\frac{AI}{ID} = \frac{AF}{FB} + \frac{AE}{EC} = \frac{b}{a} + \frac{c}{a} = \frac{b+c}{a} > 1.$$

Tien-Hsung Lin, Math. Mag. 38 (1955), 158–159.

Ch. W. Trigg, Math. Mag. 40 (1967), 28.

**14.22** Let  $d_a \leq d_b \leq d_c$ , where  $d_a, d_b, d_c$  are the distances of the circumcentre of a triangle to its excentres. Then

$$R < d_a < 2R, R < d_b < R\sqrt{5}, 2R < d_c < 3R.$$

H. J. Baron, Tôhoku Math. J. 48 (1941), 185–192.

**14.23** Let the incircle of a right triangle touch its hypotenuse at  $N$  and one of the sides of triangle at  $M$ . Then

$$MN \leq \frac{2\sqrt{3}}{9} c,$$

where  $c$  is the hypotenuse.

E. A. Bokov, Matematika v škole, 1963, No. 4, 46 and 1964, No. 4, 78.

**14.24** If  $R_1, R_2, R_3$  are the radii of the circles inscribed in the sectors  $AOB, BOC, COA$ , respectively, then

$$\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \geq \frac{3+2\sqrt{3}}{R}.$$

Ju. I. Gerasimov, Matematika v škole, 1967, No. 3, 84.

14.25 Let  $T$  and  $T'$  denote Brocard's points in a triangle, then

$$2 \cdot TT' \leq R.$$

T. Lalesco, La géométrie du triangle, Paris 1937, p. 120.

V. G. Cavallaro, Mathesis 53 (1939), 155-160.

14.26 Let  $M$  and  $N$  be two points one unit apart. With  $M$  and  $N$  as centres and with unit radii draw arcs  $ANB$  and  $AMB$ . Let  $Q$  be any point on arc  $AMB$  and  $P_1$  and  $P_2$  any points on arc  $ANB$  such that  $N$  is the midpoint of arc  $P_1P_2$ . Then

$$QP_1 + QP_2 \leq 2 \leq QP_1^2 + QP_2^2.$$

A. W. Goodman, Amer. Math. Monthly 58 (1951), 564.

14.27 If  $d$  denotes the distance between the circumcentre and the incentre of a triangle, then

$$(R-d)(3R+d)^3 \leq 4R^2s^2 \leq (R+d)(3R-d)^3.$$

Equalities hold if and only if the triangle is equilateral.

O. Bottema, Nieuw Arch. Wisk. 13 (1965), 246.

14.28 In any triangle  $ABC$

$$\omega \leq \frac{\pi}{6}$$

in which  $\omega$  denotes Brocard's angle.

Equality holds only for the equilateral triangle.

PROOF. Inequality follows from

$$\cot \omega = \cot \alpha + \cot \beta + \cot \gamma$$

by means of 2.38.

T. Lalesco, La géométrie du triangle, Paris 1937, p. 120.

14.29 Let  $P$  be an interior point of the triangle  $ABC$ . Let  $L, M, N$  be the points in which the lines  $AP, BP, CP$  intersect the sides  $BC, CA, AB$  respectively. Then

$$8 \cdot AM \cdot BN \cdot CL \leq abc.$$

Equality holds if and only if  $P$  is the centroid of the triangle.

S. I. Zetel', Novaja geometrija treugol'nika, Moskva 1962,  
pp. 12-13.

**14.30** Let  $P$  be an interior point of the triangle  $ABC$ . The lines through the midpoints of the segments  $AP$ ,  $BC$ ,  $CP$  parallel to  $BC$ ,  $CA$ ,  $AB$ , respectively form, with the sides of  $ABC$ , the triangles with areas  $F_A$ ,  $F_B$ ,  $F_C$ . Then

$$F_A + F_B + F_C \geq \frac{1}{3}F.$$

Equality holds if and only if  $P$  is the centroid of the triangle  $ABC$ .

S. I. Zetel', Novaja geometrija treugol'nika, Moskva 1962,  
pp. 32-33.

**14.31** If  $ABC$  and  $A'B'C'$  are any two equilateral triangles in a plane, their vertices being taken in the same sense of rotation, of the three lines  $AA'$ ,  $BB'$ ,  $CC'$ , the sum of any two is not less than the third.

The same theorem holds for any pair of directly similar triangles. There is equality if and only if the centre of similitude lies on the circumcircle of one (and therefore of both) triangles.

C. Tweedie, Edin. M. S. Proc. 22 (1904), 22-26.

P. Pinkerton, Edin. M. S. Proc. 22 (1904), 27. — — —

## 15. Inequalities for quadrilaterals

**15.1**  $s < p+q < 2s$ .

F. A. Andreev, Matematika v škole 1953, No. 1, 58-69.

**15.2**  $s(p^t+q^t) > p^{t+1}+q^{t+1}$ ,

where  $t$  is a positive number.

REMARK. This is a generalization of the problem proposed in Gaz. Mat. 12 (1961), 175.

**15.3** For any convex quadrilateral there exists at least one side which is smaller than the greater among its diagonals.

**15.4**  $AC \cdot BD \leq AD \cdot BC + AB \cdot CD$ .

PROOF. By the inversion with centre  $A$  and power  $AB \cdot AC \cdot AD$  the points  $E, C, D$  are transformed into  $B_1, C_1, D_1$  so that  $B_1C_1 = AD \cdot BC \dots$ . Theorem follows from the triangle inequality for  $B_1, C_1, D_1$ .

Equality holds if and only if  $A, B, C$  and  $D$  are on a circle.

