A brief overview of Dirac delta function and its derivatives

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Abstract

In this paper, Dirac delta function is revisited and we derive the N^{th} derivative of Dirac delta function and evaluate the Fourier Transform of the N^{th} derivative of Dirac delta function and derive related results. These results are used in deriving the Fourier Transform of two sided decaying exponentials and their derivatives.

1. Introduction

In this paper, Dirac delta function is revisited and we derive the N^{th} derivative of Dirac delta function and evaluate the Fourier Transform of the N^{th} derivative of Dirac delta function and derive related results. These results are used in deriving the Fourier Transform of two sided decaying exponentials and their derivatives.

The Dirac delta function or Impulse symbol $\delta(t)$ is in fact not a function at all, but a distribution such as a probability distribution, that is also a measure, in the sense that it assigns a value to a function. We could consider it a special function with infinite height at $t = 0$, zero width and an area of 1. The Dirac Delta function can be viewed as the derivative of the Heaviside unit step function $H(t)$ as follows.

$$\begin{gathered}
\frac{d}{dt}H(t) = \delta(t) \\
\int_{-\infty}^{\infty} \delta(t)dt = 1 \\
\delta(t) = 0, \quad t \neq 0
\end{gathered} \tag{1}$$

The Dirac delta has the following sifting property for a continuous compactly supported function $f(t)$.

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a) \tag{2}$$

This Dirac delta $g(t) = \delta(t)$ has a Fourier Transform given by $G(\omega) = \int_{-\infty}^{\infty} \delta(t)e^{-i\omega t}dt = 1$. Let us consider the inverse Fourier Transform of this function $G(\omega)$.

$$
\begin{aligned}
d_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{i\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}d\omega \\
= \lim_{B\to\infty} \frac{1}{2\pi}[\frac{e^{i\omega t}}{it}]_{-B}^{B} \\
d_0(t) = \lim_{B\to\infty} \frac{1}{\pi t}\sin(Bt)
\end{aligned}
\tag{3}
$$

We see that $d_0(t) = \delta(t)$ where $\delta(t)$ can be viewed as the limiting case of a decaying sinusoidal function $\frac{\sin(Bt)}{\pi t}$ which is called the sinc pulse, and as $B \to \infty$, height of the function goes to infinity at t=0, because $\lim_{B\to\infty} \frac{1}{\pi t}\sin(Bt) = \lim_{B\to\infty} \frac{B}{\pi}\cos(Bt)$ as per L'Hospital's Rule. Its zero crossings become more and more closer as $B \to \infty$. The area under the function $d_0(t)$ is unity which is shown below.

We can also show that the Fourier Transform of $d_0(t)$ is 1. We know that normalized sinc pulse function $sinc(Bt) = \frac{\sin(\pi Bt)}{\pi Bt}$ whose fourier transform is given by a rectagular pulse $\frac{1}{B}rect(\frac{f}{B})$.

$$
\begin{aligned}
d_0(t) = \lim_{B\to\infty} \frac{1}{\pi t}\sin(Bt) = \lim_{B\to\infty} \frac{B}{\pi}sinc(\frac{Bt}{\pi}) \\
D_0(\omega) = \int_{-\infty}^{\infty} d_0(t)e^{-i\omega t}dt = \lim_{B\to\infty} rect(\frac{f\pi}{B}) = 1 \\
\int_{-\infty}^{\infty} d_0(t)dt = 1
\end{aligned}
\tag{4}
$$

Hence $d_0(t)$ and $D_0(\omega)$ form a Fourier Transform pair $d_0(t) \xrightarrow{\text{F}} D_0(\omega)$. In the figure in next page, we can visualize these functions for finite value of B. We have replaced B by W in the figure.

