This is Volume 3 of the five-volume book Mathematical Inequalities, which introduces and develops the main types of elementary inequalities. The first three volumes are a great opportunity to look into many old and new inequalities, as well as elementary procedures for solving them: Volume 1 - Symmetric Polynomial Inequalities, Volume 2 - Symmetric Rational and Nonrational Inequalities, Volume 3 - Cyclic and Noncyclic Inequalities. As a rule, the inequalities in these volumes are increasingly ordered according to the number of variables: two, three, four, ... , n-variables. The last two volumes (Volume 4 – Extensions and Refinements of Jensen’s Inequality, Volume 5 – Other Recent Methods for Creating and Solving Inequalities) present beautiful and original methods for solving inequalities, such as Half/Partial convex function method, Equal variables method, Arithmetic compensation method, Highest coefficient cancellation method, pqr method etc. The book is intended for a wide audience: advanced middle school students, high school students, college and university students, and teachers. Many problems and methods can be used as group projects for advanced high school students.

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Chapter 1

Cyclic Inequalities

1.1 Applications

1.1. If a, b, c are nonnegative real numbers such that $a+b+c=3$, then

$$ab^2+bc^2+ca^2 \le 4.$$

1.2. If a, b, c are positive real numbers such that $a+b+c=3$, then

$$(ab+bc+ca)(ab^2+bc^2+ca^2) \le 9.$$

1.3. If a, b, c are nonnegative real numbers such that $a^2+b^2+c^2=3$, then

(a) $$ab^2+bc^2+ca^2 \le abc+2;$$

(b) $$\frac{a}{b+2}+\frac{b}{c+2}+\frac{c}{a+2} \le 1.$$

1.4. If $a, b, c \ge 1$, then

(a) $$2(ab^2+bc^2+ca^2)+3 \ge 3(ab+bc+ca);$$

(b) $$ab^2+bc^2+ca^2+6 \ge 3(a+b+c).$$

1.5. If a,b,c are nonnegative real numbers such that

$$a+b+c=3, \qquad a \ge b \ge c,$$

then

(a) $$a^2b+b^2c+c^2a \ge ab+bc+ca;$$

(b) $$8(ab^2+bc^2+ca^2)+3abc \le 27;$$

(c) $$\frac{18}{a^2b+b^2c+c^2a} \le \frac{1}{abc}+5.$$

1.6. If a,b,c are nonnegative real numbers such that

$$a^2+b^2+c^2=3, \qquad a \ge b \ge c,$$

then

$$ab^2+bc^2+ca^2 \le \frac{3}{4}(ab+bc+ca+1).$$

1.7. If a,b,c are nonnegative real numbers such that $a^2+b^2+c^2=3$, then

$$a^2b^3+b^2c^3+c^2a^3 \le 3.$$

1.8. If a,b,c are nonnegative real numbers such that $a+b+c=3$, then

$$a^4b^2+b^4c^2+c^4a^2+4 \ge a^3b^3+b^3c^3+c^3a^3.$$

1.9. If a,b,c are nonnegative real numbers such that $a+b+c=3$, then

(a) $$ab^2+bc^2+ca^2+abc \le 4;$$

(b) $$\frac{a}{4-b}+\frac{b}{4-c}+\frac{c}{4-a} \le 1;$$

(c) $$ab^3+bc^3+ca^3+(ab+bc+ca)^2 \le 12;$$

(d) $$\frac{ab^2}{1+a+b}+\frac{bc^2}{1+b+c}+\frac{ca^2}{1+c+a} \le 1.$$

1.10. If a,b,c are positive real numbers, then

$$\frac{1}{a(a+2b)}+\frac{1}{b(b+2c)}+\frac{1}{c(c+2a)} \ge \frac{3}{ab+bc+ca}.$$

1.11. If a, b, c are positive real numbers such that $a+b+c=3$, then

$$\frac{a}{b^2+2c}+\frac{b}{c^2+2a}+\frac{c}{a^2+2b} \geq 1.$$

1.12. If a, b, c are positive real numbers such that $a+b+c \geq 3$, then

$$\frac{a-1}{b+1}+\frac{b-1}{c+1}+\frac{c-1}{a+1} \geq 0.$$

1.13. If a, b, c are positive real numbers such that $a+b+c=3$, then

(a) $$\frac{1}{2ab^2+1}+\frac{1}{2bc^2+1}+\frac{1}{2ca^2+1} \geq 1;$$

(b) $$\frac{1}{ab^2+2}+\frac{1}{bc^2+2}+\frac{1}{ca^2+2} \geq 1.$$

1.14. If a, b, c are positive real numbers such that $a+b+c=3$, then

$$\frac{ab}{9-4bc}+\frac{bc}{9-4ca}+\frac{ca}{9-4ab} \leq \frac{3}{5}.$$

1.15. If a, b, c are positive real numbers such that $a+b+c=3$, then

(a) $$\frac{a^2}{2a+b^2}+\frac{b^2}{2b+c^2}+\frac{c^2}{2c+a^2} \geq 1;$$

(b) $$\frac{a^2}{a+2b^2}+\frac{b^2}{b+2c^2}+\frac{c^2}{c+2a^2} \geq 1.$$

1.16. Let a, b, c be positive real numbers such that $a+b+c=3$. Then,

$$\frac{1}{a+b^2+c^3}+\frac{1}{b+c^2+a^3}+\frac{1}{c+a^2+b^3} \leq 1.$$

1.17. If a, b, c are positive real numbers, then

$$\frac{1+a^2}{1+b+c^2}+\frac{1+b^2}{1+c+a^2}+\frac{1+c^2}{1+a+b^2} \geq 2.$$

1.18. If a,b,c are nonnegative real numbers, then

$$\frac{a}{4a+4b+c}+\frac{b}{4b+4c+a}+\frac{c}{4c+4a+b}\le\frac{1}{3}.$$

1.19. If a,b,c are positive real numbers, then

$$\frac{a+b}{a+7b+c}+\frac{b+c}{b+7c+a}+\frac{c+a}{c+7a+b}\ge\frac{2}{3}.$$

1.20. If a,b,c are positive real numbers, then

$$\frac{a+b}{a+3b+c}+\frac{b+c}{b+3c+a}+\frac{c+a}{c+3a+b}\ge\frac{6}{5}.$$

1.21. If a,b,c are positive real numbers, then

$$\frac{2a+b}{2a+c}+\frac{2b+c}{2b+a}+\frac{2c+a}{2c+b}\ge 3.$$

1.22. If a,b,c are positive real numbers, then

$$\frac{a(a+b)}{a+c}+\frac{b(b+c)}{b+a}+\frac{c(c+a)}{c+b}\le\frac{3(a^2+b^2+c^2)}{a+b+c}.$$

1.23. If a,b,c are real numbers, then

$$\frac{a^2-bc}{4a^2+b^2+4c^2}+\frac{b^2-ca}{4b^2+c^2+4a^2}+\frac{c^2-ab}{4c^2+a^2+4b^2}\ge 0.$$

1.24. If a,b,c are real numbers, then

(a) $$a(a+b)^3+b(b+c)^3+c(c+a)^3\ge 0;$$

(b) $$a(a+b)^5+b(b+c)^5+c(c+a)^5\ge 0.$$

1.25. If a,b,c are real numbers, then

$$3(a^4+b^4+c^4)+4(a^3b+b^3c+c^3a)\ge 0.$$

1.26. If a, b, c are positive real numbers, then

$$\frac{(a-b)(2a+b)}{(a+b)^2}+\frac{(b-c)(2b+c)}{(b+c)^2}+\frac{(c-a)(2c+a)}{(c+a)^2}\geq 0.$$

1.27. If a, b, c are positive real numbers, then

$$\frac{(a-b)(2a+b)}{a^2+ab+b^2}+\frac{(b-c)(2b+c)}{b^2+bc+c^2}+\frac{(c-a)(2c+a)}{c^2+ca+a^2}\geq 0.$$

1.28. If a, b, c are positive real numbers, then

$$\frac{(a-b)(3a+b)}{a^2+b^2}+\frac{(b-c)(3b+c)}{b^2+c^2}+\frac{(c-a)(3c+a)}{c^2+a^2}\geq 0.$$

1.29. Let a, b, c be positive real numbers such that $abc = 1$. Then,

$$\frac{1}{1+a+b^2}+\frac{1}{1+b+c^2}+\frac{1}{1+c+a^2}\leq 1.$$

1.30. Let a, b, c be positive real numbers such that $abc = 1$. Then,

$$\frac{a}{(a+1)(b+2)}+\frac{b}{(b+1)(c+2)}+\frac{c}{(c+1)(a+2)}\geq \frac{1}{2}.$$

1.31. If a, b, c are positive real numbers such that $ab + bc + ca = 3$, then

$$(a+2b)(b+2c)(c+2a)\geq 27.$$

1.32. If a, b, c are positive real numbers such that $ab + bc + ca = 3$, then

$$\frac{a}{a+a^3+b}+\frac{b}{b+b^3+c}+\frac{c}{c+c^3+a}\leq 1.$$

1.33. If a, b, c are positive real numbers such that $a \geq b \geq c$ and $ab + bc + ca = 3$, then

$$\frac{1}{a+2b}+\frac{1}{b+2c}+\frac{1}{c+2a}\geq 1.$$

1.34. If $a,b,c\in[0,1]$, then

$$\frac{a}{4b^2+5}+\frac{b}{4c^2+5}+\frac{c}{4a^2+5}\ge\frac{1}{3}.$$

1.35. If $a,b,c\in\left[\frac{1}{3},3\right]$, then

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}\ge\frac{7}{5}.$$

1.36. If $a,b,c\in\left[\frac{1}{\sqrt{2}},\sqrt{2}\right]$, then

$$\frac{3}{a+2b}+\frac{3}{b+2c}+\frac{3}{c+2a}\ge\frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{c+a}.$$

1.37. If a,b,c are nonnegative real numbers, no two of which are zero, then

$$\frac{4abc}{ab^2+bc^2+ca^2+abc}+\frac{a^2+b^2+c^2}{ab+bc+ca}\ge 2.$$

1.38. If a,b,c are nonnegative real numbers such that $a+b+c=3$, then

$$\frac{1}{ab^2+8}+\frac{1}{bc^2+8}+\frac{1}{ca^2+8}\ge\frac{1}{3}.$$

1.39. If a,b,c are nonnegative real numbers such that $a+b+c=3$, then

$$\frac{ab}{bc+3}+\frac{bc}{ca+3}+\frac{ca}{ab+3}\le\frac{3}{4}.$$

1.40. If a,b,c are nonnegative real numbers such that $a+b+c=3$, then

(a) $$\frac{a}{b^2+3}+\frac{b}{c^2+3}+\frac{c}{a^2+3}\ge\frac{3}{4};$$

(b) $$\frac{a}{b^3+1}+\frac{b}{c^3+1}+\frac{c}{a^3+1}\ge\frac{3}{2}.$$

1.41. Let a, b, c be positive real numbers, and let

$$x = a + \frac{1}{b} - 1, \quad y = b + \frac{1}{c} - 1, \quad z = c + \frac{1}{a} - 1.$$

Prove that

$$xy + yz + zx \geq 3.$$

1.42. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\left(a - \frac{1}{b} - \sqrt{2}\right)^2 + \left(b - \frac{1}{c} - \sqrt{2}\right)^2 + \left(c - \frac{1}{a} - \sqrt{2}\right)^2 \geq 6.$$

1.43. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\left|1 + a - \frac{1}{b}\right| + \left|1 + b - \frac{1}{c}\right| + \left|1 + c - \frac{1}{a}\right| > 2.$$

1.44. If a, b, c are different positive real numbers, then

$$\left|1 + \frac{a}{b-c}\right| + \left|1 + \frac{b}{c-a}\right| + \left|1 + \frac{c}{a-b}\right| > 2.$$

1.45. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\left(2a - \frac{1}{b} - \frac{1}{2}\right)^2 + \left(2b - \frac{1}{c} - \frac{1}{2}\right)^2 + \left(2c - \frac{1}{a} - \frac{1}{2}\right)^2 \geq \frac{3}{4}.$$

1.46. Let

$$x = a + \frac{1}{b} - \frac{5}{4}, \quad y = b + \frac{1}{c} - \frac{5}{4}, \quad z = c + \frac{1}{a} - \frac{5}{4},$$

where $a \geq b \geq c > 0$. Prove that

$$xy + yz + zx \geq \frac{27}{16}.$$

1.47. Let a, b, c be positive real numbers, and let

$$E = \left(a + \frac{1}{a} - \sqrt{3}\right)\left(b + \frac{1}{b} - \sqrt{3}\right)\left(c + \frac{1}{c} - \sqrt{3}\right);$$

$$F = \left(a + \frac{1}{b} - \sqrt{3}\right)\left(b + \frac{1}{c} - \sqrt{3}\right)\left(c + \frac{1}{a} - \sqrt{3}\right).$$

Prove that $E \geq F$.

1.48. If a,b,c are positive real numbers such that $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=5$, then

$$\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\geq\frac{17}{4}.$$

1.49. If a,b,c are positive real numbers, then

(a) $$1+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 2\sqrt{1+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}}\ ;$$

(b) $$1+2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\geq\sqrt{1+16\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)}\ ;$$

(c) $$3+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 2\sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}\ .$$

1.50. If a,b,c are positive real numbers, then

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+15\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)\geq 16\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right).$$

1.51. If a,b,c are positive real numbers such that $abc=1$, then

(a) $$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq a+b+c;$$

(b) $$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\frac{3}{2}(a+b+c-1);$$

(c) $$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+2\geq\frac{5}{3}(a+b+c).$$

1.52. If a,b,c are positive real numbers such that $a^2+b^2+c^2=3$, then

(a) $$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 2+\frac{3}{ab+bc+ca};$$

(b) $$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\frac{9}{a+b+c}.$$

1.53. If a,b,c are positive real numbers such that $a^2+b^2+c^2=3$, then

$$6\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+5(ab+bc+ca)\geq 33.$$

1.54. If a, b, c are positive real numbers such that $a+b+c=3$, then

(a) $$6\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+3 \ge 7(a^2+b^2+c^2);$$

(b) $$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge a^2+b^2+c^2.$$

1.55. If a, b, c are positive real numbers, then

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+2 \ge \frac{14(a^2+b^2+c^2)}{(a+b+c)^2}.$$

1.56. Let a, b, c be positive real numbers such that $a+b+c=3$, and let

$$x = 3a+\frac{1}{b}, \quad y = 3b+\frac{1}{c}, \quad z = 3c+\frac{1}{a}.$$

Prove that

$$xy+yz+zx \ge 48.$$

1.57. If a, b, c are positive real numbers such that $a+b+c=3$, then

$$\frac{a+1}{b}+\frac{b+1}{c}+\frac{c+1}{a} \ge 2(a^2+b^2+c^2).$$

1.58. If a, b, c are positive real numbers such that $a+b+c=3$, then

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}+3 \ge 2(a^2+b^2+c^2).$$

1.59. If a, b, c are positive real numbers, then

$$\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}+2(ab+bc+ca) \ge 3(a^2+b^2+c^2).$$

1.60. If a, b, c are positive real numbers such that $a^4+b^4+c^4=3$, then

(a) $$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} \ge 3;$$

(b) $$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b} \ge \frac{3}{2}.$$

1.61. If a,b,c are positive real numbers, then

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq\frac{3(a^3+b^3+c^3)}{a^2+b^2+c^2}.$$

1.62. If a,b,c are positive real numbers, then

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}+a+b+c\geq 2\sqrt{(a^2+b^2+c^2)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)}.$$

1.63. If a,b,c are positive real numbers, then

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+32\left(\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}\right)\geq 51.$$

1.64. Find the greatest positive real number K such that the inequalities below hold for any positive real numbers a,b,c:

(a) $$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\geq K\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}-\frac{3}{2}\right);$$

(b) $$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3+K\left(\frac{a}{2a+b}+\frac{b}{2b+c}+\frac{c}{2c+a}-1\right)\geq 0.$$

1.65. If $a,b,c\in\left[\frac{1}{2},2\right]$, then

(a) $$8\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\geq 5\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)+9;$$

(b) $$20\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\geq 17\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right).$$

1.66. If a,b,c are positive real numbers such that $a\leq b\leq c$, then

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\frac{2a}{b+c}+\frac{2b}{c+a}+\frac{2c}{a+b}.$$

1.67. Let a, b, c be positive real numbers such that $abc = 1$.

(a) If $a \le b \le c$, then

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge a^{3/2} + b^{3/2} + c^{3/2};$$

(b) If $a \le 1 \le b \le c$, then

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge a^{\sqrt{3}} + b^{\sqrt{3}} + c^{\sqrt{3}}.$$

1.68. If k and a, b, c are positive real numbers, then

$$\frac{1}{(k+1)a+b} + \frac{1}{(k+1)b+c} + \frac{1}{(k+1)c+a} \ge \frac{1}{ka+b+c} + \frac{1}{kb+c+a} + \frac{1}{kc+a+b}.$$

1.69. If a, b, c are positive real numbers, then

(a) $$\frac{a}{\sqrt{2a+b}} + \frac{b}{\sqrt{2b+c}} + \frac{c}{\sqrt{2c+a}} \le \sqrt{a+b+c};$$

(b) $$\frac{a}{\sqrt{a+2b}} + \frac{b}{\sqrt{b+2c}} + \frac{c}{\sqrt{c+2a}} \ge \sqrt{a+b+c}.$$

1.70. Let a, b, c be nonnegative real numbers such that $a + b + c = 3$. Prove that

$$a\sqrt{\frac{a+2b}{3}} + b\sqrt{\frac{b+2c}{3}} + c\sqrt{\frac{c+2a}{3}} \le 3.$$

1.71. If a, b, c are nonnegative real numbers such that $a + b + c = 3$, then

$$a\sqrt{1+b^3} + b\sqrt{1+c^3} + c\sqrt{1+a^3} \le 5.$$

1.72. If a, b, c are positive real numbers such that $abc = 1$, then

(a) $$\sqrt{\frac{a}{b+3}} + \sqrt{\frac{b}{c+3}} + \sqrt{\frac{c}{a+3}} \ge \frac{3}{2};$$

(b) $$\sqrt[3]{\frac{a}{b+7}} + \sqrt[3]{\frac{b}{c+7}} + \sqrt[3]{\frac{c}{a+7}} \ge \frac{3}{2}.$$

1.73. If a,b,c are positive real numbers, then

$$\left(1+\frac{4a}{a+b}\right)^2+\left(1+\frac{4b}{b+c}\right)^2+\left(1+\frac{4c}{c+a}\right)^2 \ge 27.$$

1.74. If a,b,c are positive real numbers, then

$$\sqrt{\frac{2a}{a+b}}+\sqrt{\frac{2b}{b+c}}+\sqrt{\frac{2c}{c+a}} \le 3.$$

1.75. If a,b,c are nonnegative real numbers, then

$$\sqrt{\frac{a}{4a+5b}}+\sqrt{\frac{b}{4b+5c}}+\sqrt{\frac{c}{4c+5a}} \le 1.$$

1.76. If a,b,c are positive real numbers, then

$$\frac{a}{\sqrt{4a^2+ab+4b^2}}+\frac{b}{\sqrt{4b^2+bc+4c^2}}+\frac{c}{\sqrt{4c^2+ca+4a^2}} \le 1.$$

1.77. If a,b,c are positive real numbers, then

$$\sqrt{\frac{a}{a+b+7c}}+\sqrt{\frac{b}{b+c+7a}}+\sqrt{\frac{c}{c+a+7b}} \ge 1.$$

1.78. If a,b,c are nonnegative real numbers, no two of which are zero, then

(a) $$\sqrt{\frac{a}{3b+c}}+\sqrt{\frac{b}{3c+a}}+\sqrt{\frac{c}{3a+b}} \ge \frac{3}{2};$$

(b) $$\sqrt{\frac{a}{2b+c}}+\sqrt{\frac{b}{2c+a}}+\sqrt{\frac{c}{2a+b}} \ge \sqrt[4]{8}.$$

1.79. If a,b,c are positive real numbers such that $ab+bc+ca=3$, then

(a) $$\frac{1}{(a+b)(3a+b)}+\frac{1}{(b+c)(3b+c)}+\frac{1}{(c+a)(3c+a)} \ge \frac{3}{8};$$

(b) $$\frac{1}{(2a+b)^2}+\frac{1}{(2b+c)^2}+\frac{1}{(2c+a)^2} \ge \frac{1}{3}.$$

1.80. If a, b, c are nonnegative real numbers, then

$$a^4 + b^4 + c^4 + 15(a^3b + b^3c + c^3a) \ge \frac{47}{4}(a^2b^2 + b^2c^2 + c^2a^2).$$

1.81. If a, b, c are nonnegative real numbers such that $a + b + c = 4$, then

$$a^3b + b^3c + c^3a \le 27.$$

1.82. Let a, b, c be nonnegative real numbers such that

$$a^2 + b^2 + c^2 = \frac{10}{3}(ab + bc + ca).$$

Prove that

$$a^4 + b^4 + c^4 \ge \frac{82}{27}(a^3b + b^3c + c^3a).$$

1.83. If a, b, c are positive real numbers, then

$$\frac{a^3}{2a^2 + b^2} + \frac{b^3}{2b^2 + c^2} + \frac{c^3}{2c^2 + a^2} \ge \frac{a + b + c}{3}.$$

1.84. If a, b, c are positive real numbers, then

$$\frac{a^4}{a^3 + b^3} + \frac{b^4}{b^3 + c^3} + \frac{c^4}{c^3 + a^3} \ge \frac{a + b + c}{2}.$$

1.85. If a, b, c are positive real numbers such that $abc = 1$, then

(a) $$3\left(\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}\right) + 4\left(\frac{b}{a^2} + \frac{c}{b^2} + \frac{a}{c^2}\right) \ge 7(a^2 + b^2 + c^2);$$

(b) $$8\left(\frac{a^3}{b} + \frac{b^3}{c} + \frac{c^3}{a}\right) + 5\left(\frac{b}{a^3} + \frac{c}{b^3} + \frac{a}{c^3}\right) \ge 13(a^3 + b^3 + c^3).$$

1.86. If a, b, c are positive real numbers, then

$$\frac{ab}{b^2 + bc + c^2} + \frac{bc}{c^2 + ca + a^2} + \frac{ca}{a^2 + ab + b^2} \le \frac{a^2 + b^2 + c^2}{ab + bc + ca}.$$

1.87. If a,b,c are positive real numbers, then

$$\frac{a-b}{b(2b+c)}+\frac{b-c}{c(2c+a)}+\frac{c-a}{a(2a+b)}\ge 0.$$

1.88. If a,b,c are positive real numbers, then

(a) $$\frac{a^2+6bc}{ab+2bc}+\frac{b^2+6ca}{bc+2ca}+\frac{c^2+6ab}{ca+2ab}\ge 7;$$

(b) $$\frac{a^2+7bc}{ab+bc}+\frac{b^2+7ca}{bc+ca}+\frac{c^2+7ab}{ca+ab}\ge 12.$$

1.89. If a,b,c are positive real numbers, then

(a) $$\frac{ab}{2b+c}+\frac{bc}{2c+a}+\frac{ca}{2a+b}\le\frac{a^2+b^2+c^2}{a+b+c};$$

(b) $$\frac{ab}{b+c}+\frac{bc}{c+a}+\frac{ca}{a+b}\le\frac{3(a^2+b^2+c^2)}{2(a+b+c)};$$

(c) $$\frac{ab}{4b+5c}+\frac{bc}{4c+5a}+\frac{ca}{4a+5b}\le\frac{a^2+b^2+c^2}{3(a+b+c)}.$$

1.90. If a,b,c are positive real numbers, then

(a) $$a\sqrt{b^2+8c^2}+b\sqrt{c^2+8a^2}+c\sqrt{a^2+8b^2}\le(a+b+c)^2;$$

(b) $$a\sqrt{b^2+3c^2}+b\sqrt{c^2+3a^2}+c\sqrt{a^2+3b^2}\le a^2+b^2+c^2+ab+bc+ca.$$

1.91. If a,b,c are positive real numbers, then

(a) $$\frac{1}{a\sqrt{a+2b}}+\frac{1}{b\sqrt{b+2c}}+\frac{1}{c\sqrt{c+2a}}\ge\sqrt{\frac{3}{abc}};$$

(b) $$\frac{1}{a\sqrt{a+8b}}+\frac{1}{b\sqrt{b+8c}}+\frac{1}{c\sqrt{c+8a}}\ge\sqrt{\frac{1}{abc}}.$$

1.92. If a,b,c are positive real numbers, then

$$\frac{a}{\sqrt{5a+4b}}+\frac{b}{\sqrt{5b+4c}}+\frac{c}{\sqrt{5c+4a}}\le\sqrt{\frac{a+b+c}{3}}.$$

1.93. If a, b, c are positive real numbers, then

(a) $$\frac{a}{\sqrt{a+b}}+\frac{b}{\sqrt{b+c}}+\frac{c}{\sqrt{c+a}} \geq \frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{\sqrt{2}};$$

(b) $$\frac{a}{\sqrt{a+b}}+\frac{b}{\sqrt{b+c}}+\frac{c}{\sqrt{c+a}} \geq \sqrt[4]{\frac{27(ab+bc+ca)}{4}}.$$

1.94. If a, b, c are nonnegative real numbers such that $a+b+c=3$, then

$$\sqrt{3a+b^2}+\sqrt{3b+c^2}+\sqrt{3c+a^2} \geq 6.$$

1.95. If a, b, c are nonnegative real numbers such that $a+b+c=3$, then

$$\sqrt{a^2+b^2+2bc}+\sqrt{b^2+c^2+2ca}+\sqrt{c^2+a^2+2ab} \geq 2(a+b+c).$$

1.96. If a, b, c are nonnegative real numbers, then

$$\sqrt{a^2+b^2+7bc}+\sqrt{b^2+c^2+7ca}+\sqrt{c^2+a^2+7ab} \geq 3\sqrt{3(ab+bc+ca)}.$$

1.97. If a, b, c are positive real numbers, then

$$\frac{a^2+3ab}{(b+c)^2}+\frac{b^2+3bc}{(c+a)^2}+\frac{c^2+3ca}{(a+b)^2} \geq 3.$$

1.98. If a, b, c are positive real numbers, then

$$\frac{a^2b+1}{a(b+1)}+\frac{b^2c+1}{b(c+1)}+\frac{c^2a+1}{c(a+1)} \geq 3.$$

1.99. If a, b, c are positive real numbers such that $a+b+c=3$, then

$$\sqrt{a^3+3b}+\sqrt{b^3+3c}+\sqrt{c^3+3a} \geq 6.$$

1.100. If a, b, c are positive real numbers such that $abc=1$, then

$$\sqrt{\frac{a}{a+6b+2bc}}+\sqrt{\frac{b}{b+6c+2ca}}+\sqrt{\frac{c}{c+6a+2ab}} \geq 1.$$

1.101. If a,b,c are positive real numbers such that $abc=1$, then

$$\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2 \ge 6(a+b+c-1).$$

1.102. If a,b,c are positive real numbers, then

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a} \ge \frac{a+b+c}{a+b+c-\sqrt[3]{abc}}.$$

1.103. If a,b,c are positive real numbers such that $a+b+c=3$, then

$$a\sqrt{b^2+b+1}+b\sqrt{c^2+c+1}+c\sqrt{a^2+a+1} \le 3\sqrt{3}.$$

1.104. If a,b,c are positive real numbers, then

$$\frac{1}{b(a+2b+3c)^2}+\frac{1}{c(b+2c+3a)^2}+\frac{1}{a(c+2a+3b)^2} \le \frac{1}{12abc}.$$

1.105. Let a,b,c be positive real numbers such that $a+b+c=3$. Prove that

(a) $$\frac{a^2+9b}{b+c}+\frac{b^2+9c}{c+a}+\frac{c^2+9a}{a+b} \ge 15;$$

(b) $$\frac{a^2+3b}{a+b}+\frac{b^2+3c}{b+c}+\frac{c^2+3a}{c+a} \ge 6.$$

1.106. If $a,b,c \in [0,1]$, then

(a) $$\frac{bc}{2ab+1}+\frac{ca}{2bc+1}+\frac{ab}{2ca+1} \le 1.$$

(b) $$\frac{a}{ab+1}+\frac{b}{bc+1}+\frac{c}{ca+1} \le \frac{3}{2}.$$

1.107. If a,b,c are nonnegative real numbers, then

$$a^4+b^4+c^4+5(a^3b+b^3c+c^3a) \ge 6(a^2b^2+b^2c^2+c^2a^2).$$

1.108. If a, b, c are positive real numbers, then

$$a^5 + b^5 + c^5 - a^4b - b^4c - c^4a \geq 2abc(a^2 + b^2 + c^2 - ab - bc - ca).$$

1.109. If a, b, c are positive real numbers such that $a^2 + b^2 + c^2 = 3$, then

$$\frac{a}{1+b} + \frac{b}{1+c} + \frac{c}{1+a} \geq \frac{3}{2}.$$

1.110. If a, b, c are nonnegative real numbers such that $a + b + c = 3$, then

$$a\sqrt{a+b} + b\sqrt{b+c} + c\sqrt{c+a} \geq 3\sqrt{2}.$$

1.111. If a, b, c are positive real numbers such that $a + b + c = 3$, then

$$\frac{a}{2b^2 + c} + \frac{b}{2c^2 + a} + \frac{c}{2a^2 + b} \geq 1.$$

1.112. If a, b, c are positive real numbers such that $a + b + c = ab + bc + ca$, then

$$\frac{1}{a^2 + b + 1} + \frac{1}{b^2 + c + 1} + \frac{1}{c^2 + a + 1} \leq 1.$$

1.113. If a, b, c are positive real numbers, then

$$\frac{1}{(a + 2b + 3c)^2} + \frac{1}{(b + 2c + 3a)^2} + \frac{1}{(c + 2a + 3b)^2} \leq \frac{1}{4(ab + bc + ca)}.$$

1.114. If a, b, c are positive real numbers, then

$$\sqrt{\frac{a}{a + b + 2c}} + \sqrt{\frac{b}{b + c + 2a}} + \sqrt{\frac{c}{c + a + 2b}} \leq \frac{3}{2}.$$

1.115. If a, b, c are positive real numbers, then

$$\sqrt{\frac{5a}{a + b + 3c}} + \sqrt{\frac{5b}{b + c + 3a}} + \sqrt{\frac{5c}{c + a + 3b}} \leq 3.$$

1.116. If $a, b, c \in [0, 1]$, then

$$ab^2 + bc^2 + ca^2 + \frac{5}{4} \ge a + b + c.$$

1.117. If a, b, c are nonnegative real numbers such that

$$a + b + c = 3, \qquad a \le b \le 1 \le c,$$

then

$$a^2b + b^2c + c^2a \le 3.$$

1.118. Let a, b, c be nonnegative real numbers such that

$$a + b + c = 3, \qquad a \le 1 \le b \le c.$$

Prove that

(a) $$a^2b + b^2c + c^2a \ge ab + bc + ca;$$

(b) $$a^2b + b^2c + c^2a \ge abc + 2;$$

(c) $$\frac{1}{abc} + 2 \ge \frac{9}{a^2b + b^2c + c^2a};$$

(d) $$ab^2 + bc^2 + ca^2 \ge 3.$$

1.119. If a, b, c are nonnegative real numbers such that

$$a + b + c = 3, \qquad a \le 1 \le b \le c,$$

then

(a) $$\frac{5-2a}{1+b} + \frac{5-2b}{1+c} + \frac{5-2c}{1+a} \ge \frac{9}{2};$$

(b) $$\frac{3-2b}{1+a} + \frac{3-2c}{1+b} + \frac{3-2a}{1+c} \le \frac{3}{2}.$$

1.120. If a, b, c are nonnegative real numbers such that

$$ab + bc + ca = 3, \qquad a \le 1 \le b \le c,$$

then

(a) $$a^2b + b^2c + c^2a \ge 3;$$

(b) $$ab^2 + bc^2 + ca^2 + 3(\sqrt{3} - 1)abc \ge 3\sqrt{3}.$$

1.121. If a, b, c are nonnegative real numbers such that

$$a^2+b^2+c^2=3, \qquad a \le 1 \le b \le c,$$

then

(a) $$a^2b+b^2c+c^2a \ge 2abc+1;$$

(b) $$2(ab^2+bc^2+ca^2) \ge 3abc+3.$$

1.122. If a, b, c are nonnegative real numbers such that

$$ab+bc+ca=3, \qquad a \le b \le 1 \le c,$$

then

$$ab^2+bc^2+ca^2+3abc \ge 6.$$

1.123. If a, b, c are nonnegative real numbers such that

$$a^2+b^2+c^2=3, \qquad a \le b \le 1 \le c,$$

then

$$2(a^2b+b^2c+c^2a) \le 3abc+3.$$

1.124. If a, b, c are nonnegative real numbers such that

$$a^2+b^2+c^2=3, \qquad a \le b \le 1 \le c,$$

then

$$2(a^3b+b^3c+c^3a) \le abc+5.$$

1.125. If a, b, c are real numbers, then

$$(a^2+b^2+c^2)^2 \ge 3(a^3b+b^3c+c^3a).$$

1.126. If a, b, c are real numbers, then

$$a^4+b^4+c^4+ab^3+bc^3+ca^3 \ge 2(a^3b+b^3c+c^3a).$$

1.127. If a,b,c are positive real numbers, then

(a) $$\frac{a^2}{ab+2c^2}+\frac{b^2}{bc+2a^2}+\frac{c^2}{ca+2b^2}\ge 1;$$

(b) $$\frac{a^3}{a^2b+2c^3}+\frac{b^3}{b^2c+2a^3}+\frac{c^3}{c^2a+2b^3}\ge 1.$$

1.128. If a,b,c are positive real numbers such that $a+b+c=3$, then

$$\frac{a}{ab+1}+\frac{b}{bc+1}+\frac{c}{ca+1}\ge\frac{3}{2}.$$

1.129. If a,b,c are positive real numbers such that $a+b+c=3$, then

$$\frac{a}{3a+b^2}+\frac{b}{3b+c^2}+\frac{c}{3c+a^2}\le\frac{3}{2}.$$

1.130. If a,b,c are positive real numbers such that $a+b+c=3$, then

$$\frac{a}{b^2+c}+\frac{b}{c^2+a}+\frac{c}{a^2+b}\ge\frac{3}{2}.$$

1.131. If a,b,c are positive real numbers such that $abc=1$, then

$$\frac{a}{b^3+2}+\frac{b}{c^3+2}+\frac{c}{a^3+2}\ge 1.$$

1.132. Let a,b,c be positive real numbers such that

$$a^m+b^m+c^m=3,$$

where $m>0$. Prove that

$$\frac{a^{m-1}}{b}+\frac{b^{m-1}}{c}+\frac{c^{m-1}}{a}\ge 3.$$

1.133. If a,b,c are positive real numbers, then

(a) $$\frac{1}{4a}+\frac{1}{4b}+\frac{1}{4c}+\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\ge 3\left(\frac{1}{3a+b}+\frac{1}{3b+c}+\frac{1}{3c+a}\right);$$

(b) $$\frac{1}{4a}+\frac{1}{4b}+\frac{1}{4c}+\frac{1}{a+3b}+\frac{1}{b+3c}+\frac{1}{c+3a}\ge 2\left(\frac{1}{3a+b}+\frac{1}{3b+c}+\frac{1}{3c+a}\right).$$

1.134. If a, b, c are positive real numbers such that $a^6+b^6+c^6=3$, then

$$\frac{a^5}{b}+\frac{b^5}{c}+\frac{c^5}{a}\geq 3.$$

1.135. If a, b, c are positive real numbers such that $a^2+b^2+c^2=3$, then

$$\frac{a^3}{a+b^5}+\frac{b^3}{b+c^5}+\frac{c^3}{c+a^5}\geq \frac{3}{2}.$$

1.136. If a, b, c are real numbers such that $a^2+b^2+c^2=3$, then

$$a^2b+b^2c+c^2a+9\geq 4(a+b+c).$$

1.137. If a, b, c are real numbers such that $a^2+b^2+c^2=3$, then

$$a^2b+b^2c+c^2a+3\geq a+b+c+ab+bc+ca.$$

1.138. If a, b, c are positive real numbers such that $a+b+c=3$, then

$$\frac{12}{a^2b+b^2c+c^2a}\leq 3+\frac{1}{abc}.$$

1.139. If a, b, c are positive real numbers such that $a+b+c=3$, then

$$\frac{24}{a^2b+b^2c+c^2a}+\frac{1}{abc}\geq 9.$$

1.140. Let a, b, c be nonnegative real numbers such that

$$2(a^2+b^2+c^2)=5(ab+bc+ca).$$

Prove that

(a) $$8(a^4+b^4+c^4)\geq 17(a^3b+b^3c+c^3a);$$

(b) $$16(a^4+b^4+c^4)\geq 34(a^3b+b^3c+c^3a)+81abc(a+b+c).$$

1.141. Let a,b,c be nonnegative real numbers such that

$$2(a^2+b^2+c^2)=5(ab+bc+ca).$$

Prove that

(a) $$2(a^3b+b^3c+c^3a)\geq a^2b^2+b^2c^2+c^2a^2+abc(a+b+c);$$

(b) $$11(a^4+b^4+c^4)\geq 17(a^3b+b^3c+c^3a)+129abc(a+b+c);$$

(c) $$a^3b+b^3c+c^3a\leq \frac{14+\sqrt{102}}{8}(a^2b^2+b^2c^2+c^2a^2).$$

1.142. If a,b,c are real numbers such that

$$a^3b+b^3c+c^3a\leq 0,$$

then

$$a^2+b^2+c^2\geq k(ab+bc+ca),$$

where

$$k=\frac{1+\sqrt{21+8\sqrt{7}}}{2}\approx 3.7468.$$

1.143. If a,b,c are real numbers such that

$$a^3b+b^3c+c^3a\geq 0,$$

then

$$a^2+b^2+c^2+k(ab+bc+ca)\geq 0,$$

where

$$k=\frac{-1+\sqrt{21+8\sqrt{7}}}{2}\approx 2.7468.$$

1.144. If a,b,c are real numbers such that

$$k(a^2+b^2+c^2)=ab+bc+ca,\qquad k\in\left(\frac{-1}{2},1\right),$$

then

$$\alpha_k\leq \frac{a^3b+b^3c+c^3}{(a^2+b^2+c^2)^2}\leq \beta_k,$$

where

$$27\alpha_k=1+13k-5k^2-2(1-k)(1+2k)\sqrt{\frac{7(1-k)}{1+2k}},$$

$$27\beta_k=1+13k-5k^2+2(1-k)(1+2k)\sqrt{\frac{7(1-k)}{1+2k}}.$$

1.145. If a, b, c are positive real numbers such that $a+b+c=3$, then

$$\frac{a^2}{4a+b^2}+\frac{b^2}{4b+c^2}+\frac{c^2}{4c+a^2}\ge\frac{3}{5}.$$

1.146. If a, b, c are positive real numbers, then

$$\frac{a^2+bc}{a+b}+\frac{b^2+ca}{b+c}+\frac{c^2+ab}{c+a}\le\frac{(a+b+c)^3}{3(ab+bc+ca)}.$$

1.147. If a, b, c are positive real numbers such that $a+b+c=3$, then

$$\sqrt{ab^2+bc^2}+\sqrt{bc^2+ca^2}+\sqrt{ca^2+ab^2}\le 3\sqrt{2}.$$

1.148. If a, b, c are positive real numbers such that $a^5+b^5+c^5=3$, then

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\ge 3.$$

1.149. Let $P(a, b, c)$ be a cyclic homogeneous polynomial of degree three. The inequality

$$P(a, b, c)\ge 0$$

holds for all $a, b, c\ge 0$ if and only if the following two conditions are fulfilled:

(a) $P(1, 1, 1)\ge 0$;

(b) $P(0, b, c)\ge 0$ for all $b, c\ge 0$.

1.150. If a, b, c are nonnegative real numbers such that $a+b+c=3$, then

$$8(a^2b+b^2c+c^2a)+9\ge 11(ab+bc+ca).$$

1.151. If a, b, c are nonnegative real numbers such that $a+b+c=6$, then

$$a^3+b^3+c^3+8(a^2b+b^2c+c^2a)\ge 166.$$

1.152. If a, b, c are nonnegative real numbers, then

$$a^3+b^3+c^3-3abc\ge\sqrt{9+6\sqrt{3}}\,(a-b)(b-c)(c-a).$$

1.153. If a,b,c are nonnegative real numbers, no two of which are zero, then

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+7 \ge \frac{17}{3}\left(\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}\right).$$

1.154. Let a,b,c be nonnegative real numbers, no two of which are zero. If $0 \le k \le 5$, then

$$\frac{ka+b}{a+c}+\frac{kb+c}{b+a}+\frac{kc+a}{c+b} \ge \frac{3}{2}(k+1).$$

1.155. Let a,b,c be nonnegative real numbers, no two of which are zero. If $k \le \frac{23}{8}$, then

$$\frac{ka+b}{2a+c}+\frac{kb+c}{2b+a}+\frac{kc+a}{2c+b} \ge k+1.$$

1.156. Let a,b,c be nonnegative real numbers. Prove that

(a) if $k \le 1-\frac{2}{5\sqrt{5}}$, then

$$\frac{ka+b}{2a+b+c}+\frac{kb+c}{a+2b+c}+\frac{kc+a}{a+b+2c} \ge \frac{3}{4}(k+1).$$

(b) if $k \ge 1+\frac{2}{5\sqrt{5}}$, then

$$\frac{ka+b}{2a+b+c}+\frac{kb+c}{a+2b+c}+\frac{kc+a}{a+b+2c} \le \frac{3}{4}(k+1).$$

1.157. If a,b,c are positive real numbers such that $a \le b \le c$, then

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+3 \ge 2\left(\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}\right).$$

1.158. If $a \ge b \ge c \ge 0$, then

$$\frac{3a+b}{2a+c}+\frac{3b+c}{2b+a}+\frac{3c+a}{2c+b} \ge 4.$$

1.159. Let a,b,c be nonnegative real numbers such that

$$a \ge b \ge 1 \ge c, \qquad a+b+c=3.$$

Prove that

$$\frac{1}{a^2+3}+\frac{1}{b^2+3}+\frac{1}{c^2+3} \le \frac{3}{4}.$$

1.160. Let a,b,c be nonnegative real numbers such that

$$a \ge 1 \ge b \ge c, \qquad a+b+c=3.$$

Prove that

$$\frac{1}{a^2+2}+\frac{1}{b^2+2}+\frac{1}{c^2+2} \ge 1.$$

1.161. Let a,b,c be real numbers such that

$$a \ge b \ge 1 \ge c \ge -5, \qquad a+b+c=3.$$

Prove that

$$\frac{6}{a^3+b^3+c^3}+1 \ge \frac{8}{a^2+b^2+c^2}.$$

1.162. If $a \ge 1 \ge b \ge c > -3$ such that $ab+bc+ca=3$, then

$$\frac{1}{a^2+ab+b^2}+\frac{1}{b^2+bc+c^2}+\frac{1}{c^2+ca+a^2} \ge 1.$$

1.163. If $a \ge b \ge 1 \ge c \ge 0$ such that $a+b+c=3$, then

$$\frac{1}{a^2+ab+b^2}+\frac{1}{b^2+bc+c^2}+\frac{1}{c^2+ca+a^2} \le \frac{3}{ab+bc+ca}.$$

1.164. If a,b,c are positive real numbers such that

$$a \ge 1 \ge b \ge c, \qquad abc=1,$$

then

$$\frac{1-a}{3+a^2}+\frac{1-b}{3+b^2}+\frac{1-c}{3+c^2} \ge 0.$$

1.165. If a, b, c are positive real numbers such that

$$a \geq 1 \geq b \geq c, \qquad abc = 1,$$

then

$$\frac{1}{\sqrt{3a+1}} + \frac{1}{\sqrt{3b+1}} + \frac{1}{\sqrt{3c+1}} \geq \frac{3}{2}.$$

1.166. If a, b, c are positive real numbers such that

$$a \geq 1 \geq b \geq c, \qquad abc = 1,$$

then

$$\frac{1}{a^2+4ab+b^2} + \frac{1}{b^2+4bc+c^2} + \frac{1}{c^2+4ca+a^2} \geq \frac{1}{2}.$$

1.167. Let $a \geq 1 \geq b \geq c \geq 0$ such that

$$a+b+c = 3, \qquad ab+bc+ca = q,$$

where $q \in [0, 3]$ is a fixed number. Prove that the product $r = abc$ is maximal for $b = c$, and minimal for $b = 1$ or $c = 0$.

1.168. Let p and q be fixed real numbers such that there exist three real numbers a, b, c satisfying

$$a \geq 1 \geq b \geq c \geq 0, \qquad a+b+c = p, \qquad ab+bc+ca = q.$$

Prove that

(a) the product $r = abc$ is maximal for $b = c$;

(b) the product $r = abc$ is minimal for $a = 1$ or $b = 1$ or $c = 0$.

1.169. Let p and q be fixed real numbers such that there exist three real numbers a, b, c satisfying

$$a \geq b \geq c \geq 1, \qquad a+b+c = p, \qquad ab+bc+ca = q.$$

Prove that

(a) the product $r = abc$ is maximal for $b = c$;

(b) the product $r = abc$ is minimal for $a = b$ or $c = 1$.

1.170. Let $a \geq b \geq 1 \geq c \geq 0$ such that

$$a+b+c=3, \qquad ab+bc+ca=q,$$

where $q \in [0,3]$ is a fixed number. Prove that the product $r = abc$ is maximal for $b = 1$, and minimal for $a = b$ or $c = 0$.

1.171. Let p and q be fixed real numbers such that there exist three real numbers a, b, c satisfying

$$a \geq b \geq 1 \geq c \geq 0, \qquad a+b+c=p, \qquad ab+bc+ca=q.$$

Prove that

(a) the product $r = abc$ is maximal for $b = 1$ or $c = 1$;

(b) the product $r = abc$ is minimal for $a = b$ or $c = 0$.

1.172. Let p and q be fixed real numbers such that there exist three real numbers a, b, c satisfying

$$1 \geq a \geq b \geq c \geq 0, \qquad a+b+c=p, \qquad ab+bc+ca=q.$$

Prove that

(a) the product $r = abc$ is maximal for $b = c$ or $a = 1$;

(b) the product $r = abc$ is minimal for $a = b$ or $c = 0$.

1.173. If $a \geq 1 \geq b \geq c \geq 0$ such that $a+b+c=3$, then

$$abc + \frac{9}{ab+bc+ca} \geq 4.$$

1.174. If $a \geq 1 \geq b \geq c \geq 0$ such that $a+b+c=3$, then

$$abc + \frac{2}{ab+bc+ca} \geq \frac{5}{a^2+b^2+c^2}.$$

1.175. If $a \geq b \geq 1 \geq c > 0$ such that $a+b+c=3$, then

$$\frac{1}{abc} + 2 \geq \frac{9}{ab+bc+ca}.$$

1.176. If $a \ge b \ge 1 \ge c > 0$ such that $a+b+c=3$, then

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+11 \ge 4(a^2+b^2+c^2).$$

1.177. If $a \ge b \ge 1 \ge c > 0$ such that $a+b+c=3$, then

$$\frac{1}{abc}+\frac{2}{a^2+b^2+c^2} \ge \frac{5}{ab+bc+ca}.$$

1.178. If $a \ge b \ge 1 \ge c \ge 0$ such that $a+b+c=3$, then

$$\frac{9}{a^3+b^3+c^3}+2 \le \frac{15}{a^2+b^2+c^2}.$$

1.179. If $a \ge b \ge 1 \ge c \ge 0$ such that $a+b+c=3$, then

$$\frac{36}{a^3+b^3+c^3}+9 \le \frac{65}{a^2+b^2+c^2}.$$

1.180. If $a \ge b \ge c \ge 0$ and $ab+bc+ca=2$, then

$$\sqrt{a+ab}+\sqrt{b+bc}+\sqrt{c+ca} \ge 3.$$

1.181. If $a \ge b \ge c$ are nonnegative numbers such that $ab+bc+ca=3$, then

$$\sqrt{a+2ab}+\sqrt{b+2bc}+\sqrt{c+2ca} \ge 4.$$

1.182. If a,b,c are nonnegative real numbers such that $ab+bc+ca=3$, then

$$\sqrt{a+3b}+\sqrt{b+3c}+\sqrt{c+3a} \ge 6.$$

1.183. If a,b,c are the lengths of the sides of a triangle, then

$$10\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) > 9\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right).$$

1.184. If a, b, c are the lengths of the sides of a triangle, then

$$\frac{a}{3a+b-c}+\frac{b}{3b+c-a}+\frac{c}{3c+a-b} \geq 1.$$

1.185. If a, b, c are the lengths of the sides of a triangle, then

$$\frac{a^2-b^2}{a^2+bc}+\frac{b^2-c^2}{b^2+ca}+\frac{c^2-a^2}{c^2+ab} \leq 0.$$

1.186. If a, b, c are the lengths of the sides of a triangle, then

$$a^2(a+b)(b-c)+b^2(b+c)(c-a)+c^2(c+a)(a-b) \geq 0.$$

1.187. If a, b, c are the lengths of the sides of a triangle, then

$$a^2b+b^2c+c^2a \geq \sqrt{abc(a+b+c)(a^2+b^2+c^2)}.$$

1.188. If a, b, c are the lengths of the sides of a triangle, then

$$a^2\left(\frac{b}{c}-1\right)+b^2\left(\frac{c}{a}-1\right)+c^2\left(\frac{a}{b}-1\right) \geq 0.$$

1.189. If a, b, c are the lengths of the sides of a triangle, then

(a) $$a^3b+b^3c+c^3a \geq a^2b^2+b^2c^2+c^2a^2;$$

(b) $$3(a^3b+b^3c+c^3a) \geq (ab+bc+ca)(a^2+b^2+c^2);$$

(c) $$\frac{a^3b+b^3c+c^3}{3} \geq \left(\frac{a+b+c}{3}\right)^4.$$

1.190. If a, b, c are the lengths of the sides of a triangle, then

$$2\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right) \geq \frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}+3.$$

1.191. If a,b,c are the lengths of the sides of a triangle such that $a<b<c$, then

$$\frac{a^2}{a^2-b^2}+\frac{b^2}{b^2-c^2}+\frac{c^2}{c^2-a^2}\le 0.$$

1.192. If a,b,c are the lengths of the sides of a triangle, then

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+3\ge 2\left(\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}\right).$$

1.193. Let a,b,c be the lengths of the sides of a triangle. If $k\ge 2$, then

$$a^kb(a-b)+b^kc(b-c)+c^ka(c-a)\ge 0.$$

1.194. Let a,b,c be the lengths of the sides of a triangle. If $k\ge 1$, then

$$3(a^{k+1}b+b^{k+1}c+c^{k+1}a)\ge (a+b+c)(a^kb+b^kc+c^ka).$$

1.195. Let a,b,c,d be positive real numbers such that $a+b+c+d=4$. Prove that

$$\frac{a}{3+b}+\frac{b}{3+c}+\frac{c}{3+d}+\frac{d}{3+a}\ge 1.$$

1.196. Let a,b,c,d be positive real numbers such that $a+b+c+d=4$. Prove that

$$\frac{a}{1+b^2}+\frac{b}{1+c^2}+\frac{c}{1+d^2}+\frac{d}{1+a^2}\ge 2.$$

1.197. If a,b,c,d are nonnegative real numbers such that $a+b+c+d=4$, then

$$a^2bc+b^2cd+c^2da+d^2ab\le 4.$$

1.198. If a,b,c,d are nonnegative real numbers such that $a+b+c+d=4$, then

$$a(b+c)^2+b(c+d)^2+c(d+a)^2+d(a+b)^2\le 16.$$

1.199. If a, b, c, d are positive real numbers, then

$$\frac{a-b}{b+c}+\frac{b-c}{c+d}+\frac{c-d}{d+a}+\frac{d-a}{a+b} \geq 0.$$

1.200. If a, b, c, d are positive real numbers, then

(a) $$\frac{a-b}{a+2b+c}+\frac{b-c}{b+2c+d}+\frac{c-d}{c+2d+a}+\frac{d-a}{d+2a+b} \geq 0;$$

(b) $$\frac{a}{2a+b+c}+\frac{b}{2b+c+d}+\frac{c}{2c+d+a}+\frac{d}{2d+a+b} \leq 1.$$

1.201. If a, b, c, d are positive real numbers such that $abcd = 1$, then

$$\frac{1}{a(a+b)}+\frac{1}{b(b+c)}+\frac{1}{c(c+d)}+\frac{1}{d(d+a)} \geq 2.$$

1.202. If a, b, c, d are positive real numbers, then

$$\frac{1}{a(1+b)}+\frac{1}{b(1+c)}+\frac{1}{c(1+d)}+\frac{1}{d(1+a)} \geq \frac{16}{1+8\sqrt{abcd}}.$$

1.203. If a, b, c, d are nonnegative real numbers such that $a^2+b^2+c^2+d^2=4$, then

(a) $$3(a+b+c+d) \geq 2(ab+bc+cd+da)+4;$$

(b) $$a+b+c+d-4 \geq (2-\sqrt{2})(ab+bc+cd+da-4).$$

1.204. Let a, b, c, d be positive real numbers.

(a) If $a, b, c, d \geq 1$, then

$$\left(a+\frac{1}{b}\right)\left(b+\frac{1}{c}\right)\left(c+\frac{1}{d}\right)\left(d+\frac{1}{a}\right) \geq (a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right);$$

(b) If $abcd = 1$, then

$$\left(a+\frac{1}{b}\right)\left(b+\frac{1}{c}\right)\left(c+\frac{1}{d}\right)\left(d+\frac{1}{a}\right) \geq (a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right).$$

1.205. If a,b,c,d are positive real numbers, then

$$\left(1+\frac{a}{a+b}\right)^2+\left(1+\frac{b}{b+c}\right)^2+\left(1+\frac{c}{c+d}\right)^2+\left(1+\frac{d}{d+a}\right)^2>7.$$

1.206. If a,b,c,d are positive real numbers, then

$$\frac{a^2-bd}{b+2c+d}+\frac{b^2-ca}{c+2d+a}+\frac{c^2-db}{d+2a+b}+\frac{d^2-ac}{a+2b+c}\ge 0.$$

1.207. If a,b,c,d are positive real numbers such that $a\le b\le c\le d$, then

$$\sqrt{\frac{2a}{a+b}}+\sqrt{\frac{2b}{b+c}}+\sqrt{\frac{2c}{c+d}}+\sqrt{\frac{2d}{d+a}}\le 4.$$

1.208. Let a,b,c,d be nonnegative real numbers, and let

$$x=\frac{a}{b+c},\quad y=\frac{b}{c+d},\quad z=\frac{c}{d+a},\quad t=\frac{d}{a+b}.$$

Prove that

(a) $$\sqrt{xz}+\sqrt{yt}\le 1;$$

(b) $$x+y+z+t+4(xz+yt)\ge 4.$$

1.209. If a,b,c,d are nonnegative real numbers, then

$$\left(1+\frac{2a}{b+c}\right)\left(1+\frac{2b}{c+d}\right)\left(1+\frac{2c}{d+a}\right)\left(1+\frac{2d}{a+b}\right)\ge 9.$$

1.210. Let a,b,c,d be nonnegative real numbers. If $k>0$, then

$$\left(1+\frac{ka}{b+c}\right)\left(1+\frac{kb}{c+d}\right)\left(1+\frac{kc}{d+a}\right)\left(1+\frac{kd}{a+b}\right)\ge(1+k)^2.$$

1.211. If a,b,c,d are positive real numbers such that $a+b+c+d=4$, then

$$\frac{1}{ab}+\frac{1}{bc}+\frac{1}{cd}+\frac{1}{da}\ge a^2+b^2+c^2+d^2.$$

1.212. If a, b, c, d are positive real numbers, then

$$\frac{a^2}{(a+b+c)^2} + \frac{b^2}{(b+c+d)^2} + \frac{c^2}{(c+d+a)^2} + \frac{d^2}{(d+a+b)^2} \ge \frac{4}{9}.$$

1.213. If a, b, c, d are positive real numbers such that $a+b+c+d=3$, then

$$ab(b+c)+bc(c+d)+cd(d+a)+da(a+b) \le 4.$$

1.214. If $a \ge b \ge c \ge d \ge 0$ and $a+b+c+d=2$, then

$$ab(b+c)+bc(c+d)+cd(d+a)+da(a+b) \le 1.$$

1.215. Let a, b, c, d be nonnegative real numbers such that $a+b+c+d=4$. If $k \ge \dfrac{37}{27}$, then

$$ab(b+kc)+bc(c+kd)+cd(d+ka)+da(a+kb) \le 4(1+k).$$

1.216. If a, b, c, d are nonnegative real numbers such that $a+b+c+d=4$, then

$$\sqrt{\frac{3a}{b+2}} + \sqrt{\frac{3b}{c+2}} + \sqrt{\frac{3c}{d+2}} + \sqrt{\frac{3d}{a+2}} \le 4.$$

1.217. Let a, b, c, d be positive real numbers such that $a \le b \le c \le d$. Prove that

$$2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\right) \ge 4+\frac{a}{c}+\frac{c}{a}+\frac{b}{d}+\frac{d}{b}.$$

1.218. Let a, b, c, d be positive real numbers such that

$$a \le b \le c \le d, \qquad abcd = 1.$$

Prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a} \ge ab+bc+cd+da.$$

1.219. Let a,b,c,d be positive real numbers such that

$$a \le b \le c \le d, \qquad abcd = 1.$$

Prove that

$$4 + \frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \ge 2(a+b+c+d).$$

1.220. Let $A = \{a_1, a_2, a_3, a_4\}$ be a set of real numbers such that

$$a_1 + a_2 + a_3 + a_4 = 0.$$

Prove that there exists a permutation $\{a,b,c,d\}$ of A such that

$$a^2 + b^2 + c^2 + d^2 + 3(ab + bc + cd + da) \ge 0.$$

1.221. If a,b,c,d are nonnegative real numbers such that

$$a \ge b \ge 1 \ge c \ge d, \qquad a+b+c+d = 3,$$

then

$$a^2 + b^2 + c^2 + d^2 + 10abcd \le 5.$$

1.222. If a,b,c,d are nonnegative real numbers such that

$$a \ge b \ge 1 \ge c \ge d, \qquad a+b+c+d = 6,$$

then

$$a^2 + b^2 + c^2 + d^2 + 4abcd \le 26.$$

1.223. Let a,b,c,d be nonnegative real numbers such that

$$a \ge b \ge 1 \ge c \ge d, \qquad a+b+c+d = p, \quad p \ge 2.$$

Prove that

$$\frac{p^2 - 4p + 8}{2} \le a^2 + b^2 + c^2 + d^2 \le p^2 - 2p + 2.$$

1.224. Let $a \ge b \ge 1 \ge c \ge d \ge 0$ such that

$$a+b+c+d = 4, \qquad a^2 + b^2 + c^2 + d^2 = q,$$

where $q \in [4,10]$ is a fixed number. Prove that the product $r = abcd$ is maximal when $b = 1$ and $c = d$.

1.225. If a, b, c, d are nonnegative real numbers such that

$$a \geq b \geq 1 \geq c \geq d, \qquad a + b + c + d = 4,$$

then

$$a^2 + b^2 + c^2 + d^2 + 6abcd \leq 10.$$

1.226. If a, b, c, d are nonnegative real numbers such that

$$a \geq b \geq 1 \geq c \geq d, \qquad a + b + c + d = 4,$$

then

$$a^2 + b^2 + c^2 + d^2 + 6\sqrt{abcd} \leq 10.$$

1.227. If a, b, c, d, e are positive real numbers, then

$$\frac{a}{a + 2b + 2c} + \frac{b}{b + 2c + 2d} + \frac{c}{c + 2d + 2e} + \frac{d}{d + 2e + 2a} + \frac{e}{e + 2a + 2b} \geq 1.$$

1.228. Let a, b, c, d, e be positive real numbers such that $a+b+c+d+e = 5$. Prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{e} + \frac{e}{a} \leq 1 + \frac{4}{abcde}.$$

1.229. If a, b, c, d, e are real numbers such that $a + b + c + d + e = 0$, then

$$\frac{-\sqrt{5} - 1}{4} \leq \frac{ab + bc + cd + de + ea}{a^2 + b^2 + c^2 + d^2 + e^2} \leq \frac{\sqrt{5} - 1}{4}.$$

1.230. Let a, b, c, d, e be positive real numbers such that

$$a^2 + b^2 + c^2 + d^2 + e^2 = 5.$$

Prove that

$$\frac{a^2}{b + c + d} + \frac{b^2}{c + d + e} + \frac{c^2}{d + e + a} + \frac{d^2}{e + a + b} + \frac{e^2}{a + b + c} \geq \frac{5}{3}.$$

1.231. Let a, b, c, d, e be nonnegative real numbers such that $a + b + c + d + e = 5$. Prove that

$$(a^2 + b^2)(b^2 + c^2)(c^2 + d^2)(d^2 + e^2)(e^2 + a^2) \leq \frac{729}{2}.$$

1.232. If $a,b,c,d,e \in [1,5]$, then

$$\frac{a-b}{b+c}+\frac{b-c}{c+d}+\frac{c-d}{d+e}+\frac{d-e}{e+a}+\frac{e-a}{a+b} \ge 0.$$

1.233. If $a,b,c,d,e,f \in [1,3]$, then

$$\frac{a-b}{b+c}+\frac{b-c}{c+d}+\frac{c-d}{d+e}+\frac{d-e}{e+f}+\frac{e-f}{f+a}+\frac{f-a}{a+b} \ge 0.$$

1.234. If $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ are positive real numbers, then

$$\sum_{i=1}^{n} \frac{a_i}{a_{i-1}+2a_i+a_{i+1}} \le \frac{n}{4},$$

where $a_0 = a_n$ and $a_{n+1} = a_1$.

1.235. Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be positive real numbers such that $a_1a_2\cdots a_n = 1$. Prove that

$$\frac{1}{n-2+a_1+a_2}+\frac{1}{n-2+a_2+a_3}+\cdots+\frac{1}{n-2+a_n+a_1} \le 1.$$

1.236. If $a_1, a_2, ..., a_n \ge 1$, then

$$\prod\left(a_1+\frac{1}{a_2}+n-2\right) \ge n^{n-2}(a_1+a_2+\cdots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right);$$

1.237. If $a_1, a_2, ..., a_n \ge 1$, then

$$\left(a_1+\frac{1}{a_1}\right)\left(a_2+\frac{1}{a_2}\right)\cdots\left(a_n+\frac{1}{a_n}\right)+2^n \ge 2\left(1+\frac{a_1}{a_2}\right)\left(1+\frac{a_2}{a_3}\right)\cdots\left(1+\frac{a_n}{a_1}\right).$$

1.238. Let k and n be positive integers, and let $a_1, a_2, ..., a_n$ be real numbers such that

$$a_1 \le a_2 \le \cdots \le a_n.$$

Consider the inequality

$$(a_1+a_2+\cdots+a_n)^2 \ge n(a_1a_{k+1}+a_2a_{k+2}+\cdots+a_na_{n+k}),$$

where $a_{n+i} = a_i$ for any positive integer i. Prove this inequality for

(a) $n = 2k$;

(b) $n = 4k$.

1.239. If $a_1, a_2, \ldots, a_n$ are real numbers, then

$$a_1(a_1+a_2)+a_2(a_2+a_3)+\cdots+a_n(a_n+a_1) \geq \frac{2}{n}(a_1+a_2+\cdots+a_n)^2.$$

1.240. If $a_1,\ a_2,\ \ldots,\ a_n \in [1,\ 2]$, then

$$\sum_{i=1}^{n} \frac{3}{a_i+2a_{i+1}} \geq \sum_{i=1}^{n} \frac{2}{a_i+a_{i+1}},$$

where $a_{n+1}=a_1$.

1.241. Let $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ be real numbers such that $a_1+a_2+\cdots+a_n=n$.

(a) If $a_1 \geq 1 \geq a_2 \geq \cdots \geq a_n$, then

$$a_1^3+a_2^3+\cdots+a_n^3+2n \geq 3(a_1^2+a_2^2+\cdots+a_n^2);$$

(b) If $a_1 \leq 1 \leq a_2 \leq \cdots \leq a_n$, then

$$a_1^3+a_2^3+\cdots+a_n^3+2n \leq 3(a_1^2+a_2^2+\cdots+a_n^2).$$

1.242. Let $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ be nonnegative real numbers such that $a_1+a_2+\cdots+a_n=n$.

(a) If $a_1 \geq 1 \geq a_2 \geq \cdots \geq a_n$, then

$$a_1^4+a_2^4+\cdots+a_n^4+5n \geq 6(a_1^2+a_2^2+\cdots+a_n^2);$$

(b) If $a_1 \leq 1 \leq a_2 \leq \cdots \leq a_n$, then

$$a_1^4+a_2^4+\cdots+a_n^4+6n \leq 7(a_1^2+a_2^2+\cdots+a_n^2).$$

1.243. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that

$$a_1 \geq 1 \geq a_2 \geq \cdots \geq a_n, \qquad \frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}=n,$$

then

$$a_1^2+a_2^2+\cdots+a_n^2+2n \geq 3(a_1+a_2+\cdots+a_n).$$

1.244. If $a_1, a_2, \dots, a_n$ are real numbers such that

$$a_1 \le 1 \le a_2 \le \cdots \le a_n, \qquad a_1 + a_2 + \cdots + a_n = n,$$

then

(a) $$\frac{a_1+1}{a_1^2+1} + \frac{a_2+1}{a_2^2+1} + \cdots + \frac{a_n+1}{a_n^2+1} \le n;$$

(b) $$\frac{1}{a_1^2+3} + \frac{1}{a_2^2+3} + \cdots + \frac{1}{a_1^2+3} \le \frac{n}{4}.$$

1.245. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers such that

$$a_1 \le 1 \le a_2 \le \cdots \le a_n, \qquad a_1 + a_2 + \cdots + a_n = n,$$

then

$$\frac{a_1^2-1}{(a_1+3)^2} + \frac{a_2^2-1}{(a_2+3)^2} + \cdots + \frac{a_n^2-1}{(a_n+3)^2} \ge 0.$$

1.246. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers such that

$$a_1 \ge 1 \ge a_2 \ge \cdots \ge a_n, \qquad a_1 + a_2 + \cdots + a_n = n,$$

then

$$\frac{1}{3a_1^3+4} + \frac{1}{3a_2^3+4} + \cdots + \frac{1}{3a_n^3+4} \ge \frac{n}{7}.$$

1.247. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers such that

$$a_1 \le 1 \le a_2 \le \cdots \le a_n, \qquad a_1 + a_2 + \cdots + a_n = n,$$

then

$$\sqrt{\frac{3a_1}{4-a_1}} + \sqrt{\frac{3a_2}{4-a_2}} + \cdots + \sqrt{\frac{3a_n}{4-a_n}} \le n.$$

1.248. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers such that

$$a_1 \le 1 \le a_2 \le \cdots \le a_n, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = n,$$

then

$$\frac{1}{3-a_1} + \frac{1}{3-a_2} + \cdots + \frac{1}{3-a_n} \le \frac{n}{2}.$$

1.249. If $a_1, a_2, \ldots, a_n$ are real numbers such that

$$a_1 \le 1 \le a_2 \le \cdots \le a_n, \qquad a_1 + a_2 + \cdots + a_n = n,$$

then

$$(1 + a_1^2)(1 + a_2^2)\cdots(1 + a_n^2) \ge 2^n.$$

1.250. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that

$$a_1 \ge 1 \ge a_2 \ge \cdots \ge a_n, \qquad a_1 a_2 \cdots a_n = 1,$$

then

$$\frac{1}{(a_1+1)^2} + \frac{1}{(a_2+1)^2} + \cdots + \frac{1}{(a_n+1)^2} \ge \frac{n}{4}.$$

1.251. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that

$$a_1 \ge 1 \ge a_2 \ge \cdots \ge a_n, \qquad a_1 a_2 \cdots a_n = 1,$$

then

$$\frac{1}{(a_1+2)^2} + \frac{1}{(a_2+2)^2} + \cdots + \frac{1}{(a_n+2)^2} \ge \frac{n}{9}.$$

1.252. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that

$$a_1 \ge 1 \ge a_2 \ge \cdots \ge a_n, \qquad a_1 a_2 \cdots a_n = 1,$$

then

$$a_1^n + a_2^n + \cdots + a_n^n - n \ge n^2\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - n\right).$$

1.253. If $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ are real numbers such that

$$a_1 + a_2 + \cdots + a_n = n, \qquad a_1 \ge a_2 \ge 1 \ge a_3 \ge \cdots \ge a_n,$$

then

$$a_1^4 + a_2^4 + \cdots + a_n^4 - n \ge \frac{14}{3}(a_1^2 + a_2^2 + \cdots + a_n^2 - n).$$

1.254. Let $a_1, a_2, \dots, a_n$ be positive real numbers such that

$$a_1 \ge 1 \ge a_2 \ge \cdots \ge a_n, \quad a_1a_2\cdots a_n = 1.$$

Prove that

$$\frac{1-a_1}{3+a_1^2} + \frac{1-a_2}{3+a_2^2} + \cdots + \frac{1-a_n}{3+a_n^2} \ge 0.$$

1.255. Let $a_1, a_2, \dots, a_n$ $(n \ge 3)$ be nonnegative real numbers such that

$$a_1 \ge \cdots \ge a_k \ge 1 \ge a_{k+1} \ge \cdots \ge a_n, \qquad 1 \le k \le n-1,$$

and

$$a_1 + a_2 + \cdots + a_n = p.$$

Prove that

(a) if $p \ge k$, then

$$a_1^2 + a_2^2 + \cdots + a_n^2 \le (p-k+1)^2 + k - 1;$$

(b) if $k \le p \le n$, then

$$a_1^2 + a_2^2 + \cdots + a_n^2 \ge \frac{p^2 - 2kp + kn}{n-k};$$

(c) if $p \ge n$, then

$$a_1^2 + a_2^2 + \cdots + a_n^2 \ge \frac{p^2 - 2(n-k)p + n(n-k)}{k}.$$

1.256. Let $a_1, a_2, \dots, a_n$ $(n \ge 3)$ be nonnegative real numbers such that

$$a_1 \ge \cdots \ge a_k \ge 1 \ge a_{k+1} \ge \cdots \ge a_n, \qquad 1 \le k \le n-1,$$

and

$$a_1 + a_2 + \cdots + a_n = n, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = q,$$

where q is a fixed number. Prove that the product $r = a_1a_2\cdots a_n$ is maximal when

$$a_2 = \cdots = a_k = 1, \qquad a_{k+1} = \cdots = a_n.$$

1.257. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers such that

$$a_1 \le 1 \le a_2 \le \cdots \le a_n, \qquad a_1 + a_2 + \cdots + a_n = n,$$

then

$$(a_1a_2\cdots a_n)^{\frac{2}{n}}(a_1^2 + a_2^2 + \cdots + a_n^2) \le n.$$

1.258. Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be nonnegative real numbers such that

$$a_1 \ge \cdots \ge a_k \ge 1 \ge a_{k+1} \ge \cdots \ge a_n, \qquad 1 \le k \le n-1,$$

and

$$a_1 + a_2 + \cdots + a_n = p, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = q,$$

where p and q are fixed numbers.

(a) For $p \le n$, the product $r = a_1 a_2 \cdots a_n$ is maximal when $a_2 = \cdots = a_k = 1$ and $a_{k+1} = \cdots = a_n$;

(b) For $p \ge n$ and $q \ge n-1+(p-n+1)^2$, the product $r = a_1 a_2 \cdots a_n$ is maximal when $a_2 = \cdots = a_k = 1$ and $a_{k+1} = \cdots = a_n$;

(c) For $p \ge n$ and $q < n-1+(p-n+1)^2$, the product $r = a_1 a_2 \cdots a_n$ is maximal when $a_2 = \cdots = a_k$ and $a_{k+1} = \cdots = a_n = 1$.

1.259. If $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ are nonnegative real numbers such that

$$a_1 \le a_2 \le 1 \le a_3 \le \cdots \le a_n, \qquad a_1 + a_2 + \cdots + a_n = n-1,$$

then

$$a_1^2 + a_2^2 + \cdots + a_n^2 + 10 a_1 a_2 \cdots a_n \le n+1.$$

1.260. If a, b, c, d, e are nonnegative real numbers such that

$$a \le b \le 1 \le c \le d \le e, \qquad a + b + c + d + e = 8,$$

then

$$a^2 + b^2 + c^2 + d^2 + e^2 + 3abcde \le 38.$$

1.2 Solutions

P 1.1. *If a, b, c are nonnegative real numbers such that $a+b+c=3$, then*

$$ab^2+bc^2+ca^2 \le 4.$$

(Canada, 1999)

First Solution. Assume that $a=\max\{a,b,c\}$. Since

$$ab^2+bc^2+ca^2 \le ab\cdot\frac{a+b}{2}+abc+ca^2=\frac{a(a+b)(b+2c)}{2},$$

it suffices to show that

$$a(a+b)(b+2c)\le 8.$$

By the AM-GM inequality, we have

$$a(a+b)(b+2c)\le\left[\frac{a+(a+b)+(b+2c)}{3}\right]^3=8\left(\frac{a+b+c}{3}\right)^3=8.$$

The equality holds for $a=2,\ b=0,\ c=1$ (or any cyclic permutation).

Second Solution. Let (x,y,z) be a permutation of (a,b,c) such that

$$x\ge y\ge z.$$

Since

$$xy\ge zx\ge yz,$$

by the rearrangement inequality, we have

$$\begin{aligned}ab^2+bc^2+ca^2&=b\cdot ab+c\cdot bc+a\cdot ca\\&\le x\cdot xy+y\cdot zx+z\cdot yz\\&=y(x^2+xz+z^2).\end{aligned}$$

Using this result and the AM-GM inequality, we get

$$\begin{aligned}ab^2+bc^2+ca^2&\le y(x+z)^2=4y\cdot\frac{x+z}{2}\cdot\frac{x+z}{2}\\&\le 4\left(\frac{y+\frac{x+z}{2}+\frac{x+z}{2}}{3}\right)^3\\&=4\left(\frac{x+y+z}{3}\right)^3=4.\end{aligned}$$

Third Solution. Without loss of generality, assume that b is between a and c; that is,

$$(b-a)(b-c)\le 0,\quad b^2+ac\le b(a+c).$$

Since

$$ab^2+bc^2+ca^2 = a(b^2+ac)+bc^2 \le ab(a+c)+bc^2 = b(a^2+ac+c^2)$$
$$\le b(a+c)^2 = b(3-b)^2,$$

it suffices to show that

$$b(3-b)^2 \le 4.$$

Indeed,

$$b(3-b)^2-4 = (b-1)^2(b-4) \le (b-1)^2(b-3) = -(b-1)^2(a+c) \le 0.$$

Fourth Solution. Write the inequality in the homogeneous form

$$4(a+b+c)^3 \ge 27(ab^2+bc^2+ca^2),$$

which is equivalent to

$$4(a^3+b^3+c^3)+12(a+b)(b+c)(c+a) \ge 27(ab^2+bc^2+ca^2),$$

$$4\sum a^3+12\left(\sum a^2b+\sum ab^2+2abc\right) \ge 27\sum ab^2,$$

$$4\sum a^3+12\sum a^2b+24abc \ge 15\sum ab^2.$$

On the other hand, the obvious inequality

$$\sum a(2a-pb-qc)^2 \ge 0$$

is equivalent to

$$4\sum a^3+(q^2-4p)\sum a^2b+6pqabc \ge (4q-p^2)\sum ab^2.$$

Setting $p=1$ and $q=4$ leads to the desired inequality; in addition,

$$4(a+b+c)^3-27(ab^2+bc^2+ca^2) = \sum a(2a-b-4c)^2 \ge 0.$$

□

P 1.2. *If a,b,c are positive real numbers such that $a+b+c=3$, then*

$$(ab+bc+ca)(ab^2+bc^2+ca^2) \le 9.$$

Solution. Let (x, y, z) be a permutation of (a, b, c) such that $x \ge y \ge z$. As shown in the second solution of P 1.1,

$$ab^2 + bc^2 + ca^2 \le y(x^2 + xz + z^2).$$

Consequently, it suffices to show that

$$y(xy + yz + zx)(x^2 + xz + z^2) \le 9.$$

By the AM-GM inequality, we get

$$4(xy + yz + zx)(x^2 + xz + z^2) \le (xy + yz + zx + x^2 + xz + z^2)^2$$
$$= (x + z)^2(x + y + z)^2 = 9(x + z)^2.$$

Thus, we still have to show that

$$y(x + z)^2 \le 4.$$

This follows from the AM-GM inequality, as follows:

$$2y(x + z)^2 \le \left[\frac{2y + (x + z) + (x + z)}{3}\right]^3 = 8.$$

The equality holds for $a = b = c = 1$.

□

P 1.3. *If a, b, c are nonnegative real numbers such that $a^2 + b^2 + c^2 = 3$, then*

(a) $$ab^2 + bc^2 + ca^2 \le abc + 2;$$

(b) $$\frac{a}{b+2} + \frac{b}{c+2} + \frac{c}{a+2} \le 1.$$

(Vasile C., 2005)

Solution. (a) ***First Solution.*** Without loss of generality, assume that b is between a and c; that is,

$$(b - a)(b - c) \le 0, \quad b^2 + ac \le b(a + c).$$

Since

$$ab^2 + bc^2 + ca^2 = a(b^2 + ac) + bc^2 \le ab(a + c) + bc^2 = b(a^2 + c^2) + abc,$$

it suffices to show that

$$b(a^2 + c^2) \le 2.$$

We have

$$2 - b(a^2 + c^2) = 2 - b(3 - b^2) = (b - 1)^2(b + 2) \ge 0.$$

The equality holds for $a = b = c = 1$, and also for $a = 0$, $b = 1$, $c = \sqrt{2}$ (or any cyclic permutation).

Second Solution. Let (x, y, z) be a permutation of (a, b, c) such that $x \ge y \ge z$. As shown in the second solution of P 1.1,

$$ab^2 + bc^2 + ca^2 \le y(x^2 + xz + z^2).$$

Therefore, it suffices to show that

$$y(x^2 + xz + z^2) \le xyz + 2,$$

which can be written as

$$y(x^2 + z^2) \le 2.$$

Indeed,

$$2 - y(x^2 + z^2) = 2 - y(3 - y^2) = (y - 1)^2(y + 2) \ge 0.$$

(b) Write the inequality as follows:

$$\sum a(a+2)(c+2) \le (a+2)(b+2)(c+2),$$

$$ab^2 + bc^2 + ca^2 + 2(a^2 + b^2 + c^2) \le abc + 8,$$

$$ab^2 + bc^2 + ca^2 \le abc + 2.$$

The last inequality is just the inequality in (a).

□

P 1.4. *If $a, b, c \ge 1$, then*

(a) $$2(ab^2 + bc^2 + ca^2) + 3 \ge 3(ab + bc + ca);$$

(b) $$ab^2 + bc^2 + ca^2 + 6 \ge 3(a + b + c).$$

Solution. (a) ***First Solution.*** From

$$a(b-1)^2 + b(c-1)^2 + c(a-1)^2 \ge 0,$$

we get

$$ab^2 + bc^2 + ca^2 \ge 2(ab + bc + ca) - (a + b + c).$$

Using this inequality gives

$$2(ab^2 + bc^2 + ca^2) + 3 - 3(ab + bc + ca) \ge (ab + bc + ca) - 2(a + b + c) + 3$$

$$= (a-1)(b-1) + (b-1)(c-1) + (c-1)(a-1) \ge 0.$$

The equality holds for $a = b = c = 1$.

Second Solution. From
$$\sum b(a-1)(b-1) \ge 0,$$
we get
$$ab^2 + bc^2 + ca^2 \ge a^2 + b^2 + c^2 + ab + bc + ca - (a+b+c).$$
Thus, it suffices to show that
$$2(a^2+b^2+c^2) + 2(ab+bc+ca) - 2(a+b+c) + 3 \ge 3(ab+bc+ca),$$
which is equivalent to
$$2(a^2+b^2+c^2) - 2(a+b+c) + 3 \ge ab + bc + ca,$$
$$(a-1)^2 + (b-1)^2 + (c-1)^2 + (a^2+b^2+c^2 - ab - bc - ca) \ge 0,$$
$$2(a-1)^2 + 2(b-1)^2 + 2(c-1)^2 + (a-b)^2 + (b-c)^2 + (c-a)^2 \ge 0.$$

(b) The inequality in (b) follows by summing the inequality in (a) and the obvious inequality
$$3(a-1)(b-1) + 3(b-1)(c-1) + 3(c-1)(a-1) \ge 0.$$
The equality holds for $a = b = c = 1$.

□

P 1.5. *If a, b, c are nonnegative real numbers such that*
$$a+b+c=3, \qquad a \ge b \ge c,$$
then

(a) $$a^2b + b^2c + c^2a \ge ab + bc + ca;$$

(b) $$8(ab^2 + bc^2 + ca^2) + 3abc \le 27;$$

(c) $$\frac{18}{a^2b + b^2c + c^2a} \le \frac{1}{abc} + 5.$$

Solution. (a) Write the inequality in the homogeneous form
$$3(a^2b + b^2c + c^2a) \ge (a+b+c)(ab+bc+ca),$$
which is equivalent to
$$a^2b + b^2c + c^2a - 3abc \ge ab^2 + bc^2 + ca^2 - a^2b - b^2c - c^2a.$$
This inequality is true because
$$a^2b + b^2c + c^2a - 3abc \ge 0$$

(by the AM-GM inequality) and

$$ab^2+bc^2+ca^2-a^2b-b^2c-c^2a=(a-b)(b-c)(c-a)\le 0.$$

The equality holds for $a=b=c=1$, and also for $a=3$ and $b=c=0$.

(b) Write the inequality in the homogeneous form

$$(a+b+c)^3\ge 8(ab^2+bc^2+ca^2)+3abc,$$

$$\sum a^3+3abc+3\sum a^2b\ge 5\sum ab^2,$$

$$\sum a^3+3abc-\left(\sum ab^2+\sum a^2b\right)\ge 4\left(\sum ab^2-\sum a^2b\right),$$

$$\sum a^3+3abc-\sum ab(a+b)\ge 4(a-b)(b-c)(c-a).$$

The inequality is true since

$$(a-b)(b-c)(c-a)\le 0$$

and, by Schur's inequality of degree three,

$$\sum a^3+3abc-\sum ab(a+b)\ge 0.$$

The equality holds for $a=b=c=1$, and also for $a=b=3/2$ and $c=0$.

(c) Since

$$ab^2+bc^2+ca^2-a^2b-b^2c-c^2a=(a-b)(b-c)(c-a)\le 0,$$

it suffices to prove the symmetric inequality

$$\frac{36}{(a^2b+b^2c+c^2a)+(ab^2+bc^2+ca^2)}\le\frac{1}{abc}+5,$$

which is equivalent to

$$\frac{36}{(a+b+c)(ab+bc+ca)-3abc}\le\frac{1}{abc}+5,$$

$$\frac{12}{ab+bc+ca-abc}\le\frac{1}{abc}+5,$$

$$\frac{12}{a(b+c)-(a-1)bc}\le\frac{1}{a\cdot bc}+5,$$

$$\frac{12}{a(3-a)-(a-1)bc}\le\frac{1}{a\cdot bc}+5.$$

Since $a-1\ge 0$ and

$$4bc\le(b+c)^2=(3-a)^2,$$

it suffices to show that

$$\frac{48}{4a(3-a)-(a-1)(3-a)^2} \le \frac{4}{a(3-a)^2}+5,$$

which is equivalent to

$$\frac{48}{(3-a)(3+a^2)} \le \frac{4}{a(3-a)^2}+5,$$

$$5a^5-30a^4+60a^3-38a^2-9a+12 \ge 0,$$

$$(a-1)^2(5a^3-20a^2+15a+12) \ge 9.$$

We need to show that $1 \le a \le 3$ involves

$$5a^3-20a^2+15a+12 \ge 0.$$

If $1 \le a \le 2$, then

$$5a^3-20a^2+15a+12 = 5a(a-2)^2+(12-5a) > 0.$$

If $2 \le a \le 3$, then

$$5a^3-20a^2+15a+12 = 5(a-2)^3+10a^2-45a+52 \ge 10a^2-45a+52 > 0$$
$$= 10\left(a-\frac{9}{4}\right)^2+\frac{11}{8} > 0.$$

The equality holds for $a=b=c=1$.

□

P 1.6. *If a,b,c are nonnegative real numbers such that*

$$a^2+b^2+c^2=3, \qquad a \ge b \ge c,$$

then

$$ab^2+bc^2+ca^2 \le \frac{3}{4}(ab+bc+ca+1).$$

Solution. Let us denote

$$p=a+b+c, \qquad q=ab+bc+ca.$$

From $a^2+b^2+c^2=3$, it follows that

$$2q=p^2-3.$$

In addition, from the known inequalities

$$(a+b+c)^2 \ge a^2+b^2+c^2$$

and

$$3(a^2+b^2+c^2) \ge (a+b+c)^2,$$

we get

$$\sqrt{3} \le p \le 3.$$

Since

$$ab^2+bc^2+ca^2-a^2b-b^2c-c^2a=(a-b)(b-c)(c-a) \le 0,$$

it suffices to show that

$$ab^2+bc^2+ca^2+(a^2b+b^2c+c^2a) \le \frac{3}{2}(ab+bc+ca+1).$$

which is equivalent to

$$pq \le 3abc+\frac{3}{2}(q+1),$$

$$6abc+3(q+1) \ge 2pq.$$

Consider two cases: $\sqrt{3} \le p \le \frac{12}{5}$ and $\frac{12}{5} \le p \le 3$.

Case 1: $\sqrt{3} \le p \le \frac{12}{5}$. Since

$$6abc+3(q+1)-2pq \ge 3(q+1)-2pq = 3-(2p-3)q = \frac{1}{2}[6-(2p-3)(p^2-3)],$$

it suffices to show that

$$(2p-3)(p^2-3) \le 6.$$

Indeed, we have

$$(2p-3)(p^2-3) \le \left(\frac{24}{5}-3\right)\left(\frac{144}{25}-3\right)=\frac{621}{125}<6.$$

Case 2: $\frac{12}{5} \le p \le 3$. According to Schur's inequality of degree three, we have

$$p^3+9abc \ge 4pq.$$

Thus, it suffices to prove that

$$2(4pq-p^3)+9(q+1) \ge 6pq,$$

which is equivalent to

$$(2p+9)q-2p^3+9 \ge 0,$$

$$(2p+9)(p^2-3)-4p^3+18 \ge 0,$$
$$-2p^3+9p^2-6p-9 \ge 0,$$
$$(3-p)(2p^2-3p-3) \ge 0.$$

This inequality is true since $3-p \ge 0$ and

$$2p^2-3p-3 \ge \frac{24}{5}p-3p-3 = \frac{9}{5}p-3 \ge \frac{9}{5}\cdot\frac{12}{5}-3 > 0.$$

The equality holds for $a=b=c=1$.

□

P 1.7. *If a,b,c are nonnegative real numbers such that $a^2+b^2+c^2=3$, then*

$$a^2b^3+b^2c^3+c^2a^3 \le 3.$$

(Vasile C., 2005)

Solution. Let (x,y,z) be a permutation of (a,b,c) such that

$$x \ge y \ge z.$$

Since

$$x^2y^2 \ge z^2x^2 \ge y^2z^2,$$

the rearrangement inequality yields

$$a^2b^3+b^2c^3+c^2a^3 = b\cdot a^2b^2+c\cdot b^2c^2+a\cdot c^2a^2 \le x\cdot x^2y^2+y\cdot z^2x^2+z\cdot y^2z^2$$
$$= y(x^3y+z^2x^2+yz^3) \le y\left(x^2\cdot\frac{x^2+y^2}{2}+z^2x^2+z^2\cdot\frac{y^2+z^2}{2}\right)$$
$$= \frac{y(x^2+z^2)(x^2+y^2+z^2)}{2} = \frac{3y(x^2+z^2)}{2}.$$

Thus, it suffices to show that

$$y(x^2+z^2) \le 2$$

for $x^2+y^2+z^2=3$. By the AM-GM inequality, we get

$$6 = 2y^2+(x^2+z^2)+(x^2+z^2) \ge 3\sqrt[3]{2y^2(x^2+z^2)^2}.$$

The equality holds for $a=b=c=1$.

□

P 1.8. *If a,b,c are nonnegative real numbers such that $a+b+c=3$, then*

$$a^4b^2+b^4c^2+c^4a^2+4 \ge a^3b^3+b^3c^3+c^3a^3.$$

Solution. Write the inequality as

$$a^2(a^2b^2+c^4-ab^3-ac^3)+4 \ge b^2c^2(bc-b^2).$$

Since

$$\begin{aligned}2\sum(a^2b^2+c^4-ab^3-ac^3) &= \sum[a^4+b^4+2a^2b^2-2ab(a^2+b^2)] \\ &= \sum(a^2+b^2)(a-b)^2 \ge 0,\end{aligned}$$

we may assume (without loss of generality) that

$$a^2b^2+c^4-ab^3-ac^3 \ge 0.$$

Thus, it suffices to show that

$$4 \ge b^2c^2(bc-b^2).$$

Since

$$bc-b^2 \le \frac{c^2}{4},$$

it is enough to prove that

$$16 \ge b^2c^4.$$

From

$$3 = a+b+c \ge b+\frac{c}{2}+\frac{c}{2} \ge 3\sqrt[3]{b\left(\frac{c}{2}\right)^2},$$

the conclusion follows. The equality holds for $a=0$, $b=1$, $c=2$ (or any cyclic permutation).

□

P 1.9. *If a,b,c are nonnegative real numbers such that $a+b+c=3$, then*

(a) $$ab^2+bc^2+ca^2+abc \le 4;$$

(b) $$\frac{a}{4-b}+\frac{b}{4-c}+\frac{c}{4-a} \le 1;$$

(c) $$ab^3+bc^3+ca^3+(ab+bc+ca)^2 \le 12;$$

(d) $$\frac{ab^2}{1+a+b}+\frac{bc^2}{1+b+c}+\frac{ca^2}{1+c+a} \le 1.$$

Solution. (a) ***First Solution.*** Let (x,y,z) be a permutation of (a,b,c) such that

$$x \ge y \ge z.$$

As shown in the second solution of P 1.1,

$$ab^2+bc^2+ca^2 \le y(x^2+xz+z^2);$$

hence

$$ab^2+bc^2+ca^2+abc \le y(x+z)^2.$$

Thus, it suffices to show that $x+y+z=3$ involves

$$y(x+z)^2 \le 4.$$

According to the AM-GM inequality, we have

$$\frac{1}{4}y(x+z)^2 = y\cdot\frac{x+z}{2}\cdot\frac{x+z}{2} \le \left(\frac{y+\frac{x+z}{2}+\frac{x+z}{2}}{3}\right)^3 = 1.$$

The equality holds for $a=b=c=1$, and also for $a=0$, $b=1$, $c=2$ (or any cyclic permutation).

Second Solution. Without loss of generality, assume that b is between a and c; that is,

$$(b-a)(b-c) \le 0, \qquad b^2+ca \le b(c+a).$$

Therefore,

$$ab^2+bc^2+ca^2+abc = a(b^2+ca)+bc^2+abc \le ab(c+a)+bc^2+abc$$

$$= b(a+c)^2 = b(3-b)^2 = 4+(b^3-6b^2+9b-4) = 4-(1-b)^2(4-b) \le 4.$$

Third Solution. Write the inequality in the homogeneous form

$$4(a+b+c)^3 \ge 27(ab^2+bc^2+ca^2+abc).$$

Without loss of generality, suppose that $a = min\{a,b,c\}$. Putting $b=a+x$ and $c=a+y$, where $x,y \ge 0$, the inequality can be restated as

$$9(x^2-xy+y^2)a+(2x-y)^2(x+4y) \ge 0,$$

which is obviously true.

(b) ***First Solution.*** Write the inequality in the homogeneous form

$$\sum \frac{a}{4a+b+4c} \le \frac{1}{3}.$$

Multiplying by $a+b+c$, the inequality becomes as follows:

$$\sum \frac{a^2+ab+ac}{4a+b+4c} \le \frac{a+b+c}{3},$$

$$\sum \left(\frac{a^2+ab+ac}{4a+b+4c} - \frac{a}{4} \right) \le \frac{a+b+c}{12},$$

$$\sum \frac{9ab}{4a+b+4c} \le a+b+c.$$

Since

$$\frac{9}{4a+b+4c} = \frac{9}{(2a+c)+(2a+c)+(2c+b)} \le \frac{1}{2a+c} + \frac{1}{2a+c} + \frac{1}{2c+b}$$
$$= \frac{2}{2a+c} + \frac{1}{2c+b},$$

we have

$$\sum \frac{9ab}{4a+b+4c} \le \sum \frac{2ab}{2a+c} + \sum \frac{ab}{2c+b} = \sum \frac{2ab}{2a+c} + \sum \frac{bc}{2a+c}$$
$$= \sum \frac{2ab+bc}{2a+c} = \sum b = a+b+c.$$

The equality holds for $a=b=c=1$, and also for $a=0$, $b=1$, $c=2$ (or any cyclic permutation).

Second Solution. Write the inequality as follows:

$$\sum a(4-a)(4-c) \le (4-a)(4-b)(4-c),$$

$$32 + \sum ab^2 + abc \le 4\left(\sum a^2 + 2\sum ab\right),$$

$$32 + \sum ab^2 + abc \le 4\left(\sum a\right)^2,$$

$$ab^2 + bc^2 + ca^2 + abc \le 4.$$

The last inequality is just the inequality in (a).

(c) Using the inequality in (a), we get

$$(a+b+c)(ab^2+bc^2+ca^2+abc) \le 12,$$

which is equivalent to the desired inequality

$$ab^3+bc^3+ca^3+(ab+bc+ca)^2 \le 12.$$

(d) Let $q = ab+bc+ca$. Since

$$\sum ab^2(1+b+c)(1+c+a) = \sum ab^2(4+q+c+c^2) = (4+q)\sum ab^2 + (3+q)abc$$

and

$$\prod(1+a+b) = 1+\sum(a+b)+\sum(b+c)(c+a)+\prod(a+b)$$
$$= 7+3q+\sum c^2+(3q-abc) = 16+4q-abc,$$

the inequality is equivalent to

$$(4+q)\sum ab^2+(3+q)abc \le 16+4q-abc,$$

$$(4+q)\left(\sum ab^2+abc-4\right) \le 0.$$

According to (a), the desired inequality is clearly true.

Remark. The following statement is also valid:

- *If a,b,c are nonnegative real numbers such that $a+b+c=3$, then*

$$ab^2+bc^2+ca^2+abc+(a-1)^2(b-1)^2(c-1)^2 \le 4,$$

with equality for $a=b=c=1$, and also for $a=0$, $b=1$, $c=2$ (or any cyclic permutation).

Having in view the second solution of (a), it is enough to show that

$$(a-1)^2(b-1)^2(c-1)^2 \le (4-b)(1-b)^2,$$

where b is between a and c. This is true if

$$|(a-1)(c-1)| \le \sqrt{4-b}.$$

Assuming that $a \le c$ (hence $a \le b \le c$, $a \le 1$, $c \ge 1$), the inequality can be written as follows:

$$(1-a)(c-1) \le \sqrt{4-b},$$

$$a+c-1 \le ac+\sqrt{4-b},$$

$$2-b \le ac+\sqrt{4-b}.$$

This is true if

$$2-b \le \sqrt{4-b}.$$

Indeed,

$$\sqrt{4-b}-(2-b) = \frac{4-b-(2-b)^2}{\sqrt{4-b}+2-b} = \frac{b(3-b)}{\sqrt{4-b}+2-b}$$
$$= \frac{b(a+c)}{\sqrt{4-b}+2-b}) \ge 0.$$

□

P 1.10. *If a, b, c are positive real numbers, then*

$$\frac{1}{a(a+2b)}+\frac{1}{b(b+2c)}+\frac{1}{c(c+2a)} \ge \frac{3}{ab+bc+ca}.$$

First Solution. Write the inequality as

$$\sum \frac{a(b+c)+bc}{a(a+2b)} \ge 3,$$

$$\sum \frac{b+c}{a+2b}+\sum \frac{bc}{a(a+2b)} \ge 3.$$

It suffices to show that

$$\sum \frac{b+c}{a+2b} \ge 2$$

and

$$\sum \frac{bc}{a(a+2b)} \ge 1.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{b+c}{a+2b} \ge \frac{\left[\sum(b+c)\right]^2}{\sum(b+c)(a+2b)} = \frac{4\left(\sum a\right)^2}{2\sum a^2+4\sum ab} = 2$$

and

$$\sum \frac{bc}{a(a+2b)} \ge \frac{\left(\sum bc\right)^2}{abc\sum(a+2b)} = \frac{\left(\sum bc\right)^2}{3abc\sum a} = 1+\frac{\sum a^2(b-c)^2}{6abc\sum a} \ge 1.$$

The equality holds for $a=b=c$.

Second Solution. We apply the Cauchy-Schwarz inequality in the following way

$$\sum \frac{1}{a(a+2b)} \ge \frac{\left(\sum c\right)^2}{\sum ac^2(a+2b)} = \frac{\left(\sum a\right)^2}{\sum a^2b^2+2abc\sum a}.$$

Thus, it suffices to show that

$$\frac{\left(\sum a\right)^2}{\sum a^2b^2+2abc\sum a} \ge \frac{3}{\sum ab},$$

which is equivalent to

$$\left(\sum ab\right)\left(\sum a^2+2\sum ab\right) \ge 3\sum a^2b^2+6abc\sum a,$$

$$\sum ab(a^2+b^2) \ge \sum a^2b^2+abc\sum a.$$

The last inequality follows by summing the obvious inequalities

$$\sum ab(a^2+b^2) \ge 2\sum a^2b^2$$

and

$$\sum a^2b^2 \ge abc\sum a.$$

□

P 1.11. *If a,b,c are positive real numbers such that $a+b+c=3$, then*

$$\frac{a}{b^2+2c}+\frac{b}{c^2+2a}+\frac{c}{a^2+2b} \ge 1.$$

Solution. Using the Cauchy-Schwarz inequality, we get

$$\sum \frac{a}{b^2+2c} \ge \frac{\left(\sum a\right)^2}{\sum a(b^2+2c)} = 1+\frac{\sum a^2-\sum ab^2}{\sum ab^2+2\sum ab}.$$

Thus, it suffices to show that

$$\sum a^2-\sum ab^2 \ge 0.$$

Write this inequality in the homogeneous form

$$(a+b+c)(a^2+b^2+c^2) \ge 3(ab^2+bc^2+ca^2),$$

which is equivalent to the obvious inequality

$$a(a-c)^2+b(b-a)^2+c(c-b)^2 \ge 0.$$

The equality holds for $a=b=c=1$.

□

P 1.12. *If a,b,c are positive real numbers such that $a+b+c \ge 3$, then*

$$\frac{a-1}{b+1}+\frac{b-1}{c+1}+\frac{c-1}{a+1} \ge 0.$$

Solution. Write the inequality as

$$(a^2-1)(c+1)+(b^2-1)(a+1)+(c^2-1)(b+1)\ge 0,$$

$$ab^2+bc^2+ca^2+a^2+b^2+c^2\ge a+b+c+3.$$

From

$$a(b-1)^2+b(c-1)^2+c(a-1)^2\ge 0,$$

we get

$$ab^2+bc^2+ca^2\ge 2(ab+bc+ca)-(a+b+c).$$

Using this inequality yields

$$ab^2+bc^2+ca^2+a^2+b^2+c^2-a-b-c-3\ge (a+b+c)^2-2(a+b+c)-3$$

$$=(a+b+c-3)(a+b+c+1)\ge 0.$$

The equality holds for $a=b=c=1$.

□

P 1.13. *If a,b,c are positive real numbers such that $a+b+c=3$, then*

(a) $$\frac{1}{2ab^2+1}+\frac{1}{2bc^2+1}+\frac{1}{2ca^2+1}\ge 1;$$

(b) $$\frac{1}{ab^2+2}+\frac{1}{bc^2+2}+\frac{1}{ca^2+2}\ge 1.$$

Solution. By the AM-GM inequality, we have

$$1=\left(\frac{a+b+c}{3}\right)^3\ge abc.$$

(a) Since

$$2ab^2+1\le\frac{2b}{c}+1=\frac{2b+c}{c},$$

it suffices to show that

$$\frac{c}{2b+c}+\frac{a}{2c+a}+\frac{b}{2a+b}\ge 1.$$

Using the Cauchy-Schwarz inequality, we get

$$\sum\frac{c}{2b+c}\ge\frac{\left(\sum c\right)^2}{\sum c(2b+c)}=\frac{(a+b+c)^2}{(a+b+c)^2}=1.$$

The equality holds for $a=b=c=1$.

(b) By expanding, the inequality can be restated as

$$a^3b^3c^3 + abc(a^2b + b^2c + c^2a) \le 4.$$

Applying the AM-GM inequality gives

$$\begin{aligned}(a+b+c)^3 &= \sum a^3 + 6abc + 3\sum ab^2 + 3\sum a^2b \\ &\ge 3abc + 6abc + 9abc + 3\sum a^2b,\end{aligned}$$

i.e.

$$a^2b + b^2c + c^2a \le 9 - 6abc.$$

Therefore, it suffices to show that

$$a^3b^3c^3 + abc(9 - 6abc) \le 4,$$

which is equivalent to the obvious inequality

$$(abc-1)^2(abc-4) \le 0.$$

The equality holds for $a = b = c = 1$.

□

P 1.14. *If a, b, c are positive real numbers such that $a + b + c = 3$, then*

$$\frac{ab}{9-4bc} + \frac{bc}{9-4ca} + \frac{ca}{9-4ab} \le \frac{3}{5}.$$

Solution. We have

$$\begin{aligned}\sum \frac{ab}{9-4bc} &\le \sum \frac{ab}{9-(b+c)^2} = \sum \frac{b}{3+b+c} = \sum \frac{b}{a+2b+2c} \\ &= \frac{1}{2}\sum\left[1 - \frac{a+2c}{a+2b+2c}\right] = \frac{3}{2} - \frac{1}{2}\sum \frac{a+2c}{a+2b+2c}.\end{aligned}$$

Thus, it suffices to show that

$$\sum \frac{a+2c}{a+2b+2c} \ge \frac{9}{5}.$$

Using the Cauchy-Schwarz inequality, we get

$$\sum \frac{a+2c}{a+2b+2c} \ge \frac{[\sum(a+2c)]^2}{\sum(a+2c)(a+2b+2c)} = \frac{9(a+b+c)^2}{5(a+b+c)^2} = \frac{9}{5}.$$

The equality holds for $a = b = c = 1$.

□

P 1.15. *If a, b, c are positive real numbers such that $a+b+c=3$, then*

(a) $$\frac{a^2}{2a+b^2}+\frac{b^2}{2b+c^2}+\frac{c^2}{2c+a^2} \ge 1;$$

(b) $$\frac{a^2}{a+2b^2}+\frac{b^2}{b+2c^2}+\frac{c^2}{c+2a^2} \ge 1.$$

Solution. (a) By the Cauchy-Schwarz inequality, we have

$$\sum \frac{a^2}{2a+b^2} \ge \frac{\left(\sum a^2\right)^2}{\sum a^2(2a+b^2)} = \frac{\sum a^4 + 2\sum a^2b^2}{2\sum a^3 + \sum a^2b^2}.$$

Thus, it suffices to prove that

$$\sum a^4 + \sum a^2b^2 \ge 2\sum a^3,$$

which is equivalent to the homogeneous inequalities

$$3\sum a^4 + 3\sum a^2b^2 \ge 2\left(\sum a\right)\left(\sum a^3\right),$$

$$\sum a^4 + 3\sum a^2b^2 - 2\sum ab(a^2+b^2) \ge 0,$$

$$\sum (a-b)^4 \ge 0.$$

The equality holds for $a=b=c=1$.

(b) By the Cauchy-Schwarz inequality, we get

$$\sum \frac{a^2}{a+2b^2} \ge \frac{\left(\sum a^2\right)^2}{\sum a^2(a+2b^2)} = \frac{\sum a^4 + 2\sum a^2b^2}{\sum a^3 + 2\sum a^2b^2}.$$

Thus, it suffices to prove that

$$\sum a^4 \ge \sum a^3.$$

We have

$$\sum a^4 - \sum a^3 = \sum (a^4 - a^3 - a + 1) = \sum (a-1)(a^3-1) \ge 0.$$

The equality holds for $a=b=c=1$.

$\square$

P 1.16. *Let a, b, c be positive real numbers such that $a+b+c=3$. Then,*

$$\frac{1}{a+b^2+c^3}+\frac{1}{b+c^2+a^3}+\frac{1}{c+a^2+b^3} \le 1.$$

(Vasile C., 2009)

Solution *(by Vo Quoc Ba Can)*. By the Cauchy-Schwarz inequality, we have

$$\sum \frac{1}{a+b^2+c^3} \le \sum \frac{a^3+b^2+c}{(a^2+b^2+c^2)^2} = \frac{\sum a^3 + \sum a^2 + 3}{(a^2+b^2+c^2)^2}.$$

Therefore, it suffices to show that

$$(a^2+b^2+c^2)^2 \ge a^3+b^3+c^3+(a^2+b^2+c^2)+3,$$

or, equivalently,

$$(a^2+b^2+c^2)^2 + \sum a^2(3-a) \ge 4(a^2+b^2+c^2)+3.$$

Let us denote $t = a^2+b^2+c^2$. Applying again the Cauchy-Schwarz inequality, we get

$$\sum a^2(3-a) \ge \frac{[\sum a(3-a)]^2}{\sum(3-a)} = \frac{(9-a^2-b^2-c^2)^2}{6}.$$

Thus, it is enough to show that

$$t^2 + \frac{(9-t)^2}{6} \ge 4t+3.$$

This inequality reduces to $(t-3)^2 \ge 0$. The equality occurs for $a=b=c=1$.

$\square$

P 1.17. *If a, b, c are positive real numbers, then*

$$\frac{1+a^2}{1+b+c^2} + \frac{1+b^2}{1+c+a^2} + \frac{1+c^2}{1+a+b^2} \ge 2.$$

Solution. From

$$1+b+c^2 \le 1 + \frac{1+b^2}{2} + c^2,$$

we have

$$\frac{1+a^2}{1+b+c^2} \ge \frac{2(1+a^2)}{1+b^2+2(1+c^2)}.$$

Thus, it suffices to show that

$$\frac{x}{y+2z} + \frac{y}{z+2x} + \frac{z}{x+2y} \ge 1,$$

where

$$x = 1+a^2, \quad y = 1+b^2, \quad z = 1+c^2.$$

Using the Cauchy-Schwarz inequality gives

$$\frac{x}{y+2z}+\frac{y}{z+2x}+\frac{z}{x+2y} \geq \frac{(x+y+z)^2}{x(y+2z)+y(z+2x)+z(x+2y)}$$
$$= \frac{(x+y+z)^2}{3(xy+yz+zx)} \geq 1.$$

The equality occurs for $a=b=c=1$.

□

P 1.18. *If a,b,c are nonnegative real numbers, then*

$$\frac{a}{4a+4b+c}+\frac{b}{4b+4c+a}+\frac{c}{4c+4a+b} \leq \frac{1}{3}.$$

(Pham Kim Hung, 2007)

Solution. If two of a,b,c are zero, then the inequality is trivial. Otherwise, multiplying by $4(a+b+c)$, the inequality becomes as follows:

$$\sum \frac{4a(a+b+c)}{4a+4b+c} \leq \frac{4}{3}(a+b+c),$$

$$\sum \left[\frac{4a(a+b+c)}{4a+4b+c} - a\right] \leq \frac{1}{3}(a+b+c),$$

$$\sum \frac{ca}{4a+4b+c} \leq \frac{1}{9}(a+b+c).$$

By the Cauchy-Schwarz inequality, we get

$$\frac{9}{4a+4b+c} = \frac{9}{(2b+c)+2(2a+b)} \leq \frac{1}{2b+c}+\frac{2}{2a+b}.$$

Therefore,

$$\sum \frac{ca}{4a+4b+c} \leq \frac{1}{9}\sum ca\left(\frac{1}{2b+c}+\frac{2}{2a+b}\right)$$
$$= \frac{1}{9}\left(\sum \frac{ca}{2b+c}+\sum \frac{2ab}{2b+c}\right) = \frac{1}{9}\sum a,$$

as desired. The equality occurs for $a=b=c$, and also for $a=2b$ and $c=0$ (or any cyclic permutation).

□

P 1.19. *If a,b,c are positive real numbers, then*

$$\frac{a+b}{a+7b+c}+\frac{b+c}{b+7c+a}+\frac{c+a}{c+7a+b} \geq \frac{2}{3}.$$

Solution. Write the inequality as

$$\sum\left(\frac{a+b}{a+7b+c}-\frac{1}{k}\right)\geq\frac{2}{3}-\frac{3}{k},\quad k>0,$$

$$\sum\frac{(k-1)a+(k-7)b-c}{a+7b+c}\geq\frac{2k-9}{3}.$$

Consider that all fractions in the left hand side are nonnegative and apply the Cauchy-Schwarz inequality, as follows:

$$\sum\frac{(k-1)a+(k-7)b-c}{a+7b+c}\geq\frac{[(k-1)\sum a+(k-7)\sum b-\sum c]^2}{\sum(a+7b+c)[(k-1)a+(k-7)b-c]}$$

$$=\frac{(2k-9)^2\left(\sum a\right)^2}{(8k-51)\sum a^2+2(5k-15)\sum ab}.$$

We choose $k=12$ to have $8k-51=5k-15$, hence

$$(8k-51)\sum a^2+2(5k-15)\sum ab=45\left(\sum a\right)^2.$$

For this value of k, the desired inequality

$$\sum\frac{(k-1)a+(k-7)b-c}{a+7b+c}\geq\frac{2k-9}{3}$$

can be restated as

$$\sum\frac{11a+5b-c}{a+7b+c}\geq 5.$$

Without loss of generality, assume that $a=\max\{a,b,c\}$. Consider further two cases.

Case 1: $11b+5c-a\geq 0$. By the Cauchy-Schwarz inequality, we have

$$\sum\frac{11a+5b-c}{a+7b+c}\geq\frac{[\sum(11a+5b-c)]^2}{\sum(a+7b+c)(11a+5b-c)}=\frac{225\left(\sum a\right)^2}{45\left(\sum a\right)^2}=5.$$

Case 2: $11b+5c-a<0$. We have

$$\sum\frac{a+b}{a+7b+c}>\frac{a+b}{a+7b+c}=\frac{2}{3}+\frac{a-11b-2c}{3(a+7b+c)}>\frac{2}{3}.$$

Thus, the proof is completed. The equality holds for $a=b=c$.

$\square$

P 1.20. *If a,b,c are positive real numbers, then*

$$\frac{a+b}{a+3b+c}+\frac{b+c}{b+3c+a}+\frac{c+a}{c+3a+b}\geq\frac{6}{5}.$$

(Vasile C., 2007)

Solution. Due to homogeneity, we may assume that

$$a+b+c=1,$$

when the inequality becomes

$$\sum \frac{1-c}{1+2b} \geq \frac{6}{5},$$

$$5\sum(1-c)(1+2c)(1+2a) \geq 6(2a+1)(2b+1)(2c+1),$$

$$5\left(4+6\sum ab-4\sum a^2b\right) = 6\left(3+4\sum ab+8abc\right),$$

$$1+3\sum ab \geq 10\sum a^2b+24abc,$$

$$(a+b+c)^3+3(a+b+c)(ab+bc+ca) \geq 10(a^2b+b^2c+c^a)+24abc,$$

$$\sum a^3+6\sum ab^2 \geq 4\sum a^2b+9abc,$$

$$\left[2\sum a^3-\sum ab(a+b)\right]+3\left[\sum ab(a+b)-6abc\right]+10\left(\sum ab^2-\sum a^2b\right) \geq 0,$$

$$\sum(a+b)(a-b)^2+3\sum c(a-b)^2+10\left(\sum ab^2-\sum a^2b\right) \geq 0,$$

$$\sum(a+b+3c)(a-b)^2+10(a-b)(b-c)(c-a) \geq 0.$$

Assume that

$$a=\min\{a,b,c\},$$

and use the substitution

$$b=a+x, \quad c=a+y, \qquad x,y \geq 0.$$

The inequality becomes

$$(5a+x+3y)x^2+(5a+x+y)(x-y)^2+(5a+3x+y)y^2-10xy(x-y) \geq 0.$$

Clearly, it suffices to consider the case $a=0$, when the inequality becomes

$$x^3-4x^2y+6xy^2+y^3 \geq 0.$$

Indeed, we have

$$x^3-4x^2y+6xy^2+y^3 = x(x-2y)^2+2xy^2+y^3 \geq 0.$$

The equality holds for $a=b=c$.

□

P 1.21. *If a, b, c are positive real numbers, then*

$$\frac{2a+b}{2a+c}+\frac{2b+c}{2b+a}+\frac{2c+a}{2c+b}\ge 3.$$

(Pham Kim Hung, 2007)

Solution. Without loss of generality, assume that $a = \max\{a, b, c\}$. There are two cases to consider.

Case 1: $a \le 2b+2c$. Write the inequality as

$$\sum\left(\frac{2a+b}{2a+c}-\frac{1}{2}\right)\ge\frac{3}{2},$$

$$\sum\frac{2a+2b-c}{2a+c}\ge 3.$$

Since

$$2a+2b-c>0,\quad 2b+2c-a\ge 0,\quad 2c+2a-b>0,$$

we may apply the Cauchy-Schwarz inequality to get

$$\sum\frac{2a+2b-c}{2a+c}\ge\frac{\left[\sum(2a+2b-c)\right]^2}{\sum(2a+2b-c)(2a+c)}=\frac{9\left(\sum a\right)^2}{3\left(\sum a\right)^2}=3.$$

Case 2: $a > 2b+2c$. Since

$$2a+c-(2b+a)=(a-2b-2c)+3c>0,$$

we have

$$\frac{2a+b}{2a+c}+\frac{2b+c}{2b+a}>\frac{2a+b}{2a+c}+\frac{2b+c}{2a+c}=1+\frac{3b}{2a+c}>1.$$

Therefore, it suffices to show that

$$\frac{2c+a}{2c+b}\ge 2.$$

Indeed,

$$\frac{2c+a}{2c+b}>\frac{2c+2b+2c}{2c+b}=2.$$

Thus, the proof is completed. The equality holds for $a=b=c$.

□

P 1.22. *If a, b, c are positive real numbers, then*

$$\frac{a(a+b)}{a+c}+\frac{b(b+c)}{b+a}+\frac{c(c+a)}{c+b}\le\frac{3(a^2+b^2+c^2)}{a+b+c}.$$

(Pham Huu Duc, 2007)

Solution. Write the inequality as

$$\sum \frac{a(a+b)(a+b+c)}{a+c} \le 3(a^2+b^2+c^2),$$

$$\sum \frac{ab(a+b)+a(a+b)(a+c)}{a+c} \le 3(a^2+b^2+c^2),$$

$$\sum \frac{ab(a+b)}{a+c} \le 2(a^2+b^2+c^2)-(ab+bc+ca).$$

Let (x,y,z) be a permutation of (a,b,c) such that $x \ge y \ge z$. Since

$$x+y \ge z+x \ge y+z$$

and

$$xy(x+y) \ge zx(z+x) \ge yz(y+z),$$

by the rearrangement inequality, we have

$$\sum \frac{ab(a+b)}{a+c} \le \frac{xy(x+y)}{y+z} + \frac{zx(z+x)}{z+x} + \frac{yz(y+z)}{x+y}.$$

Consequently, it suffices to show that

$$\frac{xy(x+y)}{y+z} + \frac{yz(y+z)}{x+y} \le 2(x^2+y^2+z^2) - xy - yz - 2zx.$$

Write this inequality as follows:

$$xy\left(\frac{x+y}{y+z}-1\right) + yz\left(\frac{y+z}{x+y}-1\right) \le 2(x^2+y^2+z^2-xy-yz-zx),$$

$$\frac{xy(x-z)}{y+z} + \frac{yz(z-x)}{x+y} \le (x-y)^2+(y-z)^2+(z-x)^2,$$

$$\frac{y(x+y+z)(z-x)^2}{(x+y)(y+z)} \le (x-y)^2+(y-z)^2+(z-x)^2.$$

Since

$$y(x+y+z) < (x+y)(y+z),$$

the last inequality is clearly true. The equality holds for $a=b=c$.

□

P 1.23. *If a,b,c are real numbers, then*

$$\frac{a^2-bc}{4a^2+b^2+4c^2} + \frac{b^2-ca}{4b^2+c^2+4a^2} + \frac{c^2-ab}{4c^2+a^2+4b^2} \ge 0.$$

(Vasile C., 2006)

Solution. Since
$$\frac{4(a^2-bc)}{4a^2+b^2+4c^2}=1-\frac{(b+2c)^2}{4a^2+b^2+4c^2},$$
we may rewrite the inequality as
$$\frac{(b+2c)^2}{4a^2+b^2+4c^2}+\frac{(c+2a)^2}{4b^2+c^2+4a^2}+\frac{(a+2b)^2}{4c^2+a^2+4b^2}\le 3.$$
Using the Cauchy-Schwarz inequality gives
$$\frac{(b+2c)^2}{4a^2+b^2+4c^2}=\frac{(b+2c)^2}{(2a^2+b^2)+2(2c^2+a^2)}\le\frac{b^2}{2a^2+b^2}+\frac{2c^2}{2c^2+a^2}.$$
Therefore,
$$\sum\frac{(b+2c)^2}{4a^2+b^2+4c^2}\le\sum\frac{b^2}{2a^2+b^2}+\sum\frac{2c^2}{2c^2+a^2}=\sum\frac{b^2}{2a^2+b^2}+\sum\frac{2a^2}{2a^2+b^2}=3.$$
The equality occurs when
$$a(2b^2+c^2)=b(2c^2+a^2)=c(2a^2+b^2);$$
that is, when $a=b=c$, and also when $a=2b=4c$ (or any cyclic permutation).

Remark. Similarly, we can prove the following generalization.

- *Let a,b,c be real numbers. If $k>0$, then*
$$\frac{a^2-bc}{2ka^2+b^2+k^2c^2}+\frac{b^2-ca}{2kb^2+c^2+k^2a^2}+\frac{c^2-ab}{2kc^2+a^2+k^2b^2}\ge 0,$$
with equality for $a=b=c$, and also for $a=kb=k^2c$ (or any cyclic permutation).

□

P 1.24. *If a,b,c are real numbers, then*

(a) $$a(a+b)^3+b(b+c)^3+c(c+a)^3\ge 0;$$

(b) $$a(a+b)^5+b(b+c)^5+c(c+a)^5\ge 0.$$

(Vasile C., 1989)

Solution. (a) Using the substitution
$$b+c=2x,\quad c+a=2y,\quad a+b=2z,$$
which are equivalent to
$$a=y+z-x,\quad b=z+x-y,\quad c=x+y-z,$$

the inequality becomes in succession

$$x^4+y^4+z^4+xy^3+yz^3+zx^3 \ge x^3y+y^3z+z^3x,$$

$$\sum(x^4+2xy^3-2x^3y+y^4) \ge 0,$$

$$\sum(x^2-xy-y^2)^2+\sum x^2y^2 \ge 0,$$

the last being clearly true. The equality occurs for $a=b=c=0$.

(b) Using the same substitution, the inequality turns into

$$x^6+y^6+z^6+xy^5+yz^5+zx^5 \ge x^5y+y^5z+z^5x,$$

which is equivalent to

$$\sum[x^6+y^6-2xy(x^4-y^4)] \ge 0,$$

$$\sum[(x^2+y^2)(x^4-x^2y^2+y^4)-2xy(x^2+y^2)(x^2-y^2)] \ge 0,$$

$$\sum(x^2+y^2)(x^2-xy-y^2)^2 \ge 0.$$

The equality occurs for $a=b=c=0$.

□

P 1.25. *If a,b,c are real numbers, then*

$$3(a^4+b^4+c^4)+4(a^3b+b^3c+c^3a) \ge 0.$$

(Vasile C., 2005)

Solution. If a,b,c are nonnegative, then the inequality is trivial. Since the inequality remains unchanged by replacing a,b,c with $-a,-b,-c$, respectively, it suffices to consider the case when only one of a,b,c is negative; let $c<0$. Replacing now c with $-c$, the inequality can be restated as

$$3(a^4+b^4+c^4)+4a^3b \ge 4(b^3c+c^3a),$$

where $a,b,c \ge 0$. It is enough to prove that

$$3(a^4+b^4+c^4+a^3b) \ge 4(b^3c+c^3a).$$

Case 1: $a \le b$. Since $a^3b \ge a^4$, it suffices to show that

$$6a^4+3b^4+3c^4 \ge 4(b^3c+ac^3).$$

Using the AM-GM inequality yields

$$3b^4+c^4 \ge 4\sqrt[4]{b^{12}c^4}=4b^3c.$$

Therefore, it suffices to show that

$$6a^4 + 2c^4 \ge 4ac^3.$$

Indeed, we have

$$3a^4 + c^4 = 3a^4 + \frac{1}{3}c^4 + \frac{1}{3}c^4 + \frac{1}{3}c^4 \ge 4\sqrt[4]{\frac{a^4c^{12}}{9}} = \frac{4}{\sqrt{3}}ac^3 \ge 2ac^3.$$

Case 2: $a \ge b$. Since $3a^3b \ge 3b^4$, it suffices to show that

$$3a^4 + 6b^4 + 3c^4 \ge 4(b^3c + ac^3).$$

By the AM-GM inequality, we get

$$6b^4 + \frac{c^4}{8} = 2b^4 + 2b^4 + 2b^4 + \frac{c^4}{8} \ge 4\sqrt[4]{b^{12}c^4} = 4b^3c.$$

Thus, we still have to show that

$$3a^4 + \frac{23}{8}c^4 \ge 4ac^3.$$

We will prove the sharper inequality

$$3a^4 + \frac{5}{2}c^4 \ge 4ac^3.$$

Indeed, we have

$$3a^4 + \frac{5}{2}c^4 = 3a^4 + \frac{5}{6}c^4 + \frac{5}{6}c^4 + \frac{5}{6}c^4 \ge 4\sqrt[4]{\frac{125a^4c^{12}}{72}} \ge 4ac^3.$$

The equality occurs for $a = b = c = 0$.

□

P 1.26. *If a, b, c are positive real numbers, then*

$$\frac{(a-b)(2a+b)}{(a+b)^2} + \frac{(b-c)(2b+c)}{(b+c)^2} + \frac{(c-a)(2c+a)}{(c+a)^2} \ge 0.$$

(Vasile C., 2006)

Solution. Since

$$\frac{(a-b)(2a+b)}{(a+b)^2} = \frac{2a^2 - b(a+b)}{(a+b)^2} = \frac{2a^2}{(a+b)^2} - \frac{b}{a+b},$$

we can write the inequality as

$$2\sum\left(\frac{a}{a+b}\right)^2-\sum\frac{b}{a+b}\ge 0.$$

According to P 1.1 in Volume 2, we have

$$\begin{aligned}2\sum\left(\frac{a}{a+b}\right)^2&=\sum\left(\frac{a}{a+b}\right)^2+\sum\left(\frac{b}{b+c}\right)^2\\&=\sum\left[\frac{1}{(1+b/a)^2}+\frac{1}{(1+c/b)^2}\right]\\&\ge\sum\frac{1}{1+c/a}=\sum\frac{a}{a+c}=\sum\frac{b}{b+a}.\end{aligned}$$

Therefore,

$$2\sum\left(\frac{a}{a+b}\right)^2-\sum\frac{b}{a+b}\ge\sum\frac{b}{b+a}-\sum\frac{b}{a+b}=0.$$

The equality holds for $a=b=c$.

□

P 1.27. *If a,b,c are positive real numbers, then*

$$\frac{(a-b)(2a+b)}{a^2+ab+b^2}+\frac{(b-c)(2b+c)}{b^2+bc+c^2}+\frac{(c-a)(2c+a)}{c^2+ca+a^2}\ge 0.$$

(Vasile C., 2006)

Solution. Since

$$\frac{(a-b)(2a+b)}{a^2+ab+b^2}=\frac{3a^2-(a^2+ab+b^2)}{a^2+ab+b^2}=\frac{3a^2}{a^2+ab+b^2}-1,$$

we can write the inequality as

$$\sum\frac{a^2}{a^2+ab+b^2}\ge 1,$$

$$\sum\frac{1}{1+b/a+(b/a)^2}\ge 1.$$

Clearly, this inequality follows immediately from P 1.45 in Volume 2. The equality holds for $a=b=c$.

□

P 1.28. *If a, b, c are positive real numbers, then*

$$\frac{(a-b)(3a+b)}{a^2+b^2}+\frac{(b-c)(3b+c)}{b^2+c^2}+\frac{(c-a)(3c+a)}{c^2+a^2} \ge 0.$$

(Vasile C., 2006)

Solution. Since

$$(a-b)(3a+b)=(a-b)^2+2(a^2-b^2),$$

we can write the inequality as

$$\sum \frac{(a-b)^2}{a^2+b^2}+2\sum \frac{a^2-b^2}{a^2+b^2} \ge 0.$$

Using the identity

$$\sum \frac{a^2-b^2}{a^2+b^2}+\prod \frac{a^2-b^2}{a^2+b^2}=0,$$

the inequality becomes

$$\sum \frac{(a-b)^2}{a^2+b^2} \ge 2\prod \frac{a^2-b^2}{a^2+b^2}.$$

By the AM-GM inequality, we have

$$\sum \frac{(a-b)^2}{a^2+b^2} \ge 3\sqrt[3]{\prod \frac{(a-b)^2}{a^2+b^2}}.$$

Thus, it suffices to show that

$$3\sqrt[3]{\prod \frac{(a-b)^2}{a^2+b^2}} \ge 2\prod \frac{a^2-b^2}{a^2+b^2},$$

which is equivalent to

$$27\prod \frac{(a-b)^2}{a^2+b^2} \ge 8\prod \frac{(a^2-b^2)^3}{(a^2+b^2)^3}.$$

This inequality is true if

$$27\prod (a^2+b^2)^2 \ge \prod (a-b)(a+b)^3.$$

Assume that $a=\max\{a,b,c\}$. For the nontrivial case $a>c>b$, we can get this inequality by multiplying the inequalities

$$3(a^2+b^2)^2 \ge 2(a-b)(a+b)^3,$$

$$3(c^2+b^2)^2 \ge 2(c-b)(c+b)^3,$$

$$3(a^2+c^2)^2 \ge 2(a-c)(a+c)^3.$$

These inequalities are true because

$$3(a^2+b^2)^2-2(a-b)(a+b)^3=a^2(a-2b)^2+b^2(2a^2+4ab+5b^2)>0.$$

The equality holds for $a=b=c$.

□

P 1.29. *Let a,b,c be positive real numbers such that $abc=1$. Then,*

$$\frac{1}{1+a+b^2}+\frac{1}{1+b+c^2}+\frac{1}{1+c+a^2}\le 1.$$

(Vasile C., 2005)

Solution. Using the substitution

$$a=x^3,\quad b=y^3,\quad c=z^3,$$

we have to show that $xyz=1$ involves

$$\frac{1}{1+x^3+y^6}+\frac{1}{1+y^3+z^6}+\frac{1}{1+z^3+x^6}\le 1.$$

By the Cauchy-Schwarz inequality, we have

$$\sum\frac{1}{1+x^3+y^6}\le\sum\frac{z^4+x+y^{-2}}{(z^2+x^2+y^2)^2}=\frac{\sum(z^4+x^2yz+x^2z^2)}{(x^2+y^2+z^2)^2}.$$

So, it remains to show that

$$(x^2+y^2+z^2)^2\ge\sum x^4+xyz\sum x+\sum x^2y^2,$$

which is equivalent to the known inequality

$$\sum x^2y^2\ge xyz\sum x.$$

The equality occurs for $a=b=c=1$.

Remark. Actually, the following generalization holds:

- *Let a,b,c be positive real numbers such that $abc=1$. If $k\ge 0$, then*

$$\frac{1}{1+a+b^k}+\frac{1}{1+b+c^k}+\frac{1}{1+c+a^k}\le 1.$$

□

P 1.30. *Let a,b,c be positive real numbers such that $abc=1$. Then,*

$$\frac{a}{(a+1)(b+2)}+\frac{b}{(b+1)(c+2)}+\frac{c}{(c+1)(a+2)}\ge\frac{1}{2}.$$

Solution. Using the substitution

$$a=\frac{x}{y},\quad b=\frac{y}{z},\quad c=\frac{z}{x},$$

where x, y, z are positive real numbers, the inequality can be restated as

$$\frac{zx}{(x+y)(y+2z)}+\frac{xy}{(y+z)(z+2x)}+\frac{yz}{(z+x)(x+2y)}\geq\frac{1}{2}.$$

By the Cauchy-Schwarz inequality, we have

$$\sum\frac{zx}{(x+y)(y+2z)}\geq\frac{\left(\sum zx\right)^2}{\sum zx(x+y)(y+2z)}=\frac{1}{2}.$$

The equality occurs for $a=b=c=1$.

□

P 1.31. *If a, b, c are positive real numbers such that $ab+bc+ca=3$, then*

$$(a+2b)(b+2c)(c+2a)\geq 27.$$

(Michael Rozenberg, 2007)

Solution. Write the inequality in the homogeneous form

$$A+B\geq 0,$$

where

$$\begin{aligned}A&=(a+2b)(b+2c)(c+2a)-3(a+b+c)(ab+bc+ca)\\&=(a-b)(b-c)(c-a)\end{aligned}$$

and

$$B=3(ab+bc+ca)[a+b+c-\sqrt{3(ab+bc+ca)}].$$

Since

$$\begin{aligned}B&=\frac{3(ab+bc+ca)[(a-b)^2+(b-c)^2+(c-a)^2]}{2(a+b+c+\sqrt{3(ab+bc+ca)}]}\\&\geq\frac{3(ab+bc+ca)[(a-b)^2+(b-c)^2+(c-a)^2]}{4(a+b+c)},\end{aligned}$$

it suffices to show that

$$4(a+b+c)(a-b)(b-c)(c-a)+3(ab+bc+ca)[(a-b)^2+(b-c)^2+(c-a)^2]\geq 0.$$

Consider $c = \min\{a, b, c\}$, and use the substitution

$$a = c + x, \quad b = c + y, \qquad x, y \geq 0.$$

The inequality becomes

$$-4xy(x-y)(3c+x+y) + 6(x^2 - xy + y^2)[3c^2 + 2(x+y)c + xy] \geq 0,$$

which is equivalent to

$$9(x^2 - xy + y^2)c^2 + 6Cc + D \geq 0,$$

where

$$C = x^3 - x^2y + xy^2 + y^3 \geq x(x^2 - xy + y^2),$$

$$D = xy(x^2 + 5y^2 - 3xy) \geq (2\sqrt{5} - 3)x^2y^2.$$

Since $C \geq 0$ and $D \geq 0$, the inequality is obvious. The equality holds for $a = b = c = 1$.

□

P 1.32. *If a, b, c are positive real numbers such that $ab + bc + ca = 3$, then*

$$\frac{a}{a + a^3 + b} + \frac{b}{b + b^3 + c} + \frac{c}{c + c^3 + a} \leq 1.$$

(Andrei Ciupan, 2005)

Solution. Write the inequality as

$$\frac{1}{1 + a^2 + b/a} + \frac{1}{1 + b^2 + c/b} + \frac{1}{1 + c^2 + a/c} \leq 1.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{1}{1 + a^2 + b/a} \leq \sum \frac{c^2 + 1 + ab}{(c + a + b)^2} = 1.$$

The equality holds for $a = b = c = 1$.

□

P 1.33. *If a, b, c are positive real numbers such that $a \geq b \geq c$ and $ab + bc + ca = 3$, then*

$$\frac{1}{a + 2b} + \frac{1}{b + 2c} + \frac{1}{c + 2a} \geq 1.$$

Solution. According to the well known inequality

$$x+y+z \geq \sqrt{3(xy+yz+zx)},$$

where x, y, z are positive real numbers, it suffices to prove that

$$\frac{1}{(a+2b)(b+2c)}+\frac{1}{(b+2c)(c+2a)}+\frac{1}{(c+2a)(a+2b)} \geq \frac{1}{3}.$$

This is equivalent to the following inequalities

$$9(a+b+c) \geq (a+2b)(b+2c)(c+2a),$$

$$3(a+b+c)(ab+bc+ca) \geq (a+2b)(b+2c)(c+2a),$$

$$a^2b+b^2c+c^2a \geq ab^2+bc^2+ca^2,$$

$$(a-b)(b-c)(a-c) \geq 0.$$

The last inequality is clearly true for $a \geq b \geq c$. The equality occurs for $a=b=c=1$.

□

P 1.34. *If $a, b, c \in [0,1]$, then*

$$\frac{a}{4b^2+5}+\frac{b}{4c^2+5}+\frac{c}{4a^2+5} \leq \frac{1}{3}.$$

Solution. Let

$$E(a,b,c)=\frac{a}{4b^2+5}+\frac{b}{4c^2+5}+\frac{c}{4a^2+5}.$$

We have

$$\begin{aligned}
E(a,b,c)-E(1,b,c) &= \frac{a-1}{4b^2+5}+c\left(\frac{1}{4a^2+5}-\frac{1}{9}\right) \\
&= (1-a)\left[\frac{4c(1+a)}{9(4a^2+5)}-\frac{1}{4b^2+5}\right] \\
&\leq (1-a)\left[\frac{4(1+a)}{9(4a^2+5)}-\frac{1}{9}\right] \\
&= \frac{-(1-a)(1-2a)^2}{9(4a^2+5)} \leq 0,
\end{aligned}$$

and, similarly,

$$E(a,b,c)-E(a,1,c) \leq 0, \quad E(a,b,c)-E(a,b,1) \leq 0.$$

Therefore,

$$E(a,b,c) \le E(1,b,c) \le E(1,1,c) \le E(1,1,1) = \frac{1}{3}.$$

The equality occurs for $a=b=c=1$, and also for $a=\frac{1}{2}$ and $b=c=1$ (or any cyclic permutation).