**15.5** If  $A, B, C$  and  $D$  are arbitrary points,  $BC = a, CA = b, AB = c, AD = p, BD = q, CD = r$ , then

$$(-ip + bq + cr)(ap - bq + cr)(ap + bq - cr) \geq 0.$$

Equality only if the four points are on a circle.

COROLLARY 1. If  $BC = CA = AB$ , then

$$(-p + q - r)(p - q + r)(p + q - r) \geq 0 \text{ (Pompeiu's theorem).}$$

COROLLARY 2. If the triangles  $ABC$  and  $PQR$  are given a point  $D$  exists with the property  $AD:BD:CD = p:q:r$  if, and only if

$$(-ip + bq + cr)(ap - bq + cr)(ap + bq - cr) \geq 0.$$

The two angles may be interchanged.

O. Bottemi, Euclides 38(1963/64), 129-137.

$$15.6 \quad \frac{\max(a, b, c, d, p, q)}{\min(a, b, c, d, p, q)} \geq \sqrt{2}.$$

REMARK. This theorem has been a problem at the competition for secondary school students in Hungary in 1961.

**15.7**  $I^2 - 1 \cdot F \geq 0$ .

Equality only for a square.

REMARK. This is the isoperimetric inequality for quadrilaterals. See, for example, N. D. Kazarinoff, Geometric Inequalities, New York 1961, p. 51.

**15.8**  $4F \leq (a+c)(b+d)$ .

Equality only for a rectangle.

PROOF. Since

$$4F = ad \cdot \sin \alpha + ab \cdot \sin \beta + bc \cdot \sin \gamma + cd \cdot \sin \delta,$$

and

$$\sin \alpha, \sin \beta, \sin \gamma, \sin \delta \leq 1,$$

we obtain

$$4F \leq ab + ad + bc + cd = (a+c)(b+d).$$

**15.9** If  $x$  and  $y$  are the segments connecting the midpoints of the opposite sides of a quadrilateral and if  $z$  is the segment connecting the midpoints of the diagonals, then

$$x^2 + y^2 + z^2 \geq 2F.$$

Ju. I. Gerasimov, Matematika v škole 1967, No. 1, 83.

**15.10** If  $P$  and  $Q$  denote the midpoints of the sides  $BC$  and  $AD$ , then

$$|AB - CD| < 2PQ \leq AB + CD. \quad (1)$$

PROOF. Consider the triangle  $MPQ$ ,  $M$  being the midpoint of  $AC$ . Then

$$AB + CD = 2(PM + QM) \geq 2PQ.$$

Similarly, we have

$$|AB - CD| = 2|PM - QM| < 2PQ.$$

Equality in (1) holds if and only if  $AB$  is parallel to  $CD$ .

J. I. Nassar, Problem E 1617, Amer. Math. Monthly 70 (1963), 758.

$$15.11 \quad \frac{a}{c} + \frac{c}{a} > \frac{p}{q} + \frac{q}{p}$$

unless  $c = a$ .

C. V. Durell and A. Robson, Advanced Trigonometry, London 1948, p. 281.

**15.12** If  $AB + BD = AC + CD$ , then  $AB < AC$ .

PROOF. Adding the inequalities

$$PA + PB > AB, \quad PC + PD > CD,$$

because  $AB + BD = AC + CD$ , we have

$$AC + BD > AB + AB + BD - AC,$$

i.e.,

$$AC > AB.$$

**15.13** If  $P$  be the intersection of diagonals and if the distances of  $P$  from the sides  $AB, BC, CD, DA$  are  $x, y, z$  and  $t$  respectively, then

$$2(x + y + z + t) < 3s.$$

PROOF. Let the bisectors of the angles between the diagonals  $AC$  and  $BD$  meet  $AB, BC, CD, DA$  in  $R, S, T, U$ .

By a corollary of the Erdős-Mordell theorem (see: 12.13)

$$2(PS + PT) < PB + PC + PD$$

$$2(PT + PU) < PC + PD + PA$$

$$2(PU + PR) < PD + PA + PB$$

$$2(PR + PS) < PA + PB + PC$$

or

$$4(PR + PS + PT + PU) < 3(PA + PB + PC + PD).$$

This inequality is stronger than the one proposed because

$$x + y + z + t \leq PR + PS + PT + PU$$

and

$$PA + PB + PC + PD < a + b + c + d.$$

H. Demir-L. Bankoff, Math. Mag. 39 (1966), 305 and 40 (1967), 166.

**15.14** Let  $ABCD$  be a convex quadrilateral and let  $P$  be the intersection of diagonals  $AC$  and  $BD$ . Let the lines through  $P$  parallel to  $AB, BC, CD, DA$  intersect  $AD$  and  $BC$  at  $A_1$  and  $B_2$ ,  $AB$  and  $CD$  at  $B_1$  and  $C_2$ ,  $BC$  and  $DA$  at  $C_1$  and  $D_2$ ,  $CD$  and  $AB$  at  $D_1$  and  $A_2$  respectively. Then

$$\frac{A_1B_2}{AB} \cdot \frac{B_1C_2}{BC} \cdot \frac{C_1D_2}{CD} \cdot \frac{D_1A_2}{DA} \leq 1.$$

S. Zetel', Matematika v škole 1949, No. 4, 62.

15.15 If a quadrilateral has a circumscribed circle with radius  $R$  and an inscribed circle with radius  $r$ , then

$$R \geq r\sqrt{2}.$$

Ju. I. Gerasimov and O. A. Kotii, Matematika v škole 1964, No. 1, 83.