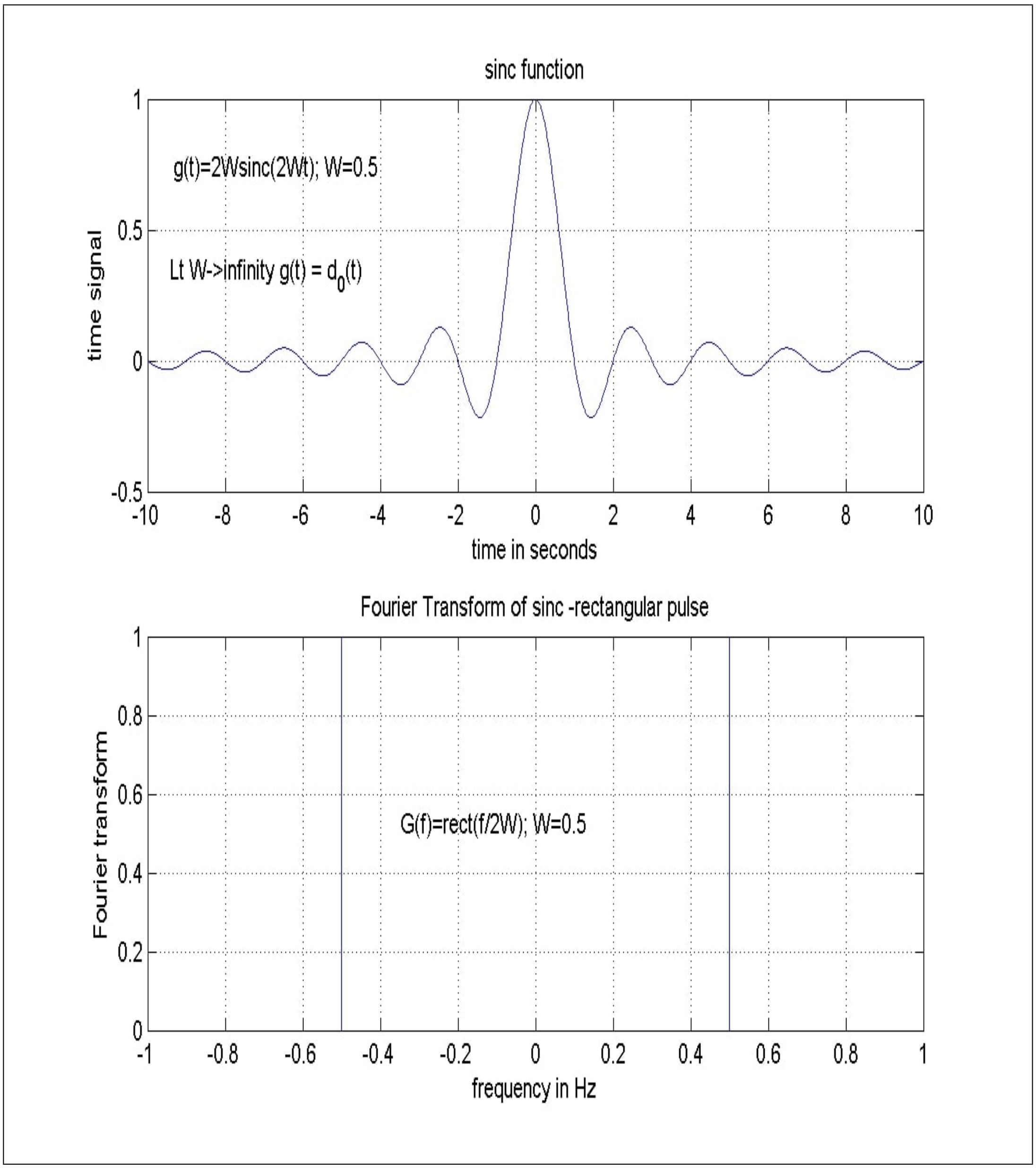
sinc function
g(t)=2Wsinc(2Wt); W=0.5
Lt W->infinity g(t) = d0(t)
time signal
time in seconds
1
0.5
0
-0.5
-10
-8
-6
-4
-2
0
2
4
6
8
10
Fourier Transform of sinc -rectangular pulse
G(f)=rect(f/2W); W=0.5
Fourier transform
frequency in Hz
1
0.8
0.6
0.4
0.2
0
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

Figure 1:

1.1. Derivatives of Dirac Delta and their Fourier Transforms

In this section, we will derive the N^{th} derivative of Dirac delta function $d_0(t)$ given by $d_N(t) = \frac{d^N}{dt^N} d_0(t)$, and show that it forms a fourier transform pair with $D_N(\omega) = (i\omega)^N$.

