GENERALIZED FUNCTIONS

Vol II

Spaces of Fundamental and Generalized Functions

I. M. Gel'fand
G. E. Shilov

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Volume 2

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MOSCOW, 1958

Preface to the Russian Edition

The constructions of Volume 1 proceeded on the basis of utilizing a few fundamental spaces: (1) the space K of infinitely differentiable functions having compact supports; (2) the space S of infinitely differentiable functions decreasing at infinity, together with all their derivatives, more rapidly than any power of $1/|x|$; (3) the space Z of analytic functions $\varphi(z)$, satisfying inequalities of the form $z^k\varphi(z) \leqslant C_k e^{a|y|}$. Generalized functions—continuous linear functionals on these spaces—were adequate for the clarification of the fundamental features of the theory and for a number of simple, but important, applications to some questions of analysis, and in particular, to the theory of differential equations.

On the other hand, although we tried there to reduce to a minimum the number of spaces utilized, we did not succeed in bypassing one pair of spaces K and K': by considering generalized functions as continuous linear functionals in K, we inevitably had to consider their Fourier transforms as continuous linear functionals on Z. The advantages of such a viewpoint will be seen particularly clearly in Volume 5, where methods of complex variable function theory will render substantial assistance in algorithmic questions of the theory of generalized functions.

We shall need a considerably more extensive circle of spaces in Volume 3, which is devoted to deeper applications of the theory of generalized functions to differential equations, than those which we encountered periodically in Volume 1, and will meet here and there in Volume 2. Namely, applications of the theory of generalized functions to the Cauchy problem and to the problem of eigenfunction expansions will be elucidated in Volume 3. Here, the fundamental peculiarity of the theory of generalized functions, in that form in which we shall understand it in this book, will be completely apparent; it is that *different classes of problems require different classes of spaces*, and, indeed classes of spaces and not individual spaces.

Thus, uniqueness and existence theorems for the solution of the Cauchy problem for different partial differential equations require different spaces, which however possess some common properties. Problems of eigenfunction expansions for different differential operators also require different spaces which, nevertheless, have a number of common features. And similarly, boundary value problems for elliptic equations require their class of fundamental spaces and spaces of generalized functions.

In the preceding stage of development of functional analysis, which was connected with the theory of integral equations, the common base for the study of the various functional spaces encountered was the theory of linear normed spaces.*

Normed spaces turned out to be inadequate for the needs of the theory of generalized functions. It must not be thought that the situation is such that much more complex constructions would be required. It is directly opposite: among the normed spaces one does not find the simplest spaces, for example the spaces K and S possessing a whole series of essential properties.

In recent years the general theory of linear topological spaces has developed considerably. However, the most general linear topological spaces are rather complicated objects possessing a whole set of "pathological" properties, and are poorly adapted to the needs of the analyst.† The basis of the theory of generalized functions is the theory of the so-called *countably normed spaces (with compatible norms), their unions (inductive limits)*, and also of the *spaces conjugate to the countably normed ones or their unions*. This set of spaces is sufficiently broad on the one hand, and sufficiently convenient for the analyst on the other.

The theory of these spaces is expounded in Chapter I. Let us note that since the countably normed spaces are very close to normed spaces, a number of important theorems is obtained almost automatically by taking them over from the normed spaces into the countably normed spaces.‡ In reading this chapter it should be kept in mind that some of the theorems proved here are actually valid for more general spaces.

In the majority of questions the class of all countably normed spaces turns out to be too broad for the theory of generalized functions. Hence,

* However, even during this period works appeared which anticipated going beyond the limits of this class of spaces, the work of Köthe-Toeplitz and Köthe on spaces of sequences in the 30's, and also the work of Mazur and Orlicz.

† To the analyst it is natural to use estimates, not neighborhoods, which he inevitably reduces to some kind of estimates.

‡ Before reading this chapter it would be useful for the reader not acquainted with the theory of normed spaces to read the first three chapters, say, of the book "Elements of Functional analysis" by L. A. Lyusternik and V. I. Sobolev, Ungar, New York, 1961 or the first volume of the lectures "Elements of the Theory of Functions and Functional Analysis" by A. N. Kolmogorov and S. V. Fomin, Moscow University Press, Moscow, USSR, 1954.

in Chapter I we study the so-called *perfect* spaces (complete countably normed spaces in which the bounded sets are compact). The reader will meet a great number of examples of such spaces in the following chapters.

The reader will also find material referring to the general theory of countably normed spaces in the first three sections of Chapter IV in Volume 3.

The expounded viewpoint certainly excludes the possibility of an *a priori* description of all classes of spaces which may be encountered in connection with various problems of the theory of generalized functions: As we have already said above, each class of problems requires its own class of spaces. Therefore, essentially two classes of spaces are introduced and studied in Chapters II and IV: spaces of the type $K\{M_p\}$ in Chapter II; spaces of the type S and similar spaces of type W in Chapter IV. The spaces of type S and W essentially satisfy the demands of Chapters II and III of Volume 3 (the Cauchy problem), and spaces of type $K\{M_p\}$ the requirements of Chapter IV of Volume 3 (the problem of eigenfunction expansions). Chapter II, and, in part, Chapter III, of the present volume are devoted primarily to transferring the results of Chapters I and II of Volume 1, almost without any difficulty, to more general spaces. The spaces $K\{M_p\}$, which are natural illustrations of the general theory, appear here. On the other hand, the results of Chapter I permit the filling in of a whole series of essential gaps, in particular, the proof of the completeness of spaces of generalized functions on K, and the establishment of a number of new results, concerning for example the structure of generalized functions.

The theory of spaces of type S is discussed in the last Chapter IV. These spaces which, as we have said already, are used in Volume 3 possess great internal orderliness, and we hope that even their independent study will give the analyst some satisfaction. The construction and utilization of these spaces is connected with results of the theory of quasi-analytic functions and the Phragmen-Lindelöf theorem. Applications of these spaces to the Cauchy problem in Volume 3 will illustrate the well-known statement of Hadamard on the relation between uniqueness theorems in the Cauchy problem on the one hand, and the theory of quasi-analytic functions and the general theory of functions of a complex variable, on the other. Spaces of type S yield natural limits for a sufficiently flexible Fourier transform theory because these spaces go over into each other under Fourier transformation; hence, Chapter IV is a natural continuation of Chapter III, devoted to Fourier transforms. Moreover, various operators of the form $f(d/dx)$, where $f(t)$ is an entire function, can be constructed in spaces of type S, and are also applicable to generalized functions. The Fourier transforms of generalized functions, considered as continuous linear functionals on spaces of type S and W, as well as the construction of operators of the form $f(d/dx)$, applicable to the generalized functions, are indeed the fundamental tools which we shall use in Volume 3 for studying the Cauchy problem.

In order not to overburden the exposition here, we have referred a summary of the results referring to spaces of type W to an appendix; proofs of these results are collected in Chapter I of Volume 3. The reader interested only in problems of eigenfunction expansions may turn to Chapter IV of Volume 3 directly after having completed Chapters I and II of the present volume.

The authors take this opportunity to express their heartfelt gratitude to all their colleagues who assisted in writing this volume. To D. A. Raikov we owe a number of essential improvements in the first chapter. B. Ya. Levin constructed the proof of some necessary theorems from the theory of entire functions (Chapter IV) at our request. G. N. Zolotarev indicated some simplifications in the exposition of Chapters II and III. The section on the Hilbert transform (Chapter III) was written according to an idea of N. Ya. Vilenkin. Finally, a multitude of improvements has been inserted in accordance with suggestions of M. S. Agranovich, who edited the entire text of this volume.

Moscow, 1958

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G. E. Shilov

Contents

Chapter III

Chapter IV

CHAPTER I

LINEAR TOPOLOGICAL SPACES

1. Definition of a Linear Topological Space[1]

1.1. System of Axioms for a Linear Topological Space

A collection Φ of elements $\varphi, \psi, \ldots$ is called a *linear topological space* if the following conditions are fulfilled:

I. Φ is a linear space with multiplication by (real or) complex numbers.

II. Φ is a topological space.

III. The operations of addition and multiplication by numbers are continuous relative to the topology of Φ.

Let us consider each of these conditions in detail.

I. *The collection Φ is a linear space with multiplication by complex numbers.*

This means that an operation of addition of elements in Φ, and an operation of multiplying elements by (complex) numbers $\lambda, \mu, \ldots$ is defined, and the following axioms are fulfilled:

I.1. $\varphi + \psi = \psi + \varphi$ (*addition is commutative*);

I.2. $\varphi + (\psi + \chi) = (\varphi + \psi) + \chi$ (*addition is associative*);

I.3. *There is an element* 0 *such that* $\varphi + 0 = \varphi$ *for any* φ;

I.4. *For every element* φ, *there is an element* ψ *such that* $\varphi + \psi = 0$ (*the negative element*);

I.5. $1 \cdot \varphi = \varphi$ for any $\varphi \in \Phi$;

I.6. $\lambda(\mu\varphi) = (\lambda\mu)\varphi$;

[1] Since Section 1 is of a preparatory nature, the reader who is familiar with the definition of a linear topological space can proceed directly to Section 2, and return to Section 1 when necessary.

I.7. $(\lambda + \mu)\varphi = \lambda\varphi + \mu\varphi$;

I.8. $\lambda(\varphi + \psi) = \lambda\varphi + \lambda\psi$.

Axiom I.6 denotes the associativity of multiplication by numbers; Axioms I.7 and I.8 express two laws of distributivity related to addition and multiplication by numbers.

It can be deduced, in turn, from Axioms I.1–I.8 that the product of 0 with any element φ is the element 0, and that the product of the number -1 with any element φ is the negative of φ, which is therefore naturally denoted by $-\varphi$.

We present some simple definitions pertaining to linear spaces.

The collection of all sums $\varphi + \psi$, where φ ranges over a set A in the linear space Φ, is called the *translate of the set A by the vector ψ*.

The collection of all sums $\varphi + \psi$, where φ ranges over a set A, and ψ ranges over a set B, is called the *sum* (more precisely, the *arithmetic sum*) of the sets A and B, and is denoted by $A + B$. The *arithmetic difference* $A - B$ is defined analogously.

The collection of all products of the elements φ of a set A by a number λ is called the λ*-tuple* (or λ*-dilation*) *of the set* A and is denoted by λA. (We remark that in general $2A \neq A + A$.) In particular, $-A$ is the collection of all the negatives of elements in A.

II. *The collection Φ is a topological space.*

This means that a system $\{U\}$ of subsets of Φ, called (open) *neighborhoods*, is specified, and the following axioms are satisfied:

II.1. *Every point $\varphi \in \Phi$ belongs to some neighborhood $U = U(\varphi)$;*

II.2. *If a point φ belongs to neighborhoods U and V, then it belongs to a neighborhood W which lies entirely in the intersection of U with V;*

II.3. *For any pair of points $\varphi \neq \psi$, there is some neighborhood U which contains φ but does not contain ψ.*

The neighborhoods and all of their unions (finite and infinite) form the system of *open sets.* An open set is characterized by the fact that *every one of its points is an interior point*, i.e., it belongs to a neighborhood which lies in the given set. It is easy to obtain from this that *the union of any number of open sets and the intersection of any finite number of open sets are open sets.*

Henceforth, by a *neighborhood of a given point* we will understand any neighborhood containing the point.

A point φ_0 is called an *adherence point of a set A*, if every neighborhood of φ_0 contains a point of A. In particular, every point of a set A is an adherence point of A. There are two possibilities for adherence points of a set A:

1. There exists a neighborhood of φ_0 which contains only a finite number of distinct points of A;
2. Every neighborhood of φ_0 contains an infinite number of distinct points of A.

In the first case, using Axioms II.2 and II.3, one can construct a neighborhood of φ_0 which contains no point of A other than φ_0 itself. In this case φ_0 belongs to A, and is called an *isolated point* of A.

In the second case φ_0 is called a *limit point* of A. An isolated adherence point of A always belongs to A; a limit point may or may not belong to A.

The collection of all adherence points of A forms the *closure* $\bar{A}$ of A; thus, the closure of a set A is obtained by adjoining to A those of its limit points which do not belong to it.

A set A is said to be *closed* if it contains all of its adherence points. One can verify that the closure of any set is closed. The closed sets can be characterized as the complements (with respect to all of Φ) of the open sets. It follows that the union of a finite number of closed sets and the intersection of any number of closed sets are closed sets.

A set A is said to be *dense* in the space Φ (more precisely, *everywhere dense*) if its closure coincides with Φ. Example: the set of rational points on the line.

A set is said to be *nowhere dense*, if its closure has no interior point. Example: the Cantor set on the line.

The collection of all open and all closed sets of a space Φ forms its *topology*.

One can arrive at the same topology in a space (i.e., the same system of open and closed sets), starting from two different systems of neighborhoods. For example, in defining the natural topology on the real line we can, on the one hand, take as neighborhoods all intervals with rational endpoints and, on the other hand, all intervals with irrational endpoints. We will call different systems of neighborhoods *equivalent*, if they lead to the same topology. The following simple condition is both necessary and sufficient for the equivalence of two given neighborhood systems $\{U\}$ and $\{V\}$: Every neighborhood U contains a neighborhood V, and every neighborhood V contains a neighborhood U.

Convergent Sequences. A sequence $\varphi_1, \varphi_2, \ldots, \varphi_\nu, \ldots$ of elements of a topological space Φ is said to *converge to an element* φ, if each neighborhood of φ contains all the points of the sequence, starting with some given one whose index in general depends upon the neighborhood.

In this case, one writes $\varphi = \lim_{\nu\to\infty} \varphi_\nu$. On the line (or in m-dimensional space), every limit point of a given set A is the limit of some convergent *sequence* of points belonging to A. The assumption, natural at first glance, that in the general case also, every limit point of a set A must be the limit of some (countable) sequence $\varphi_\nu \in A$ ($\nu = 1, 2,\ldots$) turns out, under closer examination, to be false.

Example. Let us consider the collection Φ of (all) bounded real functions $\varphi(x)$ on the interval $0 \leqslant x \leqslant 1$, with the ordinary linear operations; we define a neighborhood $U = U(\varphi_0\,; x_1,\ldots, x_n\,; \epsilon)$ of a given element $\varphi = \varphi_0(x)$ by specifying a finite number of points $x_1,\ldots, x_n$ and a number $\epsilon > 0$; this neighborhood consists of all $\varphi \in \Phi$ for which $|\varphi(x_j) - \varphi_0(x_j)| < \epsilon$, $j = 1,\ldots, n$. We form the set A of functions $\varphi(x)$, each of which equals 1 everywhere, with the exception of a finite number of points at which it equals 0. Obviously, $\varphi_0(x) \equiv 0$ is an adherence point of A. At the same time, no (countable) sequence of elements $\{\Psi_\nu(x)\}$ of A can converge to zero, since, in view of the uncountability of the continuum $0 \leqslant x \leqslant 1$, one can always find a point x_0 at which all of the $\varphi_\nu(x)$ equal 1, and consequently no one of them lies within any neighborhood of the form $U(\varphi_0\,; x_0\,; \frac{1}{2})$.

Of course, it would be very helpful in analysis if any limit point φ_0 of every set A were always the limit of some sequence of points of A. This property holds in topological spaces in which an additional condition is satisfied:

The first axiom of countability at a point φ_0. *The point φ_0 has a countable neighborhood basis.*

A system $\{U\}$ of neighborhoods $U_1, U_2,\ldots$ of φ_0 is said to be a *basis of the neighborhoods of* φ_0, if every neighborhood V of φ_0 contains at least one of the U_ν.

Let us show that *if the first axiom of countability is satisfied at a point φ_0, then from any set A which has φ_0 as a limit point, it is possible to select a sequence $\varphi_1, \varphi_2,\ldots$ which converges to φ_0.*

First we note that we can always consider a countable neighborhood basis to be decreasing, so that $U_1 \supset U_2 \supset \cdots$; indeed, if this condition is not fulfilled, then in place of U_2, we take a neighborhood U_2' lying in the intersection of U_1 and U_2; in place of U_3, we take a neighborhood U_3' lying in the intersection of U_2' and U_3, and so on. Let us now assume that φ_0 is an adherence point of some set A. We choose a point $\varphi_\nu \in A$ in each neighborhood U_ν (assuming that $U_1 \supset U_2 \supset \cdots$); then

$$\varphi_0 = \lim_{\nu\to\infty} \varphi_\nu\,.$$

Indeed, for any neighborhood V of φ_0, there is some neighborhood $U_\nu \subset V$, and since $U_{\nu+p} \subset U_\nu$, it follows that $\varphi_{\nu+p} \in U_{\nu+p} \subset V$ for any p; thus, any neighborhood of φ_0 contains all the points $\varphi_1, \varphi_2, \ldots$ starting with some one, as was required.

If the first axiom of countability is fulfilled at every point of the space Φ, then one says that *it is fulfilled everywhere in* Φ.

We now proceed to condition III.

III. *The operations of addition and multiplication by a number are continuous relative to the topology of* Φ.

Conditions I and II, which we have considered in detail just above, described separate properties of the linear operations and the operation of passing to a limit; condition III establishes the connection between these. Condition III may be divided into the following two axioms.

III.1. *The continuity of addition and subtraction. If one of the relations*

$$\varphi_0 \pm \psi_0 = \chi_0$$

holds, then for any neighborhood U *of* χ_0 *there is a neighborhood* V *of* φ_0 *and a neighborhood* W *of* ψ_0 *such that* $\varphi \in V$ *and* $\psi \in W$ *imply* $\varphi \pm \psi \in U$ (*or, briefly,* $V \pm W \subset U$).

III.2. *The continuity of multiplication by a number. If* $\lambda_0 \varphi_0 = \psi_0$, *then for any neighborhood* U *of* ψ_0 *there is a neighborhood* V *of* φ_0 *and a number* $\epsilon > 0$ *such that* $\varphi \in V$ *and* $|\lambda - \lambda_0| < \epsilon$ *imply* $\lambda\varphi \in U$.

Let us first consider some consequences of Axiom III.1.

First of all, we note that *the collection of all translates of all the neighborhoods of* 0 *defines a system of neighborhoods in* Φ *which is equivalent to the original system.* Indeed, that this collection is actually a collection of neighborhoods (i.e., satisfies II.1–II.3) is easy to show. To see that this system is equivalent to the original system of neighborhoods, let $V = \varphi_0 + U$, where U is a neighborhood of 0. If $\varphi \in V$, then $\varphi - \varphi_0 \in U$. Thus we can find neighborhoods W_1 and W_2 of φ and φ_0, respectively, such that $W_1 - W_2 \subset U$. In particular, since $\varphi_0 \in W_2$, we have $W_1 - \varphi_0 \subset U$, or $W_1 \subset \varphi_0 + U = V$. Conversely, given a neighborhood U of a point φ_0, since $\varphi_0 + 0 = \varphi_0$, we can find a neighborhood W_1 (of φ_0) and a neighborhood W_2 (of 0) such that $W_1 + W_2 \subset U$. Since $\varphi_0 \in W_1$, we have $V = \varphi_0 + W_2 \subset U$, as was required.

Thus, *the topology in* Φ *can be reconstructed from the system of neighborhoods of zero*; subjecting them to all possible translations, we obtain a complete system of neighborhoods for the entire space. This means that from the topological point of view, the structure of the space is the same at all points; a simple translation (taking the entire space onto itself) carries any point into any other, and every neighborhood of the first point is carried into a neighborhood of the second. The local properties of the topology of the space at one point are the same as at every other point. In particular, if the first axiom of countability is fulfilled at one point, for example at 0, then it is fulfilled at every other point, i.e., it is fulfilled over the entire space.

We remark, further, that *a linear topological space is always regular*, i.e., for any point φ and neighborhood U of φ there is a smaller neighborhood V of φ which lies in U, together with its closure.

For the proof it is sufficient, in view of the homogeneity of the topology in Φ, to consider the case $\varphi = 0$. In view of the continuity of subtraction, we can find two neighborhoods W_1 and W_2 of 0 such that $W_1 - W_2 \subset U$, and if we further take a neighborhood W of 0 lying in the intersection of W_1 and W_2, then we will have $W - W \subset U$. We assert that the closure $\bar{W}$ of W lies in U.

In fact, let ψ be an adherence point of W; then the neighborhood $V = \psi + W$ of the point ψ contains some point of W. Suppose that $\chi \in V \cap W$, so that $\chi = \psi + \varphi$, where $\varphi \in W$. Then

$$\psi = \chi - \varphi \in W - W \subset U,$$

as was asserted.

Let us now turn to those properties of linear topological spaces which are related to the continuity of multiplication by a number. (Axiom III.2.)

First of all, we shall show that *any dilation λU, $\lambda \neq 0$ of an open set U is an open set*. Indeed, let $\psi = \lambda\varphi$, where $\varphi \in U$; then $\varphi = (1/\lambda)\psi$ and, given a neighborhood V of φ, we can find a neighborhood W of ψ such that $(1/\lambda) W \subset V$, or $W \subset \lambda V$. Taking $V \subset U$, we see that $W \subset \lambda V \subset \lambda U$, i.e., the point ψ lies in λU together with its neighborhood W, as was required.

In particular, every dilation λU, $\lambda \neq 0$ of a neighborhood U of zero is a neighborhood of zero, and if $\lambda \neq 0$ is fixed, then the collection of sets of the form λU, where U ranges over a basis of the neighborhoods of zero, is itself a basis of the neighborhoods of zero. It is sufficient to show that for any neighborhood V of zero, one can find a set U in the basis of the neighborhoods of zero for which $\lambda U \subset V$. But the existence of such a U follows at once from the continuity of multiplication by λ and the definition of a neighborhood basis.

We can use the neighborhoods of the form λU to construct certain special systems of neighborhoods of zero, called *normal* neighborhoods.

A region V in the space Φ is called *normal*, if $\varphi \in V$, $|a| \leqslant 1$ imply that $a\varphi \in V$.

We shall show that the system of neighborhoods of zero can always be defined by means of normal neighborhoods. Indeed, if V is any neighborhood of zero, then in view of the continuity of multiplication by a number at the zero element, there exists $\epsilon > 0$ and a neighborhood W of zero such that $a\varphi \in V$ for $|a| \leqslant \epsilon$ and $\varphi \in W$. We now replace the neighborhood V by the open normal set $\bigcup_{|a| \leqslant \epsilon} aW \subset V$; doing this for every neighborhood of zero, we obtain a system of normal neighborhoods of zero which is obviously equivalent to the original system of neighborhoods.

We mention, further, two results pertaining to closed sets.

(a) *If F is a closed set, then the set λF is closed for any λ.*

If $\lambda = 0$, then $\lambda F = 0$ is closed, as its complement is seen, by Axiom II.3, to be an open set. If $\lambda \neq 0$ and G is the complement of F, then it is obvious that λG is the complement of λF. Since G, and therefore λG, is open, the set λF is closed.

(b) *A numerical factor may be taken out from under the closure sign; i.e., $\overline{\lambda A} = \lambda \bar{A}$ for any λ and any set A.*

For $\lambda = 0$, the assertion is trivial. Suppose that $\lambda \neq 0$, and φ is an adherence point of λA. Then for any neighborhood U of zero, one can find $\psi \in U$ and $\chi \in A$ such that $\varphi = \lambda\chi + \psi$, or $(1/\lambda)\varphi = \chi + \psi_1$, where $\psi_1 \in (1/\lambda)U$. Since $(1/\lambda)U$ ranges over a basis of the neighborhoods of zero as U ranges over such a system, we see that any neighborhood of $(1/\lambda)\varphi$ contains a point of A. Hence

$$\frac{1}{\lambda}\varphi \in \bar{A}, \qquad \varphi \in \lambda \bar{A};$$

thus

$$\overline{\lambda A} \subset \lambda \bar{A}.$$

Carrying out the argument in the opposite direction, we conclude that $\varphi \in \lambda \bar{A}$ implies $\varphi \in \overline{\lambda A}$. Hence $\lambda \bar{A} = \overline{\lambda A}$.

1.2. Definition of a Topology by Means of Neighborhoods of Zero

We have already shown that the topology in a linear topological space is uniquely defined by specifying the system of all neighborhoods of zero. The question arises, what properties must a system of sets in a linear space Φ have in order that it can be taken as a defining system of (open) neighborhoods of zero, and thereby transform Φ from a linear to a linear topological space.

This question is answered by the following theorem.

Theorem. *Let us suppose that we are given, in a linear space Φ, a system $\mathfrak{S} = \{U, V, W...\}$ of sets, containing the zero element and having the following properties:*

IV.1. *The intersection of any two sets U, V from $\mathfrak{S}$ contains a third set W from $\mathfrak{S}$.*

IV.2. *For any point $\varphi \neq 0$, there exists a set U from $\mathfrak{S}$ which does not contain φ.*

IV.3. *For any set U from $\mathfrak{S}$, there exists a set W from $\mathfrak{S}$ such that $W \pm W \subset U$.*

IV.4. *If $\varphi \in U \in \mathfrak{S}$, then there exists $V \in \mathfrak{S}$ such that $\varphi + V \subset U$.*

IV.5. *For any $U \in \mathfrak{S}$ and any number α, there exists a $V \in \mathfrak{S}$ such that $\alpha V \subset U$.*

IV.6. *For any $U \in \mathfrak{S}$ and any point φ, there is an $\epsilon > 0$ such that $\delta\varphi \in U$ for $|\delta| < \epsilon$.*

IV.7. *For any $U \in \mathfrak{S}$, there exists $\epsilon > 0$ such that $\delta U \subset U$ for $|\delta| < \epsilon$.*

Then there exists a topology in Φ for which Φ is a linear topological space (i.e., satisfies conditions I–III*), and the system $\mathfrak{S}$ is a basis of the neighborhoods of zero.*

Proof. We will consider all sets of the form $\psi + U$, $U \in \mathfrak{S}$, which contain a given point $\varphi \in \Phi$ as the neighborhoods of φ. Let us show that all the axioms of a linear topological space are fulfilled.

First of all, we note that if $\varphi \in \psi + U$, then $\varphi - \psi \in U$ and it follows from IV.4 that there exists $V \in \mathfrak{S}$ such that $\varphi - \psi + V \subset U$; consequently, $\varphi + V \subset \psi + U$. Thus, if φ lies in a neighborhood $\psi + U$, then there is a neighborhood of φ of the form $\varphi + V$, $V \in \mathfrak{S}$, which lies inside $\psi + U$. Therefore, in verifying the axioms of a linear topological space, we will consider only those neighborhoods of a point φ of the form

$$U(\varphi) = \varphi + U, \quad U \in \mathfrak{S}.$$

Axiom II.1— "Every point $\varphi \in \Phi$ lies in some neighborhood"—is satisfied by construction. Consider Axiom II.2: "If φ belongs to two neighborhoods $U(\varphi)$ and $V(\varphi)$, then it lies in a third neighborhood $W(\varphi)$ lying in the intersection of $U(\varphi)$ and $V(\varphi)$." By the opening remarks in the proof, this reduces to showing that the intersection of any two neighborhoods $U(\varphi)$, $V(\varphi)$ contains a neighborhood $W(\varphi)$. We take a set $W \subset U \cap V$, $W \in \mathfrak{S}$, which exists by Axiom IV.1, and consider the neighborhood $W(\varphi) = \varphi + W$; obviously, $W(\varphi)$ lies in the intersection of $U(\varphi)$ with $V(\varphi)$.

Consider Axiom II.3: "For any pair of points $\varphi \neq \psi$, there is a neighborhood $U(\varphi)$ of φ which does not contain ψ." But by Axiom IV.2, there exists $W \in \mathfrak{S}$ which does not contain $\psi - \varphi \neq 0$. Then $U(\varphi) = \varphi + W$ does not contain ψ.

Consider Axiom III.1: "If $\chi = \varphi \pm \psi$, then for any neighborhood $U(\chi)$ there are neighborhoods $V(\varphi)$ and $W(\psi)$ such that $V(\varphi) \pm W(\psi) \subset U(\chi)$." Given $U(\chi) = \chi + U$, we find, using Axiom IV.3, a subset $W \in \mathfrak{S}$ such that $W \pm W \subset U$. Then one can set $V(\varphi) = \varphi + W$, $W(\psi) = \psi + W$, and the condition will be fulfilled.

Let us verify Axiom III.2: "If $\lambda_0\varphi_0 = \psi$, then for any neighborhood $U(\psi)$ there is a neighborhood $W(\varphi_0)$ and an $\epsilon > 0$ such that $|\lambda - \lambda_0| < \epsilon$ implies $\lambda W(\varphi_0) \subset U(\psi)$." Let $U(\psi) = \psi + U$; we first consider the case $\lambda_0 \neq 0$. Using Axiom IV.3 several times, we find $V \in \mathfrak{S}$ such that $V + V + V \subset U$. Further, using Axiom IV.5, we find $W \in \mathfrak{S}$ such that $\lambda_0 W \subset U$; then, using Axioms IV.6 and IV.7, we find $\epsilon > 0$ such that $\delta\varphi_0 \in V$ and $(\delta/\lambda_0)\, V \subset V$ for $|\delta| < \epsilon$. We claim that the neighborhood $W(\varphi_0) = \varphi_0 + W$ and the number ϵ are those being sought. Indeed, if $|\lambda - \lambda_0| < \epsilon$, i.e., $\lambda = \lambda_0 + \delta$, $|\delta| < \epsilon$, and $\varphi \in W(\varphi_0)$, i.e., $\varphi = \varphi_0 + w$, $w \in W$, then $\lambda\varphi_0 = \psi + (\lambda_0 w + \delta\varphi_0 + \delta w)$; but by construction $\lambda_0 w \in V$, $\delta\varphi_0 \in V$, $\delta w = (\delta/\lambda_0)\; \lambda_0 w \in (\delta/\lambda_0) V \subset V$, and so $\lambda\varphi \in \psi + V + V + V \subset \psi + U$, as was required.

In the case $\lambda_0 = 0$, we proceed in the following way. Given $U \in \mathfrak{S}$, we have to find ϵ and $V(\varphi) = \varphi + V$ such that $|\delta| < \epsilon$ and $\psi \in V(\varphi)$ imply $\delta\psi \in U$. Using Axiom IV.5, we find $V \in \mathfrak{S}$ such that $V + V \subset U$. By Axiom IV.6, we can find $\epsilon_1 > 0$ such that $\delta\varphi \in V$ for $|\delta| < \epsilon_1$. By Axiom IV.7, we can find $\epsilon_2 > 0$ such that $\delta V \subset V$ for $|\delta| < \epsilon_2$. If we take ϵ as the smaller of ϵ_1, ϵ_2, then for $|\delta| < \epsilon$ and $\psi \in V(\varphi) = \varphi + V$ we have

$$\delta\psi = \delta\varphi + \delta(\psi - \varphi) \subset V + \delta V \subset V + V \subset U.$$

The proof of the theorem is complete.

Thus, a topology has been constructed in Φ. The sets of the system $\mathfrak{S}$ are by construction neighborhoods of zero in Φ. Since the topology in a linear topological space can be uniquely reconstructed from the system of neighborhoods of zero, we can conclude that the topology which was constructed in Φ is the only possible one having the given system $\mathfrak{S}$ of neighborhoods of zero.

It is possible that the conditions IV.1–IV.7 are not independent, and can be reduced in number. In any case, in any linear topological space there exists a system of neighborhoods of zero which satisfies all of these conditions (Axioms IV.1–IV.6 are fulfilled for any defining system of neighborhoods of zero, while Axiom IV.7 is fulfilled in a system of normal neighborhoods), so that conditions IV.1–IV.7 are necessary.

Two distinct systems of sets $\mathfrak{S}_1$ and $\mathfrak{S}_2$ can lead to the same topology in a space Φ; in this case they are called *equivalent systems*. A criterion for the equivalence of the systems $\mathfrak{S}_1$ and $\mathfrak{S}_2$ is the following: For every neighborhood $U \in \mathfrak{S}_1$, there must exist a neighborhood $V \in \mathfrak{S}_2$ such that $V \subset U$, and conversely, for every neighborhood $V \in \mathfrak{S}_2$, there must exist a neighborhood $U \in \mathfrak{S}_1$ such that $U \subset V$.

1.3. Examples

We consider two examples of linear topological spaces, consisting of functions.

1. The space $K(a)$ consists of all infinitely differentiable functions $\varphi(x)$ on the line $-\infty < x < \infty$ which vanish outside the interval $|x| \leqslant a$. The linear operations are defined here in the natural way. The neighborhoods of zero are constructed by the following rule: A number $\epsilon > 0$ and a positive integer m are given; the neighborhood $V(m, \epsilon)$ by definition consists of all functions $\varphi(x)$ for which the inequality $|\varphi^{(k)}(x)| < \epsilon$ is fulfilled for $k = 0, \ldots, m$.

The validity of Axioms IV.1–IV.7 is not hard to verify.

A sequence $\{\varphi_\nu(x)\}$ converges to a function $\varphi(x)$ in the sense of the topology defined, if for any $k = 0, 1, \ldots$ the sequence $\{\varphi_\nu^{(k)}(x)\}$ converges uniformly to $\varphi^{(k)}(x)$ as $\nu \to \infty$. Since the first axiom of countability is fulfilled in the present case (it is sufficient to restrict oneself to the neighborhoods

$$V\left(m, \frac{1}{n}\right) \qquad (n = 1, 2, \ldots, \quad m = 0, 1, \ldots)),$$

every topological relation in the space $K(a)$ can be described by means of convergent (countable) sequences.

2. The space $\mathfrak{Z}(G)$ consists of all functions $\varphi(z)$ which are defined inside the region $G = \{|z| < b\}$ in the plane of the complex variable z and are analytic in this region. The linear operations are defined in the usual way. Neighborhoods of zero are defined by the following rule:

Given a number $\epsilon > 0$ and a closed set F lying entirely in G, the neighborhood $V(\epsilon, F)$ consists of all functions $\varphi(z)$ satisfying the inequality

$$\max_{z\in F} |\varphi(z)| < \epsilon.$$

It is not hard to verify that Axioms IV.1–IV.7 are fulfilled.

A sequence $\{\varphi_\nu(z)\}$ converges to a function $\varphi(z)$ in the sense of the topology introduced, if the functions $\varphi_\nu(z)$ converge to $\varphi(z)$ uniformly on every closed set lying in the region G. The first axiom of countability is also fulfilled for $\mathfrak{Z}(G)$; as a countable neighborhood basis of zero, we can take the system

$$V\left(F_m, \frac{1}{n}\right) \qquad (m, n = 1, 2, \ldots),$$

where F_m denotes the set of points z for which $|z| \leqslant b(1 - 1/m)$.

2. Normed Spaces. Comparability and Compatibility of Norms

As has already been mentioned in the introduction, we will not consider further the properties of the most general linear topological spaces. We have introduced typical examples of the spaces which we will need at the end of Section 1. The common characteristic of such spaces consists in the fact that they are *countably normed spaces* or *unions of countably normed spaces*. These spaces are not far removed from the classical normed (Banach) spaces, and the techniques in normed spaces carry over in large measure to this broader class of spaces; on the other hand, in this class of spaces (which is much narrower than the class of all linear topological spaces), characteristic features and qualities manifest themselves, which are not possessed by infinite-dimensional normed spaces; for example, it is possible for bounded sets to be compact.

The definition of countably normed spaces will be given in Section 3 (and of the union of countably normed spaces in Section 8). The topology in these spaces will be defined by a countable family of norms. As a preliminary, we have to study relations between various norms, introduced in the same linear space; the present section is devoted to this question.

2.1. Basic Definitions

As is well known, a linear space Φ is said to be *normed*, if there is defined in Φ a nonnegative function $\|\varphi\|$ (the norm) which has the following properties:

1. $\| \varphi + \psi \| \leqslant \| \varphi \| + \| \psi \|$ for any φ, ψ.
2. $\| \alpha\varphi \| = | \alpha | \, \| \varphi \|$ for any φ and any (real or complex) number α. In particular, $\| 0 \| = 0$.
3. If $\| \varphi \| = 0$, then $\varphi = 0$.

Having a norm, one can introduce a corresponding topology. Namely, a neighborhood $U_\epsilon(0)$ of zero in a normed space is defined by a positive number ϵ and consists of all φ for which $\| \varphi \| < \epsilon$ (the open ball of radius ϵ).

The verification of Axioms IV.1–IV.7 of Section 1.2 is elementary.

It is obvious, moreover, that the first axiom of countability is fulfilled in the present instance, and therefore topological relations can be described in the language of convergent sequences. Here, a sequence φ_n $(n = 1, 2,\ldots)$ converges to an element φ if and only if

$$\| \varphi - \varphi_n \| \to 0$$

as $n \to \infty$.

A sequence of elements φ_n $(n = 1, 2,\ldots)$ of a normed space is said to be *fundamental*, if for any $\epsilon > 0$ there is an integer $n_0 = n_0(\epsilon)$ such that $\nu, \mu > n_0$ implies $\| \varphi_\nu - \varphi_\mu \| < \epsilon$. Every convergent sequence $\{\varphi_\nu\} \to \varphi$ is fundamental, since

$$\| \varphi_\nu - \varphi_\mu \| \leqslant \| \varphi_\nu - \varphi \| + \| \varphi - \varphi_\mu \|.$$

But the converse is not always true. Spaces in which every fundamental sequence converges to some element of the space are called *complete*. An incomplete space can always be completed, by adjoining to it "formal" limits of nonconvergent fundamental sequences and extending the linear operations and the norm to these formal limits in a natural way.

2.2. Comparable and Compatible Norms

Two norms $\| \varphi \|_1$ and $\| \varphi \|_2$, defined in the same space Φ, are said to be *comparable*, whereby the first is considered the *weaker*, and the second the *stronger*, if the inequality

$$\| \varphi \|_1 \leqslant C \| \varphi \|_2$$

holds for all $\varphi \in \Phi$, where C is a fixed constant.[2] Every sequence which is fundamental with respect to the stronger norm will also be fundamental with respect to the weaker norm. If the space is complete with respect

[2] It would be more exact to say that the norm $\| \varphi \|_2$ is not weaker than the norm $\| \varphi \|_1$.

to both norms, then by a well-known theorem of Banach on the boundedness of an inverse operator,[3] the comparability of the two norms leads to their equivalence: There exist constants $C' > 0$ and $C'' > 0$ such that

$$\| \varphi \|_1 \leqslant C' \| \varphi \|_2 \leqslant C'' \| \varphi \|_1 .$$

If the space Φ is incomplete with respect to at least one of the norms, then, generally speaking, comparability does not imply equivalence. In this case, we can consider the two complete spaces Φ_1 and Φ_2, which are obtained by completing Φ with respect to $\| \varphi \|_1$ and $\| \varphi \|_2$, respectively. If $\| \varphi \|_2$ is the stronger norm, then one can establish a natural mapping of Φ_2 into Φ_1 : Every element $\bar{\bar{\varphi}} \in \Phi_2$ is defined by a sequence $\{\varphi_\nu\} \in \Phi$ $(\nu = 1, 2,...)$ which is fundamental with respect to $\| \varphi \|_2$, and, as has already been said, this sequence is fundamental with respect to $\| \varphi \|_1$ and therefore defines an element $\bar{\varphi} \in \Phi_1$. It is easy to verify that $\bar{\varphi}$ is uniquely determined by $\bar{\bar{\varphi}}$. It need not be one-to-one; in other words, distinct elements $\bar{\bar{\varphi}} \in \Phi_2$ and $\bar{\bar{\psi}} \in \Phi_2$ can be mapped into the same element $\bar{\varphi} \in \Phi_1$. To exclude this possibility, we shall require that the norms $\| \varphi \|_1$ and $\| \varphi \|_2$ be compatible in the following sense.

Definition. Two norms $\| \varphi \|_1$ and $\| \varphi \|_2$, defined in a linear space Φ, are said to be *compatible* if every sequence $\varphi_\nu \in \Phi$, $\nu = 1, 2,...$, which is fundamental with respect to both norms and converges to the zero element with respect to one of them, also converges to the zero element with respect to the second.

Examples. In the linear space Φ of all continuously differentiable functions $f(x)$ on the interval $|x| \leqslant a$, the norms

$$\|f\|_1 = \max|f(x)|, \qquad \|f\|_2 = \max\{|f(x)| + |f'(x)|\}$$

are compatible.

The following two norms in Φ,

$$\|f\|_1 = \max |f(x)| \qquad \text{and} \qquad \|f\|_3 = \max |f(x)| + |f'(a)|,$$

are not compatible: One can exhibit a sequence $\{f_\nu(x)\}$ which is uniformly convergent to zero, for which $f'_\nu(a) = 1$. This sequence is fundamental

[3] See, e.g., L. A. Lyusternik and V.I. Sobolev, "Elements of Functional Analysis." Ungar, New York, 1961.

with respect to both norms, and converges to zero with respect to the first, but not with respect to the second, since $\| f_\nu \|_3 \geqslant 1$.

It is not hard to see that in the case of comparable and compatible norms $\| \varphi \|_1$ and $\| \varphi \|_2$, the mapping of Φ_2 into Φ_1, which was discussed above, is *one-to-one*.[4] It is sufficient to verify that a nonzero element $\bar{\bar{\varphi}} \in \Phi_2$ cannot be carried into the zero element of Φ_1. Assuming the contrary, we would have, for a sequence $\varphi_\nu \in \Phi$ which is fundamental with respect to both norms and defines the element $\bar{\bar{\varphi}}$, the relations

$$\lim_{\nu\to\infty} \| \varphi_\nu \|_2 = \| \bar{\bar{\varphi}} \|_2 > 0, \qquad \lim_{\nu\to\infty} \| \varphi_\nu \|_1 = 0,$$

which would contradict the compatibility of the norms.

Thus, in the case of compatible norms, the mapping of Φ_2 into Φ_1 which has been constructed is one-to-one. Therefore, if we identify the elements of Φ_2 with the corresponding elements of Φ_1, we can consider Φ_2 to be a part of Φ_1; we remark that under this identification, every element of Φ is carried into itself.

To summarize: *if two comparable and compatible norms* $\| \varphi \|_1$, $\| \varphi \|_2$, $\| \varphi \|_1 \leqslant C \| \varphi \|_2$ *are defined in a space* Φ, *then the completions* Φ_1 *and* Φ_2 *of* Φ *with respect to these norms may be considered to have the following relationship with each other and with* Φ:

$$\Phi_1 \supset \Phi_2 \supset \Phi.$$

If the norms $\| \varphi \|_1$ and $\| \varphi \|_2$ are compatible but not comparable, then we can introduce a third norm

$$\| \varphi \|_3 = \max\{\| \varphi \|_1 , \| \varphi \|_2\}$$

(it is easy to verify that $\| \varphi \|_3$ fulfills the axioms of a norm), which is obviously comparable with each of the two given norms (namely, it is stronger than each of them). The norm $\| \varphi \|_3$ is also compatible with each of the given norms; indeed, if a sequence $\varphi_\nu \in \Phi$ is fundamental with respect to $\| \varphi \|_3$, then it is fundamental with respect to $\| \varphi \|_1$ and $\| \varphi \|_2$. If it converges to zero with respect to one of these norms, then in view of their compatibility it converges to zero with respect to the other, and consequently converges to zero with respect to $\| \varphi \|_3$. The converse is obvious. In this case, the space Φ_3, obtained by completing Φ with respect to $\| \varphi \|_3$, can be mapped in a one-to-one manner into Φ_1 and into Φ_2; under these mappings Φ does not move.

[4] That is, injective (its range is a part of Φ_1, not generally speaking, all of Φ_1).

3. Countably Normed Spaces

3.1. Definition

We now proceed to define and study a class of linear topological spaces which is one of the most important in applications to analysis—the class of countably normed spaces.

Suppose that a countable system of norms $\| \varphi \|_1$, $\| \varphi \|_2$,... is defined in a linear space Φ.

By means of these norms, we introduce a topology in Φ according to the following rule. A neighborhood $U_{p,\epsilon}(0)$ of zero is defined by a positive integer p and a positive number ϵ, and consists of all $\varphi \in \Phi$ which satisfy the p inequalities

$$\| \varphi \|_1 < \epsilon, \qquad \| \varphi \|_2 < \epsilon, \qquad \ldots, \qquad \| \varphi \|_p < \epsilon. \tag{1}$$

Let us verify that Axioms IV.1–IV.7 of Section 1.2 are fulfilled.

Axiom IV.1 asserts that the intersection of any two neighborhoods of zero contains a third neighborhood of zero; in our case, if

$$U_{p_1,\epsilon_1} = \{\| \varphi \|_1 < \epsilon_1, \ldots, \| \varphi \|_{p_1} < \epsilon_1\},$$

$$U_{p_2,\epsilon_2} = \{\| \varphi \|_1 < \epsilon_2, \ldots, \| \varphi \|_{p_2} < \epsilon_2\},$$

then, as a neighborhood lying in their intersection we can take

$$U_{p,\epsilon} = \{\| \varphi \|_1 < \epsilon, \ldots, \| \varphi \|_p < \epsilon\},$$

where $\epsilon = \min(\epsilon_1, \epsilon_2)$ and $p = \max(p_1, p_2)$.

Axiom IV.2 required that for any point $\varphi_0 \neq 0$ there exist a neighborhood U of zero, not containing this point. In our case, $\| \varphi_0 \|_1 > 0$ by definition of a norm; therefore the neighborhood

$$U = \{\| \varphi \|_1 < \epsilon = \| \varphi_0 \|_1\}$$

does not contain φ_0.

Axiom IV.3 required that for any neighborhood U there exist a neighborhood W such that $W \pm W \subset U$. In our case, if

$$U = \{\| \varphi \|_1 < \epsilon, \ldots, \| \varphi \|_p < \epsilon\},$$

then one can put

$$W = \left\{\| \varphi \|_1 < \frac{\epsilon}{2}, \ldots, \| \varphi \|_p < \frac{\epsilon}{2}\right\}.$$

Axiom IV.4 required that from $\varphi_0 \in U$ there follow the existence of a neighborhood V such that $\varphi_0 + V \subset U$. Obviously, if

$$U = \{\| \varphi \|_1 < \epsilon, \dots, \| \varphi \|_p < \epsilon\},$$

it follows from $\varphi_0 \in U$ that

$$\max_{i \leqslant p} \| \varphi_0 \|_i = \eta < \epsilon;$$

then as V we can take the neighborhood defined by the inequalities

$$\| \varphi \|_1 < \epsilon - \eta, \qquad \dots, \qquad \| \varphi \|_p < \epsilon - \eta.$$

Axiom IV.5 required that for any neighborhood U and any number $\alpha \neq 0$ there exist a neighborhood W such that $\alpha W \subset U$. If

$$U = \{\| \varphi \|_1 < \epsilon, \dots, \| \varphi \|_p < \epsilon\},$$

then we put

$$W = \left\{\| \varphi \|_1 < \frac{\epsilon}{| \alpha |}, \dots, \| \varphi \|_p < \frac{\epsilon}{| \alpha |}\right\}.$$

Axiom IV.6 required that for any neighborhood U and any point φ_0 one can find a number α such that $\delta\varphi_0 \in U$ if $| \delta | < \alpha$. If

$$U = \{\| \varphi \|_1 < \epsilon, \dots, \| \varphi \|_p < \epsilon\}$$

and

$$\| \varphi_0 \|_j = a_j \qquad (j = 1, 2, \dots),$$

then one can set

$$\alpha = \frac{\epsilon}{\max(a_1, \dots, a_p)}.$$

Axiom IV.7 asserted that for any neighborhood U there exists $\epsilon > 0$ such that $\delta U \subset U$ for $| \delta | < \epsilon$. In the present case one can take $\epsilon = 1$; the neighborhoods which we have defined are normal.

Definition. A linear space Φ in which a topology is defined, in the manner described, by a countable family of *compatible* norms is called a *countably normed space*.

We note that in defining the system of norms $U_{p,\epsilon}$, it is sufficient to take $\epsilon = 1, \frac{1}{2}, \frac{1}{3}, \dots$; thus, this system is equivalent to the countable system $U_{p,1/m}$ $(m, p = 1, 2, \dots)$. Therefore, as in every space for which the first axiom of countability is fulfilled, every limit point φ_0 of a set A can be written as the limit of a (countable) convergent sequence of points $\varphi_\nu \in A$. In the present case, as is easy to see, a sequence $\{\varphi_\nu\}$ converges to zero if and only if $\| \varphi_\nu \|_p \to 0$, as $\nu \to \infty$, for every fixed p; similarly, $\varphi_\nu \to \varphi$ if and only if $\| \varphi - \varphi_\nu \|_p \to 0$, as $\nu \to \infty$, for every fixed p.

A sequence $\varphi_1, \varphi_2, \ldots$ is said to be *fundamental*, if it is fundamental with respect to each of the norms $\| \varphi \|_p$, $p = 1, 2, \ldots$. If every fundamental sequence in the (countably normed) space Φ happens to converge to some element in the space, then Φ is said to be *complete*.

One can always consider the given sequence of norms to be nondecreasing, i.e.,

$$\| \varphi \|_1 \leqslant \| \varphi \|_2 \leqslant \cdots \leqslant \| \varphi \|_p \leqslant \cdots$$

for every $\varphi \in \Phi$, for, assuming the contrary, one can replace each norm $\| \varphi \|_p$ by $\| \varphi \|_p' = \max\{\| \varphi \|_1, \ldots, \| \varphi \|_p\}$. Obviously, the sequence of norms $\| \varphi \|_p'$ is nondecreasing and generates the same topology in Φ as does the original sequence. Here, the neighborhoods $U_{p,\epsilon}(0)$ can be defined by a single inequality $\| \varphi \|_p' < \epsilon$. Together with the original norms, the new norms are pairwise compatible.

3.2. The Condition of Completeness

We assume now that the (compatible) norms $\| \varphi \|_1$, $\| \varphi \|_2, \ldots$ are nondecreasing. Completing the space Φ with respect to each of the $\| \varphi \|_p$, we obtain a system of complete normed spaces $\Phi_1, \Phi_2, \ldots$. Since all of the norms are comparable and compatible, the results of Section 2 lead to the system of inclusions

$$\Phi_1 \supset \Phi_2 \supset \cdots \supset \Phi_p \supset \cdots \supset \Phi. \tag{1}$$

A simple characterization of the completeness of Φ can be given in terms, referring to the spaces Φ_p.

Theorem. *The space Φ is complete if and only if it coincides with the intersection of all the Φ_p, i.e., if*

$$\bigcap_{p=1}^{\infty} \Phi_p = \Phi. \tag{2}$$

Proof. Suppose that Eq. (2) holds and that $\varphi_1, \varphi_2, \ldots$ is a fundamental sequence in Φ. By definition, $\{\varphi_\nu\}$ is fundamental in each Φ_p and has, therefore, a limit $\varphi^{(p)}$ in each of these spaces. In view of the mappings $\Phi_p \to \Phi_{p-1}$ which we have established, and the corresponding identification of elements of these spaces, all of the elements $\varphi^{(p)}$, $p = 1, 2, \ldots$ are, in essence, the same element, which, therefore, belongs to all of the Φ_p and by Eq. (2) belongs also to Φ. Let us denote this element, as an element of Φ, by φ. Since

$$\| \varphi_\nu - \varphi^{(p)} \|_p \to 0$$

for every p, we have

$$\| \varphi_\nu - \varphi \|_p \to 0$$

for every p, from which

$$\varphi = \lim_{\nu \to \infty} \varphi_\nu$$

in the topology of Φ. Thus, Φ is a complete space.

Conversely, let Φ be a complete space, and suppose that an element φ is contained in the intersection of all the Φ_p; we show that φ belongs to Φ. We find, for each p, an element $\varphi_p \in \Phi$ for which

$$\| \varphi - \varphi_p \|_p < \frac{1}{p};$$

this can be done, since Φ_p is the completion of Φ with respect to the norm $\| \varphi \|_p$. We assert that as $p \to \infty$, the φ_p converge to φ with respect to each of the norms. Indeed, for any k and $p > k$ we have

$$\| \varphi - \varphi_p \|_k \leqslant \| \varphi - \varphi_p \|_p < \frac{1}{p},$$

from which it follows that

$$\lim_{p \to \infty} \| \varphi - \varphi_p \|_k = 0.$$

Therefore, in particular, the sequence φ_p is fundamental with respect to each of the norms, and is therefore fundamental in Φ.

Let

$$\bar{\varphi} = \lim_{p \to \infty} \varphi_p$$

in the topology of Φ. Since, for every k, one has

$$\lim_{p \to \infty} \| \bar{\varphi} - \varphi_p \|_k = 0,$$

and $\bar{\varphi}$, as well as φ, belongs to Φ_k, it follows that $\varphi = \bar{\varphi}$. Thus, $\varphi \in \Phi$, which was to be proved.

Henceforth, by a countably normed space we will, without special mention, understand a complete countably normed space. (We emphasize that the definition of a countably normed space includes the condition of compatibility of the norms.)

3.3. Examples

A simple but nontrivial example of a countably normed space is the space $K(a)$ of all functions $\varphi(x)$ which are infinitely differentiable on the line $-\infty < x < \infty$ and vanish outside the interval $|x| \leqslant a$ (Section 1.3). We introduce in this space the system of norms

$$\| \varphi \|_p = \max_{|x| \leqslant a} \{| \varphi(x)|, | \varphi'(x)|, ..., | \varphi^{(p)}(x)|\} \qquad (p = 0, 1, 2,...).$$

The neighborhoods

$$U_{p,\epsilon}(0) = \{\| \varphi \|_p < \epsilon\}$$

obviously coincide with the neighborhoods defined in this space in Section 1.3, and therefore define the same topology.

The norms $\| \varphi \|_p$ obviously form a nondecreasing sequence, i.e., $\| \varphi \|_p \leqslant \| \varphi \|_{p+1}$ for any φ. Let us verify that these norms are pairwise compatible. For this, it suffices to verify that the pth and $(p + 1)$th norms are compatible. Let us consider a sequence of functions $\varphi_\nu(x)$ which converges to zero with respect to $\| \varphi \|_p$ and is fundamental with respect to $\| \varphi \|_{p+1}$. This means that the functions $\varphi_\nu^{(k)}(x)$ converge uniformly to zero, as $\nu \to \infty$, for $k = 0, 1,..., p$, and (uniformly) to some limit $\theta(x)$ for $k = p + 1$. But then, in view of a well-known theorem of classical analysis, $\theta(x) = 0$. It follows that

$$\| \varphi_\nu \|_{p+1} \to 0.$$

Conversely, if

$$\| \varphi_\nu \|_{p+1} \to 0,$$

then

$$\| \varphi_\nu \|_p \leqslant \| \varphi_\nu \|_{p+1} \to 0.$$

Thus, the norms $\| \varphi \|_p$ and $\| \varphi \|_{p+1}$ are indeed compatible.

Let us show that the completion $\overline{K(a)}^p$ of $K(a)$ with respect to $\| \varphi \|_p$ is just the collection $K_p(a)$ of all functions $\varphi(x)$ which vanish outside the interval $|x| \leqslant a$ and have continuous derivatives up to order p. Obviously

$$\overline{K(a)}^p \subset K_p(a),$$

and we only have to prove the opposite inclusion. First we verify that the sought-for completion $\overline{K(a)}^p$ contains every function $\varphi_0(x)$ which has continuous derivatives up to order p and vanishes outside an interval $|x| \leqslant a - \delta$, lying inside the interval $|x| \leqslant a$. One can construct a sequence of polynomials $P_n(x)$ (using Weierstrass' theorem) which converge uniformly on $|x| \leqslant a$ to $\varphi_0(x)$, and whose derivatives of order up to p converge uniformly on this interval to the corresponding derivatives of $\varphi_0(x)$. Further, let $e(x)$ be a function in $K(a)$ which equals 1 for $|x| \leqslant a - \delta$. Then the products $P_n(x)\,e(x)$ belong to $K(a)$ and converge to $\varphi_0(x)$ in the metric of $K(a)^p$, from which it follows that $\varphi_0(x)$ belongs to $\overline{K(a)}^p$. All remaining functions $\varphi(x) \in K_p(a)$ are limits, in the norm of $K(a)^p$, of functions of the type $\varphi_0(x)$ (for example, $\varphi(\lambda x) \to \varphi(x)$ in the norm of $K(a)^p$ as $\lambda \uparrow 1$), and therefore belong to the completion $\overline{K(a)}^p$ of $K(a)$. Thus,

$$K_p(a) \subset \overline{K(a)}^p;$$

hence

$$K_p(a) = \overline{K(a)}^p,$$

as was asserted.

The intersection of the spaces $K_p(a)$, $p = 1, 2, \ldots$, obviously coincides with the space $K(a)$. Therefore, in view of the criterion of the preceding Section 3.2, $K(a)$ is a complete space. This is, by the way, easy to show directly.

Thus, $K(a)$ *is a complete countably normed space.*

A somewhat more complicated, but essentially completely analogous example is the space $K(a)$ of all infinitely differentiable functions $\varphi(x) = \varphi(x_1, \ldots, x_n)$ of n variables, which vanish outside the region $G_a = \{|x_1| \leqslant a_1, \ldots, |x_n| \leqslant a_n\}$.

We introduce in this space the collection of norms

$$\|\varphi\|_p = \max_x \left\{ |\varphi(x)|, \left| \frac{\partial^n \varphi(x)}{\partial x_1 \cdots \partial x_n} \right|, \ldots, \left| \frac{\partial^{np} \varphi(x)}{\partial x_1{}^p \cdots \partial x_n{}^p} \right| \right\}.$$

Convergence with respect to the norm $\|\varphi\|_p$ is uniform convergence of the functions and their derivatives up to order $(p, \ldots, p)$; the convergence to which the collection of all the norms $\|\varphi\|_p$ gives rise is the uniform convergence of the functions as well as their derivatives of all orders. The $\|\varphi\|_p$ are nondecreasing and compatible. The completion of $K(a)$ with respect to $\|\varphi\|_p$ is the collection $K_p(a)$ of all functions which have continuous derivatives up to order $(p, \ldots, p)$ and vanish outside the region G_a. The intersection of the $K_p(a)$ coincides with the space $K(a)$. Thus, $K(a)$ is also a complete countably normed space in the n-dimensional case.

As another example, let us consider the space $\mathfrak{Z}(G)$ of all analytic functions $\varphi(z_1, \ldots, z_n)$ in the region $G = \{| z | < a\}$. For simplicity, we restrict ourselves to the case of a single independent variable z, varying in the region $| z | < a$. We introduce in $\mathfrak{Z}(G)$ the collection of norms

$$\| \varphi \|_p = \max | \varphi(z) |, \qquad | z | \leqslant a_p \qquad (a_p < a_{p+1}, \quad \lim a_p = a).$$

The convergence corresponding to $\| \varphi \|_p$ is the uniform convergence of the $\varphi(z)$ in the region $| z | \leqslant a_p$; the convergence to which the collection of all the norms gives rise is the uniform convergence of the $\varphi(z)$ in any closed region lying in the region $| z | < a$. The $\| \varphi \|_p$ are nondecreasing and compatible. The completion of $\mathfrak{Z}(G)$ with respect to $\| \varphi \|_p$ is the collection of all analytic functions in the closed disk $| z | \leqslant a_p$; the intersection of these families coincides with $\mathfrak{Z}(G)$. Thus, $\mathfrak{Z}(G)$ is a complete countably normed space.

The reader will encounter a large number of other examples further on, in particular in Chapter II, Sections 1–2, Chapter IV, Sections 1–3, and the appendix to Chapter IV.

3.4. Countably Normed Spaces as Linear Metric Spaces

One can introduce a metric in a countably normed space, i.e., define a function $\rho(\varphi, \psi)$ of pairs of points φ, ψ (the distance from φ to ψ), having the following properties:

[1] $\rho(\varphi, \varphi) = 0$, $\rho(\varphi, \psi) > 0$ for $\varphi \neq \psi$;

[2] $\rho(\varphi, \psi) = \rho(\psi, \varphi)$;

[3] $\rho(\varphi_1, \varphi_3) \leqslant \rho(\varphi_1, \varphi_2) + \rho(\varphi_2, \varphi_3)$.

Of course, introducing a metric arbitrarily would be fruitless; it is important that the distance $\rho(\varphi, \psi)$ can be taken invariant with respect to translation and continuous in the topology of the space:

[4] $\rho(\varphi, \psi) = \rho(\varphi - \psi, 0)$;

[5] if $\varphi_\nu \to 0$, then $\rho(\varphi_\nu, 0) \to 0$.

Namely, one can set

$$\rho(\varphi, \psi) = \sum_{p=1}^{\infty} \frac{1}{2^p} \frac{\| \varphi - \psi \|_p}{1 + \| \varphi - \psi \|_p}. \tag{1}$$

Let us verify that conditions [1]–[5] are fulfilled. The first two are obvious. To verify condition [3], it is sufficient to show that

$$\frac{\|\varphi+\psi\|_p}{1+\|\varphi+\psi\|_p} \leqslant \frac{\|\varphi\|_p}{1+\|\varphi\|_p} + \frac{\|\psi\|_p}{1+\|\psi\|_p}. \tag{2}$$

To do this, we note that the function

$$g(t) = \frac{t}{1+t}$$

of the real positive variable t is monotonically increasing; therefore

$$\frac{\|\varphi+\psi\|_p}{1+\|\varphi+\psi\|_p} \leqslant \frac{\|\varphi\|_p+\|\psi\|_p}{1+\|\varphi\|_p+\|\psi\|_p}$$
$$\leqslant \frac{\|\varphi\|_p}{1+\|\varphi\|_p+\|\psi\|_p} + \frac{\|\psi\|_p}{1+\|\varphi\|_p+\|\psi\|_p} \leqslant \frac{\|\varphi\|_p}{1+\|\varphi\|_p} + \frac{\|\psi\|_p}{1+\|\psi\|_p},$$

and so condition [3] is satisfied.

That condition [4] is satisfied is obvious.

Finally, to verify condition [5], it suffices to note the following. If $\varphi_\nu \to 0$, i.e., $\|\varphi_\nu\|_p \to 0$ for every p, then every term in the sum

$$\sum_{p=1}^{\infty} \frac{1}{2^p} \frac{\|\varphi_\nu\|_p}{1+\|\varphi_\nu\|_p}$$

tends to zero, and if for $\nu \geqslant \nu_0$ the sum of the first n terms is less than ϵ, then the entire sum is less than

$$\epsilon + \sum_{p=n+1}^{\infty} \frac{1}{2^p} = \epsilon + \frac{1}{2^n}.$$

Every metric space, as is well known, is a topological space; the system of sets of the form

$$\rho(\varphi, \varphi_0) < r \tag{3}$$

constitutes a complete system of neighborhoods in this topology.

It turns out that the metric defined by formula (1) defines a topology in Φ *which is identical with the original topology.*

To prove this, it suffices in view of condition [4] to consider neighborhoods of zero and sets $\rho(\varphi, 0) < r$ ($\varphi_0 = 0$). We have to show that every neighborhood of zero contains some set $\rho(\varphi, 0) < r$ and conversely.

(1) Every neighborhood of zero[5] $\{\| \varphi \|_p < \epsilon\}$ contains some set $\rho(\varphi, 0) < \delta$.
Indeed, if for some δ

$$\rho(\varphi, 0) \equiv \sum_{q=1}^{\infty} \frac{1}{2^q} \frac{\| \varphi \|_q}{1 + \| \varphi \|_q} < \delta,$$

then, in particular,

$$\frac{1}{2^p} \frac{\| \varphi \|_p}{1 + \| \varphi \|_p} < \delta,$$

whence

$$\| \varphi \|_p < \frac{2^p \delta}{1 - 2^p \delta};$$

for δ sufficiently small, this quantity will be less than ϵ, as required.

(2) Every set $\rho(\varphi, 0) < \epsilon$ contains some neighborhood of zero $\| \varphi \|_p < \delta$.

Indeed, in the contrary case we could find a sequence of points φ_ν, $\nu = 1, 2, \ldots$, not belonging to the ball $\rho(\varphi, 0) < \epsilon$, for which $\| \varphi_\nu \|_p < (1/\nu)$. This would mean that $\{\varphi_\nu\}$ tends to zero with respect to the topology of the space, but not with respect to the metric, which would contradict Property [5].

Condition [5] also implies the following property:

[5_1] If $\lambda_\nu \to 0$, then $\rho(\lambda_\nu \varphi, 0) \to 0$ for any φ.

From the equivalence of the initial and metric topologies and condition [5], we see that the following condition is satisfied:

[6] If $\rho(\varphi_\nu, 0) \to 0$, then $\rho(\lambda \varphi_\nu, 0) \to 0$ for any λ.

A linear space in which there is introduced a metric which also satisfies, in addition to conditions [1]–[3], which simply define a metric, conditions [4], [5_1], and [6] is called a *linear metric space* or a *space of type* (F).[6]

We have therefore shown that *a countably normed space is at the same time a linear metric space, and the topology defined by the metric is equivalent to the original topology.*

[5] We assume the norms to be ordered.

[6] See, for example, S. Banach, "Théorie des Opérations Linéaires." Chelsea, New York. 1955.

We remark that according to the definition of the metric in this space, the distance between any two points is less than 1.

It follows, from what has been proven, that if a countably normed space Φ is complete in its topology, then it is also a complete metric space.

It is known that in a complete metric space, a countable union of nowhere dense sets cannot coincide with the entire space.[7] It follows that *no countable union of nowhere dense sets in a countably normed space Φ can coincide with Φ.*

We now present a useful result whose proof is based upon this property. Let us adopt the following **definition:** A set F in a linear space Φ is said to be *absorbing*, if for any element $\varphi \in \Phi$ there is a real number λ such that $\lambda\varphi \in F$.

Lemma. *If F is a convex*[8] *centrally symmetric closed absorbing set in a complete countably normed space Φ, then F contains a neighborhood of zero.*

Proof. The closed set F is either nowhere dense, or it contains a neighborhood. The same is true of any dilation λF, $\lambda \neq 0$. But it follows from the conditions of the lemma that the family of all dilations mF, $m = 1, 2, \ldots$, covers the space Φ. Therefore, by the preceding remarks, F cannot be nowhere dense, and consequently contains some neighborhood $\varphi_0 + U$, where $U = -U$ is a normal neighborhood of

[7] **Proof.** Consider $\bigcup_{p=1}^{\infty} A_p$, where the A_p are nowhere dense sets. The closures $\bar{A}_p$ are also nowhere dense. Suppose that $\varphi_1 \notin \bar{A}_1$; there exists a ball

$$U_1 = \{\rho(\varphi, \varphi_1) \leqslant r_1\}$$

which does not intersect $\bar{A}_1$. We take a point φ_2 in the interior of U_1 such that $\varphi_2 \notin \bar{A}_2$, and construct a ball

$$U_2 = \{\rho(\varphi, \varphi_2) \leqslant r_2\}$$

which lies in U_1 and does not intersect $\bar{A}_2$. Continuing in this way, we obtain a system of closed balls

$$U_1 \supset U_2 \supset \cdots$$

whose radii, we may suppose, tend to zero. The sequence of centers of these balls is obviously fundamental; its limit belongs to every one of the U_p, and consequently belongs to no one of the $\bar{A}_p$. Thus, the union of the latter does not cover the entire space; hence, the union of the A_p does not cover all of Φ.

[8] We recall that a set F in a linear space Φ is said to be *convex*, if along with any two points φ_0, φ_1, it contains the entire line segment connecting them, i.e., the set of all points

$$\lambda\varphi_0 + \mu\varphi_1 \qquad (\lambda \geqslant 0, \mu \geqslant 0, \lambda + \mu = 1).$$

zero. As F is centrally symmetric, it also contains the neighborhood $-\varphi_0 - U = -\varphi_0 + U$. Now F also contains the convex hull of the union of $\varphi_0 + U$ and $-\varphi_0 + U$. But this convex hull contains U, since, for any $\psi \in U$, we have

$$\psi = \frac{(\varphi_0 + \psi) + (-\varphi_0 + \psi)}{2},$$

which proves the lemma.

Other applications of the fact that a countably normed space is a linear metric space will be encountered in Section 7.

3.5. Conditions for the Normalizability of a Countably Normed Space

We have shown, in Section 3.4, that every countably normed space is a linear metric space and that the original topology (defined by a countable family of norms) coincides with the topology defined by a (single) metric. The question arises: *When does a countably normed space not reduce to a normed linear space*, i.e., when is the introduction of a countable number of norms really justified? That the normed linear spaces are included among the countably normed spaces is obvious; one can take a countable family of norms which are either equal to or equivalent to a single norm.

In view of the theorem of Section 3.2, a countably normed space Φ is always the intersection of normed spaces:

$$\Phi = \bigcap_{p=1}^{\infty} \Phi_p, \qquad \Phi_{p+1} \subset \Phi_p,$$

where Φ_p is the completion of Φ with respect to the pth norm.

It turns out, that *if there are infinitely many different spaces among the* Φ_p , $p = 1, 2,\ldots$ (i.e., if there is a subsequence of pairwise inequivalent norms $\| \varphi \|_{p_1}, \| \varphi \|_{p_2}, \ldots, p_1 < p_2 < \cdots$), *then* Φ *does not reduce to a normed space*; *only if the* Φ_p *coincide, starting with some index* p_0 , *is* Φ *a normed space* (*which then coincides with* Φ_{p_0}).

Proof. The second assertion is obvious; we have

$$\Phi = \bigcap_{p=1}^{\infty} \Phi_p = \Phi_{p_0};$$

further, in view of the fact that $\Phi_{p_0} = \Phi_{p_0+1}$ and the comparability of $\| \varphi \|_{p_0}$ and $\| \varphi \|_{p_0+1}$, these norms are equivalent (cf. Section 2.2). In particular, convergence with respect to $\| \varphi \|_{p_0}$ implies convergence with respect to $\| \varphi \|_{p_0+1}$; hence, in the same way, it implies convergence with respect to $\| \varphi \|_{p_0+2}$, and so on, i.e., finally, convergence with respect to every norm, and so, convergence in the topology of Φ. Conversely, if a sequence $\{\varphi_\nu\}$ converges in the topology of Φ, then it converges with respect to every norm, and, in particular, with respect to $\| \varphi \|_{p_0}$.

Let us consider the first assertion. Without loss of generality, we may suppose that all of the Φ_p are distinct. We assert that in this case there does not exist a norm which gives rise to the original topology in Φ. Before proving this assertion, we prove the following lemma.

Lemma. *If all of the spaces Φ_p , entering into the construction of the countably normed space Φ, are distinct, then for any sequence m_1 , m_2 ,... of positive numbers one can exhibit an element $\varphi \in \Phi$ for which*

$$\| \varphi \|_p > m_p , \qquad p = 1, 2, \ldots .$$

Proof of lemma. First we show that for any p and any C_1 , C_2 , there exists an element φ satisfying the inequalities

$$\| \varphi \|_p \leqslant C_1 , \qquad \| \varphi \|_{p+1} > C_2 . \tag{1}$$

Indeed, if there were no such φ, this would mean that the inequality

$$\| \varphi \|_{p+1} \leqslant C_2$$

holds for any φ satisfying $\| \varphi \|_p \leqslant C_1$. But then the inequality

$$\| \varphi \|_{p+1} \leqslant \frac{C_2}{C_1} \| \varphi \|_p$$

would be satisfied for any φ. But since, on the other hand,

$$\| \varphi \|_p \leqslant \| \varphi \|_{p+1} ,$$

the norms $\| \varphi \|_p$ and $\| \varphi \|_{p+1}$ would be equivalent, which would lead to

$$\Phi_p = \Phi_{p+1} ,$$

contrary to hypothesis. This establishes the existence of an element φ satisfying inequalities (1).

We define the element, whose existence is asserted by the lemma, in the form of a series

$$\varphi = \sum_{q=1}^{\infty} \varphi_q . \tag{2}$$

For the element φ_1, we take any element satisfying the inequality

$$\| \varphi \|_1 > m_1 + 1.$$

For the second element φ_2, we take an element which satisfies the inequalities

$$\| \varphi_2 \|_1 < \frac{1}{2}, \qquad \| \varphi_2 \|_2 > m_2 + 1 + \| \varphi_1 \|_2 ;$$

the existence of such an element has just been established. Further, we take the element φ_3 as satisfying the inequalities

$$\| \varphi_3 \|_2 < \frac{1}{2^2}, \qquad \| \varphi_3 \|_3 > m_3 + 1 + \| \varphi_1 \|_3 + \| \varphi_2 \|_3 ,$$

and so on, so that φ_p satisfies the inequalities

$$\| \varphi_p \|_{p-1} < \frac{1}{2^{p-1}}, \qquad \| \varphi_p \|_p > m_p + 1 + \| \varphi_1 \|_p + \cdots + \| \varphi_{p-1} \|_p .$$

Let us show that the series (2) converges with respect to every norm (and, consequently, converges in Φ). Indeed, given p and $q > p$, we have

$$\| \varphi_q \|_p \leqslant \| \varphi_q \|_{q-1} < \frac{1}{2^{q-1}},$$

and so the tail of the series is majorized by a geometric series with ratio $\frac{1}{2}$. Further,

$$\begin{aligned} \| \varphi \|_p &\geqslant \| \varphi_p \|_p - \sum_{q>p} \| \varphi_q \|_p - \sum_{q<p} \| \varphi_q \|_p \\ &\geqslant m_p + 1 + \| \varphi_1 \|_p + \cdots + \| \varphi_{p-1} \|_p - 1 - (\| \varphi_1 \|_p + \cdots + \| \varphi_{p-1} \|_p) \\ &= m_p , \end{aligned}$$

as was required. This proves the lemma.

We now prove that if the spaces Φ_p, entering into the construction of the (complete) countably normed space Φ, are distinct, then the topology in Φ cannot be defined by any norm.

Suppose the contrary; the topology in Φ is defined by some norm $\| \varphi \|$. We consider the unit ball $E = \{\| \varphi \| \leqslant 1\}$, and assert that each norm $\| \varphi \|_p$ is bounded on E (the bound depending upon p). Indeed, if $\| \varphi \|_p$ were not bounded on E, there would exist a sequence $\varphi_q \in E$ for which $\| \varphi_q \|_p = a_q \to \infty$. For this sequence we have $\varphi_q/a_q \to 0$ in Φ, and consequently φ_q/a_q tends to zero with respect to each norm $\| \varphi \|_p$. But this contradicts the fact that $\| \varphi_q/a_q \|_p = 1$. Thus, each norm $\| \varphi \|_p$ is bounded on E by some constant, say m_p. In view of the lemma, there is an element φ such that

$$\| \varphi \|_p > pm_p .$$

Since Φ is a normed space, there is $\lambda \neq 0$ such that $\lambda\varphi \in E$. But

$$\| \lambda\varphi \|_p > | \lambda | pm_p ,$$

while

$$\| \lambda\varphi \|_p < m_p \qquad (p = 1, 2, \ldots),$$

which is not possible for any $\lambda \neq 0$. The contradiction proves our assertion.

From this result it follows at once, in particular, that the space $K(a)$ is not normalizable, since the corresponding spaces Φ_p are plainly distinct. Thus, although at first glance the normed spaces are the simplest, so simple and important a space as $K(a)$ still is not contained among them.

3.6. Comparable and Equivalent Systems of Norms

It is useful to clarify when two different systems of norms in a countably normed space lead to the same topology.

We recall that given two norms $\| \varphi \|$ and $\| \varphi \|'$, defined in a linear space Φ, the first is considered the weaker, and the second the stronger, if the inequality

$$\| \varphi \| \leqslant C \| \varphi \|'$$

holds with a constant C not depending upon φ.

Suppose that we are given two systems of norms

$$\| \varphi \|_1 \leqslant \| \varphi \|_2 \leqslant \cdots \leqslant \| \varphi \|_p \leqslant \cdots,$$

$$\| \varphi \|_1' \leqslant \| \varphi \|_2' \leqslant \cdots \leqslant \| \varphi \|_p' \leqslant \cdots$$

in a space Φ. We will say that the first of these systems is the *weaker*, and the second is the *stronger*, if each norm in the first system is weaker than some norm in the second system.[9]

[9] It would be more precise to say "not weaker" instead of "stronger."

If the second system is stronger than the first, then every sequence $\{\varphi_\nu\}$ which converges to zero with respect to all the norms in the second system also converges to zero with respect to all the norms in the first system. Indeed, for a given p, we can find $q = q(p)$ such that $\| \varphi \|_p$ is weaker than $\| \varphi \|'_q$. Then we have

$$\| \varphi_\nu \|_p \leqslant C \| \varphi_\nu \|'_q .$$

Since, by assumption, $\| \varphi_\nu \|'_q \to 0$ for any q, it follows that $\| \varphi_\nu \|_p \to 0$ for every p.

Let us show that the converse is also true.

Lemma. *If every sequence $\{\varphi_\nu\}$ which converges to zero with respect to all of the norms in the second system, also converges to zero with respect to every norm in the first system, then the first system is weaker than the second.*

Proof. Suppose the contrary. Then there is some norm $\| \varphi \|_p$ which is not weaker than any norm in the second system. This means that for any ν we can find an element φ_ν satisfying the inequality

$$\| \varphi_\nu \|_p > \nu \| \varphi_\nu \|'_\nu .$$

Multiplying each φ_ν, if necessary, by some constant, we may suppose that $\| \varphi_\nu \|_p = 1$, so that

$$\| \varphi_\nu \|'_\nu < \frac{1}{\nu} \| \varphi_\nu \|_p = \frac{1}{\nu} .$$

We shall show that the sequence $\{\varphi_\nu\}$ tends to zero with respect to every one of the norms of the second system. Indeed, given q and $\nu > q$, we have

$$\| \varphi_\nu \|'_q \leqslant \| \varphi_\nu \|'_\nu \leqslant \frac{1}{\nu} \to 0.$$

But then, according to the hypothesis of the lemma, the sequence $\{\varphi_\nu\}$ must converge to zero with respect to all of the norms in the first system. But this is not possible, as $\| \varphi_\nu \|_p = 1$. The contradiction proves the assertion of the lemma.