L. Carlitz, Math. Mag. 38 (1965), 33–35.

15.16  $4F \leq a^2 + b^2 + c^2 + d^2$ .

Equality only for a square.

Z. A. Skopec and V. A. Žanov, Zadači i teoremi po geometriji, Moskva 1962, p. 82.

15.17  $4F \leq p^2 + q^2$ .

Equality holds if and only if the diagonals are orthogonal and equal.

Z. A. Skopec and V. A. Žanov, Zadači i teoremi po geometriji, Moskva 1962, p. 82.

15.18 Let four points  $A, B, C, D$  be given.

1°. If  $D$  is in the interior or on an edge of triangle  $ABC$ , then, for all points  $P$

$$PA + PB + PC - PD \geq DA + DB + DC.$$

Equality holds only if  $P$  coincides with  $D$ .

2°. If no vertex is on or in the triangle formed by the other three points, then

$$PA + PB + PC + PD$$

is minimal for the point of intersection of the diagonals of the convex quadrilateral defined by the four points.

H. W. Guggenheimer, Plane Geometry and its Groups, San Francisco, Cambridge, London, Amsterdam 1967, p. 178.

15.19 Let  $A_1, B_1, C_1, D_1$  be the midpoints of the sides  $BC, CD, DA, AB$  respectively. If  $F_1$  is the area of the quadrilateral enclosed by the lines  $AA_1, BB_1, CC_1, DD_1$ , then

$$\frac{1}{6}F < F_1 \leq \frac{1}{5}F.$$

Equality only for a parallelogram.

T. Popoviciu, Problem 5897, Gaz. Mat. 49 (1943), 322.

**15.20** Suppose that the quadrilateral  $ABCD$  is convex and  $AD$  is not parallel to  $BC$ . If  $PQ \parallel AD$  ( $P$ -intersection of the diagonals  $AC$  and  $BD$ ;  $Q$  on  $AB$ ),  $QR \parallel BC$  ( $R$  on  $AC$ ),  $RS \parallel AD$  ( $S$  on  $CD$ ), and if  $F_1$  is the area of  $PQRS$ , then

$$F_1 < \frac{8}{27}F. \quad (1)$$

PROOF. The quadrilateral  $PQRS$  is a parallelogram.

$$\text{area } AQC = \frac{AC}{PR} \cdot \text{area } PQR = \frac{AC}{2PR} \cdot F_1 = \text{area } ASC = F_2.$$

Hence

$$\text{area } ABC = \frac{AE}{AC} \cdot F_2 = \frac{AC}{AR} \cdot F_2 = \frac{AC^2}{2PR \cdot AR} \cdot F_1$$

and

$$\text{area } ADC = \frac{CL}{CS} \cdot F_2 = \frac{AC}{RC} \cdot F_2 = \frac{AC^2}{2PR \cdot RC} \cdot F_1.$$

Adding together the above-mentioned relations, we get

$$F = \frac{AC^3}{2PR \cdot AR \cdot RC} \cdot F_1,$$

whence

$$\frac{F_1}{F} = 2PR \cdot AR \frac{RC}{AC^3} < \frac{2AR^2 \cdot RC}{AC^3};$$

or -

$$\frac{F_1}{F} < 2x^2(1-x), \text{ where } x = \frac{AR}{AC}.$$

By applying the arithmetic-geometric mean inequality to  $x$ ,  $x$  and  $2-2x$ , we get (1).

Y. Hattori, Math. Gaz. 47 (1963), 148-150.

E. Szekeres, Math. Gaz. 48 (1964), 439-440.

**15.21** Let  $ABCD$  be a convex quadrilateral. Consider a point  $A_1$  on the straight line  $AB$  so that the point  $B$  lies between the points  $A$  and  $A_1$  and the equality  $AA_1 = BC + CD + DA$  is satisfied. If  $B_1, C_1, D_1$  are determined on the lines  $BC, CD, DA$

in the same manner as point  $A_1$ , then

$$\frac{5}{3} < \frac{\text{per } A_1B_1C_1D_1}{\text{per } ABCD} < 5.$$

D. M. Batinetu, Problem E 1725, Gaz. Mat. B 12 (1961), 501.

**15.22** Let  $A_1A_2A_3A_4$  be a square and let  $P$  be an arbitrary point in its plane. Then

$$PA_1 + PA_2 + PA_3 + PA_4 \geq (1 + \sqrt{2}) \max_{(i)} PA_i + \min_{(i)} PA_i,$$

equality taking place when  $P$  lies on the circumcircle of the square.

PROOF. Without loss of generality we may assume

$$PA_4 \geq (PA_3, PA_1) \geq PA_2.$$

Then, by Ptolemy's theorem (see: 15.4)

$$(PA_1)(A_3A_4) + (PA_3)(A_1A_4) \geq (PA_4)(A_1A_3),$$

equality occurring if and only if  $A_1, P, A_3, A_4$  are concyclic.

Since  $A_3A_4 = A_1A_4 = A_1A_3/\sqrt{2}$ , we infer that

$$PA_1 + PA_3 \geq PA_4\sqrt{2}.$$

Hence

$$PA_1 + PA_2 + PA_3 + PA_4 \geq PA_4\sqrt{2} + PA_4 + PA_2,$$

equality holding when  $P$  lies on the circumcircle of the square.

