NORMAL COORDINATES IN DYNAMICAL SYSTEMS

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THE main idea of the following paper is that of the solution of dynamical problems by means of complex integrals. Such a solution was suggested by Cauchy long ago, but I am not aware that he gave any method by which the arbitrary constants of the solution can be directly and immediately expressed in terms of the initial conditions.

In § 1 this problem is solved for a system free from impressed forces, but subject to given initial disturbances: and in § 2 for given impressed forces, the initial disturbances being zero. It is proved in § 3 that the results so obtained are equivalent to those derived by the method of normal coordinates, *when the latter can be applied.* But it must be understood that the method of § 1 can be used in cases when there is no recognized method of treatment by normal coordinates.

It appears that the general method of normal coordinates, as stated in § 3, was worked out independently by Routh and by Heaviside. Starting from this, Heaviside subsequently developed a method to which he gave the name of " the method of resistance-operators" : it is proved in § 4 that this method is really equivalent to the method of § 2, for dealing with constant impressed forces.

In § 5 is given a statement of the extension of the method to deal with continuous systems; and it is verified that in certain simple problems the results agree with those found by known methods (such as Fourier's series). In § 6 is given an independent process for the interpretation of the complex integrals of § 5; it appears that for a vibrating uniform string (and allied problems) this method leads to the same conclusions as the superposition of positive and negative waves.

In § 7 it is shewn that we can apply the general methods of §§ 5, 6 very easily to deal with problems of advancing waves with a wave-front. The problems solved here refer to waves of sound: but similar methods are equally effective for electromagnetic waves. This would be anticipated from the fact that such electromagnetic problems were solved first by Heaviside, using his operational methods.

In § 8 an outline is given (for one special type of continuous systems) of a proof of the accuracy of the general method of § 5: and in § 9 the general method is compared with the method of normal functions, and it is proved that the results obtained are identical. In § 9 certain general results on normal functions are proved which I have not met with previously, although they are natural extensions of known results.

On account of the great generality of the method, I have thought it desirable to include a number of illustrative examples: most of them are taken from well-known sources, but the methods of treatment are novel.

I hope to indicate in another paper how the present method can be applied to problems in Conduction of Heat and other types of diffusion: examples of this kind have accordingly been excluded from the present paper.

1. *Solutions when Impressed Forces are absent.*

We begin by considering the solution of a system of simultaneous differential equations with constant coefficients, represented by

$$\left.\begin{aligned} e_{11}x_1+e_{12}x_2+\ldots+e_{1n}x_n &= 0 \\ e_{21}x_1+e_{22}x_2+\ldots+e_{2n}x_n &= 0 \\ \ldots \quad \ldots \quad \ldots \quad &\ldots \\ e_{n1}x_1+e_{n2}x_2+\ldots+e_{nn}x_n &= 0 \end{aligned}\right\}, \tag{1}$$

where e_{rs} denotes the differential operator

$$e_{rs} = a_{rs}D^2+b_{rs}D+c_{rs}, \tag{2}$$

following the notation used by Lord Rayleigh (*Theory of Sound*, Vol. I, § 82). Here the symbol D denotes differentiation with respect to the time t, and the coefficients a_{rs}, b_{rs}, c_{rs} are constants.

In the ordinary dynamical interpretation the coefficients a_{rs} are those of the kinetic energy, b_{rs} those of the dissipation function, and c_{rs} those of the potential energy: but in what follows we shall have no occasion to assume that the symmetrical relations

$$a_{rs} = a_{sr}, \quad b_{rs} = b_{sr}, \quad c_{rs} = c_{sr}$$

hold good, and accordingly the solution obtained can be applied to gyrostatic systems and also to systems in which the forces are not derived from a potential energy function.

The solution is found by substituting the contour integrals

$$x_r = \frac{1}{2\pi\iota}\int e^{\lambda t}\xi_r\,d\lambda, \tag{3}$$

in the equations (1), where $\xi_1, \xi_2, \ldots, \xi_n$ are certain functions of λ to be found, and the path of integration is a closed path in the λ-plane which encloses the poles of these functions ξ_r. The result of the substitution is

$$\int e^{\lambda t}p_1\,d\lambda = 0, \qquad \int e^{\lambda t}p_2\,d\lambda = 0, \qquad \ldots, \qquad \int e^{\lambda t}p_n\,d\lambda = 0, \tag{4}$$

where

$$\left.\begin{aligned} p_1 &= \lambda_{11}\xi_1+\lambda_{12}\xi_2+\ldots+\lambda_{1n}\xi_n \\ p_2 &= \lambda_{21}\xi_1+\lambda_{22}\xi_2+\ldots+\lambda_{2n}\xi_n \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \\ p_n &= \lambda_{n1}\xi_1+\lambda_{n2}\xi_2+\ldots+\lambda_{nn}\xi_n \end{aligned}\right\}, \tag{5}$$

and

$$\lambda_{rs} = a_{rs}\lambda^2+b_{rs}\lambda+c_{rs}. \tag{6}$$

If the equations (4) hold, the path of integration must contain no poles of the functions $p_1, p_2, \ldots, p_n$: and, in virtue of the equations (5), it is clear that the functions p_r have no poles other than those of the functions ξ_r. Consequently, since the path contains the poles of ξ_r, it follows that the functions p_r can have no poles: and the simplest assumption to make is that the functions p_r are simple polynomials. We have now to see that we can adjust the choice of these polynomials so as to correspond to any given initial displacements (u_r) and velocities (v_r).

On putting $t = 0$, in the integrals (3), and in their differential coefficients, we find the equations for the initial displacements and velocities

$$u_r = \frac{1}{2\pi\iota}\int \xi_r\,d\lambda, \qquad v_r = \frac{1}{2\pi\iota}\int \lambda\xi_r\,d\lambda, \tag{7}$$

where now the path of integration may be supposed to be a large circle in the λ-plane; for, by taking a sufficiently large radius, we can certainly enclose all the poles of the functions $\xi_1, \xi_2, \ldots, \xi_n$, since these functions are simply rational fractions in λ. Now, when $|\lambda|$ is sufficiently large, any rational fraction in λ can be expanded in descending powers of λ; and if we have the expansion*

$$\xi_r = \frac{X_r}{\lambda}+\frac{Y_r}{\lambda^2}+\frac{Z_r}{\lambda^3}+\ldots,$$

* It is unnecessary to include any positive powers of λ in ξ_r: for such terms will contribute zero to our original solutions (3).

it follows from equations (7) that

$$u_r = X_r, \quad v_r = Y_r.$$

Thus equations (7) lead to the inference that, when $|\lambda|$ is large,

$$\xi_r = \frac{u_r}{\lambda} + \frac{v_r}{\lambda^2} + O\left(\frac{1}{\lambda^3}\right).* \tag{8}$$

Accordingly we can now determine the polynomials $p_1, p_2, \ldots, p_n$ by substituting for ξ_r from (8) in (5), and retaining only the positive powers of λ. This gives at once

$$p_1 = \{(a_{11}\lambda + b_{11})u_1 + a_{11}v_1\} + \{(a_{12}\lambda + b_{12})u_2 + a_{12}v_2\} \\ + \ldots + \{(a_{1n}\lambda + b_{1n})u_n + a_{1n}v_n\}, \tag{9}$$

with similar formulæ for $p_2, p_3, \ldots, p_n$.

In actual calculations it is generally easier to remember the rule to substitute from (8) in (5), and to retain only positive powers of λ, than to quote the formulæ (9).

We have still to prove that the formulæ (9) do as a matter of fact lead to the equations (7): for this purpose we write the solution of (5) in the form

$$\xi_r = \frac{1}{\Delta(\lambda)}\{F_{r1}(\lambda)\,p_1 + F_{r2}(\lambda)\,p_2 + \ldots + F_{rn}(\lambda)\,p_n\}, \tag{10}$$

where $F_{rs}(\lambda)$ denotes the minor† of λ_{sr} in the determinant

$$\Delta(\lambda) = \begin{vmatrix} \lambda_{11}, & \lambda_{12}, & \ldots, & \lambda_{1n} \\ \lambda_{21}, & \lambda_{22}, & \ldots, & \lambda_{2n} \\ \ldots & \ldots & & \ldots \\ \lambda_{n1}, & \lambda_{n2}, & \ldots, & \lambda_{nn} \end{vmatrix}. \tag{11}$$

Now, if we observe that (9) can be written in the form

$$p_1 = \lambda_{11}\left(\frac{u_1}{\lambda} + \frac{v_1}{\lambda^2}\right) + \lambda_{12}\left(\frac{u_2}{\lambda} + \frac{v_2}{\lambda^2}\right) + \ldots + \lambda_{1n}\left(\frac{u_n}{\lambda} + \frac{v_n}{\lambda^2}\right) + \frac{q_1}{\lambda^2}, \tag{12}$$

where q_1 is another expression of the first degree (at most) in λ, and that

* In accordance with recent practice, the symbol $O(1/\lambda^3)$ is used to denote a function of λ, which is *of order* $1/\lambda^3$, when $|\lambda|$ is large; more precisely, if the function is multiplied by λ^3, the product remains finite, when $|\lambda|$ tends to infinity.

† Note that the order of the suffixes is reversed, intentionally, in F_{rs} and λ_{sr}: in problems for which $\lambda_{rs} = \lambda_{sr}$, the order of the two suffixes is immaterial.

$p_2, p_3, \ldots, p_n$ can be written similarly, we see that (10) leads to the result

$$\xi_r \doteq \frac{u_r}{\lambda}+\frac{v_r}{\lambda^2}+\frac{1}{\lambda^2\Delta(\lambda)}\{F_{r1}(\lambda)\,q_1+F_{r2}(\lambda)\,q_2+\ldots+F_{rn}(\lambda)\,q_n\}. \tag{13}$$

Now consider the expression in the last bracket of (13); its degree in λ is at most $(2n-1)$, since each minor F_{rs} is of degree $2(n-1)$ at most. Also the degree of $\Delta(\lambda)$ is $2n$, *provided that the determinant*

$$A = \begin{vmatrix} a_{11}, & a_{12}, & \ldots, & a_{1n} \\ a_{21}, & a_{22}, & \ldots, & a_{2n} \\ \ldots & \ldots & & \ldots \\ a_{n1}, & a_{n2}, & \ldots, & a_{nn} \end{vmatrix}, \tag{14}$$

is not zero. Consequently the formula (13) leads back to the equations (8), provided that the determinant A is not zero.

When the determinant A is zero, we can deduce from equations (1) at least one equation which contains no term in D^2. Since this equation holds for $t = 0$, there must be at least one relation amongst the initial displacements (u_r) and velocities (v_r); and accordingly it is no longer possible to solve equations (1) with a perfectly arbitrary set of displacements and velocities. This fact may suffice to indicate that the restriction that A is not to be zero does correspond to a real difficulty in the process of solution.

The question may now be naturally asked :—If A is zero, and the corresponding relations are satisfied by the u's and v's, will the general method given here still prove successful?

To this question I cannot give a complete answer; there are certain algebraic complications which have hitherto baffled me and have prevented me from constructing a general proof. But an examination of a number of special examples has convinced me that the general method is still adequate, even when $A = 0$. One of these examples is given below (Ex. 4, p. 409).

Accordingly, *the solution of equations* (1) *subject to assigned initial values of the velocities and displacements is given by the integrals* (3), *combined with equations* (5), (9) *and* (10).

To obtain the final solution in an explicit form, it is necessary to know the roots of the equation $\Delta(\lambda) = 0$; and this is the way in which the period-equation of the problem presents itself here. Knowing these roots, we have simply to calculate the sum of the residues of $e^{\lambda t}\xi_r$ at the roots of $\Delta(\lambda) = 0$, in order to have the formula for x_r. Thus, if $\lambda = a$ is any non-repeated root of $\Delta(\lambda) = 0$, we see from (10) that the corresponding contribution to x_1 will be

$$\frac{e^{at}}{\Delta'(a)}\{F_{11}(a)\bar{p}_1+F_{12}(a)\bar{p}_2+\ldots+F_{1n}\bar{p}_n\}, \tag{15}$$

where $\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_n$ denote the results of inserting the value $\lambda = a$ in the expressions for $p_1, p_2, \ldots, p_n$.

There is no difficulty in carrying out the corresponding calculations for a repeated root: but it is hardly worth while to write out a general formula corresponding to (15). The method of procedure will be easily seen from the examples given below (Exs. 2 and 3): and in many of the ordinary dynamical problems where repeated roots occur, it will be found that the corresponding roots only give *simple** poles in ξ_r (see Ex. 3).

It may, however, be well to point out explicitly that the present method is not affected by the complications which are introduced by the presence of repeated roots in $\Delta(\lambda) = 0$, when we are dealing with the closely associated problem of reducing two quadratic forms to sums of the same squares.†

It is easy to see how to modify the method of solution when the equations (1) are of the first degree only in D: it will suffice to state the corresponding results without fresh investigation.

Suppose, then, that in (1) the differential operators are given by

$$e_{rs} = a_{rs}D + b_{rs}. \tag{2'}$$

Then the equations (3), (5) remain unchanged in form, but we write

$$\lambda_{rs} = a_{rs}\lambda + b_{rs}. \tag{6'}$$

We shall now need only the initial values of $x_1, x_2, \ldots, x_n$; and these will be denoted, as before, by $u_1, u_2, \ldots, u_n$. Then, in place of (9), we find

$$p_1 = a_{11}u_1 + a_{12}u_2 + \ldots + a_{1n}u_n, \tag{9'}$$

with similar forms for $p_2, p_3, \ldots, p_n$. This solution will satisfy the required conditions, provided that A, the determinant defined in (14), is not zero.

When A is zero, we find that at least one relation must exist amongst the initial displacements.

Here also (provided that the requisite relations hold), I have found that the general method leads to correct conclusions in all special examples which I have examined: but I have failed to construct a general proof of the property.

Examples of the Process of § 1.

The following examples have been selected to illustrate the method; they are not of types likely to occur in actual dynamical or physical examples, but are easy to work out in detail.

* Because the corresponding invariant-factors of $\Delta(\lambda)$ are *linear*; so that if $(\lambda-\alpha)^r$ is a factor of $\Delta(\lambda)$, then $(\lambda-\alpha)^{r-1}$ is a factor of *every* first minor $F_{rs}(\lambda)$. Thus $(\lambda-\alpha)$ appears only to the first degree in the denominator of ξ_r.

† For the algebraic side of this problem, reference may be made to my tract on *Quadratic Forms* (No. 3 of the Cambridge University Press series).

Ex. 1.—[Routh's *Rigid Dynamics*, Vol. 2, 1892, Art. 367]

$$\left.\begin{aligned}(D^2-4D)\,x-\ (D-1)\,y &= 0,\\ (D+6)\,x+(D^2-D)\,y &= 0.\end{aligned}\right\}$$

We solve by taking
$$x = \frac{1}{2\pi\iota}\int \xi e^{\lambda t}\,d\lambda, \qquad y = \frac{1}{2\pi\iota}\int \eta e^{\lambda t}\,d\lambda,$$

where
$$\left.\begin{aligned}(\lambda^2-4\lambda)\,\xi-\ (\lambda-1)\,\eta &= p,\\ (\lambda+6)\,\xi+(\lambda^2-\lambda)\,\eta &= q,\end{aligned}\right\}$$

corresponding to (5) above.