Let us consider the first derivative of Dirac delta function $d_0(t)$ given by $d_1(t)$ and show that $d_1(t) = \frac{d}{dt} d_0(t)$ is the inverse fourier transform of $D_1(\omega) = (i\omega)$.

$$\begin{aligned} d_0(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} d\omega \\ d_1(t) = \frac{d}{dt} d_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{d}{dt} e^{i\omega t} d\omega &= \frac{1}{2\pi}\int_{-\infty}^{\infty} (i\omega) e^{i\omega t} d\omega \\ D_1(\omega) &= (i\omega) \\ D_1(\omega) &\xrightarrow{F^{-1}} d_1(t) \end{aligned} \tag{5}$$

Let us consider the second derivative of Dirac delta function $d_0(t)$ given by $d_2(t) = \frac{d^2}{dt^2} d_0(t)$ and show that $d_2(t)$ is the inverse fourier transform of $D_2(\omega) = (i\omega)^2$.

$$\begin{aligned} d_2(t) = \frac{d^2}{dt^2} d_0(t) = \frac{d}{dt} d_1(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (i\omega) \frac{d}{dt} e^{i\omega t} d\omega &= \frac{1}{2\pi}\int_{-\infty}^{\infty} (i\omega)^2 e^{i\omega t} d\omega \\ D_2(\omega) &= (i\omega)^2 \\ D_2(\omega) &\xrightarrow{F^{-1}} d_2(t) \end{aligned} \tag{6}$$

We can consider the N^{th} derivative of Dirac delta function $d_0(t)$ given by $d_N(t) = \frac{d^N}{dt^N} d_0(t)$ and show that $d_N(t)$ is the inverse fourier transform of $D_N(\omega) = (i\omega)^N$.

$$\begin{aligned} d_N(t) = \frac{d^N}{dt^N} d_0(t) = \frac{d}{dt} d_{N-1}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (i\omega)^{N-1} \frac{d}{dt} e^{i\omega t} d\omega &= \frac{1}{2\pi}\int_{-\infty}^{\infty} (i\omega)^N e^{i\omega t} d\omega \\ D_N(\omega) &= (i\omega)^N \\ D_N(\omega) &\xrightarrow{F^{-1}} d_N(t) \end{aligned} \tag{7}$$

We have derived the N^{th} derivative of Dirac delta function $d_0(t)$ given by $d_N(t) = \frac{d^N}{dt^N} d_0(t)$, and we see that it is the inverse fourier transform of $D_N(\omega) = (i\omega)^N$ and the result holds, as $N \to \infty$.

1.2. Forward Fourier Transform of Dirac Delta derivatives

Let us find the forward Fourier Transform of the N^{th} derivative of Dirac delta function given by $d_N(t)$ and show that it is indeed $D_N(\omega) = (i\omega)^N$ and that they form Fourier Transform pair.

Let us derive the first derivative of Dirac delta function $d_0(t)$ given by $d_1(t)$ and show that $d_1(t) = \frac{d}{dt}d_0(t)$ has a fourier transform given by $D_1(\omega) = (i\omega)$.

$$\begin{aligned} d_0(t) &= \lim_{B\to\infty} \frac{1}{\pi t}\sin(Bt) \\ d_1(t) = \frac{d}{dt}d_0(t) &= \lim_{B\to\infty} \frac{1}{\pi t^2}[Bt\cos(Bt) - \sin(Bt)] \end{aligned} \tag{8}$$

We take the fourier transform of $d_1(t)$ as follows.

$$D_1(\omega) = \int_{-\infty}^{\infty} d_1(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} \frac{d}{dt}d_0(t)e^{-i\omega t}dt \tag{9}$$

For a general function $g(t)$ which goes to zero as $|t| \to \infty$, we can write $\frac{d}{dt}[g(t)e^{-i\omega t}] = -(i\omega)g(t)e^{-i\omega t} + e^{-i\omega t}\frac{d}{dt}g(t)$ and hence we can write $\int_{-\infty}^{\infty} \frac{d}{dt}g(t)e^{-i\omega t}dt = (i\omega)\int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt$, assuming $[g(t)e^{-i\omega t}]_{-\infty}^{\infty} = 0$.