□

P 1.35. *If $a,b,c \in \left[\frac{1}{3},3\right]$, then*

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a} \ge \frac{7}{5}.$$

Solution. Assume that $a=\max\{a,b,c\}$ and show that

$$E(a,b,c) \ge E(a,b,\sqrt{ab}\,) \ge \frac{7}{5},$$

where

$$E(a,b,c)=\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}.$$

We have

$$\begin{aligned} E(a,b,c)-E(a,b,\sqrt{ab}\,) &= \frac{b}{b+c}+\frac{c}{c+a}-\frac{2\sqrt{b}}{\sqrt{a}+\sqrt{b}} \\ &= \frac{\left(\sqrt{a}-\sqrt{b}\right)\left(\sqrt{ab}-c\right)^2}{(b+c)(c+a)\left(\sqrt{a}+\sqrt{b}\right)} \ge 0. \end{aligned}$$

Substituting $x=\sqrt{\frac{a}{b}}$, the hypothesis $a,b,c \in \left[\frac{1}{3},3\right]$ involves $x \in \left[\frac{1}{3},3\right]$. Then,

$$\begin{aligned} E(a,b,\sqrt{ab})-\frac{7}{5} &= \frac{a}{a+b}+\frac{2\sqrt{b}}{\sqrt{a}+\sqrt{b}}-\frac{7}{5} \\ &= \frac{x^2}{x^2+1}+\frac{2}{x+1}-\frac{7}{5} \\ &= \frac{3-7x+8x^2-2x^3}{5(x+1)(x^2+1)} \\ &= \frac{(3-x)[x^2+(1-x)^2]}{5(x+1)(x^2+1)} \ge 0. \end{aligned}$$

The equality holds for $a=3$, $b=\frac{1}{3}$ and $c=1$ (or any cyclic permutation).

□

P 1.36. *If* $a, b, c \in \left[\frac{1}{\sqrt{2}}, \sqrt{2}\right]$, *then*

$$\frac{3}{a+2b}+\frac{3}{b+2c}+\frac{3}{c+2a} \geq \frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{c+a}.$$

Solution. Write the inequality as

$$\sum\left(\frac{3}{a+2b}-\frac{2}{a+b}+\frac{1}{ka}-\frac{1}{kb}\right) \geq 0, \quad k>0,$$

$$\sum \frac{-(a-b)[a^2-(k-3)ab+2b^2]}{kab(a+2b)(a+b)} \geq 0.$$

Choosing $k=6$, the inequality becomes

$$\sum \frac{(a-b)^2(2b-a)}{6ab(a+2b)(a+b)} \geq 0.$$

Since

$$2b-a \geq \frac{2}{\sqrt{2}}-\sqrt{2}=0,$$

the conclusion follows. The equality holds for $a=b=c$.

□

P 1.37. *If* a, b, c *are nonnegative real numbers, no two of which are zero, then*

$$\frac{4abc}{ab^2+bc^2+ca^2+abc}+\frac{a^2+b^2+c^2}{ab+bc+ca} \geq 2.$$

(Vo Quoc Ba Can, 2009)

First Solution. Without loss of generality, assume that b is between a and c; that is,

$$b^2+ca \leq b(c+a).$$

Then,

$$\begin{aligned} ab^2+bc^2+ca^2+abc &= a(b^2+ca)+bc^2+abc \\ &\leq ab(c+a)+bc^2+abc \\ &= b(a+c)^2, \end{aligned}$$

and it suffices to prove that

$$\frac{4ac}{(a+c)^2}+\frac{a^2+b^2+c^2}{ab+bc+ca} \geq 2.$$

This inequality is equivalent to

$$[a^2+c^2-b(a+c)]^2 \ge 0.$$

The equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

Second Solution. Let (x,y,z) be a permutation of (a,b,c) such that $x \ge y \ge z$. As we have shown in the second solution of P 1.1,

$$ab^2+bc^2+ca^2 \le y(x^2+xz+z^2);$$

hence

$$ab^2+bc^2+ca^2+abc \le y(x+z)^2.$$

Thus, it suffices to prove that

$$\frac{4xyz}{y(x+z)^2}+\frac{x^2+y^2+z^2}{xy+yz+zx} \ge 2,$$

which is equivalent to

$$\frac{x^2+y^2+z^2}{xy+yz+zx} \ge \frac{2(x^2+z^2)}{x+z)^2},$$

$$(x^2+z^2)^2-2y(x+z)(x^2+z^2)+y^2(x+z)^2 \ge 0,$$

$$(x^2+z^2-xy-yz)^2 \ge 0.$$

□

P 1.38. *If a,b,c are nonnegative real numbers such that $a+b+c=3$, then*

$$\frac{1}{ab^2+8}+\frac{1}{bc^2+8}+\frac{1}{ca^2+8} \ge \frac{1}{3}.$$

(Vasile C., 2007)

Solution. By expanding, we can write the inequality as

$$64 \ge r^3+16A+5rB,$$

$$64 \ge r^3+(16-5r)A+5r(A+B),$$

where

$$r=abc, \quad A=ab^2+bc^2+ca^2, \quad B=a^2b+b^2c+c^2a.$$

By the AM-GM inequality, we have

$$r \le \left(\frac{a+b+c}{3}\right)^3=1.$$

On the other hand, by the inequality (a) in P 1.9, we get

$$A \le 4 - r,$$

and by Schur's inequality, we have

$$(a+b+c)^3 + 9abc \ge 4(a+b+c)(ab+bc+ca),$$

which is equivalent to

$$A + B \le \frac{27-3r}{4}.$$

Therefore, it suffices to prove that

$$64 \ge r^3 + (16-5r)(4-r) + \frac{5r(27-3r)}{4}.$$

We can write this inequality in the obvious form

$$r(1-r)(9+4r) \ge 0.$$

The equality holds for $a = b = c = 1$, and also for $a = 0$, $b = 1$, $c = 2$ (or any cyclic permutation).

□

P 1.39. *If a, b, c are nonnegative real numbers such that $a + b + c = 3$, then*

$$\frac{ab}{bc+3} + \frac{bc}{ca+3} + \frac{ca}{ab+3} \le \frac{3}{4}.$$

(Vasile C., 2008)

Solution. Using the inequality (a) in P 1.9, namely

$$a^2b + b^2c + c^2a \le 4 - abc,$$

we have

$$\begin{aligned}\sum ab(ca+3)(ab+3) &= abc\sum a^2b + 9abc + 3\sum a^2b^2 + 9\sum ab \\ &\le 13abc - a^2b^2c^2 + 3\sum a^2b^2 + 9\sum ab.\end{aligned}$$

On the other hand,

$$(ab+3)(bc+3)(ca+3) = a^2b^2c^2 + 9abc + 9\sum ab + 27.$$

Therefore, it suffices to prove that

$$4\left(13abc - a^2b^2c^2 + 3\sum a^2b^2 + 9\sum ab\right) \le 3\left(a^2b^2c^2 + 9abc + 9\sum ab + 27\right),$$

which is equivalent to

$$7a^2b^2c^2+81 \ge 25abc+12\sum a^2b^2+9\sum ab,$$

$$7r^2+47r \ge 3(q+3)(4q-9),$$

where

$$q=ab+bc+ca, \quad r=abc, \quad q\le 3, \;\; r\le 1.$$

Since

$$7r^2+47r \ge 9r^2+45r,$$

it suffices to show that

$$3r^2+15r \ge (q+3)(4q-9).$$

Consider the non-trivial case

$$\frac{9}{4}<q\le 3,$$

and apply the fourth degree Schur's inequality

$$r \ge \frac{(p^2-q)(4q-p^2)}{6p}=\frac{(9-q)(4q-9)}{18}.$$

It remains to show that

$$\frac{(9-q)^2(4q-9)^2}{108}+\frac{5(9-q)(4q-9)}{6} \ge (q+3)(4q-9),$$

which is equivalent to

$$(4q-9)(3-q)(69q-4q^2-81) \ge 0.$$

This is true because

$$69q-4q^2-81=(3-q)(4q-9)+6(8q-9)>0.$$

The equality holds for $a=b=c=1$, and also for $a=0$ and $b=c=\frac{3}{2}$ (or any cyclic permutation).

□

P 1.40. *If a,b,c are nonnegative real numbers such that $a+b+c=3$, then*

(a) $$\frac{a}{b^2+3}+\frac{b}{c^2+3}+\frac{c}{a^2+3} \ge \frac{3}{4};$$

(b) $$\frac{a}{b^3+1}+\frac{b}{c^3+1}+\frac{c}{a^3+1} \ge \frac{3}{2}.$$

(Vasile Cîrtoaje and Bin Zhao, 2005)

Solution. (a) By the AM-GM inequality, we have

$$b^2+3=b^2+1+1+1\ge 4\sqrt[4]{b^2\cdot 1^3}=4\sqrt{b}.$$

Therefore,

$$\frac{3a}{b^2+3}=a-\frac{ab^2}{b^2+3}\ge a-\frac{ab^2}{4\sqrt{b}}=a-\frac{1}{4}ab\sqrt{b}.$$

Taking account of this inequality and the similar ones, it suffices to prove that

$$ab\sqrt{b}+bc\sqrt{c}+ca\sqrt{a}\le 3.$$

This inequality follows immediately by replacing a,b,c with $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ in the inequality in P 1.7. The equality holds for $a=b=c=1$.

(b) Using the AM-GM Inequality gives

$$\frac{a}{b^3+1}=a-\frac{ab^3}{b^3+1}\ge a-\frac{ab^3}{2b\sqrt{b}}=a-\frac{1}{2}ab\sqrt{b},$$

and, similarly,

$$\frac{b}{c^3+1}\ge b-\frac{1}{2}bc\sqrt{c},\quad \frac{c}{a^3+1}\ge c-\frac{1}{2}ca\sqrt{a}.$$

Thus, it suffices to show that

$$ab\sqrt{b}+bc\sqrt{c}+ca\sqrt{a}\le 3,$$

which follows from the inequality in P 1.7. The equality holds for $a=b=c=1$.

Conjecture. *Let a,b,c be nonnegative real numbers such that $a+b+c=3$. If*

$$0<k\le 3+2\sqrt{3},$$

then

$$\frac{a}{b^2+k}+\frac{b}{c^2+k}+\frac{c}{a^2+k}\ge\frac{3}{1+k}.$$

For $k=3+2\sqrt{3}$, the equality occurs when $a=b=c=1$, and again when $a=0$, $b=3-\sqrt{3}$ and $c=\sqrt{3}$ (or any cyclic permutation thereof).

□

P 1.41. *Let a,b,c be positive real numbers, and let*

$$x=a+\frac{1}{b}-1,\quad y=b+\frac{1}{c}-1,\quad z=c+\frac{1}{a}-1.$$

Prove that

$$xy+yz+zx\ge 3.$$

(Vasile C., 1991)

First Solution. Among x, y, z, there are two numbers either less than or equal to 1, or greater than or equal to 1. Let y and z be these numbers; that is,

$$(y-1)(z-1) \ge 0.$$

Since

$$xy + yz + zx - 3 = (y-1)(z-1) + (x+1)(y+z) - 4,$$

it suffices to show that

$$(x+1)(y+z) \ge 4.$$

Since

$$y + z = b + \frac{1}{a} + c + \frac{1}{c} - 2 \ge b + \frac{1}{a},$$

we have

$$(x+1)(y+z) - 4 \ge (x+1)\left(b + \frac{1}{a}\right) - 4 = ab + \frac{1}{ab} - 2 \ge 0.$$

The equality holds for $a = b = c = 1$.

Second Solution. Without loss of generality, assume that $x = \max\{x, y, z\}$. Then,

$$\begin{aligned} x \ge \frac{1}{3}(x+y+z) &= \frac{1}{3}\left[\left(a+\frac{1}{a}\right) + \left(b+\frac{1}{b}\right) + \left(c+\frac{1}{c}\right) - 3\right] \\ &\ge \frac{1}{3}(2+2+2-3) = 1. \end{aligned}$$

On the other hand, from

$$\begin{aligned} (x+1)(y+1)(z+1) &= abc + \frac{1}{abc} + a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \\ &\ge 2 + a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \\ &= 5 + x + y + z, \end{aligned}$$

we get

$$xyz + xy + yz + zx \ge 4.$$

Since

$$y + z = \frac{1}{a} + b + \frac{(c-1)^2}{c} > 0,$$

two cases are possible: $yz \le 0$ and $y, z > 0$.

Case 1: $yz \le 0$. Since $xyz \le 0$, it follows that

$$xy + yz + zx \ge 4 - xyz \ge 4 > 3.$$

Case 2: $y, z > 0$. We need to show that $d \ge 1$, where

$$d = \sqrt{\frac{xy + yz + zx}{3}}.$$

By the AM-GM inequality, we have $d^3 \geq xyz$. Thus, from $xyz + xy + yz + zx \geq 4$, we get

$$d^3 + 3d^2 \geq 4,$$

$$(d-1)(d+2)^2 \geq 0,$$

hence $d \geq 1$.

□

P 1.42. *Let a, b, c be positive real numbers such that $abc = 1$. Prove that*

$$\left(a - \frac{1}{b} - \sqrt{2}\right)^2 + \left(b - \frac{1}{c} - \sqrt{2}\right)^2 + \left(c - \frac{1}{a} - \sqrt{2}\right)^2 \geq 6.$$

Solution *(by Nguyen Van Quy)*. Using the substitution

$$a = \frac{y}{x}, \quad b = \frac{x}{z}, \quad c = \frac{z}{y}, \qquad x, y, z > 0,$$

the inequality becomes as follows:

$$\left(\frac{y-z}{x} - \sqrt{2}\right)^2 + \left(\frac{z-x}{y} - \sqrt{2}\right)^2 + \left(\frac{x-y}{z} - \sqrt{2}\right)^2 \geq 6,$$

$$\left(\frac{y-z}{x}\right)^2 + \left(\frac{z-x}{y}\right)^2 + \left(\frac{x-y}{z}\right)^2 - 2\sqrt{2}\left(\frac{y-z}{x} + \frac{z-x}{y} + \frac{x-y}{z}\right) \geq 0,$$

$$\left(\frac{y-z}{x}\right)^2 + \left(\frac{z-x}{y}\right)^2 + \left(\frac{x-y}{z}\right)^2 + \frac{2\sqrt{2}(y-z)(z-x)(x-y)}{xyz} \geq 0.$$

Assume that $x = \max\{x, y, z\}$. For $x \geq z \geq y$, the inequality is clearly true. Consider further that $x \geq y \geq z$ and write the desired inequality as

$$u^2 + v^2 + w^2 \geq 2\sqrt{2}\, uvw,$$

where

$$u = \frac{y-z}{x} \geq 0, \quad v = \frac{x-z}{y} \geq 0, \quad w = \frac{x-y}{z} \geq 0.$$

In addition, we have

$$uv = \left(1 - \frac{z}{y}\right)\left(1 - \frac{z}{x}\right) < 1 \cdot 1 = 1.$$

According to the AM-GM inequality, we get

$$u^2 + v^2 + w^2 \geq 2uv + w^2 \geq 2u^2v^2 + w^2 \geq 2\sqrt{2}\, uvw.$$

This completes the proof. The equality holds for $a = b = c$.

□

P 1.43. *Let a,b,c be positive real numbers such that $abc=1$. Prove that*

$$\left|1+a-\frac{1}{b}\right|+\left|1+b-\frac{1}{c}\right|+\left|1+c-\frac{1}{a}\right|>2.$$

Solution. Using the substitution

$$a=\frac{y}{x},\quad b=\frac{x}{z},\quad c=\frac{z}{y},\qquad x,y,z>0,$$

the inequality can be restated as

$$\left|1+\frac{y-z}{x}\right|+\left|1+\frac{x-y}{z}\right|+\left|1+\frac{z-x}{y}\right|>2.$$

Without loss of generality, assume that $x=\max\{x,y,z\}$. We have

$$\left|1+\frac{y-z}{x}\right|+\left|1+\frac{x-y}{z}\right|+\left|1+\frac{z-x}{y}\right|-2\ge\left|1+\frac{y-z}{x}\right|+\left|1+\frac{x-y}{z}\right|-2$$

$$=\frac{x+y-z}{x}+\frac{z+x-y}{z}-2=\frac{y-z}{x}+\frac{x-y}{z}\ge\frac{y-z}{x}+\frac{x-y}{x}=\frac{x-z}{x}\ge 0.$$

□

P 1.44. *If a,b,c are different positive real numbers, then*

$$\left|1+\frac{a}{b-c}\right|+\left|1+\frac{b}{c-a}\right|+\left|1+\frac{c}{a-b}\right|>2.$$

(Vasile C., 2012)

Solution. Without loss of generality, assume that $a=\max\{a,b,c\}$. It suffices to show that

$$\left|1+\frac{a}{b-c}\right|+\left|1+\frac{c}{a-b}\right|>2,$$

which is equivalent to

$$\frac{a+b-c}{|b-c|}+\frac{a-b+c}{a-b}>2.$$

For $b>c$, this inequality is true since

$$\frac{a+b-c}{|b-c|}+\frac{a-b+c}{a-b}>\frac{a+b-c}{|b-c|}=\frac{a}{b-c}+1>1+1=2.$$

Also, for $b<c$, we have

$$\frac{a+b-c}{|b-c|}+\frac{a-b+c}{a-b}=\frac{a+b-c}{c-b}+\frac{a-b+c}{a-b}$$

$$=\frac{a}{c-b}+\frac{c}{a-b}>\frac{a}{c-b}+\frac{c-b}{a-b}\ge 2\sqrt{\frac{a}{a-b}}>2.$$

□

P 1.45. *Let a, b, c be positive real numbers such that $abc = 1$. Prove that*

$$\left(2a-\frac{1}{b}-\frac{1}{2}\right)^2+\left(2b-\frac{1}{c}-\frac{1}{2}\right)^2+\left(2c-\frac{1}{a}-\frac{1}{2}\right)^2 \ge \frac{3}{4}.$$

(Vasile C., 2012)

Solution. Using the substitution

$$x = 2a-\frac{1}{b}, \quad y = 2b-\frac{1}{c}, \quad z = 2c-\frac{1}{a},$$

we can write the inequality as

$$x^2+y^2+z^2 \ge x+y+z.$$

From

$$x+y+z = 2\sum a-\sum\frac{1}{a}$$

and

$$xyz = 7-4\sum a+2\sum\frac{1}{a},$$

it follows that

$$2(x+y+z)+xyz = 7.$$

In addition, from

$$2(|x|+|y|+|z|)+\left(\frac{|x|+|y|+|z|}{3}\right)^3 \ge 2(|x|+|y|+|z|)+|xyz|$$
$$\ge 2(x+y+z)+xyz = 7,$$

we get

$$|x|+|y|+|z| \ge 3.$$

Therefore, we have

$$x^2+y^2+z^2 \ge \frac{1}{3}(|x|+|y|+|z|)^2 \ge |x|+|y|+|z| \ge x+y+z.$$

The equality holds for $a = b = c = 1$.

□

P 1.46. *Let*

$$x = a+\frac{1}{b}-\frac{5}{4}, \quad y = b+\frac{1}{c}-\frac{5}{4}, \quad z = c+\frac{1}{a}-\frac{5}{4},$$

where $a \ge b \ge c > 0$. Prove that

$$xy+yz+zx \ge \frac{27}{16}.$$

(Vasile C., 2011)

Solution. Write the inequality as

$$\sum\left(ab+\frac{1}{ab}\right)+\sum\frac{b}{a}-\frac{5}{2}\sum\left(a+\frac{1}{a}\right)+6\geq 0.$$

Since

$$\sum\frac{b}{a}-\sum\frac{a}{b}=\frac{(a-b)(b-c)(a-c)}{abc}\geq 0,$$

we have

$$2\sum\frac{b}{a}\geq\sum\frac{b}{a}+\sum\frac{a}{b}=\left(\sum a\right)\left(\sum\frac{1}{a}\right)-3.$$

Thus, it suffices to prove the symmetric inequality

$$2\sum\left(ab+\frac{1}{ab}\right)+\left(\sum a\right)\left(\sum\frac{1}{a}\right)-5\sum\left(a+\frac{1}{a}\right)+9\geq 0.$$

Setting

$$p=a+b+c,\quad q=ab+bc+ca,\quad r=abc,$$

we need to show that

$$(2q-5p+9)r+pq-5q+2p\geq 0$$

for all $a,b,c>0$. For fixed p and q, the linear function

$$f(r)=(2q-5p+9)r+pq-5q+2p$$

is minimal when r is either minimal or maximal. Thus, according to P 3.57 in Volume 1, it suffices to prove that $f(r)\geq 0$ for $a=0$ and for $b=c$.

For $a=0$, we need to show that

$$(b+c)bc-5bc+2(b+c)\geq 0.$$

Indeed, putting $x=\sqrt{bc}$, we have

$$(b+c)bc-5bc+2(b+c)\geq 2x^3-5x^2+4x>0.$$

For $b=c$, since

$$p=a+2b,\quad q=2ab+b^2,\quad r=ab^2,$$

the inequality $f(r)\geq 0$ becomes

$$(4ab+2b^2-5a-10b+9)ab^2+(a+2b)(2ab+b^2)-10ab-5b^2+2a+4b\geq 0;$$

that is,

$$Aa^2+2Ba+C\geq 0,$$

where

$$A = b(4b^2 - 5b + 2) > 0, \quad B = b^4 - 5b^3 + 7b^2 - 5b + 1, \quad C = b(2b^2 - 5b + 4) > 0.$$

Let

$$x = b + \frac{1}{b}, \quad x \geq 2.$$

The inequality $B \geq 0$ is equivalent to

$$b^2 + \frac{1}{b^2} - 5\left(b + \frac{1}{b}\right) + 7 \geq 0,$$

$$x^2 - 5x + 5 \geq 0,$$

$$x \geq \frac{5 + \sqrt{5}}{2}.$$

Consider two cases.

Case 1: $x \geq \dfrac{5+\sqrt{5}}{2}$. Since $A > 0$, $B \geq 0$, $C > 0$, we have $Aa^2 + 2Ba + C > 0$.

Case 2: $2 \leq x < \dfrac{5+\sqrt{5}}{2}$. Since $A > 0$, $B < 0$, $C > 0$ and

$$Aa^2 + 2Ba + C = (Aa^2 + C) + 2Ba \geq 2a(\sqrt{AC} + B),$$

we need to show that $AC \geq B^2$, which is equivalent to

$$8\left(b^2 + \frac{1}{b^2}\right) - 30\left(b + \frac{1}{b}\right) + 45 \geq \left[b^2 + \frac{1}{b^2} - 5\left(b + \frac{1}{b}\right) + 7\right]^2,$$

$$8x^2 - 30x + 29 \geq (x^2 - 5x + 5)^2,$$

$$(x-2)^2(x^2 - 6x - 1) \leq 0.$$

This inequality is true for $x \leq 3 + \sqrt{10}$, therefore for $x < (5+\sqrt{5})/2$. Thus, the proof is completed. The equality holds for $a = b = c = 1$.

□

P 1.47. *Let a, b, c be positive real numbers, and let*

$$E = \left(a + \frac{1}{a} - \sqrt{3}\right)\left(b + \frac{1}{b} - \sqrt{3}\right)\left(c + \frac{1}{c} - \sqrt{3}\right);$$

$$F = \left(a + \frac{1}{b} - \sqrt{3}\right)\left(b + \frac{1}{c} - \sqrt{3}\right)\left(c + \frac{1}{a} - \sqrt{3}\right).$$

Prove that $E \geq F$.

(Vasile C., 2011)

Solution. By expanding, the inequality becomes

$$\sum(a^2-bc)+\sum bc(bc-a^2)\ge\sqrt{3}\sum ab(b-c).$$

Since

$$\sum(a^2-bc)=\sum a^2-\sum ab\ge 0$$

and

$$\sum bc(bc-a^2)=\sum a^2b^2-abc\sum a\ge 0,$$

by the AM-GM inequality, we have

$$\sum(a^2-bc)+\sum bc(bc-a^2)\ge 2\sqrt{\left[\sum(a^2-bc)\right]\left[\sum bc(bc-a^2)\right]}.$$

Thus, it suffices to show that

$$2\sqrt{\left[\sum(a^2-bc)\right]\left[\sum bc(bc-a^2)\right]}\ge\sqrt{3}\sum ab(b-c),$$

which is equivalent to

$$2\sqrt{\left[\sum(a^2-bc)\right]\left[\sum\left(\frac{1}{a^2}-\frac{1}{bc}\right)\right]}\ge\sqrt{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\right),$$

$$\sqrt{[(a+c-2b)^2+3(c-a)^2]\left[3\left(\frac{1}{b}-\frac{1}{c}\right)^2+\left(\frac{2}{a}-\frac{1}{b}-\frac{1}{c}\right)^2\right]}\ge$$

$$\ge 2\sqrt{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\right).$$

Applying the Cauchy-Schwarz inequality, it suffices to show that

$$(a+c-2b)\left(\frac{1}{b}-\frac{1}{c}\right)+(c-a)\left(\frac{2}{a}-\frac{1}{b}-\frac{1}{c}\right)\ge 2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\right),$$

which is an identity. Thus, the proof is completed. The equality holds when the following two equations are satisfied:

$$a^2+b^2+c^2-ab-bc-ca=a^2b^2+b^2c^2+c^2a^2-abc(a+b+c)$$

and

$$3+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=2\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right).$$

□

P 1.48. *If a,b,c are positive real numbers such that $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=5$, then*

$$\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\geq\frac{17}{4}.$$

(Vasile C., 2007)

Solution. Making the substitution

$$x=\frac{a}{b},\quad y=\frac{b}{c},\quad z=\frac{c}{a},$$

we need to show that if x,y,z are positive real numbers satisfying

$$xyz=1,\qquad x+y+z=5,$$

then

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq\frac{17}{4}.$$

From $(y+z)^2\geq 4yz$, we get

$$(5-x)^2\geq\frac{4}{x};$$

therefore,

$$(5-x)+(5-x)+\frac{x}{4}\geq 3\sqrt[3]{(5-x)^2\frac{x}{4}}\geq 3,$$

which involves $x\leq 4$. We have

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-\frac{17}{4}=\frac{1}{x}+\frac{y+z}{yz}-\frac{17}{4}=\frac{1}{x}+x(5-x)-\frac{17}{4}$$
$$=\frac{4-17x+20x^2-4x^3}{4x}=\frac{(4-x)(1-2x)^2}{4x}\geq 4.$$

The equality holds when one of x,y,z is 4 and the others are $\frac{1}{2}$; that is, when

$$a=4b=2c$$

(or any cyclic permutation).

□

P 1.49. *If a,b,c are positive real numbers, then*

(a) $$1+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 2\sqrt{1+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}}\ ;$$

(b) $$1+2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\geq\sqrt{1+16\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)}\ ;$$

(c) $$3+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 2\sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}\ .$$

(Vasile C., 2007)

Solution. Let

$$x = \frac{a}{b}, \quad y = \frac{b}{c}, \quad z = \frac{c}{a}$$

and

$$p = x + y + z, \quad q = xy + yz + zx.$$

By the AM-GM inequality, we have

$$p \geq 3\sqrt[3]{xyz} = 3.$$

(a) We need to show that $xyz = 1$ involves

$$1 + x + y + z \geq 2\sqrt{1 + xy + yz + zx},$$

which is equivalent to

$$(1+p)^2 \geq 4 + 4q$$

or

$$p + 3 \geq 2\sqrt{p + q + 3}.$$

First Solution. By Schur's inequality of degree three, we have

$$p^3 + 9 \geq 4pq.$$

Thus,

$$(1+p)^2 - 4 - 4q \geq 1 + p)^2 - 4 - \left(p^2 + \frac{9}{p}\right) = \frac{(p-3)(2p+3)}{p} \geq 0.$$

The equality holds for $a = b = c$.

Second Solution. Without loss of generality, assume that b is between a and c. By the AM-GM inequality, we have

$$2\sqrt{p+q+3} = 2\sqrt{(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)} \leq \frac{a+b+c}{b} + b\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

Therefore,

$$\begin{aligned} p + 3 - 2\sqrt{p+q+3} &\geq \frac{a}{b} + \frac{b}{c} + \frac{c}{a} + 3 - \frac{a+b+c}{b} - b\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \\ &= \frac{(a-b)(b-c)}{ab} \geq 0. \end{aligned}$$

(b) We have to show that $xyz = 1$ involves

$$1 + 2(x + y + z) \geq \sqrt{1 + 16(xy + yz + zx)},$$

which is equivalent to

$$p^2 + p \geq 4q.$$

By Schur's inequality of degree three, we have

$$p^3 + 9 \geq 4pq.$$

Thus,

$$p^2 + p - 4q \geq p^2 + p - \left(p^2 + \frac{9}{p}\right) = \frac{(p-3)(p+3)}{9} \geq 0.$$

The equality holds for $a = b = c$.

(c) Write the inequality as follows:

$$(3 + x + y + z)^2 \geq 4(3 + x + y + z + xy + yz + zx),$$

$$(x + y + z)^2 + 2(x + y + z) \geq 3 + 4(xy + yz + zx),$$

$$(1 + x + y + z)^2 \geq 4(1 + xy + yz + zx),$$

$$1 + x + y + z \geq 2\sqrt{1 + xy + yz + zx},$$

$$1 + \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 2\sqrt{1 + \frac{b}{a} + \frac{c}{b} + \frac{a}{c}}.$$

Thus, the inequality is equivalent to the inequality in (a).

□

P 1.50. *If a, b, c are positive real numbers, then*

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} + 15\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right) \geq 16\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right).$$

Solution. Making the substitution

$$x = \frac{a}{b}, \quad y = \frac{b}{c}, \quad z = \frac{c}{a},$$

we have to show that $xyz = 1$ involves

$$x^2 + y^2 + z^2 + 15(xy + yz + zx) \geq 16(x + y + z),$$

which is equivalent to

$$(x + y + z)^2 - 16(x + y + z) + 13(xy + yz + zx) \geq 0.$$

According to P 3.58 in Volume 1, for fixed $x + y + z$ and $xyz = 1$, the expression

$$xy + yz + zx$$

is minimal when two of x, y, z are equal. Therefore, due to symmetry, it suffices to consider that $x = y$. We need to show that

$$(2x + z)^2 - 16(2x + z) + 13(x^2 + 2xz) \geq 0$$

for $x^2z=1$. Write this inequality as

$$17x^6-32x^5+30x^3-16x^2+1\ge 0,$$

or

$$(x-1)^2g(x)\ge 0,\quad g(x)=17x^4+2x^3-13x^2+2x+1.$$

Since

$$g(x)=(2x-1)^4+x(x^3+34x^2-37x+10),$$

it suffices to show that

$$x^3+34x^2-37x+10\ge 0.$$

There are two cases to consider.

Case 1: $x\in\left(0,\frac{1}{2}\right]\cup\left[\frac{10}{17},\infty\right)$. We have

$$x^3+34x^2-37x+10>34x^2-37x+10=(2x-1)(17x-10)\ge 0.$$

Case 2: $x\in\left(\frac{1}{2},\frac{10}{17}\right)$. We have

$$2(x^3+34x^2-37x+10)>2\left(\frac{1}{2}x^2+34x^2-37x+10\right)=69x^2-74x+20.$$

Since $69x^2-74x+20>0$ for all real x, the proof is completed. The equality holds for $a=b=c$.

□

P 1.51. *If a,b,c are positive real numbers such that $abc=1$, then*

(a)
$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge a+b+c;$$

(b)
$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge\frac{3}{2}(a+b+c-1);$$

(c)
$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+2\ge\frac{5}{3}(a+b+c).$$

Solution. (a) We write the inequality as

$$\left(2\frac{a}{b}+\frac{b}{c}\right)+\left(2\frac{b}{c}+\frac{c}{a}\right)+\left(2\frac{c}{a}+\frac{a}{b}\right)\ge 3(a+b+c).$$

In virtue of the AM-GM inequality, we get

$$\left(2\frac{a}{b}+\frac{b}{c}\right)+\left(2\frac{b}{c}+\frac{c}{a}\right)+\left(2\frac{c}{a}+\frac{a}{b}\right)\ge 3\sqrt[3]{\frac{a^2}{bc}}+3\sqrt[3]{\frac{b^2}{ca}}+3\sqrt[3]{\frac{c^2}{ab}}=3(a+b+c).$$

The equality holds for $a = b = c = 1$.

(b) Using the substitution

$$a = \frac{y}{x}, \quad b = \frac{z}{y}, \quad c = \frac{x}{z},$$

where $x, y, z > 0$, the inequality can be restated as

$$2(x^3 + y^3 + z^3) + 3xyz \ge 3(x^2y + y^2z + z^2x).$$

First Solution. We get the desired inequality by summing Schur's inequality of degree three

$$x^3 + y^3 + z^3 + 3xyz \ge (x^2y + y^2z + z^2x) + (xy^2 + yz^2 + zx^2)$$

and

$$x^3 + y^3 + z^3 + xy^2 + yz^2 + zx^2 \ge 2(x^2y + y^2z + z^2x).$$

The last inequality is equivalent to

$$x(x-y)^2 + y(y-z)^2 + z(z-x)^2 \ge 0.$$

The equality holds for $a = b = c = 1$.

Second Solution. Multiplying by $x + y + z$, the desired inequality in x, y, z turns into

$$2\sum x^4 - \sum x^3y - 3\sum x^2y^2 + 2\sum xy^3 \ge 0.$$

Write this inequality as

$$\sum[(1+k)x^4 - x^3y - 3x^2y^2 + 2xy^3 + (1-k)y^4] \ge 0,$$

$$\sum(x-y)[x^3 - 3xy^2 - y^3 + k(x^3 + x^2y + xy^2 + y^3)] \ge 0.$$

Choosing $k = \dfrac{3}{4}$, we get the obvious inequality

$$\sum(x-y)^2(7x^2 + 10xy + y^2) \ge 0.$$

(c) Making the substitution

$$a = \frac{y}{x}, \quad b = \frac{z}{y}, \quad c = \frac{x}{z}, \quad x, y, z > 0,$$

we need to show that

$$3(x^3 + y^3 + z^3) + 6xyz \ge 5(x^2y + y^2z + z^2x).$$

Assuming that $x=\min\{x,y,z\}$ and substituting

$$y=x+p,\quad z=x+q,\quad p,q\ge 0,$$

the inequality turns into

$$(p^2-pq+q^2)x+3p^3+3q^3-5p^2q\ge 0.$$

This is true since, by the AM-GM inequality, we get

$$6p^3+6q^3=3p^3+3p^3+6q^3\ge 3\sqrt[3]{3p^3\cdot 3p^3\cdot 6q^3}=9\sqrt[3]{2}\,p^2q\ge 10p^2q.$$

The equality holds for $a=b=c=1$.

□

P 1.52. *If a,b,c are positive real numbers such that $a^2+b^2+c^2=3$, then*

(a)
$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge 2+\frac{3}{ab+bc+ca};$$

(b)
$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge \frac{9}{a+b+c}.$$

Solution. (a) By the Cauchy-Schwarz inequality, we have

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge \frac{(a+b+c)^2}{ab+bc+ca}=2+\frac{3}{ab+bc+ca}.$$

The equality holds for $a=b=c=1$.

(b) Using the inequality in (a), it suffices to show that

$$2+\frac{3}{ab+bc+ca}\ge\frac{9}{a+b+c}.$$

Let

$$t=\frac{a+b+c}{3},\qquad t\le 1.$$

Since

$$2(ab+bc+ca)=(a+b+c)^2-(a^2+b^2+c^2)=9t^2-3,$$

the inequality becomes

$$2+\frac{2}{3t^2-1}\ge\frac{3}{t},$$

$$(t-1)^2(2t+1)\ge 0.$$

The equality holds for $a=b=c=1$.

□

P 1.53. *If a, b, c are positive real numbers such that $a^2+b^2+c^2=3$, then*

$$6\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+5(ab+bc+ca)\geq 33.$$

Solution. Write the inequality in the homogeneous form

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\geq\frac{5}{2}\left(1-\frac{ab+bc+ca}{a^2+b^2+c^2}\right).$$

We will prove the sharper inequality

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\geq m\left(1-\frac{ab+bc+ca}{a^2+b^2+c^2}\right),$$

where

$$m=4\sqrt{2}-3>\frac{5}{2}.$$

Write this inequality as follows:

$$\left(\sum a^2\right)\left(\sum ab^2\right)+mabc\sum ab-(m+3)abc\sum a^2\geq 0,$$

$$\sum ab^4+\sum a^3b^2+(m+1)abc\sum ab-(m+3)abc\sum a^2\geq 0,$$

$$\sum ab^4+\sum a^3b^2+2(2\sqrt{2}-1)abc\sum ab-4\sqrt{2}\,abc\sum a^2\geq 0,$$

On the other hand, from

$$\sum a(a-b)^2(b-kc)^2\geq 0,$$

we get

$$\sum ab^4+\sum a^3b^2+(k^2-2)\sum a^2b^3+k(4-k)abc\sum ab-4kabc\sum a^2\geq 0.$$

Choosing $k=\sqrt{2}$, we get the desired inequality. The equality holds for $a=b=c=1$.

□

P 1.54. *If a, b, c are positive real numbers such that $a+b+c=3$, then*

(a) $$6\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+3\geq 7(a^2+b^2+c^2);$$

(b) $$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq a^2+b^2+c^2.$$

Solution. (a) Write the inequality in the homogeneous form

$$2\left(\sum a\right)^2\left(\sum ab^2\right)+abc\left(\sum a\right)^2\geq 21abc\sum a^2,$$

which is equivalent to

$$\sum ab^4+\sum a^3b^2+2\sum a^2b^3+4abc\sum ab-8abc\sum a^2\geq 0.$$

On the other hand, from

$$\sum a(a-b)^2(b-kc)^2\geq 0,$$

we get

$$\sum ab^4+\sum a^3b^2+(k^2-2)\sum a^2b^3+k(4-k)abc\sum ab-4kabc\sum a^2\geq 0.$$

Choosing $k=2$, we get the desired inequality. The equality holds for $a=b=c=1$.

(b) We get the desired inequality by adding the inequality in (a) and the obvious inequality

$$a^2+b^2+c^2\geq 3.$$

The equality holds for $a=b=c=1$.

□

P 1.55. *If a,b,c are positive real numbers, then*

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+2\geq\frac{14(a^2+b^2+c^2)}{(a+b+c)^2}.$$

(Vo Quoc Ba Can, 2010)

Solution. By expanding, the inequality becomes as follows:

$$\left(\sum\frac{a}{b}\right)\left(\sum a^2+2\sum ab\right)+4\sum ab\geq 12\sum a^2,$$

$$\sum\frac{a^3}{b}+\sum\frac{a^2b}{c}+2\sum\frac{ab^2}{c}+7\sum ab\geq 10\sum a^2,$$

$$A+B\geq 10\sum a^2-10\sum ab,$$

where

$$A=\sum\frac{a^3}{b}+\sum\frac{a^2b}{c}-2\sum\frac{ab^2}{c},\quad B=4\sum\frac{ab^2}{c}-3\sum ab.$$

Since

$$A=\sum\left(\frac{b^3}{c}+\frac{a^2b}{c}-\frac{2ab^2}{c}\right)=\sum\frac{b(a-b)^2}{c}$$

and
$$B=\sum\left(\frac{4ca^2}{b}-12ca+9bc\right)=\sum\frac{c(2a-3b)^2}{b},$$
we get
$$A+B=\sum\left[\frac{b(a-b)^2}{c}+\frac{c(2a-3b)^2}{b}\right]$$
$$\geq 2\sum(a-b)(2a-3b)=10\sum a^2-10\sum ab.$$
Thus, the proof is completed. For $a\geq b\geq c$, the equality holds for
$$b(a-b)=c(2a-3b),\quad c(b-c)=a(2b-3c),\quad a(c-a)=b(2c-3a),$$
which are equivalent to
$$\frac{a}{\sqrt{7}-\tan\frac{\pi}{7}}=\frac{b}{\sqrt{7}-\tan\frac{2\pi}{7}}=\frac{c}{\sqrt{7}-\tan\frac{4\pi}{7}}.$$
Notice that the equality conditions involve
$$a^2+b^2+c^2=2ab+2bc+2ca,$$
hence
$$\sqrt{a}=\sqrt{b}+\sqrt{c}.$$

Remark. Using the inequality in P 1.55, we can prove the weaker inequality
$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{7(ab+bc+ca)}{a^2+b^2+c^2}\geq\frac{17}{2},$$
with equality for the same conditions. It suffices to show that
$$\frac{14(a^2+b^2+c^2)}{(a+b+c)^2}-2\geq\frac{17}{2}-\frac{7(ab+bc+ca)}{a^2+b^2+c^2}$$
which is equivalent to
$$(a^2+b^2+c^2-2ab-2bc-2ca)^2\geq 0.$$
Actually, the following statement is valid.

If a,b,c are positive real numbers, then
$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\frac{19(a^2+b^2+c^2)+2(ab+bc+ca)}{a^2+b^2+c^2+6(ab+bc+ca)},$$
with equality for $a=b=c$, and also for
$$\frac{a}{\sqrt{7}-\tan\frac{\pi}{7}}=\frac{b}{\sqrt{7}-\tan\frac{2\pi}{7}}=\frac{c}{\sqrt{7}-\tan\frac{4\pi}{7}}$$
(or any cyclic permutation).

This inequality is stronger than the inequality in P 1.55.

□

P 1.56. *Let a,b,c be positive real numbers such that $a+b+c=3$, and let*

$$x=3a+\frac{1}{b},\quad y=3b+\frac{1}{c},\quad z=3c+\frac{1}{a}.$$

Prove that

$$xy+yz+zx\ge 48.$$

(Vasile C., 2007)

Solution. Write the inequality as follows:

$$3(ab+bc+ca)+\frac{1}{abc}+\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)\ge 13.$$

We get this inequality by adding the inequality P 1.54-(a), namely

$$6\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)+3\ge 7(a^2+b^2+c^2),$$

and the inequality

$$18(ab+bc+ca)+\frac{6}{abc}+7(a^2+b^2+c^2)\ge 81.$$

Since

$$a^2+b^2+c^2=9-2(ab+bc+ca),$$

the last inequality is equivalent to

$$2(ab+bc+ca)+\frac{3}{abc}\ge 9.$$

By the known inequality

$$(ab+bc+ca)^2\ge 3abc(a+b+c),$$

we get

$$\frac{1}{abc}\ge\frac{9}{(ab+bc+ca)^2}.$$

Thus, it suffices to show that

$$2q+\frac{27}{q^2}\ge 9,$$

where $q=ab+bc+ca$. Indeed, by the AM-GM inequality, we have

$$2q+\frac{27}{q^2}=q+q+\frac{27}{q^2}\ge 3\sqrt[3]{q\cdot q\cdot\frac{27}{q^2}}=9.$$

The equality holds for $a=b=c=1$.

□

P 1.57. *If a, b, c are positive real numbers such that $a+b+c=3$, then*

$$\frac{a+1}{b}+\frac{b+1}{c}+\frac{c+1}{a} \geq 2(a^2+b^2+c^2).$$

Solution. We get the desired inequality by summing the inequality in P 1.54-(a), namely

$$6\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+3 \geq 7(a^2+b^2+c^2),$$

and the inequality

$$6\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 5(a^2+b^2+c^2)+3.$$

Write the last inequality as $F(a,b,c) \geq 0$, where

$$F(a,b,c)=6\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-5(a^2+b^2+c^2)-3,$$

then assume that

$$a=\max\{a,b,c\}, \qquad b+c \leq 2.$$

and show that

$$F(a,b,c) \geq F\left(a,\frac{b+c}{2},\frac{b+c}{2}\right) \geq 0.$$

Indeed, we have

$$F(a,b,c)-F\left(a,\frac{b+c}{2},\frac{b+c}{2}\right)=6\left(\frac{b+c}{bc}-\frac{4}{b+c}\right)-5\left[b^2+c^2-\frac{1}{2}(b+c)^2\right]$$

$$=(b-c)^2\left[\frac{6}{bc(b+c)}-\frac{5}{2}\right] \geq (b-c)^2\left[\frac{24}{(b+c)^3}-\frac{5}{2}\right] \geq 0.$$

Also,

$$F\left(a,\frac{b+c}{2},\frac{b+c}{2}\right)=F\left(a,\frac{3-a}{2},\frac{3-a}{2}\right)=\frac{3(a-1)^2(12-15a+5a^2)}{2a(3-a)} \geq 0.$$

The equality holds for $a=b=c=1$.

□

P 1.58. *If a, b, c are positive real numbers such that $a+b+c=3$, then*

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}+3 \geq 2(a^2+b^2+c^2).$$

(Pham Huu Duc, 2007)

First Solution. Assume that

$$a = \max\{a, b, c\},$$

then homogenize the inequality and write it as follows:

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} + a + b + c \geq \frac{6(a^2 + b^2 + c^2)}{a + b + c},$$

$$\sum\left(\frac{b^2}{c} - 2b + c\right) \geq 6\left(\frac{a^2 + b^2 + c^2}{a + b + c} - \frac{a + b + c}{3}\right),$$

$$\sum \frac{(b-c)^2}{c} \geq \frac{2}{a + b + c}\sum(b-c)^2,$$

$$(b-c)^2A + (c-a)^2B + (a-b)^2C \geq 0,$$

where

$$A = \frac{a+b}{c} - 1 > 0, \quad B = \frac{b+c}{a} - 1, \quad C = \frac{c+a}{b} - 1 > 0.$$

By the Cauchy-Schwarz inequality, we have

$$(b-c)^2A + (a-b)^2C \geq \frac{[(b-c) + (a-b)]^2}{\dfrac{1}{A} + \dfrac{1}{C}} = \frac{AC}{A+C}(a-c)^2.$$

Therefore, it suffices to show that

$$\frac{AC}{A+C} + B \geq 0.$$

Indeed, by the third degree Schur's inequality, we get

$$AB + BC + CA = 3 + \frac{a^3 + b^3 + c^3 + 3abc - ab(a+b) - bc(b+c) - ca(c+a)}{abc} \geq 3.$$

The equality holds for $a = b = c = 1$.

Second Solution (by *Michael Rozenberg*). Write the inequality in the homogeneous form

$$\left(\sum a\right)\left(\sum ab^3\right) + abc\left(\sum a\right)^2 \geq 6abc\sum a^2.$$

By expanding, we get

$$\sum(ab^4 + a^2b^3 + 2ab^2c^2 - 4a^3bc) \geq 0,$$

which is equivalent to

$$\sum a(b^2 - 2bc + ac)^2 \geq 0.$$

□

P 1.59. *If a, b, c are positive real numbers, then*

$$\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}+2(ab+bc+ca)\geq 3(a^2+b^2+c^2).$$

(Michael Rozenberg, 2010)

Solution. Write the inequality as

$$\sum\left(\frac{a^3}{b}+ab-2a^2\right)\geq a^2+b^2+c^2-ab-bc-ca,$$

$$\frac{a(a-b)^2}{b}+\frac{b(b-c)^2}{c}+\frac{c(c-a)^2}{a}\geq a^2+b^2+c^2-ab-bc-ca.$$

Assume that $a=\max\{a,b,c\}$.

Case 1: $a\geq b\geq c$. By the Cauchy-Schwarz inequality, we have

$$\frac{a(a-b)^2}{b}+\frac{b(b-c)^2}{c}\geq\frac{[(a-b)+(b-c)]^2}{\frac{b}{a}+\frac{c}{b}}=\frac{ab(a-c)^2}{b^2+ac}.$$

On the other hand,

$$a^2+b^2+c^2-ab-bc-ca=(a-c)^2+(b-a)(b-c)\leq(a-c)^2.$$

Therefore, it suffice to show that

$$\frac{ab(a-c)^2}{b^2+ac}+\frac{c(a-c)^2}{a}\geq(a-c)^2,$$

which is true if

$$\frac{ab}{b^2+ac}+\frac{c}{a}\geq 1.$$

This inequality is equivalent to

$$a^2b+b^2c+c^2a-ab^2-ca^2\geq 0,$$

$$bc^2-(a-b)(b-c)(c-a)\geq 0.$$

Case 2: $a\geq c\geq b$. By the Cauchy-Schwarz inequality, we have

$$\frac{b(b-c)^2}{c}+\frac{c(c-a)^2}{a}\geq\frac{[(b-c)+(c-a)]^2}{\frac{c}{b}+\frac{a}{c}}=\frac{bc(a-b)^2}{c^2+ab}.$$

On the other hand,

$$a^2+b^2+c^2-ab-bc-ca=(a-b)^2+(c-a)(c-b)\leq(a-b)^2.$$

Therefore, it suffice to show that

$$\frac{a(a-b)^2}{b}+\frac{bc(a-b)^2}{c^2+ab}\ge (a-b)^2,$$

which is equivalent to

$$(a-b)^2(a^2b+b^2c+c^2a-ab^2-bc^2)\ge 0,$$

$$(a-b)^2[ab(a-b)+b^2c+c^2(a-b)]\ge 0.$$

The equality holds for $a=b=c$.

□

P 1.60. *If a,b,c are positive real numbers such that $a^4+b^4+c^4=3$, then*

(a)
$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\ge 3;$$

(b)
$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\ge \frac{3}{2}.$$

(Alexey Gladkich, 2005)

Solution. (a) By Hölder's inequality, we have

$$\left(\sum\frac{a^2}{b}\right)\left(\sum\frac{a^2}{b}\right)\left(\sum a^2b^2\right)\ge\left(\sum a^2\right)^3.$$

Therefore, it suffices to show that

$$\left(\sum a^2\right)^3\ge 9\sum a^2b^2,$$

which has the homogeneous form

$$\left(\sum a^2\right)^3\ge 3\left(\sum a^2b^2\right)\sqrt{3\sum a^4}.$$

Using the notation

$$x=\sum a^2,\quad y=\sum a^2b^2,$$

the inequality can be restated as

$$x^3\ge 3y\sqrt{3(x^2-2y)}.$$

By squaring, the inequality becomes

$$x^6-27x^2y^2+54y^3\ge 0,$$

which is true because

$$x^6-27x^2y^2+54y^3=(x^2-3y)^2(x^2+6y)\geq 0.$$

The equality holds for $a=b=c=1$.

(b) By Hölder's inequality, we have

$$\left(\sum\frac{a^2}{b+c}\right)\left(\sum\frac{a^2}{b+c}\right)\left[\sum a^2(b+c)^2\right]\geq\left(\sum a^2\right)^3.$$

Thus, it suffices to prove that

$$\left(\sum a^2\right)^3\geq\frac{9}{4}\sum a^2(b+c)^2.$$

Using the inequality from the proof of (a), namely

$$\left(\sum a^2\right)^3\geq 9\sum a^2b^2,$$

we still have to show that

$$\sum a^2b^2\geq\frac{1}{4}\sum a^2(b+c)^2.$$

This inequality is equivalent to

$$\sum a^2(b-c)^2\geq 0.$$

The equality holds for $a=b=c=1$.

□

P 1.61. *If a,b,c are positive real numbers, then*

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq\frac{3(a^3+b^3+c^3)}{a^2+b^2+c^2}.$$

(Vo Quoc Ba Can, 2010)

Solution *(by Ta Minh Hoang).* Assume that

$$a=\max\{a,b,c\},$$

and write the inequality as follows:

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}-a-b-c\geq\frac{3(a^3+b^3+c^3)}{a^2+b^2+c^2}-a-b-c,$$

$$\sum \frac{(a-b)^2}{b} \geq \frac{1}{a^2+b^2+c^2} \sum (a+b)(a-b)^2,$$

$$(b-c)^2 A + (c-a)^2 B + (a-b)^2 C \geq 0,$$

where

$$A = \frac{a^2+b^2-bc}{c} > 0, \quad B = \frac{b^2+c^2-ca}{a}, \quad C = \frac{c^2+a^2-ab}{b} > 0.$$

Consider the nontrivial case $B < 0$; that is,

$$ac - b^2 - c^2 > 0.$$

From

$$ac - b^2 - c^2 = c(a-2b) - (b-c)^2,$$

it follows that

$$c(a-2b) > (b-c)^2 \geq 0,$$

hence

$$a > 2b.$$

By the Cauchy-Schwarz inequality, we have

$$(b-c)^2 A + (a-b)^2 C \geq \frac{[(b-c)+(a-b)]^2}{\dfrac{1}{A}+\dfrac{1}{C}} = \frac{AC}{A+C}\,(a-c)^2.$$

Therefore, it suffices to show that $\dfrac{AC}{A+C} + B \geq 0$; that is, $\dfrac{1}{A} + \dfrac{1}{B} + \dfrac{1}{C} \leq 0$, or

$$\frac{c}{a^2+b^2-bc} + \frac{b}{c^2+a^2-ab} \leq \frac{a}{ca-b^2-c^2}.$$

Case 1: $a \geq b \geq c$. Since

$$\begin{aligned} a^2+b^2-bc-(ca-b^2-c^2) &> a^2+b^2-bc-ca \\ &= a(a-c)+b(b-c) \geq 0, \end{aligned}$$

and

$$\begin{aligned} c^2+a^2-ab-(ca-b^2-c^2) &> a^2+b^2-a(b+c) \\ &\geq a^2+bc-a(b+c) \\ &= (a-b)(a-c) \geq 0, \end{aligned}$$

it suffices to show that $c + b \leq a$. Indeed, we have $a > 2b \geq b + c$.

Case 2: $a \ge c \ge b$. Replacing b and c by c and b, respectively, we need to show that $a \ge b \ge c$ involves

$$\frac{a^2}{c}+\frac{c^2}{b}+\frac{b^2}{a} \ge \frac{3(a^3+b^3+c^3)}{a^2+b^2+c^2}.$$

According to the preceding case, we have

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} \ge \frac{3(a^3+b^3+c^3)}{a^2+b^2+c^2}.$$

Therefore, it suffices to show that

$$\frac{a^2}{c}+\frac{c^2}{b}+\frac{b^2}{a} \ge \frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}.$$

This inequality is equivalent to

$$(a+b+c)(a-b)(b-c)(a-c) \ge 0,$$

which is clearly true for $a \ge b \ge c$.
The proof is completed. The equality holds for $a=b=c=1$.

□

P 1.62. *If a,b,c are positive real numbers, then*

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}+a+b+c \ge 2\sqrt{(a^2+b^2+c^2)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)}.$$

(Pham Huu Duc, 2006)

Solution. Without loss of generality, we may assume that b is between a and c; that is,

$$(b-a)(b-c) \le 0.$$

Since

$$\begin{aligned} 2\sqrt{(a^2+b^2+c^2)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)} &= 2\sqrt{\frac{a^2+b^2+c^2}{b}\left(a+\frac{b^2}{c}+\frac{bc}{a}\right)} \\ &\le \frac{a^2+b^2+c^2}{b}+a+\frac{b^2}{c}+\frac{bc}{a} \\ &= \frac{a^2}{b}+\frac{b^2}{c}+a+b+\frac{bc}{a}+\frac{c^2}{b}, \end{aligned}$$

it suffices to prove that

$$\frac{c^2}{a}+c \ge \frac{bc}{a}+\frac{c^2}{b}.$$

This is true because

$$\frac{c^2}{a}+c-\frac{bc}{a}-\frac{c^2}{b}=\frac{c(a-b)(b-c)}{ab}\geq 0.$$

The proof is completed. The equality holds for $a=b=c$.

□

P 1.63. *If a,b,c are positive real numbers, then*

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+32\left(\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}\right)\geq 51.$$

(Vasile C., 2009)

Solution. Write the inequality as

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+45\geq 32\left(\frac{b}{a+b}+\frac{c}{b+c}+\frac{a}{c+a}\right).$$

Using the substitution

$$x=\frac{a}{b},\quad y=\frac{b}{c},\quad z=\frac{c}{a},$$

which involves $xyz=1$, the inequality becomes

$$x+y+z+45-32\left(\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}\right)\geq 0.$$

We get this inequality by summing the inequalities

$$x-\frac{32}{x+1}+15\geq 9\ln x,$$

$$y-\frac{32}{y+1}+15\geq 9\ln y,$$

$$z-\frac{32}{z+1}+15\geq 9\ln z.$$

Let

$$f(x)=x-\frac{32}{x+1}+15-9\ln x,\quad x>0.$$

From the derivative

$$f'(x)=1+\frac{32}{(x+1)^2}-\frac{9}{x}=\frac{(x-1)(x-3)^2}{x(x+1)^2},$$

it follows that $f(x)$ is decreasing for $0<x\leq 1$ and increasing for $x\geq 1$. Therefore, we have $f(x)\geq f(1)=0$. The equality holds for $a=b=c$.

□

P 1.64. *Find the greatest positive real number K such that the inequalities below hold for any positive real numbers a,b,c:*

(a) $$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3 \geq K\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}-\frac{3}{2}\right);$$

(b) $$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3+K\left(\frac{a}{2a+b}+\frac{b}{2b+c}+\frac{c}{2c+a}-1\right) \geq 0.$$

(Vasile C., 2008)

Solution. (a) For

$$a=x^3, \quad b=x, \quad c=1,$$

the inequality becomes

$$x^2+x+\frac{1}{x^3}-3 \geq K\left(\frac{x^3}{x+1}+\frac{x}{1+x^3}+\frac{1}{x^3+x}-\frac{3}{2}\right),$$

$$\frac{(1-K)x^3}{x+1}+\frac{x^2}{x+1}+x+\frac{1}{x^3}-3-K\left(\frac{x}{1+x^3}+\frac{1}{x^3+x}-\frac{3}{2}\right) \geq 0.$$

For $x \to \infty$, we get the necessary condition $1-K \geq 0$. We will show that the original inequality is true for $K=1$; that is,

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq \frac{3}{2}+\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}.$$

Write the inequality as

$$\left(\frac{c}{a}-\frac{c}{a+b}\right)+\left(\frac{a}{b}-\frac{a}{b+c}\right)+\left(\frac{b}{c}-\frac{b}{c+a}\right) \geq \frac{3}{2},$$

$$\frac{bc}{a(a+b)}+\frac{ca}{b(b+c)}+\frac{ab}{c(c+a)} \geq \frac{3}{2}.$$

By the Cauchy-Schwarz inequality, we have

$$\frac{bc}{a(a+b)}+\frac{ca}{b(b+c)}+\frac{ab}{c(c+a)} \geq \frac{(bc+ca+ab)^2}{abc(a+b)+abc(b+c)+abc(c+a)}$$
$$=\frac{(bc+ca+ab)^2}{2abc(a+b+c)} \geq \frac{3}{2}.$$

The equality holds for $a=b=c$.

(b) For $b=1$ and $c=a^2$, the inequality becomes

$$2a+\frac{1}{a^2}-3+K\left(\frac{2a}{2a+1}+\frac{1}{a^2+2}-1\right) \geq 0,$$

$$\frac{(a-1)^2(2a+1)}{a^2}-\frac{K(a-1)^2}{(2a+1)(a^2+2)} \geq 0.$$

This inequality holds for any positive a if and only if

$$\frac{2a+1}{a^2}-\frac{K}{(2a+1)(a^2+2)}\geq 0.$$

For $a=1$, this inequality involves $K\leq 27$. We will show that the original inequality is true for $K=27$. Using the substitution

$$x=\frac{a}{b},\quad y=\frac{b}{c},\quad z=\frac{c}{a},$$

which involves $xyz=1$, the inequality can be restated as

$$x+y+z-3-\frac{27}{2}\left(\frac{1}{2x+1}+\frac{1}{2y+1}+\frac{1}{2z+1}-1\right)\geq 0.$$

First Solution. We get the desired inequality by summing the inequalities

$$x-\frac{27}{2(2x+1)}+\frac{7}{2}\geq 4\ln x,$$

$$y-\frac{27}{2(2y+1)}+\frac{7}{2}\geq 4\ln y,$$

$$z-\frac{27}{2(2z+1)}+\frac{7}{2}\geq 4\ln z.$$

Let

$$f(x)=x-\frac{27}{2(2x+1)}+\frac{7}{2}-4\ln x,\quad x>0.$$

From the derivative

$$f'(x)=1+\frac{27}{(2x+1)^2}-\frac{4}{x}=\frac{4(x-1)^3}{x(2x+1)^2},$$

it follows that $f(x)$ is decreasing for $0<x\leq 1$ and increasing for $x\geq 1$. Therefore, we have $f(x)\geq f(1)=0$. The equality holds for $a=b=c$.

Second Solution. Replacing x,y,z by e^x,e^y,e^z, respectively, we need to show that

$$x+y+z=0$$

involves

$$f(x)+f(y)+f(z)\geq 3f\left(\frac{x+y+z}{3}\right),$$

where

$$f(u)=e^u-\frac{27}{2(2e^u+1)}.$$

If f is convex on $\mathbb{R}$, then this inequality is just Jensen's inequality. Indeed, f is convex because

$$e^{-u}f''(u)=1+\frac{27(1-2e^u)}{(2e^u+1)^3}=\frac{4(e^u-1)^2(2e^u+7)}{(2e^u+1)^3}\geq 0.$$

□

P 1.65. *If $a,b,c \in \left[\frac{1}{2},2\right]$, then*

(a) $$8\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \ge 5\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)+9;$$

(b) $$20\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \ge 17\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right).$$

(Vasile C., 2008)

Solution. Without loss of generality, assume that

$$a = \max\{a,b,c\}.$$

Let

$$t = \sqrt{\frac{a}{c}}, \quad 1 \le t \le 2.$$

(a) Let

$$E(a,b,c) = 8\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) - 5\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right) - 9.$$

We will show that

$$E(a,b,c) \ge E(a,\sqrt{ac},c) \ge 0.$$

We have

$$E(a,b,c) - E(a,\sqrt{ac},c) = 8\left(\frac{a}{b}+\frac{b}{c}-2\sqrt{\frac{a}{c}}\right) - 5\left(\frac{b}{a}+\frac{c}{b}-2\sqrt{\frac{c}{a}}\right)$$

$$= \frac{(b-\sqrt{ac})^2(8a-5c)}{abc} \ge 0.$$

Also,

$$\begin{aligned}
E(a,\sqrt{ac},c) &= 8\left(2\sqrt{\frac{a}{c}}+\frac{c}{a}-3\right) - 5\left(2\sqrt{\frac{c}{a}}+\frac{a}{c}-3\right) \\
&= 8\left(2t+\frac{1}{t^2}-3\right) - 5\left(\frac{2}{t}+t^2-3\right) \\
&= \frac{8}{t^2}(t-1^2(2t+1) - \frac{5}{t}(t-1)^2(t+2) \\
&= \frac{(t-1)^2(4+5t)(2-t)}{t^2} \ge 0.
\end{aligned}$$

The equality holds for $a=b=c$, and also for $a=2$, $b=1$ and $c=\frac{1}{2}$ (or any cyclic permutation).

(b) Let

$$E(a,b,c) = 20\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) - 17\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right).$$

We will show that
$$E(a,b,c) \ge E(a,\sqrt{ac},c) \ge 0.$$
We have
$$E(a,b,c) - E(a,\sqrt{ac},c) = 20\left(\frac{a}{b}+\frac{b}{c}-2\sqrt{\frac{a}{c}}\right) - 17\left(\frac{b}{a}+\frac{c}{b}-2\sqrt{\frac{c}{a}}\right)$$
$$= \frac{(b-\sqrt{ac})^2(20a-17c)}{abc} \ge 0.$$
Also, we have
$$\begin{aligned} E(a,\sqrt{ac},c) &= 20\left(2\sqrt{\frac{a}{c}}+\frac{c}{a}\right) - 17\left(2\sqrt{\frac{c}{a}}+\frac{a}{c}\right) \\ &= 20\left(2t+\frac{1}{t^2}\right) - 17\left(\frac{2}{t}+t^2\right) \\ &= \frac{20-34t+40t^3-17t^4}{t^2} \\ &= \frac{(2-t)(17t^3-6t^2-12t+10)}{t^2}. \end{aligned}$$
We need to show that $17t^3-6t^2-12t+10 \ge 0$ for $1 \le t \le 2$. Indeed, we have
$$17t^3-6t^2-12t+10 \ge 11t^2-12t+10 > 4t^2-12t+9 = (2t-3)^2 \ge 0.$$
The equality holds for $a=2$, $b=1$ and $c=\frac{1}{2}$ (or any cyclic permutation).

□

P 1.66. *If a,b,c are positive real numbers such that $a \le b \le c$, then*
$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge \frac{2a}{b+c}+\frac{2b}{c+a}+\frac{2c}{a+b}.$$

First Solution. Since
$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} - \left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right) = \left(\frac{a}{b}-1\right)\left(\frac{b}{c}-1\right)\left(\frac{c}{a}-1\right) \ge 0,$$
it suffices to show that
$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right) \ge \frac{4a}{b+c}+\frac{4b}{c+a}+\frac{4c}{a+b}.$$
This inequality is equivalent to
$$a\left(\frac{1}{b}+\frac{1}{c}-\frac{4}{b+c}\right)+b\left(\frac{1}{c}+\frac{1}{a}-\frac{4}{c+a}\right)+c\left(\frac{1}{a}+\frac{1}{b}-\frac{4}{a+b}\right) \ge 0,$$

$$\frac{a^2(b-c)^2}{b+c}+\frac{b^2(c-a)^2}{c+a}+\frac{c^2(a-b)^2}{a+b}\ge 0.$$

The equality holds for $a=b=c$.

Second Solution. The inequality is equivalent to

$$\frac{a(c-b)}{b(b+c)}-\frac{b(c-a)}{c(c+a)}+\frac{c(b-a)}{a(a+b)}\ge 0.$$

Taking account of

$$b(c-a)=c(b-a)+a(c-b),$$

we may rewrite the inequality as

$$c(b-a)\left[\frac{1}{a(a+b)}-\frac{1}{c(c+a)}\right]+a(c-b)\left[\frac{1}{b(b+c)}-\frac{1}{c(c+a)}\right]\ge 0.$$

Since

$$\frac{1}{a(a+b)}-\frac{1}{c(c+a)}=\frac{c^2-a^2+a(c-b)}{ac(a+b)(c+a)}\ge\frac{c-b}{c(a+b)(c+a)}$$

and

$$\frac{1}{b(b+c)}-\frac{1}{c(c+a)}=\frac{c^2-b^2+c(a-b)}{bc(b+c)(c+a)}\ge\frac{a-b}{b(b+c)(c+a)},$$

it suffices to show that

$$\frac{c(b-a)(c-b)}{c(a+b)(c+a)}+\frac{a(c-b)(a-b)}{b(b+c)(c+a)}\ge 0.$$

This inequality is true if

$$\frac{1}{a+b}-\frac{a}{b(b+c)}\ge 0.$$

Indeed,

$$\frac{1}{a+b}-\frac{a}{b(b+c)}\ge\frac{1}{a+b}-\frac{1}{b+c}=\frac{c-a}{(a+b)(b+c)}\ge 0.$$

□

P 1.67. *Let a,b,c be positive real numbers such that $abc=1$.*

(a) If $a\le b\le c$, then

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge a^{3/2}+b^{3/2}+c^{3/2};$$

(b) If $a\le 1\le b\le c$, then

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge a^{\sqrt{3}}+b^{\sqrt{3}}+c^{\sqrt{3}}.$$

(Vasile C., 2008)

Solution. (a) Since

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)=\left(\frac{a}{b}-1\right)\left(\frac{b}{c}-1\right)\left(\frac{c}{a}-1\right)\geq 0,$$

it suffices to show that

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)\geq 2(a^{3/2}+b^{3/2}+c^{3/2}).$$

Indeed, by the AM-GM inequality, we have

$$\sum\frac{a}{b}+\sum\frac{b}{a}=\sum a\left(\frac{1}{b}+\frac{1}{c}\right)\geq\sum\frac{2a}{\sqrt{bc}}=2\sum a^{3/2}.$$

The equality holds for $a=b=c=1$.

(b) Let $k=\sqrt{3}$ and

$$E(a,b,c)=\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-a^k-b^k-c^k.$$

We will show that

$$E(a,b,c)\geq E(a,\sqrt{bc},\sqrt{bc})\geq 0;$$

that is,

$$E(\frac{1}{bc},b,c)\geq E(\frac{1}{bc},\sqrt{bc},\sqrt{bc})\geq 0.$$

Substituting

$$t=\sqrt{bc},\quad t\geq 1,$$

we rewrite the right inequality as $f(t)\geq 0$, where

$$f(t)=\frac{1}{t^3}+1+t^3-\frac{1}{t^{2k}}-2t^k.$$

We have the derivative

$$\frac{f'(t)}{t^2}=g(t),\quad g(t)=\frac{-3}{t^6}+3+\frac{2k}{t^{2k+3}}-\frac{2k}{t^{3-k}}.$$

Since

$$\begin{aligned}\frac{1}{2}t^{2k+4}g'(t)&=9t^{2k-3}-k(2k+3)+k(3-k)t^{3k}\\&\geq 9-k(2k+3)+k(3-k)=9-3k^2=0,\end{aligned}$$

$g(t)$ is increasing for $t\geq 1$. Therefore, $g(t)\geq g(1)=0$, $f'(t)\geq 0$, $f(t)$ is increasing for $t\geq 1$, hence $f(t)\geq f(1)=0$.

Substituting $b=x^2$ and $c=y^2$, where $1\leq x\leq y$, the left inequality becomes

$$E\left(\frac{1}{x^2y^2},x^2,y^2\right)\geq E\left(\frac{1}{x^2y^2},xy,xy\right),$$

or, equivalently,

$$\frac{1}{x^4y^2}+\frac{x^2}{y^2}+x^2y^4-\frac{1}{x^3y^3}-1-x^3y^3\geq(y^k-x^k)^2.$$

We write this inequality as

$$(y-x)\left(x^2y^3+\frac{1}{x^4y^3}-\frac{x+y}{y^2}\right)\geq(y^k-x^k)^2,$$

and then show that

$$(y-x)\left(x^2y^3+\frac{1}{x^4y^3}-\frac{x+y}{y^2}\right)\geq(y-x)(y^3-x^3)\geq(y^k-x^k)^2. \tag{*}$$

The left inequality (*) is true if $f(x,y)\geq 0$, where

$$f(x,y)=x^2y^3+\frac{1}{x^4y^3}-\frac{x+y}{y^2}-y^3+x^3.$$

We will show that

$$f(x,y)\geq f(1,y)\geq 0.$$

Since $1\leq x\leq y$, we have

$$\begin{aligned}f(x,y)-f(1,y)&=x^3-1+y^3(x^2-1)-\frac{1}{y^2}(x-1)-\frac{1}{y^3}\left(1-\frac{1}{x^4}\right)\\&\geq x^3-1+(x^2-1)-(x-1)-\left(1-\frac{1}{x^4}\right)\\&=(x^2-1)\left[\left(x-\frac{1}{x^2}\right)+\left(1-\frac{1}{x^4}\right)\right]\geq 0\end{aligned}$$

and

$$f(1,y)=\frac{1}{y^3}-\frac{1+y}{y^2}+1=\frac{(1+y)(1-y)^2}{y^3}\geq 0.$$

In order to prove the right inequality (*), we will prove that

$$(y-x)(y^3-x^3)\geq\frac{3}{4}(y^2-x^2)^2\geq(y^k-x^k)^2.$$

We have

$$4(y-x)(y^3-x^3)-3(y^2-x^2)^2=(y-x)^4\geq 0.$$

To complete the proof, we only need to show that

$$\frac{k}{2}(y^2-x^2)\geq y^k-x^k,\quad k=\sqrt{3}.$$

For fixed y, let

$$g(x)=x^k-y^k+\frac{k}{2}(y^2-x^2),\quad 1\leq x\leq y.$$

Since
$$g'(x) = kx(x^{k-2} - 1) \le 0,$$
$g(x)$ is decreasing, hence $g(x) \ge g(y) = 0$. The equality in (b) is an equality if and only if $a = b = c = 1$.

□

P 1.68. *If k and a, b, c are positive real numbers, then*
$$\frac{1}{(k+1)a+b} + \frac{1}{(k+1)b+c} + \frac{1}{(k+1)c+a} \ge \frac{1}{ka+b+c} + \frac{1}{kb+c+a} + \frac{1}{kc+a+b}.$$

(Vasile C., 2011)

First Solution. For $k = 1$, we need to show that
$$\frac{1}{2a+b} + \frac{1}{2b+c} + \frac{1}{2c+a} \ge \frac{3}{a+b+c}.$$
This follows immediately from the Cauchy-Schwarz inequality, as follows:
$$\frac{1}{2a+b} + \frac{1}{2b+c} + \frac{1}{2c+a} \ge \frac{9}{(2a+b)+(2b+c)+(2c+a)}$$
$$= \frac{3}{a+b+c}.$$
Further, consider two cases: $k > 1$ and $0 < k < 1$.

Case 1: $k > 1$. By the Cauchy-Schwarz inequality, we have
$$\frac{k-1}{(k+1)a+b} + \frac{1}{kc+a+b} \ge \frac{[(k-1)+1]^2}{(k-1)[(k+1)a+b]+(kc+a+b)}$$
$$= \frac{k}{ka+b+c}.$$
Adding this inequality and the similar ones yields the desired inequality.

Case 2: $0 < k < 1$. By the Cauchy-Schwarz inequality, we have
$$\frac{1-k}{(k+1)a+b} + \frac{k}{ka+b+c} \ge \frac{[(1-k)+k]^2}{(1-k)[(k+1)a+b]+k(ka+b+c)}$$
$$= \frac{1}{kc+a+b}.$$
Adding this inequality and the similar ones yields the desired inequality.
The equality holds for $a = b = c$.

Second Solution (by *Vo Quoc Ba Can*). By the Cauchy-Schwarz inequality, we have
$$\frac{1}{(k+1)a+b} + \frac{k}{(k+1)b+c} + \frac{k^2}{(k+1)c+a} \ge$$

$$\geq \frac{(1+k+k^2)^2}{[(k+1)a+b]+k[(k+1)b+c]+k^2[(k+1)c+a]}$$

$$= \frac{1+k+k^2}{kc+a+b}.$$

Therefore, we get in succession

$$\sum \frac{1}{(k+1)a+b} + \sum \frac{k}{(k+1)b+c} + \sum \frac{k^2}{(k+1)c+a} \geq \sum \frac{1+k+k^2}{kc+a+b},$$

$$(1+k+k^2)\sum \frac{1}{(k+1)a+b} \geq (1+k+k^2)\sum \frac{1}{ka+b+c},$$

$$\sum \frac{1}{(k+1)a+b} \geq \sum \frac{1}{ka+b+c}.$$

Third Solution. We have

$$\frac{1}{(k+1)a+b} - \frac{1}{ka+b+c} = \frac{c-a}{(ka+a+b)(ka+b+c)}$$

$$\geq \frac{c-a}{(kc+a+b)(ka+b+c)} = \frac{1}{k-1}\left(\frac{1}{ka+b+c} - \frac{1}{kc+a+b}\right),$$

hence

$$\sum \frac{1}{(k+1)a+b} - \sum \frac{1}{ka+b+c} \geq \frac{1}{k-1}\left(\sum \frac{1}{ka+b+c} - \sum \frac{1}{kc+a+b}\right) = 0.$$

□

P 1.69. *If a, b, c are positive real numbers, then*

(a) $$\frac{a}{\sqrt{2a+b}} + \frac{b}{\sqrt{2b+c}} + \frac{c}{\sqrt{2c+a}} \leq \sqrt{a+b+c};$$

(b) $$\frac{a}{\sqrt{a+2b}} + \frac{b}{\sqrt{b+2c}} + \frac{c}{\sqrt{c+2a}} \geq \sqrt{a+b+c}.$$

Solution. (a) By the Cauchy-Schwarz inequality, we have

$$\sum \frac{a}{\sqrt{2a+b}} = \sum \left(\sqrt{a} \cdot \sqrt{\frac{a}{2a+b}}\right) \leq \sqrt{\left(\sum a\right)\left(\sum \frac{a}{2a+b}\right)}.$$

Therefore, it suffices to show that

$$\sum \frac{a}{2a+b} \leq 1.$$

This inequality is equivalent to

$$\sum \frac{b}{2a+b} \ge 1.$$

Applying the Cauchy-Schwarz inequality, we get

$$\sum \frac{b}{2a+b} \ge \frac{\left(\sum b\right)^2}{\sum b(2a+b)} = 1.$$

The equality holds for $a = b = c$.

(b) By Hölder's inequality, we have

$$\left(\sum \frac{a}{\sqrt{a+2b}}\right)^2 \ge \frac{\left(\sum a\right)^3}{\sum a(a+2b)} = \sum a.$$

From this, the desired inequality follows. The equality holds for $a = b = c$.

□

P 1.70. *Let a, b, c be nonnegative real numbers such that $a + b + c = 3$. Prove that*

$$a\sqrt{\frac{a+2b}{3}} + b\sqrt{\frac{b+2c}{3}} + c\sqrt{\frac{c+2a}{3}} \le 3.$$

First Solution. By the Cauchy-Schwarz inequality, we have

$$\sum a\sqrt{\frac{a+2b}{3}} \le \sqrt{\left(\sum a\right)\left[\sum \frac{a(a+2b)}{3}\right]} = \sqrt{\frac{\left(\sum a\right)^3}{3}} = 3.$$

The equality holds for $a = b = c = 1$, and also for $a = 3$ and $b = c = 0$ (or any cyclic permutation).

Second Solution. Applying Jensen's inequality to the concave function $f(x) = \sqrt{x}$, $x \ge 0$, we have

$$a\sqrt{a+2b} + b\sqrt{b+2c} + c\sqrt{c+2a} \le$$

$$\le (a+b+c)\sqrt{\frac{a(a+2b)+b(b+2c)+c(c+2a)}{a+b+c}}$$

$$= (a+b+c)\sqrt{a+b+c} = 3\sqrt{3}.$$

□

P 1.71. *If a, b, c are nonnegative real numbers such that $a+b+c=3$, then*

$$a\sqrt{1+b^3}+b\sqrt{1+c^3}+c\sqrt{1+a^3}\le 5.$$

(Pham Kim Hung, 2007)

Solution. Using the AM-GM inequality yields

$$\sqrt{1+b^3}=\sqrt{(1+b)(1-b+b^2)}\le \frac{(1+b)+(1-b+b^2)}{2}=1+\frac{b^2}{2}.$$

Therefore,

$$\sum a\sqrt{1+b^3}\le \sum a\left(1+\frac{b^2}{2}\right)=3+\frac{ab^2+bc^2+ca^2}{2}.$$

To complete the proof, it remains to show that

$$ab^2+bc^2+ca^2\le 4.$$

But this is just the inequality in P 1.1. The equality occurs for $a=0$, $b=1$ and $c=2$ (or any cyclic permutation).

□

P 1.72. *If a, b, c are positive real numbers such that $abc=1$, then*

(a)
$$\sqrt{\frac{a}{b+3}}+\sqrt{\frac{b}{c+3}}+\sqrt{\frac{c}{a+3}}\ge \frac{3}{2};$$

(b)
$$\sqrt[3]{\frac{a}{b+7}}+\sqrt[3]{\frac{b}{c+7}}+\sqrt[3]{\frac{c}{a+7}}\ge \frac{3}{2}.$$

Solution. (a) Putting

$$a=\frac{x}{y},\quad b=\frac{z}{x},\quad c=\frac{y}{z},$$

the inequality can be restated as

$$\frac{x}{\sqrt{y(3x+z)}}+\frac{y}{\sqrt{z(3y+x)}}+\frac{z}{\sqrt{x(3z+y)}}\ge \frac{3}{2}.$$

By Hölder's inequality, we have

$$\left[\sum \frac{x}{\sqrt{y(3x+z)}}\right]^2\left[\sum xy(3x+z)\right]\ge \left(\sum x\right)^3.$$

Therefore, it suffices to show that

$$4(x+y+z)^3 \geq 27(x^2y+y^2z+z^2x+xyz).$$

This is just the inequality (a) in P 1.9. The equality holds for $a=b=c=1$.

(b) Putting

$$a=\frac{x^4}{y^4},\quad b=\frac{z^4}{x^4},\quad c=\frac{y^4}{z^4},$$

the inequality becomes

$$\sum\sqrt[3]{\frac{x^8}{y^4(7x^4+z^4)}} \geq \frac{3}{2}.$$

By Hölder's inequality, we have

$$\left[\sum\sqrt[3]{\frac{x^8}{y^4(7x^4+z^4)}}\right]^3\left[\sum(7x^4+z^4)\right] \geq \left(\sum\frac{x^2}{y}\right)^4.$$

Since $\sum(7x^4+z^4)=8\sum x^4$, it is enough to show that

$$\left(\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\right)^4 \geq 27(x^4+y^4+z^4),$$

which is just the inequality in P 1.60-(a). The equality holds for $a=b=c=1$.

□

P 1.73. *If a,b,c are positive real numbers, then*

$$\left(1+\frac{4a}{a+b}\right)^2+\left(1+\frac{4b}{b+c}\right)^2+\left(1+\frac{4c}{c+a}\right)^2 \geq 27.$$

(Vasile C., 2012)

Solution. Let

$$x=\frac{a-b}{a+b},\quad y=\frac{b-c}{b+c},\quad z=\frac{c-a}{c+a}.$$

We have

$$-1<x,y,z<1$$

and

$$x+y+z+xyz=0.$$

Since

$$\frac{2a}{a+b}=x+1,\quad \frac{2b}{b+c}=y+1,\quad \frac{2c}{c+a}=z+1,$$

we can write the inequality as follows:

$$(2x+3)^2+(2y+3)^2+(2z+3)^2 \geq 27,$$

$$x^2 + y^2 + z^2 + 3(x + y + z) \geq 0,$$

$$x^2 + y^2 + z^2 \geq 3xyz.$$

By the AM-GM inequality, we have

$$x^2 + y^2 + z^2 \geq 3\sqrt[3]{x^2y^2z^2}.$$

Thus, it suffices to show that $|xyz| \leq 1$, which is clearly true. The equality holds for $a = b = c$.

□

P 1.74. *If a, b, c are positive real numbers, then*

$$\sqrt{\frac{2a}{a+b}} + \sqrt{\frac{2b}{b+c}} + \sqrt{\frac{2c}{c+a}} \leq 3.$$

(Vasile C., 1992)

First Solution. By the Cauchy-Schwarz inequality, we have

$$\sum \sqrt{\frac{2a}{a+b}} \leq \sqrt{\left[\sum \frac{2a}{(a+b)(a+c)}\right]\left[\sum (a+c)\right]}.$$

Thus, it suffices to show that

$$\sum \frac{a}{(a+b)(a+c)} \leq \frac{9}{4(a+b+c)},$$

which is equivalent to

$$a(b-c)^2 + b(c-a)^2 + c(a-b)^2 \geq 0.$$

The equality occurs for $a = b = c$.

Second Solution. By the Cauchy-Schwarz inequality, we have

$$\sum \sqrt{\frac{2a}{a+b}} \leq \sqrt{\left[\sum \frac{1}{(a+b)(b+c)}\right]\left[\sum 2a(b+c)\right]}.$$

Thus, it suffices to show that

$$\sum \frac{1}{(a+b)(b+c)} \leq \frac{9}{4(ab+bc+ca)},$$

which is equivalent to

$$a(b-c)^2 + b(c-a)^2 + c(a-b)^2 \geq 0.$$

□

P 1.75. *If a,b,c are nonnegative real numbers, then*

$$\sqrt{\frac{a}{4a+5b}}+\sqrt{\frac{b}{4b+5c}}+\sqrt{\frac{c}{4c+5a}}\le 1.$$

(Vasile C., 2004)

Solution. If one of a,b,c is zero, then the inequality is clearly true. Otherwise, using the substitution

$$u=\frac{b}{a},\quad v=\frac{c}{b},\quad w=\frac{a}{c},$$

we need to show that $uvw=1$ involves

$$\frac{1}{\sqrt{4+5u}}+\frac{1}{\sqrt{4+5v}}+\frac{1}{\sqrt{4+5w}}\le 1.$$

Using the contradiction method, it suffices to show that

$$\frac{1}{\sqrt{4+5u}}+\frac{1}{\sqrt{4+5v}}+\frac{1}{\sqrt{4+5w}}>1$$

involves $uvw<1$. Let

$$x=\frac{1}{\sqrt{4+5u}},\quad y=\frac{1}{\sqrt{4+5v}},\quad z=\frac{1}{\sqrt{4+5w}},$$

where $x,y,z\in\left(0,\frac{1}{2}\right)$. Since

$$u=\frac{1-4x^2}{5x^2},\quad v=\frac{1-4y^2}{5y^2},\quad w=\frac{1-4z^2}{5z^2},$$

we have to prove that $x+y+z>1$ involves

$$(1-4x^2)(1-4y^2)(1-4z^2)<125x^2y^2z^2.$$

Since

$$1-4x^2<(x+y+z)^2-4x^2=(-x+y+z)(3x+y+z),$$

it suffices to prove the homogeneous inequality

$$(3x+y+z)(3y+z+x)(3z+x+y)(-x+y+z)(-y+z+x)(-z+x+y)\le 125x^2y^2z^2.$$

By the AM-GM inequality, we have

$$(3x+y+z)(3y+z+x)(3z+x+y)\le 125\left(\frac{x+y+z}{3}\right)^3.$$

Therefore, it is enough to show that

$$\left(\frac{x+y+z}{3}\right)^3(-x+y+z)(-y+z+x)(-z+x+y)\le x^2y^2z^2.$$

Using the substitution

$$a = -x + y + z, \quad b = -y + z + x, \quad c = -z + x + y,$$

where $a, b, c > 0$, the inequality can be restated as

$$64abc(a + b + c)^3 \leq 27(b + c)^2(c + a)^2(a + b)^2.$$

The known inequality

$$9(b + c)(c + a)(a + b) \geq 8(a + b + c)(ab + bc + ca),$$

equivalent to

$$a(b - c)^2 + b(c - a)^2 + c(a - b)^2 \geq 0,$$

involves

$$81(b + c)^2(c + a)^2(a + b)^2 \geq 64(a + b + c)^2(ab + bc + ca)^2.$$

Thus, it suffices to show that

$$3abc(a + b + c) \leq (ab + bc + ca)^2.$$

which is also a known inequality, equivalent to

$$a^2(b - c)^2 + b^2(c - a)^2 + c^2(a - b)^2 \geq 0.$$

Thus, the proof is completed. The equality occurs for $a = b = c$.

□

P 1.76. *If a, b, c are positive real numbers, then*

$$\frac{a}{\sqrt{4a^2 + ab + 4b^2}} + \frac{b}{\sqrt{4b^2 + bc + 4c^2}} + \frac{c}{\sqrt{4c^2 + ca + 4a^2}} \leq 1.$$

(Bin Zhao, 2006)

Solution. By the AM-GM inequality, we have

$$ab + 4b^2 \geq 5\sqrt[5]{ab \cdot b^8} = 5\sqrt[5]{ab^9},$$

$$\frac{a}{\sqrt{4a^2 + ab + 4b^2}} \leq \frac{a}{\sqrt{4a^2 + 5\sqrt[5]{ab^9}}} = \sqrt{\frac{a^{9/5}}{4a^{9/5} + 5b^{9/5}}}.$$

Therefore, it suffices to show that

$$\sqrt{\frac{a^{9/5}}{4a^{9/5} + 5b^{9/5}}} + \sqrt{\frac{b^{9/5}}{4b^{9/5} + 5c^{9/5}}} + \sqrt{\frac{c^{9/5}}{4c^{9/5} + 5a^{9/5}}} \leq 1.$$

Replacing $a^{9/5}, b^{9/5}, c^{9/5}$ by a, b, c, respectively, we get the inequality in P 1.75. The equality holds for $a = b = c$.

□

P 1.77. *If a, b, c are positive real numbers, then*

$$\sqrt{\frac{a}{a+b+7c}}+\sqrt{\frac{b}{b+c+7a}}+\sqrt{\frac{c}{c+a+7b}}\geq 1.$$

(Vasile C., 2006)

Solution. Substituting

$$x=\sqrt{\frac{a}{a+b+7c}},\quad y=\sqrt{\frac{b}{b+c+7a}},\quad z=\sqrt{\frac{c}{c+a+7b}},$$

we have

$$\begin{cases}(x^2-1)a+x^2b+7x^2c=0\\ (y^2-1)b+y^2c+7y^2a=0\\ (z^2-1)c+z^2a+7z^2b=0\end{cases},$$

which involves

$$\begin{vmatrix} x^2-1 & x^2 & 7x^2 \\ 7y^2 & y^2-1 & y^2 \\ z^2 & 7z^2 & z^2-1 \end{vmatrix}=0\ ;$$

that is,

$$F(x,y,z)=0,$$

where

$$F(x,y,z)=324x^2y^2z^2+6\sum x^2y^2+\sum x^2-1.$$

We need to show that $F(x,y,z)=0$ involves $x+y+z\geq 1$, where $x,y,z>0$. To do this, we use the contradiction method. Assume that $x+y+z<1$ and show that $F(x,y,z)<0$. Since $F(x,y,z)$ is strictly increasing in each of its arguments, it is enough to prove that $x+y+z=1$ involves $F(x,y,z)\leq 0$. We have

$$\begin{aligned}F(x,y,z)&=324x^2y^2z^2+6\Big(\sum xy\Big)^2-12xyz\sum x+\Big(\sum x\Big)^2-2\sum xy-1\\&=324x^2y^2z^2+6\Big(\sum xy\Big)^2-12xyz-2\sum xy\\&=12xyz(27xyz-1)+2\Big(\sum xy\Big)\Big(3\sum xy-1\Big).\end{aligned}$$

Because

$$27xyz\leq\Big(\sum x\Big)^3=1$$

and

$$3\sum xy\leq\Big(\sum x\Big)^2=1,$$

the conclusion follows. The equality occurs for $a=b=c$.

□

P 1.78. *If a,b,c are nonnegative real numbers, no two of which are zero, then*

(a)
$$\sqrt{\frac{a}{3b+c}}+\sqrt{\frac{b}{3c+a}}+\sqrt{\frac{c}{3a+b}}\ge\frac{3}{2};$$

(b)
$$\sqrt{\frac{a}{2b+c}}+\sqrt{\frac{b}{2c+a}}+\sqrt{\frac{c}{2a+b}}\ge\sqrt[4]{8}.$$

(Vasile Cîrtoaje and Pham Kim Hung, 2006)

Solution. Consider the inequality

$$\sqrt{\frac{(k+1)a}{kb+c}}+\sqrt{\frac{(k+1)b}{kc+a}}+\sqrt{\frac{(k+1)c}{ka+b}}\ge A_k,\quad k>0,$$

and use the substitution

$$x=\sqrt{\frac{(k+1)a}{kb+c}},\quad y=\sqrt{\frac{(k+1)b}{kc+a}},\quad z=\sqrt{\frac{(k+1)c}{ka+b}}.$$

From the identity

$$(kb+c)(kc+a)(ka+b)=(k^3+1)abc+kbc(kb+c)+kca(kc+a)+kab(ka+b),$$

written as

$$\frac{kb+c}{(k+1)a}\cdot\frac{kc+a}{(k+1)b}\cdot\frac{ka+b}{(k+1)c}=\frac{k^2-k+1}{(k+1)^2}+\frac{k}{(k+1)^2}\left[\frac{kb+c}{(k+1)a}+\frac{kc+a}{(k+1)b}+\frac{ka+b}{(k+1)c}\right],$$

we get

$$\frac{1}{x^2y^2z^2}=\frac{k^2-k+1}{(k+1)^2}+\frac{k}{(k+1)^2}\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right),$$

which is equivalent to $F(x,y,z)=0$, where

$$F(x,y,z)=k(x^2y^2+y^2z^2+z^2x^2)+(k^2-k+1)x^2y^2z^2-(k+1)^2.$$

So, we need to show that $F(x,y,z)=0$ yields $x+y+z\ge A_k$. To do this, we use the contradiction method. Assume that $x+y+z<A_k$ and show that $F(x,y,z)<0$. Since $F(x,y,z)$ is strictly increasing in each of its variables, it suffices to prove that $x+y+z=A_k$ involves $F(x,y,z)\le 0$. Let

$$k_1=\frac{49+9\sqrt{17}}{32}\approx 2.691.$$

(a) We need to show that $F(x,y,z)\le 0$ for $x+y+z=A_k=3$ and $k=3$. We will show a more general inequality, namely $F(x,y,z)\le 0$ for $k\ge k_1$ and all nonnegative numbers x,y,z satisfying $x+y+z=3$. The AM-GM inequality $x+y+z \ge 3\sqrt[3]{xyz}$ involves $xyz \le 1$. On the other hand, by Schur's inequality

$$(x+y+z)^3+9xyz \ge 4(x+y+z)(xy+yz+zx)$$

we get

$$4(xy+yz+zx) \le 9+3xyz,$$

hence

$$(xy+yz+zx)^2-9 \le \frac{(9+3xyz)^2}{16}-9=\frac{9}{16}(xyz-1)(xyz+7).$$

Therefore,

$$\begin{aligned}
F(x,y,z) &= k[(xy+yz+zx)^2-6xyz]+(k^2-k+1)x^2y^2z^2-(k+1)^2 \\
&= k[(xy+yz+zx)^2-9]+(k^2-k+1)(x^2y^2z^2-1)-6k(xyz-1) \\
&\le \frac{9k}{16}(xyz-1)(xyz+7)+(k^2-k+1)(x^2y^2z^2-1)-6k(xyz-1) \\
&= \frac{1}{16}(xyz-1)\left[(16k^2-7k+16)xyz+16k^2-49k+16\right] \le 0.
\end{aligned}$$

Since $xyz-1 \le 0$ and $16k^2-7k+16>0$, it suffices to show that $16k^2-49k+16 \ge 0$; indeed, this inequality is true for $k \ge k_1$.
The equality occurs for $a=b=c$. In addition, when $k=k_1$, the equality occurs also for $a=0$ and $b/c=\sqrt{k}$ (or any cyclic permutation).

(b) We need to show that $F(x,y,z) \le 0$ for $A_k=\sqrt[4]{72}$ and $k=2$. We will show a more general inequality, that $F(x,y,z) \le 0$ for $1 \le k \le k_1$ and all nonnegative numbers x,y,z satisfying

$$x+y+z=A_k=2\sqrt[4]{\frac{(k+1)^2}{k}}.$$

From

$$\begin{aligned}
F(x,y,z) &= k(x^2y^2+y^2z^2+z^2x^2)+(k^2-k+1)x^2y^2z^2-(k+1)^2 \\
&= k(xy+yz+zx)^2-2kA_kxyz+(k^2-k+1)x^2y^2z^2-(k+1)^2,
\end{aligned}$$

it follows that for fixed xyz, $F(x,y,z)$ is maximal when $xy+yz+zx$ is maximal; that is, according to P 3.58 in Volume 1, when two of x,y,z are equal. Due to symmetry, we only need to show that $F(x,y,z) \le 0$ for $y=z$. Write the inequality $F(x,y,z) \le 0$ as follows:

$$k(x^2y^2+y^2z^2+z^2x^2)+(k^2-k+1)x^2y^2z^2-k\left(\frac{x+y+z}{2}\right)^4 \le 0,$$

$$k\left[\left(\frac{x+y+z}{2}\right)^4-x^2y^2-y^2z^2-z^2x^2\right] \ge (k^2-k+1)x^2y^2z^2,$$

$$k\sqrt{k}\ (x+y+z)^2\left[(x+y+z)^4-16(x^2y^2+y^2z^2+z^2x^2\right]\geq 64(k^3+1)x^2y^2z^2.$$

Due to homogeneity, we may only consider the cases $y=z=0$ and $y=z=1$. In the non-trivial case $y=z=1$, the inequality becomes

$$k\sqrt{k}\ x(x+2)^2(x^3+8x^2-8x+32)\geq 64(k^3+1)x^2.$$

This is true because

$$297k\sqrt{k}\geq 64(k^3+1)$$

for $1\leq k\leq k_1$, and

$$x(x+2)^2(x^3+8x^2-8x+32)\geq 297x^2.$$

Notice that

$$x(x+2)^2(x^3+8x^2-8x+32)-297x^2=x(x-1)^2(x^3+14x^2+55x+128)\geq 0.$$

If $1\leq k<k_1$, then the equality occurs only for $a=0$ and $b/c=\sqrt{k}$ (or any cyclic permutation). Therefore, if $k=2$, then the equality holds for $a=0$ and $b/c=\sqrt{2}$ (or any cyclic permutation).

Remark. From the proof above, it follows that the following more general statement holds:

- *Let a,b,c be nonnegative real numbers, no two of which are zero. If $k>0$, then*

$$\sqrt{\frac{a}{kb+c}}+\sqrt{\frac{b}{kc+a}}+\sqrt{\frac{c}{ka+b}}\geq \min\left\{\frac{3}{\sqrt{k+1}},\frac{2}{\sqrt[4]{k}}\right\}.$$

For $k=1$, we get the known inequality

$$\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}}\geq 2,$$

with equality for $a=0$ and $b=c$ (or any cyclic permutation). We can get this inequality by summing the inequalities

$$\sqrt{\frac{a}{b+c}}\geq\frac{2a}{a+b+c},\quad \sqrt{\frac{b}{c+a}}\geq\frac{2b}{a+b+c},\quad \sqrt{\frac{c}{a+b}}\geq\frac{2c}{a+b+c}.$$

□

P 1.79. *If a,b,c are positive real numbers such that $ab+bc+ca=3$, then*

(a) $$\frac{1}{(a+b)(3a+b)}+\frac{1}{(b+c)(3b+c)}+\frac{1}{(c+a)(3c+a)}\geq\frac{3}{8};$$

(b) $$\frac{1}{(2a+b)^2}+\frac{1}{(2b+c)^2}+\frac{1}{(2c+a)^2}\geq\frac{1}{3}.$$

(Vasile Cîrtoaje and Pham Kim Hung, 2007)

Solution. (a) Using the Cauchy-Schwarz inequality and the inequality in P 1.78-(a) gives

$$\sum \frac{1}{(a+b)(3a+b)} = \sum \frac{1}{(b+c)(3b+c)}$$
$$\geq \frac{\left(\sum \sqrt{\frac{a}{3b+c}}\right)^2}{\sum a(b+c)}$$
$$\geq \frac{9}{8(ab+bc+ca)} = \frac{3}{8}.$$

The equality holds for $a = b = c$.

(b) We consider two cases (*Vo Quoc Ba Can*).

Case 1: $4(ab + bc + ca \geq a^2 + b^2 + c^2$. By the Cauchy-Schwarz inequality, we get

$$\sum \frac{1}{(2a+b)^2} \geq \frac{9\left(\sum a\right)^2}{\sum (2a+b)^2(b+2c)^2}.$$

Thus, it suffices to show that

$$9p^2q \geq \sum (2a+b)^2(b+2c)^2,$$

where $p = a + b + c$, $q = ab + bc + ca$. Since

$$(2a+b)(b+2c) = pb + q + 3ac,$$

we have

$$\sum (2a+b)^2(b+2c)^2 = p^2\sum a^2 + 3q^2 + 9\sum a^2b^2 + 2p^2q + 18abcp + 6q^2$$
$$= p^2(p^2-2q) + 9q^2 + 9(q^2 - 2abcp) + 2p^2q + 18abcp = p^4 + 18q^2,$$

and the inequality becomes

$$9p^2q \geq p^4 + 18q^2,$$
$$(p^2 - 3q)(6q - p^2) \geq 0.$$

The last inequality is true since $p^2 - 3q \geq 0$ and

$$6q - p^2 = 4(ab + bc + ca) - a^2 - b^2 - c^2 \geq 0.$$

Case 2: $4(ab + bc + ca < a^2 + b^2 + c^2$. Assume that $a = \max\{a, b, c\}$. From

$$a^2 - 4(b+c)a + (b+c)^2 > 6bc > 0,$$

we get

$$a > (2+\sqrt{3})(b+c) > 2(b+c).$$

Since

$$\frac{1}{(2a+b)^2}+\frac{1}{(2b+c)^2}+\frac{1}{(2c+a)^2}>\frac{1}{(2b+c)^2}+\frac{1}{(2c+a)^2}\geq\frac{2}{(2b+c)(2c+a)},$$

it suffices to show that

$$\frac{2}{(2b+c)(2c+a)}\geq\frac{1}{ab+bc+ca}.$$

This is equivalent to the obvious inequality

$$c(a-2b-2c)\geq 0.$$

The proof is completed. The equality holds for $a=b=c$.

Conjecture. *Let a,b,c be nonnegative real numbers, no two of which are zero. If $k>0$, then*

(a) $$\frac{1}{(a+b)(ka+b)}+\frac{1}{(b+c)(kb+c)}+\frac{1}{(c+a)(kc+a)}\geq\frac{9}{2(k+1)(ab+bc+ca)};$$

(b) $$\frac{1}{(ka+b)^2}+\frac{1}{(kb+c)^2}+\frac{1}{(kc+a)^2}\geq\frac{9}{(k+1)^2(ab+bc+ca)}.$$

For $k=1$, from (a) and (b), we get the well-known inequality (Iran 96):

$$\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\geq\frac{9}{4(ab+bc+ca)}.$$

□

P 1.80. *If a,b,c are nonnegative real numbers, then*

$$a^4+b^4+c^4+15(a^3b+b^3c+c^3a)\geq\frac{47}{4}(a^2b^2+b^2c^2+c^2a^2).$$

(Vasile C., 2011)

Solution. Without loss of generality, assume that $a=\min\{a,b,c\}$. There are two cases to consider: $a\leq b\leq c$ and $a\leq c\leq b$.

Case 1: $a\leq b\leq c$. For $a=0$, the inequality is true because is equivalent to

$$b^4+c^4+15b^3c-\frac{47}{4}b^2c^2\geq 0,$$

$$\left(b-\frac{c}{2}\right)^2(b^2+16bc+4c^2)\geq 0.$$

Based on this result, it suffices to prove that

$$a^4+15(a^3b+c^3a)\geq \frac{47}{4}a^2(b^2+c^2).$$

This inequality is true if

$$a^3b+c^3a\geq a^2(b^2+c^2).$$

Indeed,

$$\begin{aligned}a^2b+c^3-a(b^2+c^2)&=c^2(c-a)-ab(b-a)\geq c^2(b-a)-ab(b-a)\\&=(c^2-ab)(b-a)\geq 0.\end{aligned}$$

Case 2: $a\leq c\leq b$. It suffices to show that

$$a^3b+b^3c+c^3a\geq a^2b^2+b^2c^2+c^2a^2.$$

Since

$$ab^3+bc^3+ca^3-(a^3b+b^3c+c^3a)=(a+b+c)(a-b)(b-c)(c-a)\leq 0,$$

we have

$$\sum a^3b\geq \frac{1}{2}(\sum a^3b+\sum ab^3)=\frac{1}{2}\sum ab(a^2+b^2)\geq \sum a^2b^2.$$

The equality holds for $a=0$ and $2b=c$ (or any cyclic permutation).

□

P 1.81. *If a,b,c are nonnegative real numbers such that $a+b+c=4$, then*

$$a^3b+b^3c+c^3a\leq 27.$$

Solution. Assume that $a=\max\{a,b,c\}$. There are two possible cases: $a\geq b\geq c$ and $a\geq c\geq b$.
Case 1: $a\geq b\geq c$. Using the AM-GM inequality gives

$$\begin{aligned}3(a^3b+b^3c+c^3a)&\leq 3ab(a^2+ac+c^2)\leq 3ab(a+c)^2\\&=a\cdot 3b\cdot(a+c)\cdot(a+c)\leq\left[\frac{a+3b+(a+c)+(a+c)}{4}\right]^4\\&=\left(\frac{3a+3b+2c}{4}\right)^4\leq\left(\frac{3a+3b+3c}{4}\right)^4=81.\end{aligned}$$

Case 2: $a\geq c\geq b$. Since

$$ab^3+bc^3+ca^3-(a^3b+b^3c+c^3a)=(a+b+c)(a-b)(b-c)(c-a)\geq 0,$$

it suffices to prove that

$$a^3b + b^3c + c^3a + (ab^3 + bc^3 + ca^3) \le 54.$$

Indeed,

$$\begin{aligned}\sum a^3b + \sum ab^3 &\le (a^2 + b^2 + c^2)(ab + bc + ca)\\ &\le \frac{1}{8}[a^2 + b^2 + c^2 + 2(ab + bc + ca)]^2\\ &= \frac{1}{8}(a + b + c)^4 = 32 < 54.\end{aligned}$$

The equality holds for $a = 3,\ b = 1$ and $c = 0$ (or any cyclic permutation).

Remark. The following sharper inequality holds (*Michael Rozenberg*).

- *If a, b, c are nonnegative real numbers such that $a + b + c = 4$, then*

$$a^3b + b^3c + c^3a + \frac{473}{64}\, abc \le 27,$$

with equality for $a = b = c = 4/3$, and also for $a = 3,\ b = 1$ and $c = 0$ (or any cyclic permutation).

Write the inequality in the homogeneous form

$$27(a + b + c)^4 \ge 256(a^3b + b^3c + c^3a) + 473abc(a + b + c).$$

Assuming that $c = \min\{a, b, c\}$ and using the substitution

$$a = c + p, \quad b = c + q, \quad p, q \ge 0,$$

this inequality can be restated as

$$Ac^2 + Bc + C \ge 0,$$

where

$$A = 217(p^2 - pq + q^2) \ge 0,$$

$$B = 68p^3 - 269p^2q + 499pq^2 + 68q^3 \ge 60p(p^2 - 5pq + 8q^2) \ge 0,$$

$$C = (p - 3q)^2(27p^2 + 14pq + 3q^2) \ge 0.$$

□

P 1.82. *Let a, b, c be nonnegative real numbers such that*

$$a^2 + b^2 + c^2 = \frac{10}{3}(ab + bc + ca).$$

Prove that

$$a^4 + b^4 + c^4 \ge \frac{82}{27}(a^3b + b^3c + c^3a).$$

(Vasile C., 2011)

Solution *(by Vo Quoc Ba Can)*. We see that the equality holds for $a=3$, $b=1$, $c=0$. From

$$a^4+b^4+c^4+2(ab+bc+ca)^2=(a^2+b^2+c^2)^2+4abc(a+b+c),$$

we get

$$\begin{aligned}a^4+b^4+c^4 &\geq (a^2+b^2+c^2)^2-2(ab+bc+ca)^2\\ &=\frac{82}{9}(ab+bc+ca)^2.\end{aligned}$$

Therefore, it suffices to show that

$$3(ab+bc+ca)^2\geq a^3b+b^3c+c^3a.$$

In addition, since

$$ab+bc+ca=\frac{3(a^2+b^2+c^2)+6(ab+bc+ca)}{16}=3\left(\frac{a+b+c}{4}\right)^2,$$

it suffices to show that

$$27\left(\frac{a+b+c}{4}\right)^4\geq a^3b+b^3c+c^3a,$$

which is the inequality from the preceding P 1.81. The equality holds for $a=3b$ and $c=0$ (or any cyclic permutation).

□

P 1.83. *If a,b,c are positive real numbers, then*

$$\frac{a^3}{2a^2+b^2}+\frac{b^3}{2b^2+c^2}+\frac{c^3}{2c^2+a^2}\geq\frac{a+b+c}{3}.$$

(Vasile C., 2005)

Solution. We write the inequality as

$$\left(\frac{a^3}{2a^2+b^2}-\frac{a}{3}\right)+\left(\frac{b^3}{2b^2+c^2}-\frac{b}{3}\right)+\left(\frac{c^3}{2c^2+a^2}-\frac{c}{3}\right)\geq 0,$$

$$\frac{a(a^2-b^2)}{2a^2+b^2}+\frac{b(b^2-c^2)}{2b^2+c^2}+\frac{c(c^2-a^2)}{2c^2+a^2}\geq 0.$$

Taking into account that

$$\frac{a(a^2-b^2)}{2a^2+b^2}-\frac{b(a^2-b^2)}{2b^2+a^2}=\frac{(a+b)(a-b)^2(a^2-ab+b^2)}{(2a^2+b^2)(2b^2+a^2)}\geq 0,$$

it suffices to show that

$$\frac{b(a^2-b^2)}{2b^2+a^2}+\frac{b(b^2-c^2)}{2b^2+c^2}+\frac{c(c^2-a^2)}{2c^2+a^2}\ge 0.$$

Since

$$\frac{b(a^2-b^2)}{2b^2+a^2}+\frac{b(b^2-c^2)}{2b^2+c^2}=\frac{3b^2(a^2-c^2)}{(2b^2+a^2)(2b^2+c^2)},$$

the last inequality is equivalent to

$$(c^2-a^2)(c-b)[a^2(3b^2+bc+c^2)+2b^2c(c-2b)]\ge 0. \tag{*}$$

Similarly, the desired inequality is true if

$$(a^2-b^2)(a-c)[b^2(3c^2+ca+a^2)+2c^2a(a-2c)]\ge 0. \tag{**}$$

Without loss of generality, assume that

$$c=\max\{a,b,c\}.$$

According to (*), the desired inequality is true if

$$a^2(3b^2+bc+c^2)+2b^2c(c-2b)\ge 0.$$

We claim that this inequality holds for for $a\ge b$, and also for $2ac\ge\sqrt{3}\ b^2$. If $a\ge b$, then

$$\begin{aligned}a^2(3b^2+bc+c^2)+2b^2c(c-2b)&\ge b^2(3b^2+bc+c^2)+2b^2c(c-2b)\\&=3b^2[b^2+c(c-b)]>0;\end{aligned}$$

also, if $2ac\ge\sqrt{3}\ b^2$, then

$$\begin{aligned}a^2(3b^2+bc+c^2)+2b^2c(c-2b)&\ge\frac{3b^4}{4c^2}(3b^2+bc+c^2)+2b^2c(c-2b)\\&=\frac{b^2}{4c^2}(8c^4-16bc^3+3b^2c^2+3b^3c+9b^4)\\&=\frac{b^2}{4c^2}[2c(c+b)(2c-3b)^2+9b^2(c-b)^2+3b^3c]>0.\end{aligned}$$

Consequently, we only need to consider that $a<b\le c$ and $\sqrt{3}\ b^2>2ac$. According to (**), the desired inequality is true if

$$b^2(3c^2+ca+a^2)+2c^2a(a-2c)\ge 0.$$

We have

$$\begin{aligned}b^2(3c^2+ca+a^2)+2c^2a(a-2c)&>\frac{4ac}{3}(3c^2+ca+a^2)+2c^2a(a-2c)\\&=\frac{2a^2c(2a+5c)}{3}>0.\end{aligned}$$

This completes the proof. The equality occurs for $a=b=c$.

□

P 1.84. *If a,b,c are positive real numbers, then*

$$\frac{a^4}{a^3+b^3}+\frac{b^4}{b^3+c^3}+\frac{c^4}{c^3+a^3}\ge\frac{a+b+c}{2}.$$

(Vasile C., 2005)

Solution *(by Vo Quoc Ba Can).* Multiplying by $a^3+b^3+c^3$, the inequality becomes

$$\sum a^4+\sum\frac{a^4c^3}{a^3+b^3}\ge\frac{1}{2}\left(\sum a\right)\left(\sum a^3\right).$$

By the Cauchy-Schwarz inequality, we have

$$\sum\frac{a^4c^3}{a^3+b^3}\ge\frac{\left(\sum a^2c^2\right)^2}{\sum c(a^3+b^3)}=\frac{\left(\sum a^2b^2\right)^2}{\sum a(b^3+c^3)}.$$

According to the inequality

$$\frac{x^2}{y}\ge x-\frac{y}{4},\quad x,y>0,$$

we have

$$\frac{(\sum a^2b^2)^2}{\sum a(b^3+c^3)}\ge\sum a^2b^2-\frac{1}{4}\sum a(b^3+c^3).$$

Therefore, it suffices to show that

$$\sum a^4+\sum a^2b^2-\frac{1}{4}\sum a(b^3+c^3)\ge\frac{1}{2}\left(\sum a\right)\left(\sum a^3\right),$$

which is equivalent to

$$2\sum a^4+4\sum a^2b^2\ge 3\sum ab(a^2+b^2),$$

$$\sum[a^4+b^4+4a^2b^2-3ab(a^2+b^2)]\ge 0,$$

$$\sum(a-b)^2(a^2-ab+b^2)\ge 0.$$

This completes the proof. The equality occurs for $a=b=c$.

□

P 1.85. *If a,b,c are positive real numbers such that $abc=1$, then*

(a) $$3\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)+4\left(\frac{b}{a^2}+\frac{c}{b^2}+\frac{a}{c^2}\right)\ge 7(a^2+b^2+c^2);$$

(b) $$8\left(\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}\right)+5\left(\frac{b}{a^3}+\frac{c}{b^3}+\frac{a}{c^3}\right)\ge 13(a^3+b^3+c^3).$$

(Vasile C., 1992)

Solution. (a) We use the AM-GM inequality, as follows:

$$3\sum\frac{a^2}{b}+4\sum\frac{b}{a^2}=\sum\left(3\frac{a^2}{b}+\frac{c}{b^2}+3\frac{a}{c^2}\right)\geq 7\sum\sqrt[7]{\left(\frac{a^2}{b}\right)^3\cdot\frac{c}{b^2}\cdot\left(\frac{a}{c^2}\right)^3}$$

$$=7\sum\sqrt[7]{\frac{a^9}{b^5c^5}}=7\sum a^2.$$

The equality holds for $a=b=c=1$.

(b) By the AM-GM inequality, we have

$$8\sum\frac{a^3}{b}+5\sum\frac{b}{a^3}=\sum\left(8\frac{a^3}{b}+\frac{c}{b^3}+4\frac{a}{c^3}\right)\geq 13\sum\sqrt[13]{\left(\frac{a^3}{b}\right)^8\cdot\frac{c}{b^3}\cdot\left(\frac{a}{c^3}\right)^4}$$

$$=13\sum\sqrt[13]{\frac{a^{28}}{b^{11}c^{11}}}=13\sum a^3.$$

The equality holds for $a=b=c=1$.

$\square$

P 1.86. *If a,b,c are positive real numbers, then*

$$\frac{ab}{b^2+bc+c^2}+\frac{bc}{c^2+ca+a^2}+\frac{ca}{a^2+ab+b^2}\leq\frac{a^2+b^2+c^2}{ab+bc+ca}.$$

(Tran Quoc Anh, 2007)

Solution. Write the inequality as follows:

$$\sum\left(\frac{a^2}{ab+bc+ca}-\frac{ab}{b^2+bc+c^2}\right)\geq 0,$$

$$\sum\frac{ac(ac-b^2)}{b^2+bc+c^2}\geq 0,$$

$$\sum\left[\frac{ac(ac-b^2)}{b^2+bc+c^2}+ac\right]\geq\sum ac,$$

$$\sum\frac{ac^2(a+b+c)}{b^2+bc+c^2}\geq\sum ac,$$

$$\sum\frac{ac^2}{b^2+bc+c^2}\geq\frac{ab+bc+ca}{a+b+c}.$$

By the Cauchy-Schwarz inequality, we have

$$\sum\frac{ac^2}{b^2+bc+c^2}\geq\frac{\left(\sum ac\right)^2}{\sum a(b^2+bc+c^2)}=\frac{ab+bc+ca}{a+b+ca}.$$

The equality holds for $a=b=c$.

$\square$

P 1.87. *If a, b, c are positive real numbers, then*

$$\frac{a-b}{b(2b+c)}+\frac{b-c}{c(2c+a)}+\frac{c-a}{a(2a+b)} \ge 0.$$

Solution. Write the inequality as follows:

$$\sum \frac{ac(a-b)}{2b+c} \ge 0,$$

$$\sum \left[\frac{ac(a-b)}{2b+c}+ac\right] \ge ab+bc+ca,$$

$$\sum \frac{ac}{2b+c} \ge \frac{ab+bc+ca}{a+b+c}.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{ac}{2b+c} \ge \frac{\left(\sum ac\right)^2}{\sum ac(2b+c)} = \frac{\left(\sum ab\right)^2}{6abc+\sum a^2b}.$$

Thus, it suffices to prove that

$$\frac{\sum ab}{6abc+\sum a^2b} \ge \frac{1}{\sum a},$$

which is equivalent to

$$\sum ab^2 \ge 3abc.$$

Clearly, the last inequality follows immediately from the AM-GM inequality. The equality holds for $a=b=c$.

□

P 1.88. *If a, b, c are positive real numbers, then*

(a) $$\frac{a^2+6bc}{ab+2bc}+\frac{b^2+6ca}{bc+2ca}+\frac{c^2+6ab}{ca+2ab} \ge 7;$$

(b) $$\frac{a^2+7bc}{ab+bc}+\frac{b^2+7ca}{bc+ca}+\frac{c^2+7ab}{ca+ab} \ge 12.$$

(Vasile C., 2012)

Solution. (a) Write the inequality as follows:

$$\sum ac(a^2+6bc)(b+2a)(c+2b) \ge 7abc(a+2c)(b+2a)(c+2b),$$

$$2\sum a^2b^4 + abc\left(72abc + 4\sum a^3 + 26\sum a^2b + 7\sum ab^2\right) \ge$$
$$\ge 7abc\left(9abc + 4\sum a^2b + 2\sum ab^2\right),$$
$$2\left(\sum a^2b^4 - abc\sum a^2b\right) + abc\left(4\sum a^3 + 9abc - 7\sum ab^2\right) \ge 0.$$

Since

$$2\left(\sum a^2b^4 - abc\sum a^2b\right) = \sum (ab^2-bc^2)^2 \ge 0,$$

it suffices to show that

$$4\sum a^3 + 9abc - 7\sum ab^2 \ge 0.$$

Assume that $a = \min\{a,b,c\}$. Using the substitution

$$b = a + x, \quad c = a + y, \quad x, y \ge 0,$$

we have

$$4\sum a^3 + 9abc - 7\sum ab^2 = 5(x^2 - xy + y^2)a + 4x^3 + 4y^3 - 7xy^2 \ge 0,$$

since

$$4x^3 + 4y^3 = 4x^3 + 2y^3 + 2y^3 \ge 3\sqrt[3]{4x^3 \cdot 2y^3 \cdot 2y^3} = 6\sqrt[3]{2}\,xy^2 \ge 7xy^2.$$

The equality holds for $a = b = c$.

(b) Write the inequality as follows:

$$\sum ac(a^2+7bc)(b+a)(c+b) \ge 12abc(a+c)(b+a)(c+b),$$

$$\sum a^2b^4 + abc\left(21abc + \sum a^3 + 15\sum a^2b + 8\sum ab^2\right) \ge$$
$$\ge 12abc\left(2abc + \sum a^2b + \sum ab^2\right),$$
$$\left(\sum a^2b^4 - abc\sum a^2b\right) + abc\left(\sum a^3 - 3abc + 4\sum a^2b - 4\sum ab^2\right) \ge 0.$$

Since

$$\sum a^2b^4 - abc\sum a^2b = \frac{1}{2}\sum (ab^2-bc^2)^2 \ge 0,$$

it suffices to show that

$$\sum a^3 - 3abc + 4\sum a^2b - 4\sum ab^2 \ge 0,$$

which is equivalent to

$$\frac{1}{2}(a+b+c)\sum(a-b)^2-4(a-b)(b-c)(c-a)\ge 0.$$

Assume that $a=\min\{a,b,c\}$. Making the substitution

$$b=a+x,\quad c=a+y,\quad x,y\ge 0,$$

we have

$$\begin{aligned}\frac{1}{2}(a+b+c)\sum(a-b)^2-4(a-b)(b-c)(c-a)=&\\ =(x^2-xy+y^2)(3a+x+y)+4xy(x-y)&\\ =3(x^2-xy+y^2)a+x^3+y^3+4xy(x-y)&\\ =3(x^2-xy+y^2)a+x^3+y(2x-y)^2\ge 0.&\end{aligned}$$

The equality holds for $a=b=c$.

□

P 1.89. *If a,b,c are positive real numbers, then*

(a) $$\frac{ab}{2b+c}+\frac{bc}{2c+a}+\frac{ca}{2a+b}\le\frac{a^2+b^2+c^2}{a+b+c};$$

(b) $$\frac{ab}{b+c}+\frac{bc}{c+a}+\frac{ca}{a+b}\le\frac{3(a^2+b^2+c^2)}{2(a+b+c)};$$

(c) $$\frac{ab}{4b+5c}+\frac{bc}{4c+5a}+\frac{ca}{4a+5b}\le\frac{a^2+b^2+c^2}{3(a+b+c)}.$$

(Vasile C., 2012)

Solution. (a) ***First Solution.*** Since

$$\frac{2ab}{2b+c}=a-\frac{ac}{2b+c},$$

we can write the inequality as

$$\sum\frac{ac}{2b+c}+\frac{2(a^2+b^2+c^2)}{a+b+c}\ge a+b+c.$$

By the Cauchy-Schwarz inequality,

$$\sum\frac{ac}{2b+c}\ge\frac{\left(\sum\sqrt{ac}\right)^2}{\sum(2b+c)}=\frac{(\sqrt{ab}+\sqrt{bc}+\sqrt{ca})^2}{3(a+b+c)}.$$

Then, it suffices to show that

$$\frac{(\sqrt{ab}+\sqrt{bc}+\sqrt{ca})^2+6(a^2+b^2+c^2)}{3(a+b+c)} \ge a+b+c,$$

which is equivalent to

$$3(a^2+b^2+c^2)+2\sqrt{abc}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right) \ge 5(ab+bc+ca).$$

Using the substitution

$$x=\sqrt{a},\quad y=\sqrt{b},\quad z=\sqrt{c},$$

the inequality can be restated as

$$3(x^4+y^4+z^4)+2xyz(x+y+z) \ge 5(x^2y^2+y^2z^2+z^2x^2).$$

We can get it by summing Schur's inequality of degree four

$$2(x^4+y^4+z^4)+2xyz(x+y+z) \ge 2\sum xy(x^2+y^2)$$

and

$$x^4+y^4+z^4+2\sum xy(x^2+y^2) \ge 5(x^2y^2+y^2z^2+z^2x^2),$$

the last being equivalent to the obvious inequality

$$(x^4+y^4+z^4-x^2y^2-y^2z^2-z^2x^2)+2\sum xy(x-y)^2 \ge 0.$$

The equality holds for $a=b=c$.

Second Solution. By the Cauchy-Schwarz inequality, we have

$$\frac{1}{2b+c}=\frac{1}{b+b+c} \le \frac{a^2/b+b+c}{(a+b+c)^2}=\frac{a^2+b^2+bc}{b(a+b+c)^2},$$

$$\frac{ab}{2b+c} \le \frac{a(a^2+b^2+bc)}{(a+b+c)^2},$$

$$\sum \frac{ab}{2b+c} \le \frac{\sum a^3+\sum ab^2+3abc}{(a+b+c)^2}.$$

Since $3abc \le \sum a^2b$ (by the AM-GM inequality), we get

$$\sum \frac{ab}{2b+c} \le \frac{\sum a^3+\sum ab^2+\sum a^2b}{(a+b+c)^2}=\frac{a^2+b^2+c^2}{a+b+c}.$$

Third Solution. Write the inequality as

$$\sum \frac{ab(a+b+c)}{2b+c} \le a^2+b^2+c^2.$$

Since
$$2ab(a+b+c)=(a^2+2ab)(2b+c)-2ab^2-a^2c,$$
we can write the inequality as
$$\sum \frac{2ab^2}{2b+c}+\sum \frac{a^2c}{2b+c}+p\ge 2q,$$
where
$$p=a^2+b^2+c^2,\ \ q=ab+bc+ca,\quad p\ge q.$$
By the Cauchy-Schwarz inequality, we have
$$\sum \frac{ab^2}{2b+c}\ge \frac{\left(\sum ab\right)^2}{\sum a(2b+c)}=\frac{q}{3}$$
and
$$\sum \frac{a^2c}{2b+c}\ge \frac{\left(\sum ac\right)^2}{\sum c(2b+c)}=\frac{q^2}{p+2q}.$$
Thus, it suffices to show that
$$\frac{2q}{3}+\frac{q^2}{p+2q}+p\ge 2q,$$
which is equivalent to the obvious inequality
$$(p-q)(3p+5q)\ge 0.$$

(b) Write the inequality as
$$\frac{3}{2}(a^2+b^2+c^2)\ge \sum \frac{ab(a+b+c)}{b+c}.$$
Since
$$\frac{ab(a+b+c)}{b+c}=\frac{a^2b}{b+c}+ab=a^2+ab-\frac{a^2c}{b+c},$$
the inequality can be written as
$$\sum \frac{a^2c}{b+c}+\frac{1}{2}(a^2+b^2+c^2)\ge ab+bc+ca.$$
By the Cauchy-Schwarz inequality,
$$\sum \frac{a^2c}{b+c}\ge \frac{\left(\sum ac\right)^2}{\sum c(b+c)}=\frac{q^2}{p+q},$$
where
$$p=a^2+b^2+c^2,\ \ q=ab+bc+ca,\quad p\ge q.$$

Therefore, we have

$$\sum \frac{a^2c}{b+c} + \frac{1}{2}(a^2+b^2+c^2) - (ab+bc+ca) \geq \frac{q^2}{p+q} + \frac{p}{2} - q = \frac{p(p-q)}{2(p+q)} \geq 0.$$

The equality holds for $a=b=c$.

(c) Since

$$\frac{4ab}{4b+5c} = a - \frac{5ac}{4b+5c},$$

we can write the inequality as

$$5\sum \frac{ac}{4b+5c} + \frac{4(a^2+b^2+c^2)}{3(a+b+c)} \geq a+b+c.$$

By the Cauchy-Schwarz inequality,

$$\sum \frac{ac}{4b+5c} \geq \frac{\left(\sum ac\right)^2}{\sum ac(4b+5c)} = \frac{(ab+bc+ca)^2}{12abc+5(a^2b+b^2c+c^2a)}.$$

Therefore, it suffices to show that

$$\frac{5(ab+bc+ca)^2}{12abc+5(a^2b+b^2c+c^2a)} + \frac{4(a^2+b^2+c^2)}{3(a+b+c)} \geq a+b+c.$$

Due to homogeneity, we may assume that $a+b+c=3$. Using the notation

$$q = ab+bc+ca, \quad q \leq 3,$$

this inequality becomes

$$\frac{5q^2}{5(a^2b+b^2c+c^2a+abc)+7abc} + \frac{4(9-2q)}{9} \geq 3.$$

According to the inequality (a) in P 1.9, we have

$$a^2b+b^2c+c^2a+abc \leq 4.$$

On the other hand, from

$$(ab+bc+ca)^2 \geq 3abc(a+b+c),$$

we get

$$abc \leq \frac{q^2}{9}.$$

Thus, it suffices to prove that

$$\frac{5q^2}{20+7q^2/9} + \frac{4(9-2q)}{9} \geq 3,$$

which is equivalent to

$$(q-3)(14q^2-75q+135) \le 0.$$

This is true since $q-3 \le 0$ and

$$14q^2-75q+135 > 3(4q^2-25q+39) = 3(3-q)(13-4q) \ge 0.$$

The equality holds for $a=b=c$.

□

P 1.90. *If a,b,c are positive real numbers, then*

(a) $$a\sqrt{b^2+8c^2}+b\sqrt{c^2+8a^2}+c\sqrt{a^2+8b^2} \le (a+b+c)^2;$$

(b) $$a\sqrt{b^2+3c^2}+b\sqrt{c^2+3a^2}+c\sqrt{a^2+3b^2} \le a^2+b^2+c^2+ab+bc+ca.$$

(Vo Quoc Ba Can, 2007)

Solution. (a) By the AM-GM inequality, we have

$$\sqrt{b^2+8c^2} = \frac{\sqrt{(b^2+8c^2)(b+2c)^2}}{b+2c} \le \frac{(b^2+8c^2)+(b+2c)^2}{2(b+2c)}$$

$$= \frac{b^2+2bc+6c^2}{b+2c} = b+3c-\frac{3bc}{b+2c},$$

hence

$$a\sqrt{b^2+8c^2} \le ab+3ac-\frac{3abc}{b+2c},$$

$$\sum a\sqrt{b^2+8c^2} \le 4\sum ab-3abc\sum\frac{1}{b+2c}.$$

Therefore, it suffices to show that

$$\left(\sum a\right)^2+3abc\sum\frac{1}{b+2c} \ge 4\sum ab.$$

Since

$$\sum\frac{1}{b+2c} \ge \frac{9}{\sum(b+2c)} = \frac{3}{\sum a},$$

it is enough to prove that

$$\left(\sum a\right)^3+9abc \ge 4\left(\sum a\right)\left(\sum ab\right).$$

This is Shur's inequality of degree three. The equality holds for $a=b=c$.

(b) Similarly, we have

$$\sqrt{b^2+3c^2}=\frac{\sqrt{(b^2+3c^2)(b+c)^2}}{b+c}\le\frac{(b^2+3c^2)+(b+c)^2}{2(b+c)}$$

$$=\frac{b^2+bc+2c^2}{b+c}=b+2c-\frac{2bc}{b+c},$$

hence

$$a\sqrt{b^2+3c^2}\le ab+2ac-\frac{2abc}{b+c},$$

$$\sum a\sqrt{b^2+3c^2}\le 3\sum ab-2abc\sum\frac{1}{b+c}.$$

Thus, it suffices to show that

$$\left(\sum a\right)^2+2abc\sum\frac{1}{b+c}\ge 4\sum ab.$$

Since

$$\sum\frac{1}{b+c}\ge\frac{9}{\sum(b+c)}=\frac{9}{2\sum a},$$

it is enough to prove that

$$\left(\sum a\right)^3+9abc\ge 4\left(\sum a\right)\left(\sum ab\right),$$

which is just Shur's inequality of degree three. The equality holds for $a=b=c$.

□

P 1.91. *If a,b,c are positive real numbers, then*

(a) $$\frac{1}{a\sqrt{a+2b}}+\frac{1}{b\sqrt{b+2c}}+\frac{1}{c\sqrt{c+2a}}\ge\sqrt{\frac{3}{abc}};$$

(b) $$\frac{1}{a\sqrt{a+8b}}+\frac{1}{b\sqrt{b+8c}}+\frac{1}{c\sqrt{c+8a}}\ge\sqrt{\frac{1}{abc}}.$$

(Vasile C., 2007)

Solution. (a) Write the inequality as

$$\sum\sqrt{\frac{bc}{3a(a+2b)}}\ge 1.$$

Replacing a,b,c by $\frac{1}{x},\frac{1}{y},\frac{1}{z}$, respectively, the inequality can be restated as

$$\sum\frac{x}{\sqrt{3z(2x+y)}}\ge 1.$$

Since

$$\sqrt{3z(2x+y)} \le \frac{3z+(2x+y)}{2},$$

it suffices to show that

$$\sum \frac{x}{2x+y+3z} \ge \frac{1}{2}.$$

Indeed, using the Cauchy-Schwarz inequality gives

$$\sum \frac{x}{2x+y+3z} \ge \sum \frac{(\sum x)^2}{\sum x(2x+y+3z)} = \frac{1}{2}.$$

The equality holds for $a=b=c$.

(b) Write the inequality as

$$\sum \sqrt{\frac{bc}{a(a+8b)}} \ge 1.$$

Replacing a,b,c by $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$, respectively, the inequality becomes

$$\sum \frac{x^2}{z\sqrt{8x^2+y^2}} \ge 1.$$

Applying the Cauchy-Schwarz inequality yields

$$\sum \frac{x^2}{z\sqrt{8x^2+y^2}} \ge \frac{\left(\sum x\right)^2}{\sum z\sqrt{8x^2+y^2}}.$$

Therefore, it suffices to show that

$$\sum z\sqrt{8x^2+y^2} \le (x+y+z)^2,$$

which is just the inequality in P 1.90-(a). The equality holds for $a=b=c$.

□

P 1.92. *If a,b,c are positive real numbers, then*

$$\frac{a}{\sqrt{5a+4b}} + \frac{b}{\sqrt{5b+4c}} + \frac{c}{\sqrt{5c+4a}} \le \sqrt{\frac{a+b+c}{3}}.$$

(Vasile C., 2012)

Solution. By the Cauchy-Schwarz inequality, we have

$$\left(\sum \frac{a}{\sqrt{5a+4b}}\right)^2 \le \left(\sum \frac{a}{4a+4b+c}\right)\left(\sum \frac{a(4a+4b+c)}{5a+4b}\right).$$

It suffices to show that

$$\sum \frac{a}{4a+4b+c} \le \frac{1}{3}$$

and

$$\sum \frac{a(4a+4b+c)}{5a+4b} \le a+b+c.$$

The first is just the inequality in P 1.18, while the second is equivalent to

$$\sum a\left(1-\frac{4a+4b+c}{5a+4b}\right) \ge 0,$$

$$\sum \frac{a(a-c)}{5a+4b} \ge 0,$$

$$\sum a(a-c)(5b+4c)(5c+4a) \ge 0,$$

$$\sum a^2b^2 + 4\sum ab^3 \ge 5abc\sum a.$$

The last inequality follows from the well-known inequality

$$\sum a^2b^2 \ge abc\sum a$$

and the known inequality

$$\sum ab^3 \ge abc\sum a,$$

which follows from the Cauchy-Schwarz inequality, as follows:

$$\left(\sum c\right)\left(\sum ab^3\right) \ge \left(\sum \sqrt{ab^3c}\right)^2 = abc\left(\sum b\right)^2.$$

The equality holds for $a=b=c$.