Assuming that the initial values of x, y, and Dx, Dy, are represented by x_0, y_0, and x_1, y_1, the expansions corresponding to (8) will be

$$\xi = \frac{x_0}{\lambda}+\frac{x_1}{\lambda^2}+O\left(\frac{1}{\lambda^3}\right), \qquad \eta = \frac{y_0}{\lambda}+\frac{y_1}{\lambda^2}+O\left(\frac{1}{\lambda^3}\right).$$

On substituting and rejecting negative powers of λ, we find that

$$\left.\begin{aligned}p &= (\lambda-4)\,x_0+x_1-y_0,\\ q &= x_0+(\lambda-1)\,y_0+y_1.\end{aligned}\right\}$$

Then, on solving, we see that

$$= \frac{p\,(\lambda^2-\lambda)+q\,(\lambda-1)}{(\lambda^2-1)(\lambda-2)(\lambda-3)} = \frac{p\lambda+q}{(\lambda+1)(\lambda-2)(\lambda-3)},$$

and
$$\eta = \frac{-p\,(\lambda+6)+q\,(\lambda^2-4\lambda)}{(\lambda^2-1)(\lambda-2)(\lambda-3)}.$$

It is now an easy matter to evaluate x and y completely by writing down the sum of the residues of $\xi e^{\lambda t}$ and of $\eta e^{\lambda t}$ at all the poles. We shall content ourselves with writing out the value of x only.

Corresponding to $\lambda = -1$, we have the residue

$$\begin{aligned} e^{-t}\left(\frac{-p+q}{12}\right) &= \tfrac{1}{12}e^{-t}\left\{-(-5x_0+x_1-y_0)+(x_0-2y_0+y_1)\right\}\\ &= \tfrac{1}{12}e^{-t}\,(6x_0-x_1-y_0+y_1)\,;\end{aligned}$$

and corresponding to $\lambda = +2$, we have

$$\begin{aligned} e^{2t}\left(\frac{2p+q}{-3}\right) &= -\tfrac{1}{3}e^{2t}\left\{2\,(-2x_0+x_1-y_0)+(x_0+y_0+y_1)\right\}\\ &= -\tfrac{1}{3}e^{2t}\,(-3x_0+2x_1-y_0+y_1).\end{aligned}$$

Corresponding to $\lambda = 3$, we find

$$\begin{aligned} e^{3t}\left(\frac{3p+q}{4}\right) &= \tfrac{1}{4}e^{3t}\left\{3\,(-x_0+x_1-y_0)+(x_0+2y_0+y_1)\right\}\\ &= \tfrac{1}{4}e^{3t}\,(-2x_0+3x_1-y_0+y_1).\end{aligned}$$

The complete formula for x is then the sum of these three residues; it is easy to verify that as a matter of fact the initial values of x and Dx are then x_0 and x_1. It will be seen also that the first and second terms agree with those found by Routh (*l.c.*) using the "method of isolation".

Ex. 2.—To illustrate *the effect of a multiple root in the determinant* $\Delta(\lambda)$, consider the equations

$$\left.\begin{aligned}(D^2-2D)\,x-\ y &= 0\\ (2D-1)\,x+D^2y &= 0\end{aligned}\right\} \quad \text{[Routh, } l.c.\text{, Art. 373].}$$

Proceeding as in Ex. 1 above, we find the equations

$$\left.\begin{aligned}(\lambda^2-2\lambda)\,\xi - \quad \eta &= p = (\lambda-2)\,x_0 + x_1,\\ (2\lambda-1)\,\xi + \lambda^2\eta &= q = \;\; 2x_0 + \lambda y_0 + y_1.\end{aligned}\right\}$$

Hence, on solving, we have

$$(\lambda-1)^3(\lambda+1)\,\xi = \lambda^2 p + q = (\lambda^3-2\lambda^2+2)\,x_0 + \lambda^2 x_1 + \lambda y_0 + y_1,$$

with a similar formula for η.

To obtain the residue corresponding to the multiple root $\lambda = 1$, we write $\lambda = 1+\mu$, and arrange ξ in the form

$$\frac{M_2}{\mu^3} + \frac{M_1}{\mu^2} + \frac{M_0}{\mu} + \frac{N}{2+\mu}.$$

Thus on multiplying up by $\mu^3(2+\mu)$, we find the identity

$$2M_2 + \mu\,(2M_1 + M_2) + \mu^2\,(2M_0 + M_1) + (M_0 + N)\,\mu^3$$
$$\equiv (1-\mu+\mu^2+\mu^3)\,x_0 + (1+2\mu+\mu^2)\,x_1 + (1+\mu)\,y_0 + y_1.$$

Hence $$2M_2 = x_0 + x_1 + y_0 + y_1, \qquad 2M_0 + M_1 = x_0 + x_1,$$
$$2M_1 + M_2 = -x_0 + 2x_1 + y_0, \qquad M_0 + N = x_0.$$

Then the residue for x, corresponding to $\lambda = 1$, is found from the coefficient of $1/\mu$ in the expansion of

$$\left(\frac{M_2}{\mu^3} + \frac{M_1}{\mu^2} + \frac{M_0}{\mu} + \frac{N}{2+\mu}\right) e^{t(1+\mu)},$$

in powers of μ. This coefficient is seen to be

$$(M_0 + M_1 t + \tfrac{1}{2}M_2 t^2)\, e^t,$$

agreeing with Routh's result (*l.c.*).

The complete expression for x is seen to be

$$(M_0 + M_1 t + \tfrac{1}{2}M_2 t^2)\, e^t + Ne^{-t}.$$

Ex. 3.—Another example of repeated roots in $\Delta(\lambda)$ is given by

$$(D^2-1)\,x + y + z = 0, \qquad x + (D^2-1)\,y + z = 0, \qquad x + y + (D^2-1)\,z = 0,$$

[Routh, *l.c.*, Art. 396].

Proceeding as above we use three symmetric equations, of which the first is

$$(\lambda^2-1)\,\xi + \eta + \zeta = p = \lambda x_0 + x_1.$$

Hence $$(\lambda^2+1)(\xi+\eta+\zeta) = p + q + r;$$

and so, finally, $$3\xi = \frac{1}{\lambda^2+1}(p+q+r) + \frac{1}{\lambda^2-2}(2p-q-r).$$

On calculating the residues of $\xi e^{\lambda t}$, we find

$$3x = (x_0+y_0+z_0)\cos t + (x_1+y_1+z_1)\sin t + (2x_0-y_0-z_0)\cosh(t\sqrt{2}) + (2x_1-y_1-z_1)\sinh(t\sqrt{2})/\sqrt{2}.$$

The result may be verified at once by noticing that the given equations lead to

$$(D^2+1)(x+y+z) = 0, \qquad (D^2-2)(x-y) = 0, \qquad (D^2-2)(x-z) = 0.$$

This example illustrates the effect of *linear* invariant factors: the value of ξ has only simple poles, and consequently the value of x is of a simpler character than in such a problem as Ex. 2 above.

Ex. 4.—To illustrate *the effect of supposing the determinant A to be zero*, let us consider the equations

$$\left.\begin{aligned}(D^2+1)\,x+(D^2-2D)\,y &= 0,\\ (D^2+D)x+\quad D^2y \quad &= 0.\end{aligned}\right\}$$

Here

$$A = \begin{vmatrix} 1, & 1 \\ 1, & 1 \end{vmatrix} = 0;$$

and there must accordingly be a relation connecting the initial displacements and velocities. If we subtract the first equation from the second, we see that

$$(D-1)x+2Dy = 0.$$

Hence

$$x_1-x_0+2y_1 = 0$$

is the required relation, since the last equation is true for $t = 0$, as well as for all other values of t.

We shall now shew that (subject to this relation amongst the initial values) our general process leads to a correct conclusion. In fact the auxiliary equations are seen to be

$$\left.\begin{aligned}(\lambda^2+1)\,\xi+(\lambda^2-2\lambda)\,\eta &= \lambda x_0+x_1+(\lambda-2)\,y_0+y_1,\\ (\lambda^2+\lambda)\,\xi+\quad \lambda^2\eta \quad &= (\lambda+1)\,x_0+x_1+\lambda y_0+y_1.\end{aligned}\right\}$$

Hence $\lambda(\lambda+3)\,\xi = (\lambda+2)\,x_0+2x_1+2y_1,$

$$\lambda^2(\lambda+3)\,\eta = (\lambda+1)\,x_0-(\lambda-1)\,x_1+\lambda(\lambda+3)\,y_0-(\lambda-1)\,y_1.$$

Thus, when $|\lambda|$ is large, we obtain the expansions

$$\left.\begin{aligned}\xi &= \frac{x_0}{\lambda}+\frac{-x_0+2x_1+2y_1}{\lambda^2}+O\left(\frac{1}{\lambda^3}\right),\\ \eta &= \frac{y_0}{\lambda}+\quad\frac{x_0-x_1-y_1}{\lambda^2}\quad+O\left(\frac{1}{\lambda^3}\right).\end{aligned}\right\}$$

At first sight, these expressions do not appear to agree with (8); but nevertheless, on writing $x_0 = x_1+2y_1$ (in accordance with the relation already found amongst the initial values), we find that

$$\xi = \frac{x_0}{\lambda}+\frac{x_1}{\lambda^2}+O\left(\frac{1}{\lambda^3}\right), \qquad \eta = \frac{y_0}{\lambda}+\frac{y_1}{\lambda^2}+O\left(\frac{1}{\lambda^3}\right),$$

so that as a matter of fact they do agree with (8).

The final solution is found to be

$$\left.\begin{aligned} x &= (x_0+\tfrac{1}{3}x_1)-\tfrac{1}{3}x_1 e^{-3t}, \\ y &= (y_0-\tfrac{2}{9}x_1)+(y_1+\tfrac{2}{3}x_1)\, t+\tfrac{2}{9}x_1 e^{-3t}, \end{aligned}\right\}$$

where the relation $x_0 \doteq x_1+2y_1$ must be remembered.

2. *Effect of Impressed Forces.*

We pass next to the consideration of cases in which the equations contain impressed forces: in practically all physical problems these forces are either constants or simple harmonic functions of t, with or without damping factors. Any force of this type is expressible as the sum of terms of the type $Pe^{\mu t}$, where P and μ are real or complex constants. We consider, therefore, instead of (1), the equations

$$\left.\begin{array}{l} e_{11}\, x_1+e_{12}\, x_2+\ldots+e_{1n}\, x_n = P_1\, e^{\mu t} \\ e_{21}\, x_1+e_{22}\, x_2+\ldots+e_{2n}\, x_n = P_2\, e^{\mu t} \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ e_{n1}\, x_1+e_{n2}\, x_2+\ldots+e_{nn}\, x_n = P_n\, e^{\mu t} \end{array}\right\}. \tag{16}$$

As in § 1, we substitute the complex integrals (3), and we obtain the results

$$\frac{1}{2\pi\iota}\int e^{\lambda t} p_r\, d\lambda = P_r e^{\mu t}, \tag{17}$$

which replace (4); the values of $p_1, p_2, \ldots, p_n$ being defined by (5) as before.

In order to satisfy (17), the functions $p_1, p_2, \ldots, p_n$ must have $\lambda = \mu$ as a pole, and the path must surround this point. The simplest hypothesis accordingly is to write simply,

$$p_r = P_r/(\lambda-\mu). \tag{18}$$

If the path of integration encloses only the pole $\lambda = \mu$ and none of the roots of $\Delta(\lambda) = 0$, the solution (18) simply represents the *forced oscillation.* Of course this solution may be obtained almost at a glance by more elementary methods. But another solution which is often of more physical interest is *the solution corresponding to zero initial displacements and velocities; the forces being applied at* $t = 0$.

This solution is given by the same algebraic formulæ (5) and (18): but now the path of integration must enclose all the roots of $\Delta(\lambda) = 0$, as well as $\lambda = \mu$.

For, if the path is so chosen, the initial displacements and velocities are again given by the integrals (7), and the path of integration can be taken to be a large circle in the λ-plane.* If we now consider the result of substituting from (18) in the solution (10), and apply reasoning similar to that used on p. 405 above, we see that, when $|\lambda|$ is large, the expression for ξ_r takes the form

$$\xi_r = \frac{X_r}{\lambda^3} + \frac{Y_r}{\lambda^4} + \frac{Z_r}{\lambda^5} + \dots,$$

provided that the determinant A is not zero. Thus

$$\int \xi_r d\lambda = 0, \qquad \int \lambda \xi_r d\lambda = 0,$$

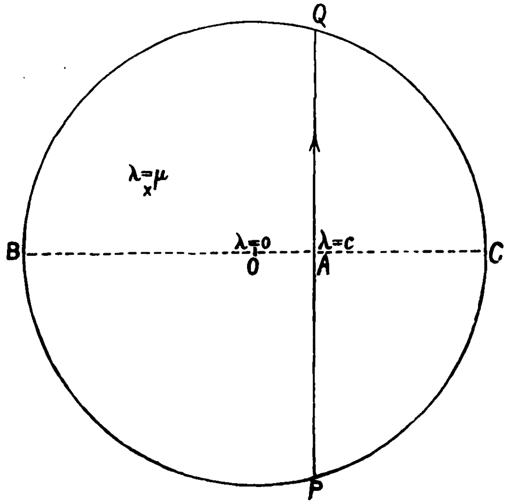
and accordingly the initial displacements and velocities, given by (7), are zero as stated.

The special case of *constant impressed forces* is of interest (see also § 4 below): the solution is then found by putting $\mu = 0$, or

$$p_r = P_r/\lambda, \tag{18a}$$

where the path of integration must now enclose $\lambda = 0$, as well as the roots of $\Delta(\lambda) = 0$.

In dealing with problems which involve impressed forces, it is often more convenient to replace the closed paths hitherto used by a straight line: this line will be parallel to the imaginary axis and on the *positive* side of the origin, so that the limits of integration can be written as $c - \iota\infty$ to $c + \iota\infty$, where c is any *positive constant*.



* Because *all* the poles of the integrands lie within this path of integration: and so the path can be extended as far out as we please, in virtue of Cauchy's theorem.

To prove that this straight path is equivalent to the closed paths, we start from a circle, with its centre at $\lambda = 0$, as in the figure; by taking a sufficiently large circle, we can ensure that all the poles to be considered* lie within the region $PAQBP$; and so we can replace the circle $PCQBP$ by the path $PAQBP$. Now *when t is positive*, it is easy to prove that any integral of the type $\int \xi_r e^{\lambda t} d\lambda$, taken along the arc QBP, will tend to zero as the radius tends to infinity.† Accordingly we are left with the limit of the path PAQ, and the solution takes the form

$$x_r = \frac{1}{2\pi\iota}\int_{c-\iota\infty}^{c+\iota\infty} e^{\lambda t}\xi_r d\lambda \quad (c > 0). \tag{3'}$$

The form (3′) can also be applied to the problem of § 1, except when $t = 0$; for, if ξ_r is of the type (8), it is easy to see that

$$\frac{1}{2\pi\iota}\int_{QBP} \xi_r d\lambda = \tfrac{1}{2}u_r,$$

and that the integral $\frac{1}{2\pi\iota}\int_{QBP} \lambda\xi_r d\lambda$ is divergent. Thus, *to get the correct values of u_r, v_r, we must not write $t = 0$ in* (3′); *but we must take the limit of* (3′) *as t tends to zero through positive values.*

By using (3′) we can extend our method, so as to include impressed forces which are not expressed explicitly in the forms assumed in (16) above. Suppose that they are given as functions of t, say $\Phi_r(t)$, and that they are applied from $t = 0$ to $t = \tau$; then Fourier's theorem enables us to write‡

$$\left.\begin{aligned} \Phi_r(t) &= \frac{1}{2\pi\iota}\int_{c-\iota\infty}^{c+\iota\infty} p_r e^{\lambda t} d\lambda \\ \text{where} \qquad p_r &= \int_0^{\tau} e^{-\lambda\theta}\Phi_r(\theta)\,d\theta \end{aligned}\right\}. \tag{19}$$

Thus the value of p_r defined by (19), used in connexion with the form (3′), enables us to solve the equations with the impressed forces $\Phi_r(t)$, by exactly the same algebraic process as in the simpler cases given in (16).

* In practically all cases of interest, the poles have real parts which are either *zero* or *negative*, corresponding to undamped or to damped simple harmonic vibrations.

† Because ξ_r is at most of order $1/\lambda^3$, when $|\lambda|$ is large.