If every sequence which converges to zero with respect to all the norms of one of the systems also converges to zero with respect to all the norms of the other system, then the two systems are said to be *equivalent.* From the assertion just proven, it follows at once that a necessary and sufficient condition for equivalence is the following requirement: Every norm of the first system is weaker than some norm in the second system, and conversely, every norm in the second system is weaker than some norm in the first system.

3.7. Bounded Sets in Countably Normed Spaces

A set A in a normed space is said to be *bounded*, if the norms of all the elements of A are bounded (by a fixed constant). We will say that a set A in a countably normed space Φ is *bounded*, if it is bounded with respect to each of the norms defining Φ, i.e.,

$$\| \varphi \|_p \leqslant C_p \qquad (p = 1, 2, \ldots) \quad \text{for all} \quad \varphi \in A.$$

Thus, a set $A = \{\varphi(x)\}$ in the space $K(a)$ of infinitely differentiable functions (see Section 3.3) is bounded if and only if, for each p, the set of numbers

$$\max_x | \varphi^{(p)}(x)|, \qquad \varphi \in A,$$

is bounded (by a constant depending upon p).

Despite the outward resemblance of this definition with the corresponding definition in a normed space, the sets described by it are profoundly different. For example, in a normed space the ball $\| \varphi \| \leqslant 1$ is bounded, and its dilations cover the entire space. As opposed to this, in a countably normed space which is not normalizable, there exists no bounded set whose dilations cover the entire space. Indeed, if A is bounded, i.e., we have

$$\| \varphi \|_p \leqslant C_p \quad (p = 1, 2, \ldots)$$

for all $\varphi \in A$, then it is obvious that no dilation of A can contain an element φ_0 for which

$$\| \varphi_0 \|_p > pC_p ;$$

but the existence of such an element follows from Lemma 5.

Further, it follows from this that *every bounded set in a nonnormalizable countably normed space is nowhere dense.* Indeed, if a bounded set A were dense in some ball $\| \varphi - \varphi_0 \|_p < C$, then its closure $\bar{A}$, which is of course also bounded, would contain the ball. Then the translate A_1 of $\bar{A}$ by the vector φ_0 (the translate is also a bounded set) would contain the ball $\| \varphi \|_p < C$. Now the dilations of this ball cover all of Φ. But then the dilations of the bounded set A_1 would cover all of Φ, which, as we have seen, is not possible.[10]

From the foregoing, and from what was proved in Section 3.4, we can conclude that *a complete countably normed space Φ which is nonnormalizable cannot be represented as a countable union of bounded subsets.*

3.8. Bounded Sets in General Spaces

Further on we will also consider linear topological spaces more general than countably normed spaces; it is therefore appropriate to give here a definition of bounded sets which will also apply to more general cases.

Definition. A set A in a linear topological space Φ is said to be *bounded*, if it is absorbed by any neighborhood of zero, i.e., if for any neighborhood U of zero there is a number $\lambda > 0$ such that $\lambda A \subset U$.

It is easy to see that for a countably normed space this definition coincides with the preceding one: A set A is bounded if each of the norms $\| \varphi \|_p$ is bounded on it. Indeed, if $U_p = \{\| \varphi \|_p < \epsilon\}$, then the inclusion $\lambda_p A \subset U_p$ is equivalent to the inequality $\| \varphi \|_p < \epsilon/\lambda_p$ holding on the set A.

A single point φ is always a bounded set: Since $\lambda\varphi \rightarrow 0$ for $\lambda \rightarrow 0$, then for any neighborhood U of zero we will have $\lambda\varphi \in U$ for $| \lambda |$ sufficiently small.

The following assertions are also easy to verify.

1. *If A_1 and A_2 are bounded sets in a linear topological space, then their union $A = A_1 \cup A_2$ and their algebraic sum $B = A_1 + A_2$ are also bounded sets.*

Indeed, if U is a given neighborhood of zero, there exist λ_1 and λ_2 such that $\lambda_1 A_1 \subset U$ and $\lambda_2 A_2 \subset U$. Setting $\lambda = \min(\lambda_1 , \lambda_2)$, we will have $\lambda A \subset U$. Further, we construct a neighborhood W of zero such that $W + W \subset U$. There exist λ_1 and λ_2 for which $\lambda_1 A_1 \subset W$ and $\lambda_2 A_2 \subset W$; then, obviously, for $\lambda = \min(\lambda_1 , \lambda_2)$, we have $\lambda B \subset U$.

In particular, for A_2 reducing to a single point, we conclude that *every translate of a bounded set by a fixed element is a bounded set.*

2. *A sequence $\varphi_1 , \varphi_2 ,\ldots$ which converges to an element φ in a linear topological space Φ is a bounded set.*

[10] In particular, we see that every ball $\| \varphi \|_p < C$ is unbounded in Φ.

For the proof, we use the fact that *there exists, inside any neighborhood V of zero, a normal neighborhood U*, i.e., a neighborhood which contains all of its dilations aU with $|a| \leqslant 1$ (cf. Section 1.1). (It follows from this, that in verifying the boundedness of one set or another, it is always sufficient to consider only normal neighborhoods of zero.)

Let us assume first that $\varphi = 0$. If U is some normal neighborhood of zero, then for some ν_0 we have $\varphi_\nu \in U$ for $\nu > \nu_0$. We find a positive number $\lambda < 1$ such that $\lambda\varphi_1 \in U, \ldots, \lambda\varphi_{\nu_0} \in U$. Since U is normal, we will also have $\lambda\varphi_\nu \in U$ for $\nu > \nu_0$. Therefore $\lambda\varphi_\nu \in U$ for all ν, as required. Since the translates of a bounded set are bounded sets, we conclude that when $\varphi \neq 0$, the sequence $\{\varphi_\nu\}$ is still a bounded set, as was asserted.

3. *The closure of a bounded set in a linear topological space is a bounded set.*

For the proof, we use the property of regularity of a linear topological space (Section 1.1), i.e., the property that we can find, inside any neighborhood U of zero, a neighborhood V of zero whose closure also lies in U. Let A be a bounded set, $\bar{A}$ its closure, U a given neighborhood of zero, and V a neighborhood of zero whose closure lies in U. We find a number λ such that $\lambda A \subset V$. Then $\lambda\bar{A} = \overline{\lambda A} \subset \bar{V} \subset U$, from which it follows that $\bar{A}$ is bounded.

A bounded set in any linear topological space is characterized by the following condition.

Theorem. *A set A in a linear topological space Φ is bounded if and only if, for any sequence $\{\varphi_\nu\} \subset A$, the sequence $\{\varphi_\nu/\nu\}$ converges to zero.*

Proof. Suppose that A is bounded, $\varphi_\nu \in A$ is any sequence, and U is a fixed normal neighborhood of zero. We find an integer ν_0 such that $(1/\nu)A \subset U$ for $\nu \geqslant \nu_0$. In particular, then, for such ν we have $(1/\nu)\varphi_\nu \in (1/\nu)A \subset U$. Consequently, the sequence $\{\varphi_\nu/\nu\}$ tends to zero in Φ.

Assume now that A is not bounded. Then for some neighborhood U of zero and any ν, one can find an element $\varphi_\nu \in A$ not belonging to νU. In other words, φ_ν/ν does not lie in U for any ν. But this means that the sequence $\{\varphi_\nu/\nu\}$ does not tend to zero in Φ, as was asserted.

4. Continuous Linear Functionals and the Conjugate Space

We recall that generalized functions, which we defined in the first volume, were defined as continuous linear functionals on certain basic spaces. *The problem of studying generalized functions, is, therefore, the* *problem of studying continuous linear functionals.* In this section, we consider continuous linear functionals on general linear topological spaces and, in particular, on countably normed spaces.

4.1. Definition

A numerical function $f(\varphi) = (f, \varphi)$, defined on a linear topological space Φ, is called a *continuous linear functional* if the following conditions are fulfilled:

(a) For any elements φ_1, φ_2 and numbers α_1, α_2, we have

$$(f, \alpha_1\varphi_1 + \alpha_2\varphi_2) = \alpha_1(f, \varphi_1) + \alpha_2(f, \varphi_2) \qquad \text{(linearity)};$$

in particular, $(f, 0) = 0$;

(b) for any $\epsilon > 0$ there is a neighborhood U of zero such that

$$|(f, \varphi)| < \epsilon \quad \text{for} \quad \varphi \in U \qquad \text{(continuity[11])}.$$

This last inequality shows that *every continuous linear functional is bounded on some neighborhood of zero.* Conversely, *any linear functional which is bounded on some neighborhood of zero is continuous.* Indeed, if the moduli of the values of a linear functional f on the neighborhood U of zero are bounded by M, then given $\epsilon > 0$ it is easy to exhibit a neighborhood of zero in which these moduli do not exceed ϵ; as such a neighborhood it is sufficient to take $(\epsilon/M)U$.

Example. We consider, on the space $K(a)$ of all infinitely differentiable functions $\varphi(x)$ which vanish outside the interval $|x| \leqslant a$, the functional

$$(f, \varphi) = \int_{-a}^{a} \varphi^{(m)}(x)\, d\mu(x), \tag{1}$$

where $\mu(x)$ is a function of bounded variation, and m is a fixed number. Evidently the functional (1) is linear.

Let us show that the functional (1) is continuous. We consider the neighborhood of zero

$$\| \varphi \|_m = \max\{| \varphi(x)|, \ldots, | \varphi^{(m)}(x)|\} < \epsilon.$$

[11] This actually defines the continuity of f at the point $\varphi = 0$. But a linear functional which is continuous at 0 is continuous at any point φ_0, since

$$(f, \varphi) - (f, \varphi_0) = (f, \varphi - \varphi_0).$$

Within this neighborhood the values of the functional f are bounded:

$$|(f, \varphi)| \leqslant \max | \varphi^{(m)}(x)| \cdot \operatorname{var} \mu < \epsilon \operatorname{var} \mu;$$

hence, as we have seen, the continuity of the functional follows.

We shall show somewhat later (Section 4.3) that the formula (1) yields the general form of a continuous linear functional on the space $K(a)$.

As we know, the balls $\| \varphi \|_p < \epsilon$ constitute a basis of the neighborhoods of zero in a countably normed space Φ. But the boundedness of a functional on such a neighborhood of zero is equivalent to its boundedness relative to the norm $\| \cdot \|_p$, i.e., to the inequality

$$|(f, \varphi)| \leqslant C \| \varphi \|_p .$$

Therefore, *every continuous linear functional f on a countably normed space Φ is bounded relative to some norm $\| \cdot \|_p$* . The least p is called the *order of the functional f*. The converse is also evident: *If a linear functional f on the space Φ is bounded relative to some norm, then it is continuous.*

In spaces with the first axiom of countability, where the topology is described by means of (countable) convergent sequences, it is natural to define the continuity of a linear functional "by means of sequences." The following simple theorem establishes this possibility.

Theorem. *For the continuity of a linear functional f, it is necessary and (in spaces with the first axiom of countability) sufficient that* $\lim_{\nu\to\infty} (f, \varphi_\nu) = 0$ *follow from* $\lim_{\nu\to\infty} \varphi_\nu = 0$.

Proof. *Necessity.* Let $\varphi_\nu \to 0$ and let $\epsilon > 0$ be given; moreover, let U be a neighborhood of zero of the space Φ in which $|(f, \varphi)| < \epsilon$; starting with some $\nu = \nu_0$, we will have $\varphi_\nu \in U$, and hence $|(f, \varphi_\nu)| < \epsilon$, q.e.d.

Sufficiency. Let $\lim_{\nu\to\infty}(f, \varphi_\nu) = 0$ always follow from $\lim_{\nu\to\infty} \varphi_\nu = 0$; we will show that the functional f is continuous. If this were not so, the functional f would not be bounded in any neighborhood of zero. Let us consider a countable basis $U_1 \supset U_2 \supset \cdots \supset U_\nu \supset \cdots$ of the neighborhoods of zero and find an element φ_ν in the neighborhood U_ν for which $|(f, \varphi_\nu)| > 1$. The sequence $\{\varphi_\nu\}$ tends to zero, but $\lim_{\nu\to\infty}(f, \varphi_\nu) \neq 0$, in contradiction with the assumption. This means that the functional f is bounded in some neighborhood of zero, and, therefore, is bounded.

4.2. Question of the Existence of Continuous Linear Functionals

In general, no nontrivial (i.e., not identically equal to zero) continuous linear functional can exist on an arbitrary linear topological space.

In *normed* spaces, the existence of a nontrivial (different from zero at a given element $\varphi \neq 0$) functional is a consequence of the well-known Hahn–Banach theorem[12] on the extension of a bounded linear functional, defined on an arbitrary subspace, to the whole space.

On a countably normed space Φ, for each nonzero element φ, there also exists a continuous linear functional which is nonzero on φ. This follows from the fact that the Hahn–Banach process may also be applied to a countably normed space. Indeed, let $H \subset \Phi$ be a subspace on which a bounded linear functional f is given. The space H, as a part of the countably normed space Φ, is also a countably normed space (with the same norms). The bounded functional f is bounded relative to some norm in the space H, say the norm $\| \varphi \|_p$ (see Section 4.1). But this norm is also defined on the whole space Φ. Applying the Hahn–Banach theorem here, we may extend the functional f from the subspace H to the whole space Φ so that it remains bounded relative to the pth norm. It is thus bounded in Φ, and therefore continuous.

4.3. Conjugate Space

Linear functionals on a linear topological space Φ may, in turn, be added and multiplied by numbers according to the natural formula

$$(\alpha_1 f_1 + \alpha_2 f_2 , \varphi) = \alpha_1(f_1 , \varphi) + \alpha_2(f_2 , \varphi). \tag{1}$$

Evidently the functional $\alpha_1 f_1 + \alpha_2 f_2$, defined by (1), is again a linear functional, and moreover continuous if f_1 and f_2 are continuous.

Thus, the set of all continuous linear functionals on the space Φ again forms a linear space. We will say that it is *conjugate* to Φ and denote it by Φ'.

If Φ is a normed space, then Φ', as is known, is a complete normed space with the norm

$$\| f \| = \sup_{\| \varphi \| \leqslant 1} |(f, \varphi)|.$$

If Φ is not a normed, but a countably normed space, then the structure of the conjugate space Φ' may be described as follows.

As we have elucidated in Section 4.1, each continuous linear functional on the space Φ has finite order, i.e., is bounded relative to some norm $\| \cdot \|_p$:

$$|(f, \varphi)| \leqslant C\| \varphi \|_p .$$

[12] See, for example, L. A. Lyusternik and V. I. Sobolev, "Elements of Functional Analysis." Ungar, New York, 1961.

The set of all continuous linear functionals of order $\leqslant p$, i.e., functionals continuous relative to the norm of the space Φ_p, form a subspace Φ'_p in Φ'; this subspace is conjugate to the normed space Φ_p, and hence, is a complete normed space. We therefore have

$$\Phi' = \bigcup_{p=1}^{\infty} \Phi'_p. \tag{2}$$

Evidently, a linear functional of order p is bounded on the balls $\| \varphi \|_{p+1} \leqslant 1$, $\| \varphi \|_{p+2} \leqslant 1$,...; hence, a functional of order p also belongs to the spaces Φ'_{p+1}, Φ'_{p+2},.... We obtain a chain of inclusions

$$\Phi'_1 \subset \Phi'_2 \subset \cdots \subset \Phi'_p \subset \cdots \subset \Phi'. \tag{3}$$

As an element of the normed spaces Φ'_p, Φ'_{p+1},... the functional f of order p has the respective norms

$$\| f \|_p = \sup_{\| \varphi \|_p \leqslant 1} |(f, \varphi)|, \qquad \| f \|_{p+1} = \sup_{\| \varphi \|_{p+1} \leqslant 1} |(f, \varphi)|, ...$$

in them; evidently

$$\| f \|_p \geqslant \| f \|_{p+1} \geqslant \cdots. \tag{4}$$

It would be possible to start this chain of inequalities with the value $p = 1$, by setting (by convention) $\| f \|_1 = \| f \|_2 = \cdots = \| f \|_{p-1} = \infty$.
In summary, we obtain: *The space Φ', conjugate to the countably normed space Φ, is the union of an increasing sequence of complete normed spaces with weaker and weaker norms.*

Example. Let us find the general form of a continuous linear functional on the space $K(a)$ of all infinitely differentiable functions $\varphi(x)$ on the line $-\infty < x < \infty$, which vanish outside the interval $| x | \leqslant a$. As we have seen in Section 3.3, $K(a)$ is a complete countably normed space with the norms

$$\| \varphi \|_p = \max_{|x| \leqslant a} \{ | \varphi(x) |, ..., | \varphi^{(p)}(x) | \} \qquad (p = 0, 1, 2, ...).$$

According to what has been proved, each continuous linear functional on the space $K(a)$ is simultaneously a continuous linear functional on the normed space $K_p(a)$, which has been obtained by providing the space $K(a)$ with some norm $\| \varphi \|_p$. Hence, the problem reduces to determining the general form of a continuous linear functional on the space $K_p(a)$. As has already been shown in Section 3, this space consists of all functions $\varphi(x)$, possessing continuous derivatives up to order p and vanishing outside the interval $|x| \leqslant a$. If we associate, with each element $\varphi(x)$ of the space $K_p(a)$, the continuous function $\psi(x) = \varphi^{(p)}(x)$, we obtain a mapping of the space $K_p(a)$ onto part of the space $C(a)$ of all continuous functions on the interval $|x| \leqslant a$. The inequality

$$\begin{aligned} \| \varphi \|_p &= \max\{|\varphi(x)|, \dots, |\varphi^{(p)}(x)|\} \\ &\leqslant C \max\{|\varphi^{(p)}(x)|\} = C \max |\psi(x)| \leqslant C \| \varphi \|_p \end{aligned}$$

shows that this mapping is one-to-one and bicontinuous, and thus the image of the space $K_p(a)$ is a closed subspace of the space $C(a)$. Hence, a continuous linear functional (f, φ), defined on the space $K_p(a)$, will generate, at the same time, a continuous linear functional $(g, \psi) = (f, \varphi)$ on the indicated subspace of the space $C(a)$.

According to the Hahn–Banach theorem, this functional may be extended to the whole space $C(a)$. Now applying the Riesz theorem,[13] we conclude that the functional in question may be written as

$$(g, \psi) = \int_{-a}^{a} \psi(x)\, d\mu(x),$$

where $\mu(x)$ is some function of bounded variation. We thus obtain the formula

$$(f, \varphi) = (g, \psi) = \int_{-a}^{a} \varphi^{(p)}(x)\, d\mu(x),$$

which yields the general form of a continuous linear functional on the space $K_p(a)$. For different p and $\mu(x)$, this same formula yields the general form of a continuous linear functional on the space $K(a)$.

Utilizing the representation of the space $K(a)$ as the intersection of normed spaces of other types, other forms for the linear functionals on this space may be obtained. We shall return to this question from a more general point of view in Chapter II.

For the space $K(a)$ of all infinitely differentiable functions of n variables $x_1, \dots, x_n$ (Section 3.3) which vanish outside the domain $G_a = \{|x_1| \leqslant a_1, \dots |x_n| \leqslant a_n\}$ a similar argument[14] leads to the following general form of a continuous linear functional:

$$(f, \varphi) = \int_{G_a} \frac{\partial^{pn}\varphi(x)}{\partial x_1{}^p \cdots \partial x_n{}^p}\, d\mu,$$

where μ is some completely additive set function in the domain G_a.

[13] L. A. Lyusternik and V. I. Sobolev, "Elements of Functional Analysis." Ungar, New York, 1961.

[14] With regard to application of the Riesz–Radon theorem, see F. Riesz and B. Sz.-Nagy, "Lectures on Functional Analysis." Ungar, New York, 1955.

4.4. Connection between the Continuity of a Linear Functional and Its Boundedness on Bounded Sets

As is known, the continuity condition in the definition of a continuous linear functional on a normed space may be replaced by the condition that the set of values of the functional on the unit sphere be bounded. In other words, a linear functional on a normed space is continuous if it is bounded on the unit sphere, and conversely.

With the intent of transferring this property to a more general space, let us first note that here, in general, it is impossible to specify a single such subset on which boundedness would guarantee continuity of the linear functional. A whole family of bounded subsets must here be considered instead of just one subset.

Let us recall that in a general linear topological space, a set B is said to be bounded if it lies in any previously assigned neighborhood of zero after having been multiplied by some number $\lambda > 0$.

Let us now examine how continuous linear functionals are related to bounded sets of the space Φ.

(a) *Every continuous linear functional f is bounded on each bounded set B.*

In fact, the functional f is bounded on some neighborhood U of zero, so that

$$|(f, \varphi)| \leqslant C$$

for $\varphi \in U$. Since B is bounded, we have, for some $\lambda > 0$,

$$\lambda B \subset U, \qquad B \subset \frac{1}{\lambda} U.$$

Hence, for $\varphi \in B \subset (1/\lambda) U$,

$$|(f, \varphi)| \leqslant \frac{1}{\lambda} C,$$

q.e.d.

(b) *If the first axiom of countability (each point has a countable neighborhood basis) is satisfied in the space Φ, then every linear functional which is bounded on every bounded set is continuous.*

In fact, let $U_1 \supset U_2 \supset \cdots \supset U_p \supset \cdots$ be a countable basis of the neighborhoods of zero in the space Φ. If the functional f is not continuous, then it is unbounded on each of these neighborhoods; a point φ_ν may be found in the neighborhood U_ν such that $|(f, \varphi_\nu)| > \nu$. The sequence φ_ν tends to zero, and hence is bounded; by assumption, the sequence of numbers (f, φ_ν) should be bounded, but by construction it is not. The contradiction shows that the functional f is continuous, q.e.d.

We shall call a functional *bounded* if it is bounded on each bounded set. Therefore, in a space satisfying the first axiom of countability, a linear functional is continuous if and only if it is bounded.

Example. Let us consider the question of the continuity of the direct product of the functionals $f(x)$ and $g(y)$ (Vol. 1, Ch. I, Section 4). Let us recall the definition referred to here. We consider the spaces $K_x(a)$, $K_y(a)$, and $K_{xy}(a)$ consisting of all infinitely differentiable functions of x, y, or the pair x, y which vanish outside the intervals $|x| \leqslant a$, $|y| \leqslant a$, or $|x|, |y| \leqslant a$, respectively. Let $f(x)$ and $g(y)$ be continuous linear functionals on $K_x(a)$ and $K_y(a)$, respectively. The functional $h(x, y)$, defined on the space $K_{xy}(a)$ by the formula

$$(h, \varphi(x, y)) = (f(x), (g(y), \varphi(x, y))), \tag{1}$$

is called the direct product of the functionals $f(x)$ and $g(y)$. We note that (1) has meaning since the function $(g(y), \varphi(x, y)) = \psi(x)$, as a function of x, is evidently infinitely differentiable. It is also evident that the functional h, defined by (1), is a linear functional. Let us show that this functional is also continuous. It is sufficient to show that it is bounded, i.e., that it transforms a bounded set of functions $\varphi(x, y)$ into a bounded set of numbers. To do this it is sufficient to show that the set of functions $(g(y), \varphi(x, y)) = \psi(x)$ is bounded in the space $K_x(a)$, which, in turn, means that for any p the expressions $D_x{}^p\psi(x) = (g(y), D_x{}^p\varphi(x, y))$ are uniformly bounded in x. But the boundedness of the set of functions $\varphi(x, y) \in K_{xy}(a)$ means the boundedness of the quantities $\sup_{x,y}| D^p\varphi(x, y)|$ for any p. It follows that the set of functions $D_x{}^p\varphi(x, y)$, as a subset of the space $K_y(a)$, is bounded in this space; hence, it is transformed by the bounded functional $g(y)$ into a uniformly bounded (in x) set of numbers $(g(y), D_x{}^p\varphi(x, y))$, q.e.d.

4.5. Characterization of a Bounded Set in a Countably Normed Space

We saw in the previous paragraph that a continuous linear functional on a linear topological space is bounded on each bounded set. In this paragraph we prove the converse theorem for countably normed spaces, which will thereby give us a new characterization of bounded sets in a countably normed space.

Let us show first that *a set B in a normed space Φ is bounded if every continuous linear functional is bounded on B.*

It is sufficient to show that *all functionals whose norms do not exceed some* $\epsilon > 0$ *are bounded on* B *by the constant* 1. Indeed, by using the Hahn–Banach theorem, for any element $\varphi \in B$ a continuous linear functional f may be constructed such that $\| f \| = \epsilon$, $(f, \varphi) = \epsilon \| \varphi \|$; if the statement formulated above is proved, we would then have $(f, \varphi) = \epsilon \| \varphi \| \leqslant 1$, whence $\| \varphi \| \leqslant (1/\epsilon)$, q.e.d.

Let us consider the set F of all functionals f for which the inequality

$$|(f, \varphi)| \leqslant 1$$

holds for all $\varphi \in B$.

The set F possesses the following properties:

(1) F is closed (in the norm) in the space Φ';

(2) F is convex: If $f = \frac{1}{2}(f_1 + f_2), f_1 \in F, f_2 \in F$, then

$$\begin{aligned} |(f, \varphi)| &= |(\tfrac{1}{2}(f_1 + f_2), \varphi)| \\ &= \tfrac{1}{2}|(f_1, \varphi) + (f_2, \varphi)| \leqslant \tfrac{1}{2}|(f_1, \varphi)| + \tfrac{1}{2}|(f_2, \varphi)| \leqslant 1; \end{aligned}$$

(3) Each functional $f \in \Phi'$, after being multiplied by a sufficiently small constant $\lambda > 0$, will be in the set F.

Indeed, the function (f, φ) is by hypothesis bounded on the set B; if, for definiteness, $|(f, \varphi)| \leqslant C$ on B, then evidently $|(1/C)f, \varphi)| \leqslant 1$ on B, from which $(1/C)f \in F$.

Since the space Φ' is normed, the lemma of Section 3.4 may be applied, which leads to the conclusion that the set F contains some neighborhood of zero of the space Φ'. If, say, F contains the ball $\| f \| \leqslant \epsilon$, this means that every functional f with norm not exceeding ϵ is bounded by unity on the set B. As we saw, this fact is sufficient to prove the validity of the theorem.

Let us now show that the analogous fact is also valid for a countably normed space.

If the values of every continuous linear functional are bounded on a subset B *of a countably normed space* Φ, *then* B *is bounded in* Φ.

It is sufficient to show that the norm $\| \cdot \|_p$ is bounded on B for any fixed p. We may consider B a subset of the normed space Φ_p. By hypothesis, all continuous linear functionals are bounded on B, in particular, all functionals of order p, which form the space Φ_p' conjugate to Φ_p. But then the set B is bounded in Φ_p; this means that the $\| \varphi \|_p$ are bounded for $\varphi \in B$, q.e.d.

5. Topology in a Conjugate Space

One of the basic operations which we carried out with generalized functions (Volume 1) was the passage to the limit. We said that a sequence of generalized functions f_ν converges to the generalized function f, if for any fundamental function φ, the relationship

$$(f, \varphi) = \lim_{\nu\to\infty} (f_\nu , \varphi)$$

holds.

In this section we consider the question of defining a topology, and in particular, of defining passage to the limit in the space Φ' of continuous linear functions on the linear topological space Φ.

This general formulation will, in particular, enable us to prove the theorem on the completeness of a space of generalized functions relative to the above-mentioned convergence, a theorem which was used repeatedly in Volume 1, but was not proved there. In this section, we obtain a proof of an analogous theorem for spaces conjugate to the countably normed spaces; we shall prove it in Section 8 for the more general case of a union of countably normed spaces as well; in particular, its validity for the space K' of generalized functions considered in Volume 1 will thereby follow.

Thus, our problem is to introduce a topology in the space Φ' of continuous linear functionals on a given linear space Φ.

It turns out that a topology may be introduced by various methods in the space Φ'. The *strong* and *weak* topologies are the principal ones.

5.1. Strong Topology

Let us first consider the case of the space Φ' conjugate to a normed space Φ. In this case, the space Φ' is also a normed (complete) space with the norm

$$\|f\| = \sup_{\|\varphi\|\leqslant 1} |(f, \varphi)|. \tag{1}$$

The norm (1) defines the strong topology in the space Φ'.

A neighborhood of zero in this topology may be defined as the set of all $f \in \Phi'$ for which $\sup|(f, \varphi)| < \epsilon$, when φ runs over the unit ball $\{\| \varphi \| \leqslant 1\}$ in the space Φ.

In the general case of a linear topological space Φ, *all possible bounded sets* in the space Φ should be considered, instead of the unit ball $\| \varphi \| \leqslant 1$. This leads us naturally to the following definition of a system of neighborhoods in the space Φ' which defines the strong topology.

Definition. *A strong neighborhood of the zero functional* 0 *is defined, by means of any bounded set* $A \subset \Phi$ *and any number* $\epsilon > 0$, *as the set of all* $f \in \Phi'$ *for which*

$$\sup_{\varphi \in A} |(f, \varphi)| < \epsilon.$$

Let us verify that Axioms IV.1–IV.7 for the neighborhoods of zero of a linear topological space (Section 1.2) are fulfilled.

Axiom IV.1 requires that the intersection of two neighborhoods U and V should contain a third neighborhood W. Let U be defined by the bounded set $A_U \subset \Phi$ and the number $\epsilon_U > 0$ and, analogously, let V be defined by the set $A_V \subset \Phi$ and the number $\epsilon_V > 0$. We construct a bounded set A, the union of A_U and A_V, and define the neighborhood W as the set of those $f \in \Phi'$ for which

$$|(f, \varphi)| < \min(\epsilon_U, \epsilon_V) \qquad \text{for all} \quad \varphi \in A.$$

Evidently W is contained in the intersection of U and V.

Axiom IV.2 requires that for any point $f_0 \neq 0$, there be a neighborhood which does not contain the functional f_0. Such a neighborhood may be defined as follows: Take an element φ_0, for which $(f_0, \varphi_0) \neq 0$, for example $(f_0, \varphi_0) = a \neq 0$, and define the neighborhood U by the inequality

$$|(f, \varphi_0)| < \frac{|a|}{2}.$$

Evidently the neighborhood U has the required property.

Axiom IV.3 requires that for any neighborhood U there be a neighborhood V such that $V \pm V \subset U$. If U is defined by the bounded set $A_U \subset \Phi$ and the number $\epsilon > 0$, then the neighborhood V may be defined by the same set A_U and the number $\epsilon/2$.

Axiom IV.4 requires that for any neighborhood U and any $f \in U$, there exist a neighborhood V such that $f + V \subset U$. Let U be defined by the bounded set A_U and the number ϵ_U. Since $\sup_{\varphi \in A_U} |(f, \varphi)| = \eta < \epsilon_U$, it is sufficient to define the neighborhood V by the same set A_U and the number $\epsilon_U - \eta$.

Axiom IV.5 requires that for any neighborhood U and any number α, there exist a neighborhood V such that $\alpha V \subset U$. If U is defined by the set $A_U \subset \Phi$ and the number $\epsilon > 0$, then V may be defined by the same set A_U and the number ϵ/α.

Axiom IV.6 requires that for any neighborhood U and any functional f there be a number $\eta > 0$ such that $\delta f \in U$ for $|\delta| < \eta$. Let the neighborhood U be defined by the bounded set $A_U \subset \Phi$ and the number $\epsilon > 0$.

The functional f, as a continuous functional, is bounded on the set A_U by some number C. It is evident that the functional $(\epsilon/C)f$ is bounded on the set A_U by the number ϵ. Hence, for $|\delta| < \eta = (\epsilon/C)$, the functional δf lies in the neighborhood U.

Axiom IV.7 requires that for any neighborhood U there exist a number $\eta > 0$ such that $\delta U \subset U$ for $|\delta| < \eta$. This axiom is satisfied in the present case with $\eta = 1$; evidently all the neighborhoods of zero we introduced in the space Φ' are normal.

In general, the strong topology cannot be defined by specifying a countable basis of neighborhoods of zero, and therefore the description of the passage to the limit by means of countable sequences will not be the complete equivalent of topological relationships;[15] nevertheless, countable sequences will play an important part in the sequel.

The convergence of (countable) sequences, defined by the strong topology in Φ', is called *strong convergence*. In other words, a sequence of functionals $f_1, f_2, \ldots, f_\nu, \ldots$ converges strongly to the functional f, if $\{(f_\nu, \varphi)\}$ converges uniformly to (f, φ) on each bounded set of elements φ from Φ.

As is known, in the case of the space Φ' conjugate to a normed space Φ, one has completeness relative to strong convergence. The analogous result is valid in the case of the space Φ' conjugate to a general linear topological space Φ, under the assumption that the first axiom of countability is satisfied in Φ. We naturally say that *the space Φ' is complete with respect to strong convergence* if every strongly fundamental (countable) sequence $f_1, f_2, \ldots, f_\nu, \ldots$ converges strongly to some functional f. Here, a sequence f_ν is said to be *strongly fundamental*, if the sequence of numbers (f_ν, φ) converges for each element φ, and indeed uniformly on each bounded set.

Theorem. *If the first axiom of countability is satisfied in the space Φ, then the space Φ' is complete with respect to strong convergence.*

Proof. Let $f_1, f_2, \ldots, f_\nu, \ldots$ be a strongly convergent sequence of functionals. Then, in particular, for each φ from Φ the sequence (f_ν, φ) $(\nu = 1, 2, \ldots)$ has a limit; we denote this limit by (f, φ).

The quantity (f, φ) is evidently a linear functional on Φ; let us verify that this functional is continuous. It is sufficient to show that (f, φ) is bounded on every bounded set $A \subset \Phi$. But the functions (f_ν, φ) are bounded and converge uniformly to (f, φ) on each such set A; hence, the limit function (f, φ) is also bounded on A, q.e.d.

[15] That is, in general f_0 may be an accumulation point of the set $F \subset \Phi'$ and yet not be the limit of some (countable) sequence of points $f_\nu \in F$.

5.2. Strongly Bounded Sets

The strong topology in the space Φ' enables us to distinguish the corresponding class of bounded sets in this space, which we shall call *strongly bounded.* Namely, in conformity with the general definition of a bounded set (Section 3.8), we shall call a set $B \subset \Phi'$ strongly bounded if every strong neighborhood of zero $U \subset \Phi'$ absorbs the set B, i.e., there exists $\lambda > 0$ for which $\lambda B \subset U$.

A somewhat less formal definition of a strongly bounded set may also be given. To do this, let us agree to say that a given set B of functionals is *bounded on the set A of fundamental elements* if the set of numbers $|(f, \varphi)|$, where f runs through the set B and φ runs through the set A, is bounded.

Let us now show that *a set B is strongly bounded if and only if it is bounded on each bounded set A.*

In fact, let $B \subset \Phi'$ be strongly bounded, and let a bounded set $A \subset \Phi$ be given. We consider the strong neighborhood $U \subset \Phi'$ consisting of those $f \in \Phi'$ for which

$$\sup_{\varphi \in A} |(f, \varphi)| < 1.$$

By hypothesis, there exists a $\lambda > 0$ for which $\lambda B \subset U$. This means that for any $f \in B$ we have $|(\lambda f, \varphi)| < 1$. But then, for all $f \in B$, $\varphi \in A$,

$$|(f, \varphi)| < \frac{1}{\lambda},$$

that is, the set B is bounded on the bounded set A.

Conversely, let the set $B \subset \Phi'$ be bounded on every bounded set $A \subset \Phi$. We shall show that B is strongly bounded. Let U be a strong neighborhood of zero in the space Φ'. This means that a bounded set $A \subset \Phi$ and a number $\epsilon > 0$ are given; the neighborhood U consists of those $f \in \Phi'$ for which, with $\varphi \in A$,

$$\sup_{\varphi} |(f, \varphi)| < \epsilon.$$

By hypothesis, the numbers $|(f, \varphi)|$ are bounded on the set A for $f \in B$, say by the constant C. But then for $f \in B$

$$\left| \left(\frac{\epsilon}{2C} f, \varphi \right) \right| < \frac{\epsilon}{2C} C = \frac{\epsilon}{2}.$$

Therefore, for $\lambda = (\epsilon/2C)$ we obtain $\lambda B \subset U$.

The following lemma, which is valid when the first axiom of countability holds for the space Φ, is quite useful; it shows that the family of functionals f belonging to a strongly bounded set $B \subset \Phi'$ is bounded not only on the bounded sets $A \subset \Phi$, *but also on some neighborhood of zero in the space* Φ (however, not on every neighborhood). Let us recall that the neighborhoods of zero are generally unbounded sets.

Lemma. *If Φ satisfies the first axiom of countability, then every strongly bounded set $B \subset \Phi'$ is bounded on some neighborhood of zero of the space Φ.*

Proof. Let $U_1 \supset U_2 \supset \cdots$ be a basis of the neighborhoods of zero in the space Φ. If the assertion of the lemma is not fulfilled, then for each ν we can find a point φ_ν in the neighborhood U_ν and a functional $f_\nu \in B$ such that $|(f_\nu, \varphi_\nu)| > \nu$. The sequence φ_ν $(\nu = 1, 2,...)$ tends to zero, and is therefore bounded. By virtue of the characterization of bounded sets in the space Φ' just obtained, we must have $|(f, \varphi_\nu) \leqslant C$ for all $f \in B$. The contradiction convinces us of the validity of the lemma.

The converse of this statement is valid without any assumptions of countability: If a set $B \subset \Phi'$ is bounded on some neighborhood U of zero of the space Φ, then it is bounded on every set λU, and, consequently, on any bounded set of the space Φ.

5.3. Strongly Bounded Sets in a Space Conjugate to a Countably Normed Space

The results obtained permit, in particular, a complete description of strongly bounded sets in a space conjugate to a countably normed space. Let us recall that every (complete) countably normed space Φ is the intersection of a decreasing sequence of normed spaces

$$\Phi_1 \supset \Phi_2 \supset \cdots \supset \Phi_p \supset \cdots \supset \Phi,$$

and the conjugate space Φ' is the union of the increasing sequence of complete normed spaces Φ'_p, conjugate to the spaces Φ_p:

$$\Phi'_1 \subset \Phi'_2 \subset \cdots \subset \Phi'_p \subset \cdots \subset \Phi'.$$

The structure of strongly bounded sets in the space Φ' is described by the following theorem.

Theorem. *If Φ is a countably normed space, then a set $B \subset \Phi'$ is strongly bounded if and only if B is contained in some Φ'_p and is bounded in its norm.*

Proof. Let the set B satisfy the mentioned condition, i.e., belong to the space $\Phi'_{p'}$ and be bounded in its norm. We consider the neighborhood U of zero of the space Φ defined by the inequality

$$\| \varphi \|_p \leqslant 1.$$

If B is bounded in the norm of Φ'_p by the number M, then according to the definition of the norm in a space conjugate to a normed space, the inequality

$$|(f, \varphi)| \leqslant M$$

holds if $\varphi \in U, f \in B$. Thus, B is bounded on some neighborhood of zero of the space Φ and is therefore strongly bounded (Section 5.2).

Conversely, let B be strongly bounded. Then by virtue of Lemma 1, there is a neighborhood U of zero of the space Φ defined, say, by the inequality

$$\| \varphi \|_p < \epsilon,$$

on which the function $\sup_{f \in B} |(f, \varphi)|$ is bounded by the number M, say. This means that each functional $f \in B$ is bounded by the number M on the neighborhood U. But then the functional f belongs to the space Φ'_p and has norm $< M/\epsilon$ in this space, q.e.d.

5.4. Weak Topology

Let us turn now to the definition of the weak topology in the space Φ'. This topology corresponds to convergence of the functionals on each element of the fundamental space Φ.

Definition. *A weak neighborhood of zero in the space Φ' is defined by a finite set $\varphi_1, \varphi_2, \dots, \varphi_m$ of elements of the space Φ and a number $\epsilon > 0$, and consists of all $f \in \Phi'$ for which*

$$|(f, \varphi_1)| < \epsilon, \dots, |(f, \varphi_m)| < \epsilon.$$

The verification of compliance with Axioms IV.1–IV.7 of Section 1.2 is carried out exactly as for the strong topology, with the replacement of bounded subsets of the space Φ by finite subsets.

Every weakly open set is also open in the strong sense. It is sufficient to verify this for a weak neighborhood of zero. If any bounded set A of elements $\varphi \in \Phi$ is given, then the set of all functionals f satisfying the inequalities

$$\sup_{\varphi \in A} |(f, \varphi)| < \epsilon \tag{1}$$

is, by definition, a strong neighborhood of zero in the space Φ'; in particular, if we take a finite set of elements $\varphi_1, \varphi_2, \ldots, \varphi_m$ as the set A, then some strong neighborhood of zero is obtained. Hence, the weak neighborhood of zero which is defined by the inequalities (1) when φ runs through a finite set of elements is a strong neighborhood of zero and, hence, a strongly open set.

With the exception of trivial cases, this topology does not satisfy the first axiom of countability, so that the limit relations in this topology cannot be completely described by the language of sequences. The appropriate convergence of (countable) sequences plays a very large role; it is called *weak convergence*. Evidently, weak convergence is convergence on each fundamental element; in other words, a sequence of functionals f_ν converges weakly to a functional f if and only if the limit relationship

$$(f_\nu, \varphi) \to (f, \varphi)$$

holds for every $\varphi \in \Phi$.

The following criterion for the weak convergence of linear functionals is valid in normed spaces: a sequence $f_\nu \in \Phi'$ converges weakly to zero if and only if the functionals f_ν ($\nu = 1, 2, \ldots$) are bounded in norm, and the relationship $\lim_{\nu\to\infty}(f_\nu, \varphi) = 0$ is satisfied at least for the elements φ belonging to some set A which is dense in Φ.

It turns out that an analogous criterion is valid in countably normed spaces. The sufficiency of the corresponding conditions is asserted by the following lemma.

Lemma. *Let $\{f_\nu\}$ be a strongly bounded sequence of continuous linear functionals defined on a countably normed space Φ. Suppose we know that for all elements φ of some dense set $A \subset \Phi$, the numbers (f_ν, φ) tend to zero. Then the numbers (f_ν, φ) tend to zero for any $\varphi \in \Phi$.*

Proof. As we have shown in Section 5.3, a strongly bounded sequence $\{f_\nu\}$ is contained in some Φ_p' and is bounded in the norm of Φ_p'. For example, let $\| f_\nu \|_p \leqslant M$. The set A is dense in Φ in the topology of Φ, in particular, in the norm $\| \varphi \|_p$; for a given $\varphi \in \Phi$ and $\epsilon > 0$, a $\varphi_\epsilon \in A$ may always be found such that $\| \varphi - \varphi_\epsilon \|_p \leqslant (\epsilon/2M)$. We find a number ν_0 such that for $\nu > \nu_0$ we will have $|(f_\nu, \varphi_\epsilon)| \leqslant (\epsilon/2)$. Then for $\nu > \nu_0$, we obtain

$$|(f_\nu, \varphi)| = |(f_\nu, \varphi_\epsilon) + (f_\nu, \varphi - \varphi_\epsilon)| \leqslant \frac{\epsilon}{2} + M\frac{\epsilon}{2M} = \epsilon,$$

from which

$$\lim_{\nu\to\infty} (f_\nu, \varphi) = 0.$$

We shall establish the necessity of these conditions somewhat later.

5.5. Weakly Bounded Sets

The weak topology in the space Φ' enables us to distinguish the corresponding class of "weakly bounded" sets: A set $B \subset \Phi'$ is said to be *weakly bounded* if for any weak neighborhood of zero $U \subset \Phi'$, there exists $\lambda > 0$ for which $\lambda B \subset U$.

But it turns out that for *spaces conjugate to countably normed spaces*, we do not here obtain a new class of sets: *The class of weakly bounded sets coincides with the class of strongly bounded sets.* A similar fact holds for normed spaces, where it plays a fundamental role in the study of conjugate spaces.

Evidently every strongly bounded set $B \subset \Phi'$ is always weakly bounded, since each weak neighborhood of zero $U \subset \Phi'$ into which the set B must be carried by multiplication by some number is also a strong neighborhood of zero. We shall prove that the converse is valid in the case of a countably normed space Φ: *Every weakly bounded set $B \subset \Phi'$ is strongly bounded.*

It is sufficient to show that a weakly bounded set B is bounded on some neighborhood of zero of the space Φ. Let us consider the set A of all elements $\varphi \in \Phi$, for which

$$|(f, \varphi)| \leqslant 1 \qquad \text{for} \quad f \in B.$$

The set A has the following properties:

(1) A is closed in the topology of the space Φ (as the intersection, over all $f \in B$, of theconvex sets $|(f, \varphi)| \leqslant 1$);

(2) A is convex in Φ (as the intersection, over all $f \in B$, of the convex sets $|(f, \varphi)| \leqslant 1$) and is centrally symmetric;

(3) Each element φ, on being multiplied by a sufficiently small $\lambda > 0$, will lie in the set A. In fact, since the set B is weakly bounded, the numbers (f, φ) are bounded for any fixed φ; for example, let

$$|(f, \varphi)| \leqslant C,$$

for a given φ and all $f \in B$; then

$$\left| \left(f, \frac{1}{C} \varphi\right) \right| \leqslant 1, \qquad \text{i.e.,} \quad \frac{1}{C} \varphi \in A.$$

We now apply the lemma of Section 3.4; by virtue of this lemma, the set A contains some neighborhood of zero of the space Φ.

Thus, there exists a neighborhood of zero of the space Φ on which the functionals f in the set B are uniformly bounded (by the number 1). It follows that B is strongly bounded, q.e.d.

As a corollary we obtain the necessity of the condition formulated above for the weak convergence of functionals.

If a sequence of functionals $f_1, f_2, \ldots, f_\nu, \ldots$ *converges on each element* $\varphi \in \Phi$, *then this sequence is strongly bounded.*

Indeed, the sequence $f_1, f_2, \ldots, f_\nu, \ldots$, by converging on each element φ, is thereby weakly bounded. But, by what has been proved, every weakly bounded set in the space Φ' is also strongly bounded; therefore, the sequence $\{f_\nu\}$ is strongly bounded, q.e.d.

5.6. Theorem on the Completeness of the Conjugate Space of a Countably Normed Space, Relative to Weak Convergence

We can now turn to the proof of the theorem on the completeness of the conjugate space of a countably normed space, relative to weak convergence.

Theorem. *If* Φ *is a countably normed space, then the space* Φ' *is complete relative to weak convergence.*

In other words, if a (countable) sequence of continuous linear functionals $f_\nu \in \Phi'$ ($\nu = 1, 2, \ldots$) is such that the quantities (f_ν, φ) form a convergent sequence for every $\varphi \in \Phi$, then there exists a continuous linear functional f which is the weak limit of the sequence f_ν as $\nu \to \infty$: $(f, \varphi) = \lim_{\nu \to \infty}(f_\nu, \varphi)$.

Proof. The set $\{f_\nu\}$ is weakly bounded, and by what has been proved, is also strongly bounded; therefore (see the lemma of Section 5.2), the set of functionals $\{f_\nu\}$ is bounded on some neighborhood U of zero of the space Φ:

$$|(f_\nu, \varphi)| \leqslant C, \qquad \varphi \in U \qquad (\nu = 1, 2, \ldots). \tag{1}$$

We now define the functional f by the equality

$$(f, \varphi) = \lim_{\nu \to \infty} (f_\nu, \varphi).$$

Evidently the limit functional f is linear along with the functionals f_ν. Let us show that it is also a continuous functional. Since the inequality (1) is satisfied in the neighborhood U, it follows that

$$|(f, \varphi)| = \lim_{\nu \to \infty} |(f_\nu, \varphi)| \leqslant C$$

in this neighborhood, i.e., the functional f is bounded on the neighborhood U. But then, in accordance with Section 4.1, the functional f is continuous, q.e.d.

Using the results thus far obtained, the following criterion may be established for the weak convergence of functionals in the space Φ' conjugate to a countably normed space Φ.

Theorem. *A sequence of functionals f_ν ($\nu = 1, 2,\ldots$) converges weakly to a functional f if and only if all the f_ν are functionals on the same normed space Φ_p and converge weakly in this space, i.e., for every $\varphi \in \Phi_p$ the relation*

$$(f_\nu, \varphi) \to (f, \varphi)$$

holds.

Proof. If $\{f_\nu\}$ converges weakly to the functional f, then it is also strongly bounded, and according to the theorem of Section 5.3, all the f_ν lie in the same space Φ_p' and are bounded in its norm. Moreover, on a dense portion of the space Φ_p, namely, on the elements of the space Φ, the convergence of (f_ν, φ) to (f, φ) holds. Since Φ_p is a normed space, the relation

$$(f, \varphi) = \lim_{\nu\to\infty} (f_\nu, \varphi)$$

is valid for all $\varphi \in \Phi_p$.

Conversely, if the functionals f_ν all lie in the same space Φ_p', and the relation

$$(f, \varphi) = \lim_{\nu\to\infty} (f_\nu, \varphi)$$

holds for every $\varphi \in \Phi_p$, then it also holds, in particular, for all elements of the space $\Phi \subset \Phi_p$; but this means that the sequence $\{f_\nu\}$ converges weakly to the functional f in the space Φ'.

5.7. The Weak and Strong Topologies in the Initial Space

After having just considered two kinds of topology "induced" in the conjugate space by the initial space Φ, it is natural that the question should arise as to what is obtained if this construction is "turned around": To define a weak and strong topology in the fundamental space Φ by utilizing bounded sets of the conjugate space. (In the case of a countably normed space Φ, there is no difference between strongly and weakly bounded sets in Φ'; in the general case, by bounded sets in Φ' we shall have in mind strongly bounded sets.) Acting by analogy with Section 5.1, we introduce the following definitions.

(a) The *strong topology* is defined by the following system of neighborhoods of zero. Each neighborhood of zero is defined by a number $\epsilon > 0$ and a bounded set B in the conjugate space, and consists of all fundamental elements φ for which

$$\sup_{f \in B} |(f, \varphi)| < \epsilon.$$

(b) The *weak topology* is defined analogously, but with the bounded sets in the conjugate space replaced by finite sets. In other words, a weak neighborhood of zero is defined by a number $\epsilon > 0$ and fixed functionals $f_1, \ldots, f_m$, and consists of all $\varphi \in \Phi$ for which

$$(f_1, \varphi)| < \epsilon, \ldots, |(f_m, \varphi)| < \epsilon.$$

In the general case, the three topologies in the space Φ, the initial, the strong, and the weak, are different.

We shall show that *if Φ is a countably normed space, then the strong topology in Φ agrees with the initial topology.*

Let us recall that a (complete) countably normed space is representable as the intersection of normed spaces

$$\Phi_1 \supset \Phi_2 \supset \cdots \supset \Phi_p \supset \cdots \supset \Phi,$$

where Φ_p is the completion of the space Φ relative to the pth norm. At the same time, the conjugate space Φ' is the union of conjugate spaces

$$\Phi_1' \subset \Phi_2' \subset \cdots \subset \Phi_p' \subset \cdots \subset \Phi'.$$

Each bounded set $B \subset \Phi'$ is contained entirely in one of the spaces Φ_p' and is bounded in Φ_p' in its norm.

Let us consider a neighborhood of zero in the initial topology of the space Φ; it is defined by the inequality

$$\| \varphi \|_p < \epsilon \tag{1}$$

(the norms are assumed to be ordered; $\| \varphi \|_1 \leqslant \| \varphi \|_2 \leqslant \cdots$). We assert that this same neighborhood can also be described by the inequality

$$\sup_f |(f, \varphi)| < \epsilon, \tag{2}$$

where the functional f runs over the unit ball in the space Φ_p'.

Indeed, if $\| f \|_p \leqslant 1$, $\| \varphi \|_p < \epsilon$, then

$$\sup_f |(f, \varphi)| \leqslant \sup_f \| f \|_p \| \varphi \|_p < \epsilon;$$

on the other hand, for every φ with norm $\|\varphi\|_p \geqslant \epsilon$ it is possible, by the Hahn-Banach theorem, to construct a functional $f \in \Phi_p'$ with $\|f\|_p = 1$, for which

$$|(f, \varphi)| = \|f\|_p \|\varphi\|_p \geqslant \epsilon,$$

so that *only* elements φ from the chosen neighborhood of zero (1) are described by the inequality (2). Thus, the neighborhood of zero (1) in the initial topology coincides with the neighborhood of zero (2) in the strong topology.

Now, let us show that every strong neighborhood of zero contains a neighborhood of zero of the initial topology. In fact, a strong neighborhood of zero $U \subset \Phi$ is defined by the inequality

$$\sup_f |(f, \varphi)| < \epsilon, \tag{3}$$

where f runs over a bounded set in Φ', i.e., as we saw, a set in the space Φ_p' which is bounded in norm by the number M, say.

We can only make the set distinguished by the inequality (3) smaller if we require the functional f to run over the entire ball of radius M in the space Φ_p'. Let V denote the set of fundamental elements distinguished by the inequality (3) under this condition. This same set V is obtained if the functional f runs over the unit ball in the space Φ_p', and the inequality (3) is replaced by the inequality

$$\sup_f |(f, \varphi)| < \frac{\epsilon}{M}. \tag{4}$$

But as we have seen already, the inequality (4) defines some neighborhood of zero of the space Φ in the initial topology. Thus, an initial neighborhood of zero can be found within any strong neighborhood of zero.

By virtue of the criterion for the equivalence of two systems of neighborhoods of zero (Section 1.2), the strong topology coincides with the initial topology, q.e.d.

In particular, *strong convergence of a sequence of elements* φ_ν $(\nu = 1, 2, \ldots)$ *agrees with initial convergence*. In other words, *a sequence* $\{\varphi_\nu\}$ *of elements of a countably normed space* Φ *converges to zero if and only if, for each bounded set* $B \subset \Phi'$, *the sequence* $\{(f, \varphi_\nu)\}$ *tends to zero uniformly on* B. In the next section we shall indicate a class of spaces in which this convergence agrees with the convergence corresponding to the weak topology (i.e., with weak convergence).

Each of the topologies introduced in the space Φ leads to its own definition of bounded sets; a set $A \subset \Phi$ is considered bounded in some topology, in accordance with the general definition of Section 3.8, if for any neighborhood of zero U (in this topology) there exists a number $\lambda > 0$ such that $\lambda A \subset U$.

In the case of a *countably normed* space, it follows from the proved identity of the strong and initial topologies that the *families of bounded sets in the strong and initial topologies agree.*

It is sometimes convenient to call the initial topology in a linear topological space the *strong* topology by contrast with the weak topology. This agrees with the facts for countably normed spaces, since the initial and strong topologies coincide; in other cases, where such usage of the term "strong topology" is inaccurate, we shall make special comment.

Let us show that *in a countably normed space, the family of weakly bounded sets coincides with the family of strongly bounded sets.*

Evidently each strongly bounded set $A \subset \Phi$ is also weakly bounded, since each weak neighborhood of zero $U \subset \Phi$, into which it is necessary to carry the set A by multiplication by λ, is also a strong neighborhood of zero.

Conversely, let A be a strongly bounded set in the countably normed space Φ. This means that for any weak neighborhood of zero U, defined, say, by the inequality

$$|(f_0, \varphi)| < \epsilon,$$

one can find $\lambda > 0$ for which $\lambda A \subset U$. It follows that the functional f_0 takes on bounded values (bounded by the number ϵ/λ) in the set A. Since the functional f_0 is arbitrary, we may state that *each functional $f \in \Phi'$ takes on a bounded set of values on the set A.* But as we have seen in Section 4.5, it follows that the set A is bounded in the space Φ in its initial topology.

Thus, our assertion is completely proved.

6. Perfect Spaces

6.1. Fundamental Definition

As is known, a set F in a topological space Φ is said to be *compact in Φ*, if each infinite subset $A \subset F$ has a limit point in Φ.[16] It is also known how important a role the Bolzano–Weierstrass theorem on the compactness of bounded sets plays in analysis. On the other hand, in the theory of normed spaces one has the Riesz theorem which asserts that

[16] In Western literature, this property is usually called *relative sequential compactness.* (Translator's note.)

if all bounded sets in some normed space are compact, then this space is finite-dimensional. Hence, normed spaces with compact bounded sets are of no interest from the point of view of functional analysis.

But if we turn from normed spaces to linear topological spaces, then we discover *classes of spaces in which all the bounded sets are compact*, and which are of importance for analysis.

Countably normed (complete) spaces in which all the bounded sets are compact will be called *perfect*.

It turns out that perfect spaces possess a number of remarkable properties, which, naturally, do not and cannot hold in infinite-dimensional normed spaces. Thus, in a perfect space strong and weak convergence agree; bounded sets in the space Φ', conjugate to a perfect space Φ, are also compact, and weak convergence in the space Φ' agrees with strong convergence. These properties will be proved below; it can also be proved that a perfect space is always reflexive (i.e., the space Φ'', conjugate to the space Φ', coincides with the space Φ).

Let us now verify that *in any linear topological space every compact set is bounded.*

Indeed, suppose that the set A in the linear topological space Φ is not bounded; we shall show that it is not compact. We can find a neighborhood U of zero and a sequence of elements $\varphi_\nu \in A$ $(\nu = 1, 2, \ldots)$, for which $\psi_\nu = (1/\nu)\varphi_\nu \notin U$. Evidently, neither the sequence $\{\varphi_\nu\}$ itself, nor any of its subsequences are bounded. But in this case, *the sequence* $\{\varphi_\nu\}$ *cannot have limit points.* Indeed, let φ be a limit point of the sequence φ_ν and V a normal neighborhood of zero of the space Φ such that $V + V \subset U$. There are surely points of the sequence φ_ν with arbitrarily large index in the neighborhood $\varphi + V$ of the point φ. On the other hand, the sequence $\{(1/\nu)\varphi\}$ tends to zero; hence, for sufficiently large ν we have $(1/\nu)\varphi \in V$. Therefore, there are arbitrarily large numbers ν for which $\varphi_\nu \in \varphi + V$, i.e., $\varphi_\nu - \varphi \in V$ and $(1/\nu)\varphi \in V$. Since V is a normal neighborhood of zero, it follows that $(1/\nu)(\varphi_\nu - \varphi) \in V$. We obtain

$$\frac{1}{\nu}\varphi_\nu = \frac{1}{\nu}(\varphi_\nu - \varphi) + \frac{1}{\nu}\varphi \in V + V \subset U,$$

contrary to construction. Therefore, A is not compact, q.e.d.

6.2. A Condition for the Perfectness of a Countably Normed Space

Let us present a simple sufficient condition assuring the perfectness of a given countably normed space.

Theorem. *Let Φ be a countably normed (complete) space with the ordered norms*

$$\| \varphi \|_1 \leqslant \| \varphi \|_2 \leqslant \cdots \leqslant \| \varphi \|_p \leqslant \cdots,$$

and let $p_1 < p_2 < \cdots < p_k < \cdots$ be a sequence of subscripts.

If, from each set $A \subset \Phi$ which is bounded in the norm $\| \varphi \|_{p_{j+1}}$, one can select a sequence which is fundamental in the norm $\| \varphi \|_{p_j}$, then the space Φ is perfect.

Proof. We must show that every bounded set $A \subset \Phi$ is compact. The set A is bounded, in particular, in the norm $\| \varphi \|_{p_2}$; by hypothesis, it contains a sequence $\varphi_{11}, \varphi_{12}, \ldots, \varphi_{1p}, \ldots$ which is fundamental in the norm $\| \varphi \|_{p_1}$. This sequence is bounded in the norm $\| \varphi \|_{p_3}$, and hence, contains a sub-sequence $\varphi_{21}, \varphi_{22}, \ldots, \varphi_{2p}, \ldots$ which is fundamental in the norm $\| \varphi \|_{p_2}$. Continuing in this way, we obtain a system of sequences

$$\begin{array}{ccccc} \varphi_{11}, & \varphi_{12}, & \ldots, & \varphi_{1p}, & \ldots, \\ \varphi_{21}, & \varphi_{22}, & \ldots, & \varphi_{2p}, & \ldots, \\ & & \vdots & & \\ \varphi_{m1}, & \varphi_{m2}, & \ldots, & \varphi_{mp}, & \ldots, \\ & & \vdots & & \end{array}$$

of which the mth is fundamental in the norm $\| \varphi \|_{p_m}$. The diagonal sequence $\{\varphi_{nn}\}$ is fundamental in each of the norms $\| \varphi \|_{p_j}$ ($j = 1, 2, \ldots$), and therefore in each of the norms $\| \varphi \|_p$. Thus, it is fundamental in the space Φ and, since Φ is complete, has some limit $\varphi_0 \in \Phi$. Thus, A is compact, q.e.d.

Example. Let us verify, using the criterion just found, that the countably normed space $K(a)$ of all infinitely differentiable functions which vanish outside the interval $| x | \leqslant a$, is perfect.

Let $A \subset K(a)$ be a set which is bounded in the norm

$$\| \varphi \|_p = \max\{| \varphi(x)|, \ldots, | \varphi^{(p)}(x)|\}, \qquad p \geqslant 1.$$

We show that it is compact in the norm $\| \varphi \|_{p-1}$. Since the first derivatives of the function $\varphi^{(p-1)}(x)$ ($\varphi \in A$) are bounded by hypothesis, it follows from Arzela's theorem that a uniformly convergent sequence $\varphi_1^{(p-1)}(x), \ldots, \varphi_\nu^{(p-1)}(x), \ldots$ can be selected from these functions. Since all the derivatives of the function $\varphi_\nu(x)$ up to order $< p - 1$ are obtained by integrating the functions $\varphi_\nu^{(p-1)}(x)$, they too form uniformly convergent sequences. Thus, the sequence $\{\varphi_\nu(x)\}$ converges in the norm $\| \varphi \|_{p-1}$, as required. It follows, according to the theorem just proved, that $K(a)$ is perfect.

A similar result is also valid for the countably normed space $K(a)$ of all infinitely differentiable functions $\varphi(x_1, \ldots, x_n)$, which vanish outside the domain $G_a = \{| x_1 | < a_1, \ldots, | x_n | < a_n\}$ (Section 3.3).

6.3. Equivalence of Strong and Weak Convergence

Theorem. *Strong and weak convergence coincide in a perfect space Φ.*

Proof. It is sufficient to consider a sequence $\varphi_\nu \in \Phi$ which converges to zero. If $\{\varphi_\nu\}$ converges to zero in the initial topology, then $(f, \varphi_\nu) \to 0$ for every $f \in \Phi'$ by virtue of the continuity of the functional f; thus, $\{\varphi_\nu\}$ converges to zero weakly.

Conversely, if the sequence $\{\varphi_\nu\}$ converges weakly to zero, then for any $f \in \Phi'$ the numbers (f, φ_ν) are certainly bounded. The set $A = \{\varphi_1, \varphi_2, \ldots\}$ is therefore weakly bounded. But then it is also strongly bounded (Section 5.7); finally, since the space Φ is perfect, the set A is compact. But any strongly convergent subsequence of the elements φ_ν can have only zero as limit, since its weak limit is zero. Hence, the sequence $\{\varphi_\nu\}$ converges strongly to zero, q.e.d.

Corollary. *A perfect space Φ is complete relative to weak convergence.*

Proof. Let the sequence $\varphi_\nu \in \Phi$ be weakly fundamental, i.e., for any $f \in \Phi'$ the numbers (f, φ_ν) form a convergent sequence. This means the set $A = \{\varphi_1, \varphi_2, \ldots\}$ is certainly weakly bounded.

According to Section 5.5, the set A is strongly bounded, and therefore (since Φ is perfect) compact; since the first axiom of countability is satisfied in the space Φ, it contains a subsequence $\varphi_{\nu_1}, \ldots, \varphi_{\nu_\mu}, \ldots$, which converges strongly to some element $\varphi \in \Phi$. By virtue of the continuity of the functional f we have $(f, \varphi_{\nu_\mu}) \to (f, \varphi)$. Since (f, φ_ν) is a convergent sequence, $(f, \varphi_\nu) \to (f, \varphi)$, i.e., the sequence $\{\varphi_\nu\}$ converges weakly to φ; therefore, Φ is complete relative to weak convergence, q.e.d.

6.4. Weak and Strong Convergence in the Conjugate Space

Theorem. *If Φ is a perfect space, then weak and strong convergence coincide in the space Φ'.*

Proof. It is sufficient to verify that a sequence $f_\nu \in \Phi'$ which converges weakly to a functional f also converges strongly to f. This sequence $\{f_\nu\}$ is evidently weakly bounded; by Section 5.5 it is also strongly bounded. Furthermore, we may suppose that $f = 0$ (by otherwise replacing f_ν

by $f_\nu - f$). We must show that $(f_\nu, \varphi) \to 0$ uniformly on each bounded set $A \subset \Phi$. Let this not be satisfied for some bounded set A. Then for some $\epsilon > 0$ one can find $\varphi_\nu \in A$ such that $|(f_\nu, \varphi_\nu)| > \epsilon$. Since A is compact, we may suppose that $\varphi_\nu \to \varphi_0 \in \Phi$. Since the $\psi_\nu = \varphi_\nu - \varphi_0$ tend to zero in the initial topology of the space Φ, and therefore tend strongly to zero, for any bounded set $B \subset \Phi'$ we have $(f, \psi_\nu) \to 0$ uniformly on the set B. As B, let us take the sequence $f_1, f_2, \ldots, f_\nu, \ldots$. Then in particular, $(f_\nu, \psi_\nu) \to 0$, and since on the other hand $(f_\nu, \varphi_0) \to 0$, we obtain

$$(f_\nu, \varphi_\nu) = (f_\nu, \psi_\nu) + (f_\nu, \varphi_0) \to 0,$$

which, by construction, cannot hold. The obtained contradiction convinces us of the validity of the theorem.

The following more precise description of (strongly or weakly) convergent sequences in the space Φ' can be given for a perfect space Φ in which the condition of Section 6.2 is satisfied.

If the condition of Section 6.2 *is fulfilled, then a sequence* $f_1, f_2, \ldots, f_\nu, \ldots$ *converges* (*strongly or weakly*) *in the space* Φ' *if and only if all the* f_ν *belong to some fixed space* Φ_p' (as we know, the union of the spaces Φ_p' is the whole space Φ'), *and* $\{f_\nu\}$ *converges in the norm of* Φ_p'.

Indeed, if $f_\nu \in \Phi_p'$ and $\| f_\nu - f \|_p \to 0$, then for any φ we have $|(f_\nu - f, \varphi)| \leqslant \| f_\nu - f \|_p \| \varphi \|_p \to 0$, so that the sequence $\{f_\nu\}$ converges (weakly and, consequently, strongly) in the space Φ'.

On the other hand, let $f_\nu \to f$ in the space Φ'. Without loss of generality, we may suppose that $f = 0$. Since the sequence $\{f_\nu\}$ is bounded in Φ', it lies entirely in the space Φ_r' for some r and is bounded in the norm of Φ_r' (Section 5.3). We find a subscript $p > r$ such that the boundedness of any sequence $\varphi_\nu \in \Phi$ in the norm $\| \cdot \|_p$ implies its compactness in the norm $\| \cdot \|_r$. The sequence $\{f_\nu\}$ is bounded in the norm $\| \cdot \|_r$ and *a fortiori* bounded in the norm $\| \cdot \|_p$ (since for $p > r$ we always have $\| f \|_r \geqslant \| f \|_p$). We shall show that $f_\nu \to 0$ in this pth norm, in other words, that $(f_\nu, \varphi) \to 0$ uniformly on the unit ball of the space Φ_p. If this were not so, then we could find a number $\epsilon > 0$ and a sequence $\{\varphi_\nu\}$, $\| \varphi_\nu \|_p \leqslant 1$, such that $|(f_\nu, \varphi_\nu)| \geqslant \epsilon$ $(\nu = 1, 2, \ldots)$. The sequence $\{\varphi_\nu\}$ is compact in the rth norm; we may suppose that $\{\varphi_\nu\}$ converges to some element φ_0 in the rth norm. But then

$$\begin{aligned} |(f_\nu, \varphi_\nu)| &\leqslant |(f_\nu, \varphi_\nu - \varphi_0)| + |(f_\nu, \varphi_0)| \\ &\leqslant \| f_\nu \|_r \| \varphi_\nu - \varphi_0 \|_r + |(f_\nu, \varphi_0)|. \end{aligned}$$

By hypothesis and by construction, both members of the right side tend to zero. But then $(f_\nu, \varphi_\nu) \to 0$, in contradiction with the definition of $\{\varphi_\nu\}$. Our assertion is thereby completely proved.

6.5. Bounded Sets in the Conjugate Space

The space Φ', conjugate to a perfect space Φ, is not itself perfect, first of all because it is not countably normed. However, *the bounded sets in the space Φ', conjugate to a perfect space Φ, are also compact relative to weak and strong convergence.*

Let us first establish the compactness of the bounded sets of the space Φ' relative to weak convergence. It turns out that this compactness holds not only for the space conjugate to a perfect space Φ, but also for the space conjugate to any separable countably normed space Φ. Let us recall that a topological space is called *separable*, if there is a countable everywhere-dense set in it. We shall show below that there is always such a set in a perfect space.

Theorem. *If a countably normed space Φ is separable, then each bounded sequence $f_1, f_2, \ldots, f_\nu, \ldots \in \Phi'$ contains a weakly convergent subsequence.*

Proof. By using the diagonal process, we can select a subsequence of functionals $f_{\nu_1}, f_{\nu_2}, \ldots, f_{\nu_k}, \ldots$ which converges on each of the elements φ_μ of a countable everywhere-dense set. Since the sequence $\{f_{\nu_k}\}$ is weakly bounded, and therefore also strongly bounded (Section 5.5), all of these functionals belong to some space Φ_p' and are bounded therein in norm (Section 5.3). Moreover, we know that these functionals converge on the set $A = \{\varphi_\mu\}$ which is dense in Φ, and therefore, dense in Φ_p. Hence, as has already been remarked in Section 5.4, it follows that the functionals f_{ν_k} converge weakly to some functional $f \in \Phi_p'$. This means that for any $\varphi \in \Phi_p$, in particular, for any $\varphi \in \Phi$, the limit relation

$$(f_{\nu_k}, \varphi) \to (f, \varphi)$$

holds, q.e.d.

To establish the weak compactness of bounded sets in the space conjugate to a perfect space, it remains for us to prove the following theorem.

Theorem. *A perfect space Φ is separable.*

Proof. Among the complete normed spaces $\Phi_1 \supset \Phi_2 \supset \cdots$, whose intersection yields the space Φ, either all are separable or there is at least one which is not separable.

In the first case, since Φ is a subset of the separable space Φ_1, we can find a countable set $S_1 \subset \Phi$ which is dense in Φ in the norm $\| \varphi \|_1$; similarly, we can find a countable set $S_2 \subset \Phi$ which is dense in Φ in the norm $\| \varphi \|_2$, etc. We shall show that the union S of all these countable sets is dense in Φ in any metric. Indeed, let φ be any point of Φ; we can find a point $\varphi_p \in S_p$ such that

$$\| \varphi - \varphi_p \|_p < \frac{1}{p} \qquad (p = 1, 2, \ldots);$$

then for any k with $p > k$ we will have

$$\| \varphi - \varphi_p \|_k \leqslant \| \varphi - \varphi_p \|_p < \frac{1}{p},$$

and, therefore, the selected sequence $\varphi_p \in S$ will converge to the element φ in the topology of the space Φ.

In the second case, we can assume without loss of generality that the space Φ_1 is nonseparable. Using Zermelo's axiom,[17] a nondenumerable set Z_1 of points bounded in the norm $\| \cdot \|_1$ such that the distance between any two of them (i.e., the norm of the difference) exceeds a positive constant, can be constructed in the space Φ. This set may turn out to be unbounded in the norm $\| \cdot \|_2$; but in any event, it possesses a nondenumerable subset Z_2 which is bounded in the norm $\| \cdot \|_2$ (since Φ lies in the union of the balls $\| \cdot \|_2 < \nu$ $(\nu = 1, 2, \ldots)$). Furthermore, Z_2 possesses a nondenumerable subset which is bounded in the norm $\| \cdot \|_3$, etc; let us note that the distances between points of the set Z_1 can only increase on passing to the norms $\| \cdot \|_2, \| \cdot \|_3, \ldots$.

Let us select an arbitrary point φ_p from the set Z_p $(p = 1, 2, \ldots)$. We obtain a bounded set in the space Φ. However, *it does not contain any fundamental sequence, since the distances between its points in any of the norms* $\| \cdot \|_p$ *are greater than a fixed constant.* Therefore, in this case the space Φ cannot be perfect.

The theorem is thereby proved completely.

We now show that every bounded set B in the space conjugate to a perfect space is not only weakly but also strongly compact, i.e., contains a strongly convergent sequence. This is now obvious: We have proved that there exists a sequence $f_\nu \in B$, converging weakly to some functional f; but by Section 7.3, this sequence also converges strongly to f, q.e.d.

[17] Or we may use Zorn's lemma, which is equivalent to Zermelo's axiom. See, for example, L. Loomis, "Introduction to Abstract Harmonic Analysis." Van Nostrand, Princeton, New Jersey, 1953.

7. Continuous Linear Operators

7.1. Definition

An operator A, mapping a linear topological space Φ into a linear topological space Ψ (in particular, into itself if $\Psi = \Phi$), is called a *continuous linear operator* if the following conditions are satisfied:

(1) $A(\alpha_1\varphi_1 + \alpha_2\varphi_2) = \alpha_1 A\varphi_1 + \alpha_2 A\varphi_2$ (*linearity*), in particular $A(0) = 0$;

(2) For any neighborhood of zero $V \subset \Psi$, a neighborhood of zero $U \subset \Phi$ can be found such that $A\varphi \in V$ for $\varphi \in U$ (*continuity*).

If Ψ is the complex plane with its customary topology, the operator A will simply be a functional.

We have seen in Section 4 that the class of continuous linear functionals coincides with the class of linear functionals bounded on each bounded set. A similar result is valid for linear operators.

(a) *Every continuous linear operator transforms each bounded set of the space Φ into a bounded set of the space Ψ.*

Proof. Let A be a continuous linear operator, $F \subset \Phi$ a given bounded set, and let $G = AF \subset \Psi$ be an unbounded set. There exists a neighborhood of zero $V \subset \Psi$ in which none of the sets $(1/\nu)G$ ($\nu = 1, 2,\ldots$) is entirely contained. This means that a sequence $\varphi_\nu \in F$ can be found such that $(1/\nu)A\varphi_\nu = (1/\nu)\psi_\nu$ does not belong to the neighborhood V. But $(1/\nu)\varphi_\nu \to 0$ in Φ (Section 3.8), and hence the elements $(1/\nu)\varphi_\nu$, starting with some number $\nu = \nu_0$, lie in any given neighborhood of zero $U \subset \Phi$. Using the continuity of the operator A, we select U such that $AU \subset V$. A contradiction is obtained, which shows that AF is bounded, q.e.d.

(b) *If the first axiom of countability is satisfied in the space Φ, then a linear operator A which transforms every bounded set of the space Φ into a bounded set of the space Ψ is continuous.*

Proof. If A is not a continuous operator, then for some neighborhood of zero $V \subset \Psi$ and any basis neighborhood of zero $U_\nu \subset \Phi$, an element $\varphi_\nu \in (1/\nu)U_\nu$ can be found such that $A\varphi_\nu$ does not lie in the neighborhood V. The sequence $\nu\varphi_\nu \in U_\nu$ ($\nu = 1, 2,\ldots$) is bounded in Φ (even tends to zero), the sequence $\psi_\nu = \nu A\varphi_\nu = A(\nu\varphi_\nu)$ is not bounded in Ψ (since none of its multiples lies in the neighborhood V), and we arrive at a contradiction with the condition of the lemma. Hence, the lemma is proved.

A linear operator A which transforms every bounded set of the space Φ into a bounded set of the space Ψ is said to be *bounded.* Lemmas (a) and (b) show that *in spaces with the first axiom of countability, the class of continuous operators coincides with the class of bounded operators.*

Corollary. *In order for the linear operator A to be continuous, it is necessary and, in spaces with the first axiom of countability, also sufficient that $\psi_\nu = A\varphi_\nu \to 0$ (in Ψ) follow from $\varphi_\nu \to 0$ (in Φ).*

Proof. Necessity. Let $\varphi_\nu \to 0$ and let $V \subset \Psi$ be a neighborhood of zero; furthermore, let $U \subset \Phi$ be a neighborhood of zero such that $AU \subset V$. Starting with some term, the φ_ν lie in U, from which $A\varphi_\nu \in V$; consequently, $A\varphi_\nu \to 0$ in Ψ.

Sufficiency. Let $F \subset \Phi$ be a bounded set; if $G = AF$ were an unbounded set in Ψ, we could find a neighborhood of zero $V \subset \Psi$ and a sequence $\psi_\nu = A\varphi_\nu \in G$ for which $(1/\nu)\psi_\nu \notin V$; but $(1/\nu)\varphi_\nu \in (1/\nu)F$, and therefore $(1/\nu)\varphi_\nu \to 0$, which contradicts our hypothesis.

Example. In the fundamental space $K(a)$ of infinitely differentiable functions $\varphi(x)$ which vanish outside the interval $|x| \leqslant a$, the operators of multiplication by x and of differentiation are continuous operators. We use the criterion just established for the proof.

Let $\varphi_\nu(x) \to 0$ in the topology of $K(a)$; this means that for any $q = 0, 1, 2, \ldots$, the functions $\varphi_\nu^{(q)}(x)$ converge uniformly to zero as $\nu \to 0$. We must show that the sequences

$$\psi_\nu(x) = x\varphi_\nu(x)$$

and

$$\chi_\nu(x) = \varphi_\nu'(x)$$

tend to zero in this same sense. But this is evident, since for any q

$$\psi_\nu^{(q)}(x) = x\varphi_\nu^{(q)}(x) + q\varphi_\nu^{(q-1)}(x),$$
$$\chi_\nu^{(q)}(x) = \varphi_\nu^{(q+1)}(x),$$

and both of the sequences obtained converge uniformly to zero as $\nu \to \infty$, for any q.

