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Wave–current interaction on a free surface

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Abstract

The classical water wave equations (CWWEs) comprise two boundary conditions for the two-dimensional flow on the free surface of a bulk three-dimensional (3D) incompressible potential flow in the volume bounded by the free surface, which itself moves under the restoring force of gravity. One of these two boundary conditions provides the kinematic definition of the vertical velocity of the surface elevation. The other boundary condition is the dynamic Bernoulli law that governs the evaluation of the bulk velocity potential on the free surface. The present paper applies these two boundary conditions as constraints in the action integral for Hamilton's variational principle, along with a non-hydrostatic pressure constraint that imposes incompressible flow on the free surface. The stationary variations in Hamilton's principle then yield closed dynamical equations of free surface flow whose divergence-free velocity admits nonzero vorticity and whose nonhydrostatic pressure matches the pressure of the 3D bulk flow when evaluated on the free surface. A minimal coupling approach is proposed to model the mutual interactions of the waves and currents. The dynamical effects of horizontal buoyancy gradients are also considered in this context. For any combination of these model variables, the resulting system of variational equations admits a Lie–Poisson Hamiltonian formulation. Finally, stochastic versions of these model equations are derived by assuming that the material loop for their Kelvin circulation theorem in each case follows stochastic Lagrangian histories in a Stratonovich sense.

KEYWORDS
free surface fluid dynamics, geometric mechanics, nonlinear water waves

> *The rest … is a series of speculations, which we hope to verify eventually*.– H. Segur et al.[1]

1 | INTRODUCTION

1.1 | Background

Waves are disturbances in a medium that propagate due to a restoring force, such as gravity. Currents are flows that transport physical properties, such as mass and heat. When waves propagate in a moving medium, the motion of the medium can affect the waves, and vice versa, the waves can affect the motion of the medium, as they both respond to the same force. The primary example is wave–current interaction on the free surface of a fluid flow under the influence of gravity. This mutual wave–current interaction is the province of nonlinear water wave theory. In nonlinear water wave dynamics on a free surface, the distinction between waves and currents is clear: the vertical velocity and surface elevation are wave variables, whereas the horizontal fluid velocity components and areal mass density are current variables. This is particularly clear in the Hamiltonian formulation of wave–current interaction dynamics, in which the symplectic Poisson operator for the two independent degrees of freedom separates into block diagonal form.

Water waves—waves on the surface of a body of water—have fascinated observers over the ages, not only because water waves are so easily observed, and not only because they move; but primarily because they form coherent moving deformations of the water surface that can interact with each other in a multitude of ways. Any disturbance—even scooping your hand in a narrow channel of shallow water, for example—will resolve itself into a train of coherent solitary waves with a few extra ripples that are left behind as the coherent solitary waves propagate away from the disturbance. The fascination in observing the creation of coherent water waves from arbitrary disturbances was captured in the famous report by the Victorian engineer John Scott Russell in August 1834, when he saw a solitary wave create itself from an impulse of current and then start propagating along a Scottish canal. The wave was considerably faster than the flow in the canal. As Russell wrote,[2]

> I followed it on a horseback, and … after a chase of one or two miles I lost it in the windings of the channel. Such, … was my first chance interview with that singular and beautiful phenomenon.

Although the classic water wave (CWW) theory introduced in 1847 by Stokes[3] now has a long history, see, for example, Refs. 4, 5, the excitement in the chase for mathematical understanding of water waves still continues. In particular, John Scott Russell's "singular and beautiful phenomenon" is now called a *soliton*. The word "soliton" was coined in a 1965 paper by Zabusky and Kruskal[6] and this word has more than 6 million Google hits, as of this writing. The sequence of approximate shallow water equations exhibiting soliton behavior includes the Korteweg-de Vries (KdV) equation in 1D,[7] as well as the Kadomtsev–Petviashvili (KP)[8] equation, which extends the KdV equation to allow weak transverse spatial dependence. Indeed, the solution behavior of soliton water wave equations still inspires mathematical progress in the theory of integrable Hamiltonian systems of nonlinear partial differential equations and their discretizations in space and time. For a good summary of the early developments of soliton theory, see Ablowitz and Segur.[9] For historical discussions of water wave theory, see Refs. 4, 5, 10. For modern mathematical discussions of the CWW theory introduced by Stokes,[3] see, for example, Refs. 11–15. A few references among modern treatments of water wave theory that are similar in spirit to the present work are[12,16–19]. A classic review of the various historical formulations of the wave–current interaction problem is given in Ref. 20.

1.2 | Objectives and methodology

Within the framework of wave–current interaction, one notes that "waves" can propagate along the free surface either with the flow, or relative to the flow as a travelling shift in the phase of the elevation that does not carry mass as it propagates on the surface of the flow. For example, if one were to place dye within a wave elevation, the wave need not carry that dye along with it (although waves with this property certainly can exist). Indeed, the wave reported by Russell in 1834 was propagating along the canal at a speed that required a horse to keep up with it, although it is a safe assumption that the "current" flow velocity in the canal was much slower. In fact, modern water wave experiments such as those of T.Y. Wu[21] report observations of periodic emission of waves that propagate in the *opposite direction* of the current in shallow-water flow over a submerged obstacle. In a situation where the free surface elevation follows the currents, the waves would be associated with mass transport and their rate of propagation would equal the fluid transport velocity. The present work will develop models in which the free surface waves can either propagate with the fluid transport velocity as in the CWW theory, or propagate in either direction on the background flow of the fluid transport velocity.

The CWW equations (CWWEs) for potential flow on a free surface comprise the kinematic constraint at the free boundary and the horizontal gradient of Bernoulli's law for potential flow restricted to the surface. This paper has two primary objectives based on the CWWE.

The first objective is to augment the CWWE to include fundamental physical aspects of wave–current interaction on a free surface (WCIFS). These physical aspects include vorticity, wave–current coupling in which the wave activity creates fluid circulation, nonhydrostatic pressure, incompressibility, and horizontal gradients of buoyancy.

The multiscale, fast–slow aspects of the wave–current interaction comprise a grand challenge for modern computational simulation. This challenge is particularly important in computational simulations of global ocean circulation. In ocean physics, the fast–slow aspects of wave–current interaction tend to introduce irreducible imprecision even beyond computational uncertainty because of unresolvable, or even unobservable, processes.[22] This situation leads to the paper's second objective.

The paper's second objective aims to introduce stochastic transport of wave activity by fluid circulation that is intended to be used in combination with data assimilation to model uncertainty due to the effects of fast, computationally unresolvable, or unknown effects of WCI on its slower, computationally resolvable aspects.

To pursue these two objectives, we will begin by using the Dirichlet–Neumann operator (DNO) for 3D potential flow of a homogeneous Euler fluid to impose the kinematic and dynamic boundary conditions of CWWE as *constraints* on the motion of the free surface in the Euler–Poincaré (EP) variational principle for ideal fluids.[23] The EP formulation is an extension of earlier variational principles for the CWWE.[24,25] In using the CWWE as constraints, the EP variational principle introduces additional dynamical equations for the Lagrange multipliers. The Lagrange multipliers are interpreted as the vertical velocity $\widehat{w}$ and the areal mass density D, arising as Hamiltonian variables canonically conjugate to the elevation ζ and the surface velocity potential $\widehat{\phi}$, respectively. The resulting Hamiltonian equations are referred to as extended CWWE (ECWWE).

After a discussion of alternative formulations that elicit a variety of properties of the ECWWE solutions, we use the EP approach to add further aspects of wave–current interaction, which include vorticity, as well as nonhydrostatic pressure and buoyancy gradients in the free surface flow.

We also introduce a wave–current minimal coupling (WCMC) term into the action integral for the EP variational principle that generates fluid circulation from wave activity and vice versa. The EP variational equations are referred to here as WCIFS equations. Finally, to model the uncertainties associated with the computations of these multiscale fast–slow WCIFS equations, we introduce stochastic advection by Lie transport (SALT) of the wave activity by the current flow, again following the EP variational approach, as in Ref. 26. In the analytical sections of the paper, both the deterministic and SALT versions of our WCIFS equations are shown to be locally well posed in the sense of existence, uniqueness, and continuous dependence on initial conditions.

1.3 | Plan of the paper

- Section 2 reviews the problem statement, boundary conditions, and key relations in the classical framework of three-dimensional (3D) fluid flows under gravity with a free surface. The free surface elevation is measured from its rest position, which defines the origin of the vertical coordinate $z = 0$. The elevation, $z = \zeta(\boldsymbol{r}, t)$, is a function of the horizontal position vector $\boldsymbol{r} = (x, y, 0)$ and time t. A key relation is stated in Equation (7). Namely, when evaluated on the free surface, the material time derivative of a function $f(\boldsymbol{r}, z, t)$ and its projection onto the free surface $f(\boldsymbol{r}, \zeta(\boldsymbol{r}, t), t)$ are equal. In combination with the kinematic boundary condition, the projection relation in Equation (7) leads to *Choi's relation* (9) for the dynamics of the free surface.[17] However, the resulting dynamical system in the motion of the free surface *is not yet closed* because: (i) the horizontal pressure gradient on the free surface is still unknown and (ii) an evolutionary equation for the vertical velocity is still missing.[17,19]
- Section 3.1 reviews the formulation of the CWWEs for free surface dynamics in terms of the DNO. Section 3.2 then derives the ECWWE that include equations for the vertical velocity $\widehat{w}$ and the preserved area measure D on the free surface. The derivation of ECWWE proceeds by regarding the CWWE as constraints imposed by Lagrange multipliers $\widehat{w}$ and D in a new variational principle in Equation (37), defined in terms of functions on the horizontal mean level of the free surface. The Lie–Poisson Hamiltonian form of the ECWWE is derived in Section 3.5. In Section 3.3, nonhydrostatic pressure is incorporated into the ECWWE and the comparison to the key relation in (9) is shown in Theorem 1. The ECWW theory satisfies a sort of time-dependent nonacceleration theorem, by which the fluid and wave circulations are preserved

separately. Hence, the ECWWE does not really qualify as genuine wave–current interaction, because the time-dependent flows of real fluids allow exchange of circulation between waves and currents.

The nonacceleration property of the ECWWE may be rectified by introducing a term into the Lagrangian that represents the dependence of the kinetic energy on the wave slope. The new kinetic energy term results in a system where the transport velocity is unaffected and the wave dynamics creates circulation in the current flow when the gradients of vertical velocity and surface elevation are not aligned.

- Section 4 introduces the *augmented* CWW system (ACWW) that includes a different wave–current interaction that rectifies the nonacceleration property of the ECWWE via a minimal coupling (WCMC) construction. The additional term involved in this construction produces a shift in the transport velocity for the wave elevation which depends on the wave slope. The result is that the waves can move relative to the fluid parcels on the free surface, and hence, the system can support waves that do not transport mass. The Kelvin–Noether circulation theorem for the ACWW model in Theorem 3 shows that the wave dynamics of the ACWW model can create circulation, as shown locally in Theorem 4 and Corollary 2. In addition, as shown in Theorem 5, the ACWW model with genuine wave–current interaction preserves the same physical energy as the ECWW model does, for which the fluid and wave circulations are preserved separately. That is, the ACWW model with its WCMC term preserves the same energy as the ECWW model. With the addition of the coupling term, the equations admit a potential vorticity (PV) formulation, which is usually rendered impossible in water wave theory by the irrotational flow assumption. See, however, Refs. 12, 17, 27 for other water wave equations that admit irrotational flow.

  Section 5 includes additional physical properties into the ACWWE to produce our final model of WCIFS. The WCIFS model is derived by modifying Hamilton's principle for ACWW with its wave–fluid coupling term, to add an advected scalar buoyancy variable ρ with nonzero horizontal gradients, as well as nonhydrostatic pressure in (99). The Kelvin–Noether circulation theorem for the WCIFS model is given in Theorem 6.
- Section 6 introduces a method of incorporating stochastic noise into this theory that preserves its variational structure. This is achieved by applying the method of SALT.[26] Within this section, we derive a stochastically perturbed version of the classical water wave equations, as well as a stochastic variational models of the ECWW, ACWW, and WCIFS models of wave–current interaction of free surfaces.
- Section 7 lays out the analytical framework for dealing with the compressible and incompressible ECWWE equations discussed in Sections 3.1 and 3.3, respectively.
- Section 8 discusses potential future research directions and identifies new problems opened up by this work.
- Appendix A discusses transformation theory for ideal fluid dynamics and derives the Kelvin circulation theorem by using the Lie chain rule.
- Appendix B reviews the 3D inhomogeneous Euler equations for incompressible fluid flow under gravity with a free upper surface and a fixed bottom topography. This is done by deriving these equations using a constrained variational principle. In particular, Appendix B reviews the theory of Langmuir circulations due to Craik and Liebovich[28].
- Appendix C treats the reduced Legendre transformation from Hamilton's principle to the Hamiltonian formulation for ECWWE in the Eulerian fluid representation.

Abbreviations

- ACWWE augmented CWWE
- CWWEs classical water wave equations
- DNO Dirichlet–Neumann operator
- ECWWE extended CWWE;
- EP Euler-Poincaré
- SALT stochastic advection by Lie transport
- WCI wave–current interaction
- WCIFS wave–current interaction on a free surface
- WCMC wave–current minimal coupling;

2 | PROBLEM STATEMENT, BOUNDARY CONDITIONS, AND KEY RELATIONS

2.1 | Problem statement

We study the dynamics of fluid parcels that are constrained to remain on the free surface of a 3D fluid with coordinates $\boldsymbol{x} = (\boldsymbol{r}, z)$. Here $\boldsymbol{r} = (x, y)$ (respectively, z) denotes horizontal (respectively, vertical) Eulerian spatial coordinates in an inertial (fixed) domain. The fluid domain is bounded below by a rigid bottom at $z = -B(\boldsymbol{r})$ and is bounded above by the free surface of the fluid at $z = \zeta(\boldsymbol{r}, t)$, which is measured from its rest position at $z = 0$ as a function of the horizontal position vector $\boldsymbol{r} = (x, y, 0)$ and time t.

2.2 | Three-dimensional fluid equations

The fluid moves in three dimensions with velocity $\boldsymbol{u}(\boldsymbol{x}, t) = (\boldsymbol{v}(\boldsymbol{x}, t), w(\boldsymbol{x}, t))$ in which $\boldsymbol{v}(\boldsymbol{x}, t)$ and $w(\boldsymbol{x}, t)$ denote, respectively, the horizontal and vertical velocity fields. Incompressible and inviscid fluid motion is governed by the Euler equations of horizontal and vertical momentum dynamics under the constant acceleration of gravity, g. The equations are given by

$$
\begin{aligned}
\mathcal{D}\boldsymbol{v} &:= \boldsymbol{v}_t + \boldsymbol{v}\cdot\nabla_{\boldsymbol{r}}\boldsymbol{v} + w\boldsymbol{v}_z = -\frac{1}{\rho}\nabla_{\boldsymbol{r}}\pi,\\
\mathcal{D}w &:= w_t + \boldsymbol{v}\cdot\nabla_{\boldsymbol{r}}w + ww_z = -\frac{1}{\rho}\pi_z - g,\\
\text{with}\quad \mathcal{D} &:= \partial_t + \boldsymbol{v}\cdot\nabla_{\boldsymbol{r}} + w\partial_z \quad\text{and}\quad \nabla_{\boldsymbol{r}}\cdot\boldsymbol{v} + w_z = 0.
\end{aligned}
\tag{1}
$$

We denote by π the pressure with 3D spatial dependence. The volume element is $d^3x = d^2r \wedge dz$, and its measure Dd^3x is preserved under the incompressible fluid flow. The mass density is given by $\rho = \rho_0(1 + b(\boldsymbol{x}, t)) > 0$, in which $b(\boldsymbol{x}, t) := (\rho - \rho_0)/\rho_0$ is the fluid buoyancy and $\rho_0 > 0$ is the (constant, positive) reference value of mass density. The mass in each fluid volume element is given by ρDd^3x. The condition $\zeta(\boldsymbol{r}, t) - z = 0$, which defines the free surface, is assumed to be preserved under the flow. This condition ensures that a particle initially on the free surface will remain on it.

These three preservation relationships may be expressed as the following three *advection relations*:

$$
\begin{aligned}
(\partial_t + \mathcal{L}_{\boldsymbol{u}})(Dd^3x) &= (\partial_t D + \nabla\cdot(D\boldsymbol{u}))d^3x = 0,\\
(\partial_t + \mathcal{L}_{\boldsymbol{u}})\rho &= \partial_t\rho + \boldsymbol{u}\cdot\nabla\rho = 0,\\
(\partial_t + \mathcal{L}_{\boldsymbol{u}})(\zeta(\boldsymbol{r},t) - z) &= (\partial_t + \boldsymbol{u}\cdot\nabla)(\zeta(\boldsymbol{r},t) - z) = 0,
\end{aligned}
\tag{2}
$$

where the operator $(\partial_t + \mathcal{L}_{\boldsymbol{u}})$ is the advection operator (see Appendix A). Requiring the volume measure Dd^3x to remain constant in the first advection relation in (2) implies that the flow velocity remains divergence-free, $\nabla\cdot\boldsymbol{u} = 0$. The preservation of the divergence-free condition under the fluid flow then implies a Poisson equation for the fluid pressure π in the motion equation (1).

The motion equations in (1) and the initial values for the advected quantities $\rho(\boldsymbol{r}, z, t)$ and $(\zeta(\boldsymbol{r}, t) - z)$ in the advection relations in (2) provide a complete specification of the initial value problem for the fluid motion with appropriate boundary conditions in three dimensions.

2.3 | Boundary conditions

Although the horizontal boundary conditions are yet to be specified and can be chosen to suit specific problems, the vertical boundary conditions must be carefully defined.

The kinematic boundary condition on the free surface is given by

$$
\widehat{w} = \widehat{D}\zeta \qquad (\widehat{D} = \partial_t + \widehat{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{r}}),
\tag{3}
$$

where the $\widehat{f}$ notation in $\widehat{D}$, $\widehat{\boldsymbol{v}}$, and $\widehat{w}$ is defined for an arbitrary flow variable f to represent evaluation on the free surface, namely,

$$
\widehat{f}(\boldsymbol{r}, t) = f(\boldsymbol{r}, z, t) \quad \text{on} \quad z = \zeta(\boldsymbol{r}, t).
\tag{4}
$$

Notice that evaluating on the free surface before taking derivatives is not equivalent to taking the derivative before evaluating. In particular, $\partial_t\widehat{f}(\boldsymbol{r}, t) \neq \widehat{\partial_t f}(\boldsymbol{r}, t)$ and $\nabla_{\boldsymbol{r}}\widehat{f}(\boldsymbol{r}, t) \neq \widehat{\nabla_{\boldsymbol{r}} f}(\boldsymbol{r}, t)$, where

$$
\widehat{\partial_t f}(\boldsymbol{r}, t) = [\partial_t f(\boldsymbol{r}, z, t)]_{z=\zeta(\boldsymbol{r},t)} \quad \text{and} \quad \widehat{\nabla_{\boldsymbol{r}} f}(\boldsymbol{r}, t) = [\nabla_{\boldsymbol{r}} f(\boldsymbol{r}, z, t)]_{z=\zeta(\boldsymbol{r},t)}.
\tag{5}
$$

Instead, from the chain rule, we have

$$
\begin{aligned}
\partial_t\widehat{f}(\boldsymbol{r}, t) &= [\partial_t f + f_z\partial_t\zeta]|_{z=\zeta(\boldsymbol{r},t)} = \widehat{\partial_t f} + \widehat{\partial_z f}\partial_t\zeta,\\
\nabla_{\boldsymbol{r}}\widehat{f}(\boldsymbol{r}, t) &= [\nabla_{\boldsymbol{r}} f + f_z\nabla_{\boldsymbol{r}}\zeta]|_{z=\zeta(\boldsymbol{r},t)} = \widehat{\nabla_{\boldsymbol{r}} f} + \widehat{\partial_z f}\nabla_{\boldsymbol{r}}\zeta.
\end{aligned}
\tag{6}
$$

Consequently, we have the following remarkable proposition.

Proposition 1 (T.Y. Wu[19]). *The advection operator on the free surface satisfies the identity*

$$
\widehat{Df} = \widehat{D}\widehat{f},
\tag{7}
$$

where $D := (\partial_t + \boldsymbol{u}\cdot\nabla)$ *and* $\widehat{D} := (\partial_t + \widehat{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{r}})$.

In words, Equation (7) means that $\mathcal{D}f$ (the material time derivative of the function f in 3D) is equal to the material time derivative $\widehat{\mathcal{D}}$ applied to the function $\widehat{f}$, when all three are evaluated on the 2D moving surface.

Proof. Applying the chain rules in (6) leads to

$$\begin{aligned}
\widehat{\mathcal{D}f} &= \widehat{\partial_t f} + \widehat{\boldsymbol{v}} \cdot \widehat{\nabla_{\boldsymbol{r}} f} + \widehat{w}\widehat{\partial_z f}. \\
\text{By (6)} &= \partial_t \widehat{f} - \widehat{\partial_z f}\partial_t\zeta + \widehat{\boldsymbol{v}} \cdot (\nabla_{\boldsymbol{r}}\widehat{f} - \widehat{\partial_z f}\nabla_{\boldsymbol{r}}\zeta) + \widehat{w}\widehat{\partial_z f} \\
&= \partial_t \widehat{f} + \widehat{\boldsymbol{v}} \cdot \nabla_{\boldsymbol{r}}\widehat{f} + \widehat{\partial_z f}(\widehat{w} - \partial_t\zeta - \widehat{\boldsymbol{v}} \cdot \nabla_{\boldsymbol{r}}\zeta) \\
&= \partial_t \zeta + \widehat{\boldsymbol{v}} \cdot \nabla_{\boldsymbol{r}}\widehat{f} =: \widehat{\mathcal{D}}\widehat{f},
\end{aligned}$$

in which the final line is implied by the kinematic condition (3). ■

2.4 | Choi's relation at the free surface

One may use the relation (7) to evaluate the horizontal and vertical coordinates of the motion equation (1) onto the free surface. Hence, one finds for constant buoyancy $\rho = \rho_0$ on the free surface $\zeta(x, y, t) - z = 0$ that

$$\begin{aligned}
\widehat{\mathcal{D}}\widehat{\boldsymbol{v}} = -\left[\frac{1}{\rho_0}\nabla_{\boldsymbol{r}}\pi\right]\Bigg|_{z=\zeta} &= -\left[\frac{1}{\rho_0}\nabla_{\boldsymbol{r}}\widehat{\pi} - \frac{1}{\rho_0}\pi_z\nabla_{\boldsymbol{r}}\zeta\right]\Bigg|_{z=\zeta} \\
&= -\frac{1}{\rho_0}\nabla_{\boldsymbol{r}}\widehat{\pi} - (\widehat{\mathcal{D}}\widehat{w} + g)\nabla_{\boldsymbol{r}}\zeta,
\end{aligned} \tag{8}$$

where, in the last step, we have used the vertical motion equation in (1) to evaluate π_z for $\rho = \rho_0$ and the relation (7) for the vertical acceleration of the free surface, $dw/dt_{z=\zeta(\boldsymbol{r},t)} = \widehat{\mathcal{D}w} = \widehat{\mathcal{D}}\widehat{w}$. In conclusion, upon using the kinematic boundary condition $\widehat{w} = \widehat{\mathcal{D}}\zeta$ in (3), we find *Choi's relation* at the free surface,[17]

$$\widehat{\mathcal{D}}\widehat{\boldsymbol{v}} + (\widehat{\mathcal{D}}^2\zeta + g)\nabla_{\boldsymbol{r}}\zeta = -\frac{1}{\rho_0}\nabla_{\boldsymbol{r}}\widehat{\pi}. \tag{9}$$

Remark 1 (Closing Choi's relation (9)). The fundamental relation in (9) is not restricted to irrotational flows. However, at this stage, the dynamical system comprising Equations (3) and (9) for the motion of the free surface *is not yet closed* because (i) the pressure gradient $\nabla_{\boldsymbol{r}}\widehat{\pi}$ is still unknown and (ii) an evolutionary equation for $\widehat{\mathcal{D}}\widehat{w}$ is missing.

2.5 | Choi's relation at the bottom boundary

One may also consider boundary conditions at either a lower free surface, $z = -B(\boldsymbol{r}, t)$, or at fixed bathymetry, $z = -B(\boldsymbol{r})$. Denote by $\check{f}$ the evaluation on the bathymetry, that is,

$$\check{f}(\boldsymbol{r}, t) = f(\boldsymbol{r}, z, t) \quad \text{on} \quad z = -B(\boldsymbol{r}, t). \tag{10}$$

The bottom boundary condition is

$$-\check{D}B(\boldsymbol{r},t) = -(\partial_t + \check{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{r}})B(\boldsymbol{r},t) = \check{w}, \quad \text{on} \quad z = -B(\boldsymbol{r},t). \tag{11}$$

and we have, by the same chain-rule calculations as on the upper free surface,

$$\check{D}f = \check{D}\check{f}. \tag{12}$$

We may now evaluate Equations (1) onto the lower surface $z = -B(\boldsymbol{r},t)$ in the same manner as we have evaluated onto the upper surface $z = \zeta(\boldsymbol{r},t)$ to give

$$\check{D}\check{\boldsymbol{v}} + (\check{D}\check{w} + g)\nabla_{\boldsymbol{r}}B(\boldsymbol{r},t) = -\frac{1}{\rho_0}\nabla_{\boldsymbol{r}}\check{\pi}, \quad \text{on} \quad z = -B(\boldsymbol{r},t). \tag{13}$$

By the bottom boundary condition (11), we then find

$$\check{D}\check{\boldsymbol{v}} + (-\check{D}^2\check{B}(\boldsymbol{r},t) + g)\nabla_{\boldsymbol{r}}B(\boldsymbol{r},t) = -\frac{1}{\rho_0}\nabla_{\boldsymbol{r}}\check{\pi}, \quad \text{on} \quad z = -B(\boldsymbol{r},t). \tag{14}$$

When $B(\boldsymbol{r},t)$ a time-dependent variable, then Equation (14) is not closed. However, when the bottom boundary is taken to be time-independent, so that $z = -B(\boldsymbol{r})$ and $\check{D} = \check{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{r}}$ there, then Equation (14) would be closed, provided that either the bottom pressure $\check{\pi}$ was prescribed, or the bottom velocity $\check{\boldsymbol{v}}$ was taken to be divergence-free.

2.6 | Mean continuity relation

Another exact result about the dynamics of the free surface elevation $\zeta(x,y,t)$ should be mentioned. This result is the following mean continuity relation for the elevation in terms of the vertically averaged horizontal velocity components.[18,29]

Proposition 2 (Mean continuity relation)**.** *For uniform mass density, ρ_0, the boundary conditions (3), as well as incompressibility and Equation (7) for advection of the free surface together imply the following* vertically integrated *continuity relation for the wave elevation on the free surface, ζ,*

$$\partial_t\zeta(\boldsymbol{r},t) + \partial_x\int_{-B}^{\zeta(\boldsymbol{r},t)} u(\boldsymbol{r},z,t)dz + \partial_y\int_{-B}^{\zeta(\boldsymbol{r},t)} v(\boldsymbol{r},z,t)dz = 0. \tag{15}$$

In terms of vertically averaged quantities, denoted by

$$\overline{f}(\boldsymbol{r},z,t) := \frac{1}{\zeta + B}\int_{-B}^{\zeta(\boldsymbol{r},t)} f(\boldsymbol{r},z,t)dz. \tag{16}$$

Equation (15) may be written equivalently as a mean (i.e., vertically averaged) continuity equation,

$$\partial_t(\zeta(\boldsymbol{r},t) + B) + \partial_x((\zeta + B)\overline{u}) + \partial_y((\zeta + B)\overline{v}) = 0. \tag{17}$$

Remark 2 (Physical interpretation). Essentially, the continuity Equation (17) arises because the incompressible flow conserves the fluid volume measure, Dd^3x. In particular, the vertically integrated continuity relation (17) in Proposition 2 proved below represents volume preservation of the divergence-free 3D Euler fluid equations in (B2) of Appendix B for the advective boundary relations in (2) and (9), see, for example, Refs. 16–18.