□

P 1.93. *If a,b,c are positive real numbers, then*

(a)
$$\frac{a}{\sqrt{a+b}} + \frac{b}{\sqrt{b+c}} + \frac{c}{\sqrt{c+a}} \ge \frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{\sqrt{2}};$$

(b)
$$\frac{a}{\sqrt{a+b}} + \frac{b}{\sqrt{b+c}} + \frac{c}{\sqrt{c+a}} \ge \sqrt[4]{\frac{27(ab+bc+ca)}{4}}.$$

(Lev Buchovsky - 1995, Pham Huu Duc - 2007)

Solution. (a) By squaring, the inequality becomes

$$\sum \frac{a^2}{a+b} + 2\sum \frac{ab}{\sqrt{(a+b)(b+c)}} \ge \frac{1}{2}\sum a + \sum \sqrt{ab}.$$

The sequences

$$\left\{ \frac{1}{\sqrt{a+b}}, \quad \frac{1}{\sqrt{b+c}}, \quad \frac{1}{\sqrt{c+a}} \right\}$$

and

$$\left\{ \frac{ab}{\sqrt{a+b}}, \quad \frac{bc}{\sqrt{b+c}}, \quad \frac{ca}{\sqrt{c+a}} \right\}$$

are always reversely ordered; therefore, according to the rearrangement inequality, we have

$$\frac{1}{\sqrt{a+b}} \cdot \frac{ab}{\sqrt{a+b}} + \frac{1}{\sqrt{b+c}} \cdot \frac{bc}{\sqrt{b+c}} + \frac{1}{\sqrt{c+a}} \cdot \frac{ca}{\sqrt{c+a}} \le$$

$$\le \frac{1}{\sqrt{a+b}} \cdot \frac{ca}{\sqrt{c+a}} + \frac{1}{\sqrt{b+c}} \cdot \frac{ab}{\sqrt{a+b}} + \frac{1}{\sqrt{c+a}} \cdot \frac{bc}{\sqrt{b+c}},$$

$$\sum \frac{ab}{a+b} \le \sum \frac{ab}{\sqrt{(a+b)(b+c)}}.$$

Thus, it suffices to show that

$$\sum \frac{a^2}{a+b} + 2\sum \frac{ab}{a+b} \ge \frac{1}{2}\sum a + \sum \sqrt{ab}.$$

Since

$$\sum \frac{a^2}{a+b} + \sum \frac{ab}{a+b} = \sum a,$$

the inequality becomes as follows:

$$\sum a + \sum \frac{ab}{a+b} \ge \frac{1}{2}\sum a + \sum \sqrt{ab},$$

$$\sum \frac{a+b}{2} + \sum \frac{2ab}{a+b} \ge 2\sum \sqrt{ab},$$

$$\sum \left(\sqrt{\frac{a+b}{2}} - \sqrt{\frac{2ab}{a+b}} \right)^2 \ge 0.$$

The equality holds for $a = b = c$.

(b) By Hölder's inequality, we have

$$\left(\sum \frac{a}{\sqrt{a+b}} \right)^2 \sum a(a+b) \ge \left(\sum a \right)^3.$$

Thus, it suffices to show that

$$\left(\sum a\right)^3 \geq \frac{3}{2}\left(\sum a^2 + \sum ab\right)\sqrt{3(ab+bc+ca)},$$

which is equivalent to

$$2p^3 + q^3 \geq 3p^2q,$$

where $p = a+b+c$ and $q = \sqrt{3(ab+bc+ca)}$. By the AM-GM inequality, we have

$$2p^3 + q^3 \geq 3\sqrt[3]{p^6q^3} = 3p^2q.$$

The equality holds for $a = b = c$.

□

P 1.94. *If a, b, c are nonnegative real numbers such that $a+b+c=3$, then*

$$\sqrt{3a+b^2} + \sqrt{3b+c^2} + \sqrt{3c+a^2} \geq 6.$$

First Solution. Assume that $a = \max\{a, b, c\}$. We can get the desired inequality by summing the inequalities

$$\sqrt{3b+c^2} + \sqrt{3c+a^2} \geq \sqrt{3a+c^2} + b + c$$

and

$$\sqrt{3a+b^2} + \sqrt{3a+c^2} \geq 2a+b+c.$$

By squaring two times, the first inequality becomes in succession

$$\sqrt{(3b+c^2)(3c+a^2)} \geq (b+c)\sqrt{3a+c^2},$$

$$[b(a+b+c)+c^2][c(a+b+c)+a^2] \geq (b+c)^2[a(a+b+c)+c^2],$$

$$b(a-b)(a-c)(a+b+c) \geq 0.$$

Similarly, the second inequality becomes

$$\sqrt{(3a+b^2)(3a+c^2)} \geq (a+b)(a+c),$$

$$[a(a+b+c)+b^2][a(a+b+c)+c^2] \geq (a+b)^2(a+c)^2,$$

$$a(a+b+c)(b-c)^2 \geq 0.$$

The original inequality becomes an equality when $a = b = c$, and also when two of a, b, c are zero.

Second Solution. Write the inequality as

$$\sqrt{X} + \sqrt{Y} + \sqrt{Z} \leq \sqrt{A} + \sqrt{B} + \sqrt{C},$$

where
$$X=(b+c)^2, \quad Y=(c+a)^2, \quad Z=(a+b)^2,$$
$$A=3a+b^2, \quad B=3b+c^2, \quad C=3c+a^2.$$
According to Lemma from the proof of P 2.11 in Volume 2, since
$$X+Y+Z=A+B+C,$$
it suffices to show that
$$\max\{X,Y,Z\}\ge\max\{A,B,C\}, \quad \min\{X,Y,Z\}\le\min\{A,B,C\}.$$
To show that $\max\{X,Y,Z\}\ge\max\{A,B,C\}$, we assume that
$$a=\min\{a,b,c\}, \quad \max\{X,Y,Z\}=X.$$
From
$$X-A=(c^2-a^2)+b(c-a)+c(b-a)\ge 0,$$
$$X-B=b(c-a)\ge 0,$$
$$X-C=(b^2-a^2)+c(b-a)\ge 0,$$
the conclusion follows. Similarly, to show that $\min\{X,Y,Z\}\le\min\{A,B,C\}$, we assume that
$$a=\max\{a,b,c\}, \quad \min\{X,Y,Z\}=X,$$
when
$$A-X=(a^2-c^2)+b(a-c)+c(a-b)\ge 0,$$
$$B-X=b(a-c)\ge 0,$$
$$C-X=(a^2-b^2)+c(a-b)\ge 0.$$

□

P 1.95. *If a,b,c are nonnegative real numbers, then*
$$\sqrt{a^2+b^2+2bc}+\sqrt{b^2+c^2+2ca}+\sqrt{c^2+a^2+2ab}\ge 2(a+b+c).$$

(Vasile C., 2012)

First Solution *(by Nguyen Van Quy).* Assume that $a=\max\{a,b,c\}$. We can get the desired inequality by summing the inequalities
$$\sqrt{a^2+b^2+2bc}+\sqrt{b^2+c^2+2ca}\ge\sqrt{a^2+b^2+2ca}+b+c$$
and
$$\sqrt{c^2+a^2+2ab}+\sqrt{a^2+b^2+2ca}\ge 2a+b+c.$$

By squaring two times, the first inequality becomes

$$\sqrt{(a^2+b^2+2bc)(b^2+c^2+2ca)} \ge (b+c)\sqrt{a^2+b^2+2ca},$$

$$c(a-b)(a^2-c^2) \ge 0.$$

Similarly, the second inequality becomes

$$\sqrt{(c^2+a^2+2ab)(a^2+b^2+2ca)} \ge (a+b)(a+c),$$

$$a(b+c)(b-c)^2 \ge 0.$$

The original inequality becomes an equality when $a=b=c$, and also when two of a,b,c are zero.

Second Solution. Let $\{x,y,z\}$ be a permutation of $\{ab,bc,ca\}$. We will prove that

$$2(a+b+c) \le \sqrt{b^2+c^2+2x} + \sqrt{c^2+a^2+2y} + \sqrt{a^2+b^2+2z}.$$

Due to symmetry, assume that $a \ge b \ge c$. Using the substitution

$$X = a^2+b^2+2ab, \quad Y = c^2+a^2+2ca, \quad Z = b^2+c^2+2bc,$$

$$A = b^2+c^2+2x, \quad B = c^2+a^2+2y, \quad C = a^2+b^2+2z,$$

we can write the inequality as

$$\sqrt{X}+\sqrt{Y}+\sqrt{Z} \le \sqrt{A}+\sqrt{B}+\sqrt{C}.$$

Since $X+Y+Z = A+B+C$, $X \ge Y \ge Z$ and

$$X \ge \max\{A,B,C\}, \qquad Z \le \min\{A,B,C\},$$

the conclusion follow by Lemma from the proof of P 2.11 in Volume 2.

□

P 1.96. *If a,b,c are nonnegative real numbers, then*

$$\sqrt{a^2+b^2+7bc} + \sqrt{b^2+c^2+7ca} + \sqrt{c^2+a^2+7ab} \ge 3\sqrt{3(ab+bc+ca)}.$$

(Vasile C., 2012)

Solution. Assume that $a = \max\{a,b,c\}$. We can get the desired inequality by summing the inequalities

$$\sqrt{a^2+b^2+7bc} + \sqrt{b^2+c^2+7ca} \ge \sqrt{a^2+b^2+7ca} + \sqrt{b^2+c^2+7bc}$$

and

$$\sqrt{a^2+c^2+7ab}+\sqrt{a^2+b^2+7ac}\ge 3\sqrt{3(ab+bc+ca)}-\sqrt{b^2+c^2+7bc}.$$

By squaring, the first inequality becomes

$$(a^2+b^2+7b)(b^2+c^2+7ca)\ge(a^2+b^2+7ca)(b^2+c^2+7bc),$$

$$c(a-b)(a^2-c^2)\ge 0.$$

Similarly, the second inequality becomes

$$a^2+\sqrt{E}+3\sqrt{3F}\ge 10a(b+c)+17bc,$$

where

$$\begin{aligned}E&=(a^2+c^2+7ab)(a^2+b^2+7ac)\\&=a^4+7(b+c)a^3+(b^2+c^2+49bc)a^2+7(b^3+c^3)a+b^2c^2\end{aligned}$$

and

$$F=(ab+bc+ca)(b^2+c^2+7bc).$$

Due to homogeneity, we may assume that $b+c=1$. Let us denote $x=bc$. We need to show that $f(x)\ge 0$ for $0\le x\le\dfrac{1}{4}$ and $a\ge\dfrac{1}{2}$, where

$$f(x)=a^2-10a-17x+\sqrt{g(x)}+3\sqrt{3h(x)},$$

with

$$\begin{aligned}g(x)&=a^4+7a^3+(1+47x)a^2+7(1-3x)a+x^2\\&=x^2+a(47a-21)x+a^4+7a^3+a^2+7a,\end{aligned}$$

$$h(x)=(a+x)(1+5x)=5x^2+(5a+1)x+a.$$

We have the derivatives

$$\begin{aligned}f'(x)&=-17+\frac{g'}{2\sqrt{g}}+\frac{3\sqrt{3}h'}{2\sqrt{h}}\\&=-17+\frac{2x+a(47a-21)}{2\sqrt{g}}+\frac{3\sqrt{3}(10x+5a+1)}{2\sqrt{h}},\end{aligned}$$

$$\begin{aligned}f''(x)&=\frac{2g''g-(g')^2}{4g\sqrt{g}}+\frac{3\sqrt{3}[2h''h-(h')^2]}{4h\sqrt{h}}\\&=\frac{a(28-45a)(7a-1)^2}{4g\sqrt{g}}-\frac{3\sqrt{3}(5a-1)^2}{4h\sqrt{h}}.\end{aligned}$$

We will show that $g \geq 3h$. Since $0 \leq x \leq \dfrac{1}{4}$ and $a \geq \dfrac{1}{2}$, we have

$$\begin{aligned} g - 3h &= -14x^2 + (47a^2 - 36a - 3)x + a^4 + 7a^3 + a^2 + 4a \\ &\geq -\frac{7}{8} + (47a^2 - 36a - 3)x + a^4 + 7a^3 + a^2 + 4a. \end{aligned}$$

For the non-trivial case $47a^2 - 36a - 3 < 0$, we get

$$\begin{aligned} g - 3h &\geq -\frac{7}{8} + \frac{47a^2 - 36a - 3}{4} + a^4 + 7a^3 + a^2 + 4a \\ &= \frac{(2a-1)(4a^3 + 30a^2 + 66a + 13)}{8} \geq 0. \end{aligned}$$

We will prove now that $f''(x) < 0$. This is clearly true for $a \geq \dfrac{28}{45}$. Otherwise, for $\dfrac{1}{2} \leq a \leq \dfrac{28}{45}$, we have

$$f''(x) \leq \frac{a(28 - 45a)(7a-1)^2 - 27(5a-1)^2}{4g\sqrt{g}} < 0,$$

since

$$a(28 - 45a)(7a-1)^2 - 27(5a-1)^2 < \left(28 - \frac{45}{2}\right)(7a-1)^2 - 27(5a-1)^2$$

$$< \frac{27}{4}(7a-1)^2 - 27(5a-1)^2 = \frac{27(1-3a)(17a-3)}{4} < 0.$$

Since f is concave, it suffices to show that $f(0) \geq 0$ and $f\left(\dfrac{1}{4}\right) \geq 0$.

From

$$f(0) = \sqrt{a}\left(a\sqrt{a} - 10\sqrt{a} + 3\sqrt{3} + \sqrt{a^3 + 7a^2 + a + 7}\right),$$

it follows that $f(0) \geq 0$ for all $a \geq \dfrac{1}{2}$ if and only if

$$\sqrt{a^3 + 7a^2 + a + 7} \geq -a\sqrt{a} + 10\sqrt{a} - 3\sqrt{3}.$$

This is true if

$$a^3 + 7a^2 + a + 7 \geq (-a\sqrt{a} + 10\sqrt{a} - 3\sqrt{3})^2,$$

which is equivalent to

$$(\sqrt{3a} - 2)^2(9a + 10\sqrt{a} - 5) \geq 0.$$

Clearly, this inequality holds for $a \geq \dfrac{1}{2}$.

Since

$$g\left(\frac{1}{4}\right)=\left(\frac{4a^2+14a+1}{4}\right)^2$$

and

$$h\left(\frac{1}{4}\right)=\frac{9(4a+1)}{16},$$

we get

$$f\left(\frac{1}{4}\right)=\frac{8a^2-26a-16+9\sqrt{3(4a+1)}}{4}.$$

Using the substitution

$$x=\sqrt{\frac{4a+1}{3}},\quad x\ge 1,$$

we find

$$f\left(\frac{1}{4}\right)=\frac{9x^4-45x^2+54x-18}{8}=\frac{(x-1)^2(9x^2+18x-18)}{8}\ge 0.$$

Thus, the proof is completed. The equality holds for $a=b=c$, and also for $3a=4b$ and $c=0$ (or any cyclic permutation).

□

P 1.97. *If a,b,c are positive real numbers, then*

$$\frac{a^2+3ab}{(b+c)^2}+\frac{b^2+3bc}{(c+a)^2}+\frac{c^2+3ca}{(a+b)^2}\ge 3.$$

Solution. Write the inequality as

$$\sum\frac{a(a+b)}{(b+c)^2}+2\sum\frac{ab}{(b+c)^2}\ge 3.$$

The sequences

$$\{bc,\ \ ca,\ \ ab\}$$

and

$$\left\{\frac{1}{(b+c)^2},\ \ \frac{1}{(c+a)^2},\ \ \frac{1}{(a+b)^2}\right\}$$

are reversely ordered. Thus, by the rearrangement inequality, we have

$$\sum\frac{bc}{(b+c)^2}\le\sum\frac{ab}{(b+c)^2}.$$

Therefore, it suffices to show that

$$\sum\frac{a(a+b)}{(b+c)^2}+\sum\frac{b(c+a)}{(b+c)^2}\ge 3,$$

which is equivalent to

$$\sum a\left[\frac{a+b}{(b+c)^2}+\sum\frac{b+c}{(a+b)^2}\right]\ge 3.$$

By the AM-GM inequality, we have

$$\frac{a+b}{(b+c)^2}+\frac{b+c}{(a+b)^2}\ge\frac{2}{\sqrt{(a+b)(b+c)}}\ge\frac{4}{(a+b)+(b+c)}.$$

Thus, it is enough to prove that

$$\sum\frac{a}{a+2b+c}\ge\frac{3}{4}.$$

Indeed, by the Cauchy-Schwarz inequality, we get

$$\sum\frac{a}{a+2b+c}\ge\frac{\left(\sum a\right)^2}{\sum a(a+2b+c)}=\frac{\sum a^2+2\sum ab}{\sum a^2+3\sum ab}\ge\frac{3}{4}.$$

The equality holds for $a=b=c$.

□

P 1.98. *If a,b,c are positive real numbers, then*

$$\frac{a^2b+1}{a(b+1)}+\frac{b^2c+1}{b(c+1)}+\frac{c^2a+1}{c(a+1)}\ge 3.$$

Solution. By the Cauchy-Schwarz inequality, we have

$$(a^2b+1)\left(\frac{1}{b}+1\right)\ge(a+1)^2,$$

hence

$$\frac{a^2b+1}{a(b+1)}\ge\frac{b(a+1)^2}{a(b+1)^2}.$$

Therefore, it suffices to prove that

$$\sum\frac{b(a+1)^2}{a(b+1)^2}\ge 3.$$

This inequality follows immediately from the AM-GM inequality:

$$\sum\frac{b(a+1)^2}{a(b+1)^2}\ge 3\sqrt[3]{\prod\frac{b(a+1)^2}{a(b+1)^2}}=3.$$

The equality holds for $a=b=c=1$.

□

P 1.99. *If a,b,c are positive real numbers such that $a+b+c=3$, then*

$$\sqrt{a^3+3b}+\sqrt{b^3+3c}+\sqrt{c^3+3a}\geq 6.$$

Solution. By the Cauchy-Schwarz inequality, we have

$$(a^3+3b)(a+3b)\geq (a^2+3b)^2.$$

Thus, it suffices to show that

$$\sum \frac{a^2+3b}{\sqrt{a+3b}}\geq 6.$$

By Hölder's inequality, we have

$$\left(\sum \frac{a^2+3b}{\sqrt{a+3b}}\right)^2\left[\sum (a^2+3b)(a+3b)\right]\geq \left[\sum (a^2+3b)\right]^3=\left(\sum a^2+9\right)^3.$$

Therefore, it is enough to show that

$$\left(\sum a^2+9\right)^3\geq 36\sum (a^2+3b)(a+3b).$$

Let

$$p=a+b+c=3,\qquad q=ab+bc+ca,\qquad q\leq 3.$$

We have

$$\sum a^2+9=p^2-2q+9=2(9-q),$$

$$\begin{aligned}\sum (a^2+3b)(a+3b)&=\sum a^3+3\sum a^2b+9\sum a^2+3\sum ab\\&=(p^3-3pq+3abc)+3\sum a^2b+9(p^2-2q)+3q\\&=108-24q+3\left(abc+\sum a^2b\right).\end{aligned}$$

Since $abc+\sum a^2b\leq 4$ (see the inequality (a) in P 1.9), we get

$$\sum (a^2+3b)(a+3b)\leq 24(5-q).$$

Thus, it suffices to show that

$$(9-q)^3\geq 108(5-q),$$

which is equivalent to

$$(3-q)^2(21-q)\geq 0.$$

The equality holds for $a=b=c=1$.

□

P 1.100. *If a, b, c are positive real numbers such that $abc = 1$, then*

$$\sqrt{\frac{a}{a+6b+2bc}}+\sqrt{\frac{b}{b+6c+2ca}}+\sqrt{\frac{c}{c+6a+2ab}} \ge 1.$$

(Nguyen Van Quy and Vasile Cîrtoaje, 2013)

Solution. By Hölder's inequality, we have

$$\left(\sum\sqrt{\frac{a}{a+6b+2bc}}\right)^2\left[\sum a(a+6b+2bc)\right] \ge \left(\sum a^{2/3}\right)^3.$$

Therefore, it suffices to show that

$$\left(\sum a^{2/3}\right)^3 \ge \sum a^2 + 6\sum ab + 6,$$

which is equivalent to

$$3\sum(ab)^{2/3}(a^{2/3}+b^{2/3}) \ge 6\sum ab.$$

Since

$$a^{2/3}+b^{2/3} \ge 2(ab)^{1/3},$$

the desired conclusion follows. The equality holds for $a = b = c = 1$.

□

P 1.101. *If a, b, c are positive real numbers such that $abc = 1$, then*

$$\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2 \ge 6(a+b+c-1).$$

(Marius Stanean, 2014)

Solution *(by Michael Rozenberg)*. By the AM-GM inequality, we have

$$\begin{aligned}
\sum\left(a+\frac{1}{b}\right)^2+6 &= \sum(a+ac)^2+6 \\
&= \sum(a^2+a^2c^2+2a^2c)+6 \\
&= \sum(a^2+a^2b^2+2a^2c+2) \\
&\ge 6\sum\sqrt[6]{a^2\cdot a^2b^2\cdot a^2c\cdot a^2c\cdot 1\cdot 1} = 6\sum a.
\end{aligned}$$

The equality holds for $a = b = c = 1$.

□

P 1.102. *If a,b,c are positive real numbers, then*

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}\ge\frac{a+b+c}{a+b+c-\sqrt[3]{abc}}.$$

(Michael Rozenberg, 2014)

Solution. There are two cases to consider.

Case 1: $ab+bc+ca\ge\sqrt[3]{abc}\,(a+b+c)$. By the Cauchy-Schwarz inequality, we have

$$\sum\frac{a}{a+b}\ge\frac{\left(\sum a\right)^2}{\sum a(a+b)}=\frac{(a+b+c)^2}{(a+b+c)^2-(ab+bc+ca)}.$$

Therefore, it suffices to show that

$$\frac{(a+b+c)^2}{(a+b+c)^2-(ab+bc+ca)}\ge\frac{a+b+c}{a+b+c-\sqrt[3]{abc}},$$

which is equivalent to

$$ab+bc+ca-\sqrt[3]{abc}\,(a+b+c)\ge 0.$$

Case 2: $\sqrt[3]{abc}\,(a+b+c)\ge ab+bc+ca$. By the Cauchy-Schwarz inequality, we have

$$\sum\frac{a}{a+b}\ge\frac{\left(\sum ac\right)^2}{\sum ac^2(a+b)}=\frac{(ab+bc+ca)^2}{(ab+bc+ca)^2-abc(a+b+c)}.$$

Thus, it suffices to show that

$$\frac{(ab+bc+ca)^2}{(ab+bc+ca)^2-abc(a+b+c)}\ge\frac{a+b+c}{a+b+c-\sqrt[3]{abc}},$$

which is equivalent to

$$\left[\sqrt[3]{abc}\,(a+b+c)\right]^2\ge(ab+bc+ca)^2,$$

$$\sqrt[3]{abc}\,(a+b+c)\ge ab+bc+ca.$$

The proof is completed. The equality does not hold.

□

P 1.103. *If a,b,c are positive real numbers such that $a+b+c=3$, then*

$$a\sqrt{b^2+b+1}+b\sqrt{c^2+c+1}+c\sqrt{a^2+a+1}\le 3\sqrt{3}.$$

(Nguyen Van Quy, 2014)

Solution. From

$$4(b^2+b+1)=2(b+1)^2+2(b^2+1)\geq 3(b+1)^2,$$

we get

$$\sqrt{b^2+b+1}\geq \frac{\sqrt{3}}{2}(b+1),$$

hence

$$\sum a\sqrt{b^2+b+1}=\sum \frac{a(b^2+b+1)}{\sqrt{b^2+b+1}}\leq \sum \frac{2a(b^2+b+1)}{\sqrt{3}(b+1)}.$$

Therefore, it suffices to prove that

$$\sum \frac{a(b^2+b+1)}{b+1}\leq \frac{9}{2},$$

which is equivalent to

$$\sum \frac{ab^2}{b+1}\leq \frac{3}{2}.$$

In addition, since $b+1\geq 2\sqrt{b}$, it is enough to show that

$$\sum ab^{3/2}\leq 3.$$

Replacing a,b,c by a^2,b^2,c^2, respectively, we need to show that $a^2+b^2+c^2=3$ involves $a^2b^3+b^2c^3+c^2a^3\leq 3$, which is just the inequality in P 1.7. The equality holds for $a=b=c$.

□

P 1.104. *If a,b,c are positive real numbers, then*

$$\frac{1}{b(a+2b+3c)^2}+\frac{1}{c(b+2c+3a)^2}+\frac{1}{a(c+2a+3b)^2}\leq \frac{1}{12abc}.$$

(Vo Quoc Ba Can, 2012)

Solution. Assume that $a=\max\{a,b,c\}$, and write the inequality as

$$\frac{ca}{(a+2b+3c)^2}+\frac{ab}{(b+2c+3a)^2}+\frac{bc}{(c+2a+3b)^2}\leq \frac{1}{12}.$$

Case 1: $a\geq b\geq c$. By the AM-GM inequality, we have

$$(a+2b+3c)^2\geq 4(2b+c)(2c+a);$$

thus, it suffices to show that

$$\sum \frac{ca}{(2b+c)(2c+a)}\leq \frac{1}{3},$$

which is equivalent to

$$3\sum ca(2a+b)\le(2a+b)(2b+c)(2c+a),$$

$$ab^2+bc^2+ca^2\le a^2b+b^2c+c^2a,$$

$$(a-b)(b-c)(c-a)\le 0.$$

Clearly, the last inequality is true.

Case 2: $a\ge c\ge b$. Since, by the AM-GM inequality,

$$(a+2b+3c)^2\ge 12c(a+2b),$$

$$(b+2c+3a)^2\ge 4(2a+b)(2c+a),$$

$$(c+2a+3b)^2\ge 4(a+2b)(a+b+c),$$

it suffices to show that

$$\frac{a}{3(a+2b)}+\frac{ab}{(2a+b)(2c+a)}+\frac{bc}{(a+2b)(a+b+c)}\le\frac{1}{3},$$

which is equivalent to

$$\frac{ab}{(2a+b)(2c+a)}+\frac{bc}{(a+2b)(a+b+c)}\le\frac{2b}{3(a+2b)},$$

$$\frac{a}{(2a+b)(2c+a)}+\frac{c}{(a+2b)(a+b+c)}\le\frac{2}{3(a+2b)},$$

$$\frac{a(a+2b)}{(2a+b)(2c+a)}+\frac{c}{a+b+c}\le\frac{2}{3},$$

$$\frac{a(a+2b)}{2a+b}+\frac{c(2c+a)}{a+b+c}\le\frac{2(2c+a)}{3},$$

$$\frac{c(2c+a)}{a+b+c}-\frac{2(2c+a)}{3}\le\frac{3a^2}{2a+b}-2a,$$

$$f(c)\le f(a),$$

where

$$f(x)=\frac{x(2x+a)}{a+b+x}-\frac{2(2x+a)}{3}.$$

We have

$$\begin{aligned}f(a)-f(c)&=(a-c)\left[\frac{3a^2+4ac+b(3a+2c)}{(a+b+c)(2a+b)}-\frac{4}{3}\right]\\&=\frac{(a-c)[a^2-3ab-4b^2+2c(2a+b)]}{3(a+b+c)(2a+b)}\ge 0,\end{aligned}$$

because

$$a^2-3ab-4b^2+2c(2a+b)\ge a^2-3ab-4b^2+2b(2a+b)=(a-b)(a+2b)\ge 0.$$

The equality holds for $a=b=c$.

□

P 1.105. *Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that*

(a) $$\frac{a^2+9b}{b+c}+\frac{b^2+9c}{c+a}+\frac{c^2+9a}{a+b}\ge 15;$$

(b) $$\frac{a^2+3b}{a+b}+\frac{b^2+3c}{b+c}+\frac{c^2+3a}{c+a}\ge 6.$$

Solution. (a) Write the inequality as follows:

$$\sum\frac{a^2+3b(a+b+c)}{b+c}\ge 5(a+b+c),$$

$$\sum\left[\frac{a^2+3b(a+b+c)}{b+c}-3b\right]\ge 2(a+b+c),$$

$$\sum\frac{a^2+3ab}{b+c}\ge 2(a+b+c),$$

$$\sum\left(\frac{a^2+3ab}{b+c}-2a\right)\ge 0,$$

$$\sum\frac{a(a+b-2c)}{b+c}\ge 0,$$

$$\sum\frac{a(a-c)}{b+c}+\sum\frac{a(b-c)}{b+c}\ge 0,$$

$$\sum\frac{a(a-c)}{b+c}+\sum\frac{b(c-a)}{c+a}\ge 0,$$

$$\sum(a-c)\left(\frac{a}{b+c}-\frac{b}{c+a}\right)\ge 0,$$

$$(a+b+c)\sum\frac{(a-b)(a-c)}{(b+c)(c+a)}\ge 0.$$

Therefore, we need to show that

$$\sum(a^2-b^2)(a-c)\ge 0,$$

which is equivalent to the obvious inequality

$$\sum a(a-c)^2\ge 0.$$

The equality holds for $a=b=c$.

(b) Write the inequality as follows:

$$\sum\frac{a^2+b(a+b+c)}{a+b}\ge 2(a+b+c),$$

$$\sum \frac{a^2+bc}{a+b} \ge a+b+c,$$

$$\sum \left(\frac{a^2+bc}{a+b} - a \right) \ge 0,$$

$$\sum \frac{b(c-a)}{a+b} \ge 0,$$

$$\sum \frac{bc}{a+b} \ge \sum \frac{ab}{a+b}.$$

Since the sequences

$$\{ab, \quad bc, \quad ca\}$$

and

$$\left\{ \frac{1}{a+b}, \quad \frac{1}{b+c}, \quad \frac{1}{c+a} \right\}$$

are reversely ordered, the inequality follows from the rearrangement inequality. The equality holds for $a=b=c$.

□

P 1.106. *If $a,b,c \in [0,1]$, then*

(a) $$\frac{bc}{2ab+1} + \frac{ca}{2bc+1} + \frac{ab}{2ca+1} \le 1.$$

(b) $$\frac{a}{ab+1} + \frac{b}{bc+1} + \frac{c}{ca+1} \le \frac{3}{2}.$$

(Vasile C., 2010)

Solution. (a) ***First Solution.*** It suffices to prove that

$$\frac{bc}{2abc+1} + \frac{ca}{2abc+1} + \frac{ab}{2abc+1} \le 1;$$

that is,

$$2abc+1 \ge ab+bc+ca,$$

$$1-bc \ge a(b+c-2bc).$$

Since $a \le 1$ and

$$b+c-2bc = b(1-c)+c(1-b) \ge 0,$$

it suffices to show that

$$1-bc \ge b+c-2bc,$$

which is equivalent to

$$(1-b)(1-c) \ge 0.$$

The equality holds for $a = b = c = 1$, and for $a = 0$ and $b = c = 1$ (or any cyclic permutation).

Second Solution. Assume that $a = \max\{a, b, c\}$. It suffices to show that

$$\frac{bc}{2bc+1} + \frac{ca}{2bc+1} + \frac{ab}{2bc+1} \le 1;$$

that is,

$$a(b+c) \le 1 + bc.$$

We have

$$1 + bc - a(b+c) \ge 1 + bc - (b+c) = (1-b)(1-c) \ge 0.$$

(b) We will show that

$$E(a,b,c) \le E(1,b,c) \le E(1,1,c) = \frac{3}{2},$$

where

$$E(a,b,c) = \frac{a}{ab+1} + \frac{b}{bc+1} + \frac{c}{ca+1}.$$

Write the inequality $E(a,b,c) \le E(1,b,c)$ as follows:

$$\frac{a}{ab+1} + \frac{c}{ca+1} \le \frac{1}{b+1} + \frac{c}{c+1},$$

$$(1-a)\left[\frac{1}{(b+1)(ab+1)} - \frac{c^2}{(c+1)(ca+1)}\right] \ge 0,$$

$$(1-a)[(c+1)(ca+1) - (b+1)(ab+1)c^2] \ge 0.$$

Since $1 - a \ge 0$ and $c \le 1$, it suffices to show that

$$(c+1)(ca+1) - (b+1)(ab+1)c \ge 0,$$

which is true because

$$\begin{aligned}(c+1)(ca+1) - (b+1)(ab+1)c &\ge (c+1)(ca+1) - 2(a+1)c \\ &= (1-c)(1-ac) \ge 0.\end{aligned}$$

Setting $a = 1$ in the similar inequality

$$E(a,b,c) \le E(a,1,c),$$

it follows that

$$E(1,b,c) \le E(1,1,c).$$

Finally,

$$E(1,1,c) = \frac{1}{2} + \frac{1}{c+1} + \frac{c}{c+1} = \frac{3}{2}.$$

The equality holds for $a = b = 1$ (or any cyclic permutation).

□

P 1.107. *If a,b,c are nonnegative real numbers, then*

$$a^4+b^4+c^4+5(a^3b+b^3c+c^3a)\ge 6(a^2b^2+b^2c^2+c^2a^2).$$

Solution. Assume that $a=\min\{a,b,c\}$ and use the substitution

$$b=a+p,\quad c=a+q,\qquad p,q\ge 0.$$

The inequality becomes

$$9Aa^2+3Ba+C\ge 0,$$

where

$$A=p^2-pq+q^2,\quad B=3p^3+p^2q-4pq^2+3q^3,$$
$$C=p^4+5p^3q-6p^2q^2+q^4.$$

Since

$$A\ge 0,$$
$$B=3p(p-q)^2+q(7p^2-7pq+3q^2)\ge 0,$$
$$C=(p-q)^4+pq(3p-2q)^2\ge 0,$$

the inequality is obviously true. The equality occurs for $a=b=c$.

□

P 1.108. *If a,b,c are positive real numbers, then*

$$a^5+b^5+c^5-a^4b-b^4c-c^4a\ge 2abc(a^2+b^2+c^2-ab-bc-ca).$$

(Vasile C., 2006)

Solution. Since

$$5\left(\sum a^5-\sum a^4b\right)=\sum(4a^5+b^5-5a^4b)=\sum(a-b)^2(4a^3+3a^2b+2ab^2+b^3)$$

and

$$2\left(\sum a^2-\sum ab\right)=\sum(a-b)^2,$$

we can write the inequality in the form

$$A(a-b)^2+B(b-c)^2+C(c-a)^2\ge 0,$$

where

$$A=4a^3+3a^2b+2ab^2+b^3-5abc,$$
$$B=4b^3+3b^2c+2bc^2+c^3-5abc,$$
$$C=4c^3+3c^2a+2ca^2+a^3-5abc.$$

Without loss of generality, assume that $a = \max\{a, b, c\}$. We have

$$A > a(4a^2 + 3ab - 5bc) > a(4c^2 + 3b^2 - 5bc) > 0,$$

$$C > a(3c^2 + 2ca + a^2 - 5bc) > a(3c^2 - 3ca + a^2) > 0,$$

$$\begin{aligned} A + B &> 4a^3 + 5b^3 + c^3 + 3a^2b + 2bc^2 - 10abc \\ &\geq 3\sqrt[3]{4a^3 \cdot 5b^3 \cdot c^3} + 2\sqrt{3a^2b \cdot 2bc^2} - 10abc \\ &= (3\sqrt[3]{20} + 2\sqrt{6} - 10)abc > 0, \end{aligned}$$

$$\begin{aligned} B + C &> a^3 + 4b^3 + 5c^3 + 3b^2c + 2ca^2 - 10abc \\ &\geq 3\sqrt[3]{a^3 \cdot 4b^3 \cdot 5c^3} + 2\sqrt{3b^2c \cdot 2ca^2} - 10abc \\ &= (3\sqrt[3]{20} + 2\sqrt{6} - 10)abc > 0. \end{aligned}$$

If $a \geq b \geq c$, then

$$\sum A(a-b)^2 \geq B(b-c)^2 + C(a-c)^2 \geq (B+C)(b-c)^2 \geq 0.$$

If $a \geq c \geq b$, then

$$\sum A(a-b)^2 \geq A(a-b)^2 + B(c-b)^2 \geq (A+B)(c-b)^2 \geq 0.$$

The equality holds for $a = b = c$.

□

P 1.109. *If a, b, c are positive real numbers such that $a^2 + b^2 + c^2 = 3$, then*

$$\frac{a}{1+b} + \frac{b}{1+c} + \frac{c}{1+a} \geq \frac{3}{2}.$$

(Vasile C., 2005)

Solution. Let

$$p = a + b + c, \quad q = ab + bc + ca, \quad p^2 = 3 + 2q.$$

First Solution. By the Cauchy-Schwarz inequality, we have

$$\sum \frac{a}{1+b} \geq \frac{\left(\sum a\right)^2}{\sum a(1+b)} = \frac{3+2q}{p+q}.$$

Thus, it suffices to prove that

$$6 + q \geq 3p.$$

Indeed,

$$2(6+q-3p) = 12+(p^2-3)-6p = (p-3)^2 \ge 0.$$

The equality holds for $a = b = c = 1$.

Second Solution. By the AM-GM inequality, we have

$$\sum \frac{a}{1+b} = \sum \frac{a(a+c)}{(1+b)(a+c)} \ge \sum \frac{4a(a+c)}{[(1+b)+(a+c)]^2}$$
$$= \frac{4\left(\sum a^2 + \sum ac\right)}{(1+p)^2} = \frac{4(3+q)}{(1+p)^2} = \frac{6+2p^2}{(1+p)^2}.$$

Therefore, it suffices to show that

$$\frac{6+2p^2}{(1+p)^2} \ge \frac{3}{2},$$

which is equivalent to $(p-3)^2 \ge 0$.

Conjecture. *If a,b,c are positive real numbers such that $a^2+b^2+c^2 = 3$, then*

$$\frac{a}{5+4b} + \frac{b}{5+4c} + \frac{c}{5+4a} \ge \frac{1}{3}.$$

□

P 1.110. *If a,b,c are nonnegative real numbers such that $a+b+c = 3$, then*

$$a\sqrt{a+b} + b\sqrt{b+c} + c\sqrt{c+a} \ge 3\sqrt{2}.$$

(Hong Ge Chen, 2011)

First Solution. Denote

$$q = \sqrt{\frac{ab+bc+ca}{3}}, \qquad q \le 1.$$

By squaring, the inequality turns into

$$\sum a^3 + \sum a^2 b + 2\sum ac\sqrt{a^2+3q^2} \ge 18.$$

Since

$$2\sqrt{a^2+3q^2} \ge a+3q,$$

we have

$$2\sum ac\sqrt{a^2+3q^2} \ge \sum ac(a+3q) = \sum ab^2 + 9q^3.$$

Thus, it suffices to show that

$$\sum a^3 + \sum ab(a+b) + 9q^3 \geq 18,$$

which is equivalent to

$$(a+b+c)(a^2+b^2+c^2) + 9q^3 \geq 18,$$

$$3(9-6q^2) + 9q^3 \geq 0,$$

$$1 - 2q^2 + q^3 \geq 0,$$

$$(1-q^2)^2 + q^3(1-q) \geq 0.$$

Clearly, the last inequality is true. The equality holds for $a = b = c = 1$.

Second Solution. Using the substitution

$$\sqrt{\frac{a+b}{2}} = \frac{x+y}{2}, \quad \sqrt{\frac{b+c}{2}} = \frac{y+z}{2}, \quad \sqrt{\frac{c+a}{2}} = \frac{z+x}{2}$$

gives

$$x = \sqrt{\frac{a+b}{2}} + \sqrt{\frac{a+c}{2}} - \sqrt{\frac{b+c}{2}} \geq 0,$$

$$a = \left(\frac{x+y}{2}\right)^2 + \left(\frac{x+z}{2}\right)^2 - \left(\frac{y+z}{2}\right)^2 = \frac{x(x+y+z)-yz}{2}.$$

In addition, $a + b + c = 3$ involves

$$x^2 + y^2 + z^2 + xy + yz + zx = 6,$$

which is equivalent to

$$p^2 - q = 6,$$

where

$$p = x + y + z, \qquad q = xy + yz + zx.$$

From

$$\begin{aligned} 18 - 2p^2 &= 3(x^2+y^2+z^2+xy+yz+zx) - 2(x+y+z)^2 \\ &= x^2+y^2+z^2-xy-yz-zx \geq 0, \end{aligned}$$

it follows that

$$p \leq 3.$$

The desired inequality is equivalent to

$$\sum (xp - yz)(x+y) \geq 12,$$

$$p\sum (x^2+xy) \geq 3xyz + \sum y^2 z + 12,$$

$$6p \ge 3xyz + \sum y^2z + 12,$$
$$6p + \sum yz^2 \ge pq + 12.$$

Since

$$\left(\sum yz^2\right)\left(\sum y\right) \ge \left(\sum yz\right)^2$$

(by the Cauchy-Schwarz inequality), it suffices to show that

$$6p + \frac{q^2}{p} \ge pq + 12.$$

Indeed,

$$6p + \frac{q^2}{p} - pq = \frac{p^2(6-q) + q^2}{p} = \frac{(6+q)(6-q) + q^2}{p} = \frac{36}{p} \ge 12.$$

Conjecture. *If a, b, c are nonnegative real numbers such that $a + b + c = 3$, then*

$$a\sqrt{4a+5b} + b\sqrt{4b+5c} + c\sqrt{4c+5a} \ge 9.$$

□

P 1.111. *If a, b, c are positive real numbers such that $a + b + c = 3$, then*

$$\frac{a}{2b^2+c} + \frac{b}{2c^2+a} + \frac{c}{2a^2+b} \ge 1.$$

(Vasile Cîrtoaje and Nguyen Van Quy, 2007)

Solution. By the Cauchy-Schwarz inequality, we have

$$\sum \frac{a}{2b^2+c} \ge \frac{\left(\sum a\sqrt{a+c}\right)^2}{\sum a(a+c)(2b^2+c)}.$$

Since $\sum a\sqrt{a+c} \ge 3\sqrt{2}$ (see the preceding P 1.110), it suffices to prove that

$$\sum a(a+c)(2b^2+c) \le 18,$$

which is equivalent to

$$2\sum a^2b^2 + 6abc + \sum ac(a+c) \le 18,$$
$$2\sum a^2b^2 + 3abc + \left(\sum a\right)\left(\sum ab\right) \le 18.$$

Denoting

$$q = ab + bc + ca,$$

the inequality becomes

$$9abc + 18 \geq 2q^2 + 3q.$$

This inequality is true for $q < 2$, because $18 > 2q^2 + 3q$. Since $q \leq p^2/3 = 3$, consider further the case $2 \leq q \leq 3$. By Schur's inequality of degree three, we have

$$9abc \geq 4pq - p^3 = 12q - 27.$$

Therefore,

$$\begin{aligned} 9abc + 18 - (2q^2 + 3q) &\geq (12q - 27) + 18 - (2q^2 + 3q) \\ &= -2q^2 + 9q - 9 = (3 - q)(2q - 3) \geq 0. \end{aligned}$$

This completes the proof. The equality holds for $a = b = c = 1$.

□

P 1.112. *If a, b, c are positive real numbers such that $a + b + c = ab + bc + ca$, then*

$$\frac{1}{a^2 + b + 1} + \frac{1}{b^2 + c + 1} + \frac{1}{c^2 + a + 1} \leq 1.$$

Solution. By the Cauchy-Schwarz inequality, we have

$$\frac{1}{a^2 + b + 1} \leq \frac{1 + b + c^2}{(a + b + c)^2},$$

hence

$$\sum \frac{1}{a^2 + b + 1} \leq \sum \frac{1 + b + c^2}{(a + b + c)^2} = \frac{3 + a + b + c + a^2 + b^2 + c^2}{(a + b + c)^2}.$$

It suffices to show that

$$3 + a + b + c \leq 2(ab + bc + ca),$$

which is equivalent to

$$a + b + c \geq 3.$$

We can get this inequality from the known inequality

$$(a + b + c)^2 \geq 3(ab + bc + ca).$$

The equality holds for $a = b = c = 1$.

□

P 1.113. *If a,b,c are positive real numbers, then*

$$\frac{1}{(a+2b+3c)^2}+\frac{1}{(b+2c+3a)^2}+\frac{1}{(c+2a+3b)^2}\le\frac{1}{4(ab+bc+ca)}.$$

Solution. By the AM-GM inequality, we have

$$\begin{aligned}(a+2b+3c)^2&=[(a+c)+2(b+c)]^2=(a+c)^2+4(b+c)^2+4(a+c)(b+c)\\&\ge 3(b+c)^2+6(a+c)(b+c)=3(b+c)(2a+b+3c).\end{aligned}$$

Thus, it suffices to show that

$$\sum\frac{1}{(b+c)(2a+b+3c)}\le\frac{3}{4(ab+bc+ca)}.$$

Write this inequality as follows:

$$\frac{3}{4}-\sum\frac{ab+bc+ca}{(b+c)(2a+b+3c)}\ge 0,$$

$$\sum\left[1-\frac{2(ab+bc+ca)}{(b+c)(2a+b+3c)}\right]\ge\frac{3}{2},$$

$$\sum\frac{(b+c)^2+2c^2}{(b+c)(2a+b+3c)}\ge\frac{3}{2},$$

$$\sum\frac{b+c}{2a+b+3c}+\sum\frac{2c^2}{(b+c)(2a+b+3c)}\ge\frac{3}{2}.$$

Applying the Cauchy-Schwarz inequality, we get

$$\sum\frac{b+c}{2a+b+3c}\ge\frac{[\sum(b+c)]^2}{\sum(b+c)(2a+b+3c)}=\frac{4\left(\sum a\right)^2}{4\left(\sum a\right)^2}=1$$

and

$$\sum\frac{c^2}{(b+c)(2a+b+3c)}\ge\frac{\left(\sum c\right)^2}{\sum(b+c)(2a+b+3c)}=\frac{1}{4},$$

from where the conclusion follows. The equality holds for $a=b=c$.

□

P 1.114. *If a,b,c are positive real numbers, then*

$$\sqrt{\frac{a}{a+b+2c}}+\sqrt{\frac{b}{b+c+2a}}+\sqrt{\frac{c}{c+a+2b}}\le\frac{3}{2}.$$

Solution. Apply the Cauchy-Schwarz inequality as follows:

$$\left(\sum\sqrt{\frac{a}{a+b+2c}}\right)^2 \le \left[\sum(b+c+2a)\right]\left[\sum\frac{a}{(b+c+2a)(a+b+2c)}\right]$$
$$= \frac{4\left(\sum a\right)\left[\sum a(c+a+2b)\right]}{(b+c+2a)(c+a+2b)(a+b+2c)}.$$

Thus, it suffices to show that

$$16\left(\sum a\right)\left[\sum a(c+a+2b)\right] \le 9(b+c+2a)(c+a+2b)(a+b+2c).$$

Denoting

$$p = a+b+c, \qquad q = ab+bc+ca,$$

the inequality becomes

$$16p(p^2+q) \le 9(p+a)(p+b)(p+c),$$

$$16p(p^2+q) \le 9(2p^3+pq+abc),$$
$$2p^3 - 7pq + 9abc \ge 0.$$

Using Schur's inequality of degree three

$$p^3 + 9abc \ge 4pq,$$

we have

$$2p^3 - 7pq + 9abc = (p^3 + 9abc - 4pq) + p(p^2 - 3q) \ge 0.$$

The equality holds for $a = b = c$.

□

P 1.115. *If a, b, c are positive real numbers, then*

$$\sqrt{\frac{5a}{a+b+3c}} + \sqrt{\frac{5b}{b+c+3a}} + \sqrt{\frac{5c}{c+a+3b}} \le 3.$$

Solution. Substituting

$$x = \sqrt{\frac{5a}{a+b+3c}}, \quad y = \sqrt{\frac{5b}{b+c+3a}}, \quad z = \sqrt{\frac{5c}{c+a+3b}},$$

we have

$$\begin{cases} (x^2-5)a + x^2b + 3x^2c = 0 \\ 3y^2a + (y^2-5)b + y^2c = 0 \\ z^2a + 3z^2b + (z^2-5)c = 0 \end{cases},$$

which involves

$$\begin{vmatrix} x^2-5 & x^2 & 3x^2 \\ 3y^2 & y^2-5 & y^2 \\ z^2 & 3z^2 & z^2-5 \end{vmatrix} = 0\ ;$$

that is,

$$F(x,y,z)=0,$$

where

$$F(x,y,z)=4x^2y^2z^2+2\sum x^2y^2+5\sum x^2-25.$$

We need to show that $F(x,y,z)=0$ involves $x+y+z\le 3$, where $x,y,z>0$. According to the contradiction method, assume that $x+y+z>3$ and show that $F(x,y,z)>0$. Since $F(x,y,z)$ is strictly increasing in each of its arguments, it is enough to prove that

$$x+y+z=3$$

involves

$$F(x,y,z)\ge 0.$$

Denote

$$q=xy+yz+zx, \qquad r=xyz.$$

Since

$$\sum x^2y^2=q^2-6r, \qquad \sum x^2=9-2q,$$

we have

$$F(x,y,z)=4r^2+2(q^2-6r)+5(9-2q)-25=2(2r^2-6r+q^2-5q+10),$$

$$\frac{1}{2}F(x,y,z)=2(r-1)^2+q^2-5q+8-2r.$$

It suffices to show that

$$q^2-5q+8\ge 2r.$$

From the known inequality

$$(xy+yz+zx)^2\ge 3xyz(x+y+z),$$

it follows that $q^2\ge 9r$. Therefore, it suffices to prove that

$$q^2-5q+8\ge \frac{2q^2}{9},$$

which is equivalent to

$$(3-q)(24-7q)\ge 0.$$

Since

$$q\le \frac{1}{3}(x+y+z)^2=3,$$

the conclusion follows. The original inequality is an equality for $a=b=c$.

□

P 1.116. *If $a,\ b,\ c \in [0,\ 1]$, then*

$$ab^2 + bc^2 + ca^2 + \frac{5}{4} \geq a + b + c.$$

(Ji Chen, 2007)

Solution. We use the substitution

$$a = 1 - x, \quad b = 1 - y, \quad c = 1 - z,$$

where $x, y, z \in [0,\ 1]$. Since

$$\begin{aligned}\sum a(1-b^2) &= \sum y(1-x)(2-y) = \sum y(2-2x-y+xy)\\ &= 2\sum x - \left(\sum x\right)^2 + \sum xy^2,\end{aligned}$$

the inequality can be written as

$$\frac{5}{4} \geq 2\sum x - \left(\sum x\right)^2 + \sum xy^2.$$

According to the known inequality in P 1.1, we have

$$\sum xy^2 \leq \frac{4}{27}\left(\sum x\right)^3.$$

Thus, it suffices to prove the following inequality

$$\frac{5}{4} \geq 2t - t^2 + \frac{4}{27}t^3,$$

where

$$t = x + y + z \leq 3.$$

This inequality is equivalent to

$$(15-4t)(3-2t)^2 \geq 0,$$

which is obviously true for $t \leq 3$. The proof is completed. The equality occurs for $a = 0$, $b = 1$ and $c = \frac{1}{2}$ (or any cyclic permutation thereof).

□

P 1.117. *If $a,\ b,\ c$ are nonnegative real numbers such that*

$$a + b + c = 3, \qquad a \leq b \leq 1 \leq c,$$

then

$$a^2b + b^2c + c^2a \leq 3.$$

Solution. Since

$$ab^2+bc^2+ca^2-(a^2b+b^2c+c^2)=(a-b)(b-c)(c-a)\ge 0,$$

it suffices to prove that

$$a^2b+b^2c+c^2+(ab^2+bc^2+ca^2)\le 6;$$

that is,

$$(a+b+c)(ab+bc+ca)-3abc\le 6,$$

$$ab+bc+ca-abc\le 2,$$

$$1-(a+b+c)+ab+bc+ca-abc\le 0,$$

$$(1-a)(1-b)(1-c)\le 0.$$

The equality occurs for $a=b=c=1$.

□

P 1.118. *Let a, b, c be nonnegative real numbers such that*

$$a+b+c=3,\qquad a\le 1\le b\le c.$$

Prove that

(a) $$a^2b+b^2c+c^2a\ge ab+bc+ca;$$

(b) $$a^2b+b^2c+c^2a\ge abc+2;$$

(c) $$\frac{1}{abc}+2\ge\frac{9}{a^2b+b^2c+c^2a};$$

(d) $$ab^2+bc^2+ca^2\ge 3.$$

(Vasile C., 2008)

Solution. (a) We have

$$\begin{aligned}a^2b+b^2c+c^2a-ab-bc-ca&=ab(a-1)+bc(b-1)+ca(c-1)\\&=-ab[(b-1)+(c-1)]+bc(b-1)+ca(c-1)\\&=b(b-1)(c-a)+a(c-1)(c-b)\ge 0.\end{aligned}$$

The equality holds for $a=b=c=1$, and also for $a=0$, $b=1$ and $c=2$.

(b) Since

$$a(b-a)(b-c)\le 0,$$

we have

$$a^2b+b^2c+c^2a \geq a^2b+b^2c+c^2a+a(b-a)(b-c)$$
$$= b^2(a+c)+ac(a+c-b).$$

Thus, it suffices to prove that

$$b^2(a+c)+ac(a+c-b) \geq abc+2.$$

This inequality is equivalent to

$$b^2(a+c)-2 \geq ac(2b-a-c),$$

$$b^2(3-b)-2 \geq ac(3b-3).$$

From $(b-a)(b-c) \leq 0$, it follows that

$$ac \leq b(a+c-b) = b(3-2b).$$

Thus, it suffices to show that

$$b^2(3-b)-2 \geq b(3-2b)(3b-3),$$

which is equivalent to the obvious inequality

$$(5b-2)(b-1)^2 \geq 0.$$

The equality holds for $a=b=c=1$, and also for $a=0$, $b=1$ and $c=2$.

(c) According to the inequality in (a), it suffices to show that

$$\frac{1}{abc}+2 \geq \frac{9}{abc+2},$$

which is equivalent to

$$(abc-1)^2 \geq 0.$$

The equality holds for $a=b=c=1$.

(d) Since

$$ab^2+bc^2+ca^2-(a^2b+b^2c+c^2) = (a-b)(b-c)(c-a) \geq 0,$$

it suffices to prove that

$$ab^2+bc^2+ca^2+(a^2b+b^2c+c^2) \geq 6;$$

that is,

$$(a+b+c)(ab+bc+ca)-3abc \geq 6,$$

$$ab+bc+ca-abc \geq 2,$$

$$1-(a+b+c)+ab+bc+ca-abc \ge 0,$$

$$(1-a)(1-b)(1-c) \ge 0.$$

The equality holds for $a=b=c=1$.

Remark 1. For

$$a+b+c=3, \qquad 0<a \le 1 \le b \le c,$$

the following *open inequality* holds

$$\frac{1}{abc}+6 \ge \frac{21}{a^2b+b^2c+c^2a},$$

which is sharper than the inequality in (c).

Remark 2. From the proof of the inequality in (d), the following identity follows for $a+b+c=3$:

$$2(ab^2+bc^2+ca^2-3)=3(1-a)(1-b)(1-c)+(a-b)(b-c)(c-a).$$

□

P 1.119. *If a, b, c are nonnegative real numbers such that*

$$a+b+c=3, \qquad a \le 1 \le b \le c,$$

then

(a)
$$\frac{5-2a}{1+b}+\frac{5-2b}{1+c}+\frac{5-2c}{1+a} \ge \frac{9}{2};$$

(b)
$$\frac{3-2b}{1+a}+\frac{3-2c}{1+b}+\frac{3-2a}{1+c} \le \frac{3}{2}.$$

(Vasile C., 2008)

Solution. (a) Write the inequality as follows:

$$2\sum(5-2a)(1+c)(1+a) \ge 9(1+a)(1+b)(1+c),$$

$$2\left(21+7\sum ab-2\sum ab^2\right) \ge 9\left(4+\sum ab+abc\right),$$

$$6+5\sum ab \ge 9abc+4\sum ab^2.$$

By P 1.9-(a), we have

$$\sum ab^2 \le 4-abc.$$

Therefore, it suffices to prove that

$$6+5\sum ab \ge 9abc+4(4-abc),$$

which is equivalent to

$$\sum ab \geq 2 + abc,$$

$$(1-a)(1-b)(1-c) \geq 0.$$

The equality holds for $a = b = c = 1$, and also for $a = 0$, $b = 1$, $c = 2$.

(b) Write the inequality as follows:

$$2\sum(3-2b)(1+b)(1+c) \leq 3(1+a)(1+b)(1+c),$$

$$2\left(3+5\sum ab - 2\sum a^2b\right) \leq 3\left(4+\sum ab + abc\right),$$

$$6 + 3abc + 4\sum a^2b \geq 7\sum ab,$$

$$6 + 3abc + 4\sum ab(a+b) \geq 7\sum ab + 4\sum ab^2,$$

$$6 + 3abc + 4\left(\sum a\right)\left(\sum ab\right) - 12abc \geq 7\sum ab + 4\sum ab^2,$$

$$6 + 5\sum ab \geq 9abc + 4\sum ab^2.$$

By P 1.9-(a), we have

$$\sum ab^2 \leq 4 - abc.$$

Therefore, it suffices to prove that

$$6 + 5\sum ab \geq 9abc + 4(4 - abc),$$

which is equivalent to

$$\sum ab \geq 2 + abc,$$

$$(1-a)(1-b)(1-c) \geq 0.$$

The equality holds for $a = b = c = 1$, and also for $a = 0$, $b = 1$, $c = 2$.

□

P 1.120. *If a, b, c are nonnegative real numbers such that*

$$ab + bc + ca = 3, \qquad a \leq 1 \leq b \leq c,$$

then

(a)
$$a^2b + b^2c + c^2a \geq 3;$$

(b)
$$ab^2 + bc^2 + ca^2 + 3(\sqrt{3}-1)abc \geq 3\sqrt{3}.$$

(Vasile C., 2008)

Solution. (a) Since

$$a(b-a)(b-c) \le 0,$$

we have

$$\begin{aligned}a^2b+b^2c+c^2a &\ge a^2b+b^2c+c^2a+a(b-a)(b-c)\\ &= b^2(a+c)+ac(a+c-b).\end{aligned}$$

Thus, it suffices to prove that

$$b^2(a+c)+ac(a+c-b) \ge 3.$$

Denote

$$x = a+c.$$

From $ab+bc+ca=3$, we get

$$ac = 3-bx$$

and

$$x = \frac{3-ac}{b} \le \frac{3}{b} \le 3.$$

Thus, we need to show that

$$b^2x+(3-bx)(x-b) \ge 3,$$

$$2b^2x-(x^2+3)b+3x-3 \ge 0.$$

Since

$$\begin{aligned}2b^2x-(x^2+3)b+3x-3 &= 2(b^2-2b+1)x+2(2b-1)x-(x^2+3)b+3x-3\\ &= 2(b-1)^2x+(3-x)(bx-b-1)\\ &\ge (3-x)(bx-b-1),\end{aligned}$$

it is enough to prove that

$$bx-b-1 \ge 0.$$

From the inequality $(b-a)(b-c) \le 0$, we get

$$bx \ge b^2+ac = b^2+3-bx, \quad bx \ge \frac{b^2+3}{2}.$$

Therefore,

$$bx-b-1 \ge \frac{b^2+3}{2}-b-1 = \frac{(b-1)^2}{2} \ge 0.$$

The proof is completed. The equality holds for $a=b=c=1$, and for $a=0$, $b=1$ and $c=3$.

(b) Since

$$ab^2+bc^2+ca^2-(a^2b+b^2c+c^2)=(a-b)(b-c)(c-a)\geq 0,$$

it suffices to prove that

$$ab^2+bc^2+ca^2+(a^2b+b^2c+c^2)+6(\sqrt{3}-1)abc\geq 6\sqrt{3};$$

that is,

$$(a+b+c)(ab+bc+ca)+3(2\sqrt{3}-3)abc\geq 6\sqrt{3},$$

$$a+b+c+(2\sqrt{3}-3)abc\geq 2\sqrt{3},$$

$$a[1+(2\sqrt{3}-3)bc]+b+c\geq 2\sqrt{3},$$

$$a[1+(2\sqrt{3}-3)p]+2(s-\sqrt{3})\geq 0,$$

where

$$s=\frac{b+c}{2},\quad p=bc,\qquad s^2\geq p\geq 1.$$

From $ab+bc+ca=3$, we get

$$a=\frac{3-p}{2s},\qquad p\leq 3.$$

Therefore, we need to show that $F(s,p)\geq 0$, where

$$F(s,p)=(3-p)[1+(2\sqrt{3}-3)p]+4s(s-\sqrt{3}).$$

Since the inequality $F(s,p)\geq 0$ is true for $s-\sqrt{3}\geq 0$, consider further the case

$$s\leq\sqrt{3}.$$

We will show that

$$F(s,p)\geq F(s,s^2)\geq 0.$$

We have

$$\begin{aligned}F(s,p)-F(s,s^2)&=(2\sqrt{3}-3)(s^4-p^2)-(6\sqrt{3}-10)(s^2-p)\\&=(s^2-p)[(2\sqrt{3}-3)(s^2+p)-6\sqrt{3}+10].\end{aligned}$$

Since $s^2-p\geq 0$ and

$$(2\sqrt{3}-3)(s^2+p)-6\sqrt{3}+10\geq(2\sqrt{3}-3)(1+1)-6\sqrt{3}+10=4-2\sqrt{3}>0,$$

the left inequality is true. The right inequality is also true because

$$\begin{aligned}F(s,s^2)&=(3-s^2)[1+(2\sqrt{3}-3)s^2]+4s(s-\sqrt{3})\\&=(\sqrt{3}-s)[(\sqrt{3}+s)(1+(2\sqrt{3}-3)s^2)-4s]\\&=(\sqrt{3}-s)[\sqrt{3}(1-s)^2(1+2s)-3s(1-s)^2]\\&=(\sqrt{3}-s)(1-s)^2[\sqrt{3}+(2\sqrt{3}-3)s]\geq 0.\end{aligned}$$

The equality holds for $a=b=c=1$, and also for $a=0$ and $b=c=\sqrt{3}$.

□

P 1.121. *If a, b, c are nonnegative real numbers such that*

$$a^2+b^2+c^2=3, \qquad a \le 1 \le b \le c,$$

then

(a) $$a^2b+b^2c+c^2a \ge 2abc+1;$$

(b) $$2(ab^2+bc^2+ca^2) \ge 3abc+3.$$

(Vasile C., 2008)

Solution. (a) Let

$$x=a+c, \qquad x \ge b.$$

From $a^2+b^2+c^2=3$, we get

$$ac=\frac{b^2+x^2-3}{2},$$

and from $(b-a)(b-c) \le 0$, we get

$$bx \ge b^2+ac,$$

$$bx \ge b^2+\frac{x^2+b^2-3}{2},$$

$$(x-b)^2 \le 3-2b^2, \qquad b \le \sqrt{\frac{3}{2}},$$

$$x \le b+d, \qquad d=\sqrt{3-2b^2}.$$

Since

$$a(b-a)(b-c) \le 0,$$

we have

$$\begin{aligned} a^2b+b^2c+c^2a &\ge a^2b+b^2c+c^2a+a(b-a)(b-c) \\ &= b^2x-ac(b-x). \end{aligned}$$

Thus, it suffices to prove that

$$b^2x-ac(3b-x) \ge 1,$$

which is equivalent to $f(x,b) \ge 0$, where

$$\begin{aligned} f(x,b) &= 2b^2x-(x^2+b^2-3)(3b-x)-2 \\ &= x^3-3bx^2+3(b^2-1)x-3b^3+9b-2. \end{aligned}$$

We will show that

$$f(x,b) \ge f(b+d,b) \ge 0.$$

Since $x \le b+d$ and

$$f(x,b)-f(b+d,b) = (x-b-d)[x^2+x(b+d)+(b+d)^2-3b(x+b+d)+3b^2-3]$$
$$= (x-b-d)[x^2-(2b-d)x-b^2-bd],$$

we need to show that $g(x) \le 0$, where

$$g(x) = x^2-(2b-d)x-b^2-bd = (x-2b)(x+d)+b(d-b).$$

Since $d-b \le 0$, it suffices to show that $x-2b \le 0$. Indeed, we have

$$x^2 = (a+c)^2 \le 2(a^2+c^2) = 2(3-b^2) \le 4,$$

hence

$$x \le 2 \le 2b.$$

To prove the right inequality $f(b+d,b) \ge 0$, we have

$$f(b+d,b) = 2b^2(b+d)-2bd(2b-d)-2 = 2(3b-b^3-1-b^2d).$$

We need to show that

$$3b-b^3-1 \ge b^2\sqrt{3-2b^2}$$

for

$$1 \le b \le \sqrt{\frac{3}{2}}.$$

We have

$$3b-b^3-1 \ge 3b-\frac{3b}{2}-1 = \frac{3b-2}{2} \ge 0.$$

By squaring, the inequality becomes

$$(3b-b^3-1)^2 \ge b^4(3-2b^2),$$

$$3b^6-9b^4+2b^3+9b^2-6b+1 \ge 0,$$

$$(b-1)^2(3b^4+6b^3-4b+1) \ge 0.$$

The original inequality is an equality for $a=b=c=1$.

(b) Denote

$$p = a+b+c, \qquad q = ab+bc+ca.$$

Since

$$ab^2+bc^2+ca^2-(a^2b+b^2c+c^2) = (a-b)(b-c)(c-a) \ge 0,$$

it suffices to prove that

$$ab^2+bc^2+ca^2+(a^2b+b^2c+c^2) \ge 3abc+3;$$

that is,

$$pq \ge 6abc + 3.$$

From

$$(a-1)(b-1)(c-1) \ge 0,$$

we get

$$abc \ge 1 - p + q,$$

therefore

$$\begin{aligned} pq - 6abc - 3 &\ge pq - 6(1 - p + q) - 3 \\ &= (p-6)q + 6p - 9 \\ &= \frac{(p-6)(p^2-3)}{2} + 6p - 9 \\ &= \frac{p(p-3)^2}{2} \ge 0. \end{aligned}$$

The equality holds for $a = b = c = 1$.

□

P 1.122. *If a, b, c are nonnegative real numbers such that*

$$ab + bc + ca = 3, \qquad a \le b \le 1 \le c,$$

then

$$ab^2 + bc^2 + ca^2 + 3abc \ge 6.$$

(Vasile C., 2008)

Solution. Denote

$$p = a + b + c.$$

Since

$$ab^2 + bc^2 + ca^2 - (a^2b + b^2c + c^2) = (a-b)(b-c)(c-a) \ge 0,$$

it suffices to prove that

$$ab^2 + bc^2 + ca^2 + (a^2b + b^2c + c^2) + 6abc \ge 12;$$

that is,

$$(a + b + c)(ab + bc + ca) + 3abc \ge 12,$$

$$a + b + c + abc \ge 4,$$

which is equivalent to

$$(a-1)(b-1)(c-1) \ge 0.$$

The equality holds for $a = b = c = 1$.

□

P 1.123. *If a, b, c are nonnegative real numbers such that*

$$a^2+b^2+c^2=3, \qquad a \le b \le 1 \le c,$$

then

$$2(a^2b+b^2c+c^2a) \le 3abc+3.$$

(Vasile C., 2008)

Solution. Consider two cases.

Case 1: $a+c \ge 2b$. Denote

$$x=a+c, \qquad x \ge 2b.$$

From $a^2+b^2+c^2=3$ and $(b-a)(b-c) \le 0$, we get in succession

$$ac=\frac{b^2+x^2-3}{2},$$

$$bx \ge b^2+ac,$$

$$bx \ge b^2+\frac{x^2+b^2-3}{2},$$

$$(x-b)^2 \le 3-2b^2,$$

$$x \le b+d, \qquad d=\sqrt{3-2b^2}.$$

Since

$$ab^2+bc^2+ca^2-(a^2b+b^2c+c^2)=(a-b)(b-c)(c-a) \ge 0,$$

it suffices to prove that

$$a^2b+b^2c+c^2a+(ab^2+bc^2+ca^2) \le 3abc+3;$$

that is,

$$(a+b+c)(ab+bc+ca) \le 6abc+3,$$

$$(x+b)(bx+ac) \le 6abc+3,$$

$$ac(x-5b)+bx(x+b)-3 \le 0.$$

Thus, we need to show that $f(x,b) \le 0$, where

$$\begin{aligned} f(x,b) &= (x^2+b^2-3)(x-5b)+2bx(x+b)-6 \\ &= x^3-3bx^2+3(b^2-1)x-5b^3+15b-6. \end{aligned}$$

We will show that

$$f(x,b) \le f(b+d,b) \le 0.$$

Since $x \le b+d$ and

$$f(x,b)-f(b+d,b) = (x-b-d)[x^2+x(b+d)+(b+d)^2-3b(x+b+d)+3b^2-3]$$
$$= (x-b-d)[x^2-(2b-d)x-b^2-bd],$$

we need to show that $g(x) \ge 0$, where

$$g(x) = x^2-(2b-d)x-b^2-bd.$$

Since $x-2b \ge 0$ and $d-b \ge 0$, we have

$$g(x) = (x-2b)(x+d)+b(d-b) \ge 0.$$

To prove the right inequality $f(b+d,b) \le 0$, from

$$f(b+d,b) = 2bd(d-4b)+2b(b+d)(2b+d)-6 = 2(6b-2b^3-3-b^2d),$$

it follows that we need to show that

$$6b-2b^3-3 \le b^2\sqrt{3-2b^2}$$

for $0 \le b \le 1$. This inequality is true for $b \le \dfrac{1}{2}$ because

$$6b-2b^3-3 \le 3(2b-1) \le 0.$$

So, it suffices to prove the inequality for $1/2 < b \le 1$. By squaring, the inequality becomes

$$(6b-2b^3-3)^2 \le b^4(3-2b^2),$$
$$2b^6-9b^4+4b^3+12b^2-12b+3 \le 0,$$
$$(b-1)^3(2b^3+6b^2+3b-3) \le 0.$$

We only need to show that

$$2b^3+6b^2+3b-3 \ge 0.$$

Indeed,

$$2b^3+6b^2+3b-3 > 3(2b^2+b-1) = 3(2b-1)(b+1) > 0.$$

Case 2: $a+c \le 2b$. Consider the nontrivial case $a<c$, denote

$$b_1 = \frac{a+c}{2}, \quad b_2 = \sqrt{\frac{a^2+c^2}{2}} \quad (b_1 < b_2),$$

and write the inequality in the homogeneous form $E(a,b,c) \le 0$, where

$$E(a,b,c) = 2(a^2b+b^2c+c^2a)-3abc-3\left(\frac{a^2+b^2+c^2}{3}\right)^{3/2}.$$

From $a^2+b^2+c^2=3$ and $b \le 1$, it follows that $b \le b_2$. For fixed a and c, consider the function

$$f(b)=E(a,b,c), \qquad b \in [b_1,b_2].$$

We will show that

$$f(b) \le f(b_2)) \le 0.$$

The left inequality is true if $f'(b) \ge 0$ for $b \in [b_1,b_2]$. Since

$$\begin{aligned}
f'(b) &= 2a^2+4bc-3ac-3b\left(\frac{a^2+b^2+c^2}{3}\right)^{1/2} \\
&= 2a^2+4bc-3ac-3b = 2a^2-3ac+b(4c-3) \\
&\ge 2a^2-3ac+\frac{(a+c)(4c-3)}{2} \\
&= \frac{(a-c)^2+3(a^2+c^2-a-c)}{2} \\
&\ge \frac{3(a^2+c^2-a-c)}{2},
\end{aligned}$$

it suffices to show that

$$a^2+c^2 \ge a+c.$$

From $a^2+b^2+c^2=3$ and $b \le 1$, it follows that $a^2+c^2 \ge 2$. If $a+c \le 2$, then

$$a^2+b^2 \ge 2 \ge a+c.$$

Also, if $a+c \ge 2$, then

$$a^2+b^2 \ge \frac{1}{2}(a+c)^2 \ge a+c.$$

To prove the right inequality $f(b_2) \le 0$, we see that

$$\begin{aligned}
f(b_2) &= 2a^2b_2+(a^2+c^2)c+2c^2a-3ab_2c-3b_2\frac{a^2+c^2}{2} \\
&= c(a+c)^2-\frac{(3c^2+6ac-a^2)}{2}b_2 \\
&= c(a+c)^2-\frac{(3c^2+6ac-a^2)}{2}\sqrt{\frac{a^2+c^2}{2}}.
\end{aligned}$$

Thus, we need to show that

$$c^2(c+a)^4 \le \frac{(3c^2+6ac-a^2)^2(c^2+a^2)}{8},$$

which is equivalent to

$$c^6+4ac^5-9a^2c^4-8a^3c^3+23a^4c^2-12a^5c+a^6 \ge 0,$$

$$(c-a)^3(c^3+7c^2a+9ca^2-a^3) \le 0.$$

The proof is completed. The equality holds for $a=b=c=1$.

$\square$

P 1.124. *If a, b, c are nonnegative real numbers such that*

$$a^2+b^2+c^2=3, \qquad a \le b \le 1 \le c,$$

then

$$2(a^3b+b^3c+c^3a) \le abc+5.$$

(Vasile C., 2008)

Solution. Let

$$p=a+b+c, \qquad q=ab+bc+ca.$$

Since

$$ab^3+bc^3+ca^3-(a^3b+b^3c+c^3)=(a+b+c)(a-b)(b-c)(c-a) \ge 0,$$

it suffices to prove that

$$(a^3b+b^3c+c^3a)+(ab^3+bc^3+ca^3) \le abc+5,$$

which is equivalent to

$$(a^2+b^2+c^2)(ab+bc+ca) \le abc(a+b+c+1)+5,$$

$$3q \le abc(p+1)+5.$$

From

$$(a-1)(b-1)(c-1) \ge 0,$$

we get

$$abc \ge q-p+1.$$

Therefore, it suffices to show that

$$3q \le (q-p+1)(p+1)+5,$$

which is equivalent to

$$6-p^2 \ge q(2-p),$$

$$12-2p^2 \ge (p^2-3)(2-p),$$

$$p^3-4p^2-3p+18 \ge 0,$$

$$(p-3)^2(p+2) \ge 0.$$

The proof is completed. The equality holds for $a=b=c=1$.

$\square$

P 1.125. *If a, b, c are real numbers, then*

$$(a^2+b^2+c^2)^2 \ge 3(a^3b+b^3c+c^3a).$$

(Vasile C., 1992)

First Solution. Write the inequality as

$$E_1 - 2E_2 \geq 0,$$

where

$$E_1 = a^3(a-b) + b^3(b-c) + c^3(c-a),$$
$$E_2 = a^2b(a-b) + b^2c(b-c) + c^2a(c-a).$$

Using the substitution

$$b = a + p, \quad c = a + q,$$

we have

$$\begin{aligned} E_1 &= a^3(a-b) + b^3[(b-a)+(a-c)] + c^3(c-a) \\ &= (a-b)^2(a^2+ab+b^2) + (a-c)(b-c)(b^2+bc+c^2) \\ &= p^2(a^2+ab+b^2) - q(p-q)(b^2+bc+c^2) \\ &= 3(p^2-pq+q^2)a^2 + 3(p^3-p^2q+q^3)a + p^4 - p^3q + q^4 \end{aligned}$$

and

$$\begin{aligned} E_2 &= a^2b(a-b) + b^2c[(b-a)+(a-c)] + c^2a(c-a) \\ &= (a-b)b(a^2-bc) + (a-c)c(b^2-ca) \\ &= pb(bc-a^2) + qc(ca-b^2) \\ &= (p^2-pq+q^2)a^2 + (p^3+p^2q-2pq^2+q^3)a + p^3q - p^2q^2. \end{aligned}$$

Thus, the inequality can be rewritten as

$$Aa^2 + Ba + C \geq 0,$$

where

$$A = p^2 - pq + q^2,$$
$$B = p^3 - 5p^2q + 4pq^2 + q^3,$$
$$C = p^4 - 3p^3q + 2p^2q^2 + q^4.$$

For the non-trivial case $A > 0$, it is enough to show that $\delta \leq 0$, where $\delta = B^2 - 4AC$ is the discriminant of the quadratic function $Aa^2 + Ba + C$. Indeed, we have

$$\begin{aligned} \delta &= -3(p^6 - 2p^5q - 3p^4q^2 + 6p^3q^3 + 2p^2q^4 - 4pq^5 + q^6) \\ &= -3(p^3 - p^2q - 2pq^2 + q^3)^2 \leq 0. \end{aligned}$$

The equality holds for $a = b = c$, and also for

$$\frac{a}{\sin^2 \frac{4\pi}{7}} = \frac{b}{\sin^2 \frac{2\pi}{7}} = \frac{c}{\sin^2 \frac{\pi}{7}}$$

(or any cyclic permutation).

Second Solution. Let us denote

$$x = a^2 - ab + bc,$$

$$y = b^2 - bc + ca,$$

$$z = c^2 - ca + ab.$$

We have

$$x^2 + y^2 + z^2 = \sum a^4 + 2\sum a^2b^2 - 2\sum a^3b$$

and

$$xy + yz + zx = \sum a^3b.$$

From the known inequality

$$x^2 + y^2 + z^2 \ge xy + yz + zx,$$

the desired inequality follows.

Third Solution. Let us denote

$$x = a(a - 2b - c),$$

$$y = b(b - 2c - a),$$

$$z = c(c - 2a - b).$$

We have

$$x^2 + y^2 + z^2 = \sum a^4 + 5\sum a^2b^2 + 4abc\sum a - 4\sum a^3b - 2\sum ab^3$$

and

$$xy + yz + zx = 3\sum a^2b^2 + 4abc\sum a - \sum a^3b - 2\sum ab^3.$$

The known inequality

$$x^2 + y^2 + z^2 \ge xy + yz + zx$$

leads to the desired inequality.

Remark 1. Let

$$E = (a^2 + b^2 + c^2)^2 - 3(a^3b + b^3c + c^3a).$$

Using the notations from the first solution, the formula

$$4A(Aa^2 + Ba + C) = (2Aa + B)^2 - \delta,$$

leads to the following identity

$$4E_1E = (A_1 - 5B_1 + 4C_1)^2 + 3(A_1 - B_1 - 2C_1 + 2D_1)^2,$$

where

$$A_1 = a^3 + b^3 + c^3, \quad B_1 = a^2b + b^2c + c^2a, \quad C_1 = ab^2 + bc^2 + ca^2, \quad D_1 = 3abc,$$

$$E_1 = a^2 + b^2 + c^2 - ab - bc - ca.$$

Remark 2. Let

$$E = (a^2 + b^2 + c^2)^2 - 3(a^3b + b^3c + c^3a),$$

The identity

$$x^2 + y^2 + z^2 - xy - yz - zx = \frac{1}{2}\sum(x-y)^2,$$

where x, y, z are defined in the second or third solution, leads to the identity

$$2E = \sum(a^2 - b^2 - ab + 2bc - ca)^2.$$

In addition, the following similar identities hold:

$$6E = \sum(2a^2 - b^2 - c^2 - 3ab + 3bc)^2,$$

$$4E = (2a^2 - b^2 - c^2 - 3ab + 3bc)^2 + 3(b^2 - c^2 - ab - bc + 2ca)^2.$$

Remark 3. The inequality in P 1.125 is known as *Vasc's inequality*, after the author's username on the Art of Problem Solving website.

□

P 1.126. *If a, b, c are real numbers, then*

$$a^4 + b^4 + c^4 + ab^3 + bc^3 + ca^3 \ge 2(a^3b + b^3c + c^3a).$$

(Vasile C., 1992)

First Solution. Making the substitution

$$b = a + p, \quad c = a + q,$$

the inequality turns into

$$Aa^2 + Ba + C \ge 0,$$

where

$$A = 3(p^2 - pq + q^2), \quad B = 3(p^3 - 2p^2q + pq^2 + q^3), \quad C = p^4 - 2p^3q + pq^3 + q^4.$$

Since the discriminant of the quadratic trinomial $Aa^2 + Ba + C$ is nonpositive,

$$\begin{aligned}\delta = B^2 - 4AC &= -3(p^6 - 6p^4q + 2p^3q^3 + 9p^2q^4 - 6pq^5 + q^6)\\ &= -3(p^3 - 3pq^2 + q^3)^2 \le 0,\end{aligned}$$

the conclusion follows. The equality holds for $a = b = c$, and also for

$$\frac{a}{\sin\frac{\pi}{9}} = \frac{b}{\sin\frac{7\pi}{9}} = \frac{c}{\sin\frac{13\pi}{9}}$$

(or any cyclic permutation).

Second Solution. Let us denote

$$x = a(a-b),$$

$$y = b(b-c),$$

$$z = c(c-a).$$

We have

$$x^2 + y^2 + z^2 = \sum a^4 + \sum a^2b^2 - 2\sum a^3b$$

and

$$xy + yz + zx = \sum a^2b^2 - \sum ab^3.$$

Applying the known inequality

$$x^2 + y^2 + z^2 \geq xy + yz + zx,$$

the desired inequality follows.

Third Solution. Let

$$x = a^2 + bc + ca,$$

$$y = b^2 + ca + ab,$$

$$z = c^2 + ab + bc.$$

We have

$$x^2 + y^2 + z^2 = \sum a^4 + 2\sum a^2b^2 + 4abc\sum a + 2\sum ab^3$$

and

$$xy + yz + zx = 2\sum a^2b^2 + 4abc\sum a + 2\sum a^3b + \sum ab^3.$$

The known inequality

$$x^2 + y^2 + z^2 \geq xy + yz + zx$$

leads to the desired inequality.

Remark 1. The inequality is more interesting in the case $abc < 0$. If a, b, c are positive, then the inequality is less sharp than Vasc's inequality in P 1.125, because it can be obtained by adding Vasc's inequality and

$$ab(a-b)^2 + bc(b-c)^2 + ca(c-a)^2 \geq 0.$$

On the other hand, if a, b, c are positive, then the inequality

$$3(a^4+b^4+c^4)+4(ab^3+bc^3+ca^3) \geq 7(a^3b+b^3c+c^3a)$$

is a refinement of the inequality in P 1.126. To prove this inequality, we write it as

$$3(a^4+b^4+c^4-a^3b-b^3c-c^3a)+4(ab^3+bc^3+ca^3-a^3b-b^3c-c^3a) \geq 0,$$

consider $a = \min\{a, b, c\}$ and use the substitution

$$b = a+p, \qquad c = a+q, \qquad a > 0,\ p \geq 0,\ q \geq 0.$$

Since

$$\begin{aligned}\sum a^4 - \sum a^3b &= \sum a^3(a-b) \\ &= 3(p^2-pq+q^2)a^2+3(p^3-p^2q+q^3)a+p^4-p^3q+q^4\end{aligned}$$

and

$$\begin{aligned}\sum ab^3 - \sum a^3b &= (a+b+c)(a-b)(b-c)(c-a) \\ &= pq(q-p)(3a+p+q),\end{aligned}$$

the inequality becomes

$$Aa^2+Ba+C \geq 0,$$

where

$$A = 9(p^2-pq+q^2), \quad B = 3(3p^3-7p^2q+4pq^2+3q^3),$$
$$C = 3p^4-7p^3q+4pq^3+3q^4.$$

The inequality $Aa^2+Ba+C \geq 0$ is true for $a > 0$ and $p, q \geq 0$, because

$$A \geq 0,$$

$$B = p(3p-4q)^2+q(p-3q)^2+2pq(p+q) \geq 0,$$

$$3C = p(p+q)(3p-5q)^2+5q^2\left(p-\frac{13q}{10}\right)^2+\frac{11}{20}q^4 \geq 0.$$

Remark 2. Let

$$E = a^4+b^4+c^4+ab^3+bc^3+ca^3-2(a^3b+b^3c+c^3a).$$

Using the notations from the first solution, the formula

$$4A(Aa^2+Ba+C) = (2Aa+B)^2-\delta$$

leads to the following identity

$$4E_1E = (A_1-3C_1+2D_1)^2+3(A_1-2B_1+C_1)^2,$$

where

$$A_1 = a^3 + b^3 + c^3, \quad B_1 = a^2b + b^2c + c^2a, \quad C_1 = ab^2 + bc^2 + ca^2, \quad D_1 = 3abc,$$

$$E_1 = a^2 + b^2 + c^2 - ab - bc - ca.$$

Remark 3. Let

$$E = a^4 + b^4 + c^4 + ab^3 + bc^3 + ca^3 - 2(a^3b + b^3c + c^3a).$$

The identity

$$x^2 + y^2 + z^2 - xy - yz - zx = \frac{1}{2}\sum(x-y)^2,$$

where x, y, z are defined in the second or third solution, leads to the identity

$$2E = \sum(a^2 - b^2 - ab + bc)^2.$$

In addition, the following similar identities hold:

$$6E = \sum(2a^2 - b^2 - c^2 - 2ab + bc + ca)^2,$$

$$4E = (2a^2 - b^2 - c^2 - 2ab + bc + ca)^2 + 3(b^2 - c^2 - bc + ca)^2.$$

Remark 4. The inequalities in P 1.125 and P 1.126 are particular cases of the following more general statement (*Vasile Cîrtoaje*, 2007).

- *Let*

$$f_4(a,b,c) = \sum a^4 + A\sum a^2b^2 + Babc\sum a + C\sum a^3b + D\sum ab^3,$$

where A, B, C, D are real constants such that

$$1 + A + B + C + D = 0, \qquad 3(1+A) \ge C^2 + CD + D^2.$$

If a, b, c are real numbers, then

$$f_4(a,b,c) \ge 0.$$

Note that the following identity holds:

$$4Sf_4(a,b,c) = [U+V+(C+D)S]^2 + 3\left(U - V + \frac{C-D}{3}S\right)^2 + \frac{4}{3}(3+3A-C^2-CD-D^2)S^2,$$

where

$$S = \sum a^2b^2 - \sum a^2bc,$$

$$U = \sum a^3b - \sum a^2bc,$$

$$V = \sum ab^3 - \sum a^2bc.$$

For the main case

$$3(1+A) = C^2 + CD + D^2,$$

the inequality $f_4(a,b,c) \geq 0$ is equivalent to each of the following two inequalities

$$\sum[2a^2 - b^2 - c^2 + Cab - (C+D)bc + Dca]^2 \geq 0,$$

$$\sum[3b^2 - 3c^2 + (C+2D)ab + (C-D)bc - (2C+D)ca]^2 \geq 0.$$

□

P 1.127. *If a,b,c are positive real numbers, then*

(a) $$\frac{a^2}{ab+2c^2} + \frac{b^2}{bc+2a^2} + \frac{c^2}{ca+2b^2} \geq 1;$$

(b) $$\frac{a^3}{a^2b+2c^3} + \frac{b^3}{b^2c+2a^3} + \frac{c^3}{c^2a+2b^3} \geq 1.$$

Solution. (a) By the Cauchy-Schwarz inequality, we have

$$\sum \frac{a^2}{ab+2c^2} \geq \frac{\left(\sum a^2\right)^2}{\sum a^2(ab+2c^2)} = \frac{\left(\sum a^2\right)^2}{\sum a^3b + 2\sum a^2b^2}.$$

Therefore, it suffices to show that

$$\left(\sum a^2\right)^2 \geq 2\sum a^2b^2 + \sum a^3b.$$

We get this inequality by summing the known inequality

$$\frac{2}{3}\left(\sum a^2\right)^2 \geq 2\sum a^2b^2$$

and Vasc's inequality

$$\frac{1}{3}\left(\sum a^2\right)^2 \geq \sum a^3b.$$

The equality holds for $a=b=c=1$.

(b) By the Cauchy-Schwarz inequality, we have

$$\sum \frac{a^3}{a^2b+2c^3} \geq \frac{\left(\sum a^2\right)^2}{\sum a(a^2b+2c^3)} = \frac{\left(\sum a^2\right)^2}{\sum a^3b + 2\sum ac^3} = \frac{\left(\sum a^2\right)^2}{3\sum a^3b}.$$

Therefore, it suffices to show that

$$\left(\sum a^2\right)^2 \geq 3\sum a^3b,$$

which is just Vasc's inequality. The equality holds for $a=b=c=1$.

□

P 1.128. *If a,b,c are positive real numbers such that $a+b+c=3$, then*

$$\frac{a}{ab+1}+\frac{b}{bc+1}+\frac{c}{ca+1}\ge\frac{3}{2}.$$

Solution. We use the following hint

$$\frac{a}{ab+1}=a-\frac{a^2b}{ab+1},\quad \frac{b}{bc+1}=b-\frac{b^2c}{bc+1},\quad \frac{c}{ca+1}=c-\frac{c^2a}{ca+1},$$

which transforms the desired inequality into

$$\frac{a^2b}{ab+1}+\frac{b^2c}{bc+1}+\frac{c^2a}{ca+1}\le\frac{3}{2}.$$

By the AM-GM inequality, we have

$$ab+1\ge 2\sqrt{ab},\quad bc+1\ge 2\sqrt{bc},\quad ca+1\ge 2\sqrt{ca}.$$

Consequently, it suffices to show that

$$\frac{a^2b}{2\sqrt{ab}}+\frac{b^2c}{2\sqrt{bc}}+\frac{c^2a}{2\sqrt{ca}}\le\frac{3}{2},$$

which is equivalent to

$$a\sqrt{ab}+b\sqrt{bc}+c\sqrt{ca}\le 3,$$

$$3(a\sqrt{ab}+b\sqrt{bc}+c\sqrt{ca})\le(a+b+c)^2.$$

Replacing $\sqrt{a},\sqrt{b},\sqrt{c}$ by a,b,c, respectively, we get Vasc's inequality in P 1.125. The equality holds for $a=b=c=1$.

□

P 1.129. *If a,b,c are positive real numbers such that $a+b+c=3$, then*

$$\frac{a}{3a+b^2}+\frac{b}{3b+c^2}+\frac{c}{3c+a^2}\le\frac{3}{2}.$$

(Vasile C., 2007)

Solution. Since

$$\frac{a}{3a+b^2}=\frac{1}{3}-\frac{b^2}{3(3a+b^2)},\quad \frac{b}{3b+c^2}=\frac{1}{3}-\frac{c^2}{3(3b+c^2)},\quad \frac{c}{3c+a^2}=\frac{1}{3}-\frac{a^2}{3(3c+a)},$$

the desired inequality can be rewritten as

$$\frac{b^2}{3a+b^2}+\frac{c^2}{3b+c^2}+\frac{a^2}{3c+a^2}\ge\frac{3}{2}.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{b^2}{3a+b^2} \ge \frac{\left(\sum b^2\right)^2}{\sum b^2(3a+b^2)} = \frac{\left(\sum a^2\right)^2}{\sum a^4 + \left(\sum a\right)\left(\sum ab^2\right)}$$

$$= \frac{\left(\sum a^2\right)^2}{\sum a^4 + \sum a^2b^2 + abc\sum a + \sum ab^3} \ge \frac{\left(\sum a^2\right)^2}{\left(\sum a^2\right)^2 + \sum ab^3}.$$

Thus, it is enough to show that

$$\left(\sum a^2\right)^2 \ge 3\sum ab^3,$$

which is Vasc's inequality. The equality holds for $a = b = c = 1$.

□

P 1.130. *If a, b, c are positive real numbers such that $a + b + c = 3$, then*

$$\frac{a}{b^2+c} + \frac{b}{c^2+a} + \frac{c}{a^2+b} \ge \frac{3}{2}.$$

(Pham Kim Hung, 2007)

Solution. By the Cauchy-Schwarz inequality, we have

$$\sum \frac{a}{b^2+c} \ge \frac{\left(\sum a^{3/2}\right)^2}{\sum a^2(b^2+c)} = \frac{\sum a^3 + 2\sum a^{3/2}b^{3/2}}{\sum a^2b^2 + \sum ab^2}.$$

Thus, it is enough to show that

$$2\sum a^3 + 4\sum a^{3/2}b^{3/2} \ge 3\sum a^2b^2 + 3\sum ab^2,$$

which is equivalent to the homogeneous inequality

$$2\left(\sum a\right)\left(\sum a^3\right) + 4\left(\sum a\right)\left(\sum a^{3/2}b^{3/2}\right) \ge 9\sum a^2b^2 + 3\left(\sum a\right)\left(\sum ab^2\right).$$

In order to get a symmetric inequality, we use Vasc's inequality. We have

$$\begin{aligned} 3\left(\sum a\right)\left(\sum ab^2\right) &= 3\sum a^2b^2 + 3abc\sum a + 3\sum ab^3 \\ &\le 3\sum a^2b^2 + 3abc\sum a + \left(\sum a^2\right)^2 \\ &= \sum a^4 + 5\sum a^2b^2 + 3abc\sum a. \end{aligned}$$

Therefore, it suffices to prove the symmetric inequality

$$2\left(\sum a\right)\left(\sum a^3\right) + 4\left(\sum a\right)\left(\sum a^{3/2}b^{3/2}\right) \ge 9\sum a^2b^2 + \sum a^4 + 5\sum a^2b^2 + 3abc\sum a,$$

which is equivalent to

$$\sum a^4 + 2\sum ab(a^2+b^2) + 4abc\sum\sqrt{ab} + 4A \ge 14\sum a^2b^2 + 3abc\sum a,$$

where

$$A = \sum (ab)^{3/2}(a+b).$$

Since

$$A \ge 2\sum a^2b^2,$$

it suffices to prove that

$$\sum a^4 + 2\sum ab(a^2+b^2) + 4abc\sum\sqrt{ab} \ge 6\sum a^2b^2 + 3abc\sum a.$$

According to Schur's inequality of degree four

$$\sum a^4 \ge \sum ab(a^2+b^2) - abc\sum a,$$

it is enough to show that

$$3\sum ab(a^2+b^2) + 4abc\sum\sqrt{ab} \ge 6\sum a^2b^2 + 4abc\sum a.$$

Write this inequality as

$$3\sum ab(a-b)^2 \ge 2abc\sum\left(\sqrt{a}-\sqrt{b}\right)^2,$$

$$\sum ab\left(\sqrt{a}-\sqrt{b}\right)^2\left[3\left(\sqrt{a}+\sqrt{b}\right)^2 - 2c\right] \ge 0.$$

We will prove the stronger inequality

$$\sum ab\left(\sqrt{a}-\sqrt{b}\right)^2\left[\left(\sqrt{a}+\sqrt{b}\right)^2 - c\right] \ge 0,$$

which is equivalent to

$$\sum\left(\frac{\sqrt{a}-\sqrt{b}}{\sqrt{c}}\right)^2\left(\sqrt{a}+\sqrt{b}-\sqrt{c}\right) \ge 0.$$

Substituting $x=\sqrt{a}$, $y=\sqrt{b}$, $z=\sqrt{c}$, the inequality becomes

$$\sum\left(\frac{x-y}{z}\right)^2(x+y-z) \ge 0.$$

Without loss of generality, assume that $x \ge y \ge z$. It suffices to show that

$$\left(\frac{y-z}{x}\right)^2(y+z-x) + \left(\frac{x-z}{y}\right)^2(z+x-y) \ge 0.$$

Since

$$\left(\frac{x-z}{y}\right)^2 \geq \left(\frac{y-z}{x}\right)^2,$$

we have

$$\left(\frac{y-z}{x}\right)^2 (y+z-x) + \left(\frac{x-z}{y}\right)^2 (z+x-y) \geq$$

$$\geq \left(\frac{y-z}{x}\right)^2 (y+z-x) + \left(\frac{y-z}{x}\right)^2 (z+x-y)$$

$$= 2z\left(\frac{y-z}{x}\right)^2 \geq 0.$$

The equality holds for $a = b = c = 1$.

□

P 1.131. *If a, b, c are positive real numbers such that $abc = 1$, then*

$$\frac{a}{b^3+2} + \frac{b}{c^3+2} + \frac{c}{a^3+2} \geq 1.$$

Solution. Using the substitution

$$a = \frac{x}{y}, \quad b = \frac{z}{x}, \quad c = \frac{y}{z}, \qquad x, y, z > 0,$$

the inequality turns into

$$\sum \frac{x^4}{y(2x^3+z^3)} \geq 1.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{x^4}{y(2x^3+z^3)} \geq \frac{\left(\sum x^2\right)^2}{\sum y(2x^3+z^3)} = \frac{\left(\sum x^2\right)^2}{2\sum x^3y + \sum xy^3}.$$

Thus, it is enough to show that

$$\left(\sum x^2\right)^2 \geq 2\sum x^3y + \sum xy^3.$$

According to Vasc's inequality, we have

$$\left(\sum x^2\right)^2 \geq 3\sum x^3y$$

and

$$\left(\sum x^2\right)^2 \geq 3\sum xy^3.$$

Thus, the conclusion follows. The equality holds for $a = b = c = 1$.

□

P 1.132. *Let a, b, c be positive real numbers such that*

$$a^m + b^m + c^m = 3,$$

where $m > 0$. Prove that

$$\frac{a^{m-1}}{b} + \frac{b^{m-1}}{c} + \frac{c^{m-1}}{a} \geq 3.$$

Solution. Making the substitution

$$x = a^{\frac{1}{k}}, \quad y = b^{\frac{1}{k}}, \quad z = c^{\frac{1}{k}},$$

where

$$k = \frac{2}{m}, \quad k > 0,$$

we need to show that $x^2 + y^2 + z^2 = 3$ yields

$$\frac{x^{2-k}}{y^k} + \frac{y^{2-k}}{z^k} + \frac{z^{2-k}}{x^k} \geq 3,$$

which is equivalent to

$$\frac{x^2}{(xy)^k} + \frac{y^2}{(yz)^k} + \frac{z^2}{(zx)^k} \geq 3.$$

Applying Jensen's inequality to the convex function $f(u) = \dfrac{1}{u^k}$, we get

$$\frac{x^2}{(xy)^k} + \frac{y^2}{(yz)^k} + \frac{z^2}{(zx)^k} \geq \frac{x^2 + y^2 + z^2}{\left(\dfrac{x^2 \cdot xy + y^2 \cdot yz + z^2 \cdot zx}{x^2 + y^2 + z^2}\right)^k}$$
$$= \frac{3^{k+1}}{(x^3 y + y^3 z + z^3 x)^k}.$$

Thus, it suffices to show that $x^3 y + y^3 z + z^3 x \leq 3$. This is just Vasc's inequality in P 1.125. The equality holds for $a = b = c = 1$.

$\square$

P 1.133. *If a, b, c are positive real numbers, then*

(a) $$\frac{1}{4a} + \frac{1}{4b} + \frac{1}{4c} + \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq 3\left(\frac{1}{3a+b} + \frac{1}{3b+c} + \frac{1}{3c+a}\right);$$

(b) $$\frac{1}{4a} + \frac{1}{4b} + \frac{1}{4c} + \frac{1}{a+3b} + \frac{1}{b+3c} + \frac{1}{c+3a} \geq 2\left(\frac{1}{3a+b} + \frac{1}{3b+c} + \frac{1}{3c+a}\right).$$

(Gabriel Dospinescu and Vasile Cîrtoaje, 2004)

Solution. We will prove that the following more general inequalities hold for $t \geq 0$:

$$\frac{t^{4a}}{4a}+\frac{t^{4b}}{4b}+\frac{t^{4c}}{4c}+\frac{t^{2a+2b}}{a+b}+\frac{t^{2b+2c}}{b+c}+\frac{t^{2c+2a}}{c+a}-3\left(\frac{t^{3a+b}}{3a+b}+\frac{t^{3b+c}}{3b+c}+\frac{t^{3c+a}}{3c+a}\right)\geq 0,$$

$$\frac{t^{4a}}{4a}+\frac{t^{4b}}{4b}+\frac{t^{4c}}{4c}+\frac{t^{a+3b}}{a+3b}+\frac{t^{b+3c}}{b+3c}+\frac{t^{c+3a}}{c+3a}-2\left(\frac{t^{3a+b}}{3a+b}+\frac{t^{3b+c}}{3b+c}+\frac{t^{3c+a}}{3c+a}\right)\geq 0.$$

For $t = 1$, we get the desired inequalities.

(a) Denoting the left hand side of the former inequality by $f(t)$, the inequality becomes $f(t) \geq f(0)$. This is true if $f'(t) \geq 0$ for $t > 0$. We have the derivative

$$tf'(t)=t^{4a}+t^{4b}+t^{4c}+2(t^{2a+2b}+t^{2b+2c}+t^{2c+2a})-3(t^{3a+b}+t^{3b+c}+t^{3c+a}).$$

Using the substitution $x = t^a$, $y = t^b$, $z = t^c$, the inequality $f'(t) \geq 0$ turns into

$$x^4+y^4+z^4+2(x^2y^2+y^2z^2+z^2x^2)\geq 3(x^3y+y^3z+z^3x),$$

which is Vasc's inequality in P 1.125. The equality holds for $a = b = c$.

(b) Similarly, we have the derivative

$$tf'(t)=t^{4a}+t^{4b}+t^{4c}+t^{a+3b}+t^{b+3c}+t^{c+3a}-2(t^{3a+b}+t^{3b+c}+t^{3c+a}).$$

Denoting $x = t^a$, $y = t^b$, $z = t^c$, the inequality $f'(t) \geq 0$ turns into

$$x^4+y^4+z^4+xy^3+yz^3+zx^3\geq 2(x^3y+y^3z+z^3x),$$

which is the the inequality in P 1.126. The equality holds for $a = b = c$.

□

P 1.134. *If a, b, c are positive real numbers such that $a^6 + b^6 + c^6 = 3$, then*

$$\frac{a^5}{b}+\frac{b^5}{c}+\frac{c^5}{a}\geq 3.$$

(Tran Quoc Anh, 2007)

Solution. By Hölder's inequality, we have

$$\left(\frac{a^5}{b}+\frac{b^5}{c}+\frac{c^5}{a}\right)^3\geq\frac{(a^6+b^6+c^6)^4}{a^9b^3+b^9c^3+c^9a^3}=\frac{81}{a^9b^3+b^9c^3+c^9a^3}.$$

Therefore, it suffices to show that

$$a^9b^3+b^9c^3+c^9a^3\leq 3.$$

This is equivalent to

$$3(a^9b^3+b^9c^3+c^9a^3)\leq(a^6+b^6+c^6)^2,$$

which is Vasc's inequality (see P 1.125). The equality holds for $a = b = c$.

□

P 1.135. *If a, b, c are positive real numbers such that $a^2 + b^2 + c^2 = 3$, then*

$$\frac{a^3}{a+b^5} + \frac{b^3}{b+c^5} + \frac{c^3}{c+a^5} \ge \frac{3}{2}.$$

(Marin Bancos, 2010)

Solution. Write the inequality as

$$\sum \left(\frac{a^3}{a+b^5} - a^2 \right) + \frac{3}{2} \ge 0,$$

$$\sum \frac{a^2b^5}{a+b^5} \le \frac{3}{2}.$$

Since

$$a + b^5 \ge 2\sqrt{ab^5},$$

it suffices to show that

$$\sum ab^2\sqrt{ab} \le 3.$$

In addition, since $2\sqrt{ab} \le a + b$, it suffices to prove that

$$\sum a^2b^2 + \sum ab^3 \le 6.$$

This is true since

$$\sum a^2b^2 \le \frac{1}{3}(a^2 + b^2 + c^2)^2 = 3,$$

and, according to Vasc's inequality,

$$\sum ab^3 \le \frac{1}{3}(a^2 + b^2 + c^2)^2 = 3.$$

The equality holds for $a = b = c = 1$.

□

P 1.136. *If a, b, c are real numbers such that $a^2 + b^2 + c^2 = 3$, then*

$$a^2b + b^2c + c^2a + 9 \ge 4(a + b + c).$$

(Vasile C., 2007)

First Solution *(by Nguyen Van Quy)*. Since

$$2a^2b = a^2(b^2 + 1) - a^2(b - 1)^2,$$

we have

$$\begin{aligned}4\sum a^2b &= 2\sum a^2b^2+2\sum a^2-2\sum a^2(b-1)^2\\ &= \left(\sum a^2\right)^2-\sum a^4+2\sum a^2-2\sum c^2(a-1)^2\\ &= 15-\sum a^4-2\sum c^2(a-1)^2.\end{aligned}$$

Therefore, we can write the desired inequality as follows:

$$\left[15-\sum a^4-2\sum c^2(a-1)^2\right]+36\ge 16\sum a,$$

$$\sum(17-16a-a^4)\ge 2\sum c^2(a-1)^2,$$

$$\sum(17-16a-a^4)+10\sum(a^2-1)\ge 2\sum c^2(a-1)^2,$$

$$\sum(7-16a+10a^2-a^4)\ge 2\sum c^2(a-1)^2,$$

$$\sum(a-1)^2(7-2a-a^2)\ge 2\sum c^2(a-1)^2,$$

$$\sum(a-1)^2(7-2a-a^2-2c^2)\ge 0.$$

Since

$$7-2a-a^2-2c^2=(a-1)^2+2(3-a^2-c^2)=(a-1)^2+2b^2\ge 0,$$

the conclusion follows. The equality holds for $a=b=c=1$.

Second Solution. Consider only the case where a,b,c are nonnegative and $a+b+c>0$. Multiplying both sides by $a+b+c$, the inequality can be restated as

$$(a+b+c)(a^2b+b^2c+c^2a)+9(a+b+c)\ge 4(a+b+c)^2.$$

Using the known inequality $\sum a^2b^2\ge\frac{1}{3}\left(\sum ab\right)^2$ and Vasc's inequality $\sum ab^3\le\frac{1}{3}\left(\sum a^2\right)^2$, we have

$$\begin{aligned}\left(\sum a\right)\left(\sum a^2b\right) &= \sum a^3b+\sum a^2b^2+abc\sum a\\ &= \left(\sum a^2\right)\left(\sum ab\right)+\sum a^2b^2-\sum ab^3\\ &\ge \left(\sum a^2\right)\left(\sum ab\right)+\frac{1}{3}\left(\sum ab\right)^2-\frac{1}{3}\left(\sum a^2\right)^2\\ &= 3\sum ab+\frac{1}{3}\left(\sum ab\right)^2-3.\end{aligned}$$

Therefore, it suffices to prove the symmetric inequality

$$3\sum ab+\frac{1}{3}\left(\sum ab\right)^2-3+9\sum a\ge 4\left(\sum a\right)^2.$$

Setting $\sum a = p$, which involves

$$\sum ab = \frac{p^2-3}{2},$$

the inequality becomes

$$\frac{3(p^2-3)}{2} + \frac{(p^2-3)^2}{12} - 3 + 9p \geq 4p^2,$$

$$(p-3)^2(p^2+6p-9) \geq 0.$$

The last inequality is true since

$$p^2 + 6p - 9 > 6p - 9 \geq 6\sqrt{a^2+b^2+c^2} - 9 = 6\sqrt{3} - 9 > 0.$$

$\square$

P 1.137. *If a, b, c are real numbers such that $a^2+b^2+c^2 = 3$, then*

$$a^2b + b^2c + c^2a + 3 \geq a + b + c + ab + bc + ca.$$

(Vasile C., 2007)

Solution. Write the inequality as follows:

$$\sum(1-ab) - \sum a(1-ab) \geq 0,$$

$$\sum(a^2+b^2+c^2-3ab) - \sum a(a^2+b^2+c^2-3ab) \geq 0,$$

$$3\left(\sum a^2 - \sum ab\right) - \sum a(a-b)^2 - \sum a(c^2-ab) \geq 0,$$

$$\frac{3}{2}\sum(a-b)^2 - \sum a(a-b)^2 \geq 0,$$

$$\sum(a-b)^2(3-2a) \geq 0.$$

Assume that

$$a = \max\{a, b, c\}.$$

For $3-2a \geq 0$, the inequality is clearly true. Consider now that $3-2a < 0$. Since

$$(a-b)^2 = [(a-c)+(c-b)]^2 \leq 2[(a-c)^2+(c-b)^2],$$

it suffices to show that

$$2[(a-c)^2+(c-b)^2](3-2a) + (b-c)^2(3-2b) + (c-a)^2(3-2c) \geq 0,$$

which can be rewritten as

$$(a-c)^2(9-4a-2c)+(b-c)^2(9-4a-2b)\geq 0.$$

This inequality is true because $9>4a+2c$ and $9>4a+2b$. For instance, the last inequality is true if $81>4(2a+b)^2$; indeed, we have

$$\frac{81}{4}-(2a+b)^2>15-(2a+b)^2=5(a^2+b^2+c^2)-(2a+b)^2=(a-2b)^2+5c^2\geq 0.$$

The equality holds for $a=b=c=1$.

Remark. The inequality in P 1.137 is sharper than the inequality in P 1.136, namely

$$a^2b+b^2c+c^2a+9\geq 4(a+b+c).$$

This claim is true if

$$a+b+c+ab+bc+ca-3\geq 4(a+b+c)-9;$$

that is,

$$ab+bc+ca+6\geq 3(a+b+c),$$

which is equivalent to

$$(a+b+c-3)^2\geq 0.$$

□

P 1.138. *If a,b,c are positive real numbers such that $a+b+c=3$, then*

$$\frac{12}{a^2b+b^2c+c^2a}\leq 3+\frac{1}{abc}.$$

(Vasile Cîrtoaje and ShengLi Chen, 2009)

Solution. Let

$$p=a+b+c=3,\quad q=ab+bc+ca,\quad r=abc\leq 1.$$

Write the inequality as

$$2(a^2b+b^2c+c^2a)\geq \frac{24r}{3r+1}.$$

From

$$\begin{aligned}(a-b)^2(b-c)^2(c-a)^2&=-27r^2+2(9pq-2p^3)r+p^2q^2-4q^3\\&=-27r^2+54(q-2)r+9q^2-4q^3,\end{aligned}$$

we get

$$(a-b)(b-c)(c-a) \le \sqrt{-27r^2+54(q-2)r+9q^2-4q^3},$$

hence

$$\begin{aligned}2(a^2b+b^2c+c^2a) &= \sum ab(a+b)-(a-b)(b-c)(c-a)\\ &= pq-3r-(a-b)(b-c)(c-a)\\ &\ge 3q-3r-\sqrt{-27r^2+54(q-2)r+9q^2-4q^3}.\end{aligned}$$

Therefore, it suffices to show that

$$3q-3r-\sqrt{-27r^2+54(q-2)r+9q^2-4q^3} \ge \frac{24r}{3r+1}.$$

which is equivalent to

$$3[(3r+1)q-3r^2-9r] \ge (3r+1)\sqrt{-27r^2+54(q-2)r+9q^2-4q^3}.$$

Before squaring this inequality, we need to show that $(3r+1)q-3r^2-9r \ge 0$. Using the known inequality $q^2 \ge 3pr$, we have

$$\begin{aligned}(3r+1)q-3r^2-9r &\ge 3(3r+1)\sqrt{r}-3r^2-9r\\ &= 3\sqrt{r}\,\left(1-\sqrt{r}\right)^3 \ge 0.\end{aligned}$$

By squaring, the desired inequality can be restated as

$$Aq^3+C \ge 3Bq,$$

where

$$A=4(3r+1)^2, \quad B=72r(3r+1)(r+1), \quad C=108r(r+1)(3r^2+12r+1).$$

By the AM-GM inequality,

$$Aq^3+C=Aq^3+\frac{C}{2}+\frac{C}{2} \ge 3\sqrt[3]{Aq^3\left(\frac{C}{2}\right)^2};$$

so, it is enough to show that

$$AC^2 \ge 4B^3,$$

which is equivalent to

$$(3r^2+12r+1)^2 \ge 32r(3r+1)(r+1).$$

Indeed,

$$(3r^2+12r+1)^2-32r(3r+1)(r+1)=(r-1)^2(3r-1)^2 \ge 0.$$

The equality holds for $a=b=c=1$, and also for $r=\frac{1}{3}$ and $q=\sqrt[3]{\frac{C}{2A}}=2$; that is, when a,b,c are the roots of the equation

$$x^3-3x^2+2x-\frac{1}{3}=0$$

such that $a\le b\le c$ or $b\le c\le a$ or $c\le a\le b$.

□

P 1.139. *If a,b,c are positive real numbers such that $a+b+c=3$, then*

$$\frac{24}{a^2b+b^2c+c^2a}+\frac{1}{abc}\ge 9.$$

(Vasile C., 2009)

Solution *(by Vo Quoc Ba Can).* Let us denote

$$p=a+b+c=3,\quad q=ab+bc+ca,\quad r=abc.$$

Write the inequality as

$$24r\ge(9r-1)(a^2b+b^2c+c^2a),$$

and consider further the nontrivial case

$$r\ge\frac{1}{9}.$$

From

$$\begin{aligned}(a-b)^2(b-c)^2(c-a)^2&=-27r^2+2(9pq-2p^3)r+p^2q^2-4q^3\\&=-27r^2+54(q-2)r+9q^2-4q^3,\end{aligned}$$

we get

$$-(a-b)(b-c)(c-a)\le\sqrt{-27r^2+54(q-2)r+9q^2-4q^3},$$

hence

$$\begin{aligned}2(a^2b+b^2c+c^2a)&=\sum ab(a+b)-(a-b)(b-c)(c-a)\\&=pq-3r-(a-b)(b-c)(c-a)\\&\le 3q-3r+\sqrt{-27r^2+54(q-2)r+9q^2-4q^3}.\end{aligned}$$

Therefore, it suffices to show that

$$48r\ge(9r-1)\left[3q-3r+\sqrt{-27r^2+54(q-2)r+9q^2-4q^3}\right],$$

which is true if

$$3[9r^2+15r-(9r-1)q] \ge (9r-1)\sqrt{-27r^2+54(q-2)r+9q^2-4q^3}.$$

We need first to show that $9r^2+15r-(9r-1)q \ge 0$. From Schur's inequality

$$p^3+9r \ge 4pq,$$

we get

$$q \le \frac{3(r+3)}{4},$$

hence

$$9r^2+15r-(9r-1)q \ge 9r^2+15r-\frac{3(r+3)(9r-1)}{4} = \frac{9(r-1)^2}{4} \ge 0.$$

By squaring the desired inequality, we get

$$Aq^3+C \ge 3Bq,$$

where

$$A=(9r-1)^2, \quad B=18r(9r-1)(3r+1), \quad C=27r(27r^3+99r^2+r+1).$$

Using the AM-GM inequality, we have

$$Aq^3+C = Aq^3+\frac{C}{2}+\frac{C}{2} \ge 3\sqrt[3]{Aq^3\left(\frac{C}{2}\right)^2};$$

thus, it is enough to show that

$$AC^2 \ge 4B^3,$$

which is equivalent to

$$(27r^3+99r^2+r+1)^2 \ge 32r(9r-1)(3r+1)^3,$$

$$729r^6-2430r^5+2943r^4-1476r^3+199r^2+34r+1 \ge 0,$$

$$(r-1)^2(27r^2-18r-1)^2 \ge 0.$$

The equality holds for $a=b=c=1$, and also for $r=\frac{3+2\sqrt{3}}{9}$ and $q=1+\sqrt{3}$; that is, when a,b,c are the roots of the equation

$$x^3-3x^2+(1+\sqrt{3})x-\frac{3+2\sqrt{3}}{9}=0$$

such that $a \ge b \ge c$ or $b \ge c \ge a$ or $c \ge a \ge b$.

□

P 1.140. *Let a,b,c be nonnegative real numbers such that*

$$2(a^2+b^2+c^2)=5(ab+bc+ca).$$

Prove that

(a) $$8(a^4+b^4+c^4)\ge 17(a^3b+b^3c+c^3a);$$

(b) $$16(a^4+b^4+c^4)\ge 34(a^3b+b^3c+c^3a)+81abc(a+b+c).$$

(Vasile C., 2011)

Solution. (a) Let

$$x=a^2+b^2+c^2,\quad y=ab+bc+ca,\qquad 2x=5y.$$

Since the equality holds for $a=2$, $b=1$, $c=0$ (when $abc=0$), we will use the inequality

$$a^2b^2+b^2c^2+c^2a^2\le y^2$$

to get

$$a^4+b^4+c^4=x^2-2(a^2b^2+b^2c^2+c^2a^2)\ge x^2-2y^2,$$

hence

$$a^4+b^4+c^4\ge x^2-2y^2=\frac{17}{144}(2x+y)^2.$$

Therefore, it suffices to prove that

$$(2x+y)^2\ge 18(a^3b+b^3c+c^3a).$$

We will show that this inequality holds for all nonnegative real numbers a,b,c. Assume that $a=\max\{a,b,c\}$. There are two possible cases: $a\ge b\ge c$ and $a\ge c\ge b$.

Case 1: $a\ge b\ge c$. Using the AM-GM inequality gives

$$2(a^3b+b^3c+c^3a)\le 2ab(a^2+bc+c^2)\le\left[\frac{2ab+(a^2+bc+c^2)}{2}\right]^2.$$

Therefore, it suffices to show that

$$2x+y\ge\frac{3}{2}(2ab+a^2+bc+c^2),$$

which is equivalent to the obvious inequality

$$(a-2b)^2+c(2a-b+c)\ge 0.$$

Case 2: $a\ge c\ge b$. Since

$$ab^3+bc^3+ca^3-(a^3b+b^3c+c^3a)=(a+b+c)(a-b)(b-c)(c-a)\ge 0,$$

we have

$$2(a^3b+b^3c+c^3a)\le (a^3b+b^3c+c^3a)+(ab^3+bc^3+ca^3)\le xy.$$

Thus, it suffices to prove that

$$(2x+y)^2\ge 9xy.$$

Since $x\ge y$, we get

$$(2x+y)^2-9xy=(x-y)(4x-y)\ge 0.$$

Thus, the proof is completed. The equality holds for $a=2b$ and $c=0$ (or any cyclic permutation).

(b) For $a=b=c=0$, the inequality is trivial. Otherwise, let us denote

$$p=a+b+c,\quad q=ab+bc+ca,\quad r=abc,$$

and write the inequality as

$$16\sum a^4\ge 17\sum ab(a^2+b^2)+17\left(\sum a^3b-\sum ab^3\right)+81abc\sum a.$$

Due to homogeneity, we may assume that $p=3$, which involves $q=2$. Since

$$abc\sum a=3r,$$

$$\begin{aligned}\sum a^4&=\left(\sum a^2\right)^2-2\sum a^2b^2\\&=(p^2-2q)^2-2q^2+4pr=17+12r,\end{aligned}$$

$$\begin{aligned}\sum ab(a^2+b^2)&=\left(\sum ab\right)\left(\sum a^2\right)-abc\sum a\\&=q(p^2-2q)-pr=10-3r,\end{aligned}$$

$$\begin{aligned}\sum a^3b-\sum ab^3&=-p(a-b)(b-c)(c-a)\\&\le p\sqrt{(a-b)^2(b-c)^2(c-a)^2}\\&=p\sqrt{p^2q^2-4q^3+2p(9q-2p^2)r-27r^2}\\&=3\sqrt{4-27r^2},\end{aligned}$$

it suffices to prove that

$$16(17+12r)\ge 17(10-3r)+51\sqrt{4-27r^2}+243r,$$

which is equivalent to the obvious inequality

$$2\ge\sqrt{4-27r^2}.$$

The equality holds for $a=2b$ and $c=0$ (or any cyclic permutation).

□

P 1.141. *Let a, b, c be nonnegative real numbers such that*

$$2(a^2+b^2+c^2)=5(ab+bc+ca).$$

Prove that

(a) $$2(a^3b+b^3c+c^3a)\ge a^2b^2+b^2c^2+c^2a^2+abc(a+b+c);$$

(b) $$11(a^4+b^4+c^4)\ge 17(a^3b+b^3c+c^3a)+129abc(a+b+c);$$

(c) $$a^3b+b^3c+c^3a\le \frac{14+\sqrt{102}}{8}(a^2b^2+b^2c^2+c^2a^2).$$

Solution. For $a=b=c=0$, the inequalities are trivial. Otherwise, let us denote

$$p=a+b+c,\quad q=ab+bc+ca,\quad r=abc.$$

Due to homogeneity, we may assume that $p=3$, which involves $q=2$. From

$$\begin{aligned}\left|\sum a^3b-\sum ab^3\right| &= |-p(a-b)(b-c)(c-a)|\\ &= p\sqrt{(a-b)^2(b-c)^2(c-a)^2}\\ &= p\sqrt{p^2q^2-4q^3+2p(9q-2p^2)r-27r^2}\\ &= 3\sqrt{4-27r^2},\end{aligned}$$

it follows that

$$-3\sqrt{4-27r^2}\le \sum a^3b-\sum ab^3\le 3\sqrt{4-27r^2}.$$

In addition, we have

$$abc\sum a=3r,$$

$$\sum a^2b^2=q^2-2pr=4-6r,$$

$$\sum ab(a^2+b^2)=q(p^2-2q)-pr=10-3r,$$

$$\sum a^4=p^4-4p^2q+2q^2+4pr=17+12r.$$

(a) Write the inequality as

$$\sum ab(a^2+b^2)+\left(\sum a^3b-\sum ab^3\right)\ge \sum a^2b^2+abc\sum a.$$

It suffices to prove that

$$10-3r-3\sqrt{4-27r^2}\ge 4-6r+3r,$$

which is equivalent to the obvious inequality

$$2 \geq \sqrt{4-27r^2}.$$

The equality holds for $a=0$ and $2b=c$ (or any cyclic permutation).

(b) Write the inequality as

$$22\sum a^4 \geq 17\sum ab(a^2+b^2)+17\left(\sum a^3b-\sum ab^3\right)+258abc\sum a.$$

It suffices to prove that

$$22(17+12r) \geq 17(10-3r)+51\sqrt{4-27r^2}+774r$$

for $0 \leq r \leq \dfrac{2}{3\sqrt{3}}$. Write this inequality as

$$4-9r \geq \sqrt{4-27r^2}.$$

We have $4-9r \geq 4-2\sqrt{3} > 0$. By squaring, the inequality becomes

$$(4-9r)^2 \geq 4-27r^2,$$

$$(3r-1)^2 \geq 0.$$

For $p=3$, the equality holds when $q=2$, $r=\dfrac{1}{3}$ and $(a-b)(b-c)(c-a) \leq 0$. In general, the equality holds when a,b,c are proportional to the roots of the equation

$$3x^3-9x^2+6x-1=0$$

and satisfy

$$(a-b)(b-c)(c-a) \leq 0.$$

This occurs when (*Wolfgang Berndt*)

$$a\sin^2\frac{\pi}{9}=b\sin^2\frac{2\pi}{9}=c\sin^2\frac{4\pi}{9}.$$

(c) Write the inequality as

$$\sum ab(a^2+b^2)+\left(\sum a^3b-\sum ab^3\right) \leq k(a^2b^2+b^2c^2+c^2a^2),$$

where

$$k=\frac{14+\sqrt{102}}{4}.$$

It suffices to prove that

$$10-3r+3\sqrt{4-27r^2} \leq k(4-6r),$$

where $r \le \dfrac{2}{3\sqrt{3}}$. Write this inequality as

$$3\sqrt{4-27r^2} \le 4k-10-3(2k-1)r.$$

We have

$$4k-10-3(2k-1)r \ge 4k-10-\frac{2(2k-1)}{\sqrt{3}} = 4\left(1-\frac{1}{\sqrt{3}}\right)k-10+\frac{2}{\sqrt{3}} > 0.$$

By squaring, the inequality becomes

$$9(4-27r^2) \le [4k-10-3(2k-1)r]^2,$$

which is equivalent to

$$(r-k_1)^2 \ge 0,$$

where

$$k_1 = \frac{2}{129}\sqrt{\frac{787+72\sqrt{102}}{3}} \approx 0.3483.$$

For $p=3$, the equality holds when $q=2$, $r=k_1$ and $(a-b)(b-c)(c-a) \le 0$. In general, the equality holds when a,b,c are proportional to the roots of the equation

$$x^3-3x^2+2x-k_1=0$$

and satisfy

$$(a-b)(b-c)(c-a) \le 0.$$

□

P 1.142. *If a,b,c are real numbers such that*

$$a^3b+b^3c+c^3a \le 0,$$

then

$$a^2+b^2+c^2 \ge k(ab+bc+ca),$$

where

$$k = \frac{1+\sqrt{21+8\sqrt{7}}}{2} \approx 3.7468.$$

(Vasile C., 2012)

Solution. Let us denote

$$p=a+b+c, \quad q=ab+bc+ca, \quad r=abc.$$

If $p=0$, then

$$3(ab+bc+ca) \le (a+b+c)^2 = 0,$$

hence
$$a^2+b^2+c^2 \ge 0 \ge k(ab+bc+ca).$$
Consider now that $p \ne 0$ and use the contradiction method. It suffices to prove that
$$a^2+b^2+c^2 < k(ab+bc+ca)$$
involves
$$a^3b+b^3c+c^3a > 0.$$
Since the statement remains unchanged by replacing a,b,c with $-a,-b,-c$, respectively, we may consider that $p>0$. In addition, due to homogeneity, we may assume that $p=1$. From the hypothesis $a^2+b^2+c^2 < k(ab+bc+ca)$, we get
$$q > \frac{1}{k+2}.$$
Write the desired inequality as
$$\sum ab(a^2+b^2)+\sum a^3b-\sum ab^3 > 0.$$
Since
$$\sum ab(a^2+b^2) = q(p^2-2q)-pr = q-2q^2-r$$
and
$$\sum a^3b-\sum ab^3 = -p(a-b)(b-c)(c-a) \ge -p\sqrt{(a-b)^2(b-c)^2(c-a)^2}$$
$$= -p\sqrt{p^2q^2-4q^3+2p(9q-2p^2)r-27r^2} = -\sqrt{q^2-4q^3+2(9q-2)r-27r^2},$$
it suffices to prove that
$$q-2q^2-r > \sqrt{q^2-4q^3+2(9q-2)r-27r^2}.$$
From $p^2 \ge 3q$, we get
$$\frac{1}{k+2} < q \le \frac{1}{3},$$
and from $q^2 \ge 3pr$, we get $r \le q^2/3$; therefore,
$$q-2q^2-r \ge q-2q^2-\frac{q^2}{3} = q\left(1-\frac{7q}{3}\right) > 0.$$
By squaring, the desired inequality can be restated as
$$(q-2q^2-r)^2 > q^2-4q^3+2(9q-2)r-27r^2,$$
$$7r^2+(1-5q+q^2)r+q^4 > 0.$$
This is true if the discriminant
$$D = (1-5q+q^2)^2-28q^4 = [1-5q+(1+2\sqrt{7})q^2][1-5q+(1-2\sqrt{7})q^2]$$

is negative. Since

$$1-5q+(1+2\sqrt{7})q^2=\left(1-\frac{5q}{2}\right)^2+\frac{8\sqrt{7}-21}{4}q^2>0,$$

we only need to show that $f(q)>0$, where

$$f(q)=(2\sqrt{7}-1)q^2+5q-1.$$

Since $q>\dfrac{1}{k+2}$, we have

$$f(q)>\frac{2\sqrt{7}-1}{(k+2)^2}+\frac{5}{k+2}-1=0.$$

For $p=1$, the equality holds when $(a-b)(b-c)(c-a)>0$ and

$$q=\frac{1}{k+2},\quad r=\frac{-q^2}{\sqrt{7}}=-\frac{1}{\sqrt{7}(k+2)^2}.$$

In general, the equality holds when a,b,c are proportional to the roots of the equation

$$w^3-w^2+\frac{1}{k+2}w+\frac{1}{\sqrt{7}(k+2)^2}=0$$

and satisfy $(a-b)(b-c)(c-a)>0$.

□

P 1.143. *If a,b,c are real numbers such that*

$$a^3b+b^3c+c^3a\ge 0,$$

then

$$a^2+b^2+c^2+k(ab+bc+ca)\ge 0,$$

where

$$k=\frac{-1+\sqrt{21+8\sqrt{7}}}{2}\approx 2.7468.$$

(Vasile C., 2012)

Solution. Let us denote

$$p=a+b+c,\quad q=ab+bc+ca,\quad r=abc.$$

At least two of a,b,c have the same sign; let b and c be these numbers. If $p=0$, then the hypothesis $a^3b+b^3c+c^3a\ge 0$ can be written as

$$-(b+c)^3b+b^3c-c^3(b+c)\ge 0.$$

Clearly, this inequality is satisfied only for $a = b = c = 0$, when the desired inequality is trivial. Consider further that $p \neq 0$ and use the contradiction method. It suffices to prove that

$$a^2 + b^2 + c^2 + k(ab + bc + ca) < 0$$

involves

$$a^3b + b^3c + c^3a < 0.$$

Since the statement remains unchanged by replacing a, b, c with $-a, -b, -c$, respectively, we may consider $p > 0$. In addition, due to homogeneity, we may assume $p = 1$. From the hypothesis $a^2 + b^2 + c^2 + k(ab + bc + ca) < 0$, we get

$$q < \frac{-1}{k-2} \approx -1.339.$$

Write the desired inequality as

$$\sum ab(a^2 + b^2) + \sum a^3b - \sum ab^3 < 0,$$

Since

$$\sum ab(a^2 + b^2) = q(p^2 - 2q) - pr = q - 2q^2 - r$$

and

$$\sum a^3b - \sum ab^3 = -p(a-b)(b-c)(c-a) \le p\sqrt{(a-b)^2(b-c)^2(c-a)^2}$$

$$= p\sqrt{p^2q^2 - 4q^3 + 2p(9q - 2p^2)r - 27r^2} = \sqrt{q^2 - 4q^3 + 2(9q-2)r - 27r^2},$$

it suffices to prove that

$$\sqrt{q^2 - 4q^3 + 2(9q-2)r - 27r^2} < r + 2q^2 - q.$$

Since $q < -1$, we have

$$\frac{1-2q}{3} > 1,$$

hence

$$r^2 = a^2b^2c^2 \le \left(\frac{a^2 + b^2 + c^2}{3}\right)^3 = \left(\frac{1-2q}{3}\right)^3 < \left(\frac{1-2q}{3}\right)^4,$$

which implies

$$r > -\left(\frac{1-2q}{3}\right)^2.$$

Therefore,

$$r + 2q^2 - q > -\left(\frac{1-2q}{3}\right)^2 + 2q^2 - q = \frac{(2q-1)(7q+1)}{9} > 0.$$

By squaring, the desired inequality becomes

$$q^2 - 4q^3 + 2(9q-2)r - 27r^2 < (r + 2q^2 - q)^2,$$

$$7r^2 + (1 - 5q + q^2)r + q^4 > 0.$$

This is true if the discriminant

$$D = (1 - 5q + q^2)^2 - 28q^4 = [1 - 5q + (1 + 2\sqrt{7})q^2][1 - 5q + (1 - 2\sqrt{7})q^2]$$

is negative. Since

$$1 - 5q + (1 + 2\sqrt{7})q^2 > 0,$$

we only need to show that $f(q) > 0$, where

$$f(q) = (2\sqrt{7} - 1)q^2 + 5q - 1.$$

Since the derivative

$$f'(q) = 2(2\sqrt{7} - 1)q + 5 < 2(2\sqrt{7} - 1)(-1) + 5 = 7 - 4\sqrt{7} < 0,$$

$f(q)$ is strictly decreasing, hence

$$f(q) > f\left(\frac{-1}{k-2}\right) = 0.$$

For $p = 1$, the equality holds when $(a-b)(b-c)(c-a) < 0$ and

$$q = \frac{-1}{k-2}, \quad r = \frac{-q^2}{\sqrt{7}} = \frac{-1}{\sqrt{7}(k-2)^2}.$$

In general, the equality holds when a, b, c are proportional to the roots of the equation

$$w^3 - w^2 - \frac{1}{k-2}w + \frac{1}{\sqrt{7}(k-2)^2} = 0$$

and satisfy $(a-b)(b-c)(c-a) < 0$.

□

P 1.144. *If a, b, c are real numbers such that*

$$k(a^2 + b^2 + c^2) = ab + bc + ca, \qquad k \in \left(\frac{-1}{2}, 1\right),$$

then

$$\alpha_k \le \frac{a^3b + b^3c + c^3}{(a^2 + b^2 + c^2)^2} \le \beta_k,$$

where

$$27\alpha_k = 1 + 13k - 5k^2 - 2(1-k)(1+2k)\sqrt{\frac{7(1-k)}{1+2k}},$$

$$27\beta_k = 1 + 13k - 5k^2 + 2(1-k)(1+2k)\sqrt{\frac{7(1-k)}{1+2k}}.$$

(Vasile C., 2012)

Solution. Let us denote

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc.$$

The case $p = 0$ is not possible, because $p = 0$ and $k(a^2 + b^2 + c^2) = ab + bc + ca$ lead to

$$ab + bc + ca = 0,$$

$$a(b + c) + bc = 0,$$

$$-(b + c)^2 + bc = 0,$$

$$b^2 + bc + c^2 = 0,$$

which involves $a = b = c = 0$. Consider further that $p \neq 0$. Since the statement remains unchanged by replacing a, b, c with $-a, -b, -c$, respectively, it suffices to consider the case $p > 0$. In addition, due to homogeneity, we may assume $p = 1$, which implies

$$q = \frac{k}{1+2k}.$$

(a) Write the desired left inequality as

$$2\alpha_k(a^2 + b^2 + c^2)^2 \le \sum ab(a^2 + b^2) + \left(\sum a^3b - \sum ab^3\right).$$

Since

$$\sum a^2 = p^2 - 2q = 1 - 2q,$$

$$\sum ab(a^2 + b^2) = q(p^2 - 2q) - pr = q - 2q^2 - r,$$

$$\sum a^3b - \sum ab^3 = -p(a-b)(b-c)(c-a) \ge -p\sqrt{(a-b)^2(b-c)^2(c-a)^2}$$

$$= -p\sqrt{\frac{4(p^2-3q)^3 - (2p^3 - 9pq + 27r)^2}{27}} = -\sqrt{\frac{4(1-3q)^3 - (2-9q+27r)^2}{27}},$$

it suffices to prove that

$$2\alpha_k(1-2q)^2 \le q - 2q^2 - r - \sqrt{\frac{4(1-3q)^3 - (2-9q+27r)^2}{27}}.$$

Applying Lemma below for

$$\alpha = \frac{1}{\sqrt{27}}, \quad \beta = \frac{-1}{27}, \quad x = 2(1-3q)\sqrt{1-3q}, \quad y = 2-9q+27r,$$

we get

$$\sqrt{\frac{4(1-3q)^3-(2-9q+27r)^2}{27}} + r + \frac{2-9q}{27} \le \frac{4(1-3q)\sqrt{7(1-3q)}}{27},$$

with equality for

$$(1-3q)\sqrt{\frac{1-3q}{7}} - 2 + 9q - 27r = 0.$$

Thus, it suffices to show that

$$2\alpha_k(1-2q)^2 \le q - 2q^2 + \frac{2-9q}{27} - \frac{4(1-3q)\sqrt{7(1-3q)}}{27},$$

which is equivalent to

$$27\alpha_k \le 1 + 13k - 5k^2 - 2(1-k)(1+2k)\sqrt{\frac{7(1-k)}{1+2k}}.$$

For $p = 1$, the equality holds when $(a-b)(b-c)(c-a) \ge 0$, $q = k/(1+2k)$ and

$$27r = (1-3q)\sqrt{\frac{1-3q}{7}} - 2 + 9q = \frac{r_1}{1+2k},$$

where

$$r_1 = 5k - 2 + (1-k)\sqrt{\frac{1-k}{7(1+2k)}}.$$

Therefore, the equality holds when a, b, c are proportional to the roots of the equation

$$w^3 - w^2 + \frac{k}{1+2k}w - \frac{r_1}{27(1+2k)} = 0$$

and satisfy $(a-b)(b-c)(c-a) \ge 0$.