I. S. Gál-L. Bankoff, Problem E 1308, Amer. Math. Monthly 65 (1958), 205 and 710.

**15.23** Let  $ABCD$  be a quadrilateral. It need not be convex and may even possess a double-point. Let

$$P_1 = (-a+b+c+d)(a-b+c+d)(a+b-c+d)(a+b-c-d)$$

and

$$P_2 = (a+b+c+d)(-a+b+c-d)(a-b+c-d)(a+b-c-d).$$

If  $P_1 > 0$ , then

$$P_2 \leq 16F^2 \leq P_1. \quad (1)$$

Equality at the right side occurs if  $ABCD$  is a convex cyclic quadrilateral (Steiner's theorem). Such a quadrilateral exists always.

If  $P_2 \geq 0$  there exists a non-convex cyclic quadrilateral the sides of which are (in a certain order)  $a, b, c, d$ . In that case the equality sign at the left-hand side (1) holds for that quadrilateral.

If  $P_2 < 0$  no such quadrilateral exists. The inequalities (1) can be replaced by

$$0 \leq 16F^2 \leq P_1$$

and equality at the left takes place for the quadrilateral (always existing in this case) for which the diagonals are parallel.

O. Bottema, Euclides 15 (1938), 1-13.

**15.24**  $0 \leq \cos^2 \frac{1}{2}(\alpha + \gamma) = \cos^2 \frac{1}{2}(\beta + \delta) \leq 1$  if  $P_2 \geq 0$ ,

$$0 \leq \cos^2 \frac{1}{2}(\alpha + \gamma) = \cos^2 \frac{1}{2}(\beta + \delta) \leq \frac{P_1}{16abcd} \text{ if } P_2 \leq 0,$$

where  $P_1$  and  $P_2$  are defined in 15.23.

O. Bottema, Euclides 15 (1938), 1-13.

**15.25**  $pq \geq |ac - bd|$  if  $P_2 > 0$ ,

$$pq \geq |a^2 - b^2 + c^2 - d^2| \text{ if } P_2 \leq 0,$$

where  $P_2$  is defined in 15.23.

Equality for the non-convex cyclic quadrilateral, resp. for that with parallel diagonals.

O. Bottema, Euclides 15 (1938), 1-13.

**15.26** If  $\theta$  is one of the angles between the diagonals, then

$$\sin^2 \theta = 1 \text{ if } a^2 - b^2 + c^2 - d^2 = 0.$$

In the other cases

$$\frac{P_2}{4(ac - bd)^2} \leq \sin^2 \theta \leq \frac{P_1}{4(ac + bd)^2} \text{ if } P_2 \geq 0,$$

$$0 \leq \sin^2 \theta \leq \frac{P_1}{4(ac + bd)^2} \text{ if } P_2 < 0,$$

where  $P_1$  and  $P_2$  are defined in 15.21.

O. Bottema, Euclides 15 (1938), 1-13.

## 16. Inequalities for polygons

16.1 For any pentagon  $A_1A_2A_3A_4A_5$

$$\sum_{i=1}^5 (A_iA_{i+2} + A_{i+1}A_{i+4})A_iA_{i+1} > \sum_{i=1}^5 A_iA_{i+2}^2 \quad (A_{i+5} = A_i). \quad (1)$$

PROOF. Since

$$A_iA_{i+1} + A_{i+1}A_{i+2} > A_iA_{i+2} \quad (i = 1, \dots, 5)$$

or

$$(A_iA_{i+1} + A_{i+1}A_{i+2})A_iA_{i+2} > A_iA_{i+2}^2 \quad (i = 1, \dots, 5),$$

we have (1).

16.2 Let  $A'$ ,  $B'$ ,  $C'$ ,  $D'$ ,  $E'$  be the midpoints of sides of a convex pentagon  $ABCDE$ . Then

$$\text{area } A'B'C'D'E' \geq \frac{1}{2} \text{ area } ABCDE.$$

16.3 Let  $ABCDE$  be a convex pentagon inscribed within a unit circle with  $AE$  as diameter, then

$$a^2 + b^2 + c^2 + d^2 + ab + cd < 4,$$

where  $a = AB$ ,  $b = BC$ ,  $c = CD$ ,  $d = DE$ .

H. Demir, Problem 1877, Amer. Math. Monthly 73 (1966), 410.

16.4 Consider a regular hexagon of area  $F$ , which contains non-intersecting circles of radii  $r_1, \dots, r_n$ . Then

$$(r_1 + \dots + r_n)^2 \leq \frac{nF}{\sqrt{12}}.$$

L. Fejes Tóth, Lagerungen in der Ebene, auf der Kugel und im Raum, Berlin 1953, p. 75.

16.5 Let  $(n-1)s$  be the perimeter of an  $n$ -gon with sides  $a_i$  ( $i = 1, \dots, n$ ), and let  $a_i \leq s$  ( $i = 1, \dots, n$ ). Then

$$\begin{aligned} \frac{n}{n-1} &\leq \frac{a_1}{a_2+\dots+a_n} + \frac{a_2}{a_3+\dots+a_n+a_1} + \dots \\ &+ \frac{a_n}{a_1+\dots+a_{n-1}} \leq \frac{n-1}{n-2}. \end{aligned} \quad (1)$$

PROOF. Denote the expression whose bounds we want by  $S$ . Since

$$a_i \leq s \text{ for } i = 1, \dots, n,$$

we have

$$a_1 + \dots + a_{i-1} + a_{i+1} + \dots + a_n = (n-1)s - a_i \geq (n-2)s,$$

so that

$$S \leq \frac{1}{(n-2)s} (a_1 + \dots + a_n) = \frac{(n-1)s}{(n-2)s},$$

which represents the right-hand side of (1).

