Inequalities and the Triangular Notation

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Abstract

The **Triangular Notation** is a convenient, though not standard, way of representing three-variable homogeneous polynomials because it simplifies polynomial manipulations. It helps us visualize patterns of polynomial coefficients and provides us an intuitive understanding of polynomial inequalities.

Keywords: 3-variable homogeneous polynomials, majorization, AM-GM, Muirhead's inequality, Schur's inequality

1 The Triangular Notation

It's easy to miss a term when we do polynomial multiplications. The triangular notation gives us a plain and less error-prone way to deal with three-variable homogeneous polynomials. For example, we represent $a + 4b + 3c$ as:

$$\begin{array}{ccc} & 1 & \\ 4 & & 3 \end{array}$$

A quadratic expression is represented as:

$$\begin{array}{ccccc} & & [a^2] & & \\ & [ab] & & [ac] & \\ [b^2] & & [bc] & & [c^2] \end{array}$$

Here $[a^x b^y c^z]$ is the coefficient of the term $a^x b^y c^z$. So in the triangular notation, $a^2 + b^2 + c^2 + ab + bc + ca$ is just:

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 1 & & 1 & & 1 \end{array}$$

Similarly, a quartic expression is represented as:

$$\begin{array}{ccccccccc} & & & & [a^4] & & & & \\ & & & [a^3b] & & [a^3c] & & & \\ & & [a^2b^2] & & [a^2bc] & & [a^2c^2] & & \\ & [ab^3] & & [ab^2c] & & [abc^2] & & [ac^3] & \\ [b^4] & & [b^3c] & & [b^2c^2] & & [bc^3] & & [c^4] \end{array}$$

We do polynomial multiplications by shifting and adding.

Example 1. *Expand* $(a + 4b + 3c)(a^2 + b^2 + c^2 + ab + bc + ca)$.

Solution.

$$\left(\begin{array}{ccc} & 1 & \\ 4 & & 3 \end{array}\right)\left(\begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 1 & & 1 & & 1 \end{array}\right)$$

$$= 1\left(\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 1 & & 1 & \\ 0 & & 0 & & 0 & & 0 \end{array}\right) + 4\left(\begin{array}{ccccccc} & & & 0 & & & \\ & & 1 & & 0 & & \\ & 1 & & 1 & & 0 & \\ 1 & & 1 & & 1 & & 0 \end{array}\right) + 3\left(\begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & & 1 & & \\ & 0 & & 1 & & 1 & \\ 0 & & 1 & & 1 & & 1 \end{array}\right)$$

$$= \begin{array}{ccccccc} & & & 1 & & & \\ & & 5 & & 4 & & \\ & 5 & & 8 & & 4 & \\ 4 & & 7 & & 7 & & 3 \end{array}$$

That is,

$$\begin{aligned}&(a + 4b + 3c)(a^2 + b^2 + c^2 + ab + bc + ca)\\ &= a^3 + 5a^2b + 4a^2c + 5ab^2 + 8abc + 4ac^2 + 4b^3 + 7b^2c + 7bc^2 + 3c^3.\end{aligned}$$

2 First Inequalities

It is a fact of elementary algebra that any square is nonnegative. In particular, $(a - b)^2 \geq 0$. Expanding the LHS gives $a^2 + b^2 \geq 2ab$. Recall that the arithmetic mean of two numbers, say a^2 and b^2, is their average, or $\dfrac{a^2 + b^2}{2}$. Also, the geometric mean of two nonnegative numbers is the square root of their product, or $\sqrt{a^2b^2} = ab$. So the above inequality says that the arithmetic mean of two nonnegative numbers is at least their geometric mean. In fact, that is true for n numbers, which is the result below.