‡ The formulæ (19) may be deduced, as a matter of simple formal transformation, from the ordinary form of Fourier's theorem. Formulæ substantially equivalent were obtained, in this way, by Riemann in his paper on the distribution of primes (*Ges. Werke*, p. 140): and the actual formulæ (19) were deduced similarly by Macdonald (these *Proceedings*, Vol. xxxv, 1902, p. 428). From the point of view of the Theory of Functions of the Complex Variable, more complete discussions have been given by Pincherle and Mellin; see, for instance, a paper by the latter in the *Math. Annalen*, Bd. 68, p. 305, where references to earlier investigations will be found.

Examples of the Effect of Impressed Forces.

To avoid unnecessary algebra, we take two examples involving only a single coordinate. In more complicated problems, the method of procedure is precisely similar.

Ex. 5.—*A dynamical system with one degree of freedom is acted on by a force having the same damping and free period as the system.*

We can represent the equation of motion by

$$\frac{d^2x}{dt^2}+2\nu\frac{dx}{dt}+(\nu^2+n^2)x = Fe^{-\nu t}\cos(nt+\omega),$$

where $2\pi/n$ is the period, and $e^{-\nu t}$ is the damping factor. The force is supposed applied at $t=0$, when the system is at rest in an undisturbed position. Then the solution is

$$x = \frac{1}{2\pi\iota}\int \xi e^{\lambda t}\,d\lambda,$$

where $$\{(\lambda+\nu)^2+n^2\}\,\xi = \frac{F}{2}\left\{\frac{e^{\iota\omega}}{\lambda+\nu-\iota n}+\frac{e^{-\iota\omega}}{\lambda+\nu+\iota n}\right\},$$

or $$\xi = \frac{F\{(\lambda+\nu)\cos\omega-n\sin\omega\}}{\{(\lambda+\nu)^2+n^2\}^2}.$$

The residues of $\xi e^{\lambda t}$ have to be calculated at the points $\lambda=-\nu\pm\iota n$, and on addition we obtain

$$x = \frac{Fe^{-\nu t}}{2n^2}\{nt\sin(nt+\omega)+\cos(nt+\omega)-\cos nt\cos\omega\}$$

$$= \frac{Fe^{-\nu t}}{2n^2}\{nt\sin(nt+\omega)-\sin\omega\sin nt\}.$$

When this formula has been obtained, it is easy to verify the result by elementary methods: but a direct determination of the arbitrary constants is somewhat tedious, although perfectly straightforward. The result for the special case $\omega=0$ was given by Bjerknes (*Annalen der Physik*, Bd. 55, 1892, p. 132).

Ex. 6.—*The system is acted on by a force having the same period; but the force is applied for a half period only.*

Here we omit the damping factors (for brevity merely), and take the equation

$$\frac{d^2x}{dt^2}+n^2x = F\sin nt,$$

which holds from $t=0$ to π/n. If we apply (19), with $\tau=\pi/n$, we find that

$$p = F\int_0^{\pi/n} e^{-\lambda\theta}\sin n\theta\,d\theta = \frac{Fn}{\lambda^2+n^2}(1+e^{-\lambda\pi/n}).$$

Thus $$\xi = \frac{Fn}{(\lambda^2+n^2)^2}(1+e^{-\lambda\pi/n}).$$

Calculating the residues of $\xi e^{\lambda t}$ at $\lambda=\pm\iota n$, we obtain the result

$$x = -\frac{F\pi}{2n^2}\cos nt \quad (t>\pi/n).$$

This conclusion is easily verified by observing that Ex. 5 (with $\nu=0$, $\omega=-\frac{1}{2}\pi$) gives the solution up to $t=\pi/n$, and that x and dx/dt are continuous at $t=\pi/n$.

3. *Comparison with the Solution in Terms of Normal Functions.*

The customary method of solving the problem proposed in § 1 above, is to use *normal functions*. That is, corresponding to each root $\lambda=a$ of the determinantal equation $\Delta(\lambda) = 0$, a solution $l_1, l_2, \ldots, l_n$ is found for the equations

$$\left.\begin{array}{l} \lambda_{11} l_1 + \lambda_{12} l_2 + \ldots + \lambda_{1n} l_n = 0 \\ \lambda_{21} l_1 + \lambda_{22} l_2 + \ldots + \lambda_{2n} l_n = 0 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \\ \lambda_{n1} l_1 + \lambda_{n2} l_2 + \ldots + \lambda_{nn} l_n = 0 \end{array}\right\}, \tag{20}$$

where

$$\lambda = a.$$

It is then evident that if the initial displacements (u) and velocities (v_r) are adjusted so that

$$\left.\begin{array}{l} \dfrac{u_1}{l_1} = \dfrac{u_2}{l_2} = \ldots = \dfrac{u_n}{l_n} = A \\ \dfrac{v_1}{l_1} = \dfrac{v_2}{l_2} = \ldots = \dfrac{v_n}{l_n} = Aa \end{array}\right\}, \tag{21}$$

and

a possible solution of the problem in § 1 is given by

$$x_1 = A l_1 e^{at}, \quad x_2 = A l_2 e^{at}, \quad \ldots, \quad x_n = A l_n e^{at}.$$

But it will be observed that this does *not* prove that no other solution exists under the given initial conditions: although it may be possible to infer, from general considerations of a physical kind, that the solution (if any) is unique.

This special type of solution gives a *normal mode of motion* (or *principal mode*) in the customary terminology.

It is then usual to assume that *every* motion possible can be obtained by the superposition of normal modes: that is, that we can satisfy the equations (1) of § 1 by taking

$$x_r = \Sigma A l_r e^{at}, \tag{22}$$

where the summation is extended to all roots a of $\Delta(\lambda) = 0$. To determine the $2n$ constants A we have the $2n$ equations

$$u_r = \Sigma A l_r, \qquad v_r = \Sigma A a l_r. \tag{23}$$

But it is by no means obvious that these $2n$ equations (23) are always algebraically capable of solution: it is conceivable that the $2n$ normal modes might not be algebraically independent.* The most direct proof

* As a matter of fact, it can be proved that they are algebraically independent when the roots of $\Delta(\lambda) = 0$ are *all different* (provided that the determinant A of § 1 is not zero). But failure may occur with equal roots: consider, for example, the equations

$$(2D^2-3)x - y = 0. \qquad x + (2D^2-1)y = 0.$$

Here the determinantal equation reduces to $(\lambda^2-1)^2 = 0$, so that $a = \pm 1$; and for all these that the constants A *can* be found is to obtain formulæ giving them explicitly in terms of the initial values, u_r, v_r. Such formulæ have been given only, as far as I know, in certain special cases; and of these, the only simple case is that of symmetry for which

$$a_{rs} = a_{sr}, \quad b_{rs} = b_{sr}, \quad c_{rs} = c_{sr},$$

so that $$\lambda_{rs} = \lambda_{sr}.$$

Under these conditions Routh* and Heaviside† have shewn that there is a conjugate relation between the normal solutions (l_r), (m_r) corresponding to any two unequal roots α, β of the determinantal equation $\Delta(\lambda) = 0$.

Following the notation used by Heaviside in the second paper quoted, the conjugate relation may be written ‡

$$T_{12} = U_{12},$$

where $$T_{12} = a_{11}(\alpha l_1)(\beta m_1) + a_{12}\{(\alpha l_1)(\beta m_2) + (\alpha l_2)(\beta m_1)\} + \ldots,$$

$$U_{12} = c_{11} l_1 m_1 + c_{12}(l_1 m_2 + l_2 m_1) + \ldots,$$

so that T_{12} is what may be called the relative kinetic energy of the two normal solutions, while U_{12} is their relative potential energy.

From this property it is easy to deduce that the value of the coefficient A is given by Heaviside's formula

$$A = \frac{T_{10} - U_{10}}{T_{11} - U_{11}}, \tag{24}$$

where T_{10}, U_{10} are constructed similarly to T_{12}, U_{12}, but with the initial velocities and displacements taking the place of those belonging to the

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roots the relations analogous to (20) reduce to $l + m = 0$. Thus the corresponding normal modes are subject to the relation $x + y = 0$, but the solution given by

$$x = te^t, \quad y = (4-t)\,e^t$$

corresponds to the initial values $x_0 = 0$, $x_1 = 1$, $y_0 = 4$, $y_1 = 3$; and clearly this solution is not expressible as a sum of normal modes.

* Routh, *Rigid Dynamics*, Vol. 2, 1892, Arts. 383, 384; Routh published his method first in 1883.

† Heaviside, *Electrical Papers*, Vol. 1, pp. 520–531; and Vol. 2, p. 202; these papers were published in 1885 and 1886 respectively. It is clear that Heaviside was unaware of Routh's work; and his results are easier to state than Routh's.

‡ When there are no frictional terms ($b_{rs} = 0$), the conjugate property takes the form

$$T_{12} = 0, \quad U_{12} = 0,$$

which is a familiar result.

second normal solution. Thus

$$\left.\begin{aligned} T_{10} &= a_{11}(al_1)\,v_1 + a_{12}\{(al_1)\,v_2 + (al_2)\,v_1\} + \dots \\ U_{10} &= c_{11}\,l_1 u_1 + c_{12}(l_1 u_2 + l_2 u_1) + \dots \\ \text{and} \qquad T_{11} &= a_{11}\,(al_1)^2 + 2a_{12}\,(al_1)(al_2) + \dots \\ U_{11} &= c_{11}\,l_1^2 + 2c_{12}\,l_1\,l_2 + \dots \end{aligned}\right\}. \tag{25}$$

We proceed now to identify (24) with our former results. As a preliminary we note that in virtue of equations (20) we can write

$$c_{11}\,l_1 + c_{12}\,l_2 + \dots + c_{1n}\,l_n$$
$$= -\alpha^2(a_{11}\,l_1 + a_{12}\,l_2 + \dots + a_{1n}\,l_n) - \alpha\,(b_{11}\,l_1 + b_{12}\,l_2 + \dots + b_{1n}\,l_n), \tag{26}$$

with similar results for the suffixes 2, 3, ..., n.

If we multiply equations (26) by $u_1, u_2, \dots, u_n$ in order, and add, we see that

$$U_{10} = -\alpha^2\,\{a_{11}\,l_1 u_1 + a_{12}(l_1 u_2 + l_2 u_1) + \dots\}$$
$$-\alpha\,\{b_{11}\,l_1 u_1 + b_{12}(l_1 u_2 + l_2 u_1) + \dots\}.$$

Thus the coefficient of l_1 in $T_{10} - U_{10}$ can be written

$$\alpha\,(a_{11}\,v_1 + a_{12}\,v_2 + \dots + a_{1n}\,v_n)$$
$$+\alpha^2(a_{11}u_1 + a_{12}u_2 + \dots + a_{1n}u_n)$$
$$+\alpha\,(b_{11}\,u_1 + b_{12}\,u_2 + \dots + b_{1n}u_n),$$

which is equal to $\alpha\bar{p}_1$, by equations (9), where, as in (15), $\bar{p}_1$ denotes the result of writing $\lambda = \alpha$ in p_1.

Consequently we can write

$$T_{10} - U_{10} = \alpha\,(l_1\,\bar{p}_1 + l_2\,\bar{p}_2 + \dots + l_n\,\bar{p}_n). \tag{27}$$

Again, if we take equations (26) and multiply them in order by $l_1, l_2, \dots, l_n$ and add, we see that

$$U_{11} = -(a_{11}\,\alpha^2 + b_{11}\,\alpha)\,l_1^2 - 2\,(a_{12}\,\alpha^2 + b_{12}\,\alpha)\,l_1\,l_2 - \dots,$$

Thus we have the result

$$T_{11} - U_{11} = \alpha\,\{(2a_{11}\,\alpha + b_{11})\,l_1^2 + 2\,(2a_{12}\,\alpha + b_{12})\,l_1\,l_2 + \dots\}. \tag{28}$$

Again using equations (20), we see that, in the notation of § 1, we can write (for any suffix r)

$$\frac{l_1}{F_{1r}(\alpha)} = \frac{l_2}{F_{2r}(\alpha)} = \dots = \frac{l_n}{F_{nr}(\alpha)}.$$

It follows that the minor $F_{rs}(a)$ is proportional to the product $l_r l_s$: and so we can write

$$F_{rs}(a) = kl_r l_s, \tag{29}$$

where k is the same for all pairs of suffixes r, s.

Now from the ordinary rule for differentiating a determinant, we see that

$$\Delta'(\lambda) = (2a_{11}\lambda + b_{11})\, F_{11}(\lambda) + 2(2a_{12}\lambda + b_{12})\, F_{12}(\lambda) + \ldots,$$

because now the determinant $\Delta(\lambda)$ is symmetrical.

Consequently, using (28) and (29), we have

$$a\Delta'(a) = k(T_{11} - U_{11}). \tag{30}$$

Also, from (27) and (29), we find that

$$kl_1(T_{10} - U_{10}) = a\,\{F_{11}(a)\,\bar{p}_1 + F_{12}(a)\,\bar{p}_2 + \ldots + F_{1n}(a)\,\bar{p}_n\}. \tag{31}$$

Combining (30) and (31), we see that Heaviside's formula (24) gives

$$Al_1 = \frac{T_{10} - U_{10}}{T_{11} - U_{11}}\, l_1 = \frac{1}{\Delta'(a)}\,\{F_{11}(a)\,\bar{p}_1 + F_{12}(a)\,\bar{p}_2 + \ldots + F_{1n}(a)\,\bar{p}_n\},$$

or, comparing with (15), we now infer that $Al_1 e^{at}$ is equal to the residue at $\lambda = a$, calculated in § 1.

In other words, *we have identified Heaviside's formula* (24) *with the solution found in* § 1. But it should be borne in mind that (unless the determination of the normal modes is extremely easy) it is usually quicker to apply the process of § 1, rather than (24).*

There is only one other class of problems for which a rule similar to (24) has been devised; this is Routh's case of a dynamical system with gyrostatic forces, but no resistances. Thus we have the relations

$$a_{rs} = a_{sr}, \qquad b_{rs} = -b_{sr}, \qquad c_{rs} = c_{sr}.$$

It is not easy to translate Routh's rule (*l.c.*, p. 415, Art. 390) into a form similar to (24); and I have not constructed a general proof to shew that this rule is included in the general process given in § 1. I have, however, verified that in a special numerical example, worked out by Routh (*l.c.*, Art. 391), the equations given for finding the coefficients can be derived from the results given by the general process of § 1. But it should be noticed that Routh's equations are left in an unsolved form; one of the equations is a quadratic, and so the process of solution would be tedious. On the other hand, § 1 gives the coefficients at once (see Ex. 7 below).

* This fact will be observed in Heaviside's later writings, where the method of § 4 is used as a rule in preference to (24).

To sum up, it has been proved that *the method of* § 1 *includes all the results which can be derived by the use of normal coordinates : and, in general, it will give these results in a more convenient form.*

Ex. 7.—*Routh's Example of a Gyrostatic System* (*l.c.*, Art. 391),

$$\left.\begin{aligned}(D^2-8)\,x + \quad \sqrt{6}\,Dy &= 0,\\ -\sqrt{6}\,Dx + (D^2+2)\,y &= 0.\end{aligned}\right\}$$

Here $$(\lambda^2-8)\,\xi + \quad \sqrt{6}\,\lambda\eta = p = \lambda x_0 + x_1 + \sqrt{6}\,y_0,$$
$$-\sqrt{6}\,\lambda\xi + (\lambda^2+2)\,\eta = q = -\sqrt{6}\,x_0 + \lambda y_0 + y_1.$$

Thus $$(\lambda^4-16)\,\xi = (\lambda^2+2)\,p - \sqrt{6}\,\lambda q = (\lambda^2+2)(\lambda x_0+x_1) + 6\lambda x_0 + \sqrt{6}\,(2y_0 - \lambda y_1),$$

with a similar formula for η.