Proof. By direct computation, using the advection condition for ζ and the vertical integral of the divergence free condition $\operatorname{div}\boldsymbol{u} = 0$, and upon noticing that no contribution arises from the flat bottom boundary, one finds that

$$
\begin{aligned}
&\partial_t\zeta(\boldsymbol{r},t) + \partial_x \int_{-B}^{\zeta(\boldsymbol{r},t)} u(\boldsymbol{r},z,t)dz + \partial_y \int_{-B}^{\zeta(\boldsymbol{r},t)} v(\boldsymbol{r},z,t)dz \\
&\quad = \partial_t\zeta(\boldsymbol{r},t) + u(\boldsymbol{r},z,t)\partial_x\zeta(\boldsymbol{r},t) + v(\boldsymbol{r},z,t)\partial_y\zeta(\boldsymbol{r},t) \\
&+ \int_{-B}^{\zeta(\boldsymbol{r},t)} \big(u_x + v_y + w_z\big)(\boldsymbol{r},z,t)dz - w(\boldsymbol{r},\zeta(\boldsymbol{r},t),t) = 0,
\end{aligned}
\tag{18}
$$

where we have added and subtracted $w(\boldsymbol{r},\zeta(\boldsymbol{r},t),t)$ and applied a tangential flow condition at the bottom boundary. Thus, the boundary conditions and the divergence-free nature of the 3D flow combine to produce the mean continuity relation in (15). ■

2.7 | The route to a closure scheme for (3) and (9) of T.Y. Wu and W. Choi

The ideas, methods, and problem statements presented in papers[17,19] comprise a launchpad for the present paper, in which we introduce similar principles into a variational framework. The closure problem for Equations (3) and (9) will be resolved in Section 3.2 in the context of the CWWE, which will imply $\widehat{D}\widehat{w} = -g$. In Section 3.5, the pair of wave variables $\widehat{w}$ and ζ will be understood as a canonically conjugate subset of a Hamiltonian system of Eulerian equations for planar fluid motion. This system will also contain the hydrostatic CWWE introduced in Section 3.1. In Section 3.3, nonhydrostatic pressure $\widehat{\pi}$ will be incorporated into the CWW problem to complete the closure of Choi's relation in (9). The rest of the paper will then build additional physics into the resulting system of planar fluid equations, for example, by including horizontal gradients of buoyancy on the free surface. Refer to Figure 1 for more perspective.

3 | FREE SURFACE DYNAMICS

3.1 | The classic water wave equations

3.1.1 | The Dirichlet–Neumann operator

In this section, we consider the much studied potential flow governed by the CWWE. The qualitative information obtained here from the CWWE will inspire our derivation of a constrained variational principle below. The CWWEs are derived from the free surface 3D Euler equations

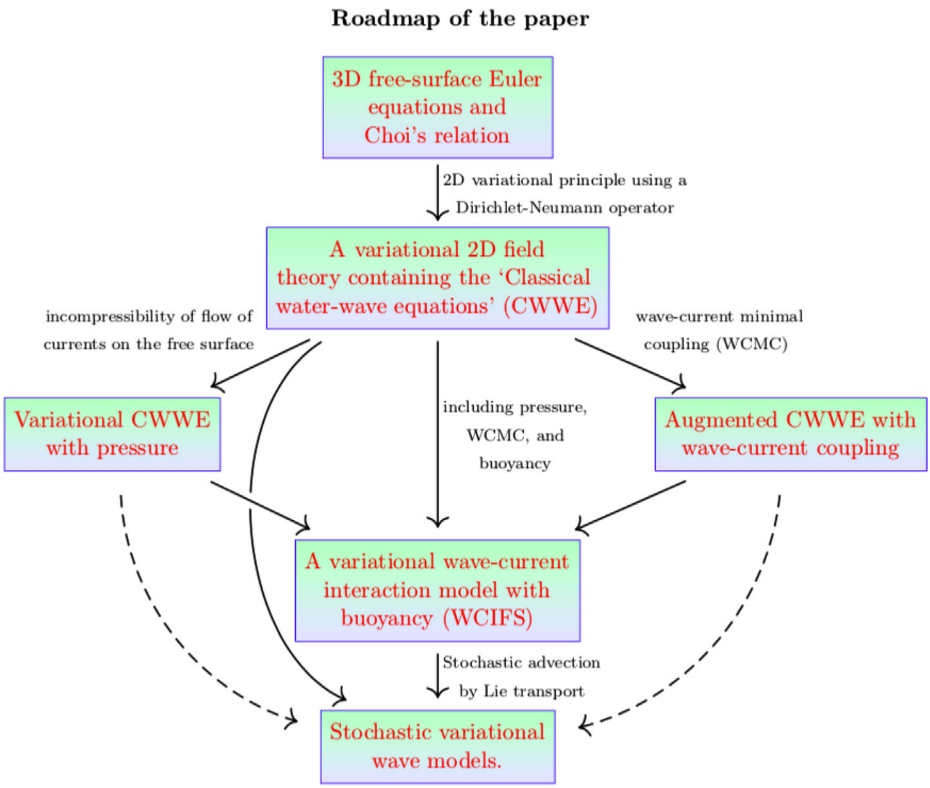

FIGURE 1 The figure sketches the relationships among the different models that are derived in the remainder of this article. Dashed arrows represent connections of these exact models to their stochastic versions that are not derived explicitly. However, the missing derivations follow the same patterns as that described in full for the stochastic wave–current interaction model in Section 6.3

via the *DNO*. The DNO maps the solution of Laplace's equation in an external domain with a Dirichlet boundary conditions to its solution on the boundary with a Neumann flux condition (see, e.g., Refs. 30 and 13). In particular, the CWWEs assume that the flow is incompressible and irrotational, and thus, there exists some $\phi(\boldsymbol{r}, z, t)$ such that $\boldsymbol{u} = \nabla\phi$, where $\boldsymbol{u}$ is the 3D velocity field throughout the domain.

In the hat-notation of (4), the variable $\widehat{\phi}(\boldsymbol{r}, t) = \phi(\boldsymbol{r}, \zeta(\boldsymbol{r}, t), t)$ evaluates the velocity potential $\phi(\boldsymbol{r}, z, t)$ on the free surface $z = \zeta(\boldsymbol{r}, t)$. The action of the DNO $G(\zeta)$ on $\widehat{\phi}(\boldsymbol{r}, t)$ is defined as the normal component of the 3D velocity field for the potential flow $\boldsymbol{u} = \nabla\phi$ evaluated at the free surface $z = \zeta(\boldsymbol{r}, t)$. Namely,

$$G(\zeta)\widehat{\phi} := (-\nabla_{\boldsymbol{r}}\zeta, 1) \cdot \widehat{\nabla\phi} := -\nabla_{\boldsymbol{r}}\zeta(\boldsymbol{r}, t) \cdot \widehat{\nabla_{\boldsymbol{r}}\phi} + \widehat{w}, \tag{19}$$

in which the horizontal gradient of the velocity potential $\phi(\boldsymbol{r}, z, t)$ is first taken, and then evaluated at the surface $z = \zeta(\boldsymbol{r}, t)$, cf. Equation (8).

Thus, the DNO in (19) takes Dirichlet data for $\widehat{\phi}$ on $z = \zeta(\boldsymbol{r}, t)$, solves Laplace's equation $\Delta\phi = 0$ for $\phi(\boldsymbol{r}, z, t)$ together with the condition that the velocity $\boldsymbol{u} = \nabla\phi$ have no normal component on the fixed parts of the boundary of the full domain volume, and then returns the corresponding Neumann data, that is, the 3D fluid normal velocity on the free surface, $z = \zeta(\boldsymbol{r}, t)$.

3.1.2 | The classical water wave equations

The classical water wave equations (CWWEs), as stated in Ref. 13, can be written in terms of the DNO as

$$\partial_t \zeta - G(\zeta)\widehat{\phi} = 0, \tag{20}$$

$$\partial_t \widehat{\phi} + g\zeta + \frac{1}{2}|\nabla_{\boldsymbol{r}} \widehat{\phi}|^2 - \frac{1}{2(1+|\nabla_{\boldsymbol{r}}\zeta|^2)}(G(\zeta)\widehat{\phi} + \nabla_{\boldsymbol{r}}\zeta \cdot \nabla_{\boldsymbol{r}}\widehat{\phi})^2 = 0. \tag{21}$$

From the chain rules in (6), we have, in the hat notation of Equation (4), that

$$\widehat{\nabla_{\boldsymbol{r}}\phi} = \nabla_{\boldsymbol{r}}\widehat{\phi}(\boldsymbol{r},t) - \widehat{\partial_z \phi}\nabla_{\boldsymbol{r}}\zeta, \tag{22}$$

$$\widehat{\partial_t \phi} = \partial_t \widehat{\phi} - \widehat{\partial_z \phi}\partial_t \zeta. \tag{23}$$

In terms of the DNO, these are expressed as

$$G(\zeta)\widehat{\phi} := -\nabla_{\boldsymbol{r}}\zeta(\boldsymbol{r},t) \cdot \nabla_{\boldsymbol{r}}\widehat{\phi}(\boldsymbol{r},t) + \widehat{\partial_z\phi}|\nabla_{\boldsymbol{r}}\zeta|^2 + \widehat{w}. \tag{24}$$

We consider (22) together with $\widehat{w} = \widehat{D}\zeta$. Observe that in the hat notation $\widehat{\boldsymbol{v}} := \widehat{\nabla_{\boldsymbol{r}}\phi}$, $\boldsymbol{V} := \nabla_{\boldsymbol{r}}\widehat{\phi}$, and $\widehat{w} := \widehat{\partial_z\phi}$, we have [1]

$$\widehat{\nabla_{\boldsymbol{r}}\phi} = \nabla_{\boldsymbol{r}}\widehat{\phi} - \widehat{w}\nabla_{\boldsymbol{r}}\zeta \quad \Longrightarrow \quad \widehat{\boldsymbol{v}} = \boldsymbol{V} - \widehat{w}\nabla_{\boldsymbol{r}}\zeta, \tag{25}$$

and hence,

$$\widehat{w} = \widehat{D}\zeta = \partial_t\zeta + \widehat{\boldsymbol{v}} \cdot \nabla_{\boldsymbol{r}}\zeta = \partial_t\zeta + (\nabla_{\boldsymbol{r}}\widehat{\phi} - \widehat{w}\nabla_{\boldsymbol{r}}\zeta) \cdot \nabla_{\boldsymbol{r}}\zeta, \tag{26}$$

which after rearranging is equivalent to

$$\partial_t\zeta + \widehat{\boldsymbol{v}} \cdot \nabla_{\boldsymbol{r}}\zeta = \widehat{w} = \frac{\partial_t\zeta + \nabla_{\boldsymbol{r}}\widehat{\phi} \cdot \nabla_{\boldsymbol{r}}\zeta}{1+|\nabla_{\boldsymbol{r}}\zeta|^2} = \frac{\partial_t\zeta + \boldsymbol{V} \cdot \nabla_{\boldsymbol{r}}\zeta}{1+|\nabla_{\boldsymbol{r}}\zeta|^2}. \tag{27}$$

Thus, applying the chain rule in the DNO appearing in the kinematic boundary condition has implied the alternative expression for $\widehat{w}$ in Equation (27). The alternative equations for $\widehat{w}$ in (27) will be used next in Section 3.2 to close the system defined by Choi's relation (9) by using a variational principle reminiscent of the approach in Ref. 25 to derive an evolutionary equation for $\widehat{w}$. The alternative expressions in (27) obtained from the DNO will also inspire a wave–current coupling term in Section 4.1.

Remark 3 (Direct derivation of the CWWE). The CWWE (20) and (21) may be derived from the standard 3D form of the Euler motion equation with constant $\rho = \rho_0$. In the case of 3D irrotational

[1] The distinction between velocities $\widehat{\boldsymbol{v}}$ and $\boldsymbol{V}$ is standard.[12,15]

flow, one finds Bernoulli's integrated form of the Euler equation

$$\partial_t \phi + \frac{1}{2}|\nabla\phi|^2 + gz = -\frac{1}{\rho_0}\pi. \tag{28}$$

Evaluating (28) on $z = \zeta(\boldsymbol{r}, t)$ with the boundary condition that the nonhydrostatic pressure vanishes on the free surface, $\pi|_{z=\zeta} = 0$, yields

$$\partial_t\widehat{\phi} - \widehat{\partial_z\phi}\,\partial_t\zeta + \frac{1}{2}|\nabla_{\boldsymbol{r}}\widehat{\phi}|^2 + \frac{1}{2}\widehat{\partial_z\phi}^2\left(1 + |\nabla_{\boldsymbol{r}}\zeta|^2\right) - \widehat{\partial_z\phi}\nabla_{\boldsymbol{r}}\zeta\cdot\nabla_{\boldsymbol{r}}\widehat{\phi} + g\zeta = 0. \tag{29}$$

Upon adding and subtracting $\widehat{\partial_z\phi}^2|\nabla_{\boldsymbol{r}}\zeta|^2$ in the previous equation, one finds

$$\partial_t\widehat{\phi} + g\zeta + \frac{1}{2}|\nabla_{\boldsymbol{r}}\widehat{\phi}|^2 + \frac{1}{2}\widehat{\partial_z\phi}^2\left(1 + |\nabla_{\boldsymbol{r}}\zeta|^2\right) - \widehat{\partial_z\phi}(\partial_t\zeta + \nabla_{\boldsymbol{r}}\widehat{\phi}\cdot\nabla_{\boldsymbol{r}}\zeta - \widehat{\partial_z\phi}|\nabla_{\boldsymbol{r}}\zeta|^2) - \widehat{\partial_z\phi}^2|\nabla_{\boldsymbol{r}}\zeta|^2 = 0. \tag{30}$$

Considering this in tandem with Equation (26) for $\widehat{w}$ yields

$$\partial_t\widehat{\phi} + g\zeta + \frac{1}{2}|\nabla_{\boldsymbol{r}}\widehat{\phi}|^2 - \frac{1}{2}\widehat{\partial_z\phi}^2\left(1 + |\nabla_{\boldsymbol{r}}\zeta|^2\right) = 0, \tag{31}$$

which one observes is equivalent to (21).

Remark 4. Equation (27) expresses $\widehat{w}$ in terms of the time derivative of ζ in the frame of reference moving with horizontal velocity $\nabla_{\boldsymbol{r}}\widehat{\phi}$ rather than with velocity $\widehat{\nabla_{\boldsymbol{r}}\phi}$. We recall the relation (25) and write

$$\boldsymbol{V} := \nabla_{\boldsymbol{r}}\widehat{\phi} = \widehat{\nabla_{\boldsymbol{r}}\phi} + \widehat{w}\nabla_{\boldsymbol{r}}\zeta =: \widehat{\boldsymbol{v}} + \boldsymbol{s}. \tag{32}$$

Physically, $\widehat{\boldsymbol{v}} := \widehat{\nabla_{\boldsymbol{r}}\phi}$ may be interpreted as the fluid transport velocity relative to a Galilean frame moving with velocity $\boldsymbol{s} := \widehat{w}\nabla_{\boldsymbol{r}}\zeta$, whereas the quantity $\boldsymbol{V} := \nabla_{\boldsymbol{r}}\widehat{\phi}$ is the total fluid velocity in the inertial frame of the Eulerian fluid description. In fact, the variational formulation taken below will show that the quantity $\boldsymbol{V}$ is the momentum per unit mass given by the variational derivative with respect to transport velocity $\widehat{\boldsymbol{v}}$ of the Lagrangian in Hamilton's principle for the wave–current dynamics. Likewise, the quantity $\boldsymbol{s} = \widehat{w}\nabla_{\boldsymbol{r}}\zeta$ will turn out to be the CWW wave momentum per unit fluid mass derived from Hamilton's principle.

The surface boundary condition (27) yields the evolution equation for the elevation ζ written in the two different frames of motion as,

$$\partial_t\zeta + \widehat{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{r}}\zeta = \widehat{w} \quad \text{and} \tag{33}$$

$$\partial_t\zeta + \boldsymbol{V}\cdot\nabla_{\boldsymbol{r}}\zeta = \widehat{w}\left(1 + |\nabla_{\boldsymbol{r}}\zeta|^2\right) = \widehat{w} - \boldsymbol{s}\cdot\nabla_{\boldsymbol{r}}\zeta. \tag{34}$$

Equating $\widehat{w}$ in these two expressions then yields

$$\partial_t\zeta + \widehat{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{r}}\zeta = \widehat{w} = \frac{\partial_t\zeta + \boldsymbol{V}\cdot\nabla_{\boldsymbol{r}}\zeta}{1 + |\nabla_{\boldsymbol{r}}\zeta|^2}. \tag{35}$$

Likewise, Equations (31) and (32) yield the Bernoulli evolution equation for zero nonhydrostatic pressure in terms of the rotational fluid velocities $\widehat{\boldsymbol{v}}$ and $\boldsymbol{V}$,

$$\partial_t \widehat{\phi} + g\zeta + \frac{1}{2}|\boldsymbol{V}|^2 - \frac{1}{2}\widehat{w}^2\big(1 + |\nabla_{\boldsymbol{r}}\zeta|^2\big) = 0 = \partial_t \widehat{\phi} + \widehat{\boldsymbol{v}} \cdot \nabla_{\boldsymbol{r}}\widehat{\phi} + g\zeta - \frac{1}{2}|\widehat{\boldsymbol{v}}|^2 - \frac{1}{2}\widehat{w}^2. \tag{36}$$

This completes the direct derivation of the CWWE.

3.2 | Imposing CWWE as constraints in Hamilton's principle

Let us introduce the following *dimension-free* action integral for a variational principle, $\delta S = 0$,

$$\begin{aligned}
S &= \int \ell(\widehat{\boldsymbol{v}}, D, \widehat{\phi}, \widehat{w}, \zeta, \lambda)dt \\
&= \int\int (\sigma^2\lambda)(\partial_t\zeta + \widehat{\boldsymbol{v}} \cdot \nabla_{\boldsymbol{r}}\zeta - \widehat{w}) - D\left(\partial_t\widehat{\phi} + \widehat{\boldsymbol{v}} \cdot \nabla_{\boldsymbol{r}}\widehat{\phi} + \frac{\zeta}{Fr^2} - \frac{1}{2}|\widehat{\boldsymbol{v}}|^2 - \frac{\sigma^2}{2}\widehat{w}^2\right) d^2r\,dt \\
&= \int\int D\left(\frac{1}{2}(|\widehat{\boldsymbol{v}}|^2 + \sigma^2\widehat{w}^2) - \frac{\zeta}{Fr^2}\right) + \sigma^2\lambda(\partial_t\zeta + \widehat{\boldsymbol{v}} \cdot \nabla_{\boldsymbol{r}}\zeta - \widehat{w}) \\
&\quad + \widehat{\phi}(\partial_t D + \mathrm{div}_{\boldsymbol{r}}(D\widehat{\boldsymbol{v}}))\,d^2r\,dt.
\end{aligned} \tag{37}$$

In the action integral (37), the two fundamental CWWE conditions (33) and (36) are imposed as constraints by the two Lagrange multipliers λ and D, respectively. From this action integral, Hamilton's principle will imply the evolution equations for λ and D. The equation for λ will yield the equation for the vertical velocity that is missing from the CWWE. An incompressible flow condition added later to the action integral in (47) will provide the missing equation for the non-hydrostatic pressure.

For spatial integration by parts, we take natural boundary conditions, so the boundary terms vanish. The temporal integration by parts introduces a total time derivative, so it also does not contribute to the equations of motion. In Equation (37), $\boldsymbol{r} = (x, y)$ denotes horizontal Eulerian spatial coordinates in an inertial (fixed) domain. We have integrated by parts in time and in space after the second line, dropping boundary terms both times. The constants σ^2 and Fr^2 here are squares of the aspect ratio and the Froude number, respectively, which are obtained in making the expression dimension-free. Finally, we make the distinction between the wave variables $\widehat{w}$ and ζ, and the current variables D and $\widehat{\boldsymbol{v}}$. The remaining variables λ and $\widehat{\phi}$ in the final form of the action integral (37) are Lagrange multipliers, to be determined from the others.

Remark 5 (Nondimensional parameters). Explicitly, the action integral for free surface motion in (37) has been cast into dimension-free form by introducing natural units for horizontal length, $[L]$, horizontal velocity, $[V]$, time, $[T] = [L]/[V]$, vertical velocity, $[W]$, and vertical wave elevation, $[\zeta]$. In terms of these units, we have defined the following dimension-free parameters: aspect ratio, $[W]/[V] = \sigma = [\zeta]/[L]$ and Froude number, $Fr^2 = [V]^2/([g][\zeta])$, for typical wave elevation scale $[\zeta]$.

Interpreting the two equivalent forms of the action integral in (37)

- The second line of the action integral in (37) may be regarded as a variant of the action integral in Luke.[25] An action integral for CWWE in Ref. 25 was derived in terms of vertically integrated expressions. In contrast, here the action integral in (37) has been made two-dimensional (2D) by using the DNO relation to project out the third (vertical) dimension. The Lagrange multiplier λ enforces the kinematic boundary condition for the elevation ζ in (33). Likewise, the Lagrange multiplier D enforces the zero-pressure Bernoulli law (36) obtained from the DNO.
- In the last line of (37), we rearrange the constraints in the action integral in the second line into the standard Clebsch advection form for 2D fluid motion, by integrating by parts in time and (horizontal) space. We may then regard the quantity $D\, d^2r$ as the area measure on the horizontal domain. That is, the area measure $D\, d^2r$ is advected by $\widehat{\boldsymbol{v}}$, which is imposed in the last line by regarding the trace of the velocity potential on the free surface $\widehat{\phi}$ as a Lagrange multiplier.

Remark 6 (The velocity $\widehat{\boldsymbol{v}}$ can have nonzero vorticity). The momentum map in Equation (55) makes it clear that the advective transport velocity $\widehat{\boldsymbol{v}}$ has nonzero vorticity

$$\widehat{\omega} := \mathrm{curl}_{\boldsymbol{r}}\, \widehat{\boldsymbol{v}} = -\widehat{\boldsymbol{z}} \cdot \nabla_{\boldsymbol{r}} \widehat{w} \times \nabla_{\boldsymbol{r}} \zeta =: -J(\widehat{w}, \zeta) \neq 0. \tag{38}$$

Consequently, the canonical constraint equations appearing in the last line of (37) are not potential flows. In contrast, Equation (32) shows that $\boldsymbol{V} = \nabla_{\boldsymbol{r}} \widehat{\phi}$ is indeed a potential velocity.

Variational formulas

Taking variations of the action integral (37) yields

$$\begin{aligned}
\delta \widehat{\boldsymbol{v}} &: \quad D\widehat{\boldsymbol{v}} \cdot d\boldsymbol{r} + \sigma^2 \lambda\, d\zeta = D d\widehat{\phi} \quad \Longrightarrow \quad \boldsymbol{V} \cdot d\boldsymbol{r} := \widehat{\boldsymbol{v}} \cdot d\boldsymbol{r} + \sigma^2 \widehat{w}\, d\zeta = d\widehat{\phi}, \\
\delta \widehat{w} &: \quad D\widehat{w} - \lambda = 0, \\
\delta \lambda &: \quad \partial_t \zeta + \widehat{\boldsymbol{v}} \cdot \nabla_{\boldsymbol{r}} \zeta = \widehat{w}, \\
\delta \zeta &: \quad \partial_t \lambda + \mathrm{div}_{\boldsymbol{r}}(\lambda \widehat{\boldsymbol{v}}) = -\frac{D}{\sigma^2 Fr^2} \quad \Longrightarrow \quad \partial_t \widehat{w} + \widehat{\boldsymbol{v}} \cdot \nabla_{\boldsymbol{r}} \widehat{w} = -\frac{1}{\sigma^2 Fr^2}, \\
\delta \widehat{\phi} &: \quad \partial_t D + \mathrm{div}_{\boldsymbol{r}}(D\widehat{\boldsymbol{v}}) = 0, \\
\delta D &: \quad (\partial_t + \widehat{\boldsymbol{v}} \cdot \nabla_{\boldsymbol{r}})\widehat{\phi} = \frac{1}{2}(|\widehat{\boldsymbol{v}}|^2 + \sigma^2 \widehat{w}^2) - \frac{\zeta}{Fr^2} =: \varpi.
\end{aligned} \tag{39}$$

Applying the Lagrangian time derivative $(\partial_t + \mathcal{L}_{\widehat{v}})$ to the first relation in (39) yields the ECWW motion equation,

$$(\partial_t + \mathcal{L}_{\widehat{v}})(\widehat{\boldsymbol{v}} \cdot d\boldsymbol{r} + \sigma^2 \widehat{w} d\zeta) = (\partial_t + \mathcal{L}_{\widehat{v}})(\boldsymbol{V} \cdot d\boldsymbol{r}) = (\partial_t + \mathcal{L}_{\widehat{v}}) d\widehat{\phi} = d\varpi. \tag{40}$$

See Appendix A for more discussion of the Lie derivative notation (as in $\mathcal{L}_{\widehat{v}}$) that is defined by the Lagrangian time derivative. The quantity $d\varpi$ is the spatial differential (i.e., the gradient) of Bernoulli's law in the last line of (39).

Kelvin circulation theorems for ECWWE in their dimensional form

Moving to the dimensional form and continuing to calculate from the ECWW motion equation in (40), we have

$$\begin{aligned} 0 &= (\partial_t + \mathcal{L}_{\widehat{v}})(\widehat{\boldsymbol{v}}\cdot d\boldsymbol{r} + \widehat{w}d\zeta) - d\varpi \\ &= (\partial_t + \mathcal{L}_{\widehat{v}})(\widehat{\boldsymbol{v}}\cdot d\boldsymbol{r}) - \frac{1}{2}d|\widehat{\boldsymbol{v}}|^2 \\ &\quad + (\partial_t + \mathcal{L}_{\widehat{v}})(\widehat{w}d\zeta) - d\left(\frac{1}{2}\widehat{w}^2 - g\,\zeta\right). \end{aligned} \tag{41}$$

Remarkably, the $(\widehat{w},\zeta)$ equations in (39) imply that the previous equation separates into two transport equations, namely,

$$\begin{aligned} &(\partial_t + \mathcal{L}_{\widehat{v}})(\widehat{\boldsymbol{v}}\cdot d\boldsymbol{r}) - \frac{1}{2}d|\widehat{\boldsymbol{v}}|^2 = 0, \\ &(\partial_t + \mathcal{L}_{\widehat{v}})(\widehat{w}d\zeta) - d\left(\frac{1}{2}\widehat{w}^2 - g\,\zeta\right) = 0. \end{aligned} \tag{42}$$

Thus, the wave and current circulations are conserved separately, in a *mutual nonacceleration pact*,

$$\begin{aligned} \frac{d}{dt}\oint_{c(\widehat{\boldsymbol{v}})} \widehat{\boldsymbol{v}}\cdot d\boldsymbol{r} &= \oint_{c(\widehat{\boldsymbol{v}})} \frac{1}{2}d|\widehat{\boldsymbol{v}}|^2 = 0, \\ \frac{d}{dt}\oint_{c(\widehat{\boldsymbol{v}})} \widehat{w}d\zeta &= \oint_{c(\widehat{\boldsymbol{v}})} d\left(\frac{1}{2}\widehat{w}^2 - g\,\zeta\right) = 0. \end{aligned} \tag{43}$$

The separation of conservation laws in (43) means that the two degrees of freedom do not influence each other's circulation. Actually, this separation is a general feature of wave–current interaction theories that arise from Hamilton's principle with a phase-space Lagrangian.[31]

Reduction of the ECWW motion equation to the pressureless Euler fluid equation

Because of a cancellation of $\frac{1}{2}d|\widehat{\boldsymbol{v}}|^2$ in Equation (42) with the Lie derivative term, the $\widehat{\boldsymbol{v}}$-equation simplifies further to produce the following *pressureless Euler fluid equation* for the transport velocity $\widehat{\boldsymbol{v}} = \nabla\widehat{\phi} - \widehat{w}\nabla\zeta$,

$$\partial_t\widehat{\boldsymbol{v}} + \widehat{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{r}}\widehat{\boldsymbol{v}} = 0 \quad \text{and} \quad \partial_t D + \mathrm{div}_{\boldsymbol{r}}(D\widehat{\boldsymbol{v}}). \tag{44}$$

Thus, although the vector $\widehat{\boldsymbol{v}}$ transports the density D, it also transports itself as though it were an array of two advected scalars, $(\widehat{v}_1, \widehat{v}_2)$. This feature further simplifies the interpretation of the $\widehat{\boldsymbol{v}}$-equation, because it can now be seen as an *inviscid Burgers equation*. However, note that the compressible "Burgers velocity" $\widehat{\boldsymbol{v}}$ in (44) has vorticity $\widehat{\omega} := \widehat{\boldsymbol{z}}\cdot\mathrm{curl}\widehat{\mathrm{v}} = -J(\widehat{w},\zeta)$ that does not vanish, in general. However, the relation $\widehat{\boldsymbol{v}} = \nabla\widehat{\phi} - \widehat{w}\nabla\zeta$ and the second equation in (42) do imply that

$$\partial_t\widehat{\omega} + \widehat{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{r}}\widehat{\omega} = 0. \tag{45}$$

Hence, if the vorticity $\widehat{\omega}$ vanishes initially, it will remain so. In this case, the pressureless 2D Euler equation in (45) reduces to the well-studied 2D Hamilton–Jacobi equation for $\widehat{\phi}$. See, for example, Refs. 32, 33 for reviews.

Back to the ECWWE in their dimensional forms
We may restore the ECWWE to their dimensional forms as

$$\begin{aligned}\partial_t\widehat{\phi} + \widehat{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{r}}\widehat{\phi} &= \frac{1}{2}(|\widehat{\boldsymbol{v}}|^2 + \widehat{w}^2) - g\zeta,\\ \partial_t\zeta + \widehat{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{r}}\zeta &= \widehat{w},\\ \partial_t\widehat{w} + \widehat{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{r}}\widehat{w} &= -g,\\ \partial_t D + \mathrm{div}_{\boldsymbol{r}}(D\widehat{\boldsymbol{v}}) &= 0,\end{aligned} \tag{46}$$

where $\widehat{\boldsymbol{v}}$ evolves according to (44). One observes that the first two equations in (46) are equivalent to the CWWE discussed in Section 3.1, since $\widehat{\boldsymbol{v}} = \widehat{\nabla_{\boldsymbol{r}}\phi}$.