7.2. A Theorem on the Inverse Operator

The following theorem on the inverse operator holds for countably normed (complete) spaces.

Theorem 1. *Let the continuous linear operator A be a one-to-one mapping of a countably normed (complete) space Φ onto a countably normed (complete) space Ψ. Then the inverse operator A^{-1} is also continuous.*

This theorem results from the following theorem of Banach.[18]

If the continuous linear operator A maps a complete linear metric space Φ one-to-one onto a complete linear metric space Ψ, then the inverse operator A^{-1} is also continuous.

Let us recall that a countably normed space is a linear metric space (Section 3.4), in which the metric defines a topology identical with the initial topology. Hence, if a countably normed space Φ is complete in its topology, then it is complete as a metric space.

Corollary. *If two countable systems of norms $\| \varphi \|_1, \ldots, \| \varphi \|_p, \ldots$ and $\| \varphi \|_1', \ldots, \| \varphi \|_p', \ldots$, each of which transforms Φ into a complete countably normed space, are introduced into the same space Φ, and these systems are comparable (i.e., each of the norms of one of the systems is weaker than some norm of the second system), then they are equivalent.*

Proof. Let us retain the notation Φ for the space in question, topologized by using the first system of norms, and let Ψ denote this space with the topology generated by the second system of norms. If the first system of norms is stronger than the second, this means continuity of the identity operator which maps Φ into Ψ. According to Theorem 1, the inverse operator is continuous; therefore, the second system of norms turns out to be stronger than the first. Hence, these systems are equivalent, q.e.d.

The assumption of the *comparability* of the systems of norms may be weakened, in turn, by replacing it by the assumption of *compatibility*. Let us imagine that we are given two linear topological spaces Φ and Ψ, where Ψ is a linear subspace of the space Φ. Let us say that the topologies in the spaces Φ and Ψ are *compatible* if, for every sequence $\varphi_\nu \in \Psi$ which converges to zero in the topology of Ψ and simultaneously converges to an element $\varphi_0 \in \Phi$ in the topology of Φ, we have $\varphi_0 = 0$.

Theorem 2. *If Φ and $\Psi \subset \Phi$ are (complete) countably normed spaces and their topologies are compatible, then from the convergence $\varphi_\nu \to \varphi$ in the topology of Ψ there follows the convergence $\varphi_\nu \to \varphi$ in the topology of Φ.*

[18] See, e.g., N. Dunford and J. T. Schwartz, "Linear Operators," Part. 1, p. 57. Wiley (Interscience), New York, 1958.

Remark. This theorem shows that the imbedding of countably normed spaces with the condition of compatibility of the topologies is imbedding with comparability of the topologies. Further on, the compatibility of topologies will be an easily verifiable fact in the analysis of fundamental spaces of functions; this theorem enables us to make useful deductions concerning the comparability of topologies and their equivalence. It may be summarized briefly thus: the smaller the space, the stronger its topology.

Proof. Let $\| \varphi \|_p$ and $\| \varphi \|'_p$, respectively, be the systems of norms in the spaces Φ and Ψ ($p = 1, 2, \ldots$). We introduce a new system of norms $\| \varphi \|''_p = \max(\| \varphi \|_p, \| \varphi \|'_p)$ in the space Ψ. Let us verify that the space Ψ is complete relative to the new system of norms. If $\varphi_\nu \in \Psi$ is a fundamental sequence in the new system of norms, then it is obviously fundamental in both of the old systems; because of the completeness of the spaces Ψ and Φ, we have $\varphi_\nu \to \psi_0$ (in Ψ), $\varphi_\nu \to \varphi_0$ (in Φ).

The difference $\varphi_\nu - \psi_0$ tends to zero in Ψ and tends to $\varphi_0 - \psi_0$ in Φ. By hypothesis, $\varphi_0 - \psi_0 = 0$, so that $\varphi_0 = \psi_0$; therefore $\varphi_\nu \to \psi_0$ in both systems of norms. But then $\varphi_\nu \to \psi_0$ in the new system of norms as well. Therefore, Ψ is complete relative to the new system of norms.

Taking into account the comparability of the systems of norms $\| \varphi \|'_p$ and $\| \varphi \|''_p$, defined in the space Ψ, we obtain their equivalence by virtue of what was proved above.

Now, let $\varphi_\nu \to \varphi_0$ in the topology of Ψ. This means that $\| \varphi_\nu - \varphi_0 \|'_p \to 0$ ($p = 1, 2, \ldots$). But by what has been proved, $\| \varphi_\nu - \varphi_0 \|''_p \to 0$, and it follows that $\| \varphi_\nu - \varphi_0 \|_p \to 0$, i.e., $\varphi_\nu \to \varphi_0$ in the topology of Φ. The theorem is proved.

Thus, *under the condition of the compatibility of the systems of norms, convergence in the larger space follows from convergence in the smaller space.* As a corollary we obtain: *Every continuous linear functional on the larger space Φ is a continuous linear functional on the smaller space Ψ.*

7.3. Operations with Linear Operators

Let us now introduce addition, multiplication by a number, and multiplication by each other for operators.

For any two linear operators A and B which map a space Φ into a space Ψ, the sum and product by a number are defined naturally by

$$(A + B)\,\varphi = A\varphi + B\varphi,$$

$$(\lambda A)\,\varphi = \lambda A\varphi,$$

where the results are again linear operators mapping the space Φ into the space Ψ. It is easy to see that the continuity of A implies the continuity of λA, and the continuity of A and B implies the continuity of $A + B$.

If the operator A maps the space Φ into the space Ψ, and the operator B maps the space Ψ into the space X, then the product BA may be defined by the formula

$$(BA)\,\varphi = B(A\varphi);$$

the operator BA is also linear. If A and B are continuous operators, then BA is likewise continuous.

7.4. Sequences of Operators

An operator A, which maps a space Φ into a space Ψ, is called *the strong (weak) limit of a sequence of operators* A_ν which map the space Φ into the same space Ψ ($\nu = 1, 2, \ldots$) if, for any $\varphi \in \Phi$, the relation

$$\lim A_\nu \varphi = A\varphi$$

holds in the strong[19] (weak) sense. In perfect spaces where these convergences agree, both of the definitions presented agree.

Theorem. *If continuous linear operators A_ν, mapping the countably normed space Φ into the countably normed space Ψ, converge weakly to some operator A as $\nu \to \infty$, then A is also a linear and continuous operator.*

Proof. The linearity of the operator A is trivial; it is essential to prove its continuity. The functional $(f_\nu, \varphi) = (g, A_\nu\varphi)$ with fixed $g \in \Psi'$ is continuous in φ by virtue of the assumed continuity of the operator A_ν. As $\nu \to \infty$, this functional converges weakly to the functional $(f, \varphi) = (g, A\varphi)$; by the theorem of Section 5.6, the functional (f, φ) is also continuous in φ. Hence, it is bounded on each bounded set F of the space Φ (Section 4.4). Therefore, for any functional $g \in \Psi'$ and any bounded set $F \subset \Phi$, the set of numbers $(g, A\varphi)$ is bounded when φ runs through the set F. We see that the set $\{A\varphi\}$, $\varphi \in F$ is weakly bounded in Ψ. It follows that the set $\{A\varphi\} \subset \Psi$ is strongly bounded (Section 5.7). Therefore, the operator A transforms any bounded set $F \subset \Phi$ into a bounded set $F_1 = \{A\varphi\} \subset \Psi$, and by virtue of Lemma (b) of Section 7.1 is continuous.

[19] We here understand, by the strong topology in the space Ψ, the initial topology in this space (if, of course, we are not speaking of a conjugate space).

7.5. The Adjoint Operator

Let a continuous linear operator A which maps the space Φ into the space Ψ be given, and let Φ' and Ψ' be the respective conjugate spaces. Let us introduce the operator A^*, which maps the space Ψ' into the space Φ' according to the formula

$$(A^*g, \varphi) = (g, A\varphi),$$

where $\varphi \in \Phi$, $g \in \Psi'$. From the continuity of the operator A and the functional g it follows that A^* is a continuous operator; it is called the *adjoint* of the operator A.

Example. Let us determine the adjoints of the operations of multiplication by x and differentiation in the space $K(a)$ of infinitely differentiable functions which vanish outside the interval $-a \leqslant x \leqslant a$. According to the definition,

$$(x^*f, \varphi) = (f, x\varphi). \tag{1}$$

If the functional f is given by the integral

$$(f, \varphi) = \int_{-a}^{a} f(x)\, \varphi(x)\, dx,$$

then, as is seen from (1), the functional x^*f is given by the integral

$$(x^*f, \varphi) = (f, x\varphi) = \int_{-a}^{a} f(x) \cdot x\varphi(x)\, dx.$$

Therefore, in this case, the operation x^* consists of multiplying the function $f(x)$ by x. This gives us the right to call the operation x^* in the conjugate space $K'(a)$ multiplication by x.

Analogously,

$$\left(\frac{d^*}{dx} f, \varphi\right) = \left(f, \frac{d}{dx} \varphi\right).$$

But we know from Volume 1 that for functionals f of the type of a differentiable function $f(x)$, one has

$$\left(f, \frac{d}{dx} \varphi\right) = \left(-\frac{d}{dx} f, \varphi\right).$$

Therefore, in this case the operator (d^*/dx) agrees with the operator which we have called $-d/dx$.

Let us note that the operator $i(d/dx)$ is self-adjoint:

$$\left(i\frac{d}{dx}\right)^* = i\frac{d}{dx}.$$

It is easy to verify the formulas

$$(A + B)^* = A^* + B^*, \qquad (\lambda A)^* = \bar{\lambda} A^*, \qquad (BA)^* = A^*B^*. \quad (2)$$

The last equality has meaning, naturally, when the operator BA is defined, i.e., when the operator B is defined on that space Ψ, into which the operator A maps the space Φ.

All these relationships are retained, of course, when the operators A and B map the space Φ into itself, so that one can take $\Psi = \Phi$. In this case, the adjoint operator A^* is defined on the space Φ' and transforms this space into itself, and we can adjoin to the formulas (2) the evident formula

$$E^* = E,$$

which says that the adjoint of the unit (identity) operator in the space Φ is the unit operator in the space Φ'. Furthermore, if the operator A in the space Φ possesses an inverse operator A^{-1}, then the adjoint operator A^* also possesses an inverse; namely, the operator $(A^{-1})^*$ is this inverse.

Indeed, for any $\varphi \in \Phi$, $f \in \Phi'$ we have

$$((A^{-1})^* A^*f, \varphi) = (A^*f, A^{-1}\varphi) = (f, AA^{-1}\varphi) = (f, \varphi),$$

from which $(A^{-1})^* A^* = E$, q.e.d.

8. Union of Countably Normed Spaces

8.1. Definition

Let there be given an increasing sequence of linear topological spaces

$$\Phi^{(1)} \subset \Phi^{(2)} \subset \cdots \subset \Phi^{(m)} \subset \cdots.$$

It is assumed that each imbedding preserves the convergence of sequences, i.e., if a sequence $\varphi_\nu \in \Phi^{(m)}$ converges to zero, then it also converges to zero when considered in the larger space $\Phi^{(m+1)}$.

Let $\Phi^{(\omega)}$ denote the union of all the spaces $\Phi^{(m)}$ ($m = 1, 2, \ldots$). The set $\Phi^{(\omega)}$ is a linear space with the natural linear operations.

We introduce the following definition of the convergence of a sequence in $\Phi^{(\omega)}$. We shall say that a sequence $\varphi_1, \varphi_2, \ldots, \varphi_\nu, \ldots$ of elements of $\Phi^{(\omega)}$ *converges* to an element $\varphi \in \Phi^{(\omega)}$, if all the elements φ_ν and φ are contained in some fixed space $\Phi^{(m)}$ and $\varphi_\nu \to \varphi$ in the topology of this space.

The space $\Phi^{(\omega)}$, obtained by this construction, is called the *union of the spaces* $\Phi^{(m)}$. It is not a topological space since no system of open or closed sets has been defined in it.

As an example, let us consider the union of the countably normed spaces $K(a)$. As we have already said (Section 3.3), the space $K(a)$, consists of all infinitely differentiable functions $\varphi(x_1, \ldots, x_n)$ which vanish outside the domain $G_a = \{|x_1| \leqslant a_1, \ldots, |x_n| \leqslant a_n\}$; let us assume that $a_1 = a_2 = \cdots = a_n = a$; $a = 1, 2, \ldots$.

The union of the spaces $K(a)$ consists of all infinitely differentiable functions $\varphi(x_1, \ldots, x_\nu)$, which vanish outside a bounded domain (which depends on the function φ).

By definition, a sequence $\{\varphi_\nu(x)\}$ of such functions converges to zero if and only if, first, all the $\varphi_\nu(x)$ belong to the same space $K(a)$, in other words, all the $\varphi_\nu(x)$ vanish outside the same domain, and second, the functions $\varphi_\nu(x)$ converge to zero in this space $K(a)$ in its topology, i.e., in the present case, converge uniformly to zero together with their derivatives of all orders.

We have thus arrived at the definition of the fundamental space K, which played a principal role in the constructions of Volume I.

Thus, *the fundamental space K is the union of the countably normed spaces $K(a)$.*

8.2. Bounded Sets and Linear Functionals

A set $B \subset \Phi^{(\omega)}$ is said to be *bounded* if it is entirely contained in some $\Phi^{(m)}$ and is bounded in the topology of $\Phi^{(m)}$.

This definition actually coincides with the second definition of bounded sets in a linear topological space (Section 3.8; it is unnecessary to speak of the first definition, since it makes essential use of topology): a set B is bounded if for any sequence $\varphi \in B$ the elements $(1/\nu)\varphi_\nu$ tend to zero as $\nu \to \infty$.

A linear functional f, defined on the space $\Phi^{(\omega)}$, is said to be *continuous* if it is continuous on each $\Phi^{(m)}$. It is easily seen that when $\Phi^{(m)}$ is a space satisfying the first axiom of countability, this definition is equivalent to the following: A linear functional f is continuous if $(f, \varphi_\nu) \to 0$ follows from $\varphi_\nu \to 0$.

If the first axiom of countability is satisfied in each of the spaces $\Phi^{(m)}$, then the definition of a continuous linear functional is equivalent to the following: A linear functional f is continuous, if it is bounded on each bounded set in the space $\Phi^{(\omega)}$.

If $\Phi^{(\omega)}$ is the space K, the union of the spaces $K(a)$ (see above), then the continuous linear functionals on $\Phi^{(\omega)}$ are the generalized functions which we have introduced in Volume 1.

The set of all continuous linear functionals on the space $\Phi^{(\omega)}$ will be denoted by $\Phi^{(\omega)\prime}$. Evidently this set is a linear space. Let us introduce the concept of convergence in this space. We will say that a sequence of functionals $f_\nu \in \Phi^{(\omega)\prime}$ converges to the functional f, if for every $\varphi \in \Phi^{(\omega)}$

$$\lim_{\nu\to\infty} (f_\nu , \varphi) = (f, \varphi).$$

Precisely this convergence was introduced in Volume 1 for functionals on the space K.

As before, we call such convergence *weak* (although we shall not introduce any other). The following important theorem holds.

Theorem. *If the $\Phi^{(m)}$ are countably normed spaces, then the space $\Phi^{(\omega)\prime}$ is complete relative to (weak) convergence.*

Proof. Let the sequence f_ν ($\nu = 1, 2,...$) possess the property that for each $\varphi \in \Phi^{(\omega)}$ the limit $(f, \varphi) = \lim_{\nu\to\infty}(f_\nu , \varphi)$ exists. The functional (f, φ), considered for $\varphi \in \Phi^{(m)}$, is a continuous linear functional on $\Phi^{(m)}$ by virtue of the theorem in Section 5.6, but then, by definition, f is a continuous linear functional on all of $\Phi^{(\omega)}$, q.e.d.

In particular, the space of generalized functions (functionals on the space K) is complete relative to weak convergence. This fact underlies many constructions in Volume 1.

Let us further introduce the concept of a bounded set for the space $\Phi^{(\omega)\prime}$. A set $F \subset \Phi^{(\omega)\prime}$ is said to be *bounded*, if for every $\varphi \in \Phi^{(\omega)}$ the set of numbers (f, φ), $f \in F$ is bounded. We are thus speaking of weak boundedness. (An equivalent definition is: A set F is bounded if for any sequence $f_1 , f_2 ,..., f_\nu ,... \in F$ the functionals $(1/\nu)f_\nu$ tend to zero in $\Phi^{(\omega)\prime}$.)

We shall show that in the case of countably normed spaces $\Phi^{(m)}$, not only are the numbers (f, φ), $f \in F$, bounded for any bounded set $F \in \Phi^{(\omega)\prime}$ and any element $\varphi \in \Phi^{(\omega)}$, they are bounded (uniformly) as φ runs through any bounded set $A \subset \Phi^{(\omega)}$.

Indeed, if A is a bounded set in $\Phi^{(\omega)}$, then by definition it is contained entirely in some $\Phi^{(m)}$ and is bounded in the topology of $\Phi^{(m)}$. The functionals $f \in F$ are continuous linear functionals, in particular, on the space $\Phi^{(m)}$. They are bounded on each element $\varphi \in \Phi^{(m)}$ and, therefore, constitute a weakly bounded set in the space $\Phi^{(m)\prime}$. But then (see Section 5.5), the set F is also strongly bounded in the space $\Phi^{(m)\prime}$, i.e., the numbers (f, φ) are uniformly bounded on any bounded set of the space $\Phi^{(m)}$. In particular, these numbers are uniformly bounded on the set A, q.e.d.

8.3. Linear Operators

A linear operator A, defined on the space $\Phi^{(\omega)}$ and carrying $\Phi^{(\omega)}$ into itself or into another analogous space $\Psi^{(\omega)}$ is said to be *continuous*, if $\varphi_\nu \to 0$ implies $A\varphi_\nu \to 0$, and *bounded*, if it transforms every bounded set $F \subset \Phi^{(\omega)}$ into a bounded set $AF \subset \Psi^{(\omega)}$.

We shall assume that the first axiom of countability is satisfied in each of the spaces $\Phi^{(m)}$.

In this case, we can assert that *a linear operator A is continuous if and only if it is bounded.*

To prove this assertion, it is sufficient to show that each of the assumptions reduces to the statement that the operator A transforms any one of the spaces $\Phi^{(m)}$ into some space $\Psi^{(p)}$. Then the operator A can be studied on the countably normed space $\Phi^{(m)}$; since its values also lie in the countably normed space $\Psi^{(p)}$, there remains to apply the appropriate results of Section 7.1.

Let us first note that in a space Φ with the first axiom of countability, for any sequence of elements $\varphi_1, \ldots, \varphi_\nu, \ldots$, we can find a sequence of constants $\lambda_1, \ldots, \lambda_\nu, \ldots$ such that the elements $\lambda_1\varphi_1, \ldots, \lambda_\nu\varphi_\nu, \ldots$ will tend to zero. In fact, if $U_1 \supset \cdots \supset U_\nu \supset \cdots$ is a basis of the neighborhoods of zero in Φ, it is sufficient to choose the numbers λ_ν so that $\lambda_\nu\varphi_\nu \in U_\nu$.

Let us now assume that the linear operator A, acting in $\Phi^{(\omega)}$, does not map some given $\Phi^{(m)}$ entirely into some $\Psi^{(p)}$ $(p = 1, 2, \ldots)$. Then for any ν, we can find an element $\varphi_\nu \in \Phi^{(m)}$ which the operator A transforms into $\psi_\nu \notin \Psi^{(\nu)}$.

Let the λ_ν be such that $\lambda_\nu\varphi_\nu \to 0$. If the operator A is *continuous*, the elements $A(\lambda_\nu\varphi_\iota) = \lambda_\nu A\varphi_\nu$ must also tend to zero, and in particular, must all belong to the same $\Psi^{(p)}$, which contradicts the assumption.

If the operator A is *bounded*, the elements $\lambda_\nu A\varphi_\nu$ must from a bounded set and also must all belong to the same $\Psi^{(p)}$, which again contradicts the assumption.

Therefore, in both cases A transforms any one of the spaces $\Phi^{(m)}$ into some $\Psi^{(p)}$. Hence, as we have seen, it follows that a linear operator is continuous if and only if it is bounded.

For a continuous linear operator A which carries a given space $\Phi^{(\omega)}$

into an analogous $\Psi^{(\omega)}$, it is possible to define *the adjoint operator* A^*, which carries $\Psi^{(\omega)'}$ into $\Phi^{(\omega)'}$ by means of the formula

$$(A^*g, \varphi) = (g, A\varphi),$$

where $\varphi \in \Phi^{(\omega)}$, $g \in \Psi^{(\omega)'}$. By virtue of the continuity of the functional g and the operator A, the operator A^* also turns out to be continuous.

Appendix 1

Elements, Functionals, Operators Depending on a Parameter[20]

A1.1. Abstract Functions

We shall call an element φ_ν of a linear topological space Φ or of the union of linear topological spaces (Section 8), which depends on a numerical parameter ν, an *(abstract) function of* ν.

If ν runs through the set of integer values 1, 2,..., then we are dealing simply with a sequence of elements φ_ν. In the general case, ν runs through some set of values in the complex plane.

An element φ_0 is called the *limit* of the (abstract) function φ_ν as $\nu \to \nu_0$, if for any neighborhood U of zero we can find a $\delta > 0$ such that $|\nu - \nu_0| < \delta$ implies $\varphi_\nu - \varphi_0 \in U$. There is also an equivalent definition: φ_0 is the limit of φ_ν as $\nu \to \nu_0$, if for any sequence $\nu_n \to \nu_0$, we have $\varphi_{\nu_n} \to \varphi_0$. We shall use the second definition in unions of linear topological spaces (Section 8), where we have not introduced a topology but have only introduced the concept of convergence of sequences.

The neighborhoods of zero in the first formulation may be strong or weak, and, correspondingly, the convergence $\varphi_{\nu_n} \to \varphi_0$ should be called strong or weak; the limit which we obtain is called *strong* or *weak*, respectively.[21] For perfect spaces and their conjugates, this difference disappears since, as we know, weak and strong convergence agree in these spaces.

In particular, an abstract function φ_ν is said to be *strongly* (*weakly*) *continuous* at the point ν_0 if $\varphi_\nu \to \varphi_{\nu_0}$ in the strong (weak) sense.

[20] Compare with Appendix 2 to Chapter I of Volume 1.

[21] Here, by the strong topology in a linear topological space we have in mind the original topology in this space if, of course, we are not speaking of a conjugate space.

In the case of a space conjugate to a union of linear topological spaces, we can speak only of weak convergence so that the "strong" versions of the definitions and results in this Appendix do not refer to this case.

In this Appendix we shall consider mainly countably normed spaces and their unions (for brevity, we shall call these and others *fundamental spaces*[22]) and also their conjugates.

Let us first establish some properties of numerical functions of the form (f_ν, φ_ν), where the φ_ν are elements of the fundamental space Φ, and the f_ν are continuous linear functionals on Φ.

Lemma. *If the sequence of elements $\{\varphi_\nu\}$ converges strongly in the space Φ to the element φ_0 as $\nu \to \nu_0$, and the sequence of functionals $\{f_\nu\}$ converges weakly to the functional f_0, then the sequence $\{(f_\nu, \varphi_\nu)\}$ converges to (f_0, φ_0).*

Proof. First let Φ be a countably normed space. We have

$$(f_\nu, \varphi_\nu) - (f_0, \varphi_0) = (f_\nu, \varphi_\nu - \varphi_0) + (f_\nu - f_0, \varphi_0).$$

The sequence $\{f_\nu\}$, being weakly convergent, is weakly bounded, and therefore (see Section 5.5) strongly bounded. The sequence $\{\varphi_\nu - \varphi_0\}$ tends to zero; therefore (see Section 5.7) we have $(f_\nu, \varphi_\nu - \varphi_0) \to 0$. The second member $(f_\nu - f_0, \varphi_0)$ also tends to zero by the definition of weak convergence of functionals. Hence

$$(f_\nu, \varphi_\nu) - (f_0, \varphi_0) \to 0.$$

Now, let $\Phi = \Phi^{(\omega)}$ be the union of countably normed spaces $\Phi^{(m)}$. In this case the convergent sequence $\varphi_\nu \to \varphi_0$ is contained in some $\Phi^{(m)}$, and the functionals f may also be regarded as functionals on $\Phi^{(m)}$; by what has been proved, $(f_\nu, \varphi_\nu) \to (f_0, \varphi_0)$.

For a weakly convergent sequence $\varphi_\nu \to \varphi_0$ the assertion of the lemma fails to hold even in normed spaces. For example, in Hilbert space any sequence of orthogonal and normalized vectors $e_1, e_2, \ldots, e_\nu, \ldots$ converges weakly to zero; however, $(e_\nu, e_\nu) = 1$ does not tend to zero.

But if it is assumed that the space Φ is perfect, or is the union of perfect spaces, then the assertion of the lemma remains true when the strong convergence $\varphi_\nu \to \varphi_0$ is replaced by weak convergence (since weak and strong convergence coincide in a perfect space). This is still another confirmation of the expediency of singling out perfect spaces.

[22] The concept of a fundamental space will be made more precise at the beginning of the next chapter.

Appendix 2

Differentiable Abstract Functions

An abstract function φ_ν is said to be *strongly* (*weakly*) *differentiable at the point* ν_0, if there exists the strong (weak) limit

$$\left.\frac{d\varphi_\nu}{d\nu}\right|_{\nu=\nu_0} = \lim_{h\to 0} \frac{\varphi_{\nu_0+h} - \varphi_{\nu_0}}{h}.$$

Naturally, this limit is called the *strong* (*weak*) *derivative* (*with respect to the parameter*).

Let us show that *the weak differentiability at a point* ν_0 *of the function* φ_ν *with values in a fundamental space or its conjugate implies its strong and weak continuity at this point.*

In fact, for any $h_n \to 0$, the sequence $\{(\varphi_{\nu_0+h_n} - \varphi_{\nu_0})/h_n\}$ converges weakly (to φ'_{ν_0}) and hence is weakly and strongly bounded. Multiplying this sequence by the number $h_n \to 0$, we conclude that the sequence $\{\varphi_{\nu_0+h_n} - \varphi_{\nu_0}\}$ converges strongly and weakly to zero; this also means strong continuity of the function φ_ν at the point ν_0.

In turn, *the weak continuity of the function* φ_ν *at a point* ν_0 *implies weak and strong boundedness of this function in some interval containing the point* ν_0.

Lemma. *If the element* φ_ν *of a fundamental space and the continuous linear functional* f_ν *on it are weakly differentiable functions of the parameter* ν, *then* (f_ν, φ_ν) *is a differentiable numerical function, and*

$$\frac{d}{d\nu}(f_\nu, \varphi_\nu) = \left(\frac{df_\nu}{d\nu}, \varphi_\nu\right) + \left(f_\nu, \frac{d\varphi_\nu}{d\nu}\right). \tag{1}$$

Proof. We have

$$\begin{aligned}\frac{\Delta(f_\nu, \varphi_\nu)}{\Delta\nu} &= \frac{(f_{\nu+\Delta\nu}, \varphi_{\nu+\Delta\nu}) - (f_\nu, \varphi_\nu)}{\Delta\nu} \\ &= \frac{(f_{\nu+\Delta\nu}, \varphi_{\nu+\Delta\nu}) - (f_\nu, \varphi_{\nu+\Delta\nu})}{\Delta\nu} + \frac{(f_\nu, \varphi_{\nu+\Delta\nu}) - (f_\nu, \varphi_\nu)}{\Delta\nu} \\ &= \left(\frac{f_{\nu+\Delta\nu} - f_\nu}{\Delta\nu}, \varphi_{\nu+\Delta\nu}\right) + \left(f_\nu, \frac{\varphi_{\nu+\Delta\nu} - \varphi_\nu}{\Delta\nu}\right).\end{aligned}$$

The ratio $(f_{\nu+\Delta\nu} - f_\nu)/\Delta\nu$ in the first term converges weakly to f'_ν, and $\varphi_{\nu+\Delta\nu}$ converges strongly to φ_ν (because of the proven strong continuity of a weakly differentiable function); therefore, by the lemma of Appendix 1, the first term has a limit (f'_ν, φ_ν). The second term has a limit

(f_ν, φ_ν') because of the assumption of the weak differentiability of the function φ_ν. The entire expression has the right side of (1) as limit for $\Delta\nu \to 0$, q.e.d.

Appendix 3

Operators Depending on a Parameter

An operator A_ν, mapping a space Φ into a space Ψ, which depends on a parameter ν is said to be *strongly* (*weakly*) *continuous* (*in the parameter* ν) for $\nu = \nu_0$, if for any $\varphi \in \Phi$

$$\lim_{\nu\to\nu_0} A_\nu\varphi = A_{\nu_0}\varphi$$

in the strong (weak) sense.

An operator A_ν is said to be (*strongly*) *weakly bounded* (*in the parammeter* ν) on the set Λ of values of ν, if the set of elements $\{A_\nu\varphi_0;\ \nu \in \Lambda\}$ is strongly (weakly) bounded for any fixed φ_0.

If the operator A_ν is weakly continuous in a closed bounded domain Λ, then it is weakly bounded in this domain. Indeed, the numerical function $(f, A_\nu\varphi)$ is continuous in the domain Λ for any $f \in \Phi'$, and therefore is weakly bounded; the set of elements $\{A_\nu\varphi;\ \nu \in \Lambda\}$ is thus weakly bounded. Naturally, if Ψ is a fundamental space or conjugate to one, then since weak and strong boundedness agree in this case, the weak continuity of the operator A_ν implies its strong boundedness.

An operator B is called *the strong* (*weak*) *derivative of the operator* A_ν at $\nu = \nu_0$, if the equality

$$B\varphi = \lim_{h\to 0} \frac{A_{\nu_0+h} - A_{\nu_0}}{h}\varphi$$

holds in the strong (weak) sense for any $\varphi \in \Phi$. In this case the operator A_ν is said to be *strongly* (*weakly*) *differentiable at* $\nu = \nu_0$.

A weakly differentiable operator is strongly and weakly continuous. (The proof is analogous to that presented above for functionals.)

If the operator $A = A_\nu$, which maps the space Φ into the space Ψ, depends on a parameter ν, then the conjugate operator $A^* = A_\nu^*$, which maps Ψ' into Φ', also depends on the parameter ν.

Lemma 1. *If the operator A_ν, which maps the fundamental space Φ into the fundamental space Ψ, is a bounded (weakly continuous, weakly differentiable) function of ν, then A_ν^* is also a bounded (weakly continuous, weakly differentiable) function of ν, and*

$$\lim_{\nu\to\nu_0} A_\nu^* = (\lim_{\nu\to\nu_0} A_\nu)^*, \qquad \frac{dA_\nu^*}{d\nu} = \left(\frac{dA_\nu}{d\nu}\right)^*. \tag{1}$$

Proof. Let A_ν be bounded for $\nu \in \Lambda$. Furthermore, let A_ν^* be the conjugate operator, and $f \in \Psi'$; then $(A_\nu^* f, \varphi) = (f, A_\nu \varphi)$ is bounded for any $\varphi \in \Phi$. Therefore, the set of functionals $A_\nu^* f \in \Phi'$ is weakly bounded for $\nu \in \Lambda$; it follows that it is strongly bounded.

Further, let the operator A_ν converge weakly to the operator A_{ν_0} as $\nu \to \nu_0$. Then for any $f \in \Psi'$ and $\varphi \in \Phi$

$$(A_\nu^* f, \varphi) = (f, A_\nu \varphi) \to (f, A_{\nu_0} \varphi) = (A_{\nu_0}^* f, \varphi),$$

so that A_ν^* converges weakly to $A_{\nu_0}^*$ as $\nu \to \nu_0$.

Applying this result to the ratio $(A_{\nu+h} - A_\nu)/h$, we arrive at the validity of the last assertion of the lemma.

Lemma 2. *If A_ν is a bounded operator which maps the fundamental space Φ into the fundamental space Ψ, and φ_ν is a bounded element in the space Φ (for $\nu \in \Lambda$) then the elements $A_\nu \varphi_\nu$ form a bounded set in the space Ψ.*

Proof. For any $f \in \Psi'$ we have

$$(f, A_\nu \varphi_\nu) = (A_\nu^* f, \varphi_\nu);$$

this quantity is bounded since the set $\{A_\nu^* f\} \subset \Phi'$ is bounded.[23]

Lemma 3. *If the operator A_ν, which maps the fundamental space Φ into the fundamental space Ψ, converges weakly to A and the element φ_ν in the space Φ converges strongly to φ as $\nu \to \nu_0$, then $A_\nu \varphi_\nu$ converges weakly to $A\varphi$.*

Proof. Let $f \in \Psi'$ be an arbitrary functional. Then

$$(f, A_\nu \varphi_\nu) = (A_\nu^* f, \varphi_\nu) \to (A^* f, \varphi) = (f, A\varphi)$$

by Lemma 1 and the lemma of Section A.1; hence, $A_\nu \varphi_\nu$ converges weakly to $A\varphi$.

Lemma 4. *If the operator A_ν, which maps the fundamental space Φ into the fundamental space Ψ, is strongly continuous in ν, and the operator B_ν, which maps Ψ into the fundamental space X, is weakly continuous in ν, then their product $C_\nu = B_\nu A_\nu$ is weakly continuous in ν.*

Indeed, let $A_\nu \to A_0$ strongly, $B_\nu \to B_0$ weakly as $\nu \to \nu_0$; let us verify that $B_\nu A_\nu \to B_0 A_0$ weakly. We put $\psi_\nu = A_\nu \varphi$ for any fixed $\varphi \in \Phi$;

[23] For unions of countably normed spaces, we use the result of Section 8.2, that a bounded set of functionals is uniformly bounded on every bounded set of elements.

by hypothesis, $\psi_\nu \to A_0\varphi$ strongly as $\nu \to \nu_0$. According to Lemma 3, the quantity $(B_\nu A_\nu)\varphi = B_\nu(A_\nu\varphi) = B_\nu\psi_\nu$ converges weakly to $B_0(A_0\varphi) = (B_0A_0)\varphi$, which leads to the desired limit relation.

We can now establish the principal result.

Theorem. *If the operator A_ν, which maps the fundamental space Φ into the fundamental space Ψ, and the element φ_ν of the space Φ are weakly differentiable at $\nu = \nu_0$, then the element $A_\nu\varphi_\nu$ of the space Ψ is also weakly differentiable at $\nu = \nu_0$, and*

$$\frac{d}{d\nu}(A_\nu\varphi_\nu) = A_\nu \frac{d\varphi_\nu}{d\nu} + \frac{dA_\nu}{d\nu}\varphi_\nu . \tag{2}$$

Proof. For any $f \in \Psi'$ we have, by applying Lemma 1,

$$\begin{aligned}\left(f, \frac{A_{\nu+h}\varphi_{\nu+h} - A_\nu\varphi_\nu}{h}\right) &= \left(f, \frac{A_{\nu+h}\varphi_{\nu+h} - A_\nu\varphi_{\nu+h}}{h}\right) + \left(f, \frac{A_\nu\varphi_{\nu+h} - A_\nu\varphi_\nu}{h}\right)\\ &= \left(\frac{A^*_{\nu+h} - A^*_\nu}{h} f, \varphi_{\nu+h}\right) + \left(A^*_\nu f, \frac{\varphi_{\nu+h} - \varphi_\nu}{h}\right)\\ &\to \left(\frac{dA^*_\nu}{d\nu} f, \varphi_\nu\right) + \left(A^*_\nu f, \frac{d\varphi_\nu}{d\nu}\right)\\ &= \left(f, \frac{dA_\nu}{d\nu}\varphi + A_\nu \frac{d\varphi_\nu}{d\nu}\right),\end{aligned}$$

from which (2) follows.

Appendix 4

Integration of Continuous Abstract Functions with Respect to the Parameter

Let φ be a continuous abstract function in the perfect space Φ for the values $a \leqslant \nu \leqslant b$. We assert that the limit of the Riemann sums

$$\lim \sum \varphi_{\nu_j} \Delta\nu_j = \int_a^b \varphi_\nu \, d\nu \tag{1}$$

exists in the space Φ.

Indeed, for any functional $f \in \Phi'$ the function (f, φ_ν), as a continuous function of ν, is integrable so that $\lim \sum (f, \varphi_{\nu_j}) \Delta\nu_j = \int_a^b (f, \varphi_\nu)\, d\nu$ exists.

By virtue of the theorem on the completeness of a perfect space relative to the weak convergence of sequences (Section 6.3), the limit (1) exists. It is easy to verify that all the customary properties are satisfied for the integral defined in this way; let us mention, in particular, the *theorem of the mean*:

$$\lim_{b \to a} \frac{1}{b - a} \int_a^b \varphi_\nu \, d\nu = \varphi_a . \tag{2}$$

The equality (2) is easily proved in the weak sense. Then it is transformed into the classical theorem; there remains to state that (2) is also valid in the strong sense, since strong and weak convergence coincide in a perfect space.

CHAPTER II

FUNDAMENTAL AND GENERALIZED FUNCTIONS

1. Definition of Fundamental and Generalized Functions

1.1. Fundamental Functions

A linear topological space Φ, formed from the functions $\varphi(x)$ defined in some set R is called a *fundamental space.*

With regard to the topological nature of the fundamental space, we shall assume that it is a countably normed space (Chapter I, Section 3), or the union of such spaces (Chapter I, Section 8).

Moreover, we assume the following natural condition to be satisfied: The convergence of the numerical sequence $\varphi_\nu(x_0)$ at any fixed point $x_0 \in R$ follows from the convergence of the sequence of elements $\varphi_\nu(x)$ of this space in topology.

Functions in the fundamental space are called *fundamental functions.*

As a rule, the set R will be an n-dimensional real space R_n or an n-dimensional complex space C_n.

We shall denote a point of the space R_n by $x = (x_1, \ldots, x_n)$ (or y, ξ, η) and a point of the space C_n by $z = (z_1, \ldots, z_n) = x + iy$ (or $\zeta = \xi + i\eta$) and a set of integers by $q = (q_1, \ldots, q_n)$ (or k, etc.). Moreover, we shall systematically use the following convenient notation. The symbol $|q|$ will denote the sum $q_1 + \cdots + q_n$, the symbol D^q the differential operator $\partial^{|q|}/(\partial x^{q_1}, \ldots, \partial x^{q_n})$, the symbol x^k the product $x_1^{k_1} \ldots x_n^{k_n}$. By analogy, we shall understand $\exp(a \mid z \mid^k)$ to denote $\exp(a(\mid z_1 \mid^{k_1} + \cdots + \mid z_n \mid^{k_n}))$. The reader, who has not yet met these notations, will quickly see that they are very convenient and that their meaning will always be clear from the context.

1.2. Examples

Let us consider some examples of fundamental spaces constructed from functions of real arguments.

1. The space $K(a)$ consisting of all infinitely differentiable functions $\varphi(x)$

defined in the n-dimensional space R_n and with support in the domain

$$G_a = \{| x_1 | \leqslant a_1 , | x_2 | \leqslant a_2 ,..., | x_n | \leqslant a_n\} \qquad (a \equiv (a_1 , a_2 ,..., a_n)).$$

We have repeatedly encountered this space earlier. The topology is given so that convergence of the sequence $\varphi_\nu(x) \in K(a)$ in topology would agree with uniform convergence of the functions $\varphi_\nu(x)$ and all their derivatives in the domain G_a . Such a topology may be defined by giving a countable set of norms

$$\| \varphi \|_p = \sup_{\substack{x \\ |q| \leqslant p}} | D^q\varphi(x)| \qquad (p = 0, 1, 2,...), \tag{1}$$

which is equivalent, as is easily verified, to the system of norms presented in Chapter I, Section 3,3. The norms (1) are also compatible. We verified in Chapter I, Section 6,2 that $K(a)$ is a perfect space.

2. The space K, consisting of all infinitely differentiable functions $\varphi(x)$, with compact support depending on $\varphi(x)$. This space, as we saw in Chapter I, Section 8 is the union of the spaces $K(a)$, where $a = (1,..., 1)$, $(2,..., 2)$,.... Each $K(a)$ is imbedded in any $K(a')$ with $a' > a$ with convergence retained. Let us recall that according to the definition of convergence in a union, the sequence $\varphi_\nu(x) \in K$ converges to zero if and only if all the $\varphi_\nu(x)$ vanish outside some finite domain B of R^n (independent of ν) and converge uniformly to zero together with all the derivatives in B. This is precisely how we defined convergence in the space K in Volume I.

3. The space S, formed from all infinitely differentiable functions $\varphi(x)$, defined in the n-dimensional space R_n , which tend to zero for $| x | \to \infty$ as well as their derivatives of all orders, more rapidly than any power of $1/| x |$.

To define the topology in the space S, we introduce a countable system of norms

$$\| \varphi \|_p = \sup_{\substack{x \\ |k|,|q| \leqslant p}} | x^kD^q\varphi(x)| \qquad (p = 0, 1, 2,...). \tag{2}$$

Evidently convergence of the sequence $\varphi_\nu(x)$ in the topology defined by these norms agrees with the convergence defined in S in Volume 1 (see Chapter I, Section 1,10). In Section 2 below, we show, in particular, that the norms (2) are pairwise comparable and that S is a perfect space.

4. The next class of spaces, containing both the spaces $K(a)$ and S as particular cases, is convenient, in its generality, for many applications.

Let be given the functions $1 \leqslant M_0(x) \leqslant M_1(x) \leqslant \cdots \leqslant M_p(x) \leqslant \cdots$, defined for all $x \in R_n$ and taking on finite or infinite values. We shall assume that these functions may vanish only simultaneously at infinity for all p and that they are continuous functions at all points where they are finite.

The fundamental space $\Phi = K\{M_p\}$ is defined as follows: It consists of all infinitely differentiable functions $\varphi(x)$, for which the products $M_p(x)\, D^q\varphi(x)$ are continuous and bounded in all space for $|q| \leqslant p$ ($p = 0, 1, 2,\ldots$). In particular, it hence follows that for each $\varphi(x) \in K\{M_p\}$, all the expressions

$$\| \varphi \|_p = \sup_{\substack{x \\ |q| \leqslant p}} M_p(x)\, | D^q\varphi(x)| \qquad (p = 0, 1, 2,\ldots) \tag{3}$$

are finite, and that where $M_p(x) = \infty$, there necessarily all $D^q\varphi(x) = 0$.

It will be shown in Section 2 that the space $K\{M_p\}$ is a complete countably normed space; furthermore, it is also a perfect space (i.e., bounded sets in this space are compact) if the functions $M_p(x)$ satisfy some additional conditions. In order to obtain the space $K(a)$ as a particular case of the space $K\{M_p\}$, it is necessary to put

$$M_p(x) = \begin{cases} 1 & \text{for} \quad |x| < a, \\ \infty & \text{for} \quad |x| \geqslant a \end{cases} \tag{4}$$

(so that $M_p(x)$ is actually independent of p); formulas (3) hence go over into formulas (1). To obtain the space S as a particular case of the space $K\{M_p\}$, it is necessary to put

$$M_p(x) = \sup_{|k| \leqslant p} | x^k | \qquad (= \sup_{|k| \leqslant p} | x_1^{k_1} \cdots x_n^{k_n} |). \tag{5}$$

This system of norms may be replaced by an equivalent system defined by the functions

$$M_p'(x) = (1 + | x_1 |)^p \cdots (1 + | x_n |)^p, \tag{6}$$

(apropos of this see Section 2.3).

Other important examples of spaces $K\{M_p\}$ will be considered in Chapter IV.

1.3. Relationships between Convergences

The following general relationships hold between the convergence in the space $K(a)$ of infinitely differentiable functions which are zero for $|x| \geqslant a$, and the convergence in any broader fundamental countably normed space.

Theorem. *If a fundamental countably normed space Φ contains the space $K(a)$ of infinitely differentiable functions with support in the sphere $|x| \leqslant a$, then from the convergence of a sequence of functions $\varphi_\nu(x)$ to zero* *in the space $K(a)$ follows the convergence of this sequence to zero in the space Φ.*

Proof. We will reduce this theorem to Theorem 2 of Chapter I, Section 7.2. A pair of complete countably normed (perfect) spaces $\Psi \subset \Phi$, were considered in this theorem, where the topologies in them were compatible in such a manner that $\varphi_0 = 0$ resulted from the relationships $\varphi_\nu \to 0$ in Ψ, $\varphi_\nu \to \varphi_0$ in Φ. The theorem stated that in this case the convergence of φ_ν to zero in Φ results from the convergence of the sequence φ_ν to zero in Ψ. Let us verify that the assumption of this theorem is satisfied in our case: To do this, let us consider the sequence φ_ν, convergent to zero in $K(a)$. Hence, the functions $\varphi_\nu(x)$ converge to zero at each point $x \in R^n$. If the functions $\varphi_\nu(x)$ converge to the function $\varphi_0(x)$ in Φ, then $\varphi_0(x) \equiv 0$. Hence, Theorem 2 of Chapter I, Section 7.2 may be applied; by applying it, we obtain what is required.

The proved theorem is valid even for the case when the space Φ is not itself countably normed, but is the union of countably normed spaces $\Phi_m (m = 1, 2,...)$; it is just necessary to require that all the spaces Φ_m, beginning with some number, contain the space $K(a)$, and to apply to these the theorem just proved.

1.4. Further Examples

Let us turn to examples of fundamental spaces constructed from functions of complex arguments.

Let us agree to say that the function $f(z)$ has an *order of growth* $\leqslant \rho$ and a *type* $\leqslant a$, if it is known that for any $\epsilon > 0$, the inequality[1]

$$|f(z)| \leqslant C_\epsilon \exp((a + \epsilon) \mid z \mid^\rho) \qquad (= C_\epsilon \exp((a + \epsilon)[\mid z_1 \mid^\rho + \cdots + \mid z_n \mid^\rho]). \tag{1}$$

is satisfied.

[1] In the theory of functions of a complex variable, the greatest lower bound of numbers ρ, i.e., the quantity

$$\overline{\lim_{z \to \infty}}[\ln \ln \max_{|z|=r} | f(z)|/\ln r],$$

is usually called the *order of an entire function* $f(z)$, and the *type of an entire function* $f(z)$ *of order* ρ is the greatest lower bound of the numbers $a + \epsilon$, that is, the quantity

$$\overline{\lim_{r \to \infty}}[\ln \max_{|z|=r} | f(z)|/r^\rho].$$

But since we shall always deal with inequalities, we shall use the definition given in the text in order to avoid constant duplication of the same stipulations.

1. The space $Z(a)$ consists of all entire analytic functions $\psi(z)$ of n complex variables $z_1 = x_1 + iy_1, \ldots, z_n = x_n + iy_n$, which satisfy the inequalities

$$| z^k\psi(z)| \leqslant C_k(\psi)\, e^{a|y|} \qquad (=C_{k_1,\ldots,k_n}(\psi) \exp[a(| y_1 | + \cdots + | y_n |)],$$
$$(k_j = 0, 1, 2,\ldots). \tag{2}$$

It is seen from the inequalities (2) that the functions $\psi(z) \in Z(a)$ have an order of growth $\leqslant 1$ and a type $\leqslant a$.

Let us prescribe the topology in the space $Z(a)$ by using the countable system of norms

$$\| \psi \|_p = \sup_{\substack{z \\ |k| \leqslant p}} | z^k\psi(x + iy)|\, e^{-a|y|} \qquad (p = 0, 1, 2,\ldots). \tag{3}$$

As we obtain from general considerations in Section 2, the space $Z(a)$ will be a perfect complete countably normed space in this topology.

2. By definition, the space Z consists of all entire functions $\psi(x + iy)$, satisfying the inequalities

$$| z^k\psi(x + iy)| \leqslant C_k(\psi)\, e^{a(\psi)|y|} \tag{4}$$

with some constant $a(\psi)$ (dependent on ψ). Hence, the space Z is, in its supply of functions, the union of all spaces $Z(a)$ $(a = 1, 2,\ldots)$. For $a < a'$, the space $Z(a)$ is imbedded in the space $Z(a')$ with convergence of the sequences retained. Hence, the space Z may be considered as the union of spaces $Z(a)$ in the sense of Chapter I, Section 8. This means that the sequence $\psi_\nu(z) \in Z$ is considered convergent to zero if and only if all the $\psi_\nu(z)$ belong to the same $Z(a)$ and converge to zero in it according to its topology.

3. Let us now give the general construction of the class of spaces of functions of complex argument, of which the space $Z(a)$ is a particular case.

Let there be given the continuous functions $M_p(z) \geqslant C(y) > 0$ $(p = 0, 1, 2,\ldots)$, which are defined for all $z = (z_1, z_2, \ldots, z_n) \in C^n$. By definition, the fundamental space $Z\{M_p\}$ consists of all entire analytic functions $\psi(z)$, for which all the expressions

$$\| \psi \|_p = \sup_z M_p(z)\, | \psi(z)| \tag{5}$$

are finite.

Evidently these expressions satisfy the customary axioms of a norm. We show in Section 2 that the space $Z\{M_p\}$ is a complete countably normed space and is also a perfect space if the functions $M_p(z)$ satisfy certain additional conditions.

In order to obtain the space $Z(a)$ as a particular case of the space $Z\{M_p\}$, we should put

$$M_p(z) = e^{-a|y|} \max_{|k|\leqslant p} | z^k | \quad (=\exp[-a(| y_1 | + \cdots + | y_n |)] \max_{|k|\leqslant p} | z_1^{k_1} \cdots z_n^{k_n} |)$$

This system of norms may be replaced by an equivalent system defined by the functions

$$M_p'(z) = e^{-a|y|} \prod_{j=1}^{n} (1 + | z_j |)^p.$$

Other important examples of spaces will be considered in Chapter IV.

1.5. Definition of Generalized Functions

Definition. A continuous linear functional (f, φ) in some fundamental space Φ is called a *generalized function.*

Hence, in contrast to the customary functions, generalized functions are not defined in themselves, but *depend on a selected fundamental space* Φ.

The set of all generalized functions on a certain fundamental space Φ, evidently coincides with the conjugate space Φ'.

Many generalized functions may be given by a formula such as

$$(f, \varphi) = \int_R f(x)\, \varphi(x)\, dx, \tag{1}$$

where $f(x)$ is a fixed function, integrable in each finite domain $G \subset R$ ("locally integrable"). The continuous linear functional given by (1) is called a *regular functional*, or a *functional of the function* $f(x)$ *type*. We shall simply identify such a functional with the corresponding function $f(x)$.

In general, the integral (1) is improper: It extends over an infinite domain; hence, it is necessary to agree on the method of summing it. We shall take a very simple condition, namely, we require that the integral (1) converge absolutely for every $\varphi(x) \in \Phi$.

This condition is satisfied automatically for any locally integrable function in the space K, since the integral (1) extends over a bounded domain because the fundamental functions $\varphi(x)$ are of bounded support.

For absolute convergence of the integral (1) in the space S for every fundamental function $\varphi(x)$, it is sufficient (and necessary) that the locally integrable function $f(x)$ have a growth not higher than a power type at infinity, i.e., that the inequality

$$|f(x)| \leqslant C(1 + |x|)^k$$

be satisfied for some $k \geqslant 0$. If this condition is satisfied, the functional (1) will be continuous.

For absolute convergence of the integral (1) for each fundamental function $\varphi(x)$ in the space $K\{M_p\}$, it is sufficient that the locally integrable function $f(x)$ have the following property: For some p, the ratio

$$r(x) = \frac{|f(x)|}{M_p(x)}$$

is a summable function of x. Let us show that the equality (1) defines a continuous linear functional in the space Φ in this case.

Actually, by virtue of the estimate

$$\left|\int f(x)\,\varphi(x)\,dx\right| \leqslant \|\varphi\|_p \int |f(x)| \frac{dx}{M_p(x)} = \|\varphi\|_p \int r(x)\,dx,$$

the integral converges absolutely for any $\varphi(x) \in K\{M_p\}$. Furthermore, if the functions $\varphi_\nu(x)$ tend to zero in the space $K\{M_p\}$, then, in particular $\|\varphi_\nu\|_p \to 0$, also, and thus

$$|(f, \varphi_\nu)| = \left|\int f(x)\,\varphi_\nu(x)\,dx\right| \leqslant \|\varphi_\nu\|_p \int r(x)\,dx \to 0;$$

hence the linear functional (1) is continuous.

Let us note that *the values of the regular functional* (f, φ) *in the space of fundamental functions determine the corresponding function* $f(x)$ *uniquely* (to within a set of measure zero); *this holds for each fundamental space* Φ *containing all infinitely-differentiable functions of bounded support.*[2]

It is sufficient to consider the regular functionals, identically null,

$$(f, \varphi) = \int f(x)\,\varphi(x)\,dx \equiv 0.$$

Let us first consider the case of functions of one variable x. Let us show that the function $f(x)$ equals zero almost everywhere in an arbitrarily assigned interval $a \leqslant x \leqslant b$. Let us take fundamental functions of support in this interval as $\varphi(x)$. Furthermore, let

$$F(x) = \int_a^x f(\xi)\,d\xi$$

[2] And generally this does not hold in spaces consisting of analytic functions; apropos of this, see below (Chapter III, Section 2.3).

be the primitive of the function $f(x)$; this is a continuous (even absolutely continuous) function, and the integration by parts formula

$$\int_a^b f(x)\,\varphi(x)\,dx = -\int_a^b F(x)\,\varphi'(x)\,dx = 0$$

holds. Since $F(x)$ is continuous, the conventional reasoning of variational calculus may be used (the DuBois-Raymond lemma), which shows that $F(x) =$ constant in $[a, b]$. But then $f(x) = F'(x) = 0$ (almost everywhere), q.e.d.

In the general case, when $x = (x_1, x_2, \ldots, x_n)$, it is possible to reason as follows. Let the equality

$$(f, \varphi) = \int \cdots \int f(x_1, \ldots, x_n)\,\varphi(x_1, \ldots, x_n)\,dx_1 \cdots dx_n = 0 \tag{2}$$

hold for each infinitely differentiable function $\varphi(x)$ of bounded support. Let us consider first functions $\varphi(x)$ of the form $\varphi_1(x_1) \cdot \varphi_2(x_2, \ldots, x_n)$. Furthermore, let

$$f_1(x_1) = \int f(x_1, \ldots, x_n)\,\varphi_2(x_2, \ldots, x_n)\,dx_2 \cdots dx_n\,;$$

when integrated in a product with any fundamental function $\varphi_1(x_1)$ the function $f_1(x_1)$ yields zero; according to what was proved, $f_1(x_1)$ equals zero for almost all x_1. Fixing some value of x_1 in (2), we arrive at an analogous equality in an $n - 1$ space, so that the proof is completed by simple induction. Thus for almost all $x_1, x_2, \ldots, x_n$, the function $f(x_1, \ldots, x_n)$ is zero, q.e.d.

By virtue of what has been proved, we may identify regular functionals with their corresponding functions. Hence, the set of those customary functions which define regular functionals may be considered as a subset of the whole set of generalized functions.

It is understood that generalized functions exist which cannot be written in the form

$$(f, \varphi) = \int f(x)\,\varphi(x)\,dx = (f, \varphi);$$

they are called *singular*.

For example, the *delta function*

$$(\delta(x - x_0), \varphi(x)) = \varphi(x_0) \tag{3}$$

is a continuous linear functional (since, as we have assumed, convergence of the sequence of values of the fundamental functions $\varphi_\nu \in \Phi$ at any fixed point x_0 follows from the convergence of the sequence of these functions in the topology of the space Φ).

Let us emphasize that the point x_0 may even be complex in the spaces of analytic functions.

In general, the functional (3) may not be written in the form (1).[3] (See, e.g., Volume 1, Chapter I, Section 1.3.)

Other examples of generalized functions which may not be written in the form (1) are given by formulas of the type

$$(f, \varphi) = \int f_0(x) \frac{\partial \varphi}{\partial x_q} \, dx$$

or, more generally

$$(f, \varphi) = \int f_0(x) \, P(D) \, \varphi(x) \, dx, \tag{4}$$

where $P(D) = \sum_{q=0}^{p} a_q D^q$ is a differential operator, and $f_0(x)$ a locally integrable function.

In reality, (4) defines a continuous linear functional in the space $K\{M_p\}$, say, if all ratios

$$r_q(x) = \frac{|f_0(x)|}{M_q(x)} \qquad (q \leqslant p)$$

are summable functions of x. The proof of this fact is analogous to that presented above for the case $p = 0$.

If the function $f_0(x)$ is not differentiable in the conventional sense to order p (which would afford the possibility of being released from application of the operator $P(D)$ to the fundamental function $\varphi(x)$ by integration by parts), the functional (4) can then not be written as

$$\int f(x) \, \varphi(x) \, dx, \tag{5}$$

i.e., does not reduce to a customary function.

Such is the situation with the delta function $\delta(x - x_0)$, say, which may not generally be written in the form (5), but may always be written in the form (4):

$$\begin{aligned} (\delta(x - x_0), \varphi) &= \varphi(x_1{}^0, \ldots, x_n{}^0) \\ &= \int_{x_j \geqslant x_j^0} \frac{\partial^n \varphi(x_1, \ldots, x_n)}{\partial x_1 \cdots \partial x_n} \, dx_1 \cdots dx_n \\ &= \int_R f_0(x) \frac{\partial^n \varphi(x)}{\partial x_1 \cdots \partial x_n} \, dx_1 \cdots dx_n \, , \end{aligned}$$

where $f_0(x)$ equals 1 for $x_j \geqslant x_j{}^0$ and equals zero elsewhere.

[3] Representation of the delta function in the form (1) is possible in some analytic-function spaces; see below (Chapter III, Section 2.3.).

Furthermore, we shall show (Section 4) that under broad assumptions, *expressions of the form* (4) *yield the general form of a continuous linear functional.*

2. Topology in the Spaces $K\{M_p\}$ and $Z\{M_p\}$

2.1. Introductory Remarks

We start this section with a brief exposition of the results which will be obtained therein.

Let us recall that the space $K\{M_p\}$ is defined by assigning a sequence of functions $M_p(x)$, satisfying the inequalities $1 \leqslant M_0(x) \leqslant M_1(x) \leqslant \cdots$, taking on finite or simultaneously infinite values, and continuous everywhere where they are finite. By definition, the space $K\{M_p\}$ consists of all infinitely differentiable functions $\varphi(x) = \varphi(x_1, \ldots, x_n)$, for which the products $M_p(x)\, D^q\varphi(x)$ ($|q| \leqslant p$) are everywhere continuous and bounded in the whole space. As we already know, the norms therein are defined by the formulas

$$\| \varphi \|_p = \sup_{\substack{x \\ |q| \leqslant p}} M_p(x)\, |D^q\varphi(x)| \qquad (p = 0, 1, 2, \ldots). \tag{1}$$

We shall show that with these norms the space $K\{M_p\}$ is a complete countably normed space.

Each sequence of functions $\{M_p(x)\}$ satisfying the conditions listed above defines some space $K\{M_p\}$. It is natural to pose the question of when two different sequences $\{M_p(x)\}$ and $\{M_p'(x)\}$ define the same space (in store of elements and in topology); in this case we agree to call the systems $\{M_p(x)\}$ and $\{M_p'(x)\}$ *equivalent*. It can be proved that the following interesting fact holds: If the spaces $K\{M_p\}$ and $K\{M_p'\}$ agree in store of elements, then they also agree in topology, i.e., the topology of the space $K\{M_p\}$ is determined uniquely by the store of its functions. For equivalence of the systems of functions $\{M_p(x)\}$ and $\{M_p'(x)\}$, it is sufficient that there exist positive constants $C(p)$ and $C'(p)$, for which

$$C(p) \leqslant \frac{M_p(x)}{M_p'(x)} \leqslant C'(p) \qquad (p = 0, 1, \ldots), \tag{2}$$

if it is assumed that this inequality is satisfied even where $M_p(x) = M_p'(x) = \infty$.

In different considerations, it is often convenient to be limited to functions of bounded support. We shall show that in the general case infinitely differentiable functions of bounded support generate a dense set in the space $K\{M_p\}$.

As we know, among the countably normed spaces, the most important are the perfect, complete countably normed spaces in which all bounded sets are relatively compact. Compliance with the following condition, which we denote by[4] (P) is sufficient for a space $K\{M_p\}$ to be perfect:

(P) For a given $\epsilon > 0$ and any p, a $p' > p$, and an N can be found such that for all x, for which at least one of the inequalities

$$|x| > N, \qquad M_p(x) > N,$$

is satisfied, the following inequality is valid

$$M_p(x) < \epsilon M_{p'}(x).$$

In particular, if the $M_p(x)$ are finite in any bounded domain, the condition (P) may be written then as

$$\lim_{|x|\to\infty} \frac{M_p(x)}{M'_p(x)} = 0. \tag{3}$$

Using this sufficient condition, it is easy to verify that, in particular, the spaces $K(a)$ and S are perfect.

Let us formulate the criterion for the convergence of the sequence $\{\varphi_\nu(x)\}$ in the space $K\{M_p\}$, which satisfies the condition (P). To do this, let us introduce the following convenient definition: We shall call a sequence of infinitely differentiable functions $\{\varphi_\nu(x)\}$ *regularly convergent* if the sequence $\{D^q\varphi_\nu(x)\}$ converges uniformly in each bounded domain for any q.

We shall see that the sequence $\{\varphi_\nu(x)\}$ converges (in topology) in the space $K\{M_p\}$, satisfying the condition (P), if and only if it is bounded in this space (i.e., $\| \varphi_\nu \|_p \leqslant C_p$) and converges regularly.

As we know, the space $Z\{M_p\}$ is defined by assigning a sequence of functions $M_p(x)$ ($p = 0, 1, \ldots$), which are assumed to be continuous everywhere and not less than 1. It consists of all entire analytic functions $\psi(z) = \psi(z_1, \ldots, z_n)$, for which all the expressions

$$\| \psi \|_p = \sup_z M_p(z) \mid \psi(z) \mid \tag{4}$$

are finite. These expressions are taken as the norms in the space $Z\{M_p\}$. We shall show below that with these norms, the space $Z\{M_p\}$ is a complete, countably normed space.

[4] (P) for perfect (parfait).

Furthermore, two systems of functions $\{M_p(z)\}$ and $\{M'_p(z)\}$ are *equivalent*, i.e., define coincident spaces $Z\{M_p\}$ and $Z\{M'_p\}$ (in store of elements and in topology), if a condition analogous to (2),

$$0 < C(p) \leqslant \frac{M_p(z)}{M'_p(z)} \leqslant C'(p) \tag{5}$$

is satisfied.

Finally, in order for the space $Z\{M_p\}$ to be perfect, it is sufficient that a condition analogous to (P) be satisfied, namely:

For each $p \in N$, a p' exists such that

$$\lim_{|z|\to\infty} \frac{M_p(z)}{M'_p(z)} = 0. \tag{P'}$$

Utilizing this condition, we verify that $Z(a)$ is a perfect space.

Let us formulate the criterion for convergence in the space $Z\{M_p\}$, which satisfies the condition (P′).

Let us call a sequence of analytic functions $\{\psi_\nu(z)\}$ *regularly convergent* if it converges uniformly in any bounded domain of the real-valued arguments $(x_1, \ldots, x_n)$. As in the case of the spaces $K\{M_p\}$, in order for the sequence $\{\psi_\nu(z)\}$ of elements of the space $Z\{M_p\}$ to converge (in its topology), it is necessary and sufficient that it converge regularly and be bounded (in the norms) in $Z\{M_p\}$.

(Let us note that regular convergence for a bounded sequence $\psi_\nu(z)$ follows from uniform convergence in a fixed domain.)

2.2. The Space $K\{M_p\}$ as a Complete Countably Normed Space

Let us turn to the proof of the results expounded in Section 2.1. We will show in this paragraph that with the norms

$$\|\varphi\|_p = \sup_{\substack{x \\ |q| \leqslant p}} M_p(x) \, |D^q\varphi(x)| \qquad (p = 0, 1, 2, \ldots), \tag{1}$$

$K\{M_p\}$ is a complete countably normed space.

Let Φ_p denote the set of all functions $\varphi(x)$, having continuous derivatives to order p, for which the functions $M_p(x)\, D^q\varphi(x)$ are continuous and bounded in the whole space. Evidently Φ_p is a normed linear space if (1) defines the norm in it. The intersection of the spaces Φ_p over all p in N coincides, as is easy to see, with the space $\Phi = K\{M_p\}$.

Our immediate aim is to prove the completeness of the space Φ_p. The lemmas needed for this theorem will also be utilized later.

Let be given the sequence of functions $\varphi_\nu(x) \in \Phi_p$ such that the functions $\varphi_\nu(x)$ themselves and their derivatives to order p converge uniformly in each compact domain to some limit functions. If

$$\varphi_0(x) = \lim_{\nu\to\infty} \varphi_\nu(x),$$

then by virtue of the classical theorem on differentiation of a uniformly convergent sequence, the function $\varphi_0(x)$ also has derivatives to order p and for any q, $|q| \leqslant p$,

$$D^q\varphi_0(x) = \lim_{\nu\to\infty} D^q\varphi_\nu(x).$$

Now, let $\varphi_\nu(x)$ ($\nu = 1, 2,\ldots$) be a fundamental sequence in the norm of the space Φ_p. Then, as follows from the definition of the norm (1), this sequence will converge uniformly, together with its derivatives to order p, in each compact domain. We will show that the *limit function* $\varphi_0(x)$ *belongs to the space* Φ_p *and is the limit of the sequence* $\varphi_\nu(x)$ *in the norm* Φ_p, *i.e.*,

$$\| \varphi_0 - \varphi_\nu \|_p \to 0 \qquad \text{for} \quad \nu \to \infty$$

Lemma 1. *The limit function* $\varphi_0(x)$ *of the sequence* $\varphi_\nu(x) \in \Phi_p$, *which converges uniformly for* $\nu \to \infty$ *together with its derivatives to order* p *in each compact domain and is bounded in the norm* (1) *by a constant* C, *also belongs to the space* Φ_p *and has a norm not exceeding* C *therein.*

Proof. As has been mentioned, the function $\varphi_0(x)$ possesses derivatives of all orders q, $|q| \leqslant p$. Let us show that the expression (1) for $\varphi_0(x)$ is finite.

Let x_0 be a point at which $M_p(x_0) < \infty$. Since the function $M_p(x)$ is continuous at such a point, there is a neighborhood of the point x_0 in which the function $M_p(x)$ deviates from $M_p(x_0)$ by not more than $\epsilon/2$, where ϵ is some previously assigned positive number. Hence, it is possible to find a number ν_0 for this neighborhood such that for $\nu > \nu_0$

$$M_p(x) \mid D^q\varphi_0(x)| \leqslant M_p(x) \mid D^q\varphi_\nu(x)| + \epsilon.$$

But by assumption

$$\sup_{\substack{x \\ |q|\leqslant p}} M_p(x) \mid D^q\varphi_\nu(x)| = \| \varphi_\nu \|_p \leqslant C;$$

hence, in this neighborhood

$$M_p(x) \mid D^q\varphi_0(x) \mid \leqslant C + \epsilon.$$

At points x_0, where $M_p(x_0) = \infty$, the functions $\varphi_\nu(x)$ vanish together with all derivatives of orders q, $|q| \leqslant p$. Hence, the function $\varphi_0(x)$ together with all the corresponding derivatives also equals zero at such points.

We see that the expression (1) exists for the function $\varphi_0(x)$ and does not exceed $C + \epsilon$. Since ϵ is arbitrary, $\| \varphi_0 \|_p \leqslant C$, q.e.d.

Lemma 2. *Every fundamental sequence $\varphi_\nu(x) \in \bar{\Phi}_p$ in the norm $\| \varphi \|_p$, which converges to zero at each point x, will converge in this norm to the function $\varphi_0(x) \equiv 0$.*

Proof. Since $\varphi_\nu(x)$ is fundamental, then in conformity with the above, a function $\varphi_0(x)$ exists which has continuous derivatives to order p and such that

$$D^q\varphi_\nu(x) \to D^q\varphi_0(x) \qquad (|q| \leqslant p)$$

uniformly in each compact domain for $\nu \to \infty$. From the assumption of the lemma, it follows that $\varphi_0(x) \equiv 0$.

For a given $\epsilon > 0$, we can find a $\nu_0(\epsilon)$ such that for every $\nu \geqslant \nu_0$, $\mu \geqslant \nu_0$, we have $\| \varphi_\nu - \varphi_\mu \|_p \leqslant \epsilon$. The sequence of differences $\varphi_\nu - \varphi_\mu$ converges uniformly for $\mu \to \infty$ to a function $\varphi_\nu(x)$ in each compact domain, together with its derivatives to the order p. According to the preceding lemma,

$$\| \varphi_\nu \|_p \leqslant \epsilon,$$

from which what is needed results.

Theorem 1. *The space $\bar{\Phi}_p$ is complete relative to the norm $\| \varphi \|_p$.*

Proof. Let $\varphi_\nu(x) \in \bar{\Phi}_p$ be a fundamental sequence in the norm $\| \varphi \|_p$. As we have already remarked, for $\nu \to \infty$ this sequence converges uniformly, together with its derivatives to order p, to some function $\varphi_0(x)$ in each compact domain. Since the norms $\| \varphi_\nu \|_p$ are bounded, then $\varphi_0(x) \in \bar{\Phi}_p$ according to Lemma 1. Again the difference $\varphi_\nu - \varphi_0$ is a fundamental sequence which converges to zero at each point; $\| \varphi_\nu - \varphi_0 \|_p \to 0$, according to Lemma 2, from which it follows that

$$\varphi_0 = \lim_{\nu \to \infty} \varphi_\nu$$

in the norm $\| \varphi \|_p$. The theorem is thus proved.

Corollary. *The completion Φ_p* of the space $\Phi = K\{M_p\}$ in the norm $\| \varphi \|_p$ is a subspace of the space $\bar{\Phi}_p$. (This is a particular case of a well-known fact: The completion of a metric space isomorphic to a part M' of a complete space M, is isometric to the closure of M' in the space M.)

In particular, we hence obtain that *the intersection of the spaces Φ_p for all $p = 0, 1, 2,\ldots$ coincides with the space $\Phi = K\{M_p\}$.*

In connection with the compatibility of the norm $\| \varphi \|_p$, which we verify at once, this result will yield completeness of the space $K\{M_p\}$. In fact, as has been shown in Chapter I, Section 3.2 the condition of completeness of the countably normed space Φ is again compliance with the equality

$$\Phi = \bigcap_{p=1}^{\infty} \Phi_p .$$

Now, let us show that every two norms $\| \varphi \|$ and $\| \varphi \|'$ of those which have been established in the space $K\{M_p\}$ are compatible in pairs, i.e., that every sequence $\varphi_\nu \in \Phi$, which is fundamental in both norms and converges to zero in one of the norms, will also converge to zero in the other norm.

Lemma 3. *Let the two norms*

$$\| \varphi \|_1 = \sup_{\substack{x \\ |q| \leqslant p_1}} M_1(x) \mid D^q \varphi(x) \mid, \qquad \| \varphi \|_2 = \sup_{\substack{x \\ |q| \leqslant p_2}} M_2(x) \mid D^q \varphi(x) \mid$$

be given in the space Φ of the functions $\varphi(x)$. Then, these norms are mutually compatible.

Proof. Let the sequence $\varphi_\nu(x) \in \Phi$ be fundamental in both norms, and tend to zero in one of them, in $\| \varphi \|_1$ for definiteness:

$$\| \varphi_\nu \|_1 = \sup_{\substack{x \\ |q| \leqslant p_1}} M_1(x) \mid D^q \varphi_\nu(x) \mid_{\nu \to \infty} \to 0, \tag{2}$$

$$\| \varphi_\nu - \varphi_\mu \|_2 = \sup_{\substack{x \\ |q| \leqslant p_2}} M_2(x) \mid D^q [\varphi_\nu(x) - \varphi_\mu(x)] \mid_{\nu \to \infty} \to 0. \tag{3}$$

Setting $q = 0$ in (2), we obtain, in particular, that the functions $\varphi_\nu(x)$ tend uniformly to zero at each point. Now, let us apply Lemma 2 to the norm $\| \cdot \|_2$; we obtain that $\| \varphi_\nu \|_2 \to 0$, q.e.d.

Hence, the following assertion has been proved.

Theorem 2. *The space $K\{M_p\}$ is a complete countably normed space.*

2.3. Condition for Perfection of the Space $K\{M_p\}$

Let us turn to an analysis of the question of the perfection of the space $K\{M_p\}$.

Let us prove that the space $K\{M_p\}$ is perfect if the following condition is satisfied.

(P) For a given $\epsilon > 0$ and any p, we can find a $p' > p$ and an $N > 0$, such that if x satisfies at least one of the inequalities

$$|x| > N, \qquad M_p(x) > N,$$

then

$$M_p(x) < \epsilon M_{p'}(x).$$

It may first be deduced from condition (P) that *all products $M_p(x)\, D^q\varphi(x)$ ($|q| \leqslant p$) are not only bounded in the whole space, but also tend to zero as $|x| \to \infty$ or $M_p(x) \to \infty$.*

Indeed, assuming the opposite, for some subscripts q and p, $|q| \leqslant p$, we could find a sequence x_ν, going to infinity or such that $M_p(x_\nu) \to \infty$, for which

$$M_p(x_\nu)\, |D^q\varphi(x_\nu)| \geqslant C > 0.$$

For the subscript p', corresponding to the subscript p by the condition (P), we have

$$M_p(x_\nu) \leqslant \epsilon_\nu M_{p'}(x_\nu),$$

where $\epsilon_\nu \to 0$. Hence

$$M_{p'}(x_\nu)\, |D^q\varphi(x_\nu)| \geqslant \frac{C}{\epsilon_\nu} \to \infty$$

which contradicts the assumption that the function $\varphi(x)$ belongs to the space $K\{M_p\}$.

The following lemma is the basis of the proof of the perfection of the space $K\{M_p\}$.

Lemma. *Every sequence $\varphi_\nu(x) \in K\{M_p\}$, bounded in each of the norms $\|\varphi\|_p$ and converging regularly to zero, also converges to zero in each of the norms $\|\varphi\|_{p'}$.*

Proof. Fixing p, we find the subscript p' from the condition (P). Since the sequence φ_ν is bounded, the quantities $\|\varphi_\nu\|_{p'}$ do not exceed a constant C.

For a given $\epsilon > 0$, we find a number N such that

$$M_p(x) \leqslant \frac{\epsilon}{C} M_{p'}(x). \tag{1}$$

for both $|x| > N$, and $M_p(x) > N$ at those points where $M_p(x)$ and $M_{p'}(x)$ are finite. Then for the mentioned x and any q, $|q| \leqslant p$, we have

$$M_p(x) \mid D^q\varphi_\nu(x)| \leqslant \frac{\epsilon}{C} M_{p'}(x) \mid D^q\varphi_\nu(x)| \leqslant \frac{\epsilon}{C} \| \varphi_\nu \|_{p'} \leqslant \epsilon. \tag{2}$$

At those points x, where $M_p(x) = M_{p'}(x) = \infty$, we have

$$M_p(x) \mid D^q\varphi_\nu(x)| = 0.$$

For all the remaining x (forming a compact set) a ν_0 exists such that for $\nu > \nu_0$ and all q, $|q| \leqslant p$, we have

$$M_p(x) \mid D^q\varphi_\nu(x)| \leqslant \epsilon. \tag{3}$$

Utilizing (2), we see that (3) is actually satisfied for $\nu > \nu_0$ and for all $x \in R$. Hence, for $\nu > \nu_0$,

$$\| \varphi_\nu \|_p = \sup_x M_p(x) \mid D^q\varphi_\nu(x)| \leqslant \epsilon.$$

Therefore, $\| \varphi_\nu \|_p \to 0$, q.e.d.

Corollary. *If the sequence $\varphi_\nu \in K\{M_p\}$ is bounded in each of the norms $\| \varphi \|_p$ and converges regularly to some function $\varphi_0(x)$ as $\nu \to \infty$, then this function $\varphi_0(x)$ belongs to the space $K\{M_p\}$ and is the limit function of the sequence $\varphi_\nu(x)$ in the topology of the space $K\{M_p\}$.*

In fact, by virtue of Lemmas 1 and 2, the function $\varphi_0(x)$ belongs to each Φ_p and thereby, also to the space

$$K\{M_p\} = \bigcap_{p=1}^{\infty} \Phi_p .$$

The difference $\varphi_0 - \varphi_\nu$ is bounded in each of the norms $\| \varphi \|_p$ and converges regularly to zero. Applying the lemma just proved, we obtain that for any $p \in N^*$,

$$\| \varphi_\nu - \varphi_0 \|_p \to 0,$$

that is, that φ_0 is the limit of the sequence $\{\varphi_\nu\}$ in the topology of the space Φ, q.e.d.

We are now prepared to prove the fundamental theorem on the perfection of the countably normed space $K\{M_p\}$ with a system of norms satisfying condition (P).

Theorem. *If the functions* $M_p(x)$ $(p = 0, 1, 2,...)$ *satisfy condition* (P), *then the space* $K\{M_p\}$ *is perfect.*

The condition of this theorem is satisfied in the spaces $K(a)$ and S. This holds for the space $K(a)$ because of the fact that the corresponding functions $M_p(x)$ become infinite outside some ball. For the space S, it is possible to set (see the next paragraph)

$$M_p(x) = \prod_{j=1}^{n} (1 + | x_j |)^p,$$

and it is evidently sufficient to take $p' = p + 1$ in order to satisfy condition (P). Hence, in particular, we obtain from this theorem that the spaces $K(a)$ and S are perfect. However, we had established the perfection of the space $K(a)$ in Chapter I, Section 6; the perfection of the space S could also have been verified directly.