Since we can write  $S$  in the form

$$\begin{aligned} S &= \left( \frac{1}{a_2 + \dots + a_n} + \frac{1}{a_3 + \dots + a_n + a_1} + \dots \right. \\ &\quad \left. + \frac{1}{a_1 + \dots + a_{n-1}} \right) (n-1)s - n \end{aligned} \quad (2)$$

and since by the arithmetic-harmonic mean inequality we have

$$\begin{aligned} \frac{1}{a_2 + \dots + a_n} + \frac{1}{a_3 + \dots + a_n + a_1} + \dots + \frac{1}{a_1 + \dots + a_{n-1}} \\ \geq n^2 [(a_2 + \dots + a_n) + (a_3 + \dots + a_n + a_1) + \dots \\ + (a_1 + \dots + a_{n-1})^{-1}] \\ = \frac{n^2}{(n-1)^2 s}, \end{aligned}$$

the left-hand side of (1) follows from (2).

This proof is due to P. M. Vasić.

M. Petrović, Bulletin Mathématique de la Société Roumaine des Sciences 40 (1938), 205–205.

S. Pavlović, Uvodjenje mladih u naučni rad, I, Matematička biblioteka, sv. 18, Beograd 1961, 139–142.

**16.6** For every convex  $n$ -gon, with area  $F$ ,

$$\sum_{i=1}^n a_i^2 \geq 4F \operatorname{tg} \frac{\pi}{n}.$$

**PROOF.** We use the fact that of all  $n$ -gons with the same perimeter, the regular  $n$ -gon has greatest area. Thus

$$F \leq \frac{na^2}{4 \operatorname{tg} \pi/n}$$

where  $na = \sum_{i=1}^n a_i$ . The Cauchy-Schwarz inequality, however, gives

$$na^2 = n \cdot (\sum_{i=1}^n a_i)^2 \leq n^{-1} (\sum_{i=1}^n a_i^2) (\sum_{i=1}^n 1) = \sum_{i=1}^n a_i^2,$$

whence the desired inequality follows.

E. Just-N. Schramberger, Problem 1634, Amer. Math. Monthly 70 (1963), 1005 and 71 (1964), 796.

**16.7** Let  $P$  be an internal point of a convex  $n$ -gon. If  $r_k$  is the radius of the incircle of the triangle  $A_kPA_{k+1}$  ( $k = 1, \dots, n$  and  $A_{n+1} = A_1$ ), then

$$r_1 + \dots + r_n \leq \sqrt{\frac{nF}{3\sqrt{3}}}.$$

This inequality is due to Chr. Karanikolov.

**16.8** Let  $R_i$  be the distances from an internal point  $P$  of an  $n$ -gon to its vertices, and  $r_i$  be the distances from  $P$  to its sides. Then

$$R_1 \dots R_n \geq \left( \sec \frac{\pi}{n} \right)^n r_1 \dots r_n.$$

L. Fejes Tóth, Lagerungen in der Ebene, auf der Kugel und im Raum, Berlin 1953, p. 33.

**REMARK.** For  $n = 3$  we have 12.25.

**16.9** Let  $A_1, \dots, A_n$  be the vertices of a convex  $n$ -gon and  $P$  an internal point. Let  $R_k = PA_k$  and let  $r_k$  be the distance from  $P$  to the side  $A_kA_{k+1}$ .

L. Fejes Tóth has conjectured the following inequality

$$\left(\cos \frac{\pi}{n}\right) \sum_{k=1}^n R_k \geq \sum_{k=1}^n r_k. \quad (1)$$

For  $n = 3$ , (1) reduces to the Erdős-Mordell's inequality (see: 12.13).

Inequality (1) for  $n = 4$  was proved by A. Florian.

Later, Lenhard proved (1) for every natural number  $n \geq 3$  and deduced the following stronger inequality

$$\left(\cos \frac{\pi}{n}\right) \sum_{k=1}^n R_k \geq \sum_{k=1}^n w_k,$$

where  $w_k$  is the segment of the bisector of the angle  $A_kPA_{k+1} = 2\varphi_i$  from  $P$  to its intersection with the side  $A_kA_{k+1}$ .

Lenhard proved also that the following inequality holds:

$$\sqrt{R_k R_{k+1}} \cos \varphi_i \geq w_k \geq r_k.$$

L. Fejes Tóth, *Lagerungen in der Ebene, auf der Kugel und im Raum*, Berlin 1953, p. 33.

A. Florian, Elem. Math. 13 (1958), 55–58.

H. C. Lenhard, Arch. Math. 12 (1961), 311–314.

**16.10** Let  $A_1, \dots, A_n$  be the vertices of a regular polygon and let  $P$  be an point in its interior. Then at least one of the angles  $A_iPA_j$  satisfies the following inequalities

$$\pi \left(1 - \frac{1}{n}\right) \leq A_iPA_j \leq \pi.$$

P. Erdős, Problem 4086, Amer. Math. Monthly 50 (1943), 391; 51 (1944), 480 and 54 (1947), 117.

**16.11** Let  $A_1, \dots, A_n$  be the vertices of a regular polygon,  $P$  any point, then

$$\sum_{k=1}^n A_k P \geq \left[ \left( \cos \frac{\pi}{2n} \right)^{\frac{1+(-1)^n}{2}} \cdot \operatorname{cosec} \frac{\pi}{2n} \right] \max_k A_k P + \frac{1+(-1)^n}{2} \min_k A_k P.$$

This result is the best possible.