(b) Write the desired right inequality as

$$2\beta_k(a^2+b^2+c^2)^2 \ge \sum ab(a^2+b^2) + \left(\sum a^3b - \sum ab^3\right).$$

Since

$$\sum a^2 = p^2 - 2q = 1 - 2q,$$

$$\sum ab(a^2+b^2) = q(p^2-2q) - pr = q - 2q^2 - r,$$

$$\sum a^3b - \sum ab^3 = -p(a-b)(b-c)(c-a) \le p\sqrt{(a-b)^2(b-c)^2(c-a)^2}$$

$$= p\sqrt{\frac{4(p^2-3q)^3-(2p^3-9pq+27r)^2}{27}} = \sqrt{\frac{4(1-3q)^3-(2-9q+27r)^2}{27}},$$

it suffices to prove that

$$2\beta_k(1-2q)^2 \ge q-2q^2-r+\sqrt{\frac{4(1-3q)^3-(2-9q+27r)^2}{27}}.$$

Applying Lemma below for

$$\alpha = \frac{1}{\sqrt{27}}, \quad \beta = \frac{1}{27}, \quad x = 2(1-3q)\sqrt{1-3q}, \quad y = 2-9q+27r,$$

we get

$$\sqrt{\frac{4(1-3q)^3-(2-9q+27r)^2}{27}} - r - \frac{2-9q}{27} \le \frac{4(1-3q)\sqrt{7(1-3q)}}{27},$$

with equality for

$$(1-3q)\sqrt{\frac{1-3q}{7}}+2-9q+27r=0.$$

Thus, it suffices to show that

$$2\beta_k(1-2q)^2 \ge q-2q^2+\frac{2-9q}{27}+\frac{4(1-3q)\sqrt{7(1-3q)}}{27},$$

which is equivalent to

$$27\beta_k \ge 1+13k-5k^2+2(1-k)(1+2k)\sqrt{\frac{7(1-k)}{1+2k}}.$$

For $p=1$, the equality holds when $(a-b)(b-c)(c-a)\le 0$, $q=k/(1+2k)$ and

$$27r = 9q-2-(1-3q)\sqrt{\frac{1-3q}{7}} = \frac{r_0}{1+2k},$$

where

$$r_0 = 5k-2-(1-k)\sqrt{\frac{1-k}{7(1+2k)}}.$$

Therefore, the equality holds when a,b,c are proportional to the roots of the equation

$$w^3-w^2+\frac{k}{1+2k}w-\frac{r_0}{27(1+2k)}=0$$

and satisfy $(a-b)(b-c)(c-a)\le 0$.

Lemma. *If α,β,x,y are real numbers such that*

$$\alpha \ge 0, \quad x \ge 0, \quad x^2 \ge y^2,$$

then

$$\alpha\sqrt{x^2-y^2} \le x\sqrt{\alpha^2+\beta^2}+\beta y,$$

with equality if and only if

$$\beta x + y\sqrt{\alpha^2+\beta^2} = 0.$$

Proof. Since

$$x\sqrt{\alpha^2+\beta^2}+\beta y \ge |\beta|x+\beta y \ge |\beta||y|+\beta y \ge 0,$$

we can write the inequality as

$$\alpha^2(x^2-y^2) \le \left(x\sqrt{\alpha^2+\beta^2}+\beta y\right)^2,$$

which is equivalent to

$$\left(\beta x + y\sqrt{\alpha^2+\beta^2}\right)^2 \ge 0.$$

□

P 1.145. *If a, b, c are positive real numbers such that $a+b+c=3$, then*

$$\frac{a^2}{4a+b^2}+\frac{b^2}{4b+c^2}+\frac{c^2}{4c+a^2} \ge \frac{3}{5}.$$

(Michael Rozenberg, 2008)

Solution. By the Cauchy-Schwarz inequality, we have

$$\sum \frac{a^2}{4a+b^2} \ge \frac{\left[\sum a(2a+c)\right]^2}{\sum(4a+b^2)(2a+c)^2} = \frac{\left(2\sum a^2+\sum ab\right)^2}{\sum(4a+b^2)(2a+c)^2}.$$

Therefore, it suffices to show that

$$5\left(2\sum a^2+\sum ab\right)^2 \ge 3\sum(4a+b^2)(2a+c)^2,$$

which is equivalent to the homogeneous inequalities

$$5\left(2\sum a^2+\sum ab\right)^2 \ge \sum[4a(a+b+c)+3b^2](2a+c)^2,$$

$$5\left(2\sum a^2+\sum ab\right)^2 \ge \sum(4a^2+3b^2+4ab+4ac)(4a^2+c^2+4ac),$$

$$2\sum a^4+5\sum a^2b^2 \ge abc\sum a+6\sum ab^3.$$

Using Vasc's inequality

$$3\sum ab^3 \le \left(\sum a^2\right)^2,$$

it is enough to prove the symmetric inequality

$$2\sum a^4+5\sum a^2b^2 \ge abc\sum a+2\left(\sum a^2\right)^2,$$

which is equivalent to the well-known inequality

$$\sum a^2b^2 \ge abc\sum a.$$

The equality holds for $a=b=c=1$.

□

P 1.146. *If a,b,c are positive real numbers, then*

$$\frac{a^2+bc}{a+b}+\frac{b^2+ca}{b+c}+\frac{c^2+ab}{c+a} \le \frac{(a+b+c)^3}{3(ab+bc+ca)}.$$

(Michael Rozenberg, 2013)

Solution *(by Manlio Marangelli)*. Write the inequality as

$$\sum\left(\frac{a^2+bc}{a+b}-a\right) \le \frac{(a+b+c)^3}{3(ab+bc+ca)}-(a+b+c),$$

$$\sum\frac{b(c-a)}{a+b} \le \frac{(a+b+c)^3}{3(ab+bc+ca)}-(a+b+c),$$

$$\frac{\sum b(c^2-a^2)(b+c)}{(a+b)(b+c)(c+a)} \le \frac{(a+b+c)^3}{3(ab+bc+ca)}-(a+b+c),$$

$$\frac{3\sum ab^3-3abc\sum a}{(a+b)(b+c)(c+a)} \le \frac{(a+b+c)^3}{ab+bc+ca}-3(a+b+c).$$

By the known Vasc's inequality

$$3\sum ab^3 \le \left(\sum a^2\right)^2,$$

it suffices to prove the symmetric inequality

$$\frac{\left(\sum a^2\right)^2-3abc\sum a}{(a+b)(b+c)(c+a)} \le \frac{(a+b+c)^3}{ab+bc+ca}-3(a+b+c).$$

Using the notation

$$p=a+b+c, \quad q=ab+bc+ca, \quad r=abc,$$

this inequality can be written as

$$\frac{(p^2-2q)^2-3pr}{pq-r} \le \frac{p^3}{q} - 3p,$$

which is equivalent to

$$q^2(p^2-4q)-(p^2-6q)pr \ge 0.$$

Case 1: $p^2-6q \ge 0$. Since $3pr \le q^2$, we have

$$q^2(p^2-4q)-(p^2-6q)pr \ge q^2(p^2-4q) - \frac{q^2(p^2-6q)}{3} = \frac{2q^2(p^2-3q)}{3} \ge 0.$$

Case 2: $p^2-6q \le 0$. Using Schur's inequality of fourth degree

$$6pr \ge (p^2-q)(4q-p^2),$$

we get

$$\begin{aligned} q^2(p^2-4q)-(p^2-6q)pr &\ge q^2(p^2-4q) - \frac{(p^2-6q)(p^2-q)(4q-p^2)}{6} \\ &= \frac{(p^2-3q)(p^2-4q)^2}{6} \ge 0. \end{aligned}$$

The equality holds for $a=b=c=1$.

□

P 1.147. *If a,b,c are positive real numbers such that $a+b+c=3$, then*

$$\sqrt{ab^2+bc^2} + \sqrt{bc^2+ca^2} + \sqrt{ca^2+ab^2} \le 3\sqrt{2}.$$

(Nguyen Van Quy, 2013)

Solution *(by Michael Rozenberg)*. By the Cauchy-Schwarz inequality, we have

$$\left(\sum \sqrt{ab^2+bc^2}\right)^2 \le \sum \frac{ab+c^2}{a+c} \cdot \sum b(a+c).$$

Therefore, it suffices to show that

$$\sum \frac{ab+c^2}{a+c} \le \frac{9}{ab+bc+ca},$$

which is equivalent to the homogeneous inequality

$$\sum \frac{ab+c^2}{a+c} \le \frac{(a+b+c)^3}{3(ab+bc+ca)},$$

which is the inequality from the preceding P 1.146. The equality holds for $a=b=c=1$.

□

P 1.148. *If a,b,c are positive real numbers such that $a^5+b^5+c^5=3$, then*

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\ge 3.$$

Solution. We will prove the inequality under the more general condition $a^m+b^m+c^m=3$, where $0<m\le 21/4$. First, write the inequality in the homogeneous form

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\ge 3\left(\frac{a^m+b^m+c^m}{3}\right)^{1/m}.$$

By the Power Mean inequality, we have

$$\left(\frac{a^m+b^m+c^m}{3}\right)^{1/m}\le\left(\frac{a^{21/4}+b^{21/4}+c^{21/4}}{3}\right)^{4/21}.$$

Thus, it suffices to show that

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\ge 3\left(\frac{a^{21/4}+b^{21/4}+c^{21/4}}{3}\right)^{4/21}.$$

By the known Vasc's inequality in P 1.125, namely

$$(x^2+y^2+z^2)^2\ge 3(x^3y+y^3z+z^3x),\quad x,y,z\in\mathbb{R},$$

we have

$$\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)^2\ge 3\left(\frac{a^3}{\sqrt{bc}}+\frac{b^3}{\sqrt{ca}}+\frac{c^3}{\sqrt{ab}}\right).$$

Therefore, it suffices to prove the symmetric inequality

$$\frac{a^3}{\sqrt{bc}}+\frac{b^3}{\sqrt{ca}}+\frac{c^3}{\sqrt{ab}}\ge 3\left(\frac{a^{21/4}+b^{21/4}+c^{21/4}}{3}\right)^{8/21},$$

which is equivalent to

$$\left(\frac{\dfrac{a^3}{\sqrt{bc}}+\dfrac{b^3}{\sqrt{ca}}+\dfrac{c^3}{\sqrt{ab}}}{3}\right)^{21/4}\ge 3\left(\frac{a^{21/4}+b^{21/4}+c^{21/4}}{3}\right)^2,$$

Setting

$$a=x^{2/7},\quad b=y^{2/7},\quad c=z^{2/7},\quad x,y,z>0,$$

the inequality becomes

$$\left(\frac{x+y+z}{3}\right)^{21/4}\ge 3(xyz)^{3/4}\left(\frac{x^{3/2}+y^{3/2}+z^{3/2}}{3}\right)^2.$$

By the Cauchy-Schwarz inequality, we have

$$(x+y+z)(x^2+y^2+z^2) \geq (x^{3/2}+y^{3/2}+z^{3/2})^2.$$

Thus, it is enough to prove that

$$\left(\frac{x+y+z}{3}\right)^{17/4} \geq \frac{1}{3}(xyz)^{3/4}(x^2+y^2+z^2).$$

Due to homogeneity, we may assume that $x+y+z=3$, when the inequality becomes

$$(xyz)^{3/4}(x^2+y^2+z^2) \leq 3.$$

Since

$$\frac{3}{4} > \frac{1}{\sqrt{2}},$$

this inequality follows from the inequality in P 2.89 from Volume 2:

$$(xyz)^k(x^2+y^2+z^2) \leq 3, \quad k \geq \frac{1}{\sqrt{2}}$$

The proof is completed. The equality holds for $a=b=c=1$.

□

P 1.149. *Let $P(a,b,c)$ be a cyclic homogeneous polynomial of degree three. The inequality*

$$P(a,b,c) \geq 0$$

holds for all $a,b,c \geq 0$ if and only if the following two conditions are fulfilled:

(a) $P(1,1,1) \geq 0$;

(b) $P(0,b,c) \geq 0$ *for all* $b,c \geq 0$.

(Pham Kim Hung, 2007)

Solution. The conditions (a) and (b) are clearly necessary. Therefore, we will prove further that these conditions are also sufficient to have $P(a,b,c) \geq 0$. The polynomial $P(a,b,c)$ has the general form

$$P(a,b,c) = A(a^3+b^3+c^3) + B(a^2b+b^2c+c^2a) + C(ab^2+bc^2+ca^2) + 3Dabc.$$

Since

$$P(1,1,1) = 3(A+B+C+D), \quad P(0,1,1) = 2A+B+C, \quad P(0,0,1) = A,$$

the conditions (a) and (b) involves

$$A+B+C+D \geq 0, \quad 2A+B+C \geq 0, \quad A \geq 0.$$

Assume that $a = \min\{a, b, c\}$, and denote

$$b = a + p, \quad c = a + q, \quad p, q \geq 0.$$

For fixed p and q, define the function

$$f(a) = P(a, a + p, a + q), \quad a \geq 0.$$

Since

$$a' = b' = c' = 1,$$

we have the derivative

$$\begin{aligned} f'(a) &= 3A(a^2 + b^2 + c^2) + (B + C)(a + b + c)^2 + 3D(ab + bc + ca) \\ &= (3A + B + C)(a^2 + b^2 + c^2) + (2B + 2C + 3D)(ab + bc + ca) \\ &= (3A + B + C)(a^2 + b^2 + c^2 - ab - bc - ca) + 3(A + B + C + D)(ab + bc + ca). \end{aligned}$$

Because $f'(a) \geq 0$, f is increasing, hence $f(a) \geq f(0)$, which is equivalent to

$$P(a, b, c) \geq P(0, p, q) = P(0, b, c).$$

According to the condition (b), we have $P(0, b, c) \geq 0$, hence $P(a, b, c) \geq 0$.

Remark 1. From the proof of P 1.149, the following statement follows:

- *Let $P(a, b, c)$ be a cyclic homogeneous polynomial of degree three. The inequality*

$$P(a, b, c) \geq 0$$

holds for all nonnegative real numbers a, b, c satisfying

$$a \leq b \leq c$$

if and only if $P(1, 1, 1) \geq 0$ and $P(0, b, c) \geq 0$ for all $0 \leq b \leq c$.

Remark 2. From P 1.149, using the substitution

$$a = y + z, \quad b = z + x, \quad c = x + y, \qquad x, y, z \geq 0,$$

we get the following statement:

- *Let $P(a, b, c)$ be a cyclic homogeneous polynomial of degree three, where a, b, c are the lengths of the sides of a triangle. The inequality*

$$P(a, b, c) \geq 0$$

holds if and only if $P(1, 1, 1) \geq 0$ and $P(b + c, b, c) \geq 0$ for all $b, c \geq 0$.

$\square$

P 1.150. *If a,b,c are nonnegative real numbers such that $a+b+c=3$, then*

$$8(a^2b+b^2c+c^2a)+9 \geq 11(ab+bc+ca).$$

Solution. Write the inequality in the homogeneous form $P(a,b,c) \geq 0$, where

$$P(a,b,c)=24(a^2b+b^2c+c^2a)+(a+b+c)^3-11(a+b+c)(ab+bc+ca).$$

According to P 1.149, it suffices to show that $P(1,1,1) \geq 0$ and $P(0,b,c) \geq 0$ for all $b,c \geq 0$. We have

$$P(1,1,1)=0$$

and

$$\begin{aligned}P(0,b,c) &= 24b^2c+(b+c)^3-11bc(b+c)\\ &= b^3+16b^2c-8bc^2+c^3\\ &\geq 16b^2c-8bc^2+c^3=c(4b-c)^2 \geq 0.\end{aligned}$$

The equality holds for $a=b=c=1$.

□

P 1.151. *If a,b,c are nonnegative real numbers such that $a+b+c=6$, then*

$$a^3+b^3+c^3+8(a^2b+b^2c+c^2a) \geq 166.$$

(Vasile C., 2010)

Solution. Write the inequality in the homogeneous form $P(a,b,c) \geq 0$, where

$$P(a,b,c)=a^3+b^3+c^3+8(a^2b+b^2c+c^2a)-166\left(\frac{a+b+c}{6}\right)^3.$$

According to P 1.149, it suffices to show that $P(1,1,1) \geq 0$ and $P(0,b,c) \geq 0$ for all $b,c \geq 0$. We have

$$P(1,1,1)=27-\frac{83}{4}=\frac{25}{4}>0$$

and

$$\begin{aligned}P(0,b,c) &= b^3+c^3+8b^2c-\frac{83}{108}(b+c)^3\\ &= \frac{1}{108}(25b^3+615b^2c-249bc^2+25c^3)\\ &= \frac{1}{108}(5b-c)^2(b+25c) \geq 0.\end{aligned}$$

The equality holds for $a=0$, $b=1$, $c=5$ (or any cyclic permutation).

□

P 1.152. *If a, b, c are nonnegative real numbers, then*

$$a^3+b^3+c^3-3abc \ge \sqrt{9+6\sqrt{3}}\,(a-b)(b-c)(c-a).$$

First Solution. Write the inequality as $P(a,b,c) \ge 0$. According to P 1.149, it suffices to show that $P(1,1,1) \ge 0$ and $P(0,b,c) \ge 0$ for all $b,c \ge 0$. We have $P(1,1,1) = 0$ and

$$P(0,b,c) = b^3+c^3+\sqrt{9+6\sqrt{3}}\,bc(b-c).$$

The inequality $P(0,b,c) \ge 0$ is true if

$$(b^3+c^3)^2 \ge (9+6\sqrt{3}\,)b^2c^2(b-c)^2,$$

which is equivalent to

$$(b+c)^2(b^2-bc+c^2)^2 \ge (9+6\sqrt{3}\,)b^2c^2(b-c)^2.$$

For the non-trivial case $bc \ne 0$, denoting

$$x = \frac{b}{c}+\frac{c}{b}-1,$$

we can write this inequality as

$$(x+3)x^2 \ge (9+6\sqrt{3}\,)(x-1),$$

$$(x-\sqrt{3}\,)^2(x+3+2\sqrt{3}\,) \ge 0.$$

The equality holds for $a=b=c$, and also for $a=0$ and $b/c+c/b = 1+\sqrt{3}$, $b<c$ (or any cyclic permutation).

Second Solution. Assume that $a = \min\{a,b,c\}$. Since the case $a \le c \le b$ is trivial, consider further that $a \le b \le c$. Write the inequality as

$$(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2] \ge 2\sqrt{9+6\sqrt{3}}\,(a-b)(b-c)(c-a).$$

Using the substitution $b = a+p$, $c = a+q$, where $q \ge p \ge 0$, the inequality becomes

$$(3a+p+q)(p^2-pq+q^2) \ge \sqrt{9+6\sqrt{3}}\,pq(q-p).$$

Since $p^2-pq+q^2 \ge 0$, it suffices to consider the case $a=0$ (as in the first solution).

□

P 1.153. *If a,b,c are nonnegative real numbers, no two of which are zero, then*

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+7\geq\frac{17}{3}\left(\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}\right).$$

(Vasile C., 2007)

Solution. Write the inequality as $P(a,b,c)\geq 0$, where

$$\begin{aligned}P(a,b,c)&=\sum(3a-17b)(a+b)(a+c)+21(a+b)(b+c)(c+a)\\&=3(a^3+b^3+c^3)-10(a^2b+b^2c+c^2a)+7(ab^2+bc^2+ca^2).\end{aligned}$$

According to P 1.149, it suffices to show that $P(1,1,1)\geq 0$ and $P(0,b,c)\geq 0$ for all $b,c\geq 0$. We have $P(1,1,1)=0$ and

$$P(0,b,c)=3(b^3+c^3)-10b^2c+7bc^2.$$

Consider the nontrivial case $b,c>0$. Setting $c=1$, we need to show that $f(b)\geq 0$, where

$$f(b)=3b^3-10b^2+7b+3.$$

Case 1: $b\geq 3$. We have

$$f(b)>3b^3-10b^2+7b=(b-1)(3b-7)>0.$$

Case 2: $2\leq b\leq 3$. We have

$$f(b)\geq 3b^3-10b^2+8b=b(b-2)(3b-4)\geq 0.$$

Case 3: $0<b\leq 2$. We have

$$f(b)\geq 3b^3-10b^2+7b+1.5b=b(3b^2-10b+8.5)>3b(b-5/3)^2\geq 0.$$

The equality holds for $a=b=c$.

□

P 1.154. *Let a,b,c be nonnegative real numbers, no two of which are zero. If $0\leq k\leq 5$, then*

$$\frac{ka+b}{a+c}+\frac{kb+c}{b+a}+\frac{kc+a}{c+b}\geq\frac{3}{2}(k+1).$$

(Vasile C., 2007)

First Solution. Write the inequality as

$$\frac{b}{a+c}+\frac{c}{b+a}+\frac{a}{c+b}-\frac{3}{2}+k\left(\frac{a}{a+c}+\frac{b}{b+a}+\frac{c}{c+b}-\frac{3}{2}\right)\geq 0.$$

Since

$$\frac{b}{a+c}+\frac{c}{b+a}+\frac{a}{c+b}-\frac{3}{2}\geq 0,$$

it suffices to consider the case $k=5$, when the inequality can be written as follows:

$$\sum(5a+b)(b+a)(c+b)\geq 9(a+c)(b+a)(c+b),$$

$$2\sum ab^2+\sum a^3\geq 3\sum a^2b,$$

$$2\sum ab^2+\frac{4}{3}\sum a^3-\frac{1}{3}\sum b^3\geq 3\sum a^2b,$$

$$\sum(6ab^2+4a^3-b^3-9a^2b)\geq 0,$$

$$(a-b)^2(4a-b)+(b-c)^2(4b-c)+(c-a)^2(4c-a)\geq 0.$$

Assume that $a=\min\{a,b,c\}$, and use the substitution

$$b=a+p,\quad c=a+q,\quad p,q\geq 0.$$

The inequality becomes

$$p^2(3a-p)+(p-q)^2(3a+4p-q)+q^2(3a+4q)\geq 0,$$

$$2Aa+B\geq 0,$$

where

$$A=p^2-pq+q^2,\quad B=p^3-3p^2q+2pq^2+q^3.$$

Since $A\geq 0$, we only need to show that $B\geq 0$. For $q=0$, we have $B=p^3\geq 0$, while for $q>0$, the inequality $B\geq 0$ is equivalent to

$$1\geq x(x-1)(2-x),$$

where $x=p/q\geq 0$. For the non-trivial case $x\in[1,2]$, we get this inequality by multiplying the obvious inequalities

$$1\geq x-1$$

and

$$1\geq x(2-x).$$

The proof is completed. The equality holds for $a=b=c$.

Second Solution. We can write the inequality in the form $P(a,b,c)\geq 0$, where $P(a,b,c)$ is a cyclic homogeneous polynomial of degree three. According to P 1.149, it suffices to show that the desired inequality holds for $a=b=c$, and also for $a=0$. If $a=0$, then the inequality becomes

$$x+k+\frac{1}{x}+\frac{k}{1+x}\geq\frac{3}{2}(k+1),$$

$$2(x-1)^2 + x \geq \frac{kx(x-1)}{x+1},$$

where

$$x = \frac{b}{c} > 0.$$

For $0 < x \leq 1$, we have

$$2(x-1)^2 + x > 0 \geq \frac{kx(x-1)}{x+1}.$$

For $1 \leq x \leq 5$, it suffices to consider the case $k = 5$, when the inequality is equivalent to

$$2(x-1)^2 + x \geq \frac{5x(x-1)}{x+1},$$

$$x^3 - 3x^2 + 2x + 1 \geq 0,$$

$$x(x-2)^2 + (x-1)^2 \geq 0.$$

Remark. As in the second solution, we can prove that the inequality in P 1.154 holds for

$$0 \leq k \leq k_0, \qquad k_0 = \sqrt{13 + 16\sqrt{2}} \approx 5.969.$$

For $a = 0$ and $k = k_0$, the inequality becomes

$$2(x-1)^2 + x \geq \frac{kx(x-1)}{x+1}, \qquad x = \frac{b}{c} > 0,$$

$$2x^3 - (k_0+1)x^2 + (k_0-1)x + 2 \geq 0,$$

$$(x-x_0)^2\left(x + \frac{1}{x_0^2}\right) \geq 0,$$

where

$$x_0 = \frac{1 + \sqrt{2} + \sqrt{2\sqrt{2} - 1}}{2} \approx 1.883.$$

If $k = k_0$, then the equality holds for $a = b = c$, and also for $a = 0$ and $\frac{b}{c} + \frac{c}{b} = 1 + \sqrt{2}$ (or any cyclic permutation).

□

P 1.155. *Let a, b, c be nonnegative real numbers. Prove that*

(a) *if $k \leq 1 - \frac{2}{5\sqrt{5}}$, then*

$$\frac{ka+b}{2a+b+c} + \frac{kb+c}{a+2b+c} + \frac{kc+a}{a+b+2c} \geq \frac{3}{4}(k+1).$$

(b) if $k \ge 1 + \dfrac{2}{5\sqrt{5}}$*, then*

$$\frac{ka+b}{2a+b+c} + \frac{kb+c}{a+2b+c} + \frac{kc+a}{a+b+2c} \le \frac{3}{4}(k+1).$$

(Vasile C., 2007)

Solution. (a) Write the inequality in the form $P(a,b,c) \ge 0$, where $P(a,b,c)$ is a cyclic homogeneous polynomial of degree three. According to P 1.149, it suffices to show that the desired inequality holds for $a = b = c$, and also for $a = 0$. For $a = 0$, the inequality becomes

$$\frac{x}{x+1} + \frac{kx+1}{2x+1} + \frac{k}{x+2} \ge \frac{3}{4}(k+1),$$

$$(x+2)(2x^2 - x + 1) \ge k(x+1)(2x^2 - x + 2),$$

where

$$x = \frac{b}{c} \ge 0.$$

It suffices to consider the case $k = 1 - \dfrac{2}{5\sqrt{5}}$, when the inequality is equivalent to

$$(x - x_0)^2 \left(x + \frac{2}{5\sqrt{5}\, x_0^2} \right) \ge 0,$$

where

$$x_0 = \frac{3 - \sqrt{5}}{2}.$$

The equality holds for $a = b = c$. If $k = 1 - \dfrac{2}{5\sqrt{5}}$, then the equality holds also for $a = 0$ and $\dfrac{b}{c} + \dfrac{c}{b} = 3$ (or any cyclic permutation).

(b) According to P 1.149, it suffices to show that the desired inequality holds for $a = b = c$, and also for $a = 0$. If $a = 0$, then the inequality becomes

$$\frac{x}{x+1} + \frac{kx+1}{2x+1} + \frac{k}{x+2} \le \frac{3}{4}(k+1),$$

$$(x+2)(2x^2 - x + 1) \le k(x+1)(2x^2 - x + 2),$$

where

$$x = \frac{b}{c} \ge 0.$$

It suffices to consider the case $k = 1 + \dfrac{2}{5\sqrt{5}}$, when the inequality is equivalent to

$$(x - x_1)^2 \left(x + \frac{2}{5\sqrt{5}\, x_1^2} \right) \ge 0,$$

where

$$x_1 = \frac{3+\sqrt{5}}{2}.$$

The equality holds for $a = b = c$. If $k = 1 + \frac{2}{5\sqrt{5}}$, then the equality holds also for $a = 0$ and $\frac{b}{c} + \frac{c}{b} = 3$ (or any cyclic permutation).

□

P 1.156. *Let a, b, c be nonnegative real numbers, no two of which are zero. If $k \le \frac{23}{8}$, then*

$$\frac{ka+b}{2a+c} + \frac{kb+c}{2b+a} + \frac{kc+a}{2c+b} \ge k+1.$$

(Vasile C., 2007)

Solution. We can write the inequality in the form $P(a,b,c) \ge 0$, where $P(a,b,c)$ is a cyclic homogeneous polynomial of degree three. According to P 1.149, it suffices to show that the desired inequality holds for $a = b = c$, and also for $a = 0$. For $a = 0$, the inequality becomes

$$x + \frac{k}{2} + \frac{1}{2x} + \frac{k}{2+x} \ge k+1,$$

$$x^2 + (x-1)^2 \ge \frac{kx^2}{x+2},$$

where

$$x = \frac{b}{c} > 0.$$

It suffices to consider that $k = 23/8$, when the inequality is equivalent to

$$2x^2 - 2x + 1 \ge \frac{23x^2}{8(x+2)},$$

$$16x^3 - 7x^2 - 24x + 16 \ge 0,$$

$$16x(x-1)^2 + (5x-4)^2 \ge 0.$$

The equality holds for $a = b = c$.

Remark. For $k = 2$, we get the inequality in P 1.21.

□

P 1.157. *If a, b, c are positive real numbers such that $a \le b \le c$, then*

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + 3 \ge 2\left(\frac{a+b}{b+c} + \frac{b+c}{c+a} + \frac{c+a}{a+b}\right).$$

Solution. Write the inequality as follows:

$$\sum\left(\frac{a}{b}-1\right)\ge 2\sum\left(\frac{b+c}{c+a}-1\right),$$

$$\sum(a-b)\left(\frac{1}{b}+\frac{2}{c+a}\right)\ge 0,$$

$$(a-b)\left(\frac{1}{b}+\frac{2}{c+a}\right)+(b-c)\left(\frac{1}{c}+\frac{2}{a+b}\right)+[(c-b)+(b-a)]\left(\frac{1}{a}+\frac{2}{b+c}\right)\ge 0,$$

$$(b-a)\left(\frac{1}{a}+\frac{2}{b+c}-\frac{1}{b}-\frac{2}{c+a}\right)+(c-b)\left(\frac{1}{a}+\frac{2}{b+c}-\frac{1}{c}-\frac{2}{a+b}\right)\ge 0,$$

$$(b-a)^2\left[\frac{1}{ab}-\frac{2}{(b+c)(c+a)}\right]+(c-b)(c-a)\left[\frac{1}{ac}-\frac{2}{(b+c)(a+b)}\right]\ge 0.$$

The inequality is true since

$$\frac{1}{ab}-\frac{2}{(b+c)(c+a)}=\frac{c(a+b+c)-ab}{(b+c)(c+a)}>\frac{a(c-b)}{(b+c)(c+a)}\ge 0$$

and

$$\frac{1}{ac}-\frac{2}{(b+c)(a+b)}=\frac{b(a+b+c)-ac}{(b+c)(a+b)}>\frac{c(b-a))}{(b+c)(a+b)}\ge 0.$$

The equality holds for $a=b=c$.

□

P 1.158. *If $a\ge b\ge c\ge 0$, then*

$$\frac{3a+b}{2a+c}+\frac{3b+c}{2b+a}+\frac{3c+a}{2c+b}\ge 4.$$

(Vasile C., 2007)

First Solution. Write the inequality as follows:

$$\sum(3a+b)(2b+a)(2c+b)\ge 4(2a+c)(2b+a)(2c+b),$$

$$2\sum a^3+13\sum ab^2+7\sum a^2b+42abc\ge 4(4\sum ab^2+2\sum a^2b+9abc),$$

$$2\sum a^3+6abc\ge 3\sum ab^2+\sum a^2b,$$

$$2E(a,b,c)\ge F(a,b,c),$$

where

$$E(a,b,c)=\sum a^3+3abc-\sum ab^2-\sum a^2b,$$

$$F(a,b,c)=\sum ab^2-\sum a^2b.$$

The inequality is true since $E(a,b,c) \geq 0$ (by Schur's inequality of degree three) and

$$F(a,b,c) = (a-b)(b-c)(c-a) \leq 0.$$

The equality holds for $a = b = c$, and also for $a = b$ and $c = 0$.

Second Solution. Denote

$$x = a - b \geq 0, \qquad y = b - c \geq 0,$$

and write the inequality as follows

$$\sum \left(\frac{3a+b}{2a+c} - \frac{4}{3} \right) \geq 0,$$

$$\sum \frac{a+3b-4c}{2a+c} \geq 0,$$

$$\frac{a+3b-4c}{2a+c} + \frac{b+3c-4a}{2b+a} + \frac{c+3a-4b}{2c+b} \geq 0,$$

$$\frac{x+4y}{2a+c} - \frac{4x+3y}{2b+a} + \frac{3x-y}{2c+b} \geq 0,$$

$$xA + yB \geq 0,$$

where

$$\begin{aligned} A &= \frac{1}{2a+c} - \frac{4}{2b+a} + \frac{3}{2c+b} \\ &= \left(\frac{1}{2a+c} - \frac{1}{2b+a} \right) + 3\left(\frac{1}{2c+b} - \frac{1}{2b+a} \right) \\ &= \frac{-x+y}{(2a+c)(2b+a)} + \frac{3(x+2y)}{(2b+a)(2c+b)} \end{aligned}$$

and

$$\begin{aligned} B &= \frac{4}{2a+c} - \frac{3}{2b+a} - \frac{1}{2c+b} \\ &= 3\left(\frac{1}{2a+c} - \frac{1}{2b+a} \right) + \left(\frac{1}{2a+c} - \frac{1}{2c+b} \right) \\ &= \frac{3(-x+y)}{(2a+c)(2b+a)} - \frac{2x+y}{(2a+c)(2c+b)}. \end{aligned}$$

Thus, the inequality is equivalent to

$$x[(-x+y)(2c+b)+3(x+2y)(2a+c)+y[3(-x+y)(2c+b)-(2x+y)(2b+a)] \geq 0,$$

$$x^2(6a-b+c) + xy(10a-6b+2c) - y^2(a-b-6c) \geq 0,$$

It suffices to show that

$$xy(10a-6b+2c)-y^2(a-b-6c)\ge 0,$$

which is true is

$$x(10a-6b+2c)-y(a-b-6c)\ge 0.$$

We have

$$\begin{aligned} x(10a-6b+2c)-y(a-b-6c) &= x(10x+4y+6c)-y(x-6c) \\ &= 10x^2+3xy+6c(x+y)\ge 0. \end{aligned}$$

Third Solution. According to Remark 1 from P 1.149, it suffices to prove that the inequality holds for $c=0$ and $a\ge b$; that is, to show that

$$\frac{3}{2}+\frac{1}{2x}+\frac{3}{2+x}+x\ge 4,$$

where

$$x=\frac{a}{b}\ge 1.$$

The inequality is equivalent to

$$2x^3-x^2-3x+2\ge 0,$$

$$(x-1)(2x^2+x-2)\ge 0.$$

□

P 1.159. *Let a,b,c be nonnegative real numbers such that*

$$a\ge b\ge 1\ge c,\qquad a+b+c=3.$$

Prove that

$$\frac{1}{a^2+3}+\frac{1}{b^2+3}+\frac{1}{c^2+3}\le\frac{3}{4}.$$

(Vasile C., 2005)

First Solution. Let

$$r=abc,\qquad q=ab+bc+ca.$$

From

$$(a-1)(b-1)(c-1)\le 0,$$

we get

$$r\le q-2.$$

The desired inequality is equivalent to

$$3a^2b^2c^2+5(a^2b^2+b^2c^2+c^2a^2)+3(a^2+b^2+c^2)-27\ge 0,$$

$$3r^2 - 30r + 5q^2 - 6q \geq 0,$$

$$3(5-r)^2 + 5q^2 - 6q - 75 \geq 0.$$

Since

$$3q \leq (a+b+c)^2 = 9$$

and

$$5 - r \geq 5 - (q-2) = 7 - q > 0,$$

it suffices to show that

$$3(7-q)^2 + 5q^2 - 6q - 75 \geq 0.$$

This is equivalent to the obvious inequality

$$(q-3)^2 \geq 0.$$

The proof is completed. The equality holds for $a = b = c = 1$.

Second Solution (by *Nguyen Van Quy*). Write the inequality as follows:

$$\left(\frac{1}{a^2+3} - \frac{3-a}{8}\right) + \left(\frac{1}{b^2+3} - \frac{3-b}{8}\right) + \left(\frac{1}{c^2+3} - \frac{3-c}{8}\right) \leq 0,$$

$$\frac{(a-1)^3}{a^2+3} + \frac{(b-1)^3}{b^2+3} \leq \frac{(1-c)^3}{c^2+3}.$$

Indeed, we have

$$\frac{(1-c)^3}{c^2+3} = \frac{(a-1+b-1)^3}{c^2+3} \geq \frac{(a-1)^3 + (b-1)^3}{c^2+3} \geq \frac{(a-1)^3}{a^2+3} + \frac{(b-1)^3}{b^2+3}.$$

Third Solution. Denoting

$$d = 2 - c,$$

we have

$$a + b = 1 + d, \quad d \geq a \geq b \geq 1.$$

We claim that

$$\frac{1}{c^2+3} + \frac{1}{d^2+3} \leq \frac{1}{2}.$$

Indeed,

$$\frac{1}{2} - \frac{1}{c^2+3} - \frac{1}{d^2+3} = \frac{(cd-1)^2}{2(c^2+3)(d^2+3)} \geq 0.$$

Thus, it suffices to show that

$$\frac{1}{a^2+3} + \frac{1}{b^2+3} \leq \frac{1}{d^2+3} + \frac{1}{4}.$$

Since

$$\frac{1}{a^2+3}-\frac{1}{d^2+3}=\frac{(d-a)(d+a)}{(a^2+3)(d^2+3)}=\frac{(b-1)(d+a)}{(a^2+3)(d^2+3)},$$

$$\frac{1}{4}-\frac{1}{b^2+3}=\frac{(b-1)(b+1)}{4(b^2+3)},$$

we need to prove that

$$\frac{d+a}{(a^2+3)(d^2+3)}\le\frac{b+1}{4(b^2+3)}.$$

We can get this inequality by multiplying the inequalities

$$\frac{d+a}{d^2+3}\le\frac{a+1}{4},$$

$$\frac{a+1}{a^2+3}\le\frac{b+1}{b^2+3}.$$

We have

$$\frac{a+1}{4}-\frac{d+a}{d^2+3}=\frac{(d-1)(ad+a+d-3)}{4(d^2+3)}\ge 0,$$

$$\frac{b+1}{b^2+3}-\frac{a+1}{a^2+3}=\frac{(a-b)(ab+a+b-3)}{(a^2+3)(b^2+3)}\ge 0.$$

□

P 1.160. *Let a,b,c be nonnegative real numbers such that*

$$a\ge 1\ge b\ge c,\quad a+b+c=3.$$

Prove that

$$\frac{1}{a^2+2}+\frac{1}{b^2+2}+\frac{1}{c^2+2}\ge 1.$$

(Vasile C., 2005)

First Solution. Let

$$r=abc,\quad q=ab+bc+ca.$$

From

$$(a-1)(b-1)(c-1)\ge 0,$$

we get

$$r\ge q-2.$$

Also, we have

$$r\le\frac{(a+b+c)^3}{27}=1.$$

$$q \le \frac{1}{3}(a+b+c)^3 = 3.$$

The desired inequality is equivalent to

$$3 \ge a^2b^2c^2 + a^2b^2 + b^2c^2 + c^2a^2,$$

$$4 \ge r^2 - 6r + q^2,$$

$$(3-r)^2 + q^2 \le 13.$$

Consider further two cases: $q \le 2$ and $2 \le q \le 3$.

Case 1: $q \le 2$. We have

$$(3-r)^2 + q^2 \le 3^2 + 2^2 = 13.$$

Case 2: $2 \le q \le 3$. From $r \le q-2$, we get

$$(3-r)^2 + q^2 \le (5-q)^2 + q^2 = 2(q-3)(q-2) \le 0.$$

The proof is completed. The equality holds for $a = b = c = 1$, as well as for $a = 2$, $b = 1$ and $c = 0$.

Second Solution. First, we can check that the desired inequality becomes an equality for $a = b = c = 1$, and also for $a = 2$, $b = 1$, $c = 0$. Consider then the inequality $f(x) \ge 0$, where

$$f(x) = \frac{1}{x^2+2} - A - Bx.$$

We have the derivative

$$f'(x) = \frac{-2x}{(x^2+2)^2} - B.$$

From the conditions $f(1) = 0$ and $f'(1) = 0$, we get $A = 5/9$ and $B = -2/9$. Also, from the conditions $f(2) = 0$ and $f'(2) = 0$, we get $A = 7/18$ and $B = -1/9$. Using these values of A and B, we obtain the relations

$$\frac{1}{x^2+2} - \frac{5-2x}{9} = \frac{(x-1)^2(2x-1)}{9(x^2+2)},$$

$$\frac{1}{x^2+2} - \frac{7-2x}{18} = \frac{(x-2)^2(2x+1)}{18(x^2+2)},$$

which involve

$$\frac{1}{x^2+2} \ge \frac{5-2x}{9}, \quad x \ge \frac{1}{2},$$

$$\frac{1}{x^2+2} \ge \frac{7-2x}{18}, \quad x \ge 0.$$

Consider further two cases: $c \ge 1/2$ and $c \le 1/2$.

Case 1: $c \ge \frac{1}{2}$. By summing the inequalities

$$\frac{1}{a^2+2} \ge \frac{5-2a}{9}, \quad \frac{1}{b^2+2} \ge \frac{5-2b}{9}, \quad \frac{1}{c^2+2} \ge \frac{5-2c}{9},$$

we get

$$\frac{1}{a^2+2} + \frac{1}{b^2+2} + \frac{1}{c^2+2} \ge \frac{15-2(a+b+c)}{9} = 1.$$

Case 2: $c \le \frac{1}{2}$. We have

$$\frac{1}{a^2+2} \ge \frac{7-2a}{18}.$$

Consider now the similar inequalities

$$\frac{1}{b^2+2} \ge \frac{B-2b}{18},$$

$$\frac{1}{c^2+2} \ge \frac{C-2c}{18},$$

which are satisfied as equalities for $b=1$ and $c=0$ if $B=8$ and $C=9$:

$$\frac{1}{b^2+2} \ge \frac{8-2b}{18},$$

$$\frac{1}{c^2+2} \ge \frac{9-2c}{18}$$

Since

$$\frac{1}{b^2+2} - \frac{8-2b}{18} = \frac{(1-b)(1+3b-b^2)}{9(b^2+2)}$$

and

$$\frac{1}{c^2+2} - \frac{9-2c}{18} = \frac{c(1-2c)(4-c)}{18(c^2+2)},$$

these inequalities holds for $0 \le b \le 1$ and $0 \le c \le 1/2$. Therefore, we have

$$\frac{1}{a^2+2} + \frac{1}{b^2+2} + \frac{1}{c^2+2} \ge \frac{7-2a}{18} + \frac{8-2b}{18} + \frac{9-2c}{18} = 1.$$

□

P 1.161. *Let a,b,c be real numbers such that*

$$a \ge b \ge 1 \ge c \ge -5, \qquad a+b+c=3.$$

Prove that

$$\frac{6}{a^3+b^3+c^3} + 1 \ge \frac{8}{a^2+b^2+c^2}.$$

(Vasile C., 2015)

Solution. First, we will show that

$$a^3 + b^3 + c^3 > 0.$$

Indeed, for the nontrivial case $-5 \le c \le -2$, we have

$$\begin{aligned} 4(a^3 + b^3 + c^3) &\ge (a + b)^3 + 4c^3 = (3 - c)^3 + 4c^3 \\ &= 3c^3 + 9c^2 - 27c + 27 \ge -15c^2 + 9c^2 - 27c + 27 \\ &= 3(-2c^2 - 9c + 9) > 3(-2c^2 - 9c + 5) = 3(c + 5)(1 - 2c) > 0. \end{aligned}$$

From

$$(a - 1)(b - 1)(c - 1) \le 0,$$

we get

$$r \le q - 2,$$

where $q = ab + bc + ca$ and $r = abc$. Write the desired inequality as follows:

$$\frac{2}{r + 9 - 3q} + 1 \ge \frac{8}{9 - 2q}.$$

Since

$$r + 9 - 3q \le (q - 2) + 9 - 3q = 7 - 2q,$$

it suffices to show that

$$\frac{2}{7 - 2q} + 1 \ge \frac{8}{9 - 2q}.$$

This is equivalent to the obvious inequality

$$(2q - 5)^2 \ge 0.$$

The equality holds for $a = 1 + \frac{1}{\sqrt{2}}$, $b = 1$, $c = 1 - \frac{1}{\sqrt{2}}$.

□

P 1.162. *If $a \ge 1 \ge b \ge c > -3$ such that $ab + bc + ca = 3$, then*

$$\frac{1}{a^2 + ab + b^2} + \frac{1}{b^2 + bc + c^2} + \frac{1}{c^2 + ca + a^2} \ge 1.$$

(Vasile C., 2015)

Solution. We will show first that $c > -1$ and $p > 0$, where $p = a + b + c$. We have

$$p \ge 1 + c + c = 1 + 2c,$$

hence

$$p - c \ge c + 1.$$

On the other hand, from

$$(a-1)(b-1) \le 0,$$

we find

$$ab-(a+b)+1 \le 0,$$

$$3-c(a+b)-(a+b)+1 \le 0,$$

$$4 \le (c+1)(a+b),$$

$$4 \le (c+1)(p-c),$$

hence

$$p(c+1) \ge c^2+c+4 > 0.$$

From $p(c+1) > 0$, it follows that $c > -1$ involves $p > 0$. To show that $c > -1$, we use the contradiction method. The case $c = -1$ contradicts the inequality $(c+1)(p-c) \ge 4$, and the case $c < -1$ leads to

$$p-c \le \frac{4}{c+1},$$

$$c+1 \le \frac{4}{c+1},$$

$$(c+1)^2 \ge 4,$$

hence $c \le -3$, which is false. Therefore, we have $c > -1$ and $p > 0$. According Lemma below, we can write the inequality as

$$p^3abc-27+(p^2-9)^2 \ge 0.$$

From $(a-1)(b-1)(c-1) \ge 0$, we get

$$abc \ge 4-p.$$

Thus,

$$p^3abc-27+(p^2-9)^2 \ge p^3(4-p)-27+(p^2-9)^2 = 2(2p+3)(p-3)^2 \ge 0.$$

The equality holds for $a = b = c = 1$.

Lemma. *Let a,b,c be real numbers, $p = a+b+c$ and $q = ab+bc+ca$. If $q > 0$, then the inequality*

$$\frac{1}{a^2+ab+b^2}+\frac{1}{b^2+bc+c^2}+\frac{1}{c^2+ca+a^2} \ge \frac{3}{ab+bc+ca}$$

is equivalent to

$$3(p^3abc-q^3)+q(p^2-3q)^2 \ge 0.$$

Proof. Write the inequality as

$$q\sum(x+ab-c^2)(x+ac-b^2)\ge 3\prod(x+bc-a^2),$$

where

$$x=a^2+b^2+c^2=p^2-2q.$$

From

$$\sum(ab-c^2)(ac-b^2)=q^2-xq,$$

$$\sum(x+ab-c^2)(x+ac-b^2)=x^2+xq+q^2$$

and

$$\prod(bc-a^2)=q^3-p^3abc,$$

$$\prod(x+bc-a^2)=xq^2+q^3-p^3abc,$$

the conclusion follows.

□

P 1.163. *If $a\ge b\ge 1\ge c\ge 0$ such that $a+b+c=3$, then*

$$\frac{1}{a^2+ab+b^2}+\frac{1}{b^2+bc+c^2}+\frac{1}{c^2+ca+a^2}\le\frac{3}{ab+bc+ca}.$$

(Vasile C., 2015)

Solution. By Lemma from the preceding P 1.162, we need to show that

$$3(p^3abc-q^3)+q(p^2-3q)^2\le 0,$$

where $p=3$ and $q=ab+bc+ca$; that is

$$27abc-q^3+3q(3-q)^2\le 0.$$

From $p^2\ge 3q$, we get $q\le 3$, and from $(a-1)(b-1)(c-1)\le 0$, we get

$$abc\le q-2,\quad q\ge 2.$$

Thus,

$$27abc-q^3+3q(3-q)^2\le 27(q-2)-q^3+3q(3-q)^2=2(q-3)^3\le 0.$$

Thus, the proof is completed. The equality holds for $a=b=c=1$.

Remark. Actually, the inequality holds for

$$a\ge b\ge 1\ge c\ge 1-\sqrt{3}.$$

To prove this, it suffices to show that $ab+bc+ca\ge 0$. Indeed, we have

$$\begin{aligned}ab+bc+ca&=(a-1)(b-1)-1+a+b+c(a+b)\ge -1+(1+c)(a+b)\\&=-1+(1+c)(3-c)\ge 0.\end{aligned}$$

□

P 1.164. *If a,b,c are positive real numbers such that*

$$a \ge 1 \ge b \ge c, \qquad abc = 1,$$

then

$$\frac{1-a}{3+a^2} + \frac{1-b}{3+b^2} + \frac{1-c}{3+c^2} \ge 0.$$

(Vasile C., 2009)

First Solution. Denote the left side of the inequality by $E(a,b,c)$. We will show that

$$E(a,b,c) \ge E(ab,1,c) \ge 0.$$

Let

$$a+b=s, \quad ab=p.$$

We have

$$p \ge abc = 1, \quad s \ge 2\sqrt{p} \ge 2.$$

Therefore,

$$\begin{aligned} E(a,b,c) - E(ab,1,c) &= \frac{1-a}{3+a^2} + \frac{1-b}{3+b^2} + \frac{ab-1}{3+a^2b^2} \\ &= \frac{s^2-(3+p)s+2(3-p)}{3s^2+(p-3)^2} + \frac{p-1}{3+p^2} \\ &= \frac{(3+p)(s-p-1)(ps+p-3)}{(3+p^2)[3s^2+(p-3)^2]}. \end{aligned}$$

Since

$$s-p-1 = (a-1)(1-b) \ge 0, \quad ps+p-3 \ge 2p+p-3 \ge 0,$$

it follows that

$$E(a,b,c) - E(ab,1,c) \ge 0.$$

Also, we have

$$E(ab,1,c) = E(1/c,1,c) = \frac{(1-c)^4}{(3c^2+1)(3+c^2)} \ge 0.$$

The equality holds for $a=b=c=1$.

Second Solution. Let $p=a+b+c$ and $q=ab+bc+ca$. From

$$(a-1)(b-1)(c-1) \ge 0,$$

we get

$$p \ge q.$$

The desired inequality is true because it is equivalent to

$$\sum (1-a)(9+3b^2+3c^2+b^2c^2) \ge 0,$$

$$27+6\sum a^2+\sum b^2c^2-9p-3pq+9-q\ge 0,$$
$$27+6(p^2-2q)+(q^2-2p)-9p-3pq+9-q\ge 0,$$
$$6p^2+q^2-3pq-11p-13q+36\ge 0,$$
$$(p+q-6)^2+5p^2-5pq+p-q\ge 0,$$
$$(p+q-6)^2+(5p+1)(p-q)\ge 0.$$

□

P 1.165. *If a,b,c are positive real numbers such that*

$$a\ge 1\ge b\ge c,\qquad abc=1,$$

then

$$\frac{1}{\sqrt{3a+1}}+\frac{1}{\sqrt{3b+1}}+\frac{1}{\sqrt{3c+1}}\ge\frac{3}{2}.$$

(Vasile C., 2007)

Solution. Let

$$b_1=1/b,\qquad b_1\ge 1.$$

We claim that

$$\frac{1}{\sqrt{3b+1}}+\frac{1}{\sqrt{3b_1+1}}\ge\frac{1}{2}.$$

This inequality is equivalent to

$$\frac{1}{\sqrt{3b+1}}+\sqrt{\frac{b}{b+3}}\ge\frac{1}{2}.$$

Making the substitution

$$\frac{1}{\sqrt{3b+1}}=t,\quad \frac{1}{2}\le t<1,$$

the inequality becomes

$$\sqrt{\frac{1-t^2}{1+8t^2}}\ge 1-t.$$

By squaring, we get

$$t(1-t)(1-2t)^2\ge 0,$$

which is clearly true. Similarly, we have

$$\frac{1}{\sqrt{3c+1}}+\frac{1}{\sqrt{3c_1+1}}\ge\frac{1}{2},$$

where

$$c_1 = 1/c, \qquad c_1 \ge 1.$$

Using these inequalities, it suffices to show that

$$\frac{1}{\sqrt{3a+1}} + \frac{1}{2} \ge \frac{1}{\sqrt{3b_1+1}} + \frac{1}{\sqrt{3c_1+1}},$$

which is equivalent to

$$\frac{1}{\sqrt{3b_1c_1+1}} + \frac{1}{2} \ge \frac{1}{\sqrt{3b_1+1}} + \frac{1}{\sqrt{3c_1+1}}.$$

According to P 2.88 in Volume 2, the conclusion follows. The equality holds for $a = b = c = 1$.

□

P 1.166. *If a, b, c are positive real numbers such that*

$$a \ge 1 \ge b \ge c, \qquad abc = 1,$$

then

$$\frac{1}{a^2+4ab+b^2} + \frac{1}{b^2+4bc+c^2} + \frac{1}{c^2+4ca+a^2} \ge \frac{1}{2}.$$

(Vasile C., 2015)

Solution. Write the inequality as

$$2E \ge F,$$

where

$$E = \sum (a^2+4ab+b^2)(a^2+4ac+c^2), \qquad F = \prod (b^2+4bc+c^2).$$

Using Lemma below for $k = 4$ and $r = 1$, we get

$$E = 18pr + p^4 - 3q^2 = 18p + p^4 - 3q^2,$$

$$F = 27r^2 + 2p^3r + p^2q^2 + 2q^3 = 27 + 2p^3 + p^2q^2 + 2q^3,$$

hence

$$2E - F = 2p^4 - 2p^3 + 36p - 27 - (p^2+6)q^2 - 2q^3.$$

From $(a-1)(b-1)(c-1) \ge 0$, we get

$$p \ge q.$$

Thus,

$$\begin{aligned}2E-F &\geq 2p^4-2p^3+36p-27-(p^2+6)p^2-2p^3\\ &= p^4-4p^3-6p^2+36p-27=(p-1)(p-3)^2(p+3)\geq 0.\end{aligned}$$

Thus, the proof is completed. The equality holds for $a=b=c=1$.

Lemma. *If a,b,c are real numbers,*

$$p=a+b+c, \quad q=ab+bc+ca, \quad r=abc$$

and

$$E=\sum(a^2+kab+b^2)(a^2+kac+c^2), \qquad F=\prod(b^2+kbc+c^2),$$

then

$$E=(k-1)(k+2)pr+p^4+(k-4)p^2q+(5-2k)q^2,$$

$$F=(k-1)^3r^2+[(k-2)p^2+(k-1)(k-4)q]pr+p^2q^2+(k-2)q^3.$$

Proof. Let

$$x=a^2+b^2+c^2=p^2-2q.$$

Since

$$\begin{aligned}E&=\sum(x+kab-c^2)(x+kac-b^2)\\ &=x^2+kxq+(k-1)(k+2)pr+q^2\end{aligned}$$

and

$$\begin{aligned}F&=\prod(x+kbc-a^2)\\ &=x[(k-1)(k+2)pr+q^2]+(k-1)^3r^2-k[kp^2-3(k-1)q]pr+kq^3,\end{aligned}$$

the conclusion follows.

□

P 1.167. *Let $a\geq 1\geq b\geq c\geq 0$ such that*

$$a+b+c=3, \qquad ab+bc+ca=q,$$

where $q\in[0,3]$ is a fixed number. Prove that the product $r=abc$ is maximal for $b=c$, and minimal for $b=1$ or $c=0$.

(Vasile C., 2015)

Solution. For $q = 3$, from $(a+b+c)^2 = 3(ab+bc+ca)$, which is equivalent to

$$(a-b)^2 + (b-c)^2 + (c-a)^2 = 0,$$

we get $a = b = c = 1$. Consider further that $q \in [0,3)$, when $a > 1 \geq b \geq c \geq 0$. We will show first that $c \in [c_1,\ c_2]$, where

$$c_1 = \begin{cases} 1-\sqrt{3-q}, & 2 \leq q < 3 \\ 0, & 0 \leq q \leq 2 \end{cases}$$

and

$$c_2 = 1 - \sqrt{1-\frac{q}{3}}.$$

From

$$(a-1)(b-1) \leq 0,$$

which is equivalent to

$$ab-(a+b)+1 \leq 0, \quad q-(a+b)(c+1)+1 \leq 0, \quad q-(3-c)(c+1)+1 \leq 0,$$

we get

$$c^2 - 2c + q - 2 \leq 0,$$

hence $c \geq 1-\sqrt{3-q}$. In the case $2 \leq q < 3$, when $1-\sqrt{3-q} \geq 0$, the equality $c = 1-\sqrt{3-q}$ is possible because it implies

$$b = 1, \quad a = 1+\sqrt{3-q} \geq 1.$$

In the case $0 \leq q \leq 2$, the equality $c = 0$ is possible because it implies $a+b = 3$ and $ab = q$, hence

$$a = \frac{3+\sqrt{9-4q}}{2} \geq 1, \quad b = \frac{3-\sqrt{9-4q}}{2} \in [0,1].$$

In conclusion, we have $c \geq c_1$ in all cases, with equality for $b = 1$ or $c = 0$. Also, from

$$(b-c)(a-c) = c^2 - 2c(a+b) + q = c^2 - 2c(3-c) + q = 3c^2 - 6c + q \geq 0,$$

we get $c \leq c_2$, with equality for $b = c$. On the other hand, from

$$abc = c[q-(a+b)c] = c[q-(3-c)c],$$

we get

$$r(c) = c^3 - 3c^2 + qc.$$

Since

$$r'(c) = 3c^2 - 6c + q = 3c^2 - 2(a+b+c)c + q = (c-a)(c-b) \geq 0,$$

$r(c)$ is strictly increasing on $[c_1, c_2]$, and hence $r(c)$ is minimal for $c = c_1$, when $b = 1$ or $c = 0$, and is maximal for $c = c_2$, when $b = c$.

$\square$

P 1.168. *Let p and q be fixed real numbers such that there exist three real numbers a,b,c satisfying*

$$a \ge 1 \ge b \ge c \ge 0, \qquad a+b+c=p, \qquad ab+bc+ca=q.$$

Prove that

(a) the product $r = abc$ is maximal for $b = c$;

(b) the product $r = abc$ is minimal for $a = 1$ or $b = 1$ or $c = 0$.

(Vasile C., 2015)

Solution. (a) According to P 3.57 in Volume 1, under the weaker condition $a \ge b \ge c \ge 0$ instead of $a \ge 1 \ge b \ge c \ge 0$, the product $r = abc$ is maximal for $b = c$, when

$$a = \frac{p+2\sqrt{p^2-3q}}{3}, \quad b = c = \frac{p-\sqrt{p^2-3q}}{3}.$$

Thus, it suffices to show that

$$\frac{p+2\sqrt{p^2-3q}}{3} \ge 1 \ge \frac{p-\sqrt{p^2-3q}}{3}.$$

The left inequality is true if

$$4(p^2-3q) \ge (3-p)^2,$$

which is equivalent to

$$(p+1)^2 \ge 4(q+1);$$

indeed,

$$(p+1)^2 - 4(q+1) = (b-c)^2 + (a-1)(a+3-2b-2c) \ge 0.$$

The right inequality is equivalent to

$$\sqrt{p^2-3q} \ge p-3.$$

This is true if $p^2-3q \ge (p-3)^2$ for $p \ge 3$; indeed,

$$\begin{aligned}\frac{p^2-3q-(p-3)^2}{3} &= 2p-q-3\\ &= (a-1)(1-b)+(1-c)(a+b-2)\\ &= (a-1)(1-b)+(1-c)[(1-c)+(p-3)] \ge 0.\end{aligned}$$

(b) We will show that abc is minimal for $a = 1$ or $b = 1$ if $p \le q+1$, and for $c = 0$ if $p \ge q+1$.

Case 1: $p \le q+1$. From

$$(a-1)(b-1)(c-1) \ge 0,$$

we get

$$abc \geq ab + bc + ca - a - b - c + 1 = q - p + 1 \geq 0,$$

with equality for $a = 1$ or $b = 1$. If one of a, b is 1, then the other two of a, b, c are

$$x = \frac{p - 1 + \sqrt{D}}{2}, \quad c = \frac{p - 1 - \sqrt{D}}{2},$$

where

$$\begin{aligned} D &= (p+1)^2 - 4(q+1) \\ &= (b-c)^2 + (a-1)(a+3-2b-2c) \geq 0. \end{aligned}$$

We only need to show that $c \geq 0$, which is equivalent to

$$p - 1 \geq \sqrt{D},$$

$$p \leq q + 1.$$

Case 2: $p \geq q + 1$. We will show that abc is minimal for $c = 0$. For this, we only need to prove that there exist two real numbers a and b such that

$$a \geq 1 \geq b \geq 0, \quad a + b = p, \ \ ab = q.$$

Since

$$a = \frac{p + \sqrt{p^2 - 4q}}{2}, \quad b = \frac{p - \sqrt{p^2 - 4q}}{2},$$

where

$$p^2 - 4q \geq (q+1)^2 - 4q = (q-1)^2 \geq 0,$$

the inequality $a \geq 1$ is equivalent to

$$\sqrt{p^2 - 4q} \geq 2 - p,$$

while the inequality $b \leq 1$ is equivalent to

$$\sqrt{p^2 - 4q} \geq p - 2.$$

These inequalities are true if

$$p^2 - 4q \geq (p-2)^2,$$

which reduces to $p \geq q + 1$.

$\square$

P 1.169. *Let p and q be fixed real numbers such that there exist three real numbers a,b,c satisfying*

$$a \ge b \ge c \ge 1, \qquad a+b+c=p, \qquad ab+bc+ca=q.$$

Prove that

(a) the product $r=abc$ is maximal for $b=c$;

(b) the product $r=abc$ is minimal for $a=b$ or $c=1$.

(Vasile C., 2015)

Solution. From $a \ge b \ge c \ge 1$, it follows that

$$p=a+b+c \ge 3.$$

(a) According to P 3.57 in Volume 1, under the weaker condition $a \ge b \ge c \ge 0$ instead of $a \ge b \ge c \ge 1$, the product $r=abc$ is maximal for $b=c$, when

$$a=\frac{p+2\sqrt{p^2-3q}}{3}, \quad b=c=\frac{p-\sqrt{p^2-3q}}{3}.$$

Thus, it suffices to show that

$$\frac{p-\sqrt{p^2-3q}}{3} \ge 1,$$

which is equivalent to

$$p-3 \ge \sqrt{p^2-3q},$$

$$(p-3)^2 \ge p^2-3q,$$

$$q+3 \ge 2p.$$

We have

$$q+3-2p=(a-1)(b-1)+(b-1)(c-1)+(c-1)(a-1) \ge 1.$$

(b) We will show that abc is minimal for $a=b$ if $p+1 \le 2\sqrt{q+1}$, and for $c=1$ if $p+1 \ge 2\sqrt{q+1}$.

Case 1: $p+1 \le 2\sqrt{q+1}$. According to P 2.53 in Volume 1, under the weaker condition $a \ge b \ge c$ instead of $a \ge b \ge c \ge 1$, the product $r=abc$ is minimal for $a=b$, when

$$a=b=\frac{p+\sqrt{p^2-3q}}{3}, \quad c=\frac{p-2\sqrt{p^2-3q}}{3}.$$

Thus, it suffices to show that

$$\frac{p-2\sqrt{p^2-3q}}{3} \ge 1,$$

which is equivalent to

$$p-3 \geq 2\sqrt{p^2-3q},$$

$$(p-3)^2 \geq 4(p^2-3q),$$

$$(p+1)^2 \leq 4(q+1),$$

$$p+1 \leq 2\sqrt{q+1}.$$

Case 2: $p+1 \geq 2\sqrt{q+1}$. From

$$(a-1)(b-1)(c-1) \geq 0,$$

we get

$$abc \geq ab+bc+ca-a-b-c+1 = q-p+1 \geq 0,$$

with equality for $c=1$. In addition, $c=1$ involves

$$a = \frac{p-1+\sqrt{D}}{2}, \quad b = \frac{p-1-\sqrt{D}}{2},$$

where

$$D = (p+1)^2 - 4(q+1) \geq 0.$$

To end the proof, it suffices to show that

$$\frac{p-1-\sqrt{D}}{2} \geq 1,$$

which is equivalent to

$$p-3 \geq \sqrt{D},$$

$$(p-3)^2 \geq (p+1)^2 - 4(q+1)),$$

$$q+3 \geq 2p,$$

$$(a-1)(b-1)+(b-1)(c-1)+(c-1)(a-1) \geq 0.$$

□

P 1.170. *Let $a \geq b \geq 1 \geq c \geq 0$ such that*

$$a+b+c=3, \quad ab+bc+ca=q,$$

where $q \in [0,3]$ is a fixed number. Prove that the product $r=abc$ is maximal for $b=1$, and minimal for $a=b$ or $c=0$.

(Vasile C., 2015)

Solution. From
$$ab+bc+ca \le \frac{1}{3}(a+b+c)^2 = 3$$
and
$$q-3 = ab+(a+b)c-a-b-c = (a-1)(b-1)+(a+b-1)c-1 \ge -1,$$
it follows that $2 \le q \le 3$. Since $q=2$ involves $b=1$ and $c=0$, and $q=3$ involves $a=b=c=1$, we consider further that $q \in (2,3)$, when $a \ge b \ge 1 > c \ge 0$. We will show first that $c \in [c_1,\ c_2]$, where
$$c_1 = \begin{cases} 1-2\sqrt{1-q/3}, & 9/4 \le q < 3 \\ 0, & 2 < q \le 9/4 \end{cases}$$
and
$$c_2 = 1-\sqrt{3-q}.$$
From
$$\begin{aligned}(a-b)^2 &= (a+b)^2-4ab = (a+b)^2+4c(a+b)-4q \\ &= (3-c)^2+4c(3-c)-4q = -3c^2+6c+9-4q,\end{aligned}$$
it follows that
$$3c^2-6c+4q-9 \le 0,$$
hence $c \ge 1-2\sqrt{1-q/3}$. In the case $9/4 \le q < 3$, when $1-2\sqrt{1-q/3} \ge 0$, the equality $c = 1-2\sqrt{1-q/3}$ is possible because it implies
$$a = b = 1+\sqrt{1-q/3} \ge 1.$$
In the case $2 < q \le 9/4$, the equality $c=0$ is possible because it implies $a+b=3$ and $ab=q$, hence
$$a = \frac{3+\sqrt{9-4q}}{2}, \qquad b = \frac{3-\sqrt{9-4q}}{2} > 1.$$
In conclusion, we have $c \ge c_1$ in all cases, with equality for $a=b$ or $c=0$. Also, from
$$(a-1)(b-1) \ge 0,$$
which is equivalent to
$$ab-(a+b)+1 \ge 0, \quad q-(a+b)(c+1)+1 \ge 0, \quad q-(3-c)(c+1)+1 \ge 0,$$
we get
$$c^2-2c+q-2 \ge 0,$$

hence $c \le c_2$, with equality for $b = 1$. On the other hand, from

$$abc = c[q - (a+b)c] = c[q - (3-c)c],$$

we get

$$r(c) = c^3 - 3c^2 + qc.$$

Since

$$r'(c) = 3c^2 - 6c + q = 3c^2 - 2(a+b+c)c + q = (c-a)(c-b) \ge 0,$$

$r(c)$ is strictly increasing on $[c_1, c_2]$, and hence $r(c)$ is minimal for $c = c_1$, when $a = b$ or $c = 0$, and is maximal for $c = c_2$, when $b = 1$.

□

P 1.171. *Let p and q be fixed real numbers such that there exist three real numbers a, b, c satisfying*

$$a \ge b \ge 1 \ge c \ge 0, \qquad a+b+c = p, \qquad ab+bc+ca = q.$$

Prove that

(a) *the product $r = abc$ is maximal for $b = 1$ or $c = 1$;*

(b) *the product $r = abc$ is minimal for $a = b$ or $c = 0$.*

(Vasile C., 2015)

Solution. (a) From

$$(a-1)(b-1)(c-1) \le 0,$$

we get

$$abc \le q - p + 1,$$

with equality for $b = 1$ or $c = 1$. If one of $b,\ c$ is 1, then the other two of a, b, c are

$$a = x = \frac{p - 1 + \sqrt{D}}{2}, \quad y = \frac{p - 1 - \sqrt{D}}{2},$$

where

$$D = (p+1)^2 - 4(q+1).$$

Notice that

$$D = (a-b)^2 + (1-c)(2a+2b-c-3) \ge 0,$$
$$x \ge 1,$$
$$xy = q - p + 1 = (a-1)(b-1) + c(a+b-1) \ge 0, \quad y \ge 0.$$

The inequality $x \ge 1$ is equivalent to $\sqrt{D} \ge 3-p$, which is true if $p \le 3$ involves

$$D \ge (3-p)^2.$$

Indeed,

$$\begin{aligned}\frac{D-(3-p)^2}{4} &= 2p-q-3\\ &= (b-1)(1-c)+(a-1)(2-b-c)\\ &= (b-1)(1-c)+(a-1)[(a-1)+(3-p)] \ge 0.\end{aligned}$$

Also, we have $y \le 1$ for $p \le 3$ or $p \ge (q+3)/2$, and $y \ge 1$ for $3 \le p \le (q+3)/2$. Therefore, there is a unique point (a,b,c) such that the product $r = abc$ is maximal:

$$(a,b,c) = \left(\frac{p-1+\sqrt{D}}{2}, 1, \frac{p-1-\sqrt{D}}{2}\right)$$

for $2 \le p \le 3$ or $p \ge \dfrac{q+3}{2}$;

$$(a,b,c) = \left(\frac{p-1+\sqrt{D}}{2}, \frac{p-1-\sqrt{D}}{2}, 1\right)$$

for $3 \le p \le \dfrac{q+3}{2}$.

(b) According to P 3.57 in Volume 1, under the weaker condition $a \ge b \ge c \ge 0$ instead of $a \ge b \ge 1 \ge c \ge 0$, the product $r = abc$ is minimal for $a = b$ (if $p^2 \le 4q$) or $c = 0$ (if $p^2 \ge 4q$).

For $a = b$, we have

$$a = b = \frac{p+\sqrt{p^2-3q}}{3}, \qquad c = \frac{p-2\sqrt{p^2-3q}}{3}.$$

Thus, it suffices to show that

$$\frac{p+\sqrt{p^2-3q}}{3} \ge 1,$$

$$\frac{p-2\sqrt{p^2-3q}}{3} \le 1.$$

The first inequality holds if $p \le 3$ involves

$$(p^2-3q) \ge (3-p)^2,$$

that is

$$2p-q-3 \ge 0.$$

We have

$$\begin{aligned}
2p-q-3 &= 2(a+b)-ab-3-(a+b-2)c \\
&\ge 2(a+b)-\frac{1}{4}(a+b)^2-3-(a+b-2)c \\
&= \frac{(a+b-2)(6-a-b)-4(a+b-2)c}{4} \\
&= \frac{(a+b-2)(6-a-b-4c)}{4} \\
&= \frac{(a+b-2)[(3-p)+3(1-c)]}{4} \ge 0.
\end{aligned}$$

The second inequalities holds if $p \ge 3$ implies

$$4(p^2-3q) \ge (p-3)^2,$$

which is equivalent to the obvious inequality

$$(a-b)^2+(1-c)(2a+2b-c-3) \ge 0.$$

For $c=0$, we have

$$a = \frac{p+\sqrt{p^2-4q}}{2}, \quad b = \frac{p-\sqrt{p^2-4q}}{2}, \quad c = 0.$$

Thus, it suffices to show that

$$\frac{p-\sqrt{p^2-4q}}{2} \ge 1,$$

that is

$$p-2 \ge \sqrt{p^2-4q}.$$

Since $p-2=(a-1)+(b-1)+c \ge 0$, we only need to show that

$$(p-2)^2 \ge p^2-4q,$$

which is equivalent to

$$q+1-p \ge 0,$$

$$(a-1)(b-1)+(a+b-1)c \ge 0.$$

□

P 1.172. *Let p and q be fixed real numbers such that there exist three real numbers a, b, c satisfying*

$$1 \ge a \ge b \ge c \ge 0, \qquad a+b+c=p, \qquad ab+bc+ca=q.$$

Prove that

(a) *the product $r = abc$ is maximal for $b = c$ or $a = 1$;*

(b) *the product $r = abc$ is minimal for $a = b$ or $c = 0$.*

(Vasile C., 2015)

Solution. We have $p \le 3$ because

$$p-3=(a-1)+(b-1)+(c-1) \le 0.$$

(a) We will show that abc is maximal for $b = c$ if $p+1 \le 2\sqrt{q+1}$, and for $a=1$ if $p+1 \ge 2\sqrt{q+1}$.

Case 1: $p+1 \le 2\sqrt{q+1}$. According to P 3.57 in Volume 1, under the weaker condition $a \ge b \ge c \ge 0$ instead of $1 \ge a \ge b \ge c \ge 0$, the product $r = abc$ is maximal for $b = c$, when

$$a=\frac{p+2\sqrt{p^2-3q}}{3}, \quad b=c=\frac{p-\sqrt{p^2-3q}}{3}.$$

Thus, it suffices to show that

$$\frac{p-\sqrt{p^2-3q}}{3} \ge 0$$

and

$$\frac{p+2\sqrt{p^2-3q}}{3} \le 1.$$

The first inequality is clearly true, and the second inequality is equivalent to

$$3-p \ge 2\sqrt{p^2-3q},$$

$$(3-p)^2 \ge 4(p^2-3q),$$

$$(p+1)^2 \le 4(q+1),$$

$$p+1 \le 2\sqrt{q+1}.$$

Case 2: $p+1 \ge 2\sqrt{q+1}$. From

$$(a-1)(b-1)(c-1) \ge 0,$$

we get

$$abc \ge ab+bc+ca-a-b-c+1 = q-p+1 \ge 0,$$

with equality for $a=1$. In addition, $a=1$ involves

$$b=\frac{p-1+\sqrt{D}}{2}, \quad c=\frac{p-1-\sqrt{D}}{2},$$

where

$$D=(p+1)^2-4(q+1)\geq 0.$$

To end the proof, it suffices to show that

$$\frac{p-1-\sqrt{D}}{2}\geq 0$$

and

$$\frac{p-1+\sqrt{D}}{2}\leq 1.$$

Write the first inequality as

$$p-1\geq\sqrt{D}.$$

Since

$$p\geq -1+2\sqrt{q+1}\geq -1+2=1,$$

the inequality is equivalent to

$$(p-1)^2\geq D,$$

$$1-p+q\geq 0,$$

$$(1-a)(1-b)(1-c)+abc\geq 0.$$

Write the second inequality as

$$3-p\geq\sqrt{D},$$

$$(3-p)^2\geq D,$$

$$q+3\geq 2p,$$

$$(1-a)(1-b)+(1-b)(1-c)+(1-c)(1-a)\geq 0.$$

(b) We will show that abc is minimal for $a=b$ if $p^2\leq 4q$, and for $c=0$ if $p^2\geq 4q$.

Case 1: $p^2\leq 4q$. According to P 2.53 in Volume 1, under the weaker condition $a\geq b\geq c$ instead of $1\geq a\geq b\geq c\geq 0$, the product $r=abc$ is minimal for $a=b$, when

$$a=b=\frac{p+\sqrt{p^2-3q}}{3}, \quad c=\frac{p-2\sqrt{p^2-3q}}{3}.$$

Thus, it suffices to show that

$$\frac{p-2\sqrt{p^2-3q}}{3}\geq 0$$

and
$$\frac{p+\sqrt{p^2-3q}}{3} \le 1.$$
Write the first inequality as
$$p \ge 2\sqrt{p^2-3q},$$
$$p^2 \ge 4(p^2-3q),$$
$$p^2 \le 4q.$$
Write now the second inequality as
$$3-p \ge \sqrt{p^2-3q},$$
$$(3-p)^2 \ge p^2-3q,$$
$$q+3 \ge 2p,$$
$$(1-a)(1-b)+(1-b)(1-c)+(1-c)(1-a) \ge 0.$$

Case 1: $p^2 \ge 4q$. From
$$0 \le p^2-4q = (a-b)^2 - c(a+b-c) \le (a-b)^2 - c^2 = (a-b-c)(a-b+c),$$
we get $a \ge b+c$, hence
$$p = a+b+c \le 2a \le 2.$$
For $c=0$, we have
$$a = \frac{p+\sqrt{p^2-4q}}{2}, \quad b = \frac{p-\sqrt{p^2-4q}}{2}, \quad c = 0.$$
Since $p-\sqrt{p^2-4q} \ge 0$, we only need to show that
$$\frac{p+\sqrt{p^2-4q}}{2} \le 1,$$
which is equivalent to
$$2-p \ge \sqrt{p^2-4q},$$
$$(2-p)^2 \ge p^2-4q,$$
$$1-p+q \ge 0,$$
$$(1-a)(1-b)(1-c)+abc \ge 0.$$

$\square$

P 1.173. *If $a \ge 1 \ge b \ge c \ge 0$ such that $a+b+c=3$, then*

$$abc + \frac{9}{ab+bc+ca} \ge 4.$$

(Vasile C., 2015)

Solution. Let

$$q = ab + bc + ca.$$

First Solution. According to P 1.167, for fixed q, the product abc is minimal when $b=1$ or $c=0$. Therefore, it suffices to consider these cases. If $b=1$, then $a+c=2$, and the inequality becomes

$$ac + \frac{9}{2+ac} \ge 4,$$

$$(ac-1)^2 \ge 0.$$

For $c=0$, we need to show that $a+b=3$ involves $4ab \le 9$. Indeed,

$$4ab < (a+b)^2 = 9.$$

The equality holds for $a=b=c=1$.

Second Solution. From $(a-1)(b-1)(c-1) \ge 0$, we get

$$abc \ge q-2.$$

Therefore,

$$abc + \frac{9}{ab+bc+ca} - 4 \ge q - 2 + \frac{9}{q} - 4 = \frac{(q-3)^2}{q} \ge 0.$$

□

P 1.174. *If $a \ge 1 \ge b \ge c \ge 0$ such that $a+b+c=3$, then*

$$abc + \frac{2}{ab+bc+ca} \ge \frac{5}{a^2+b^2+c^2}.$$

(Vasile C., 2015)

Solution. Let

$$q = ab + bc + ca, \quad q \le 3.$$

First Solution. According to P 1.167, for fixed q, the product abc is minimal when $b=1$ or $c=0$. Therefore, it suffices to consider these cases. For $b=1$, when $a+c=2$, the inequality becomes

$$ac + \frac{2}{2+ac} \ge \frac{5}{5-2ac},$$

$$ac(1-ac)(1+2ac) \geq 0.$$

The last inequality is true since

$$4 = (a+c)^2 \geq 4ac.$$

For $c = 0$, we need to show that $a + b = 3$ involves

$$\frac{2}{ab} \geq \frac{5}{9-2ab},$$

that is $ab \leq 2$. Indeed,

$$ab - 2 = ab - a - b + 1 = (a-1)(b-1) \leq 0.$$

The equality holds for $a = b = c = 1$, and also for $a = 2$, $b = 1$ and $c = 0$.

Second Solution. Write the inequality as

$$abc + \frac{2}{q} \geq \frac{5}{9-2q}.$$

Case 1: $q \leq 2$. We have

$$abc + \frac{2}{q} - \frac{5}{9-2q} \geq \frac{2}{q} - \frac{5}{9-2q} = \frac{9(q-2)}{q(9-2q)} \geq 0.$$

Case 2: $2 \leq q \leq 3$. From $(a-1)(b-1)(c-1) \geq 0$, we get

$$abc \geq q - 2,$$

hence

$$abc + \frac{2}{q} - \frac{5}{9-2q} \geq q - 2 + \frac{2}{q} - \frac{5}{9-2q} = \frac{(3-q)(q-2)(2q-3)}{q(9-2q)} \geq 0.$$

□

P 1.175. *If $a \geq b \geq 1 \geq c > 0$ such that $a + b + c = 3$, then*

$$\frac{1}{abc} + 2 \geq \frac{9}{ab + bc + ca}.$$

(Vasile C., 2015)

Solution. Let
$$q = ab + bc + ca.$$

First Solution. According to P 1.170, for fixed q, the product abc is maximal for $b = 1$. Therefore, it suffices to consider the case $b = 1$, when $a + c = 2$, and the inequality becomes
$$\frac{1}{ac} + 2 \ge \frac{9}{2 + ac},$$
$$(ac - 1)^2 \ge 0.$$
The equality holds for $a = b = c = 1$.

Second Solution. From $(a-1)(b-1)(c-1) \le 0$, we get
$$abc \le q - 2, \quad q > 2.$$
Thus, it suffices to show that
$$\frac{1}{q-2} + 2 \ge \frac{9}{q},$$
which is equivalent to
$$(q - 3)^2 \ge 0.$$

□

P 1.176. *If $a \ge b \ge 1 \ge c > 0$ such that $a + b + c = 3$, then*
$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 11 \ge 4(a^2 + b^2 + c^2).$$

(Vasile C., 2015)

Solution. Let
$$q = ab + bc + ca.$$

First Solution. Write the inequality as
$$\frac{q}{abc} + 8q \ge 25.$$
According to P 1.170, for fixed q, the product abc is maximal when $b = 1$. Therefore, it suffices to consider the case $b = 1$, when $a + c = 2$, and the inequality becomes
$$\frac{1}{ac} + 4ac \ge 4,$$
$$(2ac - 1)^2 \ge 0.$$
The equality holds for $a = 1 + \frac{1}{\sqrt{2}}$, $b = 1$ and $a = 1 - \frac{1}{\sqrt{2}}$.

Second Solution. From $(a-1)(b-1)(c-1) \le 0$, we get

$$abc \le q-2, \quad q > 2.$$

Thus, it suffices to show that

$$\frac{q}{q-2} + 11 \ge 4(9-2q),$$

which is equivalent to

$$(2q-5)^2 \ge 0.$$

□

P 1.177. *If $a \ge b \ge 1 \ge c > 0$ such that $a+b+c=3$, then*

$$\frac{1}{abc} + \frac{2}{a^2+b^2+c^2} \ge \frac{5}{ab+bc+ca}.$$

(Vasile C., 2015)

Solution. Let

$$q = ab+bc+ca.$$

First Solution. According to P 1.170, for fixed q, the product abc is maximal when $b=1$. Therefore, it suffices to consider the case $b=1$, when the inequality becomes

$$\frac{1}{ac} + \frac{2}{5-2ac} \ge \frac{5}{2+ac},$$

$$(ac-1)^2 \ge 0.$$

The equality holds for $a=b=c=1$.

Second Solution. From $(a-1)(b-1)(c-1) \le 0$, we get

$$abc \le q-2, \quad q > 2.$$

Thus, it suffices to show that

$$\frac{1}{q-2} + \frac{2}{9-2q} \ge \frac{5}{q},$$

which is equivalent to

$$(q-3)^2 \ge 0.$$

□

P 1.178. *If $a \ge b \ge 1 \ge c \ge 0$ such that $a+b+c=3$, then*

$$\frac{9}{a^3+b^3+c^3}+2 \le \frac{15}{a^2+b^2+c^2}.$$

(Vasile C., 2015)

Solution. Write the inequality as

$$\frac{3}{abc+9-3q}+2 \le \frac{15}{9-2q},$$

where

$$q = ab+bc+ca.$$

From

$$3q \le (a+b+c)^2 = 9$$

and

$$q = (1-a)(1-b)(1-c)+abc-1+a+b+c \ge -1+a+b+c = 2,$$

it follows that

$$2 \le q \le 3.$$

First Solution. Consider the following two cases.

Case 1: $2 \le q \le 9/4$. Since $abc \ge 0$, it suffices to prove that

$$\frac{1}{3-q}+2 \le \frac{15}{9-2q},$$

which is equivalent to the obvious inequality

$$(4q-9)(q-2) \le 0.$$

Case 2: $9/4 \le q \le 3$. By Schur's inequality of third degree

$$(a+b+c)^3+9abc \ge 4(a+b+c)(ab+bc+ca),$$

we get

$$3abc \ge 4q-9.$$

Therefore, it suffices to show that

$$\frac{9}{4q-9+3(9-3q)}+2 \le \frac{15}{9-2q},$$

which is equivalent to

$$\frac{9}{18-5q}+2 \le \frac{15}{9-2q},$$

$$4q^2 - 21q + 27 \le 0,$$

$$(q-3)(4q-9) \le 0.$$

The equality holds for $a = b = c = 1$, for $a = b = 3/2$ and $c = 0$, and also for $a = 2$, $b = 1$ and $c = 0$.

Second Solution. According to P 1.170, for fixed q, the product abc is minimal when $a = b$ or $c = 0$. Therefore, it suffices to consider these cases.

Case 1: $a = b \in [1, 3/2]$. The desired inequality is equivalent to

$$\frac{9}{2a^3 + (3-2a)^3} + 2 \le \frac{15}{2a^2 + (3-2a)^2}.$$

$$(a-1)^2(3-2a)(9a - 2a^2 - 3) \ge 0,$$

which is true since

$$9a - 2a^2 - 3 > 3(3a - a^2 - 2) = 3(a-1)(2-a) \ge 0.$$

Case 2: $c = 0$. We have $2 \le q \le 9/4$, because

$$q = ab \le \frac{1}{4}(a+b)^2 = \frac{9}{4}.$$

The desired inequality is equivalent to

$$\frac{1}{3-q} + 2 \le \frac{15}{9-2q},$$

$$(4q-9)(q-2) \le 0.$$

Clearly, the last inequality is true.

□

P 1.179. *If $a \ge b \ge 1 \ge c \ge 0$ such that $a + b + c = 3$, then*

$$\frac{36}{a^3 + b^3 + c^3} + 9 \le \frac{65}{a^2 + b^2 + c^2}.$$

(Vasile C., 2015)

Solution. Write the inequality as

$$\frac{12}{abc + 9 - 3q} + 9 \le \frac{65}{9 - 2q},$$

where

$$q = ab + bc + ca.$$

From
$$3q \le (a+b+c)^2 = 9$$
and
$$q = (1-a)(1-b)(1-c) + abc - 1 + a + b + c \ge -1 + a + b + c = 2,$$
it follows that
$$2 \le q \le 3.$$

First Solution. Consider the following two cases.
Case 1: $2 \le q \le 7/3$. Since $abc \ge 0$, it suffices to prove that
$$\frac{4}{3-q} + 9 \le \frac{65}{9-2q},$$
which is equivalent to the obvious inequality
$$(3q-7)(q-2) \le 0.$$
Case 2: $7/3 \le q \le 3$. By Schur's inequality of third degree
$$(a+b+c)^3 + 9abc \ge 4(a+b+c)(ab+bc+ca),$$
we get
$$3abc \ge 4q - 9.$$
Therefore, it suffices to show that
$$\frac{36}{4q-9+3(9-3q)} + 9 \le \frac{65}{9-2q},$$
which is equivalent to
$$\frac{198-45q}{18-5q} \le \frac{65}{9-2q}.$$
We will prove the sharper inequality
$$\frac{200-45q}{18-5q} \le \frac{65}{9-2q},$$
which is equivalent to
$$\frac{40-9q}{18-5q} \le \frac{13}{9-2q},$$
$$(q-3)(3q-7) \le 0.$$
The last inequality is clearly true. The equality holds for $a = 2$, $b = 1$ and $c = 0$.

Second Solution. According to the preceding P 1.178, it suffices to show that
$$4\left(\frac{15}{a^2+b^2+c^2} - 2\right) + 9 \le \frac{65}{a^2+b^2+c^2},$$

which is equivalent to

$$a^2 + b^2 + c^2 \geq 5,$$

$$ab + bc + ca \geq 2.$$

□

P 1.180. *If* $a \geq b \geq c \geq 0$ *and* $ab + bc + ca = 2$ *, then*

$$\sqrt{a + ab} + \sqrt{b + bc} + \sqrt{c + ca} \geq 3.$$

(KaiRain, 2020)

Proof. Consider the main case $a \geq b \geq c$ and show that

$$\sqrt{a + ab} + \sqrt{b + bc} + \sqrt{c + ca} \geq 3.$$

For $c = 0$, we need to show that $ab = 2$ involves

$$\sqrt{a + ab} + \sqrt{b} \geq 3,$$

that is

$$\sqrt{a + 2} + \sqrt{\frac{2}{a}} \geq 3.$$

Denoting $x = \sqrt{\frac{a}{2}}$, we need to show that

$$\sqrt{2x^2 + 2} \geq 3 - \frac{1}{x}.$$

This is true if

$$2(x^2 + 1) \geq \left(3 - \frac{1}{x}\right)^2$$

for $x \geq 1/3$, which is equivalent to the obvious inequality

$$(x - 1)^2(2x^2 + 4x - 1) \geq 0.$$

Using this result, it suffices to show that

$$\sqrt{a + ab} + \sqrt{b + bc} + \sqrt{c + ca} \geq \sqrt{a + 2} + \sqrt{\frac{2}{a}},$$

that is equivalent to

$$\sqrt{c + ca} \geq \sqrt{a + 2} - \sqrt{a + ab} + \sqrt{\frac{2}{a}} - \sqrt{b + bc},$$

$$\sqrt{c+ca} \ge \frac{2-ab}{\sqrt{a+2}+\sqrt{a+ab}} + \frac{2-ab-abc}{\sqrt{2a}+a\sqrt{b+bc}},$$

$$\sqrt{c+ca} \ge \frac{c(a+b)}{\sqrt{a+2}+\sqrt{a+ab}} + \frac{c(a+b-ab)}{\sqrt{2a}+a\sqrt{b+bc}}.$$

So, we need to show that

$$\sqrt{1+a} \ge \frac{\sqrt{c}(a+b)}{\sqrt{a+2}+\sqrt{a+ab}} + \frac{\sqrt{c}(a+b-ab)}{\sqrt{2a}+a\sqrt{b+bc}}.$$

We get this inequality by summing the inequalities

$$\frac{\sqrt{1+a}}{2} \ge \frac{\sqrt{c}(a+b)}{\sqrt{a+2}+\sqrt{a+ab}}, \qquad \frac{\sqrt{1+a}}{2} \ge \frac{\sqrt{c}(a+b-ab)}{\sqrt{2a}+a\sqrt{b+bc}}.$$

From $ab+bc+ca=2$, it follows $\frac{2}{3} \le ab \le 2$ and $b \le \sqrt{2}$. Since

$$\sqrt{a+ab} \le \sqrt{a+2}$$

and

$$a\sqrt{b} \le \sqrt{2a}, \qquad a\sqrt{b} \le a\sqrt{b+bc},$$

it suffice to prove the inequalities

$$\sqrt{1+a} \ge \frac{\sqrt{c}(a+b)}{\sqrt{a+ab}}, \qquad \sqrt{1+a} \ge \frac{\sqrt{c}(a+b-ab)}{a\sqrt{b}}.$$

By squaring, the first inequality becomes

$$a(1+a)(1+b) \ge c(a+b)^2,$$

$$a(1+a)(1+b) \ge (a+b)(2-ab).$$

Since $2a \ge a+b$, it suffices to show that

$$(1+a)(1+b) \ge 2(2-ab),$$

that is

$$a+b+3ab \ge 3.$$

Indeed, we have

$$a+b+3ab \ge 2\sqrt{ab}+3ab \ge 2\sqrt{\frac{2}{3}}+2 > 3.$$

Since $\sqrt{b} \ge \sqrt{c}$, the second inequality is true if

$$a\sqrt{1+a} \ge a+b-ab,$$

that is

$$a(\sqrt{1+a}-1) \geq b(1-a).$$

For the nontrivial case $a \leq 1$, it suffices to show that

$$a(\sqrt{1+a}-1) \geq a(1-a),$$

that is

$$\sqrt{1+a}+a \geq 2.$$

Since $3a^2 \geq ab+bc+ca=2$, we have

$$\sqrt{1+a}+a \geq \sqrt{1+\sqrt{\frac{2}{3}}}+\sqrt{\frac{2}{3}} > 2.$$

The inequality is an equality for $a=2$, $b=1$, $c=0$.

Remark. The following sharper inequality holds in the same conditions:

$$\sqrt{a+ab}+\sqrt{b}+\sqrt{c} \geq 3,$$

with equality for $a=2$, $b=1$, $c=0$.

For fixed b, according to the relation $ab+bc+ca=2$, we may consider that a is a function of c. Differentiating this equation, we get

$$a' = -\frac{a+b}{b+c},$$

$$a'' = \frac{(a+b+(b-c)a'}{(a+c)^2} = \frac{(a+b)(a-b+2c)}{(a+c)^3}.$$

Write the required inequality as $f(c) \geq 0$, where

$$f(c) = \sqrt{a+ab}+\sqrt{b}+\sqrt{c}-3, \qquad c \in [0,b].$$

We have

$$f'(c) = \frac{a'\sqrt{1+b}}{2\sqrt{a}} + \frac{1}{2\sqrt{c}},$$

$$f''(c) = \frac{(2aa''-(a')^2)\sqrt{1+b}}{4a^{3/2}} - \frac{1}{4c^{3/2}}$$

$$= \frac{(a+b)(a^2+3ac-3ab-bc)\sqrt{1+b}}{4a^{3/2}(a+c)^3} - \frac{1}{4c^{3/2}}.$$

Since

$$a^2+3ac-3ab-bc = a^2-3a(b-c)-bc < a^2,$$

we have

$$f''(c) < \frac{(a+b)\sqrt{a(1+b)}}{4(a+c)^3} - \frac{1}{4c^{3/2}}.$$

From $b^2 \le ab \le ab+bc+ca=2$, we get $b \le \sqrt{2}$, $\sqrt{1+b} < 4$, hence

$$f''(c) < \frac{(a+b)\sqrt{a}}{(a+c)^3} - \frac{1}{4c^{3/2}} \le 2\left(\frac{\sqrt{a}}{a+c}\right)^3 - \frac{1}{4(\sqrt{c})^3} \le 0.$$

Since f is concave and $0 \le c \le b$, it is enough to show that $f(0) \ge 0$ (for $c=0$ and $ab=2$) and $f(b) \ge 0$ (for $c=b$ and $2ab+b^2=2$). We have

$$f(0) = \sqrt{\frac{2+2b}{b}} + \sqrt{b} - 3 = \frac{(1-\sqrt{b})^2(2+4\sqrt{b}-b)}{\sqrt{b(2+2b)} - b\sqrt{b} + 3b} \ge 0.$$

For $c=b$, when $2 = 2ab+b^2 \ge 3b^2$, hence $b \le \sqrt{\dfrac{2}{3}}$, we have

$$f(b) = \sqrt{\frac{(1+b)(2-b^2)}{2b}} + 2\sqrt{b} - 3 = \frac{A}{\sqrt{2b(1+b)(2-b^2)} - 4b\sqrt{b} + 6b},$$

where, for $x = \sqrt{b} \le \sqrt[4]{\dfrac{2}{3}} < 1$,

$$A = (1+x^2)(2-x^4) - 2x^2(3-2x)^2 = (1-x)(2+2x-14x^2+10x^3+x^4+x^5).$$

Since

$$2+2x-14x^2+10x^3+x^4+x^5 = 2-13x^2+13x^3+(1-x)^2x(2+3x+x^2)$$

$$> 2+13x^3-13x^2 = 2 + \frac{13x^3}{2} + \frac{13x^3}{2} - 13x^2$$

$$\ge 3\sqrt[3]{2\cdot\frac{13x^3}{2}\cdot\frac{13x^3}{2}} - 13x^2 = \left(3\sqrt[3]{\frac{169}{2}} - 13\right)x^2 > 0,$$

we have $A > 0$, hence $f(b) > 0$.

□

P 1.181. *If $a \ge b \ge c$ are nonnegative numbers such that $ab+bc+ca=3$, then*

$$\sqrt{a+2ab} + \sqrt{b+2bc} + \sqrt{c+2ca} \ge 4.$$

(Vasile C., 2020)

Proof. We will prove the sharper inequality

$$\sqrt{a+2ab} + \sqrt{b+bc} + \sqrt{c+ca} \ge 4.$$

For $c = 0$, we need to show that $ab = 3$ involves

$$\sqrt{a+2ab} + \sqrt{b} \ge 4,$$

that is

$$\sqrt{a+6} + \sqrt{\frac{3}{a}} \ge 4.$$

It is easy to show that this inequality is true for all $a > 0$. Using this result, it suffices to show that

$$\sqrt{a+2ab} + \sqrt{b+bc} + \sqrt{c+ca} \ge \sqrt{a+6} + \sqrt{\frac{3}{a}},$$

that is equivalent to

$$\sqrt{c+ca} \ge \sqrt{a+6} - \sqrt{a+2ab} + \sqrt{\frac{3}{a}} - \sqrt{b+bc},$$

$$\sqrt{c+ca} \ge \frac{2(3-ab}{\sqrt{a+6}+\sqrt{a+2ab}} + \frac{3-ab-abc}{\sqrt{3a}+a\sqrt{b+bc}},$$

$$\sqrt{c+ca} \ge \frac{2c(a+b)}{\sqrt{a+6}+\sqrt{a+2ab}} + \frac{c(a+b-ab)}{\sqrt{3a}+a\sqrt{b+bc}}.$$

So, we need to show that

$$\sqrt{1+a} \ge \frac{2\sqrt{c}(a+b)}{\sqrt{a+6}+\sqrt{a+2ab}} + \frac{\sqrt{c}(a+b-ab)}{\sqrt{3a}+a\sqrt{b+bc}}.$$

We get this inequality by summing the inequalities

$$k\sqrt{1+a} \ge \frac{2\sqrt{c}(a+b)}{\sqrt{a+6}+\sqrt{a+2ab}}, \qquad (1-k)\sqrt{1+a} \ge \frac{\sqrt{c}(a+b-ab)}{\sqrt{3a}+a\sqrt{b+bc}},$$

where

$$k = \sqrt{\frac{2}{3}}.$$

From $ab + bc + ca = 3$, it follows $1 \le ab \le 3$ and $b \le \sqrt{3}$. Since

$$\sqrt{a+2ab} \le \sqrt{a+6},$$

the first inequality is true if

$$k\sqrt{1+a} \ge \frac{\sqrt{c}(a+b)}{\sqrt{a+2ab}},$$

that is

$$2a(1+a)(1+2b) \ge 3c(a+b)^2,$$

$$2a(1+a)(1+2b) \ge 3(3-ab)(a+b).$$

Since $2a \ge a+b$, it suffices to show that

$$(1+a)(1+2b) \ge 3(3-ab),$$

that is

$$(5b+1)a+2b \ge 8.$$

For $a \ge b \ge 1$, this inequality is obvious. For $0 \le b \le 1$, from

$$b \ge c = \frac{3-ab}{a+b}$$

we get

$$a \ge \frac{3-b^2}{2b}.$$

Therefore,

$$(5b+1)a+2b-8 \ge \frac{(5b+1)(3-b^2)}{2b}+2b$$

$$= \frac{3-b+3b^2-5b^3}{2b} = \frac{(1-b)(3+2b+5b^2)}{2b} \ge 0.$$

Since $1-k > \frac{1}{4}$, the second inequality is true if

$$\sqrt{1+a} \ge \frac{4\sqrt{c}(a+b-ab)}{\sqrt{3a}+a\sqrt{b+bc}},$$

Consider the nontrivial case $a+b-ab \ge 0$, and claim that $\sqrt{3a} \ge a\sqrt{b+bc}$, which is equivalent to $3 \ge ab+abc$. Indeed, we have

$$3-ab-abc = 3-ab-\frac{ab(3-ab)}{a+b} = \frac{(3-ab)(a+b-ab)}{a+b} \ge 0.$$

Thus, it suffices to show that

$$\sqrt{1+a} \ge \frac{2\sqrt{c}(a+b-ab)}{a\sqrt{b+bc}}.$$

Since

$$\frac{a+b-ab}{a} \le 1,$$

it suffices to show that

$$\sqrt{1+a} \ge 2\sqrt{\frac{c}{b(1+c)}},$$

that is

$$b(1+a)(1+c) \ge 4c.$$

Since $ab \geq 1$, we have
$$b(1+a) \geq b+1 \geq c+1,$$
therefore,
$$b(1+a)(1+c)-4c \geq (1+c)^2-4c=(1-c)^2 \geq 0.$$

The inequality is an equality for $a=3,\ b=1,\ c=0$.

□

P 1.182. *If a,b,c are nonnegative real numbers such that $ab+bc+ca=3$, then*
$$\sqrt{a+3b}+\sqrt{b+3c}+\sqrt{c+3a} \geq 6.$$

Solution. Use the substitution
$$\sqrt{a+3b}=2x, \quad \sqrt{b+3c}=2y, \quad \sqrt{c+3a}=2z,$$
which yields
$$a=\frac{x^2-3y^2+9z^2}{7}, \quad a=\frac{y^2-3z^2+9x^2}{7}, \quad a=\frac{z^2-3x^2+9y^2}{7},$$
$$ab+bc+ca=\frac{-3(x^4+y^4+z^4)+10(x^2y^2+y^2z^2+z^2x^2)}{7}.$$
So, we need to show that
$$x+y+z \geq 3$$
for
$$3(x^4+y^4+z^4)+21=10(x^2y^2+y^2z^2+z^2x^2).$$
By the contradiction method, we need to prove that
$$x+y+z<3$$
involves
$$3(x^4+y^4+z^4)+21>10(x^2y^2+y^2z^2+z^2x^2).$$
It suffices to prove the homogeneous inequality $f(x,y,z) \geq 0$, where
$$f(x,y,z)=81(x^4+y^4+z^4)+7(x+y+z)^4-270(x^2y^2+y^2z^2+z^2x^2).$$
According to P 3.68 from Volume 1, it is enough to show that $f(0,y,z) \geq 0$ and $f(x,1,1) \geq 0$ for $x,y,z \geq 0$. We have
$$f(0,y,z)=81(y^4+z^4)+7(y+z)^4-270y^2z^2$$
$$\geq 162y^2z^2+112y^2z^2-270y^2z^2=4y^2z^2 \geq 0$$

and

$$f(x,1,1) = 81(x^4+2)+7(x+2)^4-540x^2 = 4(22x^4+14x^3-93x^2+56x+1)$$

$$= (x-1)^2(22x^2+58x+1) \geq 0.$$

The equality occurs for $a = b = c = 1$.

□

P 1.183. *If a,b,c are the lengths of the sides of a triangle, then*

$$10\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) > 9\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right).$$

Solution. According to Remark 2 from the proof of P 1.149, it suffices to show that $P(1,1,1) \geq 0$ and $P(b+c,b,c) \geq 0$ for $b,c \geq 0$, where

$$P(a,b,c) = 10(ab^2+bc^2+ca^2)-9(a^2b+b^2c+c^2a).$$

We have $P(1,1,1) = 3 > 0$ and

$$P(b+c,b,c) = b^3-7b^2c+12bc^2+c^3.$$

We need to show that

$$x^3-7x^2+12x+1 > 0,$$

where $x = b/c$, $x > 0$. For $x \in (0,3] \cup [4,\infty)$, we have

$$x^3-7x^2+12x+1 > x^3-7x^2+12x = x(3-x)(4-x) \geq 0.$$

For $x \in (3,4)$, we have

$$x^3-7x^2+12x+1 > x^3-7x^2+12x+\frac{x}{4} = \frac{x(2x-7)^2}{4} \geq 0.$$

□

P 1.184. *If a,b,c are the lengths of the sides of a triangle, then*

$$\frac{a}{3a+b-c}+\frac{b}{3b+c-a}+\frac{c}{3c+a-b} \geq 1.$$

Solution. Write the inequality as follows:

$$\sum\left(\frac{a}{3a+b-c}-\frac{1}{4}\right)\ge\frac{1}{4},$$

$$\sum\frac{a-b+c}{3a+b-c}\ge 1.$$

Applying the Cauchy-Schwarz inequality, we get

$$\sum\frac{a-b+c}{3a+b-c}\ge\frac{\left[\sum(a-b+c)\right]^2}{\sum(a-b+c)(3a+b-c)}=\frac{\left(\sum a\right)^2}{\sum a^2+2\sum ab}=1.$$

The equality holds for $a=b=c$.

□

P 1.185. *If a,b,c are the lengths of the sides of a triangle, then*

$$\frac{a^2-b^2}{a^2+bc}+\frac{b^2-c^2}{b^2+ca}+\frac{c^2-a^2}{c^2+ab}\le 0.$$

(Vasile C., 2007)

First Solution. Suppose that $a=\max\{a,b,c\}$. Since

$$c^2-a^2=-(a^2-b^2)-(b^2-c^2),$$

the inequality can be written as follows:

$$(a^2-b^2)\left(\frac{1}{a^2+bc}-\frac{1}{c^2+ab}\right)+(b^2-c^2)\left(\frac{1}{b^2+ca}-\frac{1}{c^2+ab}\right)\le 0,$$

$$-\frac{(a^2-b^2)(a-c)(a-b+c)}{a^2+bc}-\frac{(b^2-c^2)(b-c)(b+c-a)}{a^2+bc}\le 0.$$

The equality holds for an equilateral triangle, and also for a degenerate triangle having a side equal to zero.

Second Solution. The sequences

$$\{a^2,\quad b^2,\quad c^2\}$$

and

$$\left\{\frac{1}{a^2+bc},\quad\frac{1}{b^2+ca},\quad\frac{1}{c^2+ab}\right\}$$

are reversely ordered. Indeed, if $a\ge b\ge c$, then

$$\frac{1}{a^2+bc}\le\frac{1}{b^2+ca}\le\frac{1}{c^2+ab},$$

because

$$\frac{1}{b^2+ca}-\frac{1}{a^2+bc}=\frac{(a-b)(a+b-c)}{(b^2+ca)(a^2+bc)}\geq 0,$$

$$\frac{1}{c^2+ab}-\frac{1}{b^2+ca}=\frac{(b-c)(b+c-a)}{(c^2+ab)(b^2+ca)}\geq 0.$$

Then, by the rearrangement inequality, we have

$$a^2\cdot\frac{1}{a^2+bc}+b^2\cdot\frac{1}{b^2+ca}+c^2\cdot\frac{1}{c^2+ab}\leq$$

$$\leq b^2\cdot\frac{1}{a^2+bc}+c^2\cdot\frac{1}{b^2+ca}+a^2\cdot\frac{1}{c^2+ab},$$

which is the desired inequality.

□

P 1.186. *If a,b,c are the lengths of the sides of a triangle, then*

$$a^2(a+b)(b-c)+b^2(b+c)(c-a)+c^2(c+a)(a-b)\geq 0.$$

(Vasile C., 2006)

First Solution. Assume that

$$a=\max\{a,b,c\},$$

use the substitution

$$a=x+p+q,\quad b=x+p,\quad c=x+q,\qquad x,p,q\geq 0,$$

and write the inequality as

$$a^2b^2+b^2c^2+c^2a^2-abc(a+b+c)\geq ab^3+bc^3+ca^3-a^3b-b^3c-c^3a,$$

$$a^2(b-c)^2+b^2(c-a)^2+c^2(a-b)^2\geq 2(a+b+c)(a-b)(b-c)(c-a),$$

$$(x+p+q)^2(p-q)^2+(x+p)^2p^2+(x+q)^2q^2\geq 2(3x+2p+2q)pq(q-p),$$

which is equivalent to

$$Ax^2+2Bx+C\geq 0,$$

where

$$A=p^2-pq+q^2\geq 0,$$

$$B=p^3+q(p-q)^2\geq 0,$$

$$C=(p^2+pq-q^2)^2\geq 0.$$

The equality holds for an equilateral triangle, and also for a degenerate triangle with

$$\frac{a}{2} = \frac{b}{1+\sqrt{5}} = \frac{c}{3+\sqrt{5}}$$

(or any cyclic permutation).

Second Solution. Using the substitution

$$x = \sqrt{\frac{ca}{b}}, \quad y = \sqrt{\frac{ab}{c}}, \quad z = \sqrt{\frac{bc}{a}},$$

we can write the inequality as follows:

$$b^2c^2 + c^2a^2 + a^2b^2 \geq ab(b^2 + c^2 - a^2) + bc(c^2 + a^2 - b^2) + ca(a^2 + b^2 - c^2),$$

$$\frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c} \geq 2b\cos A + 2c\cos B + 2a\cos C,$$

$$x^2 + y^2 + z^2 \geq 2yz\cos A + 2zx\cos B + 2xy\cos C,$$

$$(x - y\cos C - z\cos B)^2 + (y\sin C - z\sin B)^2 \geq 0.$$

□

P 1.187. *If a, b, c are the lengths of the sides of a triangle, then*

$$a^2b + b^2c + c^2a \geq \sqrt{abc(a+b+c)(a^2+b^2+c^2)}.$$

(Vasile Cîrtoaje and Vo Quoc Ba Can, 2005)

Solution. Without loss of generality, assume that b is between a and c; that is

$$(b-a)(b-c) \leq 0.$$

First Solution. By the AM-GM inequality, we have

$$4abc(a+b+c)(a^2+b^2+c^2) \leq [ac(a+b+c) + b(a^2+b^2+c^2)]^2.$$

Thus, we only need to show that

$$2(a^2b + b^2c + c^2a) \geq ac(a+b+c) + b(a^2+b^2+c^2),$$

which is equivalent to

$$b[a^2 - (b-c)^2] - ac(a+b-c) \geq 0,$$

$$(a+b-c)(a-b)(b-c) \geq 0.$$

The equality holds for an equilateral triangle, and also for a degenerate triangle with
$$c = a + b, \qquad b^3 = a^2(a + b)$$
(or any cyclic permutation).

Second Solution. The desired inequality is equivalent to $D \ge 0$, where D is the discriminant of the quadratic function

$$f(x) = (a^2 + b^2 + c^2)x^2 - 2(a^2b + b^2c + c^2a)x + abc(a + b + c).$$

For the sake of contradiction, assume that $D < 0$ for some a, b, c. Then, $f(x) > 0$ for all real x. This is not true, because

$$f(b) = b(b - a)(b - c)(a + b - c) \le 0.$$

□

P 1.188. *If a, b, c are the lengths of the sides of a triangle, then*

$$a^2\left(\frac{b}{c} - 1\right) + b^2\left(\frac{c}{a} - 1\right) + c^2\left(\frac{a}{b} - 1\right) \ge 0.$$

(Vasile Cîrtoaje, Moldova TST, 2006)

First Solution. Using the substitution

$$a = \frac{1}{x}, \quad b = \frac{1}{y}, \quad c = \frac{1}{z},$$

the inequality becomes
$$E(x, y, z) \ge 0,$$
where
$$E(x, y, z) = yz^2(z - y) + zx^2(x - z) + xy^2(y - x).$$
Without loss of generality, assume that

$$a = \min\{a, b, c\}, \qquad x = \max\{x, y, z\}.$$

We will show that
$$E(x, y, z) \ge E(y, y, z) \ge 0.$$
We have

$$\begin{aligned} E(x, y, z) - E(y, y, z) &= z(x^3 - y^3) - z^2(x^2 - y^2) + y^3(x - y) - y^2(x^2 - y^2) \\ &= (x - y)(x - z)(xz + yz - y^2) \ge 0, \end{aligned}$$

because

$$xz + yz - y^2 \geq y(2z - y) = \frac{2b - c}{b^2c} = \frac{(b-a) + (a+b-c)}{b^2c} \geq 0.$$

Also,

$$E(y, y, z) = yz(y - z)^2 \geq 0.$$

The equality holds for $a = b = c$.

Second Solution. Write the inequality as $F(a, b, c) \geq 0$, where

$$F(a, b, c) = a^3b^2 + b^3c^2 + c^3a^2 - abc(a^2 + b^2 + c^2).$$

Since

$$\begin{aligned} 2E(a, b, c) &= \left(\sum a^3b^2 + \sum a^2b^3 - 2abc\sum a^2\right) - \left(\sum a^2b^3 - \sum a^3b^2\right) \\ &= \left(\sum a^3b^2 + \sum a^3c^2 - 2abc\sum a^2\right) - \left(\sum a^2b^3 - \sum a^2c^3\right) \\ &= \sum a^3(b-c)^2 - \sum a^2(b^3 - c^3) \end{aligned}$$

and

$$\sum a^2(b^3 - c^3) = \sum a^2(b-c)^3,$$

we get

$$E(a, b, c) = \sum a^3(b-c)^2 - \sum a^2(b-c)^3 = \sum a^2(b-c)^2(a - b + c) \geq 0.$$

Third Solution. By the Cauchy-Schwarz inequality, we have

$$\sum \frac{a^2b}{c} \geq \frac{\left(\sum a^2b\right)^2}{\sum a^2bc}.$$

Therefore, it suffices to show that

$$\left(\sum a^2b\right)^2 \geq abc(a + b + c)(a^2 + b^2 + c^2),$$

which is the inequality from the preceding P 1.187.

□

P 1.189. *If a, b, c are the lengths of the sides of a triangle, then*

(a)
$$a^3b + b^3c + c^3a \geq a^2b^2 + b^2c^2 + c^2a^2;$$

(b)
$$3(a^3b + b^3c + c^3a) \geq (ab + bc + ca)(a^2 + b^2 + c^2);$$

(c)
$$\frac{a^3b + b^3c + c^3}{3} \geq \left(\frac{a + b + c}{3}\right)^4.$$

Solution. (a) ***First Solution.*** Write the inequality as

$$a^2b(a-b)+b^2c(b-c)+c^2a(c-a)\geq 0.$$

Using the substitution

$$a=y+z,\quad b=z+x,\quad c=x+y,\qquad x,y,z\geq 0,$$

the inequality turns into

$$xy^3+yz^3+zx^3\geq xyz(x+y+z),$$

which follows from the Cauchy-Schwarz inequality

$$(xy^3+yz^3+zx^3)(z+x+y)\geq xyz(y+z+x)^2.$$

The equality holds for an equilateral triangle, and also for a degenerate triangle with $a=0$ and $b=c$ (or any cyclic permutation).

Second Solution. Multiplying by $a+b+c$, the inequality becomes as follows:

$$\sum a^4b+abc\sum a^2\geq \sum a^2b^3+abc\sum ab,$$

$$\sum b^4c+abc\sum a^2\geq \sum b^2c^3+abc\sum ab,$$

$$\sum \frac{b^3}{a}+\sum a^2\geq \sum \frac{bc^2}{a}+\sum ab,$$

$$\sum a^2\geq \sum \frac{b}{a}(c^2+a^2-b^2),$$

$$a^2+b^2+c^2\geq 2bc\cos B+2ca\cos C+2ab\cos A,$$

$$(a-b\cos A-c\cos C)^2+(b\sin A-c\sin C)^2\geq 0.$$

(b) Write the inequality as

$$\sum a^2b(a-b)+\sum b^2(a-b)(a-c)\geq 0.$$

Since $\sum a^2b(a-b)\geq 0$ (according to the inequality in (a)), it suffices to show that

$$\sum b^2(a-b)(a-c)\geq 0.$$

This is a particular case ($x=c$, $y=a$, $z=b$) of the following inequality

$$(x-y)(x-z)a^2+(y-z)(y-x)b^2+(z-x)(z-y)c^2\geq 0,$$

where x,y,z are real numbers. If two of x,y,z are equal, then the inequality is trivial. Otherwise, assume that $x>y>z$ and write the inequality as

$$\frac{a^2}{y-z}+\frac{c^2}{x-y}\geq \frac{b^2}{x-z}.$$

Applying the Cauchy-Schwarz inequality, we get

$$\frac{a^2}{y-z}+\frac{c^2}{x-y}\geq\frac{(a+c)^2}{(y-z)+(x-y)}=\frac{(a+c)^2}{x-z}\geq\frac{b^2}{x-z}.$$

The equality holds for $a=b=c$.

(c) According to the inequality (b), it suffices to show that

$$9(ab+bc+ca)(a^2+b^2+c^2)\geq(a+b+c)^4.$$

This is equivalent to

$$(A-B)(4B-A)\geq 0,$$

where

$$A=a^2+b^2+c^2,\qquad B=ab+bc+ca.$$

Since $A\geq B$ and

$$\begin{aligned}4B-A&>2(ab+bc+ca)-a^2-b^2-c^2\\&=a(2b+2c-a)-(b-c)^2\\&\geq a^2-(b-c)^2\\&=(a-b+c)(a+b-c)\geq 0.\end{aligned}$$

the conclusion follows. The equality holds for $a=b=c$.

□

P 1.190. *If a,b,c are the lengths of the sides of a triangle, then*

$$2\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)\geq\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}+3.$$

Solution. Write the inequality as follows:

$$\sum\frac{a^2}{b^2}\geq 3+\sum\frac{b^2}{a^2}-\sum\frac{a^2}{b^2},$$

$$\sum\frac{b^2}{c^2}\geq 3+\sum\frac{c^2}{b^2}-\sum\frac{a^2}{b^2},$$

$$\sum\frac{b^2}{c^2}\geq\sum\left(1+\frac{c^2}{b^2}-\frac{a^2}{b^2}\right),$$

$$\sum\frac{b^2}{c^2}\geq 2\sum\frac{c}{b}\cos A.$$

Putting

$$x=\frac{b}{c},\quad y=\frac{c}{a},\quad z=\frac{a}{b},$$

we have $xyz = 1$ and

$$\frac{c}{b} = \frac{1}{x} = yz, \quad \frac{a}{c} = \frac{1}{y} = zx, \quad \frac{b}{a} = \frac{1}{z} = xy.$$

Therefore, we can write the inequality as

$$x^2 + y^2 + z^2 \geq 2yz\cos A + 2zx\cos B + 2xy\cos C,$$

which is equivalent to the obvious inequality

$$(x - y\cos C - z\cos B)^2 + (y\sin C - z\sin B)^2 \geq 0.$$

The equality occurs for $a = b = c$.

□

P 1.191. *If a, b, c are the lengths of the sides of a triangle such that $a < b < c$, then*

$$\frac{a^2}{a^2 - b^2} + \frac{b^2}{b^2 - c^2} + \frac{c^2}{c^2 - a^2} \leq 0.$$

(Vasile C., 2003)

Solution. Write the inequality as

$$\frac{a^2}{b^2 - a^2} + \frac{b^2}{c^2 - b^2} \geq \frac{c^2}{c^2 - a^2}.$$

Since $c \leq a + b$, it suffices to show that

$$\frac{a^2}{b^2 - a^2} + \frac{b^2}{c^2 - b^2} \geq \frac{(a+b)^2}{c^2 - a^2},$$

which is equivalent to

$$a^2\left(\frac{1}{b^2 - a^2} - \frac{1}{c^2 - a^2}\right) + b^2\left(\frac{1}{c^2 - b^2} - \frac{1}{c^2 - a^2}\right) \geq \frac{2ab}{c^2 - a^2},$$

$$\frac{a^2(c^2 - b^2)}{b^2 - a^2} + \frac{b^2(b^2 - a^2)}{c^2 - b^2} \geq 2ab,$$

$$\left(a\sqrt{\frac{c^2 - b^2}{b^2 - a^2}} - b\sqrt{\frac{b^2 - a^2}{c^2 - b^2}}\right)^2 \geq 0.$$

The equality occurs for a degenerate triangle with $c = a + b$ and $a = xb$, where $x \approx 0.53209$ is the positive root of the equation $x^3 + 3x^2 - 1 = 0$.

□

P 1.192. *If a, b, c are the lengths of the sides of a triangle, then*

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+3 \ge 2\left(\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}\right).$$

(Manlio Marangelli, 2008)

First Solution. Assume that $c = \max\{a, b, c\}$. If $a \le b \le c$, then the inequality follows from P 1.157. Consider further that

$$b \le a \le c.$$

Write the inequality as follows:

$$\sum\left(\frac{a}{b}-1\right) \ge 2\sum\left(\frac{b+c}{c+a}-1\right),$$

$$\sum(a-b)\left(\frac{1}{b}+\frac{2}{c+a}\right) \ge 0,$$

$$(a-b)\left(\frac{1}{b}+\frac{2}{c+a}\right)+[(b-a)+(a-c)]\left(\frac{1}{c}+\frac{2}{a+b}\right)+(c-a)\left(\frac{1}{a}+\frac{2}{b+c}\right) \ge 0,$$

$$(a-b)\left(\frac{1}{b}+\frac{2}{c+a}-\frac{1}{c}-\frac{2}{a+b}\right)+(c-a)\left(\frac{1}{a}+\frac{2}{b+c}-\frac{1}{c}-\frac{2}{a+b}\right) \ge 0,$$

$$(a-b)(c-b)\left[\frac{1}{bc}-\frac{2}{(a+b)(a+c)}\right]+(c-a)^2\left[\frac{1}{ac}-\frac{2}{(a+b)(b+c)}\right] \ge 0.$$

Since

$$\frac{1}{bc}-\frac{2}{(a+b)(a+c)}=\frac{c(a-b)+a(a+b)}{bc(a+b)(a+c)} \ge \frac{a(a+b)}{bc(a+b)(a+c)}=\frac{a}{bc(a+c)}$$

and

$$\frac{1}{ac}-\frac{2}{(a+b)(b+c)}=\frac{-c(a-b)+b(a+b)}{ac(a+b)(b+c)} > \frac{-c(a-b)}{ac(a+b)(b+c)}=\frac{-(a-b)}{a(a+b)(b+c)},$$

it suffices to show that

$$\frac{(a-b)(c-b)a}{bc(a+c)}-\frac{(c-a)^2(a-b)}{a(a+b)(b+c)} \ge 0,$$

which is true if

$$\frac{(c-b)a}{bc(a+c)} \ge \frac{(c-a)^2}{a(a+b)(b+c)}.$$

We can get this by multiplying the inequalities

$$c-b \ge c-a,$$

$$\frac{1}{b} \ge \frac{1}{a},$$
$$\frac{1}{c} \ge \frac{1}{a+b},$$
$$\frac{a}{a+c} \ge \frac{c-a}{b+c}.$$

The last inequality is true since

$$\frac{a}{a+c} - \frac{c-a}{b+c} \ge \frac{a}{a+c} - \frac{b}{b+c} = \frac{c(a-b)}{(a+c)(b+c)} \ge 0.$$

The equality holds for $a = b = c$.

Second Solution (*by Vo Quoc Ba Can*). Since

$$\sum \frac{a+b}{b+c} = \sum \left(1 + \frac{a-c}{b+c}\right) = 3 + \sum \frac{a-c}{b+c},$$

we can write the desired inequality as

$$\sum \frac{a}{b} - 3 \ge 2 \sum \frac{a-c}{b+c}.$$

Since

$$(ab+bc+ca)\left(\sum \frac{a}{b} - 3\right) = \sum a^2 - 2\sum ab + \sum \frac{a^2c}{b}$$

and

$$\begin{aligned}(ab+bc+ca)\sum \frac{a-c}{b+c} &= [a(b+c)+bc]\sum \frac{a-c}{b+c} \\ &= \sum a^2 - \sum ab + \sum \frac{bc(a-c)}{b+c},\end{aligned}$$

the inequality is equivalent to

$$\sum \frac{a^2c}{b} + 2\sum \frac{bc(c-a)}{b+c} \ge \sum a^2.$$

Since

$$\sum \frac{a^2c}{b} \ge \sum a^2$$

(see the inequality in P 1.188), we only need to show that

$$\sum \frac{bc(c-a)}{b+c} \ge 0.$$

Write this inequality as follows:

$$\sum bc(c^2-a^2)(a+b) \ge 0,$$

$$\sum(c^2-a^2)\left(1+\frac{b}{a}\right)\ge 0,$$

$$\sum(c^2-a^2)\frac{b}{a}\ge 0,$$

$$\sum\frac{bc^2}{a}\ge\sum ab.$$

According to P 1.188, we have

$$\sum\frac{bc^2}{a}\ge\sum a^2\ge\sum ab.$$

□

P 1.193. *Let a, b, c be the lengths of the sides of a triangle. If $k\ge 2$, then*

$$a^kb(a-b)+b^kc(b-c)+c^ka(c-a)\ge 0.$$

(Vasile C., 1986)

Solution *(by Darij Grinberg)*. For $k=2$, we get the known inequality (a) in P 1.189:

$$a^2b(a-b)+b^2c(b-c)+c^2a(c-a)\ge 0.$$

We will prove the following more general statement: if f is an increasing nonnegative function defined on $[0,\infty)$, then

$$E(a,b,c)\ge 0,$$

where

$$E(a,b,c)=a^2bf(a)(a-b)+b^2cf(b)(b-c)+c^2af(c)(c-a).$$

For $f(x)=x^{k-2}$, $k\ge 2$, we get the original inequality. In order to prove the claimed generalization, assume that $a=\max\{a,b,c\}$. There are two cases to consider.

Case 1: $a\ge b\ge c$. Since

$$f(a)\ge f(b)\ge f(c)\ge 0,$$

we have

$$\begin{aligned}E(a,b,c)&\ge a^2bf(c)(a-b)+b^2cf(c)(b-c)+c^2af(c)(c-a)\\&=f(c)[a^2b(a-b)+b^2c(b-c)+c^2a(c-a)]\ge 0.\end{aligned}$$

Case 2: $a\ge c\ge b$. Since

$$f(a)\ge f(c)\ge f(b)\ge 0,$$

we have

$$\begin{aligned}E(a,b,c) &\geq a^2bf(a)(a-b)+b^2cf(a)(b-c)+c^2af(a)(c-a)\\ &= f(a)[a^2b(a-b)+b^2c(b-c)+c^2a(c-a)]\geq 0.\end{aligned}$$

The equality holds for $a=b=c$, and also for a degenerate triangle with $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.194. *Let a,b,c be the lengths of the sides of a triangle. If $k\geq 1$, then*

$$3(a^{k+1}b+b^{k+1}c+c^{k+1}a)\geq (a+b+c)(a^kb+b^kc+c^ka).$$

Solution. For $k=1$, the inequality is equivalent to

$$2(a^2b+b^2c+c^2a)\geq ab^2+bc^2+ca^2+3abc,$$

$$(2c-a)b^2+(2a^2-3ac-c^2)b-ac(a-2c)\geq 0.$$

Assuming that $a=\min\{a,b,c\}$ and making the substitution

$$b=x+\frac{a+c}{2},$$

this inequality becomes

$$(2c-a)x^2+\left(x+\frac{3a}{4}\right)(a-c)^2\geq 0.$$

It is true since

$$4x+3a=a+4b-2c=2(a+b-c)+(2b-a)>0.$$

In order to prove the desired inequality for $k>1$, we rewrite it as

$$a^kb(2a-b-c)+b^kc(2b-c-a)+c^ka(2c-a-b)\geq 0.$$

We will prove that if f is an increasing nonnegative function defined on $[0,\infty)$, then $E(a,b,c)\geq 0$, where

$$E(a,b,c)=ab(2a-b-c)f(a)+bc(2b-c-a)f(b)+ca(2c-a-b)f(c).$$

For $f(x)=x^{k-1}$, $k\geq 1$, we get the original inequality. In order to prove this generalization, assume that $a=\max\{a,b,c\}$. There are two cases to consider.

Case 1: $a \ge b \ge c$. Since $f(a) \ge f(b) \ge f(c) \ge 0$, we have

$$\begin{aligned}
E(a,b,c) &\ge ab(2a-b-c)f(b) + bc(2b-c-a)f(b) + ca(2c-a-b)f(c) \\
&= b[2(a-b)(a-c) + ab - c^2]f(b) + ca(2c-a-b)f(c) \\
&\ge b[2(a-b)(a-c) + ab - c^2]f(c) + ca(2c-a-b)f(c) \\
&= [2(a^2b + b^2c + c^2a) - ab^2 - bc^2 - ca^2 - 3abc]f(c) \ge 0.
\end{aligned}$$

Case 2: $a \ge c \ge b$. Since $f(a) \ge f(c) \ge f(b) \ge 0$, we have

$$\begin{aligned}
E(a,b,c) &\ge ab(2a-b-c)f(c) + bc(2b-c-a)f(b) + ca(2c-a-b)f(c) \\
&= a[(c-b)(2c-a) + b(a-b)]f(c) + bc(2b-c-a)f(b).
\end{aligned}$$

Since

$$(c-b)(2c-a) + b(a-b) \ge (c-b)(b+c-a) + b(a-b) \ge 0,$$

we get

$$\begin{aligned}
E(a,b,c) &\ge a[(c-b)(2c-a) + b(a-b)]f(b) + bc(2b-c-a)f(b) \\
&= [2(a^2b + b^2c + c^2a) - ab^2 - bc^2 - ca^2 - 3abc]f(b) \ge 0.
\end{aligned}$$

The equality holds for $a = b = c$.

Remark. For $k = 1$, the inequality has the form

$$2\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right) \ge \frac{a}{b} + \frac{b}{c} + \frac{c}{a} + 3.$$

A sharper inequality is the following

$$3\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right) \ge 2\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) + 3.$$

Using the substitution

$$b = x + \frac{a+c}{2},$$

this inequality turns into

$$(3c-2a)x^2 + \left(x + a - \frac{c}{4}\right)(a-c)^2 \ge 0,$$

which is true since, on the assumption $a = \min\{a,b,c\}$, we have $3c - 2a > 0$ and

$$4x + 4a - c = 2a + 4b - 3c = 3(a+b-c) + (b-a) > 0.$$

□

P 1.195. *Let a,b,c,d be positive real numbers such that $a+b+c+d=4$. Prove that*

$$\frac{a}{3+b}+\frac{b}{3+c}+\frac{c}{3+d}+\frac{d}{3+a}\geq 1.$$

Solution. By the Cauchy-Schwarz inequality, we have

$$\sum\frac{a}{3+b}\geq\frac{\left(\sum a\right)^2}{\sum a(3+b)}=\frac{16}{12+\sum ab}.$$

Therefore, it suffices to show that

$$ab+bc+cd+da\leq 4.$$

Indeed,

$$ab+bc+cd+da=(a+c)(b+d)\leq\left[\frac{(a+c)+(b+d)}{2}\right]^2=2.$$

The equality occurs for $a=b=c=d=1$.

□

P 1.196. *Let a,b,c,d be positive real numbers such that $a+b+c+d=4$. Prove that*

$$\frac{a}{1+b^2}+\frac{b}{1+c^2}+\frac{c}{1+d^2}+\frac{d}{1+a^2}\geq 2.$$

Solution. Since

$$\frac{a}{1+b^2}=a-\frac{ab^2}{1+b^2},$$

the inequality is equivalent to

$$\frac{ab^2}{1+b^2}+\frac{bc^2}{1+c^2}+\frac{cd^2}{1+d^2}+\frac{da^2}{1+a^2}\leq 2.$$

Since

$$\frac{ab^2}{1+b^2}\leq\frac{ab^2}{2b}=\frac{ab}{2},$$

it suffices to show that

$$ab+bc+cd+da\leq 4.$$

Indeed, we have

$$ab+bc+cd+da=(a+c)(b+d)\leq\left[\frac{(a+c)+(b+d)}{2}\right]^2=2.$$

The equality occurs for $a=b=c=d=1$.

□

P 1.197. *If a, b, c, d are nonnegative real numbers such that $a+b+c+d=4$, then*

$$a^2bc + b^2cd + c^2da + d^2ab \le 4.$$

(Song Yoon Kim, 2006)

Solution. Let (x, y, z, t) be a permutation of (a, b, c, d) such that

$$x \ge y \ge z \ge t,$$

hence

$$xyz \ge xyt \ge xzt \ge yzt.$$

By the rearrangement inequality, we have

$$\begin{aligned} a^2bc + b^2cd + c^2da + d^2ab &= a \cdot abc + b \cdot bcd + c \cdot cda + d \cdot dab \\ &\le x \cdot xyz + y \cdot xyt + z \cdot xzt + t \cdot yzt \\ &= (xy + zt)(xz + yt). \end{aligned}$$

Consequently, it suffices to show that $x + y + z + t = 4$ involves

$$(xy + zt)(xz + yt) \le 4.$$

Indeed, by the AM-GM inequality, we have

$$(xy + zt)(xz + yt) \le \frac{1}{4}(xy + zt + xz + yt)^2 = \frac{1}{4}(x + t)^2(y + z)^2 \le 4,$$

because

$$(x + t)(y + z) \le \frac{1}{4}(x + t + y + z)^2 = 4.$$

The equality holds for $a = b = c = d = 1$, and also for $a = 2$, $b = c = 1$ and $d = 0$ (or any cyclic permutation).