Theorem 1 (AM-GM)**.** *Let a_1, ..., a_n be positive real numbers. We have*

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n}.$$

Before we move on, let's introduce the **symmetric sum** notation $\sum_{\text{sym}}$, as it comes handy when we need to deal with more complex inequalities. If we have three variables x, y, z, we can write them as x_1, x_2, and x_3. Note that there are six permutations of (1, 2, 3). Namely: (1, 2, 3), (2, 3, 1), (3, 1, 2), (1, 3, 2), (3, 2, 1), and (2, 1, 3). Then the symmetric sum of a function $f(x_1, x_2, x_3)$ applies the function to all possible permutations of x_1, x_2, x_3. For example,

$$\begin{aligned}&\sum_{\text{sym}} x^3 = x^3 + y^3 + z^3 + x^3 + y^3 + z^3 = 2x^3 + 2y^3 + 2z^3,\\ &\sum_{\text{sym}} x^2y = x^2y + y^2z + z^2x + x^2z + y^2x + z^2y,\\ &\sum_{\text{sym}} xyz = 6xyz.\end{aligned}$$

3 Weighted AM-GM

It takes practice to get used to the triangular notation. This notation may seem clumsy, but it nevertheless helps us see patterns in the coefficients of a 3-variable homogeneous polynomial. Next let's look at some polynomial inequalities. We are now all familiar with AM-GM. For any real numbers a and b, AM-GM states that $a^2 + b^2 \geq 2ab$. Representing it in the triangular notation, we have:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 0 & & 0 \end{array} \quad \geq \quad \begin{array}{ccccc} & & 0 & & \\ & 2 & & 0 & \\ 0 & & 0 & & 0 \end{array}$$

We can think of positive coefficients of a polynomial as "weights". When we slide them to their center of mass, AM-GM tells us that the total sum of the expression will not increase. The coefficients do not have to be integers. For example,

$$\begin{array}{ccccccc} & & & \frac{1}{3} & & & \\ & & 0 & & 0 & & \\ & 0 & & 0 & & 0 & \\ 0 & & 0 & & 0 & & \frac{2}{3} \end{array} \quad \geq \quad \begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & & 0 & & \\ & 0 & & 0 & & 1 & \\ 0 & & 0 & & 0 & & 0 \end{array}$$

That is, $\dfrac{1}{3}a^3 + \dfrac{2}{3}c^3 \geq ac^2$. This is just the weighted AM-GM. Indeed, AM-GM on the three variables $\frac{1}{3}a^3, \frac{1}{3}c^3, \frac{1}{3}c^3$ yields

$$\frac{1}{3}(a^3 + c^3 + c^3) \geq 3\sqrt[3]{\frac{a^3c^6}{27}} = ac^2.$$

Try the following example.

Example 2. *Prove that for all positive reals a, b, c*

$$a^3b + b^3c + c^3a \geq abc(a + b + c).$$

Proof. We will use weighted AM-GM. That is, we seek 3 nonnegative real weights w_1, w_2, w_3 with $w_1 + w_2 + w_3 = 1$ such that

$$(a^3b)^{w_1}(b^3c)^{w_2}(c^3a)^{w_3} = a^2bc$$

since then adding cyclic versions of the AM-GM inequality with these weights will yield the desired result. Comparing exponents, we have the following system of equations:

$$\begin{aligned} w_1 + w_2 + w_3 &= 1, \\ 3w_1 + w_3 &= 2, \\ 3w_2 + w_1 &= 1, \\ 3w_3 + w_2 &= 1, \end{aligned}$$

Solving gives $w_1 = \frac{4}{7}$, $w_2 = \frac{1}{7}$, $w_3 = \frac{2}{7}$. Now, by weighted AM-GM, we have

$$\begin{aligned}
\frac{4a^3b + b^3c + 2c^3a}{7} &\geq \sqrt[7]{a^{14}b^7c^7} = a^2bc, \\
\frac{4b^3c + c^3a + 2a^3b}{7} &\geq \sqrt[7]{a^7b^{14}c^7} = ab^2c, \\
\frac{4c^3a + a^3b + 2b^3c}{7} &\geq \sqrt[7]{a^7b^7c^{14}} = abc^2.
\end{aligned}$$

Taking the sum of these inequalities gives the desired result. □

Using barycentric coordinates yields a succinct representation of the above solution. Simply sum the cyclic versions of the following inequality:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & \frac{4}{7} & & 0 & & & \\
 & & 0 & & 0 & & 0 & & \\
 & 0 & & 0 & & 0 & & \frac{2}{7} & \\
0 & & \frac{1}{7} & & 0 & & 0 & & 0
\end{array}
\quad \geq \quad
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
0 & & 0 & & 0 & & 0 & & 0
\end{array}$$

The n-variable version of weighted AM-GM is as follows:

Theorem 2 (Weighted AM-GM)**.** *Let w_1, ..., w_n be nonnegative real numbers such that $w_1 + \cdots + w_n = 1$. For all positive real numbers x_1, ..., x_n, we have*

$$w_1x_1 + w_2x_2 + \cdots + w_nx_n \geq x_1^{w_1}x_2^{w_2}\cdots x_n^{w_n}.$$

4 Majorization and Muirhead's Inequality

In this section, we introduce Muirhead's inequality, a generalized form of AM-GM. Let's first define the term **majorization**.