I. S. Gál, Bull. Ecole Polytech. Jassy 3 (1948), 75-106.

**REMARK.** This inequality is a generalization of 15.22.

**16.12** Let  $A_1, \dots, A_n$  be the vertices of a convex  $n$ -gon and  $P$  an internal point. Then

$$\frac{L}{2} < \sum_{k=1}^n A_k P < (n-1) \frac{L}{2},$$

where  $L$  is perimeter of the  $n$ -gon.

A. Natucci, Periodico Mat. (4) 29 (1951), 98-101.

**16.13** Let  $P$  be an internal point of the convex polygon  $A_1, \dots, A_n$ . Let  $R_k$  denote the distance from  $P$  to the vertex  $A_k$  and  $r_k$  its distance to the side  $A_k A_{k+1}$ . Let  $\alpha_k = \angle_{A_{k-1} A_k A_{k+1}}$ . Then

$$\sum_{k=1}^n R_k \sin \frac{\alpha_k}{2} \geq \sum_{k=1}^n r_k. \quad (1)$$

Equality holds if and only if all sides of the polygon are tangent to a circle with centre  $P$ .

**PROOF.** Let  $\beta_k = \angle P A_k A_{k+1}$ . From the triangle  $P A_k P_k$ , where  $P_k$  is the orthogonal projection of  $P$  on the sides  $A_k A_{k+1}$ , we get

$$r_k = R_k \sin \beta_k \quad (k = 1, \dots, n). \quad (2)$$

From the triangle  $P A_{k+1} P_k$  we obtain

$$r_k = R_{k+1} \sin (\alpha_{k+1} - \beta_{k+1}) \quad (k = 1, \dots, n-1). \quad (3)$$

$$r_n = R_1 \sin (\alpha_1 - \beta_1).$$

Adding together (2) and (3), we get

$$2 \sum_{k=1}^n r_k = \sum_{k=1}^n R_k [\sin (\alpha_k - \beta_k) + \sin \beta_k].$$

i.e.

$$2 \sum_{k=1}^n r_k = 2 \sum_{k=1}^n R_k \sin \frac{\alpha_k}{2} \cos \left( \frac{\alpha_k}{2} - \beta_k \right). \quad (4)$$

Using  $\cos[(\alpha_k/2) - \beta_k] \leq 1$ , (4) gives (1).

Equality holds if and only if  $\cos[(\alpha_k/2) - \beta_k] = 1$ , i.e., if and only if  $\alpha_k = 2\beta_k$ . This condition is satisfied if the sides of the polygon are tangent to a circle with centre  $P$ .

This result is due to R. R. Janić.

**16.14** If  $F$  is the area of an  $n$ -gon of diameter 1, then

$$F \leq \frac{n}{2} \cos \frac{\pi}{n} \operatorname{tg} \frac{\pi}{2n}.$$

N. D. Kazarinoff, Analytic Inequalities, New York 1961, p. 84.

**16.15** Let  $A_1 \dots A_n$  be a convex polygon and let

$$\alpha_k = \angle A_{k-1} A_k A_{k+1} \quad (k = 2, \dots, n-1)$$

$$\varphi = \angle A_{n-1} A_n A_1, \quad \psi = \angle A_n A_1 A_2$$

be its internal angles. If

$$A_1 A_2 = A_2 A_3 = \dots = A_{n-1} A_n = A_n A_1$$

and

$$\alpha_2 \geq \alpha_3 \geq \dots \geq \alpha_{n-1} \quad (1)$$

then

$$\varphi \geq \psi. \quad (2)$$

Inequality holds in (2) if and only if at least one inequality occurs in (1).

S. Bilinski, Glasnik matematičko-fizički i astronomski, 16 (1961), 195–201.

**16.16** Let a polygon contain a circle with radius  $r$ . Then

$$\min \frac{\alpha_1 + \alpha_2 + \dots + \alpha_n}{n} \geq \min \frac{n}{1/\alpha_1 + 1/\alpha_2 + \dots + 1/\alpha_n}.$$

Equality holds only for  $n = 3$ .

F. Leuenberger, Elem. Math. 15 (1960), 77–79.

**16.17** Let  $a_1, \dots, a_n$  be the distances of any point on the circle, circumscribed with radius  $R$  about a regular  $n$ -gon, to its vertices. Then

$$R\sqrt{2n} \leq a_1 + \dots + a_n \leq n\sqrt{2}. \quad (1)$$

**PROOF.** Since

$$a_1^2 + \dots + a_n^2 = 2R^2n,$$

$$\frac{(a_1 + \dots + a_n)^2}{n} \leq a_1^2 + \dots + a_n^2 \leq (a_1 + \dots + a_n)^2,$$

inequality (1) is indeed true.

M. Petrović, Računanje sa brojnim razmacima, Beograd 1932, p. 82.

**16.18** With the notation of 16.17, we have

$$2R \cot \frac{\pi}{2n} \leq a_1 + \dots + a_n \leq 2R \cosec \frac{\pi}{2n}.$$

P. M. Vasić and Ž. Živanović, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 181-No. 196 (1967), 67-68.

**16.19** About a circle is circumscribed a regular polygon of  $n$  sides each of length  $a_n$ , and within this circle is inscribed a regular polygon of  $k$  sides each of length  $b_k$ . Then the following inequalities hold:

$$\begin{aligned} a_{n-1} &> b_n & (n = 3, 4), \\ a_{n-1} &< b_n & (n = 5, 6, \dots), \\ 2a_{n-1} &> b_n + b_{n+1} & (n = 3, \dots, 8), \\ 2a_{n-1} &< b_n + b_{n+1} & (n = 9, 10, \dots). \end{aligned}$$