Proof. We have already seen that $K\{M_p\}$ is a complete countably normed space. Let us prove that each bounded set $A \subset K\{M_p\}$ is compact.

Let $\varphi_\nu \in A$ $(\nu = 1, 2,...)$ be an arbitrary bounded sequence; it is sufficient to show that it contains a convergent sub-sequence.

Because of the boundedness of the norm $\| \varphi_\nu \|_1$, the functions $| \partial\varphi_\nu(x)/\partial x_j |$ $(j = 1, 2,... n)$ are uniformly bounded. Hence, by virtue of the Arzela theorem, a sub-sequence $\varphi_{11}, \varphi_{12},...$ exists which converges uniformly for $| x | \leqslant 1$. Because of the boundedness of the norm $\| \varphi_{1\nu} \|_2$ for $| x | < 2$, the values of $| \partial^2\varphi_{1\nu}(x)/\partial x_i \partial x_j |$ are bounded. Hence, according to the same Arzela theorem, it contains a sub-sequence $\varphi_{21}, \varphi_{22},...$, for which the values of the first derivatives $\partial\varphi_{2\nu}(x)/\partial x_j$ converge uniformly in the domain $| x | \leqslant 2$. From the convergence of the functions $\varphi_{2\nu}$ for $| x | \leqslant 1$ and the uniform convergence of their derivatives for $| x | \leqslant 2$, there results the uniform convergence of these functions for $| x | \leqslant 2$ also. Continuing further in this manner, and then applying a diagonalization process, we obtain a bounded sub-sequence $\varphi_{11}, \varphi_{22},...$ which converges uniformly together with all its derivatives to some limit function $\varphi_0(x)$ in any bounded domain. Applying the corollary of Lemma 4, we obtain that the sequence $\varphi_{\nu\nu}$ converges to the element φ_0 in the topology of the space $K\{M_p\}$, q.e.d.

Later, speaking of the spaces $K\{M_p\}$, we shall, as a rule, assume that the condition (P) is satisfied, without stating it explicitly.[5]

[5] There is no necessity for this in Section 2.4.

2.4. Equivalent Systems of Norms

Let there be given two systems of functions $M_p(x)$ and $M_p'(x)$ ($p = 0, 1, 2,\ldots$), satisfying the conditions

$$0 < C(p) \leqslant \frac{M_p(x)}{M_p'(x)} \leqslant C'(p), \tag{1}$$

where $C(p)$ and $C'(p)$ are some constants. We shall assume this inequality to be satisfied even where $M_p(x) = M_p'(x) = \infty$.

Let us show that the countably normed spaces $K\{M_p\}$ and $K\{M_p'\}$, constructed respectively by means of the functions $M_p(x)$ and $M_p'(x)$, coincide (in store of elements and in topology).

In fact, if the expression

$$\| \varphi \|_p = \sup_{\substack{x \\ |q| \leqslant p}} M_p(x) \mid D^q \varphi(x) \mid,$$

is finite for some function $\varphi(x)$, then the expression

$$\begin{aligned} \| \varphi \|_p' &= \sup_{\substack{x \\ |q| \leqslant p}} M_p'(x) \mid D^q \varphi(x) \mid \\ &\leqslant \frac{1}{C(p)} \sup_{\substack{x \\ |q| \leqslant p}} M_p(x) \mid D^q \varphi(x) \mid = \frac{1}{C(p)} \| \varphi \|_p \end{aligned} \tag{2}$$

is also finite; the converse is also true since

$$\| \varphi \|_p \leqslant C'(p) \cdot \| \varphi \|_p' . \tag{3}$$

Therefore, the spaces $K\{M_p\}$ and $K\{M_p'\}$ consist of the same functions. The inequalities (2), (3) show, moreover, that the systems of norms $\| \varphi \|$ and $\| \varphi \|'$ are equivalent in the sense of Chapter I, Section 3.6, it hence follows that the spaces $K\{M_p\}$ and $K\{M_p'\}$ coincide as topological spaces. As we have already stated in Section 2.1, we shall call the systems of functions $M_p(x)$ and $M_p'(x)$, satisfying conditions (1), *equivalent.*

Example. Let

$$M_p(x) = \sup_{|k| \leqslant p} \mid x_1^{k_1} \cdots x_n^{k_n} \mid$$

and

$$M_p'(x) = (1 + \mid x_1 \mid)^p \cdots (1 + \mid x_n \mid)^p$$

(the space S). Then evidently

$$M_p(x) \leqslant M'_p(x).$$

On the other hand, since

$$1 + |x_i| \leqslant \begin{cases} 2 & \text{for} \quad |x_i| \leqslant 1, \\ 2\,|x_i| & \text{for} \quad |x_i| \geqslant 1, \end{cases}$$

then

$$M'_p(x) \leqslant 2^{np} M_p(x).$$

Therefore, the systems of functions $\{M_p(x)\}$ and $\{M'_p(x)\}$ are equivalent.

2.5. Functions of Bounded Support in the Space $K\{M_p\}$

Let us show that *the set of all functions of bounded support is dense in any space* $K\{M_p\}$.[6]

It is evident that if the functions $M_p(x)$ are finite everywhere, the space $K\{M_p\}$ then contains all infinitely differentiable functions of bounded support.

Let us construct an arbitrary infinitely differentiable function $h(x)$, which equals 1 for $|x| \leqslant 1$ and zero for $|x| \geqslant 2$. Let us set

$$m_p = \max_{\substack{x \\ |q| \leqslant p}} |D^q h(x)|.$$

It is evident that for any $\nu = 1, 2, \ldots$, the inequality

$$\max_{\substack{x \\ |q| \leqslant p}} \left| D^q h\left(\frac{x}{\nu}\right) \right| \leqslant m_p$$

holds. Let us determine a sequence of infinitely differentiable functions of bounded support $\varphi_\nu(x) = \varphi(x) \cdot h(x/\nu)$ $(\nu = 1, 2, \ldots)$ for a given $\varphi(x) \in \Phi = K\{M_p\}$. Let us show that these functions converge to the function $\varphi(x)$ in the topology of the space Φ as $\nu \to \infty$. In conformity with the results of Section 2.7, it is sufficient to verify that the sequence of functions $\varphi_\nu(x)$ converges regularly to $\varphi(x)$ and is bounded in each of the norms $\|\varphi\|_p$.

The regular convergence of the sequence $\varphi_\nu(x)$ results from the fact that, starting with some number $\nu > \nu_0$, the functions $\varphi_\nu(x)$ coincide

[6] We assume condition (P) to be satisfied.

with the function $\varphi(x)$ in any bounded domain. Let us now estimate the numbers $\| \varphi_\nu \|_p$. For $| q | \leqslant p$ we have[7]

$$M_p(x) \mid D^q\varphi_\nu(x)| = M_p(x) \left| \sum_k C_q{}^k D^k h \left(\frac{x}{\nu}\right) D^{q-k}\varphi(x)\right|$$

$$\leqslant \sum_k C_q{}^k m_p M_p(x) \mid D^{q-k}\varphi(x)| \leqslant C_p m_p \| \varphi \|_p ,$$

from whence it results that the numbers $\| \varphi_\nu \|_p$ are bounded by a constant independent of ν. Therefore, the sequence $\varphi_\nu(x)$ is bounded in the space $K\{M_p\}$. As we have seen, this is sufficient for the validity of our assertion.

2.6. The Spaces $Z\{M_p\}$

Now, let us turn to the spaces $\Psi = Z\{M_p\}$.

We let $\overline{\Psi}_p$ denote the set of entire analytic functions for which $\| \psi \|_p < \infty$ for fixed p.

Lemma 1. *If the sequence $\psi_\nu(z) \in \overline{\Psi}_p$ is bounded in the norm $\| \psi \|_p$ by a constant C and converges regularly, then its limit $\psi_0(z)$ also belongs to the space $\overline{\Psi}_p$ and has therein a norm not exceeding C.*

Proof. The sequence $\psi_\nu(z)$ is bounded in modulus in each bounded domain. Since it converges to some limit function $\psi_0(z)$ for real $z = x$, then by virtue of the fundamental theorem for analytic functions it converges to the limit $\psi_0(z)$ in each bounded domain, where $\psi_0(z)$ is the analytic continuation of the function $\psi_0(x)$.

Furthermore, we may apply reasoning analogous to that used in the proof of Section 2.2, Lemma 1; it shows that the quantity $\| \psi_0 \|_p$ is bounded by a number C, as is required.

Now repeating the reasoning of Section 2.2, we find that the space $\overline{\Psi}_p$ is complete relative to the norm $\| \psi \|_p$ and, therefore, the completion Ψ_p of the space Ψ in the norm $\| \psi \|_p$ is some subspace of the space $\overline{\Psi}_p$. Hence, it furthermore follows that the intersection of all Ψ_p coincides with our Ψ.

Exactly as in Section 2.2, we prove that *the norms $\| \psi \|_p$ are pairwise compatible.* Therefore, *$Z\{M_p\}$ is a complete, countably normed space.* Let

[7] In the n-dimensional case,

$$q = (q_1 ,..., q_n), \quad k = (k_1 ,..., k_n), \qquad \text{and} \quad C_q{}^k = C_{q_1}^{k_1} \cdots C_{q_n}^{k_n}$$

where $C_{q_i}^{k_i}$ are the customary binomial coefficients.

us impose the additional condition on the functions $M_p(z)$: *For any p, there exists a $p' > p$ for which*

$$\lim_{|z|\to\infty} \frac{M_p(z)}{M_{p'}(z)} = 0. \tag{1}$$

Then exactly as in Section 2.3, we arrive at the following result: If a sequence $\psi_\nu(z) \in \Psi$ is bounded in each of the norms $\| \psi \|_p$ and hence converges regularly to some function $\psi_0(z)$ (given initially only for real $z = x$), then $\psi_0(z) \in \Psi$ is indeed the limit function of the sequence $\{\psi_0(z)\}$ in the topology of the space Ψ.

Exactly as in Section 2.3, this result leads us to the concluding theorem.

Theorem 1. *The space $Z\{M_p\}$ is complete under the condition* (1).

Exactly as in Section 2.4, it may be established that two systems of functions $M_p(z)$ and $M_p'(z)$, satisfying the inequalities

$$0 < C(p) \leqslant \frac{M_p(z)}{M_p'(z)} \leqslant C'(p), \tag{2}$$

determine coincident spaces $Z\{M_p\}$ and $Z\{M_p'\}$ (in store of elements and topology); the corresponding systems of norms $\| \psi \|_p$ and $\| \psi \|_p'$ and the systems of functions $M_p(z)$ and $M_p'(z)$ themselves are called *equivalent*.

Condition (1) is known to be satisfied in the space $Z(a)$. Indeed, because of the preceding theorem, in this case it is possible to put

$$M_p(z) = e^{-a|y|} \prod_{j=1}^{n} (1 + | z_j |^p);$$

it is evidently sufficient to take $p' = p + 1$ to comply with condition (1). Therefore, because of the last theorem, the space $Z(a)$ is perfect.

3. Operations with Generalized Functions

3.1. Linear Operations and Passage to the Limit

We have defined generalized functions as continuous linear functionals on some fundamental space Φ. Therefore, the set of generalized functions on the space Φ is the conjugate space Φ'.

Linear operations and the passage to the limit are defined naturally for generalized functions as elements of the conjugate space.

Linear operations in Φ'—addition and multiplication by a number—are given by the formulas[8]

$$(f_1 + f_2, \varphi) = (f_1, \varphi) + (f_2, \varphi),$$
$$(\alpha f, \varphi) = \alpha(f, \varphi).$$

As regards the passage to the limit, in this chapter we shall use only "weak" convergence, namely, we shall say that the sequence $f_\nu \in \Phi'$ converges to the functional $f \in \Phi'$, if for each φ there holds the relationship

$$(f_\nu, \varphi) \to (f, \varphi).$$

We have proved in Chapter I, Sections 5 and 8, that the conjugate space Φ' is complete relative to weak convergence of sequences. This means that the space of generalized functions has the following property: If for each $\varphi \in \Phi$, the numerical sequence (f_ν, φ) has the limit $f(\varphi)$, then $f(\varphi)$ is also a continuous linear functional.

Let us indicate a simple sufficient condition for convergence of the sequence of functionals $f_\nu \in \Phi'$, corresponding to the functions $f_\nu(x)$ $(\nu = 1, 2, \ldots)$ (we call such functionals regular).

Theorem. *Let it be known that a sequence of functions $\{f_\nu(x)\}$ satisfies the following conditions:*

(a) $|f_\nu(x)| \leqslant g(x)$, *where $g(x)$ is a locally integrable function also defining a functional on the space Φ;*

(b) *The limit relationship*

$$\lim f_\nu(x) = f_0(x)$$

holds almost everywhere.

Then the function $f_0(x)$ also defines a continuous linear functional f_0 on the space Φ and $\lim f_\nu = f_0$ *in the weak topology of the space Φ'.*

Proof. Applying the Lebesgue theorem to the sequence $f_\nu(x)\,\varphi(x)$, which is majorized by the integrable function $g(x)\,|\varphi(x)|$ in the domain $|x| \leqslant a$, we obtain the estimate

$$\int_{|x|\leqslant a} |f_0(x)\,\varphi(x)|\,dx = \lim_{\nu\to\infty} \int_{|x|\leqslant a} |f_\nu(x)\,\varphi(x)|\,dx$$
$$\leqslant \int_{|x|\leqslant a} g(x)\,|\varphi(x)|\,dx \leqslant \int_R g(x)\,|\varphi(x)|\,dx,$$

[8] In the case of complex numbers, we have

$$(\alpha f, \varphi) = (f, \bar{\alpha}\varphi) = \bar{\alpha}(f, \varphi).$$

which guarantees absolute convergence of the integral

$$\int_R f_0(x)\,\varphi(x)\,dx = (f_0\,,\varphi).$$

As the (weak) limit of continuous functionals f_ν , the functional f_0 is also continuous, q.e.d.

Let us note that regular functionals may converge to singular functionals. Thus, in Volume 1, Chapter I, Section 2 we constructed various sequences of regular functionals, which converge to the delta function. Furthermore, we shall see in Section 4 that in a broad class of spaces regular functionals form a dense set among all the functionals.

Further operations in the space Φ' may be defined in terms of the operations existing in the space Φ as conjugate operations to the latter (see Chapter I, Sections 7 and 8). In the space Φ', the designations of operations obtained by such means depend on which (classical) operations of analysis the given conjugate operation will correspond to if it is applied to functionals of the function type.

3.2. Multiplication by a Function

Let us assume that multiplication by some function $g(x)$ is a continuous linear operation in the fundamental space Φ. In conformity with Chapter I, Section 7, this means that for any $\varphi \in \Phi$, the product $g\varphi$ again belongs to Φ and that $g\varphi_\nu \to 0$ in the topology of Φ results from $\varphi_\nu \to 0$. Such a function $g(x)$ is called a *multiplier in the space Φ.*

An operation in the space Φ', defined by the formula

$$(gf, \varphi) = (f, g\varphi), \tag{1}$$

that is, an operation conjugate to the operation of multiplication by $g(x)$ in the space Φ, is called *multiplication by the function $g(x)$*[9]

Let us confirm the soundness of this terminology. If f is a functional of the type of the function $f(x)$, we then have

$$(gf, \varphi) = (f, g\varphi) = \int f(x)\,g(x)\,\varphi(x)\,dx$$

$$= \int [f(x)\,g(x)]\,\varphi(x)\,dx = (g(x)f(x), \varphi),$$

[9] In the complex case, multiplication by the function $g(x)$ in the space Φ' is defined by the formula

$$(gf, \varphi) = (f, \bar{g}\varphi),$$

under the assumption that $\bar{g}$ is a multiplier in Φ.

that is, gf is actually a functional of the function type—the product $g(x)f(x)$.

Examples. Any infinitely differentiable functions (in the domain $|x| \leqslant a$ or in the whole space, respectively) are multipliers in the spaces $K(a)$ and K.

A multiplier in the space S may be any infinitely differentiable function $g(x)$, each of whose derivatives has a growth not higher than a power type at infinity:

$$|D^q g(x)| \leqslant C_q(1 + |x|)^{k_q} \qquad (|q| = 0, 1, 2, \ldots).$$

For the proof, it is necessary to estimate the expression

$$x^k D^q[g(x)\varphi(x)] \qquad \text{for} \qquad |q| \leqslant p, \quad |k| \leqslant p.$$

For simplicity, let us limit ourselves to the case of one independent variable. In this case, we have

$$\begin{aligned} |x^k D^q g\varphi| &\leqslant \sum_j |x^k| \, |D^j g| \, |D^{q-j}\varphi| \, C_q{}^j \\ &\leqslant |x^k| \sum_j C_q{}^j C_j (1 + |x|)^{k_j} |D^{q-j}\varphi| \\ &\leqslant \sum_j C_q{}^j C_j (1 + |x|)^{k_j + k} |D^{q-j}\varphi| \\ &\leqslant \sum_j C_q{}^j C_j \|\varphi\|_{k+p} = C\|\varphi\|_{2p}\,. \end{aligned} \tag{2}$$

Hence, the function $|x^k D^q g\varphi|$ is bounded for $|k| \leqslant p$, $|q| \leqslant p$. This means that $g\varphi \in S$. Moreover, the inequality (2) shows that the operation of multiplication by the function $g(x)$ transforms a bounded set in the space S again into a bounded set; it is therefore bounded, and therefore, also continuous.

An analogous discussion may also be carried out in the space $K\{M_p\}$ under the following condition: *For every two subscripts p and r, $p \geqslant r$, a subscript $s \geqslant p$ exists such that*

$$M_p(x) \cdot M_r(x) \leqslant C_{pr} M_s(x). \tag{3}$$

Then every infinitely differentiable function $g(x)$ satisfying the inequalities

$$|D^q g(x)| \leqslant C_q M_{k_q}(x) \qquad (\text{for each } q)$$

is a multiplier in the space $K\{M_p\}$.

The proof proceeds according to the scheme presented for the space S.

Multipliers in the space $Z(a)$ are polynomials in z; and in the space Z they are analytic functions $g(z)$, satisfying inequalities of the form

$$|g(z)| \leqslant C(1 + |z^m|)\, e^{b|y|} \tag{4}$$

for some $b > 0$ and $m \geqslant 0$.

The first assertion is evident; let us prove the second.

It is easy to see that the product of an entire function $g(z)$, satisfying the inequality (4), and the function $\varphi(z) \in Z$ is again a function of the space Z. More accurately, if $\varphi(z)$ belongs to $Z(a)$, then the product $g\varphi$ belongs to $Z(a + b)$. Hence, if $\| \varphi \|_p^{(a)}$ denotes the p-th norm in the space $Z(a)$, we then obtain

$$\begin{aligned} \| g\varphi \|_p^{(a+b)} &= \sup_{\substack{z \\ |k| \leqslant p}} | z^k g(z)\, \varphi(z)|\, e^{-(a+b)|y|} \\ &\leqslant \sup_{\substack{z \\ |k| \leqslant p}} | z |^k\, C(1 + | z^m |)\, e^{b|y|}\, | \varphi(z)|\, e^{-(a+b)|y|} \\ &\leqslant C(\| \varphi \|_{p+m}^{(a)} + \| \varphi \|_p^{(a)}), \end{aligned}$$

and therefore, bounded sets in the space Z, when multiplied by g, go over again into bounded sets; hence, multiplication by g is a bounded, and consequently, continuous operation.

Multiplication by functions of corresponding types is defined in the corresponding conjugate spaces $K'(a)$, K', S', $K'\{M_p\}$, $Z'(a)$, Z'.

The reader will encounter a number of other examples in Chapter IV.

3.3. Division of Unity by a Polynomial in the Space Z'

In many questions, it turns out to be necessary to solve the equation

$$\bar{P}(z) f = 1, \tag{1}$$

in terms of generalized functions, where $P(z) = P(z_1, z_2, \ldots, z_n)$ is a polynomial and f is the unknown functional. We show here that *Eq.* (1) *is always solvable in the space Z'*.

Let us assume first that $P(z)$ has a rather special form

$$P(z) = a z_1{}^m + \sum_{k=0}^{m-1} P_k(z_2, \ldots, z_n)\, z_1{}^k, \qquad a \neq 0. \tag{2}$$

Let us consider a space of $n + 1$ (real) dimensions, defined by the n real components $x_1, \ldots, x_n$ and the imaginary component y_1. We construct a discontinuous manifold T in this space, which we shall call the "Hörmander staircase." To do this, let us divide the $(n-1)$-dimensional real space $x_2, \ldots, x_n$ into a locally finite (i.e., finite in each strip) number of parts $\Delta_1, \ldots, \Delta_j, \ldots$ by means of $(n-2)$-dimensional hyperplanes parallel to the coordinate hyperplanes, the parts Δ_j being limited by some value $y_1 = y_1^{(j)}$. We shall call the Hörmander staircase, the set of all points $(x_1, \ldots, x_n, y_1)$, where $-\infty < x_1 < \infty$, and such that if $(x_2, \ldots, x_n) \in \Delta_j$, then $y_1 = y_1^{(j)}$ $(j = 1, 2, \ldots)$. Shown in Fig. 1 is the staircase for the case of the coordinates x_1, x_2, y_1. We shall prove below that for a given polynomial $P(z)$, it is always possible to construct a staircase T_P, on which $|P(z)| \geqslant |a|$, all the $|y_1^{(j)}|$ being bounded by an identical constant C_0.

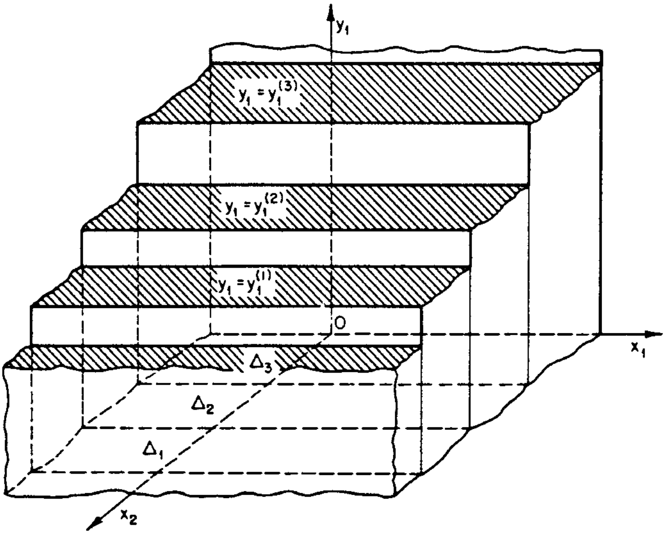

FIG. 1. The Hörmander staircase for coordinates x_1, x_2, y_1.

Having such a staircase, we set

$$(f, \varphi) = \int_{T_P} \frac{\varphi(x_1 + iy_1, x_2, \ldots, x_n)}{P(x_1 + iy_1, x_2, \ldots, x_n)} \, dx_1 \cdots dx_n . \tag{3}$$

This integral exists because the denominator exceeds $|a|$ in absolute value, and the function $\varphi(x_1 + iy_1, x_n)$, as a fundamental function in the space Z, tends to zero uniformly in y_1 (for $|y_1| \leqslant C_0$) more rapidly than any power of $1/|x|$ (when $|x| \to \infty$) and is therefore integrable. It is also evident that the functional (3) is continuous in the space Z.

Let us show that this functional transforms into 1 when multiplied by $P(z)$. Indeed, we have

$$(\bar{P}(z)f, \varphi) = (f, P(z)\,\varphi(z))$$
$$= \int_{T_P} \varphi(x_1 + iy_1, x_2, \dots, x_n)\, dx_1 \cdots dx_n$$
$$= \sum_{\Delta_j} \int \cdots \int_{(x_2,\dots,x_n)\in\Delta_j} \left[\int_{-\infty}^{\infty} \varphi(x_1 + iy_1^{(j)}, x_2, \dots, x_n)\, dx_1\right] dx_2 \cdots dx_n\,.$$

Because of the Cauchy formula, the inner integral, which is evaluated along a line in the $z_1 = x_1 + iy_1$ plane parallel to the real axis and a distance $|\, y_1^{(j)} \,|$ removed from it, may be replaced by an integral on the real axis itself, without changing its value. We hence obtain

$$(\bar{P}(z)f, \varphi) = \sum_{\Delta_j} \int \cdots \int_{(x_2,\dots,x_n)\in\Delta_j} \int_{-\infty}^{\infty} \varphi(x_1, \dots, x_n)\, dx_1 \cdots dx_n = \int \varphi(x)\, dx,$$

from which it follows that

$$\bar{P}(z)f = 1,$$

q.e.d.

Now let us show that the staircase on which $|\, P(z) \,| \geqslant |\, a \,|$ exists. Let us consider the polynomial

$$P(z) = a z_1{}^m + \sum_{k=0}^{m-1} P_k(z_2, \dots, z_n)\, z_1{}^k$$

for arbitrarily fixed values $z_2 = x_2, \dots, z_n = x_n$. It has not more than m roots $z_1^{(1)}, \dots, z_1^{(m)}$ on the z_1-plane and admits the expansion

$$P(z) = a(z - z_1^{(1)})(z - z_1^{(2)}) \cdot \dots \cdot (z - z_1^{(m)}). \tag{4}$$

A line $y_1 =$ const, which is removed by not more than 1 from all the roots of the polynomial may always be drawn in a strip $|\, y_1 \,| \leqslant m + 1$ of width $2m + 2$; it is seen from (4) that on this line, we have

$$|\, P(z)| > |\, a \,|.$$

Since the roots of a polynomial with constant highest coefficient depend continuously on the remaining coefficients, then for a sufficiently small neighborhood of the point $(z_2, \dots, z_n)$ the corresponding roots $z_1^{(1)}, \dots, z_1^{(m)}$

will be disposed in the z_1 plane in circles as small as desired around their initial positions. Hence, the inequality

$$| P(z)| > | a |$$

will hold along the line y_1 = const, in some neighborhood of the point $z_2 = x_2, \ldots, z_n = x_n$.

Then we may associate with each point in the space $(x_2, \ldots, x_n)$, some neighborhood which satisfies the above conditions. It may be considered that these neighborhoods are bounded by hyperplanes parallel to the coordinate hyperplanes. Applying the Heine–Borel Lemma, from all the coverings of the space $(x_2, \ldots, x_n)$ by such neighborhoods, it is possible to select a locally finite covering $\bar{\Delta}_1, \bar{\Delta}_2, \ldots, \bar{\Delta}_j, \ldots$; furthermore, replacing each $\bar{\Delta}_j$ by $\Delta_j = \bar{\Delta}_j - \Delta_1 - \cdots - \Delta_{j-1}$, we obtain a sequence of disjoint domains which together with the corresponding values $y_1^{(j)}$ also determine the corresponding staircase.

We have proved the theorem on the existence of a functional f for a polynomial $P(z)$ of the special form (2).

Now let $P(z)$ be an arbitrary polynomial. Let us show that *there exists a nondegenerate linear real coordinate transformation*

$$x_j = \sum_{k=1}^{n} c_{jk}\xi_k, \qquad \det \| c_{jk} \| \neq 0, \tag{5}$$

which reduces the polynomial $P(x)$ to

$$P(x_1, \ldots, x_n) = a\xi_1{}^m + \sum_{k=0}^{m-1} P_k(\xi_2, \ldots, \xi_n)\, \xi_1{}^k, \qquad a \neq 0. \tag{5'}$$

Proof. The linear transformation (5) with the still undetermined matrix $C = \| c_{jk} \|$ transforms the polynomial $P(x_1, \ldots x_n)$ into some new mth degree polynomial in the variables $\xi_1, \ldots, \xi_n$. The coefficient of $\xi_1{}^m$ is obtained from the highest degree terms of the polynomial P and is some polynomial of elements of the first column of the matrix C. More exactly, if

$$\sum_r \alpha_r x_1^{q_{1r}} x_2^{q_{2r}} \cdot \ldots \cdot x_n^{q_{nr}}$$

is the set of terms of degree m, the coefficient of $\xi_1{}^m$ has the form

$$\sum_r \alpha_r c_{11}^{q_{1r}} c_{21}^{q_{1r}} \cdots c_{n1}^{q_{nr}}. \tag{6}$$

The terms of this polynomial are all distinct; hence, it is not degenerate and is capable of taking on nonzero values. A system of real numbers, not all equal to zero, which guarantee a nonzero value a for (6) may be taken as the $c_{11}, \ldots, c_{n1}$. The remaining elements of the matrix are selected arbitrarily so that $\det \| c_{jk} \|$ is different from zero.

The way to prove the existence of the functional f, satisfying the equation

$$\bar{P}(z)f = 1, \tag{7}$$

in the general case is now clear.

Let us make the linear transformation of the arguments $z = Cz'$, reducing the polynomial $P(z)$ to the form (7). In this transformation, the space Z goes over into itself, and therefore, the space Z' also remains invariant.

We solve the equation

$$\bar{P}(Cz')g = 1$$

for the transformed polynomial. The existence of the functional $g = g(z')$ has been proved above. Furthermore, we determine the functional f from the condition

$$f(z) = g(C^{-1}z).$$

We will then have

$$\bar{P}(z)f(z) = \bar{P}(Cz')f(Cz') = \bar{P}(Cz')\,g(z') = 1,$$

q.e.d.

The existence of the functional f satisfying (7) has thereby been proved in the general case.

3.4. Differentiation

Let us assume that the operation $\partial/\partial x_j$ is defined and continuous in the fundamental space Φ: For any function $\varphi(x) \in \Phi$, the derivative $\partial\varphi/\partial x_j$ also belongs to Φ, and from $\varphi_\nu(x) \to 0$ it follows that $\partial\varphi_\nu/\partial x_j \to 0$ in the topology of Φ. The operation defined by the formula

$$\left(\frac{\partial}{\partial x_j} f, \varphi\right) = \left(f, -\frac{\partial \varphi}{\partial x_j}\right), \tag{1}$$

that is, the operation conjugate to the operation $-\partial/\partial x_j$ in the space Φ, is called the operation $\partial/\partial x_j$ in the space Φ'.

Let us verify the soundness of this definition. If f is a regular functional of the type of a function continuously differentiable with respect to x_j and finite, then by integration by parts we obtain

$$\left(\frac{\partial}{\partial x_j} f, \varphi\right) = \left(f, -\frac{\partial\varphi}{\partial x_j}\right) = -\int f(x) \frac{\partial \varphi}{\partial x_j}\, dx$$
$$= \int \frac{\partial f}{\partial x_j} \varphi(x)\, dx = \left(\frac{\partial f(x)}{\partial x_j}, \varphi(x)\right),$$

that is, the operation $\partial/\partial x_j$ transforms the function $f(x)$ into its conventional derivative with respect to x_j.

The operation of differentiation in the space Φ', as the conjugate to a continuous operation, is itself a continuous operation; if $f_\nu \to f$ in the (weak) topology of the space Φ', then

$$\frac{\partial f_\nu}{\partial x_j} \to \frac{\partial f}{\partial x_j}$$

also in the same sense.

Since the differentiation process may be repeated, each generalized function has derivatives of all orders, and each differential operator of finite order (where $a_q(x)$ are multipliers in Φ)

$$P(D) \equiv \sum a_q(x)\, D^q \equiv \sum a_q(x) \frac{\partial^{q_1+\cdots+q_n}}{\partial x_1^{q_1} \cdots \partial x_n^{q_n}},$$

is a continuous operator in Φ'. The formula by which the operator $P(D)$ operates is the following:

$$\begin{aligned}(P(D) f, \varphi) &= \left(\sum_q a_q(x) \frac{\partial^{q_1+\cdots+q_n} f}{\partial x_1^{q_1} \cdots \partial x_n^{q_n}}, \varphi\right) \\ &= \sum_q \left(\frac{\partial^{q_1+\cdots+q_n} f}{\partial x_1^{q_1} \cdots \partial x_n^{q_n}}, a_q(x)\, \varphi\right) \\ &= \sum_q \left(f, (-1)^{q_1+\cdots+q_n} \frac{\partial^{q_1+\cdots+q_n}}{\partial x_1^{q_1} \cdots \partial x_n^{q_n}} a_q(x)\, \varphi\right) \\ &= (f, P^*(D)\, \varphi), \end{aligned} \tag{2}$$

where $P^*(D)$ denotes the conjugate differential operator defined by

$$P^*(D)\, \varphi = \sum (-1)^{|q|} \frac{\partial^q (a_q \varphi)}{\partial x_1^{q_1} \cdots \partial x_n^{q_n}}. \tag{3}$$

If the coefficients $a_q(x) = a_q$ are constants, then $P^*(D) = P(-D)$ and the formula simplifies

$$(P(D)f, \varphi) = (f, P(-D)\varphi). \tag{4}$$

In particular, (4) shows that for mixed derivatives in the space Φ', the result is independent of the order of differentiation.

Examples. Evidently the operations $\partial/\partial x_j$ are everywhere defined, and continuous for any $j = 1, 2,\ldots, n$ in the spaces $K(a)$ and K.

These operations are also everywhere defined and continuous in the space S. In fact

$$\left\| \frac{\partial\varphi}{\partial x_j} \right\|_p = \sup_{\substack{x \\ |k|,|q|\leqslant p}} \left| x^k D^q \frac{\partial\varphi}{\partial x_j} \right| \leqslant \sup_{\substack{x \\ |k|,|q|\leqslant p+1}} | x^k D^q \varphi | \leqslant \| \varphi \|_{p+1},$$

from which the boundedness results, and therefore, also the continuity of the operations $\partial/\partial x_j$ in the space S.

Corresponding conclusions turn out to be valid for the space $K\{M_p\}$ also, if the following condition is satisfied: *For any subscript p, a subscript $p' > p$ exists such that the inequality*

$$M_p(x) \leqslant C_{pp'} M_{p'}(x) \tag{5}$$

holds. The proof is analogous to that presented for the space S.

Let us verify that the operation $\partial/\partial z_j$ is also continuous in the space $Z(a)$.

Applying the Cauchy formula in the plane of the complex variable z_j, we obtain ($| k | \leqslant p$):

$$\begin{aligned} z^k \frac{\partial\varphi(z)}{\partial z_j} &= z_1^{k_1} \cdots z_n^{k_n} \frac{\partial\varphi(z)}{\partial z_j} \\ &= \frac{\partial(z^k\varphi(z))}{\partial z_j} - k_j z_1^{k_1} \cdots z_j^{k_j-1} \cdots z_n^{k_n}\varphi(z) \\ &= \frac{1}{2\pi i} \int_{|\xi-z_j|=1} \frac{z_1^{k_1} \cdots \xi_j^{k_j} \cdots z_n^{k_n}\varphi(z_1,\ldots, \xi_j,\ldots, z_n)}{(\xi_j - z_j)^2} \, d\xi_j \\ &\quad - k_j z_1^{k_1} \cdots z_j^{k_j-1} \cdots z_n^{k_n}\varphi(z), \end{aligned}$$

from which

$$\left| z^k \frac{\partial\varphi}{\partial z_j} \right| \leqslant \| \varphi \|_p \, e^{a(|y|+1)} + k_j \| \varphi \|_p \, e^{a(|y|+1)};$$

therefore

$$\left\| \frac{\partial \varphi}{\partial z_j} \right\| = \sup_{\substack{z \\ |k| \leqslant p}} \left| z^k \frac{\partial \varphi}{\partial z_j} \right| e^{-a|y|} \leqslant e^a(1 + k_j) \| \varphi \|_p ,$$

which proves the boundedness, and thus the continuity of the operation $\partial/\partial z_j$ in the space $Z(a)$. Meanwhile, we also obtain continuity of the operation $\partial/\partial z_j$ in the space Z.

The reader will find other examples in Chapter IV.

4. Structure of Generalized Functions

4.1. Structure of Generalized Functions in the Space $K\{M_p\}$

First, let us find the general form of linear continuous functionals in the space $\Phi = K\{M_p\}$, defined by a system of functions $M_p(x)$ with the norms

$$\| \varphi \|_p = \sup_{\substack{x \\ |q| \leqslant p}} M_p(x) \mid D^q \varphi(x) \mid. \tag{1}$$

Here, as above, we shall assume the condition (P) of Section 2 to be satisfied: For a given $\epsilon > 0$ and any number p, a $p' > p$ and an N exist such that if

$$| x | > N \qquad \text{or} \quad M_p(x) > N,$$

then

$$M_p(x) \leqslant \epsilon M_{p'}(x).$$

As we have seen, there follows from this condition in particular, that for any natural integer q,

$$M_p(x) D^q \varphi(x) \to 0$$

for $| x | \to \infty$ or $M_p(x) \to \infty$.

By virtue of the theorem on the structure of the space conjugate to the countably normed space (Chapter I, Section 4), it is sufficient to find the general form of the linear functional on the normed space Φ_p, obtained by completion of the space Φ in the norm $\| \varphi \|_p$. As we have seen in Section 2, Φ_p is a closed subspace of the space $\bar{\Phi}_p$ of all functions $\varphi(x)$, having derivatives to order p, for which the norm (1) exists; hence, applying the Hahn–Banach theorem on the extension of a linear functional, every linear functional $f \in \Phi_p'$ can be extended in the space $\bar{\Phi}_p$. Therefore, it is sufficient to describe functionals in the space Φ_p.

For each function $\varphi(x) \in \bar{\Phi}_p$, let us consider the set of all functions

$$\psi_q(x) = M_p(x)\, D^q\varphi(x), \qquad (\| q \| \leqslant p).$$

We obtain thus a mapping of the space $\bar{\Phi}_p$ into the direct sum Ψ_p of a finite number of spaces of continuous functions $\psi_q(x)$ $(| q | \leqslant p)$. Evidently the mapping $\varphi(x) \leftrightarrow \{\psi_q(x)\}$ is one-to-one. Defining the norm of the element $\{\psi_q(x)\}$ as $\sup_{q,x} | \psi_q(x) |$, we see that the norm is also conserved for this mapping. Hence, we may assume that $\bar{\Phi}_p$ is a closed subspace of the space Ψ_p.

Applying the Hahn–Banach theorem, the functional $f \in \Phi_p'$ may be extended into the whole space Ψ_p. Afterwards, by the Riesz–Radon theorem,[10] we may write its general form:

$$(f, \varphi) = \sum_{|q| \leqslant p} \int M_p(x)\, D^q\varphi(x)\, d\sigma_q(x), \tag{2}$$

where $\sigma_q(x)$ is a measure in the space R_n, concentrated in the set of all points where the functions $M_p(x)$ are finite.

The norm of this functional as a functional on the space Ψ_p, equals the sum of the variations of the function $\sigma_q(x)$. Let us note that the functions $\sigma_q(x)$ are not generally uniquely defined by the values of the functional f on the space Φ_p. But since the Hahn–Banach theorem guarantees the possibility of extending the functional without changing its norm, the sum of the variations of these functions exactly equals the norm of the functional f in Φ_p.

Thus, we have established the following theorem.

Theorem. *Every linear continuous functional* (f, φ) *in the space* $\Phi = K\{M_p\}$ *may be written in the form*

$$(f, \varphi) = \sum_{|q| \leqslant p} \int M_p(x)\, D^q\varphi(x)\, d\sigma_q(x),$$

where the norm of the functional f *(extended to the space* Ψ_p*) equals the sum of the variations of the function* $\sigma_q(x)$.

Let us note that the least value of p is none other than the order of the functional f (see Chapter I, Section 4).

[10] See, for example, F. Riesz and B. Szekefalvy-Nagy. "Lectures on Functional Analysis," p. 143. Gostekhizdat, Moscow, 1954.

4.2. Simplification of the Writing in Spaces with the Condition (N).

With additional conditions on the functions $M_p(x)$, the obtained result may be written in a rather different, more simple form in terms of the customary integrals in place of the Stieltjes integrals.

This is the case, e.g., if the system $\{M_p(x)\}$ satisfies the following conditions:

(M) In the limits of any n-eder, where the coordinates x_j are of constant sign, the functions $M_p(x)$ are quasimonotonic with respect to each of the coordinates, i.e., in each n-eder

$$(|x_j'| \leqslant |x_j''|) \Rightarrow M(x_1, \ldots, x_j', \ldots, x_n) \leqslant M(x_1, \ldots, x_j'', \ldots, x_n)$$

for every fixed point $(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$;

(N) For each p a $p' > p$, may be given such that the ratio

$$\frac{M_p(x)}{M_{p'}(x)} = m_{pp'}(x) \tag{1}$$

tends to zero for $|x| \to \infty$ and is a summable function of x. (Where $M_p(x) = M_{p'}(x) = \infty$, we set this ratio equal to zero).[11]

The condition (M) has a technical character (and can be weakened); the condition (N) is fundamental.

If the condition (N) is satisfied, then for each p the expression

$$\| \varphi \|_p' = \sup_{|q| \leqslant p} \int M_p(x) \, | D^q \varphi(x) | \, dx. \tag{2}$$

has meaning. In fact, by virtue of the condition (N), we have:

$$M_p(x) \, | D^q \varphi(x) | \leqslant m_{pp'}(x) \, M_{p'}(x) \, | D^q \varphi(x) | \leqslant m_{pp'}(x) \, \| \varphi \|_{p'} ,$$

from which results the existence of the integral (2) as well, in addition to the inequality

$$\| \varphi \|_p' = \sup_{|q| \leqslant p} \int M_p(x) \, | D^q \varphi(x) | \, dx \leqslant B_p \, \| \varphi \|_{p'} ,$$

where B_p is an integral of the summable functions $m_{pp'}(x)$.

Now let us estimate $\| \varphi \|_p$ in terms of $\| \varphi \|_{p+1}'$. Let x_0 and q_0 be those values of x and q ($|q| \leqslant p$), for which the least upper bound is obtained

[11] (N)—Nucleaire (Fr.). We shall see in Chapter IV of Volume 3 that the condition (N) guarantees that the space $K\{M_p\}$ will belong to an important class of *nuclear* spaces.

in the expression[12] $M_p(x) \mid D^q\varphi(x) \mid$. Let us then denote by $\int_x^{+\infty}$ the integral taken over the n-eder of summit x_0 in whose limits all the coordinates have their sign constant. In view of the condition (M), we have

$$\begin{aligned} \| \varphi \|_p &= \sup_{\substack{x \\ |q| \leqslant p}} M_p(x_0) \mid D^{q_0}\varphi(x_0)| \\ &\leqslant \sup_x M_p(x) \int_x^{+\infty} D^{q+1}\varphi(\xi)\, d\xi \\ &= C_p \sup_{\substack{x \\ |q| \leqslant p}} \int_x^{\infty} M_p(\xi) \mid D^{q_0+1}\varphi(\xi)|\, d\xi \leqslant C_p \| \varphi \|'_{p+1} \,. \end{aligned}$$

Thus the inequalities

$$\| \varphi \|'_p \leqslant B_p \| \varphi \|_{p'} \,; \qquad \| \varphi \|_p \leqslant C_p \| \varphi \|'_{p+1}$$

hold.

This means that the system of norms $\| \varphi \|'_p$ is equivalent to the system of norms $\| \varphi \|_p$. The space Φ may be represented now as the intersection of a sequence of normed spaces Φ^p, each of which is obtained by completing the space Φ in the corresponding norm $\| \varphi \|'_p$. Each continuous linear functional in the space Φ is a continuous linear functional, for some p, in the space Φ^p. Hence, it is sufficient for us to find the general form of a continuous linear functional in the space Φ^p. The space Φ^p is the isometric part of the space $\bar{\Phi}^p$, formed by all functions $\varphi(x)$, for which the expression

$$\| \varphi \|'_p = \sup_{|q| \leqslant p} \int M_p(x) \mid D^q\varphi(x)|\, dx$$

has a meaning. Finally the space $\bar{\Phi}^p$ is a closed subspace of the direct sum Ψ of a finite number (equal to the number of subscripts q, $\mid q \mid \leqslant p$) of spaces of functions which are integrable with fixed weight $M_p(x)$. The continuous linear functional f may be extended into the whole of this space Ψ with the norm conserved. The general form of any continuous linear functional in the space Ψ is

$$\sum_{|q| \leqslant p} \int M_p(x)\, D^q\varphi(x) f_q(x)\, dx,$$

where $f_q(x)$ ($\mid q \mid \leqslant p$) is a bounded measurable function. It hence follows

[12] Such finite values of x and q always exist, since the functions $M_p(x)\, D^q\varphi(x)$ tend to zero for $\mid x \mid \to \infty$ and are continuous.

that *the general form of a continuous linear functional in the space* Φ^p is

$$(f, \varphi) = \sum_{|q| \leqslant p} \int M_p(x) \, D^q\varphi(x) f_p(x) \, dx.$$

Also defined by this same expression for any p *and* $f_q(x)$, *is the general form of a continuous linear functional in the space* $\Phi = K\{M_p\}$ *under the assumption of the validity of the condition* (N).

The norm of the functional $f \in \Psi'$ equals $\sum_q \sup_x |f_q(x)|$ (under the condition that the values of the functions $f_q(x)$ in sets of measure zero are neglected). Since the Hahn–Banach theorem guarantees the possibility of extending the functional with the norm conserved, then for the given functional $f \in \Phi'$, it is always possible to select these functions $f_q(x)$ so that the norm of this functional in the space Φ^p is equal to $\sum_q \sup_x |f_q(x)|$ ($|q| \leqslant p$).

Thus, we have obtained the second theorem on functionals in the space $K\{M_p\}$.

Theorem. *In the countably normed space* $\Phi = K\{M_p\}$ *with the condition* (N),

$$\frac{M_p(x)}{M_{p'}(x)} = m_{pp'}(x) \in L_1(R_n),$$

each continuous linear functional has the form

$$(f, \varphi) = \sum_{|q| \leqslant p} \int M_p(x) \, D^q\varphi(x) f_q(x) \, dx,$$

where the $f_q(x)$ *are bounded measurable functions. The norm of this functional, extended into the normed space* Φ_p, *is equal to*

$$\sum_{|q| \leqslant p} \sup_x |f_q(x)|.$$

4.3. Case of the Spaces K and S

In the space $K(a)$ of infinitely differentiable functions $\varphi(x)$, of support in the domain $|x| \leqslant a$, the functions $M_p(x)$ equal 1 in the domain $\{|x| \leqslant a\} = G_a$ and ∞ outside this domain; hence, the general form of the functional in the space $K(a)$ is the following:

$$(f, \varphi) = \sum_{|q| \leqslant p} \int f_q(x) \, D^q\varphi(x) \, dx, \tag{1}$$

where $f_q(x)$ are bounded measurable functions in the domain $\{|x| \leqslant a\}$.

Integrating by parts, the derivative with respect to x_j to any order may be taken, hence

$$(f, \varphi) = \int_{|x| \leqslant a} f(x) \, D^q \varphi(x) \, dx \qquad \left(D^p = \frac{\partial^{np}}{\partial x_1{}^p \, \partial x_2{}^p \cdots \partial x_n{}^p} \right),$$

where $f(x)$ is again a bounded measurable function. Finally, still another integration by parts reduces (1) to

$$(f, \varphi) = \int_{|x| \leqslant a} F(x) \, D^{p+1} \varphi(x) \, dx, \tag{2}$$

where $F(x)$ is a continuous function in the domain G_a. Using the definition of the derivative of a functional, the expression (2) may be written as

$$(f, \varphi) = \pm (D^{p+1} F(x), \varphi(x));$$

in other words, *each generalized function in the space $K(a)$ is the derivative of some continuous function.*

It is not easy to write the norm of the obtained functional as a general formula. In every case, on the basis of the results of Chapter I, Section 5, it may be stated that if a sequence of functionals $\{f_\nu\}$ tends to zero, they all have the same order, i.e., belong to the same normed space Φ'_p, and tend to zero in the norm in this space; then the corresponding continuous functions $F_\nu(x)$ may be chosen so that they will tend uniformly to zero in the domain G_a. The converse is evident: If the functions $F_\nu(x)$ tend uniformly to zero in the domain G_a, then for any fundamental function $\varphi \in \Phi$, we have

$$(f_\nu, \varphi) = \int_{|x| \leqslant a} F_\nu(x) \, D^{p+1} \varphi(x) \, dx \to 0.$$

In the space S of infinitely differentiable functions which decrease to 0 more rapidly than any power of $1/|x|$ as $|x| \to \infty$, the functions $M_p(x)$ have the form $\prod_{j=1}^n (1 + |x_j|)^p$. Hence, in conformity with the theorem of Section 2, the general form of the continuous linear functional is given by the formula

$$(f, \varphi) = \sum_{|q| \leqslant p} \int_{R^n} f_q(x) \prod_{j=1}^n (1 + |x_j|)^p \, D^q \varphi(x) \, dx.$$

With the aid of integration by parts, this expression may also be reduced to an integral of just derivatives of order p:

$$(f, \varphi) = \int_{R^n} f(x)\, D^p\varphi(x)\, dx, \tag{3}$$

where the function $f(x)$ is a linear combination of functions

$$f_q(x) \prod_{j=1}^{n} (1 + |x_j|)^p$$

and their primitives (to order p) and, hence, in every case it is a measurable function which increases less rapidly than $|x|^{pn}$. Still another integration by parts allows us to obtain a continuous integrand $f(x)$ by raising the order of the derivative of the function $\varphi(x)$ by 1.

Formula (3) may also be written as

$$(f, \varphi) = (f, D^p\varphi) = \pm(D^pf, \varphi); \tag{4}$$

in other words, *each generalized function in the space S is the derivative of a continuous function of power growth.*

4.4. Structure of Functionals of Bounded Support

In the case of any fundamental space Φ, the formulas

$$(f, \varphi) = \sum_{|q| \leqslant p} \int D^q\varphi(x)\, d\sigma_q(x)$$

or

$$(f, \varphi) = \sum_{|q| \leqslant p} \int D^q\varphi(x) f_q(x)\, dx$$

do not already yield the general form of the continuous linear functional; in general, the functional f depends on the derivatives of all orders. But in a broad class of spaces it is possible to obtain an analogous representation for the functionals of bounded support.

In conformity with the definitions given in Volume 1 (Chapter I, Section 1.4), we agree to say that the functional $f \in \Phi'$ *has its support contained in the closed set $F \in R$, if for any function* $\varphi \in \Phi$, which vanishes in the neighborhood of the set F, the equality $(f, \varphi) = 0$ holds.

Thus, if a functional f is regular and generated by a function $f(x)$ of support in F, then the functional f has evidently the same support.

The validity of the converse holds, in every case, in spaces containing all infinitely differentiable functions of bounded support.

If a functional f of the type of (locally integrable) functions $f(x)$, of support contained in the set F, is given on the space $\Phi \supset K$, the function $f(x)$ has the same support.

For the proof, we take any point x_0, interior to the complement of F, and we consider a neighborhood U of x_0, which has no points in common with F. Since for each $\varphi(x) \in K$, of support in U, we have by assumption

$$(f, \varphi) = \int_U f(x)\, \varphi(x)\, dx = 0,$$

the function $f(x)$ is zero almost everywhere in the neighborhood U. Hence, $f(x)$ vanishes at almost all the interior points of the complement of the set F. But then $f(x) = 0$ almost everywhere outside F, as has been stated.

If there is a sequence of functionals $f_\nu \in \Phi'$, of support in the set F, which converges to the functional $f \in \Phi'$, then the equality

$$(f, \varphi) = \lim_{\nu \to \infty} (f_\nu, \varphi)$$

shows that the functional f also has its support in the set F; hence, *the set of all functionals $f \in \Phi'$, of support in the set F, is closed in Φ'.*

We call a functional of support contained in a bounded set, a *functional of bounded support.* In some spaces, *the functionals of bounded support form a dense subset in the set of all functionals.* For example, it is so *in spaces of the type $K\{M_p\}$.* In fact, let us consider the general form of a linear functional on the space $K\{M_p\}$

$$(f, \varphi) = \sum_{|q| \leqslant p} \int_R M_p(x)\, D^q\varphi(x)\, d\sigma_q(x),$$

or, equivalently,

$$(f, \varphi) = \lim_{r \to \infty} \sum_{|q| \leqslant p} \int_{|x| \leqslant r} M_p(x)\, D^q\varphi(x)\, d\sigma_q(x).$$

Under the limit symbol is an expression defining a continuous linear functional on the space Φ, evidently of bounded support (contained in a ball of radius r). Therefore, every functional $f \in K'\{M_p\}$ is the limit of functionals of bounded support, as we have asserted.

Theorem. *Let us assume that a fundamental space Φ contains all infinitely differentiable functions $\varphi(x)$ of bounded support. Furthermore, let the functional $f \in \Phi'$ be of support contained in the parallelepiped*

$$|x_j| \leqslant a_j \qquad (j = 1, 2, \dots, n).$$

Then for any $\epsilon > 0$, *there exist continuous functions* $f_{q\epsilon}(x)$, $q = (q_1, ..., q_n)$, $|q| \leqslant p$, *which go to* 0 *for* $|x_j| \geqslant a_j + \epsilon$, *such that the functional* f *may be represented as*

$$f = \sum_{|q| \leqslant p} D^q f_{q\epsilon}. \tag{1}$$

Proof. Let $G_{a+\epsilon}$ denote the domain defined by the inequalities $|x_j| \leqslant a_j + \epsilon$, an "$\epsilon$-extension of the domain G_a." Let $h_\epsilon(x)$ be an infinitely differentiable function, equal to one in the domain $G_a + \delta$ and to zero in the complement of the domain $G_{a+\epsilon}$ ($\delta < \epsilon$). The decomposition

$$\varphi(x) = \varphi(x)\, h_\epsilon(x) + \varphi(x)[1 - h_\epsilon(x)]$$

permits each function $\varphi \in \Phi$ to be represented as a sum whose first member will vanish outside the domain $G_{a+\epsilon}$, and the second outside the domain $G_{a+\delta}$. By assumption, the functional f equals zero in the second member, so that for any $\varphi \in \Phi$, we have

$$(f, \varphi) = (f, \varphi h_\epsilon).$$

But the function φh is an element of the space $K(a + \epsilon)$. The space $K(a + \epsilon)$, is contained, by assumption, in the space Φ. By virtue of the theorem of Chapter I, Section 1.3, convergence of the sequence convergent in $K(a + \epsilon)$ is conserved in this imbedding; hence f is a continuous linear functional on the space $K(a + \epsilon)$ also. According to Section 4.3, the value of the functional f for the function $\varphi h_\epsilon \in K(a + \epsilon)$ may be written as

$$(f, \varphi h_\epsilon) = \int_{G_{a+\epsilon}} f_0(x)\, D^p(\varphi h_\epsilon)\, dx = \pm(D^p f_0, \varphi h_\epsilon),$$

where $f_0(x)$ is some continuous function in the domain $G_{a+\epsilon}$. Expanding $D^p(\varphi h_\epsilon)$ by the Leibniz rule, we find:

$$\begin{aligned}(f, \varphi) = (f, \varphi h_\epsilon) = (f_0, D^p \varphi h_\epsilon) &= \Big(f_0, \sum_{|q| \leqslant p} C_p^q\, D^q \varphi\, D^{p-q} h_\epsilon\Big) \\ &= \sum_{|q| \leqslant p} C_p^q (f_0, D^q \varphi\, D^{p-q} h_\epsilon) \\ &= \sum_{|q| \leqslant p} C_p^q (f_0\, D^{p-q} h_\epsilon, D^q \varphi) \\ &= \sum_{|q| \leqslant p} C_p^q (D^q[(-1)^q f_0\, D^{p-q} h_\epsilon], \varphi) \\ &= \Big(\sum_{|q| \leqslant p} D^q f_{q\epsilon}, \varphi\Big),\end{aligned}$$

where $f_{q\epsilon}(x) = (-1)^q f_0(x)\, D^{p-q}\, h_\epsilon(x)$ is a continuous function which vanishes outside the domain $G_{a+\epsilon}$. This result agrees with the required equality (1).

Corollary. *If the space Φ contains all infinitely differentiable functions of bounded support then each generalized function of bounded support in the space Φ is the limit (in the weak convergence sense) of ordinary infinitely-differentiable functions of bounded support.*

Proof. It follows from the theorem proved above that the generalized function f is represented as

$$f = \sum_{|q| \leqslant p} D^q f_q(x),$$

where the $f_q(x)$ are continuous functions of bounded support. A sequence of infinitely differentiable functions $g_{\nu q}(x)$, of support contained in the same ball in R^n and which converge uniformly to $f_q(x)$ for each q may be formed. The convergence $g_{\nu q}(x) \to f_q(x)$ holds in this case, even in the topology of the generalized functions space. But then the infinitely differentiable functions of bounded support

$$g_\nu(x) = \sum_{|q| \leqslant p} D^q g_{\nu q}(x)$$

converge also in the sense of generalized functions to

$$\sum_{|q| \leqslant p} D^q f_q(x),$$

q.e.d.

Moreover, if the set of all generalized functions of bounded support, is everywhere dense in the set of all generalized functions on the space Φ, this latter condition is satisfied, e.g., in the spaces $K\{M_p\}$ (Section 2), then the statement of the corollary may be strengthened: In this case, *every generalized function on the space Φ is the limit of infinitely differentiable functions of bounded support.*

Remark. Let there be a sequence of functionals $\{f_\nu\}$ having their supports contained in the same bounded set F, which tend to zero in the sense of generalized functions. Then for a given $\epsilon > 0$, the representation

$$f_\nu = \sum_{|q| \leqslant p} D^q f^\nu_{q\epsilon},$$

may be obtained, where the functions $f^{\nu}_{q\epsilon}(x)$ are continuous, vanish outside an ϵ-extension of the set F, and tend uniformly to zero for $\nu \to \infty$, and p fixed.

For the proof, let us note that the functionals f_ν converge weakly to zero in the space conjugate to the space $K(a + \epsilon)$; hence, as has been mentioned in Section 4.3, appropriate functions $F_\nu(x)$ may be selected which converge uniformly to zero; in addition to these, the functions

$$f^{\nu}_{q\epsilon}(x) = (-1)^q F_\nu(x)\, D^{p-q} h_\epsilon(x)$$

will also converge uniformly to zero; q.e.d.

4.5. Structure of a Functional Having a Point Support

A functional having a point support has a particularly simple structure.

Theorem. *If the fundamental space Φ contains all infinitely differentiable functions of bounded support at least in some neighborhood of a given point x_0, then every generalized function having the point x_0 as support has the form*

$$f = \sum_{|q| \leqslant p} a_q D^q \delta(x - x_0). \tag{1}$$

Proof. According to Theorem 4 of Section 4.4 the functional f may be represented as

$$(f, \varphi) = \sum_{|q| \leqslant p} \int_{|x - x_0| \leqslant \epsilon} f_{q\epsilon}(x)\, D^q \varphi(x)\, dx, \tag{2}$$

where the $f_{q\epsilon}(x)$ are continuous functions in the domain $G = \{|\, x - x_0 \,| \leqslant \epsilon\}$, where ϵ is an arbitrary positive number.

Formula (2) permits extension of the functional f into the space $K^p(G)$ of all functions having continuous derivatives to order p, in G; evidently the functional f hence extended remains a continuous functional.

By assumption, the functional f equals zero on every function $\varphi \in K$, equal to zero in a neighborhood of the point x_0. By continuity, the functional f will also be zero in every function $\varphi \in K^p(G)$, equal to zero in the neighborhood of the point x_0. The closure J of this set of functions $\varphi \in K^p(G)$ in the topology of the space $K^p(G)$ contains, as is easy to verify, all functions $\varphi \in K^p(G)$ equal to zero, together with their derivatives to the order p, at $x = x_0$; by continuity, the functional f also equals zero on all the functions $\varphi \in J$.

Every function $\varphi(x) \in K^p(G)$ may be represented as

$$\varphi(x) = P(x) + Q(x),$$

where $Q(x)$ has derivatives to order p equal to zero at $x = x_0$ and

$$P(x) = \sum_{|q| \leqslant p} \frac{(x - x_0)^q}{q!} D^q \varphi(x_0).$$

Multiplying this decomposition by the function $h(x) \in K^p(G)$, which equals 1 in the neighborhood of the point x_0, we obtain a new decomposition whose members belong to the class $K^p(G)$:

$$h(x)\,\varphi(x) = h(x)\,P(x) + h(x)\,Q(x);$$

here

$$(f, h\varphi) = (f, \varphi),$$

since the difference $h\varphi - \varphi$ equals zero in the neighborhood of the point x_0. By what has been proved, $(f, hQ) = 0$ and therefore

$$(f, \varphi) = (f, hP)$$

Let us put

$$c_q = \left(f, h\frac{(x - x_0)^q}{q!}\right).$$

Then (f, Ph) takes the form

$$(f, \varphi) = (f, Ph) = \sum_{|q| \leqslant p} \left(f, h\frac{(x - x_0)^q}{q!}\right) D^q \varphi(x_0) = \sum_{|q| \leqslant p} c_q\, D^q \varphi(x_0),$$

which agrees with (1) if we use the notation $a_q = (-1)^q c_q$. The theorem is thus proved.

4.6. Example: Solution of the Laplace Equation with a Power Singularity

The example pertains to the following well-known theorem.

Theorem. *Let it be known that a function $f(x)$ satisfies the Laplace equation everywhere except at the origin, where it has a singularity of order not higher than a certain power:*

$$|f(x)| \leqslant \frac{C}{r^p} \qquad (p \text{ fixed}).$$

Then, to the accuracy of a harmonic member, $f(x)$ is the result of applying some differential polynomial operator $P(D)$ to the fundamental solution of the Laplace equation.

Proof. Let us consider the generalized function $f \in K'$, which agrees with the function $f(x)$ everywhere except at the origin. Such a generalized function f may be constructed by means of the formula

$$(f, \varphi) = \int f(x)[\varphi(x) - P(x)\, e(x)]\, dx,$$

where $P(x)$ is the Taylor polynomial for the function $\varphi(x)$:

$$P(x) = \sum_{|k| \leqslant p-1} \frac{D^k \varphi(0)}{k!} x^k,$$

and $e(x) \in K$ is a function equal to one in the ball $|x| \leqslant 1$ and equal to zero, say, outside the ball $|x| \geqslant 2$.

Applying the Laplace operator to the functional f, we obtain a functional which is zero everywhere outside the origin. It hence follows that Δf is a functional having for support the origin 0 of R^n; by virtue of the theorem in Section 4.5 Δf may be written as

$$\Delta f = \sum_q a_q D^q \delta(x).$$

Furthermore, let us consider the functional $g = \sum a_q D^q E$, where E is a fundamental solution of the Laplace equation. Since $\Delta E = \delta(x)$, then

$$\Delta g = \sum a_q D^q \Delta E = \sum a_q D^q \delta(x) = \Delta f.$$

Hence $\Delta(f - g) = 0$, i.e., the functionals f and g differ by a harmonic function, q.e.d.

More accurately, the difference $f - g$ is a generalized function which is a solution of the Laplace equation. But we shall soon see (Ch. III, Section 3.6) that every solution of the equation $\Delta u = 0$ in generalized functions is an ordinary harmonic function.

This theorem remains valid for any partial differential equation with constant coefficients, since for any such equation there exists a fundamental solution (Volume 1, Chapter I, Section 5; Volume 2, Chapter II, Section 3.3, and Chapter III, Section 2.4). It is understood that the harmonic function in the formulation of this theorem should now be replaced by the solution (in generalized functions) of the appropriate homogeneous equation.

CHAPTER III

FOURIER TRANSFORMATIONS OF FUNDAMENTAL AND GENERALIZED FUNCTIONS

1. Fourier Transformations of Fundamental Functions

In Volume 1, we considered Fourier transformations of generalized functions on the fundamental space K of infinitely differentiable functions of bounded support. Now we shall consider Fourier transformations of generalized functions on any fundamental space. In Volume 1, a functional on the space Z composed of some entire analytic functions, Fourier transforms of fundamental functions of the space K, was the Fourier transformation of the generalized function on the space K. In the general case also, *a functional in the space $\tilde{\Phi}$ of Fourier transforms of functions of the space Φ will be the Fourier transform of a generalized function in the space Φ.* Hence, we should first consider Fourier transforms of the fundamental functions of the space Φ; Section 1 is devoted to this. Fourier transforms of generalized functions are considered in Section 2, and some applications of Fourier transforms to differential equations are mentioned. The convolution operation is studied in Section 3; the results are used to obtain new theorems on Fourier transforms of generalized functions. Finally, Fourier transforms of entire analytic functions, the generalization of the classical Wiener–Paley type theorems, etc., are considered in Section 4.

Chapter IV, devoted to the special fundamental spaces of the type S (and W), is a continuation of this chapter. As already mentioned in the introduction, within these spaces the Fourier-transform apparatus becomes extremely flexible, because the Fourier transforms carry these spaces into each other, and permit important general theorems to be obtained on the Cauchy problem (uniqueness of the solutions and correctness of the statement of the problem; see Chapters II and III of Volume 3).

In this chapter we shall consider complex spaces of fundamental and generalized functions. In particular, let us recall that a functional operating according to the formula

$$(f, \varphi) = \int \overline{f(x)}\, \varphi(x)\, dx,$$

is called a functional of the type of the function $f(x)$, and that multiplication of the functional f by the number α or the function $\alpha(x)$ is given by the formula

$$(\alpha f, \varphi) = (f, \bar{\alpha}\varphi) = \bar{\alpha}(f, \varphi).$$

The theorems from the previous chapter which we shall use are proved for real spaces, but as it is easy to verify, they also remain valid in the complex case.

1.1. Fourier Operators in the Space S

Let us consider the Fourier transform of the function $\varphi(x)$ of the complex space S, i.e., of the differentiable, complex-valued function all of whose derivatives approach zero more rapidly than any power of $1/|x|$ as $|x| \to \infty$. We will show that the Fourier transform of the function $\varphi(x)$

$$F[\varphi] \equiv \psi(\sigma) \equiv \widetilde{\varphi(x)} \equiv \int e^{i(x,\sigma)}\varphi(x)\, dx \left((x, \sigma) = \sum_{i=1}^{n} x_i\sigma_i\right), \tag{1}$$

also belongs, as a function of σ, to the space S (a function of σ), i.e., $\psi(\sigma)$ is infinitely differentiable, and each of its derivatives approaches zero more rapidly than any power of $1/|\sigma|$ as $|\sigma| \to \infty$.

The integral in (1) admits of differentiation with respect to the parameter σ_j, since the integral obtained after formal differentiation remains absolutely convergent:

$$\frac{\partial\psi(\sigma)}{\partial\sigma_j} = \int_R ix_j e^{i(x,\sigma)}\varphi(x)\, dx.$$

The properties of the function $\varphi(x)$ permit this differentiation to be continued without limit. This means that *the function $\psi(\sigma)$ is infinitely* *differentiable.* Hence, the following formula holds[1]

$$P(D)\,F[\varphi(x)] \equiv P(D)\,\psi(\sigma) = \int_R P(ix)\,e^{i(x,\sigma)}\varphi(x)\,dx = F[P(ix)\,\varphi(x)] \tag{2}$$

for any differential operator $P(D)$.

Now, let us consider the Fourier transform of the partial derivative $(\partial\varphi/\partial x_j)$:

$$F\left[\frac{\partial\varphi(x)}{\partial x_j}\right] = \int_R \frac{\partial\varphi(x)}{\partial x_j}\,e^{i(x,\sigma)}\,dx.$$

Integration by parts, taking into account that $\varphi(x)$ tends to zero as $|x| \to \infty$, leads to the expression

$$F\left[\frac{\partial\varphi(x)}{\partial x_j}\right] = -i\sigma_j \int_R \varphi(x)\,e^{i(x,\sigma)}\,dx = -i\sigma_j F[\varphi(x)].$$

Repeating this operation we obtain

$$F[P(D)\,\varphi(x)] = P(-i\sigma_j)\,F[\varphi(x)]. \tag{3}$$

As a Fourier transform of an integrable function, the function $P(-i\sigma_j)\,F[\varphi(x)]$ is bounded. Since P is any polynomial, we see that $F[\varphi(x)] = \psi(\sigma)$ *tends to zero more rapidly than any power of* $1/|\sigma|$ *as* $|\sigma| \to \infty$. The same is true also for any derivative of $\psi(\sigma)$ since, as we have seen, the expression $\partial\psi/\partial\sigma_j$ say, is the Fourier transform of the function $ix\varphi(x)$, which also belongs to S.

Therefore, any derivative of $\psi(\sigma)$ tends to zero more rapidly than any power of $1/|\sigma|$ as $|\sigma| \to \infty$, q.e.d.

Thus, *if a function* $\varphi(x)$ *belongs to the space* S (*a function of* x), *then* $\psi(\sigma) = F[\varphi(x)]$ *also belongs to the space* S (a function of σ).

An analogous statement is proved in exactly the same manner for the inverse Fourier transform F^{-1}, which, as is known, is defined by the formula

$$\varphi(x) = F^{-1}[\psi(\sigma)] = \frac{1}{(2\pi)^n}\int e^{-i(x,\sigma)}\psi(\sigma)\,d\sigma; \tag{4}$$

[1] Let us recall that

$$P(D) = \sum a_k D^k = \sum a_{k_1\cdots k_n}\frac{\partial^{k_1+\cdots+k_n}}{\partial x_1^{k_1}\cdots\partial x_n^{k_n}};$$

analogously

$$P(ix) = \sum a(ix)^k \sum a_{k_1\cdots k_n}(ix_1)^{k_1}\cdots(ix_n)^{k_n}.$$

if $\psi(\sigma)$ *belongs to the space* S (a function of σ), *then* $\varphi(x) = F^{-1}[\psi(\sigma)]$ *also belongs to the space* S (a function of x).

Let us note that by applying the operator F^{-1} to (2) and (3), and replacing $F[\varphi]$ everywhere by ψ, and φ by $F^{-1}[\psi]$, we obtain the following formulas for the operator F^{-1}:

$$F^{-1}[P(D)\,\psi(\sigma)] = P(ix)\,F^{-1}[\psi(\sigma)]; \tag{5}$$

$$P(D)\,F^{-1}[\psi(\sigma)] = F^{-1}[P(-i\sigma)\,\psi(\sigma)]. \tag{6}$$

From the proved assumptions, it follows that *the operators* F *and* F^{-1} *map the space* S *conformally one-to-one into itself.* These operators are evidently linear.

Let us show that convergence in S, defined as convergence in a dual space, hence agrees with the original convergence in S. It is sufficient to prove that the operators F and F^{-1} are bounded. Let us verify this for the operator F. For $k_1 + \cdots + k_n + q_1 + \cdots + q_n \leqslant p$, we have

$$\begin{aligned}(-i\sigma)^k\, D^q\psi(\sigma) &= \int_R D^k[(ix)^q\, \varphi(x)]\, e^{i(x,\sigma)}\, dx \\ &= \sum C_k{}^j i^q \int_R D^j x^q\, D^{k-j}\varphi(x)\, e^{i(x,\sigma)}\, dx,\end{aligned}$$

from whence

$$\begin{aligned}|\sigma^k\, D^q\psi(\sigma)| &\leqslant \sum A_{ik} \int |\, x^{q-j} D^{k-j}\varphi(x)|\, dx \\ &= \sum A_{jk} \int \frac{|\, x^{q-j+2} D^{k-j}\varphi(x)|}{\prod_{i=1}^n (x_i^2+1)}\, dx \leqslant \sum A'_{jk}\, \|\, \varphi\,\|_{p+2}\, .\end{aligned}$$

Hence

$$\|\,\psi\,\|_p \leqslant A_p\, \|\,\varphi\,\|_{p+2}\, ,$$

which also proves boundedness of the Fourier operator in the case under consideration.

An analogous calculation may be carried out for the operator F^{-1} but there is no need, since we may refer to the theorem on the inverse operator (Chapter I, Section 7.2).

Let us summarize: *the Fourier operators* F *and* F^{-1} *are continuous linear operators in the space* S *which map this space into itself.*

More precisely we have:

Theorem. *The Fourier operators* F *and* F^{-1} *establish two reciprocal isomorphisms between the topological spaces* $(S)_x$ *and* $(S)_\sigma$, i.e., *two automorphisms on* S, *if we identify the variables* x *and* σ.

1.2. Fourier Operators on the Spaces K and Z

Let us now recall the connection between the spaces K and Z considered in Volume 1.

The Fourier transform of the function $\varphi(x) \in K(a)$,

$$\psi(\sigma) = F[\varphi(x)] = \int_{G_a} \varphi(x)\, e^{i(x,\sigma)}\, dx$$

may be continued into the complex domain $s = \sigma + i\tau$ by means of the formula

$$\psi(s) = \int_{G_a} \varphi(x)\, e^{i(x,s)}\, dx = \int_{G_a} \varphi(x)\, e^{i(x,\sigma)-(x,\tau)}\, dx.$$

Hence, a differentiable, and therefore, also entire analytic function of s is obtained; for any k, it satisfies the inequality

$$| s^k\psi(s)| = \left| \int_{G_a} D^k\varphi(x)\, e^{i(x,s)}\, dx \right| \leqslant e^{a|\tau|} \int_{G_a} | D^k\varphi(x)|\, dx$$

$$= C_k(\psi)\, e^{a|\tau|} \qquad (= C_k(\psi)\, e^{a_1|\tau_1|+\cdots+a_n|\tau_n|}).$$

The functions $\psi(s)$ satisfying this inequality belong to the space $Z(a)$. Evidently, any bounded manifold in the space $K(a)$ (the $| D^k\varphi |$ are bounded) goes over into a bounded manifold in the space $Z(a)$ (the $C_k(\psi)$ are bounded). Therefore, the Fourier operator is a bounded, and hence, a continuous operator transforming $K(a)$ one-to-one into $Z(a)$. As has been shown in Volume 1, this mapping is realized in the whole space $Z(a)$.

Thus, the space K of infinitely-differentiable functions of bounded support is mapped by a Fourier transform into the space Z of entire analytic functions $\psi(s)$ ($s = \sigma + i\tau$), satisfying the conditions

$$| s^k\psi(s + i\tau)| \leqslant C_k e^{a|\tau|}.$$

By virtue of the general theorems on the inverse operator (Chapter I, Section 7.2), the operator F^{-1} is a continuous operator mapping $Z(a)$ into $K(a)$. Therefore, *we have*

$$F[K(a)] = Z(a), \qquad F^{-1}[Z(a)] = K(a), \tag{1}$$

where the Fourier operators are linear and continuous in both cases. Furthermore, since

$$K = \bigcup_a K(a), \qquad Z = \bigcup_a Z(a),$$

we have

$$F[K] = Z, \qquad F^{-1}[Z] = K, \tag{2}$$

where the Fourier operators are also linear and continuous here.

1.3. General Case

Now, let us consider any fundamental space Φ contained in the complex space S. (In particular, the spaces $K(a)$, K, $Z(a)$, and Z are contained in S.)

The set of all Fourier transforms of the function $\varphi(x) \in \Phi$ will be designated *a dual space with respect to* Φ and will be denoted by $\tilde{\Phi} = F[\Phi] = \Psi$. Evidently, Ψ is a linear space, linearly isomorphic to the space Φ. Let us introduce a topology in Ψ in conformity with this isomorphism; in particular, let us consider the sequence $\psi_\nu(\sigma) = F[\varphi_\nu(x)]$ to approach zero in the space Ψ, if the sequence $\varphi_\nu(x)$ tends to zero in Φ. The space $\Psi = F[\Phi]$ thereby also turns out to be a fundamental space. The Fourier operator $\psi = F[\varphi]$ is a continuous operator, mapping Φ one-to-one (and linearly isomorphically) into Ψ; by virtue of the same Theorem 1 of Chapter I, Section 7.2, the inverse operator is also continuous. Let us note that formulas (2)–(3) and (5)–(6) of Section 1.1 are understandably retained.

In concluding this section, let us make the following general remark: If a function $\varphi(-x)$ is contained together with each function $\varphi(x)$ in the spaces Φ and Ψ, then together with the relationship

$$F[\Phi] = \Psi,$$

there also holds

$$F^{-1}[\Phi] = \Psi.$$

In fact, if the function $\psi(\sigma)$ is the direct Fourier transform of the function $\varphi(x) \in \Phi$, then the inverse Fourier transform of the same function $\varphi(x)$ is the function $[1/(2\pi)^n]\, \psi(-\sigma)$.

For example, it is possible to write

$$F^{-1}[K(a)] = Z(a), \qquad F[Z(a)] = K(a),$$

$$F^{-1}[K] = Z, \qquad F[Z] = K,$$

together with relations (1) and (2) of the preceding paragraph.

2. Fourier Transforms of Generalized Functions

2.1. Fundamental Definition

Let us consider some fundamental space Φ of functions $\varphi(x)$ and its dual space Ψ of Fourier transforms $\psi(\sigma)$ of the function $\varphi(x)$.

The Fourier transform $F(f)$ of the generalized function $f \in \Phi'$ is defined by the formula

$$(F(f), F(\varphi)) = (2\pi)^n (f, \varphi). \tag{1}$$

We always introduce operations on generalized functions as conjugates to the corresponding operations on the fundamental functions.[2]

In this case, putting $F(\varphi) = \psi$, $\varphi = F^{-1}(\psi)$, we see that the constructed operator is the conjugate to the inverse Fourier transform operator[3] in the space Ψ and is linear and continuous together with it.

Hence, the Fourier transform of a generalized function on the space Φ is a generalized function in the space Ψ. For example, the Fourier transform of a generalized function on the space K is a generalized function on the space Z. In the latter case, the definition of the Fourier transform of a generalized function given here understandably agrees literally with the definition of the Fourier transform given in Chapter II of Volume 1.