The results given by Routh correspond to finding the residues of $\xi e^{\lambda t}$ for $\lambda = -2$ and for $\lambda = \pm 2\iota$. Take first the former, which gives

$$-\tfrac{1}{32}e^{-2t}\left\{6\,(-2x_0+x_1) - 12x_0 + 2\sqrt{6}\,(y_0+y_1)\right\} = -\tfrac{1}{16}e^{-2t}\left\{-12x_0 + 3x_1 + \sqrt{6}\,(y_0+y_1)\right\},$$

agreeing with one of Routh's results.

With regard to the residues for $\pm 2\iota$, suppose that they are expressed in the forms

$$\tfrac{1}{2}\,(X_1 - \iota X_2)\,e^{2\iota t} + \tfrac{1}{2}\,(X_1 + \iota X_2)\,e^{-2\iota t}.$$

Then their sum is $X_1 \cos 2t + X_2 \sin 2t$, which is Routh's form.

Now $\frac{1}{2}(X_1 - \iota X_2)$ is the residue of ξ at $\lambda = +2\iota$, and so we find

$$-16\iota\,(X_1 - \iota X_2) = -2\,(2\iota x_0 + x_1) + 12\iota x_0 + 2\sqrt{6}\,(y_0 - \iota y_1),$$

or $$8\,(X_1 - \iota X_2) = -4x_0 - \iota x_1 + \sqrt{6}\,(\iota y_0 + y_1).$$

The last equation gives X_1 and X_2 by direct inspection.

Routh's equations can be derived by multiplying up by $X_1 + \iota X_2$, and then equating real and imaginary parts; clearly these equations must (in the long run) lead to the same results. But it is equally certain that the labour involved in deriving X_1, X_2 from Routh's equations is very great.

The final formula for x can be written

$$\begin{aligned}&\tfrac{1}{8}\,(-4x_0 + \sqrt{6}\,y_1)\cos 2t \;+ \tfrac{1}{8}\,(\;x_1 - \sqrt{6}\,y_0)\sin 2t\\ +&\tfrac{1}{8}\,(\;12x_0 - \sqrt{6}\,y_1)\cosh 2t + \tfrac{1}{8}\,(3x_1 + \sqrt{6}\,y_0)\sinh 2t.\end{aligned}$$

4. *Comparison with Heaviside's Method of Resistance Operators.*

Heaviside's method is specially adapted for handling problems in which the impressed forces are steady, and are applied at $t = 0$, to a system which is then entirely free from disturbance. Heaviside has used his method particularly in the theory of currents set up in a network of linear conductors* (allowing for inductances and capacities, as well as for resistances). The original equations of the network can then be reduced to a system of the type (16), given in § 2 above, with $\mu = 0$; and with the special feature of symmetry. The coefficients a_{rs} involve

* See, in particular, *Electrical Papers*, Vol. 1, p. 412, and Vol. 2, p. 259.

the inductances, b_{rs} the resistances, and c_{rs} the reciprocals of capacities of condensers. Heaviside replaces the operator $D(=d/dt)$ by a symbol p, which is regarded as subject to the ordinary laws of algebra: and he then solves the resulting equations, as a system of linear algebraic equations in x_r. Following the notation adopted in § 1, we see that the result so obtained is*

$$\Delta(p)\,x_r = F_{r1}(p)P_1+F_{r2}(p)P_2+\ldots+F_{rn}(p)\,P_n;$$

or, supposing that one force only (P_s) is applied, we can write our result as

$$\Delta(p)\,x_r = F_{rs}(p)\,P_s. \tag{32}$$

Up to this stage the work is much the same as may be found in almost any elementary discussion of simultaneous linear differential equations; but we now encounter the essential novelty of Heaviside's process, which lies in the interpretation of the symbolic equation (32), or rather of the algebraically equivalent one

$$x_r = \frac{F_{rs}(p)}{\Delta(p)}P_s. \tag{33}$$

The result of Heaviside's interpretation is

$$x_r = P_s\left\{\frac{F_{rs}(0)}{\Delta(0)}+\Sigma_a\,\frac{F_{rs}(a)}{a\Delta'(a)}e^{at}\right\}, \tag{34}$$

where a is any root of $\Delta(p)=0$, and the summation extends to all such roots. The investigations leading up to the formula (34) are most instructive and will repay careful study:† but I am not sure that they have been sufficiently appreciated in the past. It is almost certain that few readers have fully grasped the complete and general character of the solution: and it is for this reason that I wish to call attention to it here.

To compare the results of § 2 with those of Heaviside's method, we note in the first place, that in (18a) all the P's are zero except P_s: and accordingly equation (10) gives

$$\xi_r = \frac{F_{rs}(\lambda)}{\Delta(\lambda)}\,\frac{P_s}{\lambda}. \tag{35}$$

Then, using the complex integrals of §§ 1, 2, the value of x_r is given by

$$x_r = \frac{1}{2\pi\iota}\int\frac{F_{rs}(\lambda)P_s}{\lambda\Delta(\lambda)}\,e^{\lambda t}\,d\lambda, \tag{36}$$

* Heaviside, *Electrical Papers*, Vol. 2, p. 259; his symbol $F(p)$ corresponds to $\Delta(p)$, and $f_{rs}(p)$ to $F_{rs}(p)$.

† *Ibid.*, p. 226 (where the method in the footnote supersedes the text); and p. 373.

where the path is to enclose $\lambda = 0$ and all points $\lambda = \alpha$, where α is a root of $\Delta(\lambda) = 0$. It will be remembered that (as proved in § 2) the integral (36) then solves the problem for zero initial disturbances in *all* the coordinates $x_1, x_2, \ldots, x_n$.

On forming the sum of the residues of (36), it will be seen that the result agrees precisely with Heaviside's formula (34); and accordingly we have now provided an independent proof of the latter formula.

Ex. 8.—To discuss the effect of switching a steady electromotive force E into a primary coil in the presence of a secondary coil.

This is a familiar problem, but gives a simple illustration of Heaviside's method: to shorten the algebra, suppose the coils equal in all respects, having inductances L, and resistances R; and let their mutual inductance be M.

Then, if x, y denote the currents, we have the equations

$$L\frac{dx}{dt} + M\frac{dy}{dt} + Rx = E, \qquad M\frac{dx}{dt} + L\frac{dy}{dt} + Ry = 0.$$

According to Heaviside's process we take the auxiliary equations

$$\left.\begin{aligned}(Lp+R)\,x + Mpy &= E,\\ Mpx + (Lp+R)\,y &= 0.\end{aligned}\right\}.$$

Hence
$$x = \frac{(Lp+R)\,E}{(Lp+R)^2 - M^2p^2}, \qquad y = \frac{-MpE}{(Lp+R)^2 - M^2p^2}.$$

Let α, β denote the roots of the denominator, so that

$$\alpha = -R/(L+M), \qquad \beta = -R/(L-M).$$

Then (34) gives the results

$$x = \frac{E}{R} + \frac{E\,(L\alpha+R)\,e^{\alpha t}}{\alpha\,(L^2-M^2)(\alpha-\beta)} + \frac{E\,(L\beta+R)\,e^{\beta t}}{\beta\,(L^2-M^2)(\beta-\alpha)},$$

$$y = 0 - \frac{EM\,e^{\alpha t}}{(L^2-M^2)(\alpha-\beta)} - \frac{EM\,e_{\beta t}}{(L^2-M^2)(\beta-\alpha)},$$

which simplify down to

$$x = (E/R)\left\{1-\tfrac{1}{2}(e^{\alpha t}+e^{\beta t})\right\}, \qquad y = -\tfrac{1}{2}(E/R)(e^{\alpha t}-e^{\beta t}).$$

5. *Continuous Systems.*

When the equations of motion of a vibrating continuous system are formed, the equations corresponding to those of § 1 above will usually be differential equations (one or more in number) combined with certain boundary conditions. There is usually no particular difficulty in applying the same rules as have been given in §§ 1, 2 to such systems.

For instance, suppose that the differential equation takes the form

$$\rho\frac{\partial^2 u}{\partial t^2} + \sigma\frac{\partial u}{\partial t} - \Delta u = 0, \tag{37}$$

where Δ denotes a differential operator (with respect to the coordinates employed), and ρ, σ will in general be functions of the coordinates appearing in Δ.

Then our general process suggests the following method:—

Write $$u = \frac{1}{2\pi\iota}\int ve^{\lambda t}d\lambda,$$

and then the equation for v will be

$$(\rho\lambda^2+\sigma\lambda)\,v-\Delta v = (\rho\lambda+\sigma)\,u_0+\rho u_1, \tag{38}$$

where u_0 and u_1 are the initial values of u and $\partial u/\partial t$.

The boundary conditions may or may not involve differential coefficients with respect to the time: when they do, care must be taken to modify them (by introducing terms containing u_0 and u_1) in the same way as (38) is a modified form of (37).* This point may easily be overlooked; but unless the appropriate terms are inserted, errors will be found in the final solution which may not be easy to detect.

The determination of the path of integration is perhaps less obvious; with a continuous system, the number of poles of the integrand v will, as a rule, be infinite; and these poles usually tend towards infinity (in the λ-plane) in two directions. Very often these directions correspond to the two ends of the imaginary axis. Further, in all problems which have any physical interest, the poles are distributed either *on* the imaginary axis or on the negative side of that axis. The former arrangement represents undamped simple harmonic normal modes of oscillation: and the latter corresponds to damped oscillations.

It is evident, therefore, that the closed paths of integration used in §§ 1, 2 must now be replaced by an unclosed path. But we have seen in § 2 that the path from $c-\iota\infty$ to $c+\iota\infty$ (see Fig., p. 411) is equivalent to the paths of §§ 1, 2, and so will be the natural path to adopt here. For, just as in § 2, the poles of v all lie on the negative side of the path; and this is the property corresponding to the statement of § 1, that all the poles are to be inside the closed path. Accordingly the formula for u will be

$$u = \frac{1}{2\pi\iota}\int_{c-\iota\infty}^{c+\iota\infty} e^{\lambda t}v\,d\lambda \quad (c>0), \tag{39}$$

where v is obtained from (38) and from the boundary conditions.

It will be observed that the foregoing does not profess to give a complete proof of (39): all that we have done is to establish an analogy be-

* The reason is, of course, that *the boundary conditions really form part of the equations of motion*: and they must accordingly be put on the same footing as the other equations.

tween (39) and the formulæ which we have proved for discrete systems. Anything approaching a complete proof for a continuous system would necessarily be very long, on account of the great variety of geometrical figures to be considered: and it would further require very careful handling as a matter of pure mathematics.

Fortunately, in the problems involving only one geometrical coordinate, we can appeal to the results of a paper by Prof. A. C. Dixon;* we shall trace the connexion between his paper and the present method in § 8 below. His results justify the use of our method in almost all the problems which are capable of easy solution: and I feel no doubt that, in other problems, the final justification of our process will be found fairly simple when the solution v can be actually worked out.

We have spoken so far of the determination of u in terms of u_0 and u_1: but it is easy to see that a similar modification of the method of § 2 will enable us to deal with problems involving impressed forces. Thus, if the given equation is

$$\rho \frac{\partial^2 u}{\partial t^2} + \sigma \frac{\partial u}{\partial t} - \Delta u = P e^{\mu t}, \tag{40}$$

the auxiliary equation for v becomes

$$(\rho\lambda^2 + \sigma\lambda)\, v - \Delta v = P/(\lambda - \mu). \tag{41}$$

The form (41) corresponds to the case in which the initial disturbance is zero, and the impressed force is applied at $t = 0$.

When the integral (39) has been determined, the expression of u by means of a sum of residues will lead *to the expansion of u in a series of normal functions*.

To illustrate the process, three examples are worked out below, taken from various problems in the Theory of Sound.

Ex. 9.—**A uniform string is plucked aside at its centre and let go from rest, with the ends fixed.**

If we choose the unit of velocity to be the velocity of transverse vibrations (u), the differential equation for u is

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0.$$

Thus, from $x = 0$ to $x = \tfrac{1}{2}l$, we have the auxiliary equation for v,

$$\lambda^2 v - \frac{\partial^2 v}{\partial x^2} = \lambda k x,$$

because $u_0 = kx$, $u_1 = 0$; and the boundary conditions are

$$v = 0 \quad \text{at} \quad x = 0, \qquad \frac{\partial v}{\partial x} = 0 \quad \text{at} \quad x = \tfrac{1}{2}l.$$

* These *Proceedings*, Ser. 2, Vol. 3, 1905, p. 83.

It is easily verified that the solution v (from $x = 0$ to $x = \frac{1}{2}l$) is given by

$$v = \frac{kx}{\lambda} - \frac{k}{\lambda^2}\frac{\sinh \lambda x}{\cosh \frac{1}{2}\lambda l}.$$

In order to obtain a series of normal functions for u, we note that $\lambda = 0$ is not a pole of v: and that the poles are given by $\lambda l = (2n+1)\pi\iota$, where n is an integer, positive or negative. The corresponding residue of $ve^{\lambda t}$ is

$$(-1)^n \frac{kl^2}{(2n+1)^2\pi^2 . \frac{1}{2}l} \sin\left\{(2n+1)\frac{\pi x}{l}\right\} \exp\left\{(2n+1)\frac{\pi\iota t}{l}\right\},$$

and so, on combining corresponding positive and negative terms, we find that

$$u = \frac{4kl}{\pi^2}\sum_0^\infty \frac{(-1)^n}{(2n+1)^2} \sin\left\{(2n+1)\frac{\pi x}{l}\right\} \cos\left\{(2n+1)\frac{\pi t}{l}\right\},$$

which is the formula derived by applying Fourier's theorem.*

For another method of treatment see below (§ 6, p. 429).

The problem of a string plucked at any point offers no fresh difficulty in calculation; but the results are rather longer to state, because two formulæ are needed for v, one for each of the two parts into which the string is divided by the plucking.

Ex. 10.—A pipe, open at one end, and closed at the other, is suddenly brought to rest at time $t = 0$, after having been for some time in motion, with uniform velocity u_1 parallel to the length of the pipe.†

Let u denote the displacement along the length of the tube; then taking the velocity of sound as unity (as in Ex. 9), we have

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0,$$

subject to $u = 0$ at $x = 0$, $\partial u/\partial x = 0$ at $x = l$. Thus the equation for v is

$$\lambda^2 v - \frac{\partial^2 v}{\partial x^2} = u_1,$$

because $u_0 = 0$; and so $$v = \frac{u_1}{\lambda^2}\left\{1 - \frac{\cosh \lambda(l-x)}{\cosh \lambda l}\right\}.$$

As in Ex. 9, $\lambda = 0$ is not a pole; the poles are given by $\lambda l = \pm(n+\frac{1}{2})\pi\iota$, or say $\lambda = \pm k\iota$, if $k = (n+\frac{1}{2})\pi/l$. The residue at the positive pole of $ve^{\lambda t}$ is then found to be

$$\frac{u_1}{\iota k^2 l}\sin(kx)\, e^{\iota kt}.$$

Combining the positive and negative poles, and summing, we find the formula

$$u = \frac{2u_1}{l}\sum \frac{1}{k^2}\sin(kx)\sin(kt).$$

To identify this with Lord Rayleigh's result for the velocity-potential ϕ, we note that

$$\frac{\partial u}{\partial t} = \frac{\partial \phi}{\partial x}.$$

---

* Rayleigh, *Theory of Sound*, Vol. 1, § 127; to get the required formula, put $b = \frac{1}{2}l$, and $\gamma = kb$.

† *Ibid.*, Vol. 2, Art. 258.

Ex. 11.—As an example of a slightly more difficult type, suppose that a uniform force P per unit area is applied to the surface of a uniform circular membrane, which is initially undisturbed.