□

P 1.198. *If a, b, c, d are nonnegative real numbers such that $a+b+c+d=4$, then*

$$a(b + c)^2 + b(c + d)^2 + c(d + a)^2 + d(a + b)^2 \le 16.$$

Solution *(by Vo Quoc Ba Can).* Write the inequality as

$$(a + b + c + d)^3 \ge 4[a(b + c)^2 + b(c + d)^2 + c(d + a)^2 + d(a + b)^2].$$

Since

$$(a + b + c + d)^2 \ge 4(a + b)(c + d),$$

we have

$$\begin{aligned}(a+b+c+d)^3 &\geq 4(a+b)(c+d)(a+b+c+d)\\ &= 4(c+d)(a+b)^2+4(a+b)(c+d)^2.\end{aligned}$$

Therefore, it suffices to show that

$$(c+d)(a+b)^2+(a+b)(c+d)^2 \geq a(b+c)^2+b(c+d)^2+c(d+a)^2+d(a+b)^2,$$

which is equivalent to

$$c(a+b)^2+a(c+d)^2 \geq a(b+c)^2+c(d+a)^2,$$

$$a[(c+d)^2-(b+c)^2]+c[(a+b)^2-(d+a)^2] \geq 0,$$

$$(b+d)(b-d)(c-a) \geq 0.$$

Similarly, due to cyclicity, the desired in equality is true if

$$(c+a)(c-a)(d-b) \geq 0.$$

Since one of the inequalities $(b-d)(c-a) \geq 0$ and $(c-a)(d-b) \geq 0$ is true, the conclusion follows. The equality holds for $a=c$ and $b=d$.

□

P 1.199. *If a,b,c,d are positive real numbers, then*

$$\frac{a-b}{b+c}+\frac{b-c}{c+d}+\frac{c-d}{d+a}+\frac{d-a}{a+b} \geq 0.$$

Solution. We have

$$\begin{aligned}\frac{a-b}{b+c}+\frac{c-d}{d+a}+2 &= \frac{a+c}{b+c}+\frac{a+c}{d+a}\\ &= (a+c)\left(\frac{1}{b+c}+\frac{1}{d+a}\right)\\ &\geq (a+c)\frac{4}{(b+c)+(d+a)}\\ &= \frac{4(a+c)}{a+b+c+d}.\end{aligned}$$

Similarly,

$$\frac{b-c}{c+d}+\frac{d-a}{a+b}+2 \geq \frac{4(b+d)}{a+b+c+d}.$$

Adding these inequalities yields the desired inequality. The equality holds for $a=c$ and $b=d$.

Conjecture. *If a,b,c,d,e are positive real numbers, then*

$$\frac{a-b}{b+c}+\frac{b-c}{c+d}+\frac{c-d}{d+e}+\frac{d-e}{e+a}+\frac{e-a}{a+b} \geq 0.$$

□

P 1.200. *If a,b,c,d are positive real numbers, then*

$$\text{(a)}\qquad \frac{a-b}{a+2b+c}+\frac{b-c}{b+2c+d}+\frac{c-d}{c+2d+a}+\frac{d-a}{d+2a+b}\ge 0;$$

$$\text{(b)}\qquad \frac{a}{2a+b+c}+\frac{b}{2b+c+d}+\frac{c}{2c+d+a}+\frac{d}{2d+a+b}\le 1.$$

Solution. (a) Write the inequality as

$$\sum\left(\frac{a-b}{a+2b+c}+\frac{1}{2}\right)\ge 2,$$

$$\sum\frac{3a+c}{a+2b+c}\ge 4.$$

By the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\sum\frac{3a+c}{a+2b+c}&\ge\frac{\left[\sum(3a+c)\right]^2}{\sum(3a+c)(a+2b+c)}\\&=\frac{16\left(\sum a\right)^2}{4\left(\sum a^2+2\sum ab+\sum ac\right)}\\&=\frac{4\left(\sum a\right)^2}{\left(\sum a\right)^2}=4.\end{aligned}$$

The equality holds for $a=b=c=d$.

(b) Write the inequality as

$$\sum\left(\frac{1}{2}-\frac{a}{2a+b+c}\right)\ge 1,$$

$$\sum\frac{b+c}{2a+b+c}\ge 2.$$

By the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\sum\frac{b+c}{2a+b+c}&\ge\frac{\left[\sum(b+c)\right]^2}{\sum(b+c)(2a+b+c)}\\&=\frac{4\left(\sum a\right)^2}{2\left(\sum a^2+2\sum ab+\sum ac\right)}\\&=\frac{2\left(\sum a\right)^2}{\left(\sum a\right)^2}=2.\end{aligned}$$

The equality holds for $a=b=c=d$.

Conjecture 1. *If a,b,c,d,e are positive real numbers, then*

$$\frac{a-b}{a+2b+c}+\frac{b-c}{b+2c+d}+\frac{c-d}{c+2d+e}+\frac{d-e}{d+2e+a}+\frac{e-a}{e+2a+b}\ge 0.$$

Conjecture 2 (by *Ando*). *If $a_1,a_2,\dots,a_n$ ($n\ge 4$) are positive real numbers, then*

$$\frac{a_1}{(n-2)a_1+a_2+a_3}+\frac{a_2}{(n-2)a_2+a_3+a_4}+\cdots+\frac{a_n}{(n-2)a_n+a_1+a_2}\le 1.$$

□

P 1.201. *If a,b,c,d are positive real numbers such that $abcd=1$, then*

$$\frac{1}{a(a+b)}+\frac{1}{b(b+c)}+\frac{1}{c(c+d)}+\frac{1}{d(d+a)}\ge 2.$$

(Vasile C., 2007)

Solution. Making the substitution

$$a=\sqrt{\frac{y}{x}},\quad b=\sqrt{\frac{z}{y}},\quad c=\sqrt{\frac{t}{z}},\quad d=\sqrt{\frac{x}{t}},$$

where x,y,z,t are positive real numbers, the inequality can be rewritten as

$$\frac{x}{y+\sqrt{xz}}+\frac{y}{z+\sqrt{yt}}+\frac{z}{t+\sqrt{zx}}+\frac{t}{x+\sqrt{ty}}\ge 2.$$

Since

$$2\sqrt{xz}\le x+z,\qquad 2\sqrt{yt}\le y+t,$$

it suffices to show that

$$\frac{x}{x+2y+z}+\frac{y}{y+2z+t}+\frac{z}{z+2t+x}+\frac{t}{t+2x+y}\ge 1.$$

By the Cauchy-Schwarz inequality, we have

$$\sum\frac{x}{z+2y+z}\ge\frac{\left(\sum x\right)^2}{\sum x(x+2y+z)}=\frac{\left(\sum x\right)^2}{\sum x^2+2\sum xy+\sum xz}=1.$$

The equality holds for $a=c=\dfrac{1}{b}=\dfrac{1}{d}$.

Conjecture 1. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1 a_2 \cdots a_n = 1$, then*

$$\frac{1}{a_1^2 + a_1 a_2} + \frac{1}{a_2^2 + a_2 a_3} + \cdots + \frac{1}{a_n^2 + a_n a_1} \ge \frac{n}{2}.$$

Conjecture 2. *If $a_1, a_2, \ldots, a_n$ are positive real numbers, then*

$$\frac{1}{a_1^2 + a_1 a_2} + \frac{1}{a_2^2 + a_2 a_3} + \cdots + \frac{1}{a_n^2 + a_n a_1} \ge \frac{n^2}{2(a_1 a_2 + a_2 a_3 + \cdots + a_n a_1)}.$$

Remark 1. Using the substitution

$$a_1 = \frac{x_2}{x_1}, \quad a_2 = \frac{x_3}{x_2}, \; \ldots \;, \quad a_n = \frac{x_1}{x_n},$$

the inequality in Conjecture 1 becomes

$$\frac{x_1^2}{x_2^2 + x_1 x_3} + \frac{x_2^2}{x_3^2 + x_2 x_4} + \cdots + \frac{x_n^2}{x_1^2 + x_n x_2} \ge \frac{n}{2},$$

where $x_1, x_2, \ldots, x_n > 0$. This cyclic inequality is like Shapiro's inequality

$$\frac{x_1}{x_2 + x_3} + \frac{x_2}{x_3 + x_4} + \cdots + \frac{x_n}{x_1 + x_2} \ge \frac{n}{2},$$

which is true for even $n \le 12$ and for odd $n \le 23$.

Remark 2. By the AM-GM inequality, we have

$$a_1 a_2 + a_2 a_3 + \cdots + a_n a_1 \ge n \sqrt[n]{a_1^2 a_2^2 \cdots a_n^2}.$$

Thus, the inequality in Conjecture 2 is weaker than the inequality in Conjecture 1. Therefore, if Conjecture 1 is true, then Conjecture 2 is also true.

□

P 1.202. *If a, b, c, d are positive real numbers, then*

$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+d)} + \frac{1}{d(1+a)} \ge \frac{16}{1 + 8\sqrt{abcd}}.$$

(Pham Kim Hung, 2007)

Solution. Let $p = \sqrt[4]{abcd}$. Putting

$$a = p\frac{x_2}{x_1}, \quad b = p\frac{x_3}{x_2}, \quad c = p\frac{x_4}{x_3}, \quad d = p\frac{x_1}{x_4},$$

where x_1,x_2,x_3,x_4 are positive real numbers, the inequality turns into

$$\sum \frac{x_1}{x_2+px_3} \ge \frac{16p}{1+8p^2}.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{x_1}{x_2+px_3} \ge \frac{\left(\sum x_1\right)^2}{\sum x_1(x_2+px_3)} = \frac{\left(\sum x_1\right)^2}{(x_1+x_3)(x_2+x_4)+2p(x_1x_3+x_2x_4)}.$$

Since

$$x_1x_3+x_2x_4 \le \left(\frac{x_1+x_3}{2}\right)^2+\left(\frac{x_2+x_4}{2}\right)^2,$$

it suffices to show that

$$\frac{(A+B)^2}{2AB+p(A^2+B^2)} \ge \frac{8p}{1+8p^2},$$

where

$$A=x_1+x_3, \qquad B=x_2+x_4.$$

This inequality is equivalent to

$$A^2+B^2+2(8p^2-8p+1)AB \ge 0,$$

which is true because

$$\begin{aligned} A^2+B^2+2(8p^2-8p+1)AB &\ge 2AB+2(8p^2-8p+1)AB \\ &= 4(2p-1)^2AB \ge 0. \end{aligned}$$

The equality holds for $a=b=c=d=\dfrac{1}{2}$.

□

P 1.203. *If a,b,c,d are nonnegative real numbers such that $a^2+b^2+c^2+d^2=4$, then*

(a) $$3(a+b+c+d) \ge 2(ab+bc+cd+da)+4;$$

(b) $$a+b+c+d-4 \ge (2-\sqrt{2})(ab+bc+cd+da-4).$$

(Vasile C., 2006)

Solution. Let $p = a+b+c+d$. By the Cauchy-Schwarz inequality

$$(1+1+1+1)(a^2+b^2+c^2+d^2) \ge (a+b+c+d)^2,$$

we get $p \le 4$, and by the inequality

$$(a+b+c+d)^2 \ge a^2+b^2+c^2+d^2,$$

we get $p \ge 2$. In addition, we have

$$ab+bc+cd+da = (a+c)(b+d) \le \frac{(a+c+b+d)^2}{4} = \frac{p^2}{4}.$$

(a) It suffices to show that

$$3p \ge \frac{p^2}{2} + 4.$$

Indeed,

$$3p - \frac{p^2}{2} - 4 = \frac{(4-p)(p-2)}{2} \ge 0.$$

The equality holds for $a=b=c=d=1$.

(b) It suffices to show that

$$p - 4 \ge (2-\sqrt{2})\left(\frac{p^2}{4} - 4\right).$$

This inequality is equivalent to

$$(4-p)(p-2\sqrt{2}) \ge 0,$$

which is true for $p \ge 2\sqrt{2}$. So, it remains to consider the case $2 \le p < 2\sqrt{2}$. Since

$$2(ab+bc+cd+da) \le (a+b+c+d)^2 - (a^2+b^2+c^2+d^2) = p^2 - 4,$$

it is enough to prove that

$$p - 4 \ge (2-\sqrt{2}\,)\left(\frac{p^2-4}{2} - 4\right).$$

Write this inequality as

$$(2+\sqrt{2}\,)(p-4) \ge p^2 - 12,$$

$$(2\sqrt{2}-p)(p-2+\sqrt{2}\,) \ge 0.$$

The equality holds for $a=b=c=d=1$, and also for $a=b=0$ and $c=d=\sqrt{2}$ (or any cyclic permutation).

□

P 1.204. *Let a,b,c,d be positive real numbers.*

(a) If $a,b,c,d \ge 1$, then

$$\left(a+\frac{1}{b}\right)\left(b+\frac{1}{c}\right)\left(c+\frac{1}{d}\right)\left(d+\frac{1}{a}\right) \ge (a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right);$$

(b) If $abcd = 1$, then

$$\left(a+\frac{1}{b}\right)\left(b+\frac{1}{c}\right)\left(c+\frac{1}{d}\right)\left(d+\frac{1}{a}\right) \le (a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right).$$

(Vasile Cîrtoaje and Ji Chen, 2011)

Solution. Let

$$\begin{aligned}
A =&(1+ab)(1+bc)(1+cd)(1+da)\\
=&1+\sum ab+\sum a^2bd+2abcd+abcd\sum ab+a^2b^2c^2d^2\\
=&(1-abcd)^2+4abcd+(1+abcd)\sum ab+\sum a^2bd\\
=&(1-abcd)^2+4abcd+(1+abcd)(a+c)(b+d)+\sum a^2bd
\end{aligned}$$

and

$$\begin{aligned}
B =&(a+b+c+d)(abc+bcd+cda+dab)\\
=&4abcd+\sum a^2(bc+cd+db)\\
=&4abcd+\sum a^2c(b+d)+\sum a^2bd\\
=&4abcd+(ac+bd)(a+c)(b+d)+\sum a^2bd.
\end{aligned}$$

Thus,

$$\begin{aligned}
A-B &= (1-abcd)^2+(1+abcd)(a+c)(b+d)-(ac+bd)(a+c)(b+d)\\
&= (1-abcd)^2+(1-ac)(1-bd)(a+c)(b+d).
\end{aligned}$$

(a) The inequality $A \ge B$ is clearly true for $a,b,c,d \ge 1$. The equality holds for $a=b=c=d=1$.

(b) For $abcd = 1$, we have

$$B-A = \frac{1}{ac}(1-ac)^2(a+c)(b+d) \ge 0.$$

The equality holds for $ac = bd = 1$.

□

P 1.205. *If a,b,c,d are positive real numbers, then*

$$\left(1+\frac{a}{a+b}\right)^2+\left(1+\frac{b}{b+c}\right)^2+\left(1+\frac{c}{c+d}\right)^2+\left(1+\frac{d}{d+a}\right)^2>7.$$

(Vasile C., 2012)

First Solution. Assume that $d=\max\{a,b,c,d\}$. We get the desired inequality by summing the inequalities

$$\left(1+\frac{a}{a+b}\right)^2+\left(1+\frac{b}{b+c}\right)^2+\left(1+\frac{c}{c+a}\right)^2>6$$

and

$$\left(1+\frac{c}{c+d}\right)^2+\left(1+\frac{d}{d+a}\right)^2>1+\left(1+\frac{c}{c+a}\right)^2.$$

Let

$$x=\frac{a-b}{a+b},\quad y=\frac{b-c}{b+c},\quad z=\frac{c-a}{c+a}.$$

We have $-1<x,y,z<1$ and

$$x+y+z+xyz=0.$$

Since

$$\frac{a}{a+b}=\frac{x+1}{2},\quad \frac{b}{b+c}=\frac{y+1}{2},\quad \frac{c}{c+a}=\frac{z+1}{2},$$

we can write the first inequality as follows:

$$(x+3)^2+(y+3)^2+(z+3)^2>24,$$

$$x^2+y^2+z^2+6(x+y+z)+3>0,$$

$$x^2+y^2+z^2+3>6xyz.$$

By the AM-GM inequality, we have

$$x^2+y^2+z^2+3\ge 6\sqrt[6]{x^2y^2z^2}>6xyz.$$

Write now the second inequality as

$$\left(1+\frac{c}{c+d}\right)^2-1>\left(\frac{c}{c+a}-\frac{d}{d+a}\right)\left(2+\frac{c}{c+a}+\frac{d}{d+a}\right).$$

Since

$$\frac{c}{c+a}-\frac{d}{d+a}=\frac{a(c-d)}{(c+a)(d+a)}\le 0,$$

we have

$$\left(1+\frac{c}{c+d}\right)^2-1>0\ge\left(\frac{c}{c+a}-\frac{d}{d+a}\right)\left(2+\frac{c}{c+a}+\frac{d}{d+a}\right).$$

Second Solution. Using the inequality

$$(1+x)^2 > 1+3x^2, \quad 0<x<1,$$

we have

$$\left(1+\frac{a}{a+b}\right)^2+\left(1+\frac{b}{b+c}\right)^2+\left(1+\frac{c}{c+d}\right)^2+\left(1+\frac{d}{d+a}\right)^2 >$$

$$> 4+3\left[\left(\frac{a}{a+b}\right)^2+\left(\frac{b}{b+c}\right)^2+\left(\frac{c}{c+d}\right)^2+\left(\frac{d}{d+a}\right)^2\right].$$

Therefore, it suffices to prove that

$$\left(\frac{a}{a+b}\right)^2+\left(\frac{b}{b+c}\right)^2+\left(\frac{c}{c+d}\right)^2+\left(\frac{d}{d+a}\right)^2 \ge 1,$$

which is equivalent to the known inequality in P 1.191 from Volume 2:

$$\frac{1}{(1+x)^2}+\frac{1}{(1+y)^2}+\frac{1}{(1+z)^2}+\frac{1}{(1+t)^2} \ge 1,$$

where

$$x=\frac{a}{b}, \quad y=\frac{b}{c}, \quad z=\frac{c}{d}, \quad t=\frac{d}{a}, \qquad xyzt=1.$$

□

P 1.206. *If a,b,c,d are positive real numbers, then*

$$\frac{a^2-bd}{b+2c+d}+\frac{b^2-ca}{c+2d+a}+\frac{c^2-db}{d+2a+b}+\frac{d^2-ac}{a+2b+c} \ge 0.$$

(Vo Quoc Ba Can, 2009)

Solution. Write the inequality as follows:

$$\sum\left(\frac{4a^2-4bd}{b+2c+d}+b+d-2a\right) \ge 0,$$

$$\sum\frac{(b-d)^2+2(a-c)(2a-b-d)}{b+2c+d} \ge 0.$$

It suffices to show that

$$\sum\frac{(a-c)(2a-b-d)}{b+2c+d} \ge 0.$$

This inequality is equivalent to

$$(a-c)\left(\frac{2a-b-d}{b+2c+d}-\frac{2c-d-b}{d+2a+b}\right)+(b-d)\left(\frac{2b-c-a}{c+2d+a}-\frac{2d-a-c}{a+2b+c}\right)\geq 0,$$

which can be written as

$$\frac{(a-c)(a^2-c^2)}{(b+2c+d)(d+2a+b)}+\frac{(b-d)(b^2-d^2)}{(c+2d+a)(a+2b+c)}\geq 0.$$

The equality occurs for $a=c$ and $b=d$.

□

P 1.207. *If a,b,c,d are positive real numbers such that $a\leq b\leq c\leq d$, then*

$$\sqrt{\frac{2a}{a+b}}+\sqrt{\frac{2b}{b+c}}+\sqrt{\frac{2c}{c+d}}+\sqrt{\frac{2d}{d+a}}\leq 4.$$

(Vasile C., 2009)

Solution. According to the inequality in P 1.74, we have

$$\sqrt{\frac{2a}{a+b}}+\sqrt{\frac{2b}{b+c}}+\sqrt{\frac{2c}{c+a}}\leq 3.$$

Therefore, it suffices to show that

$$\sqrt{\frac{2c}{c+d}}+\sqrt{\frac{2d}{d+a}}\leq 1+\sqrt{\frac{2c}{c+a}}.$$

By squaring, this inequality becomes

$$\frac{2c}{c+d}+\frac{2d}{d+a}+2\sqrt{\frac{4cd}{(c+d)(d+a)}}\leq 1+\frac{2c}{c+a}+2\sqrt{\frac{2c}{c+a}}.$$

We can get it by summing the inequalities

$$\frac{2c}{c+d}+\frac{2d}{d+a}\leq 1+\frac{2c}{c+a},$$

$$2\sqrt{\frac{4cd}{(c+d)(d+a)}}\leq 2\sqrt{\frac{2c}{c+a}}.$$

The former inequality is true since

$$\frac{2c}{c+d}+\frac{2d}{d+a}-1-\frac{2c}{c+a}=\frac{(a-d)(d-c)(c-a)}{(c+d)(d+a)(a+c)}\leq 0,$$

while the second inequality reduces to

$$c(a-d)(d-c)\leq 0.$$

The equality holds for $a=b=c=d$.

□

P 1.208. *Let a,b,c,d be nonnegative real numbers, and let*

$$x=\frac{a}{b+c},\quad y=\frac{b}{c+d},\quad z=\frac{c}{d+a},\quad t=\frac{d}{a+b}.$$

Prove that

(a) $$\sqrt{xz}+\sqrt{yt}\le 1;$$

(b) $$x+y+z+t+4(xz+yt)\ge 4.$$

(Vasile C., 2004)

Solution. (a) Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\sqrt{xz}+\sqrt{yt}=&\frac{\sqrt{ac}}{\sqrt{(b+c)(d+a)}}+\frac{\sqrt{bd}}{\sqrt{(c+d)(a+b)}}\\ \le&\frac{\sqrt{ac}}{\sqrt{ac}+\sqrt{bd}}+\frac{\sqrt{bd}}{\sqrt{ac}+\sqrt{bd}}=1.\end{aligned}$$

The equality holds for $a=b=c=d$, for $a=c=0$, and for $b=d=0$

(b) Write the inequality as

$$A+B\ge 6,$$

where

$$\begin{aligned}A=x+z+4xz+1=&\frac{(a+b)(c+d)+(a+c)^2+ab+2ac+cd}{(b+c)(d+a)}\\ =&\frac{(a+b)(c+d)}{(b+c)(d+a)}+\frac{(a+c)^2}{(b+c)(d+a)}+\frac{a}{d+a}+\frac{c}{b+c},\end{aligned}$$

$$B=y+t+4yt+1=\frac{(b+c)(d+a)}{(c+d)(a+b)}+\frac{(b+d)^2}{(c+d)(a+b)}+\frac{b}{a+b}+\frac{d}{c+d}.$$

Since

$$\frac{(a+b)(c+d)}{(b+c)(d+a)}+\frac{(b+c)(d+a)}{(c+d)(a+b)}\ge 2,$$

it suffices to show that

$$\frac{(a+c)^2}{(b+c)(d+a)}+\frac{(b+d)^2}{(c+d)(a+b)}+\sum\frac{a}{d+a}\ge 4.$$

By the Cauchy-Schwarz inequality, we have

$$\frac{(a+c)^2}{(b+c)(d+a)}+\frac{(b+d)^2}{(c+d)(a+b)}\ge\frac{(a+b+c+d)^2}{C},$$

$$\sum\frac{a}{d+a}\ge\frac{(a+b+c+d)^2}{D},$$

where

$$C=(b+c)(d+a)+(c+d)(a+b),$$

$$D=\sum a(d+a)=a^2+b^2+c^2+d^2+ab+bc+cd+da,$$

$$C+D=(a+b+c+d)^2.$$

Thus, it is enough to show that

$$(C+D)\left(\frac{1}{C}+\frac{1}{D}\right)\geq 4,$$

which is clearly true. The equality holds for $a=b=c=d$.

□

P 1.209. *If a,b,c,d are nonnegative real numbers, then*

$$\left(1+\frac{2a}{b+c}\right)\left(1+\frac{2b}{c+d}\right)\left(1+\frac{2c}{d+a}\right)\left(1+\frac{2d}{a+b}\right)\geq 9.$$

(Vasile C., 2004)

Solution. We can rewrite the inequality as

$$\left(1+\frac{a+c}{a+b}\right)\left(1+\frac{a+c}{c+d}\right)\left(1+\frac{b+d}{b+c}\right)\left(1+\frac{b+d}{d+a}\right)\geq 9.$$

Using the Cauchy-Schwarz inequality and the AM-GM inequality yields

$$\left(1+\frac{a+c}{a+b}\right)\left(1+\frac{a+c}{c+d}\right)\geq\left[1+\frac{a+c}{\sqrt{(a+b)(c+d)}}\right]^2\geq\left(1+\frac{2a+2c}{a+b+c+d}\right)^2,$$

$$\left(1+\frac{b+d}{b+c}\right)\left(1+\frac{b+d}{d+a}\right)\geq\left[1+\frac{b+d}{\sqrt{(b+c)(d+a)}}\right]^2\geq\left(1+\frac{2b+2d}{a+b+c+d}\right)^2.$$

Thus, it suffices to show that

$$\left(1+\frac{2a+2c}{a+b+c+d}\right)\left(1+\frac{2b+2d}{a+b+c+d}\right)\geq 3.$$

This is equivalent to the obvious inequality

$$\frac{4(a+c)(b+d)}{(a+b+c+d)^2}\geq 0.$$

The equality holds for $a=c=0$ and $b=d$, as well as for $b=d=0$ and $a=c$.

□

P 1.210. *Let a, b, c, d be nonnegative real numbers. If $k > 0$, then*

$$\left(1+\frac{ka}{b+c}\right)\left(1+\frac{kb}{c+d}\right)\left(1+\frac{kc}{d+a}\right)\left(1+\frac{kd}{a+b}\right) \ge (1+k)^2.$$

(Vasile C., 2004)

Solution. Let us denote

$$x = \frac{a}{b+c}, \quad y = \frac{b}{c+d}, \quad z = \frac{c}{d+a}, \quad t = \frac{d}{a+b}.$$

Since

$$\prod(1+kx) \ge 1 + k(x+y+z+t) + k^2(xy+yz+zt+tx+xz+yt),$$

it suffices to show that

$$x+y+z+t \ge 2$$

and

$$xy+yz+zt+tx+xz+yt \ge 1.$$

The inequality $x+y+z+t \ge 2$ is the well-known Shapiro's inequality for 4 positive real numbers. This can be proved by the Cauchy-Schwarz inequality, as follows:

$$\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b} \ge \frac{(a+b+c+d)^2}{a(b+c)+b(c+d)+c(d+a)+d(a+b)} \ge 2.$$

The right inequality reduces to the obvious inequality

$$(a-c)^2 + (b-d)^2 \ge 0.$$

To prove the inequality $xy+yz+zt+tx+xz+yt \ge 1$, we will use the inequalities

$$\frac{x+z}{2} \ge xz,$$

$$\frac{y+t}{2} \ge yt,$$

and the identity

$$xz(1+y+t) + yt(1+x+z) = 1.$$

If these are true, then

$$\begin{aligned} xy+yz+zt+tx+xz+yt &= \frac{x+z}{2}(y+t) + \frac{y+t}{2}(x+z) + xz + yt \\ &\ge xz(y+t) + yt(x+z) + xz + yt \\ &= xz(1+y+t) + yt(1+x+z) = 1. \end{aligned}$$

We have

$$\frac{x+z}{2} - xz = \frac{bc+da+(a-c)^2}{2(b+c)(d+a)} \geq 0$$

and

$$\frac{y+t}{2} - yt = \frac{ab+cd+(b-d)^2}{2(a+b)(c+d)} \geq 0.$$

To prove the identity above, we rewrite it as

$$\sum xyz + xz + yt = 1,$$

and see that

$$\sum xyz = \frac{\sum abc(a+b)}{A} = \frac{\sum a^2bc + \sum a^2bd}{A}$$

and

$$xz + yt = \frac{ac(a+b)(c+d) + bd(b+c)(d+a)}{A} = \frac{\sum a^2cd + (ac+bd)^2}{A},$$

where

$$A = \prod(a+b) = \sum a^2bc + \sum a^2bd + \sum a^2cd + (ac+bd)^2.$$

Thus, the proof is completed. The equality holds for $a = c = 0$ and $b = d$, as well as for $b = d = 0$ and $a = c$.

Remark. For $k = 2$, we get the inequality in P 1.209. For $k = 1$, we get the following known inequality

$$(a+b+c)(b+c+d)(c+d+a)(d+a+b) \geq 4(a+b)(b+c)(c+d)(d+a).$$

A proof of this inequality starts from the inequalities

$$(a+b+c)^2 \geq (2a+b)(2c+b)$$

and

$$(2a+b)(2b+a) \geq 2(a+b)^2.$$

We have

$$\begin{aligned}\prod(a+b+c)^2 &\geq \prod(2a+b) \cdot \prod(2c+b) \\ &= \prod(2a+b)(2b+a) \\ &\geq 2^4 \prod(a+b)^2,\end{aligned}$$

hence

$$\prod(a+b+c) \geq 4\prod(a+b).$$

□

P 1.211. *If a, b, c, d are positive real numbers such that $a+b+c+d=4$, then*

$$\frac{1}{ab}+\frac{1}{bc}+\frac{1}{cd}+\frac{1}{da} \ge a^2+b^2+c^2+d^2.$$

(Vasile C., 2007)

Solution. Write the inequality as

$$(a+c)(b+d) \ge abcd(a^2+b^2+c^2+d^2).$$

From $(a-c)^4 \ge 0$ and $(b-d)^4 \ge 0$, we get

$$(a+c)^4 \ge 8ac(a^2+c^2), \quad (b+d)^4 \ge 8bd(b^2+d^2),$$

hence

$$bd(a+c)^4 + ac(b+d)^4 \ge 8abcd(a^2+b^2+c^2+d^2).$$

Therefore, it suffices to show that

$$8(a+c)(b+d) \ge bd(a+c)^4 + ac(b+d)^4.$$

Since $4bd \le (b+d)^2$ and $4ac \le (a+c)^2$, we only need to show that

$$32(a+c)(b+d) \ge (b+d)^2(a+c)^4 + (a+c)^2(b+d)^4.$$

This inequality is true if

$$32 \ge xy(x^2+y^2)$$

for all positive x, y satisfying $x+y=4$. Indeed,

$$8[32 - xy(x^2+y^2)] = (x+y)^4 - 8xy(x^2+y^2) = (x-y)^4 \ge 0.$$

The equality occurs for $a=b=c=d=1$.

□

P 1.212. *If a, b, c, d are positive real numbers, then*

$$\frac{a^2}{(a+b+c)^2}+\frac{b^2}{(b+c+d)^2}+\frac{c^2}{(c+d+a)^2}+\frac{d^2}{(d+a+b)^2} \ge \frac{4}{9}.$$

(Pham Kim Hung, 2006)

First Solution. By Hölder's inequality, we have

$$\sum \frac{a^2}{(a+b+c)^2} \ge \frac{\left(\sum a^{4/3}\right)^3}{\left[\sum a(a+b+c)\right]^2}.$$

Since

$$\sum a(a+b+c) = (a+c)^2 + (b+d)^2 + (a+c)(b+d)$$

and

$$\sum a^{4/3} = \left(a^{4/3} + c^{4/3}\right) + \left(b^{4/3} + d^{4/3}\right) \ge 2\left(\frac{a+c}{2}\right)^{4/3} + 2\left(\frac{b+d}{2}\right)^{4/3},$$

it suffices to show that

$$9\left[(a+c)^{4/3} + (b+d)^{4/3}\right]^3 \ge 8[(a+c)^2 + (b+d)^2 + (a+c)(b+d)]^2.$$

Due to homogeneity, we may assume that $b+d=1$. Putting $a+c=t^3$, $t>0$, the inequality becomes

$$9(t^4+1)^3 \ge 8(t^6+1+t^3)^2,$$

$$9\left(t^2+\frac{1}{t^2}\right)^3 \ge 8\left(t^3+\frac{1}{t^3}+1\right)^2.$$

Setting

$$x = t+\frac{1}{t}, \qquad x \ge 2,$$

the inequality turns into

$$9(x^2-2)^3 \ge 8(x^3-3x+1)^2,$$

which is equivalent to

$$(x-2)^2(x^4+4x^3+6x^2-8x-20) \ge 0.$$

This is true since

$$x^4+4x^3+6x^2-8x-20 = x^4+4x^2(x-2)+4x(x-2)+10(x^2-2) > 0.$$

Thus, the proof is completed. The equality holds for $a=b=c=d$.

Second Solution. Due to homogeneity, we may assume that

$$a+b+c+d=1.$$

In this case, we write the inequality as

$$\left(\frac{a}{1-d}\right)^2 + \left(\frac{b}{1-a}\right)^2 + \left(\frac{c}{1-b}\right)^2 + \left(\frac{d}{1-c}\right)^2 \ge \frac{4}{9}.$$

Let (x,y,z,t) be a permutation of (a,b,c,d) such that

$$x \ge y \ge z \ge t.$$

Since
$$\frac{1}{(1-t)^2} \le \frac{1}{(1-z)^2} \le \frac{1}{(1-y)^2} \le \frac{1}{(1-x)^2},$$
by the rearrangement inequality, we have
$$\left(\frac{x}{1-t}\right)^2 + \left(\frac{y}{1-z}\right)^2 + \left(\frac{z}{1-y}\right)^2 + \left(\frac{t}{1-x}\right)^2 \le$$
$$\le \left(\frac{a}{1-d}\right)^2 + \left(\frac{b}{1-a}\right)^2 + \left(\frac{c}{1-b}\right)^2 + \left(\frac{d}{1-c}\right)^2.$$
Therefore, it suffices to show that $x+y+z+t=1$ involves
$$U+V \ge \frac{4}{9},$$
where
$$U = \left(\frac{x}{1-t}\right)^2 + \left(\frac{t}{1-x}\right)^2,$$
$$V = \left(\frac{y}{1-z}\right)^2 + \left(\frac{z}{1-y}\right)^2.$$
Let
$$s = x+t, \quad p = xt, \quad s \in (0,1),$$
Since
$$x^2+t^2 = s^2-2p, \quad x^3+t^3 = s^3-3ps, \quad x^4+t^4 = s^4-4ps^2+2p^2,$$
we get
$$U = \frac{x^2+t^2-2(x^3+t^3)+x^4+t^4}{(1-s+p)^2}$$
$$= \frac{2p^2-2(1-s)(1-2s)p+s^2(1-s)^2}{p^2+2(1-s)p+(1-s)^2},$$
$$(2-U)p^2-2(1-s)(1-2s+U)p+(1-s)^2(s^2-U)=0.$$
The quadratic trinomial in p has the discriminant
$$D = (1-s)^2[(1-2s+U)^2-(2-U)(s^2-U)].$$
From the necessary condition $D \ge 0$, we get
$$U \ge \frac{4s-1-2s^2}{(2-s)^2}.$$
Analogously,
$$V \ge \frac{4r-1-2r^2}{(2-r)^2},$$

where $r = y + z$. Taking into account that

$$s + r = 1,$$

we get

$$\begin{aligned} U + V &\geq \frac{4s - 1 - 2s^2}{(2-s)^2} + \frac{4r - 1 - 2r^2}{(2-r)^2} \\ &= \frac{4s - 1 - 2s^2}{(1+r)^2} + \frac{4r - 1 - 2r^2}{(1+s)^2} \\ &= \frac{5(s^2 + r^2) - 2(s^4 + r^4)}{(2+sr)^2} \\ &= \frac{5(s^2 + r^2) - 2(s^2 + r^2)^2 + 4s^2r^2}{(2+sr)^2}, \end{aligned}$$

hence

$$\begin{aligned} U + V - \frac{4}{9} &\geq \frac{5(s^2 + r^2) - 2(s^2 + r^2)^2 + 4s^2r^2}{(2+sr)^2} - \frac{4}{9} \\ &= \frac{5(s^2 + r^2) - 2(s^2 + r^2)^2}{(2+sr)^2} + \frac{2(1 - 4sr)^2 - 18}{9(2+sr)^2} \\ &\geq \frac{5(s^2 + r^2) - 2(s^2 + r^2)^2 - 2}{(2+sr)^2} \\ &= \frac{(2 - s^2 - r^2)(2s^2 + 2r^2 - 1)}{(2+sr)^2}. \end{aligned}$$

Thus, we need to show that $(2 - s^2 - r^2)(2s^2 + 2r^2 - 1) \geq 0.$ This is true since since

$$2 - s^2 - r^2 > 2 - (s + r)^2 = 1,$$

$$2s^2 + 2r^2 - 1 \geq (s + r)^2 - 1 = 0.$$

□

P 1.213. *If a, b, c, d are positive real numbers such that $a + b + c + d = 3$, then*

$$ab(b + c) + bc(c + d) + cd(d + a) + da(a + b) \leq 4.$$

(Pham Kim Hung, 2007)

Solution. Write the inequality as

$$\sum ab^2 + \sum abc \leq 4,$$

$$(ab^2 + cd^2 + bcd + dab) + (bc^2 + da^2 + abc + cda) \leq 4,$$

$$(b+d)(ab+cd)+(a+c)(bc+da)\le 4.$$

Without loss of generality, assume that $a+c\le b+d$. Since

$$(ab+cd)+(bc+da)=(a+c)(b+d),$$

we can rewrite the inequality as

$$(a+c)(b+d)^2+(a+c-b-d)(bc+da)\le 4.$$

Since $a+c-b-d\le 0$, it suffices to show that

$$(a+c)(b+d)^2\le 4.$$

Indeed, by the AM-GM inequality, we have

$$(a+c)\left(\frac{b+d}{2}\right)\left(\frac{b+d}{2}\right)\le\frac{1}{27}\left(a+c+\frac{b+d}{2}+\frac{b+d}{2}\right)^3=1.$$

The equality holds for $a=b=0$, $c=1$ and $d=2$ (or any cyclic permutation).

□

P 1.214. *If $a\ge b\ge c\ge d\ge 0$ and $a+b+c+d=2$, then*

$$ab(b+c)+bc(c+d)+cd(d+a)+da(a+b)\le 1.$$

(Vasile C., 2007)

Solution. Write the inequality as

$$\sum ab^2+\sum abc\le 1.$$

Since

$$\sum ab^2-\sum a^2b=(ab^2+bc^2+ca^2-a^2b-b^2c-c^2a)+(cd^2+da^2+ac^2-c^2d-d^2a-a^2c)$$
$$=(a-b)(b-c)(c-a)+(c-d)(d-a)(a-c)\le 0,$$

it suffices to show that

$$\sum ab^2+\sum a^2b+2\sum abc\le 2.$$

Indeed,

$$\begin{aligned}\sum ab^2+\sum a^2b+2\sum abc&=\sum(ab^2+a^2b+abc+abd)\\&=(a+b+c+d)\sum ab\\&=2(a+c)(b+d)\\&\le 2\left[\frac{(a+c)+(b+d)}{2}\right]^2=2.\end{aligned}$$

The equality holds for $a=b=t$ and $c=d=1-t$, where $t\in\left[\frac{1}{2},1\right]$.

□

P 1.215. *Let a,b,c,d be nonnegative real numbers such that $a+b+c+d=4$. If $k \ge \frac{37}{27}$, then*

$$ab(b+kc)+bc(c+kd)+cd(d+ka)+da(a+kb) \le 4(1+k).$$

(Vasile C., 2007)

Solution. Write the inequality in the homogeneous form

$$ab(b+kc)+bc(c+kd)+cd(d+ka)+da(a+kb) \le \frac{(1+k)(a+b+c+d)^3}{16}.$$

Assume that $d=\min\{a,b,c,d\}$ and use the substitution

$$a=d+x, \quad b=d+y, \quad c=d+z,$$

where $x,y,z \ge 0$. The inequality can be restated as

$$4Ad+B \ge 0,$$

where

$$A=(3k-1)(x^2+y^2+z^2)-2(k+1)y(x+z)+(6-2k)xz,$$
$$B=(1+k)(x+y+z)^3-16(xy^2+yz^2+kxyz).$$

It suffices to show that $A \ge 0$ and $B \ge 0$. We have

$$\begin{aligned}
A &= (3k-1)y^2+(3k-1)(x+z)^2-2(k+1)y(x+z)-8(k-1)xz\\
&\ge (3k-1)y^2+(3k-1)(x+z)^2-2(k+1)y(x+z)-2(k-1)(x+z)^2\\
&= (3k-1)y^2+(k+1)(x+z)^2-2(k+1)y(x+z)\\
&\ge 2\sqrt{(3k-1)(k+1)}y(x+z)-2(k+1)y(x+z)\\
&= 2\sqrt{k+1}\left(\sqrt{3k-1}-\sqrt{k+1}\right)y(x+z) \ge 0.
\end{aligned}$$

Since

$$(x+y+z)^3-16xyz \ge 0,$$

the inequality $B \ge 0$ holds for all $k \ge \frac{37}{27}$ if it holds for $k=\frac{37}{27}$. In this particular case, the inequality $B \ge 0$ can be written as

$$4\left(\frac{x+y+z}{3}\right)^3 \ge xy^2+yz^2+\frac{37}{27}xyz.$$

Actually, the following sharper inequality holds (see P 2.31)

$$4\left(\frac{x+y+z}{3}\right)^3 \ge xy^2+yz^2+\frac{3}{2}xyz.$$

Thus, the proof is completed. The equality holds for $a=b=c=d=1$. If $k=\frac{37}{27}$, then the equality holds also for $a=\frac{4}{3}$, $b=\frac{8}{3}$ and $c=d=0$ (or any cyclic permutation).

□

P 1.216. *If a,b,c,d are nonnegative real numbers such that $a+b+c+d=4$, then*

$$\sqrt{\frac{3a}{b+2}}+\sqrt{\frac{3b}{c+2}}+\sqrt{\frac{3c}{d+2}}+\sqrt{\frac{3d}{a+2}}\le 4.$$

(Vasile Cîrtoaje, 2020)

Solution. (after an idea of *Michael Rozenberg*) Let $(a1,a_2,a_3,a_4)$ be an increasing permutation of (a,b,c,d). Since the sequences

$$(a1,a_2,a_3,a_4) \qquad and \qquad \left(\frac{1}{a_4+2},\frac{1}{a_3+2},\frac{1}{a_2+2},\frac{1}{a_1+2}\right)$$

are increasing, according to the rearrangement inequality, we have

$$\sqrt{\frac{3a}{b+2}}+\sqrt{\frac{3b}{c+2}}+\sqrt{\frac{3c}{d+2}}+\sqrt{\frac{3d}{a+2}}\le$$

$$\le\sqrt{\frac{3a_1}{a_4+2}}+\sqrt{\frac{3a_2}{a_3+2}}+\sqrt{\frac{3a_3}{a_2+2}}+\sqrt{\frac{3a_4}{a_1+2}}=A+B,$$

where

$$A=\sqrt{\frac{3a_1}{a_4+2}}+\sqrt{\frac{3a_4}{a_1+2}}, \qquad B=\sqrt{\frac{3a_2}{a_3+2}}++\sqrt{\frac{3a_3}{a_2+2}}.$$

We need to show that $A+B\le 2$. According to Lemma below, we have

$$A+B\le\frac{a_1+a_4+4}{3}+\frac{a_2+a_3+4}{3}=4.$$

The equality holds for $a=b=c=d=1$.

Lemma. If a,b are nonnegative real numbers, then

$$\sqrt{\frac{3a}{b+2}}+\sqrt{\frac{3b}{a+2}}\le\frac{a+b+4}{3}.$$

Proof. Use the substitution

$$x=\sqrt{\frac{3a}{b+2}}, \quad y=\sqrt{\frac{3b}{a+2}},$$

which yields $xy<3$ and

$$a=\frac{2x^2(y^2+3)}{9-x^2y^2}, \qquad b=\frac{2y^2(x^2+3)}{9-x^2y^2}, \qquad a+b=\frac{4x^2y^2+6(x^2+y^2)}{9-x^2y^2}.$$

Thus, we need to show that

$$3(x+y)\le\frac{4x^2y^2+6(x^2+y^2)}{9-x^2y^2}+4,$$

which is equivalent to

$$2(x+y)^2-(9-x^2y^2)(x+y)+12-4xy \ge 0,$$

$$\left(4x+4y-9+x^2y^2\right)^2+15-32xy+18x^2y^2-x^4y^4 \ge 0,$$

$$\left(4x+4y-9+x^2y^2\right)^2+(1-xy)^2(3-xy)(5+xy) \ge 0.$$

The equality holds for $a=b=1$.

□

P 1.217. *Let a,b,c,d be positive real numbers such that $a \le b \le c \le d$. Prove that*

$$2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\right) \ge 4+\frac{a}{c}+\frac{c}{a}+\frac{b}{d}+\frac{d}{b}.$$

(Vasile C., 2012)

First Solution. Let

$$E(a,b,c,d)=2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\right)-4-\frac{a}{c}-\frac{c}{a}-\frac{b}{d}-\frac{d}{b}.$$

We show that

$$E(a,b,c,d) \ge E(b,b,c,d) \ge E(b,b,c,c).$$

We have

$$E(a,b,c,d)-E(b,b,c,d)=(b-a)\left(\frac{1}{c}+\frac{2d}{ab}-\frac{2}{b}-\frac{c}{ab}\right) \ge 0,$$

since

$$\frac{1}{c}+\frac{2d}{ab}-\frac{2}{b}-\frac{c}{ab} \ge \frac{1}{c}+\frac{2c}{ab}-\frac{2}{b}-\frac{c}{ab}$$
$$=\frac{1}{c}+\frac{c}{ab}-\frac{2}{b} \ge \frac{1}{c}+\frac{c}{b^2}-\frac{2}{b}=\frac{(b-c)^2}{b^2c} \ge 0.$$

Also,

$$E(b,b,c,d)-E(b,b,c,c)=(d-c)\left(\frac{1}{b}-\frac{2c-b}{cd}\right) \ge 0,$$

since

$$\frac{1}{b}-\frac{2c-b}{cd} \ge \frac{1}{b}-\frac{2c-b}{c^2}=\frac{(b-c)^2}{bc^2} \ge 0.$$

Because $E(b,b,c,c)=0$, the proof is completed. The equality holds for $a=b$ and $c=d$.

Second Solution. Using the substitution

$$x=\frac{a}{b}, \quad y=\frac{b}{c}, \quad z=\frac{c}{d}, \quad 0<x,y,z \le 1,$$

the inequality becomes as follows:

$$2\left(x+y+z+\frac{1}{xyz}\right)\geq 4+xy+\frac{1}{xy}+yz+\frac{1}{yz},$$

$$y(2-x-z)+\frac{1}{y}\left(\frac{2}{xz}-\frac{1}{x}-\frac{1}{z}\right)-2(2-x-z)\geq 0,$$

$$(2-x-z)\left(y+\frac{1}{xyz}-2\right)\geq 0.$$

The last inequality is true since $2-x-y\geq 0$ and

$$y+\frac{1}{xyz}-2\geq y+\frac{1}{y}-2\geq 0.$$

□

P 1.218. *Let a,b,c,d be positive real numbers such that*

$$a\leq b\leq c\leq d,\qquad abcd=1.$$

Prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\geq ab+bc+cd+da.$$

(Vasile C., 2012)

Solution. Write the inequality as follows:

$$a^2cd+b^2da+c^2ab+d^2bc\geq ab+bc+cd+da,$$

$$ac(ad+bc)+bd(ab+cd)\geq(ad+bc)+(ab+cd),$$

$$(ac-1)(ad+bc)+(bd-1)(ab+cd)\geq 0.$$

Since

$$ac-1=\frac{1}{bd}-1\geq 1-bd$$

and

$$bd\geq\sqrt{abcd}=1,$$

we have

$$\begin{aligned}(ac-1)(ad+bc)+(bd-1)(ab+cd)&\geq(1-bd)(ad+bc)+(bd-1)(ab+cd)\\&=(bd-1)(a-c)(b-d)\geq 0.\end{aligned}$$

The equality holds for $a=b=\dfrac{1}{c}=\dfrac{1}{d}\leq 1$.

□

P 1.219. *Let a,b,c,d be positive real numbers such that*

$$a \le b \le c \le d, \qquad abcd = 1.$$

Prove that

$$4 + \frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \ge 2(a+b+c+d).$$

(Vasile C., 2012)

Solution. Making the substitution

$$x = \sqrt[4]{\frac{a}{b}}, \quad y = \sqrt{\frac{b}{c}}, \quad z = \sqrt[4]{\frac{c}{d}}, \quad 0 < x,y,z \le 1,$$

we need to show that $E(x,y,z) \ge 0$, where

$$E(x,y,z) = 4 + x^4 + z^4 + y^2 + \frac{1}{x^4y^2z^4} - 2\left(x^3yz + \frac{yz}{x} + \frac{z}{xy} + \frac{1}{xyz^3}\right).$$

We will show that

$$E(x,y,z) \ge E(x,1,z) \ge E(x,1,1) \ge 0. \tag{*}$$

The left inequality is equivalent to

$$(1-y)E_1(x,y,z) \ge 0,$$

where

$$E_1(x,y,z) = -1 - y + \frac{1+y}{x^4y^2z^4} + 2\left(x^3z + \frac{z}{x}\right) - \frac{2}{y}\left(\frac{z}{x} + \frac{1}{xz^3}\right).$$

To prove it, we show that

$$E_1(x,y,z) \ge E_1(x,1,z) \ge 0.$$

We have

$$E_1(x,1,z) = 2(1-x^3z)\left(\frac{1}{x^4z^4} - 1\right) \ge 0.$$

Since

$$E_1(x,y,z - E_1(x,1,z) = (1-y)E_2(x,y,z),$$

where

$$E_2(x,y,z) = 1 + \frac{1+2y}{x^4y^2z^4} - \frac{2}{y}\left(\frac{z}{x} + \frac{1}{xz^3}\right),$$

we need to show $E_2(x,y,z)\ge 0$. Indeed,

$$\begin{aligned}
E_2(x,y,z) &= 1+\frac{1}{x^4y^2z^4}-\frac{2}{y}\left(\frac{z}{x}+\frac{1}{xz^3}-\frac{1}{x^4z^4}\right)\\
&\ge \frac{2}{x^2yz^2}-\frac{2}{y}\left(\frac{z}{x}+\frac{1}{xz^3}-\frac{1}{x^4z^4}\right)\\
&= \frac{2}{xyz}\left(\frac{1}{xz}-z^2-\frac{1}{z^2}+\frac{1}{x^3z^3}\right)\\
&\ge \frac{2}{xyz}\left(\frac{1}{z}-z^2-\frac{1}{z^2}+\frac{1}{z^3}\right)\\
&= \frac{2}{xyz}\left(\frac{1-z^3}{z}+\frac{1-z}{z^3}\right)\ge 0.
\end{aligned}$$

The middle inequality in (*) is equivalent to

$$(1-z)F(x,z)\ge 0,$$

where

$$F(x,z)=(1+z+z^2+z^3)\left(\frac{1}{x^4z^4}-1\right)+2\left(x^3+\frac{2}{x}\right)-\frac{1+z+z^2}{xz}.$$

It is true since

$$\begin{aligned}
F(x,z) &> \frac{1}{x^4z^4}-1+\frac{3}{x}-\frac{1+z+z^2}{xz}\\
&\ge \frac{1}{xz}-1+\frac{3}{x}-\frac{1+z+z^2}{xz}\\
&= \frac{2-x-z}{x}\ge 0.
\end{aligned}$$

The right inequality in (*) is also true since

$$\begin{aligned}
x^4E(x,1,1) &= x^8-2x^7+6x^4-6x^3+1\\
&= (x-1)^2(x^6-x^4-2x^3+3x^2+2x+1)\\
&\ge (x-1)^2(x^6-x^4-2x^3+2x^2)\\
&= x^2(x-1)^4(x^2+2x+2)\ge 0.
\end{aligned}$$

The proof is completed. The equality holds for $a=b=c=d=1$.

□

P 1.220. *Let $A=\{a_1,a_2,a_3,a_4\}$ be a set of real numbers such that*

$$a_1+a_2+a_3+a_4=0.$$

Prove that there exists a permutation $B=\{a,b,c,d\}$ of A such that

$$a^2+b^2+c^2+d^2+3(ab+bc+cd+da)\ge 0.$$

Solution. Write the desired inequality as

$$a^2+b^2+c^2+d^2+3(ab+bc+cd+da) \ge (a+b+c+d)^2,$$

$$ab+bc+cd+da \ge 2(ac+bd),$$

$$(ab+cd-ac-bd)+(bc+da-ac-bd) \ge 0.$$

$$(a-d)(b-c)+(a-b)(d-c) \ge 0.$$

Clearly, this inequality is true for $a \le b \le d \le c$. The equality occurs when A has three equal elements.

□

P 1.221. *If a,b,c,d are nonnegative real numbers such that*

$$a \ge b \ge 1 \ge c \ge d, \qquad a+b+c+d=3,$$

then

$$a^2+b^2+c^2+d^2+10abcd \le 5.$$

(Vasile C., 2015)

First Solution. Let

$$E(a,b,c,d) = a^2+b^2+c^2+d^2+10abcd.$$

We will show that

$$E(a,b,c,d) \le E(a,b,x,x) \le 5,$$

where

$$x=(c+d)/2, \qquad a+b+2x=3.$$

The left inequality is true since

$$E(a,b,c,d)-E(a,b,x,x) = \frac{1}{2}(c-d)^2(1-5ab) \le 0.$$

The right inequality can be written as follows:

$$a^2+b^2+2x^2+10abx^2 \le 5,$$

$$(a+b)^2+2x^2+2ab(5x^2-1) \le 5,$$

$$2s^2+(3-s)^2+ab[5(3-s)^2-4] \le 10,$$

where

$$s=a+b, \quad s \in [2,3].$$

Case 1: $5(3-s)^2-4 \ge 0$. Since $ab \le s^2/4$, it suffices to show that

$$2s^2+(3-s)^2+\frac{1}{4}s^2[5(3-s)^2-4] \le 10,$$

which is equivalent to the obvious inequality

$$(s-1)(s-2)[5s(s-3)-2] \le 0.$$

Case 2: $5(3-s)^2-4 \le 0$. From $(a-1)(b-1) \ge 0$, we get $ab \ge s-1$. Therefore, it suffices to show that

$$2s^2+(3-s)^2+(s-1)[5(3-s)^2-4] \le 10,$$

which is equivalent to the obvious inequality

$$(s-2)(s-3)(5s-7) \le 0.$$

The equality holds for $a=b=1$, $c=d=1/2$, and for $a=2$, $b=1$, $c=d=0$.

Second Solution. From

$$(a-1)(b-1)(c-1)(d-1) \ge 0,$$

we have

$$-2+\sum_{sym} ab-\sum abc+abcd \ge 0.$$

Since

$$2\sum_{sym} ab = 9-a^2-b^2-c^2-d^2,$$

we get

$$-2+\frac{9-a^2-b^2-c^2-d^2}{2}-\sum abc+abcd \ge 0,$$

$$a^2+b^2+c^2+d^2 \le 5-2\sum abc+2abcd.$$

Therefore, it suffices to show that

$$(5-2\sum abc+2abcd)+10abcd \le 5,$$

which is equivalent to

$$\sum abc \ge 6abcd.$$

For the non-trivial case $d \ne 0$, this inequality is equivalent to

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} \ge 6.$$

Since

$$\frac{1}{a}+\frac{1}{b} \ge \frac{4}{a+b}$$

and

$$\frac{1}{c}+\frac{1}{d} \ge \frac{4}{c+d},$$

it suffices to show that

$$\frac{2}{a+b}+\frac{2}{c+d}\geq 3,$$

which is equivalent to

$$(a+b-1)(a+b-2)\geq 0.$$

□

P 1.222. *If a,b,c,d are nonnegative real numbers such that*

$$a\geq b\geq 1\geq c\geq d,\qquad a+b+c+d=6,$$

then

$$a^2+b^2+c^2+d^2+4abcd\leq 26.$$

(Vasile C., 2015)

First Solution. Let

$$E(a,b,c,d)=a^2+b^2+c^2+d^2+10abcd.$$

We will show that

$$E(a,b,c,d)\leq E(a,b,x,x)\leq 5,$$

where

$$x=(c+d)/2,\qquad a+b+2x=3.$$

The left inequality is true since

$$E(a,b,c,d)-E(a,b,x,x)=\frac{1}{2}(c-d)^2(1-2ab)\leq 0.$$

The right inequality can be written as follows:

$$a^2+b^2+2x^2+4abx^2\leq 26,$$

$$(a+b)^2+2x^2+2ab(2x^2-1)\leq 26,$$

$$2s^2+(6-s)^2+2ab[(6-s)^2-2]\leq 52,$$

where

$$s=a+b,\qquad s\in[4,6].$$

Case 1: $(6-s)^2-2\geq 0$. Since $ab\leq s^2/4$, it suffices to show that

$$2s^2+(6-s)^2+\frac{1}{2}s^2[(6-s)^2-2]\leq 52,$$

which is equivalent to the obvious inequality

$$(s-2)(s-4)[s(s-6)-4]\leq 0.$$

Case 2: $(6-s)^2-2 \le 0$. From $(a-1)(b-1) \ge 0$, we get $ab \ge s-1$. Therefore, it suffices to show that

$$2s^2+(6-s)^2+2(s-1)[(6-s)^2-2] \le 52,$$

which is equivalent to the obvious inequality

$$(s-2)(s-6)(2s-7) \le 0.$$

The equality holds for $a=b=2$, $c=d=1$, and for $a=5$, $b=1$, $c=d=0$.

Second Solution. From

$$(a-1)(b-1)(c-1)(d-1) \ge 0,$$

we have

$$-5+\sum_{sym} ab-\sum abc+abcd \ge 0.$$

Since

$$2\sum_{sym} ab = 36-a^2-b^2-c^2-d^2,$$

we get

$$-5+\frac{36-a^2-b^2-c^2-d^2}{2}-\sum abc+abcd \ge 0,$$

$$a^2+b^2+c^2+d^2 \le 26-2\sum abc+2abcd.$$

Therefore, it suffices to show that

$$(26-2\sum abc+2abcd)+4abcd \le 26,$$

which is equivalent to

$$\sum abc \ge 3abcd.$$

For the non-trivial case $d \ne 0$, this inequality is equivalent to

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} \ge 3.$$

Since

$$\frac{1}{a}+\frac{1}{b} \ge \frac{4}{a+b}$$

and

$$\frac{1}{c}+\frac{1}{d} \ge \frac{4}{c+d},$$

it suffices to show that

$$\frac{4}{a+b}+\frac{4}{c+d} \ge 3,$$

which is equivalent to

$$(a+b-2)(a+b-4) \ge 0,$$

$$(a+b-2)(2-c-d) \ge 0.$$

$\square$

P 1.223. *Let a,b,c,d be nonnegative real numbers such that*

$$a \ge b \ge 1 \ge c \ge d, \qquad a+b+c+d=p, \quad p \ge 2.$$

Prove that

$$\frac{p^2-4p+8}{2} \le a^2+b^2+c^2+d^2 \le p^2-2p+2.$$

Solution. Write the right inequality as follows:

$$(p-1)^2-a^2+(1-b^2)-c^2-d^2 \ge 0,$$

$$(p-1-a)(p-1+a)+(1-b)(1+b)-c^2-d^2 \ge 0.$$

Since $p-1-a=(b-1)+c+d \ge 0$ and

$$(p-1+a)-(1+b)=2(a-1)+c+d \ge 0,$$

it suffices to show that

$$(p-1-a)(1+b)+(1-b)(1+b)-c^2-d^2 \ge 0,$$

which is equivalent to

$$(c+d)(1+b)-c^2-d^2 \ge 0.$$

Indeed,

$$(c+d)(1+b) \ge c+d \ge c^2+d^2.$$

The right inequality is an equality for

$$(a,b,c,d)=(p-1,1,0,0).$$

Since $(a+b)^2 \le 2(a^2+b^2)$ and $(c+d)^2 \le 2(c^2+d^2)$, the left inequality is true if

$$p^2-4p+8 \le (a+b)^2+(c+d)^2,$$

which is equivalent to

$$[(a+b)+(c+d)]^2-4[(a+b)+(c+d)]+8 \le (a+b)^2+(c+d)^2,$$

$$(a+b)(c+d)-2(a+b)-2(c+d)+4 \le 0,$$

$$(a+b-2)(c+d-2) \le 0.$$

The left inequality is an equality for

$$(a,b,c,d)=\left(1,1,\frac{p-2}{2},\frac{p-2}{2}\right), \quad 2 \le p \le 4,$$

$$(a,b,c,d)=\left(\frac{p-2}{2},\frac{p-2}{2},1,1\right), \quad p \ge 4.$$

□

P 1.224. *Let $a \ge b \ge 1 \ge c \ge d \ge 0$ such that*

$$a+b+c+d=4, \qquad a^2+b^2+c^2+d^2=q,$$

where $q \in [4,10]$ is a fixed number. Prove that the product $r = abcd$ is maximal when $b=1$ and $c=d$.

(Vasile C., 2015)

Solution. The condition $q \ge 4$ follows from the Cauchy-Schwarz inequality

$$4(a^2+b^2+c^2+d^2) \ge (a+b+c+d)^2.$$

The condition $q \le 10$ follows from the inequality $(a-1)(b-1) \ge 0$, which is equivalent to

$$ab \ge s-1,$$

where

$$s=a+b, \quad s \in [2,4].$$

Indeed,

$$\begin{aligned} q &\le (a+b)^2 - 2ab + (c+d)^2 \le s^2 - 2(s-1) + (4-s)^2 \\ &= 2(s-1)(s-4)+10 \le 10. \end{aligned}$$

Notice that $q=4$ for $a=b=c=d=1$, and $q=10$ for $a=3$, $b=1$, $c=d=0$.

We will show that for any fixed $q \in [4,10]$, we have

$$abcd \le f(d) \le f(d_1),$$

where

$$f(d) = d\left(d^2-3d+5-\frac{q}{2}\right),$$

$$d_1 = 1-\sqrt{\frac{q-4}{6}}, \qquad d_1 \in [0,1].$$

The left inequality $abcd \le f(d)$ is a consequence of the inequality

$$(a-1)(b-1)(c-1) \le 0,$$

which leads to

$$\begin{aligned} abc &\le 1-(a+b+c)+(ab+bc+ca) \\ &= 1-(4-d)+\frac{1}{2}[(a+b+c)^2-(a^2+b^2+c^2)] \\ &= -3+d+\frac{1}{2}[(4-d)^2-(q-d^2)] \\ &= d^2-3d+5-\frac{q}{2}, \end{aligned}$$

hence
$$abcd \le f(d),$$
with equality for $b = 1$.

The right inequality $f(d) \le f(d_1)$ follows immediately from
$$f(d) - f(d_1) = (d - d_1)^2(d + 2d_1 - 3) \le 0.$$
This inequality is an equality for $d = d_1$. In conclusion, for any fixed $q \in [4, 10]$, we have
$$abcd \le f(d_1),$$
with equality for $b = 1$ and $d = d_1$. These equality conditions are equivalent to $b = 1$ and $c = d$. Indeed, from $b = 1$, $d = d_1$, $a + b + c + d = 3$ and $a^2 + b^2 + c^2 + d^2 = q$, we get
$$a = 1 + \sqrt{\frac{2(q-4)}{3}}, \quad b = 1, \quad c = d = 1 - \sqrt{\frac{q-4}{6}}.$$
□

P 1.225. *If a, b, c, d are nonnegative real numbers such that*
$$a \ge b \ge 1 \ge c \ge d, \qquad a + b + c + d = 4,$$
then
$$a^2 + b^2 + c^2 + d^2 + 6abcd \le 10.$$
(Vasile C., 2015)

First Solution. According to P 1.224, it suffices to prove the inequality for $b = 1$ and $c = d$. Thus, we need to show that $a^2 + 2c^2 + 6ac^2 \le 9$ for $a + 2c = 3$; that is,
$$c(1 - c)^2 \ge 0.$$
The equality holds for $a = b = c = d = 1$, and for $a = 3$, $b = 1$, $c = d = 0$.

Second Solution. Let
$$E(a, b, c, d) = a^2 + b^2 + c^2 + d^2 + 6abcd.$$
We will show that
$$E(a, b, c, d) \le E(a, b, x, x) \le 10,$$
where
$$x = (c + d)/2.$$
The left inequality is true since
$$E(a, b, c, d) - E(a, b, x, x) = \frac{1}{2}(c - d)^2(1 - 3ab) \le 0.$$

The right inequality can be written as follows:

$$a^2+b^2+2x^2+6abx^2 \le 10, \quad a+b+2x=4,$$

$$(a+b)^2+2x^2+2ab(3x^2-1) \le 10,$$
$$2s^2+(4-s)^2+ab[3(4-s)^2-4] \le 20,$$

where
$$s=a+b, \quad s \in [2,4].$$

Case 1: $3(4-s)^2-4 \ge 0$. Since $ab \le s^2/4$, it suffices to show that

$$2s^2+(4-s)^2+\frac{1}{4}s^2[3(4-s)^2-4] \le 20,$$

which is equivalent to the obvious inequality

$$(s-2)^2[3s(s-4)-4] \le 0.$$

Case 2: $3(4-s)^2-4 \le 0$. From $(a-1)(b-1) \ge 0$, we get $ab \ge s-1$. Therefore, it suffices to show that

$$2s^2+(4-s)^2+(s-1)[3(4-s)^2-4] \le 20,$$

which is equivalent to the obvious inequality

$$(s-2)^2(s-4) \le 0.$$

Third Solution (by *Linqaszayi*). From

$$(a-1)(b-1)(c-1)(d-1) \ge 0,$$

we have
$$-3+\sum_{sym} ab - \sum abc + abcd \ge 0.$$

Since
$$2\sum_{sym} ab = 16-a^2-b^2-c^2-d^2,$$

we get
$$10-a^2-b^2-c^2-d^2 \ge 2\sum abc - 2abcd.$$

Therefore, it suffices to show that

$$2\sum abc - 2abcd \ge 6abcd,$$

which is equivalent to
$$\sum abc \ge 4abcd.$$

For the non-trivial case $d > 0$, this inequality is equivalent to the Cauchy-Schwarz inequality
$$(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \ge 16.$$

Fourth Solution (by *Nguyen Van Quy*). Write the inequality as

$$a^2+(b+c+d)^2+6abcd-2(bc+cd+db) \le 10,$$

$$3abcd-(bc+cd+db) \le (a-1)(3-a).$$

By the AM-GM inequality or the Cauchy-Schwarz inequality, we have

$$bc+cd+db \ge \frac{9bcd}{b+c+d},$$

hence

$$3abcd-(bc+cd+db) \le 3abcd - \frac{9bcd}{b+c+d} = \frac{3bcd(a-1)(3-a)}{b+c+d}.$$

Since
$$3-a \ge 4-a-b = c+d \ge 0,$$
it suffices to show that
$$\frac{3bcd}{b+c+d} \le 1.$$
Indeed, using the AM-GM inequality and $b+c+d = 4-a \le 3$, we get

$$\frac{3bcd}{b+c+d} \le \frac{(b+c+d)^2}{9} \le 1.$$

□

P 1.226. *If a,b,c,d are nonnegative real numbers such that*

$$a \ge b \ge 1 \ge c \ge d, \qquad a+b+c+d = 4,$$

then
$$a^2+b^2+c^2+d^2+6\sqrt{abcd} \le 10.$$

(Vasile C., 2015)

First Solution. According to P 1.224, it suffices to prove the inequality for $b = 1$ and $c = d$. Thus, we need to show that $a+2c = 3$ implies $a^2+2c^2+6c\sqrt{a} \le 9$; that is,
$$a^2+2c^2+6c\sqrt{a} \le (a+2c)^2,$$
$$c(c+2a-3\sqrt{a}\,) \ge 0,$$

$$\frac{3c(\sqrt{a}-1)^2}{c+2a+3\sqrt{a}} \ge 0.$$

The equality holds for $a=b=c=d=1$, and for $a=3,\ b=1,\ c=d=0$.

Second Solution. Let

$$E(a,b,c,d)=a^2+b^2+c^2+d^2+6\sqrt{abcd}.$$

We will show that

$$E(a,b,c,d) \le E(a,b,x,x) \le 10,$$

where

$$x=\frac{c+d}{2}=\frac{4-a-b}{2}.$$

The left inequality can be reduces to the obvious form

$$\left(\sqrt{c}-\sqrt{d}\right)^2\left[6\sqrt{ab}-\left(\sqrt{c}+\sqrt{d}\right)^2\right] \ge 0,$$

while the right inequality is equivalent to

$$a^2+b^2+2x^2+6x\sqrt{ab} \le 10.$$

Since $2\sqrt{ab} \le a+b$, it suffices to show that

$$a^2+b^2+2x^2+3x(a+b) \le 10.$$

which can be rewritten as

$$2(a^2+b^2)+(4-a-b)^2+3(4-a-b)(a+b) \le 20,$$

$$2(a+b)^2-4ab+16-8(a+b)+(a+b)^2+12(a+b)-3(a+b)^2 \le 20,$$

$$4(a-1)(b-1) \ge 0.$$

□

P 1.227. *If a,b,c,d,e are positive real numbers, then*

$$\frac{a}{a+2b+2c}+\frac{b}{b+2c+2d}+\frac{c}{c+2d+2e}+\frac{d}{d+2e+2a}+\frac{e}{e+2a+2b} \ge 1.$$

Solution. The inequality follows by applying the Cauchy-Schwarz inequality:

$$\sum \frac{a}{a+2b+2c} \ge \frac{\left(\sum a\right)^2}{\sum a(a+2b+2c)} = \frac{\left(\sum a\right)^2}{\sum a^2+2\sum ab+2\sum ac}=1.$$

The equality holds for $a=b=c=d=e$.

□

P 1.228. *Let a,b,c,d,e be positive real numbers such that $a+b+c+d+e=5$. Prove that*

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{e}+\frac{e}{a}\le 1+\frac{4}{abcde}.$$

Solution. Let (x,y,z,t,u) be a permutation of (a,b,c,d,e) such that $x\ge y\ge z\ge t\ge u$. By the rearrangement inequality, we have

$$\begin{aligned}\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{e}+\frac{e}{a}&\le \frac{x}{u}+\frac{y}{t}+\frac{z}{z}+\frac{t}{y}+\frac{u}{x}\\ &=\left(\frac{x}{u}+\frac{u}{x}+2\right)+\left(\frac{y}{t}+\frac{t}{y}+2\right)-3\\ &=4(p+q)-3,\end{aligned}$$

where

$$p=\frac{1}{4}\left(\frac{x}{u}+\frac{u}{x}+2\right)\ge 1,\quad q=\frac{1}{4}\left(\frac{y}{t}+\frac{t}{y}+2\right)\ge 1.$$

From $(p-1)(q-1)\ge 0$, we get

$$p+q\le 1+pq,$$

$$4(p+q)-3\le 1+4pq,$$

hence

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{e}+\frac{e}{a}\le 1+4pq.$$

Thus, it suffices to show that

$$pq\le\frac{1}{xyztu},$$

which is is equivalent to

$$z\left(\frac{x+u}{2}\right)^2\left(\frac{y+t}{2}\right)^2\le 1.$$

Indeed, by the AM-GM inequality, we get

$$z\left(\frac{x+u}{2}\right)^2\left(\frac{y+t}{2}\right)^2\le\left(\frac{z+\frac{x+u}{2}+\frac{x+u}{2}+\frac{y+t}{2}+\frac{y+t}{2}}{5}\right)^5=1.$$

The equality holds for $a=b=c=d=e=1$.

Remark. Similarly, we can prove the following generalization (*Michael Rozenberg*).

- *If $a_1,a_2,\ldots,a_n$ are positive real numbers such that $a_1+a_2+\cdots+a_n=n$, then*

$$n-4+\frac{4}{a_1a_2\cdots a_n}\ge\frac{a_1}{a_2}+\frac{a_2}{a_3}+\cdots+\frac{a_n}{a_1}.$$

□

P 1.229. *If a,b,c,d,e are real numbers such that $a+b+c+d+e=0$, then*

$$\frac{-\sqrt{5}-1}{4} \le \frac{ab+bc+cd+de+ea}{a^2+b^2+c^2+d^2+e^2} \le \frac{\sqrt{5}-1}{4}.$$

Solution. From

$$(a+b+c+d+e)^2=0,$$

we get

$$\sum a^2+2\sum ab+2\sum ac=0.$$

Therefore, for any real k, we have

$$\sum a^2+(2k+2)\sum ab=\sum 2a(kb-c).$$

By the AM-GM inequality, we get

$$2a(kb-c)\le a^2+(kb-c)^2,$$

hence

$$\sum a^2+(2k+2)\sum ab \le \sum [a^2+(kb-c)^2]=(k^2+2)\sum a^2-2k\sum ab,$$

which is equivalent to

$$\sum a^2 \ge \frac{2(2k+1)}{k^2+1}\sum ab.$$

Choosing $k=\dfrac{-1-\sqrt{5}}{2}$ and $k=\dfrac{-1+\sqrt{5}}{2}$, we get the desired inequalities. The equality in both inequalities occurs when

$$a=kb-c, \quad b=kc-d, \quad c=kd-e, \quad d=ke-a, \quad e=ka-b;$$

that is, when

$$a=x, \quad b=y, \quad c=-x+ky, \quad d=-k(x+y), \quad e=kx-y,$$

where x and y are real numbers.

□

P 1.230. *Let a, b, c, d, e be positive real numbers such that*

$$a^2+b^2+c^2+d^2+e^2=5.$$

Prove that

$$\frac{a^2}{b+c+d}+\frac{b^2}{c+d+e}+\frac{c^2}{d+e+a}+\frac{d^2}{e+a+b}+\frac{e^2}{a+b+c}\ge\frac{5}{3}.$$

(Pham Van Thuan, 2005)

Solution. By the AM-GM Inequality, we get

$$2(b+c+d) \le (b^2+1)+(c^2+1)+(d^2+1) = 8-a^2-e^2.$$

Therefore, it suffices to show that

$$\sum \frac{a^2}{8-a^2-e^2} \ge \frac{5}{6}.$$

By the Cauchy-Schwarz Inequality, we have

$$\sum \frac{a^2}{8-a^2-e^2} \ge \frac{\left(\sum a^2\right)^2}{\sum a^2(8-a^2-e^2)} = \frac{25}{40-\sum a^4-\sum a^2e^2}$$
$$= \frac{50}{80-\sum (a^2+e^2)^2} \ge \frac{50}{80-\frac{1}{5}\left[\sum (a^2+e^2)\right]^2} = \frac{5}{6}.$$

The equality holds for $a=b=c=d=e=1$.

□

P 1.231. *Let a,b,c,d,e be nonnegative real numbers such that $a+b+c+d+e=5$. Prove that*

$$(a^2+b^2)(b^2+c^2)(c^2+d^2)(d^2+e^2)(e^2+a^2) \le \frac{729}{2}.$$

(Vasile C., 2007)

Solution. Write the inequality as

$$E(a,b,c,d,e) \le 0,$$

and, without loss of generality, assume that

$$e = \min\{a,b,c,d,e\}.$$

We claim that it suffices to prove the desired inequality for the case $e=0$. To prove this, it suffices to show that

$$E(a,b,c,d,e) \le E\left(a+\frac{e}{2}, b, c, d+\frac{e}{2}, 0\right), \tag{*}$$

which is equivalent to

$$(a^2+b^2)(c^2+d^2)(d^2+e^2)(e^2+a^2) \le$$
$$\le \left[\left(a+\frac{e}{2}\right)^2+b^2\right]\left[c^2+\left(d+\frac{e}{2}\right)^2\right]\left(d+\frac{e}{2}\right)^2\left(a+\frac{e}{2}\right)^2.$$

Since

$$a^2+b^2 \le \left(a+\frac{e}{2}\right)^2+b^2,$$

$$c^2+d^2 \le c^2+\left(d+\frac{e}{2}\right)^2,$$

$$d^2+e^2 \le d^2+de \le \left(d+\frac{e}{2}\right)^2,$$

$$e^2+a^2 \le ae+a^2 \le \left(a+\frac{e}{2}\right)^2,$$

the conclusion follows. Thus, we only need to show that

$$a+b+c+d=5$$

involves

$$E(a,b,c,d,0) \le 0,$$

where

$$E(a,b,c,d,0)=a^2d^2(a^2+b^2)(b^2+c^2)(c^2+d^2)-\frac{729}{2}.$$

Without loss of generality, assume that

$$c=\min\{b,c\}.$$

We claim that it suffices to prove the inequality $E(a,b,c,d,0) \le 0$ for the case $c=0$. To prove this, it suffices to show that

$$E(a,b,c,d,0) \le E\left(a,b+\frac{c}{2},0,d+\frac{c}{2},0\right), \tag{**}$$

which is equivalent to

$$d^2(a^2+b^2)(b^2+c^2)(c^2+d^2) \le \left(d+\frac{c}{2}\right)^2\left[a^2+\left(b+\frac{c}{2}\right)^2\right]\left(b+\frac{c}{2}\right)^2\left(d+\frac{c}{2}\right)^2.$$

This is true since

$$d^2(c^2+d^2) \le \left(d+\frac{c}{2}\right)^4,$$

$$a^2+b^2 \le a^2+\left(b+\frac{c}{2}\right)^2,$$

$$b^2+c^2 \le b^2+bc \le \left(b+\frac{c}{2}\right)^2.$$

Thus, we only need to show that

$$a+b+d=5$$

involves

$$E(a,b,0,d,0) \le 0,$$

where

$$E(a,b,0,d,0)=a^2b^2d^4(a^2+b^2)-\frac{729}{2}.$$

We will show that

$$E(a,b,0,d,0)\le E\left(\frac{a+b}{2},\frac{a+b}{2},0,d,0\right)\le 0. \qquad (***)$$

The left inequality is true if

$$32a^2b^2(a^2+b^2)\le(a+b)^6.$$

Indeed, we have

$$(a+b)^6-32a^2b^2(a^2+b^2)\ge 4ab(a+b)^4-32a^2b^2(a^2+b^2)=4ab(a-b)^4\ge 0.$$

To prove the right inequality, denote

$$u=\frac{a+b}{2}.$$

We need to show that

$$2u+d=5$$

implies

$$E(u,u,0,d,0)\le 0;$$

that is,

$$u^6d^4\le\frac{729}{4},$$

$$u^3d^2\le\frac{27}{2}.$$

By the AM-GM inequality, we have

$$5=\frac{2u}{3}+\frac{2u}{3}+\frac{2u}{3}+\frac{d}{2}+\frac{d}{2}\ge 5\sqrt[5]{\left(\frac{2u}{3}\right)^3\left(\frac{t}{2}\right)^2},$$

from which the conclusion follows. The equality holds for $a=b=\frac{3}{2}$, $c=0$, $d=2$ and $e=0$ (or any cyclic permutation).

□

P 1.232. *If $a,b,c,d,e\in[1,5]$, then*

$$\frac{a-b}{b+c}+\frac{b-c}{c+d}+\frac{c-d}{d+e}+\frac{d-e}{e+a}+\frac{e-a}{a+b}\ge 0.$$

(Vasile C., 2002)

Solution. Write the inequality as

$$\sum\left(\frac{a-b}{b+c}+\frac{2}{3}\right)\geq\frac{10}{3},$$

$$\sum\frac{3a-b+2c}{b+c}\geq 10.$$

Since

$$3a-b+2c\geq 3-5+2=0,$$

we may apply the Cauchy-Schwarz inequality to get

$$\sum\frac{3a-b+2c}{b+c}\geq\frac{\left[\sum(3a-b+2c)\right]^2}{\sum(b+c)(3a-b+2c)}=\frac{16\left(\sum a\right)^2}{\sum a^2+4\sum ab+3\sum ac}.$$

Therefore, it suffices to show that

$$8\left(\sum a\right)^2\geq 5\sum a^2+20\sum ab+15\sum ac.$$

Since

$$\left(\sum a\right)^2=\sum a^2+2\sum ab+2\sum ac,$$

this inequality is equivalent to

$$3\sum a^2+\sum ac\geq 4\sum ab.$$

Indeed,

$$3\sum a^2+\sum ac-4\sum ab=\frac{1}{2}\sum(a-2b+c)^2\geq 0.$$

The equality holds for $a=b=c=d=e$.

□

P 1.233. *If* $a,b,c,d,e,f\in[1,3]$, *then*

$$\frac{a-b}{b+c}+\frac{b-c}{c+d}+\frac{c-d}{d+e}+\frac{d-e}{e+f}+\frac{e-f}{f+a}+\frac{f-a}{a+b}\geq 0.$$

(Vasile C., 2002)

Solution. Write the inequality as

$$\sum\left(\frac{a-b}{b+c}+\frac{1}{2}\right)\geq 3,$$

$$\sum\frac{2a-b+c}{b+c}\geq 6.$$

Since

$$2a-b+c \geq 2-3+1=0,$$

we may apply the Cauchy-Schwarz inequality to get

$$\sum \frac{2a-b+c}{b+c} \geq \frac{\left[\sum(2a-b+c)\right]^2}{\sum(b+c)(2a-b+c)} = \frac{2\left(\sum a\right)^2}{\sum ab+\sum ac}.$$

Thus, we still have to show that

$$\left(\sum a\right)^2 \geq 3\left(\sum ab+\sum ac\right).$$

Let

$$x=a+d, \quad y=b+e, \quad z=c+f.$$

Since

$$\sum ab+\sum ac = xy+yz+zx,$$

we have

$$\left(\sum a\right)^2 - 3\left(\sum ab+\sum ac\right) = (x+y+z)^2-3(xy+yz+zx) \geq 0.$$

The equality holds for $a=c=e$ and $b=d=f$.

□

P 1.234. *If $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ are positive real numbers, then*

$$\sum_{i=1}^{n} \frac{a_i}{a_{i-1}+2a_i+a_{i+1}} \leq \frac{n}{4},$$

where $a_0=a_n$ and $a_{n+1}=a_1$.

(Vasile C., 2008)

Solution. Applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\sum_{i=1}^{n} \frac{a_i}{a_{i-1}+2a_i+a_{i+1}} &= \sum_{i=1}^{n} \frac{a_i}{(a_{i-1}+a_i)+(a_i+a_{i+1})} \\
&\leq \frac{1}{4}\sum_{i=1}^{n} a_i\left(\frac{1}{a_{i-1}+a_i}+\frac{1}{a_i+a_{i+1}}\right) \\
&= \frac{1}{4}\left(\sum_{i=1}^{n}\frac{a_i}{a_{i-1}+a_i}+\sum_{i=1}^{n}\frac{a_i}{a_i+a_{i+1}}\right) \\
&= \frac{1}{4}\left(\sum_{i=1}^{n}\frac{a_{i+1}}{a_i+a_{i+1}}+\sum_{i=1}^{n}\frac{a_i}{a_i+a_{i+1}}\right) = \frac{n}{4}.
\end{aligned}$$

The equality holds for $a_1=a_2=\cdots=a_n$.

□

P 1.235. *Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be positive real numbers such that $a_1a_2\cdots a_n = 1$. Prove that*

$$\frac{1}{n-2+a_1+a_2}+\frac{1}{n-2+a_2+a_3}+\cdots+\frac{1}{n-2+a_n+a_1} \le 1.$$

(Vasile C., 2008)

First Solution. Let $r = \dfrac{n-2}{n}$. We can get the desired inequality by summing the following inequalities

$$\frac{n-2}{n-2+a_1+a_2} \le \frac{a_3^r+a_4^r+\cdots+a_n^r}{a_1^r+a_2^r+\cdots+a_n^r},$$

$$\frac{n-2}{n-2+a_2+a_3} \le \frac{a_1^r+a_4^r+\cdots+a_n^r}{a_1^r+a_2^r+\cdots+a_n^r},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{n-2}{n-2+a_n+a_1} \le \frac{a_2^r+a_3^r+\cdots+a_{n-1}^r}{a_1^r+a_2^r+\cdots+a_n^r}.$$

The first inequality is equivalent to

$$(a_1+a_2)(a_3^r+a_4^r+\cdots+a_n^r) \ge (n-2)(a_1^r+a_2^r).$$

By the AM-GM inequality, we have

$$a_3^r+a_4^r+\cdots+a_n^r \ge (n-2)(a_3a_4\cdots a_n)^{\frac{r}{n-2}} = \frac{n-2}{(a_1a_2)^{\frac{r}{n-2}}}.$$

Therefore, it suffices to show that

$$a_1+a_2 \ge (a_1a_2)^{\frac{r}{n-2}}(a_1^r+a_2^r),$$

or, equivalently,

$$a_1+a_2 \ge (a_1a_2)^{\frac{1}{n}}\left(a_1^{\frac{n-2}{n}}+a_2^{\frac{n-2}{n}}\right).$$

This is equivalent to the obvious inequality

$$\left(a_1^{\frac{n-1}{n}}-a_2^{\frac{n-1}{n}}\right)\left(a_1^{\frac{1}{n}}-a_2^{\frac{1}{n}}\right) \ge 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n$.

Second Solution. Since

$$\frac{n-2}{n-2+a_1+a_2} = 1-\frac{a_1+a_2}{n-2+a_1+a_2},$$

we can write the desired inequality as

$$\sum_{i=1}^{n} \frac{a_i + a_{i+1}}{a_i + a_{i+1} + n - 2} \ge 2,$$

where $a_{n+1} = a_1$. Using the Cauchy-Schwarz inequality, we get

$$\sum_{i=1}^{n} \frac{a_i + a_{i+1}}{a_i + a_{i+1} + n - 2} \ge \frac{\left(\sum_{i=1}^{n} \sqrt{a_i + a_{i+1}}\right)^2}{\sum_{i=1}^{n} (a_i + a_{i+1} + n - 2)}$$

$$= \frac{2\sum_{i=1}^{n} a_i + 2\sum_{1\le i<j\le n} \sqrt{(a_i + a_{i+1})(a_j + a_{j+1})}}{2\sum_{i=1}^{n} a_i + n(n-2)}.$$

Therefore, it suffices to prove that

$$\sum_{1\le i<j\le n} \sqrt{(a_i + a_{i+1})(a_j + a_{j+1})} \ge \sum_{i=1}^{n} a_i + n(n-2).$$

Setting $a_{n+2} = a_2$, by the Cauchy-Schwarz inequality and the AM-GM inequality, we have

$$\sum_{1\le i<j\le n} \sqrt{(a_i + a_{i+1})(a_j + a_{j+1})} =$$

$$= \sum_{i=1}^{n} \sqrt{(a_i + a_{i+1})(a_{i+1} + a_{i+2})} + \sum_{\substack{1\le i<j\le n \\ j\ne i+1}} \sqrt{(a_i + a_{i+1})(a_j + a_{j+1})}$$

$$\ge \sum_{i=1}^{n} \left(a_{i+1} + \sqrt{a_i a_{i+2}}\right) + n(n-3)\sqrt[n]{a_1 a_2 \cdots a_n}$$

$$= \sum_{i=1}^{n} a_i + n(n-3) + \sum_{i=1}^{n} \sqrt{a_i a_{i+2}}$$

$$\ge \sum_{i=1}^{n} a_i + n(n-3) + n\sqrt[n]{a_1 a_2 \cdots a_n} = \sum_{i=1}^{n} a_i + n(n-2).$$

□

P 1.236. *If $a_1, a_2, \ldots, a_n \ge 1$, then*

$$\prod \left(a_1 + \frac{1}{a_2} + n - 2\right) \ge n^{n-2}(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).$$

(Vasile C., 2011)

Solution. Write the inequality as $E(a_1,a_2,...,a_n) \ge 0$, and denote

$$A = \left(a_2 + \frac{1}{a_3} + n - 2\right)\left(a_3 + \frac{1}{a_4} + n - 2\right)\cdots\left(a_{n-1} + \frac{1}{a_n} + n - 2\right).$$

We will prove that

$$E(a_1,a_2,...,a_n) \ge E(1,a_2,...,a_n).$$

If this is true, then

$$E(a_1,a_2,...,a_n) \ge E(1,a_2,...,a_n) \ge E(1,1,a_3,...,a_n) \ge \cdots \ge E(1,1,...,1,a_n) = 0.$$

We have

$$E(a_1,a_2,...,a_n) - E(1,a_2,...,a_n) = (a_1 - 1)\left(B - \frac{C}{a_1}\right),$$

where

$$B = A(a_n + n - 2) - n^{n-2}\left(\frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_n}\right),$$

$$C = A\left(\frac{1}{a_2} + n - 2\right) - n^{n-2}(a_2 + a_3 + \cdots + a_n).$$

Since $a_1 - 1 \ge 0$, we need to show that

$$a_1 B - C \ge 0.$$

According to the AM-GM inequality, we have

$$A \ge \left(n\sqrt[n]{\frac{a_2}{a_3}}\right)\left(n\sqrt[n]{\frac{a_3}{a_4}}\right)\cdots\left(n\sqrt[n]{\frac{a_{n-1}}{a_n}}\right) = n^{n-2}\sqrt[n]{\frac{a_2}{a_n}},$$

$$a_n + n - 2 \ge (n-1)\sqrt[n-1]{a_n},$$

$$A(a_n + n - 2) \ge (n-1)n^{n-2}\sqrt[n]{a_2 a_n^{\frac{1}{n-1}}} \ge (n-1)n^{n-2},$$

therefore

$$B \ge n^{n-2}\left(n - 1 - \frac{1}{a_2} - \frac{1}{a_3} - \cdots - \frac{1}{a_n}\right) \ge 0$$

and

$$a_1 B - C \ge B - C = A\left(a_n - \frac{1}{a_2}\right) + n^{n-2}\left(a_2 - \frac{1}{a_2}\right) + \cdots + n^{n-2}\left(a_n - \frac{1}{a_n}\right) \ge 0.$$

The equality holds when $n-1$ of the numbers $a_1, a_2, ..., a_n$ are equal to 1.

$\square$

P 1.237. *If $a_1, a_2, ..., a_n \ge 1$, then*

$$\left(a_1+\frac{1}{a_1}\right)\left(a_2+\frac{1}{a_2}\right)\cdots\left(a_n+\frac{1}{a_n}\right)+2^n \ge 2\left(1+\frac{a_1}{a_2}\right)\left(1+\frac{a_2}{a_3}\right)\cdots\left(1+\frac{a_n}{a_1}\right).$$

(Vasile C., 2011)

Solution. Write the inequality as $E(a_1, a_2, ..., a_n) \ge 0$, and denote

$$A = \left(a_2+\frac{1}{a_2}\right)\cdots\left(a_n+\frac{1}{a_n}\right),$$

$$B = \left(1+\frac{a_2}{a_3}\right)\cdots\left(1+\frac{a_{n-1}}{a_n}\right).$$

We will prove that

$$E(a_1, a_2, ..., a_n) \ge E(1, a_2, ..., a_n).$$

If this is true, then

$$E(a_1, a_2, ..., a_n) \ge E(1, a_2, ..., a_n) \ge E(1, 1, a_3, ..., a_n) \ge \cdots \ge E(1, 1, ..., 1, a_n) = 0.$$

We have

$$E(a_1, a_2, ..., a_n) - E(1, a_2, ..., a_n) = (a_1 - 1)\left(C - \frac{D}{a_1}\right),$$

where

$$C = A - \frac{2B}{a_2},$$

$$D = A - 2Ba_n.$$

Since $a_1 - 1 \ge 0$, we need to show that

$$a_1 C - D \ge 0.$$

First, we prove that $C \ge 0$; that is,

$$(a_2^2+1)\cdots(a_n^2+1) \ge 2(a_2+a_3)\cdots(a_{n-1}+a_n).$$

By squaring, this inequality becomes

$$(a_2^2+1)[(a_2^2+1)(a_3^2+1)]\cdots[(a_{n-1}^2+1)(a_n^2+1)](a_n^2+1) \ge$$

$$\ge 4(a_2+a_3)^2\cdots(a_{n-1}+a_n)^2.$$

By the Cauchy-Schwarz inequality, we have

$$(a_2^2+1)(a_3^2+1) \ge (a_2+a_3)^2, \ \ \ldots\ , \ (a_{n-1}^2+1)(a_n^2+1) \ge (a_{n-1}+a_n)^2.$$

Therefore, we still have to show that

$$(a_2^2+1)(a_n^2+1) \ge 4,$$

which is clearly true for $a_2 \ge 1$ and $a_n \ge 1$. Finally, we have

$$a_1C - D \ge C - D = 2B\left(a_n - \frac{1}{a_2}\right) \ge 0.$$

The equality holds when $n-1$ of $a_1, a_2, ..., a_n$ are equal to 1.

□

P 1.238. *Let k and n be positive integers, and let $a_1, a_2, ..., a_n$ be real numbers such that*

$$a_1 \le a_2 \le \cdots \le a_n.$$

Consider the inequality

$$(a_1 + a_2 + \cdots + a_n)^2 \ge n(a_1a_{k+1} + a_2a_{k+2} + \cdots + a_na_{n+k}),$$

where $a_{n+i} = a_i$ for any positive integer i. Prove this inequality for

(a) $n = 2k$;

(b) $n = 4k$.

(Vasile C., 2004)

Solution. (a) We need to prove that

$$(a_1 + a_2 + \cdots + a_{2k})^2 \ge 4k(a_1a_{k+1} + a_2a_{k+2} + \cdots + a_ka_{2k}).$$

If x is a real number such that

$$a_k \le x \le a_{k+1},$$

then

$$(x - a_1)(a_{k+1} - x) + (x - a_2)(a_{k+2} - x) + \cdots + (x - a_k)(a_{2k} - x) \ge 0.$$

Expanding and multiplying by $4k$, we get

$$4kx(a_1 + a_2 + \cdots + a_{2k}) \ge 4k^2x^2 + 4k(a_1a_{k+1} + a_2a_{k+2} + \cdots + a_ka_{2k}).$$

On the other hand, by the AM-GM inequality, we have

$$(a_1 + a_2 + \cdots + a_{2k})^2 + 4k^2x^2 \ge 4kx(a_1 + a_2 + \cdots + a_{2k}).$$

Adding these inequalities yields the desired inequality. The equality holds for

$$a_{j+1} = a_{j+2} = \cdots = a_{j+k} = \frac{a_1 + a_2 + \cdots + a_{2k}}{2k},$$

where $j \in \{1, 2, \cdots, k-1\}$.

(b) We need to show that

$$(a_1 + a_2 + \cdots + a_{4k})^2 \geq 4k(a_1 a_{k+1} + a_2 a_{k+2} + \cdots + a_{4k} a_k).$$

Using the substitution

$$b_i = a_i + a_{2k+i}, \quad i = 1, 2, ..., 2k,$$

this inequality becomes

$$(b_1 + b_2 + \cdots + b_{2k})^2 \geq 4k(b_1 b_{k+1} + b_2 b_{k+2} + \cdots + b_k b_{2k}),$$

which is just the inequality in (a). The equality holds for

$$\begin{cases} a_{j+1} = a_{j+2} = \cdots = a_{j+k} = a \\ a_{j+2k+1} = a_{j+2k+2} = \cdots = a_{j+3k} = b \\ a_1 + a_2 + \cdots + a_{4k} = 2k(a + b) \end{cases},$$

where $a \leq b$ are real numbers, and $j \in \{1, 2, \cdots, k-1\}$.

Remark. Actually, the inequality holds for any integer k satisfying $\dfrac{n}{4} \leq k \leq \dfrac{n}{2}$.

□

P 1.239. *If $a_1, a_2, \ldots, a_n$ are real numbers, then*

$$a_1(a_1 + a_2) + a_2(a_2 + a_3) + \cdots + a_n(a_n + a_1) \geq \frac{2}{n}(a_1 + a_2 + \cdots + a_n)^2.$$

Solution. Making the substitution

$$a = \frac{1}{n}(a_1 + a_2 + \cdots + a_n)$$

and

$$x_i = a_i - a, \quad i = 1, 2, ..., n,$$

we have

$$x_1 + x_2 + \cdots + x_n = 0$$

and

$$\sum a_1(a_1 + a_2) - \frac{2}{n}(a_1 + a_2 + \cdots + a_n)^2 = \sum (x_1 + a)(x_1 + x_2 + 2a) - 2na^2$$

$$= \sum x_1(x_1 + x_2) = \frac{1}{2} \sum (x_1 - x_2)^2 \geq 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n$ - if n is odd, and for $a_1 = a_3 = \cdots = a_{n-1}$ and $a_2 = a_4 = \cdots = a_n$ - if n is even.

□

P 1.240. *If* $a_1, a_2, \ldots, a_n \in [1, 2]$, *then*

$$\sum_{i=1}^{n} \frac{3}{a_i + 2a_{i+1}} \ge \sum_{i=1}^{n} \frac{2}{a_i + a_{i+1}},$$

where $a_{n+1} = a_1$.

(Vasile C., 2005)

Solution. Rewrite the inequality as follows

$$\sum_{i=1}^{n} \frac{a_i - a_{i+1}}{(a_i + a_{i+1})(a_i + 2a_{i+1})} \ge 0,$$

$$\sum_{i=1}^{n} \left[\frac{k(a_i - a_{i+1})}{(a_i + a_{i+1})(a_i + 2a_{i+1})} + \frac{1}{a_i} - \frac{1}{a_{i+1}} \right] \ge 0, \qquad k > 0,$$

$$\sum_{i=1}^{n} \frac{(a_i - a_{i+1})[(k-3)a_i a_{i+1} - a_i^2 - 2a_{i+1}^2]}{a_i a_{i+1}(a_i + a_{i+1})(a_i + 2a_{i+1})} \ge 0,$$

Setting $k = 6$, the inequality becomes

$$\sum_{i=1}^{n} \frac{(a_i - a_{i+1})^2(2a_{i+1} - a_i)}{a_i a_{i+1}(a_i + a_{i+1})(a_i + 2a_{i+1})} \ge 0.$$

Since $1 \le a_i \le 2$, we have $2a_{i+1} - a_i \ge 0$ for all $i = 1, 2, \ldots, n$. Thus, the inequality is proved. The equality holds for $a_1 = a_2 = \cdots = a_n$.

□

P 1.241. *Let* $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ *be real numbers such that* $a_1 + a_2 + \cdots + a_n = n$.

(a) If $a_1 \ge 1 \ge a_2 \ge \cdots \ge a_n$, *then*

$$a_1^3 + a_2^3 + \cdots + a_n^3 + 2n \ge 3(a_1^2 + a_2^2 + \cdots + a_n^2);$$

(b) If $a_1 \le 1 \le a_2 \le \cdots \le a_n$, *then*

$$a_1^3 + a_2^3 + \cdots + a_n^3 + 2n \le 3(a_1^2 + a_2^2 + \cdots + a_n^2).$$

(Vasile C., 2007)

Solution. (a) Write the inequality as

$$\sum (a_1^3 - 3a_1^2 + 3a_1 - 1) \ge 0,$$

$$\sum (a_1 - 1)^3 \ge 0,$$

$$(a_1 - 1)^3 \ge (1 - a_2)^3 + \cdots + (1 - a_n)^3,$$

$$[(1-a_2)+\cdots+(1-a_n)]^3 \ge (1-a_2)^3+\cdots+(1-a_n)^3.$$

Clearly, the last inequality is true. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = 2$, $a_2 = \cdots = a_{n-1} = 1$, $a_n = 0$.

(b) Similarly, write the inequality as

$$\sum(a_1^3 - 3a_1^2 + 3a_1 - 1) \le 0,$$

$$\sum(1-a_1)^3 \ge 0,$$

$$(1-a_1)^3 \ge (a_2-1)^3+\cdots+(a_n-1)^3,$$

$$[(a_2-1)+\cdots+(a_n-1)]^3 \ge (a_2-1)^3+\cdots+(a_n-1)^3.$$

The last inequality is obviously true. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = 0$, $a_2 = \cdots = a_{n-1} = 1$, $a_n = 2$.

□

P 1.242. *Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$.*

(a) If $a_1 \ge 1 \ge a_2 \ge \cdots \ge a_n$, then

$$a_1^4 + a_2^4 + \cdots + a_n^4 + 5n \ge 6(a_1^2 + a_2^2 + \cdots + a_n^2);$$

(b) If $a_1 \le 1 \le a_2 \le \cdots \le a_n$, then

$$a_1^4 + a_2^4 + \cdots + a_n^4 + 6n \le 7(a_1^2 + a_2^2 + \cdots + a_n^2).$$

(Vasile C., 2007)

Solution. (a) Write the inequality as

$$\sum(a_1^4 - 6a_1^2 + 8a_1 - 3) \ge 0,$$

$$\sum(a_1-1)^3(a_1+3) \ge 0,$$

$$(a_1-1)^3(a_1+3) \ge (1-a_2)^3(a_2+3)+\cdots+(1-a_n)^3(a_n+3).$$

Since

$$(a_1-1)^3 = [(1-a_2)+\cdots+(1-a_n)]^3 \ge (1-a_2)^3+\cdots+(1-a_n)^3,$$

it suffices to show that

$$[(1-a_2)^3+\cdots+(1-a_n)^3](a_1+3) \ge (1-a_2)^3(a_2+3)+\cdots+(1-a_n)^3(a_n+3),$$

which is equivalent to the obvious inequality

$$(1-a_2)^3(a_1-a_2)+\cdots+(1-a_n)^3(a_1-a_n)\geq 0.$$

The equality holds for $a_1=a_2=\cdots=a_n=1$.

(b) Write the inequality as

$$\sum(a_1^4-7a_1^2+10a_1-4)\leq 0,$$

$$\sum(a_1-1)^2(a_1^2+2a_1-4)\leq 0,$$

$$(a_2-1)^2(a_2^2+2a_2-4)+\cdots+(a_n-1)^2(a_n^2+2a_n-4)\leq(1-a_1)^2(4-2a_1-a_1^2).$$

Since

$$(1-a_1)^2=[(a_2-1)+\cdots+(a_n-1)]^2\geq(a_2-1)^2+\cdots+(a_n-1)^2,$$

it suffices to show that

$$(a_2-1)^2(a_2^2+2a_2-4)+\cdots+(a_n-1)^2(a_n^2+2a_n-4)\leq[(a_2-1)^2+\cdots+(a_n-1)^2](4-2a_1-a_1^2),$$

which is equivalent to

$$(a_2-1)^2(a_1^2+a_2^2+2a_1+2a_2-8)+\cdots+(a_n-1)^2(a_1^2+a_n^2+2a_1+2a_n-8)\leq 0.$$

This inequality is true if

$$a_1^2+a_n^2+2a_1+2a_n-8\leq 0.$$

Since

$$a_1+a_n=n-(a_2+\cdots+a_{n-1})=2+(1-a_2)+\cdots+(1-a_{n-1})\leq 2,$$

we have

$$a_1^2+a_n^2+2a_1+2a_n-8=(a_1+a_n+1)^2-9-2a_1a_n\leq(a_1+a_n+1)^2-9\leq 0.$$

The equality holds for $a_1=a_2=\cdots=a_n=1$, and also for $a_1=0$, $a_2=\cdots=a_{n-1}=1$, $a_n=2$.

Remark. The inequality in (a) remains valid for all real $a_1,a_2,\ldots,a_n$ such that

$$a_1+a_2+\cdots+a_n=n,\qquad a_1\geq 1\geq a_2\geq\cdots\geq a_n.$$

□

P 1.243. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that*

$$a_1 \ge 1 \ge a_2 \ge \cdots \ge a_n, \qquad \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = n,$$

then

$$a_1^2 + a_2^2 + \cdots + a_n^2 + 2n \ge 3(a_1 + a_2 + \cdots + a_n).$$

(Vasile C., 2008)

Solution. Write the inequality as follows:

$$(a_1 - 1)(a_1 - 2) + (a_2 - 1)(a_2 - 2) + \cdots + (a_n - 1)(a_n - 2) \ge 0,$$

$$(a_1 - 1)(a_1 - 2) \ge (1 - a_2)(a_2 - 2) + \cdots + (1 - a_n)(a_n - 2),$$

$$\left(1 - \frac{1}{a_1}\right)(a_1^2 - 2a_1) \ge \left(\frac{1}{a_2} - 1\right)(a_2^2 - 2a_2) + \cdots + \left(\frac{1}{a_n} - 1\right)(a_n^2 - 2a_n),$$

$$\left[\left(\frac{1}{a_2} - 1\right) + \cdots + \left(\frac{1}{a_n} - 1\right)\right](a_1^2 - 2a_1) \ge \left(\frac{1}{a_2} - 1\right)(a_2^2 - 2a_2) + \cdots + \left(\frac{1}{a_n} - 1\right)(a_n^2 - 2a_n),$$

$$\left(\frac{1}{a_2} - 1\right)(a_1^2 - 2a_1 - a_2^2 + 2a_2) + \cdots + \left(\frac{1}{a_n} - 1\right)(a_1^2 - 2a_1 - a_n^2 + 2a_n) \ge 0,$$

$$\left(\frac{1}{a_2} - 1\right)(a_1 - a_2)(a_1 + a_2 - 2) + \cdots + \left(\frac{1}{a_n} - 1\right)(a_1 - a_n)(a_1 + a_n - 2) \ge 0.$$

Clearly, it suffices to prove that $a_1 + a_n - 2 \ge 0$. Indeed,

$$a_1 + a_n - 2 = n - 2 - (a_2 + \cdots + a_{n-1}) = (1 - a_2) + \cdots + (1 - a_{n-1}) \ge 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 1.244. *If $a_1, a_2, \ldots, a_n$ are real numbers such that*

$$a_1 \le 1 \le a_2 \le \cdots \le a_n, \qquad a_1 + a_2 + \cdots + a_n = n,$$

then

(a)
$$\frac{a_1 + 1}{a_1^2 + 1} + \frac{a_2 + 1}{a_2^2 + 1} + \cdots + \frac{a_n + 1}{a_n^2 + 1} \le n;$$

(b)
$$\frac{1}{a_1^2 + 3} + \frac{1}{a_2^2 + 3} + \cdots + \frac{1}{a_1^2 + 3} \le \frac{n}{4}.$$

(Vasile C., 2009)

Solution. (a) Write the inequality as

$$\left(1-\frac{a_1+1}{a_1^2+1}\right)+\left(1-\frac{a_2+1}{a_2^2+1}\right)+\cdots+\left(1-\frac{a_n+1}{a_n^2+1}\right)\ge 0,$$

$$\frac{a_1(a_1-1)}{a_1^2+1}+\frac{a_2(a_2-1)}{a_2^2+1}+\cdots+\frac{a_n(a_n-1)}{a_n^2+1}\ge 0,$$

$$\frac{a_2(a_2-1)}{a_2^2+1}+\cdots+\frac{a_n(a_n-1)}{a_n^2+1}\ge\frac{a_1(1-a_1)}{a_1^2+1},$$

$$\frac{a_2(a_2-1)}{a_2^2+1}+\cdots+\frac{a_n(a_n-1)}{a_n^2+1}\ge\frac{a_1[(a_2-1)+\cdots+(a_n-1)]}{a_1^2+1},$$

$$(a_2-1)\left(\frac{a_2}{a_2^2+1}-\frac{a_1}{a_1^2+1}\right)+\cdots+(a_n-1)\left(\frac{a_n}{a_n^2+1}-\frac{a_1}{a_1^2+1}\right)\ge 0,$$

$$\frac{(a_2-1)(a_2-a_1)(1-a_1a_2)}{a_2^2+1}+\cdots+\frac{(a_n-1)(a_n-a_1)(1-a_1a_n)}{a_n^2+1}\ge 0.$$

For $a_1\ge 0$, it suffices to show that $1-a_1a_n\ge 0$. Indeed,

$$2\sqrt{a_1a_n}\le a_1+a_n=2+(1-a_2)+\cdots+(1-a_{n-1})\le 2.$$

For $a_1\le 0$, the inequality is also true because

$$1-a_1a_2>0,\ \ \cdots,\ \ 1-a_1a_n>0.$$

The equality holds for $a_1=a_2=\cdots=a_n=1$.

(b) As in the case (a), we write the inequality as

$$\frac{(a_2-1)(a_2-a_1)(3-a_1a_2-a_1-a_2)}{a_2^2+3}+\cdots+\frac{(a_n-1)(a_n-a_1)(3-a_1a_n-a_1-a_n)}{a_2^2+3}\ge 0.$$

For $a_1\ge 0$, it suffices to show that $3-a_1a_n-a_1-a_n\ge 0$. From $(1-a_1)(a_n-1)\ge 0$, we get $3-a_1a_n\ge 4-a_1-a_n$, hence

$$\frac{1}{2}(3-a_1a_n-a_1-a_n)\ge 2-a_1-a_n=(a_2-1)+\cdots+(a_{n-1}-1)\ge 0.$$

For $a_1\le 0$, the inequality is also true because

$$3-a_1a_2-a_1-a_2>2-a_1-a_2=(a_3-1)+\cdots+(a_n-1)\ge 0,$$

$$\cdots$$

$$3-a_1a_n-a_1-a_n>2-a_1-a_n=(a_2-1)+\cdots+(a_{n-1}-1)\ge 0.$$

The equality holds for $a_1=a_2=\cdots=a_n=1$.

□

P 1.245. *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that*

$$a_1 \le 1 \le a_2 \le \cdots \le a_n, \qquad a_1 + a_2 + \cdots + a_n = n,$$

then

$$\frac{a_1^2-1}{(a_1+3)^2} + \frac{a_2^2-1}{(a_2+3)^2} + \cdots + \frac{a_n^2-1}{(a_n+3)^2} \ge 0.$$

(Vasile C., 2009)

Solution. Write the inequality as follows:

$$\frac{a_2^2-1}{(a_2+3)^2} + \cdots + \frac{a_n^2-1}{(a_n+3)^2} \ge \frac{1-a_1^2}{(a_1+3)^2},$$

$$\frac{a_2^2-1}{(a_2+3)^2} + \cdots + \frac{a_n^2-1}{(a_n+3)^2} \ge \frac{[(a_2-1)+\cdots+(a_n-1)](1+a_1)}{(a_1+3)^2},$$

$$(a_2-1)\left[\frac{a_2+1}{(a_2+3)^2} - \frac{a_1+1}{(a_1+3)^2}\right] + \cdots + (a_n-1)\left[\frac{a_n+1}{(a_n+3)^2} - \frac{a_1+1}{(a_1+3)^2}\right] \ge 0,$$

$$\frac{(a_2-1)(a_2-a_1)(3-a_1-a_2-a_1a_2)}{(a_1+3)^2(a_2+3)^2} + \cdots + \frac{(a_n-1)(a_n-a_1)(3-a_1-a_n-a_1a_n)}{(a_1+3)^2(a_n+3)^2} \ge 0.$$

It suffices to show that $3-a_1-a_n-a_1a_n \ge 0$. Since

$$3-a_1-a_n-a_1a_n \ge 3-a_1-a_n-\frac{1}{4}(a_1+a_n)^2 = \frac{1}{4}(2-a_1-a_n)(6+a_1+a_n) \ge 0,$$

we only need to show that $2-a_1-a_n \ge 0$. Indeed, we have

$$2-a_1-a_n = (a_2-1)+\cdots+(a_{n-1}-1) \ge 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 1.246. *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that*

$$a_1 \ge 1 \ge a_2 \ge \cdots \ge a_n, \qquad a_1 + a_2 + \cdots + a_n = n,$$

then

$$\frac{1}{3a_1^3+4} + \frac{1}{3a_2^3+4} + \cdots + \frac{1}{3a_n^3+4} \ge \frac{n}{7}.$$

(Vasile C., 2009)

Solution. Write the inequality as follows:

$$\left(\frac{1}{3a_2^3+4}-\frac{1}{7}\right)+\cdots+\left(\frac{1}{3a_n^3+4}-\frac{1}{7}\right)\ge\frac{1}{7}-\frac{1}{3a_1^3+4},$$

$$\frac{1-a_2^3}{3a_2^3+4}+\cdots+\frac{1-a_n^3}{3a_n^3+4}\ge\frac{a_1^3-1}{3a_1^3+4},$$

$$\frac{1-a_2^3}{3a_2^3+4}+\cdots+\frac{1-a_n^3}{3a_n^3+4}\ge\frac{[(1-a_2)+\cdots+(1-a_n)](1+a_1+a_1^2)}{3a_1^3+4},$$

$$(1-a_2)\left(\frac{1+a_2+a_2^2}{3a_2^3+4}-\frac{1+a_1+a_1^2}{3a_1^3+4}\right)+\cdots+(1-a_n)\left(\frac{1+a_n+a_n^2}{3a_n^3+4}-\frac{1+a_1+a_1^2}{3a_1^3+4}\right)\ge 0.$$

It suffices to show that

$$\frac{1+a_i+a_i^2}{3a_i^3+4}-\frac{1+a_1+a_1^2}{3a_1^3+4}\ge 0$$

for $i=2,\ldots,n$. Write these inequalities as

$$(a_1-a_i)E_i\ge 0,$$

where

$$\begin{aligned}E_i&=3a_1^2a_i^2+3a_1a_i(a_1+a_i)+3(a_1^2+a_1a_i+a_i^2)-4(a_1+a_i)-4\\&=(a_1+a_i)(3a_1+3a_i-4+3a_1a_i)+3a_1^2a_i^2-3a_1a_i-4.\end{aligned}$$

Since

$$a_1+a_i\ge a_1+a_n=2+(1-a_2)+\cdots+(1-a_{n-1})\ge 2,$$

we have

$$E_i\ge 2(6-4+3a_1a_i)+3a_1^2a_i^2-3a_1a_i-4=3a_1a_i+3a_1^2a_i^2\ge 0.$$

The equality holds for $a_1=a_2=\cdots=a_n=1$, and also for $a_1=2$, $a_2=\cdots=a_{n-1}=1$, $a_n=0$.

□

P 1.247. *If $a_1,a_2,\ldots,a_n$ are nonnegative real numbers such that*

$$a_1\le 1\le a_2\le\cdots\le a_n,\qquad a_1+a_2+\cdots+a_n=n,$$

then

$$\sqrt{\frac{3a_1}{4-a_1}}+\sqrt{\frac{3a_2}{4-a_2}}+\cdots+\sqrt{\frac{3a_n}{4-a_n}}\le n.$$

(Vasile C., 2009)

Solution. Write the inequality as follows:

$$\left(\sqrt{\frac{3a_1}{4-a_1}}-1\right)+\left(\sqrt{\frac{3a_2}{4-a_2}}-1\right)+\cdots+\left(\sqrt{\frac{3a_n}{4-a_n}}-1\right)\le 0.$$

$$\frac{a_1-1}{4-a_1+\sqrt{3a_1(4-a_1)}}+\frac{a_2-1}{4-a_2+\sqrt{3a_2(4-a_2)}}+\cdots+\frac{a_n-1}{4-a_n+\sqrt{3a_n(4-a_n)}}\le 0,$$

$$\frac{a_2-1}{4-a_2+\sqrt{3a_2(4-a_2)}}+\cdots+\frac{a_n-1}{4-a_n+\sqrt{3a_n(4-a_n)}}\le\frac{(a_2-1)+\cdots+(a_n-1)}{4-a_1+\sqrt{3a_1(4-a_1)}},$$

$$(a_2-1)E_2+\cdots+(a_n-1)E_n\ge 0,$$

where

$$E_j=\frac{1}{4-a_1+\sqrt{3a_1(4-a_1)}}-\frac{1}{4-a_j+\sqrt{3a_j(4-a_j)}},\qquad j=2,\ldots,n.$$

It suffices to show that all $E_j\ge 0$. The inequality $E_j\ge 0$ is equivalent to

$$\sqrt{3a_j(4-a_j)}-\sqrt{3a_1(4-a_1)}\ge a_j-a_1,$$

$$\frac{3(a_j-a_1)(4-a_1-a_j)}{\sqrt{3a_j(4-a_j)}+\sqrt{3a_1(4-a_1)}}\ge a_j-a_1.$$

This is true if

$$\sqrt{3a_1(4-a_1)}+\sqrt{3a_j(4-a_j)}\le 3(4-a_1-a_j).$$

We have

$$a_1+a_j-2\le a_1+a_n-2=(1-a_2)+\cdots+(1-a_{n-1})\le 0.$$

Denote

$$x=a_1+a_j,\qquad x\le 2.$$

Since

$$\sqrt{3a_1(4-a_1)}+\sqrt{3a_j(4-a_j)}\le\sqrt{2[3a_1(4-a_1)+3a_j(4-a_j)]}\le\sqrt{24x-3x^2},$$

it suffices to show that

$$\sqrt{24x-3x^2}\le 3(4-x),$$

which is equivalent to the obvious inequality

$$(2-x)(6-x)\ge 0.$$

The equality holds for $a_1=a_2=\cdots=a_n=1$.

□

P 1.248. *If $a_1, a_2, \dots, a_n$ are nonnegative real numbers such that*

$$a_1 \le 1 \le a_2 \le \cdots \le a_n, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = n,$$

then

$$\frac{1}{3-a_1} + \frac{1}{3-a_2} + \cdots + \frac{1}{3-a_n} \le \frac{n}{2}.$$

(Vasile C., 2009)

Solution. Write the inequality as follows:

$$\left(\frac{2}{3-a_1} - 1\right) + \left(\frac{2}{3-a_1} - 1\right) + \cdots + \left(\frac{2}{3-a_1} - 1\right) \le 0.$$

$$\frac{a_1-1}{3-a_1} + \frac{a_2-1}{3-a_2} + \cdots + \frac{a_n-1}{3-a_n} \le 0,$$

$$\frac{a_2-1}{3-a_2} + \cdots + \frac{a_n-1}{3-a_n} \le \frac{1-a_1}{3-a_1},$$

$$\frac{a_2^2-1}{(1+a_2)(3-a_2)} + \cdots + \frac{a_n^2-1}{(1+a_n)(3-a_n)} \le \frac{1-a_1^2}{(1+a_1)(3-a_1)},$$

$$\frac{a_2^2-1}{(1+a_2)(3-a_2)} + \cdots + \frac{a_n^2-1}{(1+a_n)(3-a_n)} \le \frac{(a_2^2-1) + \cdots + (a_n^2-1)}{(1+a_1)(3-a_1)},$$

$$(a_2^2-1)E_2 + \cdots + (a_n^2-1)E_n \le 0,$$

where

$$E_j = \frac{1}{(1+a_j)(3-a_j)} - \frac{1}{(1+a_1)(3-a_1)}, \qquad j = 2, \dots, n.$$

It suffices to show that $E_j \le 0$, which is equivalent to

$$(a_j - a_1)(a_1 + a_j - 2) \le 0.$$

This is true because

$$a_1 + a_i - 2 \le a_1 + a_n - 2 = (1-a_2) + \cdots + (1-a_{n-1}) \le 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 1.249. *If $a_1, a_2, \dots, a_n$ are real numbers such that*

$$a_1 \le 1 \le a_2 \le \cdots \le a_n, \qquad a_1 + a_2 + \cdots + a_n = n,$$

then

$$(1+a_1^2)(1+a_2^2)\cdots(1+a_n^2) \ge 2^n.$$

(Vasile C., 2009)

Solution. Use the substitutions $a_1 = 1 - S$ and

$$a_2 = b_2 + 1, \ \ldots, \ a_n = b_n + 1,$$

where S and $b_2, \ldots, b_n$ are nonnegative real numbers such that

$$S = b_2 + \cdots + b_n.$$

We have

$$\frac{1}{2}(1 + a_1^2) = 1 - S + \frac{1}{2}S^2,$$

$$\frac{1}{2}(1 + a_i^2) = 1 + b_i + \frac{1}{2}b_i^2, \qquad i = 2, \ldots, n,$$

and, by Lemma below,

$$\frac{1}{2^{n-1}}(1 + a_2^2)\cdots(1 + a_n^2) = \left(1 + b_2 + \frac{1}{2}b_2^2\right)\cdots\left(1 + b_n + \frac{1}{2}b_n^2\right) \geq 1 + S + \frac{1}{2}S^2.$$

Therefore, it suffices to show that

$$\left(1 - S + \frac{1}{2}S^2\right)\left(1 + S + \frac{1}{2}S^2\right) \geq 1,$$

which is equivalent to $S^4 \geq 0$. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Lemma. If $c_1, c_2, \ldots, c_k$ are nonnegative real numbers such that $c_1 + c_2 + \cdots + c_k = S$, then

$$\left(1 + c_1 + \frac{1}{2}c_1^2\right)\left(1 + c_2 + \frac{1}{2}c_2^2\right)\cdots\left(1 + c_k + \frac{1}{2}c_k^2\right) \geq 1 + S + \frac{1}{2}S^2.$$

Proof. We have

$$\begin{aligned}\prod_{1\leq i\leq k}\left(1 + c_i + \frac{1}{2}c_i^2\right) &\geq 1 + \sum_{1\leq i\leq k}\left(c_i + \frac{1}{2}c_i^2\right) + \sum_{1\leq i<j\leq k}\left(c_i + \frac{1}{2}c_i^2\right)\left(c_j + \frac{1}{2}c_j^2\right) \\ &\geq 1 + \sum_{1\leq i\leq k}\left(c_i + \frac{1}{2}c_i^2\right) + \sum_{1\leq i<j\leq k} c_i c_j \\ &= 1 + S + \frac{1}{2}S^2.\end{aligned}$$

$\square$

P 1.250. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that*

$$a_1 \geq 1 \geq a_2 \geq \cdots \geq a_n, \qquad a_1 a_2 \cdots a_n = 1,$$

then

$$\frac{1}{a_1 + 1} + \frac{1}{a_2 + 1} + \cdots + \frac{1}{a_n + 1} \geq \frac{n}{2}.$$

(Vasile C., 2009)

Solution. We use the induction method. For $n = 2$, the desired inequality is an identity. Let us denote

$$E_n(a_1, a_2, \ldots, a_n) = \frac{1}{a_1+1} + \frac{1}{a_2+1} + \cdots + \frac{1}{a_n+1} - \frac{n}{2}.$$

We will show that

$$E_n(a_1, a_2, a_3, \ldots, a_n) \ge E_n(a_1a_2, 1, a_3, \ldots, a_{n-1}, a_n) \ge 0$$

for $n \ge 3$.

The right inequality can be written as

$$E_{n-1}(a_1a_2, a_3, \ldots, a_{n-1}, a_n) \ge 0.$$

Since

$$a_1a_2 = \frac{1}{a_3 \cdots a_{n-1}a_n} \ge 1$$

and

$$(a_1a_2)a_3 \cdots a_{n-1}a_n = 1,$$

the right inequality follows by the induction hypothesis.

The left inequality is equivalent to

$$\frac{1}{a_1+1} + \frac{1}{a_2+1} \ge \frac{1}{a_1a_2+1} + \frac{1}{2},$$

$$\frac{1-a_2}{2(a_2+1)} \ge \frac{a_1(1-a_2)}{(a_1+1)(a_1a_2+1)},$$

which is true if

$$(a_1+1)(a_1a_2+1) \ge 2a_1(a_2+1).$$

This inequality can be written in the obvious form

$$(a_1-1)(a_1a_2-1) \ge 0.$$

The equality holds for $a_1 \ge 1 = a_2 = \cdots = a_{n-1} \ge a_n$.

□

P 1.251. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that*

$$a_1 \ge 1 \ge a_2 \ge \cdots \ge a_n, \qquad a_1a_2\cdots a_n = 1,$$

then

$$\frac{1}{(a_1+2)^2} + \frac{1}{(a_2+2)^2} + \cdots + \frac{1}{(a_n+2)^2} \ge \frac{n}{9}.$$

(Vasile C., 2009)

Solution. We use the induction method. For $n = 2$, the desired inequality is equivalent to

$$(a_1 - 1)^4 \ge 0.$$

Let us denote

$$E_n(a_1, a_2, \ldots, a_n) = \frac{1}{(a_1+2)^2} + \frac{1}{(a_2+2)^2} + \cdots + \frac{1}{(a_n+2)^2} - \frac{n}{9}.$$

To end the proof, it suffices to show that

$$E_n(a_1, a_2, a_3, \ldots, a_{n-1}, a_n) \ge E_n(a_1, 1, a_3, \ldots, a_{n-1}, a_2 a_n) \ge 0$$

for $n \ge 3$.

The right inequality can be written as

$$E_{n-1}(a_1, a_3, \ldots, a_{n-1}, a_2 a_n) \ge 0.$$

Since

$$a_2 a_n \le a_n \le a_{n-1}$$

and

$$a_1 a_3 \ldots a_{n-1}(a_2 a_n) = 1,$$

the inequality follows by the induction hypothesis.

The left inequality is equivalent to

$$\frac{1}{(a_2+2)^2} + \frac{1}{(a_n+2)^2} \ge \frac{1}{9} + \frac{1}{(a_2 a_n+2)^2}.$$

Denoting

$$s = a_2 + a_n, \quad p = a_2 a_n, \qquad s \le 2, \quad p \le 1,$$

the inequality becomes

$$\frac{s^2 + 4s + 8 - 2p}{(2s + 4 + p)^2} \ge \frac{p^2 + 4p + 13}{9(p+2)^2},$$

$$(1 + p - s)(As + B) \ge 0,$$

where

$$A = 16 - 20p - 5p^2, \qquad B = 80 - 32p - 29p^2 - p^3 > 0.$$

Since

$$1 + p - s = (1 - a_2)(1 - a_n) \ge 0,$$

we only need to show that $As + B \ge 0$. For the nontrivial case $A < 0$, we get

$$As + B \ge 2A + B = 112 - 72p - 39p^2 - p^3 = (1-p)(112 + 40p + p^2) \ge 0.$$

This completes the proof. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Remark. Similarly, we can prove the following generalization:

- *Let $a_1, a_2, \dots, a_n$ be positive real numbers such that*

$$a_1 \ge 1 \ge a_2 \ge \cdots \ge a_n, \qquad a_1a_2\cdots a_n = 1.$$

If $k \ge 1$, then

$$\frac{1}{(a_1+k)^k} + \frac{1}{(a_2+k)^k} + \cdots + \frac{1}{(a_n+k)^k} \ge \frac{n}{(1+k)^k}.$$

For $n = 2$, the desired inequality is true if $g(x) \ge 0$ for $x \ge 1$, where

$$g(x) = \frac{1}{(x+k)^k} + \frac{x^k}{(kx+1)^k} - \frac{2}{(1+k)^k},$$

$$\frac{g'(x)}{k} = \frac{x^{k-1}(x+k)^{k+1} - (kx+1)^{k+1}}{(x+k)^{k+1}(kx+1)^{k+1}}.$$

It suffices to show that $h(x) \ge 0$ for $x \ge 1$, where

$$h(x) = (k-1)\ln x + (k+1)\ln(x+k) - (k+1)\ln(kx+1),$$

$$h'(x) = \frac{k-1}{x} + \frac{k+1}{x+k} - \frac{k(k+1)}{kx+1} = \frac{k(k-1)(x-1)^2}{(x+k)(kx+1)}.$$

Since $h'(x) \ge 0$, $h(x)$ is increasing for $x \ge 1$, hence

$$h(x) \ge h(1) = 0.$$

Let

$$E_n(a_1, a_2, \dots, a_n) = \frac{1}{(a_1+k)^k} + \frac{1}{(a_2+k)^k} + \cdots + \frac{1}{(a_n+k)^k} - \frac{n}{(1+k)^k}.$$

It suffices to show that

$$E_n(a_1, a_2, a_3, \dots, a_{n-1}, a_n) \ge E_n(a_1, 1, a_3, \dots, a_{n-1}, a_2a_n) \ge 0.$$

The right inequality follows by the induction hypothesis, while the left inequality is equivalent to

$$f_1(a_2) + f_1(a_n) \ge f_1(1) + f_2(a_2a_n),$$

where

$$f_1(x) = \frac{1}{(x+k)^k}.$$

Using the substitution

$$a_2 = e^a, \quad a_n = e^b,$$

the inequality becomes

$$f(a)+f(b) \geq f(0)+f(a+b),$$

where

$$f(x)=\frac{1}{(e^x+k)^k}.$$

From

$$f''(x)=\frac{k^2e^x(e^x-1)}{(e^x+k)^{k+2}},$$

it follows that f is concave on $(-\infty,0]$. Since

$$0 \geq a \geq b \geq a+b,$$

the inequality $f(a)+f(b) \geq f(0)+f(a+b)$ follows from Karamata's inequality.

□

P 1.252. *If $a_1,a_2,\ldots,a_n$ are positive real numbers such that*

$$a_1 \geq 1 \geq a_2 \geq \cdots \geq a_n, \qquad a_1a_2\cdots a_n = 1,$$

then

$$a_1^n+a_2^n+\cdots+a_n^n-n \geq n^2\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}-n\right).$$

(Vasile C., 2009)

Solution. We use the induction method. For $n=2$, the desired inequality is equivalent to

$$(a_1-1)^4 \geq 0.$$

Let us denote

$$E_n(a_1,a_2,\ldots,a_n)=a_1^n+a_2^n+\cdots+a_n^n-n-n^2\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}-n\right).$$

We will show that

$$E_n(a_1,a_2,a_3,\ldots,a_{n-1},a_n) \geq E_n(a_1,1,a_3,\ldots,a_{n-1},a_2a_n) \geq 0.$$

The right inequality can be written as

$$E_{n-1}(a_1,a_3,\ldots,a_{n-1},a_2a_n) \geq 0.$$

Since

$$a_2a_n \leq a_n \leq a_{n-1}$$

and

$$a_1a_3\cdots a_{n-1}(a_2a_n)=1,$$

the inequality follows by the induction hypothesis.

The left inequality is equivalent to

$$a_2^n+a_n^n-1-a_2^na_n^n\ge n^2\left(\frac{1}{a_2}+\frac{1}{a_n}-1-\frac{1}{a_2a_n}\right),$$

$$n^2\left(\frac{1}{a_2}-1\right)\left(\frac{1}{a_n}-1\right)\ge(1-a_2^n)(1-a_n^n),$$

which is true if

$$\frac{n^2}{a_2a_n}\ge(1+a_2+\cdots+a_2^{n-1})(1+a_n+\cdots+a_n^{n-1}).$$

Since $a_2\le 1$ and $a_n\le 1$, this inequality is clearly true. The equality holds for $a_1=a_2=\cdots=a_n=1$.

□

P 1.253. *If $a_1,a_2,\ldots,a_n$ $(n\ge 3)$ are real numbers such that*

$$a_1+a_2+\cdots+a_n=n,\qquad a_1\ge a_2\ge 1\ge a_3\ge\cdots\ge a_n,$$

then

$$a_1^4+a_2^4+\cdots+a_n^4-n\ge\frac{14}{3}(a_1^2+a_2^2+\cdots+a_n^2-n).$$

(Vasile C., 2009)

Solution *(by Linqaszayi)*. Using the substitution

$$a_i=1+x_i,\quad i=1,2,\ldots,n,$$

which implies

$$x_1\ge x_2\ge 0\ge x_3\ge\cdots\ge x_n,\qquad x_1+x_2+\cdots+x_n=0,$$

we need to show that $E(x_1,x_2,\ldots,x_n)\ge 0$, where

$$E(x_1,x_2,x_3,\ldots,x_n)=3\sum_{i=1}^n {x_i}^4+12\sum_{i=1}^n {x_i}^3+4\sum_{i=1}^n {x_i}^2.$$

We will prove that

$$E(x_1,x_2,\ldots,x_n)\ge E\left(\frac{x_1+x_2}{2},\frac{x_1+x_2}{2},x_3,\ldots,x_n\right)\ge 0.$$

The left inequality is true because

$$x_1^4 + x_2^4 \geq 2\left(\frac{x_1 + x_2}{2}\right)^4, \quad x_1^3 + x_2^3 \geq 2\left(\frac{x_1 + x_2}{2}\right)^3, \quad x_1^2 + x_2^2 \geq 2\left(\frac{x_1 + x_2}{2}\right)^2.$$

To prove the right inequality, we replace $x_3, \ldots, x_n$ with $-x_3, \ldots, -x_n$. So, we need to show that

$$x_1 + x_2 = x_3 + \cdots + x_n, \qquad x_1, x_2, x_3, \ldots, x_n \geq 0,$$

involves

$$E + 3(x_3^4 + \cdots + x_n^4) - 12(x_3^3 + \cdots + x_n^3) + 4(x_3^2 + \cdots + x_n^2) \geq 0,$$

where

$$E = 6\left(\frac{x_1 + x_2}{2}\right)^4 + 24\left(\frac{x_1 + x_2}{2}\right)^3 + 8\left(\frac{x_1 + x_2}{2}\right)^2 = 6A^4 + 24A^3 + 8A^2,$$

with

$$A = \frac{x_3 + \cdots + x_n}{2}.$$

Since

$$A^4 \geq \frac{x_3^4 + \cdots + x_n^4}{16}, \quad A^3 \geq \frac{x_3^3 + \cdots + x_n^3}{8}, \quad A^2 \geq \frac{x_3^2 + \cdots + x_n^2}{4},$$

we have

$$E \geq \frac{3}{8}(x_3^4 + \cdots + x_n^4) + 3(x_3^3 + \cdots + x_n^3) + 2(x_3^2 + \cdots + x_n^2).$$

Therefore, it suffices to show that

$$\left(\frac{3}{8} + 3\right)(x_3^4 + \cdots + x_n^4) + (3 - 12)(x_3^3 + \cdots + x_n^3) + (2 + 4)(x_3^2 + \cdots + x_n^2) \geq 0,$$

which is equivalent to the obvious inequality

$$x_3^2(3x_3 - 4)^2 + \cdots + x_n^2(3x_n^2 - 4)^2 \geq 0.$$

This completes the proof. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = a_2 = \frac{5}{3}, \quad a_3 = \cdots = a_{n-1} = 1, \quad a_n = \frac{-1}{3}.$$

□

P 1.254. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that*

$$a_1 \geq 1 \geq a_2 \geq \cdots \geq a_n, \quad a_1 a_2 \cdots a_n = 1.$$

Prove that

$$\frac{1 - a_1}{3 + a_1^2} + \frac{1 - a_2}{3 + a_2^2} + \cdots + \frac{1 - a_n}{3 + a_n^2} \geq 0.$$

(Vasile C., 2013)

Solution. We use the induction method. For $n = 2$, the desired inequality is equivalent to

$$(a_1 - 1)^4 \geq 0.$$

Let us denote

$$E_n(a_1, a_2, \ldots, a_n) = \frac{1 - a_1}{3 + a_1^2} + \frac{1 - a_2}{3 + a_2^2} + \cdots + \frac{1 - a_n}{3 + a_n^2}.$$

We will show that

$$E_n(a_1, \ldots, a_{n-2}, a_{n-1}, a_n) \geq E_n(a_1, \ldots, a_{n-2}, 1, a_{n-1}a_n) \geq 0.$$

The right inequality can be written as

$$E_{n-1}(a_1, a_2, \ldots, a_{n-2}, a_{n-1}a_n) \geq 0.$$

Since

$$a_1 \geq 1 \geq a_2 \geq \cdots \geq a_{n-2} \geq a_{n-1}a_n,$$

and

$$a_1 a_2 \cdots a_{n-2}(a_{n-1}a_n) = 1,$$

the inequality follows by the induction hypothesis.