3.3 | Derivation of ECWW equations with nonhydrostatic pressure

In the standard derivation of the CWW equations (20) and (21), the 3D pressure π is taken to be zero on the surface, and thus, the resulting equations of motion have no pressure term. In order for the variational equations, we have derived in Section 3.2 to match Equations (20) and (21), we have derived *compressible* equations, and thus, the system also contains the additional equation for D. Should we want to model an incompressible flow, and avoid having an equation for D, we must introduce pressure as a Lagrange multiplier which enforces that $D = 1$. Of course, such a nonhydrostatic pressure would be incompatible with assuming that the pressure is zero on the surface. We derive the ECWW equations with nonhydrostatic pressure and incompressible transport velocity by varying the action integral defined in its dimensional form by

$$\begin{aligned}S = \int\int D\left(\frac{1}{2}(|\widehat{\boldsymbol{v}}|^2 + \widehat{w}^2) - g\zeta\right) + \lambda(\partial_t\zeta + \widehat{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{r}}\zeta - \widehat{w})\\ + \widehat{\phi}(\partial_t D + \mathrm{div}_{\boldsymbol{r}}(D\widehat{\boldsymbol{v}})) - p(D-1)\,d^2r\,dt.\end{aligned} \tag{47}$$

Proceeding in the same manner as in Section 3.2, and omitting the calculations because they are very much alike, we derive the following system of equations in their dimensional forms:

$$\begin{aligned}\widehat{D}\widehat{\phi} &:= \partial_t\widehat{\phi} + \widehat{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{r}}\widehat{\phi} = \frac{1}{2}(|\widehat{\boldsymbol{v}}|^2 + \widehat{w}^2) - g\zeta - p,\\ \widehat{D}\zeta &:= \partial_t\zeta + \widehat{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{r}}\zeta = \widehat{w},\\ \widehat{D}\widehat{w} &:= \partial_t\widehat{w} + \widehat{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{r}}\widehat{w} = -g.\end{aligned} \tag{48}$$

Here, the divergence-free transport velocity $\widehat{\boldsymbol{v}}$ satisfies a 2D Euler equation

$$\partial_t\widehat{\boldsymbol{v}} + \widehat{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{r}}\widehat{\boldsymbol{v}} = -\nabla_{\boldsymbol{r}}p, \quad \text{where} \quad D = 1 \quad \text{implies} \quad \mathrm{div}_{\boldsymbol{r}}\widehat{\boldsymbol{v}} = 0. \tag{49}$$

We may understand the structure of this problem further by comparing it to Equation (8) with $\rho_0 = 1$. Noting that $\widehat{D}\widehat{w} = -g$, Equations (8) and (49) together imply

$$\widehat{D}\widehat{\boldsymbol{v}} = -\nabla_{\boldsymbol{r}}\widehat{\pi} = -\nabla_{\boldsymbol{r}} p. \tag{50}$$

The comparison implies the following remarkable observation. This observation turns out to be one of our main conclusions about this approach because it provides a closure of the system defined by Choi's relation (9).

Theorem 1. *The pressure, p, in the 2D model (48) is equivalent to the pressure of the 3D fluid evaluated on the free surface, π, up to the addition of a spatial constant.*

3.4 | Conservation laws for the compressible ECWW dynamical system

From here, we return to the compressible ECWW equations by removing the incompressibility constraint imposed by the pressure.

3.4.1 | Eulerian conservation laws for the ECWWE

The system of ECWWE in (44) and (46) possesses the following fundamental Eulerian conservation laws in a domain Ω with fixed boundaries.

1. The last equation in (46) implies conservation of *mass*, $\mathbb{D} := \int_\Omega D\, d^2r$,

$$\frac{d\mathbb{D}}{dt} := \frac{d}{dt}\int_\Omega D\, d^2r = \int_\Omega \partial_t D\, d^2r = -\int_\Omega \mathrm{div}_{\boldsymbol{r}}(D\widehat{\boldsymbol{v}})\, d^2r = -\oint_{\partial\Omega} D\widehat{\boldsymbol{v}}\cdot\widehat{\boldsymbol{n}}\, ds = 0,$$

   for $\widehat{\boldsymbol{v}}\cdot\widehat{\boldsymbol{n}}$ on the boundary $\partial\Omega$ with normal vector $\widehat{\boldsymbol{n}}$.
2. Equation (44) and the last equation in (46) imply conservation of *momentum*, defined by

$$\frac{d\mathbb{M}_j}{dt} := \frac{d}{dt}\int_\Omega D\widehat{v}_j\, d^2r = -\int_\Omega \partial_k(D\widehat{v}_j\widehat{v}^k)\, d^2r = -\oint_{\partial\Omega} D\widehat{v}_j\widehat{\boldsymbol{v}}\cdot\widehat{\boldsymbol{n}}\, ds = 0.$$

3. Combining the $\mathrm{curl}_{\boldsymbol{r}}$ of Equation (44) and the last equation in (46) implies conservation of *mass-weighted enstrophy*, defined by

$$\mathbb{C}_\Phi := \int_\Omega D\Phi(\widehat{\omega})\, d^2r,$$

   for any differentiable function Φ of vorticity, $\widehat{\omega}$, which itself is defined by

$$\widehat{\omega} := \widehat{\boldsymbol{z}}\cdot\mathrm{curl}_{\boldsymbol{r}}\widehat{\boldsymbol{v}} = -\widehat{\boldsymbol{z}}\cdot\nabla_{\boldsymbol{r}}\widehat{w}\times\nabla_{\boldsymbol{r}}\zeta =: -J(\widehat{w},\zeta).$$

Thus, upon noticing that vorticity $\widehat{\omega}$ is advected as a scalar by the flow of $\widehat{\boldsymbol{v}}$, we also find advection of any function of $\Phi(\widehat{\omega})$, by the chain rule and linearity of the advection operator for scalars. Namely,

$$(\partial_t + \widehat{\boldsymbol{v}} \cdot \nabla_{\boldsymbol{r}})\Phi(\widehat{\omega}) = 0.$$

Thus, we obtain conservation of mass-weighted enstrophy from the continuity equation, the chain rule and integration by parts, as follows,

$$\frac{d\mathbb{C}_\Phi}{dt} := \int_\Omega \partial_t(D\Phi(\widehat{\omega}))\, d^2r = -\int_\Omega \partial_k(D\Phi(\widehat{\omega})\widehat{v}^k)\, d^2r = -\oint_{\partial\Omega} D\Phi(\widehat{\omega})\, \widehat{\boldsymbol{v}} \cdot \widehat{\boldsymbol{n}}\, ds = 0,$$

for $\widehat{\boldsymbol{v}} \cdot \widehat{\boldsymbol{n}}$ on the fixed boundary $\partial\Omega$. Thus, the D-weighted L^p norm of the vorticity $\widehat{\omega} = \mathrm{curl}\widehat{\boldsymbol{v}}$ is controlled.

4. The corresponding conserved energy is given by

$$\begin{aligned} E(\widehat{\boldsymbol{v}}, \widehat{w}, \zeta, D) &= \int \left(\frac{1}{2}|\widehat{\boldsymbol{v}}|^2 + \frac{1}{2}\widehat{w}^2 + g\zeta\right) D\, d^2r \\ &= \int \left(\frac{1}{2}|\widehat{\nabla_{\boldsymbol{r}}\phi}|^2 + \frac{1}{2}\widehat{w}^2 + g\zeta\right) D\, d^2r. \end{aligned} \tag{51}$$

This expression follows quite easily from the Legendre transformation of the Lagrangian in (47).

3.4.2 | Moment dynamics of a Lagrangian fluid blob under the ECWWE

We rewrite the continuity equation in (46) and its associated motion equation in (44) as Lagrangian conservation laws for mass and momentum,

$$\begin{aligned} (\partial_t + \mathcal{L}_{\widehat{v}})(D\, d^2r) &= (\partial_t D + \partial_k P^k)\, d^2r = 0 \quad \text{with} \quad P^k := D\widehat{v}^k, \\ (\partial_t + \mathcal{L}_{\widehat{v}})(P^j\, d^2r) &= \left(\partial_t P^j + \partial_k(P^j\widehat{v}^k)\right) d^2r = 0. \end{aligned} \tag{52}$$

Consider a 2D "blob" of fluid mass occupying a Lagrangian domain of fluid $\Omega(t)$ that is deforming under the ECWWE flow of the free-surface fluid velocity $\widehat{\boldsymbol{v}}$ so that no fluid material enters or leaves through its moving boundary $\partial\Omega(t)$. In this situation, we have the following Reynolds transport relations for the dynamics of the spatial moments of the mass distribution within the blob.

1. The total mass of a Lagrangian blob is conserved:

$$\frac{d}{dt}\int_{\Omega(t)} D\, d^2r = \int_{\Omega(t)} (\partial_t + \mathcal{L}_{\widehat{v}})(D\, d^2r) = \int_{\Omega(t)} \left(\partial_t D + \partial_k(D\widehat{v}^k)\right) d^2r = 0.$$

2. The rate of change of the center of mass of the blob is its conserved momentum:

$$\frac{d}{dt}\int_{\Omega(t)} r^j D\, d^2r = \int_{\Omega(t)} (\partial_t + \mathcal{L}_{\widehat{v}})(r^j D\, d^2r) = \int_{\Omega(t)} \widehat{v}^j\, D\, d^2r = \int_{\Omega(t)} P^j\, d^2r.$$

Conservation of the blob momentum is shown by a direct computation,

$$\frac{d^2}{dt^2}\int_{\Omega(t)} r^j D\, d^2r = \frac{d}{dt}\int_{\Omega(t)} P^j\, d^2r = \int_{\Omega(t)} (\partial_t + \mathcal{L}_{\widehat{v}})(P^j\, d^2r) = 0.$$

3. The moment of inertia $I^{ij} = \int_{\Omega(t)} r^i r^j D\, d^2r$ represents the elliptical shape of the blob. Its rate of change may be computed as

$$\frac{d}{dt}I^{ij} = \frac{d}{dt}\int_{\Omega(t)} r^i r^j D\, d^2r = \int_{\Omega(t)} (\widehat{v}^i r^j + r^i \widehat{v}^j) D\, d^2r.$$

4. The acceleration of the elliptical shape of the blob is governed by

$$\frac{d^2}{dt^2}I^{ij} = \frac{d}{dt}\int_{\Omega(t)} (P^i\, d^2r) r^j + r^i (P^j\, d^2r) = \int_{\Omega(t)} (P^i \widehat{v}^j + \widehat{v}^i P^j)\, d^2r = \int_{\Omega(t)} (\widehat{v}^i \widehat{v}^j + \widehat{v}^i \widehat{v}^j)\, D\, d^2r.$$

5. Remarkably, the acceleration of the trace of the moment of inertia $\mathrm{tr}(I)$ is positive-definite

$$\frac{d^2}{dt^2}\mathrm{tr}(I) = 2\int_{\Omega(t)} D|\widehat{\boldsymbol{v}}|^2\, d^2r > 0.$$

This is a simple version of the tensor virial theorem.[34] Here, the tensor virial theorem implies that under ECWWE flow equations in (52), any initial distribution of mass will expand outward at an acceleration rate proportional to the kinetic energy within its Lagrangian boundary. Because this result holds for every Lagrangian blob of fluid undergoing this motion, it follows that the mass density cannot become singular in an infinite flow domain. This means that the measure Dd^2r cannot become a Dirac measure.

6. Finally, we notice that blob angular momentum $L^{ij} := \int_{\Omega(t)} (\widehat{v}^i r^j - r^i \widehat{v}^j) D\, d^2r$ is conserved under the ECWWE flow, because

$$\frac{d}{dt}L^{ij} := \frac{d}{dt}\int_{\Omega(t)} (P^i\, d^2r) r^j - r^i (P^j\, d^2r) = \int_{\Omega(t)} (\widehat{v}^i \widehat{v}^j - \widehat{v}^i \widehat{v}^j)\, D\, d^2r = 0.$$

3.5 | Three Hamiltonian formulations of the ECWWE using free-surface variables

This section derives three equivalent Hamiltonian formulations of the system of ECWWE in (44) and (46). To set the stage, let us first remark on the previous literature concerning Hamiltonian formulations of fluid dynamics with free boundaries.

Remark 7 (Previous Hamiltonian formulations of fluid dynamics with free boundaries). The ECWWE model extends the CWW model to permit rotational flow. Before investigating its Hamiltonian formulation, we recall here the result of Zakharov[35] that the CWWEs also have a Hamiltonian structure with similarities to the Hamiltonian (124). Indeed, the water wave equations have canonical variables ζ and ϕ, and a Hamiltonian defined by

$$\frac{1}{2}\int\int|\nabla\phi|^2\,d^2x\,dz+\frac{g}{2}\int\zeta^2\,d^2x,$$

in the case of zero surface tension. There are some similarities between this Hamiltonian structure of the water wave equations and the full system of equations we have derived. However, the Hamiltonian for the water wave equations and one of the canonical variables are vertically integrated compared to (51), which is evaluated on the free surface.

Lewis et al[36] generalized the previous canonical structure of Zakharov[35] for irrotational flow to obtain Hamiltonian structures for 2D or 3D incompressible flows with a free boundary. The Poisson bracket in Ref. 36 was determined using reduction from canonical variables in the Lagrangian (material) description. The corresponding Hamiltonian form for the equations of a liquid drop with a free boundary having surface tension was also demonstrated, as was the structure of the bracket in terms of a reduced cotangent bundle of a principal bundle was explained. In the case of 2D flows, a vorticity bracket was determined and the generalized enstrophy was shown to be a Casimir function.

A Hamiltonian description of free boundary fluids has also been studied in Mazer and Ratiu.[37] In Ref. 37, the Hamiltonian formulation of adiabatic free boundary inviscid fluid flow using only physical variables was presented in both the material and spatial formulation. By using the symmetry of particle relabeling, the noncanonical Poisson bracket in Eulerian representation was derived as a reduction from the canonical bracket in the Lagrangian representation. When the free boundary of the fluid was specified as the zero level set of an array of advected functions (e.g., Lagrangian labels carried by the fluid flow), the formulation of Ref. 37 recovered the Lie Poisson bracket of Ref. 38, as well as the corresponding PV and other conserved quantities found in Ref. 39.

In a tour-de-force, Gay-Balmaz et al[40] carried out Lagrangian reduction for free boundary fluids and deduced both the equations of motion and their associated constrained variational principles in both the convective and spatial representations. To follow up, Gay-Balmaz and Vizman[41] constructed dual pairs for free boundary fluids

Finally, we mention that Castro and Lannes[12] present a set of vertically integrated free surface equations that include vorticity in the bulk of the fluid and they prove well-posedness conditions for their equations, provided that a certain time scale is long enough. This extension of CWWE differs from the present work by combining vertically integrated variables with free-surface variables possessing a what they called a “formal” noncanonical Poisson bracket. Although they expressed reservations about whether their formal Poisson bracket would satisfy the Jacobi identity and they cited,[42] which describes potentially problematic technical pitfalls in this regard, the “formal” Poisson bracket in Castro and Lannes[12] actually does satisfy the Jacobi identity. This is because their Poisson bracket is equivalent to that in Lewis et al,[36] which does satisfy the Jacobi identity for *admissible functionals* A such that for every triple of functionals $f, g, h \in A$, the bracket of any of two of them lies in A.

None of the previous Hamiltonian formulations of fluid dynamics with free boundaries described above have represented the free-boundary dynamics in terms of projection/evaluation properties of the DNO representation of CWW theory, as is done in the present approach.

In contrast, the late Walter Craig and his collaborators in Refs. 43, 44 used asymptotic expansions of the DNO for CWWE to derive Hamiltonian formulations of certain soliton equations. The efforts of Craig et al[43,44] took advantage of the DNO representation of the CWWE to formulate Hamiltonian equations that do not involve vertically integrated variables. These Hamiltonian equations also enabled the study of interesting bathymetry by introducing a more general DNO. See also Ref. 45 for a review and bibliography of previous work in Hamiltonian formulations of the wave–current interaction based on the DNO. In contrast, the present work uses the DNO map to extend the CWWE to ECWWE.

3.5.1 | Canonical Hamiltonian formulation of the ECWWE

To consider the Hamiltonian formulation of this problem, we define a *third* form of the Lagrangian (37) by performing a Legendre transform as follows:

$$\begin{aligned} S &= \int\int (\sigma^2\lambda)(\partial_t\zeta + \widehat{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{r}}\zeta - \widehat{w}) - D\left(\partial_t\widehat{\phi} + \widehat{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{r}}\widehat{\phi} + \frac{\zeta}{Fr^2} - \frac{1}{2}|\widehat{\boldsymbol{v}}|^2 - \frac{\sigma^2}{2}\widehat{w}^2\right) d^2r\,dt \\ &= \int\int (\sigma^2\lambda)\partial_t\zeta + \widehat{\phi}\partial_t D - \left(\frac{1}{2D}(|D\widehat{\boldsymbol{v}}|^2 + (\sigma\lambda)^2) + \frac{D\zeta}{Fr^2}\right) d^2r\,dt, \end{aligned} \tag{53}$$

where, to go from the first line to the second, we have integrated by parts in time and made use of the first two relations in (39).

We have now expressed the Lagrangian within the action integral (37), $\ell(\widehat{\boldsymbol{v}}, D, \widehat{\phi}, \widehat{w}, \zeta; \lambda)$, as a phase-space Lagrangian, by rewriting it as a Legendre transform. The phase-space form of the Lagrangian immediately identifies the canonically conjugate pairs of field variables $(\widehat{\phi}, D)$ and $(\sigma^2\lambda, \zeta)$ and determines the Hamiltonian as

$$H(\widehat{\phi}, D; \lambda, \zeta) = \int \frac{1}{2D}(|D\nabla\widehat{\phi} - (\sigma^2\lambda)\nabla\zeta|^2 + (\sigma\lambda)^2) + \frac{D\zeta}{Fr^2}\, d^2r. \tag{54}$$

The variation of the Lagrangian in any of its equivalent representations in (37) with respect to the vector-field velocity $\widehat{\boldsymbol{v}}$ yields the momentum density relation

$$\boldsymbol{m} := D\widehat{\boldsymbol{v}} = D\nabla\widehat{\phi} - D\widehat{w}\nabla\zeta = D\nabla\widehat{\phi} - (\sigma^2\lambda)\nabla\zeta. \tag{55}$$

This expression provides a (cotangent lift) momentum map from the canonically conjugate pairs of field variables $(\widehat{\phi}, D)$ and $(\sigma^2\lambda, \zeta)$ to the momentum density $\boldsymbol{m} := D\widehat{\boldsymbol{v}}$ that is in concert with Equation (25).

Restoring the dimensions, in the canonical Hamiltonian field variables for currents $(\widehat{\phi}, D)$ and for waves (λ, ζ), the Bernoulli function ϖ in (39) is expressed as

$$\varpi := \frac{1}{2}(|\widehat{\boldsymbol{v}}|^2 + \widehat{w}^2) - g\zeta = \frac{1}{2}\left|\nabla\widehat{\phi} - \frac{\lambda}{D}\nabla\zeta\right|^2 + \frac{\lambda^2}{2D^2} - g\zeta. \tag{56}$$

The corresponding energy Hamiltonian in these variables is given by

$$H(\widehat{\boldsymbol{v}}, \widehat{w}, \zeta, D) = \int \left(\frac{1}{2} \left| \nabla \widehat{\phi} - \frac{\lambda}{D} \nabla \zeta \right|^2 + \frac{\lambda^2}{2D^2} + g\zeta \right) D \, d^2 r. \tag{57}$$

The canonical Hamiltonian equations for ECWWE in terms of the two degrees of freedom comprising wave variables (λ, ζ) and current variables $(\widehat{\phi}, D)$ are given by

$$\frac{\partial}{\partial t} \begin{bmatrix} \widehat{\phi} \\ D \\ \lambda \\ \zeta \end{bmatrix} = - \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \delta h / \delta \widehat{\phi} = -\operatorname{div}_{\boldsymbol{r}}(D \widehat{\boldsymbol{v}}) \\ \delta h / \delta D = \widehat{\boldsymbol{v}} \cdot \nabla_{\boldsymbol{r}} \widehat{\phi} - \varpi \\ \delta h / \delta \lambda = -\widehat{\boldsymbol{v}} \cdot \nabla_{\boldsymbol{r}} \zeta + \lambda / D \\ \delta h / \delta \zeta = \operatorname{div}_{\boldsymbol{r}}(\lambda \widehat{\boldsymbol{v}}) + gD \end{bmatrix}. \tag{58}$$

One observes that the symplectic Poisson operator for the two independent degrees of freedom in (58) appears in the canonical block-diagonal form. The wave–current interactions between these two independent degrees of freedom (waves (λ, ζ) with $\lambda = D\widehat{w}$ and currents $(\widehat{\phi}, D)$) are determined by the Hamiltonian in (57).

3.5.2 | Entangled Hamiltonian formulation of the ECWWE

In terms of the canonical Hamiltonian field variables $(\widehat{\phi}, D)$ and (λ, ζ), the total momentum density of the fluid $\widehat{\boldsymbol{m}} := D\widehat{\boldsymbol{v}}$ is defined as the sum

$$\boldsymbol{m} = D\widehat{\boldsymbol{v}} = D\nabla\widehat{\phi} - \lambda\nabla\zeta, \tag{59}$$

of both wave and current variables.

The Legendre transform with respect to both pairs of canonical wave variables defining the momentum density $\boldsymbol{m}$ leads to the following Hamiltonian:

$$\begin{aligned} h(\boldsymbol{m}, D, \lambda, \zeta) &= \int \boldsymbol{m} \cdot \widehat{\boldsymbol{v}} \, d^2 r - \ell(\widehat{\boldsymbol{v}}, D, \widehat{\phi}, \widehat{w}, \zeta; \lambda) \\ &= \int \frac{1}{2D} |\boldsymbol{m}|^2 + \frac{\lambda^2}{2D} + gD\zeta \, d^2 r. \end{aligned} \tag{60}$$

The corresponding conserved energy was already mentioned in (51) as

$$\begin{aligned} E(\widehat{\boldsymbol{v}}, \widehat{w}, \zeta, D) &= \int \left(\frac{1}{2} |\widehat{\boldsymbol{v}}|^2 + \frac{1}{2} \widehat{w}^2 + g\zeta \right) D \, d^2 r \\ &= \int \left(\frac{1}{2} |\widehat{\nabla_{\boldsymbol{r}} \phi}|^2 + \frac{1}{2} \widehat{w}^2 + g\zeta \right) D \, d^2 r. \end{aligned} \tag{61}$$

This change of variables leads to the following Lie–Poisson Hamiltonian formulation:

$$\frac{\partial}{\partial t}\begin{bmatrix} m_i \\ D \\ \lambda \\ \zeta \end{bmatrix} = -\begin{bmatrix} \partial_j m_i + m_j\partial_i & D\partial_i & \lambda\partial_i & -\zeta_{,i} \\ \partial_j D & 0 & 0 & 0 \\ \partial_j \lambda & 0 & 0 & 1 \\ \zeta_{,j} & 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} \delta h/\delta m_j = \hat{v}^j \\ \delta h/\delta D = -\varpi \\ \delta h/\delta\lambda = \lambda/D \\ \delta h/\delta\zeta = gD \end{bmatrix}, \tag{62}$$

where ϖ is defined in Equation (56). Here, the Poisson operator is the direct sum of the usual semidirect-product Lie–Poisson bracket for ideal fluids[23] and a symplectic Poisson bracket for the canonical wave variables, (λ, ζ).

The Poisson operator in (62) is said to *entangle* the dynamics of the wave variables (λ, ζ) with the combined variables (m_i, D). Next, we will *untangle* the entangled Poisson operator to put it back into block-diagonal form as before in (58) by considering only the momentum density corresponding to the potential part of the fluid flow.

3.5.3 | Untangled Hamiltonian formulation of the ECWWE

The momentum density of only the purely potential part of the fluid flow is given in terms of the canonical wave variables and the transport velocity $\hat{\boldsymbol{v}} := \widehat{\nabla_{\boldsymbol{r}}\phi}$ by

$$\boldsymbol{M} = D\hat{\boldsymbol{v}} + \lambda\nabla_{\boldsymbol{r}}\zeta = D\nabla\hat{\phi} = D\boldsymbol{V}. \tag{63}$$

The Legendre transform with respect to only the D and $\hat{\phi}$ variables corresponding to the potential part of the flow leads to the following Hamiltonian:

$$\begin{aligned} h(\boldsymbol{M}, D, \lambda, \zeta) &= \int \boldsymbol{M}\cdot\hat{\boldsymbol{v}} + \lambda\partial_t\zeta \, d^2r - \ell(\hat{\boldsymbol{v}}, D, \hat{\phi}, \hat{w}, \zeta; \lambda) \\ &= \int \frac{1}{2D}|\boldsymbol{M} - \lambda\nabla_{\boldsymbol{r}}\zeta|^2 + \frac{\lambda^2}{2D} + gD\zeta \, d^2r = E(\hat{\boldsymbol{v}}, \hat{w}, \zeta, D). \end{aligned} \tag{64}$$

where the energy $E(\hat{\boldsymbol{v}}, \hat{w}, \zeta, D)$ is defined in (68). Thus, the Hamiltonian in (64) is yet another representation of the conserved energy for the ECWWE system in (20) and (21).

Variations of the Hamiltonian in (64)
In the Hamiltonian variables, the Bernoulli function ϖ in (39) is denoted as

$$\varpi := \frac{1}{2}(|\hat{\boldsymbol{v}}|^2 + \hat{w}^2) - g\zeta = \frac{1}{2D^2}|\boldsymbol{M} - \lambda\nabla_{\boldsymbol{r}}\zeta|^2 + \frac{\lambda^2}{2D^2} - g\zeta. \tag{65}$$

After evaluating the corresponding variational derivatives of the Hamiltonian in (64), the system of equations in (62) may be written in block-diagonal form, as

$$\frac{\partial}{\partial t}\begin{bmatrix} M_i \\ D \\ \lambda \\ \zeta \end{bmatrix} = -\begin{bmatrix} \partial_j M_i + M_j\partial_i & D\partial_i & 0 & 0 \\ \partial_j D & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} \delta h/\delta M_j = \hat{v}^j \\ \delta h/\delta D = -\varpi \\ \delta h/\delta\lambda = -\hat{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{r}}\zeta + \lambda/D \\ \delta h/\delta\zeta = \mathrm{div}_{\boldsymbol{r}}(\lambda\hat{\boldsymbol{v}}) + gD \end{bmatrix}. \tag{66}$$

This *untangled* form of the Poisson operator comprises a direct product of the standard Lie–Poisson bracket for fluid variables $(\boldsymbol{M}, D)$ and a symplectic Poisson bracket for the canonical wave variables, (λ, ζ).

Remark 8 (Physical meaning of the model). The dual entangled and untangled forms of the Lie–Poisson brackets seen in (62) and (66) are familiar in Hamiltonian formulations of wave–current interactions and other compound Eulerian–Lagrangian fluid systems, as well as body-space mechanical systems. These dual formulations are particularly well known in the investigations of systems whose dynamics is governed by variational principles that are averaged over time, phase, or some other fluctuating or stochastic parameter. See, for example, Ref. 46 for a recent review and bibliography relevant to the current investigation.

The nonacceleration theorem for ECWWE

The Lie–Poisson Hamiltonian structure in (66) provides insight into the physical interactions occurring in the ECWWE. Namely, the Eulerian fluid variables are Lie-Poisson in the total momentum in (63) and the area element D, whereas the canonically conjugate wave variables undergo symplectic dynamics in the elevation ζ and its canonical momentum density $\lambda = D\widehat{w}$. As the Poisson structure block-diagonalizes for the two types of fields, both fields are seen to contribute on the same footing to the Hamiltonian formulation of the combined motion.