The differential equation for the displacement u is then*

$$\rho\left\{\frac{\partial^2 u}{\partial t^2}-c^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)\right\}=P,$$

where r is the distance from the centre. Thus the equation for v becomes

$$\rho\left\{\lambda^2 v-c^2\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}\right)\right\}=P/\lambda.$$

Hence $$v=\frac{P}{\rho\lambda^3}\left\{1-\frac{I_0(\lambda r/c)}{I_0(\lambda a/c)}\right\},$$

because $v=0$ at $r=a$. Here I_0 is the modified Bessel function, defined by

$$I_0(z)=1+\frac{z^2}{2^2}+\frac{z^4}{2^2.4^2}+\frac{z^6}{2^2.4^2.6^2}+\dots.$$

The residue of $ve^{\lambda t}$ at $\lambda=0$ is thus

$$u_0=\frac{P}{\rho}\left(\frac{a^2}{4c^2}-\frac{r^2}{4c^2}\right),$$

which represents the statical displacement produced by the application of the force P.

The other poles of v are given by the roots of $I_0(\lambda a/c)=0$; these roots are all purely imaginary, and we can write them in the form

$$\lambda a/c=\pm\omega_n\iota,$$

where ω_n is the nth root of $$J_0(\omega)=0.$$

The corresponding residue of $ve^{\lambda t}$ is then

$$-\frac{P}{\rho(\omega c/a)^3}\,\frac{J_0(\omega r/a)}{(a/c)\,J_1(\omega)}\,e^{\iota\omega ct/a}.$$

And so, combining the positive and negative poles, we find

$$-\frac{2Pa^2}{\rho c^2\omega^3}\,\frac{J_0(\omega r/a)}{J_1(\omega)}\cos(\omega ct/a).$$

Accordingly the complete displacement produced is

$$u=\frac{P}{\rho c^2}\left\{\tfrac{1}{4}(a^2-r^2)-\Sigma\,\frac{2a^2}{\omega^3}\,\frac{J_0(\omega r/a)}{J_1(\omega)}\cos(\omega ct/a)\right\}.$$

This result agrees with one given by Lord Rayleigh† for the associated problem of the membrane held at rest in the statical position $u=u_0$, and then released. Thus Lord Rayleigh's formula will give the value of u_0-u, in the present problem.

To identify the formulæ, we note that in Lord Rayleigh's notation

$$p_{m0}=k_{m0}c=\omega c/a$$

in the above formula. Thus $$k_{m0}p_{m0}^2\,\rho a=\omega^3\rho c^2/a^2;$$

also we have $$J_0'(\omega)=-J_1(\omega),$$

and the identification becomes evident.

* Rayleigh, *Theory of Sound*, Vol. 1, Art. 200: the vibration is clearly symmetrical about the centre, and so is independent of θ.

† *Ibid.*, Art. 204, (9)–(11).

It may seem, at a casual glance, that the present solution is longer than Lord Rayleigh's (who uses the method of normal coordinates): but to obtain a fair comparison of the two, account must be taken of the preliminary analysis, carried out by Lord Rayleigh in Art. 203. In the present solution, no such analysis is needed.

6. *Alternative method for interpreting the formulæ obtained as in* § 5.

It is often possible to replace the expression for v (found by the method of § 5) by a series of terms such as

$$e^{-x\lambda}/\lambda^{n+1},$$

where x is real and n is an integer.

The corresponding contribution to the value of the integral

$$\frac{1}{2\pi\iota}\int ve^{\lambda t}\,d\lambda$$

is then
$$\frac{(t-x)^n}{n!} \quad (t > x),$$
or
$$0 \quad (t < x).$$

To prove the accuracy of these statements we need only notice that when $t > x$ (so that the index in the exponential is a *positive* multiple of λ), the path PAQ of the diagram on p. 411 can be completed by the additional piece QBP, which then contributes zero. Thus the value of the integral along PAQ is equal to the residue at $\lambda = 0$, which may be calculated in the usual way and gives the result just stated.

But, when $t < x$, we may add on the additional piece QCP; and since no pole of the integrand is contained within $PAQCP$, the result is zero.

When constructing the series to represent a given function, care must be taken to remember that *the real part of* λ *is positive* on the fundamental path PAQ; and this enables us to settle questions of convergence without trouble, as a rule. For example,

$$\frac{1}{\sinh\lambda} = \frac{2}{e^{\lambda}-e^{-\lambda}} = 2e^{-\lambda}(1-e^{-2\lambda})^{-1},$$

and this function can accordingly be expanded as

$$2e^{-\lambda}(1+e^{-2\lambda}+e^{-4\lambda}+\ldots),$$

since $|e^{-\lambda}|$ is less than unity.

In order to bring out the points of this method of interpretation, we shall work out in some detail the solution of a known problem (Ex. 12) in the theory of extensional vibrations of a thin rod. The problems already treated in § 5 (Exs. 9, 10) will then be solved, more briefly, below; it will be noticed that in these cases, the results obtained by the present process are equivalent to those derived by the familiar process of superposing positive and negative waves.*

Ex. 12.—*Bar with one end fixed, struck at the other end by a particle moving in the direction of the length of the bar.*†

Without loss of generality we may suppose the velocity of propagation of extensional waves in the bar to be the unit of velocity; and the length of the bar to be the unit of length: thus the unit of time is the interval occupied by a wave in travelling along the bar.

Then if u denotes the displacement along the bar, and if y is the displacement of the particle, the equations of motion will reduce to‡

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad \frac{\partial^2 y}{\partial t^2} = -k\left(\frac{\partial u}{\partial x}\right)_{x=1},$$

and the end-conditions become

$$u = 0 \text{ at } x = 0, \qquad u = y \text{ at } x = 1.$$

The first steps in the solution are precisely similar to those adopted in Exs. 9–11 in § 5, and need only brief indications. We assume the solutions

$$u = \frac{1}{2\pi\iota}\int v e^{\lambda t}\, d\lambda, \qquad y = \frac{1}{2\pi\iota}\int \eta e^{\lambda t}\, d\lambda,$$

where v, η are functions of λ to be determined. The equations for v, η are found to be

$$\lambda^2 v - \frac{\partial^2 v}{\partial x^2} = 0, \qquad \lambda^2\eta + k\left(\frac{\partial v}{\partial x}\right)_{x=1} = -V,$$

because the initial conditions are

$$u = 0, \quad \frac{\partial u}{\partial t} = 0, \quad \text{and} \quad y = 0, \quad \frac{\partial y}{\partial t} = -V.$$

Then using the end conditions

$$v = 0 \quad \text{at} \quad x = 0, \qquad v = \eta \quad \text{at} \quad x = 1,$$

we have $$v = \eta\,(\sinh \lambda x)/(\sinh \lambda),$$

which leads to the result $$(\lambda^2 + k\lambda \coth \lambda)\,\eta = -V.$$

Thus the values of v, η are now completely known; and we proceed to obtain the interpreta-

* Rayleigh, *Theory of Sound*, Vol. 1, Art. 145; of course this simple process cannot easily be extended to harder problems, such as Ex. 12.

† The problem was considered first by Boussinesq and Saint-Venant; see Love's *Elasticity* (Art. 281, 2nd edition; Arts. 275–277, 1st edition).

‡ These equations follow at once from those given by Love, on writing $a = 1$, $l = 1$; it should be noticed that here k denotes the ratio of the mass of the bar to the mass of the particle, so that $k = 1/m$ in Love's notation.

tion of u by means of the method described above. For the sake of comparison with the results given by Love, we shall write u in the form

$$u = \frac{1}{2\pi\iota}\int \eta \frac{\sinh \lambda x}{\sinh \lambda} e^{\lambda t}\,d\lambda = \frac{1}{2\pi\iota}\int \frac{\eta}{2\sinh\lambda}\left\{e^{\lambda(t+x)} - e^{\lambda(t-x)}\right\}d\lambda.$$

Thus we can express our solution by the equation

$$u = f(t-x) - f(t+x),$$

where $$f(\theta) = \frac{1}{2\pi\iota}\int \frac{-\eta e^{\lambda\theta}\,d\lambda}{2\sinh\lambda} = \frac{1}{2\pi\iota}\int \frac{Ve^{\lambda\theta}\,d\lambda}{2\lambda(\lambda\sinh\lambda + k\cosh\lambda)}.$$

In accordance with our general process we now proceed to expand the integrand according to powers of $e^{-\lambda}$, using the identity*

$$\frac{1}{2(\lambda\sinh\lambda + k\cosh\lambda)} = \frac{e^{-\lambda}}{(\lambda+k)-(\lambda-k)e^{-2\lambda}}$$
$$= \frac{e^{-\lambda}}{\lambda+k}\left\{1 + \frac{\lambda-k}{\lambda+k}e^{-2\lambda} + \left(\frac{\lambda-k}{\lambda+k}\right)^2 e^{-4\lambda} + \dots\right\}.$$

Thus our formula for $f(\theta)$ takes the shape

$$f(\theta) = \frac{1}{2\pi\iota}\int \frac{Ve^{\lambda(\theta-1)}d\lambda}{\lambda(\lambda+k)}\left\{1 + \frac{\lambda-k}{\lambda+k}e^{-2\lambda} + \left(\frac{\lambda-k}{\lambda+k}\right)^2 e^{-4\lambda} + \dots\right\}.$$

Now, as explained in our general remarks on p. 425, it is only necessary to retain the exponentials with *positive* indices: those with negative indices simply contribute zero to the result.

It is therefore clear that

$$f(\theta) = 0 \quad (\theta < 1),$$

and that $$f(\theta) = \frac{1}{2\pi\iota}\int \frac{Ve^{\lambda(\theta-1)}\,d\lambda}{\lambda(\lambda+k)} \quad (1 < \theta < 3).$$

Now, calculating the residues at $\lambda = 0$ and $\lambda = -k$, we see at once that

$$f(\theta) = (V/k)\left\{1 - e^{-k(\theta-1)}\right\} \quad (1 < \theta < 3).$$

Similarly, when $3 < \theta < 5$, we take

$$f(\theta) = \frac{1}{2\pi\iota}\int \frac{Ve^{\lambda(\theta-1)}\,d\lambda}{\lambda(\lambda+k)}\left(1 + \frac{\lambda-k}{\lambda+k}e^{-2\lambda}\right),$$

---

* Convergence is assured because the real part of λ is positive, and so

$$|e^{-2\lambda}| < 1, \quad |\lambda - k| < |\lambda + k|.$$

and here the residue for $\lambda = 0$ is zero, and so, calculating the residue for $\lambda = -k$, we find

$$f(\theta) = -(V/k)\, e^{-k(\theta-1)} + (V/k)\,\{1+2k(\theta-3)\}\, e^{-k(\theta-3)} \quad (3 < \theta < 5).$$

As a last example, suppose that $5 < \theta < 7$, so that

$$f(\theta) = \frac{1}{2\pi\iota} \int \frac{V e^{\lambda(\theta-1)}\, d\lambda}{\lambda(\lambda+k)} \left\{ 1 + \frac{\lambda-k}{\lambda+k} e^{-2\lambda} + \left(\frac{\lambda-k}{\lambda+k}\right)^2 e^{-4\lambda} \right\}.$$

Then, on calculating the residues for $\lambda = 0$ and $\lambda = -k$, we find the result

$$\begin{aligned} f(\theta) = \ & (V/k)\,\{1-e^{-k(\theta-1)}\} + (V/k)\,\{1+2k(\theta-3)\}\, e^{-k(\theta-3)} \\ & -(V/k)\,\{1+2k^2(\theta-5)^2\}\, e^{-k(\theta-5)} \qquad (5 < \theta < 7). \end{aligned}$$

These three formulæ will be seen to agree with those given by Love.*

It is sometimes of special interest, in problems of this type, to determine the displacement of one particular point: for instance, here we might wish to find the value of y (the displacement of the particle). When we have already found the forms of $f(\theta)$, this displacement is given by

$$y = f(t-1) - f(t+1).$$

But supposing that $f(\theta)$ has not been already worked out, we can find y by expanding η in the same way, and writing

$$\begin{aligned} -\eta &= \frac{V \sinh \lambda}{\lambda(\lambda \sinh \lambda + k \cosh \lambda)} \\ &= \frac{V(1-e^{-2\lambda})}{\lambda\,\{(\lambda+k)-(\lambda-k)\, e^{-2\lambda}\}} \\ &= \frac{V}{\lambda(\lambda+k)} \left\{ 1 - \frac{2k}{\lambda+k} e^{-2\lambda} - \frac{2k(\lambda-k)}{(\lambda+k)^2} e^{-4\lambda} - \dots \right\}. \end{aligned}$$

From this series the values of y can be determined in exactly the way in which we found the values of $f(\theta)$: but instead of writing out the results, we shall deal similarly with the problem of finding $(\partial u/\partial x)_{x=1}$, from which the duration of the impact may be found.

It will be seen that

$$\left(\frac{\partial u}{\partial x}\right)_{x=1} = \frac{1}{2\pi\iota} \int \lambda \eta \coth \lambda\, e^{\lambda t}\, d\lambda,$$

* See the table on p. 414 of the second edition: the corresponding table in the first edition is arranged a little differently, and gives the values of $f(1+\theta)$. It should be remembered that, in the units used here, $a = 1$, $l = 1$, and that $k = 1/m$.

and so we write

$$-\lambda\eta\coth\lambda = \frac{V\cosh\lambda}{\lambda\sinh\lambda+k\cosh\lambda}$$
$$= \frac{V}{\lambda+k}\left\{1+\frac{2\lambda}{\lambda+k}e^{-2\lambda}+\frac{2\lambda(\lambda-k)}{(\lambda+k)^2}e^{-4\lambda}+\dots\right\}.$$

We have now to write down the residues for $\lambda = -k$, and it will be found that

$$-\left(\frac{\partial u}{\partial x}\right)_{x=1} = Ve^{-kt} \qquad (0 < t < 2),$$
$$= Ve^{-kt}+2V\{1-k(t-2)\}e^{-k(t-2)} \qquad (2 < t < 4),$$
$$= Ve^{-kt}+2V\{1-k(t-2)\}e^{-k(t-2)}$$
$$+2V\{1-3k(t-4)+k^2(t-4)^2\}e^{-k(t-4)} \quad (4 < t < 6).$$

The first and second of these results agree with those given by Love; the third is not actually given by Love, but can be verified from results given there.

Before leaving this problem it may be useful to remark that any of the results can be expanded in terms of normal functions, if desired; suppose, for instance, that we want a series for y. The poles of η are given by $\lambda = \pm\iota\nu$, where ν is a positive root of

$$\nu-k\cot\nu = 0.$$

The residue of $\eta e^{\lambda t}$ at $\lambda = \iota\nu$ is then seen to be

$$-Ve^{\iota\nu t}/\{\iota\nu(1+k\,\mathrm{cosec}^2\nu)\};$$

and so we find that $\quad y = -2V\Sigma\dfrac{\sin\nu t}{\nu(1+k\,\mathrm{cosec}^2\nu)},$

where $\qquad \nu-k\cot\nu = 0.$

This result may be confirmed by Lord Rayleigh's process.*

We pass now to a brief consideration of Exs. 9, 10 of § 5.

Ex. 9 (*bis*).—It was proved above that, when $0 \leqslant x \leqslant \frac{1}{2}l$,

$$v = \frac{kx}{\lambda}-\frac{k}{\lambda^2}\frac{\sinh\lambda x}{\cosh\frac{1}{2}\lambda l} = \frac{kx}{\lambda}-\frac{k}{\lambda^2}\{e^{-(\frac{1}{2}l-x)\lambda}-e^{-(\frac{1}{2}l+x)\lambda}\}(1-e^{-l\lambda}+e^{-2l\lambda}-\dots),$$

the convergence being assured because $|e^{-l\lambda}| < 1$, on account of the real part of λ being positive.

In applying the formulæ given above to find the residues of $ve^{\lambda t}$, it may be noticed that each successive result is derived from the preceding by adding on an extra term—corresponding to a new index of the exponential which becomes positive as t increases.

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* *Theory of Sound*, Vol. 1, Arts. 93, 94.