The left inequality reduces to

$$\frac{1 - a_{n-1}}{3 + a_{n-1}^2} + \frac{1 - a_n}{3 + a_n^2} \geq \frac{1 - a_{n-1}a_n}{3 + a_{n-1}^2 a_n^2},$$

which is equivalent to the obvious inequality

$$(1 - a_{n-1})(1 - a_n)(3 + a_{n-1}a_n)(3 - a_{n-1}a_n - a_{n-1}^2 a_n - a_{n-1}a_n^2) \geq 0.$$

Thus, the proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 1.255. *Let $a_1, a_2, \ldots, a_n$ ($n \geq 3$) be nonnegative real numbers such that*

$$a_1 \geq \cdots \geq a_k \geq 1 \geq a_{k+1} \geq \cdots \geq a_n, \qquad 1 \leq k \leq n - 1,$$

and

$$a_1 + a_2 + \cdots + a_n = p.$$

Prove that

(a) if $p \geq k$, then

$$a_1^2 + a_2^2 + \cdots + a_n^2 \leq (p - k + 1)^2 + k - 1;$$

(b) if $k \le p \le n$, then

$$a_1^2 + a_2^2 + \cdots + a_n^2 \ge \frac{p^2 - 2kp + kn}{n-k};$$

(c) if $p \ge n$, then

$$a_1^2 + a_2^2 + \cdots + a_n^2 \ge \frac{p^2 - 2(n-k)p + n(n-k)}{k}.$$

(Vasile C., 2015)

First Solution. (a) For $k = 1$, the inequality is equivalent to $a_1^2 + a_2^2 + \cdots + a_n^2 \le p^2$, which is clearly true. For $k \ge 2$, write the inequality as

$$[(p-k+1)^2 - a_1^2] + (1-a_2^2) + \cdots + (1-a_k^2) - a_{k+1}^2 - \cdots - a_n^2 \ge 0,$$

$$(p-k+1-a_1)(n-k+1+a_1) \ge (a_2-1)(a_2+1)+\cdots+(a_k-1)(a_k+1)+a_{k+1}^2+\cdots+a_n^2.$$

Since

$$p-k+1-a_1 = (a_2-1) + \cdots + (a_k-1) + a_{k+1} + \cdots + a_n \ge 0$$

and

$$(p-k+1+a_1) - (a_2+1) = (p-k+1-a_1) + (a_1-a_2) + (a_1-1) \ge 0,$$

we have

$$(p-k+1-a_1)(p-k+1+a_1) \ge (p-k+1-a_1)(a_2+1).$$

In addition, we have

$$\begin{aligned}(a_2-1)(a_2+1) + \cdots + (a_k-1)(a_k+1) &\le (a_2-1)(a_2+1) + \cdots + (a_k-1)(a_2+1)\\ &= (a_2 + \cdots + a_k - k + 1)(a_2+1).\end{aligned}$$

Thus, it suffices to show that

$$(p-k+1-a_1)(a_2+1) \ge (a_2 + \cdots + a_k - k + 1)(a_2+1) + a_{k+1}^2 + \cdots + a_n^2,$$

which is equivalent to

$$(a_{k+1} + \cdots + a_n)(a_2+1) \ge a_{k+1}^2 + \cdots + a_n^2.$$

Indeed, we have

$$(a_{k+1} + \cdots + a_n)(a_2+1) \ge a_{k+1} + \cdots + a_n \ge a_{k+1}^2 + \cdots + a_n^2.$$

The equality holds for

$$a_1 = p-k+1, \qquad a_2 = \cdots = a_k = 1, \qquad a_{k+1} = \cdots = a_n = 0.$$

(b) Let

$$A = a_1 + \cdots + a_k, \quad B = a_{k+1} + \cdots + a_n, \qquad A \ge k, \quad A + B = p \le n.$$

We have

$$A^2 \le k(a_1^2 + \cdots + a_k^2), \quad B^2 \le (n-k)(a_{k+1}^2 + \cdots + a_n^2),$$

hence

$$\frac{A^2}{k} + \frac{B^2}{n-k} \le a_1^2 + a_2^2 + \cdots + a_n^2.$$

Thus, it suffices to show that

$$\frac{n-k}{k}A^2 + B^2 \ge p^2 - 2kp + kn,$$

which is equivalent to

$$\frac{n-k}{k}A^2 + B^2 \ge (A+B)^2 - 2k(A+B) + kn,$$

$$\frac{n-2k}{k}A^2 + 2kA - kn \ge 2kB(A-k),$$

$$(A-k)\left(\frac{n-2k}{k}A + n\right) \ge 2kB(A-k),$$

$$(A-k)\left(\frac{n-2k}{k}A + n - 2kB\right) \ge 0,$$

$$(A-k)\left[\frac{n}{k}(A-k) + 2(n-A-B)\right] \ge 0.$$

The equality holds for

$$a_1 = \cdots = a_k = 1, \qquad a_{k+1} = \cdots = a_n = \frac{p-k}{n-k}.$$

(c) Let

$$A = a_1 + \cdots + a_k, \quad B = a_{k+1} + \cdots + a_n, \qquad B \le n-k, \quad A + B = p \ge n.$$

We have

$$A^2 \le k(a_1^2 + \cdots + a_k^2), \quad B^2 \le (n-k)(a_{k+1}^2 + \cdots + a_n^2),$$

hence

$$\frac{A^2}{k} + \frac{B^2}{n-k} \le a_1^2 + a_2^2 + \cdots + a_n^2.$$

Thus, it suffices to show that

$$A^2 + \frac{k}{n-k}B^2 \ge p^2 - 2(n-k)p + (n-k)n,$$

which is equivalent to

$$A^2+\frac{k}{n-k}B^2\ge (A+B)^2-2(n-k)(A+B)+(n-k)n,$$

$$2A(n-k-B)+\frac{2k-n}{n-k}B^2+2(n-k)B-(n-k)n\ge 0,$$

$$2A(n-k-B)-(n-k-B)\left(n+\frac{2k-n}{n-k}B\right)\ge 0,$$

$$(n-k-B)\left(2A-n-\frac{2k-n}{n-k}B\right)\ge 0,$$

$$(n-k-B)\left[2(A+B-n)+\frac{n(n-k-B)}{n-k}\right]\ge 0.$$

The equality holds for

$$a_1=\cdots=a_k=\frac{p-n+k}{k},\qquad a_{k+1}=\cdots=a_n=1.$$

Second Solution. The desired inequalities can be proved by applying Karamata's inequality to the convex function $f(u)=u^2$. In the case (a), the decreasingly ordered sequence $(p-k+1,1,\ldots,1,0,\ldots,0)$ majorizes the decreasingly ordered sequence $(a_1,a_2,\ldots,a_n)$; that is

$$(p-k+1,1,\ldots,1,0,\ldots,0)\succ(a_1,a_2,\ldots,a_k,a_{k+1},\ldots,a_n).$$

Also, in the cases (b) and (c), we have

$$(a_1,a_2,\ldots,a_k,a_{k+1},\ldots,a_n)\succ\left(1,1,\ldots,1,\frac{p-k}{n-k},\ldots,\frac{p-k}{n-k}\right)$$

and

$$(a_1,a_2,\ldots,a_k,a_{k+1},\ldots,a_n)\succ\left(\frac{p-n+k}{k},\frac{p-n+k}{k},\ldots,\frac{p-n+k}{k},1,\ldots,1\right),$$

respectively

□

P 1.256. *Let $a_1,a_2,\ldots,a_n$ $(n\ge 3)$ be nonnegative real numbers such that*

$$a_1\ge\cdots\ge a_k\ge 1\ge a_{k+1}\ge\cdots\ge a_n,\qquad 1\le k\le n-1,$$

and

$$a_1+a_2+\cdots+a_n=n,\qquad a_1^2+a_2^2+\cdots+a_n^2=q,$$

where q is a fixed number. Prove that the product $r=a_1a_2\cdots a_n$ is maximum when

$$a_2=\cdots=a_k=1,\qquad a_{k+1}=\cdots=a_n.$$

(Vasile C., 2015)

Solution. We show first that there exists a unique n-tuple $(a_1, a_2, \ldots, a_n)$ such that

$$a_1 \ge a_2 = \cdots = a_k = 1 \ge a_{k+1} = \cdots = a_n.$$

By the Cauchy-Schwarz inequality

$$n(a_1^2 + a_2^2 + \cdots + a_n^2) \ge (a_1 + a_2 + \cdots + a_n)^2,$$

we get $q \ge n$. Since $q = n$ involves $a_1 = a_2 = \cdots = a_n = 1$, consider further that $q > n$. For

$$a_1 := x, \quad a_2 = \cdots = a_k = 1, \quad a_{k+1} = \cdots = a_n := y,$$

we get

$$x = 1 + \sqrt{\frac{(n-k)(q-n)}{n-k+1}}, \quad y = 1 - \sqrt{\frac{q-n}{(n-k)(n-k+1)}}.$$

Since $x \ge 1$ and $y \le 1$, we only need to show that $y \ge 0$. This is equivalent to

$$q \le (n-k+1)^2 + k - 1,$$

which is the inequality (a) in P 1.255.

Consider that r is maximum at $(b_1, b_2, \ldots, b_n)$, where

$$b_1 \ge \cdots \ge b_k \ge 1 \ge b_{k+1} \ge \cdots \ge b_n.$$

We will show now, by the contradiction method, that

$$b_2 = \cdots = b_k = 1, \quad b_{k+1} = \cdots = b_n.$$

To show that $b_{k+1} = \cdots = b_n$ for $1 \le k \le n-2$, assume that

$$b_{k+1} \ne b_n.$$

For

$$a_2 = b_2, \ \ldots, \ a_k = b_k, \qquad a_{k+2} = b_{k+2}, \ \ldots, \ a_{n-1} = b_{n-1},$$

we have $a_1 + a_{k+1} + a_n = constant$ and $a_1^2 + a_{k+1}^2 + a_n^2 = constant$, where

$$a_1 \ge 1 \ge a_{k+1} \ge a_n.$$

According to P 1.168, the product $a_1 a_{k+1} a_n$ is maximum for $a_{k+1} = a_n$, which contradicts the assumption that $b_{k+1} \ne b_n$. From this contradiction, it follows that $b_{k+1} = \cdots = b_n$.

To show that $b_2 = \cdots = b_k = 1$ for $2 \le k \le n-1$, assume that

$$b_2 \ne 1.$$

For

$$a_3 = b_3, \ \ldots, \ a_{n-1} = b_{n-1},$$

we have $a_1+a_2+a_n = constant$ and $a_1^2+a_2^2+a_n^2 = constant$, where

$$a_1 \ge a_2 \ge 1 \ge a_n.$$

According to P 1.171, the product $a_1a_2a_n$ is maximum for $a_2 = 1$ or $a_n = 1$. The first case contradicts the assumption that $b_2 \ne 1$, while the second case involves $b_n = 1$, hence $b_1 = b_2 = \cdots = b_n = 1$ (because $b_1 \ge b_2 \ge \cdots \ge b_n$ and $b_1+b_2+\cdots+b_n = n$), which also contradicts the assumption that $b_2 \ne 1$; as a consequence, we have $b_2 = 1$, which involves $b_2 = \cdots = b_k = 1$.

□

P 1.257. *If $a_1, a_2, \ldots, a_n$ are nonnegtive real numbers such that*

$$a_1 \le 1 \le a_2 \le \cdots \le a_n, \qquad a_1+a_2+\cdots+a_n = n,$$

then

$$(a_1a_2\cdots a_n)^{\frac{2}{n}}(a_1^2+a_2^2+\cdots+a_n^2) \le n.$$

(Vasile C., 2015)

Solution. For $n = 2$, we need to show that $a_1+a_2 = 2$ implies

$$a_1a_2(a_1^2+a_2^2) \le 2.$$

Indeed, we have

$$16-8a_1a_2(a_1^2+a_2^2) = (a_1+a_2)^4-8a_1a_2(a_1^2+a_2^2) = (a_1-a_2)^4 \ge 0.$$

For $n \ge 3$, according to the preceding P 1.256, it suffices to consider the case $a_2 = \cdots = a_{n-1} = 1$. Thus, we only need to show that $a_1+a_n = 2$ involves

$$(a_1a_n)^{\frac{2}{n}}(a_1^2+a_n^2+n-2) \le n.$$

This is true if $f(x) \le \ln n$ for $x \in (0,2)$, where

$$f(x) = \frac{2}{n}[\ln x+\ln(2-x)]+\ln(2x^2-4x+n+2).$$

From the derivative

$$f'(x) = \frac{2}{n}\left(\frac{1}{x}-\frac{1}{2-x}\right)+\frac{4(x-1)}{2x^2-4x+n+2} = \frac{4(n+2)(1-x)^3}{nx(2-x)(2x2-4x+n+2)},$$

it follows that $f(x)$ is increasing on $(0,1]$ and decreasing on $[1,2)$; therefore,

$$f(x) \le f(1) = \ln n.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 1.258. *Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be nonnegative real numbers such that*

$$a_1 \ge \cdots \ge a_k \ge 1 \ge a_{k+1} \ge \cdots \ge a_n, \qquad 1 \le k \le n-1,$$

and

$$a_1 + a_2 + \cdots + a_n = p, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = q,$$

where p and q are fixed numbers.

(a) For $p \le n$, the product $r = a_1 a_2 \cdots a_n$ is maximum when $a_2 = \cdots = a_k = 1$ and $a_{k+1} = \cdots = a_n$;

(b) For $p \ge n$ and $q \ge n-1+(p-n+1)^2$, the product $r = a_1 a_2 \cdots a_n$ is maximum when $a_2 = \cdots = a_k = 1$ and $a_{k+1} = \cdots = a_n$;

(c) For $p \ge n$ and $q < n-1+(p-n+1)^2$, the product $r = a_1 a_2 \cdots a_n$ is maximum when $a_2 = \cdots = a_k$ and $a_{k+1} = \cdots = a_n = 1$.

(Vasile Cîrtoaje and Linqaszayi, 2015)

Solution. (a) For $p = k$, we have

$$a_1 = \cdots = a_k = 1, \qquad a_{k+1} = \cdots = a_n = 0.$$

Consider further that $p > k$. We show first that there exists a unique n-tuple $(a_1, a_2, \ldots, a_n)$ such that

$$a_1 \ge a_2 = \cdots = a_k = 1 \ge a_{k+1} = \cdots = a_n.$$

According to P 1.255, we have

$$\frac{p^2 - 2pk + kn}{n-k} \le q \le (p-k+1)^2 + k - 1.$$

For $a_1 := x$, $a_2 = \cdots = a_k = 1$ and $a_{k+1} = \cdots = a_n := y$, from

$$a_1 + a_2 + \cdots + a_n = p, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = q,$$

we get

$$x + (n-k)y = p - k + 1, \qquad x^2 + (n-k)y^2 = q - k + 1.$$

We need to show that this system has a unique solution (x, y) such that $x \ge 1 \ge y \ge 0$. From the system equations, we get $f(x) = 0$, where

$$f(x) = (n-k+1)x^2 - 2(p-k+1)x + (p-k+1)^2 - (n-k)(q-k+1).$$

We have

$$f(1) = (n-k)\left(\frac{p^2 - 2pk + kn}{n-k} - q\right) \le 0$$

and

$$f(p-k+1) = (n-k)[(p-k+1)^2 + k - 1 - q] \ge 0.$$

Therefore, the equation $f(x)=0$ has a single root $x\in[1,p-k+1]$. From $1\le x\le p-k+1$, we get

$$1\le p-k+1-(n-k)y\le p-k+1,$$

hence

$$1\ge \frac{p-k}{n-k}\ge y\ge 0.$$

Consider now that r is maximum at $(b_1,b_2,\ldots,b_n)$, where

$$b_1\ge\cdots\ge b_k\ge 1\ge b_{k+1}\ge\cdots\ge b_n.$$

Applying the contradiction method as in P 1.256, we get

$$b_2=\cdots=b_k=1,\quad b_{k+1}=\cdots=b_n.$$

(b) We show first that there exists a unique n-tuple $(a_1,a_2,\ldots,a_n)$ such that

$$a_1\ge a_2=\cdots=a_k=1\ge a_{k+1}=\cdots=a_n.$$

By hypothesis and P 1.255-(a), we have

$$n-1+(p-n+1)^2\le q\le (p-k+1)^2+k-1.$$

As in the case (a), we need to show that the system

$$x+(n-k)y=p-k+1,\quad x^2+(n-k)y^2=q-k+1,$$

has a unique solution (x,y) such that $x\ge 1\ge y\ge 0$. From the system equations, we get $g(y)=0$, where

$$g(y)=(n-k)(n-k+1)y^2-2(n-k)(p-k+1)y+(p-k+1)^2+k-1-q.$$

We have

$$g(0)=(p-k+1)^2+k-1-q\ge 0$$

and

$$g(1)=(p-n+1)^2+n-1-q\le 0.$$

Therefore, the equation $g(y)=0$ has a single root $y\in[0,1]$. From $y\le 1$, we get

$$y=\frac{p-k+1-x}{n-k}\le 1,$$

hence

$$x\ge p-n+1\ge 1.$$

Consider now that r is maximum at $(b_1,b_2,\ldots,b_n)$, where

$$b_1\ge\cdots\ge b_k\ge 1\ge b_{k+1}\ge\cdots\ge b_n.$$

Applying the contradiction method as in P 1.256, we get

$$b_{k+1} = \cdots = b_n.$$

We still need to show that

$$b_2 = \cdots = b_k = 1$$

for $k \geq 2$. Assume for the sake of contradiction, that

$$b_2 \neq 1.$$

For

$$a_3 = b_3, \ \cdots, \ a_{n-1} = b_{n-1},$$

we have $a_1 + a_2 + a_n = constant$ and $a_1^2 + a_2^2 + a_n^2 = constant$, where $a_1 \geq a_2 \geq 1 \geq a_n$. According to P 1.171, the product $a_1 a_2 a_{k+1}$ is maximum for $a_2 = 1$ or $a_n = 1$. The first case contradicts the assumption that $b_2 \neq 1$. The second case leads to $b_n = 1$, hence $b_{k+1} = \cdots = b_n = 1$. From the hypothesis $q \geq n-1+(p-n+1)^2$ and

$$q = b_1^2 + \cdots + b_k^2 + n - k, \qquad p = b_1 + \cdots + b_k + n - k,$$

we get

$$b_1^2 + \cdots + b_k^2 - k + 1 \geq (b_1 + \cdots + b_k - k + 1)^2,$$

which is equivalent to

$$(b_1 - 1)^2 + \cdots + (b_k - 1)^2 \geq [(b_1 - 1) + \cdots + (b_k - 1)]^2.$$

This is true only if

$$b_2 - 1 = \cdots = b_k - 1 = 0,$$

that is,

$$b_2 = \cdots = b_k.$$

This result contradicts also the assumption that $b_2 \neq 1$; as a consequence, we have $b_2 = 1$, which involves $b_2 = \cdots = b_k = 1$.

(c) By hypothesis and P 1.255-(c), we have

$$\frac{p^2 - 2(n-k)p + n(n-k)}{k} \leq q \leq n - 1 + (p - n + 1)^2.$$

For $k = 1$, these inequalities become

$$n - 1 + (p - n + 1)^2 \leq q < n - 1 + (p - n + 1)^2,$$

which is not possible. Consider further that

$$k \geq 2.$$

We show first that there exists a unique n-tuple $(a_1, a_2, \ldots, a_n)$ such that

$$a_1 \geq a_2 = \cdots = a_k \geq a_{k+1} = \cdots = a_n = 1.$$

For $a_1 := x$, $a_2 = \cdots = a_k = y$ and $a_{k+1} = \cdots = a_n = 1$, from

$$a_1 + a_2 + \cdots + a_n = p, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = q,$$

we get

$$x + (k-1)y = p - n + k, \quad x^2 + (k-1)y^2 = q - n + k.$$

We need to show that this system has a unique solution (x, y) such that $x \geq y \geq 1$. From the system equations, we get $h(y) = 0$, where

$$h(y) = (k-1)ky^2 - 2(k-1)(p-n+k)y + (p-n+k)^2 + n - k - q.$$

We have

$$h(1) = (p-n+1)^2 + n - 1 - q > 0$$

and

$$h\left(\frac{p-n+k}{k}\right) = \frac{p^2 - 2(n-k)p + n(n-k)}{k} - q \leq 0.$$

Therefore, the equation $hy) = 0$ has a single root

$$y \in \left(1, \frac{p-n+k}{k}\right].$$

From

$$y = \frac{p-n+k-x}{k-1} \leq \frac{p-n+k}{k},$$

we get

$$x \geq \frac{p-n+k}{k} \geq y.$$

Consider now that r is maximum at $(b_1, b_2, \ldots, b_n)$, where

$$b_1 \geq \cdots \geq b_k \geq 1 \geq b_{k+1} \geq \cdots \geq b_n.$$

We need to show that

$$b_2 = \cdots = b_k, \qquad b_{k+1} = \cdots = b_n = 1.$$

To show that $b_2 = \cdots = b_k$ for $k \geq 3$, assume for the sake of contradiction that

$$b_2 \neq b_k.$$

For

$$a_3 = b_3, \cdots, a_{k-1} = b_{k-1}, \qquad a_{k+1} = b_{k+1}, \cdots, a_n = b_n,$$

we have $a_1 + a_2 + a_k = constant$ and $a_1^2 + a_2^2 + a_k^2 = constant$, where

$$a_1 \geq a_2 \geq a_k \geq 1.$$

According to P 1.169 the product $a_1a_2a_k$ is maximum for $a_2 = a_k$, which contradicts the assumption that $b_2 \neq b_k$.

To show that $b_{k+1} = \cdots = b_n$ for $k \le n-2$, assume for the sake of contradiction that

$$b_{k+1} \neq b_n.$$

For

$$a_2 = b_2, \cdots, a_k = b_k, \qquad a_{k+2} = b_{k+2}, \cdots, a_{n-1} = b_{n-1},$$

we have $a_1 + a_{k+1} + a_n = constant$ and $a_1^2 + a_{k+1}^2 + a_n^2 = constant$, where

$$a_1 \ge 1 \ge a_{k+1} \ge a_n.$$

According to P 1.168, the product $a_1 a_{k+1} a_n$ is maximum for $a_{k+1} = a_n$, which contradicts the assumption that $b_{k+1} \neq b_n$. Therefore, we have

$$b_2 = \cdots = b_k := x, \qquad b_{k+1} = \cdots = b_n := y.$$

To end the proof, we still need to show that $y = 1$. Assume, for the sake of contradiction that

$$y \neq 1.$$

For

$$a_3 = b_3, \cdots, a_{n-1} = b_{n-1},$$

we have $a_1 + a_2 + a_n = constant$ and $a_1^2 + a_2^2 + a_n^2 = constant$, where

$$a_1 \ge a_2 \ge 1 \ge a_n.$$

According to P 1.171, the product $a_1 a_2 a_n$ is maximum for $a_n = 1$ or $a_2 = 1$, hence for $y = 1$ or $x = 1$. The first case contradicts the assumption that $y \neq 1$. The second case leads to

$$b_2 = \cdots = b_k = 1, \qquad b_{k+1} = \cdots = b_n := y < 1.$$

From the hypothesis $q \le n - 1 + (p - n + 1)^2$ and

$$q = b_1^2 + k - 1 + (n-k)y^2, \qquad p = b_1 + k - 1 + (n-k)y,$$

we get

$$b_1^2 + (n-k)(y^2 - 1) \le [b_1 + (n-k)(y-1)]^2,$$

which is equivalent to

$$(1-y)[(n-k-1)(1-y) - 2(b_1 - 1)] \ge 0.$$

Under the assumption that $y < 1$, this inequality implies

$$(n-k-1)(1-y) \ge 2(b_1 - 1).$$

On the other hand, the condition $p \ge n$ is equivalent to

$$b_1 - 1 \ge (n-k)(1-y).$$

Thus, we have

$$(n-k-1)(1-y) \geq 2(b_1-1) \geq 2(n-k)(1-y),$$

which involves

$$-(n-k+1)(1-y) \geq 0.$$

This result contradicts also the assumption $y \neq 1$.
Remark 1. For $p = n$, from P 1.258 we get P 1.256.

Remark 2. From P 1.258, we get the following simplified statement.

- *Let $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ be nonnegative real numbers such that*

$$a_1 \geq \cdots \geq a_k \geq 1 \geq a_{k+1} \geq \cdots \geq a_n, \quad 1 \leq k \leq n-1,$$

and

$$a_1 + a_2 + \cdots + a_n = p, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = q,$$

where p and q are fixed numbers. Then, the product $r = a_1 a_2 \cdots a_n$ is maximum when

$$a_2 = \cdots = a_k = 1, \quad a_{k+1} = \cdots = a_n$$

or

$$a_2 = \cdots = a_k, \quad a_{k+1} = \cdots = a_n = 1.$$

□

P 1.259. *If $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ are nonnegative real numbers such that*

$$a_1 \leq a_2 \leq 1 \leq a_3 \leq \cdots \leq a_n, \quad a_1 + a_2 + \cdots + a_n = n-1,$$

then

$$a_1^2 + a_2^2 + \cdots + a_n^2 + 10a_1 a_2 \cdots a_n \leq n+1.$$

(Vasile C., 2015)

Solution. According to P 1.258-(a), it suffices to prove the inequality for

$$a_1 = a_2, \quad a_3 = \cdots = a_{n-1} = 1.$$

Thus, we need to show that

$$2a + b = 2, \quad 0 \leq a \leq 1/2, \quad b \geq 1,$$

implies

$$2a^2 + (n-3) + b^2 + 10a^2 b \leq n+1,$$

which is equivalent to

$$2a^2 + b^2 + 10a^2 b \leq 4,$$

$$2a^2+(2-2a)^2+10a^2(2-2a)\le 4,$$

$$2a(1-2a)(4-5a)\ge 0.$$

The equality holds for

$$a_1=a_2=0,\quad a_3=\cdots=a_{n-1}=1,\quad a_n=2,$$

and for

$$a_1=a_2=1/2,\quad a_3=\cdots=a_n=1.$$

□

P 1.260. *If a,b,c,d,e are nonnegative real numbers such that*

$$a\le b\le 1\le c\le d\le e,\qquad a+b+c+d+e=8,$$

then

$$a^2+b^2+c^2+d^2+e^2+3abcde\le 38.$$

(Vasile C., 2015)

Solution. According to Remark 2 from P 1.258, it suffices to prove the inequality for

$$a=b,\quad c=d=1,$$

and for

$$a=b=1,\quad c=d.$$

Case 1: $a=b,\ \ c=d=1$. We need to show that

$$2a+e=6,\qquad 0\le a\le 1,\ \ e\ge 4,$$

implies

$$2a^2+2+e^2+3a^2e\le 38,$$

which is equivalent to

$$2a^2+e^2+3a^2e\le 36,$$

$$a^2+2(3-a)^2+3a^2(3-a)\le 18,$$

$$3a(a-2)^2\ge 0.$$

The equality holds for

$$a=b=0,\quad c=d=1,\quad e=6.$$

Case 2: $a=b=1,\ \ c=d$. We need to show that

$$2c+e=6,\qquad 1\le c\le 2\le e\le 4,$$

implies

$$2 + 2c^2 + e^2 + 3c^2e \leq 38,$$

which is equivalent to

$$2c^2 + e^2 + 3c^2e \leq 36,$$

$$c^2 + 2(3-c)^2 + 3c^2(3-c) \leq 18,$$

$$3c(c-2)^2 \geq 0.$$

The equality holds for

$$a = b = 1, \quad c = d = e = 2.$$

$\square$

Chapter 2

Noncyclic Inequalities

2.1 Applications

2.1. If a, b are positive real numbers, then

$$\frac{1}{4a^2+b^2}+\frac{3}{b^2+4ab}\geq\frac{16}{5(a+b)^2}.$$

2.2. If a, b are positive real numbers, then

$$3a\sqrt{3a}+3b\sqrt{6a+3b}\geq 5(a+b)\sqrt{a+b}.$$

2.3. If a, b, c are nonnegative real numbers such that $a+b+c=3$, then

$$(ab+c)(ac+b)\leq 4.$$

2.4. If a, b, c are nonnegative real numbers, then

$$a^3+b^3+c^3-3abc\geq\frac{1}{4}(b+c-2a)^3.$$

2.5. Let a, b, c be nonnegative real numbers such that $a\geq b\geq c$. Prove that

(a) $$a^3+b^3+c^3-3abc\geq 2(2b-a-c)^3;$$

(b) $$a^3+b^3+c^3-3abc\geq (a-2b+c)^3.$$

2.6. Let a,b,c be nonnegative real numbers such that $a \ge b \ge c$. Prove that

(a) $$a^3+b^3+c^3-3abc \ge 3(a^2-b^2)(b-c);$$

(b) $$a^3+b^3+c^3-3abc \ge \frac{9}{2}(a-b)(b^2-c^2).$$

2.7. If a,b,c are nonnegative real numbers such that

$$c=\min\{a,b,c\}, \qquad a^2+b^2+c^2=3,$$

then

(a) $$5b+2c \le 9;$$

(b) $$5(b+c) \le 9+3a.$$

2.8. Let a,b,c be nonnegative real numbers such that $a=\max\{a,b,c\}$. Prove that

$$a^6+b^6+c^6-3a^2b^2c^2 \ge 2(b^4+c^4+4b^2c^2)(b-c)^2.$$

2.9. Let a,b,c be nonnegative real numbers such that $a=\max\{a,b,c\}$. Prove that

$$a^2+b^2+c^2 \ge \frac{9abc}{a+b+c}+\frac{5}{3}(b-c)^2.$$

2.10. If a,b,c are nonnegative real numbers, no two of which are zero, then

$$\frac{1}{(a+b)^2}+\frac{1}{(a+c)^2}+\frac{16}{(b+c)^2} \ge \frac{6}{ab+bc+ca}.$$

2.11. If a,b,c are nonnegative real numbers, no two of which are zero, then

$$\frac{1}{(a+b)^2}+\frac{1}{(a+c)^2}+\frac{2}{(b+c)^2} \ge \frac{5}{2(ab+bc+ca)}.$$

2.12. If a,b,c are nonnegative real numbers, no two of which are zero, then

$$\frac{1}{(a+b)^2}+\frac{1}{(a+c)^2}+\frac{25}{(b+c)^2} \ge \frac{8}{ab+bc+ca}.$$

2.13. If a, b, c are positive real numbers, then

$$(a+b)^3(a+c)^3 \geq 4a^2bc(2a+b+c)^2.$$

2.14. If a, b, c are positive real numbers such that $abc = 1$, then

(a) $$\frac{a}{b}+\frac{b}{c}+\frac{1}{a} \geq a+b+1;$$

(b) $$\frac{a}{b}+\frac{b}{c}+\frac{1}{a} \geq \sqrt{3(a^2+b^2+1)}.$$

2.15. If a, b, c are positive real numbers such that $abc \geq 1$, then

$$a^{\frac{a}{b}} b^{\frac{b}{c}} c^c \geq 1.$$

2.16. If a, b, c are positive real numbers such that $ab + bc + ca = 3$, then

$$ab^2c^3 < 4.$$

2.17. If a, b, c are positive real numbers such that $ab + bc + ca = \frac{5}{3}$, then

$$ab^2c^2 \leq \frac{1}{3}.$$

2.18. Let a, b, c be positive real numbers such that

$$a \leq b \leq c, \qquad ab+bc+ca = 3.$$

Prove that

(a) $$ab^2c \leq \frac{9}{8};$$

(b) $$ab^4c \leq 2;$$

(c) $$a^2b^3c \leq 2.$$

2.19. Let a, b, c be positive real numbers such that

$$a \le b \le c, \qquad a+b+c = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Prove that

$$b \ge \frac{1}{a+c-1}.$$

2.20. Let a, b, c be positive real numbers such that

$$a \le b \le c, \qquad a+b+c = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Prove that

$$ab^2c^3 \ge 1.$$

2.21. Let a, b, c be positive real numbers such that

$$a \le b \le c, \qquad a+b+c = abc+2.$$

Prove that

$$(1-b)(1-ab^3c) \ge 0.$$

2.22. Let a, b, c be real numbers, no two of which are zero. Prove that

(a) $$\frac{(a-b)^2}{a^2+b^2}+\frac{(a-c)^2}{a^2+c^2} \ge \frac{(b-c)^2}{2(b^2+c^2)};$$

(b) $$\frac{(a+b)^2}{a^2+b^2}+\frac{(a+c)^2}{a^2+c^2} \ge \frac{(b-c)^2}{2(b^2+c^2)}.$$

2.23. Let a, b, c be real numbers, no two of which are zero. If $bc \ge 0$, then

(a) $$\frac{(a-b)^2}{a^2+b^2}+\frac{(a-c)^2}{a^2+c^2} \ge \frac{(b-c)^2}{(b+c)^2};$$

(b) $$\frac{(a+b)^2}{a^2+b^2}+\frac{(a+c)^2}{a^2+c^2} \ge \frac{(b-c)^2}{(b+c)^2}.$$

2.24. Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{|a-b|^3}{a^3+b^3}+\frac{|a-c|^3}{a^3+c^3}\ge\frac{|b-c|^3}{(b+c)^3}.$$

2.25. Let a,b,c be positive real numbers, $b\ne c$. Prove that

$$\frac{ab}{(a+b)^2}+\frac{ac}{(a+c)^2}\le\frac{(b+c)^2}{4(b-c)^2}.$$

2.26. Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{3bc+a^2}{b^2+c^2}\ge\frac{3ab-c^2}{a^2+b^2}+\frac{3ac-b^2}{a^2+c^2}.$$

2.27. Let a,b,c be nonnegative real numbers such that $a+b>0$. Prove that

$$abc\ge(b+c-a)(c+a-b)(a+b-c)+\frac{ab(a-b)^2}{a+b}.$$

2.28. Let a,b,c be nonnegative real numbers such that $a\ge b\ge c$. Prove that

(a) $$abc\ge(b+c-a)(c+a-b)(a+b-c)+\frac{2ab(a-b)^2}{a+b};$$

(b) $$abc\ge(b+c-a)(c+a-b)(a+b-c)+\frac{27b(a-b)^4}{4a^2}.$$

2.29. Let a,b,c be nonnegative real numbers such that $a+b>0$. Prove that

$$\sum a^2(a-b)(a-c)\ge a^2b^2\left(\frac{a-b}{a+b}\right)^2.$$

2.30. Let a,b,c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$ab^2+bc^2+2ca^2\le 8.$$

2.31. Let a,b,c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$ab^2+bc^2+\frac{3}{2}abc\le 4.$$

2.32. Let a,b,c be nonnegative real numbers such that $a+b+c=5$. Prove that

$$ab^2+bc^2+2abc \le 20.$$

2.33. If a,b,c are nonnegative real numbers, then

$$a^3+b^3+c^3-a^2b-b^2c-c^2a \ge \frac{8}{9}(a-b)(b-c)^2.$$

2.34. Let a,b,c be nonnegative real numbers such that $a \ge b \ge c$. Prove that

(a) $$\sum a^2(a-b)(a-c) \ge 4a^2b^2\left(\frac{a-b}{a+b}\right)^2;$$

(b) $$\sum a^2(a-b)(a-c) \ge \frac{27b(a-b)^4}{4a}.$$

2.35. If a,b,c are real numbers such that

$$a \ge b \ge 1 \ge c, \qquad a^2+b^2+c^2=3,$$

then

(a) $$1-abc \le 2(b-c)^2;$$

(b) $$1-abc \ge 2(a-b)^2;$$

(c) $$1-abc \ge \frac{1}{2}(a-c)^2;$$

(d) $$1-abc \le \frac{3}{4}(a-c)^2.$$

2.36. If a,b,c are real numbers such that

$$a \ge 1 \ge b \ge c, \qquad a^2+b^2+c^2=3,$$

then

$$1-abc \le \frac{2}{3}(a-c)^2.$$

2.37. If $a \ge 1 \ge b \ge c \ge 0$ and $a^2+b^2+c^2=3$, then

$$1-abc \le \frac{1}{\sqrt{2}}(a-c).$$

2.38. If $a \ge 1 \ge b \ge c \ge 0$ and $a^2+b^2+c^2=3$, then

$$1-abc \le (1+\sqrt{2})(a-b).$$

2.39. If $a \ge 1 \ge b \ge c \ge 0$ and $a^2+b^2+c^2=3$, then

$$1-abc \le (3+2\sqrt{2})(a-b)^2.$$

2.40. If a,b,c are positive real numbers, then

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge 3+\frac{(a-c)^2}{ab+bc+ca}.$$

2.41. If a,b,c are positive real numbers, then

(a) $$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge 3+\frac{4(a-c)^2}{(a+b+c)^2};$$

(b) $$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge 3+\frac{5(a-c)^2}{(a+b+c)^2}.$$

2.42. If $a \ge b \ge c > 0$, then

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge 3+\frac{3(b-c)^2}{ab+bc+ca}.$$

2.43. Let a,b,c be positive real numbers such that $abc=1$. Prove that

(a) if $a \ge b \ge 1 \ge c$, then

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge 3+\frac{2(a-b)^2}{ab};$$

(b) if $a \ge 1 \ge b \ge c$, then

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge 3+\frac{2(b-c)^2}{bc}.$$

2.44. Let a, b, c be positive real numbers such that

$$a \ge 1 \ge b \ge c, \qquad abc = 1.$$

prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge 3 + \frac{9(b-c)^2}{ab+bc+ca}.$$

2.45. Let a, b, c be positive real numbers such that

$$a \ge 1 \ge b \ge c, \qquad a+b+c = 3.$$

prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge 3 + \frac{4(b-c)^2}{b^2+c^2}.$$

2.46. Let a, b, c be positive real numbers such that

$$a \ge b \ge 1 \ge c, \qquad a+b+c = 3.$$

Prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge 3 + \frac{3(a-b)^2}{ab}.$$

2.47. If a, b, c are positive real numbers, then

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge 3 + \frac{2(a-c)^2}{(a+c)^2}.$$

2.48. If a, b, c are positive real numbers, then

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \ge a+b+c + \frac{4(a-c)^2}{a+b+c}.$$

2.49. If $a \ge b \ge c > 0$, then

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \ge a+b+c + \frac{6(b-c)^2}{a+b+c}.$$

2.50. If $a \ge b \ge c > 0$, then

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} > 5(a-b).$$

2.51. Let a, b, c be positive real numbers such that

$$a \ge b \ge 1 \ge c, \qquad a+b+c=3.$$

Prove that

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \ge 3 + \frac{11(a-c)^2}{4(a+c)}.$$

2.52. If a, b, c are positive real numbers, then

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{3}{2} + \frac{27(b-c)^2}{16(a+b+c)^2}.$$

2.53. Let a, b, c be positive real numbers such that $a = \min\{a, b, c\}$. Prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{3}{2} + \frac{9(b-c)^2}{4(a+b+c)^2}.$$

2.54. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{3}{2} + \frac{(b-c)^2}{2(b+c)^2}.$$

2.55. Let a, b, c be positive real numbers such that $a = \min\{a, b, c\}$. Prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{3}{2} + \frac{(b-c)^2}{4bc}.$$

2.56. Let a, b, c be positive real numbers such that

$$a \le 1 \le b \le c, \qquad a+b+c=3,$$

then

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{3}{2} + \frac{3(b-c)^2}{4bc}.$$

2.57. Let a, b, c be nonnegative real numbers such that

$$a \ge 1 \ge b \ge c, \qquad a+b+c=3,$$

then

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \ge \frac{3}{2}+\frac{(b-c)^2}{(b+c)^2}.$$

2.58. Let a, b, c be positive real numbers such that $a=\min\{a,b,c\}$. Prove that

(a) $$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{2(b-c)^2}{3(b^2+c^2)} \le 1;$$

(b) $$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{(b-c)^2}{b^2+bc+c^2} \le 1;$$

(c) $$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{(a-b)^2}{2(a^2+b^2)} \le 1.$$

2.59. Let a, b, c be positive real numbers such that

$$a \le 1 \le b \le c, \qquad a+b+c=3,$$

then

$$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{(b-c)^2}{bc} \le 1.$$

2.60. Let a, b, c be nonnegative real numbers such that $a=\max\{a,b,c\}$ and $b+c>0$. Prove that

(a) $$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{(b-c)^2}{2(ab+bc+ca)} \le 1;$$

(b) $$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{2(b-c)^2}{(a+b+c)^2} \le 1.$$

2.61. Let a, b, c be positive real numbers. Prove that

(a) if $a \ge b \ge c$, then

$$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{(a-c)^2}{a^2-ac+c^2} \ge 1;$$

(b) if $a \ge 1 \ge b \ge c$ and $abc=1$, then

$$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{(b-c)^2}{b^2-bc+c^2} \le 1.$$

2.62. Let a,b,c be positive real numbers such that $a=\min\{a,b,c\}$. Prove that

(a) $$\frac{a^2+b^2+c^2}{ab+bc+ca} \geq 1+\frac{4(b-c)^2}{3(b+c)^2};$$

(b) $$\frac{a^2+b^2+c^2}{ab+bc+ca} \geq 1+\frac{(a-b)^2}{(a+b)^2}.$$

2.63. If a,b,c are positive real numbers, then

$$\frac{a^2+b^2+c^2}{ab+bc+ca} \geq 1+\frac{9(a-c)^2}{4(a+b+c)^2}.$$

2.64. Let a,b,c be nonnegative real numbers, no two of which are zero. If $a=\min\{a,b,c\}$, then

$$\frac{1}{\sqrt{a^2-ab+b^2}}+\frac{1}{\sqrt{b^2-bc+c^2}}+\frac{1}{\sqrt{c^2-ca+a^2}} \geq \frac{6}{b+c}.$$

2.65. If $a \geq 1 \geq b \geq c \geq 0$ such that

$$ab+bc+ca=abc+2,$$

then

$$ac \leq 4-2\sqrt{2}.$$

2.66. If a,b,c are nonnegative real numbers such that

$$ab+bc+ca=3, \qquad a \leq 1 \leq b \leq c,$$

then

(a) $$a+b+c \leq 4;$$

(b) $$2a+b+c \leq 4.$$

2.67. Let a,b,c be nonnegative real numbers such that $a \leq b \leq c$. Prove that

(a) if $a+b+c=3$, then

$$a^4(b^4+c^4) \leq 2;$$

(b) if $a+b+c=2$, then

$$c^4(a^4+b^4) \leq 1.$$

2.68. If a,b,c are nonnegative real numbers such that $ab+bc+ca=3$, then

(a) $$a^2+b^2+c^2-a-b-c \ge \frac{5}{8}(a-c)^2;$$

(b) $$a^2+b^2+c^2-a-b-c \ge \frac{5}{2}\min\{(a-b)^2,(b-c)^2,(c-a)^2\}.$$

2.69. If a,b,c are nonnegative real numbers such that $ab+bc+ca=3$, then

$$\frac{a^3+b^3+c^3}{a+b+c} \ge 1+\frac{5}{9}(a-c)^2.$$

2.70. If a,b,c are nonnegative real numbers such that

$$a \ge b \ge c, \quad ab+bc+ca=3,$$

then

(a) $$\frac{a^3+b^3+c^3}{a+b+c} \ge 1+\frac{7}{9}(a-b)^2;$$

(b) $$\frac{a^3+b^3+c^3}{a+b+c} \ge 1+\frac{2}{3}(b-c)^2.$$

(c) $$\frac{a^3+b^3+c^3}{a+b+c} \ge 1+\frac{7}{3}\min\{(a-b)^2,(b-c)^2\}.$$

2.71. If a,b,c are nonnegative real numbers such that $ab+bc+ca=3$, then

$$a^4+b^4+c^4-a^2-b^2-c^2 \ge \frac{11}{4}(a-c)^2.$$

2.72. If a,b,c are nonnegative real numbers such that

$$a \ge b \ge c, \quad ab+bc+ca=3,$$

then

(a) $$a^4+b^4+c^4-a^2-b^2-c^2 \ge \frac{11}{3}(a-b)^2;$$

(b) $$a^4+b^4+c^4-a^2-b^2-c^2 \ge \frac{10}{3}(b-c)^2.$$

2.73. Let a,b,c be nonnegative real numbers such that

$$a \le b \le c, \quad a+b+c=3.$$

Find the greatest real number k such that

$$\sqrt{(56b^2+25)(56c^2+25)}+k(b-c)^2 \le 14(b+c)^2+25.$$

2.74. If $a \ge b \ge c > 0$ such that $abc=1$, then

$$3(a+b+c) \le 8+\frac{a}{c}.$$

2.75. If $a \ge b \ge c > 0$, then

$$(a+b-c)(a^2b-b^2c+c^2a) \ge (ab-bc+ca)^2.$$

2.76. If $a \ge b \ge c \ge 0$, then

$$\frac{(a-c)^2}{2(a+c)} \le a+b+c-3\sqrt[3]{abc} \le \frac{2(a-c)^2}{a+5c}.$$

2.77. If $a \ge b \ge c \ge d \ge 0$, then

$$\frac{(a-d)^2}{a+3d} \le a+b+c+d-4\sqrt[4]{abcd} \le \frac{3(a-d)^2}{a+5d}.$$

2.78. If $a \ge b \ge c > 0$, then

(a) $$a+b+c-3\sqrt[3]{abc} \ge \frac{3(a-b)^2}{5a+4b};$$

(b) $$a+b+c-3\sqrt[3]{abc} \ge \frac{64(a-b)^2}{7(11a+24b)}.$$

2.79. If $a \ge b \ge c > 0$, then

(a) $$a+b+c-3\sqrt[3]{abc} \ge \frac{3(b-c)^2}{4b+5c};$$

(b) $$a+b+c-3\sqrt[3]{abc} \ge \frac{25(b-c)^2}{7(3b+11c)}.$$

2.80. If $a \ge b \ge c > 0$, then

$$a+b+c-3\sqrt[3]{abc} \ge \frac{3(a-c)^2}{4(a+b+c)}.$$

2.81. If $a \ge b \ge c > 0$, then

(a) $$a^6+b^6+c^6-3a^2b^2c^2 \ge 12a^2c^2(b-c)^2;$$

(b) $$a^6+b^6+c^6-3a^2b^2c^2 \ge 10a^3c(b-c)^2.$$

2.82. If $a \ge b \ge c > 0$, then

$$\frac{ab+bc}{a^2+b^2+c^2} \le \frac{1+\sqrt{3}}{4}.$$

2.83. If $a \ge b \ge c \ge d > 0$, then

$$\frac{ab+bc+cd}{a^2+b^2+c^2+d^2} \le \frac{2+\sqrt{7}}{6}.$$

2.84. If

$$a \ge 1 \ge b \ge c \ge d \ge 0, \qquad a+b+c+d=4,$$

then

$$ab+bc+cd \le 3.$$

2.85. Let k and a,b,c be positive real numbers, and let

$$E=(ka+b+c)\left(\frac{k}{a}+\frac{1}{b}+\frac{1}{c}\right), \quad F=(ka^2+b^2+c^2)\left(\frac{k}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right).$$

(a) If $k \ge 1$, then

$$\sqrt{\frac{F-(k-2)^2}{2k}}+2 \ge \frac{E-(k-2)^2}{2k};$$

(b) If $0 < k \le 1$, then

$$\sqrt{\frac{F-k^2}{k+1}}+2 \ge \frac{E-k^2}{k+1}.$$

2.86. If a, b, c are positive real numbers, then

$$\frac{a}{2b+6c}+\frac{b}{7c+a}+\frac{25c}{9a+8b}>1.$$

2.87. If a, b, c are positive real numbers such that

$$\frac{1}{a}\geq\frac{1}{b}+\frac{1}{c},$$

then

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\geq\frac{55}{12(a+b+c)}.$$

2.88. If a, b, c are positive real numbers such that

$$\frac{1}{a}\geq\frac{1}{b}+\frac{1}{c},$$

then

$$\frac{1}{a^2+b^2}+\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2}\geq\frac{189}{40(a^2+b^2+c^2)}.$$

2.89. Find the best real numbers k, m, n such that

$$(\sqrt{a}+\sqrt{b}+\sqrt{c})\sqrt{a+b+c}\geq ka+mb+nc$$

for all $a\geq b\geq c\geq 0$.

2.90. Let $a, b\in(0,1]$, $a\leq b$.

(a) If $a\leq\dfrac{1}{e}$, then

$$2a^a\geq a^b+b^a;$$

(b) If $b\geq\dfrac{1}{e}$, then

$$2b^b\geq a^b+b^a.$$

2.91. If $0\leq a\leq b$ and $b\geq\dfrac{1}{2}$, then

$$2b^{2b}\geq a^{2b}+b^{2a}.$$

2.92. If $a \ge b \ge 0$, then

(a)
$$a^{b-a} \le 1 + \frac{a-b}{\sqrt{a}};$$

(b)
$$a^{a-b} \ge 1 - \frac{3(a-b)}{4\sqrt{a}}.$$

2.93. If a, b, c are positive real numbers such that

$$a \ge b \ge c, \qquad ab^2c^3 = 1,$$

then

$$a + 2b + 3c \ge \frac{1}{a} + \frac{2}{b} + \frac{3}{c}.$$

2.94. If a, b, c are positive real numbers such that

$$a + b + c = 3, \qquad a \le b \le c,$$

then

$$\frac{1}{a} + \frac{2}{b} \ge a^2 + b^2 + c^2.$$

2.95. If a, b, c are positive real numbers such that

$$a + b + c = 3, \qquad a \le b \le c,$$

then

$$\frac{2}{a} + \frac{3}{b} + \frac{1}{c} \ge 2(a^2 + b^2 + c^2).$$

2.96. If a, b, c are positive real numbers such that

$$a + b + c = 3, \qquad a \le b \le c,$$

then

$$\frac{31}{a} + \frac{25}{b} + \frac{25}{c} \ge 27(a^2 + b^2 + c^2).$$

2.97. If a, b, c are the lengths of the sides of a triangle, then

$$a^3(b + c) + bc(b^2 + c^2) \ge a(b^3 + c^3).$$

2.98. If a, b, c are the lengths of the sides of a triangle, then

$$\frac{(a+b)^2}{2ab+c^2} + \frac{(a+c)^2}{2ac+b^2} \geq \frac{(b+c)^2}{2bc+a^2}.$$

2.99. If a, b, c are the lengths of the sides of a triangle, then

$$\frac{a+b}{ab+c^2} + \frac{a+c}{ac+b^2} \geq \frac{b+c}{bc+a^2}.$$

2.100. If a, b, c are the lengths of the sides of a triangle, then

$$\frac{b(a+c)}{ac+b^2} + \frac{c(a+b)}{ab+c^2} \geq \frac{a(b+c)}{bc+a^2}.$$

2.101. If a, b, c, d are positive real numbers such that

$$a \geq b \geq c \geq d, \qquad ab^2c^3d^6 = 1,$$

then

$$a + 2b + 3c + 6d \geq \frac{1}{a} + \frac{2}{b} + \frac{3}{c} + \frac{6}{d}.$$

2.102. If a, b, c, d are positive real numbers such that

$$a \geq b \geq c \geq d, \qquad abc^2d^4 \geq 1,$$

then

$$a + b + 2c + 4d \geq \frac{1}{a} + \frac{1}{b} + \frac{2}{c} + \frac{4}{d}.$$

2.103. If a, b, c, d are positive real numbers such that

$$abcd \geq 1, \qquad a \geq b \geq c \geq d, \qquad ad \geq bc,$$

then

$$a + b + c + d \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}.$$

2.104. If a,b,c,d,e,f are positive real numbers such that

$$abcdef \ge 1, \qquad a \ge b \ge c \ge d \ge e \ge f, \qquad af \ge be \ge cd,$$

then

$$a+b+c+d+e+f \ge \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}+\frac{1}{f}.$$

2.105. Let a,b,c,d be nonnegative real numbers such that

$$a^2-ab+b^2=c^2-cd+d^2.$$

Prove that

$$(a+b)(c+d) \ge 2(ab+cd).$$

2.106. Let a,b,c,d be nonnegative real numbers such that

$$a^2-ab+b^2=c^2-cd+d^2.$$

Prove that

$$\frac{1}{a^2+ab+b^2}+\frac{1}{c^2+cd+d^2} \le \frac{8}{3(a+b)(c+d)}.$$

2.107. Let a,b,c,d be nonnegative real numbers such that

$$a^2-ab+b^2=c^2-cd+d^2.$$

Prove that

$$\frac{1}{a^2+ab+b^2}+\frac{1}{c^2+cd+d^2} \le \frac{8}{3(a+b)(c+d)}.$$

2.108. Let a,b,c,d be nonnegative real numbers such that

$$a^2-ab+b^2=c^2-cd+d^2.$$

Prove that

$$\frac{1}{(ac+bd)^4}+\frac{1}{(ad+bc)^4} \le \frac{2}{(ab+cd)^4}.$$

2.109. Let a,b,c,d be nonnegative real numbers such that $a \ge b \ge c \ge d$ and

$$a+b+c+d=13, \qquad a^2+b^2+c^2+d^2=43.$$

Prove that

$$ab \ge cd+3.$$

2.110. Let a, b, c, d be nonnegative real numbers such that $a \ge b \ge c \ge d$ and

$$a+b+c+d=13, \qquad a^2+b^2+c^2+d^2=43.$$

Prove that

$$ab-cd \ge \frac{7}{2}$$

(a) for $a \le \dfrac{39}{10}$;

(b) for $d \le \dfrac{31}{11}$.

2.111. Let a, b, c, d be nonnegative real numbers such that $a \ge b \ge c \ge d$ and

$$a+b+c+d=13, \qquad a^2+b^2+c^2+d^2=43.$$

Prove that

$$\frac{83}{4} \le ac+bd \le \frac{169}{8}.$$

2.112. If a, b, c, d are positive real numbers such that

$$a+b+c+d=4, \qquad a \le b \le 1 \le c \le d,$$

then

$$9\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \ge 4+8(a^2+b^2+c^2+d^2).$$

2.113. If a, b, c, d are positive real numbers such that

$$a^2+b^2+c^2+d^2=4, \qquad a \le b \le c \le d,$$

then

$$\frac{1}{a}+a+b+c+d \ge 5.$$

2.114. If a, b, c, d are real numbers, then

$$6(a^2+b^2+c^2+d^2)+(a+b+c+d)^2 \ge 12(ab+bc+cd).$$

2.115. If a, b, c, d are positive real numbers, then

$$\frac{1}{a^2+ab}+\frac{1}{b^2+bc}+\frac{1}{c^2+cd}+\frac{1}{d^2+da}\ge \frac{4}{ac+bd}.$$

2.116. If a, b, c, d are positive real numbers, then

$$\frac{1}{a(1+b)}+\frac{1}{b(1+a)}+\frac{1}{c(1+d)}+\frac{1}{d(1+c)}\ge \frac{16}{1+8\sqrt{abcd}}.$$

2.117. If a, b, c, d are positive real numbers such that $a \ge b \ge c \ge d$ and

$$a+b+c+d=4,$$

then

$$ac+bd \le 2.$$

2.118. If a, b, c, d are positive real numbers such that $a \ge b \ge c \ge d$ and

$$a+b+c+d=4,$$

then

$$2\left(\frac{1}{b}+\frac{1}{d}\right)\ge a^2+b^2+c^2+d^2.$$

2.119. Let a, b, c, d be positive real numbers such that $a \ge b \ge c \ge d$ and

$$ab+bc+cd+da=3.$$

Prove that

$$a^3bcd < 4.$$

2.120. Let a, b, c, d be positive real numbers such that $a \ge b \ge c \ge d$ and

$$ab+bc+cd+da=6.$$

Prove that

$$acd \le 2.$$

2.121. Let a, b, c, d be positive real numbers such that $a \ge b \ge c \ge d$ and

$$ab + bc + cd + da = 9.$$

Prove that

$$abd \le 4.$$

2.122. Let a, b, c, d be positive real numbers such that $a \ge b \ge c \ge d$ and

$$a^2 + b^2 + c^2 + d^2 = 10.$$

Prove that

$$2b + 4d \le 3c + 5.$$

2.123. Let a, b, c, d be positive real numbers such that $a \le b \le c \le d$ and

$$abcd = 1.$$

Prove that

$$4 + \frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \ge 2(a+b)(c+d).$$

2.124. Let a, b, c, d be positive real numbers such that $a \ge b \ge c \ge d$ and

$$3(a^2 + b^2 + c^2 + d^2) = (a+b+c+d)^2.$$

Prove that

(a) $$\frac{a+d}{b+c} \le 2;$$

(b) $$\frac{a+c}{b+d} \le \frac{7+2\sqrt{6}}{5};$$

(c) $$\frac{a+c}{c+d} \le \frac{3+\sqrt{5}}{2}.$$

2.125. Let a, b, c, d be nonnegative real numbers such that $a \ge b \ge c \ge d$ and

$$2(a^2 + b^2 + c^2 + d^2) = (a+b+c+d)^2.$$

Prove that

$$a \ge b + 3c + (2\sqrt{3} - 1)d.$$

2.126. If $a \ge b \ge c \ge d \ge 0$, then

(a) $$a+b+c+d-4\sqrt[4]{abcd} \ge \frac{3}{2}\left(\sqrt{b}-2\sqrt{c}+\sqrt{d}\right)^2;$$

(b) $$a+b+c+d-4\sqrt[4]{abcd} \ge \frac{2}{9}\left(3\sqrt{b}-2\sqrt{c}-\sqrt{d}\right)^2;$$

(c) $$a+b+c+d-4\sqrt[4]{abcd} \ge \frac{4}{19}\left(3\sqrt{b}-\sqrt{c}-2\sqrt{d}\right)^2;$$

(d) $$a+b+c+d-4\sqrt[4]{abcd} \ge \frac{3}{8}\left(\sqrt{b}-3\sqrt{c}+2\sqrt{d}\right)^2;$$

(e) $$a+b+c+d-4\sqrt[4]{abcd} \ge \frac{1}{2}\left(2\sqrt{b}-3\sqrt{c}+\sqrt{d}\right)^2;$$

(f) $$a+b+c+d-4\sqrt[4]{abcd} \ge \frac{1}{6}\left(2\sqrt{b}+\sqrt{c}-3\sqrt{d}\right)^2.$$

2.127. If $a \ge b \ge c \ge d \ge 0$, then

(a) $$a+b+c+d-4\sqrt[4]{abcd} \ge \left(\sqrt{a}-\sqrt{d}\right)^2;$$

(b) $$a+b+c+d-4\sqrt[4]{abcd} \ge 2\left(\sqrt{b}-\sqrt{c}\right)^2;$$

(c) $$a+b+c+d-4\sqrt[4]{abcd} \ge \frac{4}{3}\left(\sqrt{b}-\sqrt{d}\right)^2;$$

(d) $$a+b+c+d-4\sqrt[4]{abcd} \ge \frac{3}{2}\left(\sqrt{c}-\sqrt{d}\right)^2.$$

2.128. If $a \ge b \ge c \ge d \ge e \ge 0$, then

$$a+b+c+d+e-5\sqrt[5]{abcde} \ge 2\left(\sqrt{b}-\sqrt{d}\right)^2.$$

2.129. If a, b, c, d, e are real numbers, then

$$\frac{ab+bc+cd+de}{a^2+b^2+c^2+d^2+e^2} \le \frac{\sqrt{3}}{2}.$$

2.130. If a, b, c, d, e are positive real numbers, then

$$\frac{a^2b^2}{bd+ce}+\frac{b^2c^2}{cd+ae}+\frac{c^2a^2}{ad+be} \ge \frac{3abc}{d+e}.$$

2.131. If a, b, c, d, e, f are nonnegative real numbers such that

$$a \geq b \geq c \geq d \geq e \geq f,$$

then

$$(a+b+c+d+e+f)^2 \geq 8(ac+bd+ce+df).$$

2.132. If $a \geq b \geq c \geq d \geq e \geq f \geq 0$, then

$$a+b+c+d+e+f-6\sqrt[6]{abcdef} \geq 2\left(\sqrt{b}-\sqrt{e}\right)^2.$$

2.133. Let a, b, c and x, y, z be positive real numbers such that

$$x+y+z=a+b+c.$$

Prove that

$$ax^2+by^2+cz^2+xyz \geq 4abc.$$

2.134. Let a, b, c and x, y, z be positive real numbers such that

$$x+y+z=a+b+c.$$

Prove that

$$\frac{x(3x+a)}{bc}+\frac{y(3y+b)}{ca}+\frac{z(3z+c)}{ab} \geq 12.$$

2.135. Let a, b, c be given positive numbers. Find the minimum value $F(a, b, c)$ of

$$E(x,y,z)=\frac{ax}{y+z}+\frac{by}{z+x}+\frac{cz}{x+y},$$

where x, y, z are nonnegative real numbers, no two of which are zero.

2.136. Let a, b, c and x, y, z be real numbers.

(a) If $ab+bc+ca>0$, then

$$[(b+c)x+(c+a)y+(a+b)z]^2 \geq 4(ab+bc+ca)(xy+yz+zx);$$

(b) If $a, b, c \geq 0$, then

$$[(b+c)x+(c+a)y+(a+b)z]^2 \geq 4(a+b+c)(ayz+bzx+cxy).$$

2.137. Let a,b,c and x,y,z be positive real numbers such that

$$\frac{a}{yz}+\frac{b}{zx}+\frac{c}{xy}=1.$$

Prove that

(a) $$x+y+z\ge\sqrt{4(a+b+c+\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\,)+3\sqrt[3]{abc}};$$

(b) $$x+y+z>\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}.$$

2.138. If a,b,c and x,y,z are nonnegative real numbers, then

$$\frac{2}{(a+b)(x+y)}+\frac{2}{(b+c)(y+z)}+\frac{2}{(c+a)(z+x)}\ge\frac{9}{(b+c)x+(c+a)y+(a+b)z}.$$

2.139. Let a,b,c be the lengths of the sides of a triangle. If x,y,z are real numbers, then

$$(ya^2+zb^2+xc^2)(za^2+xb^2+yc^2)\ge(xy+yz+zx)(a^2b^2+b^2c^2+c^2a^2).$$

2.140. If $a_1\ge a_2\ge\cdots\ge a_8\ge 0$, then

$$a_1+a_2+\cdots+a_8-8\sqrt[8]{a_1a_2\cdots a_8}\ge 3\left(\sqrt{a_6}-\sqrt{a_7}\right)^2.$$

2.141. Let $a_1,a_2,\ldots,a_n$ and $b_1,b_2,\ldots,b_n$ be real numbers. Prove that

$$a_1b_1+\cdots+a_nb_n+\sqrt{(a_1^2+\cdots+a_n^2)(b_1^2+\cdots+b_n^2)}\ge\frac{2}{n}(a_1+\cdots+a_n)(b_1+\cdots+b_n).$$

2.142. Let $a_1,a_2,\ldots,a_n$ be positive real numbers such that $a_1\ge 2a_2$. Prove that

$$(5n-1)(a_1^2+a_2^2+\cdots+a_n^2)\ge 5(a_1+a_2+\cdots+a_n)^2.$$

2.143. If $a_1,a_2,\ldots,a_n$ are positive real numbers such that $a_1\ge 4a_2$, then

$$(a_1+a_2+\cdots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right)\ge\left(n+\frac{1}{2}\right)^2.$$

2.144. If $a_1 \ge a_2 \ge \cdots \ge a_n > 0$ such that $a_1 + a_2 + \cdots + a_n = n$, then

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - n \ge \frac{4(n-1)^2}{n^3}(a_1 - a_2)^2.$$

2.145. If $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ are real numbers such that

$$a_1 \le a_2 \le \cdots \le a_n, \quad a_1 + a_2 + \cdots + a_n = 0,$$

then

$$a_1^2 + a_2^2 + \cdots + a_n^2 + n a_1 a_n \le 0.$$

2.146. Let $a_1, a_2, \ldots, a_n$ $(n \ge 4)$ be nonnegative real numbers such that

$$a_1 \ge a_2 \ge \cdots \ge a_n$$

and

$$(a_1 + a_2 + \cdots + a_n)^2 = 4(a_1^2 + a_2^2 + \cdots + a_n^2).$$

Prove that

$$1 \le \frac{a_1 + a_2}{a_3 + a_4 + \cdots + a_n} \le 1 + \sqrt{\frac{2n-8}{n-2}}.$$

2.147. If $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$, then

(a) $$a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1 a_2 \cdots a_n} \ge \frac{1}{3}\left(\sqrt{a_1} + \sqrt{a_2} - 2\sqrt{a_n}\right)^2;$$

(b) $$a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1 a_2 \cdots a_n} \ge \frac{1}{4}\left(2\sqrt{a_1} - \sqrt{a_{n-1}} - \sqrt{a_n}\right)^2.$$

2.148. If $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0,\ n \ge 3$, then

$$a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1 a_2 \cdots a_n} \ge \frac{n-1}{2n}\left(\sqrt{a_{n-2}} + \sqrt{a_{n-1}} - 2\sqrt{a_n}\right)^2.$$

2.149. Let $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$. If

$$\frac{n}{2} \le k \le n-1,$$

then

$$a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1 a_2 \cdots a_n} \ge \frac{2k(n-k)}{n}\left(\sqrt{a_k} - \sqrt{a_{k+1}}\right)^2.$$

2.150. Let $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$. If

$$1 \le k < j \le n, \qquad k+j \ge n+1,$$

then

$$a_1+a_2+\cdots+a_n-n\sqrt[n]{a_1a_2\cdots a_n} \ge \frac{2k(n-j+1)}{n+k-j+1}\left(\sqrt{a_k}-\sqrt{a_j}\right)^2.$$

2.151. If $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0,\ n \ge 4$, then

(a) $$a_1+a_2+\cdots+a_n-n\sqrt[n]{a_1a_2\cdots a_n} \ge \frac{1}{2}\left(1-\frac{1}{n}\right)\left(\sqrt{a_{n-2}}-3\sqrt{a_{n-1}}+2\sqrt{a_n}\right)^2;$$

(b) $$a_1+a_2+\cdots+a_n-n\sqrt[n]{a_1a_2\cdots a_n} \ge \left(1-\frac{2}{n}\right)\left(2\sqrt{a_{n-2}}-3\sqrt{a_{n-1}}+\sqrt{a_n}\right)^2.$$

2.2 Solutions

P 2.1. *If a, b are positive real numbers, then*

$$\frac{1}{4a^2+b^2}+\frac{3}{b^2+4ab} \ge \frac{16}{5(a+b)^2}.$$

Solution. Using the Cauchy-Schwarz inequality gives

$$\frac{1}{4a^2+b^2}+\frac{3}{b^2+4ab} \ge \frac{(1+3)^2}{(4a^2+b^2)+3(b^2+4ab)} = \frac{4}{a^2+b^2+3ab}.$$

Thus, we only need to show that

$$\frac{1}{a^2+b^2+3ab} \ge \frac{4}{5(a+b)^2},$$

which reduces to $(a-b)^2 \ge 0$. The equality holds for $a = b$.

□

P 2.2. *If a, b are positive real numbers, then*

$$3a\sqrt{3a}+3b\sqrt{6a+3b} \ge 5(a+b)\sqrt{a+b}.$$

Solution. Due to homogeneity, we may assume that $a+b=3$. Thus, we need to show that

$$a\sqrt{a}+(3-a)\sqrt{3+a} \ge 5$$

for $0 < a < 3$. Substituting

$$\sqrt{a}=x, \qquad 0 < x < \sqrt{3},$$

the inequality becomes

$$(3-x^2)\sqrt{3+x^2} \ge 5-x^3.$$

For $\sqrt[3]{5} \le x < \sqrt{3}$, the inequality is trivial. For $0 < x < \sqrt[3]{5}$, squaring both sides of the inequality gives

$$(3-x^2)(9-x^4) \ge (5-x^3)^2,$$

$$3x^4-10x^3+9x^2-2 \le 0,$$

$$(x-1)^2(3x^2-4x-2) \le 0.$$

Since $3x^2-4x-2 \le 0$ for $\dfrac{2-\sqrt{10}}{3} \le x \le \dfrac{2+\sqrt{10}}{3}$, we only need to prove that

$$\sqrt[3]{5} \le \frac{2+\sqrt{10}}{3}.$$

Indeed, we have

$$\left(\frac{2+\sqrt{10}}{3}\right)^3 - 5 = \frac{22\sqrt{10}-67}{27} > 0.$$

The equality holds for $a = b/2$.

□

P 2.3. *If a, b, c are nonnegative real numbers such that $a+b+c=3$, then*

$$(ab+c)(ac+b) \le 4.$$

Solution. By the AM-GM inequality, we have

$$(ab+c)(ac+b) \le \left[\frac{(ab+c)+(ac+b)}{2}\right]^2 = \frac{(a+1)^2(b+c)^2}{4}.$$

Therefore, it suffices to show that

$$(a+1)(b+c) \le 4.$$

Indeed,

$$(a+1)(b+c) \le \left[\frac{(a+1)+(b+c)}{2}\right]^2 = 4.$$

The equality holds for $a=b=c=1$, for $a=1$, $b=0$, $c=2$, and for $a=1$, $b=2$, $c=0$.

□

P 2.4. *If a, b, c are nonnegative real numbers, then*

$$a^3+b^3+c^3-3abc \ge \frac{1}{4}(b+c-2a)^3.$$

Solution. Write the inequality as

$$2(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2] \ge (b+c-2a)^3.$$

Consider the non-trivial case $b+c-2a \ge 0$. Since $(b-c)^2 \ge 0$ and

$$a+b+c \ge b+c-a,$$

it suffices to show that

$$2(a-b)^2+2(c-a)^2 \ge (b+c-2a)^2.$$

Indeed, we have

$$2(a-b)^2+2(c-a)^2-(b+c-2a)^2 = (b-c)^2 \ge 0.$$

The equality holds for $a=b=c$, and also for $a=0$ and $b=c$.

□

P 2.5. *Let a, b, c be nonnegative real numbers such that $a \ge b \ge c$. Prove that*

(a) $$a^3 + b^3 + c^3 - 3abc \ge 2(2b - a - c)^3;$$

(b) $$a^3 + b^3 + c^3 - 3abc \ge (a - 2b + c)^3.$$

Solution. (a) Write the inequality as

$$(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) \ge 2(2b - a - c)^3.$$

For the non-trivial case $2b - a - c \ge 0$, since

$$a + b + c \ge 2(2b - a - c),$$

it suffices to show that

$$a^2 + b^2 + c^2 - ab - bc - ca \ge (2b - a - c)^2.$$

This is equivalent to the obvious inequality

$$3(a - b)(b - c) \ge 0.$$

The equality holds for $a = b = c$, and also for $a = b$ and $c = 0$.

(b) Write the inequality as

$$(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) \ge (a - 2b + c)^3.$$

For the non-trivial case $a - 2b + c \ge 0$, since

$$a + b + c \ge a - 2b + c,$$

it suffices to show that

$$a^2 + b^2 + c^2 - ab - bc - ca \ge (a - 2b + c)^2,$$

which is equivalent to

$$3(a - b)(b - c) \ge 0.$$

The equality holds for $a = b = c$, and also for $b = c = 0$.

□

P 2.6. *Let a, b, c be nonnegative real numbers such that $a \ge b \ge c$. Prove that*

(a) $$a^3 + b^3 + c^3 - 3abc \ge 3(a^2 - b^2)(b - c);$$

(b) $$a^3 + b^3 + c^3 - 3abc \ge \frac{9}{2}(a - b)(b^2 - c^2).$$

Solution. (a) Write the inequality as

$$(a+b+c)(a^2+b^2+c^2-ab-bc-ca) \ge 3(a+b)(a-b)(b-c)).$$

Since

$$a+b+c \ge a+b,$$

it suffices to show that

$$a^2+b^2+c^2-ab-bc-ca \ge 3(a-b)(b-c).$$

Indeed,

$$a^2+b^2+c^2-ab-bc-ca-3(a-b)(b-c) = (a-2b+c)^2 \ge 0.$$

The equality holds for $a=b=c$, and also for $a=2b$ and $c=0$.

(b) Write the inequality as

$$(a+b+c)(a^2+b^2+c^2-ab-bc-ca) \ge \frac{9}{2}(a-b)(b-c)(b+c).$$

Since

$$a+b+c \ge \frac{3}{2}(b+c),$$

it suffices to show that

$$a^2+b^2+c^2-ab-bc-ca \ge 3(a-b)(b-c).$$

This is equivalent to the obvious inequality

$$(a-2b+c)^2 \ge 0.$$

The equality holds for $a=b=c$.

□

P 2.7. *If a,b,c are nonnegative real numbers such that*

$$c = \min\{a,b,c\}, \qquad a^2+b^2+c^2 = 3,$$

then

(a) $$5b+2c \le 9;$$

(b) $$5(b+c) \le 9+3a.$$

Solution. (a) It suffices to show that

$$5b + 2c + (a - c) \le 9;$$

that is,

$$9 \ge a + 5b + c.$$

This follows from the Cauchy-Schwarz inequality

$$(1 + 25 + 1)(a^2 + b^2 + c^2) \ge (a + 5b + c)^2.$$

The equality holds for $a = c = \frac{1}{3}$ and $b = \frac{5}{3}$.

(b) It suffices to show that

$$5(b + c) + 4(a - c) \le 9 + 3a;$$

that is,

$$9 \ge a + 5b + c.$$

As we have shown at (a), this follows from the Cauchy-Schwarz inequality

$$(1 + 25 + 1)(a^2 + b^2 + c^2) \ge (a + 5b + c)^2.$$

The equality holds for $a = c = \frac{1}{3}$ and $b = \frac{5}{3}$.

□

P 2.8. *Let a, b, c be nonnegative real numbers such that $a = \max\{a, b, c\}$. Prove that*

$$a^6 + b^6 + c^6 - 3a^2b^2c^2 \ge 2(b^4 + c^4 + 4b^2c^2)(b - c)^2.$$

Solution. Because the inequality is symmetric in b and c, we may assume that $b \ge c$; that is,

$$a \ge b \ge c.$$

We will show that

$$a^6 + b^6 + c^6 - 3a^2b^2c^2 \ge 2b^6 + c^6 - 3b^4c^2 \ge 2(b^4 + c^4 + 4b^2c^2)(b - c)^2.$$

The left inequality is equivalent to the obvious inequality

$$(a^2 - b^2)(a^4 + a^2b^2 + b^4 - 3b^2c^2) \ge 0.$$

The right inequality is equivalent to

$$(b^2 - c^2)^2(2b^2 + c^2) \ge 2(b^4 + c^4 + 4b^2c^2)(b - c)^2,$$

$$(b - c)^2[(b + c)^2(2b^2 + c^2) - 2(b^4 + c^4 + 4b^2c^2)] \ge 0,$$

$$c(b - c)^3(4b^2 - bc + c^2) \ge 0.$$

The equality holds for $a = b = c$, for $a = b$ and $c = 0$, and for $a = c$ and $b = 0$.

□

P 2.9. *Let a,b,c be nonnegative real numbers such that $a=\max\{a,b,c\}$. Prove that*

$$a^2+b^2+c^2 \ge \frac{9abc}{a+b+c}+\frac{5}{3}(b-c)^2.$$

Solution. Because the inequality is symmetric in b and c, we may assume that $b \ge c$, hence

$$a \ge b \ge c.$$

Write the inequality as follows:

$$(a^2+b^2+c^2)(a+b+c)-9abc \ge \frac{5}{3}(a+b+c)(b-c)^2;$$

$$a^3+b^3+c^3-3abc+\sum a(b-c)^2 \ge \frac{5}{3}(a+b+c)(b-c)^2;$$

$$(a+b+c)\sum(b-c)^2+2\sum a(b-c)^2 \ge \frac{10}{3}(a+b+c)(b-c)^2.$$

It suffices to show that

$$(a+b+c)[(a-c)^2+(b-c)^2]+2a(b-c)^2+2b(a-c)^2 \ge \frac{10}{3}(a+b+c)(b-c)^2.$$

This inequality is true if

$$(a+b+c)[(b-c)^2+(b-c)^2]+2a(b-c)^2+2b(b-c)^2 \ge \frac{10}{3}(a+b+c)(b-c)^2.$$

Thus, we only need to show that

$$2(a+b+c)+2a+2b \ge \frac{10}{3}(a+b+c),$$

which reduces to $a+b-2c \ge 0$. The equality holds for $a=b=c$.

□

P 2.10. *If a,b,c are nonnegative real numbers, no two of which are zero, then*

$$\frac{1}{(a+b)^2}+\frac{1}{(a+c)^2}+\frac{16}{(b+c)^2} \ge \frac{6}{ab+bc+ca}.$$

(Vasile C., 2014)

Solution *(by Nguyen Van Quy)*. Since the equality holds for $a = 0$ and $b = c$, we write the desired inequality in the form

$$\frac{16}{(b+c)^2} + \left(\frac{1}{a+b} + \frac{1}{a+c}\right)^2 \geq \frac{6}{ab+bc+ca} + \frac{2}{(a+b)(a+c)}$$

and apply then the AM-GM inequality

$$\frac{16}{(b+c)^2} + \left(\frac{1}{a+b} + \frac{1}{a+c}\right)^2 \geq \frac{8}{b+c}\left(\frac{1}{a+b} + \frac{1}{a+c}\right).$$

Therefore, it suffices to show that

$$\frac{8}{b+c}\left(\frac{1}{a+b} + \frac{1}{a+c}\right) \geq \frac{6}{ab+bc+ca} + \frac{2}{(a+b)(a+c)}.$$

Since $(a+b)(a+c) \geq ab+bc+ca$, it is enough to show that

$$\frac{8}{b+c}\left(\frac{1}{a+b} + \frac{1}{a+c}\right) \geq \frac{8}{ab+bc+ca},$$

which is equivalent to

$$(2a+b+c)(ab+bc+ca) \geq (a+b)(b+c)(c+a).$$

We have

$$\begin{aligned}(2a+b+c)(ab+bc+ca) &\geq (a+b+c)(ab+bc+ca)\\ &\geq (a+b)(b+c)(c+a).\end{aligned}$$

This completes the proof. The equality holds for $a = 0$ and $b = c$.

□

P 2.11. *If a, b, c are nonnegative real numbers, no two of which are zero, then*

$$\frac{1}{(a+b)^2} + \frac{1}{(a+c)^2} + \frac{2}{(b+c)^2} \geq \frac{5}{2(ab+bc+ca)}.$$

Solution. This inequality follows from Iran 1996 inequality (see P 1.72 in Volume 2, for $k = 2$), namely

$$\frac{1}{(a+b)^2} + \frac{1}{(a+c)^2} + \frac{1}{(b+c)^2} \geq \frac{9}{4(ab+bc+ca)},$$

and the inequality in P 2.10, namely

$$\frac{1}{(a+b)^2} + \frac{1}{(a+c)^2} + \frac{16}{(b+c)^2} \geq \frac{6}{ab+bc+ca}.$$

Indeed, summing the first inequality multiplied by 14 and the second inequality, we get the desired inequality. The equality holds for $a = 0$ and $b = c$.

□

P 2.12. *If a, b, c are nonnegative real numbers, no two of which are zero, then*

$$\frac{1}{(a+b)^2}+\frac{1}{(a+c)^2}+\frac{25}{(b+c)^2}\ge\frac{8}{ab+bc+ca}.$$

(Vasile C., 2014)

Solution. Write the inequality as

$$\left(\frac{1}{a+b}+\frac{1}{a+c}\right)^2+\frac{25}{(b+c)^2}\ge\frac{8}{ab+bc+ca}+\frac{2}{(a+b)(a+c)}.$$

By the AM-GM inequality, we have

$$\left(\frac{1}{a+b}+\frac{1}{a+c}\right)^2+\frac{25}{(b+c)^2}\ge\frac{10}{b+c}\left(\frac{1}{a+b}+\frac{1}{a+c}\right).$$

Therefore, it suffices to show that

$$\frac{10}{b+c}\left(\frac{1}{a+b}+\frac{1}{a+c}\right)\ge\frac{8}{ab+bc+ca}+\frac{2}{(a+b)(a+c)}.$$

Since $(a+b)(a+c)\ge ab+bc+ca$, it is enough to show that

$$\frac{10}{b+c}\left(\frac{1}{a+b}+\frac{1}{a+c}\right)\ge\frac{10}{ab+bc+ca},$$

which is equivalent to

$$(2a+b+c)(ab+bc+ca)\ge(a+b)(b+c)(c+a).$$

Indeed,

$$\begin{aligned}(2a+b+c)(ab+bc+ca)&\ge(a+b+c)(ab+bc+ca)\\&\ge(a+b)(b+c)(c+a).\end{aligned}$$

This completes the proof. The equality holds for $a=0$ and $\dfrac{b}{c}+\dfrac{c}{b}=3$.

□

P 2.13. *If a, b, c are positive real numbers, then*

$$(a+b)^3(a+c)^3\ge4a^2bc(2a+b+c)^2.$$

(XZLBQ, 2014)

Solution *(by Nguyen Van Quy)*. Write the inequality as

$$\frac{(a+b)^2(a+c)^2}{4a^2bc} \geq \frac{(2a+b+c)^2}{(a+b)(a+c)}.$$

Since

$$\begin{aligned}(a+b)^2(a+c)^2 &= [(a-b)^2+4ab][(a-c)^2+4ac] \\ &\geq 4ac(a-b)^2+4ab(a-c)^2+16a^2bc,\end{aligned}$$

it suffices to show that

$$\frac{(a-b)^2}{ab}+\frac{(a-c)^2}{ac}+4 \geq \frac{(2a+b+c)^2}{(a+b)(a+c)},$$

which is equivalent to

$$\frac{(a-b)^2}{ab}+\frac{(a-c)^2}{ac} \geq \frac{(b-c)^2}{(a+b)(a+c)}.$$

Indeed, by the Cauchy-Schwarz inequality, we have

$$\frac{(a-b)^2}{ab}+\frac{(a-c)^2}{ac} \geq \frac{(a-b-a+c)^2}{ab+ac} \geq \frac{(b-c)^2}{(a+b)(a+c)}.$$

The equality holds for $a=b=c$.

□

P 2.14. *If a,b,c are positive real numbers such that $abc=1$, then*

(a)
$$\frac{a}{b}+\frac{b}{c}+\frac{1}{a} \geq a+b+1;$$

(b)
$$\frac{a}{b}+\frac{b}{c}+\frac{1}{a} \geq \sqrt{3(a^2+b^2+1)}.$$

(Vasile C., 2007)

Solution. (a) ***First Solution.*** Write the inequality as

$$\left(2\frac{a}{b}+\frac{b}{c}\right)+\left(\frac{b}{c}+\frac{1}{a}\right)+\left(\frac{1}{a}+a\right) \geq 3a+2b+2.$$

By the AM-GM inequality, we have

$$\left(2\frac{a}{b}+\frac{b}{c}\right)+\left(\frac{b}{c}+\frac{1}{a}\right)+\left(\frac{1}{a}+a\right) \geq 3\sqrt[3]{\frac{a^2}{bc}}+2\sqrt{\frac{b}{ca}}+2=3a+2b+2.$$

The equality holds for $a=b=c=1$.

Second Solution. Since $c = \dfrac{1}{ab}$, the inequality becomes as follows:

$$\frac{a}{b} + ab^2 + \frac{1}{a} \ge a + b + 1,$$

$$\frac{1}{b} + b^2 + \frac{1}{a^2} \ge 1 + \frac{b}{a} + \frac{1}{a},$$

$$\frac{1}{a^2} - (b+1)\frac{1}{a} + b^2 + \frac{1}{b} - 1 \ge 0,$$

$$\left(\frac{1}{a} - \frac{b+1}{2}\right)^2 + \frac{(b-1)^2(3b+4)}{4b} \ge 0.$$

(b) Write the inequality as

$$a\left(\frac{1}{b} + b^2\right) + \frac{1}{a} \ge \sqrt{3(a^2+b^2+1)}.$$

By squaring, this inequality becomes

$$a^2\left(b^4 + 2b - 3 + \frac{1}{b^2}\right) + \frac{1}{a^2} \ge b^2 + 3 - \frac{2}{b}.$$

Since

$$b^4 + 2b - 3 + \frac{1}{b^2} > 2b - 3 + \frac{1}{b^2} = \frac{(b-1)^2(2b+1)}{b^2} \ge 0,$$

by the AM-GM inequality, we have

$$a^2\left(b^4 + 2b - 3 + \frac{1}{b^2}\right) + \frac{1}{a^2} \ge 2\sqrt{b^4 + 2b - 3 + \frac{1}{b^2}}.$$

Thus, it suffices to prove that

$$2\sqrt{b^4 + 2b - 3 + \frac{1}{b^2}} \ge b^2 + 3 - \frac{2}{b}.$$

Squaring again, we get the inequality

$$b^5 - 2b^3 + 4b^2 - 7b + 4 \ge 0,$$

which is equivalent to the obvious inequality

$$b(b^2-1)^2 + 4(b-1)^2 \ge 0.$$

The equality holds for $a = b = c = 1$.

□

P 2.15. *If a, b, c are positive real numbers such that $abc \ge 1$, then*

$$a^{\frac{a}{b}} b^{\frac{b}{c}} c^c \ge 1.$$

(Vasile C., 2011)

Solution. Write the inequality as

$$\frac{a}{b}\ln a + \frac{b}{c}\ln b + c\ln c \ge 0.$$

Since $f(x) = x\ln x$ is a convex function on $(0, \infty)$, apply Jensen's inequality to get

$$pa\ln a + qb\ln b + rc\ln c \ge (p+q+r)\left(\frac{pa+qb+rc}{p+q+r}\right)\ln\left(\frac{pa+qb+rc}{p+q+r}\right)$$

$$= (pa+qb+rc)\ln\left(\frac{pa+qb+rc}{p+q+r}\right),$$

where $p, q, r > 0$. Choosing

$$p = \frac{1}{b}, \quad q = \frac{1}{c}, \quad r = 1,$$

we get

$$\frac{a}{b}\ln a + \frac{b}{c}\ln b + c\ln c \ge (\frac{a}{b} + \frac{b}{c} + c)\ln\left(\frac{\frac{a}{b} + \frac{b}{c} + c}{\frac{1}{b} + \frac{1}{c} + 1}\right).$$

Thus, it suffices to show that

$$\frac{a}{b} + \frac{b}{c} + c \ge \frac{1}{b} + \frac{1}{c} + 1.$$

Since $a \ge \frac{1}{bc}$, we need to show that

$$\frac{1}{b^2c} + \frac{b}{c} + c \ge \frac{1}{b} + \frac{1}{c} + 1.$$

This is equivalent to

$$\frac{1}{b^2} + b + c^2 \ge \frac{c}{b} + 1 + c,$$

$$c^2 - \left(1 + \frac{1}{b}\right)c + b - 1 + \frac{1}{b^2} \ge 0,$$

$$\left(c - \frac{b+1}{2b}\right)^2 + \frac{(b-1)^2(4b+3)}{4b^2} \ge 0.$$

The equality holds for $a = b = c = 1$.

□

P 2.16. *If a, b, c are positive real numbers such that $ab+bc+ca=3$, then*

$$ab^2c^3<4.$$

(Vasile C., 2012)

Solution. From $ab+bc+ca=3$, we get

$$c=\frac{3-ab}{a+b}<\frac{3}{a+b}.$$

Therefore,

$$\begin{aligned}(a+b)^3(4-ab^2c^3)&>4(a+b)^3-27ab^2\\&=4a^3+12a^2b-15ab^2+4b^3\\&=(a+4b)(2a-b)^2\geq 0.\end{aligned}$$

□

P 2.17. *If a, b, c are positive real numbers such that $ab+bc+ca=\frac{5}{3}$, then*

$$ab^2c^2\leq\frac{1}{3}.$$

(Vasile C., 2012)

Solution. By the AM-GM inequality, we have

$$ab+ca\geq 2a\sqrt{bc}.$$

Thus, from $ab+bc+ca=\frac{5}{3}$, we get

$$2a\sqrt{bc}+bc\leq\frac{5}{3}.$$

Therefore, it suffices to show that

$$\frac{(5-3bc)b^2c^2}{6\sqrt{bc}}\leq\frac{1}{3}.$$

Setting $\sqrt{bc}=t$, this inequality becomes

$$3t^5-5t^3+2\geq 0.$$

Indeed, be the AM-GM inequality, we have

$$3t^5+2=t^5+t^5+t^5+1+1\geq 5\sqrt[5]{t^5\cdot t^5\cdot t^5\cdot 1\cdot 1}=5t^3.$$

The equality holds for $a=\frac{1}{3}$ and $b=c=1$.

□

P 2.18. *Let a, b, c be positive real numbers such that*

$$a \le b \le c, \qquad ab + bc + ca = 3.$$

Prove that

(a) $$ab^2c \le \frac{9}{8};$$

(b) $$ab^4c \le 2;$$

(c) $$ab^3c^2 \le 2.$$

(Vasile C., 2012)

Solution. From $(b-a)(b-c) \le 0$, we get

$$b^2 + ac \le b(a+c),$$

$$b^2 + ac \le 3 - ac,$$

$$b^2 + 2ac \le 3.$$

(a) We have

$$9 - 8ab^2c \ge 9 - 4b^2(3 - b^2) = (2b^2 - 3)^2 \ge 0.$$

The equality holds for $a = \frac{1}{2}\sqrt{\frac{3}{2}}$ and $b = c = \sqrt{\frac{3}{2}}$.

(b) We have

$$4 - 2ab^4c \ge 4 - b^4(3 - b^2) = (b^2 - 2)^2(b^2 + 1) \ge 0.$$

The equality holds for $a = \frac{\sqrt{2}}{4}$ and $b = c = \sqrt{2}$.

(c) Write the desired inequality as follows:

$$2(ab + bc + ca)^3 \ge 27ab^3c^2,$$

$$2\left(a + c + \frac{ca}{b}\right)^3 \ge 27ac^2.$$

Since $ca/b \ge a$, it suffices to show that

$$2(2a + c)^3 \ge 27ac^2,$$

which is equivalent to the obvious inequality

$$(a + 2c)(4a - c)^2 \ge 0.$$

The equality holds for $a = \frac{\sqrt{2}}{4}$ and $b = c = \sqrt{2}$.

□

P 2.19. *Let a,b,c be positive real numbers such that*

$$a \le b \le c, \qquad a+b+c = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Prove that

$$b \ge \frac{1}{a+c-1}.$$

(Vasile C., 2007)

Solution. Let us show that

$$a \le 1, \qquad c \ge 1.$$

From $a+b+c = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ and

$$a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-6 = \frac{(a-1)^2}{a}+\frac{(b-1)^2}{b}+\frac{(c-1)^2}{c} \ge 0,$$

we get

$$a+b+c = \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \ge 3.$$

Then,

$$\frac{1}{a} \ge \frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \ge 1, \qquad c \ge \frac{a+b+c}{3} \ge 1.$$

Further, consider the following two cases.

Case 1: $abc \ge 1$. Write the desired inequality as

$$a+c-1-\frac{1}{b} \ge 0.$$

We have

$$a+c-1-\frac{1}{b} = (1-a)(c-1)+\frac{abc-1}{b} \ge 0.$$

Case 2: $abc \le 1$. Since

$$a+c-1-\frac{1}{b} = \frac{1}{a}+\frac{1}{c}-1-b,$$

the desired inequality is equivalent to

$$\frac{1}{a}+\frac{1}{c}-1-b \ge 0.$$

We have

$$\frac{1}{a}+\frac{1}{c}-1-b = \left(\frac{1}{a}-1\right)\left(1-\frac{1}{c}\right)+\frac{1-abc}{ac} \ge 0.$$

This completes the proof. The equality holds for $a=b=c=1$.

□

P 2.20. *Let a, b, c be positive real numbers such that*

$$a \le b \le c, \qquad a+b+c = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Prove that

$$ab^2c^3 \ge 1.$$

(Vasile C., 1998)

First Solution. Write the inequality in the homogeneous form

$$ab^2c^3 \ge \left[\frac{abc(a+b+c)}{ab+bc+ca}\right]^3,$$

which is equivalent to

$$(ab+bc+ca)^3 \ge a^2b(a+b+c)^3.$$

Since

$$(ab+bc+ca)^2 \ge 3abc(a+b+c),$$

it suffices to show that

$$3c(ab+bc+ca) \ge a(a+b+c)^2.$$

Indeed,

$$3c(ab+bc+ca)-a(a+b+c)^2 \ge (a+b+c)(ab+bc+ca)-a(a+b+c)^2$$

$$=(a+b+c)(bc-a^2) \ge 0.$$

The equality holds for $a=b=c=1$.

Second Solution. Let us show that

$$a \le 1, \qquad bc \ge 1.$$

Indeed, if $a>1$, then $1<a \le b \le c$ and

$$a+b+c-\frac{1}{a}-\frac{1}{b}-\frac{1}{c} = \frac{1-a^2}{a}+\frac{1-b^2}{b}+\frac{1-c^2}{c} < 0,$$

which is false. On the other hand, from $a \le 1$ and

$$a-\frac{1}{a} = (b+c)\left(\frac{1}{bc}-1\right),$$

we get $bc \ge 1$. Similarly, we can prove that

$$c \ge 1, \qquad ab \le 1.$$

Since $bc \ge 1$, it suffices to show that

$$abc^2 \ge 1.$$

Taking account of $ab \le 1$, we have

$$c - \frac{1}{c} = (a+b)\left(\frac{1}{ab} - 1\right) \ge 2\sqrt{ab}\left(\frac{1}{ab} - 1\right) = 2\left(\frac{1}{\sqrt{ab}} - \sqrt{ab}\right) \ge \frac{1}{\sqrt{ab}} - \sqrt{ab},$$

hence

$$\left(c - \frac{1}{\sqrt{ab}}\right)\left(1 + \frac{\sqrt{ab}}{c}\right) \ge 0.$$

The last inequality involves

$$abc^2 \ge 0.$$

□

P 2.21. *Let a, b, c be positive real numbers such that*

$$a \le b \le c, \qquad a + b + c = abc + 2.$$

Prove that

$$(1-b)(1-ab^3c) \ge 0.$$

(Vasile C., 1999)

Solution. Let us show that

$$a \le 1, \qquad c \ge 1.$$

To do this, we write the hypothesis $a + b + c = abc + 2$ in the equivalent form

$$(1-a)(1-c) + (1-b)(1-ac) = 0, \tag{*}$$

If $a > 1$, then $1 < a \le b \le c$, which contradicts (*). Similarly, if $c < 1$, then $a \le b \le c < 1$, which also contradicts (*). Therefore, we have $a \le 1$ and $c \ge 1$. According to (*), we get

$$(1-b)(1-ac) = (1-a)(c-1) \ge 0. \tag{**}$$

There are two cases to consider.

Case 1: $b \ge 1$. According to (**), we have $ac \ge 1$. Therefore,

$$ab^3c = ac \cdot b^3 \ge 1,$$

hence $(1-b)(1-ab^3c) \ge 0$.

Case 2: $b \le 1$. According to (**), we have $ac \le 1$. Therefore,

$$ab^3c = ac \cdot b^3 \le 1,$$

and hence

$$(1-b)(1-ab^3c) \ge 0.$$

This completes the proof. The equality holds for $a = b = 1 \le c$ or $a \le 1 = b = c$.

□

P 2.22. *Let a,b,c be real numbers, no two of which are zero. Prove that*

(a) $$\frac{(a-b)^2}{a^2+b^2}+\frac{(a-c)^2}{a^2+c^2}\ge\frac{(b-c)^2}{2(b^2+c^2)};$$

(b) $$\frac{(a+b)^2}{a^2+b^2}+\frac{(a+c)^2}{a^2+c^2}\ge\frac{(b-c)^2}{2(b^2+c^2)}.$$

Solution. (a) Consider two cases.
Case 1: $2a^2\le b^2+c^2$. By the Cauchy-Schwarz inequality, we have

$$\frac{(a-b)^2}{a^2+b^2}+\frac{(a-c)^2)}{a^2+c^2}\ge\frac{[(b-a)+(a-c)]^2}{(a^2+b^2)+(a^2+c^2)}=\frac{(b-c)^2}{2a^2+b^2+c^2}.$$

Thus, it suffices to show that

$$\frac{1}{2a^2+b^2+c^2}\ge\frac{1}{2(b^2+c^2)},$$

which reduces to $b^2+c^2\ge 2a^2$.
Case 2: $2a^2\ge b^2+c^2$. By the Cauchy-Schwarz inequality, we have

$$\frac{(a-b)^2}{a^2+b^2}+\frac{(a-c)^2}{a^2+c^2}\ge\frac{[c(b-a)+b(a-c)]^2}{c^2(a^2+b^2)+b^2(a^2+c^2)}=\frac{a^2(b-c)^2}{a^2(b^2+c^2)+2b^2c^2}.$$

Therefore, it suffices to prove that

$$\frac{a^2}{a^2(b^2+c^2)+2b^2c^2}\ge\frac{1}{2(b^2+c^2)},$$

which reduces to $a^2(b^2+c^2)\ge 2b^2c^2$. This is true since

$$2a^2(b^2+c^2)-4b^2c^2\ge(b^2+c^2)^2-4b^2c^2=(b^2-c^2)^2.$$

The equality holds for $a=b=c$.

(b) The inequality follows from the inequality in (a) by replacing a with $-a$. The equality holds for $-a=b=c$.

□

P 2.23. *Let a,b,c be real numbers, no two of which are zero. If $bc\ge 0$, then*

(a) $$\frac{(a-b)^2}{a^2+b^2}+\frac{(a-c)^2}{a^2+c^2}\ge\frac{(b-c)^2}{(b+c)^2};$$

(b) $$\frac{(a+b)^2}{a^2+b^2}+\frac{(a+c)^2}{a^2+c^2}\ge\frac{(b-c)^2}{(b+c)^2}.$$

(Vasile C., 2011)

Solution. (a) Consider two cases: $a^2 \le bc$ and $a^2 \ge bc$.
Case 1: $a^2 \le bc$. By the Cauchy-Schwarz inequality, we have

$$\frac{(a-b)^2}{a^2+b^2}+\frac{(a-c)^2)}{a^2+c^2} \ge \frac{[(b-a)+(a-c)]^2}{(a^2+b^2)+(a^2+c^2)} = \frac{(b-c)^2}{2a^2+b^2+c^2}.$$

Thus, it suffices to show that

$$\frac{1}{2a^2+b^2+c^2} \ge \frac{1}{(b+c)^2},$$

which is equivalent to $a^2 \le bc$.
Case 2: $a^2 \ge bc$. By the Cauchy-Schwarz inequality, we have

$$\frac{(a-b)^2}{a^2+b^2}+\frac{(a-c)^2}{a^2+c^2} \ge \frac{[c(b-a)+b(a-c)]^2}{c^2(a^2+b^2)+b^2(a^2+c^2)} = \frac{a^2(b-c)^2}{a^2(b^2+c^2)+2b^2c^2}.$$

Therefore, it suffices to prove that

$$\frac{a^2}{a^2(b^2+c^2)+2b^2c^2} \ge \frac{1}{(b+c)^2},$$

which reduces to $bc(a^2-bc) \ge 0$. The equality holds for $a=b=c$, for $b=0$ and $a=c$, and for $c=0$ and $a=b$.

(b) The inequality follows from the inequality in (a) by replacing a with $-a$. The equality holds for $-a=b=c$, for $b=0$ and $a+c=0$, and for $c=0$ and $a+b=0$.

□

P 2.24. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{|a-b|^3}{a^3+b^3}+\frac{|a-c|^3}{a^3+c^3} \ge \frac{|b-c|^3}{(b+c)^3}.$$

(Vasile C., 2013)

Solution. Without loss of generality, assume that $b \ge c$. Thus, we have three cases to consider: $a \ge b \ge c$, $b \ge c \ge a$ and $b \ge a \ge c$.
Case 1: $a \ge b \ge c$. It suffices to show that

$$\frac{|a-c|^3}{(a+c)^3} \ge \frac{|b-c|)^3}{(b+c)^3},$$

which is equivalent to

$$\frac{a-c}{a+c} \ge \frac{b-c}{b+c}.$$

Indeed,
$$\frac{a-c}{a+c}-\frac{b-c}{b+c}=\frac{2c(a-b)}{(a+c)(b+c)}\geq 0.$$
Case 2: $b\geq c\geq a$. It suffices to show that
$$\frac{(b-a)^3}{a^3+b^3}\geq\frac{(b-c)^3}{(b+c)^3}.$$
Indeed,
$$\frac{(b-a)^3}{a^3+b^3}\geq\frac{(b-c)^3}{a^3+b^3}\geq\frac{(b-c)^3}{b^3+c^3}\geq\frac{b-c)^3}{(b+c)^3}.$$
Case 3: $b\geq a\geq c$. We need to prove that
$$\frac{(b-a)^3}{a^3+b^3}+\frac{(a-c)^3}{a^3+c^3}\geq\frac{(b-c)^3}{(b+c)^3}.$$
Using the substitution
$$x=\frac{b-a}{a+b},\quad y=\frac{a-c}{a+c},\quad 0\leq x<1,\quad 0\leq y\leq 1,$$
we have
$$b=\frac{1+x}{1-x}a,\quad c=\frac{1-y}{1+y}a,$$
$$(b-a)^3=\frac{8x^3}{(1-x)^3}a^3,\quad (a-c)^3=\frac{8y^3}{(1+y)^3}a^3,$$
$$a^3+b^3=\frac{2(1+3x^3)}{(1-x)^3},\quad a^3+c^3=\frac{2(1+3y^2)}{(1+y)^3},$$
$$\frac{b-c}{b+c}=\frac{x+y}{1+xy}.$$
Thus, the desired inequality becomes
$$\frac{4x^3}{1+3x^2}+\frac{4y^3}{1+3y^2}\geq\frac{(x+y)^3}{(1+xy)^3},$$
$$\frac{x^2+y^2-xy+3x^2y^2}{(1+3x^2)(1+3y^2)}\geq\frac{(x+y)^2}{4(1+xy)^3},$$
$$\frac{s-p+3p^2}{1+3s+9p^2}\geq\frac{s+2p}{4(1+p)^3},$$
where
$$s=x^2+y^2,\quad p=xy,\quad 0\leq p<1,\quad 2p\leq s\leq 1+p^2.$$
Therefore, we need to show that $f(s)\geq 0$, where
$$f(s)=4(1+p)^3(s-p+3p^2)-(s+2p)(3s+1+9p^2).$$

Since f is a concave function, it suffices to show that $f(2p) \geq 0$ and $f(1+p^2) \geq 0$. Indeed, we have

$$f(2p) = 4p^3(3p+1)(p+3) \geq 0,$$

$$f(1+p^2) = 16p^3(p+1)^2 \geq 0.$$

Thus, the proof is completed. The equality holds for $a = b = c$, for $b = 0$ and $a = c$, and for $c = 0$ and $a = b$.

□

P 2.25. *Let a,b,c be positive real numbers, $b \neq c$. Prove that*

$$\frac{ab}{(a+b)^2} + \frac{ac}{(a+c)^2} \leq \frac{(b+c)^2}{4(b-c)^2}.$$

(Vasile C., 2010)

Solution. Write the inequality in the form

$$\frac{(a-b)^2}{(a+b)^2} + \frac{(a-c)^2}{(a+c)^2} + \frac{(b+c)^2}{(b-c)^2} \geq 2.$$

Replacing a be $-a$, the inequality becomes

$$\frac{(a+b)^2}{(a-b)^2} + \frac{(a+c)^2}{(a-c)^2} + \frac{(b+c)^2}{(b-c)^2} \geq 2. \tag{*}$$

Making the substitution

$$x = \frac{a+b}{a-b}, \quad y = \frac{b+c}{b-c}, \quad z = \frac{c+a}{c-a},$$

we can write the inequality as

$$x^2 + y^2 + z^2 \geq 2.$$

From

$$x+1 = \frac{2a}{a-b}, \quad y+1 = \frac{2b}{b-c}, \quad z+1 = \frac{2c}{c-a}$$

and

$$x-1 = \frac{2b}{a-b}, \quad y-1 = \frac{2c}{b-c}, \quad z-1 = \frac{2a}{c-a},$$

we get

$$(x+1)(y+1)(z+1) = (x-1)(y-1)(z-1),$$

$$xy + yz + zx + 1 = 0.$$

Therefore, we have

$$x^2 + y^2 + z^2 - 2 = x^2 + y^2 + z^2 + 2(xy + yz + zx) = (x+y+z)^2 \geq 0.$$

The inequality (*) is an equality for $x + y + z = 0$; that is,

$$(a+b+c)(ab+bc+ca)-9abc=0.$$

Therefore, the original inequality is an equality for

$$(b+c-a)(bc-ab-ac)+9abc=0.$$

□

P 2.26. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{3bc+a^2}{b^2+c^2} \ge \frac{3ab-c^2}{a^2+b^2} + \frac{3ac-b^2}{a^2+c^2}.$$

(Vasile C., 2014)

Solution *(by Nguyen Van Quy)*. Write the inequality as

$$\frac{a^2}{b^2+c^2} + \frac{b^2}{a^2+c^2} + \frac{c^2}{a^2+b^2} + \frac{3bc}{b^2+c^2} \ge \frac{3ab}{a^2+b^2} + \frac{3ac}{a^2+c^2}.$$

By the Cauchy-Schwarz inequality, we have

$$\frac{b^2}{a^2+c^2} + \frac{c^2}{a^2+b^2} \ge \frac{(b^2+c^2)^2}{b^2(a^2+c^2)+c^2(a^2+b^2)} = \frac{(b^2+c^2)^2}{a^2(b^2+c^2)+2b^2c^2}$$
$$\ge 2 - \frac{a^2(b^2+c^2)+2b^2c^2}{(b^2+c^2)^2} = 2 - \frac{a^2}{b^2+c^2} - \frac{2b^2c^2}{(b^2+c^2)^2},$$

hence

$$\frac{a^2}{b^2+c^2} + \frac{b^2}{a^2+c^2} + \frac{c^2}{a^2+b^2} \ge 2 - \frac{2b^2c^2}{(b^2+c^2)^2}.$$

Therefore, it suffices to show that

$$2 - \frac{2b^2c^2}{(b^2+c^2)^2} + \frac{3bc}{b^2+c^2} \ge \frac{3ab}{a^2+b^2} + \frac{3ac}{a^2+c^2}.$$

This inequality is equivalent to

$$\left[\frac{1}{2} - \frac{2b^2c^2}{(b^2+c^2)^2}\right] + \left(\frac{3}{2} - \frac{3ab}{a^2+b^2}\right) + \left(\frac{3}{2} - \frac{3ac}{a^2+c^2}\right) \ge \left(\frac{3}{2} - \frac{3bc}{b^2+c^2}\right),$$

$$\frac{(b^2-c^2)^2}{3(b^2+c^2)^2} + \frac{(a-b)^2}{a^2+b^2} + \frac{(a-c)^2}{a^2+c^2} \ge \frac{(b-c)^2}{b^2+c^2}.$$

Using the inequality in P 2.23-(a), namely

$$\frac{(a-b)^2}{a^2+b^2} + \frac{(a-c)^2}{a^2+c^2} \ge \frac{(b-c)^2}{(b+c)^2},$$

it is enough to prove that

$$\frac{(b+c)^2}{3(b^2+c^2)^2}+\frac{1}{(b+c)^2}\geq\frac{1}{b^2+c^2},$$

which is equivalent to

$$\frac{1}{(b+c)^2}\geq\frac{2(b^2-bc+c^2)}{3(b^2+c^2)^2}.$$

We have

$$\begin{aligned}3(b^2+c^2)^2-2(b+c)^2(b^2-bc+c^2)&=3(b^2+c^2)^2-2(b+c)(b^3+c^3)\\&=b^4+c^4+6b^2c^2-2bc(b^2+c^2)\\&\geq(b^2+c^2)^2-2bc(b^2+c^2)\\&=(b^2+c^2)(b-c)^2\geq 0.\end{aligned}$$

The equality holds for $a=b=c$.

□

P 2.27. *Let a,b,c be nonnegative real numbers such that $a+b>0$. Prove that*

$$abc\geq(b+c-a)(c+a-b)(a+b-c)+\frac{ab(a-b)^2}{a+b}.$$

(Vasile C., 2011)

Solution. Since

$$(b+c-a)(c+a-b)(a+b-c)=\frac{2(a^2b^2+b^2c^2+c^2a^2)-a^4-b^4-c^4}{a+b+c},$$

we can rewrite the inequality as

$$a^4+b^4+c^4+abc(a+b+c)\geq 2(a^2b^2+b^2c^2+c^2a^2)+\frac{ab(a+b+c)(a-b)^2}{a+b}.$$

By Schur's inequality of fourth degree, we have

$$a^4+b^4+c^4+abc(a+b+c)\geq\sum ab(a^2+b^2).$$

Therefore, it suffices to prove that

$$\sum ab(a^2+b^2)\geq 2(a^2b^2+b^2c^2+c^2a^2)+\frac{ab(a+b+c)(a-b)^2}{a+b},$$

which is equivalent to

$$\sum ab(a-b)^2\geq\frac{ab(a+b+c)(a-b)^2}{a+b},$$

or

$$bc(b-c)^2 + ca(c-a)^2 \ge \frac{abc(a-b)^2}{a+b}.$$

This inequality follows immediately from the Cauchy-Schwarz inequality

$$(a+b)[bc(b-c)^2 + ca(c-a)^2] \ge [\sqrt{abc}(b-c) + \sqrt{abc}(c-a)]^2.$$

The equality holds for $a = b = c$, and for $a = 0$ and $b = c$ (or any cyclic permutation).

□

P 2.28. *Let a, b, c be nonnegative real numbers such that $a \ge b \ge c$. Prove that*

(a) $$abc \ge (b+c-a)(c+a-b)(a+b-c) + \frac{2ab(a-b)^2}{a+b};$$

(b) $$abc \ge (b+c-a)(c+a-b)(a+b-c) + \frac{27b(a-b)^4}{4a^2}.$$

(Vasile C., 2011)

Solution. (a) Write the inequality as

$$\sum a(a-b)(a-c) \ge \frac{2ab(a-b)^2}{a+b}.$$

Since

$$c(c-a)(c-b) \ge 0,$$

it suffices to show that

$$a(a-b)(a-c) + b(b-c)(c-a) \ge \frac{2ab(a-b)^2}{a+b}.$$

Since

$$a(a-b)(a-c) = a(a-b)[(a-b)+(b-c)] = a(a-b)^2 + a(a-b)(b-c)$$

$$\ge \frac{2ab(a-b)^2}{a+b} + a(a-b)(b-c),$$

it suffices to show that

$$a(a-b)(b-c) + b(b-c)(b-a) \ge 0.$$

This inequality is equivalent to

$$(a-b)^2(b-c) \ge 0.$$

The equality holds for $a = b = c$, and for $a = b$ and $c = 0$.

(b) Write the inequality as

$$\sum a(a-b)(a-c) \geq \frac{27b(a-b)^4}{4a^2}.$$

Since

$$c(c-a)(c-b) \geq 0,$$

it suffices to show that

$$a(a-b)(a-c)+b(b-c)(c-a) \geq \frac{27b(a-b)^4}{4a^2},$$

which is equivalent to

$$a(a-b)^2+a(a-b)(b-c)+b(b-c)(c-a) \geq \frac{27b(a-b)^4}{4a^2}.$$

Since

$$a(a-b)^2-\frac{27b(a-b)^4}{4a^2}=\frac{(a-b)^2(a-3b)^2}{4a^2},$$

it suffices to show that

$$a(a-b)(b-c)+b(b-c)(b-a) \geq 0.$$

This inequality is equivalent to

$$(a-b)^2(b-c) \geq 0.$$

The equality holds for $a=b=c$, and for $a/3=b=c$.

□

P 2.29. *Let a,b,c be nonnegative real numbers such that $a+b>0$. Prove that*

$$\sum a^2(a-b)(a-c) \geq a^2b^2\left(\frac{a-b}{a+b}\right)^2.$$

(Vasile C., 2011)

Solution. Without loss of generality, assume that $a \geq b$. There three cases to consider.

Case 1. $c \geq a \geq b$. Since

$$\begin{aligned} a^2(a-b)(a-c)+c^2(c-a)(c-b) &\geq a^2(a-b)(a-c)+c^2(c-a)(a-b) \\ &= (a-b)(c-a)^2(c+a) \geq 0, \end{aligned}$$

it suffices to show that

$$b^2(a-b)(c-b) \ge a^2b^2\left(\frac{a-b}{a+b}\right)^2.$$

Since $c-b \ge a-b$, this is true if

$$1 \ge \left(\frac{a}{a+b}\right)^2,$$

which is true.

Case 2. $a \ge b \ge c$. Since

$$c^2(c-a)(c-b) \ge 0$$

and

$$a^2(a-b)(a-c)+b^2(b-c)(b-a) = (a-b)^2[a^2+ab+b^2-c(a+b)]$$

$$\ge (a-b)^2[a^2+ab+b^2-b(a+b)] = a^2(a-b)^2,$$

it suffices to show that

$$1 \ge \left(\frac{b}{a+b}\right)^2,$$

which is true.

Case 3. $a \ge c \ge b$. Since

$$b^2(b-c)(b-a) \ge b^2(c-b)^2$$

and

$$a^2(a-b)(a-c)+c^2(c-a)(c-b) = (a-c)^2[a^2+ac+c^2-b(a+c)]$$

$$\ge (a-c)^2[a^2+ac+c^2-c(a+c)] = a^2(a-c)^2,$$

it suffices to show that

$$b^2(c-b)^2+a^2(a-c)^2 \ge a^2b^2\left(\frac{a-b}{a+b}\right)^2.$$

By the Cauchy-Schwarz inequality, we have

$$\left(\frac{1}{b^2}+\frac{1}{a^2}\right)\left[b^2(c-b)^2+a^2(a-c)^2\right] \ge [(c-b)+(a-c)]^2 = (a-b)^2.$$

Therefore, it suffices to prove that

$$\frac{a^2b^2(a-b)^2}{a^2+b^2} \ge a^2b^2\left(\frac{a-b}{a+b}\right)^2,$$

which is clearly true.

This completes the proof. The equality holds for $a=b=c$, and for $a=0$ and $b=c$ (or any cyclic permutation).

Remark. Similarly, we can prove the following generalization.

• *Let a,b,c be nonnegative real numbers such that $a+b>0$. If k is a positive natural number, then*

$$\sum a^k(a-b)(a-c) \ge \left(\frac{ab}{a+b}\right)^k (a-b)^2.$$

□

P 2.30. *Let a,b,c be nonnegative real numbers such that $a+b+c=3$. Prove that*

$$ab^2+bc^2+2ca^2 \le 8.$$

Solution. Since the equality holds for $a=2$, $b=0$, $c=1$, we apply the AM-GM inequality to get

$$\frac{ca^2}{4} = c\cdot\frac{a}{2}\cdot\frac{a}{2} \le \frac{1}{27}\left(c+\frac{a}{2}+\frac{a}{2}\right)^3 = \frac{1}{27}(c+a)^3 \le \frac{1}{27}(a+b+c)^3 = 1.$$

Therefore, it suffices to show that

$$ab^2+bc^2+ca^2 \le 4,$$

which is the inequality in P 1.1.

□

P 2.31. *Let a,b,c be nonnegative real numbers such that $a+b+c=3$. Prove that*

$$ab^2+bc^2+\frac{3}{2}abc \le 4.$$

(Vasile Cîrtoaje and Vo Quoc Ba Can, 2007)

Solution. Consider two cases.

Case 1: $c \ge 2b$. We have

$$ab^2+bc^2+\frac{3}{2}abc = b(a+c)^2 - ab\left(a-b+\frac{c}{2}\right) \le b(a+c)^2$$

$$= 4b\left(\frac{a+c}{2}\right)\left(\frac{a+c}{2}\right) \le 4\left(\frac{b+\frac{a+c}{2}+\frac{a+c}{2}}{3}\right)^3 = 4.$$

Case 2: $2b > c$. Write the desired inequality as $f(a) \geq 0$, where

$$f(a) = 4\left(\frac{a+b+c}{3}\right)^3 - ab^2 - bc^2 - \frac{3}{2}abc,$$

with the derivative

$$f'(a) = 4\left(\frac{a+b+c}{3}\right)^2 - b^2 - \frac{3}{2}bc.$$

The equation $f'(a) = 0$ has the positive root

$$a_1 = \frac{3}{2}\sqrt{\frac{b(2b+3c)}{2}} - b - c = \frac{(2b-c)(5b+8c)}{6\sqrt{2b(2b+c)} + 8(b+c)}.$$

Since $f'(a) < 0$ for $0 \leq a < a_1$ and $f'(a) > 0$ for $a > a_1$, $f(a)$ is decreasing on $[0, a_1]$ and increasing on $[a_1, \infty)$; consequently, $f(a) \geq f(a_1)$. To complete the proof, it suffices to show that $f(a_1) \geq 0$. Indeed, since

$$4\left(\frac{a_1+b+c}{3}\right)^2 = b^2 + \frac{3}{2}bc,$$

we have

$$\begin{aligned}
f(a_1) &= 4\left(\frac{a_1+b+c}{3}\right)^3 - a_1\left(b^2 + \frac{3}{2}bc\right) - bc^2 \\
&= \frac{a_1+b+c}{3}\left(b^2 + \frac{3}{2}bc\right) - a_1\left(b^2 + \frac{3}{2}bc\right) - bc^2 \\
&= \frac{b+c-2a_1}{3}\left(b^2 + \frac{3}{2}bc\right) - bc^2 \\
&= \left(b+c-\sqrt{\frac{2b^2+3bc}{2}}\right)\left(b^2 + \frac{3}{2}bc\right) - bc^2 \\
&= \frac{b}{4}\left[4b^2 + 10bc + 2c^2 - (2b+3c)\sqrt{2b(2b+3c)}\right] \\
&= \frac{bc(2b-c)^2(b+2c)}{2[4b^2 + 10bc + 2c^2 + (2b+3c)\sqrt{2b(2b+3c)}]} \geq 0.
\end{aligned}$$

Thus, the proof is completed. The equality holds for $a = 0$, $b = 1$, $c = 2$, and for $a = 1$, $b = 2$, $c = 0$.

□

P 2.32. *Let a, b, c be nonnegative real numbers such that $a + b + c = 5$. Prove that*

$$ab^2 + bc^2 + 2abc \leq 20.$$

(Vo Quoc Ba Can, 2011)

Solution. Write the inequality as

$$b(ab+c^2+2ac) \le 20.$$

We see that the equality holds for $a=1$ and $b=c=2$. From $(a-b/2)^2 \ge 0$, it follows that

$$ab \le a^2+\frac{b^2}{4}.$$

Therefore, for $b \le 4$, we have

$$\begin{aligned} b(ab+c^2+2ac)-20 &\le b\left(a^2+\frac{b^2}{4}+c^2+2ac\right)-20 = b\left[(a+c)^2+\frac{b^2}{4}\right]-20 \\ &= b\left[(5-b)^2+\frac{b^2}{4}\right]-20 = \frac{5}{4}(b-4)(b-2)^2 \le 0. \end{aligned}$$

Consider now that $b>4$. Since

$$a=5-b-c \le 5-b,$$

We have

$$\begin{aligned} ab^2+bc^2+2abc-20 &= ab^2+b(5-a-b)^2+2ab(5-a-b)-20 \\ &= b^3+ab^2-10b^2-a^2b+25b-20 \\ &\le b^3+ab^2-10b^2+25b-20 \\ &\le b^3+(5-b)b^2-10b^2+25b-20 \\ &= -5(b-4)(b-1) < 0. \end{aligned}$$

□

P 2.33. *If a,b,c are nonnegative real numbers, then*

$$a^3+b^3+c^3-a^2b-b^2c-c^2a \ge \frac{8}{9}(a-b)(b-c)^2.$$

Solution. Since

$$3(a^3+b^3+c^3-a^2b-b^2c-c^2a) = \sum(2a^3-3a^2b+b^3) = \sum(2a+b)(a-b)^2,$$

we can write the inequality as

$$(2a+b)(a-b)^2+(2b+c)(b-c)^2+(2c+a)(c-a)^2 \ge \frac{8}{3}(a-b)(b-c)^2.$$

If $a \le b$, then

$$(2a+b)(a-b)^2+(2b+c)(b-c)^2+(2c+a)(c-a)^2 \ge 0 \ge \frac{8}{3}(a-b)(b-c)^2.$$

If $a \geq b$, then there are two cases to consider: $b \geq c$ and $b \leq c$.
Case 1: $a \geq b \geq c$. It suffices to show that

$$(2c+a)(a-c)^2 \geq \frac{8}{3}(a-b)(b-c)^2.$$

By the AM-GM inequality, we have

$$\begin{aligned}(a-b)(b-c)^2 &= 4(a-b)\left(\frac{b-c}{2}\right)\left(\frac{b-c}{2}\right)\\ &\leq 4\left[\frac{(a-b)+(b-c)/2+(b-c)/2}{3}\right]^3\\ &= \frac{4}{27}(a-c)^3.\end{aligned}$$

Therefore, it suffices to show that

$$(2c+a)(a-c)^2 \geq \frac{32}{81}(a-c)^3,$$

which is obvious.
Case 2: $a \geq b$, $c \geq b$. Making the substitution

$$a = b+p, \qquad c = b+q, \qquad p,q \geq 0,$$

the inequality becomes

$$(3b+2p)p^2+(3b+q)q^2+(3b+p+2q)(p-q)^2 \geq \frac{8}{3}pq^2,$$

$$3[p^2+q^2+(p-q)^2]b+2p^3+q^3+(p+2q)(p-q)^2 \geq \frac{8}{3}pq^2.$$

It suffices to show that

$$2p^3+q^3+(p+2q)(p-q)^2 \geq \frac{8}{3}pq^2,$$

which is equivalent to

$$2p^3+2q^3 \geq \frac{34}{9}pq^2.$$

By the AM-GM inequality, we have

$$2p^3+2q^3 = 2p^3+q^3+q^3 \geq 3\sqrt[3]{2p^3q^6} \geq \frac{34}{9}pq^2,$$

because

$$3\sqrt[3]{2} > \frac{34}{9}.$$

The equality holds for $a = b = c$.

□

P 2.34. *Let a,b,c be nonnegative real numbers such that $a \ge b \ge c$. Prove that*

(a)
$$\sum a^2(a-b)(a-c) \ge 4a^2b^2\left(\frac{a-b}{a+b}\right)^2;$$

(b)
$$\sum a^2(a-b)(a-c) \ge \frac{27b(a-b)^4}{4a}.$$

(Vasile C., 2011)

Solution. (a) Since $c^2(c-a)(c-b) \ge 0$, it suffices to show that

$$a^2(a-b)(a-c)+b^2(b-c)(b-a) \ge 4a^2b^2\left(\frac{a-b}{a+b}\right)^2.$$

Since

$$a^2(a-b)(a-c) = a^2(a-b)[(a-b)+(b-c)]$$
$$= a^2(a-b)^2 + a^2(a-b)(b-c) \ge 4a^2b^2\left(\frac{a-b}{a+b}\right)^2 + a^2(a-b)(b-c),$$

it suffices to show that

$$a^2(a-b)(b-c)+b^2(b-c)(b-a) \ge 0.$$

This inequality is equivalent to

$$(a-b)^2(a+b)(b-c) \ge 0.$$

The equality holds for $a=b=c$, and for $a=b$ and $c=0$.

(b) Since $c^2(c-a)(c-b) \ge 0$, it suffices to show that

$$a^2(a-b)(a-c)+b^2(b-c)(c-a) \ge \frac{27b(a-b)^4}{4a},$$

which is equivalent to

$$a^2(a-b)^2 + a(a-b)(b-c) + b^2(b-c)(c-a) \ge \frac{27b(a-b)^4}{4a}.$$

Since

$$a^2(a-b)^2 - \frac{27b(a-b)^4}{4a} = \frac{(a-b)^2(a-3b)^2(4a-3b)}{4a} \ge 0,$$

it suffices to show that

$$a^2(a-b)(b-c)+b^2(b-c)(b-a) \ge 0.$$

This inequality is equivalent to

$$(a-b)^2(a+b)(b-c) \ge 0.$$

The equality holds for $a=b=c$, and for $a/3=b=c$.

□

P 2.35. *If a, b, c are real numbers such that*

$$a \ge b \ge 1 \ge c, \qquad a^2 + b^2 + c^2 = 3,$$

then

(a) $$1 - abc \le 2(b-c)^2;$$

(b) $$1 - abc \ge 2(a-b)^2;$$

(c) $$1 - abc \ge \frac{1}{2}(a-c)^2;$$

(d) $$1 - abc \le \frac{3}{4}(a-c)^2.$$

(Vasile Cîrtoaje, 2020)

Solution. (a) Write the inequality as follows:

$$1 - abc \le 2(3 - a^2 - 2bc),$$

$$5 - 2a^2 \ge (4-a)bc.$$

From $(b^2-1)(c^2-1) \le 0$, we get

$$b^2c^2 \le b^2 + c^2 - 1 = 2 - a^2, \qquad bc \le \sqrt{2-a^2}, \qquad a \le \sqrt{2}.$$

Thus, it is enough to show that

$$5 - 2a^2 \ge (4-a)\sqrt{2-a^2},$$

which, by squaring, becomes

$$5a^4 - 8a^3 - 6a^2 + 16a - 7 \ge 0,$$

$$(a-1)^3(5a+7) \ge 0.$$

The equality occurs for $a = b = c = 1$.

(b) From

$$3 = a^2 + b^2 + c^2 \ge 1 + 1 + c^2,$$

it follows that $c \in [-1, 1]$. Write the required inequality as follows:

$$1 - abc \ge 2(3 - c^2 - 2ab),$$

$$(4-c)ab \ge 5 - 2c^2.$$

From $(a^2-1)(b^2-1) \ge 0$, we get

$$ab \ge \sqrt{a^2 + b^2 - 1} = \sqrt{2 - c^2}.$$

Thus, it is enough to show that

$$(4-c)\sqrt{2-c^2} \ge 5-2c^2,$$

which, by squaring, becomes

$$5c^4-8c^3-6c^2+16c-7 \le 0,$$

$$(c-1)^3(5c+7) \le 0.$$

(c) Write the inequality as follows:

$$2-2abc \ge 3-b^2-2ac,$$

$$b^2-1 \ge 2ac(b-1),$$

which is true if

$$b+1 \ge 2ac.$$

It is enough to show that

$$b+1 \ge a^2+c^2,$$

which is equivalent to

$$b+1 \ge 3-b^2,$$

$$(b-1)(b+2) \ge 0.$$

The equality occurs for $a=b=c=1$.

(d) Write the inequality as follows:

$$4-4abc \le 3(3-b^2)-6ac,$$

$$2(3-2b)ac \le 5-3b^2.$$

From

$$(a^2-b^2)(b^2-c^2) \ge 0,$$

it follows that

$$ac \le \sqrt{b^2(a^2+c^2)-b^4} = b\sqrt{3-2b^2}, \quad 1 \le b \le \sqrt{\frac{3}{2}}.$$

Thus, it suffices to show that

$$2b(3-2b)\sqrt{3-2b^2} \le 5-3b^2.$$

By squaring, the inequality becomes

$$32b^6-96b^5+33b^4+144b^3-138b^2+25 \ge 0,$$

$$(b-1)^2(32b^4-32b^3-63b^2+50b+25) \ge 0.$$

It is true because

$$32b^4 - 32b^3 - 63b^2 + 50b + 25 > 32b^4 - 32b^3 - 64b^2 + 48b + 24$$

$$= 8(3 - 2b^2)(1 + 2b - 2b^2) \geq 0.$$

The equality occurs for $a = b = c = 1$.

□

P 2.36. *If a, b, c are real numbers such that*

$$a \geq 1 \geq b \geq c, \qquad a^2 + b^2 + c^2 = 3,$$

then

$$1 - abc \leq \frac{2}{3}(a - c)^2.$$

First Solution. There are two cases to consider: $b \leq 0$ and $b \geq 0$.

Case 1: $b \leq 0$. Since $0 \geq b \geq c$, hence $c^2 \geq b^2$, we have

$$3abc + 2(a - c)^2 - 3 \geq 2(a - c)^2 - 3$$

$$> a^2 + 2c^2 - 3 \geq a^2 + b^2 + c^2 - 3 = 0.$$

Case 2: $b \geq 0$. Write the inequality as follows:

$$3abc + 2(3 - b^2 - 2ac) \geq 3,$$

$$3 - 2b^2 \geq (4 - 3b)ac.$$

From $(b^2 - c^2)(b^2 - a^2) \leq 0$, we get

$$a^2c^2 \leq b^2(a^2 + c^2) - b^4 = b^2(3 - b^2) - b^4, \qquad ac \leq b\sqrt{3 - 2b^2}.$$

Thus, it is enough to show that

$$3 - 2b^2 \geq b(4 - 3b)\sqrt{3 - 2b^2},$$

which, by squaring, becomes

$$6b^6 - 16b^5 + 3b^4 + 24b^3 - 20b^2 + 3 \geq 0,$$

$$(1 - b)^3(3 + 9b - 2b^2 - 6b^3) \geq 0.$$

$$(1 - b)^3[3 + b + 2b(1 - b) + 6b(1 - b^2)] \geq 0.$$

The equality occurs for $a = b = c = 1$.

Second Solution (*by Mudok*). We will prove the stronger inequality

$$1-abc \le \frac{2}{3}\left[(a-c)^2-(a-b)(b-c)\right],$$

which is equivalent to

$$1-abc \le \frac{2}{3}(3-ab-bc-ca),$$

$$3-3abc \le 9-p^2,$$

where

$$p=a+b+c.$$

From

$$(a-1)(b-1)(c-1) \ge 0,$$

we get

$$abc \ge \frac{p^2-2p-1}{2}.$$

Thus it suffices to show that

$$3-\frac{3(p^2-2p-1)}{2} \le 9-p^2,$$

which is equivalent to

$$(p-3)^2 \ge 0.$$

□

P 2.37. *If $a \ge 1 \ge b \ge c \ge 0$ and $a^2+b^2+c^2=3$, then*

$$1-abc \le \frac{1}{\sqrt{2}}(a-c).$$

(Vasile Cîrtoaje, 2020)

Solution. Denoting $x=ac$, we need to show that $f(x) \ge 0$, where

$$f(x)=bx+\sqrt{\frac{3-b^2-2x}{2}}-1.$$

For fixed b, we have $x \in [0,M]$, where

$$M=b\sqrt{3-2b^2}.$$

Indeed, $(b^2-a^2)(b^2-c^2) \le 0$ yields

$$a^2c^2 \le b^2(a^2+c^2)-b^4=3b^2-2b^4, \qquad x \le b\sqrt{3-2b^2}=M.$$

We have $x = 0$ for $c = 0$, and $x = M$ for $a = b = 1$ or $b = c$. Since

$$f''(x) = -\frac{1}{\sqrt{2}} \cdot \frac{1}{(3 - b^2 - 2x)^{3/2}} \leq 0,$$

f is a concave function, therefore it suffices to show that $f(0) \geq 0$ and $f(M) \geq 0$. We have

$$f(0) = \sqrt{\frac{3 - b^2}{2}} - 1 \geq 0.$$

Since

$$\sqrt{3 - b^2 - 2M} = \sqrt{3 - b^2 - 2b\sqrt{3 - 2b^2}} = \sqrt{3 - 2b^2} - b,$$

we have

$$f(M) \geq bM + \frac{2}{3} \cdot \sqrt{3 - b^2 - 2M} - 1 = \frac{2}{3} \cdot \sqrt{3 - b^2 - 2M} - (1 - bM)$$

$$= \frac{2}{3} \cdot \left(\sqrt{3 - 2b^2} - b\right) - (1 - b^2\sqrt{3 - 2b^2})$$

$$= \left(b^2 + \frac{2}{3}\right)\sqrt{3 - 2b^2} - \frac{2b}{3} - 1.$$

So, we need to show that

$$(3b^2 + 2)\sqrt{3 - 2b^2} \geq 2b + 3.$$

By squaring, the inequality becomes

$$6b^6 - b^4 - 8b^2 + 4b - 1 \leq 0,$$

$$(b - 1)(6b^5 + 6b^4 + 5b^3 + 5b^2 - 3b + 1) \leq 0.$$

It is true because

$$5b^2 - 3b + 1 = b^2 + b + (2b - 1)^2 > 0.$$

The equality occurs for $a = b = c = 1$, and also for $a = \sqrt{2}$, $b = 1$, $c = 0$.