This dual wave–current contribution is already clear from the coordinate-free form of the motion equation in (40), because it immediately implies conservation of a two-component Kelvin circulation integral involving both types of fields present in the momentum density,

$$
\begin{aligned}
\frac{d}{dt}\oint_{c(\widehat{\boldsymbol{v}})} (\widehat{\boldsymbol{v}}\cdot d\boldsymbol{r} + \widehat{w}d\zeta) &= \oint_{c(\widehat{\boldsymbol{v}})} (\partial_t + \mathcal{L}_{\widehat{v}})(\widehat{\boldsymbol{v}}\cdot d\boldsymbol{r} + \widehat{w}d\zeta) \\
&= \oint_{c(\widehat{\boldsymbol{v}})} (\partial_t + \mathcal{L}_{\widehat{v}})(\boldsymbol{V}\cdot d\boldsymbol{r}) = \oint_{c(\widehat{\boldsymbol{v}})} (\partial_t + \mathcal{L}_{\widehat{v}})d\widehat{\phi} = \oint_{c(\widehat{\boldsymbol{v}})} d\varpi = 0,
\end{aligned}
\tag{67}
$$

where $c(\widehat{\boldsymbol{v}})$ denotes a closed material loop moving with the fluid transport velocity, $\widehat{\boldsymbol{v}}$. However, a closer look at the separate equations of motion for λ and ζ in the Hamiltonian form in Equation (66) shows that the wave dynamics takes place independently in the moving frame the fluid flow, without actually influencing the flow. Indeed, a closer look at the circulation dynamics in (67) verifies that the two components of the circulation are *conserved separately*, as we already know from Equation (42). In particular, this means that in solutions of ECCWE, the waves cannot generate circulation of the currents. This is known in the literature as the "nonacceleration theorem."[47]

Modifying the ECWWE to rectify the nonacceleration theorem

The nonacceleration theorem arises in time-dependent solutions of ECCWE because the model has no dependence on the combination of wave slope $\nabla\zeta$ and vertical velocity $\widehat{w}$ that would characterize a wave-slope dependence of the energy of the wave field. One natural proposal for rectifying this situation would be to introduce an additional energy density as in the last term in the following:

$$E(\widehat{\boldsymbol{v}}, \widehat{w}, \zeta, D) = \int \left(\frac{1}{2}|\widehat{\nabla_{\boldsymbol{r}}\phi}|^2 + \frac{1}{2}\widehat{w}^2 + g\zeta + \frac{\epsilon}{2}|\nabla_{\boldsymbol{r}}\widehat{\phi} - \widehat{\nabla_{\boldsymbol{r}}\phi}|^2 \right) D\, d^2r$$
$$= \int \left(\frac{1}{2}|\widehat{\boldsymbol{v}}|^2 + \frac{1}{2}\widehat{w}^2 + g\zeta + \frac{\epsilon}{2}|\boldsymbol{V} - \widehat{\boldsymbol{v}}|^2 \right) D\, d^2r \tag{68}$$
$$= \int \left(\frac{1}{2}|\widehat{\boldsymbol{v}}|^2 + \frac{1}{2}\widehat{w}^2 + g\zeta + \frac{\epsilon}{2}|\widehat{w}\nabla\zeta|^2 \right) D\, d^2r.$$

Physically, this would mean that the wave-field energy density $\epsilon D|\widehat{w}\nabla\zeta|^2/2$ would increase with mass density D, wave slope $\nabla\zeta$, and vertical velocity $\widehat{w}$. The multiplier ϵ could be varied $0 \le \epsilon \le 1$ to test the sensitivity of the model solutions to the proposed wave-field energy cost.

In nondimensional form, the wave-field energy cost could be included in the action integral as

$$S = \int \ell(\widehat{\boldsymbol{v}}, D, \widehat{\phi}, \widehat{w}; \zeta, \lambda) dt$$
$$= \int\int D\left(\frac{1}{2}\left(|\widehat{\boldsymbol{v}}|^2 + \sigma^2\widehat{w}^2 + \epsilon\sigma^4|\widehat{w}\nabla_{\boldsymbol{r}}\zeta|^2\right) - \frac{\zeta}{Fr^2} \right) + \sigma^2\lambda(\partial_t\zeta + \widehat{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{r}}\zeta - \widehat{w}) \tag{69}$$
$$+ \widehat{\phi}(\partial_t D + \mathrm{div}_{\boldsymbol{r}}(D\widehat{\boldsymbol{v}}))\, d^2r\, dt.$$

Stationarity of the modified action integral in (69) yields the variational equations,

$$\begin{aligned}
\widehat{\boldsymbol{v}}:&\quad D\widehat{\boldsymbol{v}}\cdot d\boldsymbol{r} + \sigma^2\lambda\, d\zeta = Dd\widehat{\phi} \quad\Longrightarrow\quad \boldsymbol{V}\cdot d\boldsymbol{r} := \widehat{\boldsymbol{v}}\cdot d\boldsymbol{r} + \sigma^2(\lambda/D)\, d\zeta = d\widehat{\phi},\\
\delta\widehat{w}:&\quad D\widehat{w} - \lambda + \epsilon\sigma^2 D\widehat{w}|\nabla_{\boldsymbol{r}}\zeta|^2 = 0 \quad\Longrightarrow\quad (\lambda/D) = \widehat{w}\left(1 + \epsilon\sigma^2|\nabla_{\boldsymbol{r}}\zeta|^2\right),\\
\delta\lambda:&\quad \partial_t\zeta + \widehat{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{r}}\zeta - \widehat{w} = 0,\\
\delta\zeta:&\quad \partial_t\lambda + \mathrm{div}_{\boldsymbol{r}}(\lambda\widehat{\boldsymbol{v}}) = -\frac{D}{\sigma^2 Fr^2} - \mathrm{div}_{\boldsymbol{r}}\left(\epsilon\sigma^2 D\widehat{w}^2\nabla_{\boldsymbol{r}}\zeta\right),\\
\delta\widehat{\phi}:&\quad \partial_t D + \mathrm{div}_{\boldsymbol{r}}(D\widehat{\boldsymbol{v}}) = 0,\\
\delta D:&\quad (\partial_t + \widehat{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{r}})\widehat{\phi} = \frac{1}{2}\left(|\widehat{\boldsymbol{v}}|^2 + \sigma^2\widehat{w}^2\left(1 + \epsilon\sigma^2|\nabla_{\boldsymbol{r}}\zeta|^2\right)\right) - \frac{\zeta}{Fr^2} =: \widetilde{\varpi}.
\end{aligned} \tag{70}$$

The previous wave system in (39) is recovered when one sets $\epsilon = 0$.

Theorem 2. *The system of equations in (70) implies the following Kelvin theorem for the fluid circulation:*

$$\frac{d}{dt}\oint_{c(\widehat{\boldsymbol{v}})} \widehat{\boldsymbol{v}}\cdot d\boldsymbol{r} = \epsilon\sigma^4 \oint_{c(\widehat{\boldsymbol{v}})} \frac{1}{D} div_{\boldsymbol{r}}(D\widehat{w}^2\nabla\zeta)d\zeta + |\nabla_{\boldsymbol{r}}\zeta|^2 d\widehat{w}^2/2, \tag{71}$$

in which $c(\widehat{\boldsymbol{v}})$ is a closed loop moving with the material velocity $\widehat{\boldsymbol{v}}$.

Proof. The proof follows by first integrating the $\delta \boldsymbol{v}$ equation (70) around a material loop $c(\widehat{\boldsymbol{v}})$ moving with velocity $\widehat{\boldsymbol{v}}$ to find,

$$\frac{d}{dt}\oint_{c(\widehat{\boldsymbol{v}})} (\widehat{\boldsymbol{v}} + \sigma^2 \lambda\, \nabla \zeta) \cdot d\boldsymbol{r} = \oint_{c(\widehat{\boldsymbol{v}})} (\partial_t + \mathcal{L}_{\widehat{\boldsymbol{v}}})((\widehat{\boldsymbol{v}} + \sigma^2 \lambda\, \nabla \zeta) \cdot d\boldsymbol{r}) = \oint_{c(\widehat{\boldsymbol{v}})} d\widehat{\phi} = 0,$$

upon using the well-known identity

$$\frac{d}{dt}\oint_{c(\widehat{\boldsymbol{v}})} \widehat{\boldsymbol{v}} \cdot d\boldsymbol{r} = \oint_{c(\widehat{\boldsymbol{v}})} (\partial_t + \mathcal{L}_{\widehat{\boldsymbol{v}}})(\widehat{\boldsymbol{v}} \cdot d\boldsymbol{r}).$$

One then computes

$$\begin{aligned}
\frac{d}{dt}\oint_{c(\widehat{\boldsymbol{v}})} \widehat{\boldsymbol{v}} \cdot d\boldsymbol{r} &= -\sigma^2 \oint_{c(\widehat{\boldsymbol{v}})} (\partial_t + \mathcal{L}_{\widehat{\boldsymbol{v}}})((\lambda/D)\, d\zeta) \\
&= \oint_{c(\widehat{\boldsymbol{v}})} \left(\frac{1}{Fr^2} + \frac{1}{D}\mathrm{div}_{\boldsymbol{r}}(\epsilon\sigma^4 D \widehat{w}^2 \nabla \zeta)\right) d\zeta + \sigma^2 (\lambda/D) d\widehat{w} \\
&= \epsilon\sigma^4 \oint_{c(\widehat{\boldsymbol{v}})} \frac{1}{D}\mathrm{div}_{\boldsymbol{r}}(D\widehat{w}^2 \nabla \zeta) d\zeta + |\nabla_{\boldsymbol{r}} \zeta|^2 d\widehat{w}^2/2.
\end{aligned}$$

■

Corollary 1. *The modified CWW model of wave dynamics arising via Hamilton's principle from the action integral in (69) with its additional wave energy creates circulation in the fluid whenever the gradients of the wave variables are not aligned.*

Proof. The proof follows by applying the Stokes theorem to the right-hand side of the material loop $c(\widehat{\boldsymbol{v}})$ in Equation (71) in Theorem 2. ■

Remark 9 (Wave propagation velocity). Although the modification made to the energy here does rectify the nonacceleration theorem so that the wave evolution can affect the circulation of the current, the equation in (70) corresponding to the variation in λ indicates that the surface elevation remains a Lagrangian coordinate. This feature does not allow for waves of phase that do not carry mass.

In Section 4, we will introduce a different coupling which will allow for waves that propagate at a speed different to the Lagrangian parcels on the fluid surface. For this purpose, we will make use of the two distinct 2D velocities on the free surface which appear in the energy equation in (68), $\widehat{\boldsymbol{v}} = \widehat{\nabla \phi}$ and $\boldsymbol{V} = \nabla \widehat{\phi}$. The first of them, $\widehat{\boldsymbol{v}}$, is the transport velocity of Lagrangian parcels on the fluid surface. The second of them, $\boldsymbol{V}$, is the phase velocity of a level set of the velocity potential $\widehat{\phi}$ evaluated on the free surface.

The coupling we will introduce in the next section will include a homotopy coefficient $0 \le \epsilon \le 1$ that will provide the option to set the Eulerian transport velocity of the wave elevation to a value anywhere between $\widehat{\boldsymbol{v}} = \widehat{\nabla \phi}$ for $\epsilon = 0$ and $\boldsymbol{V} = \nabla \widehat{\phi}$ for $\epsilon = 1$. For $0 < \epsilon \le 1$, the wave propagation velocity will no longer be equal to the material velocity.

4 | AUGMENTED CLASSICAL WATER WAVE EQUATIONS (ACWWE)

4.1 | Variational derivation of the ACWWE

Let us propose a less severe modification of the action integral in Equation (37) than the energy modification introduced in (69). This proposal will not introduce any change in the wave energy. Instead, it will allow a slip in the phase of the wave velocity relative to Lagrangian mass transport velocity that will be imposed by the following constraint in the Lagrangian:

$$\begin{aligned} S &= \int \ell(\widehat{\boldsymbol{v}}, D, \widehat{\phi}, \widehat{w}; \zeta, \lambda) dt \\ &= \int\int D\left(\frac{1}{2}(|\widehat{\boldsymbol{v}}|^2 + \sigma^2 \widehat{w}^2) - \frac{\zeta}{Fr^2}\right) + \sigma^2 \lambda(\partial_t \zeta + (\widehat{\boldsymbol{v}} - \epsilon\sigma^2 \widehat{w} \nabla \zeta) \cdot \nabla_{\boldsymbol{r}} \zeta - \widehat{w}) \\ &\quad + \widehat{\phi}(\partial_t D + \operatorname{div}_{\boldsymbol{r}}(D\widehat{\boldsymbol{v}}))\, d^2 r\, dt. \end{aligned} \tag{72}$$

Here, we have retained the same nondimensional parameters as in Remark 5 and the nondimensional constant parameter $0 \le \epsilon \le 1$ is the homotopy coefficient mentioned in Remark 9. According to Remark 5, each wave variable is multiplied by the aspect ratio σ relative the fluid variables.

In Equation (72), we have introduced a term that is intended to model WCMC.[48] The WCMC term modifies the transport velocity of the wave momentum density, to allow *genuine* wave–current interaction, so the free-surface water waves will no longer be passively advected by the fluid velocity. However, it leaves the wave energy unchanged.

Stationarity of the augmented action integral in (72) now yields the variational equations,

$$\begin{aligned} \delta\widehat{\boldsymbol{v}}:&\quad D\widehat{\boldsymbol{v}}\cdot d\boldsymbol{r} + \sigma^2\lambda\, d\zeta = D d\widehat{\phi} \quad\Longrightarrow\quad \boldsymbol{V}\cdot d\boldsymbol{r} := \widehat{\boldsymbol{v}}\cdot d\boldsymbol{r} + \sigma^2 \widetilde{w}\, d\zeta = d\widehat{\phi}, \\ \delta\widehat{w}:&\quad D\widehat{w} - \lambda\big(1 + \epsilon\sigma^2 |\nabla_{\boldsymbol{r}}\zeta|^2\big) = 0 \quad\Longrightarrow\quad \lambda = \frac{D\widehat{w}}{1 + \epsilon\sigma^2|\nabla_{\boldsymbol{r}}\zeta|^2} =: D\widetilde{w}, \\ \delta\lambda:&\quad \partial_t\zeta + \big(\widehat{\boldsymbol{v}} - \epsilon\sigma^2\widehat{w}\nabla_{\boldsymbol{r}}\zeta\big)\cdot\nabla_{\boldsymbol{r}}\zeta - \widehat{w} = 0, \\ \delta\zeta:&\quad \partial_t\lambda + \operatorname{div}_{\boldsymbol{r}}\big(\lambda\big(\widehat{\boldsymbol{v}} - \epsilon\sigma^2\widehat{w}\nabla_{\boldsymbol{r}}\zeta\big)\big) = -\frac{D}{\sigma^2 Fr^2} + \operatorname{div}_{\boldsymbol{r}}\big(\epsilon\sigma^2\lambda\widehat{w}\nabla_{\boldsymbol{r}}\zeta\big), \\ \delta\widehat{\phi}:&\quad \partial_t D + \operatorname{div}_{\boldsymbol{r}}(D\widehat{\boldsymbol{v}}) = 0, \\ \delta D:&\quad (\partial_t + \widehat{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{r}})\widehat{\phi} = \frac{1}{2}(|\widehat{\boldsymbol{v}}|^2 + \sigma^2\widehat{w}^2) - \frac{\zeta}{Fr^2} =: \varpi. \end{aligned} \tag{73}$$

Notice that the ϵ coupling term in the modified Lagrangian in (72) does not affect variations in the fluid variables, $(\widehat{\boldsymbol{v}}, D, \widehat{\phi})$. However, it changes the previous relationship between λ and $\widehat{w}$ by a term proportional to ϵ, which changes the relationship between $\widehat{\boldsymbol{v}}$ and $\boldsymbol{V}$, as seen in the first and second lines of (73). Of course, the previous wave system in (39) is recovered when one sets $\epsilon \to 0$. Specifically, we will see in the Kelvin circulation theorem that the total momentum per unit mass is no longer $\boldsymbol{V}$, and instead is given by

$$
\begin{aligned}
\widehat{\boldsymbol{v}} + \widetilde{\omega}\nabla_{\boldsymbol{r}}\zeta &= \widehat{\boldsymbol{v}} + \frac{\widehat{\omega}}{1+\epsilon|\nabla_{\boldsymbol{r}}\zeta|^2}\nabla_{\boldsymbol{r}}\zeta \\
&= \widehat{\boldsymbol{v}} + \widehat{\omega}\nabla_{\boldsymbol{r}}\zeta - \frac{\epsilon\widehat{\omega}|\nabla_{\boldsymbol{r}}\zeta|^2}{1+\epsilon|\nabla_{\boldsymbol{r}}\zeta|^2}\nabla_{\boldsymbol{r}}\zeta \\
&= \boldsymbol{V} - \frac{\epsilon\widehat{\omega}|\nabla_{\boldsymbol{r}}\zeta|^2}{1+\epsilon|\nabla_{\boldsymbol{r}}\zeta|^2}\nabla_{\boldsymbol{r}}\zeta .
\end{aligned} \tag{74}
$$

The right-hand side of this expression has nonzero curl, so the equations of motion can support a nontrivial PV formulation.

4.1.1 | ACWW motion equation

We may write out the ACWW motion equation obtained by substituting the variational results into the application of the advective time derivative $(\partial_t + \mathcal{L}_{\widehat{v}})$ on the first line of the system (73), to find in the notation $\lambda = D\widetilde{\omega}$ that

$$
D(\partial_t + \mathcal{L}_{\widehat{v}})(\widehat{\boldsymbol{v}}\cdot d\boldsymbol{r} + \sigma^2\widetilde{\omega}\, d\zeta) = Dd(\partial_t + \mathcal{L}_{\widehat{v}})\widehat{\phi} = Dd\varpi . \tag{75}
$$

Recall from (73) that Bernoulli function ϖ and vertical wave momentum density λ are defined as

$$
\varpi := \frac{1}{2}(|\widehat{\boldsymbol{v}}|^2 + \sigma^2\widehat{\omega}^2) - \frac{\zeta}{Fr^2}, \qquad \lambda = \frac{D\widehat{\omega}}{1+\epsilon\sigma^2|\nabla_{\boldsymbol{r}}\zeta|^2} =: D\widetilde{\omega}. \tag{76}
$$

At this point, let us collect the ACWWE in terms of (i) fluid variables, comprising velocity $\widehat{\boldsymbol{v}}$ and area density D, and (ii) wave variables, comprising surface elevation ζ, and vertical wave momentum density λ. Namely, the ACWWEs are given by

$$
\begin{aligned}
&(\partial_t + \mathcal{L}_{\widehat{v}})(\widehat{\boldsymbol{v}}\cdot d\boldsymbol{x} + \sigma^2\widetilde{\omega}\, d\zeta) = d\varpi, \\
&\partial_t D + \mathrm{div}_{\boldsymbol{r}}(D\widehat{\boldsymbol{v}}) = 0, \\
&\partial_t\zeta + \widehat{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{r}}\zeta = \widehat{\omega}(1+\epsilon\sigma^2|\nabla_{\boldsymbol{r}}\zeta|^2), \\
&\partial_t\widetilde{\omega} + \widehat{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{r}}\widetilde{\omega} = -\frac{1}{\sigma^2 Fr^2} + \frac{2\epsilon\sigma^2}{D}\,\mathrm{div}(D\widetilde{\omega}\,\widehat{\omega}\,\nabla_{\boldsymbol{r}}\zeta).
\end{aligned} \tag{77}
$$

The equation set (77) recovers the ECWWE equations (46) when $\epsilon \to 0$. The first of these equations implies Kelvin's circulation theorem for the ACWW model, as follows.

Theorem 3 (Kelvin–Noether theorem for the ACWW model)**.** *For every closed loop $c(\widehat{\boldsymbol{v}})$, moving with the ACWW transport velocity $\widehat{\boldsymbol{v}}$ for the system of ACWWE in (77) the Kelvin circulation relation* *holds. Namely,*

$$\frac{d}{dt}\oint_{c(\hat{\boldsymbol{v}})}(\hat{\boldsymbol{v}}\cdot d\boldsymbol{x}+\sigma^2\widetilde{w}d\zeta)=\oint_{c(\hat{\boldsymbol{v}})}d\varpi=0. \tag{78}$$

Proof. From the first ACWWE in (77), we have

$$(\partial_t+\mathcal{L}_{\hat{\boldsymbol{v}}})(\hat{\boldsymbol{v}}\cdot d\boldsymbol{x}+\sigma^2\widetilde{w}d\zeta)=d\varpi, \tag{79}$$

and the result (78) follows from the standard relation for the time derivative of an integral around a closed moving loop, $c(\hat{\boldsymbol{v}})$,

$$\frac{d}{dt}\oint_{c(\hat{\boldsymbol{v}})}(\hat{\boldsymbol{v}}\cdot d\boldsymbol{r}+\sigma^2\widetilde{w}d\zeta)=\oint_{c(\hat{\boldsymbol{v}})}(\partial_t+\mathcal{L}_{\hat{\boldsymbol{v}}})(\hat{\boldsymbol{v}}\cdot d\boldsymbol{r}+\sigma^2\widetilde{w}d\zeta)=\oint_{c(\hat{\boldsymbol{v}})}d\varpi=0.$$

■

Remark 10 (Transport of wave dynamics relative to the fluid velocity). A slight rearrangement of the last two equations in (77) demonstrates that the wave dynamics is no longer transported passively by the fluid velocity $\hat{\boldsymbol{v}}$. Instead, a shifted transport velocity appears; namely,

$$\hat{\boldsymbol{v}}-\epsilon\sigma^2\widehat{w}\nabla_{\boldsymbol{r}}\zeta=\hat{\boldsymbol{v}}-\epsilon\sigma^2\boldsymbol{s}. \tag{80}$$

This shift in wave velocity introduces wave dynamics into the transport velocity of the wave variables, as follows:

$$\begin{aligned}\partial_t\zeta+\left(\hat{\boldsymbol{v}}-\epsilon\sigma^2\widehat{w}\nabla_{\boldsymbol{r}}\zeta\right)\cdot\nabla_{\boldsymbol{r}}\zeta&=\widehat{w},\\ \partial_t\lambda+\mathrm{div}_{\boldsymbol{r}}\left(\lambda\left(\hat{\boldsymbol{v}}-\epsilon\sigma^2\widehat{w}\nabla_{\boldsymbol{r}}\zeta\right)\right)&=-\frac{1}{\sigma^2Fr^2}+\epsilon\sigma^2\mathrm{div}_{\boldsymbol{r}}(\lambda\widehat{w}\,\nabla_{\boldsymbol{r}}\zeta),\end{aligned} \tag{81}$$

where one recalls that the canonical momentum density λ conjugates to the elevation ζ is defined in terms of the other wave variables in (76).

Remark 11. The difference in the wave momentum transport velocity relative to the fluid velocity in Equation (80) will turn out to produce an important effect by which the waves will generate fluid circulation. To compute this effect on the circulation of the fluid, we subtract the fluid transport of the wave momentum from the total momentum transport by the fluid in (79). The fluid velocity transport of the wave momentum is found from the wave dynamical equations in (81), as

$$\begin{aligned}(\partial_t+\mathcal{L}_{\hat{\boldsymbol{v}}})(\sigma^2\widetilde{w}d\zeta)&=\sigma^2\left(-\frac{1}{\sigma^2Fr^2}+\frac{2\epsilon\sigma^2}{D}\mathrm{div}(D\widetilde{w}(\widehat{w}\nabla\zeta))\right)d\zeta+\frac{\sigma^2\widehat{w}}{1+\epsilon\sigma^2|\nabla_{\boldsymbol{r}}\zeta|^2}d\left(\widehat{w}(1+\epsilon\sigma^2|\nabla_{\boldsymbol{r}}\zeta|^2)\right)\\ &=\sigma^2\left(-\frac{1}{\sigma^2Fr^2}+\frac{2\epsilon\sigma^2}{D}\mathrm{div}(D\widetilde{w}(\widehat{w}\nabla\zeta))\right)d\zeta\\ &\quad+\frac{1}{2}\sigma^2d\widehat{w}^2+\sigma^2\widehat{w}^2d\left(\log((1+\epsilon\sigma^2|\nabla_{\boldsymbol{r}}\zeta|^2)\right).\end{aligned} \tag{82}$$

Upon subtracting Equation (82) from the total fluid momentum transport equation in (79), a short calculation yields

$$(\partial_t + \mathcal{L}_{\hat{\boldsymbol{v}}})(\hat{\boldsymbol{v}} \cdot d\boldsymbol{r}) = \frac{1}{2} d|\hat{\boldsymbol{v}}|^2 - \frac{\sigma^2}{2} \widetilde{w}^2 d\big(1 + \epsilon\sigma^2 |\nabla_{\boldsymbol{r}}\zeta|^2\big)^2 - \frac{2\epsilon\sigma^4}{D} \mathrm{div}(D\widetilde{w}(\widehat{w}\nabla_{\boldsymbol{r}}\zeta))\, d\zeta. \tag{83}$$

Theorem 4. *The corresponding Kelvin theorem for Equation (83) is given by*

$$\frac{d}{dt} \oint_{c(\hat{\boldsymbol{v}})} \hat{\boldsymbol{v}} \cdot d\boldsymbol{r} = \oint_{c(\hat{\boldsymbol{v}})} \frac{1}{2} d|\hat{\boldsymbol{v}}|^2 - \frac{\sigma^2}{2} \widetilde{w}^2 d\big(1 + \epsilon\sigma^2 |\nabla_{\boldsymbol{r}}\zeta|^2\big)^2 - \frac{2\epsilon\sigma^4}{D} \mathrm{div}(D\widetilde{w}(\widehat{w}\nabla_{\boldsymbol{r}}\zeta))\, d\zeta. \tag{84}$$

Proof. The proof follows by integrating Equation (83) around a material loop $c(\hat{\boldsymbol{v}})$ moving with velocity $\hat{\boldsymbol{v}}$, then using the well-known identity

$$\frac{d}{dt} \oint_{c(\hat{\boldsymbol{v}})} \hat{\boldsymbol{v}} \cdot d\boldsymbol{r} = \oint_{c(\hat{\boldsymbol{v}})} (\partial_t + \mathcal{L}_{\hat{\boldsymbol{v}}})(\hat{\boldsymbol{v}} \cdot d\boldsymbol{r}).$$

■

Corollary 2. *The ACWW model of wave dynamics creates circulation in the fluid whenever the gradients of the wave variables are not aligned.*

Proof. The proof follows by applying the Stokes theorem to the right-hand side of the material loop $c(\hat{\boldsymbol{v}})$ in Equation (84) in Theorem 4. ■

Theorem 5 (Total energy conservation is independent of ϵ)**.** *The energy conserved by modified equations (82) and (83) is independent of ϵ.*

Proof. The modified constrained action integral in Equation (72) may be rewritten equivalently as a phase-space Lagrangian, upon rearranging as follows:

$$\begin{aligned} S &= \int \ell(\hat{\boldsymbol{v}}, D, \hat{\phi}, \widehat{w}; \zeta, \lambda) dt \\ &= \int\int D\left(\frac{1}{2}(|\hat{\boldsymbol{v}}|^2 + \sigma^2\widehat{w}^2) - \frac{\zeta}{Fr^2}\right) + \sigma^2\lambda(\partial_t\zeta + \hat{\boldsymbol{v}} \cdot \nabla_{\boldsymbol{r}}\zeta - \widehat{w}) \\ &\quad + \hat{\phi}(\partial_t D + \mathrm{div}_{\boldsymbol{r}}(D\hat{\boldsymbol{v}})) - \epsilon\sigma^4\widehat{w}\lambda|\nabla_{\boldsymbol{r}}\zeta|^2\, d^2r\, dt. \\ &= \int\int \hat{\phi}\partial_t D + \sigma^2\lambda\partial_t\zeta - D\left(\frac{1}{2}|\hat{\boldsymbol{v}}|^2 + \frac{1}{2}\widehat{w}^2 + \frac{\zeta}{Fr^2}\right) d^2r\, dt. \end{aligned} \tag{85}$$

The last term in this equation agrees with the definition of energy for the unmodified Lagrangian in Equation (68) obtained by setting $\epsilon = 0$ in the modified Lagrangian in Equation (72). Thus, the modified and unmodified system conserve the same physical energy. ■

Theorem 5 shows that the equations resulting from the modified Lagrangian in (72) conserve the same energy as for the unmodified Lagrangian in (37). Thus, the modification in (72) that was obtained by introducing the ϵ term produces the wave–current interaction in Equations (82) and

(83) while also preserving the original physical energy density. What depends on ϵ is the definition of the vertical wave momentum density λ canonically conjugate to the elevation ζ depends on ϵ, as well as the definition of the velocity $\boldsymbol{V}$ in terms of the wave variables, as seen in the first and second lines of (73).

Remark 12 (Tensor virial theorem for a Lagrangian fluid blob under the ACWWE). Although the conserved energy remains the same for the ECWW and ACWW models for any value of the coupling constant ϵ, the tensor virial theorem for a Lagrangian fluid blob under the ACWWE is considerably more intricate than in Section 3.4.2 for the ECWW model.

4.2 | Lie–Poisson Hamiltonian formulation of the ACWWE

As discussed in Appendix C, the Legendre transformation of the augmented Lagrangian in the action integral (72) with respect to the sum of the fluid and wave momentum densities

$$\boldsymbol{M} = D\widehat{\boldsymbol{v}} + \lambda\nabla_{\boldsymbol{r}}\zeta = D\boldsymbol{V} + (\lambda - D\widehat{w})\nabla_{\boldsymbol{r}}\zeta \tag{86}$$

leads to the ECWW Hamiltonian defined now in dimensional units by

$$\begin{aligned} h(\boldsymbol{M}, D, \lambda, \zeta) &= \int \frac{1}{2D}|\boldsymbol{M} - \lambda\nabla_{\boldsymbol{r}}\zeta|^2 + \frac{\lambda^2}{2D}(1 + \epsilon|\nabla_{\boldsymbol{r}}\zeta|^2)^2 + gD\zeta\, d^2r, \\ &= \int \left(\frac{1}{2}|\widehat{\boldsymbol{v}}|^2 + \frac{1}{2}\widehat{w}^2 + g\zeta\right) D\, d^2r, \\ &= \int \left(\frac{1}{2}|\widehat{\nabla_{\boldsymbol{r}}\phi}|^2 + \frac{1}{2}\widehat{w}^2 + g\zeta\right) D\, d^2r. \end{aligned} \tag{87}$$

The Hamiltonian in (87) is also the conserved energy (68) for the system of ECWWE in (46), as proven in Theorem 5.