Just as there, the definition (1) is justified as follows. Let f be a functional of the type of the absolutely integrable function $f(x)$. Furthermore, let $g(\sigma)$ be the customary Fourier transform of the function $f(x)$, and $\psi(\sigma)$ the Fourier transform of the fundamental function $\varphi(x)$. Then the Parseval equality[4]

$$\begin{aligned}(f, \varphi) = \int \overline{f(x)}\, \varphi(x)\, dx &= \frac{1}{(2\pi)^n} \int_x \overline{f(x)} \left\{ \int_\sigma \psi(\sigma)\, e^{-i(x,\sigma)}\, d\sigma \right\} dx \\ &= \frac{1}{(2\pi)^n} \int_\sigma \psi(\sigma) \left\{ \int_x \overline{f(x)\, e^{i(x,\sigma)}}\, dx \right\} d\sigma \\ &= \frac{1}{(2\pi)^n} \int \overline{g(\sigma)}\, \psi(\sigma)\, d\sigma = \frac{1}{(2\pi)^n} (g, \psi)\end{aligned}$$

[2] With some kind of slight distortion, say, the operation d/dx on the generalized functions is conjugate to the operation $-d/dx$, but not to d/dx on the fundamental functions.

[3] This is the distortion in this case.

[4] The change in the order of integration, made in the fourth step, is valid in this case by virtue of the absolute convergence of the double integral

$$\iint |f(x)\, \psi(\sigma)|\, dx\, d\sigma = \int |f(x)|\, dx \cdot \int |\psi(\sigma)|\, d\sigma.$$

holds. Hence, using our definition of the Fourier transform for the functional f, we obtain

$$(F(f), F(\varphi)) = (2\pi)^n (f, \varphi) = (g, \psi),$$

from which $F(f) = g$; therefore, the Fourier transform of the function $f(x)$ as a functional agrees with its customary Fourier transform.

The definition of the inverse operator $F^{-1}(f)$, may be obtained if $F(f)$ is replaced by g and $F(\varphi)$ by ψ in (1):

$$(g, \psi) = (2\pi)^n (F^{-1}(g), F^{-1}(\psi));$$

hence,

$$(F^{-1}(g), F^{-1}(\psi)) = \frac{1}{(2\pi)^n} (g, \psi). \tag{2}$$

If differentiation and multiplication by x are defined in the space Φ (and, hence, also in Ψ), then the following formulas hold:

$$P(D) F(f) = F[P(ix) f], \tag{3}$$

$$F[P(D) f] = P(-i\sigma) F[f], \tag{4}$$

which are analogous to (2) and (3) from Section 1.1.

Proof. In view of (1), and utilizing (2) and (3) of Section 1, we have

$$\begin{aligned}(F[P(ix) f], F(\varphi)) &= (2\pi)^n (P(ix) f, \varphi) \\ &= (2\pi)^n (f, \bar{P}(-ix)\, \varphi) \\ &= (F(f), F[\bar{P}(-ix)\, \varphi]) \\ &= (F[f], \bar{P}(-D) F(\varphi)) \\ &= (P(D) F(f), F(\varphi)),\end{aligned}$$

which yields (3). Analogously, furthermore,

$$\begin{aligned}(F[P(D) f], F(\varphi)) &= (2\pi)^n (P(D) f, \varphi) \\ &= (2\pi)^n (f, \bar{P}(-D)\, \varphi) \\ &= (F(f), F[\bar{P}(-D)\, \varphi]) \\ &= (F(f), \bar{P}(i\sigma) F(\varphi)) \\ &= (P(-i\sigma) F(f) F(\varphi)),\end{aligned}$$

from which (4) follows.

Applying the operator F^{-1} to (3) and (4) and replacing $F(f)$ by g, we obtain the formulas

$$F^{-1}[P(D)\, g] = P(ix)\, F^{-1}(g), \tag{5}$$

$$P(D)\, F^{-1}(g) = F^{-1}[P(-i\sigma)\, g], \tag{6}$$

which are analogous to (5) and (6) of Section 1.1.

Example 1. Let us find $F(\delta)$. If $\psi = F(\varphi)$, then by definition

$$(F(\delta), \psi) = (2\pi)^n\, (\delta, \varphi) = (2\pi)^n\, \varphi(0) = \int \psi(\sigma)\, d\sigma = (1, \psi),$$

from which

$$F(\delta) = 1. \tag{7}$$

Analogously

$$F^{-1}(\delta) = \frac{1}{(2\pi)^n}\,; \qquad (2\pi)^n F(1) = F^{-1}(1) = \delta(x).$$

Example 2. By means of (3) and (7), we find

$$F[P(x)] = F[P(x) \cdot 1] = P(-iD) \cdot F(1) = P(-iD)\, \delta(\sigma).$$

In particular

$$F(x^k) = (-iD)^k\, \delta(\sigma). \tag{8}$$

We indicated many other examples in Chapter II of Volume 1.

2.2. Fourier Transform of Generalized Functions of Bounded Support

As is known, the Fourier transform of the function $f(x)$, of support in the compact domain $G_a = \{|\, x_1 \,| \leqslant a_1\,, |\, x_2 \,| \leqslant a_2\,, \ldots, |\, x_n \,| \leqslant a_n\}$, is an entire analytic function of the variable $s = \sigma + i\tau$. More accurately, this entire function of s has an order of growth $\leqslant 1$ and a type $\leqslant a$; this means that for complex values of $s = \sigma + i\tau$ the function $f(x)$ satisfies the inequality

$$|f(\sigma + i\tau)| \leqslant C_\epsilon \exp[(a + \epsilon)\, |\, \tau \,|]$$

$$(= C_\epsilon \exp[(a_1 + \epsilon)\, |\, \tau_1 \,| + \cdots + (a_n + \epsilon)\, |\, \tau_n \,|)$$

for any $\epsilon > 0$.

It turns out that an analogous theorem is valid also for generalized functions of bounded support.

Theorem. *If a generalized function f on K has its support contained in the domain $G_a = \{|x_1| \leqslant a_1, \dots, |x_n| \leqslant a_n\}$, its Fourier transform $\tilde{f}$ is then a generalized function of the type of the functions $g(\sigma)$, which can be continued analytically into the complex domain $s = \sigma + i\tau$ as an entire function of the first order of growth with type a and will grow no more rapidly than $|\sigma|^q$ with some q for $|\sigma| \to \infty$ and fixed τ.*[4]

Proof. By the theorem of Chapter II, Section 4.4, for any $\epsilon > 0$ it is possible to find continuous functions $f_{k\epsilon}(x)$ $(k = 1, 2, \dots, m)$, of support in the domain $G_{a+\epsilon} = \{|x_1| \leqslant a_1 + \epsilon, \dots, |x_n| \leqslant a_n + \epsilon\}$ and such that

$$f = \sum_k P_{k\epsilon}(D) f_{k\epsilon}(x).$$

The Fourier transform of the functions $f_{k\epsilon}(x)$ is a functional of the type of the functions

$$g_{k\epsilon}(\sigma) = \int_{G_{a+\epsilon}} f_{k\epsilon}(x)\, e^{i(x,\sigma)}\, dx.$$

The function $g_{k\epsilon}(\sigma)$ is continued analytically into the complex domain by means of the formula

$$g_{k\epsilon}(\sigma + i\tau) = \int_{G_{a+\epsilon}} f_{k\epsilon}(x)\, e^{i(x,\sigma+i\tau)}\, dx,$$

and we evidently have

$$|g_{k\epsilon}(\sigma + i\tau)| \leqslant C \exp[(a_1 + \epsilon)|\tau_1| + \cdots + (a_n + \epsilon)|\tau_n|].$$

Hence, $g_{k\epsilon}(s)$ is an entire function of order of growth $\leqslant 1$ and of type $\leqslant a + \epsilon$ for any $\epsilon > 0$, or equivalently, of type $\leqslant a$. Application of the differential operator $P_{k\epsilon}(D)$ to the function $f_{k\epsilon}$ is equivalent to multiplication of the function $g_{k\epsilon}^{(s)}$ by the polynomial $P_{k\epsilon}(is)$. But it is well known that multiplication of an entire function by a polynomial does not change its order and type. Hence, the expressions $P_{k\epsilon}(is)\, g_{k\epsilon}(s)$ are also entire functions of order $\leqslant 1$ and of the type $\leqslant a$; therefore, their sum

$$g(s) = \sum_k P_{k\epsilon}(is)\, g_{k\epsilon}(s)$$

[4] Or on any other space of generalized functions where all infinitely differentiable functions of compact support are fundamental functions.

is also an entire function of order $\leqslant 1$ and of type $\leqslant a$. Furthermore, the function $g_{k\epsilon}(\sigma + i\tau)$ is bounded for fixed τ as the Fourier transform of the integrable functions $f_{k\epsilon}(x)\, e^{-(x,\tau)}$ of bounded support; hence, the product $P_{k\epsilon}(i\sigma)\, g_{k\epsilon}(\sigma)$ and their sum $g(\sigma)$ grow no more rapidly than some power of $|\sigma|$ as $|\sigma| \to \infty$. Our theorem is thereby proved completely.

Let us note that, as has been shown in Volume 1, the following Fourier transform formula

$$\tilde{f} = (f, e^{i(x,\sigma)}), \tag{1}$$

holds for any generalized function f of bounded support, where $e^{i(x,\sigma)}$ should be understood to be any of the fundamental functions (of the space K), which agree with the functions $e^{i(x,\sigma)}$ in the neighborhood of the support of f.

Formula (1) could also underlie the proof of our theorem.

The converse also holds.

Every entire function of order $\leqslant 1$ and of finite type, which has a growth not higher than a power for real $s = \sigma$, is the Fourier transform of some generalized function of bounded support on K if it is considered as a generalized function on Z.

The proof of this theorem will be given in Section 4.

2.3. Structure of Generalized Functions on the Space $Z(a)$

By using the Fourier transform, let us find the general form of a continuous linear functional in the space $Z(a)$. Each linear continuous functional $g \in Z'(a)$ indeed defines a continuous linear functional f in $K(a)$, operating according to the formula

$$(f, \varphi) = (2\pi)^n (g, \psi) \qquad (\psi = \tilde{\varphi}).$$

But the general form of a continuous linear functional on the space $K(a)$ is known (Chapter II, Section 4.3). We hence obtain that

$$(g, \psi) = \int_{G_a} f(x)\, D^m\varphi(x)\, dx,$$

where $f(x)$ is a continuous function. Substituting

$$D^m\varphi(x) = \int_R (-i\sigma)^m\, \psi(\sigma)\, e^{-i(x,\sigma)}\, d\sigma$$

in this formula, and inverting the order of integration, we find

$$(g, \psi) = \int_R (-i\sigma)^m \psi(\sigma) \left\{ \int_{G_a} f(x)\, e^{-i(x,\sigma)}\, dx \right\} d\sigma.$$

The function

$$g(\sigma) = \int_{G_a} f(x)\, e^{-i(x,\sigma)}\, dx$$

can be continued into the complex plane as an entire analytic function of order of growth $\leqslant 1$ and type $\leqslant a$; for real σ, it is bounded. Multiplying it by $(-i\sigma)^m$, we obtain the function $G(\sigma)$, which can also be continued into the complex plane as an entire analytic function of not higher than order of growth $\leqslant 1$ and type $\leqslant a$; for real σ, it grows not more rapidly than $C \mid \sigma \mid^m$.

Thus, the general form of a continuous linear functional in the space $Z(a)$ is given by the formula

$$(f, \psi) = \int_R G(\sigma)\, \psi(\sigma)\, d\sigma,$$

where $G(s)$ is an entire function of order of growth $\leqslant 1$ and type $\leqslant a$, which increases not more rapidly than some power of $\mid \sigma \mid$ for real $s = \sigma$.

Let us note that the function $G(s)$ is not determined uniquely. For example, a functional of the type of the function $e^{i(b,\sigma)}$ coincides with the null functional in the space $Z(a)$ if $\mid b_j \mid > \mid a_j \mid$ for any j. Indeed, the inverse Fourier transform of the function $e^{i(b,\sigma)}$ is $\delta(x - b)$, the null functional on the space $K(a)$.

2.4. Fourier Transforms and Differential Equations

Let us consider two examples of applying the Fourier transform to problems of differential equations.

Example 1. *Fundamental Solutions of Differential Equations.* Let us recall that the fundamental solution of the differential equation

$$P(D)\, u = g, \tag{1}$$

is the solution $E = E(x)$ of the equation

$$P(D)\, E = \delta(x). \tag{2}$$

Explicit expressions for the fundamental solutions of several kinds of differential equations were given in Volume 1 (Chapter I, Section 4; Chapter III, Section 2; Chapter IV, Section 2). Here we shall consider the general problem of the existence of fundamental solutions and we shall prove that *a fundamental solution exists in the class of functionals in the space K, for any linear differential equation with constant coefficients.*

Equation (2) in functionals on the space K is equivalent to an equation in functionals on the space Z, obtained from the Fourier transformation of (2):

$$P(i\sigma)\,\tilde{E} = 1. \tag{3}$$

The question of the solvability of (3) in the space Z' is a question of the existence of the functional $1/P(i\sigma)$ in this space. This question was solved affirmatively in Chapter II, Section 3.3. But *the question of the existence of the solution of* (2) *is thereby also solved affirmatively*; *the desired solution is the inverse Fourier transform of the functional* $1/P(i\sigma) \in Z'$.

Example 2. Let us call the differential operator

$$P(D) = \sum a_q \left(i\,\frac{\partial}{\partial x}\right)^q$$

quasi-elliptic if the polynomial $P(\sigma) = \sum a_q \sigma^q$ does not vanish at any point outside the sphere $|\sigma| \leqslant a$. Let us show that *every solution of the equation*

$$P(D)\,u(x) = 0, \tag{4}$$

which grows no more rapidly than a polynomial for real x is an entire function of $z = x + iy$ of order of growth $\leqslant 1$ and of type $\leqslant a$. (This is true for $P(D) = \Delta + 1$.)

For the proof, let us construct a functional in the space S by using the mentioned solution $u(x)$ of (4):

$$(u, \varphi) = \int_R u(x)\,\varphi(x)\,dx.$$

Let us show that the functional u satisfies the equation $P(D)u = 0$. Indeed, for any function $\varphi(x) \in S$ of bounded support

$$(P(D)\,u, \varphi) = (u, P(-D)\,\varphi) = \int_R u(x)\,P(-D)\,\varphi(x)\,dx.$$

Integration by parts is possible here, where the boundary terms vanish because the function $\varphi(x)$ is of bounded support. Hence

$$(P(D)\,u, \varphi) = \int_R P(D)\,u(x)\,\varphi(x)\,dx = 0,$$

that is, the functional $P(D)u$ vanishes for each function $\varphi(x)$ of bounded support. Since this functional is continuous on S, and the functions of bounded support generate a dense set in S (Chapter II, Section 2), then $P(D)u$ is the null functional on the whole S; therefore, the equation $P(D)u = 0$ is satisfied even in the sense of the theory of generalized functions.

Let $v = \tilde{u} \in S'$ be the Fourier transform of the functional u. Applying the Fourier transformation to (4), we obtain the new equation

$$P(\sigma)\, v = 0. \tag{5}$$

Since, by assumption the polynomial $P(\sigma)$ does not vanish outside the sphere $|\sigma| \leqslant a$, the support of the functional v is contained in this sphere. But then, by what has been proved, the functional $u(x)$ is an ordinary function, and moreover an entire analytic function of order of growth $\leqslant 1$ with type $\leqslant a$, as has been asserted.

Remark. If the polynomial $P(\sigma) = \sum a_q \sigma^q$ vanishes only at the one point $\sigma = 0$ (in this case the operator $P(D)$ is called *elliptic*), the result may be refined. In this case the support of the functional v is equal to the one point $\sigma = 0$. Applying the theorem of Chapter II, Section 4.5, we conclude that v may be written as

$$v = P_0\left(-i\frac{\partial}{\partial\sigma}\right)\delta(\sigma),$$

where P_0 is a polynomial; hence $u = P_0(x)$, by means of formula (8) of Section 2.1, i.e., u is a polynomial in x. We have arrived at the following result: *If $P(D)$ is an elliptic operator, then every solution of the equation $P(D)u = 0$, which grows not more rapidly than some power of $|x|$ as $|x| \to \infty$ is a polynomial in x.*

If the polynomial $P(\sigma)$ does not vanish for any real σ (in this case the operator $P(D)$ is called hypo-elliptic), the functional v should evidently equal zero. Hence, *if $P(D)$ is a hypo-elliptic operator, then all solutions of the equation $P(D)u = 0$ different from the solution identically null, will grow more rapidly than any power of $|x|$ as $|x| \to \infty$.* Analogous results are valid also for Eq. (4) written in vector form; the determinant of the appropriate matrix will play the part of the polynomial $P(\sigma)$.

3. Convolution of Generalized Functions and Its Connection to Fourier Transforms

We introduced the convolution operation in Volume 1 for the generalized functions on the fundamental space K (of infinitely differentiable functions of compact support). This operation played an important part in applications of the theory to differential equations. In this section, we introduce the convolution operation for generalized functions on any fundamental space Φ with the single condition that the space Φ contains together with each function $\varphi(x)$ all its translations.

3.1. Translation Operation

Let us assume that the translation operation

$$T_h\varphi(x) = \varphi(x + h)$$

has been defined in the fundamental space Φ for all real h. Let us also assume that T_h is a bounded operator in the space Φ uniformly for all h in any bounded domain[5] $|h| \leqslant h_0$.

Hence, it follows first that *the operator* T_h *is continuous for each fixed* h: If $\varphi_\nu \to 0$ in the topology of the space Φ, then $T_h\varphi_\nu \to 0$ also in the topology of the space Φ.

Furthermore, it may be asserted that *the operator* T_h *is continuous* in h in a perfect space or in the union of such spaces, i.e., for each $\varphi \in \Phi$ and $h \to h_1$, the relationship $T_h\varphi \to T_{h_1}\varphi$ holds in the topology of the space Φ. In fact, the family $\varphi(x + h_\nu)$ $(h_\nu \to h_1)$ is bounded in Φ, and therefore, is compact; but since convergence in the topology of Φ implies convergence of the function at each point x, the single limiting point of the family $\varphi(x + h_\nu)$ is $\varphi(x + h_1)$. It hence follows that

$$\varphi(x + h_1) = \lim_{\nu\to\infty} \varphi(x + h_\nu)$$

in this topology, q.e.d.

Example 1. There is no translation operation in the space $K(a)$ of infinitely differentiable functions of support contained in the compact domain $|x| \leqslant a$. It exists in the space K of all infinitely differentiable functions of compact support and evidently possesses all the required properties.

Example 2. Let us show that the translation operation is defined and bounded in the space S. To do this, let us estimate the quantity

$$|x^k D^q\varphi(x + h)| \quad \text{for} \quad |h| \leqslant h_0, \quad k \leqslant p, \quad q \leqslant p:$$

$$\begin{aligned}\sup_x |x^k D^q\varphi(x + h)| &= \sup_x |(x - h)^k D^q\varphi(x)| \\ &\leqslant \sup_x \sum C_k{}^j\, |x^j h^{k-j} D^q\varphi(x)| \\ &\leqslant 2^p h_0{}^p \sup_x |x^k D^q\varphi(x)| = 2^p h_0{}^p \|\varphi\|_p .\end{aligned}$$

[5] That is, the union of the images $T_h A$ of any bounded manifold A for all h, $|h| \leqslant h_0$, is a bounded manifold.

It hence follows that the function $\varphi(x + h)$ belongs to the space S together with the function $\varphi(x)$, and that the translation operator is bounded uniformly in h for $|h| \leqslant h_0$, as is required.

Example 3. A similar discussion may be made in the space $K\{M_p\}$ also under the following conditions: For any p, there exists a $p' \geqslant p$, such that for $|h| \leqslant h_0$,

$$M_p(x - h) \leqslant C_{ph_0} M_{p'}(x). \tag{1}$$

Example 4. Let us show that the translation operator is defined and bounded in the space $Z(a)$. Let $\varphi(z) \in Z(a)$; the function $\varphi(z + h)$ is an analytic function together with $\varphi(z)$, and

$$\begin{aligned}\sup_z |z^k \varphi(z + h)| e^{a|y|} &\leqslant \sup_z |(z - h)^k \varphi(z)| e^{a|y|} \\ &\leqslant \sum_j C_k^j |h|^{k-j} \sup_z |z^j \varphi(z)| e^{a|y|} \\ &\leqslant \sum_j C_k^j |h|^{k-j} \|\varphi\|_k .\end{aligned}$$

Hence, it follows that $\varphi(x + h) \in Z(a)$ and $\|\varphi(x + h)\|_k \leqslant C\|\varphi\|_k$, so that the translation operator in $Z(a)$ is bounded uniformly in h for $|h| \leqslant h_0$.

The reader will meet other examples in Chapter IV.

3.2. Definition of the Convolution

Since the function $\varphi(x + \xi)$ is continuous in x in the sense of the topology of Φ by the assumptions of Section 3.1, the expression

$$(f(\xi), \varphi(x + \xi))$$

is then a continuous function of x for any $f \in \Phi'$.

Let us introduce the following definition.

For some generalized function $f_0 \in \Phi'$, let us have

$$f_0 * \varphi \equiv (f_0(\xi), \varphi(x + \xi)) = \psi(x) \in \Phi, \tag{1}$$

for any function $\varphi \in \Phi$, and from the relationship $\varphi_\nu \to 0$ imply that $f_0 * \varphi_\nu \to 0$ in the topology of Φ; then the functional f_0 is called a "*convolute*" in the space Φ.

For example, $\delta(x - a)$ and its derivatives can be translated in any space in which a translation is defined:

$$\delta(x - a) * \varphi(x) = (\delta(\xi - a), \varphi(x + \xi)) = \varphi(x + a),$$
$$\delta'(x - a) * \varphi(x) = (\delta'(\xi - a), \varphi(x + \xi)) = -\varphi'(x + a),$$

etc.

Now, let us consider the conjugate operation in the space Φ', defined by the equality

$$(f_0 * f, \varphi) = (f, f_0 * \varphi).$$

Let us clarify what the operation $f_0 * f$ is, if f and f_0 are functionals of the type of functions of compact support.[1] In this case we will have

$$\begin{aligned}(f_0 * f, \varphi) = (f, f_0 * \varphi) &= \int \overline{f(x)} \left\{\int \overline{f_0(\xi)}\, \varphi(x + \xi)\, d\xi\right\} dx \\ &= \int \overline{f(x)} \left\{\int \overline{f_0(\eta - x)}\, \varphi(\eta)\, d\eta\right\} dx \\ &= \int \left\{\int \overline{f(x) f_0(\eta - x)}\, dx\right\} \varphi(\eta)\, d\eta,\end{aligned}$$

where the change in order of integration is valid, since all the integrals are actually taken between finite limits.

Hence, $f_0 * f$ is a functional of the function type

$$\int f(\xi) f_0(x - \xi)\, d\xi. \tag{2}$$

In analysis this function is called the *convolution* of the functions f_0 and f. Hence, we shall designate the operation $f_0 *$ in the space Φ' as a *convolution with the functional* f_0. Thus, we have defined a new linear and continuous operation in the space Φ', *the convolution with the convolute* f_0 operating according to the formula

$$(f_0 * f, \varphi) = (f, f_0 * \varphi).$$

We shall designate the operation $f_0 * \varphi$ in the fundamental space Φ as a *convolution of the functional* f_0 *with the fundamental function* φ. Let us emphasize that if f_0 is a functional of the type of the function $f_0(x)$, then in contrast to (2)

$$f_0 * \varphi = \int \overline{f_0(\xi - x)}\, \varphi(\xi)\, d\xi. \tag{3}$$

[6] A rather more exact analysis using the Fubini theorem permits us to establish the validity of an analogous result even for the case of functions $f(x)$ and $f_0(x)$, which are integrable in all space.

Numerous examples of convolutions were given in Volume 1 in the space of generalized functions on the fundamental space K (Chapter I, Section 4). Even there, in the general case

$$\delta * f = f.$$

Indeed

$$(\delta * f, \varphi) = (f, \delta * \varphi) = (f, \varphi).$$

Analogously

$$\frac{\partial \delta}{\partial x_j} * f = \frac{\partial f}{\partial x_j},$$

since,

$$\left(\frac{\partial \delta}{\partial x_j} * f, \varphi\right) = \left(f, \frac{\partial \delta}{\partial x_j} * \varphi\right) = \left(f, -\frac{\partial \varphi}{\partial x_j}\right) = \left(\frac{\partial f}{\partial x_j}, \varphi\right).$$

3.3. Differentiation of Convolutions

We shall derive the formula for differentiation of convolutions. Let us assume here that the translation operation is not only continuous but also differentiable in the space Φ, i.e., the limit relation

$$\frac{\varphi(x + h_j) - \varphi(x)}{h_j} \to \frac{\partial \varphi}{\partial x_j} \qquad (h_j = (0,..., h_j ,..., 0) \to 0)$$

is satisfied for any fundamental function $\varphi(x)$ in the sense of convergence in the space Φ.

The following lemma yields a conception of the spaces in which this requirement is satisfied.

Lemma. *In a perfect space Φ with continuous translation and differentiation operations, the limit relation*

$$\frac{\varphi(x + h_j) - \varphi(x)}{h_j} \to \frac{\partial \varphi}{\partial x_j}$$

is satisfied for each fundamental function $\varphi(x)$ in the sense of convergence in in the space Φ.

It is sufficient to show that the ratios $[\varphi(x + h_j) - \varphi(x)]/h_j$ remain bounded (in the topology of Φ) as $h_j \to 0$. Indeed, under this assumption, a convergent sequence $[\varphi(x + h_{j_\nu}) - \varphi(x)]/h_{j_\nu}$ may be selected; it is clear that it can converge only to $\partial\varphi(x)/\partial x_j$. Since the limit is defined in a unique manner, the mentioned ratio has this same limit as $h_j \to 0$ according to any law.

Let us show that the ratio $[\varphi(x + h_j) - \varphi(x)]/h_j$ is actually bounded as $h_j \to 0$ in spaces with the mentioned properties. This ratio may be represented as

$$\frac{\varphi(x + h_j) - \varphi(x)}{h_j} = \frac{1}{h_j} \int_0^{h_j} \frac{\partial\varphi(x + \theta_j)}{\partial x_j} \, d\theta_j .$$

According to the assumption, the function $[\partial\varphi(x + \theta_j)]/\partial x_j$ depends continuously on the parameter θ_j. By virtue of the theorem of the mean (Appendix to Chapter I, Section 4), the integral on the right side has the limit $\partial\varphi(x)/\partial x_j$ in the topology of the space as $h_j \to 0$; it hence follows that this integral is bounded for small h_j, q.e.d.

In particular, the assumptions of the lemma are satisfied in the spaces K, S, $K\{M_p\}$ (under the conditions (5) of Chapter II, Section 3.4, and (1) of Section 3.1); therefore, the result on convergence

$$\frac{\varphi(x + h_j) - \varphi(x)}{h_j} \to \frac{\partial\varphi}{\partial x_j}$$

in the topology of the appropriate space also holds in these spaces.

The expression $(f(x), \varphi(x + \xi))$ is not only a continuous function in a space with differentiable translation for any functional f, but is also (infinitely) differentiable; it is a fundamental function if f is a convolute.

The following theorem on differentiation of a convolution holds.

Theorem. *If a functional f_0 is a convolute in the fundamental space Φ with differentiable translation, then any functional $P(D) f_0$ is also a convolute in Φ and the equality*

$$P(D)(f_0 * f) = P(D) f_0 * f = f_0 * P(D) f \tag{1}$$

holds.

Proof. Let us first consider the case of the operator $P(D) = \partial/\partial x_j$. Since

$$\frac{\varphi(x + h_j) - \varphi(x)}{h_j} \to \frac{\partial\varphi}{\partial x_j}$$

by assumption, then we have by virtue of the continuity of the operator $f_0 *$:

$$f_0 * \frac{\varphi(x + h_j) - \varphi(x)}{h_j} \to f_0 * \frac{\partial\varphi}{\partial x_j} .$$

Hence, the function $f_0 * \varphi$ is differentiable and

$$\frac{\partial}{\partial x_j}(f_0 * \varphi) = f_0 * \frac{\partial \varphi}{\partial x_j}. \tag{2}$$

Now, let us show that the functional $\partial f_0/\partial x_j$ is also a convolute. We have

$$\begin{aligned} \frac{\partial f}{\partial x_j} * \varphi &= \left(\frac{\partial f_0}{\partial \xi_j}, \varphi(x+\xi)\right) \\ &= -\left(f_0(\xi), \frac{\partial \varphi(x+\xi)}{\partial \xi_j}\right) = -f_0 * \frac{\partial \varphi}{\partial x_j}; \end{aligned}$$

according to (2), this latter expression coincides with $-(\partial/\partial x_j)(f_0 * \varphi)$ and is therefore a fundamental function.

Furthermore

$$\begin{aligned} \left(\frac{\partial}{\partial x_j}(f_0 * f), \varphi\right) &= -\left(f_0 * f, \frac{\partial \varphi}{\partial x_j}\right) \\ &= -\left(f, f_0 * \frac{\partial \varphi}{\partial x_j}\right) = \left(f, \frac{\partial f_0}{\partial x_j} * \varphi\right) = \left(\frac{\partial f_0}{\partial x_j} * f, \varphi\right), \end{aligned}$$

from which the first of the equalities (1) results for $P(D) = \partial/\partial x_j$. On the other hand, according to (2),

$$\begin{aligned} \left(\frac{\partial}{\partial x_j}(f_0 * f), \varphi\right) &= -\left(f, f_0 * \frac{\partial \varphi}{\partial x_j}\right) \\ &= -\left(f, \frac{\partial}{\partial x_j}(f_0 * \varphi)\right) \\ &= \left(\frac{\partial f}{\partial x_j}, f_0 * \varphi\right) = \left(f_0 * \frac{\partial f}{\partial x_j}, \varphi\right), \end{aligned}$$

from which the second of the equalities (1) is obtained for $P(D) = \partial/\partial x_j$.

Therefore, formula (1) has been proved completely for the case $P(D) = \partial/\partial x_j$. Iterating the obtained result, we arrive at the validity of (1) in the general case also.

3.4. Convolutions of Generalized Functions of Bounded Support

By using the theorem on the differentiability of convolutions, we determine a broad class of functionals-convolutes.

Theorem. *If Φ is a perfect space with differentiable translation operation, then every functional $f \in \Phi'$ of compact support is a convolute.*

Proof. Let us first consider the case when the functional f corresponds to some ordinary continuous function $f(x)$ of compact support:

$$(f, \varphi) = \int f(x)\, \varphi(x)\, dx.$$

In this case

$$f * \varphi = (f(\xi), \varphi(x + \xi)) = \int_R f(\xi)\, \varphi(x + \xi)\, d\xi \qquad (1)$$

may be interpreted as the integral of the abstract function $f(\xi)\, \varphi(x + \xi)$ of the parameter ξ with values in Φ (see the Appendix to Chapter I, Section 4.4).

This function is continuous as the product of a continuous (by assumption) abstract function $\varphi(x + \xi)$ and a continuous numerical function $f(\xi)$. Hence, it may be integrated with respect to the parameter ξ within the limits of the bounded domain, where $f(\xi) \neq 0$; an element of the space Φ will again be the result. Comparing the values at each point, it is easy to see the agreement between the result of integrating $f(\xi)\, \varphi(x + \xi)$ as an abstract function and the function represented by the integral (1).

By the theorem of Chapter II, Section 4.4, the functional f is represented in the general case as the sum of derivatives of certain orders of finite continuous functions $f_k(x)$. According to what has been proved, each of these functions is a convolute. Since derivatives of convolutes are also convolutes in a space with differentiable translation, each of the derivatives of the functions $f_k(x)$ will be a convolute in Φ; their linear combination, the functional f, will be a convolute together with them.

At the same time, we also obtain a formula for the convolution of the functional $f = \sum P_k(D) f_k$ of bounded support with any fundamental function $\varphi(x)$;

$$\begin{aligned} f * \varphi = \sum P_k(D) f_k * \varphi &= \sum f_k * P_k(D)\, \varphi \\ &= \sum \int f_k(\xi)\, P_k(D)\, \varphi(x + \xi)\, d\xi. \end{aligned} \qquad (2)$$

Let us note yet another important property of a convolution with a generalized function of bounded support.

If the generalized function of bounded support f_0 corresponds to an infinitely differentiable function $f_0(x)$, then its convolution with any gener-

alized function f is a generalized function of the type of an infinitely differentiable function.

Indeed, for any fundamental function of compact support φ, we have

$$(f_0 * f, \varphi) = (f, f_0 * \varphi) = \left(f(x), \int f_0(\xi)\, \varphi(x + \xi)\, d\xi\right)$$

$$= \left(f(x), \int f_0(y - x)\, \varphi(y)\, dy\right)$$

$$= \int (f(x), f_0(y - x))\, \varphi(y)\, dy.$$

The function $(f(x), f_0(y - x))$ is the convolution of the functional $f(x)$ with the function $f_0(-x)$ and is hence infinitely differentiable. Therefore, the convolution $f_0 * f$ operates on the fundamental function $\varphi(x)$ as an infinitely differentiable function, q.e.d.

3.5. Theorem on the Continuity of a Convolution

Let us now establish a useful theorem on the *continuity of the convolution operation.*

Theorem. *If Φ is a fundamental space with differentiable translation operation, which contains all infinitely differentiable functions of compact support and the sequence of generalized functions f_ν ($\nu = 1, 2, \ldots$), of support in a fixed bounded set F, converges to the generalized function f* (whose support is also contained in the set F), *then for any generalized function g*

$$f_\nu * g \to f * g. \tag{1}$$

Let us first note that by virtue of the theorem of Section 3.4, f_ν and f are convolutes, and the expressions in (1) have meaning.

To prove (1), let us show first that the relation

$$f_\nu * \varphi \to f * \varphi$$

holds for any fundamental function $\varphi(x)$ of compact support.

It is sufficient to verify this for $f = 0$.

First, let Φ be a countably normed space. The functionals f_ν, concentrated in the fixed bounded set F and converging to zero may be written, according to the remark at the end of Chapter II, Section 4.4, as

$$f_\nu = \sum_{|q| \leqslant p} D^q f_q^{\,\nu},$$

where the functions $f_q(x)$ are continuous, vanish outside a fixed set F_1, and tend uniformly to zero as $\nu \to \infty$. According to (2) of the preceding paragraph, we have

$$f_\nu * \varphi = \sum_{|q| \leqslant p} \int f_q^\nu(\xi)\, D^q\varphi(x + \xi)\, d\xi.$$

Hence, we obtain the estimate

$$\| f_\nu * \varphi \|_m \leqslant \sum_{|q| \leqslant p} \max_{\xi \in F_1} |f_q^\nu(\xi)| \max_{\xi \in F_1} \| D^q\varphi(x + \xi)\|_m \cdot \operatorname{mes} F_1 ,$$

which shows that for any m, the norm of the elements $f_\nu * \varphi$ in the space Φ_m tends to zero when $\nu \to \infty$. This means that $f_\nu * \varphi$ tends to zero in the countably normed space Φ.

Now let Φ be the union of countably normed spaces. Convergence in the union reduces to convergence in the countably normed spaces themselves; it hence follows that our statement is true even in this case.

Now the assertion in the general case is easily proved. For any fundamental function $\varphi(x)$,

$$(f_\nu * g, \varphi) = (g, f_\nu * \varphi) \to (g, f * \varphi) = (f * g, \varphi)$$

according to what has been proved, and because of the continuity of the functional g, q.e.d.

Example. Let us consider the sequence of infinitely differentiable functions $f_\nu(x)$, of support in the interval $(-1, 1)$ which tend to $\delta(x)$ in the sense of generalized functions (the "delta-transformed sequence," see Chapter I, Section 2 of Volume 1). According to the theorem on the continuity of convolutions, for any functional g,

$$f_\nu(x) * g \to \delta(x) * g = g.$$

As has been mentioned above, the convolution $f_\nu(x) * g$ is an infinitely differentiable function. Hence, we have again obtained a proof of the possibility of approximating any generalized function by using infinitely differentiable functions (Chapter II, Section 4.4); we hence also obtain the explicit form of the corresponding approximate infinitely differentiable functions.

3.6. Harmonic Generalized Functions

In this paragraph, we will consider generalized functions on the space K of infinitely differentiable functions of compact support. A generalized function f is called *harmonic* if it satisfies the equation $\Delta f = 0$ (Δ is the

Laplace operator). It is known that the *theorem of the mean* is valid for classical harmonic functions: The value of a harmonic function at the center of any sphere equals the arithmetic mean of the values on the sphere itself. The theorem of the mean may be written thus with the aid of convolutions: For any $R > 0$,

$$f = \delta_R * f, \tag{1}$$

where δ_R denotes the generalized function corresponding to a uniform distribution of mass 1 on a sphere of radius R, i.e.,

$$\delta_R = \frac{1}{\Omega_n R^{n-1}} \delta(r - R)$$

(see Vol. 1, Chapter III, Section 1). Let us show that the *equality* (1) *is valid for any harmonic function f, and moreover, that it is characteristic of harmonic generalized functions, i,e., every generalized function satisfying equation* (1) *is a harmonic generalized function.*

Let us first consider the equation

$$\Delta g = \delta_R - \delta, \tag{2}$$

where g is the unknown generalized function.

Since the generalized function $\delta_R - \delta$ is of compact support, and therefore, is a convolute in the space K, Eq. (2) may then be solved by using the fundamental solution E of the Laplace equation:

$$g = (\delta_R - \delta) * E.$$

Since the fundamental solution E is known to be a harmonic function outside the origin, its average over a sphere of radius R with center at a distance greater than R from the origin, will equal its value at the center of the sphere. It hence follows that the generalized function g is zero outside a sphere of radius R with center at the origin, and is thereby finite. But then for any given harmonic generalized function f, we have

$$(\delta_R - \delta) * f = \Delta g * f = g * \Delta f = 0,$$

i.e.,

$$\delta_R * f = \delta * f = f,$$

q.e.d.

Conversely, let the theorem of the mean (1) be satisfied. Then evidently the equation

$$\frac{\delta_R - \delta}{R^2} * f = 0 \tag{3}$$

is also satisfied.

Let us find the limit of the functional $(\delta_R - \delta)/R^2$ as $R \to 0$. Using the Pizzeti formula (see Volume 1, Chapter I, Section 3.9), we have for the given fundamental function $\varphi(x)$,

$$\left(\frac{\delta_R - \delta}{R^2}, \varphi\right) = \frac{S_\varphi(r) - \varphi(0)}{R^2} = \frac{1}{2n} \Delta\varphi(0) + \cdots,$$

where the dots replace terms containing the positive powers of R^2.

Passing to the limit, we find

$$\lim \left(\frac{\delta_R - \delta}{R^2}, \varphi\right) = \frac{1}{2n} \Delta\varphi(0) = \left(\frac{1}{2n} \Delta\delta(x), \varphi(x)\right),$$

and therefore

$$\lim \frac{\delta_R - \delta}{R^2} = \frac{1}{2n} \Delta\delta(x).$$

Passing to the limit in the equality (3) (which is admissible by virtue of the theorem in Section 3.5), we obtain

$$\frac{1}{2n} \Delta\delta * f = \frac{1}{2n} \delta * \Delta f = \frac{1}{2n} \Delta f = 0,$$

q.e.d.

As a corollary we easily obtain that *there are actually no other harmonic generalized functions except those which correspond to the customary harmonic functions.*

Theorem. *Each harmonic generalized function is a generalized function of the type of infinitely differentiable functions, harmonic in the conventional sense.*

Proof. Let us $a(R)$ denote a non-negative infinitely differentiable function of R, of support in the interval $(1, 2)$ and which has an integral equal to 1. Let us construct the functional

$$h = \int_1^2 a(R)\, \delta_R \, dR.$$

By definition, it operates on the fundamental function $\varphi(x)$ (of compact support) according to the formula

$$\begin{aligned}(h, \varphi) &= \left(\int_1^2 a(R)\, \delta_R\, dR, \varphi(x)\right) \\ &= \int_1^2 a(R)(\delta_R, \varphi(x))\, dR \\ &= \int_1^2 a(R) \left\{\frac{1}{\Omega_n R^{n-1}} \int_{|x|=R} \varphi(x)\, dx\right\} dR \\ &= \int_{1 \leqslant |x| \leqslant 2} a_1(R)\, \varphi(x)\, dx.\end{aligned}$$

We see that the functional h is regular; it corresponds to an infinitely differentiable, spherically symmetric function $a_1(|x|)$ of compact support. Furthermore, because of the continuity of the convolution operator, we have for the harmonic functional f

$$\begin{aligned}h * f &= \int_1^2 a(R)\, \delta_R\, dR * f \\ &= \int_1^2 a(R)(\delta_R * f)\, dR \\ &= \int_1^2 a(R) f\, dR = f \int_1^2 a(R)\, dR = f.\end{aligned}$$

Hence, the generalized function f is the result of a convolution of two generalized functions h and f, the former of which is an infinitely differentiable function of compact support. But then, by virtue of the result obtained at the end of Section 3.4, the generalized function f *is itself infinitely differentiable* in any bounded domain; the operator Δ is applied to such a function in the conventional sense. Since f is a harmonic functional, i.e., $\Delta f = 0$ in the sense of generalized functions, then $\Delta f = 0$ also in the conventional sense.

3.7. Fourier Transformation and Convolutions

It is proved in analysis that the Fourier transform of the convolution of two integrable functions $f(x)$ and $g(x)$ (which is also an integrable function) equals the product of the Fourier transforms of the functions[7] $f(x)$ and $g(x)$.

[7] For example, see E. Titchmarsh, "Introduction to the Theory of Fourier Integrals," New York, 1939.

It turns out that an analogous assertion may be formulated for generalized functions also under certain conditions.

Theorem 1. *Let us consider the fundamental space Φ with continuous translation, and the dual space $\Psi = F[\Phi]$. If the functional g of the type of the functions $g(\sigma)$ is a multiplier in the space Ψ', then the functional $f = F^{-1}(g)$ is a convolute in the space Φ' and the following formula holds:*

$$F(f * f_1) = F(f) \cdot F(f_1). \tag{1}$$

Proof. By assumption, the translation operation is defined and continuous in the space Φ. We assert that the equality

$$F(\varphi(x - h)) = e^{i(h,\sigma)}F(\varphi(x)) \tag{2}$$

holds. In fact, replacing $x - h$ by y, we find

$$F(\varphi(x - h)) = \int_R \varphi(x - h)\, e^{i(x,\sigma)}\, dx = \int_R \varphi(y)\, e^{i(y+h,\sigma)}\, dy$$

$$= e^{i(h,\sigma)} \int_R \varphi(y)\, e^{i(y,\sigma)}\, dy = e^{i(h,\sigma)}F(\varphi(x)).$$

Hence, the existence of a (bounded) translation in the space Φ leads to the existence of a multiplier $e^{i(h,\sigma)}$ in the space $F[\Phi]$.

We now prove that the functional $f = F^{-1}(g)$ is a convolute in the space Φ. Applying f to the translated function $\varphi(x + h)$ and using the definition of the Fourier transformation of a functional and formula (2), we obtain

$$\begin{aligned} f * \varphi &= (f(\xi), \varphi(x + \xi)) \\ &= \frac{1}{(2\pi)^n} (g(\sigma), e^{-i(x,\sigma)}\psi(\sigma)) \\ &= \frac{1}{(2\pi)^n} \int \overline{g(\sigma)}\, e^{-i(x,\sigma)}\psi(\sigma)\, d\sigma = F^{-1}[\bar{g}\psi], \end{aligned}$$

and since $\bar{g}\psi \in \Psi$, the result belongs to the space Φ.[8]

Furthermore, if $\varphi_\nu \to 0$ in Φ, then $F[\varphi_\nu] = \psi_\nu \to 0$ in Ψ, and since the operation of multiplication by $\bar{g}$ in Ψ is continuous, then $\bar{g}\psi_\nu \to 0$ in Ψ, from which in turn follows $F^{-1}[\bar{g}\psi_\nu] = f * \varphi_\nu \to 0$ in Φ.

Hence, the functional f is a convolute in the space Φ and the formula

$$F[f * \varphi] = \bar{g}\psi.$$

holds.

[8] See the remark of Chapter II, Section 3.2. p. 100.

According to the definition of the convolution of generalized functions (Section 5.2), the expression $f * f_1$ has meaning for any $f_1 \in \Phi'$.

Let us now find $F(f * f_1)$. According to the fundamental formula (1)

$$\begin{aligned}(F(f * f_1), F(\varphi)) &= (2\pi)^n (f * f_1, \varphi) \\ &= (2\pi)^n (f_1, f * \varphi) \\ &= (F(f_1), F(f * \varphi)) \\ &= (F(f_1), \bar{g}\psi) = (g(\sigma) F(f_1), F(\varphi)),\end{aligned}$$

from which formula (1) also results.

We prove the converse under more restrictive assumptions.

Theorem 2. *If Φ is a space with a differentiable translation, which contains all infinitely differentiable functions of compact support, where these functions generate a dense set in Φ, and if f is a generalized function of compact support* (and therefore, a convolute according to the Theorem of Section 3.4), *then $\tilde{f} = g$ is a multiplier in the space $\Psi' = \tilde{\Phi}'$ and the formula*

$$F(f * f_1) = g \cdot g_1$$

holds for any generalized functions f_1 and $g_1 = \tilde{f}_1$.

Proof. Let us first consider the Fourier transform of the convolution $f * \varphi$, where $\varphi(x)$ is a fundamental function of compact support. We have

$$F(f * \varphi) = F(f(\xi), \varphi(x + \xi)) = \int e^{i(x,\sigma)}(f(\xi), \varphi(x + \xi))\, dx,$$

since $f * \varphi = (f(\xi), \varphi(x + \xi))$ is a fundamental function by assumption. The obtained integral may be represented as

$$(f(\xi), \int e^{i(x,\sigma)}\varphi(x + \xi)\, dx),$$

since the function $e^{i(x,\sigma)}\, \varphi(x + \xi)$ may be considered a continuous abstract function of ξ (with values in Φ) and integration is over the (compact) support of $f * \varphi$.

Furthermore

$$(f(\xi), \int e^{i(x,\sigma)}\varphi(x + \xi)\, dx) = (f(\xi), e^{-i(\sigma,\xi)}\psi(\sigma)),$$

where $\psi(\sigma) = \widetilde{\varphi(x)}$ is the Fourier transform of the function $\varphi(x)$. But here the numerical factor $\psi(\sigma)$ may be taken out, and the remaining expression is the function $\tilde{\tilde{f}}(\sigma)$ (according to formula (1) of Section 2.2). Thus

$$\widetilde{f * \varphi} = \tilde{\tilde{f}} * \tilde{\varphi}$$

for every function of compact support $\varphi \in \Phi$.

Now, if $\varphi(x)$ is any fundamental function, then by assumption there exists a sequence of functions of compact support $\varphi_\nu(x)$, convergent to $\varphi(x)$ in the topology of the space Φ. According to what has been proved, we have

$$\widetilde{f * \varphi_\nu} = \tilde{\tilde{f}} \cdot \tilde{\varphi}_\nu .$$

Furthermore, as $\nu \to \infty$, the functions $\widetilde{\varphi_\nu(\sigma)}$ converge to the function $\widetilde{\varphi(\sigma)}$ in the topology of the space Ψ; in particular, this convergence holds at each point σ. It hence follows that at each point

$$\tilde{\tilde{f}}(\sigma) \cdot \tilde{\varphi}_\nu(\sigma) \to \tilde{\tilde{f}}(\sigma) \cdot \tilde{\varphi}(\sigma);$$

on the other hand, by virtue of the continuity of the convolution $f * \varphi_\nu \to f * \varphi$, hence $\widetilde{f * \varphi_\nu} \to \widetilde{f * \varphi}$; summarizing, the function $\widetilde{f * \varphi}$ agrees everywhere with the function $\tilde{\tilde{f}}(\sigma) \cdot \tilde{\varphi}(\sigma)$ and, therefore, the *relationship*

$$\widetilde{f * \varphi} = \tilde{\tilde{f}} \cdot \tilde{\varphi}$$

is valid for all fundamental functions $\varphi \in \Phi$. In particular, the function $\tilde{\tilde{f}}(\sigma)$ may be multiplied by any fundamental function $\tilde{\varphi} \in \tilde{\Phi}$; hence $\tilde{\tilde{f}}(\sigma)$ *is a multiplier in the space* Φ.

Now, let $f_1 \in \Phi'$ be an arbitrary functional. According to what has been proved, we have for any fundamental function $\varphi \in \Phi$,

$$\begin{aligned}(\widetilde{f * f_1}, \tilde{\varphi}) &= (2\pi)^n (f * f_1, \varphi) = (2\pi)^n (f_1, f * \varphi) \\ &= (\tilde{f}_1, \tilde{\tilde{f}} \cdot \varphi) = (\tilde{f} \cdot \tilde{f}_1, \tilde{\varphi}),\end{aligned}$$

from which

$$\widetilde{f * f_1} = \tilde{f} \cdot \tilde{f}_1 ,$$

q.e.d.

Example. The functional

$$\delta_R = \frac{1}{\Omega_n R^{n-1}} \delta(^a - R),$$

corresponding to a uniformly distributed mass 1 on a sphere of radius R, has the function

$$\int_{|x|=R} e^{i(x,\sigma)}\, dx = \Omega_{n-1} R^{n/2} \,|\, \sigma \,|^{1-(n/2)} J_{(n-2)/2}(R \,|\, \sigma \,|)$$

as its own Fourier transform, as has been shown in Vol. 1 (Chapter II, Section 3.4).

By virtue of the theorem proved, the functional $\delta_R * \delta_R * \cdots * \delta_R$ (m times) has the Fourier transform

$$\Omega_{n-1}^m R^{mn/2} \left(\frac{J_{(n-2)/2}(R \,|\, \sigma \,|)}{|\, \sigma \,|^{(n/2)-1}} \right)^m .$$

For $n \geqslant 2$ and sufficiently large m, this function vanishes at infinity with any power order of decrease. Hence, it follows in particular that the functional $\delta_R * \cdots * \delta_R$ (m times) is a functional of the type of continuous functions with arbitrarily high order of smoothness for sufficiently large m.

3.8. Hilbert Transformation

It is known[9] that the Hilbert transformation

$$h(\xi) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\varphi(x)}{\xi - x}\, dx \tag{1}$$

(the integral is understood in the sense of the Cauchy principal value) with the inversion formula

$$\varphi(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{h(\xi)\, d\xi}{x - \xi} \tag{2}$$

has meaning for functions $\varphi(x)$, defined for $-\infty < x < \infty$ and tending to zero together with the first derivative not more slowly than $1/|\, x \,|$ as $|\, x \,| \to \infty$.

In the present paragraph, we obtain formulas (1)–(2) in the natural domain of their definition, and we extend them to a certain class of generalized functions.

Let us consider the set Ψ of all functions $\psi(s)$ ($-\infty < s < \infty$), possessing the following properties:

(a) The function $s^k \psi(s)$ is absolutely continuous on the line $-\infty < s < \infty$ for any k.

[9] E. Titchmarsh, "Introduction to the Theory of Fourier Integrals," New York, 1939, Chapter V.

(b) $\psi(s)$ is continuous and has a continuous derivative $\psi'(s)$ on each of the half-lines $-\infty < s \leqslant 0, 0 \leqslant s < \infty$; the function $\psi(s)$ (and $\psi'(s)$) may have a discontinuity of the first kind at the point $s = 0$.

(c) $s^k\psi'(s)$ is absolutely integrable on the line $-\infty < s < \infty$ for any k.

A topology may be introduced in the space Ψ by defining the countable set of norms

$$\| \psi(s)\|_k = \int_{-\infty}^{\infty} | s^k\psi(s)| \, ds + \int_{-\infty}^{\infty} | s^k\psi'(s)| \, ds$$
$$+ \max_{-\infty < s < \infty} | \psi(s)| + \max_{-\infty < s < \infty} | \psi'(s)|. \tag{3}$$

With this topology, it is easy to verify that Ψ is a complete countably normed space. The operator A, consisting of the multiplication of each function $\psi(s) \in \Psi$ by the factor

$$\beta(s) = \begin{cases} -1 & \text{for} \quad s < 0, \\ \ \ 1 & \text{for} \quad s > 0. \end{cases}$$

is defined, and evidently bounded, in the space Ψ.

Let us seek the functions of the space Φ dual to Ψ. To do this, let us note that each function $\psi(s)$ may be transformed into the continuous function $\psi_0(s)$ with a continuous derivative absolutely integrable (together with the derivative) under multiplication by any power of $| s |$, if we substract from $\psi(s)$ the function

$$\psi_1(s) = \begin{cases} (C + C_1 s) \, e^{-s} & (s > 0) \\ \qquad 0 & (s < 0) \end{cases}$$

with suitably selected C and C_1.

Let Y denote the class of functions $\varphi(x)$, which are obtained by a Fourier transformation of continuous, absolutely integrable functions; in each case the class Y consists of continuous bounded functions. Then the class Y_1 of functions which are obtained by a Fourier transformation of absolutely integrable functions with a continuous and absolutely integrable derivative, consists of functions $\varphi(x)$ belonging to Y and remaining within Y upon multiplication by x. Functions with a continuous derivative, which are absolutely integrable (together with the derivative) upon multiplication by any power $| s |$, transform into infinitely differentiable functions, under Fourier transformation, for which all the derivatives belong to the class Y_1. Hence, the Fourier transform of the function $\psi_0(s)$ belongs to the class Y_1; this is an infinitely differentiable function $\varphi_0(x)$, which decreases more slowly than $1/| x |$ at infinity.

The Fourier transform of the function $\psi_1(s)$ is easily evaluated completely:

$$\widetilde{\psi_1(s)} = C\int_0^\infty e^{-s}e^{ixs}\,ds + C_1\int_0^\infty se^{-s}e^{ixs}\,ds$$
$$= \frac{C}{-1+ix} - \frac{C_1}{(-1+ix)^2}.$$

This is also an infinitely differentiable function which decreases more slowly than $1/|x|$ at infinity.

Hence, the Fourier transformation of any function $\psi(s) \in \Psi$ is also an infinitely differentiable function which decreases more slowly than $1/|x|$ at infinity.

For each function $\varphi(x) = \psi(s)$, the operator

$$H\varphi(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\varphi(x)}{\xi - x}\,dx = h(\xi) \tag{4}$$

is defined, where the integral at the point $x = \xi$ is taken in the sense of the Cauchy principal value, and is evidently absolutely convergent at infinity.

The operator H is evidently a convolution operator with the generalized function $-1/\pi x$, defined on the space Φ. The Fourier transform of $-1/\pi x$ is the generalized function $\beta(s)$, which equals $+1$ for $s < 0$ and -1 for $s > 0$, and the convolution operation with $-1/\pi x$ in Φ goes over into the operation of multiplication by $\beta(s)$ in Ψ. Since this latter operation transforms the space Ψ into itself, the operator H then transforms the space Φ into itself; therefore the function $h(\xi)$ also belongs to the space Φ.

Applying the operation of multiplication by $\beta(s)$ twice to the function $\psi(s) \in \Psi$, we will evidently return to the original function $\psi(s)$. Hence, by applying the operator H to the function $h(\xi)$ which has been obtained, we shall return to the original function $\varphi(x)$:

$$\varphi(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{h(\xi)\,d\xi}{x - \xi}.$$

Thus, formulas (1)–(2) are valid in the space Φ.

It is now clear how these formulas may be generalized. The conjugate operation $1/\pi x\, *$, which agrees with the conventional convolution with the function $1/\pi x$ in the case of ordinary, good enough functions, is defined for generalized functions on the space Φ. Since the original operation $-1/\pi x\, *$ is its own inverse in the space Φ, the conjugate operation is also its own inverse in the space Φ'; taking into account that

Φ together with each function $\varphi(x)$ contains $\varphi(-x)$, we may conclude that (1)–(2) are also retained for the generalized functions on the space Φ (with the appropriate interpretation of the integrals). Let us note that any function $f(x)$, which is locally integrable and grows no more rapidly than $|x|^{1-\epsilon}$, $\epsilon > 0$ as $|x| \to \infty$, defines a functional on the space Φ; formula (1) (which again reduces to a functional on the space Φ, although possibly not regular) and the inversion formula (2) have meaning for such a function.

4. Fourier Transformation of Entire Analytic Functions

Fourier transformations of entire analytic functions of the first order of growth will be considered in this section. In the conventional sense, it is understood, these entire functions do not generally have Fourier transforms. But if these functions are considered as generalized functions, i.e., functionals on the space K of all infinitely differentiable functions of compact support, say, then Fourier transform will exist in conformity with the general theory, as functionals in the space Z, dual to K. The assumption on the order of growth permits a simple characterization to be given of these functionals.

4.1. Fundamental Theorem on the Fourier Transformation of First Order Entire Functions

Let $f(z) = f(z_1, \ldots, z_n)$ be an entire analytic function of not greater than first order of growth with type $\leqslant b = (b_1, b_2, \ldots, b_n)$; this means that the inequality

$$|f(z)| \leqslant C_\epsilon \exp[(b_1 + \epsilon)\,|z_1| + \cdots + (b_n + \epsilon)\,|z_n|] \tag{1}$$

is satisfied for any $\epsilon > 0$.

By means of the formula

$$(f, \varphi) = \int_R \overline{f(x)}\,\varphi(x)\,dx,$$

the function $f(z)$ defines a continuous linear functional in the space K.

The Fourier transform of the functional f will be some functional in the space Z; let us find it explicitly.

The Taylor series for the function $f(z)$

$$f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu = \sum a_{\nu_1 \cdots \nu_n} z_1^{\nu_1} \cdots z_n^{\nu_n} \tag{2}$$

converges uniformly in each bounded domain, and therefore, converges in the topology of the space K'. Since the Fourier-transformation operator is continuous, the result of its application to the functional f may be calculated by applying this operation to each member of the series (2) and combining the results. Utilizing formula (8) of Section 2.1, we obtain[10]

$$F(f) = \sum_{\nu=0}^{\infty} a_\nu F(x^\nu) = \sum_{\nu=0}^{\infty} a_\nu (-iD)^\nu \delta(\sigma).$$

Applying this result to any function $\psi(\sigma) \in Z$, we find

$$\begin{aligned}(F(f), \psi) &= \sum_\nu a_\nu((-iD)^\nu \delta(\sigma), \psi(\sigma)) \\ &= \sum_\nu a_\nu(\delta(\sigma), (-iD)^\nu \psi(\sigma)) = \sum_\nu (-i)^\nu a_\nu D^\nu \psi(0). \end{aligned} \tag{3}$$

The series is known to converge for any function $\psi \in Z$. But it may be asserted that it even converges for any function $\psi(s)$, analytic in the domains $| s_1 | \leqslant b_1 ,..., | s_n | \leqslant b_n$. In fact, as follows from the n-dimensional Cauchy formula,[11] for any such function the inequalities

$$\left| \frac{\partial^{|\nu|}\psi(0)}{\partial s_1^{\nu_1} \cdots \partial s_n^{\nu_n}} \right| \leqslant C\nu\,! \left(\frac{1}{\bar{b}_1}\right)^{\nu_1} \left(\frac{1}{\bar{b}_2}\right)^{\nu_2} \cdots \left(\frac{1}{\bar{b}_n}\right)^{\nu_n}, \tag{4}$$

are satisfied, where $\bar{b}_j > b_j$ are parameters of the domain in which the function $\psi(s)$ still remains analytic, and

$$C = \max_{|s_j| \leqslant \bar{b}_j} | \psi(s)|. \tag{5}$$

[10] Let us recall the notation

$$x^\nu = x_1^{\nu_1} \cdots x_n^{\nu_n}, \qquad (-iD)^\nu = (-i)^{\nu_1+\cdots+\nu_n} \frac{\partial^{\nu_1+\cdots+\nu_n}}{\partial x_1^{\nu_1} \cdots \partial x_n^{\nu_n}}.$$

[11] We speak of the formula ($| s_1 | < \rho_1 ,..., | s_n | < \rho_n$),

$$\frac{\partial^{k_1+\cdots+k_n}\psi(s_1 ,..., s_n)}{\partial s_1^{k_1} \cdots \partial s_n^{k_n}} = \frac{1}{(2\pi i)^n} k_1\,! \cdots k_n\,! \int_{|\xi_1|=\rho_1} \cdots \int_{|\xi_n|=\rho_n} \frac{\psi(\xi_1 ,..., \xi_n)\, d\xi_1 \cdots d\xi_n}{(\xi_1 - s_1)^{k_1+1} \cdots (\xi_n - s_n)^{k_n+1}}.$$

On the other hand, the Taylor coefficients of the function $f(s)$ satisfy the inequalities[12]

$$| a_{\nu_1 \cdots \nu_n} | \leqslant C' \left(\frac{b_1 e\theta_1}{\nu_1}\right)^{\nu_1} \cdots \left(\frac{b_n e\theta_n}{\nu_n}\right)^{\nu_n} \tag{6}$$

for any $\theta_j > 1$ as is easy to obtain from (1) by application of the same Cauchy formula. Substituting the estimates (4) and (5) into (3), we obtain the majorizing series

$$CC' \prod_{j=1}^{n} \sum_{\nu_j=0}^{\infty} \frac{(b_j e\theta_j)^{\nu_j}}{\nu_j^{\nu_j}} \nu_j ! \left(\frac{1}{\bar{b}_j}\right)^{\nu_j},$$

which converges for $\theta_j < (\bar{b}_j/b_j)$ by virtue of the simplest convergence criteria.

Hence, formula (3) permits extension of the functional $F(f)$ over all functions $\psi(s)$, analytic in the domains $| s_i | \leqslant b_i \ (i = 1,..., n)$. For brevity, we shall sometimes write just one $| s | \leqslant b$ instead of all these inequalities. We have thus established the following theorem.

Theorem. *The Fourier transform of the first order entire function $f(x)$ of type $\leqslant b = (b_1, ..., b_n)$ is a functional in the space Z, which may be extended to all functions $\psi(s)$, analytic in the domains $| s_1 | \leqslant b_1, ..., | s_n | \leqslant b_n$.*

The functional $\tilde{f}$ in the set of functions $\psi(s)$, analytic in the mentioned domains, will also be continuous in the following sense: If the sequence of functions $\psi_\nu(s)$ converges uniformly to zero in the domain $| s | \leqslant b + \epsilon$, then the numbers $(F(f), \psi_\nu)$ tend to zero. In fact, in this case the constant $C = C_\nu$ in inequality (4), written for $\psi = \psi_\nu$, will tend to zero as $\nu \to \infty$, which leads to the desired result upon being substituted into the series (3).

[12] By virtue of the n-dimensional Cauchy formula

$$a_\nu = \frac{1}{\nu_1 ! \cdots \nu_n !} \frac{\partial^{\nu_1 + \cdots + \nu_n} \psi(0)}{\partial s_1^{\nu_1} \cdots \partial s_n^{\nu_n}}$$
$$= \frac{1}{(2\pi i)^n} \int_{|\xi_1| = r_1} \cdots \int_{|\xi_n| = r_n} \frac{f(\xi)\, d\xi_1 \cdots d\xi_n}{\xi_1^{\nu_1+1} \cdots \xi_n^{\nu_n+1}},$$

from which

$$| a_\nu | \leqslant \frac{1}{(2\pi i)^n} C \frac{\exp[(b_1 + \epsilon) r_1 + \cdots + (b_n + \epsilon_n) r_n]}{r_1^{\nu_1+1} \cdots r_n^{\nu_n+1}}.$$

Passing to the minimum in $r_1, ..., r_n$ in the right side, we obtain the estimate (6).

4.2. Explicit Expression of the Fourier Transform of a First Order Entire Function

An explicit formula may be given for the Fourier transform of a function $f(z)$, satisfying the inequality (1) of the preceding paragraph, namely, by writing $f(z)$ as

$$f(z) = \int e^{izs}\, d\mu(s), \tag{1}$$

where $\mu(s)$ is a complex measure of support in the domain $| s_j | \leqslant b_j + \epsilon_j$ (dependent on $\epsilon = (\epsilon_1, ..., \epsilon_n)$). To prove (1), we reason as follows:

Let $\mathfrak{Z}$ be the linear space of all entire functions. Let us introduce a family of norms in $\mathfrak{Z}$ by means of the formulas

$$\| g \|_m = \max_{|s| \leqslant m} | g(s) |.$$

These norms are the limiting values of the norms in the spaces $Z\{M_p\}$ (Chapter II, Section 1), wherein it is necessary to put

$$M_p(s) = \begin{cases} 1 & \text{for} \quad | s | \leqslant p, \\ 0 & \text{for} \quad | s | > p. \end{cases}$$

It is easy to verify that these norms agree, that the space $\mathfrak{Z}$ is complete, and finally, that it is perfect.

Let us find the general form of a linear continuous functional in the space $\mathfrak{Z}$. It is sufficient to find a general linear functional (f, g) in the normed space $\mathfrak{Z}_m$, the completion of the space $\mathfrak{Z}$ in the norm $\| g \|_m$. The space $\mathfrak{Z}_m$ consists of some continuous functions $g(s)$, defined in the domain $| s | \leqslant m$; it is closed relative to uniform convergence. Continuing the functional f into the space of all continuous functions in the domain $| s | \leqslant m$ according to the Hahn–Banach theorem, and applying the Riesz–Radon theorem, we obtain

$$(f, g) = \int_{|s| \leqslant m} g(s)\, d\mu(s), \tag{2}$$

where $\mu(s)$ is a complex, completely additive measure in the domain $| s | \leqslant m$. By virtue of the theorem in Chapter I, Section 4.3 on the structure of a space conjugate to a countably normed space, formula (2) yields the general form of a linear continuous functional in the space $\mathfrak{Z}$

for all possible m. The Taylor series $g(s) = \sum a_\nu s^\nu$ converges in the topology of the space $\mathfrak{Z}$; hence

$$(f, g) = \sum a_\nu f_\nu , \tag{3}$$

where $f_\nu = (f, s^\nu)$ is a fixed sequence of constants. Conversely, every sequence of constants f_ν, such that the series (3) converges for any entire function $g(s) \in \mathfrak{Z}$ and defines a continuous linear functional in the space $\mathfrak{Z}$ by means of (3), may be represented as

$$f_\nu = \int_{|s| \leqslant m} s^\nu \, d\mu(s),$$

which is obtained from the general formula (2) for $g(s) = s^\nu$.

Let $f(z) = \sum c_\nu z^\nu$ be an entire function of order $\leqslant 1$ and of type $\leqslant b$, i.e., one satisfying an inequality of the form (1) of Section 4.1. Let us show that the numbers $f_\nu = (-i)^\nu c_\nu \nu!$ satisfy the imposed conditions.

Indeed, we have according to inequality (6) of Section 4.1

$$| c_\nu | \leqslant C \left(\frac{e\theta b}{\nu}\right)^\nu, \tag{4}$$

where $\theta > 1$ is a constant. On the other hand, if $\sum a_\nu s^\nu = g(s)$ is any entire function, then by applying the Cauchy formula in the domain $| s | \leqslant \theta_1 \theta b$, $\theta_1 > 1$, we will have

$$| a_\nu | \leqslant \frac{M}{(\theta_1 \theta b)^\nu},$$

where $M = \max_{|s|=\theta_1\theta b} | g(s)|$. Hence[13]

$$\sum | a_\nu f_\nu | = \sum | a_\nu c_\nu | \nu ! \leqslant CM \sum_{\nu=0}^{\infty} \frac{\sqrt{\nu}}{\theta_1{}^\nu} = \text{const.} \tag{5}$$

i.e., the series (3) converges. At the same time, we obtain boundedness of the functional (3) in the norm $\| \cdot \|_m$ with $m > \theta_1 \theta b$, which also means boundedness of the functional (3) in the whole space $\mathfrak{Z}$.

According to what has been proved, there exists a measure $\mu(s)$, such that

$$f_\nu = (-i)^\nu c_\nu \nu ! = \int s^\nu \, d\mu(s);$$

[13] According to the well-known Stirling formula

$$\nu ! = \left(\frac{\nu}{e}\right)^\nu (2\pi\nu)^{1/2} E_\nu ,$$

where $E_\nu \to 1$.

hence

$$c_\nu = \int \frac{(is)^\nu}{\nu\,!}\, d\mu(s).$$

Multiplying by z^ν and adding, we obtain convergent series on the left and right, and therefore

$$f(z) = \sum c_\nu z^\nu = \sum \int \frac{(izs)^\nu}{\nu\,!}\, d\mu(s) = \int e^{izs}\, d\mu(s),$$

q.e.d.

Formula (2) which we have proved independently of the previous constructions, might itself be the basis for the proof of the theorem of Section 2.1. The value of the first proof is that it may be carried over even to the case of entire functions of higher order of growth.

4.3. Converse Theorem

Let $\mathfrak{Z}(G)$ denote the linear space of all analytic functions in a closed domain G (analytic within and on the boundary of the domain). Let us introduce the following concept of the passage to the limit in this space: A sequence $g_\nu(s) \in \mathfrak{Z}(G)$ is called *convergent to zero* if there exists a domain $G' \subset G$, in which all the functions $g_\nu(s)$ remain analytic and tend uniformly to zero.

We shall designate every continuous linear functional in the space $\mathfrak{Z}(G)$ "as belonging to the domain G."

The result obtained in Section 4.1 may be formulated as follows.

Theorem. *The Fourier transform of an entire function of order of growth $\leqslant 1$ and of type $\leqslant a$ is a functional on the space Z, which may be extended to a functional belonging to the domain*

$$G_a = \{|\, s_1 \,| \leqslant a_1 ,\dots, |\, s_n \,| \leqslant a_n\}.$$

It turns out that the converse is also valid.

Theorem. *If the functional $g \in Z'$ may be extended to a functional belonging to the domain G_b, then its Fourier transform is a functional on the space K of the type of the analytic entire function $f(z)$ of first order of growth and of type $\leqslant b$.*

Proof. Let the function $\psi(s)$ be analytic in the domain $|\, s \,| \leqslant b$. It can then be expanded in a Taylor series which converges uniformly in a broader domain:

$$\psi(s) = \sum a_\nu s^\nu ;$$

$$| a_\nu | \leqslant C\left(\frac{\theta}{b}\right)^\nu \quad \left(= C\left(\frac{\theta}{b_1}\right)^{\nu_1} \cdots \left(\frac{\theta}{b_n}\right)^{\nu_n}\right),$$

where $\theta < 1$.

If the functional $g \in Z'$ may be extended in $\mathfrak{Z}(G_b)$, then because of its continuity, we have:

$$(g, \psi) = \sum a_\nu(g, s^\nu) = \sum a_\nu g_\nu , \tag{1}$$

where g_ν is some fixed numerical sequence ($\nu = (\nu_1 ,..., \nu_n)$).

The condition of convergence of the series (1) for all $\psi(s) \in \mathfrak{Z}(G_b)$ imposes definite constraints on the order of growth of the sequence a_ν.

For example, if we set

$$a_\nu = \pm\left(\frac{\alpha_1}{b_1}\right)^{\nu_1} \cdots \left(\frac{\alpha_n}{b_n}\right)^{\nu_n},$$

where $\alpha_j < 1$, then

$$\psi(s) = \sum a_\nu s^\nu \in \mathfrak{Z}(G_b);$$

from the convergence of the series (1), it follows that

$$| g_\nu | \leqslant C\left(\frac{b_1}{\alpha_1}\right)^{\nu_1} \cdots \left(\frac{b_n}{\alpha_n}\right)^{\nu_n}. \tag{2}$$

The dual functional $F^{-1}(g) = f$, defined on the space K, operates according to the formula

$$\begin{aligned}(F^{-1}(g), F^{-1}(\psi)) &= (2\pi)^{-n}(g, \psi)\\ &= (2\pi)^{-n} \sum \frac{D^\nu\psi(0)}{\nu\,!}\, g_\nu\\ &= (2\pi)^{-n} \int \sum \frac{g_\nu}{\nu\,!} (ix)^\nu \varphi(x)\, dx.\end{aligned}$$

The function

$$f(z) = \sum \frac{g_\nu}{\nu\,!} (iz)^\nu$$

is an entire function of order of growth $\leqslant 1$ and of type $\leqslant b$. Indeed, by virtue of (2), we have

$$\sum \frac{|g_\nu|}{\nu\,!} |z|^\nu \leqslant C \sum \left(\frac{b_1}{\alpha_1}\right)^{\nu_1} \cdots \left(\frac{b_n}{\alpha_n}\right)^{\nu_n} \frac{|z_1|^{\nu_1}}{\nu_1\,!} \cdots \frac{|z_n|^{\nu_n}}{\nu_n\,!}$$

$$\leqslant C \sum_{\nu_1} \left(\frac{b_1}{\alpha_1}\right)^{\nu_1} \frac{|z_1|^{\nu_1}}{\nu_1\,!} \sum_{\nu_2} \left(\frac{b_2}{\alpha_2}\right)^{\nu_2} \frac{|z_2|^{\nu_2}}{\nu_2\,!} \cdots \sum_{\nu_n} \left(\frac{b_n}{\alpha_n}\right)^{n} \frac{|z_n|^{\nu_n}}{\nu_n\,!}$$

$$\leqslant C \exp\left(\frac{b_1}{\alpha_1} |z_1| + \cdots + \frac{b_n}{\alpha_n} |z_n|\right).$$

Since $\alpha_j < 1$ may be selected arbitrarily, the function $f(z)$ is of type $\leqslant b$, q.e.d.

4.4. Case of an Entire Square Integrable Function

Theorem (Paley-Wiener). *If an entire analytic function $f(z) = f(z_1, \ldots, z_n)$ of order of growth $\leqslant 1$ and of type $\leqslant b$ is square-integrable in the domain $R = \{-\infty < x_j < \infty\}$, then its Fourier transform is a function $g(\sigma)$ square-integrable in the domain $R = \{-\infty < \sigma_j < \infty\}$ which vanishes almost everywhere outside the domain G_b.*

Proof. According to this formula, a functional on the space K

$$(f, \varphi) = \int \overline{f(x)}\, \varphi(x)\, dx,$$

may be extended to all functions $\varphi(x) \in L_2(R)$. The dual functional $F(f)$ operates according to the formula

$$(F(f), F(\varphi)) = (2\pi)^n (f, \varphi) = (2\pi)^n \int \overline{f(x)}\, \varphi(x)\, dx = \int \overline{g(\sigma)}\, \psi(\sigma)\, d\sigma,$$

where $g(\sigma)$ and $\psi(\sigma)$ are, respectively, the Fourier transforms of the functions $f(x)$ and $\varphi(x)$. The latter is the Parseval equality, which holds for functions belonging to $L_2(R)$.[14] The function $g(\sigma)$ belongs to the space $L_2(R)$; hence, the last expression permits extension of the functional $(F(f), \psi)$ to all functions in $L_2(R)$. Let us show that the function $g(\sigma)$ equals zero almost everywhere for $|\sigma_j| \geqslant b_j$. For fixed j, let us set

$$\psi_\nu(s) = \frac{(s_j/(b_j + \epsilon))^{2\nu}}{1 + (s_j/(b_j + \epsilon))^{2\nu}}\, \psi_0(s),$$

[14] For example, see E. Titchmarsh "Introduction to the Theory of Fourier Integrals," New York, 1939, Chapter 3.

where $\psi_0(s) \in L_2(R)$ is some entire function and $\epsilon > 0$. As $\nu \to \infty$, the sequence of functions $\psi\,(s)$

(a) converges uniformly to zero in the domain $| s_j | \leqslant b_j + \epsilon/2$;

(b) converges in the mean for real $s = \sigma$ to a function equal to zero for $| \sigma_j | < b_j + \epsilon$ and equal to $\psi_0(\sigma)$ for $| \sigma_j | > b_j + \epsilon$.

According to the theorem of Section 4.1, the functional $F(f)$ can be extended to the space $\mathfrak{Z}\,(G_b)$. Since it hence remains continuous, and since the sequence $\psi_\nu(s)$ tends to zero in this space by virtue of (a), we then have $(F(f), \psi_\nu) \to 0$. On the other hand, by virtue of (b)

$$(F(f), \psi_\nu) = \int \overline{g(\sigma)}\, \psi_\nu(\sigma)\, d\sigma \to \int_{|\sigma_j| > b_j + \epsilon} \overline{g(\sigma)}\, \psi_0(\sigma)\, d\sigma.$$

Hence

$$\int_{|\sigma_j| > b_j + \epsilon} \overline{g(\sigma)}\, \psi_0(\sigma)\, d\sigma = 0$$

for any entire function $\psi_0(\sigma) \in L_2(R)$. The set of all such functions contains all the Hermite functions $P(\sigma_j) \exp(-\sigma_j^2)$, for example, and hence is complete in the space $L_2(R)$ as well as in the space $L_2\{| \sigma_j | > b_j + \epsilon\}$.[15] It follows that

$$g(\sigma) = 0$$

almost everywhere for $| \sigma_j | > b_j + \epsilon$; since $\epsilon > 0$ is arbitrary, the function $g(\sigma) = 0$ almost everywhere for $| \sigma_j | > b_j$, q.e.d.

4.5. Case of an Entire Function of Power Growth

Theorem (Paley-Wiener-Schwartz). *If an entire analytic function $f(z) = f(z_1, \ldots, z_n)$ of first order of growth and of type $\leqslant b$ does not grow more rapidly than $| x |^q$ as $| x | \to \infty$ for some q, thereby defining a functional in the space Z*

$$(f, \varphi) = \int \overline{f(x)}\, \varphi(x)\, dx,$$

then the Fourier transform[16] *$F(f) \in K'$ of the functional f has its support in the domain $G_b = \{| \sigma_j | \leqslant b_j\}$. Furthermore, for any $\epsilon > 0$, the functional $F(f)$ may be represented as the sum of a finite number* (dependent only on q and n) *of components, each of which is the result of applying a differential*

[15] For example, see E. Titchmarsh, "Introduction to the Theory of the Fourier Integral," New York, 1939.