□

P 2.38. *If* $a \geq 1 \geq b \geq c \geq 0$ *and* $a^2 + b^2 + c^2 = 3$, *then*

$$1 - abc \leq (1 + \sqrt{2})(a - b).$$

(Vasile Cîrtoaje, 2020)

Solution. Denoting $x = ab$, we need to show that $f(x) \ge 0$, where

$$f(x) = cx + k\sqrt{3 - c^2 - 2x} - 1, \qquad k = 1 + \sqrt{2}.$$

For fixed c, we have $x \in [m, M]$, where

$$m = c\sqrt{3 - 2c^2}, \qquad M = \sqrt{2 - c^2}.$$

Indeed, $(a^2 - c^2)(b^2 - c^2) \ge 0$ yields

$$a^2b^2 \ge c^2(a^2 + b^2) - c^4 = 3c^2 - 2c^4, \qquad x \ge c\sqrt{3 - 2c^2} = m,$$

and $(a^2 - 1)(b^2 - 1) \le 0$ yields

$$ab \le \sqrt{a^2 + b^2 - 1} = \sqrt{2 - c^2} = M.$$

We have $x = m$ for $b = c$, and $x = M$ for $b = 1$. Since

$$f''(x) = \frac{-k}{(3 - c^2 - 2x)^{3/2}} \le 0,$$

f is a concave function, therefore it suffices to show that $f(m) \ge 0$ and $f(M) \ge 0$. We have

$$\begin{aligned} f(m) &= c^2\sqrt{3 - 2c^2} + k\sqrt{3 - c^2 - 2c\sqrt{3 - 2c^2}} - 1 \\ &= c^2\sqrt{3 - 2c^2} + k\left(\sqrt{3 - 2c^2} - c\right) - 1 \\ &\ge c^2\sqrt{3 - 2c^2} + 2\left(\sqrt{3 - 2c^2} - c\right) - 1 \\ &= (c^2 + 2)\sqrt{3 - 2c^2} - 2c - 1 \ge (2c + 1)\left(\sqrt{3 - 2c^2} - 1\right) \ge 0. \end{aligned}$$

Also,

$$\begin{aligned} f(M) &= c\sqrt{2 - c^2} + k\sqrt{3 - c^2 - 2\sqrt{2 - c^2}} - 1 \\ &= c\sqrt{2 - c^2} - 1 + k\left(\sqrt{2 - c^2} - 1\right) \\ &= \frac{-(1 - c^2)^2}{c\sqrt{2 - c^2} + 1} + \frac{k(1 - c^2)}{\sqrt{2 - c^2} + 1}. \end{aligned}$$

So, we need to show that

$$\frac{k}{\sqrt{2 - c^2} + 1} \ge \frac{1 - c^2}{c\sqrt{2 - c^2} + 1}.$$

It is true because

$$\frac{k}{\sqrt{2 - c^2} + 1} \ge 1 \ge \frac{1 - c^2}{1 + c\sqrt{2 - c^2}}.$$

The equality occurs for $a = b = c = 1$, and also for $a = \sqrt{2}$, $b = 1$, $c = 0$.

□

P 2.39. *If $a \ge 1 \ge b \ge c \ge 0$ and $a^2+b^2+c^2=3$, then*

$$1-abc \le (3+2\sqrt{2})(a-b)^2.$$

(Vasile Cîrtoaje, 2020)

Solution. Write the inequality as follows:

$$abc+k(3-c^2-2ab)\ge 1,$$

$$3k-1-kc^2 \ge (2k-c)ab, \qquad k=3+2\sqrt{2}.$$

From $(a^2-1)(b^2-1)\le 0$, we get

$$ab \le \sqrt{a^2+b^2-1}=\sqrt{2-c^2}.$$

Thus, it is enough to show that

$$3k-1-kc^2 \ge (2k-c)\sqrt{2-c^2}.$$

Write this inequality as follows:

$$k\left(3-c^2-2\sqrt{2-c^2}\right)\ge 1-c\sqrt{2-c^2},$$

$$\frac{k(1-c^2)^2}{3-c^2+2\sqrt{2-c^2}} \ge \frac{(1-c^2)^2}{1+c\sqrt{2-c^2}},$$

which is true if

$$k\left(1+c\sqrt{2-c^2}\right)\ge 3-c^2+2\sqrt{2-c^2},$$

$$k-3+c^2 \ge (2-kc)\sqrt{2-c^2}.$$

For the nontrivial case $2-kc\ge 0$, we have

$$k-3+c^2 \ge k-3 = 2\sqrt{2} \ge (2-kc)\sqrt{2-c^2}.$$

The equality occurs for $a=b=c=1$, and also for $a=\sqrt{2},\ b=1,\ c=0$.

□

P 2.40. *If a,b,c are positive real numbers, then*

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge 3+\frac{(a-c)^2}{ab+bc+ca}.$$

(Vasile Cîrtoaje and Vo Quoc Ba Can, 2008)

First Solution. By expanding, the inequality can be written as

$$b^2+\frac{bc^2}{a}+\frac{ca^2}{b}+\frac{ab^2}{c}\ge 2ab+2bc.$$

We can get this inequality by summing the AM-GM inequalities

$$ab+\frac{bc^2}{a}\ge 2bc,$$

$$b^2+\frac{ca^2}{b}+\frac{ab^2}{c}\ge 3ab.$$

The equality holds for $a=b=c$.

Second Solution. From

$$\begin{aligned}(a+b+c)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\right)&=\sum\frac{a^2}{b}+\sum\frac{bc}{a}-2\sum a\\&=\sum\left(\frac{a^2}{b}-2a+b\right)+\sum\left(\frac{bc}{a}-b\right)\\&=\sum\left(\frac{a^2}{b}-2a+b\right)+\frac{1}{2}\sum\left(\frac{ab}{c}+\frac{ac}{b}-2a\right)\\&=\sum\frac{(a-b)^2}{b}+\frac{1}{2}\sum\frac{a(b-c)^2}{bc},\end{aligned}$$

we get

$$(a+b+c)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\right)\ge\frac{(a-b)^2}{b}+\frac{(b-c)^2}{c}+\frac{(c-a)^2}{a}.$$

By the Cauchy-Schwarz inequality, we have

$$\frac{(a-b)^2}{b}+\frac{(b-c)^2}{c}\ge\frac{(a-c)^2}{b+c}.$$

Therefore,

$$(a+b+c)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\right)\ge\frac{(a-c)^2}{b+c}+\frac{(c-a)^2}{a},$$

which is equivalent to

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\ge\frac{(a-c)^2}{a(b+c)}.$$

From this result, the desired inequality follows immediately.

□

P 2.41. *If a,b,c are positive real numbers, then*

(a) $$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge 3+\frac{4(a-c)^2}{(a+b+c)^2};$$

(b) $$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge 3+\frac{5(a-c)^2}{(a+b+c)^2}.$$

(Vo Quoc Ba Can and Vasile Cîrtoaje, 2009)

Solution. As we have shown at the second solution of the preceding problem P 2.40:

$$(a+b+c)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\right)=\sum\frac{(a-b)^2}{b}+\frac{1}{2}\sum\frac{a(b-c)^2}{bc},$$

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\ge\frac{(a-c)^2}{a(b+c)}.$$

(a) According to the upper inequality, it suffices to show that

$$\frac{1}{a(b+c)}\ge\frac{4}{(a+b+c)^2}.$$

Indeed,

$$\frac{1}{a(b+c)}-\frac{4}{(a+b+c)^2}=\frac{(a-b-c)^2}{a(b+c)(a+b+c)^2}\ge 0.$$

The equality holds for $a=b=c$.

(b) According to the upper identity, write the inequality as

$$(a+b+c)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\right)\ge\frac{5(a-c)^2}{a+b+c},$$

$$\sum\frac{(a-b)^2}{b}+\frac{1}{2}\sum\frac{a(b-c)^2}{bc}\ge\frac{5(a-c)^2}{a+b+c},$$

$$\frac{(a-b)^2}{b}+\frac{(b-c)^2}{c}+\frac{c(a-b)^2}{2ab}+\frac{a(b-c)^2}{2bc}\ge\left(\frac{5}{a+b+c}-\frac{1}{a}-\frac{b}{2ac}\right)(a-c)^2.$$

By the Cauchy-Schwarz inequality, we have

$$\frac{(a-b)^2}{b}+\frac{(b-c)^2}{c}\ge\frac{[(a-b)+(b-c)]^2}{b+c},$$

$$\frac{c(a-b)^2}{2ab}+\frac{a(b-c)^2}{2bc}\ge\frac{[(a-b)+(b-c)]^2}{\frac{2ab}{c}+\frac{2bc}{a}}=\frac{ac(a-c)^2}{2b(a^2+c^2)}.$$

Thus, we only need to show that

$$\frac{1}{b+c}+\frac{ac}{2b(a^2+c^2)}\ge\frac{5}{a+b+c}-\frac{1}{a}-\frac{b}{2ac},$$

which is equivalent to

$$\left(\frac{1}{a}+\frac{1}{b+c}\right)+\frac{ac}{2b(a^2+c^2)}+\frac{b}{2ac}\geq\frac{5}{a+b+c}.$$

This inequality is true because, by the Cauchy-Schwarz inequality and the AM-GM inequality, we have

$$\frac{1}{a}+\frac{1}{b+c}\geq\frac{4}{a+(b+c)}$$

and

$$\frac{ac}{2b(a^2+c^2)}+\frac{b}{2ac}\geq\frac{1}{\sqrt{a^2+c^2}}>\frac{1}{a+c}>\frac{1}{a+b+c}.$$

The equality holds for $a=b=c$.

□

P 2.42. *If $a\geq b\geq c>0$, then*

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 3+\frac{3(b-c)^2}{ab+bc+ca}.$$

First Solution. Since

$$\frac{a}{b}+\frac{c}{a}-1-\frac{c}{b}=\frac{(a-b)(a-c)}{ab}\geq 0,$$

it suffices to show that

$$\frac{b}{c}+\frac{c}{b}-2\geq\frac{3(b-c)^2}{ab+bc+ca}.$$

Indeed, we have

$$\frac{b}{c}+\frac{c}{b}-2-\frac{3(b-c)^2}{ab+bc+ca}=\frac{(b-c)^2(ab+ac-2bc)}{bc(ab+bc+ca)}.$$

The equality holds for $a=b=c$.

Second Solution. Since

$$ab+bc+ca\geq 3bc,$$

it suffices to show that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 3+\frac{(b-c)^2}{bc},$$

which is equivalent to

$$\frac{a}{b}+\frac{c}{a}\geq 1+\frac{c}{b},$$

$$\frac{(a-b)(a-c)}{ab}\geq 0.$$

□

P 2.43. *Let a, b, c be positive real numbers such that $abc = 1$. Prove that*

(a) if $a \ge b \ge 1 \ge c$, then

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge 3 + \frac{2(a-b)^2}{ab};$$

(b) if $a \ge 1 \ge b \ge c$, then

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge 3 + \frac{2(b-c)^2}{bc}.$$

(Vasile C., 2010)

Solution. (a) Write the inequality as

$$f(c) \ge \frac{a}{b} + 2\frac{b}{a} - 1,$$

where

$$f(c) = \frac{b}{c} + \frac{c}{a}.$$

From

$$b^3 \ge 1 = abc,$$

we find

$$b^2 \ge ac.$$

We will show that

$$f(c) \ge f\left(\frac{b^2}{a}\right) \ge \frac{a}{b} + 2\frac{b}{a} - 1.$$

The left inequality is equivalent to

$$\frac{b}{c} + \frac{c}{a} \ge \frac{a}{b} + \frac{b^2}{a^2},$$

$$\frac{b^2 - ac}{bc} \ge \frac{b^2 - ac}{a^2} \ge 0,$$

$$(a^2 - bc)(b^2 - ac) \ge 0.$$

The right inequality reduces to

$$\left(\frac{b}{a} - 1\right)^2 \ge 0.$$

The equality holds for $a = b = c = 1$.

(b) Write the inequality as

$$f(a) \geq \frac{b}{c} + 2\frac{c}{b} - 1,$$

where

$$f(a) = \frac{a}{b} + \frac{c}{a}.$$

From

$$b^3 \leq 1 = abc,$$

we find

$$b^2 \leq ac.$$

We will show that

$$f(a) \geq f\left(\frac{b^2}{c}\right) \geq \frac{b}{c} + 2\frac{c}{b} - 1$$

The left inequality is equivalent to

$$\frac{a}{b} + \frac{c}{a} \geq \frac{b}{c} + \frac{c^2}{b^2},$$

$$\frac{ac - b^2}{bc} \geq \frac{c(ac - b^2}{ab^2} \geq 0,$$

$$(ab - c^2)(ac - b^2) \geq 0.$$

The right inequality reduces to

$$\left(\frac{c}{b} - 1\right)^2 \geq 0.$$

The equality holds for $a = b = c = 1$.

□

P 2.44. *Let a, b, c be positive real numbers such that*

$$a \geq 1 \geq b \geq c, \qquad abc = 1.$$

prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3 + \frac{9(b - c)^2}{ab + bc + ca}.$$

(Vasile C., 2010)

Solution. From $b^3 \le 1 = abc$, we find $b^2 \le ac$. We will show that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge \frac{2b}{c}+\frac{c^2}{b^2} \ge 3+\frac{9(b-c)^2}{ab+bc+ca}.$$

The left inequality is equivalent to

$$\frac{a}{b}+\frac{c}{a} \ge \frac{b}{c}+\frac{c^2}{b^2},$$

$$\frac{a}{b}-\frac{b}{c}+\left(\frac{c}{a}-\frac{c^2}{b^2}\right) \ge 0,$$

$$\frac{ac-b^2}{bc}+\frac{c(b^2-ac)}{ab^2} \ge 0,$$

$$\frac{(ac-b^2)(ab-c^2)}{ab^2c} \ge 0.$$

The right inequality is equivalent to

$$\frac{2b}{c}+\frac{c^2}{b^2}-3 \ge \frac{9(b-c)^2}{ab+bc+ca}.$$

$$\frac{(b-c)^2(2b+c)}{b^2c} \ge \frac{9(b-c)^2}{ab+bc+ca}.$$

We need to show that

$$\frac{(2b+c)}{b^2c} \ge \frac{9}{a(b+c)+bc}.$$

This is true if

$$\frac{(2b+c)}{b^2c} \ge \frac{9}{b(b+c)+bc},$$

which is equivalent to

$$\frac{2(b-c)^2}{b^2c(b+2c)} \ge 0.$$

The equality holds for $a=b=c=1$.

□

P 2.45. *Let a,b,c be positive real numbers such that*

$$a \ge 1 \ge b \ge c, \qquad a+b+c=3.$$

prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge 3+\frac{4(b-c)^2}{b^2+c^2}.$$

(Vasile C., 2010)

Solution. From
$$3b \le 3 = a+b+c,$$
we find
$$2b \le a+c, \qquad a \ge 2b-c.$$
We will show that
$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge \frac{2b-c}{b}+\frac{b}{c}+\frac{c}{2b-c} \ge 3+\frac{4(b-c)^2}{b^2+c^2}.$$
The left inequality is equivalent to
$$\frac{a}{b}+\frac{c}{a} \ge \frac{2b-c}{b}+\frac{c}{2b-c},$$
$$\frac{a+c-2b}{b}-\frac{c(a+c-2b)}{a(2b-c)} \ge 0,$$
$$\frac{(a+c-2b)[a(b-c)+b(a-c)]}{ab(2b-c)} \ge 0.$$
The right inequality is equivalent to
$$\frac{(b-c)^2(2b+c)}{bc(2b-c)} \ge \frac{4(b-c)^2}{b^2+c^2}.$$
We need to show that
$$\frac{(2b+c)}{bc(2b-c)} \ge \frac{4}{b^2+c^2},$$
which is equivalent to
$$2b^3-7b^2c+6bc^2+c^3 \ge 0,$$
$$2b(b-2c)^2+(b-c)^2c \ge 0.$$
The equality holds for $a=b=c=1$.

□

P 2.46. *Let a,b,c be positive real numbers such that*
$$a \ge b \ge 1 \ge c, \qquad a+b+c=3.$$
Prove that
$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge 3+\frac{3(a-b)^2}{ab}.$$
(Vasile C., 2008)

Solution. From

$$3b \ge 3 = a + b + c,$$

we get

$$2b \ge a + c, \qquad c \le 2b - a.$$

We will show that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge \frac{a}{b} + \frac{b}{2b-a} + \frac{2b-a}{a} \ge 3 + \frac{3(a-b)^2}{ab}.$$

The left inequality is equivalent to

$$\frac{b}{c} + \frac{c}{a} \ge \frac{b}{2b-a} + \frac{2b-a}{a},$$

$$(2b-a-c)[b(a-c) + c(a-b)] \ge 0.$$

The right inequality is equivalent to

$$\frac{a}{b} + \frac{b}{2b-a} + \frac{2b-a}{a} - 3 \ge \frac{3(a-b)^2}{ab},$$

$$\frac{(a-b)^2(4b-a)}{ab(2b-a)} \ge \frac{3(a-b)^2}{ab},$$

$$\frac{2(a-b)^3}{ab(2b-a)} \ge 0.$$

The equality holds for $a = b = c = 1$.

□

P 2.47. *If a, b, c are positive real numbers, then*

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge 3 + \frac{2(a-c)^2}{(a+c)^2}.$$

Solution. Since

$$\frac{a}{b} + \frac{b}{c} \ge 2\sqrt{\frac{a}{c}},$$

it suffices to show that

$$\frac{c}{a} + 2\sqrt{\frac{a}{c}} \ge 3 + \frac{2(a-c)^2}{(a+c)^2}.$$

Using the substitution $x = \sqrt{\dfrac{a}{c}}$, this inequality becomes as follows:

$$\frac{1}{x^2} + 2x \ge 3 + \frac{2(x^2-1)^2}{(x^2+1)^2},$$

$$\frac{(x-1)^2(2x+1)}{x^2} \geq \frac{2(x^2-1)^2}{(x^2+1)^2}.$$

We need to show that

$$\frac{2x+1}{x^2} \geq \frac{2(x+1)^2}{(x^2+1)^2},$$

which is equivalent to

$$2x^5 - 3x^4 + 2x + 1 \geq 0.$$

For $0 < x \leq 1$, we have

$$2x^5 - 3x^4 + 2x + 1 > -3x^4 + 2x + 1 \geq -3x + 2x + 1 \geq 0.$$

Also, for $x \geq 1$, we have

$$2x^5 - 3x^4 + 2x + 1 > 2x^5 - 3x^4 + 2x - 1 = (x-1)^2(2x^3 + x^2 - 1) \geq 0.$$

The equality holds for $a = b = c$.

□

P 2.48. *If a, b, c are positive real numbers, then*

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq a + b + c + \frac{4(a-c)^2}{a+b+c}.$$

(Balkan MO, 2005, 2008)

Solution. Write the inequality as follows:

$$\left(\frac{a^2}{b} + b - 2a\right) + \left(\frac{b^2}{c} + c - 2b\right) + \left(\frac{c^2}{a} + a - 2c\right) \geq \frac{4(a-c)^2}{a+b+c},$$

$$\frac{(a-b)^2}{b} + \frac{(b-c)^2}{c} + \frac{(a-c)^2}{a} \geq \frac{4(a-c)^2}{a+b+c}.$$

By the Cauchy-Schwarz inequality, we have

$$\frac{(a-b)^2}{b} + \frac{(b-c)^2}{c} + \frac{(a-c)^2}{a} \geq \frac{[(a-b)+(b-c)+(a-c)]^2}{b+c+a} = \frac{4(a-c)^2}{a+b+c}.$$

The equality holds for $a = b = c$, and also for $a = b + c$ and $\dfrac{b}{c} = \dfrac{1+\sqrt{5}}{2}$.

□

P 2.49. *If $a \geq b \geq c > 0$, then*

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq a + b + c + \frac{6(b-c)^2}{a+b+c}.$$

(Vasile C., 2014)

Solution. Write the inequality as follows:

$$\left(\frac{a^2}{b}+b-2a\right)+\left(\frac{b^2}{c}+c-2b\right)+\left(\frac{c^2}{a}+a-2c\right)\ge\frac{6(b-c)^2}{a+b+c},$$

$$\frac{(a-b)^2}{b}+\frac{(b-c)^2}{c}+\frac{(a-c)^2}{a}\ge\frac{6(b-c)^2}{a+b+c},$$

$$\frac{(a-b)^2}{b}+\frac{(a-c)^2}{a}+\frac{(a+b-5c)(b-c)^2}{c(a+b+c)}\ge 0.$$

Since

$$(a-c)^2=[(a-b)+(b-c)]^2=(a-b)^2+2(a-b)(b-c)+(b-c)^2,$$

we have

$$\frac{(a-b)^2}{b}+\frac{(a-c)^2}{a}\ge\frac{(a-c)^2}{a}\ge\frac{2(a-b)(b-c)+(b-c)^2}{a}.$$

Therefore, it suffices to show that

$$\frac{2(a-b)(b-c)+(b-c)^2}{a}+\frac{(a+b-5c)(b-c)^2}{c(a+b+c)}\ge 0,$$

which can be written as

$$\frac{2(a-b)(b-c)}{a}+\frac{(a-c)^2+ab+bc-2ca}{ac(a+b+c)}(b-c)^2\ge 0.$$

Since

$$(a-c)^2+ab+bc-2ca=(a-c)^2+a(b-c)-c(a-b)\ge -c(a-b),$$

it is enough to prove that

$$\frac{2(a-b)(b-c)}{a}-\frac{a-b}{a(a+b+c)}(b-c)^2\ge 0.$$

Indeed,

$$\frac{2(a-b)(b-c)}{a}-\frac{a-b}{a(a+b+c)}(b-c)^2=\frac{(a-b)(b-c)}{a}\left(2-\frac{b-c}{a+b+c}\right)\ge 0.$$

The equality holds for $a=b=c$.

□

P 2.50. *If $a \ge b \ge c > 0$, then*

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} > 5(a-b).$$

(Vasile C., 2014)

Solution. Consider two cases: $a \le 2b$ and $a \ge 2b$.

Case 1: $a \le 2b$. It suffices to show that

$$\frac{a^2}{b}+\frac{b^2}{b} \ge 5(a-b),$$

which is equivalent to the obvious inequality

$$(2b-a)(3b-a) \ge 0.$$

Case 2: $a \ge 2b$. Since

$$\frac{b^2}{c}+\frac{c^2}{a}-b-\frac{b^2}{a} = (b-c)\left(\frac{b}{c}-\frac{b+c}{a}\right)$$

$$\ge (b-c)\left(\frac{b}{c}-\frac{b+c}{2b}\right) = \frac{(b-c)^2(2b+c)}{2bc} \ge 0,$$

it suffices to show that

$$\frac{a^2}{b}+b+\frac{b^2}{a} \ge 5(a-b),$$

which is equivalent to

$$x(x-2)(3-x) < 1,$$

where $x = a/b \ge 2$. For the non-trivial case $2 \le x \le 3$, we have

$$x(x-2)(3-x) \ge x\left[\frac{(x-2)+(3-x)}{2}\right]^2 = \frac{x}{4} < 1.$$

$\square$

P 2.51. *Let a,b,c be positive real numbers such that*

$$a \ge b \ge 1 \ge c, \qquad a+b+c=3.$$

Prove that

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} \ge 3+\frac{11(a-c)^2}{4(a+c)}.$$

(Vasile C., 2010)

Solution. We have

$$a+b+c=3\le b, \qquad 2b\ge a+c.$$

Thus, we need to prove the homogeneous inequality

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\ge a+b+c+\frac{11(a-c)^2}{4(a+c)}$$

for

$$a\ge b\ge \frac{a+c}{2}.$$

Denote

$$f(a,b,c)=\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}-a-b-c.$$

We will show that

$$f(a,b,c)\ge f\left(a,\frac{a+c}{2},c\right)\ge \frac{11(a-c)^2}{4(a+c)}.$$

Write the left inequality as follows:

$$\left(\frac{a^2}{b}-\frac{2a^2}{a+c}\right)+\left[\frac{b^2}{c}-\frac{(a+c)^2}{4c}\right]-\left(b-\frac{a+c}{2}\right)\ge 0,$$

$$(2b-a-c)\left[-\frac{a^2}{b(a+c)}+\frac{2b+a+c}{4c}-\frac{1}{2}\right]\ge 0.$$

Since $2b-a-c\ge 0$, we only need to show that

$$\frac{2b+a+c}{4c}\ge \frac{a^2}{b(a+c)}+\frac{1}{2}.$$

It suffices to prove this inequality for $b=\dfrac{a+c}{2}$. Making this, the inequality becomes

$$\frac{a(a-c)^2}{2c(a+c)^2}\ge 0.$$

To prove the right inequality, we find

$$f\left(a,\frac{a+c}{2},c\right)=\frac{(a-c)^2(a^2+7ac+4c^2)}{4ac(a+c)},$$

hence

$$f\left(a,\frac{a+c}{2},c\right)-\frac{11(a-c)^2}{4(a+c)}=\frac{(a-c)^2(a-2c)^2}{4ac(a+c)}\ge 0.$$

The equality holds for $a=b=c=1$, and also for $\dfrac{a}{4}=\dfrac{b}{3}=\dfrac{c}{2}$ (that is, for $a=\dfrac{4}{3}$, $b=1$, $c=\dfrac{2}{3}$).

□

P 2.52. *If a,b,c are positive real numbers, then*

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\ge \frac{3}{2}+\frac{27(b-c)^2}{16(a+b+c)^2}.$$

(Vasile C., 2014)

Solution. Write the inequality as follows:

$$\sum\left(\frac{a}{b+c}+1\right)\ge \frac{9}{2}+\frac{27(b-c)^2}{16(a+b+c)^2},$$

$$\left[\sum(b+c)\right]\left(\sum\frac{1}{b+c}\right)\ge 9+\frac{27(b-c)^2}{2\left[\sum(b+c)\right]^2}.$$

Replacing $b+c$, $c+a$, $a+b$ by a, b, c, respectively, we need to show that

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge 9+\frac{27(b-c)^2}{2(a+b+c)^2},$$

where a,b,c are the side-lengths of a non-degenerate triangle. Write this inequality in the form

$$\frac{a+b+c}{a}+(a+b+c)\left(\frac{1}{b}+\frac{1}{c}\right)+\frac{54bc}{(a+b+c)^2}\ge 9+\frac{27(b+c)^2}{2(a+b+c)^2}.$$

Applying the AM-GM inequality gives

$$(a+b+c)\left(\frac{1}{b}+\frac{1}{c}\right)+\frac{54bc}{(a+b+c)^2}\ge 6\sqrt{\frac{6(b+c)}{a+b+c}}.$$

Therefore, it suffices to show that

$$\frac{a+b+c}{a}+6\sqrt{\frac{6(b+c)}{a+b+c}}\ge 9+\frac{27(b+c)^2}{2(a+b+c)^2},$$

which can be rewritten as

$$\frac{1}{1-\dfrac{b+c}{a+b+c}}+6\sqrt{\frac{6(b+c)}{a+b+c}}\ge 9+\frac{27(b+c)^2}{2(a+b+c)^2}.$$

Using the substitution

$$\frac{b+c}{a+b+c}=\frac{2}{3}t^2,\quad t^2>\frac{3}{4},$$

this inequality becomes

$$\frac{1}{3-2t^2}+4t\ge 3+2t^4,$$

$$2t^6-3t^4-4t^3+3t^2+6t-4\geq 0,$$
$$(t-1)^2(2t^4+4t^3+3t^2-2t-4)\geq 0,$$
$$(t-1)^2\left[(4t^2-3)(t^2+2t+2)+t^2+2t-2\right]\geq 0.$$

Clearly, the last inequality is true for $t^2>3/4$. The original inequality is an equality for $a=b=c$.

□

P 2.53. *Let a,b,c be positive real numbers such that $a=\min\{a,b,c\}$. Prove that*

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}+\frac{9(b-c)^2}{4(a+b+c)^2}.$$

(Vasile C., 2014)

Solution. Write the inequality as

$$\sum\left(\frac{a}{b+c}+1\right)\geq\frac{9}{2}+\frac{9(b-c)^2}{4(a+b+c)^2},$$
$$\left[\sum(b+c)\right]\left(\sum\frac{1}{b+c}\right)\geq 9+\frac{18(b-c)^2}{[(b+c)+(c+a)+(a+b)]^2}.$$

Replacing $b+c$, $c+a$, $a+b$ by a, b, c, respectively, we need to show that

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 9+\frac{18(b-c)^2}{(a+b+c)^2},$$

where a,b,c are the side-lengths of a non-degenerate triangle, $a=\max\{a,b,c\}$. Since

$$(a+b+c)^2\geq\frac{9}{4}(b+c)^2\geq 9bc,$$

it suffices to show that

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 9+\frac{2(b-c)^2}{bc}.$$

Write the inequality as follows:

$$\frac{(a-b)^2}{ab}+\frac{(a-c)^2}{ac}+\frac{(b-c)^2}{bc}\geq\frac{2(b-c)^2}{bc},$$
$$c(a-b)^2+b(a-c)^2\geq a(b-c)^2,$$
$$(b+c)a^2-(b+c)^2a+bc(b+c)\geq 0,$$
$$(b+c)(a-b)(a-c)\geq 0.$$

Clearly, the last inequality is true. The original inequality is an equality for $a=b=c$.

□

P 2.54. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\ge\frac{3}{2}+\frac{(b-c)^2}{2(b+c)^2}.$$

(Vasile C., 2014)

First Solution. Write the inequality as follows:

$$\frac{2bc}{(b+c)^2}+\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\ge 2,$$

$$\frac{a(b+c)+2bc}{(b+c)^2}+\frac{b}{c+a}+\frac{c}{a+b}\ge 2,$$

By the Cauchy-Schwarz inequality, we have

$$\frac{b}{c+a}+\frac{c}{a+b}\ge\frac{(b+c)^2}{b(c+a)+c(a+b)}=\frac{(b+c)^2}{a(b+c)+2bc}.$$

Therefore, it suffices to prove that

$$\frac{a(b+c)+2bc}{(b+c)^2}+\frac{(b+c)^2}{a(b+c)+2bc}\ge 2,$$

which is obvious. The original inequality is an equality for $a=b=c$, for $a=b$ and $c=0$, and for $a=c$ and $b=0$.

Second Solution. Write the inequality as follows:

$$\sum\left(\frac{a}{b+c}+1\right)\ge\frac{9}{2}+\frac{(b-c)^2}{2(b+c)^2},$$

$$\left[\sum(b+c)\right]\left(\sum\frac{1}{b+c}\right)\ge 9+\frac{(b-c)^2}{(b+c)^2}.$$

Replacing $b+c$, $c+a$, $a+b$ by a, b, c, respectively, we need to show that

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge 9+\frac{(b-c)^2}{a^2},$$

where a,b,c are the lengths of the sides of a triangle. Write this inequality as

$$\frac{(a-b)^2}{ab}+\frac{(a-c)^2}{ac}+\frac{(b-c)^2}{bc}\ge\frac{(b-c)^2}{a^2},$$

$$a[c(a-b)^2+b(a-c)^2]\ge(bc-a^2)(b-c)^2.$$

Without loss of generality, assume that $b\ge c$. Since $a\ge b-c$, it suffices to show that

$$c(a-b)^2+b(a-c)^2\ge(bc-a^2)(b-c).$$

Indeed, we have

$$c(a-b)^2+b(a-c)^2-(bc-a^2)(b-c)=2b(a-c)^2\ge 0.$$

$\square$

P 2.55. *Let a,b,c be positive real numbers such that $a=\min\{a,b,c\}$. Prove that*

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\ge\frac{3}{2}+\frac{(b-c)^2}{4bc}.$$

(Vasile C., 2014)

First Solution *(by Nguyen Van Quy).* Notice that for $a=\min\{a,b,c\}$, we have

$$4bc=(2b)(2c)\ge(a+b)(a+c)\ge 2a(b+c),$$

hence

$$\frac{a}{b+c}\ge\frac{2a^2}{(a+b)(a+c)},\quad \frac{(b-c)^2}{4bc}\le\frac{(b-c)^2}{(a+b)(a+c)}.$$

So, it suffices to show that

$$\frac{2a^2}{(a+b)(a+c)}+\frac{b}{c+a}+\frac{c}{a+b}\ge\frac{3}{2}+\frac{(b-c)^2}{(a+b)(a+c)},$$

which is equivalent to the obvious inequality

$$(a-b)(a-c)\ge 0.$$

The proof is completed. The original inequality is an equality for $a=b=c$.

Second Solution. Let

$$E(a,b,c)=\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}.$$

Without loss of generality, assume that $b\le c$, hence $a\le b\le c$. We will show that

$$E(a,b,c)\ge E(b,b,c)\ge\frac{3}{2}+\frac{(b-c)^2}{4bc}.$$

We have

$$\begin{aligned}E(a,b,c)-E(b,b,c)&=\frac{a-b}{b+c}+\frac{b(b-a)}{(a+c)(b+c)}+\frac{c(b-a)}{2b(a+b)}\\&=(b-a)\left[\frac{(b-a)-c}{(a+c)(b+c)}+\frac{c}{2b(a+b)}\right]\\&=\frac{(b-a)[2b(b^2-a^2)+c(c-b)(a+2b+c)]}{2b(a+b)(a+c)(b+c)}\ge 0\end{aligned}$$

and

$$\begin{aligned}E(b,b,c)-\frac{3}{2}-\frac{(b-c)^2}{4bc}&=\left(\frac{2b}{b+c}+\frac{c}{2b}-\frac{3}{2}\right)-\frac{(b-c)^2}{4bc}\\&=\frac{(b-c)^2}{2b(b+c)}-\frac{(b-c)^2}{4bc}\\&=\frac{(c-b)^3}{4bc(b+c)}\ge 0.\end{aligned}$$

□

P 2.56. *Let a,b,c be positive real numbers such that*

$$a \le 1 \le b \le c, \qquad a+b+c=3,$$

then

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \ge \frac{3}{2}+\frac{3(b-c)^2}{4bc}.$$

(Vasile C., 2014)

Solution. From

$$3b \ge 3 = a+b+c,$$

we get

$$a \le 2b-c, \qquad 2b > c.$$

Let

$$E(a,b,c)=\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}.$$

We will show that

$$E(a,b,c) \ge E(2b-c,b,c) \ge \frac{3}{2}+\frac{3(b-c)^2}{4bc}.$$

We have

$$E(a,b,c)-E(2b-c,b,c)=(2b-a-c)F,$$

where

$$F=\frac{-1}{b+c}+\frac{1}{2(c+a)}+\frac{c}{(a+b)(3b-c)}.$$

Since $2b-a-c \ge 0$, we need to show that $F \ge 0$. This is true because

$$\begin{aligned}
F &= \frac{1}{2}\left(-\frac{1}{b+c}+\frac{1}{c+a}\right)-\frac{1}{2(b+c)}+\frac{c}{(a+b)(3b-c)}\\
&\ge -\frac{1}{2(b+c)}+\frac{c}{(a+b)(3b-c)} \ge -\frac{1}{2(a+b)}+\frac{c}{(a+b)(3b-c)}\\
&= \frac{3(c-b)}{2(a+b)(3b-c)} \ge 0.
\end{aligned}$$

In what concerns the right inequality, we have

$$\begin{aligned}
E(2b-c,b,c)-\frac{3}{2}-\frac{3(b-c)^2}{4bc} &= 3(b-c)^2\left[\frac{1}{(b+c)(3b-c)}-\frac{1}{4bc}\right]\\
&= \frac{-3(b-c)^3(3b+c)}{4bc(b+c)(3b-c)} \ge 0.
\end{aligned}$$

The proof is completed. The original inequality is an equality for $a=b=c=1$.

□

P 2.57. *Let a, b, c be nonnegative real numbers such that*

$$a \geq 1 \geq b \geq c, \qquad a+b+c=3,$$

then

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{3}{2}+\frac{(b-c)^2}{(b+c)^2}.$$

(Vasile C., 2014)

Solution. From

$$3b \leq 3 = a+b+c,$$

we get

$$a \geq 2b-c.$$

Let

$$E(a,b,c)=\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}.$$

We will show that

$$E(a,b,c) \geq E(2b-c,b,c) \geq \frac{3}{2}+\frac{(b-c)^2}{(b+c)^2}.$$

We have

$$E(a,b,c)-E(2b-c,b,c)=(a-2b+c)F,$$

where

$$F=\frac{1}{b+c}-\frac{1}{2(c+a)}-\frac{c}{(a+b)(3b-c)}.$$

Since $a-2b+c \geq 0$, we need to show that $F \geq 0$. This is true because

$$\begin{aligned}
F &= \frac{1}{2}\left(\frac{1}{b+c}-\frac{1}{c+a}\right)+\frac{1}{2(b+c)}-\frac{c}{(a+b)(3b-c)} \\
&\geq \frac{1}{2(b+c)}-\frac{c}{(a+b)(3b-c)} \geq \frac{1}{2(a+b)}-\frac{c}{(a+b)(3b-c)} \\
&= \frac{3(b-c)}{2(a+b)(3b-c)} \geq 0.
\end{aligned}$$

The right inequality is also true because

$$\begin{aligned}
E(2b-c,b,c)-\frac{3}{2}-\frac{(b-c)^2}{(b+c)^2} &= \frac{(b-c)^2}{b+c}\left[\frac{3}{3b-c}-\frac{1}{b+c}\right] \\
&= \frac{4c(b-c)^2}{(b+c)^2(3b-c)} \geq 0.
\end{aligned}$$

The proof is completed. The original inequality is an equality for $a=b=c=1$, and also for $a=2$, $b=1$, $c=0$.

□

P 2.58. *Let a,b,c be positive real numbers such that $a=\min\{a,b,c\}$. Prove that*

(a) $$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{2(b-c)^2}{3(b^2+c^2)}\le 1;$$

(b) $$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{(b-c)^2}{b^2+bc+c^2}\le 1;$$

(c) $$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{(a-b)^2}{2(a^2+b^2)}\le 1.$$

(Vasile C., 2014)

Solution. (a) *First Solution.* Since

$$3(b^2+c^2)\ge 2(a^2+b^2+c^2),$$

it suffices to show that

$$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{(b-c)^2}{a^2+b^2+c^2}\le 1.$$

This inequality is equivalent to

$$(a-b)(a-c)\ge 0,$$

which is clearly true. The equality holds for $a=b=c$.

Second Solution. Write the inequality as follows:

$$\frac{4(b-c)^2}{3(b^2+c^2)}\le\frac{(b-c)^2+(a-b)^2+(a-c)^2}{a^2+b^2+c^2},$$

$$3(b^2+c^2)[(a-b)^2+(a-c)^2]\ge(b-c)^2(4a^2+b^2+c^2),$$

$$3(b^2+c^2)[(b-c)^2+2(a-b)(a-c)]\ge(b-c)^2(4a^2+b^2+c^2),$$

$$6(b^2+c^2)(a-b)(a-c)+2(b-c)^2(b^2+c^2-2a^2)\ge 0.$$

The last inequality is true because $(a-b)(a-c)\ge 0$ and $b^2+c^2-2a^2\ge 0$.

(b) Without loss of generality, assume that

$$a\le b\le c.$$

Write the inequality as

$$\frac{ab+bc+ca}{a^2+b^2+c^2}\le\frac{3bc}{b^2+bc+c^2};$$

that is,

$$E(a,b,c)\ge 0,$$

where

$$E(a,b,c) = 3bca^2 - (b+c)(b^2+c^2+bc)a + bc(2b^2+2c^2-bc).$$

We will show that

$$E(a,b,c) \geq E(b,b,c) \geq 0.$$

We have

$$\begin{aligned} E(a,b,c) - E(b,b,c) &= 3bc(a^2-b^2) - (b+c)(b^2+c^2+bc)(a-b) \\ &= (b-a)[(b+c)(b^2+c^2+bc) - 3bc(a+b)] \\ &\geq (b-a)[(b+c)(b^2+c^2+bc) - 3bc(c+b)] \\ &= (b-a)(b+c)(b-c)^2 \geq 0. \end{aligned}$$

Also,

$$E(b,b,c) = b(c-b)^3 \geq 0.$$

The equality holds for $a = b = c$, and also for $a = b = 0$ or $a = c = 0$.

(c) Write the inequality as follows:

$$\frac{ab+(a+b)c}{a^2+b^2+c^2} \leq \frac{(a+b)^2}{2(a^2+b^2)},$$

$$(a+b)^2c^2 - 2(a+b)(a^2+b^2)c + (a^2+b^2)^2 \geq 0,$$

$$[(a+b)c - (a^2+b^2)]^2 \geq 0.$$

The equality holds for $c = \dfrac{a^2+b^2}{a+b}$.

□

P 2.59. *Let a,b,c be positive real numbers such that*

$$a \leq 1 \leq b \leq c, \qquad a+b+c = 3,$$

then

$$\frac{ab+bc+ca}{a^2+b^2+c^2} + \frac{(b-c)^2}{bc} \leq 1.$$

(Vasile C., 2014)

Solution. From

$$3b \geq 3 = a+b+c,$$

we get

$$a \leq 2b - c.$$

Write the inequality as follows:

$$\frac{2(b-c)^2}{bc} \le \frac{(b-c)^2+(a-b)^2+(a-c)^2}{a^2+b^2+c^2},$$

$$(b-a)^2+(c-a)^2 \ge \left(\frac{2a^2+2b^2+2c^2}{bc}-1\right)(c-b)^2,$$

$$(c-b)^2+2(b-a)(c-a) \ge \left(\frac{2a^2+2b^2+2c^2}{bc}-1\right)(c-b)^2,$$

$$(b-a)(c-a) \ge \left(\frac{a^2+b^2+c^2}{bc}-1\right)(c-b)^2.$$

Since

$$b-a \ge b-(2b-c)=c-b \ge 0, \qquad c-a \ge c-(2b-c)=2(c-b) \ge 0,$$

it suffices to show that

$$2 \ge \frac{a^2+b^2+c^2}{bc}-1,$$

which is equivalent to

$$3bc \ge a^2+b^2+c^2.$$

This is true if

$$3bc \ge (2b-c)^2+b^2+c^2,$$

which reduces to

$$7bc \ge 5b^2+2c^2,$$

$$(c-b)(5b-2c) \ge 0.$$

Thus, we only need to show that $5b-2c \ge 0.$ Indeed, we have

$$5b-2c > 2(2b-c) \ge 2a > 0.$$

The equality holds for $a=b=c=1.$

□

P 2.60. *Let a,b,c be nonnegative real numbers such that $a=\max\{a,b,c\}$ and $b+c>0.$ Prove that*

(a)
$$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{(b-c)^2}{2(ab+bc+ca)} \le 1;$$

(b)
$$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{2(b-c)^2}{(a+b+c)^2} \le 1.$$

(Vasile C., 2014)

Solution. Without loss of generality, assume that $a \ge b \ge c$.

(a) Write the inequality as follows:

$$\frac{(b-c)^2}{ab+bc+ca} \le \frac{(b-c)^2+(a-b)^2+(a-c)^2}{a^2+b^2+c^2},$$

$$(ab+bc+ca)[(a-b)^2+(a-c)^2] \ge (b-c)^2(a^2+b^2+c^2-ab-bc-ca).$$

Since

$$ab+bc+ca \ge ab \ge b^2 \ge (b-c)^2,$$

it suffices to show that

$$(a-b)^2+(a-c)^2 \ge a^2+b^2+c^2-ab-bc-ca.$$

Indeed,

$$(a-b)^2+(a-c)^2-(a^2+b^2+c^2-ab-bc-ca) = (a-b)(a-c) \ge 0.$$

The equality holds for $a=b=c$, for $a=b$ and $c=0$, and for $a=c$ and $b=0$.

(b) Write the inequality as follows:

$$\frac{4(b-c)^2}{(a+b+c)^2} \le \frac{(b-c)^2+(a-b)^2+(a-c)^2}{a^2+b^2+c^2},$$

$$(a+b+c)^2[(a-b)^2+(a-c)^2] \ge (b-c)^2[3(a^2+b^2+c^2)-2(ab+bc+ca)],$$

$$(a+b+c)^2[(b-c)^2+2(a-b)(a-c)] \ge (b-c)^2[3(a^2+b^2+c^2)-2(ab+bc+ca)],$$

$$(a+b+c)^2(a-b)(a-c) \ge (b-c)^2[a^2+b^2+c^2-2(ab+bc+ca)].$$

Since

$$a^2+b^2+c^2-2(ab+bc+ca) = (a-b)^2-c(2a+2b-c) \le (a-b)^2,$$

it suffices to show that

$$(a+b+c)^2(a-c) \ge (b-c)^2(a-b).$$

This inequality is true because

$$(a+b+c)^2 \ge (b-c)^2$$

and

$$a-c \ge a-b.$$

The equality holds for $a=b=c$, for $a=b$ and $c=0$, and for $a=c$ and $b=0$.

□

P 2.61. *Let a, b, c be positive real numbers. Prove that*

(a) if $a \ge b \ge c$, then

$$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{(a-c)^2}{a^2-ac+c^2}\ge 1;$$

(b) if $a \ge 1 \ge b \ge c$ and $abc = 1$, then

$$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{(b-c)^2}{b^2-bc+c^2}\le 1.$$

(Vasile C., 2014)

Solution. (a) Write the inequality as follows:

$$\frac{ab+bc+ca}{a^2+b^2+c^2}\ge \frac{ac}{a^2-ac+c^2},$$

$$acb^2-(a+c)(a^2-ac+c^2)b+a^2c^2\le 0,$$

$$acb^2-(a^3+c^3)b+a^2c^2\le 0,$$

$$(ab-c^2)(bc-a^2)\le 0.$$

Because $ab-c^2\ge 0$ and $bc-a^2\le 0$, the conclusion follows. The equality holds for $a=b=c$.

(b) From

$$b^3\le 1=abc,$$

it follows that

$$b^2\le ac.$$

Write the inequality as follows:

$$\frac{ab+bc+ca}{a^2+b^2+c^2}\le \frac{bc}{b^2-bc+c^2},$$

$$bca^2-(b+c)(b^2-bc+c^2)a+b^2c^2\ge 0,$$

$$bca^2-(b^3+c^3)a+b^2c^2\ge 0,$$

$$(ab-c^2)(ac-b^2)\ge 0.$$

The inequality is true because $ab-c^2\ge 0$ and $ac-b^2\ge 0$. The equality holds for $a=b=c=1$.

□

P 2.62. *Let a, b, c be positive real numbers such that $a = \min\{a, b, c\}$. Prove that*

(a) $$\frac{a^2+b^2+c^2}{ab+bc+ca} \ge 1 + \frac{4(b-c)^2}{3(b+c)^2};$$

(b) $$\frac{a^2+b^2+c^2}{ab+bc+ca} \ge 1 + \frac{(a-b)^2}{(a+b)^2}.$$

(Vasile C., 2014)

Solution. (a) *First Solution.* Since

$$3(b+c)^2 \ge 12bc \ge 4(ab+bc+ca),$$

it suffices to prove that

$$\frac{a^2+b^2+c^2}{ab+bc+ca} \ge 1 + \frac{(b-c)^2}{ab+bc+ca},$$

which is equivalent to the obvious inequality

$$(a-b)(a-c) \ge 0.$$

The equality holds for $a = b = c$.

Second Solution. Since $(b+c)^2 \ge 4bc$, it suffices to prove that

$$\frac{a^2+b^2+c^2}{ab+bc+ca} \ge 1 + \frac{(b-c)^2}{3bc}.$$

Write this inequality as follows:

$$\frac{(a-b)^2+(a-c)^2+(b-c)^2}{ab+bc+ca} \ge \frac{2(b-c)^2}{3bc},$$

$$3bc[(a-b)^2+(a-c)^2] \ge (b-c)^2(2ab+2ac-bc),$$

$$3bc[(b-c)^2+2(a-b)(a-c)] \ge (b-c)^2(2ab+2ac-bc),$$

$$6bc(a-b)(a-c)+2(b-c)^2(2bc-ab-ac) \ge 0.$$

The last inequality is true because $(a-b)(a-c) \ge 0$ and

$$2bc-ab-ac = b(c-a)+c(b-a) \ge 0.$$

(b) Write the inequality as follows:

$$\frac{a^2+b^2+c^2}{ab+(a+b)c} \ge \frac{2(a^2+b^2)}{(a+b)^2},$$

$$(a+b)^2c^2 - 2(a+b)(a^2+b^2)c + (a^2+b^2)^2 \ge 0,$$

$$[(a+b)c-(a^2+b^2)]^2 \ge 0.$$

The equality holds for $c = \dfrac{a^2+b^2}{a+b}$.

□

P 2.63. *If a, b, c are positive real numbers, then*

$$\frac{a^2+b^2+c^2}{ab+bc+ca} \ge 1+\frac{9(a-c)^2}{4(a+b+c)^2}.$$

(Vasile C., 2014)

Solution. Write the inequality as follows:

$$\frac{(b-c)^2+(a-b)^2+(a-c)^2}{ab+bc+ca} \ge \frac{9(a-c)^2}{2(a+b+c)^2},$$

$$2(a+b+c)^2[(b-c)^2+(a-b)^2] \ge (a-c)^2[5(ab+bc+ca)-2(a^2+b^2+c^2)],$$

$$2(a+b+c)^2[(a-c)^2-2(a-b)(b-c)] \ge (a-c)^2[5(ab+bc+ca)-2(a^2+b^2+c^2)],$$

$$(a-c)^2[4(a^2+b^2+c^2)-(ab+bc+ca)] \ge 4(a+b+c)^2(a-b)(a-c).$$

Consider further the nontrivial case $(a-b)(a-c) \ge 0$. Since

$$(a-c)^2 = [(a-b)+(b-c)]^2 \ge 4(a-b)(b-c),$$

it suffices to show that

$$4(a^2+b^2+c^2)-(ab+bc+ca) \ge (a+b+c)^2.$$

Indeed,

$$4(a^2+b^2+c^2)-(ab+bc+ca)-(a+b+c)^2 = 3(a^2+b^2+c^2-ab-bc-ca) \ge 0.$$

The equality holds for $a=b=c$.

□

P 2.64. *Let a, b, c be nonnegative real numbers, no two of which are zero. If $a = \min\{a, b, c\}$, then*

$$\frac{1}{\sqrt{a^2-ab+b^2}}+\frac{1}{\sqrt{b^2-bc+c^2}}+\frac{1}{\sqrt{c^2-ca+a^2}} \ge \frac{6}{b+c}.$$

Solution. Since

$$\frac{1}{\sqrt{a^2-ab+b^2}}+\frac{1}{\sqrt{b^2-bc+c^2}}+\frac{1}{\sqrt{c^2-ca+a^2}} \ge \frac{1}{b}+\frac{1}{\sqrt{b^2-bc+c^2}}+\frac{1}{c},$$

it suffices to show that

$$\frac{1}{b}+\frac{1}{\sqrt{b^2-bc+c^2}}+\frac{1}{c} \ge \frac{6}{b+c}.$$

Write this inequality as

$$\frac{b}{c}+\frac{c}{b}+\sqrt{\frac{b^2+c^2+2bc}{b^2+c^2-bc}}\ge 4,$$

which is equivalent to

$$\sqrt{\frac{x+2}{x-1}}\ge 4-x,$$

where $x=\frac{b}{c}+\frac{c}{b}$, $x\ge 2$. Consider the non-trivial case $2\le x\le 4$. The inequality is true if

$$\frac{x+2}{x-1}\ge (4-x)^2,$$

which is equivalent to

$$(x-2)(x^2-7x+9)\le 0.$$

This inequality is true because

$$x^2-7x+9<x^2-7x+10=(x-2)(x-5)\le 0.$$

The equality holds for $a=b=c$, and also $a=0$ and $b=c$.

□

P 2.65. *If* $a\ge 1\ge b\ge c\ge 0$ *such that*

$$ab+bc+ca=abc+2,$$

then

$$ac\le 4-2\sqrt{2}.$$

(Vasile C., 2012)

Solution. By hypothesis, we have

$$a=\frac{2-bc}{b+c-bc}.$$

Since

$$ac\le\frac{1}{2}a(b+c)=\frac{(2-bc)(b+c)}{2(b+c-bc)}=\frac{2-bc}{2-\frac{2bc}{b+c}}\le\frac{2-bc}{2-\sqrt{bc}},$$

it suffices to show that

$$\frac{2-bc}{2-\sqrt{bc}}\le 4-2\sqrt{2},$$

which is equivalent to

$$(\sqrt{bc}-2+\sqrt{2})^2\ge 0.$$

The equality holds for $a=2$ and $b=c=2-\sqrt{2}$.

□

P 2.66. *If a,b,c are nonnegative real numbers such that*

$$ab+bc+ca=3, \qquad a\le 1\le b\le c,$$

then

(a) $$a+b+c\le 4;$$

(b) $$2a+b+c\le 4.$$

Solution. From

$$(1-b)(1-c)\ge 0,$$

we get

$$bc\ge b+c-1.$$

Therefore, we have

$$3=a(b+c)+bc\ge a(b+c)+b+c-1=(a+1)(b+c)-1,$$

$$b+c\le \frac{4}{a+1},$$

hence

$$a+b+c-4\le a+\frac{4}{a+1}-4=\frac{a(a-3)}{a+1}\le 0,$$

$$2a+b+c-4\le 2a+\frac{4}{a+1}-4=\frac{2a(a-1)}{a+1}\le 0.$$

The equality holds for $a=0$, $b=1$ and $c=3$. In addition, the inequality (b) is also an equality for $a=b=c=1$.

□

P 2.67. *Let a,b,c be nonnegative real numbers such that $a\le b\le c$. Prove that*

(a) if $a+b+c=3$, *then*

$$a^4(b^4+c^4)\le 2;$$

(b) if $a+b+c=2$, *then*

$$c^4(a^4+b^4)\le 1.$$

(Vasile C., 2012)

Solution. (a) Let x, y be nonnegative real numbers. We claim that

$$x^4 - y^4 \geq 4y^3(x-y).$$

Indeed, this inequality follows from

$$\begin{aligned} x^4 - y^4 - 4y^3(x-y) &= (x-y)(x^3 + x^2y + xy^2 - 3y^3) \\ &= (x-y)[(x^3-y^3) + y(x^2-y^2) + y^2(x-y)]. \end{aligned}$$

Using this inequality, we can show that

$$b^4 + c^4 \leq a^4 + (b+c-a)^4.$$

Indeed, we have

$$\begin{aligned} a^4 + (b+c-a)^4 - b^4 - c^4 &= (a^4 - b^4) + (b+c-a)^4 - c^4 \\ &\geq 4b^3(a-b) + 4c^3(b+c-a-c) \\ &= 4(a-b)(b^3-c^3) \geq 0. \end{aligned}$$

Thus, it suffices to show that

$$a^4[a^4 + (b+c-a)^4] \leq 2,$$

which is equivalent to $f(a) \leq 2$, where

$$f(a) = a^8 + a^4(3-2a)^4, \quad 0 \leq a \leq 1.$$

If $f'(a) \geq 0$ for $0 \leq a \leq 1$, then $f(a)$ is increasing, hence $f(a) \leq f(1) = 2$. From the derivative

$$f'(a) = 4a^3[2a^4 - (4a-3)(3-2a)^3],$$

we need to show that

$$2a^4 \geq (4a-3)(3-2a)^3.$$

This inequality is true for the trivial case $0 \leq a \leq 3/4$. Consider further that $3/4 < a \leq 1$. We need to show that $h(a) \geq 0$, where

$$h(a) = \ln 2 + 4\ln a - \ln(4a-3) - 3\ln(3-2a), \quad 3/4 < a \leq 1.$$

From

$$h'(a) = \frac{4}{a} - \frac{4}{4a-3} + \frac{6}{3-2a} = \frac{6(7a-6)}{a(4a-3)(3-2a)},$$

it follows that $h(a)$ is decreasing on $(3/4, 6/7]$ and increasing on $[6/7, 1]$. Thus,

$$h(a) \geq h\left(\frac{6}{7}\right) = \ln 2 + 4\ln\frac{6}{7} - \ln\frac{3}{7} - 3\ln\frac{9}{7} = \ln\frac{32}{27} > 0.$$

The equality holds for $a = b = c = 1$.

(b) Since $a^4+b^4 \le (a+b)^4$, it suffices to show that

$$c^4(a+b)^4 \le 1,$$

which is true if

$$c(a+b) \le 1.$$

Indeed, we have

$$1-c(a+b) = 1-c(2-c) = (c-1)^2 \ge 0.$$

The equality holds for $a=0$ and $b=c=1$.

□

P 2.68. *If a,b,c are nonnegative real numbers such that $ab+bc+ca=3$, then*

(a) $$a^2+b^2+c^2-a-b-c \ge \frac{5}{8}(a-c)^2;$$

(b) $$a^2+b^2+c^2-a-b-c \ge \frac{5}{2}\min\{(a-b)^2,(b-c)^2,(c-a)^2\}.$$

(Vasile C., 2014)

Solution. Denote

$$E = a^2+b^2+c^2-a-b-c, \qquad S = a^2+b^2+c^2-ab-bc-ca.$$

From

$$(a+b+c)^2 \ge 3(ab+bc+ca),$$

it follows that

$$a+b+c \ge 3.$$

We have

$$a+b+c-\sqrt{3(ab+bc+ca)} = \frac{S}{a+b+c+\sqrt{3(ab+bc+ca)}},$$

$$a+b+c-3 = \frac{S}{a+b+c+3},$$

$$(a+b+c)^2-3(a+b+c) = \frac{(a+b+c)S}{a+b+c+3},$$

$$-3(a+b+c) = -(a+b+c)^2 + \frac{S}{1+\cfrac{3}{a+b+c}},$$

$$-3(a+b+c) \ge -(a+b+c)^2 + \frac{S}{2},$$

therefore

$$E \geq a^2 + b^2 + c^2 - \frac{1}{3}(a+b+c)^2 + \frac{S}{6},$$

which is equivalent to

$$E \geq \frac{5S}{6}.$$

(a) It suffices to show that

$$\frac{5S}{6} \geq \frac{5}{8}(a-c)^2,$$

which is equivalent to

$$\frac{S}{3} \geq \frac{(a-c)^2}{4},$$

$$\frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{3} \geq \frac{(a-c)^2}{2},$$

$$2(a-b)^2 + 2(b-c)^2 \geq (a-c)^2,$$

$$2(a-b)^2 + 2(b-c)^2 \geq [(a-b) + (b-c)]^2,$$

$$(a-b)^2 + (b-c)^2 \geq 2(a-b)(b-c),$$

$$(a-2b+c)^2 \geq 0.$$

The equality holds for $a = b = c = 1$.

(b) Due to symmetry, without loss of generality, assume that

$$a \geq b \geq c.$$

It suffices to show that

$$\frac{5S}{6} \geq \frac{5}{2}\min\{(a-b)^2, (b-c)^2\},$$

which is equivalent to

$$S \geq 3\min\{(a-b)^2, (b-c)^2\},$$

$$(a-b)^2 + (b-c)^2 + (a-c)^2 \geq 6\min\{(a-b)^2, (b-c)^2\},$$

$$(a-b)^2 + (b-c)^2 + [(a-b) + (b-c)]^2 \geq 6\min\{(a-b)^2, (b-c)^2\},$$

$$(a-b)^2 + (b-c)^2 + (a-b)(b-c) \geq 3\min\{(a-b)^2, (b-c)^2\}.$$

The last inequality is clearly true. The equality holds for $a = b = c = 1$.

□

P 2.69. *If a,b,c are nonnegative real numbers such that $ab+bc+ca=3$, then*

$$\frac{a^3+b^3+c^3}{a+b+c} \ge 1+\frac{5}{9}(a-c)^2.$$

(Vasile C., 2014)

Solution. It suffices to consider the case

$$a \ge b \ge c.$$

Write the inequality as

$$E \ge \frac{5}{9}(a-c)^2,$$

where

$$E = \frac{a^3+b^3+c^3}{a+b+c} - 1.$$

We have

$$\begin{aligned} E &= \frac{a^3+b^3+c^3}{a+b+c} - \frac{ab+bc+ca}{3} \\ &= \frac{3(a^3+b^3+c^3)-(a+b+c)(ab+bc+ca)}{3(a+b+c)} \\ &= \frac{A+B}{3(a+b+c)}, \end{aligned}$$

where

$$A = \sum [a^3+b^3-ab(a+b)], \qquad B = \sum a^3 - 3abc.$$

Since

$$A = \sum (a+b)(a-b)^2,$$

$$B = \frac{1}{2}(a+b+c)\sum (a-b)^2,$$

we get

$$E = \frac{\sum (3a+3b+c)(a-b)^2}{6(a+b+c)}.$$

Thus, we need to show that

$$\sum (3a+3b+c)(a-b)^2 \ge \frac{10}{3}(a+b+c)(a-c)^2,$$

which is equivalent to

$$3(3a+3b+c)(a-b)^2 + 3(a+3b+3c)(b-c)^2 \ge (a+7b+c)(a-c)^2.$$

Using the substitution

$$a = c+x, \quad b = c+y, \quad x \ge y \ge 0,$$

the inequality becomes

$$3(7c+3x+3y)(x-y)^2+3(7c+x+3y)y^2 \ge (9c+x+7y)x^2,$$

which is equivalent to

$$6c(2x^2-7xy+7y^2)+2(x+y)(2x-3y)^2 \ge 0.$$

This inequality is true since

$$2x^2-7xy+7y^2=(2x^2+7y^2)-7xy \ge (2\sqrt{14}-7)xy \ge 0.$$

The equality holds for $a=b=c=1$, and also for $a=3/\sqrt{2}$, $b=\sqrt{2}$, $c=0$. $\square$

P 2.70. *If a,b,c are nonnegative real numbers such that*

$$a \ge b \ge c, \quad ab+bc+ca=3,$$

then

(a) $$\frac{a^3+b^3+c^3}{a+b+c} \ge 1+\frac{7}{9}(a-b)^2;$$

(b) $$\frac{a^3+b^3+c^3}{a+b+c} \ge 1+\frac{2}{3}(b-c)^2.$$

(c) $$\frac{a^3+b^3+c^3}{a+b+c} \ge 1+\frac{7}{3}\min\{(a-b)^2,(b-c)^2\}.$$

(Vasile C., 2014)

Solution. As we have shown in the proof of the preceding problem P 2.69,

$$\frac{a^3+b^3+c^3}{a+b+c}-1=\frac{\sum(3a+3b+c)(a-b)^2}{6(a+b+c)}.$$

(a) Write the inequality as

$$\sum(3a+3b+c)(a-b)^2 \ge \frac{14}{3}(a+b+c)(a-b)^2,$$

$$3(a+3b+3c)(b-c)^2+3(3a+b+3c)(a-c)^2 \ge (5a+5b+11c)(a-b)^2.$$

It suffices to show that

$$3(3a+b+3c)(a-c)^2 \ge (5a+5b+11c)(a-b)^2.$$

This is true since

$$(a-c)^2 \ge (a-b)^2$$

and

$$3(3a+b+3c)-(5a+5b+11c) = 2(2a-b-c) \ge 0.$$

The equality holds for $a = b = c = 1$.

(b) Write the desired inequality as

$$\sum (3a+3b+c)(a-b)^2 \ge 4(a+b+c)(b-c)^2,$$

$$(3a+3b+c)(a-b)^2 + (3a+b+3c)(a-c)^2 \ge (3a+b+c)(b-c)^2.$$

It suffices to show that

$$(3a+b+3c)(a-c)^2 \ge (3a+b+c)(b-c)^2.$$

This is true since

$$(a-c)^2 \ge (a-b)^2$$

and

$$(3a+b+3c)-(3a+b+c) = 2c \ge 0.$$

The equality holds for $a = b = c = 1$, and also for $a = b = \sqrt{3}$, $c = 0$.

(c) Denote

$$m = \min\{(a-b)^2, (b-c)^2\},$$

then write the desired inequality as

$$\frac{\sum (3a+3b+c)(a-b)^2}{6(a+b+c)} \ge \frac{7}{3}m,$$

$$\sum (3a+3b+c)(a-b)^2 \ge 14(a+b+c)m,$$

$$(3a+3b+c)(a-b)^2+(a+3b+3c)(b-c)^2+(3a+b+3c)[(a-b)+(b-c)]^2 \ge 14(a+b+c)m,$$

$$(3a+2b+2c)(a-b)^2+(2a+2b+3c)(b-c)^2+(3a+b+3c)(a-b)(b-c) \ge 7(a+b+c)m.$$

Case 1: $a-2b+c \ge 0$. The inequality is true if

$$(3a+2b+2c)+(2a+2b+3c)+(3a+b+3c) \ge 7(a+b+c),$$

which is equivalent to $a-2b+c \ge 0$.

Case 2: $a-2b+c \le 0$. Since $a-b \le b-c$, we need to show that

$$(3a+2b+2c)(a-b)^2+(2a+2b+3c)(b-c)^2+(3a+b+3c)(a-b)(b-c) \ge 7(a+b+c)(a-b)^2,$$

which is equivalent to

$$(2a+2b+3c)(b-c)^2 + (3a+b+3c)(a-b)(b-c) \ge (4a+5b+5c)(a-b)^2.$$

Since $b-c \ge a-b \ge 0$, it suffices to show that

$$(2a+2b+3c)(a-b)(b-c)+(3a+b+3c)(a-b)(b-c) \ge (4a+5b+5c)(a-b)^2.$$

This is true if

$$(2a+2b+3c)(b-c)+(3a+b+3c)(b-c) \ge (4a+5b+5c)(a-b),$$

which is equivalent to

$$(5a+3b+6c)(b-c) \ge (4a+5b+5c)(a-b),$$

$$2(2b-a-c)(2b+2a+3c) \ge 0.$$

The equality holds for $2b = a+c$ and $a^2+4ac+c^2 = 6$.

□

P 2.71. *If a,b,c are nonnegative real numbers such that $ab+bc+ca=3$, then*

$$a^4+b^4+c^4-a^2-b^2-c^2 \ge \frac{11}{4}(a-c)^2.$$

(Vasile C., 2014)

Solution. It suffices to consider the case $a \ge b \ge c$. Denote

$$S = a^2+b^2+c^2, \quad q = ab+bc+ca.$$

Summing the identities

$$a^4+b^4+c^4-\frac{1}{3}S^2 = \frac{(a^2-b^2)^2+(b^2-c^2)^2+(c^2-a^2)^2}{3}$$

and

$$\frac{1}{3}S^2-\frac{1}{3}Sq = S\cdot\frac{(a-b)^2+(b-c)^2+(c-a)^2}{6},$$

we get

$$a^4+b^4+c^4-a^2-b^2-c^2 = \frac{(a^2-b^2)^2+(b^2-c^2)^2+(c^2-a^2)^2}{3} + S\cdot\frac{(a-b)^2+(b-c)^2+(c-a)^2}{6}.$$

Therefore, we can write the desired inequality in the homogeneous form

$$\frac{(a^2-b^2)^2+(b^2-c^2)^2+(c^2-a^2)^2}{3}+S\cdot\frac{(a-b)^2+(b-c)^2+(c-a)^2}{6} \ge \frac{11}{12}q(a-c)^2.$$

Since

$$(a-b)^2+(b-c)^2 \geq \frac{1}{2}[(a-b)+(b-c)]^2 = \frac{1}{2}(a-c)^2,$$

it suffices to prove that

$$\frac{(a^2-b^2)^2+(b^2-c^2)^2+(c^2-a^2)^2}{3}+\frac{S(a-c)^2}{4} \geq \frac{11}{12}q(a-c)^2,$$

which is equivalent to

$$4(a+b)^2(a-b)^2+4(b+c)^2(b-c)^2+E(a-c)^2 \geq 0,$$

where

$$E = 4(a+c)^2+3S-11q.$$

Using the substitution

$$b = c+x, \quad a = c+x+y, \quad x,y \geq 0,$$

the inequality becomes

$$4(2c+2x+y)^2y^2+4(2c+x)^2x^2+E(x+y)^2 \geq 0,$$

where

$$E = -8c^2-16xc-x^2+7y^2+3xy.$$

Write this inequality as

$$Ac^2+D \geq 2Bc,$$

where

$$A = 8(x-y)^2, \;\; B = 8y(x-y)(2x+y), \quad D = 3x^4+11y^4+28x^2y^2+33xy^3+x^3y.$$

Since $Ac^2+D \geq 2c\sqrt{AD}$, it suffices to show that $AD \geq B^2$. Indeed,

$$\begin{aligned} AD-B^2 &= 8(x-y)^2[3x^4+11y^4+28x^2y^2+33xy^3+x^3y-8y^2(2x+y)^2] \\ &= 8(x-y)^2[x^4+y^4+2(x^2-y^2)^2+xy(x^2+y^2)] \geq 0. \end{aligned}$$

This completes the proof. The equality holds for $a=b=c=1$.

□

P 2.72. *If a,b,c are nonnegative real numbers such that*

$$a \geq b \geq c, \quad ab+bc+ca = 3,$$

then

(a) $$a^4+b^4+c^4-a^2-b^2-c^2 \geq \frac{11}{3}(a-b)^2;$$

(b) $$a^4+b^4+c^4-a^2-b^2-c^2 \geq \frac{10}{3}(b-c)^2.$$

(Vasile C., 2014)

Solution. Denote

$$S = a^2 + b^2 + c^2, \quad q = ab + bc + ca.$$

As we have shown in the proof of the preceding problem,

$$\begin{aligned} a^4 + b^4 + c^4 - a^2 - b^2 - c^2 = &\frac{(a^2-b^2)^2 + (b^2-c^2)^2 + (c^2-a^2)^2}{3} \\ &+ S \cdot \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{6}. \end{aligned}$$

(a) Write the desired inequality in the homogeneous form

$$(a^2-b^2)^2+(b^2-c^2)^2+(c^2-a^2)^2+S\cdot\frac{(a-b)^2+(b-c)^2+(c-a)^2}{2} \ge \frac{11}{3}q(a-b)^2.$$

Since

$$(a^2-b^2)^2+(b^2-c^2)^2+(c^2-a^2)^2 \ge (a^2-b^2)^2+(a^2-c^2)^2 \ge 2(a^2-b^2)^2$$

and

$$(a-b)^2+(b-c)^2+(c-a)^2 \ge (a-b)^2+(a-c)^2 \ge 2(a-b)^2,$$

it suffices to prove that

$$2(a+b)^2 + a^2 + b^2 + c^2 \ge \frac{11}{3}(ab+bc+ca);$$

that is,

$$9(a^2+b^2) + ab + 3c^2 \ge 11c(a+b).$$

Since

$$9(a^2+b^2) + ab - \frac{19}{4}(a+b)^2 = \frac{17}{4}(a-b)^2 \ge 0,$$

we have

$$\begin{aligned} 9(a^2+b^2) + ab + 3c^2 - 11c(a+b) &\ge \frac{19}{4}(a+b)^2 + 3c^2 - 11c(a+b) \\ &= \frac{(a+b-2c)(19a+19b-6c)}{4} \ge 0. \end{aligned}$$

The equality holds for $a = b = c = 1$.

(b) Write the desired inequality in the homogeneous form

$$(a^2-b^2)^2+(b^2-c^2)^2+(c^2-a^2)^2+S\cdot\frac{(a-b)^2+(b-c)^2+(c-a)^2}{2} \ge \frac{10}{3}q(b-c)^2.$$

Since

$$\begin{aligned} (a^2-b^2)^2+(b^2-c^2)^2+(c^2-a^2)^2 &\ge (b^2-c^2)^2+(a^2-c^2)^2 \\ &\ge (b+c)^2(b-c)^2+(a+c)^2(b-c)^2 \end{aligned}$$

and

$$(a-b)^2+(b-c)^2+(c-a)^2 \ge (b-c)^2+(a-c)^2 \ge 2(b-c)^2,$$

it suffices to prove that

$$(b+c)^2+(a+c)^2+a^2+b^2+c^2 \ge \frac{10}{3}(ab+bc+ca);$$

that is,

$$6(a^2+b^2)-10ab+9c^2 \ge 4c(a+b).$$

Since

$$6(a^2+b^2)-10ab-\frac{1}{2}(a+b)^2 = \frac{11}{2}(a-b)^2 \ge 0,$$

we have

$$\begin{aligned}
6(a^2+b^2)-10ab+9c^2-4c(a+b) &\ge \frac{1}{2}(a+b)^2+9c^2-4c(a+b)\\
&\ge 2\sqrt{\frac{9}{2}}\,c(a+b)-4c(a+b)\\
&= (3\sqrt{2}-4)c(a+b) \ge 0.
\end{aligned}$$

The equality holds for $a=b=c=1$.

Remark. Similarly, we can prove the following refinement of the inequality in (b):

$$a^4+b^4+c^4-a^2-b^2-c^2 \ge \frac{1+\sqrt{33}}{2}\,(b-c)^2,$$

with equality for $a=b=c=1$, and also for $a=b=\dfrac{3+\sqrt{33}}{4}\,c$.

□

P 2.73. *Let a,b,c be nonnegative real numbers such that*

$$a \le b \le c, \quad a+b+c=3.$$

Find the greatest real number k such that

$$\sqrt{(56b^2+25)(56c^2+25)}+k(b-c)^2 \le 14(b+c)^2+25.$$

(Vasile C., 2014)

Solution. For $a=b=0$ and $c=3$, the inequality becomes

$$115+9k \le 126+25, \quad k \le 4.$$

To show that 4 is the greatest possible value of k, we need to prove the inequality

$$\sqrt{(56b^2+25)(56c^2+25)}+4(b-c)^2 \le 14(b+c)^2+25,$$

which is equivalent to

$$\sqrt{(56b^2+25)(56c^2+25)} \le 10(b^2+c^2)+36bc+25.$$

By squaring, the inequality becomes as follows:

$$(10b^2+10c^2+36bc)^2-56^2b^2c^2 \ge 50[28(b^2+c^2)-(10b^2+10c^2+36bc)],$$

$$20(b-c)^2(5b^2+5c^2+46bc) \ge 900(b-c)^2,$$

$$20(b-c)^2(5b^2+5c^2+46bc-45) \ge 0.$$

Therefore, we need to show that

$$5(b+c)^2+36bc-45 \ge 0.$$

From $(a-b)(a-c) \ge 0$, we get

$$bc \ge a(b+c)-a^2 = a(3-a)-a^2 = 3a-2a^2.$$

Thus,

$$5(b+c)^2+36bc-45 \ge 5(3-a)^2+36(3a-2a^2)-45 = a(78-67a) \ge 0.$$

The proof is completed. If $k=4$, then the equality holds for $a=b=c=1$ and also for $a=b=0$ and $c=3$.

□

P 2.74. *If $a \ge b \ge c > 0$ such that $abc=1$, then*

$$3(a+b+c) \le 8+\frac{a}{c}.$$

(Vasile C., 2009)

Solution. Write the inequality in the homogeneous form

$$\frac{3(a+b+c)}{\sqrt[3]{abc}} \le 8+\frac{a}{c},$$

which is equivalent to

$$\frac{3(x^3+y^3+z^3)}{xyz} \le 8+\frac{x^3}{z^3}, \quad x \ge y \ge z > 0.$$

We show that
$$\frac{x^3+y^3+z^3}{xyz} \le \frac{x^3+2z^3}{xz^2} \le \frac{1}{3}\left(8+\frac{x^3}{z^3}\right).$$
Write the left inequality as
$$(y-z)[x^3+z^3-yz(y+z)] \ge 0.$$
This is true since
$$x^3+z^3-yz(y+z) \ge y^3+z^3-yz(y+z) = (y+z)(y-z)^2 \ge 0.$$
Write the right inequality as
$$(x-z)(x^3-2x^2z-2xz^2+6z^3) \ge 0.$$
This is also true since
$$x^3-2x^2z-2xz^2+6z^3 = (x-z)^3+z(x^2-5xz+7z^2) \ge 0.$$
The equality holds for $a=b=c=1$.

□

P 2.75. *If $a \ge b \ge c > 0$, then*
$$(a+b-c)(a^2b-b^2c+c^2a) \ge (ab-bc+ca)^2.$$

Solution. Making the substitution
$$a=(p+1)c, \quad b=(q+1)c, \quad p \ge q \ge 0,$$
we get
$$a+b-c=(p+q+1)c,$$
$$a^2b-b^2c+c^2a=(p^2q+p^2+2pq-q^2+3p-q+1)c^3,$$
$$ab-bc+ca=(pq+2p+1)c^2.$$
Thus, the inequality becomes
$$(p+q+1)(p^2q+p^2+2pq-q^2+3p-q+1) \ge (pq+2p+1)^2,$$
which is equivalent to the obvious inequality
$$p^3(q+1)+q^2(p-q)+2q(p-q) \ge 0.$$
The equality holds for $a=b=c$.

□

P 2.76. *If $a \ge b \ge c \ge 0$, then*

$$\frac{(a-c)^2}{2(a+c)} \le a+b+c-3\sqrt[3]{abc} \le \frac{2(a-c)^2}{a+5c}.$$

(Vasile C., 2007)

Solution. (a) To prove the inequality

$$a+b+c-3\sqrt[3]{abc} \ge \frac{(a-c)^2}{2(a+c)},$$

we will show that

$$a+b+c-3\sqrt[3]{abc} \ge a+c-2\sqrt{ac} \ge \frac{(a-c)^2}{2(a+c)}. \qquad (*)$$

The left inequality is equivalent to

$$b+2\sqrt{ac} \ge 3\sqrt[3]{abc},$$

which is a consequence of the AM-GM inequality. The right inequality in (*) can be written as follows:

$$a^2+c^2+6ac \ge 4(a+c)\sqrt{ac},$$

$$\left(\sqrt{a}-\sqrt{b}\right)^4 \ge 0.$$

The equality holds for $a=b=c$.

(b) To prove the inequality

$$a+b+c-3\sqrt[3]{abc} \le \frac{2(a-c)^2}{a+5c},$$

we will show that

$$a+b+c-3\sqrt[3]{abc} \le 2a+c-3\sqrt[3]{a^2c} \le \frac{2(a-c)^2}{a+5c}. \qquad (**)$$

Write the left inequality as

$$a-b-3\sqrt[3]{ac}\left(\sqrt[3]{a}-\sqrt[3]{b}\right) \ge 0,$$

$$\left(\sqrt[3]{a}-\sqrt[3]{b}\right)\left(\sqrt[3]{a^2}+\sqrt[3]{ab}+\sqrt[3]{b^2}-3\sqrt[3]{ac}\right) \ge 0.$$

This is true since

$$\sqrt[3]{a^2}+\sqrt[3]{ab}+\sqrt[3]{b^2} \ge 3\sqrt[3]{ab} \ge 3\sqrt[3]{ac}.$$

The right inequality in (**) is an equality for $c=0$. For $c>0$, due to homogeneity, we may assume that $c=1$. In addition, making the substitution $a=x^3$, $x \ge 1$, the right inequality in (**) becomes in succession

$$(x^3+5)(2x^3-3x^2+1) \le 2(x^3-1)^2,$$

$$(x-1)^2(x^3+2x^2-2x-1) \ge 0,$$

$$(x-1)^3(x^2+3x+1) \ge 0.$$

The equality holds for $a=b=c$, and also for $a=b$ and $c=0$.

□

P 2.77. *If $a \ge b \ge c \ge d \ge 0$, then*

$$\frac{(a-d)^2}{a+3d} \le a+b+c+d-4\sqrt[4]{abcd} \le \frac{3(a-d)^2}{a+5d}.$$

(Vasile C., 2009)

Solution. (a) To prove the inequality

$$a+b+c+d-4\sqrt[4]{abcd} \ge \frac{(a-d)^2}{a+3d},$$

we will show that

$$a+b+c+d-4\sqrt[4]{abcd} \ge a+d-2\sqrt{ad} \le \frac{(a-d)^2}{a+3d}. \tag{*}$$

The left inequality is equivalent to

$$b+c+2\sqrt{ad} \ge 4\sqrt[4]{abcd},$$

which is a consequence of the AM-GM inequality. The right inequality in (*) can be written as follows:

$$(a-d)^2 \ge (a+3d)\left(\sqrt{a}-\sqrt{d}\right)^2,$$

$$2\sqrt{d}\left(\sqrt{a}-\sqrt{d}\right)^3 \ge 0.$$

The equality holds for $a=b=c=d$, and also for $b=c=d=0$.

(b) To prove the inequality

$$a+b+c+d-4\sqrt[4]{abcd} \le \frac{3(a-d)^2}{a+5d},$$

we will show that

$$a+b+c+d-4\sqrt[4]{abcd} \le 2a+c+d-4\sqrt[4]{a^2cd} \le \frac{3(a-d)^2}{a+5d}. \tag{**}$$

Write the left inequality as

$$a-b-4\sqrt[4]{acd}\left(\sqrt[4]{a}-\sqrt[4]{b}\right) \ge 0,$$

$$\left(\sqrt[4]{a}-\sqrt[4]{b}\right)\left(\sqrt[4]{a^3}+\sqrt[4]{a^2b}+\sqrt[4]{ab^2}+\sqrt[4]{b^3}-4\sqrt[4]{acd}\right)\geq 0.$$

The last inequality follows from the AM-GM inequality:

$$\begin{aligned}\sqrt[4]{a^3}+\sqrt[4]{a^2b}+\sqrt[4]{ab^2}+\sqrt[4]{b^3}-4\sqrt[4]{acd} &\geq \sqrt[4]{a^3}+\sqrt[4]{a^2b}+\sqrt[4]{b^3}-3\sqrt[4]{ab^2}\\ &\geq \sqrt[4]{a^3}+\sqrt[4]{b^3}+\sqrt[4]{b^3}-3\sqrt[4]{ab^2}\geq 0.\end{aligned}$$

Write the right inequality in (**) as

$$F(c)\geq 0,$$

where

$$F(c)=3(a-d)^2-(a+5d)\left(2a+c+d-4\sqrt[4]{a^2cd}\right).$$

Since F is a concave function and $d\leq c\leq a$, it suffices to show that $F(d)\geq 0$ and $F(a)\geq 0$. We have

$$F(d)=3(a-d)^2-2(a+5d)\left(\sqrt{a}-\sqrt{d}\right)^2=\left(\sqrt{a}-\sqrt{d}\right)^3\left(\sqrt{a}+7\sqrt{d}\right)\geq 0$$

and

$$F(a)=3(a-d)^2-(a+5d)\left(3a+d-4\sqrt[4]{a^3d}\right).$$

Setting $a=1$ (due to homogeneity) and substituting $d=x^4$, $0\leq x\leq 1$, the inequality $F(a)\geq 0$ becomes

$$3(1-x^4)^2-(1+5x^4)(3+x^4-4x)\geq 0.$$

Since $3+x^4-4x=(1-x)^2(3+2x+x^2)$, we need to show that

$$3(1+x+x^2+x^3)^2-(1+5x^4)(3+2x+x^2)\geq 0,$$

which is equivalent to

$$x(2+4x+6x^2-3x^3-2x^4-x^5)\geq 0.$$

This inequality is true since

$$2+4x+6x^2-3x^3-2x^4-x^5>6x^2-3x^3-2x^4-x^5\geq 0.$$

The equality holds for $a=b=c=d$, and also for $a=b=c$ and $d=0$.

Remark. The following generalization holds.

- *If $a_1\geq a_2\geq\cdots\geq a_n\geq 0$, then*

$$a_1+a_2+\cdots+a_n-n\sqrt[n]{a_1a_2\cdots a_n}\leq\frac{(n-1)(a_1-a_n)^2}{a_1+k_na_n},$$

where

$$k_n=\begin{cases}7-\dfrac{8}{n+1}, & n\ odd\\ 7-\dfrac{8}{n}, & n\ even\end{cases}.$$

□

P 2.78. *If $a \ge b \ge c > 0$, then*

(a) $$a+b+c-3\sqrt[3]{abc} \ge \frac{3(a-b)^2}{5a+4b};$$

(b) $$a+b+c-3\sqrt[3]{abc} \ge \frac{64(a-b)^2}{7(11a+24b)}.$$

(Vasile C., 2009)

Solution. We use the inequality

$$a+b+c-3\sqrt[3]{abc} \ge a+2b-3\sqrt[3]{ab^2},$$

which is equivalent to

$$3\sqrt[3]{ab}\left(\sqrt[3]{b}-\sqrt[3]{c}\right) \ge b-c,$$

$$\left(\sqrt[3]{b}-\sqrt[3]{c}\right)\left(3\sqrt[3]{ab}-\sqrt[3]{b^2}-\sqrt[3]{bc}-\sqrt[3]{c^2}\right) \ge 0.$$

Since $a \ge b \ge c$, the inequality is obvious.

(a) It suffices to show that

$$a+2b-3\sqrt[3]{ab^2} \ge \frac{3(a-b)^2}{5a+4b}.$$

Setting $b=1$ (due to homogeneity) and $a=x^3$, $x \ge 1$, this inequality becomes as follows:

$$(5x^3+4)(x^3-3x+2) \ge 3(x^3-1)^2,$$

$$(x-1)^2(2x^4+4x^3-9x^2-2x+5) \ge 0,$$

$$(x-1)^4(2x^2+8x+5) \ge 0.$$

The equality holds for $a=b=c$.

(b) It suffices to show that

$$a+2b-3\sqrt[3]{ab^2} \ge \frac{64(a-b)^2}{7(11a+24b)}.$$

Setting $b=1$ and $a=x^3$, $x \ge 1$, this inequality becomes in succession:

$$7(11x^3+24)(x^3-3x+2) \ge 64(x^3-1)^2,$$

$$(x-1)^2(13x^4+26x^3-192x^2+40x+272) \ge 0,$$

$$(x-1)^2(x-2)^2(13x^3+78x+68) \ge 0.$$

The equality holds for $a=b=c$, and for $\frac{a}{8}=b=c$.

□

P 2.79. *If $a \ge b \ge c > 0$, then*

(a) $$a+b+c-3\sqrt[3]{abc} \ge \frac{3(b-c)^2}{4b+5c};$$

(b) $$a+b+c-3\sqrt[3]{abc} \ge \frac{25(b-c)^2}{7(3b+11c)}.$$

(Vasile C., 2009)

Solution. We use the inequality

$$a+b+c-3\sqrt[3]{abc} \ge 2b+c-3\sqrt[3]{b^2c},$$

which is equivalent to

$$a-b \ge 3\sqrt[3]{bc}\left(\sqrt[3]{a}-\sqrt[3]{b}\right),$$

$$\left(\sqrt[3]{a}-\sqrt[3]{b}\right)\left(\sqrt[3]{a^2}+\sqrt[3]{ab}+\sqrt[3]{b^2}-3\sqrt[3]{bc}\right) \ge 0.$$

Since $a \ge b \ge c$, the inequality is obvious.

(a) It suffices to show that

$$2b+c-3\sqrt[3]{b^2c} \ge \frac{3(b-c)^2}{4b+5c}.$$

Setting $c=1$ and $b=x^3$, $x \ge 1$, this inequality becomes as follows:

$$(4x^3+5)(2x^3-3x^2+1) \ge 3(x^3-1)^2,$$

$$(x-1)^2(5x^4-2x^3-9x^2+4x+2) \ge 0,$$

$$(x-1)^4(5x^2+8x+2) \ge 0.$$

The equality holds for $a=b=c$.

(b) It suffices to show that

$$2b+c-3\sqrt[3]{b^2c} \ge \frac{25(b-c)^2}{7(3b+11c)}.$$

Setting $c=1$ and $b=x^3$, $x \ge 1$, this inequality becomes in succession:

$$7(3x^3+11)(2x^3-3x^2+1) \ge 25(x^3-1)^2,$$

$$(x-1)^2(17x^4-29x^3-75x^2+104x+52) \ge 0,$$

$$(x-1)^2(x-2)^2(17x^3+39x+13) \ge 0.$$

The equality holds for $a=b=c$, and for $a=b=8c$.

Remark. The following generalization holds.

- *If $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$, then*

$$a_1+a_2+\cdots+a_n-n\sqrt[n]{a_1a_2\cdots a_n} \ge \frac{3i(n-j+1)(a_i-a_j)^2}{2(2n+i-2j+2)a_i+2(n+2i-j+1)a_j}$$

for all $i<j$.

□

P 2.80. *If $a \ge b \ge c > 0$, then*

$$a+b+c-3\sqrt[3]{abc} \ge \frac{3(a-c)^2}{4(a+b+c)}.$$

(Vasile C., 2009)

Solution. Due to homogeneity, assume that $a+b+c=3$. Let

$$x=\left(\frac{a+c}{2}\right)^2, \quad y=ac, \quad x \ge y.$$

We have

$$x=\left(\frac{3-b}{2}\right)^2, \quad x-y=\left(\frac{a-c}{2}\right)^2.$$

The desired inequality is equivalent to

$$3-3\sqrt[3]{by} \ge x-y.$$

There are two cases to consider.

Case 1: $b \le 1$. By the AM-GM inequality, we have

$$y+2\sqrt{b} \ge 3\sqrt[3]{by}.$$

Thus, it suffices to show that

$$3-2\sqrt{b} \ge x.$$

Indeed,

$$3-2\sqrt{b}-x=3-2\sqrt{b}-\left(\frac{3-b}{2}\right)^2=\frac{1}{4}\left(1-\sqrt{b}\right)^3\left(3+\sqrt{b}\right) \ge 0.$$

Case 2: $b \ge 1$. From

$$a+b+c=b+\frac{a+c}{2}+\frac{a+c}{2} \ge 3\sqrt[3]{b\left(\frac{a+c}{2}\right)^2},$$

we get

$$3 \geq 3\sqrt[3]{bx}.$$

Therefore, it suffices to prove that

$$3\sqrt[3]{bx} - 3\sqrt[3]{by} \geq x - y,$$

which is equivalent to

$$\left(\sqrt[3]{x} - \sqrt[3]{y}\right)\left(3\sqrt[3]{b} - \sqrt[3]{x^2} - \sqrt[3]{xy} - \sqrt[3]{y^2}\right) \geq 0.$$

Since

$$y \leq x = \left(\frac{3-b}{2}\right)^2 \leq 1 \leq b,$$

the inequality is clearly true. The equality holds for $a = b = c$

□

P 2.81. *If $a \geq b \geq c > 0$, then*

(a) $$a^6 + b^6 + c^6 - 3a^2b^2c^2 \geq 12a^2c^2(b-c)^2;$$

(b) $$a^6 + b^6 + c^6 - 3a^2b^2c^2 \geq 10a^3c(b-c)^2.$$

(Vasile C., 2014)

Solution. (a) Let us denote

$$E(a,b,c) = a^6 + b^6 + c^6 - 3a^2b^2c^2 - 12a^2c^2(b-c)^2.$$

We will show that

$$E(a,b,c) \geq E(b,b,c) \geq 0.$$

We have

$$\begin{aligned} E(a,b,c) - E(b,b,c) &= (a^2 - b^2)[a^4 + a^2b^2 + b^4 - 3b^2c^2 - 12c^2(b-c)^2] \\ &\geq (a^2 - b^2)[3b^2(b^2 - c^2) - 12c^2(b-c)^2] \\ &= 3(a^2 - b^2)(b-c)[b^3 + c(b-2c)^2] \geq 0. \end{aligned}$$

Also,

$$\begin{aligned} E(b,b,c) &= 2b^6 + c^6 - 3b^4c^2 - 12b^2c^2(b-c)^2 \\ &= (b^2 - c^2)^2(2b^2 + c^2) - 12b^2c^2(b-c)^2 \\ &= (b-c)^2(2b^4 + 4b^3c - 9b^2c^2 + 2bc^3 + c^4) \\ &= (b-c)^3(2b^3 + 6b^2c^2 - 3bc^2 - c^3) \geq 0. \end{aligned}$$

The equality holds for $a = b = c$.

(b) Let

$$E(a,b,c) = a^6 + b^6 + c^6 - 3a^2b^2c^2 - 12a^2c^2(b-c)^2.$$

We will show that

$$E(a,b,c) \geq E(b,b,c) \geq 0.$$

To prove the left inequality, it suffices to show that for fixed b and c, the function

$$f(a) = E(a,b,c)$$

is increasing on $[b,\infty)$; that is, $f'a) \geq 0$. Indeed, we have the derivative

$$\begin{aligned} f'(a) &= 6a[a^4 - b^2c^2 - 5ac(b-c)^2] \geq 6a[a^4 - a^2c^2 - 5ac(a-c)^2] \\ &= 6a^2(a-c)[a(a+c) - 5c(a-c)] = 6a^2(a-c)[(a-2c)^2 + c^2] \geq 0. \end{aligned}$$

With regard to the right inequality, we have

$$\begin{aligned} E(b,b,c) &= 2b^6 + c^6 - 3b^4c^2 - 10b^3c(b-c)^2 \\ &= (b^2-c^2)^2(2b^2+c^2) - 10b^3c(b-c)^2 = (b-c)^2 g(b,c), \end{aligned}$$

where

$$g(b,c) = 2b^4 - 6b^3c + 3b^2c^2 + 2bc^3 + c^4.$$

Since

$$g(b,c) = 2b(b-c)(b-2c)^2 + c \cdot h(b,c), \quad h(b,c) = 4b^3 - 13b^2c + 10bc^2 + c^3,$$

it suffices to show that $h(b,c) \geq 0$. For $b \geq 2c$, we have

$$h(b,c) = b(b-2c)(4b-5c) + c^3 > 0.$$

Also, for $c \leq b \leq 2c$, we have

$$2h(b,c) = (2c-b)(b-c)^2 + b(3b-5c)^2 \geq 0.$$

Thus, the proof is completed. The equality holds for $a = b = c$.

□

P 2.82. *If $a \geq b \geq c > 0$, then*

$$\frac{ab+bc}{a^2+b^2+c^2} \leq \frac{1+\sqrt{3}}{4}.$$

Solution. Denote

$$k = \frac{1+\sqrt{3}}{4} \approx 0.683,$$

and write the inequality as $E(a,b,c) \geq 0$, where

$$E(a,b,c) = k(a^2+b^2+c^2) - ab - bc.$$

We show that

$$E(a,b,c) \geq E(b,b,c) \geq 0.$$

We have

$$E(a,b,c) - E(b,b,c) = (a-b)[ka-(1-k)b] \geq (2k-1)(a-b)b \geq 0$$

and

$$E(b,b,c) = (2k-1)b^2 + kc^2 - bc \geq 2\sqrt{k(2k-1)}bc - bc = 0.$$

The equality holds for $a = b = \dfrac{1+\sqrt{3}}{2}c$.

□

P 2.83. *If $a \geq b \geq c \geq d > 0$, then*

$$\frac{ab+bc+cd}{a^2+b^2+c^2+d^2} \leq \frac{2+\sqrt{7}}{6}.$$

Solution. Write the inequality as $E(a,b,c,d) \geq 0$, where

$$E(a,b,c,d) = k(a^2+b^2+c^2+d^2) - ab - bc - cd, \qquad k = \frac{2+\sqrt{7}}{6} \approx 0.774.$$

We show that

$$E(a,b,c,d) \geq E(b,b,c,d) \geq E(c,c,c,d) \geq 0.$$

We have

$$E(a,b,c,d) - E(b,b,c,d) = (a-b)[ka-(1-k)b] \geq (2k-1)(a-b)b \geq 0,$$

$$E(b,b,c,d) - E(c,c,c,d) = (b-c)[(2k-1)b-(2-2k)c] \geq (4k-3)(b-c)c \geq 0$$

and

$$E(c,c,c,d) = (3k-2)c^2 + kd^2 - cd \geq 2\sqrt{k(3k-2)}cd - cd = 0.$$

The equality holds for $a = b = c = \dfrac{2+\sqrt{7}}{3}d$.

□

P 2.84. *If*

$$a \ge 1 \ge b \ge c \ge d \ge 0, \qquad a+b+c+d=4,$$

then

$$ab+bc+cd \le 3.$$

Solution. Write the inequality in the homogeneous form $E(a,b,c,d) \ge 0$, where

$$E(a,b,c,d) = 3(a+b+c+d)^2 - 16(ab+bc+cd).$$

From

$$a+b+c+d = 4 \ge 4b,$$

we get

$$a \ge 3b-c-d.$$

We will show that

$$E(a,b,c,d) \ge E(3b-c-d,b,c,d) \ge 0.$$

We have

$$\begin{aligned} E(a,b,c,d) - E(3b-c-d,b,c,d) &= 3[(a+b+c+d)^2 - (4b)^2] - 16b(a-3b+c+d) \\ &= (a-3b+c+d)(3a-b+3c+3d) \ge 0. \end{aligned}$$

Also,

$$E(3b-c-d,b,c,d) = 48b^2 - 16(3b^2 - bd + cd) = 16d(b-c) \ge 0.$$

The equality holds for

$$a \in [2,3], \quad b=1, \quad c=3-a, \quad d=0.$$

$\square$

P 2.85. *Let k and a,b,c be positive real numbers, and let*

$$E = (ka+b+c)\left(\frac{k}{a}+\frac{1}{b}+\frac{1}{c}\right), \qquad F = (ka^2+b^2+c^2)\left(\frac{k}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right).$$

(a) If $k \ge 1$, then

$$\sqrt{\frac{F-(k-2)^2}{2k}} + 2 \ge \frac{E-(k-2)^2}{2k};$$

(b) If $0 < k \le 1$, then

$$\sqrt{\frac{F-k^2}{k+1}} + 2 \ge \frac{E-k^2}{k+1}.$$

(Vasile C., 2007)

Solution. Due to homogeneity, we may assume that $bc = 1$. Under this assumption, if we denote

$$x = a + \frac{1}{a}, \quad y = b + \frac{1}{b} = c + \frac{1}{c}$$

$(x \geq 2,\ y \geq 2)$, then

$$\begin{aligned} E &= \left(ka + b + \frac{1}{b}\right)\left(\frac{k}{a} + b + \frac{1}{b}\right) \\ &= (ka + y)\left(\frac{k}{a} + y\right) \\ &= k^2 + kxy + y^2 \end{aligned}$$

and

$$\begin{aligned} F &= \left(ka^2 + b^2 + \frac{1}{b^2}\right)\left(\frac{k}{a^2} + b^2 + \frac{1}{b^2}\right) \\ &= (ka^2 + y^2 - 2)\left(\frac{k}{a^2} + y^2 - 2\right) \\ &= k^2 + k(x^2 - 2)(y^2 - 2) + (y^2 - 2)^2. \end{aligned}$$

(a) Write the inequality as

$$2kF - 2k(k-2)^2 \geq (E - k^2 - 4)^2.$$

We have

$$E - k^2 - 4 = kxy + y^2 - 4 > 0,$$
$$(E - k^2 - 4)^2 = k^2x^2y^2 + 2kxy(y^2 - 4) + (y^2 - 4)^2,$$

and

$$F - (k-2)^2 = 4k + k(x^2 - 2)(y^2 - 2) + y^2(y^2 - 4),$$
$$2kF - 2k(k-2)^2 = 8k^2 + 2k^2(x^2 - 2)(y^2 - 2) + 2ky^2(y^2 - 4).$$

Therefore,

$$2kF - 2k(k-2)^2 - (E - k^2 - 4)^2 = (y^2 - 4)[k^2(x^2 - 4) - 2ky(x - y) - (y^2 - 4)].$$

Since $y^2 - 4 \geq 0$, we still need to show that

$$k^2(x^2 - 4) - 2ky(x - y) \geq y^2 - 4.$$

We will show that

$$k^2(x^2 - 4) - 2ky(x - y) \geq (x^2 - 4) - 2y(x - y) \geq y^2 - 4.$$

The right inequality reduces to $(x - y)^2 \geq 0$, and the left inequality is equivalent to

$$(k-1)[(k+1)(x^2 - 4) - 2y(x - y)] \geq 0.$$

This is true because

$$(k+1)(x^2-4)-2y(x-y) \geq 2(x^2-4)-2y(x-y) = 2(x-y)^2+2(xy-4) \geq 0.$$

The equality holds for $b=c$. If $k=1$, then the equality holds for $a=b$ or $b=c$ or $c=a$.

(b) Write the inequality as

$$(k+1)(F-k^2) \geq (E-k^2-2k-2)^2.$$

We have

$$E-k^2-2k-2 = k(xy-2)+y^2-2 > 0,$$

$$(E-k^2-2k-2)^2 = k^2(xy-2)^2+2k(xy-2)(y^2-2)+(y^2-2)^2,$$

and

$$(k+1)(F-k^2) = k^2(x^2-2)(y^2-2)+k(y^2-2)(x^2+y^2-4)+(y^2-2)^2.$$

Thus,

$$\begin{aligned}(k+1)(F-k^2)-(E-k^2-2k-2)^2 &= k(x-y)^2(y^2-2k-2)\\ &\geq k(x-y)^2(y^2-4) \geq 0.\end{aligned}$$

If $0<k<1$, then the equality holds for $a=b$ or $a=c$.

□

P 2.86. *If a,b,c are positive real numbers, then*

$$\frac{a}{2b+6c}+\frac{b}{7c+a}+\frac{25c}{9a+8b} > 1.$$

Solution. By the Cauchy-Schwarz inequality, we have

$$\frac{a}{2b+6c}+\frac{b}{7c+a}+\frac{25c}{9a+8b} \geq \frac{(a+b+5c)^2}{a(2b+6c)+b(7c+a)+c(9a+8b)}.$$

Therefore, it suffices to show that

$$(a+b+5c)^2 \geq 3ab+15bc+15ca,$$

which is equivalent to

$$a^2+b^2+25c^2-ab-5bc-5ca \geq 0.$$

Indeed, we have

$$\begin{aligned}2(a^2+b^2+25c^2-ab-5bc-5ca) &= (a-b)^2+a^2+b^2+50c^2-10bc-10ca\\ &= (a-b)^2+(a-5c)^2+(b-5c)^2 \geq 0.\end{aligned}$$

□

P 2.87. *If a,b,c are positive real numbers such that*

$$\frac{1}{a} \ge \frac{1}{b} + \frac{1}{c},$$

then

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \ge \frac{55}{12(a+b+c)}.$$

(Vasile C., 2014)

Solution. Denote

$$x = \frac{bc}{b+c}, \quad a \le x,$$

and write the desired inequality as

$$\sum \frac{a+b+c}{b+c} \ge \frac{55}{12},$$

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{19}{12}.$$

Using the Cauchy-Schwarz inequality

$$\frac{b}{c+a} + \frac{c}{a+b} \ge \frac{(b+c)^2}{b(c+a)+c(a+b)},$$

it suffices to show that

$$F(a,b,c) \ge \frac{19}{12},$$

where

$$F(a,b,c) = \frac{a}{b+c} + \frac{(b+c)^2}{a(b+c)+2bc}.$$

We will show that

$$F(a,b,c) \ge F(x,b,c) \ge \frac{19}{12}.$$

Since

$$F(a,b,c) - F(x,b,c) = (x-a)\left[-\frac{1}{b+c} + \frac{(b+c)^3}{(a(b+c)+2bc)(x(b+c)+2bc)}\right],$$

we need to prove that

$$(b+c)^4 \ge [a(b+c)+2bc][(x(b+c)+2bc].$$

Since

$$a(b+c)+2bc \le x(b+c)+2bc,$$

it is enough to show that

$$(b+c)^2 \ge x(b+c)+2bc,$$

which is equivalent to the obvious inequality

$$(b+c)^2 \ge 3bc.$$

Also, we have

$$F(x,b,c)-\frac{19}{12}=\frac{bc}{(b+c)^2}+\frac{(b+c)^2}{3bc}-\frac{19}{12}=\frac{(b-c)^2(4b^2+5bc+4c^2)}{12bc(b+c)^2}\ge 0.$$

The equality occurs for $2a=b=c$.

□

P 2.88. *If a,b,c are positive real numbers such that*

$$\frac{1}{a}\ge\frac{1}{b}+\frac{1}{c},$$

then

$$\frac{1}{a^2+b^2}+\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2}\ge\frac{189}{40(a^2+b^2+c^2)}.$$

(Vasile C., 2014)

Solution. Denote

$$x=\frac{bc}{b+c},\quad a\le x,$$

and write the desired inequality as

$$\sum\frac{a^2+b^2+c^2}{b^2+c^2}\ge\frac{189}{40},$$

$$\frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2}\ge\frac{69}{40}.$$

Using the Cauchy-Schwarz inequality

$$\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2}\ge\frac{(b^2+c^2)^2}{b^2(c^2+a^2)+c^2(a^2+b^2)},$$

it suffices to show that

$$F(a,b,c)\ge\frac{69}{40},$$

where

$$F(a,b,c)=\frac{a^2}{b^2+c^2}+\frac{(b^2+c^2)^2}{a^2(b^2+c^2)+2b^2c^2}.$$

We will show that

$$F(a,b,c)\ge F(x,b,c)\ge\frac{69}{40}.$$

Since

$$F(a,b,c) - F(x,b,c) = (x^2 - a^2)\left[-\frac{1}{b^2+c^2} + \frac{(b^2+c^2)^3}{(a^2(b^2+c^2)+2b^2c^2)(x^2(b^2+c^2)+2b^2c^2)}\right],$$

we need to prove that

$$(b^2+c^2)^4 \ge [a^2(b^2+c^2)+2b^2c^2][x^2(b^2+c^2)+2b^2c^2].$$

Since

$$a^2(b^2+c^2)+2b^2c^2 \le x^2(b^2+c^2)+2b^2c^2,$$

it is enough to show that

$$(b^2+c^2)^2 \ge x^2(b^2+c^2)+2b^2c^2,$$

which is equivalent to

$$(b^4+c^4)(b+c)^2 \ge b^2c^2(b^2+c^2).$$

This inequality follows from $b^4+c^4 > b^2c^2$ and $(b+c)^2 > b^2+c^2$. Also, we have

$$F(x,b,c) = \frac{x^2}{b^2+c^2} + \frac{(b^2+c^2)^2}{x^2(b^2+c^2)+2b^2c^2}.$$

Since

$$2b^2c^2 \le 4x^2(b^2+c^2),$$

we have

$$F(x,b,c) \ge \frac{x^2}{b^2+c^2} + \frac{(b^2+c^2)^2}{5x^2(b^2+c^2)} = \frac{1}{t} + \frac{t}{5},$$

where

$$t = \frac{b^2+c^2}{x^2} \ge 8.$$

Therefore,

$$F(x,b,c) - \frac{69}{40} \ge \frac{1}{t} + \frac{t}{5} - \frac{69}{40} = \frac{(t-8)(8t-5)}{40t} \ge 0.$$

The equality occurs for $2a = b = c$.

□

P 2.89. *Find the best real numbers k, m, n such that*

$$(\sqrt{a}+\sqrt{b}+\sqrt{c})\sqrt{a+b+c} \ge ka + mb + nc$$

for all $a \ge b \ge c \ge 0$.

Solution. For $a=1$ and $b=c=0$, for $a=b=1$ and $c=0$, and for $a=b=c=1$, we get respectively

$$k \le 1, \quad k+m \le 2\sqrt{2}, \quad k+m+n \le 3\sqrt{3},$$

which yield

$$\begin{aligned} ka+mb+nc &= k(a-b)+(k+m)(b-c)+(k+m+nz)c \\ &\le a-b+2\sqrt{2}\,(b-c)+3\sqrt{3}\,c \\ &= a+(2\sqrt{2}-1)b+(3\sqrt{3}-2\sqrt{2})c. \end{aligned}$$

Therefore, if the following inequality holds

$$\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\sqrt{a+b+c} \ge a+(2\sqrt{2}-1)b+(3\sqrt{3}-2\sqrt{2})c,$$

then

$$k=1, \quad m=2\sqrt{2}-1, \quad n=3\sqrt{3}-2\sqrt{2}$$

are the best real k,m,n. Since

$$\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2 = a+\left(2\sqrt{ab}+b\right)+\left(2\sqrt{ac}+2\sqrt{bc}+c\right) \ge a+3b+5c,$$

it suffices to show that

$$(a+3b+5c)(a+b+c) \ge [a+(2\sqrt{2}-1)b+(3\sqrt{3}-2\sqrt{2})c]^2,$$

which is equivalent to the obvious inequality

$$(3-2\sqrt{2})b(a-b)+(3+2\sqrt{2}-3\sqrt{3})c(a-b)+3(5-2\sqrt{6})c(b-c) \ge 0.$$

If $k=1,\ m=2\sqrt{2}-1,\ n=3\sqrt{3}-2\sqrt{2}$, then the equality holds for $a=b=c$, for $a=b$ and $c=0$, and for $b=c=0$.

□

P 2.90. *Let $a,b \in (0,1]\,,\ \ a \le b$.*

(a) If $a \le \dfrac{1}{e}$, then

$$2a^a \ge a^b + b^a;$$

(b) If $b \ge \dfrac{1}{e}$, then

$$2b^b \ge a^b + b^a.$$

(Vasile C., 2012)

Solution. (a) We need to show that $f(a) \ge f(b)$, where

$$f(x) = a^x + x^a, \quad x \in [a, b].$$

This is true if $f(x)$ is decreasing; that is, if $f'(x) \le 0$ on $[a, b]$. Since the derivative

$$f'(x) = a(x^{a-1} + a^{x-1} \ln a) \le a(x^{a-1} - a^{x-1}),$$

it suffices to show that

$$x^{a-1} \le a^{x-1}$$

for $0 < a \le x \le 1$. Consider the non-trivial case $0 < a \le x < 1$, and write the inequality as $g(x) \ge g(a)$, where

$$g(x) = \frac{\ln x}{1-x}.$$

It suffices to show that $g'(x) \ge 0$ for $0 < x < 1$. We have

$$g'(x) = \frac{h(x)}{(1-x)^2}, \quad h(x) = \frac{1}{x} - 1 + \ln x.$$

Since

$$h'(x) = \frac{x-1}{x^2} < 0,$$

$h(x)$ is strictly decreasing, $h(x) > h(1) = 0$, $g'(x) > 0$. This completes the proof. The equality holds for $a = b$.

(b) We need to show that $f(b) \ge f(a)$, where

$$f(x) = x^b + b^x, \quad x \in [a, b].$$

This is true if $f(x)$ is increasing; that is, if $f'(x) \ge 0$ on $[a, b]$. Since the derivative

$$f'(x) = b(x^{b-1} + b^{x-1} \ln b) \ge b(x^{b-1} - b^{x-1}),$$

it suffices to show that

$$x^{b-1} \ge b^{x-1}$$

for $0 < x \le b \le 1$. As we shown at (a), this inequality is true. The equality holds for $a = b$.

□

P 2.91. *If* $0 \le a \le b$ *and* $b \ge \frac{1}{2}$, *then*

$$2b^{2b} \ge a^{2b} + b^{2a}.$$

(Vasile C., 2012)

Solution. We need to show that $f(a) \le f(b)$, where

$$f(x) = x^{2b} + b^{2x}, \quad x \in [0, b].$$

From the derivative

$$f''(x) = 2b\left[2b^{2x-1}\ln^2 b + (2b-1)x^{2b-2}\right] > 0, \quad x \in (0, b],$$

it follows that $f(x)$ is convex on $[0, b]$. Therefore, we have

$$f(a) \le \max\{f(0), f(b)\}.$$

From this, it follows that $f(a) \le f(b)$ if $f(0) \le f(b)$. To prove that $f(0) \le f(b)$, we apply Bernoulli's inequality as follows:

$$\begin{aligned} f(b) - f(0) &= 2b^{2b} - 1 = 2[1 + (b-1)]^{2b} - 1 \\ &\ge 2[1 + 2b(b-1)] - 1 = (2b-1)^2 \ge 0. \end{aligned}$$

The equality holds for $a = b \ge \dfrac{1}{2}$, and also for $a = 0$ and $b = \dfrac{1}{2}$.

□

P 2.92. *If $a \ge b \ge 0$, then*

(a) $$a^{b-a} \le 1 + \frac{a-b}{\sqrt{a}};$$

(b) $$a^{a-b} \ge 1 - \frac{3(a-b)}{4\sqrt{a}}.$$

(Vasile C., 2010)

Solution. (a) Write the inequality as

$$(a-b)\ln a + \ln\left(1 + \frac{a-b}{\sqrt{a}}\right) \ge 0,$$

which follows by adding the inequalities

$$\ln\left(1 + \frac{a-b}{\sqrt{a}}\right) - \frac{a-b}{\sqrt{a}} + \frac{(a-b)^2}{2a} \ge 0,$$

$$(a-b)\ln a + \frac{a-b}{\sqrt{a}} - \frac{(a-b)^2}{2a} \ge 0.$$

Denoting

$$x = \frac{a-b}{\sqrt{a}},$$

we can write the first inequality as $f(x) \ge 0$ for $x \ge 0$, where

$$f(x) = \ln(1+x) - x + \frac{x^2}{2}.$$

From the derivative

$$f'(x) = \frac{x^2}{1+x} \ge 0,$$

it follows that f is increasing, hence $f(x) \ge f(0) = 0$.

The second inequality is true if

$$\ln a + \frac{1}{\sqrt{a}} - \frac{a-b}{2a} \ge 0.$$

It suffices to prove that $g(a) \ge 0$, where

$$g(a) = \ln a + \frac{1}{\sqrt{a}} - \frac{1}{2}.$$

From

$$g'(a) = \frac{2\sqrt{a}-1}{2a\sqrt{a}},$$

it follows that g is decreasing on $(0, 1/4]$ and increasing on $[1/4, \infty)$; therefore,

$$g(a) \ge g\left(\frac{1}{4}\right) = \frac{3}{2} - \ln 4 > 0.$$

The equality holds for $a = b$.

(b) Consider the non-trivial case $1 - \dfrac{3(a-b)}{4\sqrt{a}} > 0$, write the inequality as

$$(a-b)\ln a \ge \ln\left(1 - \frac{3a-3b}{4\sqrt{a}}\right),$$

and prove it by adding the inequalities

$$0 \ge \ln\left(1 - \frac{3a-3b}{4\sqrt{a}}\right) + \frac{3(a-b)}{4\sqrt{a}},$$

$$(a-b)\ln a + \frac{3(a-b)}{4\sqrt{a}} \ge 0.$$

Denoting

$$x = \frac{3(a-b)}{4\sqrt{a}}, \quad 0 \le x < 1,$$

we can write the first inequality as $f(x) \le 0$, where

$$f(x) = \ln(1-x) + x.$$

From the derivative
$$f'(x) = \frac{-x}{1-x} \le 0,$$
it follows that f is decreasing, hence $f(x) \le f(0) = 0$.

The second inequality is true if $g(a) \ge 0$, where
$$g(a) = \ln a + \frac{3}{4\sqrt{a}}.$$
From the derivative
$$g'(a) = \frac{8\sqrt{a}-3}{8a\sqrt{a}},$$
it follows that
$$g(a) \ge g\left(\frac{9}{64}\right) = 2\ln\frac{3e}{8} > 0.$$
The equality holds for $a = b$.

□

P 2.93. *If a,b,c are positive real numbers such that*
$$a \ge b \ge c, \qquad ab^2c^3 \ge 1,$$
then
$$a + 2b + 3c \ge \frac{1}{a} + \frac{2}{b} + \frac{3}{c}.$$
(Vasile C., 2018)

Solution. It suffices to prove the homogeneous inequality
$$a + 2b + 3c \ge \sqrt[3]{ab^2c^3}\left(\frac{1}{a} + \frac{2}{b} + \frac{3}{c}\right).$$
Replacing a,b,c with a^3,b^3,c^3, the inequality becomes as follows:
$$a^3 + 2b^3 + 3c^3 \ge \frac{b^2c^3}{a^2} + \frac{2ac^3}{b} + 3ab^2,$$
$$a^3 + 2b^3 - 3ab^2 \ge \frac{c^3}{a^2b}(2a^3 - 3a^2b + b^3),$$
$$(a-b)^2(a+2b) \ge \frac{c^3}{a^2b}(a-b)^2(2a+b).$$
Thus, we need to show that
$$a^2b(a+2b) \ge c^3(2a+b)$$
for $a \ge b \ge c$. Since $c^3 \le ab^2$, we have
$$a^2b(a+2b) - c^3(2a+b) \ge a^2b(a+2b) - ab^2(2a+b) = ab(a^2-b^2) \ge 0.$$
The equality occurs for $a = b = 1/c \ge 1$.

□

P 2.94. *If a,b,c are positive real numbers such that*

$$a+b+c=3, \qquad a \le b \le c,$$

then

$$\frac{1}{a}+\frac{2}{b} \ge a^2+b^2+c^2.$$

(Vasile C., 2020)

Solution. Let

$$f(a,b,c)=\frac{1}{a}+\frac{2}{b}-a^2-b^2-c^2.$$

We will show that

$$f(a,b,c) \ge f(a,x,x) \ge 0,$$

where

$$x=\frac{b+c}{2}=\frac{3-a}{2}.$$

Since

$$f(a,b,c)-f(a,x,x)=\frac{2}{b}-\frac{2}{x}-(b^2+c^2-2x^2)$$
$$=\frac{2(c-b)}{b(b+c)}-\frac{(c-b)^2}{2}=\frac{(c-b)(b^3-bc^2+4)}{2b(b+c)},$$

we need to show that

$$b^3-bc^2+4 \ge 0.$$

Since $b+c<3$, we have

$$b^3-bc^2+4 > b^3-b(3-b)^2+4=6b^2+4-9b \ge (4\sqrt{6}-9)b > 0.$$

Also, since $a \le 1$, we have

$$f(a,x,x)=\frac{1}{a}+\frac{2}{x}-a^2-2x^2=\frac{1}{a}+\frac{4}{3-a}-a^2-\frac{1}{2}(3-a)^2$$
$$=\frac{a^4-5a^3+9a^2-7a+2}{a(3-a)}=\frac{(1-a)^3(2-a)}{a(3-a)} \ge 0.$$

The equality occurs for $a=b=c=1$.

□

P 2.95. *If a,b,c are positive real numbers such that*

$$a+b+c=3, \qquad a \le b \le c,$$

then

$$\frac{2}{a}+\frac{3}{b}+\frac{1}{c} \ge 2(a^2+b^2+c^2).$$

(Vasile C., 2020)

Solution. From
$$a \le b = 3 - a - c,$$
we get
$$a \le \frac{3-c}{2}.$$
For fixed b, write the inequality as $f(a) \ge 0$, where
$$f(a) = \frac{2}{a} + \frac{3}{b} + \frac{1}{c} - 2(a^2 + b^2 + c^2), \quad c = 3 - a - b.$$
We have
$$f'(a) = -\frac{2}{a^2} + \frac{1}{c^2} - 4(a - c) = \frac{1}{c^2} + 4c - 2g(a), \quad g(a) = 2a + \frac{1}{a^2}.$$
Since
$$g'(a) = 2 - \frac{2}{a^3} \le 0,$$
$g(a)$ is decreasing, hence
$$g(a) \ge g\left(\frac{3-c}{2}\right)$$
and
$$f'(a) \le \frac{1}{c^2} + 4c - 2g\left(\frac{3-c}{2}\right) = 6(c-1) - \frac{7c^2 + 6c - 9}{c^2(3-c)^2}$$
$$\le 6(c-1) - \frac{16}{81}(7c^2 + 6c - 9) = \frac{-2}{81}(56c^2 + 171 - 195c)$$
$$\le \frac{-2}{27}(4\sqrt{266} - 65)c < 0.$$
Therefore, $f(a)$ is decreasing. On the other hand, from $a \le b$ and $b \le c = 3 - a - b$, we get
$$a \le b, \quad a \le 3 - 2b.$$
There are two cases to consider: $b \in (0, 1]$ and $b \in [1, 3/2)$.

Case 1: $b \in (0, 1]$. Since $a \le b$, we have
$$f(a) \ge f(b) = \frac{5}{b} + \frac{1}{c} - 2(2b^2 + c^2), \quad c = 3 - 2b,$$
hence
$$f(a) \ge \frac{5}{b} + \frac{1}{3-2b} - 4b^2 - 2(3-2b)^2$$
$$= \frac{3(5-3b)}{b(3-2b)} - 3(4b^2 - 8b + 6)$$
$$= \frac{3(8b^4 - 28b^3 + 36b^2 - 21b + 5)}{b(3-2b)}$$

$$\geq \frac{3(8b^4 - 27b^3 + 35b^2 - 21b + 5)}{b(3-2b)}$$

$$= \frac{3(b-1)^2(8b^2 - 11b + 5)}{b(3-2b)} \geq 0.$$

Case 2: $b \in [1, 3/2)$. Since $a \leq 3 - b$, we have

$$f(a) \geq f(3-b) = \frac{2}{3-2b} + \frac{3}{b} + \frac{1}{c} - 2(3-2b)^2 - 2(b^2 + c^2), \quad c = b,$$

hence

$$f(a) \geq f(3-b) = \frac{2}{3-2b} + \frac{4}{b} - 2(3-2b)^2 - 4b^2$$

$$= \frac{6(2-b)}{b(3-2b)} - 6(2b^2 - 4b + 3)$$

$$= \frac{12(2b^4 - 7b^3 + 9b^2 - 5b + 1)}{b(3-2b)}$$

$$= \frac{12(b-1)^3(2b-1)}{b(3-2b)} \geq 0.$$

The equality occurs for $a = b = c = 1$.

Remark. Since

$$\frac{2}{a} + \frac{3}{b} + \frac{1}{c} \leq 2\left(\frac{1}{a} + \frac{2}{b}\right),$$

the inequality is stronger than the one of P 2.94.

□

P 2.96. *If a, b, c are positive real numbers such that*

$$a + b + c = 3, \qquad a \leq b \leq c,$$

then

$$\frac{31}{a} + \frac{25}{b} + \frac{25}{c} \geq 27(a^2 + b^2 + c^2).$$

(Vasile C., 2020)

Solution. From

$$a \leq b = 3 - a - c,$$

we get

$$a \leq \frac{3-c}{2}.$$

For fixed $c \in [1,3)$,write the inequality as $f(a) \geq 0$, where $a \leq \dfrac{3-c}{2}$ and

$$f(a) = \frac{31}{a} + \frac{25}{b} + \frac{25}{c} - 27(a^2+b^2+c^2), \quad b = 3-a-c.$$

We will show that

$$f(a) \geq f\left(\frac{3-c}{2}\right) \geq 0.$$

Since $a+b \leq 2$, we have

$$\frac{a+b}{a^2b^2} \geq \frac{16}{(a+b)^3} \geq 2,$$

therefore

$$f'a) = -\frac{31}{a^2} + \frac{25}{b^2} - 27(2a-2b) < -\frac{27}{a^2} + \frac{27}{b^2} - 54(a-b)$$

$$= 27(a-b)\left(\frac{a+b}{a^2b^2} - 2\right) \leq 0,$$

$f(a)$ is decreasing, hence $f(a)$ is minimal for $a = \dfrac{3-c}{2}$, when

$$b = 3-a-c = \frac{3-c}{2} = a.$$

So, we have

$$f\left(\frac{3-c}{2}\right) = \frac{56}{a} + \frac{25}{c} - 27(2a^2+c^2)$$

$$= \frac{112}{3-c} + \frac{25}{c} - \frac{27(3-c)^2}{2} - 27c^2$$

$$= \frac{3(27c^4 - 135c^3 + 243c^2 - 185c + 50)}{2c(3-c)}$$

$$= \frac{3(c-1)(3c-2)(3c-5)^2}{2c(3-c)} \geq 0.$$

The equality occurs for $a = b = c = 1$, and also for $a = b = \dfrac{2}{3}$ and $c = \dfrac{5}{3}$.

Remark. Actually, the following stronger inequalities are true:

$$\frac{29}{a} + \frac{27}{b} + \frac{25}{c} \geq 27(a^2+b^2+c^2),$$

$$\frac{28}{a} + \frac{28}{b} + \frac{25}{c} \geq 27(a^2+b^2+c^2). \tag{*}$$

For (*), we have

$$f(a) = \frac{28}{a} + \frac{28}{b} + \frac{25}{c} - 27(a^2+b^2+c^2), \quad b = 3-a-c,$$

$$f'a) = -\frac{28}{a^2} + \frac{28}{b^2} - 27(2a-2b) \le -\frac{27}{a^2} + \frac{27}{b^2} - 54(a-b)$$
$$= 27(a-b)\left(\frac{a+b}{a^2b^2} - 2\right) \le 0$$

and

$$f\left(\frac{3-c}{2}\right) = \frac{56}{a} + \frac{25}{c} - 27(2a^2+c^2)$$
$$= \frac{3(c-1)(3c-2)(3c-5)^2}{2c(3-c)} \ge 0.$$

On the other hand, we can prove the inequality (*) by showing that

$$f(a,b,c) \ge f(x,x,c) \ge 0,$$

where

$$f(a,b,c) = \frac{28}{a} + \frac{28}{b} + \frac{25}{c} - 27(a^2+b^2+c^2), \qquad x = \frac{a+b}{2} = \frac{3-c}{2}.$$

We have

$$f(a,b,c) - f(x,x,c) = 28\left(\frac{1}{a} + \frac{1}{b} - \frac{2}{x}\right) - 27(a^2+b^2-2x^2)$$
$$= \frac{1}{2}(a-b)^2\left[\frac{56}{ab(a+b)} - 27\right] \ge \frac{27}{2}(a-b)^2\left[\frac{2}{ab(a+b)} - 1\right] \ge 0$$

and

$$f(x,x,c) = \frac{56}{x} + \frac{25}{c} - 27(2x^2+c^2) = \frac{3(c-1)(3c-2)(3c-5)^2}{2c(3-c)} \ge 0.$$

□

P 2.97. *If a,b,c are the lengths of the sides of a triangle, then*

$$a^3(b+c) + bc(b^2+c^2) \ge a(b^3+c^3).$$

(Vasile C., 2010)

First Solution. Because the inequality is symmetric in b and c, we may assume that $b \ge c$. Consider the following two cases.

Case 1: $a \ge b$. It suffices to show that

$$a^3(b+c) \ge a(b^3+c^3).$$

We have

$$a^3(b+c) - a(b^3+c^3) \ge ab^2(b+c) - a(b^3+c^3) = ac(b^2-c^2) \ge 0.$$

Case 2: $a \le b$. Write the inequality as

$$c(a^3+b^3)-c^3(a-b)+ab(a^2-b^2) \ge 0.$$

It suffices to show that

$$c(a^3+b^3)+ab(a^2-b^2) \ge 0.$$

We have

$$c(a^3+b^3)+ab(a^2-b^2) \ge c(a^3+b^3)-abc(a+b) = c(a+b)(a-b)^2 \ge 0.$$

The equality holds for a degenerate triangle with $a=b$ and $c=0$, or $a=c$ and $b=0$.

Second Solution. Consider two cases.

Case 1: $b^2+c^2 \ge a(b+c)$. Write the inequality as

$$bc(b^2+c^2) \ge a(b+c)(b^2+c^2-bc-a^2).$$

It suffices to show that

$$bc \ge b^2+c^2-bc-a^2,$$

which is equivalent to the obvious inequality

$$a^2 \ge (b-c)^2.$$

Case 2: $a(b+c) \ge b^2+c^2$. Write the inequality as

$$a(b+c)(a^2+bc) \ge (b^2+c^2)(ab+ac-bc).$$

It suffices to show that

$$a^2+bc \ge ab+ac-bc,$$

which is equivalent to the obvious inequality

$$bc \ge (a-c)(b-a).$$

□

P 2.98. *If a,b,c are the lengths of the sides of a triangle, then*

$$\frac{(a+b)^2}{2ab+c^2}+\frac{(a+c)^2}{2ac+b^2} \ge \frac{(b+c)^2}{2bc+a^2}.$$

(Vasile C., 2010)

Solution. By the Cauchy-Schwarz inequality, we have

$$\frac{(a+b)^2}{2ab+c^2}+\frac{(a+c)^2}{2ac+b^2}\geq\frac{(2a+b+c)^2}{2a(b+c)+b^2+c^2}.$$

Therefore, it suffices to show that

$$\frac{(2a+b+c)^2}{2a(b+c)+b^2+c^2}\geq\frac{(b+c)^2}{2bc+a^2}.$$

We will show that

$$\frac{(2a+b+c)^2}{2a(b+c)+b^2+c^2}\geq 2\geq\frac{(b+c)^2}{2bc+a^2}.$$

The left inequality reduces to $4a^2\geq(b-c)^2$, and the right inequality reduces to $2a^2\geq(b-c)^2$. These are true because $a^2\geq(b-c)^2$. The equality holds for a degenerate triangle with $a=0$ and $b=c$.

□

P 2.99. *If a,b,c are the lengths of the sides of a triangle, then*

$$\frac{a+b}{ab+c^2}+\frac{a+c}{ac+b^2}\geq\frac{b+c}{bc+a^2}.$$

(Vasile C., 2010)

Solution. Without loss of generality, assume that $b\geq c$. Since $a+b\geq a+c$ and

$$ab+c^2-(ac+b^2)=(b-c)(a-b-c)\leq 0,$$

by Chebyshev's inequality, we have

$$\begin{aligned}\frac{a+b}{ab+c^2}+\frac{a+c}{ac+b^2}&\geq\frac{1}{2}[(a+b)+(a+c)]\left(\frac{1}{ab+c^2}+\frac{1}{ac+b^2}\right)\\&\geq\frac{2(2a+b+c)^2}{a(b+c)+b^2+c^2}.\end{aligned}$$

On the other hand,

$$\frac{b+c}{bc+a^2}\leq\frac{b+c}{\frac{1}{2}(b-c)^2+bc}=\frac{2(b+c)}{b^2+c^2}.$$

Therefore, it suffices to show that

$$\frac{2(2a+b+c)}{a(b+c)+b^2+c^2}\geq\frac{2(b+c)}{b^2+c^2},$$

which is equivalent to $a(b-c)^2\geq 0$. The equality holds for a degenerate triangle with $a=0$ and $b=c$.

□

P 2.100. *If a,b,c are the lengths of the sides of a triangle, then*

$$\frac{b(a+c)}{ac+b^2}+\frac{c(a+b)}{ab+c^2}\geq\frac{a(b+c)}{bc+a^2}.$$

(Vo Quoc Ba Can, 2010)

Solution. Without loss of generality, assume that $b\geq c$. Since

$$ab+c^2-(ac+b^2)=(b-c)(a-b-c)\leq 0,$$

it suffices to prove that

$$\frac{b(a+c)}{ac+b^2}+\frac{c(a+b)}{ac+b^2}\geq\frac{a(b+c)}{bc+a^2},$$

which is equivalent to

$$\frac{2bc+a(b+c)}{ac+b^2}\geq\frac{a(b+c)}{bc+a^2},$$

$$\frac{2bc}{ac+b^2}\geq a(b+c)\left(\frac{1}{bc+a^2}-\frac{1}{ac+b^2}\right),$$

$$2bc(bc+a^2)\geq a(b+c)(b-a)(a+b-c).$$

Consider the nontrivial case $b\geq a$. Since $c\geq b-a$, it suffices to show that

$$2b(bc+a^2)\geq a(b+c)(a+b-c).$$

We have

$$2b(bc+a^2)-a(b+c)(a+b-c)=ab(a-b)+c(2b^2-a^2+ac)$$

$$\geq -abc+c(2b^2-a^2+ac)=ac(b+c-a)+2bc(b-a)\geq 0.$$

The equality holds for a degenerate triangle with $a=b$ and $c=0$, or $a=c$ and $b=0$.

□

P 2.101. *If a,b,c,d are positive real numbers such that*

$$a\geq b\geq c\geq d,\qquad ab^2c^3d^6\geq 1,$$

then

$$a+2b+3c+6d\geq\frac{1}{a}+\frac{2}{b}+\frac{3}{c}+\frac{6}{d}.$$

(Vasile C., 2018)

Solution. It suffices to prove the homogeneous inequality

$$a + 2b + 3c + 6d \ge \sqrt[6]{ab^2c^3d^6}\left(\frac{1}{a} + \frac{2}{b} + \frac{3}{c} + \frac{6}{d}\right).$$

Replacing a, b, c, d with a^6, b^6, c^6, d^6, we need to show that

$$a^6 + 2b^6 + 3c^6 \ge \left(\frac{b^2c^3}{a^5} + \frac{2ac^3}{b^4} + \frac{3ab^2}{c^3} - 6\right)d^6 + 6ab^2c^3$$

for $a \ge b \ge c \ge d$. By the AM-GM inequality, we have

$$\frac{b^2c^3}{a^5} + \frac{2ac^3}{b^4} + \frac{3ab^2}{c^3} - 6 \ge 6\sqrt[6]{\frac{b^2c^3}{a^5} \cdot \left(\frac{ac^3}{b^4}\cdot\right)^2\left(\frac{ab^2}{c^3}\right)^3} - 6 = 0.$$

Since $d^6 \le ab^2c^3$, it suffices to show that

$$a^6 + 2b^6 + 3c^6 \ge \left(\frac{b^2c^3}{a^5} + \frac{2ac^3}{b^4} + \frac{3ab^2}{c^3} - 6\right)ab^2c^3 + 6ab^2c^3,$$

which is equivalent to

$$a^6 + 2b^6 + 3c^6 \ge \frac{b^4c^6}{a^4} + \frac{2a^2c^6}{b^2} + 3a^2b^4,$$

$$a^6 + 2b^6 - 3a^2b^4 \ge \left(\frac{b^4}{a^4} + \frac{2a^2}{b^2} - 3\right)c^6,$$

$$(a^2 - b^2)^2(a^2 + 2b^2) \ge \frac{(a^2 - b^2)^2(2a^2 + b^2)c^6}{a^4b^2}.$$

We need to show that

$$a^4b^2(a^2 + 2b^2) \ge (2a^2 + b^2)c^6.$$

Since $c^6 \le a^2b^4$, we have

$$a^4b^2(a^2+2b^2)-(2a^2+b^2)c^6 \ge a^4b^2(a^2+2b^2)-(2a^2+b^2)a^2b^4 = a^2b^2(a^4-b^4) \ge 0.$$

The equality occurs for $a = b = c = d = 1$.

Remark. By induction method, we can prove the following generalization.

- *If $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ are positive real numbers such that*

$$a_1 \ge a_2 \ge \cdots \ge a_n, \qquad a_1a_2^2a_3^3a_4^6\cdots a_n^{k_n} \ge 1, \qquad k_n = 3 \cdot 2^{n-3},$$

then

$$a_1 + 2a_2 + 3a_3 + 6a_4 + \cdots + k_na_n \ge \frac{1}{a_1} + \frac{2}{a_2} + \frac{3}{a_3} + \frac{6}{a_4} + \cdots + \frac{k_n}{a_n},$$

with equality for $a_1 = a_2 = \cdots = a_n$.