Definition 1. *Let $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ be non-increasing finite sequences of real numbers. Then x **majorizes** y, denoted as $x \succ y$, if,*

$$\begin{gathered}
x_1 \geq y_1, \\
x_1 + x_2 \geq y_1 + y_2, \\
\cdots \\
x_1 + x_2 + \cdots + x_{n-1} \geq y_1 + y_2 + \cdots + y_{n-1}, \\
x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_n.
\end{gathered}$$

For example, $(4, 2) \succ (3, 3)$ and $(5, 2, 1) \succ (3, 3, 2)$.

Theorem 3 (Muirhead's Inequality)**.** *If a sequence $A = (a_1, ..., a_n)$ majorizes a sequence $B = (b_1, ..., b_n)$, then given a set of positive reals $x_1, x_2, \cdots, x_n$, we have*

$$\sum_{sym} {x_1}^{a_1} {x_2}^{a_2} \cdots {x_n}^{a_n} \geq \sum_{sym} {x_1}^{b_1} {x_2}^{b_2} \cdots {x_n}^{b_n},$$

where the symmetric sum is taken over all $n!$ permutations of x_1, x_2, ..., x_n.

Remark. *Since $(1, 0, ..., 0) \succ (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})$,*

$$\sum_{sym} x = (n-1)! \sum_i x_i \geq \sum_{sym} (x_1 x_2 ... x_n)^{1/n} = n!(x_1 x_2 ... x_n)^{1/n},$$

or

$$\sum_i x_i \geq n \sqrt[n]{x_1 x_2 ... x_n}.$$

Thus AM-GM is a special case of Muirhead's inequality.

For example, $(3, 1, 0) \succ (2, 1, 1)$, so given three positive real numbers x, y, and z, we have

$$\sum_{\text{sym}} x^3 y \geq \sum_{\text{sym}} x^2 yz.$$

Expanding out, we get the inequality:

$$x^3y + x^3z + xy^3 + xz^3 + y^3z + yz^3 \geq 2x^2yz + 2xy^2z + 2xyz^2.$$

This inequality can be visualized easily with the triangular notation:

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 1 & & 1 & & & \\ & & 0 & & 0 & & 0 & & \\ & 1 & & 0 & & 0 & & 1 & \\ 0 & & 1 & & 0 & & 1 & & 0 \end{array} \quad \geq \quad \begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 0 & & 0 & & & \\ & & 2 & & 0 & & 2 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 0 & & 0 & & 2 & & 0 & & 0 \end{array} \quad \geq \quad \begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 2 & & 0 & & \\ & 0 & & 2 & & 2 & & 0 & \\ 0 & & 0 & & 0 & & 0 & & 0 \end{array}$$

Example 3. *Let a, b, and c be positive real numbers such that $abc = 1$. Prove that*

$$a + b + c \leq a^2 + b^2 + c^2.$$

Proof. Since the given inequality is not homogeneous, we multiply the LHS by $\sqrt[3]{abc} = 1$ to homogenize it. We get the following equivalent inequality

$$a^{\frac{4}{3}} b^{\frac{1}{3}} c^{\frac{1}{3}} + a^{\frac{1}{3}} b^{\frac{4}{3}} c^{\frac{1}{3}} + a^{\frac{1}{3}} b^{\frac{1}{3}} c^{\frac{4}{3}} \leq a^2 + b^2 + c^2.$$

Since $(2, 0, 0) \succ (\frac{4}{3}, \frac{1}{3}, \frac{1}{3})$, Muirhead's inequality gives us the desired result exactly. $\square$

5 Schur's Inequality

The following easy to prove inequality is useful.

Theorem 4 (Schur's Inequality)**.** *Let $t > 0$ be a real number. Then for all nonnegative real numbers x, y, and z,*

$$x^t(x-y)(x-z) + y^t(y-z)(y-x) + z^t(z-x)(z-y) \geq 0,$$

where the equality holds only when $x = y = z \neq 0$ or some two of x, y, and z are equal and the third is zero.