N. Obreškov, Problems 30 and 31, Fiziko-matematičeskoe spisanie (Sofia) 3 (1960), 310-312.

**16.20** If  $S_k$  is the sum of all the perpendiculars from the centre to the sides of a regular polygon of  $k$  sides which is inscribed in a circle of radius  $\bar{r}$ , then

$$S_{k+1} - S_k > R.$$

**16.21** Given a circle  $C$  of circumference  $U$ . Let  $p_n$  be the perimeter of the regular  $n$ -gon inscribed in  $C$  and  $P_n$  the perimeter of the circumscribed regular  $n$ -gon ( $n > 3$ ). Then

- 1°. The sequence  $(p_n)$  is monotonously increasing, and the sequence  $(P_n)$  is monotonously decreasing;
- 2°.  $\frac{1}{2}p_n + \frac{1}{2}P_n > U$ .

A proof of this inequality, due to D. D. Adamović, can be found in: D. S. Mitrinović, Elementary Inequalities, Groningen 1964, pp. 115–117.

**16.22** Given a circle  $C$  with radius  $R$ . Let  $o_n$  be the perimeter of the regular  $n$ -gon inscribed in  $C$  and  $O_n$  the perimeter of the circumscribed regular  $n$ -gon. Then

$$o_n + O_n > 4\pi R. \quad (1)$$

PROOF. We shall first prove that

$$\sin x + \operatorname{tg} x > 2x \text{ for } 0 < x < \frac{\pi}{2}. \quad (2)$$

Consider the function  $f(x) = \sin x + \operatorname{tg} x - 2x$ . Its derivative is

$$f'(x) = \cos x + \frac{1}{\cos^2 x} - 2 = \frac{(1 - \cos x)(1 + \cos x(1 - \cos x))}{\cos^2 x}.$$

Since  $f(0) = 0$  and  $f'(x) > 0$  for  $0 < x < \pi/2$ , one gets  $f(x) > 0$ , i.e., (2).

Since  $o_n = 2Rn \sin \pi/n$  and  $O_n = 2Rn \operatorname{tg} \pi/n$ , we have

$$o_n + O_n = 2Rn \left( \sin \frac{\pi}{n} + \operatorname{tg} \frac{\pi}{n} \right).$$

For  $n \geq 3$ , we get  $0 < \pi/n < \pi/2$  and consequently

$$\sin \frac{\pi}{n} + \operatorname{tg} \frac{\pi}{n} > 2 \frac{\pi}{n}.$$

Therefore (1).

Matematička v škole, 1965, No. 3, 76.

**16.23** Let  $F$ ,  $L$ ,  $r$ ,  $R$  be area, perimeter, radius of incircle, radius of circumcircle respectively of a convex  $n$ -gon. Then

$$nr^2 \operatorname{tg} \frac{\pi}{n} \leq F \leq \frac{1}{2}nR^2 \sin \frac{2\pi}{n},$$

$$2nr \operatorname{tg} \frac{\pi}{n} \leq L \leq 2nR \sin \frac{\pi}{n}.$$

Equalities hold if and only if the  $n$ -gon is equilateral.

J. Kürschak, Math. Ann. 30 (1887), 578–581.

**16.24** The area of any  $n$ -gon, inscribed in an unit circle is at most equal to  $\frac{1}{2}n \sin 2\pi/n$ .

REMARK. This inequality was to be proved by high school students on competition in China in 1957.

**16.25** Given a convex polygon inscribed in a circle, with vertices  $A_1, \dots, A_n$ . Consider a second polygon, whose consecutive vertices are the midpoints of the arcs  $A_1A_2, A_2A_3, \dots, A_nA_1$ . Then the area of the second polygon is greater or equal to the area of the first polygon, with equality if and only if the initial polygon is regular.

M. S. Klamkin, Math. Teacher 60 (1967), 323–328.

**16.26** Let  $r$ ,  $R$ ,  $L$  be the radius of the incircle, radius of the circumcircle and the perimeter of a regular polygon, respectively. Then

$$2\pi \sqrt[3]{rR^2} < L < \frac{2}{3}\pi(r+2R).$$

E. Catalan, Question 65, Mathesis 2 (1882), 85–88 and 245–248.

**16.27** If  $t_n$  denotes the area of a regular  $n$ -sided polygon inscribed in a circle of unit radius and if  $T_n$  denotes the area of a regular  $n$ -sided polygon circumscribed about this circle, then

$$\frac{3t_nT_n}{T_n+2t_n} < \pi < \frac{t_n+2T_n}{3}.$$

P. Szász, Mat. Lapok 5 (1954), 73–78.

**16.28** Denote  $L$  and  $L'$  the perimeters of two polygons inscribed in the same circle. If the largest side of the second is less than the smallest side of the first, then

$$L < L'.$$

J. V. Uspensky, Amer. Math. Monthly 34 (1927), 27–250.

**16.29** Let  $F_i$  be the area of the convex polygon inscribed in an unit circle and let  $F_c$  be the area of the circumscribed polygon whose points of contact with the circle coincide with the vertices of the inscribed polygon. Then

$$F_i + F_c \geq 6.$$

Equality holds only for squares.

**REMARK.** This inequality was conjectured by P. Szász and has been proved by: J. Aczél and L. Fuchs, Compositio Math. 8 (1950), 61–67.

## 17. Inequalities for a circle

**17.1** Let  $P$  be a point inside the circle  $K$  of radius  $R$ . If  $\text{arc } AF = \text{arc } FB$ , with  $\text{arc } AB < R\pi$  and  $PA < PB$ , then  $\widehat{APF} > \widehat{FPB}$ .