For $n = 3$ and $n = 4$, we get the inequalities in P 2.93 and P 2.101.

□

P 2.102. *If a,b,c,d are positive real numbers such that*

$$a \ge b \ge c \ge d, \qquad abc^2d^4 \ge 1,$$

then

$$a+b+2c+4d \ge \frac{1}{a}+\frac{1}{b}+\frac{2}{c}+\frac{4}{d}.$$

(Vasile C., 2018)

Solution. It suffices to prove the homogeneous inequality

$$a+b+2c+4d \ge \sqrt[4]{abc^2d^4}\left(\frac{1}{a}+\frac{1}{b}+\frac{2}{c}+\frac{4}{d}\right).$$

Replacing a,b,c,d with a^4,b^4,c^4,d^4, we need to show that

$$a^4+b^4+2c^4 \ge \left(\frac{bc^2}{a^3}+\frac{ac^2}{b^3}+\frac{2ab}{c^2}-4\right)d^4+4abc^2$$

for $a \ge b \ge c \ge d$. By the AM-GM inequality, we have

$$\frac{bc^2}{a^3}+\frac{ac^2}{b^3}+\frac{2ab}{c^2}-4 \ge 4\sqrt[4]{\frac{bc^2}{a^3}\cdot\frac{ac^2}{b^3}\cdot\left(\frac{ab}{c^2}\right)^2}-4=0.$$

Since $d^4 \le abc^2$, it suffices to show that

$$a^4+b^4+2c^4 \ge \left(\frac{bc^2}{a^3}+\frac{ac^2}{b^3}+\frac{2ab}{c^2}-4\right)abc^2+4abc^2$$

which is equivalent to

$$a^4+b^4+2c^4 \ge \frac{b^2c^4}{a^2}+\frac{a^2c^4}{b^2}+2a^2b^2,$$

$$(a^2-b^2)^2 \ge \frac{(a^2-b^2)^2c^4}{a^2b^2},$$

$$(a^2-b^2)^2\left(1-\frac{c^4}{a^2b^2}\right) \ge 0.$$

The equality occurs for $a=b=c=d=1$.

Remark. By induction method, we can prove the following generalization.

- *If $a_1,a_2,\dots,a_n$ $(n \ge 3)$ are positive real numbers such that*

$$a_1 \ge a_2 \ge \cdots \ge a_n, \qquad a_1a_2a_3^2a_4^4\cdots a_n^{2^{n-2}} \ge 1,$$

then

$$a_1+a_2+2a_3+4a_4+\cdots+2^{n-2}a_n \ge \frac{1}{a_1}+\frac{1}{a_2}+\frac{2}{a_3}+\frac{4}{a_4}+\cdots+\frac{2^{n-2}}{a_n},$$

with equality for $a_1=a_2=\cdots=a_n$.

For $n=4$, we get the inequalities in P 2.102.

□

P 2.103. *If a,b,c,d are positive real numbers such that*

$$abcd \geq 1, \qquad a \geq b \geq c \geq d, \qquad ad \geq bc,$$

then

$$a+b+c+d \geq \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}.$$

(Vasile C., 2018)

Solution. It suffices to prove the homogeneous inequality

$$a+b+c+d \geq \sqrt{abcd}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right).$$

Replacing a,b,c,d with a^2,b^2,c^2,d^2, we need to show that

$$a^2+b^2+c^2+d^2 \geq \frac{bc}{ad}(a^2+d^2)+\frac{ad}{bc}(b^2+c^2)$$

for $a \geq b \geq c \geq d$ and $ad \geq bc$. Write the inequality as follows:

$$(a^2+d^2)\left(1-\frac{bc}{ad}\right)+(b^2+c^2)\left(1-\frac{ad}{bc}\right) \geq 0,$$

$$(ad-bc)\left(\frac{a}{d}+\frac{d}{a}-\frac{b}{c}-\frac{c}{b}\right) \geq 0,$$

$$(ad-bc)\left(\frac{ac-bd}{cd}+\frac{bd-ac}{ab}\right) \geq 0.$$

$$\frac{(ad-bc)(ac-bd)(ab-cd)}{abcd} \geq 0.$$

Clearly, the last inequality is true. The equality occurs for $ad=bc=1$.

Remark. The following extension is valid.

- *If a,b,c,d,e are positive real numbers such that*

$$abcde \geq 1, \qquad a \geq b \geq c \geq d \geq e, \qquad ae \geq bd \geq c^2,$$

then

$$a+b+c+d+e \geq \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e},$$

with equality for $af=c^2=cd=1$

□

P 2.104. *If a,b,c,d,e,f are positive real numbers such that*

$$abcdef \ge 1, \qquad a \ge b \ge c \ge d \ge e \ge f, \qquad af \ge be \ge cd,$$

then

$$a+b+c+d+e+f \ge \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}+\frac{1}{f}.$$

(Vasile C., 2018)

Solution. Write the inequality as

$$(a+f)\left(1-\frac{1}{af}\right)+(b+e)\left(1-\frac{1}{be}\right)+(c+d)\left(1-\frac{1}{cd}\right) \ge 0.$$

For

$$af = k = constant,$$

we claim that the sum $a+f$ is minimum for $a=\frac{k}{e} \ge b$ and $f=e$. Indeed, we have

$$a+f-\frac{k}{e}-e = a+f-\frac{af}{e}-e = a-e-\left(\frac{a}{e}-1\right)f = \frac{(a-e)(e-f)}{e} \ge 0.$$

In addition, for

$$cd = k = constant,$$

we claim that the sum $c+d$ is maximum for $c=\frac{k}{e} \le b$ and $d=e$. Indeed, we have

$$c+d-\frac{k}{e}-e = c+d-\frac{cd}{e}-e = c-e-\left(\frac{c}{e}-1\right)d = \frac{-(c-e)(d-e)}{e} \le 0.$$

Thus, it suffices to prove the inequality for $f=e$ and $d=e$, that is for $d=e=f$. So, we need to show that

$$a+b+c+3d \ge \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{3}{d}$$

for

$$a \ge b \ge c \ge d, \qquad abcd^3 \ge 1.$$

It suffices to prove the homogeneous inequality

$$a+b+c+3d \ge \sqrt[3]{abcd^3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{3}{d}\right).$$

Replacing a,b,c,d with a^3,b^3,c^3,d^3, we need to show that

$$a^3+b^3+c^3 \ge \left(\frac{bc}{a^2}+\frac{ca}{b^2}+\frac{ab}{c^2}-3\right)d^3+3abc$$

for $a \ge b \ge c \ge d$. By the AM-GM inequality, we have

$$\frac{bc}{a^2}+\frac{ca}{b^2}+\frac{ab}{c^2}-3 \ge 0.$$

Since $d^3 \le c^3$, it suffices to show that

$$a^3+b^3+c^3 \ge \left(\frac{bc}{a^2}+\frac{ca}{b^2}+\frac{ab}{c^2}-3\right)c^3+3abc,$$

which can be written as follows:

$$a^3+b^3+4c^3 \ge \frac{bc^4}{a^2}+\frac{ac^4}{c^2}+4abc,$$

$$(a^3+b^3)\left(1-\frac{c^4}{a^2b^2}\right)-4c(ab-c^2) \ge 0,$$

$$(ab-c^2)[(a^3+b^3)(ab+c^2)-4a^2b^2c] \ge 0.$$

It is true since

$$(a^3+b^3)(ab+c^2)-4a^2b^2c \ge 2ab\sqrt{ab}\left(ab+c^2\right)-4a^2b^2c = 2ab\sqrt{ab}\left(\sqrt{ab}-c\right)^2 \ge 0.$$

The equality occurs for $af = be = cd = 1$.

□

P 2.105. *Let a,b,c,d be nonnegative real numbers such that*

$$a^2-ab+b^2 = c^2-cd+d^2.$$

Prove that

$$(a+b)(c+d) \ge 2(ab+cd).$$

(Vasile C., 2000)

Solution. Let

$$x = a^2-ab+b^2 = c^2-cd+d^2.$$

Without loss of generality, assume that $ab \ge cd$. Then,

$$x \ge ab \ge cd, \quad (a+b)^2 = x+3ab, \quad (c+d)^2 = x+3cd.$$

By squaring, the desired inequality can be restated as

$$(x+3ab)(x+3cd) \ge 4(ab+cd)^2.$$

It is true since

$$\begin{aligned}(x+3ab)(x+3cd)-4(ab+cd)^2 &\ge (ab+3ab)(ab+3cd)-4(ab+cd)^2\\ &= 4cd(ab-cd) \ge 0.\end{aligned}$$

The equality occurs for $a = b = c = d$, and also for $a = b = c$ and $d = 0$ (or any cyclic permutation).

□

P 2.106. *Let a, b, c, d be nonnegative real numbers such that*

$$a^2 - ab + b^2 = c^2 - cd + d^2.$$

Prove that

$$\frac{1}{a^2+b^2} + \frac{1}{c^2+d^2} \le \frac{8}{(a+b)^2+(c+d)^2}.$$

(Vasile C. and Relic-93, 2021)

Solution. Let

$$x = a^2 - ab + b^2 = c^2 - cd + d^2.$$

Without loss of generality, assume that $ab \ge cd$. Then, $x \ge ab \ge cd$ and

$$a^2+b^2 = x + ab, \quad c^2+d^2 = x + cd, \quad (a+b)^2 = x + 3ab, \quad (c+d)^2 = x + 3cd.$$

The required inequality can be rewritten as

$$\frac{1}{x+ab} + \frac{1}{x+cd} \le \frac{8}{2x+3(ab+cd)},$$

$$3(a^2b^2 + c^2d^2) \le 4x^2 + 2abcd.$$

It is true if

$$3(a^2b^2 + c^2d^2) \le 4a^2b^2 + 2abcd,$$

which is equivalent to

$$(ab - cd)(ab + 3cd) \ge 0.$$

The equality occurs for $a = b = c = d$.

□

P 2.107. *Let a, b, c, d be nonnegative real numbers such that*

$$a^2 - ab + b^2 = c^2 - cd + d^2.$$

Prove that

$$\frac{1}{a^2+ab+b^2} + \frac{1}{c^2+cd+d^2} \le \frac{8}{3(a+b)(c+d)}.$$

(Anhduy98, 2021)

Solution. Without loss of generality, assume that $ab \ge cd$. Let

$$x = a^2 - ab + b^2 = c^2 - cd + d^2, \quad y = ab, \quad z = cd.$$

Then, $x \ge y \ge z$ and

$$a^2+ab+b^2 = x+2y, \quad c^2+cd+d^2 = x+2z, \quad (a+b)^2 = x+3y, \quad (c+d)^2 = x+3z.$$

The required inequality can be rewritten as $F(x,y,z) \le 0$, where

$$F(x,y,z) = \frac{1}{x+2y} + \frac{1}{x+2z} - \frac{8}{3\sqrt{(x+3y)(x+3z)}}.$$

We will show that

$$F(x,y,z) \le F(x,x,z) \le 0.$$

The left inequality is equivalent to

$$\frac{4}{\sqrt{x+3z}}\left(\frac{1}{\sqrt{x+3y}} - \frac{1}{2\sqrt{x}}\right) \ge \frac{x-y}{x(x+2y)},$$

$$\frac{6(x-y)}{\sqrt{x(x+3y)(x+3z)}\left(2\sqrt{x}+\sqrt{x+3z}\right)} \ge \frac{x-y}{x(x+2y)}.$$

It is true if

$$\frac{6}{\sqrt{(x+3y)(x+3z)}\left(2\sqrt{x}+\sqrt{x+3z}\right)} \ge \frac{1}{(x+2y)\sqrt{x}}.$$

Since $x \ge y \ge z$, we only need to show that

$$\frac{6}{(x+3y)\left(2\sqrt{x}+\sqrt{4x}\right)} \ge \frac{1}{(x+2y)\sqrt{x}},$$

which is clearly true.

The right inequality $F(x,x,z) \le 0$ is equivalent to

$$\frac{1}{3x} + \frac{1}{x+2z} \le \frac{4}{3\sqrt{x(x+3z)}},$$

$$(2x+z)^2(x+3z) \le 4x(x+2z)^2.$$

It is true because

$$4x(x+2z)^2 - (2x+z)^2(x+3z) = 3(x-z)z^2 \ge 0.$$

The equality occurs for $a=b=c=d$, and also for $a=b=c$ and $d=0$ (or any cyclic permutation).

□

P 2.108. *Let a,b,c,d be nonnegative real numbers such that*

$$a^2 - ab + b^2 = c^2 - cd + d^2.$$

Prove that

$$\frac{1}{(ac+bd)^4} + \frac{1}{(ad+bc)^4} \le \frac{2}{(ab+cd)^4}.$$

(Vasile C., 2021)

Solution. Due to homogeneity, we may set

$$a^2 - ab + b^2 = c^2 - cd + d^2 = 1.$$

Let

$$x = ab, \quad y = cd, \quad s = x + y, \quad p = xy.$$

From $1 = a^2 - ab + b^2 \ge ab$, we get $x \le 1$. Similarly, $y \le 1$, hence $p \le 1$. In addition, from

$$(1-x)1-y) \ge 0,$$

we get

$$s \le 1 + p.$$

Since

$$(ac+bd)(ad+bc) = ab(c^2+d^2) + cd(a^2+b^2) = x(1+y) + y(1+x) = s + 2p,$$

$$(ac+bd)^2 + (ad+bc)^2 = (a^2+b^2)(c^2+d^2) + 4abcd = (1+x)(1+y) + 4xy = 1 + s + 5p,$$

$$(ac+bd)^4 + (ad+bc)^4 = \left[(ac+bd)^2 + (ad+bc)^2\right]^2 - 2(ac+bd)^2(ad+bc)^2$$
$$= (1+s+5p)^2 - 2(s+2p)^2 ,$$

we need to show that

$$\frac{(1+s+5p)^2 - 2(s+2p)^2}{(s+2p)^4} \le \frac{2}{s^4} ,$$

that is equivalent to $f(s,p) \ge g(s,p)$, where

$$f(s,p) = 2\left(1 + \frac{2p}{s}\right)^4, \qquad g(s,p) = (1+s+5p)^2 - 2(s+2p)^2 .$$

Since

$$f(s,p) \ge f(1+p,p)$$

and

$$g(s,p) - g(1+p,p) = (s-1-p)(3+s+11p) - 2(s-1-p)(1+s+5p) = -(s-1-p)^2 \le 0,$$

it is enough to show that

$$f(1+p,p) \ge g(1+p,p),$$

that is

$$\frac{2(1+3p)^4}{(1+p)^4} \ge 2(1+3p)^2,$$

$$p(1-p)(1+3p)^2(2+5p+p^2) \ge 0.$$

The equality occurs for $a = b = c = d$, and also for $a = b = c$ and $d = 0$ (or any cyclic permutation).

□

P 2.109. *Let a, b, c, d be nonnegative real numbers such that $a \ge b \ge c \ge d$ and*

$$a+b+c+d=13, \qquad a^2+b^2+c^2+d^2=43.$$

Prove that

$$ab \ge cd+3.$$

(PMO, 2021)

Solution *(by Doxuantrong).* From

$$43-a^2=b^2+c^2+d^2 \ge \frac{1}{3}(b+c+d)^2=\frac{1}{3}(13-a)^2,$$

we get

$$(a-4)(2a-5) \le 0,$$

hence $\frac{5}{2} \le a, b, c, d \le 4$. On the other hand, we write the required inequality as follows:

$$2ab \ge 2cd+6,$$

$$(a+b)^2-(a^2+b^2) \ge (c+d)^2-(c^2+d^2)+6,$$

$$(13-c-d)^2-(43-c^2-d^2) \ge (c+d)^2-(c^2+d^2)+6,$$

$$c^2+d^2+60 \ge 13(c+d),$$

$$(c-d)^2+(c+d)^2+120 \ge 26(c+d),$$

$$(c-d)^2 \ge (c+d-6)(20-c-d).$$

Thus, it suffices to show that $c+d \le 6$, that is equivalent to $a+b \ge 7$. If $a=4$, then

$$a+b \ge a+\frac{b+c+d}{3}=a+\frac{13-a}{3}=7.$$

Consider further that $a<4$. From

$$(b-c)(b-d) \ge 0,$$

we get

$$b^2-(c+d)b+cd \ge 0,$$

that is equivalent to

$$2b^2-2(c+d)b+(c+d)^2-(c^2+d^2) \ge 0,$$

$$b^2+(b-c-d)^2-(c^2+d^2) \ge 0,$$

$$b^2+(a+2b-13)^2-(43-a^2-b^2) \ge 0,$$

$$3b^2-2(13-a)b+a^2-13a+63 \ge 0,$$

$$3b \ge 13-a+\sqrt{(4-a)(2a-5)}.$$

Note that we cannot have $3b \le 13 - a - \sqrt{(4-a)(2a-5)}$ because this involves a contradiction:

$$13 - a = b + c + d \le 3b \le 13 - a - \sqrt{(4-a)(2a-5)} < 13 - a.$$

From

$$3a \ge 3b \ge 13 - a + \sqrt{(4-a)(2a-5)},$$

we get

$$4a - 13 \ge \sqrt{(4-a)(2a-5)},$$

$$(2a-7)(a-3) \ge 0,$$

hence $a \ge 7/2$. As a consequence, we have

$$3(a+b-7) = 3(a-7) + 3b \ge 3(a-7) + 13 - a + \sqrt{(4-a)(2a-5)}$$

$$= \sqrt{4-a}\left(\sqrt{2a-5} - 2\sqrt{4-a}\right) = \frac{3\sqrt{4-a}\,(2a-7)}{\sqrt{2a-5} + 2\sqrt{4-a}} \ge 0.$$

The equality occurs for $a = 4$ and $b = c = d = 3$.

Second solution (by *KaiRain*) To show that $a + b \ge 7$, the key is

$$a^2+b^2+c^2+d^2+6(ab+cd) = (a+b+c+d)^2+2(a-c)(b-d)+2(a-d)(b-c)$$

$$\ge (a+b+c+d)^2,$$

which gives

$$ab + cd \ge 21,$$

$$(a+b)^2 + (c+d)^2 \ge a^2 + b^2 + c^2 + d^2 + 42,$$

$$(a+b)^2 + (13-a-b)^2 \ge 85,$$

$$(a+b-6)(a+b-7) \ge 0,$$

$$a + b \ge 7.$$

Hence,

$$ab - cd \ge ab - \frac{c^2+d^2}{2} = ab + \frac{a^2+b^2-43}{2} = \frac{(a+b)^2-43}{2} \ge 3.$$

$\square$

P 2.110. *Let a,b,c,d be nonnegative real numbers such that $a \ge b \ge c \ge d$ and*

$$a+b+c+d=13, \qquad a^2+b^2+c^2+d^2=43.$$

Prove that

$$ab-cd \ge \frac{7}{2}$$

(a) *for* $a \le \dfrac{39}{10}$;

(b) *for* $d \le \dfrac{31}{11}$.

(Vasile C., 2021)

Solution. (a) As shown at the preceding P 2.109, we have

$$\frac{7}{2} \le a \le 4$$

and

$$b \ge B, \qquad B=\frac{13-a+\sqrt{(4-a)(2a-5)}}{3}.$$

Write the required inequality as follows:

$$2ab \ge 2cd+7,$$

$$(a+b)^2-(a^2+b^2) \ge (c+d)^2-(c^2+d^2)+7,$$

$$(a+b)^2-(a^2+b^2) \ge (13-a-b)^2-(43-a^2-b^2)+7,$$

$$13a-a^2+13b-b^2 \ge \frac{133}{2}.$$

Since

$$13b-b^2-(13B-B^2)=(b-B)(13-b-B) \ge 0,$$

it suffices to show that

$$13a-a^2+13B-B^2 \ge \frac{133}{2},$$

which is equivalent to

$$2(2a+13)\sqrt{(4-a)(2a-5)} \ge 16a^2-182a+481. \tag{*}$$

Write this inequality in the form

$$2(2a+13)(4-a)(2a-5) \ge (16a^2-182a+481)\sqrt{(4-a)(2a-5)}.$$

Since

$$2\sqrt{(4-a)(2a-5)}=2\sqrt{2(4-a)\cdot\frac{2a-5}{2}} \le 2(4-a)+\frac{2a-5}{2}=\frac{11-2a}{2},$$

it suffices to show that

$$4(2a+13)(4-a)(2a-5) \geq (16a^2-182a+481)(11-2a),$$

which is equivalent to the obvious inequality

$$(2a-7)(39-10a) \geq 0.$$

The equality occurs for $a=b=c=\frac{7}{2}$ and $d=\frac{5}{2}$.

Remark 1. Actually, the inequality is true for $a \leq k$, where

$$k \approx 3.980572$$

is a root of the equation

$$16k^3-256k^2+1742k-3887=0.$$

Indeed, by squaring, the equation (*) becomes

$$(2a-7)(16a^3-256a^2+1742a-3887) \leq 0.$$

It is easy to show that his inequality holds for

$$a \leq \frac{613}{154} \approx 3.980519.$$

Indeed, we have

$$16a^3-256a^2+1742a-3887 = 16(a-4)^3-64(a-4)^2+3(154a-613)$$

$$< 3(154a-613) < 0.$$

(b) As shown at the preceding P 2.109, we have

$$d \geq \frac{5}{2}.$$

Write the required inequality as follows:

$$2ab \geq 2cd+7,$$

$$(a+b)^2-(a^2+b^2) \geq (c+d)^2-(c^2+d^2)+7,$$

$$(13-c-d)^2-(43-c^2-d^2) \geq (c+d)^2-(c^2+d^2)+7,$$

$$2c^2-26c+2d^2-26d+119 \geq 0.$$

If $d=\frac{5}{2}$, then

$$c \leq \frac{a+b+c}{3} = \frac{13-d}{3} = \frac{7}{2},$$

hence
$$2c^2-26c+2d^2-26d+119=\frac{(7-2c)(19-2c)}{2}\geq 0.$$
Consider further that $d>\frac{5}{2}$. From
$$(c-a)(c-b)\geq 0,$$
we get
$$c^2-(a+b)c+ab\geq 0,$$
that is equivalent to
$$c^2+(a+b-c)^2-a^2-b^2\geq 0,$$
$$c^2+(13-2c-d)^2+c^2+d^2-43\geq 0,$$
$$3c^2-2(13-d)c+d^2-13d+63\geq 0,$$
$$c\leq C,\qquad C=\frac{13-d-\sqrt{(4-d)(2d-5)}}{3}\ .$$
Note that we cannot have $3c\geq 13-d+\sqrt{(4-d)(2d-5)}$ because this involves a contradiction:
$$13-d=a+b+c\geq 3c\geq 13-d+\sqrt{(4-d)(2d-5)}>13-d.$$
From $d\leq c\leq C$, we get
$$\sqrt{(4-d)(2d-5)}\leq 13-4d,$$
$$(7-2d)(d-3)\leq 0,$$
hence
$$d\leq 3.$$
Since
$$2c^2-26c-(2C^2-26C)=2(c-C)(c+C-26)\geq 0,$$
it suffices to show that
$$2C^2-26C+2d^2-26d+119\geq 0,$$
which is equivalent to
$$2(2d+13)\sqrt{(4-d)(2d-5)}\geq(2d-5)(71-8d)\ .$$
This is true if
$$2(2d+13)\geq(71-8d)\sqrt{\frac{2d-5}{4-d}}\ . \tag{**}$$

Since

$$2\sqrt{\frac{2d-5}{4-d}} \le \frac{2d-5}{4-d} + 1 = \frac{d-1}{4-d},$$

it is enough to show that

$$4(2d+13)(4-d) \ge (71-8d)(d-1),$$

which is equivalent to the obvious inequality

$$31 - 11d \ge 0.$$

The equality occurs for $a = b = c = \frac{7}{2}$ and $d = \frac{5}{2}$.

Remark 2. The inequality is true for $d \le k$, where

$$k \approx 2.84647$$

is a root of the equation

$$16k^3 - 272k^2 + 1734k - 3101 = 0.$$

Indeed, by squaring, the equation (**) becomes

$$16d^3 - 272d^2 + 1734d - 3101 \le 0.$$

It is easy to show that his inequality holds for

$$d \le \frac{1517}{534} \approx 2.84082.$$

Indeed, we have

$$16d^3 - 272d^2 + 1734d - 3101 = 16(d-3)^3 - 128(d-3)^2 + 534d - 1517$$

$$< 534d - 1517 < 0.$$

□

P 2.111. *Let a, b, c, d be nonnegative real numbers such that $a \ge b \ge c \ge d$ and*

$$a + b + c + d = 13, \qquad a^2 + b^2 + c^2 + d^2 = 43.$$

Prove that

$$\frac{83}{4} \le ac + bd \le \frac{169}{8}.$$

(Vasile C., 2021)

Solution. As shown at P 2.109, we have

$$\frac{5}{2} \le a, b, c, d \le 4.$$

Since

$$2(ac+bd) = (a+c)^2 + (b+d)^2 - (a^2+b^2+c^2+d^2) = (a+c)^2 + (13-a-c)^2 - 43$$
$$= 2(a+c)^2 - 26(a+c) + 126,$$

the left required inequality is equivalent to

$$\left(a+c-\frac{13}{2}\right)^2 \ge 0 ,$$

and the right required inequality is equivalent to

$$8(a+c)^2 - 104(a+c) + 335 \ge 0.$$

Since

$$a+c \ge \frac{a+b+c+d}{2} = \frac{13}{2} ,$$

we only need yo show that

$$a+c \le \frac{26+\sqrt{6}}{4} .$$

From

$$(c-b)(c-d) \le 0,$$

we get

$$c^2 - (b+d)b + bd \le 0,$$

that is equivalent to

$$c^2 + (b+d-c)^2 - b^2 - d^2 \le 0 ,$$
$$c^2 + (13-a-2c)^2 + a^2 + c^2 - 43 \le 0 ,$$
$$3c^2 - 2(13-a)c + a^2 - 13a + 63 \le 0 ,$$
$$c \le C, \qquad C = \frac{13-a+\sqrt{(4-a)(2a-5)}}{3} .$$

So, it suffices to show that

$$a + C \le \frac{26+\sqrt{6}}{4},$$

which is equivalent to

$$26 + 3\sqrt{6} - 8a \ge 4\sqrt{(4-a)(2a-5)},$$

$$(\sqrt{6}+2)(4-a) + \frac{\sqrt{6}-2}{2}(2a-5) \ge 4\sqrt{(4-a)(2a-5)} .$$

Clearly, the last inequality is true (by the AM-GM inequality).

The left inequality is an equality for $a+c=b+d=\frac{13}{2}$ and $ac+bc=\frac{83}{4}$, while the right inequality is an equality for $a=\frac{13+\sqrt{6}}{4}$, $b=c=\frac{13}{4}$ and $d=\frac{13-\sqrt{6}}{4}$.

□

P 2.112. *If a,b,c,d are positive real numbers such that*

$$a+b+c+d=4, \qquad a\le b\le 1\le c\le d,$$

then

$$9\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\ge 4+8(a^2+b^2+c^2+d^2).$$

(Vasile C., 2021)

Solution. For fixed b and d, write the required inequality as $f(c)\ge 0$, where

$$f(c)=9\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)-4-8(a^2+b^2+c^2+d^2), \qquad a=4-b-c-d.$$

We will show that

$$f(c)\ge f(1)\ge 0.$$

Since

$$a+c\le\frac{a+b}{2}+\frac{c+d}{2}=2,$$

$$\frac{a+c}{a^2c^2}\ge\frac{16}{(a+c)^3}\ge 2,$$

we have

$$f'c)=\frac{9}{a^2}-\frac{9}{c^2}+16(a-c)=9(c-a)\left(\frac{a+c}{a^2c^2}-\frac{16}{9}\right)$$

$$\ge 9(c-a)\left(\frac{a+c}{a^2c^2}-2\right)\ge 0,$$

$f(c)$ is increasing, hence $f(c)\ge f(1)$. The inequality $f(1)\ge 0$ has the form

$$9\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{d}\right)-3-8(a^2+b^2+d^2)\ge 0,$$

where

$$a=3-b-d.$$

We may write this inequality as $g(a,b)\ge 0$, where

$$g(a,b)=9\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{d}\right)-3-8(a^2+b^2+d^2), \qquad d=3-a-b.$$

We will show that

$$g(a,b) \geq g(x,x) \geq 0,$$

where

$$x = \frac{a+b}{2}, \qquad 0 < x \leq 1.$$

We have

$$g(a,b) - g(x,x) = 9\left(\frac{1}{a} + \frac{1}{b} - \frac{2}{x}\right) - 8(a^2 + b^2 - 2x^2)$$

$$= \frac{9(a-b)^2}{2abx} - 4(a-b)^2 = \frac{(a-b)^2(9-8abx)}{2abx} \geq 0$$

and

$$g(x,x) = 9\left(\frac{2}{x} + \frac{1}{d}\right) - 3 - 8(2x^2 + d^2), \qquad d = 3 - 2x,$$

$$g(x,x) = 9\left(\frac{2}{x} + \frac{1}{3-2x}\right) - 3 - 16x^2 - 8(3-2x)^2$$

$$= \frac{6(16x^4 - 56x^3 + 73x^2 - 42x + 9)}{x(3-2x)},$$

$$= \frac{6(x-1)^2(4x-3)^2}{x(3-2x)} \geq 0.$$

The equality holds for $a = b = c = d = 1$, and also for $a = b = \frac{3}{4}$, $c = 1$, $d = \frac{3}{2}$. □

P 2.113. *If a,b,c,d are positive real numbers such that*

$$a^2 + b^2 + c^2 + d^2 = 4, \qquad a \leq b \leq c \leq d,$$

then

$$\frac{1}{a} + a + b + c + d \geq 5.$$

(Vasile C., 2021)

Solution. Write the inequality in the homogeneous form

$$\frac{a^2 + b^2 + c^2 + d^2}{4} + a(a+b+c+d) \geq 5a\sqrt{\frac{a^2 + b^2 + c^2 + d^2}{4}}.$$

For fixed a,b,d, we need to prove that $f(c) \geq 0$, where

$$f(c) = 5a^2 + b^2 + c^2 + d^2 + 4a(b+c+d) - 10a\sqrt{a^2 + b^2 + c^2 + d^2}, \quad c \in [b,d].$$

From

$$f'(c) = 2c + 4a - \frac{10ac}{\sqrt{a^2+b^2+c^2+d^2}} \ge 4a + 2c - \frac{10ac}{\sqrt{2(a^2+c^2)}}$$

$$\ge 4\sqrt{2ac} - 5\sqrt{ac} = (4\sqrt{2}-5)\sqrt{ac} > 0,$$

it follows that $f(c)$ is increasing, hence $f(c) \ge f(b)$. The inequality $f(b) \ge 0$ is equivalent to

$$5a^2 + 2b^2 + d^2 + 4a(2b+d) - 10a\sqrt{a^2+2b^2+d^2} \ge 0.$$

For fixed a and d, we need to show that $g(b) \ge 0$, where

$$g(b) = 5a^2 + 2b^2 + d^2 + 4a(2b+d) - 10a\sqrt{a^2+2b^2+d^2}, \quad b \in [a,d].$$

From

$$g'(b) = 4b + 8a - \frac{20ab}{\sqrt{a^2+2b^2+d^2}} \ge 4b + 8a - \frac{20ab}{\sqrt{a^2+3b^2}}$$

$$\ge 8\sqrt{2ab} - \frac{20\sqrt{ab}}{\sqrt{2\sqrt{3}}} = 4\left(2\sqrt{2} - \frac{5}{\sqrt{2\sqrt{3}}}\right)\sqrt{ab} > 0,$$

it follows that $g(b)$ is increasing, hence $g(b) \ge g(a)$, that is

$$g(b) \ge 15a^2 + 4ad + d^2 - 10a\sqrt{3a^2+d^2}.$$

Thus, we only need to show that

$$15a^2 + 4ad + d^2 \ge 10a\sqrt{3a^2+d^2}.$$

Due to homogeneity, we may set $a = 1$, hence $d \ge 1$. We need to show that

$$(15 + 4d + d^2)^2 \ge 100(3 + d^2),$$

which is equivalent to

$$d^4 + 8d^3 - 54d^2 + 120d - 75 \ge 0,$$

$$(d-1)(d^3 + 9d^2 - 45d + 75) \ge 0.$$

This is true because

$$d^3 + 9d^2 - 45d + 75 > 9d^2 - 45d + 63 = 9(d^2 - 5d + 7) > 0.$$

The equality holds for $a = b = c = d = 1$.

Remark. Similarly, we can prove the following stronger inequality

$$\frac{3}{4a} + a + b + c + d \ge \frac{19}{4}.$$

□

P 2.114. *If a, b, c, d are real numbers, then*

$$6(a^2 + b^2 + c^2 + d^2) + (a + b + c + d)^2 \geq 12(ab + bc + cd).$$

(Vasile C., 2005)

Solution. Let

$$E(a, b, c, d) = 6(a^2 + b^2 + c^2 + d^2) + (a + b + c + d)^2 - 12(ab + bc + cd).$$

First Solution. We have

$$E(x + a, x + b, x + c, x + d) =$$

$$= 4x^2 + 4(2a - b - c + 2d)x + 7(a^2 + b^2 + c^2 + d^2) + 2(ac + ad + bd) - 10(ab + bc + cd)$$

$$= (2x + 2a - b - c + 2d)^2 + 3(a^2 + 2b^2 + 2c^2 + d^2 - 2ab + 2ac - 2ad - 4bc + 2bd - 2cd)$$

$$= (2x + 2a - b - c + 2d)^2 + 3(b - c)^2 + 3(a - b + c - d)^2.$$

For $x = 0$, we get

$$E(a, b, c, d) = (2a - b - c + 2d)^2 + 3(b - c)^2 + 3(a - b + c - d)^2 \geq 0.$$

The equality holds for $2a = b = c = 2d$.

Second Solution. Let

$$x = a - b, \qquad y = c - d.$$

We have

$$\begin{aligned}
E &= 6[(a - b)^2 + (c - d)^2] + (a + b + c + d)^2 - 12bc \\
&= 6(x^2 + y^2) + [x + y + 2(b + c)]^2 - 12bc \\
&= 3(x - y)^2 + 3(x + y)^2 + [x + y + 2(b + c)]^2 - 12bc \\
&= 3(x - y)^2 + 4(x + y)^2 + 4(x + y)(b + c) + (b + c)^2 + 3(b - c)^2 \\
&= 3(x - y)^2 + (2x + 2y + b + c)^2 + 3(b - c)^2 \geq 0.
\end{aligned}$$

□

P 2.115. *If a, b, c, d are positive real numbers, then*

$$\frac{1}{a^2 + ab} + \frac{1}{b^2 + bc} + \frac{1}{c^2 + cd} + \frac{1}{d^2 + da} \geq \frac{4}{ac + bd}.$$

Solution. Write the inequality as follows:

$$\sum\left(\frac{ac+bd}{a^2+ab}+1\right)\ge 8,$$

$$\sum\frac{a(c+a)+b(d+a)}{a(a+b)}\ge 8,$$

$$\sum\frac{c+a}{a+b}+\sum\frac{b(d+a)}{a(a+b)}\ge 8.$$

By the AM-GM inequality, we have

$$\sum\frac{b(d+a)}{a(a+b)}\ge 4\sqrt[4]{\prod\frac{b(d+a)}{a(a+b)}}=4.$$

Therefore, it suffices to prove the inequality

$$\sum\frac{c+a}{a+b}\ge 4,$$

which is equivalent to

$$(a+c)\left(\frac{1}{a+b}+\frac{1}{c+d}\right)+(b+d)\left(\frac{1}{b+c}+\frac{1}{d+a}\right)\ge 4.$$

This inequality follows immediately from

$$\frac{1}{a+b}+\frac{1}{c+d}\ge\frac{4}{(a+b)+(c+d)}$$

and

$$\frac{1}{b+c}+\frac{1}{d+a}\ge\frac{4}{(b+c)+(d+a)}.$$

The equality occurs for $a=b=c=d$.

□

P 2.116. *If a,b,c,d are positive real numbers, then*

$$\frac{1}{a(1+b)}+\frac{1}{b(1+a)}+\frac{1}{c(1+d)}+\frac{1}{d(1+c)}\ge\frac{16}{1+8\sqrt{abcd}}.$$

(Vasile C., 2007)

Solution. Let

$$x=\sqrt{ab},\quad y=\sqrt{cd}.$$

Write the inequality as

$$\frac{a+b+2ab}{ab(1+a)(1+b)}+\frac{c+d+2cd}{cd(1+c)(1+d)}\ge\frac{16}{1+8\sqrt{abcd}}.$$

We claim that

$$x \ge 1 \quad \Longrightarrow \quad \frac{a+b+2ab}{ab(1+a)(1+b)} \ge \frac{1}{ab},$$

and

$$x \le 1 \quad \Longrightarrow \quad \frac{a+b+2ab}{ab(1+a)(1+b)} \ge \frac{2}{\sqrt{ab}+ab}.$$

The first inequality is equivalent to $ab \ge 1$, while the second inequality is equivalent to

$$\left(1-\sqrt{ab}\right)\left(\sqrt{a}-\sqrt{b}\right)^2 \ge 0.$$

Similarly, we have

$$y \ge 1 \quad \Longrightarrow \quad \frac{c+d+2cd}{cd(1+d)(1+d)} \ge \frac{1}{cd}$$

and

$$y \le 1 \quad \Longrightarrow \quad \frac{c+d+2cd}{cd(1+d)(1+d)} \ge \frac{2}{\sqrt{cd}+cd}.$$

There are four cases to consider.

Case 1: $x \ge 1$, $y \ge 1$. It suffices to show that

$$\frac{1}{x^2}+\frac{1}{y^2} \ge \frac{16}{1+8xy}.$$

Indeed, we have

$$\frac{1}{x^2}+\frac{1}{y^2} \ge \frac{2}{xy} > \frac{16}{1+8xy}.$$

Case 2: $x \le 1$, $y \le 1$. It suffices to show that

$$\frac{2}{x+x^2}+\frac{2}{y+y^2} \ge \frac{16}{1+8xy}.$$

Putting $s = x + y$ and $p = \sqrt{xy}$, this inequality becomes

$$\frac{s^2+s-2p^2}{p^2(s+p^2+1)} \ge \frac{8}{1+8p^2},$$

$$(1+8p^2)s^2+s-24p^4-10p^2 \ge 0.$$

Since $s \ge 2p$, we get

$$\begin{aligned}(1+8p^2)s^2+s-24p^4-10p^2 &\ge 4(1+8p^2)p^2+2p-24p^4-10p^2\\ &= 2p(p+1)(2p-1)^2 \ge 0.\end{aligned}$$

Case 3: $x \ge 1$, $y \le 1$. It suffices to show that

$$\frac{1}{x^2}+\frac{2}{y+y^2} \ge \frac{16}{1+8xy}.$$

This inequality is equivalent in succession to

$$(1+8xy)(2x^2+y^2+y) \ge 16x^2y(1+y),$$

$$(1+8xy)(x-y)^2+8x^3y+8xy^2-16x^2y+2xy+x^2+y \ge 0,$$

$$(1+8xy)(x-y)^2+8xy(x-1)^2+8xy^2+x^2+y \ge 6xy.$$

The last inequality is true since the AM-GM inequality yields

$$8xy^2+x^2+y \ge 3\sqrt[3]{8xy^2 \cdot x^2 \cdot y} = 3\sqrt[3]{8x^3y^3} = 6xy.$$

Case 4: $x \le 1$, $y \ge 1$. It suffices to show that

$$\frac{2}{x+x^2}+\frac{1}{y^2} \ge \frac{16}{1+8xy},$$

which is equivalent to

$$(1+8xy)(x-y)^2+8xy(y-1)^2+8x^2y+y^2+x \ge 6xy.$$

As in the case 3, we have

$$8x^2y+y^2+x \ge 3\sqrt[3]{8x^2y \cdot y^2 \cdot x} = 3\sqrt[3]{8x^3y^3} = 6xy.$$

The proof is completed. The equality holds for $a=b=c=d=\dfrac{1}{2}$.

□

P 2.117. *If a,b,c,d are positive real numbers such that $a \ge b \ge c \ge d$ and*

$$a+b+c+d=4,$$

then

$$ac+bd \le 2.$$

(Vasile C., 2019)

Solution. Write the inequality in the homogeneous form

$$(a+b+c+d)^2 \ge 8(ac+bd).$$

We have

$$(a+b+c+d)^2-8(ac+bd) = a^2+2(b+d-3c)a+(b+c+d)^2-8bd$$

$$=(a+b+d-3c)^2-(b+d-3c)^2+(b+d+c)^2-8bd$$

$$=(a+b+d-3c)^2+8(b-c)(c-d) \ge 0.$$

The equality holds for $b=c=1$ and $a+d=2$.

□

P 2.118. *If a, b, c, d are positive real numbers such that $a \ge b \ge c \ge d$ and*

$$a + b + c + d = 4,$$

then

$$2\left(\frac{1}{b} + \frac{1}{d}\right) \ge a^2 + b^2 + c^2 + d^2.$$

(Vasile C., 2019)

Solution. Write the inequality in the homogeneous form

$$(a + b + c + d)^3\left(\frac{1}{b} + \frac{1}{d}\right) - 32(a^2 + b^2 + c^2 + d^2) \ge 0.$$

For fixed b, c, d, the inequality becomes $f(a) \ge 0$, with

$$f'(a) = 3(a + b + c + d)^2\left(\frac{1}{b} + \frac{1}{d}\right) - 64a.$$

For $a + b + c + d = 4$, when $a = 4 - b - c - d \le 4 - b - 2d$, we have

$$\frac{1}{16}f'(a) \ge 3\left(\frac{1}{b} + \frac{1}{d}\right) - 4(4 - b - 2d)$$

$$= \left(\frac{3}{b} + 4b\right) + \left(\frac{3}{d} + 8d\right) - 16 \ge 4(\sqrt{3} + \sqrt{(}6) - 4) > 0.$$

Therefore, $f(a)$ is increasing, hence $f(a) \ge f(b)$. Similarly, for fixed a, b, d, the inequality becomes $g(c) \ge 0$, with

$$g'(c) = 3(a + b + c + d)^2\left(\frac{1}{b} + \frac{1}{d}\right) - 64c \ge f'(a) > 0.$$

Therefore, $g(c)$ is increasing, hence $g(c) \ge g(d)$. As a consequence, it suffices to prove the original inequality for $a = b$ and $c = d$. So, we only need to show that $b + d = 2$ involves

$$\frac{1}{b} + \frac{1}{d} \ge b^2 + d^2,$$

which is equivalent to

$$(bd - 1)^2 \ge 0.$$

The equality holds for $a = b = c = d = 1$.

□

P 2.119. *Let a, b, c, d be positive real numbers such that $a \ge b \ge c \ge d$ and*

$$ab + bc + cd + da = 3.$$

Prove that

$$a^3bcd < 4.$$

(Vasile C., 2012)

Solution. Write the desired inequality as

$$4(ab+bc+cd+da)^3 > 27a^3bcd,$$

$$4\left(b+d+\frac{bc+cd}{a}\right)^3 > 27bcd.$$

It suffices to show that

$$4(b+d)^3 \ge 27bcd.$$

Indeed, by the AM-GM inequality, we have

$$(b+d)^3 = \left(\frac{b}{2}+\frac{b}{2}+d\right)^3 \ge 27\left(\frac{b}{2}\right)^2 d \ge \frac{27bcd}{4}.$$

□

P 2.120. *Let a,b,c,d be positive real numbers such that $a \ge b \ge c \ge d$ and*

$$ab+bc+cd+da = 6.$$

Prove that

$$acd \le 2.$$

(Vasile C., 2012)

Solution. Write the desired inequality in the homogeneous form

$$(a+c)^3(b+d)^3 \ge 54a^2c^2d^2.$$

Since $b \ge c$, we only need to show that

$$(a+c)^3(c+d)^3 \ge 54a^2c^2d^2.$$

By the AM-GM inequality, we have

$$(a+c)^3 = \left(\frac{a}{2}+\frac{a}{2}+c\right)^3 \ge 27\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)c = \frac{27}{4}a^2c.$$

Thus, it suffices to show that

$$(c+d)^3 \ge 8cd^2.$$

Indeed,

$$(c+d)^3 - 8cd^2 = (c-d)(c^2+4cd-d^2) \ge 0.$$

The equality holds for $a=2$ and $b=c=d=1$.

□

P 2.121. *Let a,b,c,d be positive real numbers such that $a \ge b \ge c \ge d$ and*

$$ab+bc+cd+da=9.$$

Prove that

$$abd \le 4.$$

(Vasile C., 2012)

Solution. Write the desired inequality in the homogeneous form

$$(a+c)^3(b+d)^3 \ge \frac{729}{16}a^2b^2d^2.$$

Since $c \ge d$, we only need to show that

$$(a+d)^3(b+d)^3 \ge \frac{729}{16}a^2b^2d^2.$$

By the AM-GM inequality, we have

$$(a+d)^3 = \left(\frac{a}{2}+\frac{a}{2}+d\right)^3 \ge 27\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)d = \frac{27}{4}a^2d$$

and, similarly,

$$(b+d)^3 \ge \frac{27}{4}b^2d$$

Multiplying these inequalities, the desired inequality holds. The equality occurs for $a=b=2$ and $c=d=1$.

□

P 2.122. *Let a,b,c,d be positive real numbers such that $a \ge b \ge c \ge d$ and*

$$a^2+b^2+c^2+d^2=10.$$

Prove that

$$2b+4d \le 3c+5.$$

(Vasile C., 2012)

Solution. Write the desired inequality in the homogeneous form

$$2b-3c+4d \le \sqrt{\frac{5}{2}(a^2+b^2+c^2+d^2)}.$$

It is true if

$$5(a^2+b^2+c^2+d^2) \ge 2(2b-3c+4d)^2.$$

Since $a \ge b$, it remains to show that

$$5(2b^2 + c^2 + d^2) \ge 2(2b - 3c + 4d)^2,$$

which is equivalent to

$$2b^2 + 24bc + 48cd \ge 13c^2 + 27d^2 + 32bd.$$

Since $d^2 \le cd$, it suffices to prove that

$$2b^2 + 24bc + 48cd \ge 13c^2 + 27cd + 32bd,$$

which is equivalent to

$$2b^2 + 24bc \ge 13c^2 + (32b - 21c)d.$$

Since $32b - 21c > 0$ and $c \ge d$, it is enough to show that

$$2b^2 + 24bc \ge 13c^2 + (32b - 21c)c.$$

This reduces to the obvious inequality

$$2(b - 2c)^2 \ge 0.$$

The equality holds for $a = b = 2$ and $c = d = 1$.

□

P 2.123. *Let a, b, c, d be positive real numbers such that $a \le b \le c \le d$ and*

$$abcd = 1.$$

Prove that

$$4 + \frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \ge 2(a + b)(c + d).$$

Solution. Since

$$\frac{b}{c} + \frac{d}{a} - \frac{b}{a} - \frac{d}{c} = \frac{(d - b)(c - a)}{ca} \ge 0,$$

we only need to prove that

$$4 + \frac{a}{b} + \frac{b}{a} + \frac{c}{d} + \frac{d}{c} \ge 2(a + b)(c + d),$$

which is equivalent to

$$\frac{(a + b)^2}{ab} + \frac{(c + d)^2}{cd} \ge 2(a + b)(c + d),$$

$$\left(\frac{a + b}{\sqrt{ab}} - \frac{c + d}{\sqrt{cd}}\right)^2 \ge 0.$$

The proof is completed. The equality holds for $a = b = c = d = 1$.

□

P 2.124. *Let a,b,c,d be positive real numbers such that $a \ge b \ge c \ge d$ and*

$$3(a^2+b^2+c^2+d^2) = (a+b+c+d)^2.$$

Prove that

(a) $$\frac{a+d}{b+c} \le 2;$$

(b) $$\frac{a+c}{b+d} \le \frac{7+2\sqrt{6}}{5};$$

(c) $$\frac{a+c}{c+d} \le \frac{3+\sqrt{5}}{2}.$$

(Vasile C., 2010)

Solution. (a) Since

$$(a+d)(b+c) - 2(ad+bc) = (a-b)(c-d) + (a-c)(b-d) \ge 0,$$

we have

$$\begin{aligned} a^2+b^2+c^2+d^2 &= (a+d)^2 + (b+c)^2 - 2(ad+bc) \\ &\ge (a+d)^2 + (b+c)^2 - (a+d)(b+c), \end{aligned}$$

hence

$$\frac{1}{3}(a+b+c+d)^2 \ge (a+d)^2 + (b+c)^2 - (a+d)(b+c),$$

$$\left(\frac{a+d}{b+c} - 2\right)\left(\frac{a+d}{b+c} - \frac{1}{2}\right) \le 0,$$

from where the desired result follows. The equality holds for $a/3 = b = c = d$.

(b) From $(a-d)(b-c) \ge 0$ and the AM-GM inequality, we have

$$2(ac+bc) \le (a+d)(b+c) \le \frac{(a+b+c+d)^2}{4},$$

hence

$$\begin{aligned} a^2+b^2+c^2+d^2 &= (a+c)^2 + (b+d)^2 - 2(ac+bd) \\ &\ge (a+c)^2 + (b+d)^2 - \frac{(a+b+c+d)^2}{4}, \end{aligned}$$

$$\frac{1}{3}(a+b+c+d)^2 \ge (a+c)^2 + (b+d)^2 - \frac{(a+b+c+d)^2}{4},$$

$$\left(\frac{a+c}{b+d} - \frac{7+2\sqrt{6}}{2}\right)\left(\frac{a+c}{b+d} - \frac{7-2\sqrt{6}}{2}\right) \le 0,$$

from where the desired result follows. The equality holds for

$$(3-\sqrt{6})a = b = c = (3+\sqrt{6})d.$$

(c) Writing the hypothesis $3(a^2+b^2+c^2+d^2)=(a+b+c+d)^2$ as

$$b^2-(a+c+d)b+a^2+c^2+d^2-ac-cd-da=0,$$

$$(2b-a-c-d)^2 = 3(2ac+2cd+2da-a^2-c^2-d^2),$$

it follows that

$$2ac+2cd+2da \ge a^2+c^2+d^2,$$

$$a^2-2(c+d)a+(c-d)^2 \le 0,$$

$$a \le c+d+2\sqrt{cd}.$$

Thus, it suffices to prove that

$$\frac{2c+d+2\sqrt{cd}}{c+d} \le \frac{3+\sqrt{5}}{2},$$

which is equivalent to

$$(\sqrt{5}-1)c+(\sqrt{5}+1)d \ge 4\sqrt{cd}.$$

This inequality follows immediately from the AM-GM inequality. The equality holds for

$$\frac{a}{3+\sqrt{5}} = \frac{b}{4} = \frac{c}{2} = \frac{d}{3-\sqrt{5}}.$$

□

P 2.125. *Let a,b,c,d be nonnegative real numbers such that $a \ge b \ge c \ge d$ and*

$$2(a^2+b^2+c^2+d^2) = (a+b+c+d)^2.$$

Prove that

$$a \ge b+3c+(2\sqrt{3}-1)d.$$

(Vasile C., 2010)

First Solution. For $c=d=0$, the desired inequality is an equality. Assume further that $c>0$. From the hypothesis $2(a^2+b^2+c^2+d^2)=(a+b+c+d)^2$, we get

$$a = b+c+d \pm 2\sqrt{bc+cd+db}.$$

It is not possible to have

$$a = b+c+d-2\sqrt{bc+cd+db},$$

because this equality and $a \ge b$ involve

$$c + d \ge 2\sqrt{bc + cd + db},$$
$$(c-d)^2 \ge 4b(c+d),$$
$$(c-d)^2 \ge 4c(c+d),$$
$$d^2 \ge 3c(c+2d),$$

which is not true. Thus, we have

$$a = b + c + d + 2\sqrt{bc + cd + db}.$$

Using this equality, we can rewrite the desired inequality as

$$b + c + d - 2\sqrt{bc + cd + db} \ge b + 3c + (2\sqrt{3} - 1)d,$$
$$\sqrt{b(c+d) + cd} \ge c + (\sqrt{3} - 1)d.$$

Since $b \ge c$, it suffices to show that

$$\sqrt{c(c+d) + cd} \ge c + (\sqrt{3} - 1)d.$$

By squaring, we get the obvious inequality $d(c-d) \ge 0$. The equality holds for $a = b$ and $c = d = 0$, for $\frac{a}{4} = b = c$ and $d = 0$, and for $\frac{a}{3 + 2\sqrt{3}} = b = c = d$.

Second Solution (by *Vo Quoc Ba Can*). Write the hypothesis $2(a^2 + b^2 + c^2 + d^2) = (a + b + c + d)^2$ as

$$(a-b)^2 + (c-d)^2 \ge 2(a+b)(c+d).$$

Since

$$a + b \ge (a - b) + 2c,$$

we get

$$(a-b)^2 + (c-d)^2 \ge 2[(a-b) + 2c](c+d),$$

which is equivalent to

$$(a-b)^2 - 2(c+d)(a-b) - 3c^2 - 6cd + d^2 \ge 0.$$

From this, we get

$$a - b \ge c + d + 2\sqrt{c^2 + 2cd}.$$

Thus, the desired inequality

$$a - b \ge 3c + (2\sqrt{3} - 1)d$$

is true if

$$c + d + 2\sqrt{c^2 + 2cd} \ge 3c + (2\sqrt{3} - 1)d,$$

that is,

$$\sqrt{c^2 + 2cd} \ge c + (\sqrt{3} - 1)d.$$

By squaring, we get the obvious inequality $d(c-d) \ge 0$.

$\square$

P 2.126. *If $a \ge b \ge c \ge d \ge 0$, then*

(a) $$a+b+c+d-4\sqrt[4]{abcd} \ge \frac{3}{2}\left(\sqrt{b}-2\sqrt{c}+\sqrt{d}\right)^2;$$

(b) $$a+b+c+d-4\sqrt[4]{abcd} \ge \frac{2}{9}\left(3\sqrt{b}-2\sqrt{c}-\sqrt{d}\right)^2;$$

(c) $$a+b+c+d-4\sqrt[4]{abcd} \ge \frac{4}{19}\left(3\sqrt{b}-\sqrt{c}-2\sqrt{d}\right)^2;$$

(d) $$a+b+c+d-4\sqrt[4]{abcd} \ge \frac{3}{8}\left(\sqrt{b}-3\sqrt{c}+2\sqrt{d}\right)^2;$$

(e) $$a+b+c+d-4\sqrt[4]{abcd} \ge \frac{1}{2}\left(2\sqrt{b}-3\sqrt{c}+\sqrt{d}\right)^2;$$

(f) $$a+b+c+d-4\sqrt[4]{abcd} \ge \frac{1}{6}\left(2\sqrt{b}+\sqrt{c}-3\sqrt{d}\right)^2.$$

(Vasile C., 2010)

Solution. First, we show that

$$a-4\sqrt[4]{abcd} \ge b-4\sqrt[4]{b^2cd}.$$

Write this inequality as

$$a-b \ge 4\sqrt[4]{bcd}\left(\sqrt[4]{a}-\sqrt[4]{b}\right),$$

and prove then the following sharper inequality

$$a-b \ge 4\sqrt[4]{b^3}\left(\sqrt[4]{a}-\sqrt[4]{b}\right).$$

Indeed,

$$a-b-4\sqrt[4]{b^3}\left(\sqrt[4]{a}-\sqrt[4]{b}\right)=\left(\sqrt[4]{a}-\sqrt[4]{b}\right)\left(\sqrt[4]{a^3}+\sqrt[4]{a^2b}+\sqrt[4]{ab^2}-3\sqrt[4]{b^3}\right) \ge 0.$$

Thus, we have

$$a+b+c+d-4\sqrt[4]{abcd} \ge 2b+c+d-4\sqrt[4]{b^2cd},$$

which is equivalent to

$$a+b+c+d-4\sqrt[4]{abcd} \ge 2\left(\sqrt{b}-\sqrt[4]{cd}\right)^2+\left(\sqrt{c}-\sqrt{d}\right)^2.$$

Since

$$\sqrt{b}-\sqrt[4]{cd} \ge \sqrt{b}-\frac{\sqrt{c}+\sqrt{d}}{2} \ge 0,$$

we have

$$a+b+c+d-4\sqrt[4]{abcd} \ge \frac{1}{2}\left(2\sqrt{b}-\sqrt{c}-\sqrt{d}\right)^2+\left(\sqrt{c}-\sqrt{d}\right)^2.$$

Using the substitution

$$x = \sqrt{b} - \sqrt{c}, \quad y = \sqrt{c} - \sqrt{d}, \qquad x, y \geq 0,$$

we get

$$a + b + c + d - 4\sqrt[4]{abcd} \geq \frac{1}{2}(2x + y)^2 + y^2,$$

that is

$$a + b + c + d - 4\sqrt[4]{abcd} \geq \frac{1}{2}(4x^2 + 4xy + 3y^2). \tag{*}$$

The inequality (*) is an equality for $a = b$ and $c = d$.

(a) According to (*), it suffices to show that

$$4x^2 + 4xy + 3y^2 \geq 3(x - y)^2,$$

which is equivalent to

$$x(x + 10y) \geq 0.$$

The equality holds for $a = b = c = d$.

(b) According to (*), it suffices to show that

$$9(4x^2 + 4xy + 3y^2) \geq 4(3x + y)^2,$$

which is equivalent to

$$y(12x + 23y) \geq 0.$$

The equality holds for $a = b$ and $c = d$.

(c) According to (*), it suffices to show that

$$19(4x^2 + 4xy + 3y^2) \geq 8(3x + 2y)^2,$$

which is equivalent to

$$(2x - 5y)^2 \geq 0.$$

The equality holds for $a = b = c = d$.

(d) According to (*), it suffices to show that

$$4(4x^2 + 4xy + 3y^2) \geq 3(x - 2y)^2,$$

which is equivalent to

$$x(13x + 28y) \geq 0.$$

The equality holds for $a = b = c = d$.

(e) According to (*), it suffices to show that

$$4x^2 + 4xy + 3y^2 \geq (2x - y)^2,$$

which is equivalent to

$$y(4x+y) \ge 0.$$

The equality holds for $a=b$ and $c=d$.

(f) According to (*), it suffices to show that

$$3(4x^2+4xy+3y^2) \ge (2x+3y)^2,$$

which is equivalent to

$$x^2 \ge 0.$$

The equality holds for $a=b=c=d$.

□

P 2.127. *If $a \ge b \ge c \ge d \ge 0$, then*

(a) $$a+b+c+d-4\sqrt[4]{abcd} \ge \left(\sqrt{a}-\sqrt{d}\right)^2;$$

(b) $$a+b+c+d-4\sqrt[4]{abcd} \ge 2\left(\sqrt{b}-\sqrt{c}\right)^2;$$

(c) $$a+b+c+d-4\sqrt[4]{abcd} \ge \frac{4}{3}\left(\sqrt{b}-\sqrt{d}\right)^2;$$

(d) $$a+b+c+d-4\sqrt[4]{abcd} \ge \frac{3}{2}\left(\sqrt{c}-\sqrt{d}\right)^2.$$

(Vasile C., 2010)

Solution. (a) Write the inequality as

$$b+c+2\sqrt{ad} \ge 4\sqrt[4]{abcd},$$

which follows immediately from the AM-GM inequality. The equality holds for

$$b=c=\sqrt{ad}.$$

(b) ***First Solution.*** Since

$$a+b+c+d-4\sqrt[4]{abcd} \ge 2\sqrt{ab}+2\sqrt{cd}-4\sqrt[4]{abcd} = 2\left(\sqrt[4]{ab}-\sqrt[4]{cd}\right)^2,$$

we only need to show that

$$\sqrt[4]{ab}-\sqrt[4]{cd} \ge \sqrt{b}-\sqrt{c},$$

which is equivalent to the obvious inequality

$$\sqrt[4]{b}\left(\sqrt[4]{a}-\sqrt[4]{b}\right)+\sqrt[4]{c}\left(\sqrt[4]{c}-\sqrt[4]{d}\right) \ge 0.$$

The equality holds for $a = b$ and $c = d$.

Second Solution. According to the inequality (*) from the proof of the preceding P 2.126, it suffices to show that

$$4x^2 + 4xy + 3y^2 \geq 4x^2,$$

which is obvious.

(c) According to the inequality (*) from the proof of the preceding P 2.126, it suffices to show that

$$3(4x^2 + 4xy + 3y^2) \geq 8(x + y)^2,$$

which is equivalent to

$$(2x - y)^2 \geq 0.$$

The equality holds for $a = b = c = d$.

(d) According to the inequality (*) from the proof of the preceding P 2.126, it suffices to show that

$$4x^2 + 4xy + 3y^2 \geq 3y^2,$$

which is obvious. The equality holds for $a = b = c = d$.

□

P 2.128. *If $a \geq b \geq c \geq d \geq e \geq 0$, then*

$$a + b + c + d + e - 5\sqrt[5]{abcde} \geq 2\left(\sqrt{b} - \sqrt{d}\right)^2.$$

(Vasile C., 2010)

Solution. From the AM-GM inequality, we have

$$c + 4\sqrt[4]{abde} \geq 5\sqrt[5]{abcde},$$

which can be rewritten as

$$c - 5\sqrt[5]{abcde} \geq -4\sqrt[4]{abde}.$$

Thus, it suffices to show that

$$a + b + d + e - 4\sqrt[4]{abde} \geq 2\left(\sqrt{b} - \sqrt{d}\right)^2.$$

Since

$$a + b + d + e - 4\sqrt[4]{abde} \geq 2\sqrt{ab} + 2\sqrt{de} - 4\sqrt[4]{abde} = 2\left(\sqrt[4]{ab} - \sqrt[4]{de}\right)^2,$$

we only need to prove that

$$\sqrt[4]{ab}-\sqrt[4]{de}\ge\sqrt{b}-\sqrt{d},$$

which is equivalent to the obvious inequality

$$\sqrt[4]{b}\left(\sqrt[4]{a}-\sqrt[4]{b}\right)+\sqrt[4]{d}\left(\sqrt[4]{d}-\sqrt[4]{e}\right)\ge 0.$$

The equality holds for

$$a=b,\quad d=e,\quad c^2=ad.$$

□

P 2.129. *If a,b,c,d,e are real numbers, then*

$$\frac{ab+bc+cd+de}{a^2+b^2+c^2+d^2+e^2}\le\frac{\sqrt{3}}{2}.$$

Solution. Using the AM-GM inequality, we have

$$a^2+b^2+c^2+d^2+e^2=\left(a^2+\frac{1}{3}b^2\right)+\left(\frac{2}{3}b^2+\frac{1}{2}c^2\right)+\left(\frac{1}{2}c^2+\frac{2}{3}d^2\right)+\left(\frac{1}{3}d^2+e^2\right)$$

$$\ge 2\sqrt{a^2\cdot\frac{1}{3}b^2}+2\sqrt{\frac{2}{3}b^2\cdot\frac{1}{2}c^2}+2\sqrt{\frac{1}{2}c^2\cdot\frac{2}{3}d^2}+2\sqrt{\frac{1}{3}d^2\cdot e^2}$$

$$\ge\frac{2}{\sqrt{3}}(ab+bc+cd+da).$$

The equality holds for

$$a=\frac{b}{\sqrt{3}}=\frac{c}{2}=\frac{d}{\sqrt{3}}=e.$$

Remark. The following more general inequality holds

$$\frac{a_1a_2+a_2a_3+\cdots+a_{n-1}a_n}{a_1^2+a_2^2+\cdots+a_n^2}\le\cos\frac{\pi}{n+1},$$

with equality for

$$\frac{a_1}{\sin\frac{\pi}{n+1}}=\frac{a_2}{\sin\frac{2\pi}{n+1}}=\cdots=\frac{a_n}{\sin\frac{n\pi}{n+1}}.$$

Denoting

$$c_i=\frac{\sin\frac{(i+1)\pi}{n+1}}{2\sin\frac{i\pi}{n+1}},\quad i=1,2,\cdots,n-1,$$

we have

$$c_1 = \cos\frac{\pi}{n+1}, \qquad 4c_{n-1} = \frac{1}{\cos\frac{\pi}{n+1}},$$

$$\frac{1}{4c_i} + c_{i+1} = \cos\frac{\pi}{n+1}, \qquad i = 1, 2, \cdots, n-2,$$

hence

$$(a_1^2 + a_2^2 + \cdots + a_n^2)\cos\frac{\pi}{n+1} =$$

$$= c_1a_1^2 + \left(\frac{1}{4c_1} + c_2\right)a_2^2 + \cdots + \left(\frac{1}{4c_{n-2}} + c_{n-1}\right)a_{n-1}^2 + \frac{1}{4c_{n-1}}a_n^2$$

$$= \left(c_1a_1^2 + \frac{1}{4c_1}a_2^2\right) + \left(c_2a_2^2 + \frac{1}{4c_2}a_3^2\right) + \cdots + \left(c_{n-1}a_{n-1}^2 + \frac{1}{4c_{n-1}}a_n^2\right)$$

$$\geq 2\sqrt{c_1a_1^2 \cdot \frac{1}{4c_1}a_2^2} + 2\sqrt{c_2a_2^2 \cdot \frac{1}{4c_2}a_3^2} + \cdots + 2\sqrt{c_{n-1}a_{n-1}^2 \cdot \frac{1}{4c_{n-1}}a_n^2}$$

$$\geq a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n.$$

□

P 2.130. *If a, b, c, d, e are positive real numbers, then*

$$\frac{a^2b^2}{bd+ce} + \frac{b^2c^2}{cd+ae} + \frac{c^2a^2}{ad+be} \geq \frac{3abc}{d+e}.$$

Solution. Using the Cauchy-Schwarz inequality

$$\frac{a^2b^2}{bd+ce} + \frac{b^2c^2}{cd+ae} + \frac{c^2a^2}{ad+be} \geq \frac{(ab+bc+ca)^2}{(bd+ce)+(cd+ae)+(ad+be)},$$

it suffices to show that

$$\frac{(ab+bc+ca)^2}{(bd+ce)+(cd+ae)+(ad+be)} \geq \frac{3abc}{d+e},$$

which is equivalent to

$$\frac{(ab+bc+ca)^2}{a+b+c} \geq 3abc,$$

$$a^2(b-c)^2 + b^2(c-a)^2 + c^2(a-b)^2 \geq 0.$$

The equality holds for $a = b = c$.

□

P 2.131. *If a,b,c,d,e,f are nonnegative real numbers such that*

$$a \ge b \ge c \ge d \ge e \ge f,$$

then

$$(a+b+c+d+e+f)^2 \ge 8(ac+bd+ce+df).$$

(Vasile C., 2005)

First Solution. Let us denote

$$x = b+c+d+e+f,$$

and write the inequality as follows:

$$(a+x)^2 - 8(ac+bd+ce+df) \ge 0,$$

$$(a+x-4c)^2 + 8(a+x)c - 16c^2 - 8(ac+bd+ce+df) \ge 0,$$

$$(a+x-4c)^2 - 8[c^2-(b+d+f)c+d(b+f)] \ge 0,$$

$$(a+x-4c)^2 - 8(c-d)(c-b-f) \ge 0,$$

$$(a+x-4c)^2 + 8(c-d)(b-c+f) \ge 0.$$

The last inequality is clearly true. The equality holds for $c = d = (a+b+e+f)/2$, and for $c = b+f = (a+d+e)/2$; that is, for

$$a = b = c = d, \quad e = f = 0,$$

and for

$$a \ge d+e, \quad b = c = \frac{a+d+e}{2}, \quad f = 0.$$

□

P 2.132. *If $a \ge b \ge c \ge d \ge e \ge f \ge 0$, then*

$$a+b+c+d+e+f - 6\sqrt[6]{abcdef} \ge 2\left(\sqrt{b}-\sqrt{e}\right)^2.$$

(Vasile C., 2010)

Solution. Since

$$a+b \ge 2\sqrt{ab}, \quad c+d \ge 2\sqrt{cd}, \quad e+f \ge 2\sqrt{ef},$$

it suffices to show that

$$\sqrt{ab}+\sqrt{cd}+\sqrt{ef} - 3\sqrt[6]{abcdef} \ge \left(\sqrt{b}-\sqrt{e}\right)^2.$$

By the AM-GM inequality, we have

$$\sqrt{cd}+2\sqrt[4]{abef}\ge 3\sqrt[6]{abcdef},$$

which can be rewritten as

$$\sqrt{cd}-3\sqrt[6]{abcdef}\ge -2\sqrt[4]{abef}.$$

Thus, it suffices to show that

$$\sqrt{ab}+\sqrt{ef}-2\sqrt[4]{abef}\ge\left(\sqrt{b}-\sqrt{e}\right)^2.$$

Since

$$\sqrt{ab}+\sqrt{ef}-2\sqrt[4]{abef}=\left(\sqrt[4]{ab}-\sqrt[4]{ef}\right)^2,$$

we only need to prove that

$$\sqrt[4]{ab}-\sqrt[4]{ef}\ge\sqrt{b}-\sqrt{e},$$

which is equivalent to the obvious inequality

$$\sqrt[4]{b}\left(\sqrt[4]{a}-\sqrt[4]{b}\right)+\sqrt[4]{e}\left(\sqrt[4]{e}-\sqrt[4]{f}\right)\ge 0.$$

The equality holds for

$$a=b,\quad c=d,\quad e=f,\quad c^2=ae.$$

□

P 2.133. *Let a,b,c and x,y,z be positive real numbers such that*

$$x+y+z=a+b+c.$$

Prove that

$$ax^2+by^2+cz^2+xyz\ge 4abc.$$

(Vasile C., 1989)

First Solution. Write the inequality as $E\ge 0$, where

$$E=ax^2+by^2+cz^2+xyz-4abc.$$

Among the numbers

$$a-\frac{y+z}{2},\quad b-\frac{z+x}{2},\quad c-\frac{x+y}{2},$$

there are two of them with the same sign; let

$$pq \ge 0,$$

where

$$p = b - \frac{z+x}{2}, \quad q = c - \frac{x+y}{2}.$$

We have

$$b = p + \frac{x+z}{2}, \quad c = q + \frac{x+y}{2}, \quad a = x+y+z-b-c = \frac{y+z}{2} - p - q.$$

Then,

$$\begin{aligned}
E =& \left(\frac{y+z}{2} - p - q\right)x^2 + \left(p + \frac{x+z}{2}\right)y^2 + \left(q + \frac{x+y}{2}\right)z^2 \\
&+ xyz - 4\left(\frac{y+z}{2} - p - q\right)\left(p + \frac{x+z}{2}\right)\left(q + \frac{x+y}{2}\right) \\
=& 4pq(p+q) + 2p^2(x+y) + 2q^2(x+z) + 4pqx \\
=& 4q^2\left(p + \frac{x+z}{2}\right) + 4p^2\left(q + \frac{x+y}{2}\right) + 4pqx \\
=& 4(q^2b + p^2c + pqx) \ge 0.
\end{aligned}$$

The equality holds for $a = \dfrac{y+z}{2}$, $b = \dfrac{z+x}{2}$, $c = \dfrac{x+y}{2}$.

Second Solution. Consider the following two cases.

Case 1: $x^2 \ge 4bc$. We have

$$ax^2 + by^2 + cz^2 + xyz - 4abc > ax^2 - 4abc \ge 0.$$

Case 2: $x^2 \le 4bc$. Let

$$u = x + y + z = a + b + c.$$

Substituting

$$z = u - x - y, \quad a = u - b - c,$$

the inequality can be restated as

$$Au^2 + Bu + C \ge 0,$$

where

$$A = c,$$

$$B = (x^2 - 4bc) - 2c(x+y) + xy,$$

$$C = -(b+c)(x^2 - 4bc) + by^2 + c(x+y)^2 - xy(x+y).$$

Since the quadratic function $Au^2 + Bu + C$ has the discriminant

$$D = (x^2 - 4bc)(2c - x - y)^2 \le 0,$$

the conclusion follows.

□

P 2.134. *Let a, b, c and x, y, z be positive real numbers such that*

$$x + y + z = a + b + c.$$

Prove that

$$\frac{x(3x+a)}{bc} + \frac{y(3y+b)}{ca} + \frac{z(3z+c)}{ab} \ge 12.$$

(Vasile C., 1990)

Solution. Write the inequality as

$$ax^2 + by^2 + cz^2 + \frac{1}{3}(a^2x + b^2y + c^2z) \ge 4abc.$$

Applying the Cauchy-Schwarz inequality, we have

$$a^2x + b^2y + c^2z \ge \frac{(a+b+c)^2}{\frac{1}{x} + \frac{1}{y} + \frac{1}{z}} = \frac{xyz(x+y+z)^2}{xy+yz+zx} \ge 3xyz.$$

Therefore, it suffices to show that

$$ax^2 + by^2 + cz^2 + xyz \ge 4abc,$$

which is just the inequality in the preceding P 2.133. The equality holds for

$$x = y = z = a = b = c.$$

□

P 2.135. *Let a, b, c be given positive numbers. Find the minimum value $F(a, b, c)$ of*

$$E(x, y, z) = \frac{ax}{y+z} + \frac{by}{z+x} + \frac{cz}{x+y},$$

where x, y, z are nonnegative real numbers, no two of which are zero.

(Vasile C., 2006)

Solution. Assume that

$$a = \max\{a, b, c\}.$$

There are two cases to consider.

Case 1: $\sqrt{a} < \sqrt{b} + \sqrt{c}$. Using the Cauchy-Schwarz inequality, we get

$$E = \sum \frac{a(x+y+z) - a(y+z)}{y+z} = (x+y+z)\sum \frac{a}{y+z} - \sum a$$

$$\ge (x+y+z)\frac{\left(\sum \sqrt{a}\right)^2}{\sum (y+z)} - \sum a = \sqrt{ab} + \sqrt{bc} + \sqrt{ca} - \frac{a+b+c}{2}.$$

The equality holds for
$$\frac{y+z}{\sqrt{a}}=\frac{z+x}{\sqrt{b}}=\frac{x+y}{\sqrt{c}};$$
that is, for
$$\frac{x}{\sqrt{b}+\sqrt{c}-\sqrt{a}}=\frac{y}{\sqrt{c}+\sqrt{a}-\sqrt{b}}=\frac{z}{\sqrt{a}+\sqrt{b}-\sqrt{c}}.$$
Case 2: $\sqrt{a}\ge\sqrt{b}+\sqrt{c}$. Let us denote
$$A=(\sqrt{b}+\sqrt{c})^2,$$
$$X=\frac{y+z}{2},\quad Y=\frac{z+x}{2},\quad Z=\frac{x+y}{2},$$
hence
$$x=Y+Z-X,\quad y=Z+X-Y,\quad z=X+Y-Z.$$
We have
$$\begin{aligned}
E&\ge\frac{Ax}{y+z}+\frac{by}{z+x}+\frac{cz}{x+y}\\
&=\frac{A(Y+Z-X)}{2X}+\frac{b(Z+X-Y)}{2Y}+\frac{c(X+Y-Z)}{2Z}\\
&=\frac{1}{2}\left(A\frac{Y}{X}+b\frac{X}{Y}\right)+\frac{1}{2}\left(b\frac{Z}{Y}+c\frac{Y}{Z}\right)+\frac{1}{2}\left(c\frac{X}{Z}+A\frac{Z}{X}\right)-b-c-\sqrt{bc}\\
&\ge\sqrt{Ab}+\sqrt{bc}+\sqrt{cA}-b-c-\sqrt{bc}=2\sqrt{bc}.
\end{aligned}$$
The equality holds for $x=0$ and $\dfrac{y}{z}=\sqrt{\dfrac{c}{b}}$. Therefore, for $a=\max\{a,b,c\}$, we have
$$F(a,b,c)=\begin{cases}\sqrt{ab}+\sqrt{bc}+\sqrt{ca}-\dfrac{a+b+c}{2}, & \sqrt{a}<\sqrt{b}+\sqrt{c}\\ 2\sqrt{bc}, & \sqrt{a}\ge\sqrt{b}+\sqrt{c}\end{cases}.$$
□

P 2.136. *Let a,b,c and x,y,z be real numbers.*

(a) If $ab+bc+ca>0$, then
$$[(b+c)x+(c+a)y+(a+b)z]^2\ge 4(ab+bc+ca)(xy+yz+zx);$$
(b) If $a,b,c\ge 0$, then
$$[(b+c)x+(c+a)y+(a+b)z]^2\ge 4(a+b+c)(ayz+bzx+cxy).$$
(Vasile C., 1995)

Solution. (a) ***First Solution.*** The condition $ab+bc+ca>0$ yields $b+c\neq 0$. Indeed, if $b+c=0$, then $ab+bc+ca=-b^2\leq 0$, which is false. The desired inequality is equivalent to $D\geq 0$, where D is the discriminant of the quadratic function

$$f(t)=(at-x)(bt-y)+(bt-y)(ct-z)+(ct-z)(at-x).$$

For the sake of contradiction, assume that $D<0$ for some real numbers a,b,c and x,y,z. Since the coefficient of t^2 is positive, we have $f(t)>0$ for all real t. This is not true, because for

$$(bt-y)+(ct-z)=0,$$

we get

$$t=\frac{y+z}{b+c}$$

and

$$f\left(\frac{y+z}{b+c}\right)=-\left(\frac{bz-cy}{b+c}\right)^2\leq 0.$$

For $pqr\neq 0$, the equality holds when

$$\frac{x}{a}=\frac{y}{b}=\frac{z}{c}.$$

Second Solution. If $xy+yz+zx\leq 0$, then the inequality is obviously true. Otherwise, due to homogeneity in x,y,z, we may assume that

$$x+y+z=a+b+c.$$

Then, by the AM-GM inequality, we have

$$\begin{aligned}2\sqrt{(ab+bc+ca)(xy+yz+zx)}&\leq(ab+bc+ca)+(xy+yz+zx)\\&=\frac{(a+b+c)^2-a^2-b^2-c^2}{2}+\frac{(x+y+z)^2-x^2-y^2-z^2}{2}\\&=(a+b+c)(x+y+z)-\frac{a^2+x^2}{2}-\frac{b^2+y^2}{2}-\frac{c^2+z^2}{2}\\&\leq(a+b+c)(x+y+z)-ax-by-cz=(b+c)x+(c+a)y+(a+b)z.\end{aligned}$$

(b) Assume that x is between y and z, that is,

$$(x-y)(x-z)\leq 0.$$

Consider the non-trivial case

$$a+b+c>0.$$

The desired inequality is equivalent to $D \ge 0$, where D is the discriminant of the quadratic function

$$f(t) = a(t-y)(t-z) + b(t-z)(t-x) + c(t-x)(t-y).$$

For the sake of contradiction, assume that $D < 0$ for some $a, b, c \ge 0$ and real numbers x, y, z. Since the coefficient of t^2 is positive, we have $f(t) > 0$ for all real t. This is false, because

$$f(x) = a(x-y)(x-z) \le 0.$$

The equality holds for $x = y = z$, and also for $a = 0$ and $x = \dfrac{cy+bz}{c+b}$, or $b = 0$ and $y = \dfrac{az+cx}{a+c}$, or $c = 0$ and $z = \dfrac{bx+ay}{b+a}$.

Remark 1. For $x = b$, $y = c$, $z = a$, from the inequality in (b), we get the following cyclic inequality:

$$(a^2+b^2+c^2+ab+bc+ca)^2 \ge 4(a+b+c)(ab^2+bc^2+ca^2),$$

where $a, b, c \ge 0$. The equality holds for $a = b = c$, and also for $a = 0$ and $\dfrac{b}{c} = \dfrac{\sqrt{5}-1}{2}$ (or any cyclic permutation). Notice that this inequality is equivalent to

$$a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2 \ge 2(ab^3+bc^3+ca^3-a^3b-b^3c-c^3a),$$

which is the inequality in P 3.95 from Volume 1.

Remark 2. For $x = 1/c$, $y = 1/a$, $z = 1/b$, from the inequality in (b), we get the following cyclic inequality:

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+3\right)^2 \ge 4(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right),$$

which is the inequality in P 1.49-(c).

Remark 3. For $a = x(x-y+z)$, $b = y(y-z+x)$, $c = z(z-x+y)$, the inequality in (b) turns into

$$(x^2y+y^2z+z^2x)^2 \ge xyz(x+y+z)(x^2+y^2+z^2).$$

where x, y, z are the lengths of the sides of a triangle (see P 1.187).

$\square$

P 2.137. *Let a,b,c and x,y,z be positive real numbers such that*

$$\frac{a}{yz}+\frac{b}{zx}+\frac{c}{xy}=1.$$

Prove that

(a) $$x+y+z\ge\sqrt{4(a+b+c+\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\ )+3\sqrt[3]{abc}};$$

(b) $$x+y+z>\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}.$$

Solution. (a) Write the desired inequality in the form

$$\left(\frac{a}{yz}+\frac{b}{zx}+\frac{c}{xy}\right)(x+y+z)^2\ge 4\left(a+b+c+\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)+3\sqrt[3]{abc}.$$

We have

$$\left(\frac{a}{yz}+\frac{b}{zx}+\frac{c}{xy}\right)(x^2+y^2+z^2)=\sum\frac{ax^2}{yz}+\sum\frac{a(y^2+z^2)}{yz}.$$

In addition, by the AM-GM inequality, we get

$$\sum\frac{ax^2}{yz}\ge 3\sqrt[3]{abc},$$

$$\sum\frac{a(y^2+z^2)}{yz}\ge 2(a+b+c).$$

Therefore,

$$\left(\frac{a}{yz}+\frac{b}{zx}+\frac{c}{xy}\right)(x^2+y^2+z^2)\ge 3\sqrt[3]{abc}+2(a+b+c).$$

Adding this inequality to the Cauchy-Schwarz inequality

$$2\left(\frac{a}{yz}+\frac{b}{zx}+\frac{c}{xy}\right)(yz+zx+xy)\ge 2\left(\sqrt{a}+\sqrt{a}+\sqrt{c}\right)^2$$

yields the desired inequality. The equality holds for

$$x=y=z=\sqrt{3a}=\sqrt{3b}=\sqrt{3c}.$$

(b) According to the inequality in (a), it suffices to show that

$$4\left(a+b+c+\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)\ge\left(\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}\right)^2.$$

This inequality is equivalent to

$$\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2\ge\sqrt{(a+b)(b+c)}+\sqrt{(b+c)(c+a)}+\sqrt{(c+a)(a+b)}\ ,$$

which follows immediately from the inequality P 2.24 in Volume 2.

□

P 2.138. *If a,b,c and x,y,z are nonnegative real numbers, then*

$$\frac{2}{(b+c)(y+z)}+\frac{2}{(c+a)(z+x)}+\frac{2}{(a+b)(x+y)}\ge \frac{9}{(b+c)x+(c+a)y+(a+b)z}.$$

(Ji Chen and Vasile Cîrtoaje, 2010)

Solution. Since

$$(b+c)x+(c+a)y+(a+b)z=a(y+z)+(b+c)x+bz+cy,$$

we can write the inequality as

$$\sum \frac{2a(y+z)+2(b+c)x+2(bz+cy)}{(b+c)(y+z)}\ge 9,$$

$$\sum \frac{2a}{b+c}+\sum \frac{2x}{y+z}\ge 9-\sum \frac{2(bz+cy)}{(b+c)(y+z)},$$

$$\sum \frac{2a}{b+c}+\sum \frac{2x}{y+z}\ge 6+\sum \left[1-\frac{2(bz+cy)}{(b+c)(y+z)}\right],$$

$$\sum \frac{2a}{b+c}+\sum \frac{2x}{y+z}\ge 6+\sum \frac{(b-c)(y-z)}{(b+c)(y+z)}.$$

Since

$$\sum \frac{(b-c)(y-z)}{(b+c)(y+z)}\le \frac{1}{2}\sum \left(\frac{b-c}{b+c}\right)^2+\frac{1}{2}\sum \left(\frac{y-z}{y+z}\right)^2,$$

it suffices to show that

$$\sum \frac{2a}{b+c}+\sum \frac{2x}{y+z}\ge 6+\frac{1}{2}\sum \left(\frac{b-c}{b+c}\right)^2+\frac{1}{2}\sum \left(\frac{y-z}{y+z}\right)^2,$$

which is equivalent to

$$\sum \frac{2a}{b+c}+\sum \frac{2x}{y+z}\ge 9-\sum \frac{2bc}{(b+c)^2}-\sum \frac{2yz}{(y+z)^2},$$

$$\sum \left[\frac{2a}{b+c}+\frac{2bc}{(b+c)^2}\right]+\sum \left[\frac{2x}{y+z}+\frac{2yz}{(y+z)^2}\right]\ge 9,$$

$$2(ab+bc+ca)\sum \frac{1}{(b+c)^2}+2(xy+yz+zx)\sum \frac{1}{(y+z)^2}\ge 9.$$

This inequality can be obtained by summing the known inequalities (see P 1.72 in Volume 2, case $k=2$)

$$4(ab+bc+ca)\sum \frac{1}{(b+c)^2}\ge 9,$$

$$4(xy+yz+zx)\sum \frac{1}{(y+z)^2}\ge 9.$$

The equality holds for $a=b=c$ and $x=y=z$, and also for $a=x=0$, $b=c$ and $y=z$ (or any cyclic permutation).

Remark. For $x=a$, $y=b$ and $z=c$, we get the known inequality (Iran 1996):

$$\frac{1}{(a+b)^2}+\frac{1}{(a+c)^2}+\frac{1}{(b+c)^2}\ge\frac{9}{4(ab+bc+ca)}.$$

□

P 2.139. *Let a,b,c be the lengths of the sides of a triangle. If x,y,z are real numbers, then*

$$(ya^2+zb^2+xc^2)(za^2+xb^2+yc^2)\ge(xy+yz+zx)(a^2b^2+b^2c^2+c^2a^2).$$

(Vasile C., 2001)

First Solution. Write the inequality as follows:

$$x^2b^2c^2+y^2c^2a^2+z^2a^2b^2\ge\sum yza^2(b^2+c^2-a^2),$$

$$x^2b^2c^2+y^2c^2a^2+z^2a^2b^2\ge 2abc\sum yza\cos A,$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\ge\frac{2yz\cos A}{bc}+\frac{2zx\cos B}{ca}+\frac{2xy\cos C}{ab},$$

$$\left(\frac{x}{a}-\frac{y}{b}\cos C-\frac{z}{c}\cos B\right)^2+\left(\frac{y}{b}\sin C-\frac{z}{c}\sin B\right)^2\ge 0.$$

The equality holds for

$$\frac{x}{a^2}=\frac{y}{b^2}=\frac{z}{c^2}.$$

Second Solution. Write the inequality as

$$b^2c^2x^2-Bx+C\ge 0,$$

where

$$B=c^2(a^2+b^2-c^2)y+b^2(a^2-b^2+c^2)z,$$

$$C=a^2[c^2y^2-(b^2+c^2-a^2)yz+b^2z^2].$$

It suffices to show that

$$B^2-4b^2c^2C\le 0,$$

which is equivalent to

$$A(c^2y-b^2z)^2\ge 0,$$

where

$$A=2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4.$$

This inequality is true since

$$A = (a+b+c)(a+b-c)(b+c-a)(c+a-b) \ge 0.$$

Remark 1. For $x = 1/b$, $y = 1/c$ and $z = 1/a$, we get the well-known inequality from P 1.189-(a):

$$a^3b + b^3c + c^3a \ge a^2b^2 + b^2c^2 + c^2a^2.$$

Remark 2. For $x = 1/c^2$, $y = 1/a^2$ and $z = 1/b^2$, we get the elegant cyclic inequality of *Walker*:

$$3\left(\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2}\right) \ge (a^2+b^2+c^2)\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right).$$

□

P 2.140. *If $a_1 \ge a_2 \ge \cdots \ge a_8 \ge 0$, then*

$$a_1 + a_2 + \cdots + a_8 - 8\sqrt[8]{a_1a_2\cdots a_8} \ge 3\left(\sqrt{a_6} - \sqrt{a_7}\right)^2.$$

Solution. Let us denote

$$x = \sqrt[6]{a_1a_2\cdots a_6}, \quad y = \sqrt{a_7a_8}, \quad x \ge a_6 \ge a_7 \ge y.$$

By the AM-GM inequality, we have

$$a_1 + a_2 + \cdots + a_6 \ge 6x, \quad a_7 + a_8 \ge 2y.$$

Also, we have

$$\sqrt{a_6} - \sqrt{a_7} \le \sqrt{x} - \sqrt{y}.$$

Thus, it suffices to show that

$$6x + 2y - 8\sqrt[8]{x^6y^2} \ge 3(\sqrt{x} - \sqrt{y})^2.$$

For the nontrivial case $y \ne 0$, we can set $y = 1$ (due to homogeneity) and $x = t^4$, $t \ge 1$. The inequality can be restated as

$$6t^4 + 2 - 8t^3 \ge 3(t^2-1)^2,$$

which is equivalent to

$$(t-1)^3(3t+1) \ge 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_8$.

□

P 2.141. *Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers. Prove that*

$$a_1b_1 + \cdots + a_nb_n + \sqrt{(a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2)} \geq \frac{2}{n}(a_1 + \cdots + a_n)(b_1 + \cdots + b_n).$$

(Vasile C., 1989)

First Solution. Write the inequality as

$$\sqrt{(a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2)} \geq a_1(2b - b_1) + \cdots + a_n(2b - b_n),$$

where

$$b = \frac{1}{n}(b_1 + \cdots + b_n).$$

Using the substitution

$$x_i = 2b - b_i, \quad i = 1, 2, \ldots, n,$$

we have

$$\sum_{i=1}^n x_i = 2nb - \sum_{i=1}^n b_i = 2nb - nb = nb,$$

$$\sum_{i=1}^n b_i^2 = \sum_{i=1}^n (2b - x_i)^2 = 4nb^2 - 4b\sum_{i=1}^n x_i + \sum_{i=1}^n x_i^2 = \sum_{i=1}^n x_i^2.$$

Therefore, the desired inequality can be restated as

$$\sqrt{(a_1^2 + \cdots + a_n^2)(x_1^2 + \cdots + x_n^2)} \geq a_1x_1 + \cdots + a_nx_n,$$

which is just the Cauchy-Schwarz inequality. If $a_1a_2\cdots a_n \neq 0$, then the equality holds for

$$\frac{2b - b_1}{a_1} = \frac{2b - b_2}{a_2} = \cdots = \frac{2b - b_n}{a_n} \geq 0.$$

Second Solution. Consider the nontrivial case where $a_1^2 + \cdots + a_n^2 \neq 0$ and $b_1^2 + \cdots + b_n^2 \neq 0$, denote

$$p = \sqrt{\frac{b_1^2 + \cdots + b_n^2}{a_1^2 + \cdots + a_n^2}},$$

and use the substitution

$$b_i = px_i, \quad i = 1, 2, \ldots, n$$

to have

$$a_1^2 + \cdots + a_n^2 = x_1^2 + \cdots + x_n^2.$$

The desired inequality becomes

$$(a_1x_1 + \cdots + a_nx_n) + (a_1^2 + \cdots + a_n^2) \geq \frac{2}{n}(a_1 + \cdots + a_n)(x_1 + \cdots + x_n),$$

$$(a_1+x_1)^2+\cdots+(a_n+x_n)^2 \ge \frac{4}{n}(a_1+\cdots+a_n)(x_1+\cdots+x_n).$$

Since

$$4(a_1+\cdots+a_n)(x_1+\cdots+x_n) \le [(a_1+\cdots+a_n)+(x_1+\cdots+x_n)]^2,$$

it suffices to show that

$$(a_1+x_1)^2+\cdots+(a_n+x_n)^2 \ge \frac{1}{n}[(a_1+x_1)+\cdots+(a_n+x_n)]^2.$$

This follows immediately from the Cauchy-Schwarz inequality.

Remark. Substituting $b_i = 1/a_i$ for all i, we get the following inequality

$$n^2+n\sqrt{(a_1^2+\cdots+a_n^2)\left(\frac{1}{a_1^2}+\cdots+\frac{1}{a_n^2}\right)} \ge 2(a_1+\cdots+a_n)\left(\frac{1}{a_1}+\cdots+\frac{1}{a_n}\right).$$

If $a_1 \le a_2 \le \cdots \le a_n$ and n is even, $n=2k$, then the equality holds for

$$a_1=a_2=\cdots=a_k, \quad a_{k+1}=a_{k+2}=\cdots=a_{2k}.$$

If n is odd, then the equality holds only if $a_1=a_2=\cdots=a_n$.

Conjecture. *If $a_1,a_2,\ldots,a_n$ are positive real numbers and n is odd, then*

$$n^2+1+\sqrt{(n^2-1)(a_1^2+\cdots+a_n^2)\left(\frac{1}{a_1^2}+\cdots+\frac{1}{a_n^2}\right)-n^2+1} \ge$$

$$\ge 2(a_1+\cdots+a_n)\left(\frac{1}{a_1}+\cdots+\frac{1}{a_n}\right).$$

If $a_1 \le a_2 \le \cdots \le a_n$ and n is odd, $n=2k+1$, then the equality holds for

$$a_1=a_2=\cdots=a_k, \quad a_{k+1}=a_{k+2}=\cdots=a_{2k+1},$$

and for

$$a_1=a_2=\cdots=a_{k+1}, \quad a_{k+2}=a_{k+3}=\cdots=a_{2k+1}.$$

□

P 2.142. *Let a_1, a_2, …, a_n be positive real numbers such that $a_1 \ge 2a_2$. Prove that*

$$(5n-1)(a_1^2+a_2^2+\cdots+a_n^2) \ge 5(a_1+a_2+\cdots+a_n)^2.$$

(Vasile C., 2009)

Solution. Let

$$a_1 = ka_2, \quad k \ge 2.$$

By the Cauchy-Schwarz inequality, we have

$$a_1^2 + a_2^2 + \cdots + a_n^2 = (k^2+1)a_2^2 + a_3^2 + \cdots + a_n^2$$

$$\ge \frac{[(k+1)a_2 + a_3 + \cdots + a_n]^2}{\dfrac{(k+1)^2}{k^2+1} + n - 2} = \frac{(a_1 + a_2 + \cdots + a_n)^2}{\dfrac{2k}{k^2+1} + n - 1}.$$

Therefore, it suffices to show that

$$\frac{5n-1}{5} \ge \frac{2k}{k^2+1} + n - 1,$$

which is equivalent to the obvious inequality

$$(k-2)(2k-1) \ge 0.$$

The equality holds if and only if $k = 2$ and

$$5a_2^2 + a_3^2 + \cdots + a_n^2 = \frac{(3a_2 + a_3 + \cdots + a_n)^2}{\frac{9}{5} + n - 2};$$

that is, if and only if

$$\frac{5a_1}{6} = \frac{5a_2}{3} = a_3 = \cdots = a_n.$$

□

P 2.143. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1 \ge 4a_2$, then*

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \ge \left(n + \frac{1}{2}\right)^2.$$

Solution. Setting

$$a_1 = ka_2, \quad k \ge 4,$$

the inequality becomes

$$[(1+k)a_2 + a_3 + \cdots + a_n]\left(\frac{1+k}{ka_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_n}\right) \ge \left(n + \frac{1}{2}\right)^2.$$

By the Cauchy-Schwarz inequality, we have

$$[(1+k)a_2 + a_3 + \cdots + a_n]\left(\frac{1+k}{ka_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_n}\right) \ge \left(\frac{1+k}{\sqrt{k}} + n - 2\right)^2.$$

Thus, we only need to show that

$$\frac{1+k}{\sqrt{k}}+n-2 \geq n+\frac{1}{2},$$

which reduces to

$$\left(\sqrt{k}-2\right)\left(2\sqrt{k}-1\right) \geq 0.$$

The equality holds if and only if $k=4$ and

$$\frac{a_1}{2}=2a_2=a_3=\cdots=a_n.$$

□

P 2.144. *If $a_1 \geq a_2 \geq \cdots \geq a_n > 0$ such that $a_1+a_2+\cdots+a_n=n$, then*

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}-n \geq \frac{4(n-1)^2}{n^3}(a_1-a_2)^2.$$

(Vasile C., 2009)

Solution. Since

$$\frac{1}{a_2}+\frac{1}{a_3}+\cdots+\frac{1}{a_n} \geq \frac{(n-1)^2}{a_2+a_3+\cdots+a_n}=\frac{(n-1)^2}{n-a_1}$$

and

$$a_1-a_2 \leq a_1-\frac{a_2+a_3+\cdots+a_n}{n-1}=a_1-\frac{n-a_1}{n-1}=\frac{n(a_1-1)}{n-1},$$

it suffices to show that

$$\frac{1}{a_1}+\frac{(n-1)^2}{n-a_1}-n \geq \frac{4}{n}(a_1-1)^2.$$

This is equivalent to the obvious inequality

$$(a_1-1)^2(2a_1-n)^2 \geq 0.$$

The equality holds for

$$a_1=a_2=\cdots=a_n=1,$$

and also for

$$a_1=\frac{n}{2}, \quad a_2=a_3=\cdots=a_n=\frac{n}{2(n-1)}.$$

□

P 2.145. *If $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ are real numbers such that*

$$a_1 \le a_2 \le \cdots \le a_n, \quad a_1 + a_2 + \cdots + a_n = 0,$$

then

$$a_1^2 + a_2^2 + \cdots + a_n^2 + n a_1 a_n \le 0.$$

(Vasile C., 2009)

Solution. For the nontrivial case $a_1^2+a_2^2+\cdots+a_n^2 \ne 0$, let $a_1 = a < 0$ and $a_n = b > 0$ be fixed. We claim that for

$$a \le a_2 \le \cdots \le a_{n-1} \le b, \quad a_2 + \cdots + a_{n-1} = -a - b,$$

the sum $S = a_2^2 + \cdots + a_{n-1}^2$ is maximum when at least $n-3$ of the numbers $a_2, \ldots, a_{n-1}$ are equal to a or b. In the contrary case, if $a < a_i \le a_j < b$, then

$$a_i^2 + a_j^2 < c_i^2 + c_j^2$$

for all c_i and c_j such that

$$a \le c_i < a_i \le a_j < c_j \le b, \quad c_i + c_j = a_i + a_j;$$

indeed,

$$a_i^2 + a_j^2 - c_i^2 - c_j^2 = (a_i - c_i)(a_i + c_i) + (a_j - c_j)(a_j + c_j) = (a_i - c_i)(a_i + c_i - a_j - c_j) < 0.$$

This result confirms our claim. Therefore, it suffices to consider the case where at least $n-3$ of the numbers $a_2, \ldots, a_{n-1}$ are equal to a or b. More precisely, assume that k of $a_2, \ldots, a_{n-1}$ are equal to a and m of $a_2, \ldots, a_{n-1}$ are equal to b, where

$$k + m = n - 3, \quad k, m \ge 0.$$

Therefore, it suffices to show that

$$(k+1)a^2 + c^2 + (m+1)b^2 + (k+m+3)ab \le 0,$$

where

$$a \le c \le b, \quad (k+1)a + c + (m+1)b = 0.$$

We have

$$\begin{aligned}(k+1)a^2 + c^2 + (m+1)b^2 + (k+m+3)ab &= c^2 + (a+b)[(k+1)a + (m+1)b] + ab \\ &= c^2 - (a+b)c + ab = (c-a)(c-b) \le 0.\end{aligned}$$

The equality holds if and only if

$$a_1, a_2, \ldots, a_n \in \{a_1, a_n\}, \quad a_1 + a_2 + \cdots + a_n = 0.$$

□

P 2.146. *Let $a_1, a_2, \dots, a_n$ ($n \ge 4$) be nonnegative real numbers such that*

$$a_1 \ge a_2 \ge \cdots \ge a_n$$

and

$$(a_1 + a_2 + \cdots + a_n)^2 = 4(a_1^2 + a_2^2 + \cdots + a_n^2).$$

Prove that

$$1 \le \frac{a_1 + a_2}{a_3 + a_4 + \cdots + a_n} \le 1 + \sqrt{\frac{2n-8}{n-2}}.$$

(Vasile C., 2007)

Solution. Denote

$$A = a_1 + a_2, \qquad B = a_3 + a_4 + \cdots + a_n.$$

Since

$$2(a_1^2 + a_2^2) \ge A^2, \qquad (n-2)(a_3^2 + a_4^2 + \cdots + a_n^2) \ge B^2,$$

from the hypothesis

$$(a_1 + a_2 + \cdots + a_n)^2 = 4(a_1^2 + a_2^2) + 4(a_3^2 + \cdots + a_n^2),$$

we get

$$(A+B)^2 \ge 2A^2 + \frac{4}{n-2}B^2,$$

$$A \le \left(1 + \sqrt{\frac{2n-8}{n-2}}\right)B.$$

The right inequality is an equality for

$$a_1 = a_2 = ka_3 = \cdots = ka_n, \qquad k = \frac{n-2+\sqrt{2(n-2)(n-4)}}{2}.$$

To prove the left inequality, let

$$a_1 \ge a_2 \ge x \ge a_3 \ge \cdots \ge a_n.$$

From

$$\frac{A}{a_1 a_2} = \frac{1}{a_1} + \frac{1}{a_2} \le \frac{1}{x} + \frac{1}{x} = \frac{2}{x},$$

we get

$$2a_1 a_2 \ge Ax,$$

hence

$$a_1^2 + a_2^2 = A^2 - 2a_1 a_2 \le A^2 - Ax.$$

In addition,

$$a_3^2 + \cdots + a_n^2 \le a_3 x + \cdots + a_n x = Bx.$$

Therefore, from the hypothesis

$$(a_1+a_2+\cdots+a_n)^2 = 4(a_1^2+a_2^2)+4(a_3^2+\cdots+a_n^2),$$

we get

$$(A+B)^2 \le 4(A^2-Ax)+4Bx,$$

$$4(A-B)x-3A^2+2AB+B^2 \le 0,$$

$$(A-B)(3A+B-4x) \ge 0.$$

Since

$$3A+B-4x \ge 3A-4x \ge 6x-4x \ge 0,$$

it follows that $A-B \ge 0$. The left inequality is an equality only for $n=4$ and $a_1=a_2=a_3=a_4$.

□

P 2.147. *If $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$, then*

(a) $$a_1+a_2+\cdots+a_n-n\sqrt[n]{a_1a_2\cdots a_n} \ge \frac{1}{3}\left(\sqrt{a_1}+\sqrt{a_2}-2\sqrt{a_n}\right)^2;$$

(b) $$a_1+a_2+\cdots+a_n-n\sqrt[n]{a_1a_2\cdots a_n} \ge \frac{1}{4}\left(2\sqrt{a_1}-\sqrt{a_{n-1}}-\sqrt{a_n}\right)^2.$$

(Vasile C., 2010)

Solution. (a) For $n=2$, the inequality is equivalent to $\left(\sqrt{a_1}-\sqrt{a_2}\right)^2 \ge 0$. Consider further $n \ge 3$. By the AM-GM inequality, we have

$$a_3+\cdots+a_{n-1}+3\sqrt[3]{a_1a_2a_n} \ge n\sqrt[n]{a_1a_2\cdots a_n}.$$

Therefore, it suffices to prove that

$$a_1+a_2+a_n-3\sqrt[3]{a_1a_2a_n} \ge \frac{1}{3}\left(\sqrt{a_1}+\sqrt{a_2}-2\sqrt{a_n}\right)^2.$$

Setting

$$x=\left(\frac{\sqrt{a_1}+\sqrt{a_2}}{2}\right)^2, \quad x \ge a_n,$$

since $a_1+a_2 \ge 2x$ and $a_1a_2 \le x^2$, it suffices to show that

$$2x+a_n-3\sqrt[3]{x^2a_n} \ge \frac{4}{3}\left(\sqrt{x}-\sqrt{a_n}\right)^2.$$

For the nontrivial case $a_n \ne 0$, we may consider $a_n=1$ (due to homogeneity). In addition, substituting $x=y^6$, $y \ge 1$, the inequality can be restated as

$$2y^6+1-3y^4 \ge \frac{4}{3}(y^3-1)^2,$$

$$(y-1)^2[3(y+1)^2(2y^2+1)-4(y^2+y+1)^2] \geq 0.$$

This inequality is true if

$$(y+1)\sqrt{3(2y^2+1)} \geq 2(y^2+y+1).$$

Since

$$\sqrt{3(2y^2+1)} \geq 2y+1,$$

we have

$$(y+1)\sqrt{3(2y^2+1)}-2(y^2+y+1) \geq (y+1)(2y+1)-2(y^2+y+1) = y-1 \geq 0.$$

This completes the proof. The equality holds for $a_1 = a_2 = \cdots = a_n$.

(b) For $n = 2$, the inequality is equivalent to $\left(\sqrt{a_1}-\sqrt{a_2}\right)^2 \geq 0$. Consider further $n \geq 3$. By the AM-GM inequality, we have

$$a_2 + a_3 + \cdots + a_{n-2} + 3\sqrt[3]{a_1 a_{n-1} a_n} \geq n\sqrt[n]{a_1 a_2 \cdots a_n}.$$

Therefore, it suffices to prove that

$$a_1 + a_{n-1} + a_n - 3\sqrt[3]{a_1 a_{n-1} a_n} \geq \frac{1}{4}\left(2\sqrt{a_1}-\sqrt{a_{n-1}}-\sqrt{a_n}\right)^2.$$

Setting

$$x = \sqrt{a_{n-1}a_n}, \quad x \leq a_1,$$

since $a_{n-1} + a_n \geq 2x$ and $\sqrt{a_{n-1}} + \sqrt{a_n} \geq 2\sqrt{x}$, it suffices to show that

$$a_1 + 2x - 3\sqrt[3]{a_1 x^2} \geq \left(\sqrt{a_1}-\sqrt{x}\right)^2.$$

Due to homogeneity, we may consider $a_1 = 1$. In addition, substituting $x = y^6$, $y \leq 1$, the inequality becomes

$$1 + 2y^6 - 3y^4 \geq (1-y^3)^2,$$

which is equivalent to the obvious inequality

$$y^3(y-1)^2(y+2) \geq 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n$. If $n \geq 3$, then the equality holds also for $a_2 = \cdots = a_n = 0$.

□

P 2.148. *If $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$, $n \geq 3$, then*

$$a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{n-1}{2n}\left(\sqrt{a_{n-2}}+\sqrt{a_{n-1}}-2\sqrt{a_n}\right)^2.$$

(Vasile C., 2010)

Solution. Let us denote

$$x = \frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1}, \quad x \ge a_n.$$

By the AM-GM inequality, we have

$$a_1 a_2 \cdots a_{n-1} \le x^{n-1}.$$

Also,

$$\frac{\sqrt{a_{n-2}} + \sqrt{a_{n-1}}}{2} \le \sqrt{\frac{a_{n-2} + a_{n-1}}{2}} \le \sqrt{\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1}} = \sqrt{x}.$$

Then, it suffices to show that

$$(n-1)x + a_n - n\sqrt[n]{x^{n-1}a_n} \ge \frac{2(n-1)}{n}\left(\sqrt{x} - \sqrt{a_n}\right)^2.$$

For the nontrivial case $a_n \ne 0$, we may consider $a_n = 1$ (due to homogeneity). In addition, substituting $x = t^{2n}$, $t \ge 1$, the inequality becomes $g(t) \ge 0$, where

$$g(t) = (n-1)t^{2n} + 1 - nt^{2n-2} - \frac{2(n-1)}{n}(t^n - 1)^2.$$

We have

$$g'(t) = 2(n-1)t^{n-1}h(t),$$

where

$$h(t) = n(t^n - t^{n-2}) - 2(t^n - 1).$$

Since

$$h'(t) = n(n-2)t^{n-3}(t^2 - 1) \ge 0,$$

$h(t)$ is increasing, $h(t) \ge h(1) = 0$, $g'(t) \ge 0$, $g(t)$ is increasing, hence $g(t) \ge g(1) = 0$. This completes the proof. The equality holds for $a_1 = a_2 = \cdots = a_n$.

□

P 2.149. *Let* $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$. *If*

$$\frac{n}{2} \le k \le n-1,$$

then

$$a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1 a_2 \cdots a_n} \ge \frac{2k(n-k)}{n}\left(\sqrt{a_k} - \sqrt{a_{k+1}}\right)^2.$$

(Vasile C., 2010)

Solution. Let us denote

$$x=\sqrt[k]{a_1a_2\cdots a_k},\quad y=\sqrt[n-k]{a_{k+1}a_{k+2}\cdots a_n},\quad x\ge a_k\ge a_{k+1}\ge y.$$

By the AM-GM inequality, we have

$$a_1+a_2+\cdots+a_k\ge kx,\quad a_{k+1}+a_{k+2}+\cdots+a_n\ge (n-k)y.$$

Also, we have

$$\sqrt{a_k}-\sqrt{a_{k+1}}\le \sqrt{x}-\sqrt{y}.$$

Thus, it suffices to show that

$$kx+(n-k)y-n\sqrt[n]{x^ky^{n-k}}\ge \frac{2k(n-k)}{n}(\sqrt{x}-\sqrt{y})^2.$$

For the nontrivial case $y>0$, we can set $y=1$ (due to homogeneity). In addition, setting $x=t^{2n}$, $t\ge 1$, the inequality becomes $f(t)\ge 0$, where

$$f(t)=kt^{2n}+n-k-nt^{2k}-\frac{2k(n-k)}{n}(t^n-1)^2.$$

We have the derivative

$$f'(t)=2kt^{n-1}h(t),$$

where

$$h(t)=n(t^n-t^{2k-n})-2(n-k)(t^n-1).$$

Since

$$h'(t)=n(2k-n)(t^{n-1}-t^{2k-n-1})\ge 0,$$

$h(t)$ is increasing for $t\ge 1$, $h(t)\ge h(1)=0$, $f'(t)\ge 0$, $f(t)$ is increasing, $f(t)\ge f(1)=0$. This completes the proof. The equality holds for $a_1=a_2=\cdots=a_n$. If n is even and $2k=n$, then the equality holds for $a_1=a_2=\cdots=a_k$ and $a_{k+1}=a_{k+2}=\cdots=a_n$.

□

P 2.150. *Let $a_1\ge a_2\ge\cdots\ge a_n\ge 0$. If*

$$1\le k<j\le n,\qquad k+j\ge n+1,$$

then

$$a_1+a_2+\cdots+a_n-n\sqrt[n]{a_1a_2\cdots a_n}\ge \frac{2k(n-j+1)}{n+k-j+1}\left(\sqrt{a_k}-\sqrt{a_j}\right)^2.$$

(Vasile C., 2010)

Solution. Let us denote
$$P=\frac{k(n-j+1)}{n+k-j+1}$$
and
$$x=\sqrt[k]{a_1a_2\cdots a_k},\quad y=\sqrt[n-j+1]{a_ja_{j+1}\cdots a_n},\quad x\ge a_k\ge a_j\ge y.$$
By the AM-GM inequality, we have
$$a_1+a_2+\cdots+a_k\ge kx,\quad a_j+a_{j+1}+\cdots+a_n\ge(n-j+1)y$$
and
$$a_{k+1}+\cdots+a_{j-1}\ge(j-k-1)\sqrt[j-k-1]{a_{k+1}\cdots a_{j-1}}.$$
Also, we have
$$\sqrt{a_k}-\sqrt{a_j}\le\sqrt{x}-\sqrt{y}.$$
Thus, it suffices to show that
$$kx+(n-j+1)y+(j-k-1)\sqrt[j-k-1]{a_{k+1}\cdots a_{j-1}}-n\sqrt[n]{a_1a_2\cdots a_n}\ge 2P(\sqrt{x}-\sqrt{y})^2.$$
By the AM-GM inequality, we have
$$(j-k-1)\sqrt[j-k-1]{a_{k+1}\cdots a_{j-1}}+(n-j+k+1)\sqrt[n-j+k+1]{(a_1\cdots a_k)(a_j\cdots a_n)}\ge n\sqrt[n]{a_1a_2\cdots a_n},$$
which is equivalent to
$$\begin{aligned}(j-k-1)\sqrt[j-k-1]{a_{k+1}\cdots a_{j-1}}-n\sqrt[n]{a_1a_2\cdots a_n}&\ge-(n-j+k+1)\sqrt[n-j+k+1]{(a_1\cdots a_k)(a_j\cdots a_n)}\\&=-(n-j+k+1)x^{\frac{k}{n-j+k+1}}y^{\frac{n-j+1}{n-j+k+1}}.\end{aligned}$$
Therefore, we only need to show that
$$kx+(n-j+1)y-(n-j+k+1)x^{\frac{k}{n-j+k+1}}y^{\frac{n-j+1}{n-j+k+1}}\ge 2P\left(\sqrt{x}-\sqrt{y}\right)^2.$$
For the nontrivial case $y\ne 0$, we can set $y=1$ (due to homogeneity). Thus, we need to prove that $f(x)\ge 0$ for $x\ge 1$, where
$$f(x)=kx+n-j+1-(n-j+k+1)x^{\frac{k}{n-j+k+1}}-2P\left(\sqrt{x}-1\right)^2.$$
We have the derivatives
$$f'(x)=k-kx^{\frac{k}{n-j+k+1}-1}+2P\left(\frac{1}{\sqrt{x}}-1\right),$$
$$f''(x)=P\left(x^{\frac{k}{n-j+k+1}-2}-x^{\frac{-3}{2}}\right).$$
Since $f''(x)\ge 0$ for $x\ge 1$, f' is increasing, $f'(x)\ge f'(1)=0$, f is increasing, $f(x)\ge f(1)=0$. This completes the proof. The equality holds for $a_1=a_2=\cdots=a_n$. If n is even, $k=n/2$ and $j=k+1$, then the equality holds for $a_1=a_2=\cdots=a_k$ and $a_{k+1}=a_{k+2}=\cdots=a_n$.

Remark. For $j=k+1$, we get the inequality in P 2.149.

□

P 2.151. *If $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0,\ n \ge 4$, then*

(a) $$a_1+a_2+\cdots+a_n-n\sqrt[n]{a_1a_2\cdots a_n} \ge \frac{1}{2}\left(1-\frac{1}{n}\right)\left(\sqrt{a_{n-2}}-3\sqrt{a_{n-1}}+2\sqrt{a_n}\right)^2;$$

(b) $$a_1+a_2+\cdots+a_n-n\sqrt[n]{a_1a_2\cdots a_n} \ge \left(1-\frac{2}{n}\right)\left(2\sqrt{a_{n-2}}-3\sqrt{a_{n-1}}+\sqrt{a_n}\right)^2.$$

(Vasile C., 2010)

Solution. Let

$$x=\sqrt{a_{n-2}}-\sqrt{a_{n-1}} \ge 0, \quad y=\sqrt{a_{n-1}}-\sqrt{a_n} \ge 0.$$

For $k=n-2$ and $k=n-1$, the inequality in P 2.149 becomes respectively

$$a_1+a_2+\cdots+a_n-n\sqrt[n]{a_1a_2\cdots a_n} \ge \frac{4(n-2)x^2}{n}$$

and

$$a_1+a_2+\cdots+a_n-n\sqrt[n]{a_1a_2\cdots a_n} \ge \frac{2(n-1)y^2}{n}.$$

Therefore,

$$a_1+a_2+\cdots+a_n-n\sqrt[n]{a_1a_2\cdots a_n} \ge \frac{2}{n}\max\{2(n-2)x^2,\ (n-1)y^2\}.$$

(a) It suffices to show that

$$\max\{8(n-2)x^2,\ 4(n-1)y^2\} \ge (n-1)(x-2y)^2.$$

This is true since

$$8(n-2)x^2 \ge (n-1)x^2 \ge (n-1)(x-2y)^2$$

for $x-2y \ge 0$, and

$$4(n-1)y^2 \ge (n-1)(2y-x)^2$$

for $2y-x \ge 0$. The equality holds for $a_1=a_2=\cdots=a_n$.

(b) ***First Solution.*** It suffices to show that

$$\max\{4(n-2)x^2,\ 2(n-1)y^2\} \ge (n-2)(2x-y)^2.$$

This is true since

$$4(n-2)x^2 \ge (n-2)(2x-y)^2$$

for $2x-y \ge 0$, and

$$2(n-1)y^2 \ge (n-2)y^2 \ge (n-2)(y-2x)^2$$

for $y-2x\ge 0$. The equality holds for $a_1=a_2=\cdots=a_n$. If $n=4$, then the equality holds for $a_1=a_2$ and $a_3=a_4$.

Second Solution. Let us denote

$$A=\sqrt[n-2]{a_1a_2\cdots a_{n-2}}\,,\quad B=\sqrt{a_{n-1}a_n}\,,\quad A\ge a_{n-2}\ge B.$$

By the AM-GM inequality, we have

$$a_1+a_2+\cdots+a_{n-2}\ge(n-2)A,$$

$$a_{n-1}+a_n\ge 2B,$$

and

$$\sqrt{a_{n-1}}+\sqrt{a_n}\ge 2\sqrt{B}.$$

Then, it suffices to show that

$$(n-2)A+2B-n\sqrt[n]{A^{n-2}B^2}\ge\frac{4(n-2)}{n}\left(\sqrt{A}-\sqrt{B}\right)^2.$$

For the nontrivial case $B\ne 0$, we may consider $B=1$ (due to homogeneity). In addition, substituting $A=t^{2n}$, $t\ge 1$, the inequality becomes $g(t)\ge 0$, where

$$g(t)=(n-2)t^{2n}+2-nt^{2n-4}-\frac{4(n-2)}{n}(t^n-1)^2.$$

We have

$$g'(t)=2(n-2)t^{n-1}h(t),$$

where

$$h(t)=(n-4)t^n-nt^{n-4}+4.$$

Since

$$h'(t)=n(n-4)t^{n-5}(t^4-1)\ge 0,$$

$h(t)$ is increasing, $h(t)\ge h(1)=0$, $g'(t)\ge 0$, $g(t)$ is increasing, hence $g(t)\ge g(1)=0$. This completes the proof.

□

Appendix A

Glosar

1. AM-GM (ARITHMETIC MEAN-GEOMETRIC MEAN) INEQUALITY

If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers, then

$$a_1 + a_2 + \cdots + a_n \geq n\sqrt[n]{a_1 a_2 \cdots a_n},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

2. WEIGHTED AM-GM INEQUALITY

Let $p_1, p_2, \ldots, p_n$ be positive real numbers satisfying

$$p_1 + p_2 + \cdots + p_n = 1.$$

If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers, then

$$p_1 a_1 + p_2 a_2 + \cdots + p_n a_n \geq a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

3. AM-HM (ARITHMETIC MEAN-HARMONIC MEAN) INEQUALITY

If $a_1, a_2, \ldots, a_n$ are positive real numbers, then

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \geq n^2,$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

4. POWER MEAN INEQUALITY

The power mean of order k of positive real numbers $a_1, a_2, \dots, a_n$,

$$M_k = \begin{cases} \left(\frac{a_1^k + a_2^k + \cdots + a_n^k}{n}\right)^{\frac{1}{k}}, & k \neq 0 \\ \sqrt[n]{a_1 a_2 \cdots a_n}, & k = 0 \end{cases},$$

is an increasing function with respect to $k \in \mathbb{R}$. For instant, $M_2 \ge M_1 \ge M_0 \ge M_{-1}$ is equivalent to

$$\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} \ge \frac{a_1 + a_2 + \cdots + a_n}{n} \ge \sqrt[n]{a_1 a_2 \cdots a_n} \ge \frac{n}{\dfrac{1}{a_1} + \dfrac{1}{a_2} + \cdots + \dfrac{1}{a_n}}.$$

5. BERNOULLI'S INEQUALITY

For any real number $x \ge -1$, we have

a) $(1+x)^r \ge 1 + rx$ for $r \ge 1$ and $r \le 0$;

b) $(1+x)^r \le 1 + rx$ for $0 \le r \le 1$.

If $a_1, a_2, \dots, a_n$ are real numbers such that either $a_1, a_2, \dots, a_n \ge 0$ or

$$-1 \le a_1, a_2, \dots, a_n \le 0,$$

then

$$(1+a_1)(1+a_2)\cdots(1+a_n) \ge 1 + a_1 + a_2 + \cdots + a_n.$$

6. SCHUR'S INEQUALITY

For any nonnegative real numbers a, b, c and any positive number k, the inequality holds

$$a^k(a-b)(a-c) + b^k(b-c)(b-a) + c^k(c-a)(c-b) \ge 0,$$

with equality for $a = b = c$, and for $a = 0$ and $b = c$ (or any cyclic permutation). For $k = 1$, we get the third degree Schur's inequality, which can be rewritten as follows

$$a^3 + b^3 + c^3 + 3abc \ge ab(a+b) + bc(b+c) + ca(c+a),$$

$$(a+b+c)^3 + 9abc \ge 4(a+b+c)(ab+bc+ca),$$

$$a^2 + b^2 + c^2 + \frac{9abc}{a+b+c} \ge 2(ab+bc+ca),$$

$$(b-c)^2(b+c-a) + (c-a)^2(c+a-b) + (a-b)^2(a+b-c) \ge 0.$$

For $k = 2$, we get the fourth degree Schur's inequality, which holds for any real numbers a, b, c, and can be rewritten as follows

$$a^4+b^4+c^4+abc(a+b+c) \ge ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2),$$

$$a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2 \ge (ab+bc+ca)(a^2+b^2+c^2-ab-bc-ca),$$

$$(b-c)^2(b+c-a)^2+(c-a)^2(c+a-b)^2+(a-b)^2(a+b-c)^2 \ge 0,$$

$$6abcp \ge (p^2-q)(4q-p^2), \qquad p=a+b+c, \;\; q=ab+bc+ca.$$

A generalization of the fourth degree Schur's inequality, which holds for any real numbers a, b, c and any real number m, is the following (*Vasile Cirtoaje*, 2004)

$$\sum (a-mb)(a-mc)(a-b)(a-c) \ge 0,$$

where the equality holds for $a = b = c$, and for $a/m = b = c$ (or any cyclic permutation). This inequality is equivalent to

$$\sum a^4 + m(m+2)\sum a^2b^2 + (1-m^2)abc\sum a \ge (m+1)\sum ab(a^2+b^2),$$

$$\sum (b-c)^2(b+c-a-ma)^2 \ge 0.$$

A more general result is given by the following theorem (*Vasile Cirtoaje*, 2004).

Theorem. *Let*

$$f_4(a,b,c) = \sum a^4 + \alpha\sum a^2b^2 + \beta abc\sum a - \gamma\sum ab(a^2+b^2),$$

where α, β, γ are real constants such that $1+\alpha+\beta = 2\gamma$. Then,

(a) $f_4(a,b,c) \ge 0$ *for all $a, b, c \in \mathbb{R}$ if and only if*

$$1+\alpha \ge \gamma^2;$$

(b) $f_4(a,b,c) \ge 0$ *for all $a, b, c \ge 0$ if and only if*

$$\alpha \ge (\gamma-1)\max\{2, \gamma+1\}.$$

7. CAUCHY-SCHWARZ INEQUALITY

If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers, then

$$(a_1^2+a_2^2+\cdots+a_n^2)(b_1^2+b_2^2+\cdots+b_n^2) \ge (a_1b_1+a_2b_2+\cdots+a_nb_n)^2,$$

with equality for

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}.$$

Notice that the equality conditions are also valid for $a_i = b_i = 0$, where $1 \le i \le n$.

8. HÖLDER'S INEQUALITY

If x_{ij} $(i=1,2,\cdots,m; j=1,2,\cdots n)$ are nonnegative real numbers, then

$$\prod_{i=1}^{m}\left(\sum_{j=1}^{n} x_{ij}\right) \geq \left(\sum_{j=1}^{n} \sqrt[m]{\prod_{i=1}^{m} x_{ij}}\right)^{m}.$$

9. CHEBYSHEV'S INEQUALITY

Let $a_1 \geq a_2 \geq \cdots \geq a_n$ be real numbers.

a) If $b_1 \geq b_2 \geq \cdots b_n$, then

$$n\sum_{i=1}^{n} a_i b_i \geq \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right);$$

b) If $b_1 \leq b_2 \leq \cdots \leq b_n$, then

$$n\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right).$$

10. REARRANGEMENT INEQUALITY

(1) If $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are two increasing (or decreasing) real sequences, and $(i_1, i_2, \cdots, i_n)$ is an arbitrary permutation of $(1, 2, \cdots, n)$, then

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n \geq a_1b_{i_1} + a_2b_{i_2} + \cdots + a_nb_{i_n}$$

and

$$n(a_1b_1 + a_2b_2 + \cdots + a_nb_n) \geq (a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n).$$

(2) If $(a_1, a_2, \ldots, a_n)$ is decreasing and $(b_1, b_2, \ldots, b_n)$ is increasing, then

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n \leq a_1b_{i_1} + a_2b_{i_2} + \cdots + a_nb_{i_n}$$

and

$$n(a_1b_1 + a_2b_2 + \cdots + a_nb_n) \leq (a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n).$$

(3) Let $b_1, b_2, \ldots, b_n)$ and $(c_1, c_2, \ldots, c_n)$ be two real sequences such that

$$b_1 + \cdots + b_i \geq c_1 + \cdots + c_i, \quad i = 1, 2, \cdots, n.$$

If $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$, then

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n \geq a_1c_1 + a_2c_2 + \cdots + a_nc_n.$$

Notice that all these inequalities follow immediately from the identity

$$\sum_{i=1}^{n} a_i(b_i - c_i) = \sum_{i=1}^{n}(a_i - a_{i+1})\left(\sum_{j=1}^{i} b_j - \sum_{j=1}^{i} c_j\right), \qquad a_{n+1} = 0.$$

11. CONVEX FUNCTIONS

A function f defined on a real interval $\mathbb{I}$ is said to be *convex* if

$$f(\alpha x + \beta y) \le \alpha f(x) + \beta f(y)$$

for all $x, y \in \mathbb{I}$ and any $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$. If the inequality is reversed, then f is said to be concave.
If f is differentiable on $\mathbb{I}$, then f is (strictly) convex if and only if the derivative f' is (strictly) increasing. If $f'' \ge 0$ on $\mathbb{I}$, then f is convex on $\mathbb{I}$. Also, if $f'' \ge 0$ on (a, b) and f is continuous on $[a, b]$, then f is convex on $[a, b]$.

Jensen's inequality. *Let $p_1, p_2, \ldots, p_n$ be positive real numbers. If f is a convex function on a real interval $\mathbb{I}$, then for any $a_1, a_2, \ldots, a_n \in \mathbb{I}$, the inequality holds*

$$\frac{p_1 f(a_1) + p_2 f(a_2) + \cdots + p_n f(a_n)}{p_1 + p_2 + \cdots + p_n} \ge f\left(\frac{p_1 a_1 + p_2 a_2 + \cdots + p_n a_n}{p_1 + p_2 + \cdots + p_n}\right).$$

For $p_1 = p_2 = \cdots = p_n$, Jensen's inequality becomes

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right).$$

12. SQUARE PRODUCT INEQUALITY

Let a, b, c be real numbers, and let

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc,$$
$$s = \sqrt{p^2 - 3q} = \sqrt{a^2 + b^2 + c^2 - ab - bc - ca}.$$

From the identity

$$(a-b)^2(b-c)^2(c-a)^2 = -27r^2 + 2(9pq - 2p^3)r + p^2q^2 - 4q^3,$$

it follows that

$$\frac{-2p^3 + 9pq - 2(p^2 - 3q)\sqrt{p^2 - 3q}}{27} \le r \le \frac{-2p^3 + 9pq + 2(p^2 - 3q)\sqrt{p^2 - 3q}}{27},$$

which is equivalent to

$$\frac{p^3 - 3ps^2 - 2s^3}{27} \le r \le \frac{p^3 - 3ps^2 + 2s^3}{27}.$$

Therefore, for constant p and q, the product r is minimal and maximal when two of a, b, c are equal.

13. KARAMATA'S MAJORIZATION INEQUALITY

Let f be a convex function on a real interval $\mathbb{I}$. If a decreasingly ordered sequence

$$A = (a_1, a_2, \ldots, a_n), \quad a_i \in \mathbb{I},$$

majorizes a decreasingly ordered sequence

$$B = (b_1, b_2, \ldots, b_n), \quad b_i \in \mathbb{I},$$

then

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge f(b_1) + f(b_2) + \cdots + f(b_n).$$

We say that a sequence $A = (a_1, a_2, \ldots, a_n)$ with $a_1 \ge a_2 \ge \cdots \ge a_n$ majorizes a sequence $B = (b_1,\, b_2,\, \ldots,\, b_n)$ with $b_1 \ge b_2 \ge \cdots \ge b_n$, and write it as

$$A \succ B,$$

if

$$\begin{gathered} a_1 \ge b_1, \\ a_1 + a_2 \ge b_1 + b_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_1 + a_2 + \cdots + a_{n-1} \ge b_1 + b_2 + \cdots + b_{n-1}, \\ a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n. \end{gathered}$$

14. VASC'S EXPONENTIAL INEQUALITY

Let $0 < k \le e$.

(a) *If* $a, b > 0$*, then (Vasile Cîrtoaje, 2006)*

$$a^{ka} + b^{kb} \ge a^{kb} + b^{ka};$$

(b) *If* $a, b \in (0, 1]$*, then (Vasile Cîrtoaje, 2010)*

$$2\sqrt{a^{ka} b^{kb}} \ge a^{kb} + b^{ka}.$$

Appendix B

Bibliography

[1] Andreescu T., Cîrtoaje V., Dospinescu G., Lascu M., *Old and New Inequalities*, GIL Publishing House, 2004.

[2] Bin X., Boreico I., Can V.Q.B., Bulj A., Lascu M., *Opympiad Inequalities*, GIL Publishing House, 2015.

[3] Bin X., Boreico I., Can V.Q.B., Cîrtoaje V., Lascu M., *An Introduction to Inequalities*, GIL Publishing House, 2015.

[4] Can V.Q.B., Pohoaţă C., *Old and New Inequalities*, GIL Publishing House, 2008.

[5] Can V.Q.B., Anh T.Q., *Su Dung Phuong Phap Cauchy-Schwarz De Chung Minh Bat Dang Thuc*, Nha Xuat Ban Dai Hoc Su Pham, 2010.

[6] Cîrtoaje V., *Algebraic Inequalities-Old and New Methods*, GIL Publishing House, 2006.

[7] Cîrtoaje V., *On Jensen Type Inequalities with Ordered Variables*, Journal of Inequalities In Pure and Applied Mathematics, Volume 9, Issue 1, 2008.

[8] Cîrtoaje V., *The Proof of Three Open Inequalities*, Crux Mathematicorum, Volume 34, Issue 4, 2008, 231-238.

[9] Cîrtoaje V., Can V.Q.B., Anh T.Q., *Inequalities with Beautiful Solutions*, GIL Publishing House, 2009.

[10] Cîrtoaje V., *On the Cyclic Homogeneous Polynomial Inequalities of Degree Four*, Journal of Inequalities in Pure and Applied Mathematics, Volume 10, Issue 3, 2009.

[11] Cîrtoaje V., *The Best Lower Bound Depended on Two Fixed Variables for Jensen's Inequality with Ordered Variables*, Journal of Inequalities and Applications, Volume 2010.

[12] Cîrtoaje V., *Proofs of Three Open Inequalities with Power-Exponential Functions*, Journal of Nonlinear Sciences and Applications, Volume 4, no. 2, 2011.

[13] Cîrtoaje V., *The Best Upper Bound for Jensen's Inequality*, Australian Journal of Mathematical Analysis and Aplications, Volume 7, Issue 2, Art. 22, 2011.

[14] Cîrtoaje V., Zhou Y., *Necessary and Sufficient Conditions for Cyclic Homogeneous Polynomial Inequalities of Degree Four in Real Variables*, The Australian Journal of Mathematical Analysis and Applications, Volume 9, Issue 1, Article 15, 2012.

[15] Cîrtoaje V., Zhou Y., *Some Strong Sufficient Conditions for Cyclic Homogeneous Polynomial Inequalities of Degree Four in Real Variables*, Journal of Nonlinear Analysis and Applications, Volume 2012.

[16] Cîrtoaje V., Can V.Q.B., *On Some Cyclic Homogeneous Polynomial Inequalities of Degree Four in Real Variables*, International Journal of Pure and Applied Mathematics, Volume 80, No. 3, 2012.

[17] Cîrtoaje V., *The Best Lower Bound for Jensen's Inequality with three fixed ordered variables Journal*, Banach Journal of Mathematical Analysis, Volume 7, Issue 1, 2013.

[18] Cîrtoaje V., *Mathematical Inequalities - Volumme 1, Symmetric Polynomial Inequalities*, Lambert Academic Publishing, 2018.

[19] Cîrtoaje V., *Mathematical Inequalities - Volumme 2, Symmetric Rational and Nonrational Inequalities*, Lambert Academic Publishing, 2018.

[20] Cîrtoaje V., *Three extensions of HCF and PCF theorems*, Advances in Inequalities and Applications, no. 2, 2016.

[21] Cvetkovski, *Inequalities: Theorems, Techniques and Selected Problems*, Springer-Verlag Berlin Heidelberg, 2012.

[22] Drimbe M.O., *Inegalitati - Idei si Metode*, GIL Publishing House, 2003.

[23] Engel A., *Problem-Solving Strategies*, Springer, 1998.

[24] Hung P.K., *Secrets in Inequalities*, Volume 1: Basic Inequalities, GIL Publishing House, 2007.

[25] Hung P.K., *Secrets in Inequalities*, Volume 2: Advanced Inequalities, GIL Publishing House, 2008.

[26] Littlewood G.H., Polya J.E., *Inequalities*, Cambridge University Press, 1967.

[27] Mitrinovič D.S., Pecarič J.E., Fink A.M., *Classical and New Inequalities in Analysis*, Kluwer, 1993.

[28] Zhou Y., Cîrtoaje V., *Some strong sufficient conditions for cyclic homogeneous polynomial inequalities of degree four in nonnegative variables*, Journal of Nonlinear Science and Applications, Volume 6, Issue 2, 2013.