Thus we find

$$
\begin{array}{lll}
\text{(i)} & u = kx & (0 < t < \tfrac{1}{2}l - x). \\
\text{(ii)} & u = kx \quad -k(t-\tfrac{1}{2}l+x) = k(\tfrac{1}{2}l-t) & (\tfrac{1}{2}l - x < t < \tfrac{1}{2}l + x). \\
\text{(iii)} & u = k(\tfrac{1}{2}l-t)+k(t-\tfrac{1}{2}l-x) = -kx & (\tfrac{1}{2}l + x < t < \tfrac{3}{2}l - x). \\
\text{(iv)} & u = -kx \quad +k(t-\tfrac{3}{2}l+x) = k(t-\tfrac{3}{2}l) & (\tfrac{3}{2}l - x < t < \tfrac{3}{2}l + x). \\
\text{(v)} & u = k(t-\tfrac{3}{2}l)-k(t-\tfrac{3}{2}l-x) = kx & (\tfrac{3}{2}l + x < t < \tfrac{5}{2}l - x),
\end{array}
$$

and so on.

These results are calculated for a given value of x, corresponding to different values of t: but it is rather easier to appreciate the motion of the string by classifying the results according to the values of x, for a given t. We then arrive at the formulæ :—

$$
\begin{array}{llll}
(a) & 0 < t < \tfrac{1}{2}l, & u = kx, & \text{if } x < \tfrac{1}{2}l - t, \\
 & & u = k(\tfrac{1}{2}l - t), & \text{if } x > \tfrac{1}{2}l - t. \\
(b) & \tfrac{1}{2}l < t < l, & u = -kx, & \text{if } x < t - \tfrac{1}{2}l, \\
 & & u = -k(t-\tfrac{1}{2}l), & \text{if } x > t - \tfrac{1}{2}l. \\
(c) & l < t < \tfrac{3}{2}l, & u = -kx, & \text{if } x < \tfrac{3}{2}l - t, \\
 & & u = -k(\tfrac{3}{2}l - t), & \text{if } x > \tfrac{3}{2}l - t. \\
(d) & \tfrac{3}{2}l < t < 2l, & u = kx, & \text{if } x < t - \tfrac{3}{2}l, \\
 & & u = k(t-\tfrac{3}{2}l), & \text{if } x > t - \tfrac{3}{2}l. \\
(e) & 2l < t < \tfrac{5}{2}l, & u = kx, & \text{if } x < \tfrac{5}{2}l - t, \\
 & & u = k(\tfrac{5}{2}l - t), & \text{if } x > \tfrac{5}{2}l - t,
\end{array}
$$

and so on; where it must be borne in mind that $0 \leqslant x \leqslant \tfrac{1}{2}l$. It is not necessary to write out the formulæ which refer to the second half of the string; for the shape of the string is always symmetrical about its centre.

From these formulæ it is easy to construct a diagram to shew the motion of the string: the string in fact consists always of three straight pieces.*

Ex. 10 (*bis*).—We found that

$$v = \frac{u_1}{\lambda^2}\left\{1 - \frac{\cosh\lambda(l-r)}{\cosh\lambda l}\right\} = \frac{u_1}{\lambda^2}\left[1 - \left\{e^{-x} + e^{-(2l-x)}\right\}(1 - e^{-2l\lambda} + e^{-4l\lambda} - \dots)\right].$$

Proceeding, as before, we find for the displacements

$$
\begin{array}{lll}
\text{(i)} & u_1 t & (0 < t < x). \\
\text{(ii)} & u_1 t \quad -u_1(t-x) = u_1 x & (x < t < 2l - x). \\
\text{(iii)} & u_1 x \quad -u_1(t-2l+x) = u_1(2l-t) & (2l - x < t < 2l + x). \\
\text{(iv)} & u_1(2l-t) + u_1(t-2l-x) = -u_1 x & (2l + x < t < 4l - x). \\
\text{(v)} & -u_1 x \quad +u_1(t-4l+x) = u_1(t-4l) & (4l - x < t < 4l + x),
\end{array}
$$

and so on.

These conclusions may be summed up as follows:—A wave-front may be regarded as travelling to and fro, along the tube. The velocity of the air is zero in the part between the fixed end and the wave-front; and in the part between the wave-front and the open end, the velocity is $\pm u_1$. The latter changes sign when the wave-front is reflected at the open end.

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* See Helmholtz's diagram for the plucked string: Rayleigh's *Theory of Sound*, Vol. 1, Art. 146. In the special problem considered here, however, the central portion is always parallel to the line joining the ends.

7. *Waves with an Advancing Wave-Front.*

As another group of problems which can be readily solved by the method of § 6, we may refer to those given by Prof. Love* on wave-systems with an advancing wave-front. To illustrate the process we shall work out the simplest case: some of the other problems are rather more interesting, but the algebra involved is heavier.

We consider, then, the problem of *waves communicated to air by a sphere which vibrates radially under the action of some internal mechanism.* Let ξ denote the radial displacement of the surface of the sphere, and ϕ the velocity-potential in the air, while $2\pi/n$ is the period of oscillation of the sphere under the action of the mechanism alone. Then the equations of motion are†

$$\left.\begin{aligned}\sigma\left(\frac{\partial^2\xi}{\partial t^2}+n^2\xi\right)-\rho\frac{\partial\phi}{\partial t}=0\\ \frac{\partial\xi}{\partial t}-\frac{\partial\phi}{\partial r}=0\end{aligned}\right\},\text{ at } r=b, \tag{42}$$

and

$$\frac{\partial^2}{\partial t^2}(r\phi)=\frac{\partial^2}{\partial r^2}(r\phi),\quad\text{if}\quad r>b, \tag{43}$$

where the velocity of sound is taken as unity, and σ is the surface-density of the sphere, ρ the density of air, and b the mean radius of the sphere.

In the problem considered the air is supposed initially undisturbed, while the sphere may have both displacement and velocity initially, say ξ_0 and ξ_1.

If we solve by writing

$$\xi=\frac{1}{2\pi\iota}\int_{c-\iota\infty}^{c+\iota\infty}\eta e^{\lambda t}d\lambda,\qquad \phi=\frac{1}{2\pi\iota}\int_{c-\iota\infty}^{c+\iota\infty}\psi e^{\lambda t}d\lambda,$$

the previous methods lead to the equations, derived from (42), (43),

$$\left.\begin{aligned}\sigma(\lambda^2+n^2)\eta-\rho\lambda\psi=\sigma(\lambda\xi_0+\xi_1)\\ \lambda\eta-\frac{\partial\psi}{\partial r}=\xi_0\end{aligned}\right\},\text{ at } r=b, \tag{44}$$

and

$$\lambda^2(r\psi)-\frac{\partial^2}{\partial r^2}(r\psi)=0,\quad r>b. \tag{45}$$

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* These *Proceedings*, Ser. 2, Vol. 2, 1904, p. 88.

† *L.c.*, p. 94, equations (9) and (10).

Since the real part of λ is positive, and the last equation (45) is to hold for all values of r greater than b, we must take

$$r\psi = Ae^{-\lambda r}.$$

Then, substituting in the previous equations, we find that

$$\{(\lambda^2+n^2)(\lambda b+1)+(\rho/\sigma)\lambda^2 b\} A = b^2 e^{\lambda b}(n^2\xi_0-\lambda\xi_1). \tag{46}$$

Hence the value of ϕ at any point is given by the integral

$$r\phi = \frac{1}{2\pi\iota}\int_{c-\iota\infty}^{c+\iota\infty} \frac{e^{\lambda(b+t-r)}\, b^2(n^2\xi_0-\lambda\xi_1)}{(\lambda^2+n^2)(\lambda b+1)+(\rho/\sigma)\lambda^2 b}\, d\lambda. \tag{47}$$

By what has been explained in § 6, the value of ϕ is zero so long as $b+t<r$; and when $b+t>r$, the value is

$$r\phi = \Sigma \frac{e^{\lambda_1(b+t-r)}\, b(n^2\xi_0-\lambda_1\xi_1)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}, \tag{48}$$

where λ_1, λ_2, λ_3 are the roots of the denominator in (47). This result (48) agrees with Prof. Love's formula (19) for the special case $\xi_0 = 0$.

If we neglect the effect of the term in ρ/σ in the denominator of (47), the roots are $\lambda = \pm\iota n$ and $1/b$. It is easy to calculate the residues, and we obtain the formula

$$r\phi = \frac{b^2}{1+n^2b^2}\big[(n^2b\xi_0+\xi_1)\, e^{-(b+t-r)/b} - (n^2b\xi_0+\xi_1)\cos\{n(b+t-r)\} \\ + n(\xi_0-b\xi_1)\sin\{n(b+t-r)\}\big]. \tag{49}$$

The formula (49) agrees with that given on p. 97 of Prof. Love's paper, except that in the latter the sign of ϕ is left ambiguous. It is easy to confirm the sign of (49) by observing that, when $t=0$ and $r=b$, it leads to the correct result $\partial\phi/\partial r = \xi_1$; moreover, in the special case $\xi_0 = 0$, (49) reduces to Prof. Love's formula (20).

As remarked by Sir Joseph Larmor (see Prof. Love's footnote on p. 97), the exponential pulse in (49) disappears if

$$n^2b\xi_0+\xi_1 = 0, \tag{50}$$

and can only disappear under this special condition. This conclusion can be foreseen from the integral (47) for ϕ: for the exponential term disappears if (and only if) the factor $\lambda b+1$ cancels out of the numerator and denominator in (47); which gives the same condition (50) again. The harmonic wave is then given by

$$r\phi = b^2 n\xi_0 \sin n(b+t-r),$$

agreeing with Prof. Love's formula (21a).

It is now an easy matter to deal with the problem of *the waves sent out by the sphere when acted on by a periodic force for a definite time.* Prof. Love has given materials for the solution of this problem (*l.c.*, pp. 98–100), but has not completed the details.

To obtain a definite result, suppose that the force applied is represented by $\sigma F \sin pt$, and acts for the half period $t = 0$ to $t = \pi/p$. Then (see Ex. 6, p. 413 above) we replace equations (44) by

$$\left.\begin{aligned} \sigma(\lambda^2+n^2)\eta-\rho\lambda\psi &= \frac{pF\sigma}{\lambda^2+p^2}(1+e^{-\lambda\pi/p}) \\ \lambda\eta-\frac{\partial\psi}{\partial r} &= 0 \end{aligned}\right\}, \text{ at } r = b. \tag{51}$$

From (45) and (51) we deduce the formulæ

$$r\psi = A_1 e^{-\lambda r},$$

where $$\{(\lambda^2+n^2)(\lambda b+1)+(\rho/\sigma)\lambda^2 b\} A_1 = -\frac{pF\lambda b^2 e^{\lambda b}}{\lambda^2+p^2}(1+e^{-\lambda\pi/p}). \tag{52}$$

Hence the value of ϕ is given by

$$r\phi = -\frac{1}{2\pi\iota}\int_{c-\iota\infty}^{c+\iota\infty} \frac{e^{\lambda(b+t-r)}(1+e^{-\lambda\pi/p})}{\lambda^2+p^2}\,\frac{pF\lambda b^2 d\lambda}{(\lambda^2+n^2)(\lambda b+1)+(\rho/\sigma)\lambda^2 b}, \tag{53}$$

from which the residues can be written down as before. It should be noticed that (as remarked by Prof. Love) the waves of period $2\pi/p$ disappear as soon as $b+t-r$ exceeds π/p; for then both exponentials in the numerator of (53) must be used in forming the residues, and their contribution at the points $\lambda = \pm \iota p$ will cancel each other.*

One further point may be noticed: since the numerator in (53) cannot be made to vanish for a real value of λ (other than $\lambda = 0$), it follows that *the exponential pulse can never disappear*; this, of course, is in contrast to the special case represented by (50) above.

Ex. 13.—The corresponding problem for *a rigid sphere set moving in a straight line* has been proposed by Prof. Love, and the solution was completed by Prof. Lamb.†

It may suffice to state the results obtained by the present method, supposing the initial velocity of the sphere to be U. It will be found that the velocity potential is given by

$$\phi = \frac{\partial\chi}{\partial r}\cos\theta,$$

* Because $(1+e^{-\lambda\pi/p})$ is zero for these values of λ.

† Love, *l.c.*, pp. 100–102: Lamb, *Hydrodynamics*, 1906, p. 497.

where $$\chi = \frac{(Ub^3/r)}{2\pi\iota}\int_{c-\iota\infty}^{c+\iota\infty}\frac{e^{\lambda(t+b-r)}\lambda\,d\lambda}{(\lambda^2+n^2)(2+\lambda b+\lambda^2b^2)+\beta\lambda^2(1+\lambda b)},$$

where β is the ratio of the mass of air displaced to the mass of the sphere.

It will be seen at a glance that this leads to a formula of the type given in Lamb's equation (22): on taking $\beta = 0$ and completing the calculation of the residues, it will be found that we obtain Lamb's result (25), except for some small numerical discrepancies.*

8. *Further Details of the Theory for One Type of Continuous Systems.*

We confine the following to problems in which one geometrical coordinate only is required to specify both the dependent and the independent variables. We denote the dependent variable by u, and the independent by x. We shall suppose the system to be a dynamical one specified by a system of energy functions

$$2T = \int_0^l \rho\left(\frac{\partial u}{\partial t}\right)^2 dx, \qquad 2V = \int_0^l p\left(\frac{\partial u}{\partial x}\right)^2 dx,$$

and a dissipation function
$$F = \int_0^l \sigma\left(\frac{\partial u}{\partial t}\right)^2 dx,$$

where ρ, σ, p may be functions of x, but will be supposed to be essentially positive. Thus the system will correspond to the type of *symmetrical* systems considered in § 1.

With regard to the mechanism at the ends $x = 0$, $x = l$, the most natural assumption is to include terms representing the ends in each of the functions T, V, F. Thus, if the end-displacements are denoted by y_0, y_1 respectively,† we might include the terms

$$m_0\left(\frac{\partial y_0}{\partial t}\right)^2 + m_1\left(\frac{\partial y_1}{\partial t}\right)^2, \qquad q_0y_0^2 + q_1y_1^2, \qquad k_0\left(\frac{\partial y_0}{\partial t}\right)^2 + k_1\left(\frac{\partial y_1}{\partial t}\right)^2,$$

in $2T$, $2V$, and F, respectively. The first and third of these can, however, be regarded as included in the corresponding integrals by supposing that ρ and σ tend to infinity at the ends $x = 0$, $x = l$: and, to avoid complicated statements in reference to the end-conditions, we shall make this hypothesis in our general theory, instead of using the terms containing m_0, m_1, k_0, k_1. Of course we cannot use this device for the solution of

* It appears to me that these have arisen in the course of Lamb's work, as I have checked the results obtained from the complex integral, by using the older method.

† Note that the symbols u_0, u_1 are used below to denote the *initial values* of u and $\partial u/\partial t$ (as in § 5).

practical problems ; instead, we should first write down the actual end-conditions and then modify them according to our regular process ; for an illustration, see Ex. 12 in § 6 (in particular, the small type on p. 426).

Thus in future we shall adopt the specification

$$\left.\begin{aligned} 2T &= \int_0^l \rho \left(\frac{\partial u}{\partial t}\right)^2 dx, \qquad F = \int_0^l \sigma \left(\frac{\partial u}{\partial t}\right)^2 dx \\ 2V &= \int_0^l p \left(\frac{\partial u}{\partial x}\right)^2 dx + q_0 y_0^2 + q_1 y_1^2 \end{aligned}\right\}. \tag{54}$$

Forming the variational equation of motion in the usual way from (54), we deduce the equation

$$\rho \frac{\partial^2 u}{\partial t^2} + \sigma \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(p \frac{\partial u}{\partial x}\right) = 0, \tag{55}$$

with the end-conditions $$q_0 y_0 - \left(p \frac{\partial u}{\partial x}\right)_0 = 0, \tag{56}$$

$$q_1 y_1 + \left(p \frac{\partial u}{\partial x}\right)_1 = 0. \tag{57}$$

Proceeding as in § 5, we obtain, as a conjectural solution of (55),

$$u = \frac{1}{2\pi\iota} \int_{c-\iota\infty}^{c+\iota\infty} v e^{\lambda t} d\lambda, \tag{58}$$

where* $$(\rho\lambda^2 + \sigma\lambda) v - \frac{\partial}{\partial x}\left(p \frac{\partial v}{\partial x}\right) = (\rho\lambda + \sigma) u_0 + \rho u_1, \tag{59}$$

and v is to satisfy conditions of the same form† as (56), (57).