Variations of the Hamiltonian in (87)
In the Hamiltonian variables, the Bernoulli function ϖ in (73) is denoted as

$$\varpi = \frac{1}{2D^2}|\boldsymbol{M} - \lambda\nabla_{\boldsymbol{r}}\zeta|^2 + \frac{\lambda^2}{2D^2}(1 + \epsilon|\nabla_{\boldsymbol{r}}\zeta|^2) - g\zeta. \tag{88}$$

After evaluating the corresponding variational derivatives of the Hamiltonian in (87), the system of equations in (77) may be written in the untangled block-diagonal form, as

$$\frac{\partial}{\partial t}\begin{bmatrix} M_i \\ D \\ \lambda \\ \zeta \end{bmatrix} = -\begin{bmatrix} \partial_j M_i + M_j\partial_i & D\partial_i & 0 & 0 \\ \partial_j D & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} \delta h/\delta M_j = \widehat{v}^j \\ \delta h/\delta D = -\varpi \\ \delta h/\delta\lambda = -(\widehat{\boldsymbol{v}} - \epsilon\widehat{w}\nabla_{\boldsymbol{r}}\zeta)\cdot\nabla_{\boldsymbol{r}}\zeta + \widehat{w} \\ \delta h/\delta\zeta = \mathrm{div}_{\boldsymbol{r}}(\lambda(\widehat{\boldsymbol{v}} - 2\epsilon\widehat{w}\nabla_{\boldsymbol{r}}\zeta)) + gD \end{bmatrix}. \tag{89}$$

Casimir functions
The Casimir functions, conserved by the relation $\{C_\Phi, h\} = 0$ with any Hamiltonian $h(\boldsymbol{M}, D)$ for the block-diagonal Lie-Poisson bracket in Equation (89), are given by

$$C_\Phi := \int \Phi(q)\, D\, d^2r \quad \text{for} \quad q := D^{-1}\hat{\boldsymbol{z}} \cdot \operatorname{curl}(\boldsymbol{M}/D) \quad \text{with} \quad \partial_t q + \hat{\boldsymbol{v}} \cdot \nabla_{\boldsymbol{r}} q = 0. \tag{90}$$

As one may verify, the C_Φ are conserved for any differentiable function, Φ, provided that the velocity $\hat{\boldsymbol{v}}$ is tangent on the 2D boundary.

The proof of the constancy of the family of functions C_Φ is straightforward and well known. That the C_Φ comprise a family of Casimirs so that $\{C_\Phi, h\} = 0$ for any Hamiltonian $h(\boldsymbol{M}, D)$ is a standard result for semidirect-product Lie–Poisson brackets. See, for example, Ref. 49.

Remark 13 (Consequences of introducing the WCMC term). The consequences of introducing the slip velocity $\boldsymbol{s} := \hat{w}\nabla_{\boldsymbol{r}}\zeta$ in (32) into the variational principle in (72) as a WCMC term are evident in the Bernoulli function in (88) and in the transport velocities of the wave dynamics in (89), upon comparing them with (65) and (66), respectively. In contrast to the complexity of the separate relations for wave and current circulation laws in (82) and (83), the simplicity of the conservation of the total circulation in Equation (78) for ACWW dynamics seems to be a more meaningful statement about WCI than in the ECWWE, where the wave and current circulations are conserved separately in Equation (43), as a *mutual nonacceleration pact*.

The introduction of the WCMC coupling term with $\epsilon \neq 0$ in the modified Lagrangian in (72) has also made the PV formulation possible. Indeed, setting $\epsilon = 0$ would restore the relation (74) to potential flow. This, in turn, would set the PV q above to zero, which would then reduce the Casimirs C_Φ in (90) to conservation of mass.

In the next section of the paper, we will explore the further ramifications of introducing the WCMC term, by adding nonhydrostatic pressure, buoyancy and other physics to the ACWW system.

5 | HAMILTON'S PRINCIPLE FOR WAVE–CURRENT INTERACTION ON A FREE SURFACE (WCIFS)

5.1 | Adding buoyancy and other physics to the ACWW system

This section further augments the ACWWE set (77) to add more physical aspects to the WCIFS (WCI FS). These physical aspects include wave–current coupling, nonhydrostatic pressure, incompressibility, and horizontal gradients of buoyancy.

5.1.1 | Hamilton's principle for WCIFS

Let us modify the action integral (72) for the system of ACWWE in (77) to encompass the following aspects of WCIFS. As in (72), we will impose the surface boundary condition (35) and the continuity equation for the areal density variable D as constraints. We will also include the WCMC term via the slip velocity, as in Section 4.1. In addition, we will introduce an advected scalar buoyancy variable ρ with nonzero horizontal gradients. Finally, we will allow nonhydrostatic pressure, p.

To determine the pressure, p, we will constrain the 2D fluid transport velocity $\widehat{\boldsymbol{v}}$ to be divergence-free.[2] To include these various physical effects, we will apply Hamilton's principle with the following dimension-free action integral:

$$
\begin{aligned}
S &= \int \ell(\widehat{\boldsymbol{v}}, D, \rho, \widehat{\phi}, \widehat{w}, \zeta; \widetilde{\mu}, p)dt \\
&= \int \int D\rho\left(\frac{1}{2}(|\widehat{\boldsymbol{v}}|^2 + \sigma^2\widehat{w}^2) - \frac{\zeta}{Fr^2}\right) + \sigma^2\widetilde{\mu}(\partial_t\zeta + \widehat{\boldsymbol{v}}\cdot\nabla\zeta - \widehat{w}) - \sigma^4\epsilon\boldsymbol{s}\cdot(\widetilde{\mu}\nabla\zeta) \\
&\quad - \frac{1}{Fr^2}\,p(D-1) + \widehat{\phi}(\partial_t D + \operatorname{div}(D\widehat{\boldsymbol{v}})) + \gamma(\partial_t\rho + \widehat{\boldsymbol{v}}\cdot\nabla\rho)\,d^2r\,dt.
\end{aligned}
\tag{91}
$$

Here, we recall that the quantity $\boldsymbol{s} := \widehat{w}\nabla_{\boldsymbol{r}}\zeta$ is the slip velocity, defined in Equation (32).

Remark 14. Note that by including only certain terms in the above action integral, we may derive equations for the dynamics of subsystems with any combination of these additional properties (wave–current coupling, nonhydrostatic pressure, incompressibility, and buoyancy).

The passive wave case, $\epsilon = 0$

Taking variations of the dimensional version of action integral in (91) with $\epsilon = 0$ yields

$$
\begin{aligned}
\delta\widehat{\boldsymbol{v}} &: \quad D\rho\widehat{\boldsymbol{v}}\cdot d\boldsymbol{x} + \widetilde{\mu}\,d\zeta = Dd\widehat{\phi} - \gamma d\rho, \\
\delta\widehat{w} &: \quad D\rho\widehat{w} - \widetilde{\mu} = 0, \\
\delta\widetilde{\mu} &: \quad \partial_t\zeta + \widehat{\boldsymbol{v}}\cdot\nabla\zeta = \widehat{w}, \\
\delta\zeta &: \quad \partial_t\widetilde{\mu} + \operatorname{div}(\widetilde{\mu}\widehat{\boldsymbol{v}}) = -D\rho g, \\
\delta\rho &: \quad (\partial_t + \mathcal{L}_{\widehat{v}})\left(\frac{\gamma}{D}\right) = \frac{1}{2}(|\widehat{\boldsymbol{v}}|^2 + \widehat{w}^2) - g\zeta =: \varpi, \\
\delta D &: \quad (\partial_t + \mathcal{L}_{\widehat{v}})\widehat{\phi} = \rho\varpi - p, \\
\delta\widehat{\phi} &: \quad \partial_t D + \operatorname{div}(D\widehat{\boldsymbol{v}}) = 0, \\
\delta p &: \quad D - 1 = 0 \quad \Rightarrow \operatorname{div}\widehat{\boldsymbol{v}} = 0, \\
\delta\gamma &: \quad (\partial_t + \mathcal{L}_{\widehat{v}})\rho = 0.
\end{aligned}
\tag{92}
$$

Applying $(\partial_t + \mathcal{L}_{\widehat{v}})$ to the first relation in (92) yields

$$
(\partial_t + \mathcal{L}_{\widehat{v}})(\widehat{\boldsymbol{v}}\cdot d\boldsymbol{r} + \widehat{w}d\zeta) = d\varpi - \frac{1}{\rho}dp,
\tag{93}
$$

[2] We will reserve the hat notation for aspects of velocity $(\widehat{v}, \widehat{w}, \widehat{\phi})$ evaluated on the free surface. Hence, we will refrain from gratuitously adding hats to the pressure p and the buoyancy ρ, because it is understood that they are evaluated on the free surface. The meaning for pressure p and the buoyancy ρ will always be clear from the context. The assumption of incompressibility of the fluid flow will enable the Bernoulli law to admit finite nonhydrostatic pressure.

so we obtain the following Kelvin circulation theorem,

$$\frac{d}{dt}\oint_{c_t} \boldsymbol{V}\cdot d\boldsymbol{r} = \oint_{c_t}(\partial_t + \mathcal{L}_{\widehat{\boldsymbol{v}}})(\boldsymbol{V}\cdot d\boldsymbol{r}) = -\oint_{c_t}\frac{1}{\rho}dp. \tag{94}$$

As expected, the momentum per unit mass in the motion equation is $\widehat{\boldsymbol{v}} + \widehat{w}\nabla\zeta =: \boldsymbol{V}$. The result has the same right-hand side as for the 2D inhomogeneous Euler equation. Here, the total momentum now is the sum of the fluid momentum and the wave momentum, whose evolution is obtained as a separate degree of freedom appearing in the third and fourth lines of the equation set (92).

However, continuing to calculate from (93) yields a nonacceleration result as in Equation (42), in the sense that the wave momentum evolves passively with the flow of the fluid,

$$(\partial_t + \mathcal{L}_{\widehat{\boldsymbol{v}}})(\widehat{w}\nabla\zeta) = -g\,d\zeta + \frac{1}{2}d\widehat{w}^2,$$

and the fluid momentum evolves independently of the wave variables,

$$(\partial_t + \mathcal{L}_{\widehat{\boldsymbol{v}}})(\widehat{\boldsymbol{v}}\cdot d\boldsymbol{r}) = d\left(\frac{1}{2}|\widehat{\boldsymbol{v}}|^2\right) - \frac{1}{\rho}dp.$$

Thus, the momentum equation in the passive wave case is simply a 2D Euler equation to be considered in tandem with the remaining identities from (92). Note that if $\nabla\rho \neq 0$, then the right-hand side of the previous equation generates circulation in $\widehat{\boldsymbol{v}}$; so, in this case, $\widehat{\boldsymbol{v}}$ cannot produce potential flow.

Next, we will pursue the implications when the WCMC parameter ϵ does not vanish and the wave variables do not interact passively.

5.2 | Derivation of the WCIFS equations for active waves

To derive a WCIFS model system of equations for the motion of free surface with active waves and spatially varying buoyancy, we will apply the free-surface condition (35) and incompressibility of the $\widehat{\boldsymbol{v}}$-flow as constraints in the action integral, while also including the minimal coupling term with nondimensional parameter $\epsilon \neq 0$ in the action integral (91). Then, upon restoring dimensionality to the variables in (91), we obtain the following action principle for the free-surface motion:

$$\begin{aligned} S &= \int \ell(\widehat{\boldsymbol{v}}, \widehat{w}, D, \rho, \zeta; \widetilde{\mu}, p)dt \\ &= \int\!\!\int D\rho\left(\frac{1}{2}(|\widehat{\boldsymbol{v}}|^2 + \widehat{w}^2) - g\zeta\right) + \widetilde{\mu}(\partial_t\zeta + \widehat{\boldsymbol{v}}\cdot\nabla\zeta - \widehat{w}) - \epsilon\widehat{w}\nabla\zeta\cdot(\widetilde{\mu}\nabla\zeta) \\ &\quad - p(D-1) + \widehat{\phi}(\partial_t D + \mathrm{div}(D\widehat{\boldsymbol{v}})) + \gamma(\partial_t\rho + \widehat{\boldsymbol{v}}\cdot\nabla\rho)\,d^2r\,dt. \end{aligned} \tag{95}$$

The Lagrange multipliers $\widetilde{\mu}$, p, $\widehat{\phi}$, and γ apply, respectively, the free-surface condition (35), incompressibility of the $\widehat{\boldsymbol{v}}$-flow, mass preservation, and buoyancy advection.

Hamilton's principle, $\delta S = 0$, for the restricted free-surface action integral in (95) yields the following independent relations:

$$
\begin{aligned}
\delta \widehat{\boldsymbol{v}} &: \quad D\rho \widehat{\boldsymbol{v}} \cdot d\boldsymbol{x} + \widetilde{\mu}\, d\zeta = D d\widehat{\phi} - \gamma d\rho, \\
\delta \widehat{w} &: \quad D\rho \widehat{w} - \widetilde{\mu}\,(1 + \epsilon|\nabla\zeta|^2) = 0, \\
\delta \widetilde{\mu} &: \quad \partial_t \zeta + \widehat{\boldsymbol{v}} \cdot \nabla\zeta - \widehat{w}(1 + \epsilon|\nabla\zeta|^2) = 0, \\
\delta \zeta &: \quad \partial_t \widetilde{\mu} + \mathrm{div}(\widetilde{\mu}\widehat{\boldsymbol{v}}) = -D\rho g + 2\epsilon\, \mathrm{div}(\widehat{w}\,\widetilde{\mu}\,\nabla\zeta), \\
\delta \rho &: \quad (\partial_t + \mathcal{L}_{\widehat{v}})\Big(\frac{\gamma}{D}\Big) = \frac{1}{2}(|\widehat{\boldsymbol{v}}|^2 + \widehat{w}^2) - g\zeta =: \varpi, \\
\delta D &: \quad (\partial_t + \mathcal{L}_{\widehat{v}})\widehat{\phi} = \rho\varpi - p, \\
\delta \widehat{\phi} &: \quad \partial_t D + \mathrm{div}(D\widehat{\boldsymbol{v}}) = 0, \\
\delta p &: \quad D - 1 = 0 \quad \Rightarrow \mathrm{div}\widehat{\boldsymbol{v}} = 0, \\
\delta \gamma &: \quad (\partial_t + \mathcal{L}_{\widehat{v}})\rho = 0.
\end{aligned}
\tag{96}
$$

Before enforcing the pressure constraint $D = 1$, we write out the fluid motion equation obtained by substituting the variational results into the application of the advective time derivative $(\partial_t + \mathcal{L}_{\widehat{v}})$ on the first line of the system (96), to find, upon writing $\widetilde{\mu} = D\rho\widetilde{w}$

$$
\begin{aligned}
D\rho(\partial_t + \mathcal{L}_{\widehat{v}})(\widehat{\boldsymbol{v}} \cdot d\boldsymbol{x} + \widetilde{w}\, d\zeta) &= Dd(\partial_t + \mathcal{L}_{\widehat{v}})\widehat{\phi} - D\Big((\partial_t + \mathcal{L}_{\widehat{v}})\frac{\gamma}{D}\Big)d\rho \\
&= D\rho\Big(d\varpi - \frac{1}{\rho}dp\Big).
\end{aligned}
\tag{97}
$$

Recall that the Bernoulli function ϖ and vertical wave momentum density $\widetilde{\mu}$ are defined as

$$
\varpi := \frac{1}{2}(|\widehat{\boldsymbol{v}}|^2 + \widehat{w}^2) - g\zeta, \qquad \widetilde{\mu} := \frac{D\rho\widehat{w}}{(1 + \epsilon|\nabla\zeta|^2)} = D\rho\widetilde{w}.
\tag{98}
$$

At this point, let us collect the WCIFS equations in terms of (i) fluid variables, comprising velocity, $\widehat{\boldsymbol{v}}$, area density, D, and buoyancy, ρ, and (ii) wave variables, comprising surface elevation, ζ and vertical wave momentum density $\widetilde{\mu}$. The WCIFS equations are

$$
\begin{aligned}
(\partial_t + \mathcal{L}_{\widehat{v}})(\widehat{\boldsymbol{v}} \cdot d\boldsymbol{x} + \widetilde{w}\, d\zeta) &= d\varpi - \frac{1}{\rho}dp, \\
\partial_t D + \mathrm{div}(D\widehat{\boldsymbol{v}}) &= 0, \\
\partial_t \rho + \widehat{\boldsymbol{v}} \cdot \nabla\rho &= 0, \\
\partial_t \zeta + \widehat{\boldsymbol{v}} \cdot \nabla\zeta &= \widehat{w}(1 + \epsilon|\nabla\zeta|^2), \\
\partial_t \widetilde{w} + \widehat{\boldsymbol{v}} \cdot \nabla\widetilde{w} &= -g + \frac{2\epsilon}{D\rho}\,\mathrm{div}(\widehat{w}\,\widetilde{\mu}\,\nabla\zeta).
\end{aligned}
\tag{99}
$$

In its role as a Lagrange multiplier in the action integral (95), the pressure p enforces the constraint $D = 1$. In turn, persistence of the condition $D = 1$ along the flow implies that the fluid motion generated by $\widehat{\boldsymbol{v}}$ is incompressible. In particular, setting $D = 1$ in the continuity equation in (99) above implies that the free surface fluid velocity $\widehat{\boldsymbol{v}}$ is divergence-free, $\operatorname{div}\widehat{\boldsymbol{v}} = 0$. The pressure p is then determined by requiring that $\widehat{\boldsymbol{v}}$ remain divergence free, which implies the following elliptic equation for p,

$$
\begin{aligned}
-(\nabla \cdot \rho^{-1}\nabla)p = \operatorname{div}\bigg(& \widehat{\boldsymbol{v}} \cdot \nabla\widehat{\boldsymbol{v}} + \frac{\widehat{w}}{1+\epsilon|\nabla\zeta|^2}\nabla(\widehat{w}(1+\epsilon|\nabla\zeta|^2)) \\
& - \widehat{w}\nabla\widehat{w} + \frac{2\epsilon}{\rho}\operatorname{div}(\rho\,\widetilde{w}^2(1+\epsilon|\nabla\zeta|^2)\,\nabla\zeta)\nabla\zeta \bigg).
\end{aligned}
\tag{100}
$$

Thus, the pressure p depends on the horizontal flow velocity $\widehat{\boldsymbol{v}}$ of the surface current and fluid buoyancy ρ, as well as the wave elevation ζ and the vertical velocity w. We stress that the flow variables, $(\widehat{\boldsymbol{v}}, \rho)$, and the wave variables, $(\zeta, \widetilde{w})$, comprise two separate Eulerian degrees of freedom at each point $\boldsymbol{r} = (x, y)$ in the 2D domain of flow.

Remark 15 (Making the action integral stochastic in Section 6). In Section 6, the action integral (91) will be made stochastic, following,[26] and we will to derive a stochastic generalization of the water wave equations.

5.3 | Comparison of WCIFS system to other known systems

Remark 16 (Comparison of system (99) to the John-Sclavounos (JS) model equations [33,50]). The JS model comprises a dynamical system of ordinary differential equations for the motion of a single particle that is constrained to remain upon the free surface $\zeta(x, y, t) - z = 0$, with *prescribed* $\zeta(x, y, t)$. This dynamical system has recently been derived from a variational principle using the Euler–Lagrange methodology.[51] This variational principle raises the question of whether the particle dynamics of JS model may be associated with Lagrangian fluid trajectory dynamics in the present continuum framework.

The JS equations give the horizontal fluid particle trajectories $\boldsymbol{r}(t) = (x(t), y(t))$ driven by the free surface $z = \zeta(\boldsymbol{r}, t)$. The equations can be expressed as

$$
(1+\zeta_x^2)\ddot{x} + \zeta_x\zeta_y\ddot{y} + (\zeta_{tt} + \zeta_{xt}\dot{x} + \zeta_{yt}\dot{y} + \zeta_{xx}\dot{x}^2 + 2\zeta_{xy}\dot{x}\dot{y} + \zeta_{yy}\dot{y} + g)\zeta_x = 0, \tag{101}
$$

$$
(1+\zeta_y^2)\ddot{y} + \zeta_x\zeta_y\ddot{x} + (\zeta_{tt} + \zeta_{xt}\dot{x} + \zeta_{yt}\dot{y} + \zeta_{xx}\dot{x}^2 + 2\zeta_{xy}\dot{x}\dot{y} + \zeta_{yy}\dot{y} + g)\zeta_y = 0. \tag{102}
$$

Note that

$$
\partial_t(\zeta_t + \zeta_x\dot{x} + \zeta_y\dot{y}) + g - \zeta_x\ddot{x} - \zeta_y\ddot{y} = \zeta_{tt} + \zeta_{xt}\dot{x} + \zeta_{yt}\dot{y} + \zeta_{xx}\dot{x}^2 + 2\zeta_{xy}\dot{x}\dot{y} + \zeta_{yy}\dot{y} + g.
$$

Hence, the JS equations can be rewritten in more concise vector notation as

$$\ddot{\boldsymbol{r}} + ((\partial_t + \dot{\boldsymbol{r}}\cdot\nabla)(\partial_t\zeta + \dot{\boldsymbol{r}}\cdot\nabla\zeta) + g)\nabla\zeta =: \ddot{\boldsymbol{r}} + \left(\frac{D}{Dt}\left(\frac{D\zeta}{Dt}\right) + g\right)\nabla\zeta = 0, \tag{103}$$

with $D/Dt := \partial_t + \dot{\boldsymbol{y}}\cdot\nabla$.

Choi's relation

One may immediately make the connection between the JS equations (103) and Choi's relation (9). Naturally, as the JS equations represent a single particle's motion, whereas Choi's relation is a statement about continuum flows, (9) features a pressure term on the right-hand side. However, the two equations are otherwise strikingly similar.

Comparison with the JS equations

To make the comparison between the system of equations derived in this paper with the JS equations in vector form (103), we combine the last two equations of the system (99) to write,

$$g - \frac{2\epsilon}{D\rho}\operatorname{div}(D\rho\,\widetilde{w}^2(1+\epsilon|\nabla\zeta|^2)\,\nabla\zeta) = -(\partial_t + \widehat{\boldsymbol{v}}\cdot\nabla)\left(\frac{\partial_t\zeta + \widehat{\boldsymbol{v}}\cdot\nabla\zeta}{(1+\epsilon|\nabla\zeta|^2)^2}\right) = -(\partial_t + \widehat{\boldsymbol{v}}\cdot\nabla)\widetilde{w}. \tag{104}$$

Consequently, the motion equation in system (99) may be expressed as

$$\partial_t\widehat{\boldsymbol{v}} + \widehat{\boldsymbol{v}}\cdot\nabla\widehat{\boldsymbol{v}} + ((\partial_t + \widehat{\boldsymbol{v}}\cdot\nabla)\widetilde{w} + g)\nabla\zeta = -\frac{1}{\rho}\nabla p + \widehat{w}\nabla\widehat{w} - \frac{\widehat{w}}{1+\epsilon|\nabla\zeta|^2}\nabla(\widehat{w}(1+\epsilon|\nabla\zeta|^2)). \tag{105}$$

Thus, the present form of the WCIFS fluid equations (105) does seem to have some kinematic resemblance to the JS equations, although the two types of dynamics also have major physical and mathematical differences in their interpretations.

For example, one may write the WCIFS motion equation (105) equivalently in more compact form, as

$$\partial_t\widehat{\boldsymbol{v}} + \widehat{\boldsymbol{v}}\cdot\nabla\widehat{\boldsymbol{v}} + ((\partial_t + \widehat{\boldsymbol{v}}\cdot\nabla)\widetilde{w} + g)\nabla\zeta = -\frac{1}{\rho}\nabla p - \frac{1}{2}\widetilde{w}^2\nabla(1+\epsilon|\nabla\zeta|^2)^2. \tag{106}$$

In this compact form, which is also reminiscent of Choi's relation (9), the geometric, coordinate-free nature of the WCIFS equation begins to emerge upon writing (105) equivalently as the advective Lie derivative of a 1-form, which also arises in the Kelvin circulation theorem below, cf. (111),

$$(\partial_t + \mathcal{L}_{\widehat{v}})(\widehat{\boldsymbol{v}}\cdot d\boldsymbol{x} + \widetilde{w}d\zeta) = d\varpi - \frac{1}{\rho}dp. \tag{107}$$

In one dimension with constant buoyancy $\rho = \rho_0$ and $p = p_s = \rho_0 g\zeta$, this formula becomes

$$(\partial_t + \mathcal{L}_v)(v dx + \widetilde{w}d\zeta) = d\varpi - \frac{g}{\rho_0}d\zeta, \tag{108}$$

where ϖ is defined in (98). In this geometric form, the JS and WCIFS models look rather more distant.

5.4 | Balance relations, Kelvin theorem, and potential vorticity

Remark 17 (Dimension-free form of motion Equation (107)). In terms of the parameters in Remark 5, the dimension-free form of the motion Equation (107) is given by

$$(\partial_t + \mathcal{L}_{\widehat{\boldsymbol{v}}})(\widehat{\boldsymbol{v}} \cdot d\boldsymbol{x} + \sigma^2 \widetilde{w} d\zeta) = d\left(\frac{1}{2}(|\widehat{\boldsymbol{v}}|^2 + \sigma^2 \widehat{w}^2) - \frac{\zeta}{Fr^2}\right) - \frac{1}{Fr^2}\frac{1}{\rho}dp, \tag{109}$$

where $\partial_t \zeta + \widehat{\boldsymbol{v}} \cdot \nabla\zeta = \widetilde{w}(1 + \epsilon\sigma^2 |\nabla\zeta|^2)^2 = \widehat{w}(1 + \epsilon\sigma^2 |\nabla\zeta|^2)$.

Balance relations required for significant wave–current interaction
For small Froude number, $Fr^2 \ll 1$, Equation (109) approaches hydrostatic balance, and for small aspect ratio $\sigma^2 \ll 1$, Equation (109) suppresses wave activity. When Froude number Fr^2 and aspect ratio σ are both of order $O(1)$, then Equation (109) admits order $O(1)$ significant nonhydrostatic wave activity.

Likewise, the $\widetilde{w}$ equation in (99) in dimensionless form for the same scaling parameters becomes

$$\sigma^2(\partial_t \widetilde{w} + \widehat{\boldsymbol{v}} \cdot \nabla \widetilde{w}) = -\frac{1}{Fr^2} + \frac{2\epsilon\sigma^4}{D\rho}\operatorname{div}(D\rho\, \widehat{w}\, \widetilde{w}\, \nabla\zeta). \tag{110}$$

The balance between current and wave properties in the dimension-free $\widetilde{w}$ Equation (110) also requires both Froude number Fr^2 and aspect ratio σ to be of order $O(1)$ for significant wave activity to occur.

Only the motion equation and the $\widetilde{w}$ equation in (99) change their coefficients for these scaling parameters. The coefficients of the others remain unchanged.

Remark 18. To prove the following Kelvin–Noether circulation theorem for the system of WCIFS equations in (99), it is convenient to return to the variational equations in (96) and the notation introduced in (97) and (98).

Theorem 6 (Kelvin–Noether theorem for the WCIFS model). *For every closed loop $c(\widehat{\boldsymbol{v}})$ moving with the WCIFS velocity $\widehat{\boldsymbol{v}}$ for the system of WCIFS equations in (99), the Kelvin circulation relation holds,*

$$\frac{d}{dt}\oint_{c(\widehat{\boldsymbol{v}})}\left(\widehat{\boldsymbol{v}} \cdot d\boldsymbol{x} + \frac{\widetilde{\mu}}{D\rho}d\zeta\right) = \oint_{c(\widehat{\boldsymbol{v}})} d\varpi - \frac{1}{\rho}dp. \tag{111}$$

Proof. From the variational equations in (96), we have

$$\begin{aligned}(\partial_t + \mathcal{L}_{\widehat{\boldsymbol{v}}})(D\rho\widehat{\boldsymbol{v}} \cdot d\boldsymbol{x} + \widetilde{\mu} d\zeta) &= Dd(\partial_t + \mathcal{L}_{\widehat{\boldsymbol{v}}})\widehat{\phi} - D\left((\partial_t + \mathcal{L}_{\widehat{\boldsymbol{v}}})\frac{\gamma}{D}\right)d\rho \\ &= Dd(\rho\varpi - p) - D\varpi d\rho.\end{aligned}$$

Hence, we find

$$(\partial_t + \mathcal{L}_{\widehat{\boldsymbol{v}}})\left(\widehat{\boldsymbol{v}} \cdot d\boldsymbol{x} + \frac{\widetilde{\mu}}{D\rho}d\zeta\right) = d\varpi - \frac{1}{\rho}dp, \tag{112}$$

and the result (111) follows from the standard relation for the time derivative of an integral around a closed moving loop, $c(\hat{\boldsymbol{v}})$,

$$\frac{d}{dt}\oint_{c(\hat{\boldsymbol{v}})}\left(\hat{\boldsymbol{v}}\cdot d\boldsymbol{x}+\frac{\widetilde{\mu}}{D\rho}d\zeta\right)=\oint_{c(\hat{\boldsymbol{v}})}(\partial_t+\mathcal{L}_{\hat{v}})\left(\hat{\boldsymbol{v}}\cdot d\boldsymbol{x}+\frac{\widetilde{\mu}}{D\rho}d\zeta\right)=\oint_{c(\hat{\boldsymbol{v}})}d\varpi-\frac{1}{\rho}dp. \qquad \blacksquare$$

Remark 19 (Interpretation of WCIFS as a compound fluid system). The compound circulation of the WCIFS wave–fluid system in (99) obeys the same dynamical equations as the planar incompressible flow description of a single-component flow with horizontal buoyancy gradient, except for two features associated with the wave degrees of freedom. First, the presence of the wave field contributes to the solution for the pressure from the condition that the velocity of the fluid component remains incompressible. Second, the presence of the wave field is a source of circulation for the fluid component of this compound system. Both of these features are due to the momentum of the waves, defined using the notation $\widetilde{w}$ defined in (99) as $\widetilde{\mu}/(D\rho)\nabla\zeta=\widetilde{w}\nabla\zeta$, which is proportional to the wave slope, $\nabla\zeta$. In particular, the momentum $\widetilde{w}\nabla\zeta$ appears in both the pressure equation in (100) and the Kelvin–Noether integrand in (111).