We can use this result for $g(t) = d_0(t)$ in equation below, given that $d_1(t)$ goes to zero as $|t| \to \infty$ and we know $\int_{-\infty}^{\infty} d_0(t)e^{-i\omega t}dt = 1$ as derived in earlier section.

$$\begin{aligned} D_1(\omega) = \int_{-\infty}^{\infty} d_1(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} \frac{d}{dt}d_0(t)e^{-i\omega t}dt = (i\omega)\int_{-\infty}^{\infty} d_0(t)e^{-i\omega t}dt &= (i\omega) \\ d_1(t) \xrightarrow{\text{F}} D_1(\omega) &= (i\omega) \end{aligned} \tag{10}$$

Let us consider the second derivative of Dirac delta function $d_0(t)$ given by $d_2(t) = \frac{d^2}{dt^2}d_0(t)$ and show that $d_2(t)$ has a fourier transform given by $D_2(\omega) = (i\omega)^2$. We use result in above equation $\int_{-\infty}^{\infty} d_1(t)e^{-i\omega t}dt = (i\omega)$ in equation below.

$$\begin{aligned} D_2(\omega) = \int_{-\infty}^{\infty} d_2(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} \frac{d}{dt}d_1(t)e^{-i\omega t}dt = (i\omega)\int_{-\infty}^{\infty} d_1(t)e^{-i\omega t}dt &= (i\omega)^2 \\ d_2(t) \xrightarrow{\text{F}} D_1(\omega) &= (i\omega)^2 \end{aligned} \tag{11}$$

We can consider the N^{th} derivative of Dirac delta function $d_0(t)$ given by $d_N(t) = \frac{d^N}{dt^N} d_0(t)$ and show that $d_N(t)$ has a fourier transform given by $D_N(\omega) = (i\omega)^N$ as follows, by using results in above equations.

$$
\begin{aligned}
D_N(\omega) = \int_{-\infty}^{\infty} d_N(t) e^{-i\omega t} dt = \int_{-\infty}^{\infty} \frac{d}{dt} d_{N-1}(t) e^{-i\omega t} dt = (i\omega) \int_{-\infty}^{\infty} d_{N-1}(t) e^{-i\omega t} dt = (i\omega)^N \\
d_N(t) = \frac{d^N}{dt^N} d_0(t) \xrightarrow{\text{F}} D_N(\omega) = (i\omega)^N
\end{aligned}
\tag{12}
$$

In this section, we have derived the N^{th} derivative of Dirac delta function $d_0(t)$ given by $d_N(t) = \frac{d^N}{dt^N} d_0(t)$, and shown that it forms a fourier transform pair with $D_N(\omega) = (i\omega)^N$ for $n = 1, 2, ..N$, as $N \to \infty$.

2. Two sided decaying exponentials and their derivatives

We know that a two-sided decaying exponential function $g_0(t) = e^{at} u(-t) + e^{-at} u(t)$, where $u(t)$ is Heaviside unit step function and $a > 0$ is real, has the Fourier Transform given by $G_0(\omega)$ as follows. (link)

$$
G_0(\omega) = \int_{-\infty}^{\infty} g_0(t) e^{-i\omega t} dt = \frac{1}{a - i\omega} + \frac{1}{a + i\omega} = \frac{2a}{a^2 + \omega^2}
\tag{13}
$$

Let us compute the fourier transform of the second derivative of $g_0(t)$ as follows. We get Dirac delta $d_0(t) = \delta(t)$ in the second derivative.

$$
\begin{aligned}
g_2(t) = \frac{d^2 g_0(t)}{dt^2} = a^2 e^{at} u(-t) + a^2 e^{-at} u(t) + (-2a) d_0(t) \\
G_2(\omega) = \int_{-\infty}^{\infty} g_2(t) e^{-i\omega t} dt = \frac{2a^3}{a^2 + \omega^2} - 2a = \frac{2a^3 - 2a^3 - 2a\omega^2}{a^2 + \omega^2} = (-\omega^2) \frac{2a}{a^2 + \omega^2}
\end{aligned}
\tag{14}
$$