In practical work it is usually easiest to solve the equation (59) *directly*, just as we have done in the various examples considered above : because the functions selected for the initial values u_0, u_1 are ordinarily of a simple character. But *in theory* it is better to reduce the solution of (59) to a quadrature. Suppose, then, that we consider the associated equation

$$(\rho\lambda^2 + \sigma\lambda) w - \frac{\partial}{\partial x}\left(p \frac{\partial w}{\partial x}\right) = 0, \tag{60}$$

---

* Here u_0, u_1 denote the *initial values* of u and $\partial u/\partial t$.

† When inertia and frictional terms occur in the end-conditions, care must be taken to introduce the proper terms on the right in forming the equations for v.

2 F 2

and let $w = \phi(x)$ be a solution of (60) which satisfies the end-condition corresponding to (56), while $w = \psi(x)$ is a second solution satisfying the end-condition corresponding to (57).

Then we can solve (59) by the method of "variation of parameters," assuming that

$$v = A\phi(x) + B\psi(x), \tag{61}$$

where A, B are functions of x. If we assume further that*

$$0 = A'\phi(x) + B'\psi(x), \tag{62}$$

it is easy to see that (59) becomes

$$-p\{A'\phi'(x) + B'\psi'(x)\} = (\rho\lambda + \sigma)u_0 + \rho u_1. \tag{63}$$

Now, since $\phi(x)$, $\psi(x)$ both satisfy the same equation (60), we have

$$p\{\phi'(x)\psi(x) - \psi'(x)\phi(x)\} = \text{const.} = \Delta(\lambda), \tag{64}$$

say.† Using (62), (63) and (64), we now find

$$\frac{A'}{-\psi(x)} = \frac{B'}{\phi(x)} = \frac{(\rho\lambda + \sigma)u_0 + \rho u_1}{\Delta(\lambda)}. \tag{65}$$

We have next to consider the effect of the end-conditions: since $\phi(x)$ satisfies the condition at $x = 0$, which is also to be satisfied by v, it follows‡ that B must vanish at $x = 0$; and similarly A must vanish at $x = l$. Thus, in consequence of (65), we can write

$$A = \frac{1}{\Delta(\lambda)} \int_x^l \{(\rho\lambda + \sigma)u_0(\xi) + \rho u_1(\xi)\} \psi(\xi)\, d\xi,$$

and $$B = \frac{1}{\Delta(\lambda)} \int_0^x \{(\rho\lambda + \sigma)u_0(\xi) + \rho u_1(\xi)\} \phi(\xi)\, d\xi.$$

Hence we can reduce v to the form

$$v = \int_0^l \{(\rho\lambda + \sigma)u_0(\xi) + \rho u_1(\xi)\} G(x, \xi)\, d\xi, \tag{66}$$

---

* Accents denote differentiation with regard to x.

† Since λ occurs in both $\phi(x)$ and $\psi(x)$, it is clear that (64) is a function of λ. The notation $\Delta(\lambda)$ is used because this function now plays a part corresponding to the determinant (11) used in §§ 1 and 3 (for details see § 9).

‡ Note that $v' = A\phi'(x) + B\psi'(x)$, in consequence of (62).

where we have written

$$\left.\begin{aligned} G(x, \xi) &= \frac{\phi(x)\,\psi(\xi)}{\Delta(\lambda)} \quad (0 < x < \xi < l) \\ \text{and} \qquad &= \frac{\phi(\xi)\,\psi(x)}{\Delta(\lambda)} \quad (0 < \xi < x < l) \end{aligned}\right\}. \tag{67}$$

The function $G(x, \xi)$ is equivalent to the function often called the *Green's function** of the equation (60); this function is continuous as x varies from 0 to l, but its first differential coefficient has a discontinuity at $x = \xi$, given by

$$p\left(\frac{\partial G}{\partial x}\right)_{\xi-0} - p\left(\frac{\partial G}{\partial x}\right)_{\xi+0} = \frac{p}{\Delta(\lambda)}\{\phi'(\xi)\,\psi(\xi) - \phi(\xi)\,\psi'(\xi)\} = 1. \tag{68}$$

Further $G(x, \xi)$ is symmetrical in the two arguments x, ξ; and satisfies equation (60) with the end-conditions analogous to (56), (57).

It may be worth while to call attention to the fact that equation (66) is really analogous to equation (10) of § 1, which gives the solution of a set of linear equations by means of a sum: here we get an integral as the solution of a differential equation.

It may be convenient to mention some of the applications of the function $G(x, \xi)$ which present themselves naturally in the present theory :—

1. *If an impulse J is applied at the point $x = \xi$*, the solution is given by

$$v = J \,.\, G(x, \xi).$$

For then u_0 is everywhere zero, and u_1 is zero except near $x = \xi$: further we have the relation

$$\int_0^l \rho u_1 \, d\xi = \text{the total momentum of the blow} = J;$$

and accordingly (66) gives the required result.

2. *For a constant force P steadily maintained at the point $x = \xi$*, we may take

$$v = P \,.\, G(x, \xi)/\lambda.$$

This follows at once from the last result, using the method by which the solution of § 2 is derived from that of § 1.

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* Picard, *Traité d'Analyse*, t. 3 ; Burkhardt, *Bull. Soc. Math. de France*, t. 22, 1894 ; Hilbert, *Integralgleichungen*, 1912, pp. 40–42.

3. *For a force $Pe^{\mu t}$ applied at $x = \xi$*, we take similarly

$$v = P.G(x, \xi)/(\lambda - \mu).$$

EXAMPLES.—It will perhaps tend to clear ideas if the forms taken by $G(x, \xi)$ are stated in a few simple cases (corresponding to $\sigma = 0$, $p = \rho a^2$).

1. A string with fixed ends $(x = 0, x = l)$:

$$G(x, \xi) = \frac{\sinh(\lambda x/a)\sinh\{\lambda(l-\xi)/a\}}{\rho a\lambda \sinh(\lambda l/a)}.$$

2. A pipe with two open ends $(x = 0, x = l)$:

$$G(x, \xi) = \frac{\cosh(\lambda x/a)\cosh\{\lambda(l-\xi)/a\}}{\rho a\lambda \sinh(\lambda l/a)}.$$

3. A pipe with one end closed $(x = 0)$ and one end open $(x = l)$:

$$G(x, \xi) = \frac{\sinh(\lambda x/a)\cosh\{\lambda(l-\xi)/a\}}{\rho a\lambda \cosh(\lambda l/a)}.$$

4. A string with both ends attached to springs (contributing potential energy but not kinetic energy):

$$G(x, \xi) = \frac{\{\theta_0 \sinh(\lambda x/a) + \lambda\cosh(\lambda x/a)\}[\theta_1 \sinh\{\lambda(l-\xi)/a\} + \lambda\cosh\{\lambda(l-\xi)/a\}]}{\rho a\lambda\{(\theta_0\theta_1 + \lambda^2)\sinh(\lambda l/a) + (\theta_0+\theta_1)\lambda\cosh(\lambda l/a)\}},$$

where $\rho a\theta_0 = q_0$, $\rho a\theta_1 = q_1$ denote the strengths of the springs.

In each of these examples one only of the two forms for $G(x, \xi)$ has been written out (corresponding to $0 < x < \xi < l$). The other form is obtained by interchanging x and ξ.

5. To illustrate the application of these forms to physical cases, we might consider *the effect of an impulse J applied to a string with fixed ends.*

We have here to use the first of the foregoing functions, and we multiply by J in accordance with (1) of the previous page. Thus we take

$$v = \frac{J}{\rho a\lambda}\,\frac{\sinh(\lambda x/a)\sinh\{\lambda(l-\xi)/a\}}{\sinh(\lambda l/a)}.$$

It is obvious that $\lambda = 0$ is not a pole of $ve^{\lambda t}$; and that the poles correspond to

$$\lambda l/a = n\pi\iota$$

where n is a positive or negative integer. The residue of $ve^{\lambda t}$ is thus

$$-\frac{Je^{\lambda t}}{\rho a\lambda}\,\frac{\sinh(\lambda x/a)\sinh(\lambda\xi/a)}{l/a} = \frac{J}{l\rho}\sin\frac{n\pi x}{l}\sin\frac{n\pi\xi}{l}\,\frac{e^{\iota(n\pi at/l)}}{n\pi a\iota/l}.$$

Thus, on combining together the positive and negative terms, we obtain

$$\frac{2J}{l\rho}\Sigma\sin\frac{n\pi x}{l}\sin\frac{n\pi\xi}{l}\,\frac{\sin(n\pi at/l)}{n\pi a/l},$$

as in Rayleigh's *Theory of Sound*, Vol. 1, Art. 129.

We have still to examine the question as to whether the solution derived from (58) and (66) really does satisfy the prescribed initial conditions. As we have already remarked, anything approaching a complete proof is out of the question here ;* but, by reference to some of the results contained in Prof. A. C. Dixon's paper on expansions in series of oscillating functions,† we can at least sketch some outlines of a proof.

In the first place it must be remembered that in calculating the initial values of u and of $\partial u/\partial t$ from the integrals

$$u = \frac{1}{2\pi\iota}\int ve^{\lambda t}\,d\lambda, \qquad \frac{\partial u}{\partial t} = \frac{1}{2\pi\iota}\int \lambda ve^{\lambda t}\,d\lambda,$$

we must (as explained on p. 412) regard the initial values as obtained by making t tend to zero through positive values. We can accordingly (for this purpose) replace our standard path by the limit of a closed path such as $PAQBP$ (in the figure, p. 411) : and so the initial values are given by

$$u = \frac{1}{2\pi\iota}\int v\,d\lambda, \qquad \frac{\partial u}{\partial t} = \frac{1}{2\pi\iota}\int \lambda v\,d\lambda, \tag{69}$$

each being evaluated for a closed path such as $PAQBP$, which afterwards must be supposed to extend to infinity.

It will appear from the formulæ to be obtained for v that these integrals remain convergent when taken round the arc PCQ ;‡ and so by Cauchy's theorem (since v has no poles inside the space $PCQAP$), we can regard the integrals (69) as taken round the path which is the limit of the circular path $PCQBP$.

The advantage of this preliminary transformation of the path of integration lies in the fact that we can now restrict our discussion of v to those values of λ, for which $|\lambda|$ is large. And, for such values of λ, Prof. Dixon has proved that (at least when $\sigma = 0$) the order of magnitude of the functions can be estimated by treating ρ, p as having constant values (averaged over their actual range of values).§ It is not very easy to quote conveniently Prof. Dixon's actual estimates, on account of differences of

* To indicate the nature of the difficulties, it is perhaps enough to remark that the simplest case of all (with ρ, p constant, and $\sigma = 0$) leads to Fourier's theorem.

† These *Proceedings*, Ser. 2, Vol. 3, 1905, p. 83.

‡ It should be carefully noticed that this statement is *not* true as long as the factor $e^{\lambda t}$ forms part of the integrand.

§ As will be seen from Prof. Dixon's work, the proof of the sufficiency of this estimate is one of the chief difficulties of the discussion.

notation:* but it is easy to apply the same principle to our problem.†

Now, if ρ, σ, p are constants, the two functions $\phi(x)$, $\psi(x)$ will satisfy the differential equation

$$\frac{d^2v}{dx^2} - \mu^2 v = 0,$$

where
$$\mu^2 = (\rho\lambda^2 + \sigma\lambda)/p.$$

Thus, when $|\lambda|$ is large, we can express μ in the form

$$\mu = \frac{\lambda}{a}\left(1 + \frac{b}{\lambda} + \frac{c}{\lambda^2} + \dots\right),$$

where a is positive; and the functions $\phi(x)$, $\psi(x)$ will be of the form

$$A \cosh \mu x + B \sinh \mu x,$$

the values of the coefficients depending on the end conditions of the problem. A reference to the examples on p. 438 will show the various forms of $G(x, \xi)$; it is only necessary to replace $\lambda x/a$ by μx, and so on.

Now when the real part of μ is large and positive (so that the same is true of the real part of λ), the most important term both in $\cosh \mu x$ and in $\sinh \mu x$ will be

$$\tfrac{1}{2} e^{\mu x},$$

because x is positive. If we make this and corresponding substitutions in the formulæ of p. 438 for $G(x, \xi)$, we readily see that the leading term in

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* Prof. Dixon's value of λ is equivalent to our $-\lambda^2$: and his differential equation is expressed in a more precise standard form than our (60), involving changes of both dependent and independent variables.

† To indicate (as far as we can) the corresponding functions, the following table may be useful:—

T. J. I'A. B.	A. C. D.
$\phi(x)$	$\theta_0\phi + \psi$, with $\theta_0 = q_0/p_0$
$\psi(x)$	$\theta_1\chi - \omega$, with $\theta_1 = q_1/p_1$
$G(x, \xi)$	$\Omega(x, \xi, \lambda)$ (see p. 89),
	with $E = \theta_0\theta_1$, $G = \theta_1$, $H = \theta_0$, $K = 1$, $L = 0$

$G(x, \xi)$ is given by*

$$\frac{\frac{1}{2}e^{\mu x} \cdot \frac{1}{2}e^{\mu(l-\xi)}}{\rho a\lambda \cdot \frac{1}{2}e^{\mu l}} = \frac{1}{2\rho a\lambda}\, e^{-\mu(\xi-x)},$$

when the real part of μ is positive, and $0 < x < \xi < l$.

When the real part of μ is negative, the sign of the exponential in the last formula must be reversed; and similarly when $\xi < x$. Now, differentiating, and remembering the relation between μ and λ, we see that the leading term in $\partial G/\partial \xi$ is

$$\frac{1}{2\rho a^2}\, e^{\mp\mu(\xi-x)} = \frac{1}{2p}\, e^{\mp\mu(\xi-x)}.$$

Hence our final estimate of the order of magnitude, when $|\lambda|$ is large, is expressed by the equation

$$\left| p\,\frac{\partial G}{\partial \xi} \right| = \tfrac{1}{2}e^{-\Lambda}, \tag{70}$$

where $$\Lambda = \kappa\,|\xi - x|,$$

and κ is the real part of λ, taken with a positive sign.

Again, since $G(x, \xi)$ is a solution of equation (60), it follows that†

$$\int (\rho\lambda^2 + \sigma\lambda)\, G\, d\xi = p\,\frac{\partial G}{\partial \xi};$$

and if we take the integral between limits a, b, such that

$$0 \leqslant a < x < b \leqslant l,$$

we find $$\int_a^b (\rho\lambda^2 + \sigma\lambda)\, G\, d\xi = \left[p\,\frac{\partial G}{\partial \xi}\right]_a^{x-0} + \left[p\,\frac{\partial G}{\partial \xi}\right]_{x+0}^{b}$$

$$= 1 - \left[p\,\frac{\partial G}{\partial \xi}\right]_{\xi=a} + \left[p\,\frac{\partial G}{\partial \xi}\right]_{\xi=b},$$

where we have made use of (68) in the form obtained by interchanging x and ξ.

Now, referring to (70), it is clear that the terms given by $\xi = a$,

* A little consideration will shew that this conclusion remains correct whatever combination of end-conditions is required.

† Since $G(x, \xi)$ is symmetrical in the two variables, the parts played by x and ξ may be interchanged.

$\xi = b$ in the last equation must tend to zero exponentially as κ tends to infinity; we shall indicate this by writing the equation in the form

$$\int_a^b (\rho\lambda^2+\sigma\lambda) G\, d\xi = 1+O(e^{-\Lambda}), \tag{71}$$

where Λ denotes any positive multiple of κ, such as $\kappa(x-a)$ or $\kappa(b-x)$.