[16] Let us recall that $K = F[Z] = F^{-1}[Z]$.

operator of order $\leqslant q + (1) = (q_1 + 1, \ldots, q_n + 1)$ *to some function* $e(\sigma)$, *integrable in the domain* $G_{b+\epsilon}$, *but vanishing outside this domain.*[17]

Proof. Let $\psi_\epsilon(\sigma)$ be an infinitely differentiable function of support in the domain $G_\epsilon = \{|\sigma| < \epsilon\}$; its inverse Fourier transform $\varphi_\epsilon(x)$ is an entire analytic function of first order of growth and of type $\leqslant \epsilon$, which tends to zero more rapidly than any power of $1/|x|$ as $|x| \to \infty$. The product $\varphi_\epsilon(x) f(x)$ is again an entire function of first order of growth and of type $\leqslant b + \epsilon$, which tends to zero more rapidly than any power of $1/|x|$. The Fourier transform of this product is, according to the theorem of Section 3.7, the convolution

$$F[\varphi_\epsilon(x) f(x)] = F[\varphi_\epsilon] * F[f] = \psi_\epsilon(\sigma) * F(f)$$

and, moreover, according to the theorem of Section 4.4, it is a function which vanishes outside the domain $G_{b+\epsilon}$.

A sequence converging to $\delta(\sigma)$ in the topology of the space K' may be extracted from the family $\psi_\epsilon(\sigma)$. By virtue of the theorem on the continuity of the convolution (Section 3.5), we have

$$\psi_\epsilon(\sigma) * F(f) \to \delta(\sigma) * F(f) = F(f).$$

Since the functions $F[\varphi_\epsilon(x) f(x)]$ have their supports in the domain $G_{b+\epsilon}$, their limit $F(f)$ also has its support in this domain. Finally, since $\epsilon > 0$ is arbitrarily small, the support of the functional $F(f)$ is actually contained in the domain G_b.

It is also possible to find the Fourier transform of the function $f(z)$ by other means. Let us introduce the function

$$f_0(x) = \frac{f(x)}{(x_1 - i)^{q_1+1} \cdots (x_n - i)^{q_n+1}}. \tag{1}$$

This function belongs to the space $L_2(R)$ by virtue of the condition on the growth of $f(x)$ for real x. Let $g_0(\sigma) \in L_2(R)$ be the Fourier transform of the function $f_0(x)$. From (1), we have

$$g(\sigma) = F(f(x)) = P(iD)\, g_0(\sigma),$$
$$(P(x) = (x_1 - 1)^{q_1+1} \cdots (x_n - 1)^{q_n+1}).$$

[17] As the authors have shown later, in this variant, the proof contains gaps, namely: It is not shown that by the extension of the functional $F(f)$ to the space $Z(G_b)$ and by the extension of this functional to $L_2(R)$, we obtain for the functions $g(s)$ one and only one result. For a proof of this identification, see G. E. Shilov, Mathematical Analysis (Special Course), 1961, pp. 396–398.

Hence, the functional $F(f)$ is the result of applying a differential operator $P(iD)$ of order $q + (1) = (q_1 + 1, \ldots, q_n + 1)$ to the square-integrable function $g_0(\sigma)$. If the function $g_0(\sigma)$ has its support in the domain G_b, the theorem is proved. If $g_0(\sigma)$ has a nonzero value outside the domain G_b, in general it is possible that $g_0(\sigma)$ will be nonzero even in an unbounded domain, then we reason as follows.

Since it is known that the functional $P(iD)\, g_0(\sigma)$ has its support in the domain G_b, we have

$$(P(iD)\, g_0(\sigma), \psi(\sigma)) = (P(iD)\, g_0(\sigma), h(\sigma)\, \varphi(\sigma)),$$

where $h(\sigma) \in K$ is an arbitrary function equal to 1 in the domain G_b. For given $\epsilon > 0$, we shall select the function $h(\sigma)$ so that it would equal 1 in the domain G_b and zero outside the domain $G_{b+\epsilon}$. Then by the Leibnitz formula

$$\begin{aligned}(P(iD)\, g_0, \varphi) &= (P(iD)\, g_0, h\varphi) = (g_0, \bar{P}(iD)\, h\varphi) \\ &= \Big(g_0, \sum_{j,k} P_{j1}(iD)\, h \cdot P_{k2}(iD)\, \varphi\Big),\end{aligned}$$

where P_{j1} and P_{k2} are some new differential operators, the sum of whose orders does not exceed $q + (1)$. Furthermore, this expression is transformed as follows:

$$\begin{aligned}(P(iD)\, g_0, \varphi) &= \sum_{j,k} (g_0, P_{j1}(iD)\, h \cdot P_{k2}(iD)\, \varphi) \\ &= \sum_{j,k} (P_{j1}(iD)\, hg_0, P_{k2}(iD)\, \varphi) \\ &= \sum_{j,k} (P_{k2}(iD)[P_{j1}(iD)\, h \cdot g_0], \varphi).\end{aligned}$$

Since $h(\sigma)$ is infinitely differentiable and of support in the domain $G_{b+\epsilon}$, the product $f_j = P_{j1}(iD)\, h \cdot g_0$ also has its support in the domain $G_{b+\epsilon}$. Since $g_0 \in L_2(R)$, then $f_j \in L_2(R)$ also, where it may be asserted, since everything is now limited to bounded domains, that the function $f_j(\sigma)$ is integrable even in the first power. Finally, we have

$$(P(iD)\, g_0(\sigma), \varphi(\sigma)) = \Big(\sum_{j,k} P_{k2}(iD) f_j, \varphi\Big),$$

where the functions $f_j(\sigma)$ and the operators $P_{k2}(iD)$ possess all the requisite properties.

The theorem is thereby proved.

The converse theorem, that a functional of support in the domain $|x| \leqslant a$ has an entire analytic function of first order of growth and of type a as its Fourier transform, is much more elementary; it is proved in Section 2.2.

Combination of the results of these two theorems leads to some new assertions relative to functions of first order of growth. Namely, let an entire function $g(s)$ satisfy the conditions of the theorem just proved, i.e., be of order of growth $\leqslant 1$, of type $\leqslant a$, and increase not more rapidly than $|\sigma|^q$ for some q at $s = \sigma$. Applying the theorem, we obtain that the Fourier transform of the function $g(s)$ is the functional $f \in K'$ of support in the domain G_a, which is representable as $\sum P_k(D) f_k(x)$, where $P_k(D)$ is a differential operator of the order of $\leqslant q + 1$, and $f_k(x)$ is a continuous function.

Using the converse theorem, we obtain an estimate of the form

$$|g(\sigma + i\tau)| \leqslant C(1 + |s|)^{q+1} e^{(|a|+\epsilon)|\tau|}$$

for the function $g(s)$. Furthermore, since

$$(1 + |s|)^{q+1} \leqslant C_1[(1 + |\sigma|)^{q+1} + (1 + |\tau|)^{q+1}]$$

and the inequality

$$|\tau|^{q+1} e^{(|a|+\epsilon)|\tau|} \leqslant C_\epsilon e^{(|a|+2\epsilon)|\tau|}$$

also holds for any $\epsilon > 0$, we arrive at the following result: *Every entire analytic function $g(s) = g(s_1, \ldots, s_n)$ of order of growth $\leqslant 1$, and of type $\leqslant a$, which increases not more rapidly than $|\sigma|^q$ for $s = \sigma$, $|\sigma| \to \infty$ admits a majorization given by*:

$$|g(\sigma + i\tau)| \leqslant C_\epsilon(1 + |\sigma|)^{q+1} e^{(|a|+\epsilon)|\tau|} \tag{2}$$

for any $\epsilon > 0$.

CHAPTER IV

SPACES OF TYPE S

1. Introduction

New types of fundamental spaces will be introduced and investigated in this chapter; they will be used in Volume 3, particularly, for a study of the Cauchy problem.

In defining these spaces, we shall impose conditions not only on the decrease of the fundamental functions at infinity, but also on the growth of their derivatives as the order of the derivative increases.

All these conditions are formulated most naturally by using inequalities which the expression $\sup_x | x^k \varphi^{(q)}(x)|$ must satisfy for all k and q; these inequalities have the form

$$\sup_x | x^k \varphi^{(q)}(x)| \leqslant m_{kq} \qquad (k, q = 0, 1, 2, \ldots),$$

where m_{kq} is some double sequence of numbers.

If no special conditions are imposed on this sequence, i.e., the numbers of this sequence may vary arbitrarily together with the function $\varphi(x)$, we will then obtain the set of all infinitely differentiable functions which will decrease at infinity, together with all their derivatives, more rapidly than any power of $1/|x|$. This is indeed the space S, introduced in Volume 1 (Chapter I, Section 1.10) and encountered repeatedly in preceding chapters.

We impose some special constraints on the sequence m_{kq}, in particular, we shall consider this sequence to depend on k, or on q, or on both subscripts, in a specific manner, and we thereby obtain different concrete spaces. All these spaces have many common properties. It is very important that the Fourier transformation may be used freely in these spaces: The operators $\partial/\partial x$ and multiplication by x exchange roles under the Fourier transformation, and these spaces transform into each other. It hence follows that the Fourier transforms of generalized functions defined on a space of fixed type will be generalized functions on another space of the same fixed type. Let us recall, in particular, that the Fourier transformation transforms the space S into itself (Chapter III, Section 1).

Furthermore, various operators, functions of the differentiation operator, may be constructed in these spaces (and therefore, in their conjugates as well), where the quantity of such admissible functions depends on the nature of the space. The importance of examining functions of the differentiation operator is clarified by the following simple example. Let us consider the Cauchy problem for the heat conduction equation

$$\frac{\partial u(x, t)}{\partial t} = \frac{\partial^2 u(x, t)}{\partial x^2}, \qquad u(x, 0) = u_0(x).$$

The solution is written formally as

$$u(x, t) = \exp\left(t \frac{\partial^2}{\partial x^2}\right) u_0(x).$$

The well-known Poisson formula is obtained by evaluating the right side.

These two methods, the method of Fourier transformations and the method of constructing (entire) functions of the differentiation operator, will be our principal weapons in constructing uniqueness classes and correctness classes for the solution of the Cauchy problem (Volume 3, Chapters II and III).

Although our study of type S spaces is subordinated to the study of the Cauchy problem, the results obtained below are also of independent interest. However, the reader not interested in the Cauchy problem may omit this chapter and read Chapter IV of Volume 3.

Let us turn to the definition of those kinds of fundamental spaces which we shall examine later. At the beginning we shall limit ourselves to the case of one independent variable, leaving the general case for Section 9.

We shall designate all the spaces described below as "spaces of type S."

1. The Space S_α ($\alpha \geqslant 0$) consists of all infinitely differentiable functions $\varphi(x)$ ($-\infty < x < \infty$), satisfying the inequalities[1]

$$|x^k \varphi^{(q)}(x)| \leqslant C_q A^k k^{k\alpha} \qquad (k, q = 0, 1, 2, \ldots), \tag{1}$$

where the constants A and C_q depend on the function φ.

2. The Space S^β ($\beta \geqslant 0$) consists of all infinitely differentiable functions $\varphi(x)$ ($-\infty < x < \infty$), satisfying the inequalities

$$|x^k \varphi^{(q)}(x)| \leqslant C_k B^q q^{q\beta} \qquad (k, q = 0, 1, 2, \ldots), \tag{2}$$

where the constants B and C_k depend on the function φ.

[1] For $k = 0$, the expression $k^{k\alpha}$ is considered to equal 1. Also $q^{q\beta}$ for $q = 0$ is considered to equal 1.

3. The Space S_α^β ($\alpha \geqslant 0, \beta \geqslant 0$) consists of all infinitely differentiable functions $\varphi(x)$ ($-\infty < x < \infty$), satisfying the inequalities

$$| x^k \varphi^{(q)}(x) | \leqslant C A^k B^q k^{k\alpha} q^{q\beta}, \tag{3}$$

where the constants A, B, C depend on the function φ.

Let us compare the definitions of these spaces. In substance, the definition of the space S_α imposes a constraint on the decrease of the fundamental functions as $| x | \to \infty$. This easily seen if both sides of the inequality (1) are divided by $| x |^k$ and we pass to the minimum in k on the right side; as a result, a function of x is obtained on the right side which decreases more rapidly than any function of the form $1/| x |^k$ as $| x | \to \infty$, and the more rapidly, the smaller the number α. We shall evaluate this function somewhat later.

The definition of the space S^β imposes a constraint on the growth of the derivatives of the function $\varphi(x)$. These restrictions are the stronger, the smaller the value of the β. Later we shall show that the function $\varphi(x)$ may be continued analytically from the real axis to some strip on the complex plane when $\beta = 1$. For $\beta < 1$, the function $\varphi(x)$ is continued analytically into the complex plane as an entire analytic function; we shall compute its order of growth as a function of the number β somewhat later.

The definition of the space S_α^β imposes a constraint on both the decrease of the fundamental functions, and on the growth of their derivatives.

The fundamental functions in the space S_α^β decrease at infinity exactly as do the fundamental functions in the space S_α, and their derivatives may grow no more rapidly as $q \to \infty$ than for the functions in the space S^β. All these constraints will be the stronger, the smaller the numbers α and β.

For sufficiently small α and β, we shall see below for precisely which values, the spaee S_α^β degenerates into the single function $\varphi(x) \equiv 0$. Naturally, we shall be interested only in nontrivial spaces.

From the topological viewpoint, each of the spaces S_α, S^β, S_α^β is the union of countably normed spaces. For example, the space S_α is a union of the countably normed spaces $S_{\alpha,A}$ ($A = 1, 2, \ldots$); the space S_{α,A_0} consists of functions satisfying the inequalities (1), where any constant greater than A_0 is suitable for the constant A. The norms in the space $S_{\alpha,A}$ and in the other countably normed spaces will be indicated in Section 3; it will be shown there that all these spaces are perfect.

The spaces S_α, S^β, S may be considered limiting cases of the space $S_\alpha{}^\beta$, namely

$$S_\alpha = S_\alpha{}^\infty, \qquad S^\beta = S_\infty{}^\beta, \qquad S = S_\infty{}^\infty.$$

Many of the subsequent results referring to the spaces $S_\alpha{}^\beta$, go over into corresponding results for the spaces S_α, S^β or S when the passage to the limit $\alpha \to \infty$, $\beta \to \infty$ is interpreted correctly.

Undoubtedly even the more general types of spaces whose definitions are obtained by replacing the sequences $k^{k\alpha}$ and $q^{q\beta}$ in the definitions of S_α, S^β, $S_\alpha{}^\beta$ by the arbitrary sequences a_k and b_q $(k, q = 0, 1, 2,...)$, are of interest. Such, for example, is the space $S_{a_k b_q}$, which consists of all infinitely differentiable functions $\varphi(x)$, for which the inequalities

$$|x^k \varphi^{(q)}(x)| \leqslant CA^k B^q a_k b_q \qquad (k, q = 0, 1, 2,...).$$

are satisfied. In the appendices, we shall indicate which of the properties of spaces of type S will be retained for such more general spaces.

2. Various Modes of Defining Spaces of Type S

2.1. The Space S_α

As has already been said, the space S_α $(\alpha \geqslant 0)$ consists of infinitely differentiable functions $\varphi(x)$ $(-\infty < x < \infty)$, satisfying the inequalities

$$|x^k \varphi^{(q)}(x)| \leqslant C_q A^k k^{k\alpha}, \tag{1}$$

where the constants C_q and A depend on the function $\varphi(x)$.

As we already know, this definition subjects the function $\varphi(x)$ to some conditions for decreasing at infinity. Let us elucidate what these conditions are.

Let us divide both sides of the inequality (1) by $|x|^k$ and let us take the lower bound for k on the right side; we obtain that

$$|\varphi^{(q)}(x)| \leqslant C_q \inf_k \frac{A^k k^{k\alpha}}{|x|^k} = C_q \mu_\alpha\left(\frac{x}{A}\right),$$

where we have put

$$\mu_\alpha(\xi) = \inf_k \frac{k^{k\alpha}}{|\xi|^k}. \tag{2}$$

Let us find the expression for the function $\mu_\alpha(\xi)$ explicitly. For $\alpha = 0$, we have

$$\mu_0(\xi) = \inf_k \frac{1}{|\xi|^k} = \begin{cases} 1 & \text{for } |\xi| \leqslant 1, \\ 0 & \text{for } |\xi| > 1, \end{cases}$$

Hence, it follows that the function $\mu_0(x/A)$ equals 1 for $|x| \leqslant A$ and 0 for $|x| > A$. Hence, the function $\varphi(x)$ vanishes for $|x| > A$.

Conversely, if $\varphi(x)$ is an infinitely differentiable function which vanishes for $|x| > A$, then evidently

$$|x^k \varphi^{(q)}(x)| \leqslant C_q A^k,$$

i.e., $\varphi(x)$ belongs to the space S_0.

Since the constant A may be chosen arbitrarily, *the space* S_0 therefore, *consists of all infinitely-differentiable functions of compact support*, i.e., coincides with the space K introduced in Volume 1.

Now, let $\alpha > 0$. In this case it turns out that the function $\mu_\alpha(\xi)$ has the order of decrease of $\exp(-a |\xi|^{1/\alpha})$. More exactly, the inequality

$$\exp\left(-\frac{\alpha}{e} |\xi|^{1/\alpha}\right) \leqslant \mu_\alpha(\xi) \leqslant C \exp\left(-\frac{\alpha}{e} |\xi|^{1/\alpha}\right) \tag{3}$$

holds, where C is some constant.

For the proof, temporarily considering k to be a continuously changing variable, we find the minimum of the function $k^{k\alpha}/\xi^k = f(k)$ by the customary means of differential calculus. Taking the logarithm, differentiating, and equating the result to zero, we obtain

$$[\ln f(k_0)]' = \frac{f'(k_0)}{f(k_0)} = -\ln \xi + \alpha \ln k_0 + \alpha = 0, \tag{4}$$

where k_0 is the value of k at which the minimum of the function $f(k)$ is realized. From (4) we find $k_0 = (1/e)\, \xi^{1/\alpha}$ and therefore

$$\min_k \ln f(k) = -\frac{\alpha}{e} \xi^{1/\alpha},$$

$$\min_k f(k) = \exp\left(-\frac{\alpha}{e} \xi^{1/\alpha}\right).$$

Since k in reality only varies over the natural numbers, then $\min_k f(k)$ is actually somewhat higher than what has been found. We have

$$[\ln f(k)]'' = \alpha/k;$$

hence, at an integer point k_1, closest to the right of k_0,

$$\ln f(k_1) = \ln f(k_0) + \frac{\alpha}{2k_2}(k_1 - k_0)^2 < \ln f(k_0) + \frac{\alpha}{2k_0} \quad (k_0 < k_2 < k_1)$$

and, therefore,

$$\min_k \ln f(k) \leqslant \ln f(k_1) < \ln f(k_0) + \frac{\alpha}{2k_0} = -\frac{\alpha}{e}\xi^{1/\alpha} + \frac{\alpha e}{2}\xi^{-1/\alpha},$$

from which

$$\min_k f(k) < \exp\left(\frac{\alpha e}{2}\xi^{-1/\alpha}\right)\exp\left(-\frac{\alpha}{e}\xi^{1/\alpha}\right).$$

For $\xi \geqslant 1$, the first term is bounded by the quantity $C_1 = \exp(\alpha e/2)$. If $0 < \xi < 1$,

$$\min_k \frac{k^{k\alpha}}{\xi^k} \leqslant 1 \leqslant \exp\left(\frac{\alpha}{e}\right)\exp\left(-\frac{\alpha}{e}\xi^{1/\alpha}\right).$$

Thus, for any ξ, $0 < \xi < \infty$,

$$\exp\left(-\frac{\alpha}{e}\xi^{1/\alpha}\right) \leqslant \mu_\alpha(\xi) \leqslant C\exp\left(-\frac{\alpha}{e}\xi^{1/\alpha}\right),$$

where C may be set equal to $\exp(\alpha e/2)$, say.

For the function $\varphi(x)$, we now obtain the estimate

$$|\varphi^{(q)}(x)| \leqslant C_q' \exp\left(-\frac{\alpha}{e}\left|\frac{x}{A}\right|^{1/\alpha}\right),$$

or utilizing the notation

$$a = \frac{\alpha}{eA^{1/\alpha}}, \tag{5}$$

we finally find

$$|\varphi^{(q)}(x)| \leqslant C_q' \exp(-a|x|^{1/\alpha}). \tag{6}$$

This result may be formulated thus: *All functions $\varphi(x) \in S_\alpha$ together with all their derivatives decrease exponentially at infinity, with an order*[2] $\geqslant 1/\alpha$ *and a type $\geqslant a$, dependent on the function φ.*

[2] If it is known that the function $g(x)$ satisfies the inequality $|g(x)| \leqslant C\exp(-a|x|^p)$, $a > 0$, when then say that $g(x)$ decreases exponentially with an order $\geqslant p$ and a type $\geqslant a$.

Let us show that each function $\varphi(x)$, satisfying this condition of decrease at infinity, belongs to the space S_α. Indeed, let be satisfied the conditions

$$|\varphi^{(q)}(x)| \leqslant C_q \exp(-a|x|^{1/\alpha}) \qquad (q = 0, 1, 2, \ldots).$$

Having determined the number A from (5), we may write these conditions as

$$|\varphi^{(q)}(x)| \leqslant C_q \exp\left(-\frac{\alpha}{e}\left|\frac{x}{A}\right|^{1/\alpha}\right);$$

furthermore, applying the fundamental inequality (3), we find

$$|\varphi^{(q)}(x)| \leqslant C_q \cdot \mu_\alpha\left(\frac{x}{A}\right) = C_q \min_k \frac{A^k k^{k\alpha}}{|x|^k} \leqslant C_q \frac{A^k k^{k\alpha}}{|x|^k},$$

from which

$$|x^k \varphi^{(q)}(x)| \leqslant C_q A^k k^{k\alpha},$$

q.e.d.

Thus, we have obtained the second definition of the space S_α ($\alpha \neq 0$): *it consists of those and only those functions* $\varphi(x)$, *which satisfy inequalities of the form*

$$|\varphi^{(q)}(x)| \leqslant C_q \exp(-a|x|^{1/\alpha})$$

with constants C_q and a, dependent on the function φ.

2.2. The Space S^β

As already mentioned, the space S^β ($\beta \geqslant 0$) consists of infinitely differentiable functions $\varphi(x)$ ($-\infty < x < \infty$), satisfying the inequalities

$$|x^k \varphi^{(q)}(x)| \leqslant C_k B^q q^{q\beta} \qquad (q = 0, 1, 2, \ldots). \tag{1}$$

The constraints imposed on the growth of the derivatives of the fundamental functions $\varphi(x)$ in this definition are stricter, the smaller the number β. If $\beta > 1$, then there are still functions of bounded support among the fundamental functions $\varphi(x)$ (see Section 8, below). If $\beta \leqslant 1$, then *all the functions* $\varphi(x) \in S^\beta$ *are already analytic* and there are, thereby, no functions of bounded support among them. Indeed, the remainder term of the Taylor formula

$$\frac{h^q}{q!}\varphi^{(q)}(x + \theta h) = \varphi(x + h) - \varphi(x) - h\varphi'(x) - \cdots - \frac{h^{q-1}}{(q-1)!}\varphi^{(q-1)}(x) \tag{2}$$

admits the following estimate in this case:

$$\left| \frac{h^q}{q!} \varphi^{(q)}(x + \theta h) \right| \leqslant \frac{|h|^q}{q!} C_0 B^q q^q.$$

From the Stirling formula

$$q! = q^{q+(1/2)} e^{-q} \sqrt{2\pi}\, E_q \quad (E_q \to 1),$$

it follows that this remainder term tends to zero for $|h| < 1/Be$. Hence, in the appropriate neighborhood of the point x, the function $\varphi(x)$ is represented by the Taylor series

$$\varphi(x + h) = \sum_{q=0}^{\infty} \frac{h^q}{q!} \varphi^{(q)}(x). \tag{3}$$

Since the Taylor series remains convergent even for complex values of h, which satisfy the inequality $|h| < 1/Be$, it then follows that the function $\varphi(x)$ may be extended analytically in the strip $|y| < 1/Be$ of the complex plane $z = x + iy$. In general, the width of this strip depends on the function $\varphi(x)$.

But if $\beta < 1$, it then follows from the estimate

$$\left| \frac{h^q}{q!} \varphi^{(q)}(x + \theta h) \right| \leqslant \frac{|h|^q}{q!} C_0 B^q q^{q\beta},$$

by virtue of the Stirling formula presented above, that the *remainder term in* (2) *tends to zero for any h and the function* $\varphi(x)$ turns out to be *an entire analytic function.* The inequality

$$|x^k \varphi(x + iy)| = \left| \sum_{q=0}^{\infty} \frac{(iy)^q}{q!} x^k \varphi^{(q)}(x) \right| \leqslant C_k \sum_{q=0}^{\infty} \frac{|y|^q}{q!} B^q q^{q\beta} \tag{4}$$

is hence satisfied for the function $\varphi(z) = \varphi(x + iy)$.

Let us put

$$M_\beta(\eta) = \sum_{q=0}^{\infty} \frac{\eta^q}{q!} q^{q\beta} \qquad (\eta \geqslant 0). \tag{5}$$

It is then possible to write that

$$|x^k \varphi(x + iy)| \leqslant C_k M_\beta(B\,|y|).$$

Let us estimate the growth of the function $M_\beta(\eta)$ as $\eta \to \infty$. By virtue of the Stirling formula $q! \geqslant C q^q e^{-q}$; hence

$$M_\beta(\eta) \leqslant \frac{1}{C} \sum_{q=0}^{\infty} \frac{\eta^q e^q}{q^{q(1-\beta)}}. \tag{6}$$

The quantity

$$\sup_q \frac{\eta^q e^q}{q^{q(1-\beta)}} = \left(\inf_q \frac{q^{q(1-\beta)}}{\eta^q e^q}\right)^{-1}$$

is called the *maximum term* of the series (6). We already estimated this quantity in Section 2.1 (formula (3), where α must be replaced by $1-\beta$ and ξ by ηe). By virtue of this estimate, we have

$$\sup_q \frac{\eta^q e^q}{q^{q(1-\beta)}} \leqslant \exp\left[\frac{1-\beta}{e}\,\eta \exp\left(\frac{1}{1-\beta}\right)\right] = \exp(b\eta^{1/1-\beta}),$$

where

$$b = \frac{1-\beta}{e} \exp\left(\frac{1}{1-\beta}\right).$$

Since the terms of the series (6) tend to zero more rapidly than the terms of any geometric progression, then starting with some number, the qth term of the series becomes less than $1/2^q$. In order to find this number, it is necessary to solve the inequality

$$\frac{\eta^q e^q}{q^{q(1-\beta)}} < \frac{1}{2^q}$$

for q. Taking the logarithm, it is easy to obtain the solution of this inequality; namely, it is

$$q > q_0 = (2e\eta)^{1/1-\beta}.$$

The sum of all the terms of the series of rank $q > q_0$ does not exceed one. The sum of the rest of the terms does not, in turn, exceed

$$(2e\eta)^{1/1-\beta} \exp b\eta^{1/1-\beta} \leqslant C' \exp b'\eta^{1/1-\beta},$$

where b' is any number greater than b.

Hence, the estimate

$$M_\beta(\eta) \leqslant C^{-1} + C' \exp b'\eta^{1/1-\beta} \leqslant C'' \exp b'\eta^{1/1-\beta} \tag{7}$$

is obtained for the function $M_\beta(\eta)$. In turn, the estimate

$$|x^k\varphi(x+iy)| \leqslant C_k M_\beta(B|y|) \leqslant C_k'' \exp b'|y|^{1/1-\beta}$$

is obtained for the function $x^k\varphi(x+iy)$, where b' is any constant greater than

$$bB^{1/1-\beta} = \frac{1-\beta}{e}(Be)^{1/1-\beta}.$$

We have obtained the following important result:

Every function $\varphi(x)$, *satisfying the inequalities*

$$| x^k\varphi^{(q)}(x)| \leqslant C_k B^q q^{q\beta} \qquad (\beta < 1, \ k, q = 0, 1, 2, \ldots), \tag{8}$$

may be continued analytically in the $z = x + iy$ *plane as an entire function of order of growth* $1/1 - \beta$; *more accurately, the obtained function satisfies the inequalities*

$$| x^k\varphi(x + iy)| \leqslant C'_k \exp b' \, | y |^{1/1-\beta}, \tag{9}$$

where b' *is any constant greater than* $[(1 - \beta)/e](Be)^{1/1-\beta}$.

We shall see later (Section 7) that functions of the space S^β $(b < 1)$, which satisfy the inequalities (8) with a given value for B, are characterized completely by the estimate (9).

2.3. The Space S_α^β

As has been said, the space S_α^β consists of infinitely differentiable functions $\varphi(x)$ $(-\infty < x < \infty)$, satisfying the inequalities

$$| x^k\varphi^{(q)}(x)| \leqslant CA^k B^q k^{k\alpha} q^{q\beta} \qquad (k, q = 0, 1, 2, \ldots). \tag{1}$$

In this definition, constraints are imposed as well on the decrease of the fundamental functions as $| x | \to \infty$, as on the growth of their derivatives as $q \to \infty$. Evidently the space S_α^β is contained in the intersection of the spaces S_α and S^β.[3] Hence, every function $\varphi(x) \in S_\alpha^\beta$ satisfies the estimate of the decrease at infinity

$$| \varphi^{(q)}(x)| \leqslant CB^q q^{q\beta} \exp(-a| x |^{1/\alpha}). \tag{1}$$

If $\beta \leqslant 1$, then every function $\varphi(x) \in S_\alpha^\beta$ is continued in the $z = x + iy$ plane, in all of this plane for $\beta < 1$; the estimate

$$| x^k\varphi(x + iy)| \leqslant CA^k k^{k\alpha} \exp(b \, | y |^{1/1-\beta}) \tag{2}$$

is hence satisfied. Dividing both sides of this inequality by $| x |^k$, passing to the lower bound of k, and utilizing the inequality (3) of Section 2.1, we find

$$| \varphi(x + iy)| \leqslant C \exp(-a| x |^{1/\alpha} + b| y |^{1/(1-\beta)}), \tag{3}$$

where the coefficients a and b are expressed in a known manner in terms of A and B.

3 It is not known whether the converse is true: Will every function lying simultaneously in S_α and S^β be an element of the space S_α^β?

Later (Section 7), we shall show that every entire function $\varphi(x + iy)$, satisfying the inequality (3) belongs to the space $S_\alpha{}^\beta$.[4]

As will be shown later (Section 8), the inequality (3) imposes such strong constraints on the function $\varphi(x)$ that for sufficiently small α and β, more exactly for $\alpha + \beta < 1$, the space $S_\alpha{}^\beta$ contains only the single function $\varphi(x) \equiv 0$.

3. Topological Structure of Fundamental Spaces

3.1. The Space S_α as the Union of Countably Normed Spaces

Let $S_{\alpha,A}$ denote the set of functions $\varphi(x)$ of the space S_α, for which the inequalities

$$| x^k \varphi^{(q)}(x)| \leqslant C_{q\bar{A}} \bar{A}^k k^{k\alpha}$$

are valid, where any constant greater than a given number A may be taken as $\bar{A}$. In other words, $S_{\alpha,A}$ consists of those functions $\varphi(x)$ which satisfy the inequalities

$$| x^k \varphi^{(q)}(x)| \leqslant C_{q\delta}(A + \delta)^k \, k^{k\alpha} \tag{1}$$

for any $\delta < 0$. If we pass to the second definition of the space S_α (Section 2.1), this definition may be formulated thus: The space $S_{\alpha,A}$ consists of those functions $\varphi(x)$, which satisfy the inequalities

$$| \varphi^{(q)}(x)| \leqslant C'_{q\delta} \exp[-(a - \delta) \, | \, x \, |^{1/\alpha}],$$

for $\delta > 0$, where as before

$$a = \alpha/eA^{1/\alpha}.$$

This definition corresponds to the case $\alpha > 0$. For $\alpha = 0$, the space $S_{0,A}$ consists of infinitely differentiable functions of support in the segment $| \, x \, | \leqslant A$. As we see, it agrees with the space $K(A)$, which we have often considered earlier; in particular, we know that it is perfect (Chapter II, Section 2).

Let us assume

$$M_p(x) = \exp\left[a\left(1 - \frac{1}{p}\right) | \, x \, |^{1/\alpha}\right] \quad (p = 2, 3, \ldots.) \tag{2}$$

[4] More exactly, $\varphi(x)$ belongs to $S_\alpha{}^\beta$. However, in order not to present this stipulation each time, we agree always to understand an imbedding of the type $\varphi(x + iy) \in R$ (R some space of functions of the real argument x) in the sense of $\varphi(x) \in R$.

The functions $M_p(x)$ generate an increasing sequence ($M_p(x) \leqslant M_{p+1}(x)$), and the functions $\varphi(x) \in S_{\alpha,A}$ may be characterized as infinitely differentiable functions for which the expression

$$\sup_{\substack{x \\ q \leqslant p}} M_p(x) \mid \varphi^{(q)}(x) \mid = \| \varphi \|_p \tag{3}$$

is finite for any $p = 2, 3, \ldots$. We introduced the space of functions subject to this kind of conditions in Section 1 of Chapter II under the name of $K\{M_p\}$ spaces. Hence, the space $S_{\alpha,A}$ belongs to the class of spaces $K\{M_p\}$.

In particular, as is every $K\{M_p\}$ space, *the space* $S_{\alpha,A}$ *is a complete countably normed space.* Let us verify that even the condition (P) of Section 2 in Chapter II, assuring perfection of the space $K\{M_p\}$, is satisfied in this case. This condition is that for each subscript p, there exists a subscript $p' > p$, such that

$$\lim_{|x|\to\infty} \frac{M_p(x)}{M_{p'}(x)} = 0.$$

Indeed, in our case

$$M_p(x) = \exp\left[a\left(1 - \frac{1}{p}\right) \mid x \mid^{1/\alpha}\right]$$

and therefore, for any $p' > p$,

$$\frac{M_p(x)}{M_{p'}(x)} = \exp\left[a\left(1 - \frac{1}{p}\right) \mid x \mid^{1/\alpha} - a\left(1 - \frac{1}{p'}\right) \mid x \mid^{1/\alpha}\right]$$
$$= \exp\left[a\left(\frac{1}{p'} - \frac{1}{p}\right) \mid x \mid^{1/\alpha}\right] \to 0,$$

q.e.d.

For this cases let us also write down the criterion we obtained in Chapter II, Section 2 for the convergence of a sequence of fundamental functions. This criterion states: *A sequence* $\varphi_\nu \in S_{\alpha,A}$ *converges to zero if and only if the sequence* $\varphi_\nu(x)$ *converges correctly to zero* (i.e., for any q the functions $\varphi_\nu^{(q)}(x)$ converge uniformly to zero in any segment $\mid x \mid \leqslant x_0 < \infty$) *and the norms* $\| \varphi_\nu \|_p$ *are bounded for any* p.

It will be more convenient for us to consider another system of norms in the space $S_{\alpha,A}$ which is directly associated with the original definition of this space. Namely, let us put[5]

$$\| \varphi \|_{q\delta} = \sup_{x,k} \frac{\mid x^k \varphi^{(q)}(x) \mid}{(A + \delta)^k \, k^{k\alpha}} . \tag{4}$$

[5] The symbol $\sup_{x,k}$ means the upper bound of the last expression in all x and k, $-\infty < x < \infty$, $0 \leqslant k < \infty$.

Let us show that this system of norms is equivalent to the system (3) of norms $\|\varphi\|_p$. If we first find the sup with respect to the index k in (4), we will obtain

$$\sup_k \frac{|x|^k}{(A+\delta)^k k^{k\alpha}} = \frac{1}{\inf_k (A+\delta)^k k^{k\alpha}/|x|^k} = \frac{1}{\mu_\alpha(|x|/A+\delta)}$$

where $\mu_\alpha(\xi) = \inf_k k^{k\alpha}/|\xi|^k$ is the function defined in Section 2.1. By virtue of the inequality (3) of Section 2.1, we have

$$\frac{1}{\mu_\alpha(\xi)} \leqslant \exp\left[\frac{\alpha}{e}\left(\frac{|x|}{A+\delta}\right)^{1/\alpha}\right] \leqslant M_p(x)$$

for some p. Therefore

$$\|\varphi\|_{q\delta} \leqslant \sup_x M_p(x)\,|\varphi^{(q)}(x)| \leqslant \|\varphi\|_p ;$$

and the reverse inequality is proved analogously.

In particular, the space $S_{\alpha,A}$ with the norms (4) is also a perfect space.

For $A_1 < A_2$, the space S_{α,A_1} is contained in the space S_{α,A_2}, i.e., every sequence $\varphi_\nu(x)$ convergent in the space S_{α,A_1}, will also converge in the space S_{α,A_2}. We may construct the union of the countably normed spaces $S_{\alpha,A}$ over all subscripts $A = 1, 2, \ldots$. Since each function $\varphi(x) \in S_\alpha$ enters into some $S_{\alpha,A}$, the union of the spaces $S_{\alpha,A}$ coincides with the space S_α. Thus, *the space S_α is a union of the countably normed spaces $S_{\alpha,A}$*. This permits introduction of the definition of convergence of the sequence $\varphi_\nu(x) \in S_\alpha$, as is done in such unions (Chapter I, Section 8): a sequence $\varphi_\nu \in S_\alpha$ converges to zero if all the functions $\varphi_\nu(x)$ belong to some space $S_{\alpha,A}$, where they converge to zero in the topology of $S_{\alpha,A}$; this means that the sequence $\varphi_\nu(x)$ converges correctly to zero (i.e., for any q, the functions $\varphi_\nu^{(q)}(x)$ converge uniformly to zero in any segment $|x| \leqslant x_0 < \infty$) and for some A and C_q, not dependent on ν, the inequalities

$$|x^k \varphi^{(q)}(x)| \leqslant C_q A^k k^{k\alpha}$$

are satisfied.

It is possible to give a definition of the convergence of the sequence $\varphi_\nu(x) \in S_\alpha$ in terms of the second definition of this space also. Namely, the sequence $\varphi_\nu(x)$ converges to zero if it converges correctly to zero, and for some $a > 0$ the inequalities

$$|\varphi_\nu^{(q)}(x)| \leqslant C_q \exp(-a\,|x|^{1/\alpha})$$

are satisfied.

For $\alpha = 0$, our definitions go over into the definition of the space K of all infinitely-differentiable functions of compact support as the union of countably normed spaces $K(A)$ of functions of compact support contained in $|x| \leqslant A$.

3.2. The Space S^β as the Union of Countably Normed Spaces

As has already been stated, the space S^β ($\beta \geqslant 0$) consists of infinitely differentiable functions $\varphi(x)$ ($-\infty < x < \infty$), satisfying the inequalities

$$|x^k \varphi^{(q)}(x)| \leqslant C_k B^q q^{q\beta} \qquad (k, q = 0, 1, 2, \ldots),$$

where the constants C_k and B depend on the function φ. Let $S^{\beta, B}$ denote the set of functions $\varphi(x) \in S^\beta$, for which the inequalities

$$|x^k \varphi^{(q)}(x)| \leqslant C_{k,\bar{B}} \bar{B}^q q^{q\beta} \qquad (k, q = 0, 1, 2, \ldots),$$

are valid, in which any constant greater than a given number B may be taken as $\bar{B}$. In other words, $S^{\beta, B}$ consists of those functions $\varphi(x)$ which satisfy, for any $\rho > 0$, the inequalities

$$|x^k \varphi^{(q)}(x)| \leqslant C_{k\rho}(B + \rho)^q \, q^{q\beta} \qquad (k, q = 0, 1, 2, \ldots). \tag{1}$$

The space $S^{\beta, B}$ does not already belong to the class of spaces $K\{M_p\}$, as does the space $S_{\alpha, A}$; hence, the topological configuration of the space $S^{\beta, B}$ must be considered independently. In the space $S^{\beta, B}$, let us introduce a system of norms by means of the formulas

$$\| \varphi \|_{k\rho} = \sup_{x, q} \frac{|x^k \varphi^{(q)}(x)|}{(B + \rho)^q \, q^{q\beta}} \qquad (k = 0, 1, 2, \ldots; \ \rho = 1, \tfrac{1}{2}, \ldots). \tag{2}$$

Let us show that here *the space $S^{\beta, B}$ becomes a complete countably normed perfect space.*

The proof is broken up into several steps.

First, let us call a sequence of infinitely differentiable functions $\varphi_\nu(x)$ *correctly convergent to the function* $\varphi(x)$, if for any q, the functions $\varphi_\nu^{(q)}(x)$ converge uniformly to $\varphi^{(q)}(x)$ in any bounded interval.

(a) *If the sequence $\varphi_\nu(x)$ converges correctly to some function $\varphi(x)$ and for some k and ρ the norms $\| \varphi_\nu \|_{k\rho}$ are bounded, $\| \varphi_\nu \|_{k\rho} \leqslant C$, then the norm $\| \cdot \|_{k\rho}$ exists even for the function $\varphi(x)$ and here $\| \varphi \|_{k\rho} \leqslant C$.*

Indeed, in any bounded interval $-a \leqslant x \leqslant a$

$$\sup_{\substack{x \\ q \leqslant p}} \frac{|x^k \varphi^{(q)}(x)|}{(B + \rho)^q \, b_q} = \lim_{\nu \to \infty} \sup_{\substack{x \\ q \leqslant p}} \frac{|x^k \varphi^{(q)}(x)|}{(B + \rho)^q \, b_q} \leqslant \overline{\lim_{\nu \to \infty}} \| \varphi_\nu \|_k \leqslant C;$$

hence, passing to the limit as $a \to \infty$ and $p \to \infty$, we find

$$\| \varphi \|_{k\rho} = \sup_{x,q} \frac{| x^k \varphi^{(q)}(x)|}{(B + \rho)^q b_q} \leqslant C,$$

q.e.d.

(b) *If the sequence $\varphi_\nu(x)$ converges to zero at each point and is fundamental in the norm $\| \cdot \|_{k\rho}$, then $\| \varphi_\nu \|_{k\rho} \to 0$.*

In fact, since the sequence $\varphi_\nu(x)$ is fundamental, it converges correctly to zero. Hence, the sequence of differences $\varphi_\nu(x) - \varphi_\mu(x)$ converges correctly to $\varphi_\nu(x)$ as $\mu \to \infty$. According to the above,

$$\| \varphi_\nu \|_{k\rho} \leqslant \sup_{\mu \geqslant \nu} \| \varphi_\nu - \varphi_\mu \|_{k\rho} < \epsilon$$

for sufficiently large ν, q.e.d.

(c) *The space $S^{\beta,B}$ is complete.*

Indeed, if the sequence $\varphi_\nu(x) \in S^{\beta,B}$ is fundamental in each of the norms $\| \cdot \|_{k\rho}$, then according to (a) each of the norms $\| \cdot \|_{k\rho}$ exists for the limit function $\varphi(x)$; hence, $\varphi(x) \in S^{\beta,B}$. The difference $\varphi - \varphi_\nu$ converges correctly to zero and is bounded in each of the norms; by virtue of (b) we have $\| \varphi - \varphi_\nu \|_{k\rho} \to 0$ for any k and ρ, which indeed denotes completeness of the space $S^{\beta,B}$.

(d) *The norms $\| \cdot \|_{k\rho}$ are pairwise consistent.*

Let $\varphi_\nu \in S^{\beta,B}$ be fundamental in the norms $\| \cdot \|_{k_1\rho_1}$ and $\| \cdot \|_{k_2\rho_2}$ and tend to zero in the former. Then the functions $\varphi_\nu(x)$ are known to tend to zero at each point, and $\| \varphi_\nu \|_{k_2\rho_2} \to 0$, according to (b), q.e.d.

(e) *If the sequence $\varphi_\nu(x)$ is bounded in each of the norms $\| \cdot \|_{k\rho}$ and converges correctly to zero, it tends to zero in the topology of the space $S^{\beta,B}$* (i.e., in each of the norms).

Actually, let k, ρ, and an arbitrary $\eta > 0$ be given. Let us take $\rho' < \rho$. By assumption, the numbers $\| \varphi_\nu \|_{k\rho'}$ are bounded by the constant $C_{k\rho'}$. For sufficiently large q, say $q \geqslant q_0$, the inequality

$$\frac{(B + \rho')^q}{(B + \rho)^q} < \frac{\eta}{C_{k\rho'}}$$

holds; hence, for $q \geqslant q_0$, we have

$$| x^k \varphi_\nu^{(q)}(x)| \leqslant C_{k\rho'}(B + \rho')^q q^{q\beta} < \eta(B + \rho)^q q^{q\beta}.$$

For $q < q_0$ and $|x| > C_{k+1,\rho}/\eta$ we obtain

$$|x^k\varphi_\nu^{(q)}(x)| = \frac{1}{|x|}|x^{k+1}\varphi_\nu^{(q)}(x)|$$

$$\leqslant \frac{1}{|x|}\|\varphi_\nu\|_{k+1,\rho}(B+\rho)^q q^{q\beta} < \eta(B+\rho)^q q^{q\beta}.$$

Finally, if $q < q_0$ and $|x| \leqslant C_{k+1,\rho}/\eta$, then by virtue of the uniform convergence of the sequence $\varphi_\nu^{(q)}(x)$ to zero for $|x| \leqslant C_{k+1,\rho}/\eta$, the inequality

$$|x^k\varphi_\nu^{(q)}(x)| \leqslant \eta(B+\rho)^q q^{q\beta} \tag{3}$$

will also hold for sufficiently large ν, say $\nu > \nu_0$. Thus, for $\nu > \nu_0$, the inequality (3) holds for all x and q. For $\nu > \nu_0$, thereby

$$\|\varphi_\nu\|_{k,\rho} = \sup_{x,q}\frac{|x^k\varphi^{(q)}(x)|}{(B+\rho)^q q^{q\beta}} < \eta,$$

from whence it follows that the sequence $\varphi_\nu(x)$ tends to zero in the norm $\|\cdot\|_{k\rho}$; since k and ρ are arbitrary, $\varphi_\nu(x) \to 0$ in the topology of the space $S^{\beta,B}$, q.e.d.

(f) *If the sequence $\varphi_\nu(x)$ is bounded in each of the norms $\|\cdot\|_{k\rho}$ and converges correctly to some function $\varphi(x)$, then $\varphi(x)$ belongs to $S^{\beta,B}$ and is the limit of the sequence $\varphi_\nu(x)$ in the topology of the space $S^{\beta,B}$.*

In fact, $\varphi(x) \in S^{\beta,B}$ by virtue of (a). The difference $\varphi(x) - \varphi_\nu(x)$ is bounded in all norms, and converges correctly to zero; according to (b) this difference converges to zero in the topology of the space $S^{\beta,B}$, q.e.d.

By the same method as in Section 2 of Chapter II for spaces of $K\{M_p\}$ type, it is easy to deduce from (f) that the space $S^{\beta,B}$ is perfect.

Thus, *the space $S^{\beta,B}$ is a complete countably normed perfect space.*

For $B_1 < B_2$ the space S^{β,B_1} is contained in the space S^{β,B_2} and every sequence $\varphi_\nu(x)$, which converges in the space S^{β,B_1}, will also converge in the space S^{β,B_2}. Hence, we may construct the union of countably normed spaces $S^{\beta,B}$ for all the superscripts $B = 1, 2,\ldots$. Since each function $\varphi(x) \in S^\beta$ belongs to some $S^{\beta,B}$, then the union of the spaces $S^{\beta,B}$ coincides with the space S^β. Thus, *the space S^β is the union of the countably normed spaces $S^{\beta,B}$*. In conformity with this, the sequence $\varphi_\nu(x) \in S^\beta$ is considered convergent to zero if all functions $\varphi_\nu(x)$ belong to some space $S^{\beta,B}$ and converge to zero in its topology; this means that the sequence $\varphi_\nu(x)$ converges correctly to zero and the inequalities

$$|x^k\varphi_\nu^{(q)}(x)| \leqslant C_k B^q q^{q\beta}$$

are satisfied, where C_k and B are independent of ν.

3.3. The Space $S_\alpha{}^\beta$ as the Union of Countably Normed Spaces

As has already been stated, the space $S_\alpha{}^\beta(\alpha \geqslant 0,\ \beta \geqslant 0)$ consists of infinitely differentiable functions $\varphi(x)\ (-\infty < x < \infty)$, satisfying the inequalities

$$|x^k\varphi^{(q)}(x)| \leqslant CA^kB^qk^{k\alpha}q^{q\beta} \qquad (k, q = 0, 1, 2,\ldots),$$

where the constants A, B, C depend on the function φ.

Let $S_{\alpha,A}^{\beta,B}$ denote the set of functions $\varphi(x) \in S_\alpha{}^\beta$, for which the inequalities

$$|x^k\varphi^{(q)}(x)| \leqslant C\bar{A}^k\bar{B}^qk^{k\alpha}q^{q\beta} \qquad (k, q = 0, 1, 2,\ldots)$$

are valid for any $\bar{A} > A$, $\bar{B} > B$, where A and B are given numbers. In other words, the space $S_{\alpha,A}^{\beta,B}$ consists of those functions $\varphi(x)$, which for any $\delta > 0$, $\rho > 0$ satisfy the inequalities

$$|x^k\varphi^{(q)}(x)| \leqslant C_{\delta\rho}(A + \delta)^k (B + \rho)^q k^{k\alpha}q^{q\beta} \qquad (k, q = 0, 1, 2,\ldots). \tag{1}$$

In the space $S_{\alpha,A}^{\beta,B}$, let us introduce the system of norms[6]

$$\|\varphi\|_{\delta\rho} = \sup_{x,k,q} \frac{|x^k\varphi^{(q)}(x)|}{(A + \delta)^k B(+ \rho)^q k^{k\alpha}q^{q\beta}} \qquad (\delta, \rho = 1, \tfrac{1}{2},\ldots). \tag{2}$$

We assert that with this system of norms, the space $S_{\alpha,A}^{\beta,B}$ becomes a *complete countably normed perfect space*.

The proof may be carried out analogously to that performed for the space $S^{\beta,B}$ (Section 3.2). There will be a certain difference in the proof of the assertions analogous to (a) and (d). Let us present this somewhat modified reasoning here.

In (a) we start now from the inequalities

$$\sup_{\substack{x\\ q\leqslant p\\ k\leqslant l}} \frac{|x^k\varphi^{(q)}(x)|}{(B + \rho)^q (A + \delta)^k k^{k\alpha}q^{q\beta}}$$

$$= \varlimsup_{\nu\to\infty} \sup_x \frac{|x^k\varphi_\nu^{(q)}(x)|}{(A + \delta)^k (B + \rho)^q k^{k\alpha}q^{q\beta}} \leqslant \|\varphi_\nu\|_{\delta\rho} \leqslant C \qquad (|x| \leqslant a)$$

[6] The symbol $\sup_{x,k,q}$ means the upper bound in all x, k, and q $(-\infty < x < \infty,\ 0 \leqslant k < \infty,\ 0 \leqslant q < \infty)$.

and perform the passage to the limit for $a \to \infty, p \to \infty, l \to \infty$; we hence obtain

$$\sup_{x,k,q} \frac{|x^k \varphi^{(q)}(x)|}{(A+\delta)^k (B+\rho)^q k^{k\alpha} q^{q\beta}} \leqslant C,$$

q.e.d.

To prove the assertion analogous to (d), let us reason as follows. Let be given $\eta > 0$, $\delta > 0$, and $\rho > 0$; let us take $\delta' < \delta$, $\rho' < \rho$ arbitrarily. Since $\varphi_\nu(x)$ is a bounded sequence, then in particular, the inequality

$$\| \varphi_\nu \|_{\rho,\delta'} \leqslant C_1$$

holds, or equivalently, for any k, q, x,

$$|x^k \varphi_\nu^{(q)}(x)| \leqslant C_1 (A + \delta')^k (B+\rho)^q k^{k\alpha} q^{q\beta}.$$

For sufficiently large k, $k > k_0$, the inequality

$$(A+\delta')^k < \frac{\eta}{C_1} (A+\delta)^k$$

holds; consequently, for any q, x and $k \geqslant k_0$

$$|x^k \varphi^{(q)}(x)| \leqslant \eta (A+\delta)^k (B+\rho)^q k^{k\alpha} q^{q\beta}. \tag{3}$$

Analogously, utilizing the boundedness of the norm $\| \varphi_\nu \|_{\rho',\delta}$, we arrive at the validity of the inequality (3) for any k, x and $q \geqslant q_0$.

There remains to examine the case when $k < k_0$, $q < q_0$.

For $k < k_0$, $|x| > 1$, we have for any q and x by virtue of (3)

$$|x^k \varphi^{(q)}(x)| = \frac{|x|^{k_0}}{|x|^{k_0 - k}} |\varphi^{(q)}(x)| \leqslant \frac{1}{|x|} \eta (A+\delta)^{k_0} (B+\rho)^q k_0^{k_0\alpha} q^{q\beta}.$$

For sufficiently large $|x|$, $|x| > x_0$, we obtain

$$\frac{(A+\delta)^{k_0} k_0^{k_0\alpha} q^{q\beta}}{|x|} \leqslant (A+\delta)^k k^{k\alpha} q^{q\beta}$$

$$(k = 0, 1, \ldots, k_0 - 1, q = 0, 1, \ldots, q_0 - 1),$$

and therefore, for $q < q_0$, $k < k_0$, $|x| \geqslant x_0$ the inequality (3) is also satisfied.

Finally, if $k < k_0$, $q < q_0$, then for fixed ρ and δ, the constants $(A+\delta)^k (B+\rho)^q k^{k\alpha} q^{q\beta}$ are bounded by some number C_2. Since the sequence $\varphi_\nu^{(q)}(x)$ tends uniformly to zero in the segment $|x| \leqslant x_0$, then for any $\eta > 0$, for sufficiently large ν ($\nu \geqslant \nu_0$), the inequality (3) is known to be satisfied in this segment. Thus, for $\nu \geqslant \nu_0$ the inequality (3) is satisfied for all x, k, and q. But this means that for $\nu \geqslant \nu_0$,

$$\| \varphi_\nu \|_{\rho,\delta} < \eta,$$

from which there also results that the sequence φ_ν tends to zero in the topology of the space $S_{\alpha,A}^{\beta,B}$ as $\nu \to \infty$.

Therefore, $S_{\alpha,A}^{\beta,B}$ is *a complete, countably normed perfect space.*

If $A_1 < A_2$, $B_1 < B_2$, then the space $S_{\alpha,A_1}^{\beta,B_1}$ is contained in the space $S_{\alpha,A_2}^{\beta,B_2}$ and every sequence $\varphi_\nu(x)$, convergent in the space $S_{\alpha,A_1}^{\beta,B_1}$, will converge also in the space $S_{\alpha,A_2}^{\beta,B_2}$. Hence, we may construct the union of countably normed spaces $S_{\alpha,A}^{\beta,B}$ for all the indices A, $B = 1, 2, \ldots$. This union coincides with the original space $S_\alpha{}^\beta$. We see that the space $S_\alpha{}^\beta$ is the union of the countably normed spaces $S_{\beta,B}^{\alpha,A}$. In conformity with this, a sequence $\varphi_\nu(x) \in S_\alpha{}^\beta$ is considered *convergent to zero* if all the functions $\varphi_\nu(x)$ belong to some space $S_{\alpha,A}^{\beta,B}$ and converge to zero in its topology; this means that the sequence $\varphi_\nu(x)$ converges correctly to zero, and the inequality

$$| x^k \varphi_\nu^{(q)}(x) | \leqslant C A^k B^q k^{k\alpha} q^{q\beta}$$

is satisfied, where A, B, C are independent of ν.

3.4. The Spaces $S_{\alpha,A}^{\beta}$ and $S_\alpha^{\beta,B}$

A union in just one of the indices A or B (and not in both at once) may be constructed from the space $S_{\alpha,A}^{\beta,B}$. Naturally, the spaces which are thus obtained are denoted by $S_\alpha^{\beta,B}$ and $S_{\alpha,A}^{\beta}$. We leave it to the reader to formulate which functions belong to such spaces and what is the convergence in these spaces.

4. Simplest Bounded Operations in Spaces of Type S

Many linear operators of importance to analysis are defined and bounded (and therefore also continuous) in spaces of type S. Primarily, this is the operator of multiplication by x (and by all polynomials).

It turns out that the operators of multiplication even by some infinitely differentiable functions $f(x)$ are also defined and continuous in spaces of type S; the solution of the question in this case depends only on the relation between the rapidity of growth of the function $f(x)$ as $|x| \to \infty$ and the growth of its derivatives $f^{(q)}(x)$ as $q \to \infty$, on the one hand, and the numbers α, β, defining the space itself, on the other.

In particular, it turns out that fundamental functions may be multiplied by each other in spaces of type S so that these spaces are actually *topological algebras*; however, we shall not go deeply into this aspect in studying them. Moreover, we shall also consider the operations of translation and dilation of the argument x.

4.1. Operation of Multiplication by x

Let us show that the *operation of multiplication by x is defined and bounded in the spaces* S_α, S^β, S_α^β. Moreover, we shall show that this operation is defined and bounded even in the countably normed spaces $S_{\alpha,A}$, $S^{\beta,B}$, $S_{\alpha,A}^{\beta,B}$ (Section 3).

Let the function $\varphi(x)$ run through the bounded set F in the space $S_{\alpha,A}$. This means that for all admissible $\varphi(x)$, the inequalities

$$| x^k \varphi^{(q)}(x)| \leqslant C_{q\delta}(A + \delta)^k k^{k\alpha} \qquad (k, q = 0, 1, 2, ...)$$

are satisfied for any $\delta > 0$ with a constant $C_{q\delta}$, independent of the function φ. Let us put $\psi(x) = x\varphi(x)$. Then

$$\begin{aligned} | x^k \psi^{(q)}(x)| &= | x^k [x\varphi(x)]^{(q)} | \leqslant | x^{k+1}\varphi^{(q)}(x)| + q| x^k \varphi^{(q-1)}(x)| \\ &\leqslant C_{q\delta}(A + \delta)^{k+1}(k + 1)^{(k+1)\alpha} + qC_{q-1,\delta}(A + \delta)^k k^{k\alpha} \\ &= (A + \delta)^k k^{k\alpha} \left[C_{q\delta}(A + \delta) \frac{(k + 1)^{(k+1)\alpha}}{k^{k\alpha}} + qC_{q-1,\delta}\right]. \end{aligned}$$

But for any $\epsilon > 0$,

$$\frac{(k + 1)^{(k+1)\alpha}}{k^{k\alpha}} = \left(1 + \frac{1}{k}\right)^{k\alpha} (k + 1)^\alpha < e^\alpha(k + 1)^\alpha < C_\epsilon(1 + \epsilon)^k;$$

hence

$$C_{q\delta}(A + \delta) \frac{(k + 1)^{(k+1)\alpha}}{k^{k\alpha}} + qC_{q-1,\delta} \leqslant C'_{q\delta\epsilon}(1 + \epsilon)^k,$$

and therefore

$$| x^k \psi^{(q)}(x)| \leqslant C'_{q\delta\epsilon}(1 + \epsilon)^k (A + \delta)^k k^{k\alpha} \leqslant C'_{q\delta}(A + 2\delta)^k k^{k\alpha}$$

for sufficiently small ϵ. Since 2δ is an arbitrarily small quantity together with δ, we then obtain that the image of the bounded set F, when multiplied by x, is again a bounded set in the space $S_{\alpha,A}$, q.e.d.

An analogous computation is carried out in the spaces $S^{\beta,B}$ and $S_{\alpha,A}^{\beta,B}$.

Let $\varphi(x)$ run through the bounded set F in the space $S^{\beta,B}$. This means that for all admissible $\varphi(x)$, the inequality

$$| x^k\varphi^{(q)}(x)| \leqslant C_{k\rho}(B + \rho)^q q^{q\beta}$$

is satisfied for any $\rho > 0$ with a constant $C_{k\rho}$ independent of the function φ. We then have for $\psi(x) = x\varphi(x)$

$$\begin{aligned} | x^k\psi^{(q)}(x)| &\leqslant | x^{k+1}\varphi^{(q)}(x)| + q| x^k\varphi^{(q-1)}(x)| \\ &\leqslant C_{k+1,\rho}(B + \rho)^q q^{q\beta} + qC_{k\rho}(B + \rho)^{q-1}(q - 1)^{(q-1)\beta} \\ &\leqslant (B + \rho)^q q^{q\beta}[C_{k+1,\rho} + qC_{k\rho}(B + \rho)^{-1}]. \end{aligned}$$

But for any $\epsilon > 0$,

$$\frac{q}{B + \rho} < C_\epsilon(1 + \epsilon)^q;$$

hence

$$C_{k+1,\rho} + qC_{k\rho}(B + \rho)^{-1} \leqslant C'_{k\rho\epsilon}(1 + \epsilon)^q,$$

and therefore

$$| x^k\psi^{(q)}(x)| \leqslant C'_{k\rho\epsilon}(1 + \epsilon)^q(B + \rho)^q q^{q\beta} < C'_{k\rho}(B + 2\rho)^q q^{q\beta}$$

for sufficiently small ϵ. Since 2ρ is an arbitrarily small quantity together with ρ, we then obtain that the image of the bounded set $F \subset S^{\beta,B}$, when multiplied by x, is again a bounded set in $S^{\beta,B}$.

We leave to the reader the corresponding computation for the space $S^{\beta,B}_{\alpha,A}$.

4.2. Multiplication by an Infinitely Differentiable Function

1. THE SPACE S_α

If $\alpha = 0$, then all functions $\varphi(x)$, in the space $S_{\alpha,A}$, are of compact support. Any infinitely differentiable function $f(x)$ defines a bounded (and continuous) operator of multiplication by $f(x)$ in this space.

Now, let $\alpha > 0$. In this case we show that *the multiplier in the space S_α is the function $f(x)$, which for any $\epsilon > 0$ satisfies inequalities of the form*

$$|f^{(q)}(x)| \leqslant C_{q\epsilon} \exp(\epsilon| x |^{1/\alpha}).$$

A more exact result, with which we indeed begin, will be needed later. Let us consider the function $f(x)$, satisfying the inequalities

$$|f^{(q)}(x)| \leqslant C_q \exp(a_1| x |^{1/\alpha}). \tag{1}$$

We assert that multiplication by this function is a bounded operation transforming the space $S_{\alpha,A}$ for certain values of A (which will be given below) into a space $S_{\alpha,A'}$ with certain other values A'. In order to refine this formulation, let us recall that the space $S_{\alpha,A}$ may be defined as the set of all functions satisfying the inequalities

$$|\varphi^{(q)}(x)| \leqslant C_{q\delta} \exp[-(a-\delta)|x|^{1/\alpha}]$$

for any $\delta > 0$; hence, the constants A and a are connected by means of the relation (5) of Section 2.1:

$$a = \frac{\alpha}{eA^{1/\alpha}}.$$

Theorem. *Multiplication by a function $f(x)$, satisfying the inequalities* (1) *is a bounded operation in the space $S_{\alpha,A}$, where the constant A is such that for an appropriate constant a, the inequality $a > a_1$ is satisfied; hence, the result of multiplication by $f(x)$ lies in the (broader) space $S_{\alpha,A'}$, where A' corresponds to the constant $a - a_1$.*

If the space $S_{\alpha,A}$ is denoted, for convenience, by $K_{\alpha,a}$, our assertion then reduces to the fact that multiplication by a function $f(x)$, satisfying the inequalities (1) is defined in the space $K_{\alpha,a}$ with $a > a_1$ and transforms this space into the space $K_{\alpha,a-a_1}$.

Proof. The estimate

$$\begin{aligned}|[f(x)\varphi(x)]^q| &\leqslant \sum_{j=0}^{q} C_q^j |f^{(j)}(x)|\,|\varphi^{(q-j)}(x)| \\ &\leqslant \sum_{j=0}^{q} C_q^j C_j C_{q-j,\delta} \exp(a_1|x|^{1/\alpha} - (a-\delta)|x|^{1/\alpha}) \\ &= C'_{q\delta} \exp[-(a-a_1-\delta)|x|^{1/\alpha}],\end{aligned}$$

is valid for the product $f\varphi$, which also shows that the function $f\varphi$ belongs to the space $K_{\alpha,a-a_1}$. Furthermore, the operator of multiplication by $f(x)$ transforms a bounded set in the space $K_{\alpha,a}$ (with fixed constants $C_{q\delta}$) into a bounded set in the space $K_{\alpha,a-a_1}$ (with fixed constants $C'_{q\delta}$), q.e.d.

In the considered case, the function $f(x)$ was not a multiplier in the whole space S_α, since it was impossible to multiply the function $f(x)$ by any function $\varphi(x) \in S_\alpha$ (under the condition that the result should lie in the same space). The conditions imposed on the function $f(x)$ should be strengthened somewhat in order that it might become a multiplier in

S_α. Namely, the result with whose formulation we started, holds: *If for any* $\epsilon > 0$, *the function* $f(x)$ *satisfies an inequality of the form*

$$|f^{(q)}(x)| \leqslant C_{q\epsilon} \exp(\epsilon |x|^{1/\alpha}), \tag{2}$$

then multiplication by the function $f(x)$ *is a bounded operation in the whole space* S_α, *which transforms each of the spaces* $S_{\alpha,A}$ *into itself.*

Indeed, in this case we have for $\varphi(x) \in S_{\alpha,A}$,

$$|\varphi^{(q)}(x)| \leqslant C_{q\delta} \exp[-(a - \delta)|x|^{1/\alpha}],$$

where $a = \alpha/eA^{1/\alpha}$. Since ϵ may be chosen arbitrarily, let us put $\epsilon = \delta$; then, exactly as above, we arrive at the result

$$|[f(x)\varphi(x)]^{(q)}| \leqslant C'_{q\delta} \exp[-(a - 2\delta)|x|^{1/\alpha}];$$

since 2δ is arbitrarily small together with δ, we see that the product $f\varphi$ again belongs to $S_{\alpha,A}$ under the conditions (2). It is also evident that the operator of multiplication by $f(x)$ is bounded in this case, q.e.d.

2. THE SPACE S^β

Let us consider the function $f(x)$, satisfying the inequalities

$$|f^{(q)}(x)| \leqslant CB_0^q q^{q\beta}(1 + |x|^h). \tag{3}$$

It is asserted that *multiplication by this function is a bounded operation in the space* S^β.

Later we shall need the following refinement of this proposition: *This operation transforms any countably normed space* $S^{\beta,B}$ *into the space* $S^{\beta,B+B_0}$.

Proof. For $\varphi \in S^{\beta,B}$, we have

$$|x^k\varphi^{(q)}(x)| \leqslant C_{k\rho}(B + \rho)^q q^{q\beta};$$

hence

$$|x^k[f(x)\varphi(x)]^{(q)}| \leqslant |x^k| \sum C_q^j |f^{(j)}(x)|\,|\varphi^{(q-j)}(x)|$$

$$\leqslant \sum_{j-0}^{q} C_q^j \cdot CB_0^j j^{j\beta}(|x|^k + |x|^{k+h})|\varphi^{(q-j)}(x)|$$

$$\leqslant C \sum_{j=0}^{q} C_q^j B_0^j j^{j\beta}(C_{k\rho} + C_{k+h,\rho})(B + \rho)^{q-j}(q - j)^{(q-j)\beta}.$$

But

$$j^{j\beta}(q - j)^{(q-j)\beta} \leqslant q^{j\beta}q^{(q-j)\beta} = q^{q\beta},$$

and therefore

$$| x^k[f(x)\,\varphi(x)]^{(q)} | \leqslant C(C_{k\rho} + C_{k+h,\rho})\, q^{q\beta} \sum_{j=0}^{q} C_q{}^j B_0{}^j (B+\rho)^{q-j}$$

$$= C'_{k\rho} q^{q\beta}(B_0 + B + \rho)^q.$$

We see that the result belongs to the space $S^{\beta, B+B_0}$. Hence, a bounded set in $S^{\beta,B}$ (the constants $C_{k\rho}$ are fixed) goes over into a bounded set in the space $S^{\beta,B+B_0}$ (the constants $C'_{k\rho} = C(C_{k\rho} + C_{k+h,\rho})$ are fixed), q.e.d.

By strengthening the condition on the function $f(x)$ somewhat, it may be achieved that multiplication by this function would transform each space $S^{\beta,B}$ into itself. Namely, we demand that for any $\epsilon > 0$, the inequality

$$|f^{(q)}(x)| \leqslant C_\epsilon \epsilon^q q^{q\beta}(1 + | x |^h) \tag{4}$$

be satisfied. Then, putting $B_0 = \epsilon = \rho$ in the computation just performed, we arrive at the estimate

$$| x^k[f(x)\,\varphi(x)]^{(q)} | \leqslant C'_{k\rho} q^{q\beta}(B + 2\rho)^q,$$

and since 2ρ is arbitrarily small together with ρ, we obtain that the result belongs to the space $S^{\beta,B}$.

As an example, let us consider multiplication by the function

$$f(x) = e^{i\sigma x},$$

where σ is a real constant. We have

$$|f^{(q)}(x)| = | \sigma |^q$$

and for any $\beta > 0$,

$$|f^{(q)}(x)| \leqslant C_\epsilon \epsilon^q q^{q\beta};$$

therefore, multiplication by the function $f(x) = e^{i\sigma x}$ transforms any space $S^{\beta,B}$ ($\beta > 0$) into itself. It is impossible to assert this for $\beta = 0$. Because of the general theorem, multiplication by this function transforms the space $S^{0,B}$ into the space $S^{0,B+|\sigma|}$. But the union

$$S^0 = \bigcup_B S^{0,B}$$

is transformed into itself by multiplication by the function $e^{i\sigma x}$.

Other kinds of multiplication operators in the spaces $S^{\beta,B}$ are described in Section 7.

3. THE SPACE S_α^β

Let us consider the function $f(x)$, satisfying for any $\epsilon > 0$, the inequalities

$$|f^{(q)}(x)| \leqslant C_\epsilon \epsilon^q q^{q\beta} \exp[\epsilon \mid x \mid^{1/\alpha}] \qquad (\alpha > 0). \tag{5}$$

We assert that *multiplication by this function transforms the space* S_α^β *into itself. Moreover, it transforms each* $S_{\alpha,A}^{\beta,B}$ *into itself and is hence a bounded operator.*

If $f(x)$ *satisfies the inequalities*

$$|f^{(q)}(x)| \leqslant CB_0{}^q q^{q\beta} \exp(a_0 \mid x \mid^{1/\alpha}) \qquad (\alpha > 0), \tag{6}$$

then multiplication by this function is a bounded operation defined in those spaces $S_{\alpha,A}^{\beta,B}$, for which $a_0 < a = \alpha/eA^{1/\alpha}$, *and transforming such a* $S_{\alpha,A}^{\beta,B}$ *into* $S_{\alpha,A'}^{\beta,B'}$, *where* $\alpha/eA'^{1/\alpha} = a - a_0$, $B' = B + B_0$.

If the function $f(x)$ *satisfies the condition*

$$|f^{(q)}(x)| \leqslant C_\epsilon B_0{}^q q^{q\beta} \exp(\epsilon \mid x \mid^{1/\beta}) \tag{7}$$

for any $\epsilon > 0$, *then multiplication by this function transforms the space* $S_{\alpha,A}^{\beta,B}$ *into* $S_{\alpha,A}^{\beta,B+B_0}$.

For $\alpha = 0$, a simpler theorem holds.

Every function $f(x)$, *satisfying for* $\mid x \mid \leqslant A$ *the inequality*

$$|f^{(q)}(x)| \leqslant CB_0{}^q q^{q\beta}, \tag{8}$$

defines a (bounded) multiplication operator in the space $S_{0,A}^{\beta,B}$ *which transforms this space into* $S_{0,A}^{\beta,B+B_0}$ *(and in particular, transforms the space* $S_{0,A}^{\beta}$ *into itself). If* B_0 *in the inequality* (8) *may be taken as small as desired, then multiplication by the function* $f(x)$ *transforms the space* $S_{0,A}^{\beta,B}$ *into itself.*

The proofs of these propositions proceed along the scheme of Sections 4.2, 1.2 and we can leave them to the reader.

In particular, the function

$$f(x) = e^{i\sigma x}$$

satisfies condition (8) for any $\beta > 0$. Hence, the function $f(x)$ is a multiplier in any space $S_{\alpha,A}^{\beta,B}$ with $\beta > 0$, which transforms this space into itself.

For $\beta = 0$, the function $f(x) = e^{i\sigma x}$ does not satisfy condition (8), but satisfies condition (7) with $B_0 = |\sigma|$. Hence, *multiplication by the function $e^{i\sigma x}$ transform the space $S_{\alpha,A}^{0,B}$ into $S_{\alpha,A}^{0,B+|\sigma|}$, and the union*

$$S_{\alpha,A}^{0} = \bigcup_B S_{\alpha,A}^{0,B}$$

into itself.

Other kinds of multipliers in the spaces $S_{\alpha,A}^{\beta,B}$ are described in Section 7.

4.3. Translation Operation

For $\alpha > 0$, the translation operation

$$\varphi(x) \to \varphi(x-h)$$

is defined and bounded in any space $S_{\alpha,A}$ and transforms this space into itself.

Proof. Let $\varphi(x) \in S_{\alpha,A}$, such that

$$|x^k \varphi^{(q)}(x)| \leqslant C_{q\delta}(A+\delta)^k k^{k\alpha}.$$

Furthermore, we have

$$\sup_x |x^k \varphi^{(q)}(x-h)| = \sup_x |(x+h)^k \varphi^{(q)}(x)|.$$

The expression on the right under the symbol sup may be estimated as follows:

$$\begin{aligned}
|(x+h)^k \varphi^{(q)}(x)| &\leqslant \left| \sum C_k^j x^j h^{k-j} \varphi^{(q)}(x) \right| \leqslant \sum_j C_k^j h^{k-j} \sup_x |x^j \varphi^{(q)}(x)| \\
&\leqslant \sum_j C_k^j h^{k-j} C_{q\delta}(A+\delta)^j j^{j\alpha} \leqslant C_{q\delta} \sum_j C_k^j (A+\delta)^j k^{j\alpha} h^{k-j} \\
&= C_{q\delta}((A+\delta)k^\alpha + h)^k \\
&= C_{q\delta}(A+\delta)^k k^{k\alpha} \left(1 + \frac{h}{(A+\delta)k^\alpha}\right)^k.
\end{aligned}$$

For $\alpha > 0$, the quantity $h/(A+\delta)k^\alpha$ tends to zero when $k \to \infty$; hence, for sufficiently large k,

$$(A+\delta)\left(1 + \frac{h}{(A+\delta)k^\alpha}\right) \leqslant A + 2\delta;$$

therefore, for all k,

$$|(x+h)^k \varphi^{(q)}(x)| \leqslant C_{q\delta}(A+2\delta)^k k^{k\alpha};$$

and hence, $\varphi(x-h)$ belongs to the space $S_{\alpha,A}$, q.e.d.[7]

For $\alpha = 0$, the space $S_{0,A}$ consists of functions of compact support contained in a fixed segment, and hence, the translation operator does not transform this space into itself. However, it is evident that in this case the translation operator transforms $S_{0,A}$ into $S_{0,A+|b|}$ and, consequently, the space S_0 remains unchanged.

An analogous computation shows that the *translation operation* $\varphi(x) \to \varphi(x-h)$ *is defined and bounded also in any spaces* $S^{\beta,B}$, $S^{\beta,B}_{\alpha,A}$ $(\alpha > 0)$, $S^{\beta,B}_0$ *and also transforms these spaces into themselves.*

4.4. Dilatation Operation

Let us show that *for* $\lambda > 0$, *the dilatation operation*

$$\varphi(x) \to \varphi(\lambda x) \tag{1}$$

is defined and bounded in any space $S_{\alpha,A}$ *and transforms this space into* $S_{\alpha,A/\lambda}$.

Actually, if $\varphi(x) \in S_{\alpha,A}$, so that

$$|x^k \varphi^{(q)}(x)| \leqslant C_{q\delta}(A+\delta)^k k^{k\alpha},$$

then for $\psi(x) = \varphi(\lambda x)$, we have

$$\begin{aligned} |x^k \psi^{(q)}(x)| &= |x^k \varphi^{(q)}(\lambda x)| = |[x^k \lambda^q \varphi^{(q)}(\xi)]|_{\xi=\lambda x}| \\ &= \left| \frac{\xi^k}{\lambda^k} \lambda^q \varphi^{(q)}(\xi) \right| \leqslant C'_{q\delta} \left(\frac{A+\delta}{\lambda} \right)^k k^{k\alpha}, \end{aligned}$$

from which $\psi(x) \in S_{\alpha,A/\lambda}$, q.e.d.

Furthermore, *the operation* (1) *is defined and bounded in the space* $S^{\beta,B}$ *and transforms this space into* $S^{\beta,\lambda B}$.