Let us compute the fourier transform of the third derivative of $g_0(t)$ as follows. We use the fact that $d_1(t) = \frac{d(d_0(t))}{dt}$ whose fourier transform is given by $i\omega$.

$$
\begin{aligned}
g_3(t) = \frac{d^3 g_0(t)}{dt^3} = a^3 e^{at} u(-t) - a^3 e^{-at} u(t) + (-2a) d_1(t) \\
G_3(\omega) = i\omega \frac{2a^3}{a^2 + \omega^2} - 2a(i\omega) = i\omega(-\omega^2) \frac{2a}{a^2 + \omega^2}
\end{aligned}
\tag{15}
$$

Let us compute the fourier transform of the fourth derivative of $g_0(t)$ as follows. We use the fact that $d_2(t) = \frac{d^2(d_0(t))}{dt^2}$ whose fourier transform is given by $(i\omega)^2$. We get a new Dirac delta, in every even derivative.

$$
\begin{aligned}
g_4(t) = \frac{d^4 g_0(t)}{dt^4} = a^4 e^{at}u(-t) + a^4 e^{-at}u(t) + (-2a)d_2(t) + (-2a^3)d_0(t) \\
G_4(\omega) = \frac{2a^5}{a^2+\omega^2} - 2a(-\omega^2) - 2a^3 = (-\omega^2)^2 \frac{2a}{a^2+\omega^2}
\end{aligned}
\tag{16}
$$

Let us compute the fourier transform of the $(2N-1)^{th}$ odd order derivative of $g_0(t)$ as follows. We use the fact that $d_{2N-1}(t) = \frac{d^{2N-1}(d_0(t))}{dt^{2N-1}}$ whose fourier transform is given by $(i\omega)^{2N-1}$.

$$
\begin{aligned}
g_{2N-1}(t) = \frac{d^{2N-1} g_0(t)}{dt^{2N-1}} = a^{2N-1} e^{at}u(-t) - a^{2N-1} e^{-at}u(t) - 2\sum_{r=0}^{N-2} (a)^{2r+1} d_{2r+1}(t) \\
G_{2N-1}(\omega) = (i\omega)\frac{2a^{2N-1}}{a^2+\omega^2} + (i\omega)(-2)\sum_{r=0}^{N-2} (a)^{2r+1}(-\omega^2)^{N-2-r} = (i\omega)(-\omega^2)^{N-1}\frac{2a}{a^2+\omega^2}
\end{aligned}
\tag{17}
$$

Let us compute the fourier transform of the $2N^{th}$ even order derivative of $g_0(t)$ as follows. We use the fact that $d_{2N}(t) = \frac{d^{2N}(d_0(t))}{dt^{2N}}$ whose fourier transform is given by $(i\omega)^{2N}$. We get a new Dirac delta, in every even derivative.

$$
\begin{aligned}
g_{2N}(t) = \frac{d^{2N} g_0(t)}{dt^{2N}} = a^{2N} e^{at}u(-t) + a^{2N} e^{-at}u(t) - 2\sum_{r=0}^{N-1} (a)^{2N-1-2r} d_{2r}(t) \\
G_{2N}(\omega) = \frac{2a^{2N+1}}{a^2+\omega^2} - 2\sum_{r=0}^{N-1} (a)^{2N-1-2r}(-\omega^2)^r = (-\omega^2)^N \frac{2a}{a^2+\omega^2}
\end{aligned}
\tag{18}
$$

Thus we can compute the fourier transform of two sided decaying exponential functions using the fourier transforms of Dirac delta and its derivatives.

3. Conclusion

In this paper, Dirac delta function is revisited and we have derived the N^{th} derivative of Dirac delta function and evaluated the Fourier Transform of the N^{th} derivative of Dirac delta function and derived related results. These results are used in deriving the Fourier Transform of two sided decaying exponentials and their derivatives.