Corollary 3 (Total wave–fluid PV for WCIFS)**.** *The evolution equation for the total wave–fluid PV follows by taking the exterior derivative of Equation (112) in the proof of the Kelvin circulation theorem for WCIFS. Namely,*

$$(\partial_t+\mathcal{L}_{\hat{v}})(\mathrm{curl}\hat{\boldsymbol{v}}+\nabla\widetilde{w}\times\nabla\zeta)\cdot d\boldsymbol{S}=-\nabla\rho^{-1}\times\nabla p\cdot d\boldsymbol{S}. \tag{113}$$

If we introduce a stream function ψ, so that $\hat{\boldsymbol{v}}=\hat{\boldsymbol{z}}\times\nabla\psi$, then the previous equation can be written formally in terms of the 2D Laplacian Δ and the Jacobian $J(\psi,\rho):=\hat{\boldsymbol{z}}\cdot\nabla\psi\times\nabla\rho=\hat{\boldsymbol{v}}\cdot\nabla\rho$ between functions ψ and ρ, then we have

$$\partial_t q+J(\psi,\,q)=-J(\rho^{-1},\nabla p),\quad \textit{with PV defined as } q:=\Delta\psi+J(\widetilde{w},\zeta). \tag{114}$$

Remark 20. Note that advancing the PV quantity q in time in Equation (114) requires one to advance the entire system of WCIFS equations in (99), as well as the solution of the following elliptic Equation (100) for the pressure p to complete the evolution algorithm.

5.5 | Integral conservation laws for the WCIFS equations

5.5.1 | Spatially varying specific buoyancy

The system of equations for the PV and specific buoyancy (q,ρ^{-1}) is given by

$$\partial_t q+J(\psi,\,q)=-J(\rho^{-1},\nabla p),\quad \partial_t\rho^{-1}+J(\psi,\,\rho^{-1})=0. \tag{115}$$

This system of equations implies that the following integral quantity is conserved under the (q,ρ^{-1}) dynamics:

$$C_{\Phi,\Psi}:=\int\Phi(\rho^{-1})+q\Psi(\rho^{-1})\,d^2x, \tag{116}$$

for arbitrary differentiable functions Φ and Ψ.

5.5.2 | Spatially homogeneous specific buoyancy

In the case that the specific buoyancy is initially constant, $\rho^{-1} = \rho_0^{-1}$, then it will remain constant, and $\nabla\rho^{-1} = 0$ will persist throughout the WCIFS domain of flow. In this case, the (q, ρ^{-1}) system (115) will reduce to a single equation, $\partial_t q + J(\psi, q) = 0$, describing simple advection of the PV quantity, q. Hence, the conserved integral quantities are the familiar vorticity functionals from the 2D Euler equations, or PV functionals from the quasi-geostrophic (QG) equation. Namely, for a spatially homogeneous initial specific buoyancy, the WCIFS system in (115) will conserve the following class of integral quantities:

$$C_\Phi := \int \Phi(q)\, d^2x, \tag{117}$$

for an arbitrary differentiable function Φ. Thus, the WCIFS integral conservation laws for PV in Equations (116) and (117) depend on whether the specific buoyancy gradient $(\nabla\rho^{-1})$ vanishes at the initial time.

Energy
The conserved WCIFS integrated energy is given by

$$e(\hat{\boldsymbol{v}}, \rho, \zeta, \widetilde{w}) := \int \frac{\rho}{2}(|\hat{\boldsymbol{v}}|^2 + \widetilde{w}^2(1+\epsilon|\nabla\zeta|^2)^2) + g\rho\zeta\, d^2x. \tag{118}$$

Although the energy conservation law may be proven directly from the WCIFS equations in (99), it may be more enlightening to discover this energy via the Legendre transformation of the Lagrangian in the action integral S in (95) and thereby determine the Hamiltonian formulation and its remarkable properties for the WCIFS system. In particular, the Lie-Poisson bracket in the Hamiltonian formulation of the WCIFS system in the next section will explain the source of the WCIFS conservation laws and their relationships among each other from the viewpoint of the Hamiltonian structure for the WCIFS system.

5.6 | Hamiltonian formulation of WCIFS in terms of potential vorticity

Remark 21 (WCI FS with constant buoyancy). The simplest form of the WCIFS equations in (99) arises when the buoyancy is constant, that is, $\rho = \rho_0$. In that case, the WCIFS equations reduce to

$$\begin{aligned}
&(\partial_t + \mathcal{L}_{\hat{v}})(\hat{\boldsymbol{v}}\cdot d\boldsymbol{x} + \widetilde{w} d\zeta) = d\varpi - \frac{1}{\rho_0}\, dp,\\
&(\partial_t + \mathcal{L}_{\hat{v}})\zeta = \widehat{w}(1+\epsilon|\nabla\zeta|^2) =: \widetilde{w}(1+\epsilon|\nabla\zeta|^2)^2\\
&(\partial_t + \mathcal{L}_{\hat{v}})\widetilde{w} = -g + 2\epsilon\,\mathrm{div}(\widetilde{w}^2(1+\epsilon|\nabla\zeta|^2)\,\nabla\zeta),\\
&\text{with}\quad \hat{\boldsymbol{v}} = \hat{\boldsymbol{z}}\times\nabla\psi,\\
&\text{and } (\partial_t + \mathcal{L}_{\hat{v}})(1+\epsilon|\nabla\zeta|^2) = 2\epsilon\nabla\zeta\cdot\left(\nabla\left(\widehat{w}(1+\epsilon|\nabla\zeta|^2)\right) - \zeta_{,j}\nabla\hat{v}^j\right),\\
&\text{with}\quad \varpi := \frac{1}{2}|\hat{\boldsymbol{v}}|^2 + \frac{1}{2}\widetilde{w}^2(1+\epsilon|\nabla\zeta|^2)^2 - g\zeta.
\end{aligned} \tag{119}$$

Corollary 4 (WCI FS in PV Hamiltonian form)**.** *The fluid dynamical system (119) for WCIFS with constant buoyancy in terms of PV is a Hamiltonian system whose Poisson bracket is the following sum of a Lie–Poisson bracket for PV as* $q := \Delta\psi + J(\widetilde{w}, \zeta)$ *and a canonical bracket for the wave variables* $(\zeta, \widetilde{w})$*. Namely,*

$$\frac{df}{dt} = \{f, e\} = \int q\, J\left(\frac{\delta f}{\delta q}, \frac{\delta e}{\delta q}\right) d^2x + \int \frac{\delta f}{\delta \zeta}\frac{\delta e}{\delta \widetilde{w}} - \frac{\delta f}{\delta \widetilde{w}}\frac{\delta e}{\delta \zeta}\, d^2x. \tag{120}$$

Proof. We write the energy (118) in terms of PV defined as $q := \Delta\psi + J(\widetilde{w}, \zeta)$ in Equation (114),

$$e(q, \zeta, \widetilde{w}) := \int \frac{1}{2}((q - J(\widetilde{w}, \zeta))(-\Delta^{-1})(q - J(\widetilde{w}, \zeta)) + \widetilde{w}^2(1 + \epsilon|\nabla\zeta|^2)^2) + g\zeta\, d^2x. \tag{121}$$

The variational derivatives of the energy $e(q, \zeta, \widetilde{w})$ in (121) are given by

$$\begin{bmatrix} \delta e/\delta q \\ \delta e/\delta \widetilde{w} \\ \delta e/\delta \zeta \end{bmatrix} = \begin{bmatrix} -\psi \\ -J(\psi, \zeta) + \widetilde{w}(1 + \epsilon|\nabla\zeta|^2)^2 \\ J(\psi, \widetilde{w}) + g - 2\epsilon\, \mathrm{div}(\widetilde{w}^2(1 + \epsilon|\nabla\zeta|^2)\, \nabla\zeta) \end{bmatrix}. \tag{122}$$

Hence, Equations (119) become

$$\begin{aligned} \frac{\partial}{\partial t}\begin{bmatrix} q \\ \widetilde{w} \\ \zeta \end{bmatrix} = \begin{bmatrix} \{q, e\} \\ \{\widetilde{w}, e\} \\ \{\zeta, e\} \end{bmatrix} &= -\begin{bmatrix} J(q, \cdot\,) & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} \delta e/\delta q \\ \delta e/\delta \widetilde{w} \\ \delta e/\delta \zeta \end{bmatrix} \\ &= -\begin{bmatrix} J(\psi, q) \\ J(\psi, \widetilde{w}) + g - 2\epsilon\, \mathrm{div}(\widetilde{w}^2(1 + \epsilon|\nabla\zeta|^2)\, \nabla\zeta) \\ J(\psi, \zeta) - \widetilde{w}(1 + \epsilon|\nabla\zeta|^2)^2 \end{bmatrix}. \end{aligned} \tag{123}$$

■

Proof. One computes the PV bracket between $C_\Phi(q)$ and an arbitrary functional $F(q)$, as

$$\{F, C_\Phi(q)\} = \int q\, J\left(\frac{\delta F}{\delta q}, \frac{\delta C_\Phi}{\delta q}\right) d^2x = -\int \frac{\delta F}{\delta q} J(q, \Phi'(q)) d^2x = 0, \quad \text{for all } F(q),$$

after an integration by parts in a periodic domain, say. Thus, $C_\Phi(q)$ is a Casimir function for the PV Poisson bracket in (123). ■

Proposition 3. *Sufficient conditions for* $(q_e, \zeta_e, \widetilde{w}_e)$ *to be an equilibrium solution of (122) arise by requiring that the functional* $H_\Phi = e(q, \zeta, \widetilde{w}) + C_\Phi(q)$ *would have a critical point at* $(q_e, \zeta_e, \widetilde{w}_e)$*.*

Proof. Evaluated at $(q_e, \zeta_e, \widetilde{w}_e)$ a critical point the functional H_Φ satisfies

$$\delta H_\Phi = \int (-\psi_e + \Phi'(q_e)\delta q + \left.\frac{\delta e}{\delta \zeta}\right|_{(q_e, \zeta_e, \widetilde{w}_e)} \delta\zeta + \left.\frac{\delta e}{\delta \widetilde{w}}\right|_{(q_e, \zeta_e, \widetilde{w}_e)} \delta\widetilde{w}\, d^2x$$

This is sufficient for the right-hand side of Equation (122) to vanish and thereby produce an equilibrium solution. ■

The Lie–Poisson brackets in the Hamiltonian formulations of the system provided here for the case of constant buoyancy, $\rho = \rho_0$, and in the next section for the general case explain the sources of the conservation laws and their relationships among each other from the viewpoint of the Hamiltonian structure for the system.

5.7 | Hamiltonian formulation of the WCIFS equations

Legendre transformation

By considering the Lagrangian function in the action integral (95), $\ell(\hat{\boldsymbol{v}}, D, \rho, \zeta, w; p)$, one may define the Legendre transformation as the variation with respect to the velocity $\hat{\boldsymbol{v}}$ in (96). Namely,

$$\frac{\delta \ell}{\delta \hat{\boldsymbol{v}}} = D\rho\hat{\boldsymbol{v}} + \widetilde{\mu}\,\nabla\zeta - D\nabla\hat{\phi} + \gamma\nabla\rho, \quad \text{where } \widetilde{\mu} := \frac{D\rho\widehat{w}}{1+\epsilon|\nabla\zeta|^2} = \frac{D\rho(\partial_t + \mathcal{L}_{\hat{\boldsymbol{v}}})\zeta}{(1+\epsilon|\nabla\zeta|^2)^2}.$$

Upon defining the fluid momentum as $\boldsymbol{m} = D\rho\hat{\boldsymbol{v}}$, the Hamiltonian in these variables is obtained via the following calculation:

$$\begin{aligned} h(\boldsymbol{m}, D, \rho, \zeta, \widetilde{\mu}; p) &:= \langle \boldsymbol{m}, \hat{\boldsymbol{v}} \rangle + \widetilde{\mu}\partial_t\zeta - D\partial_t\phi + \gamma\partial_t\rho - \ell(\hat{\boldsymbol{v}}, D, \rho, \zeta, w; p) \\ &= \int \frac{|\boldsymbol{m}|^2}{2D\rho} + \frac{\widetilde{\mu}^2}{2D\rho}(1+\epsilon|\nabla\zeta|^2)^2 + gD\rho\zeta + p(D-1)\,d^2x. \end{aligned} \tag{124}$$

Conserved energy

The Hamiltonian in (124) is also the conserved energy for the system of equations in (97).

Variations of the Hamiltonian

In the Hamiltonian variables, the Bernoulli function ϖ in (98) is denoted as

$$\widetilde{\varpi} = \frac{|\boldsymbol{m}|^2}{2D^2\rho^2} + \frac{\widetilde{\mu}^2(1+\epsilon|\nabla\zeta|^2)^2}{2D^2\rho^2} - g\zeta. \tag{125}$$

The corresponding variational derivatives of the Hamiltonian in (124) for the system of equations in (97) are given by

$$\begin{bmatrix} \delta h/\delta m_j \\ \delta h/\delta D \\ \delta h/\delta \rho \\ \delta h/\delta \widetilde{\mu} \\ \delta h/\delta \zeta \end{bmatrix} = \begin{bmatrix} \frac{m^j}{D\rho} \\ -\rho\widetilde{\varpi} + p \\ -D\widetilde{\varpi} \\ \frac{1}{D\rho}\widetilde{\mu}(1+\epsilon|\nabla\zeta|^2)^2 \\ D\rho g - 2\epsilon\,\mathrm{div}(\widetilde{\mu}\widehat{w}\nabla\zeta) \end{bmatrix} = \begin{bmatrix} v^j \\ -\rho\widetilde{\varpi} + p \\ -D\widetilde{\varpi} \\ \widetilde{w}(1+\epsilon|\nabla\zeta|^2)^2 \\ D\rho g - 2\epsilon\,\mathrm{div}(\widetilde{\mu}\widehat{w}\nabla\zeta) \end{bmatrix}. \tag{126}$$

The system of equations in (97) may now be written in Lie–Poisson form, augmented by a symplectic 2-cocycle in the elevation ζ and its canonical momentum density $\widetilde{\mu}$ in its *entangled* form as

$$\frac{\partial}{\partial t}\begin{bmatrix} m_i \\ D \\ \rho \\ \widetilde{\mu} \\ \zeta \end{bmatrix} = -\begin{bmatrix} \partial_j m_i + m_j \partial_i & D\partial_i & -\rho_{,i} & \widetilde{\mu}\partial_i & -\zeta_{,i} \\ \partial_j D & 0 & 0 & 0 & 0 \\ \rho_{,j} & 0 & 0 & 0 & 0 \\ \partial_j \widetilde{\mu} & 0 & 0 & 0 & 1 \\ \zeta_{,j} & 0 & 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} \delta h/\delta m_j \\ \delta h/\delta D \\ \delta h/\delta \rho \\ \delta h/\delta \widetilde{\mu} \\ \delta h/\delta \zeta \end{bmatrix}. \tag{127}$$

Remark 22 (Physical meaning of the model). The Lie–Poisson structure in (127) reveals the physical meaning of the WCIFS system of equations. Namely, the fluid variables sweep the wave degrees of freedom along the fluid Lagrangian paths, whereas the wave subsystem evolves and acts back on the fluid circulation as an *internal* force.

Remark 23 (Transformation to the potential flow momentum). The Poisson operator in the previous formula is block diagonalized by the transformation $\boldsymbol{m} \to \boldsymbol{M} = \boldsymbol{m} + \widetilde{\mu}\nabla\zeta$, which separates it into a direct sum of a Lie–Poisson bracket in $\boldsymbol{M}, D, \rho$ and a canonical (symplectic) Poisson bracket in $\widetilde{\mu}$ and ζ. Consequently, the system of equations in (97) may now be written equivalently as a direct sum of a semidirect-product Lie–Poisson bracket in the fluid variables $(\boldsymbol{m}, D, \rho)$, plus a symplectic 2-cocycle in the wave variables $(\widetilde{\mu}, \zeta)$ in its *untangled* form, as

$$\frac{\partial}{\partial t}\begin{bmatrix} M_i \\ D \\ \rho \\ \widetilde{\mu} \\ \zeta \end{bmatrix} = -\begin{bmatrix} \partial_j M_i + M_j \partial_i & D\partial_i & -\rho_{,i} & 0 & 0 \\ \partial_j D & 0 & 0 & 0 & 0 \\ \rho_{,j} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} \delta h/\delta M_j \\ \delta h/\delta D \\ \delta h/\delta \rho \\ \delta h/\delta \widetilde{\mu} \\ \delta h/\delta \zeta \end{bmatrix}. \tag{128}$$

Thus, the Poisson bracket block-diagonalizes when it is written in terms of the total fluid plus wave momentum, $\boldsymbol{M} = \boldsymbol{m} + \widetilde{\mu}\,\nabla\zeta$.

The Poisson structure in Equation (127) yields the following motion equation:

$$\begin{aligned}(\partial_t + \mathcal{L}_{\widehat{v}})(\boldsymbol{m}\cdot d\boldsymbol{x}\otimes d^2x) = -Dd(-\rho\widetilde{\varpi} + p)\otimes d^2x - (D\widetilde{\varpi})d\rho\otimes d^2x - \widetilde{\mu}d(\widetilde{w}(1+\epsilon|\nabla\zeta|^2)^2)\otimes d^2x \\ + (Dg\rho - 2\epsilon\,\mathrm{div}(\widetilde{\mu}\widehat{w}\nabla\zeta))d\zeta\otimes d^2x.\end{aligned} \tag{129}$$

If we divide through by $D\rho$, using $(\partial_t + \mathcal{L}_{\widehat{v}})(D\rho d^2x) = 0$, then we obtain the following motion equation, which agrees with that previously obtained in (99):

$$\begin{aligned}(\partial_t + \mathcal{L}_{\widehat{v}})\widehat{\boldsymbol{v}}\cdot d\boldsymbol{x} &= -\frac{1}{\rho}dp + d\widetilde{\varpi} - \widetilde{w}d(\widetilde{w}(1+\epsilon|\nabla\zeta|^2)^2) + gd\zeta - \frac{2\epsilon}{D\rho}\mathrm{div}(\widetilde{\mu}\widehat{w}\nabla\zeta)d\zeta \\ &= -\frac{1}{\rho}dp + d\left(\frac{|\widehat{\boldsymbol{v}}|^2}{2} + \frac{\widehat{w}^2}{2}\right) - \widetilde{w}d(\widetilde{w}(1+\epsilon|\nabla\zeta|^2)^2) - \frac{2\epsilon}{D\rho}\mathrm{div}(\widetilde{\mu}\widehat{w}\nabla\zeta)d\zeta.\end{aligned}$$

5.8 | A one-dimensional WCIFS equation

We begin by deriving the one-dimensional equation by applying Hamilton's principle to the following action integral, which is the one-dimensional version of (95),

$$
\begin{aligned}
S = \int\int D\rho\left(\frac{1}{2}(\widehat{v}^2+\widehat{w}^2)-g\zeta\right) &+ \widetilde{\mu}\big(\partial_t\zeta+\widehat{v}\partial_x\zeta-\widehat{w}\big(1+\epsilon|\partial_x\zeta|^2\big)\big) \\
&+ \widehat{\phi}(\partial_t(D\rho)+\partial_x(D\rho\widehat{v}))\,dxdt,
\end{aligned}
\tag{130}
$$

where $\zeta(x,t)$, $\widehat{v}(x,t)$, and $\widehat{w}(x,t)$ are scalar functions of one-dimensional space and time, (x,t), and we consider the volume form $D\rho$ to be a single variable. Taking variations with respect to each variable gives

$$
\begin{aligned}
\delta\widehat{v}:&\quad D\rho\widehat{v}\cdot dx+\widetilde{\mu}d\zeta = D\rho d\widehat{\phi},\\
\delta(D\rho):&\quad (\partial_t+\mathcal{L}_{\widehat{v}})\widehat{\phi}=\partial_t\widehat{\phi}+\widehat{v}\partial_x\widehat{\phi}=\frac{1}{2}\widehat{v}^2+\frac{1}{2}\widehat{w}^2-g\zeta=:\varpi,\\
\delta\widehat{\phi}:&\quad (\partial_t+\mathcal{L}_{\widehat{v}})(D\rho)=\partial_t(D\rho)+\partial_x(D\rho\widehat{v})=0,\\
\delta\widehat{w}:&\quad D\rho\widehat{w}-\widetilde{\mu}\big(1+\epsilon|\partial_x\zeta|^2\big)=0,\\
\delta\widetilde{\mu}:&\quad (\partial_t+\mathcal{L}_{\widehat{v}})\zeta=\partial_t\zeta+\widehat{v}\partial_x\zeta=\widehat{w}\big(1+\epsilon|\partial_x\zeta|^2\big),\\
\delta\zeta:&\quad (\partial_t+\mathcal{L}_{\widehat{v}})\widetilde{\mu}=-D\rho g+2\epsilon\partial_x(\widetilde{\mu}\widehat{w}\partial_x\zeta).
\end{aligned}
\tag{131}
$$

These relations imply a fluid motion equation

$$
\begin{aligned}
D\rho(\partial_t+\mathcal{L}_{\widehat{v}})\widehat{v} &= D\rho\, d(\partial_t+\mathcal{L}_{\widehat{v}})\widehat{\phi}-(\partial_t+\mathcal{L}_{\widehat{v}})(\widetilde{\mu}\,d\zeta)\\
&= D\rho\, d\varpi-\widetilde{\mu}d\big(\widehat{w}\big(1+\epsilon|\partial_x\zeta|^2\big)\big)+D\rho g d\zeta-2\epsilon\partial_x\big(D\rho\widetilde{w}^2\big(1+\epsilon|\partial_x\zeta|^2\big)\partial_x\zeta\big)d\zeta,
\end{aligned}
$$

or,

$$
\begin{aligned}
(\partial_t+\mathcal{L}_{\widehat{v}})\widehat{v} &= d\varpi-\frac{\widehat{w}}{1+\epsilon|\partial_x\zeta|^2}d\big(\widehat{w}\big(1+\epsilon|\partial_x\zeta|^2\big)\big)+gd\zeta-\frac{2\epsilon}{D\rho}\partial_x\big(D\rho\widetilde{w}^2\big(1+\epsilon|\partial_x\zeta|^2\big)\partial_x\zeta\big)d\zeta\\
&= d\left(\frac{1}{2}\widehat{v}^2\right)-\frac{1}{2}\widetilde{w}^2 d\left(\big(1+\epsilon|\partial_x\zeta|^2\big)^2\right)-\frac{2\epsilon}{D\rho}\partial_x\big(D\rho\widetilde{w}^2\big(1+\epsilon|\partial_x\zeta|^2\big)\partial_x\zeta\big)d\zeta.
\end{aligned}
$$

where $\widetilde{\mu}=D\rho\widetilde{w}$. Thus, we have

$$
\partial_t\widehat{v}+\widehat{v}\partial_x\widehat{v}=-\frac{1}{2}\widetilde{w}^2\partial_x\left(\big(1+\epsilon|\partial_x\zeta|^2\big)^2\right)-\frac{2\epsilon}{D\rho}\partial_x\big(D\rho\widetilde{w}^2\big(1+\epsilon|\partial_x\zeta|^2\big)\partial_x\zeta\big)\partial_x\zeta,
\tag{132}
$$

and this equation is to be considered together with

$$
\begin{aligned}
\partial_t D+\partial_x(D\widehat{v})&=0,\\
\partial_t\rho+\widehat{v}\partial_x\rho&=0,\\
\partial_t\widetilde{w}+\widehat{v}\partial_x\widetilde{w}&=-g+\frac{2\epsilon}{D\rho}\partial_x\big(D\rho\widetilde{w}^2\big(1+\epsilon|\partial_x\zeta|^2\big)\partial_x\zeta\big).
\end{aligned}
$$

These one-dimensional WCIFS equations are of interest in their own right and they will be investigated elsewhere.

6 | STOCHASTIC WAVE MODELLING

6.1 | Stochastic advection by Lie transport (SALT)

SALT[26] provides a methodology of stochastically perturbing a continuum model at the level of the action integral. As a result, SALT both preserves the Kelvin–Noether circulation theorem and provides a platform for stochastic wave-current interaction.[52] Consider first the 3D case where we have a 3D fluid velocity field, evaluated on the free surface, denoted by $\hat{\boldsymbol{u}}$. For a deterministic (unconstrained) Lagrangian, $\ell(\hat{\boldsymbol{u}}, q)$, depending on the velocity field $\hat{\boldsymbol{u}}$ and advected quantities q, we constrain the advected quantities to follow a stochastically perturbed path via a Lagrange multiplier. For models where we are considering incompressible flow, the pressure must act as a Lagrange multiplier to enforce the advected quantity D, the volume element, to be constant. More specifically, the advection constraint enforces that the advected quantities obey a stochastic partial differential equation given by

$$(\mathrm{d} + \mathcal{L}_{\mathrm{d}\boldsymbol{x}_t})q := \mathrm{d}q + \mathcal{L}_{\hat{\boldsymbol{u}}} q\, dt + \sum_i \mathcal{L}_{\tilde{\xi}_i} q \circ dW_t^i = 0, \tag{133}$$

where the vector field $\hat{\boldsymbol{u}}$ has been perturbed in the following way:

$$\mathrm{d}\boldsymbol{x}_t = \hat{\boldsymbol{u}}\, dt + \sum_i \tilde{\xi}_i \circ dW_t^i. \tag{134}$$

After this introduction of this stochastic transport constraint, the action integral becomes a *semimartingale driven variational principle*.[53] Consequently, the pressure Lagrange multiplier must be compatible with the noise introduced in the advection. This is required because one cannot enforce a variable in a stochastic system to remain constant without also requiring the Lagrange multiplier to also be a semimartingale, to control both the deterministic part of the system as well as the random fluctuations. With these constraints, the action integral takes the form

$$S = \int \ell(\hat{\boldsymbol{v}}, q)\, dt + \langle \mathrm{d}p, D - 1 \rangle + \langle \lambda, \mathrm{d}q + \mathcal{L}_{\mathrm{d}\boldsymbol{x}_t} q \rangle. \tag{135}$$

The application of Hamilton's principle implies an EP equation and, as in Ref. 26, we have a Kelvin–Noether circulation theorem for the stochastic system that is analogous to that of the deterministic system.

For the purposes of our variational wave models, we need a notation for 2D advection as well as 3D. We recall the notation for the 2D velocity field and introduce a new notation for the first two components of the stochastic perturbation as follows:

$$\hat{\boldsymbol{u}} = (\hat{\boldsymbol{v}}, \hat{w}), \quad \text{and} \quad \tilde{\boldsymbol{\xi}}_{\boldsymbol{i}} = (\boldsymbol{\xi}_{\boldsymbol{i}}, \xi_{3i}). \tag{136}$$

The perturbation of vector field $\hat{\boldsymbol{v}}$ is therefore given by

$$\mathrm{d}\boldsymbol{r}_t = \hat{\boldsymbol{v}}(\boldsymbol{r}_t, t)\, dt + \sum_i \boldsymbol{\xi}_{\boldsymbol{i}}(\boldsymbol{r}_t) \circ dW_t^i. \tag{137}$$

6.2 | Stochastic ECWWE

First, we derive the free surface boundary condition (3) in the stochastic case by applying the operator $\mathrm{d} + \mathcal{L}_{\mathrm{d}\boldsymbol{r}_t}$ to $z - \zeta(\boldsymbol{r}, t)$ to obtain

$$0 = (\mathrm{d} + \mathcal{L}_{\mathrm{d}\boldsymbol{r}_t})(z - \zeta(\boldsymbol{r}, t)) = \widehat{w}(\boldsymbol{r}, t)\, dt + \sum_i \xi_{3i}(\boldsymbol{r}) \circ dW_t^i - \mathrm{d}\zeta(\boldsymbol{r}, t) - \mathcal{L}_{\mathrm{d}r}\zeta(\boldsymbol{r}, t),$$

and hence

$$\mathrm{d}\zeta(\boldsymbol{r}, t) + \mathcal{L}_{\mathrm{d}\boldsymbol{r}_t}\zeta(\boldsymbol{r}, t) = \widehat{w}(\boldsymbol{r}, t)\, dt + \sum_i \xi_{3i}(\boldsymbol{r}) \circ dW_t^i, \tag{138}$$

where the notation $\mathrm{d}\boldsymbol{r}_t$ in (137) is the path of a Lagrangian coordinate. When we write dependence on $\boldsymbol{r}$, we mean that $\boldsymbol{r}$ is an Eulerian point that is the pullback of the path defined by (137). In more informal language, $\boldsymbol{r}$ is an Eulerian point along the Lagrangian path $\boldsymbol{r}_t$.