Proof. Since the expressions are symmetric, without loss of generality, we may assume that $x \geq y \geq z$. Then we can rewrite the inequality as:

$$(x-y)\big(x^t(x-z) - y^t(y-z)\big) + z^t(x-z)(y-z) \geq 0.$$

Note that each term of the LHS is nonnegative, so the inequality is true. □

When $t = 1$, Schur's inequality is a 3rd degree polynomial.

$$x^3 + y^3 + z^3 + 3xyz - \big(x^2(y+z) + y^2(z+x) + z^2(x+y)\big) \geq 0.$$

For convenience, we represent 3rd degree Schur's inequality as

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & -1 & & -1 & & \\ & -1 & & 3 & & -1 & \\ 1 & & -1 & & -1 & & 1 \end{array}$$

When $t = 2$, Schur's inequality is a 4th degree polynomial.

$$a^4 + b^4 + c^4 + abc(a+b+c) \geq a^3b + b^3c + c^3a + ab^3 + bc^3 + ca^3.$$

Example 4. *Show that for any positive real numbers a, b, and c,*

$$(a+b-c)(b+c-a)(c+a-b) \leq abc.$$

Proof. The left hand side is

$$\left(\begin{array}{ccc} & 1 & \\ 1 & & -1 \end{array}\right)\left(\begin{array}{ccc} & -1 & \\ 1 & & 1 \end{array}\right)\left(\begin{array}{ccc} & 1 & \\ -1 & & 1 \end{array}\right).$$

Multiplying out, we get

$$\text{LHS} = \left(\begin{array}{ccccccc} & & & -1 & & & \\ & & 1 & & 1 & & \\ & 1 & & -2 & & 1 & \\ -1 & & 1 & & 1 & & -1 \end{array}\right).$$

Combining with the RHS, we get Schur's inequality. So we are done. □

6 Exercises

1. Prove that for any positive real numbers a, b, and c,
$$(a+b+c)^2 + \frac{9abc}{a+b+c} \geq 4(ab+bc+ca).$$

2. Let a, b, and c be nonnegative real numbers such that $a+b+c=1$. Prove that
$$a^3+b^3+c^3+6abc \geq \frac{1}{4}.$$

3. Show that for any positive real numbers a, b, and c,
$$(a+b+c)(a^3+b^3+c^3+3abc) \geq 2(a^2+b^2+c^2)(ab+bc+ca).$$

4. Show that for any positive real numbers a, b, and c,
$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + \frac{4abc}{(a+b)(b+c)(c+a)} \geq 2.$$

5. Show that for any positive real numbers a, b, and c,
$$\frac{1}{a^3+b^3+abc} + \frac{1}{b^3+c^3+abc} + \frac{1}{c^3+a^3+abc} \leq \frac{1}{abc}.$$

6. Show that for nonnegative reals a, b, and c,
$$2(a^6+b^6+c^6) + 16(a^3b^3+b^3c^3+c^3a^3) \geq 9a^4(b^2+c^2) + 9b^4(c^2+a^2) + 9c^4(a^2+b^2).$$

7. Let a, b, c be positive reals such that $a+b \geq c$; $b+c \geq a$; and $c+a \geq b$. Prove that
$$2a^2(b+c) + 2b^2(c+a) + 2c^2(a+b) \geq a^3+b^3+c^3+9abc.$$

8. Let a, b, c be real numbers such that $abc = -1$. Show that
$$a^4+b^4+c^4+3(a+b+c) \geq \sum_{\text{sym}} \frac{a^2}{b}.$$

9. (MOP 2011) Let a, b, c be positive real numbers such that $a+b+c=3$. Show that
$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 1 + 2\sqrt{\frac{a^2+b^2+c^2}{3abc}}.$$

References

[1] Evan Chen, Supersums of Square-Weights: A Dumbass's Perspective, `web.evanchen.cc/handouts/SOS_Dumbass/SOS_Dumbass.pdf`

[2] Brian Hamrick, The Art of Dumbassing, `tjhsst.edu/~2010bhamrick/files/dumbassing.pdf`

[3] Farrell Eldrian Wu, Triangular Polynomial Notation, `www.overleaf.com/articles/triangular-polynomial-notation/zpskcfqygqzx/viewer.pdf`