References

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[3] Dirac delta Function, Wikipedia

[4] Dirac delta Function, Mathworld

4. Appendix A

4.1. Heaviside unit step function

Let us consider the Heaviside Unit Step Function $H(t) = u(t)$ which is defined as follows.

$$\begin{aligned} u(t) &= 0 \quad t < 0 \\ &= \frac{1}{2} \quad t = 0 \\ &= 1 \quad t > 0 \end{aligned} \tag{19}$$

Heaviside Unit Step Function $u(t)$ is related to Dirac delta function $\delta(t)$ as follows.

$$\begin{aligned} \frac{d}{dt}u(t) &= \delta(t) \\ u(t) &= \int_{-\infty}^{t} \delta(\tau) d\tau \end{aligned} \tag{20}$$

We can derive the fourier transform of Heaviside Unit Step Function as follows.

$$\begin{aligned} U(\omega) &= \int_{-\infty}^{\infty} u(t)e^{-i\omega t} dt \\ U(\omega) &= \frac{1}{i\omega} + \frac{\delta(\omega)}{2} \end{aligned} \tag{21}$$

Let us consider the integration property of the Fourier Transform and consider the function $g(t) = \int_{-\infty}^{t} m(\tau)d\tau = m(t) * u(t)$ where $*$ denotes convolution.

The fourier transform of $g(t)$ is given by $G(f) = M(f)U(f) = \frac{M(f)}{i\omega} + \frac{M(0)\delta(f)}{2}$ where $\omega = 2\pi f$.

We consider the Nth derivative of Dirac delta given by $m(t) = d_N(t)$, we know that $M(\omega) = (i\omega)^N$ and we get $G(\omega) = (i\omega)^{N-1}$.

4.2. Sifting property of Dirc delta function

Let us now examine the sifting properties of the derivatives of delta function. For Dirac delta function $d_0(t) = \delta(t)$ and a continuous compactly supported function $f(t)$, the following equations are well known.

$$\begin{aligned} f(t)\delta(t) &= f(0)\delta(t) \\ \int_{-\infty}^{\infty} \delta(t)dt &= 1 \\ \int_{-\infty}^{\infty} \delta(\tau)f(t-\tau)d\tau &= f(t) \end{aligned} \tag{22}$$

Let us derive similar relations for the Nth derivative of Dirac delta given by $d_N(t)$. Given that $d_N(t) \Longleftrightarrow D_N(\omega) = (i\omega)^N$ form a fourier transform pair, using the notation $F[f(t)]$ and $F^{-1}[F(\omega)]$ to describe the forward and inverse Fourier Transform operations and $*$ denotes convolution, we can write

$$\begin{aligned} F[f(t)d_0(t)] = F(\omega) * 1 = \int_{-\infty}^{\infty} F(\omega)d\omega &= f(0) \\ F^{-1}[f(0)] &= f(0)d_0(t) \\ f(t)d_0(t) &= f(0)d_0(t) \end{aligned} \tag{23}$$

Similarly, let us consider the fourier transform of $f(t)d_1(t)$.

$$F[f(t)d_1(t)] = \int_{-\infty}^{\infty} \frac{d}{dt}[d_0(t)]f(t)e^{-i\omega t}dt$$

(24)

We know that $\int_{-\infty}^{\infty}\frac{d}{dt}[d_0(t)f(t)e^{-i\omega t}]dt = 0 = \int_{-\infty}^{\infty}\frac{d}{dt}[d_0(t)]f(t)e^{-i\omega t}dt+\int_{-\infty}^{\infty}d_0(t)\frac{d}{dt}[f(t)e^{-i\omega t}]dt$. Hence we can write as follows using sifting property of delta function $d_0(t)$, where $\frac{df}{dt} = f_1(t)$.

$$F[f(t)d_1(t)] = -\int_{-\infty}^{\infty}d_0(t)\frac{d}{dt}[f(t)e^{-i\omega t}]dt$$
$$= -\int_{-\infty}^{\infty}d_0(t)[(-i\omega)f(t)e^{-i\omega t}dt + e^{-i\omega t}\frac{df}{dt}]dt = (i\omega)f(0) - f_1(0)$$