In fact, if $\varphi(x) \in S^{\beta,B}$ so that

$$|x^k \varphi^{(q)}(x)| \leqslant C_{k\rho}(B+\rho)^q q^{q\beta},$$

[7] In place of the computations carried out, it would be possible to use the fact that the space $S_{\alpha,A}$ is the space $K\{M_p\}$ for $\alpha > 0$, for which the functions $M_p(x)$ satisfy condition (1) of Chapter III, Section 3.1, which guarantees the existence of a bounded translation.

then for $\psi(x) = \varphi(\lambda x)$, we have exactly as above

$$| x^k \psi^{(q)}(x)| = \left| \frac{\xi^k}{\lambda^k} \lambda^q \varphi^{(q)}(\xi) \right| \leqslant C'_{k\rho}[\lambda(B + \rho)]^q \, q^{q\beta},$$

from which $\psi(x) \in S_{\beta,\lambda B}$, q.e.d.

Finally, the operation (1) is defined and bounded in the space $S_{\alpha,A}^{\beta,B}$ and transforms this space into $S_{\alpha,A/\lambda}^{\beta,\lambda B}$. The proof is analogous to the preceding one and may be left to the reader.

As a corollary, we obtain: *The dilatation operation* (1) *is defined and bounded in all spaces* S_α, S^β, $S_\alpha{}^\beta$ *and transforms each into itself.*

5. Differential Operators

5.1. The Operator d/dx

Let us show that the differentiation operator is defined and bounded in the spaces S_α, S^β, $S_\alpha{}^\beta$. Moreover, we shall show that *this operator is defined and bounded in the countably normed spaces* $S_{\alpha,A}$, $S^{\beta,B}$, $S_{\alpha,A}^{\beta,B}$.

Let the function $\varphi(x)$ run through a bounded set in the space $S_{\alpha,A}$. As we remember, this space consists of all functions $\varphi(x)$, satisfying, for any $\delta > 0$, the inequalities

$$| x^k \varphi^{(q)}(x)| \leqslant C_{q\delta}(A + \delta)^k \, k^{k\alpha} \qquad (k, q = 0, 1, 2,...).$$

In conformity with the definition of a norm in $S_{\alpha,A}$ (Section 3), a bounded set in this space consists of functions $\varphi(x)$, satisfying these inequalities with constants $C_{q\delta}$ independent of the selection of the functions $\varphi(x)$.

Let us put $\psi(x) = \varphi'(x)$. Then

$$| x^k \psi^{(q)}(x)| = | x^k \varphi^{(q+1)}(x)| \leqslant C_{q+1,\delta}(A + \delta)^k \, k^{k\alpha},$$

i.e., the function $\psi(x)$ belongs to a bounded set of the space $S_{\alpha,A}$, q.e.d.

Let us verify an analogous property that holds in the space $S^{\beta,B}$. Let the function $\varphi(x)$ run through a bounded set in the space $S^{\beta,B}$; this means that the inequalities

$$| x^k \varphi^{(q)}(x)| \leqslant C_{k\rho}(B + \rho)^q \, q^{q\beta}$$

are satisfied for any $\delta > 0$ with a constant $C_{k\rho}$, independent of $\varphi(x)$. Let us set $\psi(x) = \varphi'(x)$; then

$$|x^k\psi^{(q)}(x)| = |x^k\varphi^{(q+1)}(x)| \leqslant C_{k\rho}(B+\rho)^{q+1}(q+1)^{(q+1)\beta}$$

$$\leqslant C_{k\rho}(B+\rho)^q q^{q\beta}\left[(B+\rho)(q+1)^\beta\left(1+\frac{1}{q}\right)^{q\beta}\right] \leqslant C'_{k\rho}(B+2\rho)^q q^{q\beta},$$

and therefore, the image of the bounded set in the space $S^{\beta,B}$ is again a bounded set in this space.

We leave a corresponding computation for the space $S^{\beta,B}_{\alpha,A}$ to the reader.

By virtue of the results of Section 4.1, we conclude that *all linear differential operators of finite order with constant or polynomial coefficients are bounded (and continuous) in spaces of type S.*

5.2. Infinite Order Differential Operators

Let

$$f(s) = \sum_0^\infty c_\nu s^\nu$$

be some entire function. We will say that the differential operator $f(d/dx) = \sum_0^\infty c_\nu\, d^\nu/dx^\nu$ is defined in some fundamental space Φ, if for any fundamental function $\varphi(x) \in \Phi$, the series

$$f\left(\frac{d}{dx}\right)\varphi(x) = \sum_0^\infty c_\nu \varphi^{(\nu)}(x) \tag{1}$$

is again a fundamental function (in the space Φ or in another space Ψ). It turns out that for some constraints on the growth of the function f, the operators $f(d/dx)$ are defined and bounded (and therefore continuous) in spaces of type S; the situation depends only on the relationship between the numbers α and β, which define spaces of type S, and the order of growth of the function $f(s)$. We shall say that the entire function $f(s)$, has an order of growth $\leqslant \lambda$ and a type $< b$, if the inequality

$$|f(s)| \leqslant C\exp(b_1|s|^\lambda),$$

is satisfied, where $b_1 < b$ is some constant.

As is proved in the theory of entire functions, the Taylor coefficients c_ν of such a function $f(s)$ satisfy the inequalities

$$|c_\nu| \leqslant C\left(\frac{b_1 e\lambda}{\nu}\right)^{\nu/\lambda} \tag{2}$$

Let us present the proof of (2). According to the Cauchy formula

$$c_\nu = \frac{1}{2\pi i} \int_{|s|=r} \frac{f(s)}{s^{\nu+1}}\, ds,$$

from which

$$|c_\nu| \leqslant C \frac{\exp(b_1 r^\lambda)}{r^\nu}. \tag{3}$$

Let us find the minimum of the right side with respect to r. Taking the logarithm, differentiating, and equating to zero, we find for the desired minimum r_0

$$b_1 \lambda r_0^{\lambda-1} - \frac{\nu}{r_0} = 0,$$

from which

$$r_0 = \left(\frac{\nu}{b_1\lambda}\right)^{1/\lambda};$$

substituting the obtained value in the right side of (3), we find

$$|c_\nu| < C \left(\frac{b_1 e\lambda}{\nu}\right)^{\lambda/\nu},$$

q.e.d.

The next theorem states that in spaces S^β and $S_\alpha{}^\beta$ with given $\beta > 0$, entire functions $f(d/dx)$ of order $1/\beta$ are defined under compliance with some additional condition connecting characteristics of type B (for the fundamental function $\varphi(x)$) and b (for the entire function $f(s)$). More precisely, we have:

Theorem. *If $f(s) = \sum c_\nu s^\nu$ is an entire analytic function of order of growth $\leqslant 1/\beta$ and type $< \beta/B^{1/\beta}e^2$, then the operator $f(d/dx)$ is defined and bounded in the space $S^{\beta,B}$, and transforms this space into the space S^{β,Be^β}.*

Proof. Let $\varphi(x) \in S^{\beta,B}$, such that

$$|x^k \varphi^{(q)}(x)| \leqslant C_{k\rho}(B+\rho)^q\, q^{q\beta}.$$

Furthermore, let us set

$$\psi(x) = \sum_{\nu=0}^{\infty} c_\nu \varphi^{(\nu)}(x);$$

we show that $\psi(x) \in S^{\beta, Be^\beta}$. We have

$$
\begin{aligned}
|x^k \psi^{(q)}(x)| &\leqslant \sum_{\nu=0}^{\infty} |c_\nu x^k \varphi^{(\nu+q)}(x)| \\
&\leqslant C_{k\rho}(B+\rho)^q \sum |c_\nu| (B+\rho)^\nu (\nu+q)^{(\nu+q)\beta} \\
&\leqslant C_{k\rho}(B+\rho)^q\, q^{q\beta} \sum |c_\nu| (B+\rho)^\nu \frac{(\nu+q)^{(\nu+q)\beta}}{q^{q\beta}} \\
&\leqslant C_{k\rho}(B+\rho)^q\, q^{q\beta} \sum |c_\nu| (B+\rho)^\nu (\nu+\rho)^{\nu\beta} \left(1+\frac{\nu}{q}\right)^{q\beta} \\
&\leqslant C_{k\rho}(B+\rho)^q\, q^{q\beta} \sum |c_\nu| (B+\rho)^\nu\, e^{\nu\beta} \nu^{\nu\beta} \left(1+\frac{q}{\nu}\right)^{\nu\beta} \\
&\leqslant C_{k\rho}(B+\rho)^q\, q^{q\beta}\, e^{q\beta} \sum |c_\nu| (B+\rho)^\nu\, e^{\nu\beta} \nu^{\nu\beta}.
\end{aligned} \tag{4}
$$

The assumption on the order of growth and type of the function $f(s)$ leads to the following inequality for the coefficients c_ν [see(2)]:

$$|c_\nu| \leqslant C \frac{\theta^\nu}{B^\nu e^{\nu\beta} \nu^{\nu\beta}},$$

where θ is some positive constant less than one.

By virtue of this inequality, the series (4) converges if $1 + (\rho/B) < 1/\theta$, which is known to take place for sufficiently small ρ. Denoting the sum of this series by C_ρ, we obtain the estimate

$$|x^k \psi^{(q)}(x)| \leqslant C_{k\rho} \cdot C_\rho (Be^\beta + \rho')^q\, q^{q\beta},$$

where $\rho' = \rho e^\beta$.

The obtained estimate shows that the function $\psi(x)$ belongs to the space S^{β, Be^β} and that the operator $f(d/dx)$, transforming $S^{\beta, B}$ into S^{β, Be^β}, is bounded.

Remark. In general, the operator $f(d/dx)$ is not bounded in the whole space S^β for the mentioned constraints on the function $f(s)$; it is not even defined in S^{β, B_1} for $B_1 > B$.

By strengthening the constraints imposed on the function $f(s)$ somewhat, it may be achieved that the operator $f(d/dx)$ be defined and bounded in all S^β. Namely, this will hold if the function $f(s)$ has the order of growth $1/\beta$ with a *minimum type*, i.e., satisfies the inequality

$$|f(s)| \leqslant C_\epsilon \exp(\epsilon\, |s|^{1/\beta})$$

for any $\epsilon > 0$.

In fact, in this case the estimate of the coefficients c_ν may be taken in the form (5) *with any value of* B. Hence, the operator $f(d/dx)$ is defined in *any* space $S^{\beta,B}$. As above, the space $S^{\beta,B}$ is transformed by the operator $f(d/dx)$ into S^{β,Be^β}

The theorem and the subsequent remark remain valid upon replacement of the spaces $S^{\beta,B}$ by $S_{\alpha,A}^{\beta,B}$ and S^{β,Be^β} by $S_{\alpha,A}^{\beta,Be^\beta}$. The computations for this case are completely analogous, and we leave them to the reader.

6. Fourier Transformations

If some fundamental space Φ is given, we may construct a *dual space* $\Psi = \tilde{\Phi}$, consisting of the Fourier transforms

$$\psi(\sigma) = \int_{-\infty}^{\infty} \varphi(x)\, e^{ix\sigma}\, dx = \widetilde{\varphi(x)} = F[\varphi]$$

of the fundamental functions $\varphi(x) \in \Phi$.

As we have already said, spaces of type S are closely interrelated by means of the Fourier transformation; namely, the formulas

$$\widetilde{S_\alpha} = S^\alpha, \qquad \widetilde{S^\beta} = S_\beta\,, \qquad \widetilde{S_\alpha^\beta} = S_\beta^\alpha$$

hold. A derivation of these (and more exact) formulas is given below.[8]

Underlying this proof is the following idea. We know that the operations of differentiation and multiplication by an independent variable change roles in the transition from the fundamental space to its dual. Hence, if the operations of differentiation and multiplication by an independent variable could be interchanged in the definition of the fundamental spaces, i.e., in inequalities such as

$$|\, x^k \varphi^{(q)}(x)| \leqslant m_{kq}\,,$$

then after the Fourier transformation, we would obtain a space of the same type, with the sole difference that the constraints on the growth of the derivatives and the decrease of the fundamental functions at infinity would exchange roles (i.e., the sequence m_{kq} would be replaced by m_{qk}). But multiplication by x and differentiation do not actually enter symme-

[8] Let us recall that in this case the relationship $F^{-1}[\Phi] = \Psi$ is true, together with each relationship $F[\Phi] = \Psi$ (F is the direct Fourier transform, Φ and Ψ fundamental spaces), since each space of type S together with every function $\varphi(x)$ contains the function $\varphi(-x)$.

trically in these inequalities: The differentiation operator operates first, and then multiplication by x. The order of these operations is changed in the Fourier transformation and we already obtain a space of formally different type. Hence, we show first that in reality *the order of the operations of multiplication by x and of differentiation is not essential in the definition of spaces of type S*, in any case for a sufficiently rapid growth of the constants m_{kq}.

Afterwards, it will already be clear that the Fourier transformation will result in spaces of the same type; we need only verify that the constants m_{kq}, taking part in the definition of the spaces S_α, S^β, S_α^β, will satisfy the mentioned growth constraints.

6.1. General Theorem

In conformity with the above, we wish to clarify first for which m_{kq} the inequalities

$$| [x^k\varphi(x)]^{(q)} | \leqslant C'A_1{}^kB_1{}^qm_{kq} \qquad (k, q = 0, 1,2,...)$$

follow from the inequalities

$$| x^k\varphi^{(q)}(x)| \leqslant CA^kB^qm_{kq} \qquad (k, q = 0, 1, 2,...), \tag{1}$$

The answer to this question is given by the following lemma, in which connections are established between the constants A and B, on the one hand, and between A_1 and B_1, on the other.

Lemma 1. *If an infinitely differentiable function $\varphi(x)$ satisfies the inequalities*

$$| x^k\varphi^{(q)}(x)| \leqslant CA^kB^qm_{kq} \qquad (k, q = 0, 1, 2,...)$$

where the numbers m_{kq} are such that

$$kq\,\frac{m_{k-1,q-1}}{m_{kq}} \leqslant \gamma(k+q)^\theta, \qquad \theta \leqslant 1, \tag{2}$$

then the function $\varphi(x)$ also satisfies the inequalities

$$| [x^k\varphi(x)]^{(q)} | \leqslant C'A_1{}^kB_1{}^qm_{kq} \qquad (k,\ q = 0, 1, 2,...), \tag{3}$$

where C', A_1, B_1 are new constants; for $\theta = 1$

$$A_1 = A \exp(\gamma/AB), \qquad B_1 = B \exp(\gamma/AB), \tag{4}$$

and for $\theta < 1$ *and any* $\epsilon > 0$, *it may be considered that*

$$A_1 = A(1+\epsilon), \qquad B_1 = B(1+\epsilon). \tag{5}$$

Proof. Applying the Leibnitz formula we find

$$\begin{aligned}
|[x^k\varphi(x)]^{(q)}| &= \Big| x^k\varphi^{(q)}(x) + kqx^{k-1}\varphi^{(q-1)}(x) \\
&\quad + \frac{1}{1\cdot 2}k(k-1)\,q(q-1)\,x^{k-2}\varphi^{(q-2)}(x) + \cdots \Big| \\
&\leqslant C\Big[A^kB^qm_{kq} + kqA^{k-1}B^{q-1}m_{k-1,q-1} \\
&\quad + \frac{1}{1\cdot 2}k(k-1)\,q(q-1)\,m_{k-2,q-2} + \cdots\Big] \\
&\leqslant CA^kB^qm_{kq}\Big[1 + \frac{1}{AB}kq\frac{m_{k-1,q-1}}{m_{kq}} + \frac{1}{1\cdot 2}\frac{1}{A^2B^2}kq\frac{m_{k-1,q-1}}{m_{kq}} \\
&\qquad \times (k-1)(q-1)\frac{m_{k-2,q-2}}{m_{k-1,q-1}} + \cdots\Big] \\
&\leqslant CA^kB^qm_{kq}\Big[1 + \frac{1}{AB}\gamma(k+q)^\theta + \frac{1}{1\cdot 2}\frac{1}{A^2B^2}\gamma^2(k+q)^{2\theta} + \cdots\Big] \\
&\leqslant CA^kB^qm_{kq}\exp\Big[\frac{\gamma}{AB}(k+q)^\theta\Big].
\end{aligned}$$

For $\theta = 1$, we obtain the estimate

$$|[x^k\varphi(x)]^{(q)}| \leqslant CA_1{}^kB_1{}^qm_{kq},$$

where

$$A_1 = A\exp(\gamma/AB), \qquad B_1 = B\exp(\gamma/AB).$$

For $\theta < 1$ and any $\epsilon > 0$, we have

$$\frac{\gamma}{AB}(k+q)^\theta \leqslant (k+q)\ln(1+\epsilon) + \ln C_\epsilon;$$

hence

$$\exp\Big[\frac{\gamma}{AB}(k+q)^\theta\Big] \leqslant C_\epsilon(1+\epsilon)^{k+q},$$

and therefore

$$|[x^k\varphi(x)]^{(q)}| \leqslant C_\epsilon A_1{}^kB_1{}^qm_{kq},$$

where

$$A_1 = A(1 + \epsilon), \qquad B(1 + \epsilon),$$

q.e.d.

Remark. In addition, we shall need the estimate

$$|\, x^2[x^k\varphi(x)]^{(q)} \,| \leqslant C'' A_1^{k+2} B_1{}^q m_{k+2,q}\,, \tag{6}$$

which is obtained by the same means as the preceding one; the constant C'' equals $C \exp(2\gamma/AB)$ for $\theta = 1$ and depends on ϵ for $\theta < 1$; the constants B_1 and A_1 retain the values mentioned above.

Henceforth, we shall assume that the numbers m_{kq} satisfy the following conditions:

(a) For any $k, q = 1, 2, \ldots$,

$$kq \frac{m_{k-1,q-1}}{m_{kq}} \leqslant \gamma(k + q)^{\theta}, \qquad \theta \leqslant 1 \tag{7}$$

(this condition constrains the growth of the sequence m_{kq} from below);

(b) For any $k, q = 0, 1, 2, \ldots$ and any $\epsilon > 0$,

$$\frac{m_{k+2,q}}{m_{kq}} \leqslant \mu_\epsilon(1 + \epsilon)^{k+q} \tag{8}$$

(this condition constrains the growth of the sequence m_{kq} from above).

Let us now formulate our fundamental theorem.

Theorem. *If an infinitely differentiable function satisfies the inequalities*

$$|\, x^k \varphi^{(q)}(x)| \leqslant C A^k B^q m_{kq} \qquad (k, q = 0, 1, 2, \ldots) \tag{9}$$

and the numbers m_{kq} are such that conditions (a)–(b) *are satisfied, then the Fourier transform $\psi(\sigma)$ of the function $\varphi(x)$ satisfies the inequalities*

$$|\, \sigma^k \psi^{(q)}(\sigma)| \leqslant C_2 A_2{}^q B_2{}^k m_{qk}\,, \tag{10}$$

where for any $\epsilon > 0$, it may be considered that $A_2 = A_1(1 + \epsilon)$, $B_2 = B_1(1 + \epsilon)$; the values of A_1 and B_1 were mentioned in the lemma.

Therefore, under conditions (a)—(b), the class of functions $\varphi(x)$ satisfying the inequality (9), will transform by Fourier transformation into an analogous class, but corresponding to the sequence m_{qk} (with commuted subscripts k and q).

Proof. Applying the result of Lemma 1 and its Remark, we may estimate the expression $[x^k\varphi(x)]^{(q)}$ as follows:

$$\begin{aligned} | [x^k\varphi(x)]^{(q)} | &\leqslant \min \left\{ C' A_1{}^k B_1{}^q m_{kq} \,, \frac{C''}{x^2} A_1^{k+2} B_1{}^q m_{k+2,q} \right\} \\ &= C''' A_1{}^k B_1{}^q m_{kq} \min \left\{ 1, \frac{A_1{}^2}{x^2} \frac{m_{k+2,q}}{m_{kq}} \right\}. \end{aligned}$$

This inequality permits us to estimate the Fourier transform of the function $[x^k\varphi(x)]^{(q)}$:

$$\begin{aligned} | F\{[x^k\varphi(x)]^{(q)}\} | &= | \sigma^q \psi^{(k)}(\sigma) | \\ &= \left| \int_{-\infty}^{\infty} [x^k\varphi(x)]^{(q)} e^{ix\sigma} \, dx \right| \leqslant \int_{-\infty}^{\infty} | [x^k\varphi(x)]^{(q)} | \, dx \\ &\leqslant C''' A_1{}^k B_1{}^q m_{kq} \left[1 + \frac{A_1{}^2 m_{k+2,q}}{m_{kq}} \int_1^{\infty} \frac{dx}{x^2} \right] \\ &\leqslant C_2 A_2{}^k B_2{}^q m_{kq} \,, \end{aligned}$$

where

$$A_2 = A_1(1 + \epsilon), \qquad B_2 = B_1(1 + \epsilon), \qquad C_2 = C'''(1 + \mu_\epsilon A_1{}^2),$$

which agrees with the required result (10), with the subscripts k and q interchanging places.

6.2. Fourier Transformation in Spaces of Type S

Let us turn now to spaces of type S, in which we are interested. In these spaces, the constants m_{kq} have a special form; we must confirm that these constants do not grow too slowly on the one hand, such that conditions (7) of Section 6.1 are satisfied, and not too rapidly on the other hand, such that inequalities (8) of Section 6.1 are satisfied.

Let us first establish one simple lemma.

Lemma 2. *If the numbers a_k and b_q satisfy the conditions*

$$\frac{a_k}{a_{k-1}} \geqslant C_a k^{1-\chi}, \tag{1}$$

$$\frac{b_q}{b_{q-1}} \geqslant C_q q^{1-\lambda}, \tag{2}$$

$$\chi, \lambda \geqslant 0, \qquad \chi + \lambda = \theta \leqslant 1, \tag{3}$$

then the sequence $m_{kq} = a_k b_q$ $(k, q = 0, 1, 2,...)$ *satisfies inequalities* (7) *of Section* 6.1 *with the same values of* θ *and* $\gamma = 1/C_a C_b$.

Proof. In this case we have

$$kq \frac{m_{k-1,q-1}}{m_{kq}} = k \frac{a_{k-1}}{a_k} q \frac{b_{q-1}}{b_q} \leqslant \frac{1}{C_a C_b} k^\chi q^\lambda.$$

Furthermore, let us use the inequality

$$k^\chi q^\lambda \leqslant [\max(k, q)]^{\chi+\lambda} \leqslant (k + q)^\theta,$$

which holds for all non-negative k and q because of (3); we then obtain the inequality (7) of Section 6.1, q.e.d.

Let us turn to the proof that the constants m_{kq}, entering in the definition of spaces of type S, satisfy the required conditions.

Let us first consider the case of the spaces $S_{\alpha,A}$, $\alpha > 0$. The function $\varphi(x)$ belonging to this space satisfies the inequalities

$$| x^k \varphi^{(q)}(x)| \leqslant C_{q\delta}(A + \delta)^k k^{k\alpha} \tag{4}$$

for any $\delta > 0$. This inequality coincides with the inequality (9) of Section 6.1 if C there is replaced by 1, A by $A + \delta$, B by 1, and m_{kq} by $C_{q\delta} k^{k\alpha}$. Let us note that, together with the constants $C_{q\delta}$, any greater constants satisfy the inequalities (4); hence by increasing the constants $C_{q\delta}$ if needed, it may be considered that they satisfy the inequalities

$$\frac{C_{q\delta}}{C_{q-1,\delta}} \geqslant q^{1-\lambda} \qquad (q = 1, 2,...), \tag{5}$$

where the choice of the constant λ will be refined below.

Let us put $a_k = k^{k\alpha}$, $b_q = C_{q\delta}$, and let us apply Lemma 2. Condition (2) of this lemma is satisfied by virtue of the inequalities (5). Let us verify compliance with condition (1):

$$\frac{a_k}{a_{k-1}} = \frac{k^{k\alpha}}{(k-1)^{(k-1)\alpha}} = \frac{(k-1)^\alpha}{(1-(1/k))^{k\alpha}} \geqslant (k-1)^\alpha \geqslant \frac{k^\alpha}{2^\alpha} \geqslant \frac{k^{1-\chi}}{2^\alpha},$$

where $\chi = 1 - \alpha$, if $1 - \alpha \geqslant 0$, and $\chi = 0$ if $1 - \alpha < 0$. In particular, we have $\chi < 1$ by virtue of the assumption $\alpha > 0$. Let us now choose $\lambda \geqslant 0$ in (5) so that we would have

$$\chi + \lambda = \theta < 1;$$

whereby condition (3) will be satisfied.

All the demands of Lemma 2 are now satisfied; moreover, the results of Lemma 1 and the general theorem of Section 6.1 are valid.

Let us now estimate the ratio $m_{k+2,q}/m_{kq}$, which enters in the theorem. Since $m_{kq} = k^{k\alpha}C_{q\delta}$, then

$$\frac{m_{k+2,q}}{m_{kq}} = \frac{(k+2)^{(k+2)\alpha}}{k^{k\alpha}} = \left(1+\frac{2}{k}\right)^{k\alpha}(k+2)^{2\alpha} \leqslant e^{2\alpha}(k+2)^{2\alpha}.$$

For any $\delta > 0$, this latter expression admits of the estimate

$$\frac{m_{k+2,q}}{m_{kq}} \leqslant C_\delta(1+\delta)^k.$$

Now applying the general theorem and taking account of the expression (5) of Section 6.1 for the coefficients A_1 and B_1, we obtain that the Fourier transform $\psi(\sigma)$ of the function $\varphi(x)$ satisfies for $\delta > 0$ the inequalities

$$|\sigma^k\psi^{(q)}(\sigma)| \leqslant C'_\delta(A+\delta)^q(B+\delta)^k\, q^{q\alpha}C_{k\delta} \leqslant C'_{k\delta}(A+\delta)^q\, q^{q\alpha},$$

where

$$C'_{k\delta} = C'_\delta(B+\delta)^k\, C_{k\delta}.$$

Hence, the function $\psi(\sigma)$ belongs to the space $S^{\alpha,A}$.

Thus, *for* $\alpha > 0$, *we have established the imbedding*

$$\widetilde{S_{\alpha,A}} \subset S^{\alpha,A}, \tag{6}$$

The Fourier operator transforming the space $S_{\alpha,A}$ into $S^{\alpha,A}$ is bounded (and therefore is also continuous) in the topology of the space $S_{\alpha,A}$, as is seen from the relationship between the constants.

We consider the $\alpha = 0$ case later.

Let us turn to the space $S^{\beta,B}$, $\beta > 0$. The functions $\varphi(x) \in S^{\beta,B}$ satisfy the inequalities

$$|x^k\varphi^{(q)}(x)| \leqslant C_{k\rho}(B+\rho)^q\, q^{q\beta} \qquad (k, q = 0, 1, 2, \ldots)$$

for any $\rho > 0$. This inequality agrees with (9) of Section 6.1 if C there is replaced by 1, A by 1, B by $B+\rho$, and m_{kq} by $C_{k\rho}q^{q\beta}$. Exactly as above for the space $S_{\alpha,A}$, it may be shown that the constants $m_{kq} = C_{k\rho}q^{q\beta}$ with $\beta > 0$ satisfy the conditions of Lemma 2; hence, the fundamental theorem may be applied. The ratio $m_{k+2,q}/m_{kq}$ in the fundamental theorem has the form

$$\frac{m_{k+2,q}}{m_{kq}} = \frac{C_{k+2,\rho}}{C_{kq}} = C'_{k\rho},$$

where $C'_{k\rho}$ is some new constant. As a result, the inequality

$$|\sigma^k \psi^{(q)}(\sigma)| \leqslant C'_\rho (A+\rho)^q (B+\rho)^k C'_{q\rho} k^{k\beta} = C''_{q\rho}(B+\rho)^k k^{k\beta},$$

where

$$C''_{q\rho} = C'_\rho (A+\rho)^q C'_{q\rho}$$

is obtained for the function $\psi(\sigma) = \widetilde{\varphi(x)}$ for any $\rho > 0$.

Therefore, the Fourier transform of the function $\varphi(x) \in S^{\beta,B}$ belongs to the space $S_{\beta,B}$, and the corresponding operator is bounded and continuous. Thus for $\beta > 0$,

$$\widetilde{S^{\beta,B}} \subset S_{\beta,B}. \tag{7}$$

Here replacing β by α, B by A, again applying the Fourier transformation, and utilizing the imbedding (6), we find

$$\widetilde{\widetilde{S^{\alpha,A}}} \subset \widetilde{S_{\alpha,A}} \subset S^{\alpha,A}. \tag{8}$$

But the doubly applied Fourier transformation transforms each fundamental function $\varphi(x)$ into $\varphi(-x)$, and therefore, the space $S^{\alpha,A}$ transforms into itself. Hence, $\widetilde{\widetilde{S^{\alpha,A}}} = S^{\alpha,A}$ and the imbedding (8) has the equality

$$\widetilde{S_{\alpha,A}} = S^{\alpha,A} \tag{9}$$

as corollary. Analogously

$$\widetilde{S^{\beta,B}} = S_{\beta,B}. \tag{10}$$

The Fourier operator is hence bounded and continuous in the topology of the corresponding spaces.

Finally, let us consider the space $S^{\beta,B}_{\alpha,A}$. It consists of functions which satisfy the inequalities

$$|x^k \varphi^{(q)}(x)| \leqslant C_{\delta\rho}(A+\delta)^k (B+\rho)^q k^{k\alpha} q^{q\beta}$$

for any δ and ρ. This inequality agrees with the inequality (9) of Section 6.1, if C there is replaced by $C_{\delta\rho}$, A by $A+\delta$, B by $B+\rho$, and m_{kq} by $k^{k\alpha}q^{q\beta}$. Let us verify compliance with the conditions of Lemma 2. As we have already seen earlier, for $a_k = k^{k\alpha}$ and $b_q = q^{q\beta}$, the inequalities

$$\frac{a_k}{a_{k-1}} \geqslant k^{1-\chi}, \qquad \frac{b_q}{b_{q-1}} \geqslant q^{1-\lambda},$$

are satisfied, where $\chi = \max(1 - \alpha, 0)$, $\lambda = \max(1 - \beta, 0)$. Let us first consider the case $\alpha + \beta > 1$, $\alpha > 0$, $\beta > 0$. Then

$$\chi + \lambda = \max(1 - \alpha + 1 - \beta, 1 - \alpha, 1 - \beta, 0) = \theta < 1$$

and the demands of the lemma are satisfied. Moreover, the results of Lemma 1 and the fundamental theorem are valid. Let us estimate the ratio $m_{k+2,q}/m_{kq}$, which enters into the fundamental theorem. As above, we have

$$\frac{m_{k+2,q}}{m_{kq}} = \frac{(k+2)^{(k+2)\alpha}}{k^{k\alpha}} \leqslant \left(1 + \frac{2}{k}\right)^{k\alpha} (k+2)^{2\alpha} \leqslant C_\delta(1+\delta)^k.$$

Now applying the fundamental theorem, we obtain that the function $\psi(\sigma) = \widetilde{\varphi(x)}$ satisfies the inequalities

$$|\sigma^k \psi^{(q)}(\sigma)| \leqslant C'_{\delta\rho}(A+\delta)^q (B+\delta)^k q^{q\alpha} k^{k\beta}$$

for any $\delta > 0$.

Therefore, the function $\psi(\sigma)$ belongs to the space $S_{\beta,B}^{\alpha,A}$.

Thus, the imbedding

$$\widetilde{S_{\alpha,A}^{\beta,B}} \subset S_{\beta,B}^{\alpha,A}$$

is valid. But in exactly the same manner, the imbedding

$$\widetilde{S_{\beta,B}^{\alpha,A}} \subset S_{\alpha,A}^{\beta,B}$$

is also valid. Hence, *for* $\alpha > 0$, $\beta > 0$, $\alpha + \beta > 1$ (exactly as above for the space $S_{\alpha,A}$), *the equality*

$$\widetilde{S_{\alpha,A}^{\beta,B}} = S_{\beta,B}^{\alpha,A}, \tag{11}$$

holds, hence the Fourier-transformation operator is, as before, bounded and continuous in the topology of the fundamental space $S_{\alpha,A}^{\beta,B}$.

Now let us turn to the cases $\alpha = 0$, or $\beta = 0$, or $\alpha + \beta \leqslant 1$, to which we have not yet paid any attention. We shall not consider the $\alpha + \beta < 1$ case here; we will see in Section 8 that in this case the space $S_\alpha{}^\beta$ consists of the single function $\varphi(x) \equiv 0$. In all the other cases (i.e., $\alpha = 0$, or $\beta = 0$, or $\alpha + \beta = 1$), Lemma 2 is satisfied with exponent $\theta = 1$ and with $\gamma = 1$. Taking into account the values of the constants A_1 and B_1, which formula (4) of Section 6.1 yields in these cases, we arrive, by the same means as above, at the following results.

$$\widetilde{S_{0,A}} \subset S^{0,A_1}, \quad A_1 = A\exp(1/A), \tag{12}$$

$$\widetilde{S^{0,B}} \subset S_{0,B_1}, \quad B_1 = B \exp(1/B), \tag{13}$$

$$\widetilde{S^{\beta,B}_{\alpha,A}} \subset S^{\alpha,A_1}_{\beta,B_1}, \quad A_1 = A \exp(1/AB), \quad B_1 = B \exp(1/AB). \tag{14}$$

The result may be improved for the $S_{0,A}$ and $S^{0,B}$ spaces[9]: namely, the *formulas*

$$\widetilde{S_{0,A}} = S^{0,A}, \qquad \widetilde{S^{0,B}} = S_{0,B},$$

hold, just as in the cases when $\alpha \neq 0, \beta \neq 0$.

For the proof, we find a number A_0 for given $\epsilon < 0$ such that $\exp(1/A_0) < 1 + \epsilon$. Then, according to what has been proved

$$\widetilde{S_{0,A}} \subset S^{0,A_0(1+\epsilon)}.$$

Let us transform from the functions $\varphi(x) \in S_{0,A}$ to the functions $\varphi_\lambda(x) = \varphi(\lambda x)$, where $\lambda = A/A_0$. According to what has been proved in Section 4.4, we have

$$\varphi_\lambda(x) = \varphi(\lambda x) \in S_{0,A_0},$$

from which

$$\widetilde{\varphi_\lambda(x)} \in S^{0,A_0(1+\epsilon)}.$$

But it is easy to show by a direct calculation that if $\widetilde{\varphi(x)} = \psi(\sigma)$, then $\widetilde{\varphi(\lambda x)} = (1/\lambda)\, \psi(\sigma/\lambda)$; in fact

$$\widetilde{\varphi(\lambda x)} = \int_{-\infty}^{\infty} e^{ix\sigma} \varphi(\lambda x)\, dx = \int_{-\infty}^{\infty} \exp\left(i \frac{y}{\lambda} \sigma\right) \varphi(y) \frac{dy}{\lambda} = \frac{1}{\lambda} \psi\left(\frac{\sigma}{\lambda}\right).$$

Hence, $\psi(\sigma/\lambda) \in S^{0,A_0(1+\epsilon)}$. Therefore we again have, because of the result of Section 4.4,

$$\psi(\sigma) \in S^{0,A(1+\epsilon)}.$$

Since the last imbedding is valid for any $\epsilon > 0$, then $\psi(\sigma) \in S^{0,A}$. Thus

$$\widetilde{S_{0,A}} \subset S^{0,A}.$$

An analogous discussion performed in reverse order shows that the reverse imbedding

$$\widetilde{S^{0,B}} \subset S_{0,B}$$

[9] Probably such an improvement can also be established for the space $S^{\beta,B}_{\alpha,A}$ with $\alpha + \beta = 1$.

is also valid. Hence

$$\widetilde{S_{0,A}} = S^{0,A}, \qquad \widetilde{S^{0,B}} = S_{0,B},$$

q.e.d.

In conclusion, let us note that in all cases it is possible to take the unions in the indices A and B; we therefore obtain:

$$\widetilde{S_\alpha} = S^\alpha, \qquad \widetilde{S^\beta} = S_\beta. \qquad \widetilde{S_\alpha^\beta} = S_\beta^\alpha,$$

where the Fourier operator in each of these formulas is continuous in the topology of the appropriate fundamental space.

In particular, the space S^0, as the dual of the space $S_0 = K$ of infinitely differentiable functions of compact support coincides with the space Z (Vol. I, Chapter II) of entire analytic functions $\psi(\sigma + i\tau)$, satisfying the inequalities

$$| s^k\psi(\sigma + i\tau)| \leqslant C_k e^{a|\tau|}.$$

However, the equality $S^0 = Z$ could also have been obtained directly from the definitions of these spaces without special difficulty.

Remark. We mentioned in Section 4 that the spaces of type S are topological algebras relative to conventional multiplication. Since the family of spaces of type S transforms into itself in the Fourier transformation, and the multiplication operation goes over into the convolution operation, we may then conclude that all spaces of type S are topological algebras relative to convolution also.

7. Entire Analytic Functions as Elements or Multipliers in Spaces of Type S

7.1. Summary of Results

We saw in Section 2 that if the function $\varphi(x)$ belongs to the space S^β, i.e., satisfies the inequalities

$$| x^k\varphi^{(q)}(x)| \leqslant C_k B^q q^{q\beta} \quad (q = 0, 1, \ldots),$$

then for $\beta < 1$ it can be continued into the complex $z = x + iy$ plane as an entire function of order of growth $1/(1 - \beta)$. More accurately, this entire analytic function satisfies inequalities of the form

$$| x^k\varphi(x + iy)| \leqslant C'_k \exp\left(b\, | y |^{\frac{1}{1-\beta}}\right). \tag{1}$$

Furthermore, we have seen that if $\varphi(x)$ belongs to the space $S_\alpha{}^\beta$, i.e. satisfies the inequalities

$$|x^k\varphi^{(q)}(x)| \leqslant CA^kB^qk^{k\alpha}q^{q\beta} \qquad (k, q = 0, 1,...),$$

then for $\alpha > 0$ and $\beta < 1$ it can be continued into the complex plane as an entire analytic function satisfying the inequality

$$|\varphi(x+iy)| \leqslant C\exp(-a\,|\,x\,|^{1/\alpha} + b\,|\,y\,|^{1/(1-\beta)}). \tag{2}$$

Here $a = \alpha/eA^{1/\alpha}$, and b is any constant greater than $([1-\beta]/e)(Be)^{1/1-\beta}$.

The inverse theorems will be obtained in this paragraph.

As is seen from the above, the fact that the entire function belongs to the space S^β or $S_\alpha{}^\beta$ is connected with specific conditions on the behavior of this function first in the plane, and secondly on the real axis; such conditions are expressed most simply by inequalities containing not $|\,z\,|$, but $|\,x\,|$ and $|\,y\,|$. For an entire function to be a multiplier in the space S^β or $S_\alpha{}^\beta$, it is sufficient that it satisfy some less stringent conditions of the same kind. We obtain below that if an entire function $f(x)$ satisfies the inequality

$$|f(x+iy)| \leqslant C\exp(a\,|\,x\,|^h + b\,|\,y\,|^\gamma) \qquad (h \leqslant \gamma), \tag{3}$$

then it is a multiplier in the space $S_{1/h}^{1-(1/\gamma)}$ (for $a > 0$, and is even an element in this space for $a < 0$). Furthermore, if

$$|f(x+iy)| \leqslant C(1+|\,x\,|)^h e^{b|y|^\gamma}, \tag{4}$$

then the function f will be a multiplier in the space $S^{1-(1/\gamma)}$. We shall formulate still another result on multipliers somewhat later.

The listed results are contained in the following chain of theorems which are of independent interest.

Theorem 1. *If an entire function $f(z)$ satisfies the inequalities*

$$|f(z)| \leqslant C_1\exp(b\,|\,z\,|^p),$$
$$|f(x)| \leqslant C_2\exp(a\,|\,x\,|^h) \qquad (a \neq 0,\ 0 < h \leqslant p),$$

then there exists a domain G_μ of the form

$$|\,y\,| \leqslant K_1(1+|\,x\,|)^\mu, \qquad \mu \geqslant 1-(p-h) \tag{5}$$

(Fig. 2) *in which the inequality*

$$|f(z)| \leqslant C_3\exp(a'\,|\,x\,|^h), \qquad C_3 = \max(C_1, C_2); \tag{6}$$

is satisfied; here a' has the same sign as a, and may be selected as close as desired to a.

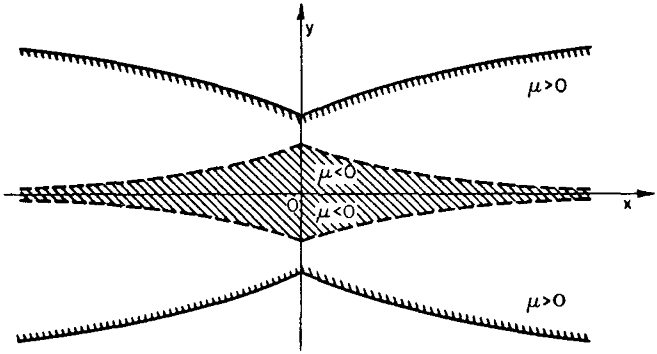

FIG. 2. The function $|y| \leqslant K_1(1 + |x|)^{\mu}$.

The estimate $\exp(a |x|^h)$ in the conditions of the theorem may be replaced by $(1 + |x|)^h$; then the inequality

$$|f(z)| \leqslant C(1 + |x|)^h$$

will be satisfied in the domain (5) with $\mu = 1 - p$.

Theorem 2. *If an entire function $f(z)$ satisfies the inequality*

$$|f(z)| \leqslant C_1 \exp(b |z|^p)$$

for all z and the inequality

$$|f(z)| \leqslant C_2 \exp(a |x|^h)$$

in a domain of the form (1) *with* $0 < \mu \leqslant 1$ ($h < p$; *even* $h = p$ *is* allowed for $a < 0$), *then the function $f(z)$ also satisfies the inequality*

$$|f(z)| \leqslant C_3 \exp(a |x|^h + b' |y|^{p/\mu}). \tag{7}$$

The function $\exp(a |x|^h)$ in the conditions and statement of the theorem may be replaced by $(1 + |x|)^h$.

Theorem 3. *If an entire function $f(z)$ satisfies the inequality*

$$|f(x + iy)| \leqslant C \exp(a |x|^h + b |y|^{\gamma}) \quad (h \leqslant \gamma), \tag{8}$$

then for any $q = 0, 1, 2, \ldots,$

$$|f^{(q)}(x)| \leqslant CB^q q^{q(1-(1/\gamma))} \exp(a_1 \mid x \mid^h),$$

wherein a_1 *differs as slightly as desired from* a.

Instead of the inequality (8), it is possible to consider the inequality

$$\mid x^k f(x + iy)| \leqslant C_k \exp(b \mid y \mid^\gamma); \tag{9}$$

then the inequalities

$$\mid x^k f^{(q)}(x)| \leqslant C'_k B^q q^{q(1-(1/\gamma))}$$

will be satisfied for the function $f(x)$ so that the function $f(x)$ will be an element of the space $S^{1-(1/\gamma)}$; if the inequality (8) is replaced by the inequality

$$|f(x + iy)| \leqslant C(1 + \mid x \mid)^h \exp(b \mid y \mid^\gamma),$$

then the function $f(x)$ will be a multiplier in this space.

Theorem 4. *If an analytic function* $f(z)$ *in the domain* (5) *with* $\mu \leqslant 0$ *satisfies the inequality*

$$|f(x + iy)| \leqslant C \exp(a \mid x \mid^h),$$

then

$$|f^{(q)}(x)| \leqslant C'B^q q^{q(1-(\mu/h))} \exp(a' \mid x \mid^h), \tag{10}$$

where a' *has the same sign as* a.

Therefore, for $a < 0$, the function $f(x)$ turns out to be an element of the space $S_{1/h}^{1-(\mu/h)}$; for $a > 0$, it will be a multiplier in this space. The function $\exp(a \mid x \mid^h)$ may be replaced by $(1 + \mid x \mid)^h$ in the conditions; then in place of (10), the inequality

$$|f^{(q)}(x)| \leqslant C'B^q q^q(1 + \mid x \mid^{h-\mu q})$$

will hold.

In all cases, the estimates given are obtained as constants.

7.2. Phragmen-Lindelöf Theorem

Now, let us turn to the proofs. The foundation for all the constructions in this paragraph will be the well-known Phragmen-Lindelöf theorem with whose exposition we indeed begin.

The Phragmen-Lindelöf theorem is based on the classical property of the maximum of analytic functions (if the analytic function does not exceed a constant on the boundary of a bounded domain, then it will also not exceed the constant within this domain) and is a generalization of this property to domains extending to infinity.

Phragmen-Lindelöf Theorem. *If an analytic function $f(z)$, defined within and on the sides of an angle G_θ with aperture $\theta < \pi/p$, satisfies the inequality*

$$|f(z)| \leqslant C \exp(b \mid z \mid^p) \tag{1}$$

(i.e., has exponential growth with order $\leqslant p$ and type $\leqslant \rho$ within the angle G_θ) *and is bounded on the sides of this angle by some constant, say C_1, then it is bounded by the same constant C_1 within the angle G_θ also.*

Proof. Without limiting the generality, we may consider the angle G_θ to be bounded by the rays $\arg z = \pm\theta/2$. Let us find a number p_1 satisfying the inequality

$$\theta < \frac{\pi}{p_1} < \frac{\pi}{p}.$$

Let us consider that branch of the function

$$F_\epsilon(z) = \exp(-\epsilon z^{p_1}) \quad (|\arg z| \leqslant \theta/2),$$

which takes positive values on the real axis.

Let us construct the function

$$f_\epsilon(z) = f(z) \cdot F_\epsilon(z).$$

Let us show that within the limits of the angle G_θ, the function $f_\epsilon(z)$ is bounded.

In fact, on the sides of the angle G_θ,

$$|f_\epsilon(z)| = |f(r \exp[\pm i(\theta/2)]) F_\epsilon(r \exp[\pm i(\theta/2)])| \leqslant C_1 \exp[-\epsilon r^{p_1} \cos p_1(\theta/2)] \leqslant C_1,$$

since by assumption $p_1(\theta/2) < \pi/2$, $\cos p_1(\theta/2) > 0$. Within the angle G_θ, we have on the arc of the circle $z = re^{i\omega}$, $|\omega| \leqslant \theta/2$:

$$\begin{aligned} |f_\epsilon(re^{i\omega})| = |f(re^{i\omega}) \cdot F_\epsilon(re^{i\omega})| &\leqslant C \exp(br^p - \epsilon r^{p_1} \cos p_1\omega) \\ &\leqslant C \exp(br^p - \epsilon r^{p_1} \cos p_1(\theta/2)) \to 0 \end{aligned}$$

as $r \to \infty$, since $p_1 > p$. Hence, for sufficiently large r, we also have

$$|f_\epsilon(re^{i\omega})| \leqslant C_1.$$

Thus, on the contour formed from segments of the two rays $\omega = \pm\theta/2$ and the arc of the circle of radius r, the function $f_\epsilon(z)$ is bounded by a constant C_1. But then, by virtue of the classical maximum principle, the function $f_\epsilon(z)$ is bounded by this same constant within the contour also.

Therefore, the inequality

$$|f(z)| = |f_\epsilon(z) \cdot F_\epsilon^{-1}(z)| \leqslant C_1 \, | \exp(\epsilon z^{p_1})|$$

is satisfied at every inner point of the angle G_θ. Since $\epsilon > 0$ is arbitrary, then

$$|f(z)| \leqslant C_1,$$

q.e.d.

Remark. We shall also need later the following generalization of the Phragmen-Lindelöf theorem.

If an analytic function $f(z)$, defined within and on the sides of the angle G_θ with aperture $\theta < \pi/p$, satisfies the inequalities

$$|f(z)| \leqslant C \exp(b \, | z |^p) \qquad \textit{within the angle} \quad G_\theta, \tag{2}$$

$$|f(z)| \leqslant C_1(1 + | z |^h) \qquad \textit{on the sides of the angle} \quad G_\theta, \tag{3}$$

then

$$|f(z)| \leqslant C_1'(1 + | z |^h) \quad \text{for} \quad z \in G_\theta. \tag{4}$$

For the proof, let us consider a polynomial $P(z)$ of degree h, which has no roots within nor on the boundary of the angle G_θ; the function

$$F(z) = f(z)/P(z)$$

satisfies all the conditions of the Phragmen-Lindelöf theorem, and therefore, is bounded in the domain G_θ. Hence, in G_θ

$$|f(z)| = |F(z) \, P(z)| \leqslant C_1'(1 + |z|^h),$$

q.e.d.

Analogously, if the inequality

$$|f(z)| \leqslant C_1 \exp(b_1 \, | z |^h), \qquad h < p, \tag{5}$$

is satisfied instead of (3) on the sides of the angle, then by applying the same recipe, we obtain that *the inequality*

$$|f(z)| \leqslant C_1' \exp(b_1' \, | z |^h) \tag{6}$$

is satisfied in the whole domain G_θ.

7.3. Theorem on the Existence of the Domain G_μ

Theorem 1. *If an entire function $f(z)$ has order of growth $\leqslant p$ with a finite type, i.e., satisfies the inequality*

$$|f(z)| \leqslant C_1 \exp(b\,|z|^p) \tag{1}$$

for all z, and in addition, satisfies for real $z = x$, the inequality

$$|f(x)| \leqslant C_2 \exp(a\,|x|^h) \quad (a \neq 0,\ 0 < h \leqslant p), \tag{2}$$

then there exists a domain G_μ, defined by the inequality

$$|y| \leqslant K_1(1 + |x|)^\mu, \qquad \mu \geqslant 1 - (p - h), \tag{3}$$

in which

$$|f(x + iy)| \leqslant C_3 \exp(a'\,|x|^h), \qquad C_3 = \max(C_1, C_2), \tag{4}$$

where the constant a' differs as little as desired from a.

Therefore, a function of finite order of growth, which has slower growth on the real axis than in the whole plane (or an exponential decrease), retains this slower growth (or decrease) in some domain of the form (3).

Proof. In the right half-plane, let us construct the analytic function

$$f_1(z) = f(z) \exp(-az^h)$$

with an arbitrarily fixed branch of the second factor. This function is bounded on the $x > 0$ half-axis (by the constant C_2). In the right half-plane, it satisfies the inequality

$$|f_1(z)| \leqslant C_1 \exp(b_1\,|z|^p),$$

where it is possible to assume $b_1 = b + |a|$.

Furthermore, let us introduce the function

$$f_2(z) = f_1(z) \exp(ib_2\, z^p), \quad b_2 = b_1 + \epsilon,$$

which is analytic in the first quadrant; this function is also bounded on the $x > 0$ half-axis (by the constant C_2). Moreover, it is also bounded on the ray $z = r \exp[i\pi/2p]$, since on this ray

$$\left| f_2\left(r \exp\left(i\frac{\pi}{2p}\right)\right)\right| \leqslant C_1 \exp(b_1 r^p - b_2 r^p) \leqslant C_1 .$$

Finally, on the limits of the angle $0 \leqslant \arg z \leqslant \pi/2p$, the function $f_2(z)$ satisfies the inequality

$$|f_2(z)| \leqslant C_1 e^{b_3 r^p}, \quad b_3 = 2b_1 + \epsilon.$$

Hence, by virtue of the Phragmen-Lindelöf theorem, the function $f_2(z)$ is also bounded within the mentioned angle.

It hence follows that within the limits of this angle ($z = re^{i\omega}$, $0 \leqslant \omega \leqslant \pi/2p$) the function $f(z)$ satisfies the inequality

$$\begin{aligned} |f(re^{i\omega})| &= |f(z)| = |f_1(z)\exp(az^h)| = |f_2(z)\exp(az^h - ib_2 z^p)| \\ &\leqslant C_3 \exp(ar^h \cos h\omega + b_2 r^p \sin p\omega). \end{aligned} \tag{5}$$

If we consider only the points $z = re^{i\omega}$ such that

$$ar^h \cos h\omega + b_2 r^p \sin p\omega \leqslant a_1 r^h \cos^h \omega \tag{6}$$

($a_1 > a$ and of the same sign), then the function $f(z)$ will also satisfy the inequality

$$|f(z)| \leqslant C_3 \exp(a_1 r^h \cos^h \omega) = C_3 \exp(a_1 x^h). \tag{7}$$

The inequality (6) may be written as

$$a_1 r^h \cos^h \omega - ar^h \cos h\omega - b_2 r^p \sin p\omega \geqslant 0,$$

from which it follows that it is satisfied in a domain having the curve

$$r^{p-h} = \frac{a_1 \cos^h \omega - a \cos h\omega}{b_2 \sin p\omega} \tag{8}$$

for its boundary.

If $h = p$, this curve is the ray $a_1 \cos^h \omega - a \cos h\omega = b_2 \sin p\omega$.

If $h < p$, then the argument ω tends to zero on this curve as $r \to \infty$, since the numerator on the right of (8) is bounded. It may hence be assumed that

$$\sin p\omega = p\frac{y}{x}E_1, \quad \cos^h \omega = E_2, \quad \cos h\omega = E_3, \quad r = xE_4,$$

where the E_i are variables approaching one; substituting into (5), we obtain the equation

$$x^{p-h} = KE \cdot \frac{x}{y} \quad \left(K = \frac{a_1 - a}{b_2}\right),$$

from which

$$y = KEx^{1-(p-h)}; \qquad E(x) \to 1 \quad \text{for} \quad x \to \infty.$$

Evidently the curve (8) has not more than one point of intersection with each ray $\omega = \text{const}$.

Approaching the origin, the curve (8) intersects the ray $\omega = \pi/2p$ at some point (possibly at the origin). Thus, for $h < p$, the inequality (7) will be satisfied in a domain bounded from above, possibly by the ray $\omega = \pi/2p$ through the origin, and then by the curve (8).

Evidently the function $f(z)$ satisfies inequality (7) in a domain defined by the inequalities

$$y \leqslant \min\left(x \operatorname{tg}\frac{\pi}{2p}, \quad K_1 x^{1-(p-h)}\right) \tag{9}$$

for some K_1, depending only on a, a_1, b_2. Let us note that the constant a_1 may be taken as close as desired to a.

A similar construction may also be made in all the remaining quadrants of the z plane; the obtained inequalities will differ from the inequality (9) in that $|x|$ and $|y|$ will figure in them in place of x and y.

From continuity considerations, it is clear that *in the domain defined by the inequality* (see Fig. 2)

$$|y| \leqslant K_1'(1 + |x|)^{1-(p-h)},$$

there will be valid an inequality analogous to the one proved, namely, that

$$|f(x + iy)| \leqslant C' \exp(a' |x|^h)$$

will hold. As the constant C', it is possible to take C_3 multiplied by $\exp(br^p + |a'| r^h)$, where r is the radius of the least circle enclosing the adjoined part of the domain; this constant is expressed in terms of p, h, a, b. Theorem 1 is thereby proved completely.

Remark. It is possible that the inequality

$$|f(x + iy)| \leqslant C \exp(a' |x|^h)$$

is also satisfied in a wider domain than that mentioned in the formulation of Theorem 1, namely in the domain

$$|y| \leqslant K(1 + |x|)^\mu,$$

where $\mu > 1 - (p - h)$. But in every case, *if* $h < p$ and p *is the exact* *order of growth of the function $f(z)$ in the plane, or if $h = p$ and $a < 0$, then the number μ may not exceed unity.*

In fact, if we had $\mu > 1$, then the entire function $f(z)$ would have a lesser order of growth on each ray through the origin (with the possible exception of the y axis) than its total order of growth in the plane. But then, by virtue of the Phragmen-Lindelöf theorem (see the remark after this theorem), the function $f(z)$ would have an order of growth $\leqslant h < p$ for $h < p$, in contradiction to the assumption. But for $h = p$, $a < 0$, by virtue of the same theorem, the function $f(z)$ would turn out to be bounded, and therefore, identically zero.

For $a = 0$, the function of exponential growth $\exp(a \mid x \mid^h)$, in the inequality (2) is naturally replaced by a function of power growth $(1 + \mid x \mid)^h$. The formulation of the theorem changes as follows.

Theorem 1′. *If an entire function $f(z)$ satisfies the inequalities*

$$|f(z)| \leqslant C_1 \exp(b \mid z \mid^p), \qquad |f(x)| \leqslant C_2(1 + \mid x \mid)^h,$$

then there exists a domain G_μ, defined by the inequality

$$\mid y \mid \leqslant K_1(1 + \mid x \mid)^\mu, \qquad \mu \geqslant 1 - p, \tag{10}$$

in which

$$|f(x + iy)| \leqslant C_3(1 + \mid x \mid)^h. \tag{11}$$

The proof proceeds according to the scheme of the proof of Theorem 1. The function $\exp(-az^h)$ is replaced by $(1 + z)^{-h}$, and the inequality (5) by the inequality

$$|f(re^{i\omega})| \leqslant C_3(1 + \mid z \mid)^h \exp(b_2 r^p \sin p\omega). \tag{12}$$

The exponent $b_2 r^p \sin p\omega$ is bounded by the curve $y \leqslant x^{1-p}$; hence, the inequality (11) is known to be satisfied in the domain (10).

Exactly as in the preceding case, the domain (10) may be replaced by the domain

$$\mid y \mid \leqslant \min\left(x \tan \frac{\pi}{2p}, K_1 x^{1-p}\right). \tag{13}$$

7.4. Behavior of an Entire Function in a Plane for $\mu > 0$

Theorem 2. *Let us consider the entire analytic function $f(z)$ of order of growth $\leqslant p$ and finite type, such that the inequality*

$$|f(z) \leqslant C \exp(b \mid z \mid^p) \tag{1}$$

is satisfied. Furthermore, let us assume that in some domain G, defined by the inequality

$$|y| \leqslant K_1(1 + |x|)^\mu \qquad (0 < \mu \leqslant 1), \tag{2}$$

the function $f(z)$ satisfies the inequality

$$|f(z)| \leqslant C_1 \exp(a|x|^h) \tag{3}$$

($h < p$; for $a < 0$ $h = p$ is also admissible).

Then the function $f(z)$ satisfies the inequality

$$|f(x + iy)| \leqslant C_2 \exp(a|x|^h + b'|y|^{p/\mu}), \tag{4}$$

for all $z = x + iy$, where the constant b' only depends on a, b and K_1, $C_2 = \max(C, C_1)$.

Proof. Let us put

$$M(y) = \sup_x |f(x + iy)| \exp(-a|x|^h). \tag{5}$$

The upper bound mentioned above exists since for $\mu > 0$, each horizontal line belongs almost entirely, except for some finite segment Δ_y, to the domain G defined by the inequality (2); the expression under the sup symbol is bounded by the constant C_1 in the domain G.

One of two things is possible: Either the mentioned upper bound is reached on the segment Δ_y for any given y or it is not reached on this segment.

Let us consider the first case. Let the upper bound in (4) be reached outside the domain G for a given $y = \bar{y}$, i.e., at the point $\bar{x} + i\bar{y}$ satisfying the inequality

$$|\bar{y}| > K_1(1 + |\bar{x}|)^\mu > K_1|\bar{x}|^\mu.$$

Hence, by utilizing the inequality $\mu \leqslant 1$, we obtain[10]

$$|\bar{x}| < K_2|\bar{y}|^{1/\mu},$$
$$|\bar{z}| = (|\bar{x}|^2 + |\bar{y}|^2)^{1/2} \leqslant (K_2^2|\bar{y}|^{2/\mu} + |\bar{y}|^2)^{1/2} \leqslant K_3|\bar{y}|^{1/\mu}.$$

Hence, replacing $-a|\bar{x}|^h$ with $a < 0$ by a greater quantity, $|a|K_4|\bar{y}|^{h/\mu}$, or simply by zero for $a \geqslant 0$, we obtain

$$M(\bar{y}) = |f(\bar{x} + i\bar{y})| \exp(-a|\bar{x}|^h) \leqslant C \exp(b|\bar{z}|^p + |a|K_4|\bar{y}|^{h/\mu}),$$

[10] It has here been assumed that $|y| \geqslant 1$. It may always be assumed that the domain G contains the strip $|y| \leqslant 1$; otherwise $f(z)$ is replaced by $f(\alpha z)$ with a suitable α.

and moreover

$$M(\bar{y}) \leqslant C \exp(b_1 \mid \bar{y} \mid^{p/\mu} + b_2 \mid \bar{y} \mid^{h/\mu}) \leqslant C \exp(b_3 \mid \bar{y} \mid^{p/\mu}),$$

since $h \leqslant p$. But then for any x,

$$\mid f(x + i\bar{y}) \mid \exp(-a \mid x \mid^h) \leqslant M(\bar{y}) \leqslant C \exp(b_3 \mid \bar{y} \mid^{p/\mu}),$$

and therefore

$$\mid f(x + i\bar{y}) \mid \leqslant C \exp(a \mid x \mid^h + b_3 \mid \bar{y} \mid^{p/\mu}).$$

There remains to consider the case when the upper bound in (5) is not reached in the segment Δ_y for given $y = \bar{y}$. This means that a point $\bar{x} + i\bar{y}$ exists in the domain G at which the quantity $\mid f(x + i\bar{y}) \mid \exp(-a \mid x \mid^h)$ exceeds its maximum in the segment Δ_y. But, as follows from (3), the mentioned quantity is bounded everywhere in the domain G by the constant C_1. Hence, for all x and given $\bar{y}$,

$$\mid f(x + i\bar{y}) \mid \exp(-a \mid x \mid^h) \leqslant C_1,$$

from which

$$\mid f(x + i\bar{y}) \mid \leqslant C_1 \exp(a \mid x \mid^h) \leqslant C_1 \exp(a \mid x \mid^h + b_3 \mid y \mid^{p/\mu}).$$

Hence, as is seen from the proof $b_3 \leqslant b_1 + b_2 \leqslant (\mid a \mid + b) K_5$, where K_5 depends only on K_1. Theorem 2 is thereby proved completely.

Exactly as in Theorem 1′, for $h = 0$, it is natural to replace the exponential estimate (11) by a power estimate

$$\mid f(z) \mid \leqslant C_2(1 + \mid x \mid^h).$$

In this case, the following theorem holds.

Theorem 2′. *If an entire function $f(z)$ satisfies the inequality*

$$\mid f(z) \mid \leqslant C \exp(b \mid z \mid^p) \tag{6}$$

for all z, and the inequality

$$\mid f(z) \mid \leqslant C_1(1 + \mid x \mid^h), \tag{7}$$

is satisfied in the domain $\mid y \mid \leqslant K_1(1 + \mid x \mid)^\mu$ $(0 < \mu < 1)$, then for all z

$$\mid f(z) \mid \leqslant C_2(1 + \mid x \mid^h) \exp(b' \mid y \mid^{p/\mu}), \tag{8}$$

where $b' \leqslant bB_3$, in which B_3 depends only on b_1 and K_1.

The proof proceeds according to the scheme of the proof of Theorem 2, with $\exp(a \mid x \mid^h)$ replaced everywhere by $(1 + \mid x \mid^h)$.

7.5. Estimates of the Derivatives of an Entire Function on the Real Axis by Its Behavior in a Plane

Theorem 3. *If an entire analytic function $f(z)$ satisfies the inequality*

$$|f(x + iy)| \leqslant C \exp(a \mid x \mid^h + b \mid y \mid^\gamma) \quad (h \leqslant \gamma), \tag{1}$$

then for any $q = 0, 1, 2, \ldots$

$$|f^{(q)}(x)| \leqslant CB^q q^{q(1-(1/\gamma))} \exp(a_1 \mid x \mid^h), \tag{2}$$

where a_1 differs as little as desired from a.

Proof. The derivatives of the function $f(z)$ may be evaluated by means of the Cauchy formula,

$$f^{(q)}(x) = \frac{q!}{2\pi i} \int_{\Gamma_R} \frac{f(\xi)\, d\xi}{(\xi - x)^{q+1}}, \tag{3}$$

where Γ_R is a circle of radius R with center at the point x. From (3) we obtain that

$$|f^{(q)}(x)| \leqslant \frac{q!}{R^q} C \exp(bR^\gamma + a \mid x_1 \mid^h), \tag{4}$$

where x_1 is a point between the values $x - R$ and $x + R$, at which the quantity $a \mid x \mid^h$ attains its maximum.[11]

Let us select the radius R so that the ratio $\exp(bR^\gamma)/R^q$ would attain its minimum. As may easily be verified by differentiation, this is realized for

$$R = \left(\frac{q}{b\gamma}\right)^{1/\gamma},$$

so that (4) reduces to

$$|f^{(q)}(x)| \leqslant CB_1{}^q q! q^{-(q/\gamma)} \exp(a \mid x_1 \mid^h), \quad B_1 = (be\gamma)^{1/\gamma}. \tag{5}$$

The last factor may be estimated as follows. Let us replace x_1 by $x + \theta R$, where $\mid \theta \mid \leqslant 1$; then

$$\exp(a \mid x_1 \mid^h) = \exp(a \mid x + \theta R \mid^h) \leqslant \exp(a_1 \mid x \mid^h) \exp(a_2 R^h),$$

[11] One of three cases is possible: $x_1 = x - R$, $x_1 = 0$, or $x_1 = x + R$.

where the constant a_1 may be chosen so that it differs as little as desired from a. Furthermore

$$R^h = \left(\frac{q}{b\gamma}\right)^{h/\gamma} = C_1 q^{h/\gamma} \leqslant C_1 q,$$

since by assumption $h \leqslant \gamma$. Hence (5) is reduced to

$$|f^{(q)}(x)| \leqslant C_2 B^q q^{q(1-(1/\gamma))} \exp(a_1 \mid x \mid^h),$$

which agrees with the required inequality (2).

If $a < 0$ (i.e., $a_1 < 0$), this means that $f(x) \in S_{1/h}^{1-1/\gamma}$.

If $a > 0$ (i.e., $a_1 > 0$) and may be taken as small as desired, then by virtue of the results of Section 4.2, *the function $f(x)$ is a multiplier in the space $S_{1/h}^{1-1/\gamma}$.*

For example, let us consider the function

$$f(z) = \exp(-z^2).$$

Since

$$|\exp(-z^2)| = \exp(-x^2 + y^2),$$

then the inequality (1) is satisfied with exponents $h = \gamma = 2$ and $a < 0$, hence, by virtue of Theorem 3, the function $\exp(-x^2)$ belongs to the space $S_{1/2}^{1/2}$ and therefore satisfies the inequalities

$$|x^k(\exp(-x^2))^{(q)}| \leqslant CA^k B^q k^{k/2} q^{q/2}.$$

If it is only known about some function $f(z)$ that it has an order of growth p in the z plane, and has an exponential decrease of order h on the x axis (i.e., inequality (1) is satisfied with $a < 0$), where this exponential decrease is retained in the domain $|y| \leqslant K(1 + |x|)^\mu$, $0 < \mu \leqslant 1$ (which is known to hold in particular for $\mu = p - h \leqslant 1$ by virtue of Theorem 1), then by combining Theorems 2 and 3, we obtain that the function $f(x)$ belongs to the space S_α^β, where $\alpha = 1/h$, $\beta = 1 - \mu/h$.

Remark. Let be given an entire function $f(z)$ satisfying the inequalities

$$|f(z)| \leqslant C \exp(b \mid z \mid^p), \qquad |f(x)| \leqslant C_1 \exp(-a \mid x \mid^p) \quad (p > 1). \qquad (6)$$

The authors first considered the set of such entire functions and denoted it by $Z_p{}^p$. By virtue of Theorem 1 for the function $f(z) \in Z_p{}^p$, the number μ may be taken equal to 1. Moreover, the number $\gamma = p/\mu$ in Theorem 2 turns out to be equal to p. Hence, according to Theorem 3, the function $f(z)$ belongs to the space $S_{1/p}^{1-1/p}$. Thus, the imbedding

$$Z_p^{\ p} \subset S_{1/p}^{1-1/p}$$

holds. On the other hand, each function $\varphi(x)$ from the space $S_{1/p}^{1-1/p}$ will satisfy according to Section 2, the inequality

$$|\varphi(x+iy)| \leqslant C \exp(b\,|y|^p - a\,|x|^p),$$

so that the inequalities (6) are known to be satisfied. Thus, the space $Z_p^{\ p}$ coincides with the space $S_{1/p}^{1-1/p}$.

A theorem analogous to Theorem 3 may be formulated also for the case when the function $f(z)$ is subject to a slower law of decrease on the x axis than an exponential law.

Theorem 3′. *If an entire analytic function $f(z)$ satisfies for any k the inequality*

$$|x^k f(x+iy)| \leqslant C_k \exp(b\,|y|^\gamma) \qquad (\gamma > 1), \tag{7}$$

then for any $q = 0, 1, 2,\ldots$

$$|x^k f^{(q)}(x)| \leqslant C_k' B^q q^{q(1-(1/\gamma))}, \tag{8}$$

where $B = (1/e)(b'e\gamma)^{1/\gamma}$, b' is any constant greater than b.

Proof. Let us put $f_k(z) = z^k f(z)$; the function $f_k(z)$ satisfies the inequality

$$\begin{aligned} |f_k(z)| = |z^k f(z)| &\leqslant 2^k[|x|^k + |y|^k]\,|f(z)| \\ &\leqslant 2^k C_k \exp(b\,|y|^\gamma) + 2^k C_0\,|y|^k \exp(b\,|y|^\gamma) \\ &\leqslant C_k' \exp(b'\,|y|^\gamma), \end{aligned}$$

where b' is any constant greater than b. Let us apply Theorem 3 to the function $f_k(z)$ by putting $h = 0$. We hence obtain

$$|f_k^{(q)}(x)| \leqslant C_h' B^q q^{q(1-(1/\gamma))}, \tag{9}$$

where $B = (1/e)(b'e\gamma)^{1/\gamma}$. But on the other hand

$$f_k^{(q)}(x) = [x^k f(x)]^{(q)} = x^k f^{(q)}(x) + kqx^{k-1} f^{(q-1)}(x) + \cdots.$$

Let us prove the inequality (8) by induction over k. For $k = 0$, it coincides with the proved inequality (9). In the general case, by putting $\beta = 1 - (1/\gamma)$ for brevity, we find

$$
\begin{aligned}
| x^k f^{(q)}(x)| &\leqslant | f_k^{(q)}(x)| + kq \, | x^{k-1} f^{(q-1)}(x)| \\
&\quad + \frac{k(k-1)\, q(q-1)}{1 \cdot 2} \, | x^{k-2} f^{(q-2)}(x)| + \cdots \\
&\leqslant C'_k B^q q^{q\beta} + kq C'_{k-1} B^{q-1} (q-1)^{(q-1)\beta} \\
&\quad + \frac{k(k-1)\, q(q-1)}{1 \cdot 2} \, C'_{k-2} B^{q-2} (q-2)^{(q-2)\beta} + \cdots \\
&\leqslant C''_k B^q q^{q\beta} \Big(1 + q \, \frac{(q-1)^{(q-1)\beta}}{q^{q\beta}} + \frac{1}{1 \cdot 2} \, q \, \frac{(q-1)^{(q-1)\beta}}{q^{q\beta}} \\
&\qquad \times (q-1) \, \frac{(q-2)^{(q-2)\beta}}{(q-1)^{(q-1)\beta}} + \cdots \Big) \\
&\leqslant C''_k B^q q^{q\beta} \Big(1 + a_q + \frac{1}{1 \cdot 2} \, a_q \cdot a_{q-1} + \cdots \Big),
\end{aligned}
$$

where we have put

$$
a_q = q \, \frac{(q-1)^{(q-1)\beta}}{q^{q\beta}} .
$$

But for $q \geqslant 2$, we have

$$
a_q = \frac{q}{(q-1)^\beta} \left(1 - \frac{1}{q}\right)^{q\beta} \leqslant \frac{q}{(q-1)^\beta} \leqslant 2q^{1-\beta};
$$

hence

$$
\begin{aligned}
1 + a_q + \frac{1}{1 \cdot 2} \, a_q a_{q-1} + \cdots &\leqslant 1 + 2q^{1-\beta} \\
&\quad + \frac{1}{1 \cdot 2} (2q^{1-\beta})^2 + \cdots \leqslant \exp(2q^{1-\beta}).
\end{aligned}
$$

But since $\beta < 1$, for any $\delta > 0$

$$
\exp(2q^{1-\beta}) \leqslant C_\delta (1 + \delta)^q,
$$

from which

$$
| x^k f^{(q)}(x)| \leqslant C'''_k (B + \delta')^q \, q^{q\beta},
$$

q.e.d.

Theorem 3′ shows that *the function* $f(x)$ *satisfying the inequality* (6) *belongs to the space* $S^{\beta, B}$, *where*

$$
\beta = 1 - \frac{1}{\gamma}, \qquad B = \frac{1}{e} (be\gamma)^{1/\gamma}.
$$

Let us recall that the converse statement was proved in Section 3: Every function $\varphi(x)$ in the space $S^{\beta,B}$ ($\beta < 1$), satisfies inequalities of the form (7). Therefore, the inequality (7) yields a complete characterization of the fundamental functions in the space $S^{\beta,B}$.

The following supplement to Theorem 3′ may now be made.

Theorem 3″. *If an entire analytic function $f(z)$ satisfies the inequality*

$$|f(x+iy)| \leqslant C(1+|x|)^h \exp(b|y|^\gamma),$$

then it is a bounded multiplication operator in the space S^{β,B_1} and transforms this space into $S^{\beta,B'}$, where

$$(B_1 e)^\gamma = b_1 e\gamma, \qquad (B'e)^\gamma = (b_1 + b)\, e\gamma.$$

Proof. Let $\varphi(x) \in S^{\beta,B_1}$; then for any $k = 0, 1, 2,\ldots$

$$|x^k \varphi(x+iy)| \leqslant C_k \exp(b_1 |y|^\gamma),$$

where b_1 and B_1 are connected by the relation $(eB_1)^\gamma = b_1' e\gamma$, $b_1' > b_1$ is arbitrary. Hence

$$|x^{k+h} f(x+iy)\,\varphi(x+iy)| \leqslant C_k'(1+|x|)^h \exp((b+b_1)|y|^\gamma)$$

and therefore, for any k,

$$|x^k f(x+iy)\,\varphi(x+iy)| \leqslant C_k'' \exp((b+b_1)|y|^\gamma).$$

Therefore, by virtue of Theorem 3, the product $f\varphi$ belongs to the space $S^{\beta,B'}$. By virtue of the relation between the constants, the operation of multiplication by f is a bounded operator on S^{β,B_1}, q.e.d.

7.6. Estimates of the Derivatives on the Real Axis for $\mu \leqslant 0$

Theorem 4. *If an analytic function $f(z)$ is defined in the domain G*

$$|y| \leqslant K(1+|x|)^\mu \quad (\mu \leqslant 0) \tag{1}$$

and satisfies the inequality

$$|f(x+iy)| \leqslant C \exp(a|x|^h) \tag{2}$$

in this domain, then its successive derivatives on the real axis satisfy the inequalities

$$|f^{(q)}(x)| \leqslant C' B^q q^{q(1-(\mu/h))} \exp(a'|x|^h), \tag{3}$$

where the constant a' has the same sign as a.

Proof. Let us calculate the value of the derivatives of the function $f(x)$ by means of the Cauchy formula by utilizing a circle lying entirely within the domain G. First of all, we assert that a circle with center at x and with radius $R = K_2(1 + |x|)^\mu$ lies, for sufficiently small K_2, entirely within the domain G. In fact, this statement is evident for $\mu = 0$; for $\mu < 0$, the situation reduces to the proof of the inequality

$$K_2(1 + |x|)^\mu \leqslant K_1(1 + |x| + R)^\mu,$$

by virtue of the monotone decrease of the function $(1 + |x|)^\mu$. Since R is bounded (it may be assumed that $R \leqslant 1$), then this inequality is known to be satisfied for sufficiently small K_2.

Let us now apply the Cauchy formula

$$f^{(q)}(x) = \frac{q!}{2\pi i} \int_{\Gamma_R} \frac{f(\xi)\, d\xi}{(\xi - x)^{q+1}},$$

where Γ_R is the circle with center at the point x and with radius R. Substituting the estimate of $f(\xi)$ in formula (2), we find

$$|f^{(q)}(x)| \leqslant Cq!R^{-q} \exp(a |x_1|^h),$$

where x_1 is the point in the interval $(x - R, x + R)$, at which the function $\exp(a |x|^h)$ attains its maximum. Since R is a bounded function of x, it may be considered that

$$\exp(a |x_1|^h) \leqslant C_3 \exp(a_1 |x|^h),$$

where a_1 is a constant of the same sign as a. For $|x| > 1$, the inequality

$$R = K_2(1 + |x|)^\mu \geqslant K_3 |x|^\mu$$

holds, so that

$$|f^{(q)}(x)| \leqslant C_4 q! |x|^{-\nu q} \exp(a_1 |x|^h). \tag{4}$$

Furthermore, it is easy to show by differentiation that

$$|x|^{-\nu q} \exp(a_1 |x|^h) \leqslant B^q q^{-\nu q/h} \exp(a_2 |x|^h),$$

where a_2 is a constant of the same sign as a_1 (and a), $B = 1/a_1$. Hence (4) is transformed to

$$|f^{(q)}(x)| \leqslant C_5 B_1{}^q q^{q(1-(\mu/h))} \exp(a_2 |x|^h). \tag{5}$$

For $|x| \leqslant 1$, it may be considered that $R \geqslant \rho > 0$, and hence

$$|f^{(q)}(x)| \leqslant C_6 \rho^{-q} q! \tag{6}$$

The estimates (5) and (6) may be combined into one common estimate valid for all x;

$$|f^{(q)}(x)| \leqslant C_7 B_2{}^q q^{q(1-(\mu/h))} \exp(a_2 \mid x \mid^h),$$

which indeed proves our theorem.

In the $a < 0$ *case* (i.e., $a_2 < 0$), the obtained inequality shows that the *function* $f(x)$ *belongs to the space* $S_{1/h}^{1-\mu/h}$.