We may derive the stochastic ECWW equations by considering the dimensional version of the action integral (37) where the transport velocity $\widehat{\boldsymbol{v}}$ has been perturbed as in (137). The stochastic action integral is then

$$\begin{aligned} S = \int\!\!\int D\left(\frac{1}{2}(|\widehat{\boldsymbol{v}}|^2 + \widehat{w}^2) - g\zeta\right) dt + \lambda\left(\mathrm{d}\zeta + \mathcal{L}_{\mathrm{d}\boldsymbol{r}_t}\zeta - \widehat{w}\, dt - \sum_i \xi_{3i} \circ dW_t^i\right) \\ + \widehat{\phi}(\mathrm{d}D + \mathcal{L}_{\mathrm{d}\boldsymbol{r}_t}D)\, d^2r. \end{aligned} \tag{139}$$

Taking variations of the action integral (139) yields

$$\begin{aligned} \delta\widehat{\boldsymbol{v}} &: \quad D\widehat{\boldsymbol{v}} \cdot d\boldsymbol{r} + \lambda\, d\zeta = Dd\widehat{\phi} \quad \Longrightarrow \quad \boldsymbol{V} \cdot d\boldsymbol{r} := \widehat{\boldsymbol{v}} \cdot d\boldsymbol{r} + \widehat{w}\, d\zeta = d\widehat{\phi}, \\ \delta\widehat{w} &: \quad D\widehat{w} - \lambda = 0, \\ \delta\lambda &: \quad \mathrm{d}\zeta + \mathcal{L}_{\mathrm{d}\boldsymbol{r}_t}\zeta = \widehat{w}\, dt + \sum_i \xi_{3i} \circ dW_t^i, \\ \delta\zeta &: \quad \mathrm{d}\lambda + \mathcal{L}_{\mathrm{d}\boldsymbol{r}_t}\lambda = -gD\, dt \quad \Longrightarrow \quad \partial_t\widehat{w} + \mathrm{d}\boldsymbol{r}_t \cdot \nabla_{\boldsymbol{r}}\widehat{w} = -g, \\ \delta\widehat{\phi} &: \quad \mathrm{d}D + \mathcal{L}_{\mathrm{d}\boldsymbol{r}_t}D = 0, \\ \delta D &: \quad \mathrm{d}\widehat{\phi} + \mathcal{L}_{\mathrm{d}\boldsymbol{r}_t}\widehat{\phi} = \mathrm{d}\widehat{\phi} + \mathrm{d}\boldsymbol{r}_t \cdot \nabla_{\boldsymbol{r}}\widehat{\phi} = \frac{1}{2}(|\widehat{\boldsymbol{v}}|^2 + \widehat{w}^2)\, dt - \zeta\, dt =: \varpi\, dt. \end{aligned} \tag{140}$$

We may therefore write the stochastic ECWW equations as

$$\begin{aligned} \mathrm{d}\widehat{\phi} + \mathrm{d}\boldsymbol{r}_t \cdot \nabla_{\boldsymbol{r}}\widehat{\phi} &= \frac{1}{2}(|\widehat{\nabla_{\boldsymbol{r}}\phi}|^2 + \widehat{w}^2)\, dt - g\zeta\, dt, \\ \mathrm{d}\zeta + \mathrm{d}\boldsymbol{r}_t \cdot \nabla_{\boldsymbol{r}}\zeta &= \widehat{w}\, dt + \sum_i \xi_{3i} \circ dW_t^i, \\ \mathrm{d}\widehat{w} + \mathrm{d}\boldsymbol{r}_t \cdot \nabla_{\boldsymbol{r}}\widehat{w} &= -g\, dt, \\ \mathrm{d}D + \mathcal{L}_{\mathrm{d}\boldsymbol{r}_t}D &= 0. \end{aligned} \tag{141}$$

As in the deterministic case, these equations imply a Kelvin–Noether theorem as follows:

$$
\begin{aligned}
\mathrm{d}\oint_{c(\mathrm{d}\boldsymbol{r}_t)} (\widehat{\boldsymbol{v}}\cdot d\boldsymbol{r} + \widehat{w}d\zeta)\cdot d\boldsymbol{r} &= \oint_{c(\mathrm{d}\boldsymbol{r}_t)} (\mathrm{d}+\mathcal{L}_{\mathrm{d}\boldsymbol{r}_t})(\widehat{\boldsymbol{v}}\cdot d\boldsymbol{r} + \widehat{w}d\zeta) \\
&= \oint_{c(\mathrm{d}\boldsymbol{r}_t)} (\mathrm{d}+\mathcal{L}_{\mathrm{d}\boldsymbol{r}_t})(\boldsymbol{V}\cdot d\boldsymbol{r}) = \oint_{c(\mathrm{d}\boldsymbol{r}_t)} (\mathrm{d}+\mathcal{L}_{\mathrm{d}\boldsymbol{r}_t})d\widehat{\phi} = \oint_{c(\mathrm{d}\boldsymbol{r}_t)} d\varpi\, dt = 0.
\end{aligned}
\tag{142}
$$

6.3 | Stochastic WCIFS equations

Similarly to the stochastic ECWW equations, we may define stochastic versions of any of the wave–current models we have derived, including the MCWW equations. Here, we will demonstrate this for our most complete wave–current model corresponding to the action integral (95). We may again define the equivalent action integral featuring SALT to derive the corresponding stochastic system of equations.

In the stochastic case, to couple the waves and currents, we consider the insertion of the stochastic vector field $\mathrm{d}x_3\nabla_{\boldsymbol{r}}\zeta$, where $\mathrm{d}x_3 = \widehat{w}\,dt + \sum_i \xi_{3i}\circ dW_t^i$, into the 1-form $\widetilde{\mu}\nabla_{\boldsymbol{r}}\zeta$.

The stochastic version of the action integral (95) is therefore given by

$$
\begin{aligned}
S = \int\int D\rho\left(\frac{1}{2}(|\widehat{\boldsymbol{v}}|^2 + \widehat{w}^2) - g\zeta\right) dt - \mathrm{d}p(D-1) + \widehat{\phi}(\mathrm{d}D + \mathcal{L}_{\mathrm{d}\boldsymbol{r}_t}D) + \gamma(\mathrm{d}\rho + \mathcal{L}_{\mathrm{d}\boldsymbol{r}_t}\rho) \\
\widetilde{\mu}\left(\mathrm{d}\zeta + \mathcal{L}_{\mathrm{d}\boldsymbol{r}_t}\zeta - \widehat{w}\big(1+\epsilon|\nabla_{\boldsymbol{r}}\zeta|^2 d\zeta\big)\,dt - \sum_i \xi_{3i}\big(1+\epsilon|\nabla_{\boldsymbol{r}}\zeta|^2\big)\circ dW_t^i\right) dxdy.
\end{aligned}
\tag{143}
$$

Similar to the application of Hamilton's principle to (95), variations of (143) are given by

$$
\begin{aligned}
\delta\widehat{\boldsymbol{v}}:&\quad D\rho\widehat{\boldsymbol{v}}\cdot d\boldsymbol{r} + \widetilde{\mu}\,d\zeta = Dd\widehat{\phi} - \gamma d\rho, \\
\delta\widehat{w}:&\quad D\rho\widehat{w} - \widetilde{\mu}\left(1+\epsilon|\nabla_{\boldsymbol{r}}\zeta|^2\right) = 0, \\
\delta\widetilde{\mu}:&\quad \mathrm{d}\zeta + \mathcal{L}_{\mathrm{d}\boldsymbol{r}_t}\zeta = \mathrm{d}x_3\big(1+\epsilon|\nabla_{\boldsymbol{r}}\zeta|^2\big) = \widehat{w}(1+\epsilon|\nabla_{\boldsymbol{r}}\zeta|^2)\,dt + \sum_i \xi_{3i}\big(1+\epsilon|\nabla_{\boldsymbol{r}}\zeta|^2\big)\circ dW_t^i, \\
\delta\zeta:&\quad \mathrm{d}\widetilde{\mu} + \mathcal{L}_{\mathrm{d}\boldsymbol{r}_t}\widetilde{\mu} = -D\rho g\,dt + 2\epsilon\,\mathrm{div}_{\boldsymbol{r}}(\mathrm{d}x_3\,\widetilde{\mu}\,\nabla_{\boldsymbol{r}}\zeta) \\
&\quad = -D\rho g\,dt + 2\epsilon\,\mathrm{div}_{\boldsymbol{r}}(\widehat{w}\,\widetilde{\mu}\,\nabla_{\boldsymbol{r}}\zeta)\,dt + \sum_i 2\epsilon\,\mathrm{div}_{\boldsymbol{r}}(\xi_{3i}\,\widetilde{\mu}\,\nabla_{\boldsymbol{r}}\zeta)\circ dW_t^i, \\
\delta\rho:&\quad (\mathrm{d}+\mathcal{L}_{\mathrm{d}\boldsymbol{r}_t})\left(\frac{\gamma}{D}\right) = \frac{1}{2}(|\widehat{\boldsymbol{v}}|^2 + \widehat{w}^2)\,dt - g\zeta\,dt =: \varpi\,dt, \\
\delta D:&\quad (\mathrm{d}+\mathcal{L}_{\mathrm{d}\boldsymbol{r}_t})\widehat{\phi} = \rho\varpi\,dt - \mathrm{d}p, \\
\delta\widehat{\phi}:&\quad \mathrm{d}D + \mathcal{L}_{\mathrm{d}\boldsymbol{r}_t}D = 0, \\
\delta p:&\quad D-1=0 \quad \Rightarrow \mathrm{div}_{\boldsymbol{r}}\widehat{\boldsymbol{v}} = 0, \\
\delta\gamma:&\quad (\mathrm{d}+\mathcal{L}_{\mathrm{d}\boldsymbol{r}_t})\rho = 0.
\end{aligned}
\tag{144}
$$

We apply the operator $\mathrm{d} + \mathcal{L}_{\mathrm{d}\boldsymbol{r}_t}$ to the first line in (144) to find

$$
\begin{aligned}
D\rho(\mathrm{d} + \mathcal{L}_{\mathrm{d}\boldsymbol{r}_t})(\widehat{\boldsymbol{v}}\cdot d\boldsymbol{r}) &= Dd(\mathrm{d} + \mathcal{L}_{\mathrm{d}\boldsymbol{r}_t})\widehat{\phi} - (\mathrm{d} + \mathcal{L}_{\mathrm{d}\boldsymbol{r}_t})\gamma d\rho - (\mathrm{d} + \mathcal{L}_{\mathrm{d}\boldsymbol{r}_t})(\widetilde{\mu} d\zeta) \\
&= D\rho d\varpi\, dt - Dd(\mathrm{d}p) - \widetilde{\mu} d\big(\mathrm{d}x_3\big(1+\epsilon|\nabla_{\boldsymbol{r}}\zeta|^2\big)\big) + D\rho g\, d\zeta\, dt \\
&\quad - 2\epsilon\,\mathrm{div}(\mathrm{d}x_3\,\widetilde{\mu}\,\nabla_{\boldsymbol{r}}\zeta)d\zeta,
\end{aligned}
\tag{145}
$$

and thus,

$$
\begin{aligned}
(\mathrm{d}t + \mathcal{L}_{\mathrm{d}\boldsymbol{r}_t})(\widehat{\boldsymbol{v}}\cdot d\boldsymbol{r}) = -\frac{1}{\rho}d(\mathrm{d}p) &+ \left[d\frac{|\widehat{\boldsymbol{v}}|^2}{2} - \frac{1}{2}\widetilde{\varpi}^2 d\big(1+\epsilon|\nabla_{\boldsymbol{r}}\zeta|^2\big)^2\right]dt \\
&- \frac{2\epsilon}{D\rho}\,\mathrm{div}_{\boldsymbol{r}}(\mathrm{d}x_3\,\widetilde{\mu}\,\nabla_{\boldsymbol{r}}\zeta)d\zeta \\
&- \widetilde{\varpi}\sum_i d\big(\xi_{3i}(1+\epsilon|\nabla_{\boldsymbol{r}}\zeta|^2)\big)\circ dW_t^i.
\end{aligned}
$$

Remark 24 (A stochastic Kelvin–Noether theorem). We have, from calculations analogous to the deterministic case performed similarly to the above, a stochastic version of the Theorem 3 In the stochastic case, this takes the form:

$$
\mathrm{d}\oint_{c(\mathrm{d}\boldsymbol{r}_t)}\left(\widehat{\boldsymbol{v}}\cdot d\boldsymbol{x} + \frac{\widetilde{\mu}}{D\rho}d\zeta\right) = \oint_{c(\mathrm{d}\boldsymbol{r}_t)} d\varpi\, dt - \frac{1}{\rho}\mathrm{d}dp.
\tag{146}
$$

We may collect the WCIFS SALT equations of motion in (144), as follows:

$$
\begin{aligned}
\mathrm{d}\widehat{\boldsymbol{v}} + (\mathrm{d}\boldsymbol{r}_t\cdot\nabla_{\boldsymbol{r}})\widehat{\boldsymbol{v}} + (\nabla_{\boldsymbol{r}}\mathrm{d}\boldsymbol{r}_t)^T\cdot\widehat{\boldsymbol{v}} = \nabla_{\boldsymbol{r}}\frac{|\widehat{\boldsymbol{v}}|^2}{2}\,dt &- \frac{1}{\rho}\nabla_{\boldsymbol{r}}\mathrm{d}p - \frac{1}{2}\widetilde{\varpi}^2\nabla_{\boldsymbol{r}}\big(1+\epsilon|\nabla_{\boldsymbol{r}}\zeta|^2\big)^2\,dt \\
&- \frac{2\epsilon}{D\rho}\,\mathrm{div}(\mathrm{d}x_3\,\widetilde{\mu}\,\nabla_{\boldsymbol{r}}\zeta)\nabla_{\boldsymbol{r}}\zeta \\
&- \widetilde{\varpi}\sum_i\nabla_{\boldsymbol{r}}\big(\xi_{3i}(1+\epsilon|\nabla_{\boldsymbol{r}}\zeta|^2)\big)\circ dW_t^i, \\
(\mathrm{d} + \mathcal{L}_{\mathrm{d}\boldsymbol{r}_t})(D\,d^2x) &= 0, \\
(\mathrm{d} + \mathcal{L}_{\mathrm{d}\boldsymbol{r}_t})\rho &= 0, \\
(\mathrm{d} + \mathcal{L}_{\mathrm{d}\boldsymbol{r}_t})\zeta &= \mathrm{d}x_3\big(1+\epsilon|\nabla_{\boldsymbol{r}}\zeta|^2\big), \\
(\mathrm{d} + \mathcal{L}_{\mathrm{d}\boldsymbol{r}_t})(\widetilde{\mu}\,d^2x) &= (-D\rho g\,dt + 2\epsilon\,\mathrm{div}_{\boldsymbol{r}}(\mathrm{d}x_3\,\widetilde{\mu}\,\nabla_{\boldsymbol{r}}\zeta))\,d^2x.
\end{aligned}
\tag{147}
$$

The properties of these equations will be studied in detail, elsewhere.

7 | ANALYTICAL REMARKS ABOUT VARIATIONAL WATER WAVE MODELS

Recall the equations in (39), found by varying the action integral (37). These equations may be written in the form

$$
\begin{aligned}
\partial_t \widehat{\phi} + \widehat{\boldsymbol{v}} \cdot \nabla_{\boldsymbol{r}} \widehat{\phi} &= \frac{1}{2}(|\widehat{\boldsymbol{v}}|^2 + \widehat{w}^2) - g\zeta, \\
\partial_t \zeta + \widehat{\boldsymbol{v}} \cdot \nabla_{\boldsymbol{r}} \zeta &= \widehat{w}, \\
\partial_t \widehat{w} + \widehat{\boldsymbol{v}} \cdot \nabla_{\boldsymbol{r}} \widehat{w} &= -g \\
\partial_t D + \mathrm{div}_{\boldsymbol{r}}(D\widehat{\boldsymbol{v}}) &= 0,
\end{aligned}
$$

where

$$
\begin{aligned}
\widehat{\boldsymbol{v}} &= \boldsymbol{V} - \widehat{w}\nabla_{\boldsymbol{r}}\zeta \\
&= \nabla_{\boldsymbol{r}}\widehat{\phi} - \widehat{w}\nabla_{\boldsymbol{r}}\zeta, \quad \text{in the irrotational case.}
\end{aligned}
\tag{148}
$$

Recall that the transport velocity $\widehat{\boldsymbol{v}}$ evolves according to (44), that is,

$$
\partial_t \widehat{\boldsymbol{v}} + \widehat{\boldsymbol{v}} \cdot \nabla \widehat{\boldsymbol{v}} = 0, \tag{149}
$$

Of course, (149) can be identified as the 2D inviscid Burger's equation. Under certain conditions on the initial velocity $\widehat{\boldsymbol{v}}_0$, it has unique solution (possibly only local in time). We sketch below the classical argument for showing this using the method of characteristics. Define the characteristic equation given by

$$
\begin{cases}
\frac{d\boldsymbol{r}_t}{dt}(\boldsymbol{r}) = \widehat{\boldsymbol{v}}(\boldsymbol{r}_t(\boldsymbol{r}), t), & t > 0, \\
\boldsymbol{r}_t(\boldsymbol{r}) = \boldsymbol{r}.
\end{cases}
\tag{150}
$$

Provided that $\widehat{\boldsymbol{v}}$ is sufficiently smooth, the system (150) will have a unique solution. Moreover, from (150) and (149), we deduce, by the chain rule, that

$$
\frac{\partial}{\partial t}[\widehat{\boldsymbol{v}}(\boldsymbol{r}_t(\boldsymbol{r}), t)] = \frac{\partial \widehat{\boldsymbol{v}}}{\partial t}(\boldsymbol{r}_t(\boldsymbol{r}), t) + \frac{d\boldsymbol{r}_t}{dt}(\boldsymbol{r}) \cdot \nabla_{\boldsymbol{r}} \widehat{\boldsymbol{v}}(\boldsymbol{r}_t(\boldsymbol{r}), t) = (\partial_t \widehat{\boldsymbol{v}} + \widehat{\boldsymbol{v}} \cdot \nabla_{\boldsymbol{r}} \widehat{\boldsymbol{v}})(\boldsymbol{r}_t(\boldsymbol{r}), t) = 0,
$$

so $\widehat{\boldsymbol{v}} = (\widehat{v}_1, \widehat{v}_2)$ is constant along the characteristics. Thus, the characteristic curves corresponding to (149) are straight lines determined by the initial conditions, given by

$$
\boldsymbol{r}_t = \varphi_t \boldsymbol{r} := \boldsymbol{r} + \widehat{\boldsymbol{v}}_0(\boldsymbol{r})t, \tag{151}
$$

and therefore the follow pull-back relation holds,

$$
\varphi_t{}^* \widehat{\boldsymbol{v}}_t(\boldsymbol{r}) := \widehat{\boldsymbol{v}}_t(\varphi_t \boldsymbol{r}) = \widehat{\boldsymbol{v}}_0(\boldsymbol{r}), \tag{152}
$$

so that

$$
0 = \frac{d\widehat{\boldsymbol{v}}_0}{dt} = \frac{d}{dt}\varphi_t^* \widehat{\boldsymbol{v}}_t(\boldsymbol{r}) = \varphi_t^* \left(\partial_t \widehat{\boldsymbol{v}}_t(\boldsymbol{r}) + \frac{\partial \widehat{\boldsymbol{v}}_t(\boldsymbol{r})}{\partial \boldsymbol{r}} \cdot \widehat{\boldsymbol{v}}_t(\boldsymbol{r}) \right). \tag{153}
$$

Equations (151) and (152) enable us to give an explicit description of the (classical) solution of (149) up to first time τ at which the characteristic lines cross. The time τ is the first time when $\nabla_{\boldsymbol{r}}(\varphi_t \boldsymbol{r})$

degenerates, in other words the Jacobian of $\varphi_t \boldsymbol{r}$ has determinant equal to 0. Note that Equation (149) may have a weak solution beyond τ. The time τ can be explicitly described in term of the eigenvalues of the Jacobian $\nabla_{\boldsymbol{r}} \widehat{\boldsymbol{v}}_0$ of the initial velocity $\widehat{\boldsymbol{v}}_0$. We will denote by $\lambda_i(\boldsymbol{r})$, $i = 1, 2$, the two eigenvalues of $\nabla_{\boldsymbol{r}} \widehat{\boldsymbol{v}}_0(\boldsymbol{r})$. We introduce the (possibly empty) subset S of the fluid domain Ω (which we assume to be a closed bounded set of $\mathbb{R}^2$) defined as

$$S = \{\boldsymbol{r} \in \Omega : \lambda_1(\boldsymbol{r}) < 0, \text{ or } \lambda_2(\boldsymbol{r}) < 0\}.$$

Define $\tau := \infty$ if S is the empty set, otherwise $\tau = -1/\bar{\lambda}$ where[3]

$$\bar{\lambda} = \min\left\{\min_{\boldsymbol{r}\in S} \lambda_1(\boldsymbol{r}), \min_{\boldsymbol{r}\in S} \lambda_2(\boldsymbol{r})\right\},$$

then $\varphi_t : \Omega \to \Omega$ is a diffeomorphism for any $t \in [0, \tau)$. This statement is immediate after observing that the eigenvalues of $\nabla_{\boldsymbol{r}}(\varphi_t \boldsymbol{r})$ are given by $(1 + t\lambda_1(\boldsymbol{r}), 1 + t\lambda_2(\boldsymbol{r}))$ and cannot become zero before time τ. In other words, the Lagrangian flow is well defined and differentiable in both the spatial and temporal variable. Let $\varphi_t^{-1} : \Omega \to \Omega$ be the inverse of the Lagrangian flow. From (152), we deduce that (149) has a (unique) solution given by the push-forward of the initial velocity $\varphi_{t_*} \widehat{\boldsymbol{v}}_0(\boldsymbol{r})$, that is,

$$\widehat{\boldsymbol{v}}(\boldsymbol{r}, t) = \widehat{\boldsymbol{v}}_0(\varphi_t^{-1} \boldsymbol{r}) =: \varphi_{t_*} \widehat{\boldsymbol{v}}_0(\boldsymbol{r}), \quad \boldsymbol{r} \in \Omega_t, t \in [0, \tau).$$

From (39), we also deduce that

$$\begin{aligned} \frac{d}{dt} \widehat{\phi}(\varphi_t \boldsymbol{r}, t) &= e(\varphi_t \boldsymbol{r}, t), \\ \frac{d}{dt} \zeta(\varphi_t \boldsymbol{r}, t) &= \widehat{w}(\varphi_t \boldsymbol{r}, t), \\ \frac{d}{dt} \widehat{w}(\varphi_t \boldsymbol{r}, t) &= -g, \\ \frac{d}{dt}(D(\varphi_t \boldsymbol{r}, t) d^2(\varphi_t \boldsymbol{r})) &= 0, \end{aligned} \tag{154}$$

where $e(\varphi_t \boldsymbol{r}, t) = \frac{1}{2}((|\widehat{\boldsymbol{v}}|^2 + \widehat{w}^2) - g\zeta)(\varphi_t \boldsymbol{r}, t)$. It follows that the solution for the full system of variables is obtained by integrations along the characteristics, as

$$\begin{aligned} \widehat{\phi}(\boldsymbol{r}, t) &= \widehat{\phi}(\varphi_t^{-1} \boldsymbol{r}, 0) + \int_0^t e(\varphi_{s-t} \boldsymbol{r}, t)\, ds, \quad \text{where } \varphi_s \varphi_t^{-1} = \varphi_{s-t}, \\ \widehat{w}(\boldsymbol{r}, t) &= \widehat{w}_0(\varphi_t^{-1} \boldsymbol{r}) - gt, \\ \zeta(\boldsymbol{r}, t) &= \zeta(\varphi_t^{-1} \boldsymbol{r}, 0) + \int_0^t \widehat{w}(\varphi_{s-t} \boldsymbol{r}, t)\, ds, \\ D(\boldsymbol{r}, t) d^2 \boldsymbol{r} &= \varphi_{t_*}\big(D(\boldsymbol{r}, 0) d^2 \boldsymbol{r}\big) = D(\varphi_t^{-1} \boldsymbol{r}, 0) d^2(\varphi_t^{-1} \boldsymbol{r}), \quad \text{since } \varphi_0 = \mathrm{Id}. \end{aligned} \tag{155}$$

[3] Note that the set Ω is compact; therefore, the two minima $\min_{\boldsymbol{r}\in S} \lambda_1(\boldsymbol{r})$ as well as $\min_{\boldsymbol{r}\in S} \lambda_2(\boldsymbol{r})$ are well defined.

Thus, the explicit solution for $\widehat{\boldsymbol{v}}$ corresponding to the characteristics also provides an explicit solution for the full system of variables.

An Eulerian approach

For an alternative Eulerian approach, notice that taking the $\mathrm{curl}_{\boldsymbol{r}}$ of the evolution equation (44) for the transport velocity $\widehat{\boldsymbol{v}}$ defined in Equation (148) implies that $\widehat{\omega} = \mathrm{curl}_{\boldsymbol{r}}\, \widehat{\boldsymbol{v}} = J(\widehat{w}, \zeta)$ satisfies

$$\partial_t \widehat{\omega} + \widehat{\boldsymbol{v}} \cdot \nabla_{\boldsymbol{r}} \widehat{\omega} = 0. \tag{156}$$

Consequently, the continuity equation for D implies

$$\partial_t (D\widehat{\omega}) + \mathrm{div}_{\boldsymbol{r}} (D\widehat{\omega}\widehat{\boldsymbol{v}}) = 0, \tag{157}$$

and the volume integral $\int D\widehat{\omega}\, d^2r$ is preserved for tangential boundary conditions on $\widehat{\boldsymbol{v}}$. Another way of writing (157) is

$$(\partial_t + \mathcal{L}_{\widehat{\boldsymbol{v}}})(Dd\widehat{w} \wedge d\zeta) = 0. \tag{158}$$

Explicit solutions for (156)–(158) can be deduced in a similar manner as above.

An alternative approach, possibly to show well-posedness in more general spaces is to attempt an analysis based on energy estimates. For this, one needs to analyze the pair of equations

$$\begin{aligned} \partial_t \widehat{\boldsymbol{v}} + \widehat{\boldsymbol{v}} \cdot \nabla \widehat{\boldsymbol{v}} &= 0 \\ \partial_t D_t + \mathrm{div}_{\boldsymbol{r}} (D_t \widehat{\boldsymbol{v}}) &= 0 \end{aligned}$$

on the domain Ω of the measure $D_t d^2r$. For this we can use a,s apriori estimates the conserved energy for this system given by

$$\begin{aligned} h(\boldsymbol{M}, D, \lambda, \zeta) &= \int \boldsymbol{M} \cdot \widehat{\boldsymbol{v}} + \lambda \partial_t \zeta\, d^2r - \ell(\widehat{\boldsymbol{v}}, D, \widehat{\phi}, \widehat{w}, \zeta; \lambda) \\ &= \int \frac{1}{2D} |\boldsymbol{M} - \lambda \nabla_{\boldsymbol{r}} \zeta|^2 + \frac{\lambda^2}{2D} + gD\zeta\, d^2r \\ &= \int \left(\frac{1}{2} |\widehat{\boldsymbol{v}}|^2 + \frac{1}{2} |\widehat{w}|^2 + g\zeta \right) D\, d^2r \\ &= \int \left(\frac{1}{2} |\widehat{\nabla_{\boldsymbol{r}} \phi}|^2 + \frac{1}{2} |\widehat{w}|^2 + g\zeta \right) D\, d^2r. \end{aligned}$$

as well as Sobolev norm estimates deduced from (39). For this, we introduce the (time-dependent) L^p norm of some function f with respect to the measure Dd^2r as

$$\|f\|_{D_t,p} = \left(\int_\Omega f^p D_t d^2r \right)^{1/p},$$

and can show that $|\widehat{\boldsymbol{v}}|_{D,p}$ is conserved. Moreover, via a standard Grönwall/Young inequality argument, one shows that $\|\widehat{\phi}\|_{D,p}$, $\|\zeta\|_{D,p}$, $\|\widehat{\omega}\|_{D,p}$ and $\|D\|_{D,2}$ are controlled. Controls on higher Sobolev norms are also possible.

For existence, one follows DiPerna and Lions,[54] to define a sequence $(\widehat{\boldsymbol{v}}^n, D_t^n)_{n\in\mathbb{N}}$ by

$$\partial_t \widehat{\boldsymbol{v}}^n + \widehat{\boldsymbol{v}}^{n-1} \cdot \nabla_{\boldsymbol{r}} \widehat{\boldsymbol{v}}^n = 0,$$
$$\partial_t D_t^n + \mathrm{div}_{\boldsymbol{r}}\left(D_t^n \widehat{\boldsymbol{v}}^n\right) = 0.$$

For each $\widehat{\boldsymbol{v}}^{n-1}$, we may apply the results from Ref. 54 (Theorem III.2) to prove existence of $\widehat{\boldsymbol{v}}^n$. We would need to prove that $\widehat{\boldsymbol{v}}^n$ satisfies the relevant bounds to allow us to apply the theorem again and iterate this process. We then show that the sequence is relatively compact in a suitably chosen Sobolev space.