(25)

We know that inverse fourier transform of $i\omega$ is $d_1(t)$ and 1 is $d_0(t)$. We can write as follows.

$$f(t)d_1(t) = F^{-1}[(i\omega)f(0) - f_1(0)] = f(0)d_1(t) - f_1(0)d_0(t)$$

(26)

Similarly, let us consider the fourier transform of $f(t)d_2(t)$.

$$F[f(t)d_2(t)] = \int_{-\infty}^{\infty}\frac{d}{dt}(d_1(t))f(t)e^{-i\omega t}dt$$

(27)

We know that $\int_{-\infty}^{\infty}\frac{d}{dt}[d_1(t)f(t)e^{-i\omega t}]dt = 0 = \int_{-\infty}^{\infty}\frac{d}{dt}[d_1(t)]f(t)e^{-i\omega t}dt+\int_{-\infty}^{\infty}d_1(t)\frac{d}{dt}[f(t)e^{-i\omega t}]dt$. Hence we can write as follows.

$$F[f(t)d_2(t)] = -\int_{-\infty}^{\infty}d_1(t)\frac{d}{dt}[f(t)e^{-i\omega t}]dt = -\int_{-\infty}^{\infty}d_1(t)[-i\omega f(t)e^{-i\omega t} + e^{-i\omega t}\frac{df}{dt}]dt$$

$$(28)$$

Using $F[f(t)d_1(t)] = (i\omega)f(0) - f_1(0)$ from Eq. 25, and using $\frac{df}{dt} = f_1(t)$, $\frac{d^2f}{dt^2} = f_2(t)$, we can write

$$\begin{aligned} F[f(t)d_2(t)] &= (i\omega)[(i\omega)f(0) - f_1(0)] - [(i\omega)f_1(0) - f_2(0)] \\ &= [(i\omega)^2 f(0) - 2(i\omega)f_1(0) + f_2(0)] = I(\omega) \end{aligned} \tag{29}$$

We can compute inverse fourier transform of $I(\omega)$ as follows. We use the result that the inverse fourier transform of $(i\omega)^N$ is given by $d_N(t)$

$$f(t)d_2(t) = F^{-1}[I(\omega)] = f(0)d_2(t) - 2f_1(0)d_1(t) + f_2(0)d_0(t) \tag{30}$$

Using results in Eq. 26 to Eq. 30, and using $\frac{d^r f}{dt^r} = f_r(t)$, we can generalize the results to case $n = N$.

$$\begin{aligned} f(t)d_N(t) &= \sum_{r=0}^{N} (-1)^r \binom{N}{r} f_r(0) d_{N-r}(t) \\ F[f(t)d_N(t)] &= \sum_{r=0}^{N} (-1)^r \binom{N}{r} f_r(0)(i\omega)^{N-r} \end{aligned} \tag{31}$$

We can prove by the Principle of Mathematical Induction that the results holds for the case $N + 1$. Given that $F[f(t)d_{N+1}(t)] = \int_{-\infty}^{\infty} f(t)\frac{d}{dt}d_N(t)e^{-i\omega t}dt =$

$-\int_{-\infty}^{\infty} d_N(t)\frac{d}{dt}[f(t)e^{-i\omega t}]dt = -\int_{-\infty}^{\infty} d_N(t)[-i\omega f(t)e^{-i\omega t}+e^{-i\omega t}\frac{df}{dt}]dt$ and using Eq. 31, we can write as follows.

$$
\begin{aligned}
F[f(t)d_{N+1}(t)] &= (i\omega)[\sum_{r=0}^{N}(-1)^r\binom{N}{r}f_r(0)(i\omega)^{N-r}] - [\sum_{r=0}^{N}(-1)^r\binom{N}{r}f_{r+1}(0)(i\omega)^{N-r}] \\
&= \sum_{r=0}^{N+1}(-1)^r\binom{N+1}{r}f_r(0)(i\omega)^{N+1-r}
\end{aligned}
\tag{32}
$$

given that $[\sum_{r=0}^{N}(-1)^r\binom{N}{r}f_{r+1}(0)(i\omega)^{N-r}] = [\sum_{r=1}^{N+1}(-1)^{r-1}\binom{N}{r-1}f_r(0)(i\omega)^{N+1-r}]$ and that $\binom{N}{r}+\binom{N}{r-1}=\binom{N+1}{r}$.