**REMARK.** This theorem was given to be proved by secondary school students at the competition in Hungary in 1950.

**17.2** Let  $A$  and  $B$  be two points outside a circle  $K$  and let the straight line segment  $AB$  not intersect this circle. If  $C$  and  $D$  are the points of contact of tangent lines drawn from  $A$  and  $B$  with opposite orientation to the circle  $K$ , then the following inequalities hold

$$|AC - BD| < AB < AC + BD. \quad (1)$$

If the straight line  $AB$  intersects the circle  $K$  neither inequality in (1) holds.

**PROOF.** Let the straight line segment  $AB$  not intersect this circle. Consider the triangle  $ADM$ ,  $M$  being the intersection of

the tangent lines. We have

$$AM + BM > AB > |AM - BM|.$$

Since

$$AC > AM, \quad BD > BM, \quad MC = MD,$$

we get

$$AC + BD > AB > |AC - BD|.$$

If the straight line  $AB$  cuts the circle we shall have two cases:

1°. the points of intersection of the straight line  $AB$  and the circle lie between  $A$  and  $B$ ;

2°. the points of intersection of the straight line  $AB$  and the circle do not lie between  $A$  and  $B$ .

If 1° holds, we have

$$AB > AE + EB > AC + BD,$$

where  $E$  and  $F$  are the intersection segment  $AB$  with  $OC$  and  $OD$  respectively.

If 2° holds, the segment  $AI$  lies in the angle  $CAC'$ , where  $C'$  is the point of contact of the second tangent line drawn from point  $A$ . Let us draw another circle, concentric to given circle, through point  $B$ . Let this new circle cut  $AC$  and  $AC'$  in points  $E$  and  $E'$ . Then

$$AB < AE = AI - EC = AC - BD.$$

V. B. Lidskii, L. V. Ovsjannikov, A. N. Tulaikov, M. I. Šabunin, Problem 362, Zadaci po elementarnoi matematike, Moskva 1962, p. 62.

**17.3** A circle is inscribed in a right angle with vertex in  $A$  such that it touches the legs of the angle in points  $M$  and  $N$  ( $AM < AB$ ,  $AN < AC$ ), then

$$\frac{1}{2}(AB + AC) < MN < \frac{1}{2}(AB + AC).$$

PROOF. If  $K$  is the point of contact of the tangent  $MN$  and circle, then we have  $MB = MK$  and  $NK = NC$ , whence

$$MN = MB + NC. \quad (1)$$

Since

$$MN < AM + AN,$$

we get

$$2MN < AM + MB + AN + NC = AB + BC.$$

On the other hand,  $MN > AM$  and  $MN > AN$ , i.e.,

$$2MN > AM + AN,$$

and because of (1) we obtain

$$3MN > AM + MB + AN + NC = AB + AC.$$

**17.4** Consider two circles of radii  $R$  with centres in points  $C_1$  and  $C_2$ . Let  $A$  be a point on the first circle, and  $B_1$  a point on the second.

If point  $B_2$  is symmetrical to point  $B_1$  in respect to the straight line  $C_1C_2$  and if  $C_1C_2 = R$ , then

$$AB_1^2 + AB_2^2 \geq 2R^2.$$

**PROOF.** From triangles  $ABB_1$  and  $ABB_2$ , where  $B$  is midpoint of the straight line  $B_1B_2$ , we have

$$AB_1^2 = AB^2 + BB_1^2 - 2AB \cdot BB_1 \cos \widehat{ABB_1}, \quad (1)$$

$$AB_2^2 = AB^2 + BB_2^2 - 2AB \cdot BB_2 \cos \widehat{ABB_2}. \quad (2)$$

Since  $\widehat{ABB_1} + \widehat{ABB_2} = \pi$ , adding together the equalities (1) and (2), we obtain

$$AB_1^2 + AB_2^2 = 2(AB^2 + BB_1^2). \quad (3)$$

From (3) and inequality  $AB \geq BC_2$ , we get

$$AB_1^2 + AB_2^2 \geq 2(BC_2^2 + BB_1^2) = 2 \cdot B_1C_2^2 = 2 \cdot R^2.$$

**17.5** If  $AC$  and  $BD$  are two mutually perpendicular chords of a circle with radius  $R$ , then

$$2R \leq AC + BD \leq 4R. \quad (1)$$

**PROOF.** Let  $E$  be the other end of the diameter which passes through the point  $A$  and let  $P$  denote the point of intersection

$AC$  and  $BD$ . Then we get the equalities:

$$PA^2 + PB^2 = AB^2, \quad PC^2 + PD^2 = CD^2 = BE^2,$$

whence

$$PA^2 + PB^2 + PC^2 + PD^2 = AB^2 + BE^2 = 4R^2. \quad (2)$$

On the basis of the inequalities

$$\frac{(x+y+z+t)^2}{4} \leq x^2 + y^2 + z^2 + t^2 \leq (x+y+z+t)^2,$$

where  $x, y, z, t$  are non-negative real numbers, we see that the sum of squares on the left-hand side of identity (2) is between

$$\frac{1}{4}(PA + PB + PC + PD)^2 = \frac{1}{4}(AC + BD)^2$$

and

$$(PA + PB + PC + PD)^2 = (AC + BD)^2.$$

i.e.

$$\frac{1}{4}(AC + BD)^2 \leq 4R^2 \leq (AC + BD)^2.$$

Whence (1) follows.

M. Petrović, Računanje sa brojnim razmacima, Beograd  
1932, pp. 84–85.

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