In the $a > 0$ *case* (i.e., $a_2 > 0$), *and if* a *may be taken arbitrarily small*, by virtue of one of the theorems of Section 4.2 *the function* $f(x)$ *is a multiplier in the space* $S_{1/h}^{1-\mu/h}$. For $a = 0$, it is natural to replace the function $\exp(a \mid x \mid^h)$ by the power function $1 + \mid x \mid^h$. We hence obtain the following result.

Theorem 4′. *If an analytic function* $f(z)$ *is defined in the domain*

$$|y| \leqslant C_1(1 + |x|)^\mu, \quad \mu < 0, \tag{7}$$

and satisfies the inequality

$$|f(x + iy)| \leqslant C(1 + |x|^h), \tag{8}$$

in this domain, then its successive derivatives on the real axis satisfy the inequalities

$$|f^{(q)}(x)| \leqslant C' B^q q^q (1 + |x|^{h-\mu q}). \tag{9}$$

The proof is analogous to the proof of Theorem 4 (see inequalities (4) and (6)).

8. The Question of the Nontriviality of Spaces of Type *S*

We speak first about the following question: *In a given space of type* S, *is there at least one function* $\varphi(x)$ *not identically zero?*

The answer is always affirmative for the space $S_{\alpha,A}$: Every infinitely differentiable function of compact support is known to belong to such a space if $\alpha > 0$; for $\alpha = 0$, every infinitely differentiable function with support in the domain $|x| \leqslant A$ belongs to the space $S_{\alpha,A}$ $(= S_{0,A})$.

Furthermore, every space of type $S^{\beta,B}$ is the image by the Fourier transformation of the space $S_{\beta,B}$ and this means that it also contains functions not identically zero.

The question of the nontriviality of the spaces S_α^β offers the greatest difficulty. In Sections 8.1 and 2, we show that *these spaces are nontrivial for*

(1) $\alpha + \beta \geqslant 1, \quad \alpha > 0, \quad \beta > 0;$

(2) $\alpha = 0, \quad \beta > 1;$

(3) $\beta = 0, \quad \alpha > 1,$

and are trivial in the remaining cases.

To prove this we use some results obtained in Section 6 and 7, as well as the well-known Carleman-Ostrovski theorem on quasi-analyticity conditions.

8.1. Case of the Spaces $S_\alpha{}^0, S_0{}^\beta$.

Let us first consider the case when one of the numbers α, β is zero.

Theorem. *The space S_0^β is nontrivial* (i.e., it contains the function $\varphi(x) \not\equiv 0$) *if and only if* $\beta > 1$.

Proof. The functions $\varphi(x) \in S_0^\beta$ are characterized by the inequalities

$$|x^k \varphi^{(q)}(x)| \leqslant CA^k B^q q^{q\beta}. \tag{1}$$

Dividing by $|x|^k$ and taking in the right side the lower bound of k, we obtain

$$|\varphi^{(q)}(x)| \leqslant CB^q q^{q\beta} \begin{cases} 1 & \text{for} \quad |x| \leqslant A, \\ 0 & \text{for} \quad |x| > A. \end{cases} \tag{2}$$

It is also evident that, conversely, every infinitely differentiable function $\varphi(x)$ satisfying the inequalities (2) will also satisfy the inequalities (1), i.e., belongs to the space S_0^β. The question of the nontriviality of S_0^β now reduces to the classical problem of quasi-analyticity: What conditions must be imposed on the numbers b_q $(= q^{q\beta})$, so that there would exist an infinitely differentiable function $\varphi(x) \not\equiv 0$, of compact support and satisfying the inequalities

$$|\varphi^{(q)}(x)| \leqslant CB^q b_q\,.$$

As is known, the answer is given by the following Carleman-Ostrovski theorem: For the desired function to exist, it is necessary and sufficient that the Ostrovski function

$$\Gamma(r) = \max_q \frac{r^q}{b_q} \tag{3}$$

possess the property that

$$\int_1^\infty \frac{\ln \Gamma(r)}{r^2}\, dr < \infty. \tag{4}$$

Let us find an estimate for the Ostrovski function, when $b_q = q^{q\beta}$, $\beta \geqslant 0$, such that

$$\Gamma(r) = \max_q \frac{r^q}{q^{q\beta}}\,.$$

For $\beta = 0$, it is evident that $\Gamma(r) = \infty$ for $r > 1$ and the integral (4) will diverge. It is hence possible to limit oneself to the case $\beta > 0$.

In this case, the function $1/\Gamma(r)$ agrees with the function $\mu_\beta(r)$, constructed in Section 2.1. We can utilize the result of the calculation made there, which is given by the inequality

$$\exp\left(-\frac{\beta}{e}\, r^{1/\beta}\right) \leqslant \mu_\beta(r) \leqslant C \exp\left(-\frac{\beta}{e}\, r^{1/\beta}\right).$$

Applying this result, we find

$$\exp(br^{1/\beta}) \leqslant \Gamma(r) \leqslant C_1 \exp(br^{1/\beta}).$$

By virtue of this estimate, the convergence of the integral (3) evidently holds if and only if $\beta > 1$. The theorem is proved.

Corollary. *Since* $\widetilde{S_0{}^\beta} = S_\beta{}^0$ (Section 6), *we simultaneously obtain the triviality of* $S_\alpha{}^0$ *for* $\alpha \leqslant 1$ *and the nontriviality of* $S_\alpha{}^0$ *for* $\alpha > 1$.

8.2. Case of the Spaces $S_\alpha{}^\beta$, $\alpha > 0$, $\beta > 0$

Let us turn to the case when both the numbers α, β are positive. Here the following theorem holds.

Theorem. *The space* $S_\alpha{}^\beta$ *with* $\alpha > 0, \beta > 0$ *is nontrivial* (i.e., will contain the function $\varphi(x) \not\equiv 0$) *if and only if* $\alpha + \beta \geqslant 1$.

Proof. Let us first consider the case $\alpha + \beta < 1$, and let us show that in this case the space S_α^β contains the single function $\varphi(x) \equiv 0$.

As has been shown in Section 2, in the case under consideration the functions $\varphi(x) \in S_\alpha^\beta$ can be continued analytically into the complex domain $z = x + iy$ as entire functions, and the estimate

$$| \varphi(x + iy)| \leqslant C \exp[-a \mid x \mid^{1/\alpha} + b \mid y \mid^{1/(1-\beta)}] \tag{1}$$

holds. Evidently the estimate

$$| \varphi(ix - y)| \leqslant C \exp[-a \mid y \mid^{1/\alpha} + b \mid x \mid^{1/(1-\beta)}]$$

holds for the entire function $\varphi(iz) = \varphi(ix - y)$. For the product $\varphi(z) \cdot \varphi(iz)$ we obtain

$$| \varphi(z) \cdot \varphi(iz)| \leqslant C^2 \exp[-a| x |^{1/\alpha} + b \mid x \mid^{1/(1-\beta)}] \exp[-a \mid y \mid^{1/\alpha} + b \mid y \mid^{1/(1-\beta)}]. \tag{2}$$

The inequality $\alpha + \beta < 1$ shows that $1/\alpha > 1/1 - \beta$. Hence, both factors on the right side of the inequality (2) tend to zero as $| x | \to \infty$, $| y | \to \infty$. Hence, according to the Liouville theorem, the function $\varphi(z) \cdot \varphi(iz)$ is identically zero. But then $\varphi(z) \equiv 0$, also, q.e.d.

There remains for us to consider the case $\alpha + \beta \geqslant 1$, $\alpha > 0$, $\beta > 0$, and to show that the corresponding space S_α^β is nontrivial. Since it is evident that $S_\alpha^\beta \subset S_{\alpha'}^{\beta'}$ follows from the inequalities $\alpha \leqslant \alpha'$, $\beta \leqslant \beta'$, it is then sufficient to limit oneself to the case $\alpha + \beta = 1$, $\alpha > 0$, $\beta > 0$.

Initially let us consider the function

$$\psi(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{n^\rho}\right) \tag{3}$$

Evidently for $\rho > 1$, the product (3) converges everywhere. Let us show that it is an entire function of order of growth $\leqslant 1/\rho$; in other words, the inequality

$$| \psi (z)| \leqslant C_\epsilon \exp[| z |^{(1/\rho)+\epsilon}] \tag{4}$$

is satisfied for any $\epsilon > 0$. First of all we have

$$| \psi(z)| = \prod_{n=1}^{\infty} \left| 1 - \frac{z}{n^\rho} \right| \leqslant \prod_{n=1}^{\infty} \left(1 + \frac{| z |}{n^\rho}\right),$$

$$\ln | \psi(z)| \leqslant \sum_{n=1}^{\infty} \ln \left(1 + \frac{| z |}{n^\rho}\right).$$

Let us utilize the two obvious estimates for the function $\ln(1 + \zeta)$:

$$\ln(1+\zeta) < \zeta \quad \text{for} \quad 0 < \zeta \leqslant 1,$$

$$\ln(1+\zeta) < C_\mu \zeta^\mu \quad \text{for} \quad \zeta \geqslant 1 \quad \text{and any} \quad \mu > 0.$$

In conformity with these estimates, we find

$$\sum_{n=1}^{\infty} \ln\left(1 + \frac{|z|}{n^\rho}\right) < \sum_{n^\rho \geqslant |z|} \frac{|z|}{n^\rho} + C_\mu \sum_{n^\rho \leqslant |z|} \frac{|z|^\mu}{n^{\rho\mu}}$$

$$= |z| \sum_{n^\rho| \geqslant |z|} \frac{1}{n^\rho} + C_\mu |z|^\mu \sum_{n^\rho \leqslant |z|} \frac{1}{n^{\rho\mu}}. \tag{5}$$

Let us estimate the first sum on the right-hand side. Since

$$\frac{1}{n^\rho} \leqslant \int_{n-1}^{n} \frac{dx}{x^\rho},$$

we have

$$\sum_{n^\rho \geqslant |z|} \frac{1}{n^\rho} \leqslant \int_{|z|^{1/\rho}-1}^{\infty} \frac{dx}{x^\rho} = \frac{1}{(\rho - 1)[|z|^{1/\rho} - 1]^{\rho-1}} \leqslant C|z|^{(1/\rho)-1},$$

in which the constant C may be selected as fixed for all sufficiently large $|z|$. The last sum on the right-hand side of the relation (5) is bounded for any $\mu > 1/\rho$:

$$\sum_{n^\rho \leqslant |z|} \frac{1}{n^{\rho\mu}} \leqslant \sum_{n=1}^{\infty} \frac{1}{n^{\rho\mu}} = C_0 < \infty.$$

As a result, we arrive at the estimate

$$\ln|\psi(z)| \leqslant C|z|^{1/\rho} + C'_\mu |z|^\mu \leqslant C''_\mu |z|^\mu.$$

Since μ may be taken in the form $(1/\rho) + (\epsilon/2)$, then the estimate

$$\ln|\psi(z)| \leqslant C_\epsilon |z|^{(1/\rho)+(\epsilon/2)} \leqslant C'_\epsilon + |z|^{(1/\rho)+\epsilon}$$

is valid, from which the required inequality (4) also follows.

Furthermore, we shall construct the growth index of the function $\psi(z)$. Let us recall how the growth index of an entire function of order of growth p is defined. Let us fix a ray through the origin at an angle θ to the x axis. If constants C and b exist such that the inequality

$$|f(re^{i\theta})| \leqslant C \exp(br^p)$$

is satisfied, then the value of the growth index $h(\theta)$ is considered to be the lower bound of the numbers b, which satisfy this inequality. Therefore, for any $\epsilon > 0$, a constant C_ϵ may be found such that

$$|f(re^{i\theta})| \leqslant C_\epsilon \exp([h(\theta) + \epsilon]r^p).$$

If such constants C and b do not exist, then we put $h(\theta) = \infty$.

The following properties of the growth index of an entire function of order p are proved in function theory.

(a) *If $h(\theta)$ takes on finite values for $\theta = \theta_1$ and $\theta = \theta_2$, where $|\theta_1 - \theta_2| < \pi/\rho$, then $h(\theta)$ has an upper bound in the whole range $\theta_1 \leqslant \theta \leqslant \theta_2$.*

(b) *If $h(\theta)$ is bounded in the range $\theta_1 \leqslant \theta \leqslant \theta_2$, then it is continuous in this range.*[12]

Let us show that the growth index of the function $\psi(z)$ is given by the equation

$$h(\theta) = \frac{\pi}{\sin(\pi/\rho)} \cos \frac{\theta - \pi}{\rho} \qquad (0 \leqslant \theta \leqslant 2\pi). \tag{6}$$

A graph of the function $h(\theta)$ is pictured in Fig. 3.

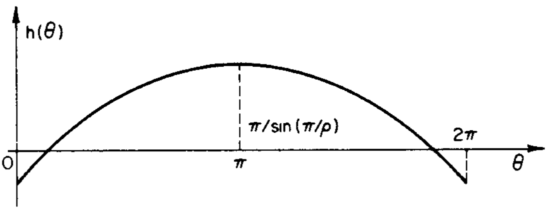

FIG. 3. The function $h(\theta)$.

Keeping in mind the properties (a)–(b) of the index, we limit ourselves to the case $\theta \neq 0$ in the proof.

Furthermore, since $|\psi(\bar{z})| = |\psi(z)|$, the function $h(\theta)$ is even; hence it is sufficient to consider the case $0 < \theta < \pi$.

The function

$$\operatorname{Ln} \psi(z) = \ln |\psi(z)| + i \operatorname{Arg} |\psi(z)|$$

with some fixed $\operatorname{Arg}|\psi(z)|$ may be represented as

$$\operatorname{Ln} \psi(re^{i\theta}) = \sum_{n=1}^{\infty} \ln \left(1 - \frac{z}{n^\rho}\right) = \int_{1-0}^{\infty} \ln \left(1 - \frac{z}{t}\right) dn(t),$$

[12] See, for example, A. I. Markushevich, "Theory of Analytic Functions," Chapter VII, Section 1.4, p. 508, Gostekhizdat, Moscow, 1950.

where the function $n(t)$ has a jump $+1$ at each point $t = n^\rho$ $(n = 1, 2, \ldots)$ and is constant between these points.

Integrating by parts, we find

$$\int_{1-0}^{\infty} \ln\left(1 - \frac{z}{t}\right) dn(t) = n(t) \ln\left(1 - \frac{z}{t}\right)\Big|_{1-0}^{\infty} - \int_{1-0}^{\infty} \frac{(z/t^2)}{1 - (z/t)}\, n(t)\, dt. \tag{7}$$

The function $n(t)$ has the form shown in Fig. 4. Evidently it may be represented as

$$n(t) = t^{1/\rho} - \omega(t),$$

where $0 \leqslant \omega(t) \leqslant 1$.

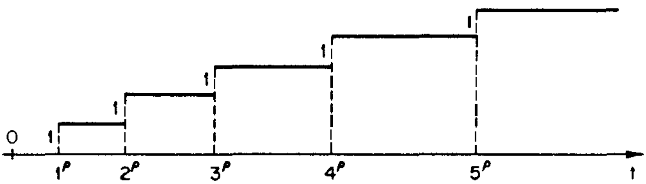

FIG. 4. The function $n(t)$.

Let us first consider the expression outside the integrand in (7). For large t, the factor $\ln(1 - (z/t))$ is of the order $|z|/t$, and hence, yields zero as a limit in the product with $n(t) \sim t^{1/\rho}$. At the lower limit, the term outside the integrand is zero, together with $n(t)$. Thus, the term outside the integral vanishes, and we have

$$\int_{1-0}^{\infty} \ln\left(1 - \frac{z}{t}\right) dn(t) = -z \int_{1}^{\infty} \frac{n(t)\, dt}{t(t - z)}$$

$$= -z \int_{1}^{\infty} \frac{t^{1/\rho}\, dt}{t(t - z)} + z \int_{1}^{\infty} \frac{\omega(t)\, dt}{t(t - z)}.$$

Let us convert both the obtained integrals by substituting $t = ur$:

$$\int_{1-0}^{\infty} \ln\left(1 - \frac{z}{t}\right) dn(t) = -r^{1/\rho} e^{i\theta} \int_{1/r}^{\infty} \frac{u^{(1/\rho)-1}}{u - e^{i\theta}}\, du + e^{i\theta} \int_{1/r}^{\infty} \frac{\omega(ur)\, du}{u(u - e^{i\theta})}. \tag{8}$$

The last member admits of the estimate

$$\left| e^{i\theta} \int_{1/r}^{\infty} \frac{\omega(ur)\, du}{u(u - e^{i\theta})} \right| \leqslant \int_{1/\rho}^{1} \frac{du}{u\, |u - e^{i\theta}|} + \int_{1}^{\infty} \frac{du}{u\, |u - e^{i\theta}|} = O(\ln r).$$

Replacing the lower limit $1/r$ by zero in the first member on the right-hand side of (8) results in an error which tends to zero as $r \to \infty$ and is hence insignificant. The obtained integral

$$I = \int_0^\infty \frac{u^{(1/\rho)-1}\,du}{u - e^{i\theta}}$$

is evaluated completely by using residue theory. Let us consider the closed contour Γ, pictured in Fig. 5. Since the integrand is single-valued within the contour Γ and has a single singularity $u = e^{i\theta}$, a first order pole, then

$$\frac{1}{2\pi i}\int_\Gamma \frac{u^{(1/\rho)-1}\,du}{u - e^{i\theta}} = e^{i\theta((1/\rho)-1)}.$$

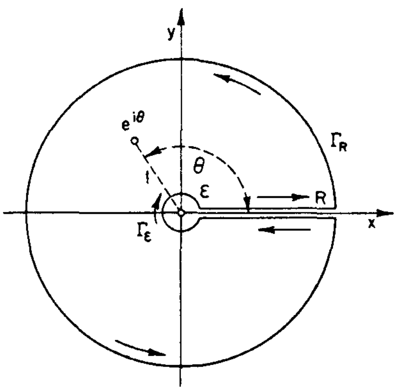

FIG. 5. The closed contour Γ.

On the other hand, integrals over the arcs of the circles Γ_ϵ and Γ_R, which make up the contour Γ, vanish in the limit (as $\epsilon \to 0$ and $R \to \infty$); hence we obtain for $0 < \theta < \pi$,

$$e^{i\theta((1/\rho)-1)} = \frac{1}{2\pi i}\int_0^\infty \frac{u^{(1/\rho)-1}\,du}{u - e^{i\theta}} + \frac{1}{2\pi i}\int_\infty^0 \frac{u^{(1/\rho)-1}\,e^{2\pi i(1/\rho)}\,du}{u - e^{i\theta}}$$

$$= \frac{1 - e^{2\pi i/\rho}}{2\pi i}\int_0^\infty \frac{u^{(1/\rho)-1}\,du}{u - e^{i\theta}} = -\frac{1}{\pi}\frac{\sin(\pi/\rho)}{e^{-(i\pi/\rho)}} \cdot I,$$

from which

$$I = -\pi e^{-i\pi/\rho}\frac{e^{i\theta((1/\rho)-1)}}{\sin(\pi/\rho)} = -e^{-i\theta}\frac{\pi}{\sin(\pi/\rho)}\,e^{i(\theta-\pi)/\rho}.$$

Therefore

$$\ln |\psi(z)| = \operatorname{Re} \ln \psi(z) = \operatorname{Re}\{-e^{i\theta} I r^{1/\rho}\} + O(\ln r)$$
$$= r^{1/\rho} \frac{\pi}{\sin(\pi/\rho)} \cos \frac{\theta - \pi}{\rho} + O(\ln r).$$

The obtained expression shows that the growth index of the function $\psi(z)$ is for $0 < \theta < \pi$,

$$h(\theta) = \overline{\lim}\, r^{-1/\rho} \ln |\psi(re^{i\theta})| = \frac{\pi}{\sin(\pi/\rho)} \cos \frac{\theta - \pi}{\rho},$$

q.e.d.

As is seen from the expression for the index, the function $\psi(z)$, having exponential growth of order $1/\rho$ in the z plane, has an exponential decrease of order $1/\rho$ on the half-axis $x > 0$ also. Let us now examine the function

$$\varphi(z) = \psi(z^2). \tag{9}$$

Evidently the function $\varphi(z)$ has an exponential order of growth in the z plane which equals $2/\rho$, and an exponential decrease of order $2/\rho$ on the whole real axis for $1 < \rho < 2$.

Hence, by virtue of Theorems 1–3 of Section 7, the function $\varphi(z)$ belongs to the space $S_\alpha{}^\beta$, where $\alpha = 1/p = \rho/2$, $\beta = 1 - \alpha = 1 - (\rho/2)$. Since ρ is any number between 1 and 2, we thereby obtain that *all spaces* $S_\alpha{}^\beta$ *with* $\frac{1}{2} < \alpha < 1$, $\beta = 1 - \alpha$ *are nontrivial.*

By virtue of the formula $\widetilde{S_\alpha{}^\beta} = S_\beta{}^\alpha$, all spaces $S_\alpha{}^\beta$ with $0 < \alpha < \frac{1}{2}$, $\beta = 1 - \alpha$ are also nontrivial.

There remains only for us to consider the case $\alpha = \beta = \frac{1}{2}$. But the space $S_{1/2}^{1/2}$ is also nontrivial: As we saw Section 7.4, it contains a nonzero function $\varphi(z) = \exp(-z^2)$. Therefore, our theorem is proved completely.

8.3. Case of the Space $S_{\alpha,A}^{\beta,B}$

To refine the previous results, let us consider the question of nontriviality of the spaces $S_{\alpha,A}^{\beta,B}$, which we shall need in Volume III. The fundamental functions in the space $S_{\alpha,A}^{\beta,B}$ satisfy, as we remember, the inequalities

$$| x^k \varphi^{(q)}(x)| \leqslant C_{\delta\rho}(A + \delta)^k (B + \rho)^q\, k^{k\alpha} q^{q\beta}.$$

Examining this inequality, we remark that for $\alpha \leqslant \alpha'$, $\beta \leqslant \beta'$, $A \leqslant A'$, $B \leqslant B'$, the imbeddings

$$S_{\alpha,A}^{\beta,B} \subset S_{\alpha',A'}^{\beta,B}, \qquad S_{\alpha,A}^{\beta,B} \subset S_{\alpha,A}^{\beta',B'} \tag{1}$$

hold, where if $\alpha < \alpha'$, then A' may be taken arbitrary (not necessarily greater than A) and analogously, any B may be taken for $\beta < \beta'$.

Since the space S_α^β is the union of spaces $S_{\alpha.A}^{\beta.B}$ in all A and B, the nontriviality of the space S_α^β is then equivalent to the nontriviality of the space $S_{\alpha.A}^{\beta.B}$ with some A and B. The nontriviality of the space S_α^β was proved for the cases

$$\alpha = 0, \beta > 1; \qquad \alpha > 1, \beta = 0; \qquad \alpha > 0, \beta > 0, \alpha + \beta \geqslant 1.$$

Therefore, in all these cases there are indeed nontrivial spaces $S_{\alpha.A}^{\beta.B}$.

By virtue of the reasoning presented above, we may assert that *all spaces* $S_{\alpha.A}^{\beta.B}$ with $\alpha > 0, \beta > 0, \alpha + \beta > 1$ and any A and B are nontrivial. There remains to be considered the extreme cases when $\alpha = 0$, or $\beta = 0$, or $\alpha + \beta = 1$.

Let us reason as follows. Let us consider some nontrivial space S_α^β and let us fix points in the A, B planes which correspond to the nontrivial spaces $S_{\alpha.A}^{\beta.B}$; the corresponding pairs of numbers A, B and point (A, B) will be called "admissible." Let us study the repartition of admissible points (A, B) on a plane with coordinates A, B. Since A and B are nonnegative numbers, it is then sufficient to consider the first quadrant of this plane. First of all, according to what has been proved, all pairs (A, B), where $A \geqslant A_0$, $B \geqslant B_0$ are also admissible together with every admissible pair (A_0, B_0).

Furthermore, let us transform the function $\varphi(x)$, in the nontrivial space $S_{\alpha.A}^{\beta.B}$, to the function $\varphi(\lambda x) = \psi(x)$. As we already know from Section 4.4, the function $\psi(x)$ belongs to the space $S_{\alpha.A/\lambda}^{\beta.\lambda B}$, which is therefore also nontrivial. Hence, together with the pair (A_0, B_0), the pair $(A_0/\lambda, \lambda B_0)$ is also admissible; both corresponding points lie on the hyperbola $AB = A_0 B_0$.

Hence, *a complete domain of all admissible pairs is a domain bounded from below by the hyperbola*

$$AB = \gamma \tag{2}$$

(where the hyperbola itself may belong to the domain of all admissible pairs or not). Evidently, for $\gamma = 0$, all pairs (A, B) with $A > 0$, $B > 0$ are admissible.

Let us show that this is precisely the situation for the spaces $S_{0.A}^{\beta.B}$ $(\beta > 1)$ and $S_{\alpha.A}^{0.B}$ $(\alpha > 1)$. Let us limit ourselves to the analysis of the first case. Let be given the numbers $\beta > 1$, $A > 0$, and $B > 0$. If the number β_1 is such that $1 < \beta_1 < \beta$, then by virtue of the nontriviality of the space $S_0^{\beta_1}$, there is a nontrivial space $S_{0.A_1}^{\beta_1.B_1}$ with some A_1 and B_1. By virtue of the imbedding (1), the space $S_{0.A_1}^{\beta.B}$ is nontrivial for any B as small as desired. But then $\gamma = \inf BA_1$ (the lower bound over all admissible pairs) evidently equals zero, q.e.d.

It can be shown that for $\alpha > 0$, $\beta > 0$, $\alpha + \beta = 1$, the constant γ is already not zero. It could even be evaluated as a function of α and β, but we do not need this.

Thus, we have established the nontriviality of the following spaces:

(1) $S_{\alpha,A}$, $S^{\beta,B}$ with any α, β, A, B;

(2) $S^{0,B}_{\alpha,A}$, $S^{\beta,B}_{0,A}$ with any $\alpha > 1$, $\beta > 1$, A, B;

(3) $S^{\beta,B}_{\alpha,A}$ with any $\alpha + \beta > 1$, A, B;

(4) $S^{\beta,B}_{\alpha,A}$ with $\alpha + \beta = 1$, $AB > \gamma$ (or $\geqslant \gamma$), where γ is some positive number.

8.4. On the Supply of Functions in Spaces of Type S

Closely connected with the questions of the nontriviality of the fundamental spaces is the *question of the sufficient abundance of the supply of functions in these spaces.* We state that the fundamental space Φ *is sufficiently rich in functions* if, for any locally integrable function $f(x)$, there results $f(x) \equiv 0$ almost everywhere from the convergence of the integral

$$\int_{-\infty}^{\infty} f(x)\, \varphi(x)\, dx \tag{1}$$

for all $\varphi(x) \in \Phi$ and the equality of this integral to zero for each $\varphi \in \Phi$.

Spaces sufficiently rich in functions possess the following important property: *Every space Φ, which is sufficiently rich in functions, is dense in every normed space of functions E, containing Φ, if the norm in E is given by a formula such as*

$$\| \varphi \| = \int_{-\infty}^{\infty} M(x)\, |\varphi(x)|\, dx, \tag{2}$$

where $M(x)$ is a fixed positive function.

In fact, assuming the opposite, we could construct a continuous linear functional $(f, \varphi) \not\equiv 0$ in the space E, by utilizing the Hahn–Banach process, which would vanish at each element $\varphi \in \Phi$. The general form of a continuous linear functional in the space E is well known; it is given by the formula

$$(f, \varphi) = \int_{-\infty}^{\infty} f(x)\, M(x)\, \varphi(x)\, dx, \tag{3}$$

where $f(x)$ is a bounded measurable function. Therefore, the integral (3) vanishes for every function $\varphi(x) \in \Phi$. But since the space Φ is sufficiently rich in functions, there would result from this fact that $f(x)\, M(x) \equiv 0$, from which also $f(x) \equiv 0$ almost everywhere, i.e., $(f, \varphi) = 0$ for any $\varphi \in E$ in contradiction to the assumption.

It is possible to establish the following criterion for the sufficiency of richness in functions in the fundamental space Φ.

Lemma. *If*

(*a*) *At least one function* $\varphi(x) \not\equiv 0$ *exists in the space* Φ;

(*b*) *Together with every function* $\varphi(x)$, *all translations* $\varphi(x - h)$, $-\infty < h < \infty$ *belong to the fundamental space* Φ[13];

(*c*) *Together with every function* $\varphi(x)$, *all products* $\varphi(x)\, e^{ix\sigma}$ *belong to the fundamental space* Φ,

then the space Φ *is sufficiently rich in functions.*

Proof. Let be given a locally integrable function $f(x)$, for which the equality

$$\int_{-\infty}^{\infty} f(x)\, \varphi(x)\, dx = 0$$

holds for any function $\varphi(x) \in \Phi$. Let us show that $f(x)$ equals zero almost everywhere. Let us consider the function $\varphi_0(x) \in \Phi$, which is not identically zero; since all translations are admissible in the space Φ, it may then be assumed that the function $\varphi_0(x)$ is not zero in the neighborhood of the given point x_0. By assumption, for any σ, the product $\varphi_0(x)\, e^{ix\sigma} \in \Phi$; hence for any σ,

$$\int_{-\infty}^{\infty} f(x)\, \varphi_0(x)\, e^{ix\sigma}\, dx = 0.$$

But this equality means that the Fourier transform of the function $f(x)\, \varphi_0(x)$ is identically zero. The function $f(x)\, \varphi_0(x)$ equals zero almost everywhere by the theorem on the uniqueness of the Fourier transform.[14] Since the function $\varphi_0(x) \neq 0$ in the neighborhood of the point x_0, the function $f(x)$ equals zero almost everywhere in this neighborhood. By virtue of the arbitrariness of x_0, the function $f(x)$ equals zero almost everywhere on the line $-\infty < x < \infty$, q.e.d.

Let us now elucidate the question of the sufficiency of the store of functions in spaces of type S.

[13] Assumption (b) could be replaced by the following: (b') There exists a function $\varphi(x) \in \Phi$, which is not zero in the neighborhood of any fixed point x_0.

[14] See E. Titchmarsh: Introduction to the Theory of the Fourier Integral, New York, 1949.

We have proved (in Section 4) for spaces $S_{\alpha,A}^{\beta,B}$ with $\alpha > 0$, $\beta > 0$, that the operations of translation and multiplication by $e^{i\sigma x}$ are defined in these spaces. Applying the lemma, we obtain that *each such space, if it is nontrivial,* is sufficiently rich in functions. *The spaces* $S_{\alpha,A}$ ($\alpha > 0$), $S^{\beta,B}$ ($\beta > 0$) are always nontrivial, and the operations of translation and multiplication by $e^{i\sigma x}$ are also defined therein; therefore these spaces are *also sufficiently rich in functions.*

If one of the numbers α, β equals zero in the nontrivial space $\Phi = S_{\alpha,A}^{\beta,B}$ (or $S_{\alpha,A}$, $S^{\beta,B}$), then this space is not sufficiently rich in functions. In this case, for $\alpha = 0$, all the functions $\varphi(x) \in \Phi$ vanish outside a fixed compact domain, and for every function $f(x) \not\equiv 0$, which equals zero in this domain, the integral (1) is equal to zero for all $\varphi \in \Phi$. For $\beta = 0$, all the $\varphi(x) \in \Phi$ are entire functions, the Fourier transforms of the preceding functions; if the function $f(x) \not\equiv 0$, which takes part in the preceding construction, possesses the classical Fourier transform $g(x)$, if the function $f(x)$ is of compact support, for example, then if the function $g(x)$ is substituted into the integral (1) instead of $f(x)$, this integral will vanish for all $\varphi \in \Phi$. But if the constants A or B are not fixed, and the unions

$$S_0^{\beta,B} = \bigcup_A S_{0,A}^{\beta,B}, \qquad S_{\alpha,A}^0 = \bigcup_B S_{\alpha,A}^{0,B}$$

are considered, then these spaces are sufficiently rich in functions. In fact, as has been shown in Section 4, both the translation operation and the operation of multiplication by $e^{ix\sigma}$ for any σ are defined in these spaces, and as we have seen, this is sufficient for the validity of our assertion.

In exactly the same way, the unions $S_0 = \bigcup_A S_{0,A}$, $S^0 = \bigcup_B S^{0,B}$ are also sufficiently rich in functions.

Thus, *the following spaces of type S are sufficiently rich in functions: the nontrivial spaces* $S_{\alpha,A}^{\beta,B}$, $\alpha > 0, \beta > 0$; $S_{\alpha,A}$, $\alpha > 0$; $S^{\beta,B}$, $\beta > 0$; $S_0^{\beta,B}$, $\beta > 1$; $S_{\alpha,A}^0$, $\alpha > 1$; S_0; S^0.

9. The Case of Several Independent Variables

Let be given the non-negative numbers $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_n$; α denotes the set $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ and β the set $(\beta_1, \beta_2, \ldots, \beta_n)$.

The space $S_\alpha = S_{\alpha_1,\alpha_2,\ldots,\alpha_n}$ consists of all infinitely differentiable functions $\varphi(x) = \varphi(x_1, x_2, \ldots, x_n)$, for which the inequalities

$$|x^k D^q \varphi(x)| \equiv \left| x_1^{k_1} \cdots x_n^{k_n} \frac{\partial^{q_1+\cdots+q_n}\varphi(x)}{\partial x_1^{q_1} \cdots \partial x_n^{q_n}} \right|$$
$$\leqslant C A_1^{k_1} \cdots A_n^{k_n} k_1^{k_1\alpha_1} \cdots k_n^{k_n\alpha_n} \quad (k_1, k_2, \ldots, k_n = 0, 1, 2, \ldots)$$

are satisfied. By definition, the sequence $\varphi_\nu(x) \in S_\alpha$ converges to zero, if this sequence converges uniformly to zero in each bounded domain together with the derivatives of any order, and if the constants $A_1, \ldots, A_n$, C in these inequalities, written for the functions $\varphi_\nu(x)$, can be selected the same for all $\nu = 1, 2, \ldots$.

The set of functions $\varphi(x) \in S_\alpha$, for which the constants $A_1 = \bar{A}_1, \ldots, A_n = \bar{A}_n$ may be selected arbitrarily greater than a fixed $A_1, A_2, \ldots, A_n$ is a countably normed space; we denote it by $S_{\alpha,A} = S_{\alpha_1,\ldots,\alpha_n;A_1,\ldots,A_n}$.

The norms in this space are given by the formulas

$$\| \varphi \|_{q\delta} = \sup_{x,k} \frac{| x_1^{k_1} \cdots x_n^{k_n} (\partial^{q_1+\cdots+q_n} \varphi(x)/\partial x_1^{q_1} \cdots \partial x_n^{q_n})|}{(A_1 + \delta_1)^{k_1} \cdots (A_n + \delta_n)^{k_n} k_1^{k_1\alpha_1} \cdots k_n^{k_n\alpha_n}},$$

where δ is the set $(\delta_1, \ldots, \delta_n)$ and q is the set $(q_1, \ldots, q_n)$.

The space $S_{\alpha,A}$ is a complete countably normed perfect space. The proofs of these facts are carried out exactly as in the case of one independent variable.

Analogous changes are made in going over to n independent variables in the remaining definitions of spaces of type S. The space $S^\beta = S^{\beta_1,\ldots,\beta_n}$ consists of all infinitely differentiable functions $\varphi(x)$, for which

$$| x^k D^q \varphi(x)| = \left| x_1^{k_1} \cdots x_n^{k_n} \frac{\partial^{q_1+\cdots+q_n}\varphi(x)}{\partial x_1^{q_1} \cdots \partial x_n^{q_n}} \right| \leqslant C B_1^{q_1} \cdots B_n^{q_n} q_1^{q_1\beta_1} \cdots q_n^{q_n\beta_n}.$$

The space $S_\alpha^\beta = S_{\alpha_1,\ldots,\alpha_n}^{\beta_1,\ldots,\beta_n}$ consists of functions $\varphi(x)$, for which

$$| x^k D^q \varphi(x)| \equiv \left| x_1^{k_1} \cdots x_n^{k_n} \frac{\partial^{q_1+\cdots+q_n}\varphi(x)}{\partial x_1^{q_1} \cdots \partial x_n^{q_n}} \right| \leqslant C A_1^{k_1} \cdots A_n^{k_n} B_1^{q_1} \cdots B_n^{q_n} k_1^{k_1\alpha_1} \cdots k_n^{k_n\alpha_n} q_1^{q_1\beta_1} \cdots q_n^{q_n\beta_n}.$$

Just as the space S_α, these spaces are represented as the union of countably normed spaces

$$S^\beta = \bigcup_B S^{\beta,B}, \qquad S_\alpha^\beta = \bigcup_{A,B} S_{\alpha,A}^{\beta,B},$$

defined analogously to $S_{\alpha,A}$.

After the definitions have been given, all the results of Sections 2–8 referring to one variable, may be extended to the obtained spaces. In particular, the operations of multiplication by the independent variables (and by any polynomial in them) and of differentiation are defined and continuous in these spaces. Every function $f(x) = f(x_1, \ldots, x_n)$, satisfying the inequalities

$$| D^q f(x)| \leqslant C_q \exp[a_1 | x_1 |^{1/\alpha_1} + \cdots + a_n | x_n |^{1/\alpha_n}]$$

defines, for sufficiently small a_i, a multiplication operator in the space $S_{\alpha,A}$ by transforming it into $S_{\alpha,A'}$, where the components $A' = (A_1', \ldots, A_n')$ are related to the corresponding components $A = (A_1, \ldots, A_n)$ by the same formulas as in Section 4. Every function $f(x) = f(x_1, \ldots, x_n)$, satisfying the inequalities

$$| D^q f(x)| \leqslant C\bar{B}_1^{q_1} \cdots \bar{B}_n^{q_n} q_1^{q_1\beta_1} \cdots q_n^{q_n\beta_n},$$

defines a multiplication operator in the space $S^{\beta,B}$ by transforming it into $S^{\beta,B+\bar{B}}$, where

$$B + \bar{B} = (B_1 + \bar{B}_1, \ldots, B_n + \bar{B}_n).$$

Every function $f(x)$, satisfying the inequalities

$$| D^q f(x)| \leqslant C\bar{B}_1^{q_1} \cdots \bar{B}_n^{q_n} q_1^{q_1\beta_1} \cdots q_n^{q_n\beta_n} \exp(\bar{a}_1 | x_1 |^{1/\alpha_1} + \cdots + \bar{a}_n | x_n |^{1/\alpha_n}),$$

defines a multiplication operator in the space $S_{\alpha,A}^{\beta,B}$ for sufficiently small $\bar{a}$ $(\bar{a}_j < \alpha_k/eA_k^{1/\alpha_k})$, by transforming this space into the space $S_{\alpha,A'}^{\beta,B+\bar{B}}$, as in Section 4.

If the entire function

$$f(z) = f(z_1, \ldots, z_n) = \sum a_{m_1 \ldots m_n} z_1^{m_1} \cdots z_n^{m_n}$$

has the order of growth $\leqslant 1/\beta$ and type $< \beta/Be^{1/\beta}e^2$, i.e., if the inequality

$$|f(z_1, \ldots, z_n)| \leqslant C \exp(b_1 | z_1 |^{1/\beta_1} + \cdots + b_n | z_n |^{1/\beta_n}),$$

is satisfied in which

$$b_k < \frac{\beta}{B_k^{1/\beta_k} e^2} \qquad (k = 1, 2, \ldots, n),$$

then the operator

$$f\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) = \sum a_{m_1 \ldots m_n} \frac{\partial^{m_1 + \cdots + m_n}}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}}$$

is defined in the space $S_{\alpha,A}^{\beta,B}$; it transforms the space $S_{\alpha,A}^{\beta,B}$ into the space $S_{\alpha,A}^{\beta,Be^\beta}$, where $Be^\beta = (B_1 e^{\beta_1}, \ldots, B_n e^{\beta_n})$.

Reciprocity theorems completely analogous to the theorems established in Section 6 also hold:

$$\widetilde{S_\alpha} = S^\alpha, \qquad \widetilde{S^\beta} = S_\beta, \qquad \widetilde{S_\alpha^\beta} = S_\beta^\alpha;$$

$$\widetilde{S_{\alpha,A}} = S^{\alpha,A}, \qquad \widetilde{S^{\beta,B}} = S_{\beta,B};$$

$\widetilde{S_{\alpha,A}^{\beta,B}} = S_{\beta,B}^{\alpha,A}$ for $\alpha > 0$ (i.e., all $\alpha_j > 0$), $\beta > 0$, $\alpha + \beta > 1$; $S_{\alpha,A}^{\beta,B} \subset S_{\beta,B'}^{\alpha,A'}$ in the remaining cases; where $A' = (A_1', \dots, A_n')$, with $A_k' = A_k \exp(1/A_k B_k)$ $(k = 1, 2, \dots, n)$; the B_k' are defined analogously.

All the theorems of Section 7 are also carried over to the case of n independent variables without substantial changes. Let us only indicate the fundamental ideas here.

Phragmen-Lindelöf Theorem for Functions of n Variables. *Let be given an analytic function $f(z) = f(z_1, \dots, z_n)$, defined for values of the variables $z_1, \dots, z_n$, each of which runs through the angle G_j of aperture $\omega_j < \pi/p_j$ in its plane, independently of the values of the remaining variables. The boundary of the angle G_j is denoted by Γ_j. Furthermore, let the function $f(z)$ satisfy the inequalities*

$$|f(z_1, \dots, z_n)| \leqslant C \exp(b_1 |z_1|^{p_1} + \cdots + b_n |z_n|^{p_n}) \qquad (z_j \in G_j),$$

$$|f(z_1, \dots, z_n)| \leqslant C_1 \quad (z_1 \in \Gamma_1, \dots, z_n \in \Gamma_n).$$

Then the inequality

$$|f(z_1, \dots, z_n)| \leqslant C_1 \qquad (z_1 \in G_1, \dots, z_n \in G_n) \tag{1}$$

is valid.

Proof. Let us arbitrarily fix $z_2 \in \Gamma_2, \dots, z_n \in \Gamma_n$; the function of z_1 obtained satisfies the inequality

$$|f(z_1, z_2, \dots, z_n)| \leqslant C_1 \quad (z_1 \in G_1, z_2 \in \Gamma_2, \dots, z_n \in \Gamma_n),$$

by virtue of the Phragmen–Lindelöf theorem. Let us now fix $z_1 \in G_1$, $z_3 \in \Gamma_3, \dots, z_n \in \Gamma_n$; we then obtain in the same way that

$$|f(z_1, \dots, z_n)| \leqslant C_1 \quad (z_1 \in G_1, z_2 \in G_2, z_3 \in \Gamma_3, \dots, z_n \in \Gamma_n).$$

Continuing in the same manner, we arrive at the desired inequality (1) after n steps.

Theorem. *If an entire function $f(z_1, \dots, z_n)$ has an order of growth $\leqslant p = (p_1, \dots, p_n)$ with a finite type, i.e., for all z satisfies the inequality*

$$|f(z)| \leqslant C_1 \exp(b_1 |z_1|^{p_1} + \cdots + b_n |z_n|^{p_n})$$

and moreover, for real values $z_j = x_j$ satisfies the inequality

$$|f(x_1, \dots, x_n)| \leqslant C_2 \exp[a_1 |x_1|^{h_1} + \cdots + a_n |x_n|^{h_n}] \qquad (a_i \neq 0, 0 < h_i \leqslant p_i),$$

then there exists a domain G, defined by the inequalities

$$|y_j| \leqslant K_j'(1 + |x_j|)^{1-(p_j - h_j)} \qquad (j = 1, \dots, n),$$

in which

$$|f(x + iy)| \leqslant C_3 \exp(a_1' |x_1|^{h_1} + \cdots + a_n' |x_n|^{h_n}) \qquad (C_3 = \max(C_1, C_2)).$$

The proof proceeds entirely according to the scheme of the proof for one variable presented in Section 7. The functions

$$f_1(z) = f(z) \cdot \exp(-a_1 z_1^{h_1} - \cdots - a_n z_n^{h_n})$$
$$f_2(z) = f_1(z) \cdot \exp(ib_1 z_1^{p_1} + \cdots + ib_n z_n^{p_n})$$

are introduced; it is proved that the function $f_2(z)$ is bounded in the "skeleton" of the domain

$$0 \leqslant \omega_j \leqslant \pi/2p_j \qquad (\arg z_j = \omega_j),$$

(i.e., when the equality $\omega_j = 0$ or $\omega_j = \pi/2p_j$ holds for each j in place of these inequalities), and the Phragmen–Lindelöf theorem is applied in the form formulated above, we consequently obtain that within the limits of this domain

$$|f(z_1, \dots, z_n)|$$
$$\leqslant C \exp(a_1 r_1^{h_1} \cos h_1\omega_1 + b_1 r_1^{p_1} \sin p_1\omega_1 + \cdots + a_n r_n^{h_n} \cos h_n\omega_n + b_n r_n^{p_n} \sin p_n\omega_n)$$

If $z_j = r_j \exp(i\omega_j)$ and r_j and ω_j are such that

$$a_j r_j^{h_j} \cos h_j\omega_j + b_j r_j^{p_j} \sin p_j\omega_j \leqslant a_j' r_j^{h_j} \cos h_j\omega_j \tag{2}$$

($a_j' > a_j$ and of the same sign), then the function $f(z)$ will satisfy the inequality

$$|f(z)| \leqslant C \exp(a_1'' x_1^{h_1} + \cdots + a_n'' x_n^{h_n})$$

(a_j'' is of the same sign as a_j'). The inequality (2) separates in the z_j plane a domain with the boundary

$$r_j^{p_j-h_j} = \frac{a_j' \cos h_j\omega_j - a_j \cos h_j\omega_j}{b_j \sin p_j\omega_j}\,; \tag{3}$$

just as in the one variable case, it may be given by the inequalities

$$| y_j | \leqslant \min \left[| x_j | \tan \frac{\pi}{2p_j}, \quad K_j \, | x_j |^{1-(p_j-h_j)} \right]. \tag{4}$$

Let us now show that the obtained domain may be replaced by the simpler domain

$$G' = \{ | y_j | \leqslant K_j'(1 + | x_j |)^{1-(p_j-h_j)} \} \qquad (j = 1, 2, \dots, n). \tag{5}$$

If the passage to the new domain could be motivated by continuity considerations in the one-variable case, such considerations are now inadequate since the union of these points is not a compact manifold. We hence use another method.

The preceding reasoning could be applied only to some variables z_j rather than to all. For example, if it was applied to the variables $z_2, \dots, z_n$, then we would obtain the inequality

$$| f(z) | \leqslant C \exp[b_1 \, | z_1 |^p] \exp[a_2 \, | x_2 |^{h_2} + \cdots + a_n \, | x_n |^{h_n}] \tag{6}$$

in the domain

$$| y_j | \leqslant \min \left\{ | x_j | \tan \frac{\pi}{2p_j}, \quad K_j \, | x_j |^{1-(p_j-h_j)} \right\} \qquad (j = 2, \dots, n).$$

Moreover, it has been proved that the inequality

$$| f(z) | \leqslant C \exp(a_1 \, | x_1 |^{h_1} + a_2 \, | x_2 |^{h_2} + \cdots + a_n \, | x_n |^{h_n}) \tag{7}$$

is valid in the domain (4). It is now clear that the inequality

$$| f(z) | \leqslant C \exp(a_1 \, | x_1 |^{h_1} + a_2 \, | x_2 |^{h_2} + \cdots + a_n \, | x_n |^{h_n})$$

is satisfied in the domain

$$| y_1 | \leqslant K_1'(1 + | x_1 |)^{1-(p_1-h_1)},$$

$$| y_j | \leqslant \min \left\{ | x_j | \tan \frac{\pi}{2p_j}, \quad K_j \, | x_j |^{1-(p_j-h_j)} \right\} \qquad (j = 2, \dots, n),$$

where, as in the one-variable case, the C is obtained from the previous constant C by multiplying it by a number which depends only on the constants a and b.

Therefore, the inequality

$$|f(z)| \leqslant C \exp(a_1 \mid x_1 \mid^h) \exp(b_2 \mid z_2 \mid^{p_2} + \cdots + b_n \mid z_n \mid^{p_n})$$

is proved in the domain

$$|y_1| \leqslant K_1'(1 + |x_1|)^{1-(p_1-h_1)}. \tag{8}$$

Let us now fix the coordinate z_1 in this domain, and let us repeat the considerations presented above for the coordinate z_2 ; we hence obtain the inequality (7) in the domain

$$|y_j| \leqslant K_j'(1 + |x_j|)^{1-(p_j-h_j)} \qquad (j = 1, 2);$$

$$|y_j| \leqslant \min \left\{ |x_j| \tan \frac{\pi}{2p_j}, \quad K_j \mid x_j \mid^{1-(p_j-h_j)} \right\} \qquad (j = 3, \ldots, n).$$

Continuing further in the same manner, we arrive at the desired result after n steps.

In all the remaining formulas and proofs of Section 7, it now remains to insert a change consisting of replacing the single coordinate by n coordinates. All these theorems rely on Theorem 1 of Section 7 and the Remark, whose validity in n space has now been established. Taking account of this circumstance, the proofs of the remaining theorems obviously go over into the n-dimensional case.

The question of the nontriviality of spaces of type S is solved by utilizing the following remark: *The space*

$$S_{\alpha,A}^{\beta,B} = S_{\alpha_1,\ldots\alpha_n,A_1,\ldots,A_n}^{\beta_1,\ldots\beta_n,B_1,\ldots,B_n}$$

is nontrivial if and only if all the spaces $S_{\alpha_k,A_k}^{\beta_k,B_k}$ $(k = 1, 2, \ldots, n)$ *are nontrivial.*

Indeed, if the function $\varphi_k(x_k)$ belongs to $S_{\alpha_k,A_k}^{\beta_k,B_k}$ and is not identically zero, then $\varphi(x) = \varphi_1(x_1) \ldots \varphi_n(x_n)$ is also not identically zero and belongs to the space $S_{\alpha,A}^{\beta,B}$. Conversely, if some $S_{\alpha_k,A_k}^{\beta_k,B_k}$ is trivial, i.e., contains only the single function $\varphi(x_k) \equiv 0$, then $S_{\alpha,A}^{\beta,B}$ is also trivial, since any function $\varphi(x)$ from $S_{\alpha,A}^{\beta,B}$ belongs to the space $S_{\alpha_k,A_k}^{\beta_k,B_k}$ for any fixed values $x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n$ and, therefore, is identically zero.

It is easy to see that the nontrivial spaces $S_{\alpha,A}^{\beta,B}$ $(\alpha > 0, \beta > 0)$ and the spaces $S_{\alpha,A}$ $(\alpha > 0)$, $S^{\beta,B}$ $(\beta > 0)$, $S_0^{\beta,B}$ $(\beta > 1)$, $S_{\alpha,A}^0$ $(\alpha > 1)$, S_0, S^0 are sufficiently rich in functions, as in the one-variable case.

Appendix 1

Generalization of Spaces of Type S

The class of spaces of type S may be generalized significantly if the sequences $k^{k\alpha}$ ($k = 0, 1, 2,...,$) and $q^{q\beta}$ ($q = 0, 1, 2,...$) in the definitions of these spaces are replaced by arbitrary sequences a_k and b_q. We thus obtain the spaces S_{a_k}, S^{b_q}, $S_{a_k}^{b_q}$, defined by the following systems of inequalities for ($k, q = 0, 1, 2,...$):

$$S_{a_k}: \qquad |x^k\varphi^{(q)}(x)| \leqslant C_q A^k a_k, \tag{1}$$

$$S^{b_q}: \qquad |x^k\varphi^{(q)}(x)| \leqslant C_k B^q b_q, \tag{2}$$

$$S_{a_k}^{b_q}: \qquad |x^k\varphi^{(q)}(x)| \leqslant C A^k B^q a_k b_q. \tag{3}$$

The results of the theory of spaces of type S, developed in Sections 2–8, may be carried over to the case of the generalized spaces (1)–(3) only if the sequences a_k and b_q satisfy specific conditions which are presented below.

The functions $\varphi(x) \in S_{a_k}$ may be characterized by their decrease at infinity in conformity with the formula

$$|\varphi^{(q)}(x)| \leqslant C_q l\left(\frac{x}{A}\right), \quad \text{where} \quad l(x) = \inf_k \frac{a_k}{|x|^k}. \tag{4}$$

By definition, the growth of the derivatives of the functions $\varphi(x) \in S^{b_k}$ is restricted. For a sufficiently slow growth of the b_k, the functions $\varphi(x) \in S^{b_k}$ are entire, and satisfy the estimate on their growth

$$|x^k\varphi(x + iy)| \leqslant C_k \Lambda(B_y), \quad \text{where} \quad \Lambda(y) = \sum_{q=0}^{\infty} \frac{b_q |y|^q}{q}. \tag{5}$$

Now the operations of multiplication by x and of differentiation are not always defined. Compliance with the inequality

$$\frac{a_{k+1}}{a_k} \leqslant Ch^k, \tag{6}$$

where C and h are constants, is a sufficient condition for the possibility of multiplication by x in the space S_{a_k} wherein differentiation is always possible.

Compliance with the inequality

$$C_1 h_1{}^q \leqslant \frac{b_{q+1}}{b_q} \leqslant C_2 h_2{}^q \tag{7}$$

with certain constants C_1, C_2, h_1, h_2, is a sufficient condition for the possibility of multiplication by x and for differentiation in the space S^{b_q}. Analogous conditions are sufficient for execution of the corresponding operations in the space $S_{a_k}^{b_q}$ also.

The proofs of these propositions proceed along the same lines as for the spaces of type S.

Some conditions for the possibility of differential operations of infinite order in $S_{a_k}^{b_q}$ could also be formulated. The question of the complete determination of the topological ring of operators in $S_{a_k}^{b_q}$ generated by the operators x and d/dx is probably of greatest interest; this question is still open.

The reciprocity theorems are retained even in the generalized spaces in the form which is given in the general Theorem of Section 6, namely: If the function $\varphi(x)$ satisfies the inequalities

$$| x^k \varphi^{(q)}(x)| \leqslant C A^k B^q a_k b_q \,, \tag{8}$$

where the numbers a_k and b_q are such that

$$\frac{a_k}{a_{k-1}} \geqslant C k^{1-\mu}, \qquad \frac{b_q}{b_{q-1}} \geqslant C q^{1-\lambda}, \qquad \mu + \lambda \leqslant 1, \qquad \frac{a_{k+2}}{a_k} \leqslant A_0{}^k, \tag{9}$$

then the Fourier transform $\psi(\sigma)$ of the function $\varphi(x)$ satisfies the inequalities

$$| \sigma^k \psi^{(q)}(\sigma)| \leqslant C' A_1{}^q B_1{}^k a_q b_k \tag{10}$$

and therefore, the formula

$$\widetilde{S_{a_k}^{b_q}} = S_{b_k}^{a_q} \tag{11}$$

holds.

The problem of the nontriviality of the spaces $S_{a_k}^{b_q}$ is considerably more complex than for the spaces $S_\alpha{}^\beta$. The classical problem of quasianalyticity is a particular case of this problem corresponding to the values $a_k \equiv 1$. A more natural condition on the order of decrease as $| x | \to \infty$ is imposed in our case in place of the condition that the function $\varphi(x)$ vanish (together with all its derivatives) at the ends of a given segment in the classical problem of quasianalyticity.

K. I. Babenko proved the following general theorem. Let us put

$$L(x) = \sup_k \frac{|x|^k}{a_k} \quad (=1/l(x) \text{ in our notations}), \tag{12}$$

$$M(y) = \sup_q \frac{|y|^q}{b_q}, \tag{13}$$

$$\lambda(x) = \ln \int_0^\infty \frac{\cosh tx}{L(t)}\, dt, \tag{14}$$

$$\mu(x) = \ln \int_0^\infty \frac{\cosh ty}{M(t)}\, dt. \tag{15}$$

Let us assume compliance with one of the two conditions for the correctness of the growth of the function $L(x)$:

$$\underline{\lim}_{x\to\infty} \frac{\ln \ln L(x)}{\ln x} > \frac{3}{2} \tag{16}$$

or

$$\overline{\lim}_{x\to\infty} \frac{\ln \ln L(x)}{\ln x} < 3. \tag{17}$$

If condition (16) is satisfied, then it is sufficient for the nontriviality of the space $S_{a_k}^{b_q}$ that

$$\underline{\lim}_{y\to\infty} \frac{\lambda(y)}{y^3 \int_0^\infty \frac{\ln M(\xi)}{\xi^2} \frac{d\xi}{\xi^2 + y^2}} > 0, \tag{18}$$

and it is necessary that

$$\overline{\lim}_{y\to\infty} \frac{\lambda(y)}{y^3 \int_0^\infty \frac{\ln M(\xi)}{\xi^2} \frac{d\xi}{\xi^2 + y^2}} > 0. \tag{19}$$

If condition (17) is satisfied, then $\lambda(y)$ should be replaced by $\mu(y)$ and $M(\xi)$ by $L(\xi)$ in inequalities (18) and (19).

Appendix 2

Spaces of Type W

In this appendix, the theory of spaces of type W, which is contained in Chapter I of Volume 3 is expounded without proofs and rather concisely. These spaces are analogous to spaces of type S, corresponding to values $\alpha < 1$ and $\beta < 1$ of the indices; however, the spaces of type W are able more exactly to discern singularities in the growth or decrease of functions at infinity, because of the use of arbitrary convex, rather than power, functions.

For simplicity, we shall limit ourselves herein to the case of one independent variable.

A2.1. The Space W_M

Let be given an increasing continuous function $\mu(\xi)$ $(0 \leqslant \xi < \infty)$ such that $\mu(0) = 0$ and $\mu(\infty) = \infty$. Furthermore, let us define the function $M(x)$ by the equality

$$M(x) = \int_0^x \mu(\xi)\, d\xi, \qquad M(-x) = M(x). \tag{13}$$

This is a convex function which increases more rapidly than any linear function as $x \to +\infty$ or $-\infty$. Then the space W_M is defined as the set of all infinitely differentiable functions $\varphi(x)$ $(-\infty < x < \infty)$, satisfying inequalities of the form

$$|\varphi^{(q)}(x)| \leqslant C_q e^{-M(ax)} \qquad (q = 0, 1, \ldots), \tag{2}$$

where the positive constants C_q and a depend on the function $\varphi(x)$. From these inequalities, it follows that $\varphi^{(q)}(x)$ decreases more rapidly than any exponential for any q. Linear operations are defined in W_M in a natural manner. The sequence of elements φ_ν is called *convergent to zero* if (1) it converges correctly to zero (i.e., $\varphi_\nu^{(q)}(x) \to 0$ for any q uniformly in each bounded interval) and (2) $\varphi_\nu(x)$ satisfies the inequalities

$$|\varphi_\nu^{(q)}(x)| \leqslant C_q \exp(-M(ax)) \qquad (q, \nu = 0, 1, \ldots) \tag{3}$$

with constants C_q and a independent of ν. A set is called *bounded* in the space W_M if it consists of functions satisfying the same inequality (2) with fixed C_q and a.

The space W_M is the union (in the sense of Section 8 of Chapter 1) of countably-normed spaces $W_{M,a}$. The space $W_{M,a}$ is defined as the set of functions $\varphi(x)$, which for any $\delta > 0$ satisfy the inequalities

$$|\varphi^{(q)}(x)| \leqslant C_{q\delta} \exp(-M[(a-\delta)\, x]) \qquad (q = 0, 1, \ldots). \tag{4}$$

The space $W_{M,a}$ belongs to the class of spaces $K\{M_p\}$; the functions $M_p(x)$ in this case are

$$M_p(x) = \exp\left(M\left[a\left(1 - \frac{1}{p}\right) x\right]\right) \qquad (p = 2, 3, \ldots). \tag{5}$$

As in every space $K\{M_p\}$, the topology in the space $W_{M,a}$ is defined by the norms

$$\| \varphi \|_p = \sup_{\substack{x \\ |q| \leqslant p}} M_p(x) | \varphi^{(q)}(x)|. \tag{6}$$

Furthermore, the space $W_{M,a}$ is complete; the condition (P) of Section 2 in Chapter II is satisfied so that $W_{M,a}$ is perfect. Therefore, $W_{M,a}$ is the union of perfect spaces.

The spaces S_α with $0 < \alpha < 1$ are examples of the spaces W_M. In this case $M(x) = x^{1/\alpha}$ $(x \geqslant 0)$; $\mu(\xi) = (1/\alpha)\, \xi^{(1/\alpha)-1}$ $(\xi > 0)$.

A space W_M, which differs from the spaces S_α, is obtained if we take

$$\mu(\xi) = \ln(\xi + 1) \quad (\xi \geqslant 0).$$

The corresponding function $M(x)$ is written awkwardly enough as

$$M(x) = (x + 1) \ln(x + 1) - x.$$

But since the space W_M may be constructed formally by means of any non-negative continuous function which is not necessarily convex, the function written down above may be replaced by the following:

$$M_1(x) = x \ln x.$$

These two functions are equivalent in the following sense: There exist constants γ_1, γ_2, and γ_3, such that for sufficiently large $x \geqslant 0$,

$$M_1(\gamma_1 x) \leqslant M(\gamma_2 x) \leqslant M_1(\gamma_3 x). \tag{7}$$

Equivalent functions define coincident spaces (both in the store of elements and in convergence).

A2.2. The Space W^Ω

The space W^Ω is constructed from entire analytic functions $\varphi(z)$. A function $\omega(\eta)$, which possesses the same properties as the function $\mu(\xi)$ of the preceding paragraph is given; the function $\Omega(y)$ is constructed by means of it, exactly as $M(x)$ was constructed by means of $\mu(\xi)$:

$$\Omega(y) = \int_0^y \omega(\eta)\, d\eta \qquad (y \geqslant 0), \qquad \Omega(-y) = \Omega(y). \tag{1}$$

The space W^{Ω} is defined as the set of entire functions satisfying the inequalities

$$|z^k \varphi(z)| \leqslant C_k \exp(\Omega(by)), \tag{2}$$

with the customary linear operations. The constants C_k and b are positive and depend on the functions. The sequence φ_ν of elements of the space W^{Ω} is called *convergent to zero* if (1) $\varphi_\nu(z)$ converges correctly to zero (i.e., $\varphi_\nu(z) \to 0$ uniformly in any bounded domain of the z plane), and (2) the estimates

$$|z^k \varphi_\nu(z)| \leqslant C_k \exp(\Omega(by))$$

are valid, where C_k and b are independent of ν. A set in the space W^{Ω} is called *bounded* if elements of this set satisfy the same inequality (2) with fixed constants C_k and b.

The space W^{Ω} is the union of countably-normed spaces $W^{\Omega,b}$. The space $W^{\Omega,b}$ is defined by the inequalities

$$|z^k \varphi(z)| \leqslant C_{k\rho} \exp(\Omega[(b+\rho)\, y]) \qquad (\rho > 0 \text{ arbitrary}). \tag{3}$$

Norms are given in this space by the formulas

$$\|\varphi\|_{k\rho} = \sup_z |z^k \varphi(z)| \exp(-\Omega[(b+\rho)\, y]). \tag{4}$$

With these norms, $W^{\Omega,b}$ is a complete, countably normed perfect space.

For

$$\Omega(y) = y^{1/1-\beta} \qquad (\beta < 1),$$

the space W^{Ω} agrees with the space S^{β}.

It is also possible to take $\omega(\eta) = e^{\eta} - 1$; the corresponding function

$$\Omega(y) = e^{y} - y - 1$$

is equivalent to the function $\Omega_1(y) = e^{y}$. The equivalent functions define coincident spaces here also.

A2.3. The Space $W_M{}^{\Omega}$

The space $W_M{}^{\Omega}$ is defined by giving two functions $\mu(\xi)$ and $\omega(\eta)$ $(0 \leqslant \xi,\ \eta < \infty)$ which increase from 0 to ∞; as before, they are used to construct functions $M(x)$ and $\Omega(y)$, which are convex and grow more rapidly than any linear function as $|x| \to \infty$. The space $W_M{}^\Omega$ is defined as the set of entire analytic functions $\varphi(z)$, satisfying inequalities of the form

$$|\varphi(x+iy)| \leqslant C\exp(-M(ax)+\Omega(by)). \tag{1}$$

The linear operations are the customary ones. The sequence $\varphi_\nu \in W_M{}^\Omega$ *converges to zero*, by definition, if (1) it converges correctly to zero; (2) its elements satisfy the same inequality (1) with constants independent of ν. A set in the space $W_M{}^\Omega$ is called *bounded* if its elements satisfy the inequality (1) with fixed constants.

The space $W_M{}^\Omega$ is the union of (complete, countably normed) perfect spaces $W_{M,a}^{\Omega,b}$. The space $W_{M,a}^{\Omega,b}$ is defined by the inequalities

$$\varphi(x+iy)| \leqslant C\exp(-M[(a-\delta)\,x]+\Omega[(b+\rho)\,y])\;(\delta>0, \rho>0 \text{ -- arbitrary}), \tag{2}$$

in which the norms are given by the formulas

$$\|\varphi\|_{\delta\rho} = \sup_z |\varphi(z)|\exp(M[(a-\delta)\,x]-\Omega[(b+\rho)\,y]). \tag{3}$$

We may obtain examples of the spaces $W_M{}^\Omega$ corresponding to the functions

$$M_1(x) = x^{1/\alpha}, \qquad M_2(x) = x\ln x,$$

$$\Omega_1(y) = y^{1/(1-\beta)}, \qquad \Omega_2(y) = e^y.$$

In particular, $W_{M_1}^{\Omega_1}$ coincides with $S_\alpha{}^\beta$ $(0<\alpha<1, \beta<1)$.

A2.4. The Question of the Nontriviality of Spaces of Type W

The space $W_M{}^\Omega$ is trivial if

$$\lim_{x\to\infty}[\Omega(by)-M(ax)] = -\infty \tag{1}$$

for any a and b.

We know one class of nontrivial spaces $W_M{}^\Omega$: It is the space $S_\alpha{}^\beta$ for $0<\alpha<1, \beta<1$.

Let us mention still another class of nontrivial spaces $W_M{}^\Omega$.

Let us call the continuous function $l(x)$ $(x>0)$ *slow* if for any $\epsilon>0$ and sufficiently large x, we have:

$$C'_\epsilon x^{-\epsilon} < l(x) < C_\epsilon x^\epsilon. \tag{2}$$

The space $W_M{}^\Omega$ is nontrivial if

$$M(x) = l(x) \cdot x^p, \qquad \Omega(x) \geqslant l(x) \cdot x^p,$$

where $l(x)$ is a slow function.

If the space $W_M{}^\Omega$ is nontrivial, then pairs (a, b) are known to exist for which the space $W_{M,a}^{\Omega,b}$ is nontrivial. Let such pairs be called "admissible." The domain of admissible pairs is an angle in the first quadrant of the (a, b) plane defined by an inequality of the form $\tan (b/a) \geqslant \gamma$ (or $\tan(b/a) > \gamma$).

All nontrivial spaces of type W are sufficiently rich in functions in the sense mentioned at the end of Section 8.

A2.5. Bounded Operators

The simplest bounded operators in the spaces of type W are the operators of differentiation and of multiplication by the independent variable.

For the entire analytic function $f(z)$ to be a multiplier in the space W^Ω, it is sufficient that it satisfy an inequality of the form

$$|f(z)| \leqslant C \exp[\Omega(b_0 y)](1 + |x|^h). \tag{1}$$

Multiplication by such a function hence transforms $W^{\Omega,b}$ into $W^{\Omega,b+b_0}$. If as small a number as desired may be taken as b_0 (here C depends on b_0), then $W^{\Omega,b}$ is transformed into itself.

An entire analytic function $f(z)$ will define the bounded multiplication operation in the space $W_{M,a}^{\Omega,b}$, if it satisfies the inequality

$$|f(z)| \leqslant C \exp[M(a_0 x) + \Omega(b_0 y)]\,. \tag{2}$$

Here $W_{M,a}^{\Omega,b}$ is transformed into $W_{M,a-a_0}^{\Omega,b+b_0}$. In order for the space $W_{M,a}^{\Omega,b}$ to be transformed into itself by multiplication by $f(z)$, it is sufficient that the constants a_0 and b_0 in the inequality (2) may be taken as small as desired (here $C = C_{a_0,b_0}$). In this case $f(z)$ will be a multiplier in the space $W_M{}^\Omega$ also.

In particular, the function $f(z) = e^{i\sigma z}$ with any real σ is a multiplier in each space $W^{\Omega,b}$ and $W_{M,a}^{\Omega,b}$.

A2.6. Fourier Transformation

Just as spaces of type S, the spaces of type W are transformed into each other by Fourier transformations. In order to clarify the connection existing here, let us present the Young definition of reciprocity. Let the functions $M(x)$ and $\Omega(y)$ be defined as in Sections A2.1–2. If the functions $\mu(\xi)$ and $\omega(\eta)$ in these definitions are mutually reversible so that

$$\mu[\omega(\eta)] = \eta, \qquad \omega[\mu(\xi)] = \xi, \tag{1}$$

then the functions $M(x)$ and $\Omega(x)$ are called *reciprocal according to Young*. In this case, the geometrically obvious Young inequality holds:

$$xy \leqslant M(x) + \Omega(y) \qquad (x \geqslant 0, y \geqslant 0), \tag{2}$$

where for each x there exists a $y = y(x)$, which together with the given x turns the inequality (1) into an equality.

Examples of pairs of mutually reciprocal functions are:

(1) $M(x) = x^p/p$, $\Omega(y) = y^q/q$, where $(1/p) + (1/q) = 1$,

(2) $M(x) = (x + 1)\ln(x + 1) - x$, $\Omega(y) = e^y - y - 1$.

It turns out that the following relationships hold:

$$\widetilde{W_M} = W^\Omega, \qquad \widetilde{W^\Omega} = W_M\,, \tag{3}$$

if $M(x)$ and $\Omega(y)$ are functions reciprocal according to Young. These relations are refined as follows:

$$\widetilde{W_{M,a}} = W^{\Omega,1/a}, \qquad \widetilde{W^{\Omega,b}} = W_{M,1/b}\,. \tag{4}$$

Furthermore, the relationship

$$\widetilde{W_M^{\ \Omega}} = W_{M_1}^{\Omega_1} \tag{5}$$

is valid, where Ω_1 is a function reciprocal to M according to Young, and M_1 is a function reciprocal to Ω according to Young. More exactly,

$$\widetilde{W_{M,a}^{\Omega,b}} = W_{M_1,1/b}^{\Omega_1,1/a}\,. \tag{6}$$

In all these relations, the wavy line $\sim$ may be interpreted as both the direct or the inverse Fourier transformation operator; the Fourier operators are bounded in all these cases.

In particular, the relationships

$$\widetilde{S_\alpha} = S^\beta, \qquad \widetilde{S_\alpha^{\ \beta}} = S_\beta^{\ \alpha}$$

are contained in these relations for the cases when α and β are strictly included between 0 and 1.

NOTES AND REFERENCES

Chapter I

The theory of linear metric and normed spaces was developed by F. Riesz and the school of S. Banach, starting during the 1920's. See the fundamental book by Banach [26], as well as the work of Mazur and Orlicz [17].

The first work on general linear topological spaces (A. N. Kolmogorov [32], J. von Neumann [18] A. N. Tikhonov [37]) was performed during the middle thirties. Specific linear topological spaces formed from number sequences are first considered by Köthe and Töplitz [13], and manifolds of linear continuous functionals in these spaces are first examined by them. Many authors were concerned with the construction of conjugate spaces in the general case and with the problem of reflexivity: V. L. Shmul'ian [39], Dieudonné [5], Mackey [15], Arens [1], Dieudonné and Schwartz [7]. In the last work in particular, spaces with compact bounded sets as well as the union (inductive limits) of metrizable spaces, were separated out and studied. The Bourbaki paper [3] was a continuation of this work, wherein the role of "T-spaces," spaces in which the Lemma of Section 3.5 is satisfied, is isolated. Some of the problems posed in [7] have been solved by Grothendieck [11]. Sebastião e Silva [23] and D. A. Raikov [35] studied the inductive and projective limits of normed spaces with completely continuous mappings (as in the "conditions of perfection" in Section 6.2). Köthe [14] and Grothendieck [10] first considered linear topological spaces of analytic functions.

The most detailed exposition of the general theory of linear topological spaces exists in Bourbaki [2]. See also the very complete survey of Dieudonné [6], with its detailed bibliography. Spaces with a countable number (field) of norms were considered by many authors, particularly Mazur and Orlicz [17]; countably normed spaces with compatible norms have apparently been introduced first by the authors [30].

Chapter II

Sections 1–2. S. L. Sobolev [36] first introduced generalized functions as continuous linear functionals in spaces of functions; he applied generalized functions to the uniqueness problem of the solution of the Cauchy problem for a hyperbolic equation. L. Schwartz first used the spaces $K(a)$, K, and S as fundamental spaces in his "Theory of Distributions" [21]. The authors proposed the first kinds of spaces with conditions of a strong decrease in the fundamental functions (and their derivatives) at infinity ($\infty \exp(-x^p)$) in [28]. Of the later work, let us mention L. Schwartz [22], and A. IA. Lepin and A. D. Myshkis [33]. G. E. Shilov constructed the theory of countably normed spaces $K\{M_p\}$ in preparing the present edition. The union of spaces $K\{M_p\}$ will also be considered in Volume 4.

Section 3. The possibility of dividing unity into polynomials in the space Z' was first established by Malgrange [16] and Ehrenpreis [8]. The proof with the "Hörmander staircase" was mentioned by Trève [24]; we present it with simplifications proposed by G. N. Zolotarev.

Section 4. Schwartz [21] mentioned the general form of the continuous linear functional in the spaces $K(a)$ and S; he obtained a theorem on the general form of a functional ("distribution") with bounded and single-point support. The scheme of determining continuous linear functionals in countably normed spaces was proposed by the authors [30].

Chapter III

Sections 1–2. The definition of the Fourier transform for functionals in the space S was given by Schwartz [21]; he thereby defined the Fourier transform for functions (and generalized functions) having a growth no greater than a power. The general scheme (functional in one space, Fourier transform in dual space) was constructed by the authors [28].

The necessity of considering many kinds of fundamental spaces and their conjugates with the corresponding Fourier transforms in various problems of analysis was clearly formulated by the authors in [28] and [30]. In [28], the authors introduced the specific spaces K_p , Z^p, $Z_p^{p'}$ corresponding to prescribed conditions of decrease at infinity for fundamental functions or their Fourier transforms; it hence turned out to be possible to write the Fourier transform for functions (and generalized functions) of arbitrary growth or with a given order of growth at infinity.

At approximately the same time Malgrange [16] and Ehrenpreis [8] constructed the Fourier transform for the particular case of a K space.

The results of Section 2.4.1 (fundamental solutions) are due to Malgrange and Ehrenpreis, and the results of Section 2.4.2 (the solutions of quasi-elliptic equations) to Schwartz [21].

Section 3. The definition of convolution utilized in our exposition is a modification of the Schwartz definition [21] (see Volume 1, Chapter I, Section 4, also), which could not be used in the original form with the degree of generality we needed. The theorem on harmonic functionals is due to Schwartz [21]. The theorem on the passage to the product after the convolution of the Fourier transform was proved by Schwartz [21] for functionals in S space; the proof in the general case was carried out by V. M. Borok [27]. Section 3.8 (Hilbert transform) is written according to an idea of N. Ya. Vilenkin.

Section 4. See [19] for the classical Paley–Wiener theorem. Its generalization to the case of first-order entire functions with power growth on the axis was given by Schwartz [21]; in our exposition, this theorem is formulated in a more complete form. The general scheme of Fourier transforms of first-order entire functions without any growth constraints on the axis was constructed by the authors in [28] (for the case of one variable). The theorem of Section 4.2 is due to Ehrenpreis [9]; the proof in our exposition is presented with simplifications proposed by G. N. Zolotarev.

Chapter IV

As already stated, the authors introduced the spaces of fundamental functions K_p, Z_p, $Z_p{}^p$ in [28] and applied them to the clarification of the question of classes of uniqueness of the solution of the Cauchy problem for systems of partial differential equations. Spaces of type S, later introduced and studied by G. E. Shilov [38], are a broader, and moreover, more natural class of spaces. The authors used them in [29] to construct an operator method in the problem of uniqueness of the solution of the Cauchy problem. The idea of considering the still broader class of generalized spaces of type S (with replacement of the sequences $k^{k\alpha}$ and $q^{q\beta}$ by a_k and b_q) is due to I. M. Gel'fand. A detailed exposition of the theory of spaces of type S is published here for the first time.

Section 7. See [20] for the classical Phragmen–Lindelöf theorem. Theorem 1 was proved initially for entire functions of a special kind needed in the investigation of solutions of the Cauchy problem; the general proof presented here was indicated by B. IA. Levin.

Section 8. The intial proof of the nontriviality of the spaces $Z_p{}^p$ $(= S_{1/p}^{1-(1/p)})$ was given by the authors in [28] for a dense set of values $p > 1$. The subsequent proof of G. E. Shilov [38], giving necessary and sufficient conditions for the nontriviality of spaces S_α^β is based on the V. Bernstein theorem [4] on the existence of an entire function of prescribed order of growth with a given index. The elementary proof on the nontriviality of the spaces S_α^β (Section 8.2) presented here, which is based on the construction and investigation of the special function $\psi(z)$, is due to B. IA. Levin.

See Mandelbrojt [34] for the Carleman–Ostrovski theorem.

Appendix 1. The idea of considering generalized spaces of type S issues from I. M. Gel'fand. See [25] for the K. I. Babenko theorem (on the nontriviality of generalized spaces).

Appendix 2. B. L. Gurevich [31] constructed spaces of type W. Later, L. Hörmander proposed a rather more general scheme.

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INDEX

A
B 7
C 8
D 9
E 0
F 1
G 2
H 3
I 4
J 5