Remark 25. According to Remark 3.3, when considering the ECWWE equations with pressure as in Section 3.3, the transport velocity evolves according to the 2D Euler equation and the system thus inherits the analytical properties of this equation.

8 | FUTURE WORK

The augmented water wave problem that has been introduced here opens doors for new analytical results, as well as interesting directions for numerical studies of wave–current interaction. For example:

1. Analytical properties of the wave–current interaction equations introduced here are unknown, and the extended version of the CWWEs opens the door for new analytical results for CWWE.
2. The incorporation of surface tension into this framework is a potentially interesting issue. Other physical approximations commonly made to derive other well-known water wave equations (KdV, KP, etc.) may also be considered within this framework.
3. Further study of the stochastic ECWW equations in Section 6 and their stochastic ACWW and WCI FS versions would be worthwhile. In particular, Section 6 introduces a stochastic version of the well-studied classical water wave model that has been stochastically perturbed in a way that preserves many of its desirable fluid dynamics properties. In particular, this model would allow us to consider a stochastic CWW theory based on a DNO for 3D irrotational SALT flows. Introducing noise into the parameter ϵ in the ACWW and WCI FS models may also be interesting, since doing so would enable investigations of the probabilistic nature of wave–current interactions using stochastic versions of WCMC.

We expect to pursue all of these research directions in future work.

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APPENDIX A: TRANSFORMATION THEORY FOR FLUID DYNAMICS—KELVIN THEOREM

The Kelvin–Noether theorem is the statement of Newton's law for fluid mass distributed on a material loop.

$$\frac{d}{dt}\oint_{c_t=\phi_t c_0} \mathbf{v}(t,\mathbf{x})\cdot d\mathbf{x} = \oint_{c_t} \underbrace{\mathbf{f}\cdot d\mathbf{x}}_{\text{Newton's Law}} . \tag{A1}$$

For a discussion of the geometric mechanics underlying the deterministic case, see, for example, Ref. 23. For a discussion of the geometric mechanics underlying the stochastic case, see, for example, Refs. 26, 29.

Proof. The *deterministic* Kelvin–Noether theorem may be proved, as follows. Consider a closed loop c_t moving with the material flow as

$$c_t = \phi_t c_0.$$

The Eulerian velocity of the loop is

$$\frac{d}{dt}\phi_t(x) = \phi_t^* u(t,x) = u(t,\phi_t(x)).$$

This equation illustrates the operation of "pull-back" $\phi_t^* u(t,x)$ of the Eulerian fluid velocity $u(t,x)$ by the material flow map ϕ_t.

Compute the time derivative of the integral of the momentum/mass (impulse) around a time-dependent loop $c_t = \phi_t c_0$ moving with the flow map, ϕ_t, as

$$\begin{aligned}
\frac{d}{dt}\oint_{c_t=\phi_t c_0} \mathbf{v}(t,\mathbf{x})\cdot d\mathbf{x} &= \oint_{c_0} \frac{d}{dt}(\phi_t^*(\mathbf{v}(t,\mathbf{x})\cdot d\mathbf{x}))\\
&= \oint_{c_0} \underbrace{\phi_t^*((\partial_t + \mathcal{L}_{u(t,\mathbf{x})})(\mathbf{v}\cdot d\mathbf{x}))}_{\text{Lie derivative defined via chain rule}}\\
&= \oint_{\phi_t c_0 = c_t} (\partial_t + \mathcal{L}_{u(t,\mathbf{x})})(\mathbf{v}\cdot d\mathbf{x})\\
&= \oint_{c_t} \underbrace{\mathbf{f}\cdot d\mathbf{x}}_{\text{Newton's Law}} = \oint_{c_0} \phi_t^*(\underbrace{\mathbf{f}\cdot d\mathbf{x}}_{\text{Motion eqn}}).
\end{aligned} \tag{A2}$$

This is the Kelvin–Noether theorem of Ref. 23. When the covector field $\mathbf{v}(t, \mathbf{x})$ is interpreted as the momentum per unit mass in the fixed Eulerian inertial frame, then the last line states Newton's law for fluid mass distributed on a material loop. When the covector field $\mathbf{f}(t, \mathbf{x}) = -\nabla p$ is a pressure-gradient force per unit mass in the Eulerian inertial frame, then the last line states Kelvin's theorem for the conservation of circulation in ideal Euler fluid dynamics with spatially homogeneous density. ■

Let us delve more deeply into the statement in the second line of the proof of the Kelvin–Noether theorem that "the Lie derivative is defined via the chain rule." More specifically, the Lie derivative is defined by the time derivative of the pull-back ϕ_t^* of the flow map ϕ_t acting on the circulation integrand (which is a 1-form) by using the chain rule. The pull-back is also used in the discussion of the Burgers equation in Section 7. Let us do the corresponding calculation for the Kelvin–Noether theorem.

Integration in time of the pull-back relation in the proof,

$$\frac{d}{dt}\phi_t(x) = \phi_t^* u(t, x) = u(t, \phi_t(x)),$$

yields the smooth invertible map, $\phi_t \in \mathrm{Diff}(M)$, by integration of the characteristic curves of the smooth time-dependent vector field $u_t \in \mathfrak{X}(M)$ acting on smooth functions $f \in C^\infty(M)$ defined on a smooth manifold, M. In this situation, one says that the map ϕ_t is *generated* by the vector field u_t. The pull-back relation can be written equivalently, as a push-forward, denoted as

$$u_t = \phi_{t*}\dot{\phi}_t = {\phi_t^{-1}}^* \dot{\phi}_t = \dot{\phi}_t \phi_t^{-1},$$

in which the operation of push-forward of a smooth function f by a smooth invertible map ϕ_t depending on a parameter t is defined as the inverse of the pull-back, which may be written as $\phi_{t*} = {\phi_t^{-1}}^*$.

We may now understand the first step in the proof above as the change of variables in the loop integral to transform the loop $c_t = \phi_t c_0$ moving under the flow map ϕ_t in the fixed frame, into a fixed loop ${\phi_t}^{-1} c_t = c_0$ in the moving frame of the flow map; while also transforming the integrand $(\mathbf{v}(t, \mathbf{x}) \cdot d\mathbf{x})$ in the loop integral from the fixed frame into the moving frame of the flow map. This transformation of the Kelvin circulation loop integral into the frame in which the moving loop is fixed allows the time derivative to commute with integration around the loop. Consequently, the time derivative comes inside the integral to act on the transformed integrand, which is now in the moving frame of the flow map ϕ_t.

The second step in the proof above defines the Lie derivative as Ref. 23

$$\begin{aligned}
\frac{d}{dt}(\phi_t^*(\mathbf{v}(t, \mathbf{x}) \cdot d\mathbf{x})) &=: \phi_t^*\Big(\Big(\partial_t + \mathcal{L}_{\dot{\phi}_t \phi_t^{-1}}\Big)(\mathbf{v} \cdot d\mathbf{x})\Big) \\
&=: \phi_t^*((\partial_t + \mathcal{L}_{u(t,\mathbf{x})})(\mathbf{v} \cdot d\mathbf{x})) \\
&=: \phi_t^*\big(\big(\partial_t \mathbf{v} + (\mathbf{u} \cdot \nabla)\mathbf{v} + v_j \nabla u^j\big)\big) \cdot d\mathbf{x}.
\end{aligned} \tag{A3}$$

To finish the proof of the Kelvin–Noether theorem in (A2), one transforms the loop integral back into the fixed frame, in which the loop moves with the flow map and the integrand is fixed.

Remark A1. The Lie derivative of a differential k-form has the same expression in any coordinate system, even in a moving coordinate system. In particular, this is true for functions (0-forms), circulation 1-forms, and mass density 2-forms in 2D.

Lie derivatives in the hat formulation. The transformation to the $\widehat{f}$ notation in (4) evaluates an arbitrary flow variable f on the free surface,

$$\widehat{f}(\boldsymbol{r},t) = f(\boldsymbol{r},z,t) \quad \text{on } z = \zeta(\boldsymbol{r},t). \tag{A4}$$

For functions (0-forms), the hat-transformation evolves according to

$$\begin{aligned}
\frac{d}{dt}\phi_t^*(\widehat{f}(\boldsymbol{r},t)) &= \phi_t^*((\partial_t + \mathcal{L}_{\widehat{\boldsymbol{v}}})\widehat{f}) = \phi_t^*(\partial_t \widehat{f} + \widehat{\boldsymbol{v}}(\boldsymbol{r},t)\cdot\nabla_{\boldsymbol{r}}\widehat{f}) \\
\left[\frac{d}{dt}\phi_t^*(f(\boldsymbol{x},t))\right]_{z=\zeta(\boldsymbol{r},t)} &= [\phi_t^*(\partial_t f + f_z\,\partial_t\zeta + \boldsymbol{v}(\boldsymbol{x},t)\cdot(\nabla_{\boldsymbol{r}} f + f_z\,\nabla_{\boldsymbol{r}}\zeta(\boldsymbol{r},t)))]_{z=\zeta(\boldsymbol{r},t)} \\
&= [\phi_t^*(\partial_t f + \boldsymbol{v}(\boldsymbol{x},t)\cdot\nabla_{\boldsymbol{r}} f + f_z\,(\partial_t\zeta + \boldsymbol{v}(\boldsymbol{x},t)\cdot\nabla_{\boldsymbol{r}}\zeta(\boldsymbol{r},t)))]_{z=\zeta(\boldsymbol{r},t)} \\
&= [\phi_t^*(\partial_t f + \boldsymbol{v}(\boldsymbol{x},t)\cdot\nabla_{\boldsymbol{r}} f + f_z\,\widehat{w})]_{z=\zeta(\boldsymbol{r},t)} \\
&= [\phi_t^*((\partial_t + \mathcal{L}_{\boldsymbol{v}})f)]_{z=\zeta(\boldsymbol{r},t)} \\
&= \phi_t^*((\partial_t + \mathcal{L}_{\widehat{\boldsymbol{v}}})\widehat{f}) \\
&= \frac{d}{dt}\phi_t^*(\widehat{f}(\boldsymbol{r},t)).
\end{aligned} \tag{A5}$$

Thus, Equation (7) is recovered for 0-forms

$$(\partial_t + \mathcal{L}_{\widehat{\boldsymbol{v}}})\widehat{f} = [(\partial_t + \mathcal{L}_{\boldsymbol{v}})f]_{z=\zeta(\boldsymbol{r},t).}$$

By the product rule for the pull-back, this calculation also applies to 1-forms and 2-forms, so we have

$$\begin{aligned}
\frac{d}{dt}\phi_t^*(\widehat{\boldsymbol{V}}(\boldsymbol{r},t)\cdot d\boldsymbol{r}) &= \phi_t^*((\partial_t + \mathcal{L}_{\widehat{\boldsymbol{v}}})(\widehat{\boldsymbol{V}}\cdot d\boldsymbol{r})) \\
&= \phi_t^*((\partial_t\widehat{\boldsymbol{V}} + (\widehat{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{r}})\widehat{\boldsymbol{V}} + \widehat{V}_j\nabla_{\boldsymbol{r}}\widehat{v}^j)\cdot d\boldsymbol{r}) \\
\left[\frac{d}{dt}\phi_t^*(\boldsymbol{V}(\boldsymbol{x},t)\cdot d\boldsymbol{x})\right]_{z=\zeta(\boldsymbol{r},t)} &= [\phi_t^*((\partial_t + \mathcal{L}_{\boldsymbol{v}})(\boldsymbol{V}(\boldsymbol{x},t)\cdot d\boldsymbol{x}))]_{z=\zeta(\boldsymbol{r},t)} \\
&= \left[\phi_t^*(\left(\partial_t\boldsymbol{V} + \boldsymbol{v}\big(\boldsymbol{x},t\big)\cdot\nabla_{\boldsymbol{r}}\boldsymbol{V} + \boldsymbol{V}_z\,\widehat{w} + V_j\nabla v^j\right)\cdot d\boldsymbol{x})\right]_{z=\zeta(\boldsymbol{r},t)} \\
&= \phi_t^*((\partial_t + \mathcal{L}_{\widehat{\boldsymbol{v}}})(\widehat{\boldsymbol{V}}\cdot d\boldsymbol{r})) \\
&= \frac{d}{dt}\phi_t^*(\widehat{\boldsymbol{V}}(\boldsymbol{r},t)\cdot d\boldsymbol{r})
\end{aligned} \tag{A6}$$

Thus, for 1-forms we have a formula which will project the Kelvin theorem onto the free surface. Namely,

$$(\partial_t + \mathcal{L}_{\widehat{\boldsymbol{v}}})(\widehat{\boldsymbol{V}} \cdot d\boldsymbol{r}) = [((\partial_t + \mathcal{L}_{\boldsymbol{v}})(\boldsymbol{V}(\boldsymbol{x}, t) \cdot d\boldsymbol{x})]_{z=\zeta(\boldsymbol{r},t)}$$

Finally, for 2-forms we have the continuity equation on the free surface,

$$\begin{aligned}\left[\frac{d}{dt}\phi_t^*(\widehat{\rho}\, d^2r)\right]_{t=0} &= (\partial_t + \mathcal{L}_{\widehat{\boldsymbol{v}}})(\widehat{\rho}\, d^2r) = (\partial_t\widehat{\rho} + \widehat{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{r}}\widehat{\rho} + \widehat{\rho}\,\nabla_{\boldsymbol{r}}\cdot\widehat{\boldsymbol{v}})d^2r \\ &= (\partial_t\widehat{\rho} + \nabla_{\boldsymbol{r}}\cdot(\widehat{\rho}\widehat{\boldsymbol{v}}))d^2r = 0.\end{aligned} \tag{A7}$$

APPENDIX B: HAMILTON'S PRINCIPLE FOR 3D FLUID DYNAMICS WITH A FREE SURFACE

In this appendix, we will re the 3D Euler fluid equations (1) and (2) from a constrained variational approach for dynamics on a free surface. In this setting, we will be able to continue modelling the free surface equations. In particular, in Hamilton's principle, we will *constrain* the action integral by applying what we have learned in the present section about the Eulerian equations of irrotational free-surface motion to obtain the ECWWE (73) for the 2D velocity fields $\boldsymbol{V}$ and $\widehat{\boldsymbol{v}}$ on the free surface.

This approach via Hamilton's principle will also enable us to derive equations for fluid dynamic flows on a free surface with vorticity, nonhydrostatic pressure, and spatially varying buoyancy. In the action integral, the wave variables will be regarded as field variables interacting with the fluid variables. After introducing a wave–current "minimal-coupling" Ansatz reminiscent of the coupling of a charged fluid to an electromagnetic field,[23] we will show that this system of equations can be closed and that the wave variables will be able to generate circulation in the fluid. The resulting coupled equations will model a sort of Craik–Leibovich[28] wave–current interaction on the free surface.

Consider an action integral defined by

$$\begin{aligned}S = \int\int D\rho\left(\frac{1}{2}|\boldsymbol{u}|^2 - gz\right) - p(D-1) - \mu(\partial_t + \boldsymbol{u}\cdot\nabla)(\zeta - z) \\ + \varphi(\partial_t D + \mathrm{div}(D\boldsymbol{u})) + \gamma(\partial_t\rho + \boldsymbol{u}\cdot\nabla\rho)\, d^3x\, dt.\end{aligned} \tag{B1}$$

The Lagrange multipliers μ, φ, and γ impose the dynamical constraints in (2) as pioneered in Clebsch.[55] From left to right, the terms in (B1) are: the difference between kinetic and potential energies, the incompressible flow constraint, the Clebsch constraint that the quantity $(\zeta - z)$ is advected (i.e., particles on the surface remain so), and two more Clebsch constraints that impose advection dynamics on D as a density and ρ as a scalar function, respectively.

Remark B1. The action integral in (B1) makes sense physically, as long as the free surface $z = \zeta(\boldsymbol{r}, t)$ is a graph, so that the magnitude of the elevation slope $|\nabla_{\boldsymbol{r}}\zeta|$ remains bounded. Hence, we assume that no wave breaking will occur in the underlying fluid model during the temporal interval of the flow.

Hamilton's principle. Applying Hamilton's principle $\delta S = 0$ to the constrained action integral in (B1) yields the following variations:

$$
\begin{aligned}
0 = \delta S = \int\int & \delta D\left(\rho\left(\frac{1}{2}|\boldsymbol{u}|^2 - gz\right) - p - (\partial_t\varphi + \boldsymbol{u}\cdot\nabla\varphi)\right) + \delta p(D-1) \\
&+ \delta\rho\left(D\left(\frac{1}{2}|\boldsymbol{u}|^2 - gz\right) - (\partial_t\gamma + \mathrm{div}(\gamma\boldsymbol{u}))\right) \\
&+ \delta\boldsymbol{u}\cdot(D\rho\boldsymbol{u} - \mu\nabla(\zeta - z) - D\nabla\varphi + \gamma\nabla\rho) \\
&+ \delta(\zeta - z)(\partial_t\mu + \mathrm{div}(\mu\boldsymbol{u})) - (\partial_t(\zeta - z) + \boldsymbol{u}\cdot\nabla(\zeta - z))\delta\mu \\
&+ \delta\varphi(\partial_t D + \mathrm{div}(D\boldsymbol{u})) + \delta\gamma(\partial_t\rho + \boldsymbol{u}\cdot\nabla\rho)\, d^3x\, dt.
\end{aligned}
$$

The variations with respect to each dynamical variable yield the following independent relations, written in the coordinate-free Lie derivative notation discussed in Appendix A,

$$
\begin{aligned}
&\delta D: \quad (\partial_t + \mathcal{L}_{\boldsymbol{u}})\varphi = \rho\left(\frac{1}{2}|\boldsymbol{u}|^2 - gz\right) - p \\
&\delta\rho: \quad (\partial_t + \mathcal{L}_{\boldsymbol{u}})\left(\frac{\gamma}{D}\right) = \frac{1}{2}|\boldsymbol{u}|^2 - gz \\
&\left.\begin{aligned}
&\delta\varphi: \quad (\partial_t + \mathcal{L}_{\boldsymbol{u}})(D\, d^3x) = (\partial_t D + \mathrm{div}(D\boldsymbol{u}))d^3x = 0 \\
&\delta p: \quad D - 1 = 0,
\end{aligned}\right\} \quad \Rightarrow \mathrm{div}\boldsymbol{u} = 0, \\
&\delta\zeta: \quad (\partial_t + \mathcal{L}_{\boldsymbol{u}})\left(\frac{\mu}{D}\right) = 0, \\
&\delta\gamma: \quad (\partial_t + \mathcal{L}_{\boldsymbol{u}})\rho = 0, \\
&\delta\mu: \quad (\partial_t + \mathcal{L}_{\boldsymbol{u}})(\zeta - z) = 0, \\
&\delta\boldsymbol{u}: \quad \rho\boldsymbol{u}\cdot d\boldsymbol{x} = \frac{\mu}{D}d(\zeta - z) + d\varphi - \frac{\gamma}{D}d\rho,
\end{aligned}
$$

where $\mathcal{L}_{\boldsymbol{u}}$ denotes Lie derivative with respect to the 3D velocity vector field. We have also imposed natural homogeneous boundary conditions. Assembling these variational equations leads to the following fluid motion equation:

$$
\begin{aligned}
\rho(\partial_t + \mathcal{L}_{\boldsymbol{u}})(\boldsymbol{u}\cdot d\boldsymbol{x}) &= 0 + d(\partial_t + \mathcal{L}_{\boldsymbol{u}})\varphi - (\partial_t + \mathcal{L}_{\boldsymbol{u}})\left(\frac{\gamma}{D}\right)d\rho \\
&= d\left(\rho\left(\frac{1}{2}|\boldsymbol{u}|^2 - gz\right) - p\right) - \left(\frac{1}{2}|\boldsymbol{u}|^2 - gz\right)d\rho \\
&= \rho d\left(\frac{1}{2}|\boldsymbol{u}|^2 - gz\right) - dp.
\end{aligned}
$$

From these considerations, we have the following set of coordinate-free dynamical equations for the incompressible flow of an inhomogeneous fluid:

$$
\begin{aligned}
(\partial_t + \mathcal{L}_{\boldsymbol{u}})(\boldsymbol{u}\cdot d\boldsymbol{x}) + \frac{1}{\rho}dp - d\left(\frac{1}{2}|\boldsymbol{u}|^2 + gz\right) &= 0,\\
(\partial_t + \mathcal{L}_{\boldsymbol{u}})(D\, d^3x) = 0, \quad \text{with } D &= 1,\\
(\partial_t + \mathcal{L}_{\boldsymbol{u}})\rho &= 0,\\
(\partial_t + \mathcal{L}_{\boldsymbol{u}})(\zeta(x,y,t) - z) &= 0.
\end{aligned}
\tag{B2}
$$

These equations impose the conditions for incompressible flow $\mathrm{div}\boldsymbol{u} = 0$ and the constraint that fluid parcels initially on the free surface remain on it.

The system of equations in (B2) may be evaluated on the free surface immediately by using the coordinate-free identities derived for Lie derivatives in Appendix A. This evaluation results in the following system of equations in the hat notation from the previous section:

$$
\begin{aligned}
(\partial_t + \mathcal{L}_{\hat{\boldsymbol{v}}})(\hat{\boldsymbol{v}}\cdot d\boldsymbol{r}) + \frac{1}{\hat{\rho}}d\hat{p} - d\left(\frac{1}{2}|\hat{\boldsymbol{v}}|^2 + \frac{1}{2}\hat{w}^2 + g\zeta\right) &= 0,\\
(\partial_t + \mathcal{L}_{\hat{\boldsymbol{v}}})[D\, d^3x]_{z=\zeta(x,y,t)} = 0, \quad \text{with} \quad D &= 1,\\
(\partial_t + \mathcal{L}_{\hat{\boldsymbol{v}}})\hat{\rho} &= 0,\\
(\partial_t + \mathcal{L}_{\hat{\boldsymbol{v}}})(\zeta(x,y,t) - \zeta) &= 0.
\end{aligned}
\tag{B3}
$$

Again, one sees that the free surface fluid equations are not closed. As before, they are missing an evolution equation for $\hat{w}$ and a method of computing, $d\hat{p}$, the gradient of the nonhydrostatic pressure. Also, we see that the evaluation of the 3-form volume element on the free surface $z = \zeta(\boldsymbol{r}, t)$ vanishes identically, so the connection of the pressure to 3D volume preservation vanishes there, too.

Vorticity dynamics. Taking the differential (i.e., the curl) of the motion equation in (B2) yields the following equation for the vorticity dynamics: $\boldsymbol{\omega} := \mathrm{curl}\boldsymbol{u}$,

$$
(\partial_t + \mathcal{L}_{\boldsymbol{u}})(\boldsymbol{\omega}\cdot d\boldsymbol{S}) = -d(\rho^{-1}) \wedge dp. \tag{B4}
$$

Together, the buoyancy equation in (B2) and the vorticity Equation (B4) yield an advection equation for the PV defined as $q = \boldsymbol{\omega}\cdot\nabla\rho$. Now, $\boldsymbol{\omega}\cdot d\boldsymbol{S}\wedge d\rho = \boldsymbol{\omega}\cdot\nabla\rho\, d^3x$ and $\mathrm{div}\boldsymbol{u} = 0$ imply,

$$
(\partial_t + \mathcal{L}_{\boldsymbol{u}})(\boldsymbol{\omega}\cdot d\boldsymbol{S}\wedge d\rho) = -d(\rho^{-1}) \wedge dp \wedge d\rho = 0. \tag{B5}
$$

Thus, nonalignment of gradients of pressure and density results in local creation of vorticity.

One expands Equation (B5) to find the advection equation $(\partial_t + \mathcal{L}_{\boldsymbol{u}})q = 0$ for PV $q = \boldsymbol{\omega}\cdot\nabla\rho$, by computing

$$
(\partial_t + \mathcal{L}_{\boldsymbol{u}})(\boldsymbol{\omega}\cdot d\boldsymbol{S}\wedge d\rho) = (\partial_t + \mathcal{L}_{\boldsymbol{u}})(\boldsymbol{\omega}\cdot\nabla\rho\, d^3x) = ((\partial_t + \mathcal{L}_{\boldsymbol{u}})q)\, d^3x = 0. \tag{B6}
$$

The last line in deriving the PV equation (B6) uses the product rule for the Lie derivative and enforces volume preservation arising from the divergence-free condition,

$$
(\partial_t + \mathcal{L}_{\boldsymbol{u}})d^3x = (\mathrm{div}\boldsymbol{u})d^3x = 0. \tag{B7}
$$

Thus, the issues of preservation of volume and PV are linked in projecting the 3D fluid motion onto the free surface; so, they should be solved together using similar considerations.

APPENDIX C: LEGENDRE TRANSFORMATION TO THE HAMILTONIAN FOR ACWWE

This appendix explains how the Legendre transformation of the augmented Lagrangian in the action integral (72) with respect to the sum of the fluid and wave momentum densities

$$\boldsymbol{M} = D\widehat{\boldsymbol{v}} + \lambda \nabla_{\boldsymbol{r}}\zeta = D\boldsymbol{V} \tag{C1}$$

leads to the ACWW Hamiltonian in (87), which is also the conserved energy for the system of ACWWE in (77).

The Lagrangian in the action integral (72) is given in dimensional form by

$$\begin{aligned}\ell(\widehat{\boldsymbol{v}}, D, \widehat{\phi}, \widehat{w}; \zeta, \lambda) = \int D\left(\frac{1}{2}(|\widehat{\boldsymbol{v}}|^2 + \widehat{w}^2) - g\zeta\right) + \lambda(\partial_t \zeta + \widehat{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{r}}\zeta - \widehat{w}) \\ + \widehat{\phi}(\partial_t D + \mathrm{div}_{\boldsymbol{r}}(D\widehat{\boldsymbol{v}})) - \epsilon\widehat{w}\lambda|\nabla_{\boldsymbol{r}}\zeta|^2\, d^2r.\end{aligned} \tag{C2}$$

Recall from (73) and (76) that that Bernoulli function ϖ and vertical wave momentum density λ are defined as

$$\varpi := \frac{1}{2}(|\widehat{\boldsymbol{v}}|^2 + \widehat{w}^2) - g\zeta, \qquad \lambda = \frac{D\widehat{w}}{1+\epsilon|\nabla_{\boldsymbol{r}}\zeta|^2} =: D\widetilde{w}. \tag{C3}$$

Legendre transforming yields the Hamiltonian,

$$\begin{aligned} h(\boldsymbol{M}, D, \lambda, \zeta) &= \int \left(\frac{\delta \ell}{\delta \widehat{\boldsymbol{v}}}\cdot\widehat{\boldsymbol{v}} + \lambda\partial_t\zeta - D\partial_t\widehat{\phi}\right) d^2r - \ell(\widehat{\boldsymbol{v}}, D, \widehat{\phi}, \widehat{w}; \zeta, \lambda) \\ &= \int \left(\frac{1}{2}|\widehat{\boldsymbol{v}}|^2 - \frac{1}{2}\widehat{w}^2 + \frac{\lambda}{D}\widehat{w}(1+\epsilon|\nabla_{\boldsymbol{r}}\zeta|^2) + g\zeta\right) D\, d^2r \\ &= \int \left(\frac{1}{2}|\widehat{\boldsymbol{v}}|^2 + \frac{1}{2}\widehat{w}^2 + g\zeta\right) D\, d^2r \\ &= \int \frac{1}{2D}|\boldsymbol{M} - \lambda\nabla_{\boldsymbol{r}}\zeta|^2 + \frac{\lambda^2}{2D}(1+\epsilon|\nabla_{\boldsymbol{r}}\zeta|^2)^2 + gD\zeta\, d^2r \\ &= \int \left(\frac{1}{2}|\widehat{\nabla_{\boldsymbol{r}}\phi}|^2 + \frac{1}{2}\widehat{w}^2 + g\zeta\right) D\, d^2r. \end{aligned} \tag{C4}$$

The corresponding variational derivatives of the Hamiltonian $h(\boldsymbol{M}, D, \lambda, \zeta)$ applied in the Lie–Poisson formulation in Equation (89) are now given by

$$\begin{aligned}\delta h(\boldsymbol{M}, D, \lambda, \zeta) = \int \widehat{\boldsymbol{v}}\cdot\delta\boldsymbol{M} + \left(-\frac{1}{2}|\widehat{\boldsymbol{v}}|^2 - \frac{1}{2}\widehat{w}^2 + g\zeta\right)\delta D \\ + (-(\widehat{\boldsymbol{v}} - \epsilon\widehat{w}\nabla_{\boldsymbol{r}}\zeta)\cdot\nabla_{\boldsymbol{r}}\zeta + \widehat{w})\,\delta\lambda + (\mathrm{div}_{\boldsymbol{r}}(\lambda(\widehat{\boldsymbol{v}} - 2\epsilon\widehat{w}\nabla_{\boldsymbol{r}}\zeta)) + gD)\,\delta\zeta\, d^2r.\end{aligned} \tag{C5}$$