Similarly we prove that

$$\int_0^a (\rho\lambda^2+\sigma\lambda) G\, d\xi = O(e^{-\Lambda}), \quad \int_b^l (\rho\lambda^2+\sigma\lambda) G\, d\xi = O(e^{-\Lambda}), \tag{72}$$

We can now estimate the order of v as derived from (66): assuming that the functions $u_0(\xi)$ and $u_1(\xi)$ satisfy Dirichlet's conditions (as laid down for Fourier's theorem), we can apply the second theorem of the mean* (first dividing the range of integration at the point $\xi = x$). The result is

$$\begin{aligned} v = \; & u_0(0) \int_0^a (\rho\lambda+\sigma)\, G\, d\xi + u_0(x) \int_a^x (\rho\lambda+\sigma)\, G\, d\xi \\ & + u_0(x) \int_x^b (\rho\lambda+\sigma)\, G\, d\xi + u_0(l) \int_b^l (\rho\lambda+\sigma)\, G\, d\xi \\ & + \text{similar terms arising from } u_1. \end{aligned}$$

Applying (71) and (72), we deduce that

$$v = \frac{1}{\lambda}\left\{u_0(x)+O(e^{-\Lambda})\right\} + \frac{1}{\lambda^2}\left\{u_1(x)+O(e^{-\Lambda})+O\left(\frac{1}{\lambda}\right)\right\}. \tag{73}$$

If we substitute from (73) in (69), and then integrate round a circle whose radius afterwards tends to infinity, we now see that the initial values of u and $\partial u/\partial t$ are respectively equal to $u_0(x)$ and $u_1(x)$. This justifies the claims made for the solution given by (58) and (66).

One further point should be mentioned in which this discussion is not quite complete. The terms which have been written as $O(e^{-\Lambda})$ tend to zero when κ tends to infinity: but there are certain minor arcs† of the circle on which these terms need not tend to zero: these arcs are (comparatively speaking) short, but their effect ought not to be overlooked. Prof. Dixon (*l.c.*, §§14–19) has shewn that in his problem the contributions of the minor arcs tend to zero: and I anticipate that a similar proof could be made here,

* Strictly speaking the functions u_0 and u_1 must be monotonic from $x = 0$ to $x = l$, to allow the use of the second mean value theorem; but under Dirichlet's conditions u_0 and u_1 can each be expressed as the difference of two monotonic functions. Accordingly the final conclusion (73) will remain valid.

† In the figure of p. 411, these arcs surround the points P and Q.

but I have not actually gone into the details of a general proof—although in special examples the discussion of the minor arcs can always be carried out by reasoning of a type which is of constant occurrence in elementary work with complex integrals.*

9. *Comparison of the Results obtained in § 8 with those of the Method of Normal Functions.*

In view of what has been proved already (in § 3) for the case of discrete systems, it is natural to expect that the formulæ of § 8 will again lead to the same results as are found by the use of normal functions. It will be instructive, however, to trace the corresponding steps in the proof.

We begin then with the roots $\alpha_1, \alpha_2, \alpha_3, \ldots$ of the equation $\Delta(\lambda) = 0$, where $\Delta(\lambda)$ is defined by equation (64). For each such root we have special values of the functions ϕ, ψ: denote them in order by $\phi_1, \phi_2, \phi_3, \ldots$ and $\psi_1, \psi_2, \psi_3, \ldots$. Then, in virtue of equation (64), we have

$$\phi_1/\psi_1 = \text{const.} = c_1, \qquad \phi_2/\psi_2 = \text{const.} = c_2, \quad \&\text{c.} \tag{74}$$

Accordingly the special functions $\phi_1, \phi_2, \phi_3, \ldots$ (or $\psi_1, \psi_2, \psi_3, \ldots$) satisfy equation (60) with $\lambda = \alpha_1, \alpha_2, \alpha_3, \ldots$, and moreover satisfy *both* the end-conditions (56) and (57). *These functions are the normal functions of the problem.*

We shall now obtain *the conjugate property of the normal functions.* In fact we have

$$\left.\begin{aligned} (\rho\alpha_1^2+\sigma\alpha_1)\,\phi_1 - \frac{\partial}{\partial x}\left(p\,\frac{\partial\phi_1}{\partial x}\right) &= 0, \\ (\rho\alpha_2^2+\sigma\alpha_2)\,\phi_2 - \frac{\partial}{\partial x}\left(p\,\frac{\partial\phi_2}{\partial x}\right) &= 0, \end{aligned}\right\}$$

and so $$(\alpha_1^2-\alpha_2^2)\,\rho\phi_1\phi_2+(\alpha_1-\alpha_2)\sigma\phi_1\phi_2 = \frac{\partial}{\partial x}\left\{p\left(\phi_2\frac{\partial\phi_1}{\partial x}-\phi_1\frac{\partial\phi_2}{\partial x}\right)\right\}.$$

Now integrate from 0 to l, and we have

$$(\alpha_1^2-\alpha_2^2)\int_0^l \rho\phi_1\phi_2\,dx+(\alpha_1-\alpha_2)\int_0^l \sigma\phi_1\phi_2\,dx = \left[p\left(\phi_2\frac{\partial\phi_1}{\partial x}-\phi_1\frac{\partial\phi_2}{\partial x}\right)\right]_0^l.$$

* Such as the deduction of $\int_0^\infty \frac{\sin x}{x}\,dx$ from the complex integral $\int \frac{e^{iz}}{z}\,dz$; the minor arcs here would be two arcs of the circle close to the real axis.

In virtue of the fact that ϕ_1, ϕ_2 satisfy *both* the conditions (56), (57), at $x = 0$ and $x = l$, it follows that

$$\phi_2 \frac{\partial \phi_1}{\partial x} - \phi_1 \frac{\partial \phi_2}{\partial x} = 0,$$

both at $x = 0$ and at $x = l$. Using this fact, we see (after division by $a_1 - a_2$) that

$$(a_1 + a_2) \int_0^l \rho \phi_1 \phi_2 dx + \int_0^l \sigma \phi_1 \phi_2 dx = 0, \tag{75}$$

care being taken to include contributions from the terminal mechanism, if any.

The equation (75) constitutes *the conjugate property of normal functions* in the form most convenient for our present purpose; it can also be transformed into the form

$$\int_0^l \rho (a_1 \phi_1)(a_2 \phi_2)\, dx = \int_0^l p \frac{\partial \phi_1}{\partial x} \frac{\partial \phi_2}{dx} dx + [q \phi_1 \phi_2]_{x=0} + [q \phi_1 \phi_2]_{x=l},$$

which corresponds to Heaviside's relation $T_{12} = U_{12}$, given in § 3 above.

In the special case $\sigma = 0$, (75) reduces to Lord Rayleigh's form*

$$\int_0^l \rho \phi_1 \phi_2 dx = 0. \tag{76}$$

The equation (76) also holds when σ/ρ is *constant*: but care must be taken to notice that *the same constant must apply to the terminal mechanism as to the main parts of the system.* Various examples of the type (75) will be found amongst Heaviside's earlier investigations on cables;† a simple example is given by supposing a constant resistance in the cable, with an inductance coil at an end. Here $\rho = 0$ in general, and σ is constant: but to get the correct result, the terminal inductance must be allowed for, and thus the equation $\int_0^l \phi_1 \phi_2 dx = 0$ is no longer correct.

Suppose next that a_1, a_2 are conjugate complex roots of $\Delta(\lambda) = 0$: then $\int_0^l \rho \phi_1 \phi_2 dx$ and $\int_0^l \sigma \phi_1 \phi_2 dx$ are essentially *positive*;‡ and so (75) shews that $a_1 + a_2$ is *negative*. Thus *the real part of any complex root of* $\Delta(\lambda) = 0$ *is negative*: this is a property which has been constantly

* *Theory of Sound*, Vol. 1, Arts. 93, 94.

† See, for instance, *Electrical Papers*, Vol. 1, pp. 71 (see p. 81); p. 123; p. 141, &c.

‡ Because ρ, σ are positive, and ϕ_1, ϕ_2 are conjugate complexes.

assumed in the foregoing work, and may, perhaps, be regarded as obvious. Of course, *when* $\sigma = 0$, *the real parts are all zero*, a well-known result.

We shall need one more general formula to complete our preliminary work. We have defined $\phi(x)$ by the equation (60),

$$(\rho\lambda^2+\sigma\lambda)\phi - \frac{\partial}{\partial x}\left(p\frac{\partial\phi}{\partial x}\right) = 0,$$

along with the end-condition (56). Differentiate with regard to λ, and write

$$\frac{\partial\phi}{\partial\lambda} = \omega;$$

then we have $$(2\rho\lambda+\sigma)\phi + (\rho\lambda^2+\sigma\lambda)\omega - \frac{\partial}{\partial x}\left(p\frac{\partial\omega}{\partial x}\right) = 0.$$

Thus, since ψ also satisfies (60), we have

$$(2\rho\lambda+\sigma)\phi\psi = \frac{\partial}{\partial x}\left\{p\left(\psi\frac{\partial\omega}{\partial x} - \omega\frac{\partial\psi}{\partial x}\right)\right\}$$

or $$\int_0^l (2\rho\lambda+\sigma)\phi\psi\,dx = \left[p\left(\psi\frac{\partial\omega}{\partial x} - \omega\frac{\partial\psi}{\partial x}\right)\right]_0^l. \tag{77}$$

Now since ϕ satisfies (56) *for all values of* λ, the same condition will be satisfied by ω; and for the special values $\lambda = a_1, a_2, \ldots$ this condition is also satisfied by ψ. Accordingly we find that, for these special values of λ, $\psi\frac{\partial\omega}{\partial x} - \omega\frac{\partial\psi}{\partial x} = 0$ at $x = 0$; and so (77) yields

$$\int_0^l (2\rho a_1+\sigma)\phi_1\psi_1\,dx = \left[p\left(\psi_1\frac{\partial\omega_1}{\partial x} - \omega_1\frac{\partial\psi_1}{\partial x}\right)\right]_{x=l}. \tag{78}$$

Now we have, from (64),

$$\Delta(\lambda) = \left[p\left(\psi\frac{\partial\phi}{\partial x} - \phi\frac{\partial\psi}{\partial x}\right)\right]_{x=l},$$

and so, if we differentiate with respect to λ, and then put $\lambda = a_1$, we get*

$$\Delta'(a_1) = \left[p\left(\psi_1\frac{\partial\omega_1}{\partial x} - \omega_1\frac{\partial\psi_1}{\partial x}\right)\right]_{x=l}. \tag{79}$$

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* Observe that since ψ satisfies (57) for all values of λ, the same is true of $\partial\psi/\partial\lambda$: and so

$$\frac{\partial\phi_1}{\partial x}\frac{\partial\psi_1}{\partial\lambda} - \phi_1\frac{\partial}{\partial x}\left(\frac{\partial\psi_1}{\partial\lambda}\right) = 0,$$

when $\lambda = a_1$ and $x = l$.

Combining (78) and (79), we have the result

$$\int_0^l (2\rho a_1+\sigma)\phi_1\psi_1\,dx = \Delta'(a_1)\,; \tag{80}$$

it will be noticed that equation (80) corresponds to equations (28) and (30) of § 3.

It is now easy to find formulæ for the residues of v, as defined in § 8; in fact, from (66) and (67), we deduce that *the residue of v corresponding to $\lambda = a_1$ is*

$$\frac{\phi_1(x)}{c_1\Delta'(a_1)}\int_0^l \{(\rho a_1+\sigma)\,u_0(\xi)+\rho u_1(\xi)\}\,\phi_1(\xi)\,d\xi,$$

where c_1 has the value given in (74). Now, substituting from (80), we obtain the residue of v, namely,

$$\phi_1(x)\,\frac{\int_0^l \{(\rho a_1+\sigma)\,u_0(\xi)+\rho u_1(\xi)\}\,\phi_1(\xi)\,d\xi}{\int_0^l (2\rho a_1+\sigma)\{\phi_1(\xi)\}^2\,d\xi}.$$

Thus, finally, *the solution u corresponding to the given initial values can be written in the form*

$$u = \Sigma A_1\phi_1(x)e^{a_1t},$$

where

$$A_1\int_0^l (2\rho a_1+\sigma)\{\phi_1(x)\}^2dx = \int_0^l \{(\rho a_1+\sigma)\,u_0(x)+\rho u_1(x)\}\,\phi_1(x)\,dx. \tag{81}$$

It is now to be proved that the solution (81) agrees with that which would be obtained by the method of normal functions. According to that method we *assume* that the solution must take the form (81), and then find the constants A from the two initial conditions

$$\left.\begin{aligned} u_0(x) &= \Sigma A_1\phi_1(x) \\ u_1(x) &= \Sigma A_1a_1\phi_1(x) \end{aligned}\right\}. \tag{82}$$

If we thus assume the possibility of expanding $u_0(x)$, $u_1(x)$, in terms of normal functions, we can obtain the coefficients fairly quickly, as follows. Multiply the first of equations (82) by $(\rho a_1+\sigma)$, and the second by ρ, and add: this gives

$$(\rho a_1+\sigma)\,u_0(x)+\rho u_1(x) = A_1\phi_1(x)(2\rho a_1+\sigma)+A_2\phi_2(x)\{\rho(a_1+a_2)+\sigma\}+\dots. \tag{83}$$

Now multiply (83) by $\phi_1(x)$ and integrate: then, in virtue of (75), all the coefficients A_2, A_3, A_4, ... disappear from the equation: and so we obtain the formula

$$\int_0^l \{(\rho a_1+\sigma)\, u_0(x)+\rho u_1(x)\}\, \phi_1(x)\,dx = A_1 \int_0^l (2\rho a_1+\sigma)\{\phi_1(x)\}^2 dx,$$

which is the same result as we obtained in (81) by the method of residues.

We have accordingly verified the conclusion that *the method of normal functions is equivalent to the method of* § 8, *as regards results*. But, from the point of view of theory, the method of § 8 gives some indication as to the necessary restrictions on the functions $u_0(x)$, $u_1(x)$, in order that the results shall be correct; whereas the method of normal functions gives no such indication. Moreover, as we have seen in some special examples, it may be possible to obtain more precise information from the complex integrals than is possible from the series of normal functions; and the problems solved in §§ 6 and 7 indicate that our general process is successful in cases which have hitherto proved insoluble by the usual forms of the method of normal functions.

[*Added November* 14*th*, 1916.

While the preceding pages were in the press, I have been reminded that various authors have applied complex integrals, as a means for the summation of series of normal functions: the whole problem being regarded mainly from the point of view of Pure Mathematics. This method goes back to Cauchy;* it was afterwards applied to series of more general types of normal functions by Dini,† and Dini's work has been carried further by W. B. Ford.‡

In all these discussions, it will be seen that the complex integrals are developed from the other end of the problem. Thus, starting from a series such as (81) above, a complex integral is devised which yields the terms of the series as its residues; but the series is found first, and not the integral. I believe that it will be found that the integrals so invented are more complicated than those derived directly from the formulæ (58), (66), (67), and (69), of § 8 above.

When the functions $u_0(x)$, $u_1(x)$ do not satisfy Dirichlet's conditions,

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* *Œuvres Complètes*, t. 7 (2 sér.), p. 393; see also Picard, *Traité d'Analyse*, t. 2, pp. 167–183.

† *Serie di Fourier, &c.*, §§ 61–64, 90–109.

‡ *Studies on Divergent Series and Summability*, New York, 1916, pp. 123-183. It was through the study of this monograph that I was reminded of the earlier use of complex integrals as a means of summation.

it is possible (at least in certain special series) to establish the fact that the corresponding series are summable (instead of convergent), provided that the functions do satisfy certain less restrictive conditions; and Prof. Ford has shewn (*l.c.*, pp. 134–183) that his complex integrals can be used for this purpose. It seems likely that a corresponding investigation could be carried out, starting from the complex integrals of § 8 above; but I have made no attempt to consider this problem—which is mainly of interest from the side of Pure Mathematics, because Dirichlet's conditions are satisfied by almost all functions which are used in Applied Mathematics.—T. J. I'A. B.]