So we can write as follows using the result that the inverse fourier transform of $(i\omega)^N$ is given by $d_N(t)$.

$$
\begin{aligned}
F[f(t)d_{N+1}(t)] &= \sum_{r=0}^{N+1}(-1)^r\binom{N+1}{r}f_r(0)(i\omega)^{N+1-r} \\
f(t)d_{N+1}(t) &= \sum_{r=0}^{N+1}(-1)^r\binom{N+1}{r}f_r(0)d_{N+1-r}(t)
\end{aligned}
\tag{33}
$$

We also know that the area under the function $d_N(t)$ is given as follows.

$$
\begin{aligned}
\int_{-\infty}^{\infty} d_N(t)dt &= [(i\omega)^N]_{\omega=0} \\
\int_{-\infty}^{\infty} d_N(t)dt &= 1 \quad N=0 \\
&= 0 \quad N>0
\end{aligned}
\tag{34}
$$

4.3. Convolution property of Dirac delta and its derivatives

Let us consider the convolution of $f(t)$ and $d_0(t)$ given by $f(t) * d_0(t) = \int_{-\infty}^{\infty} d_0(\tau)f(t-\tau)d\tau$ using Eq. 31. Putting $g(\tau) = f(t-\tau)$ in equations below, we get

$$f(t)d_0(t) = f(0)d_0(t)$$

$$f(t) * d_0(t) = \int_{-\infty}^{\infty} d_0(\tau)f(t-\tau)d\tau = \int_{-\infty}^{\infty} d_0(\tau)g(\tau)d\tau = g(0)\int_{-\infty}^{\infty} d_0(\tau)d\tau = g(0) = f(t) \tag{35}$$

Let us consider the convolution of $f(t)$ and $d_1(t)$. We use the fact that $\int_{-\infty}^{\infty} d_1(\tau)d\tau = 0$.

$$f(t)d_1(t) = f(0)d_1(t) - f_1(0)d_0(t)$$

$$\begin{aligned} f(t) * d_1(t) = \int_{-\infty}^{\infty} d_1(\tau)f(t-\tau)d\tau = \int_{-\infty}^{\infty} d_1(\tau)g(\tau)d\tau &= g(0)\int_{-\infty}^{\infty} d_1(\tau)d\tau - g_1(0)\int_{-\infty}^{\infty} d_0(\tau)d\tau \\ &= -g_1(0) = -f_1(t) \end{aligned} \tag{36}$$

Let us consider the convolution of $f(t)$ and $d_N(t)$ given by $f(t) * d_N(t) = \int_{-\infty}^{\infty} d_N(\tau)f(t-\tau)d\tau$ using Eq. 31 and above results. We use the fact that $\int_{-\infty}^{\infty} d_N(\tau)d\tau = 0$ for $N > 0$.

$$
\begin{aligned}
f(t)d_N(t) &= \sum_{r=0}^{N-1}(-1)^r\binom{N}{r}f_r(0)d_{N-r}(t) + (-1)^N f_N(0)d_0(t)\\
f(t)*d_N(t) &= \int_{-\infty}^{\infty} d_N(\tau)f(t-\tau)d\tau = \int_{-\infty}^{\infty} d_N(\tau)g(\tau)d\tau\\
= \sum_{r=0}^{N-1}(-1)^r\binom{N}{r}g_r(0)\int_{-\infty}^{\infty} d_{N-r}(\tau)d\tau &+ (-1)^N g_N(0)\int_{-\infty}^{\infty} d_0(\tau)d\tau = (-1)^N g_N(0) = (-1)^N f_N(t)\\
f(t)*d_N(t) &= (-1)^N f_N(t)
\end{aligned}
\tag{37}
$$

Thus we have derived the familiar convolution properties of the derivatives of the Dirac delta function.

4.4. Dirac delta comb function

Let us consider the Fourier Transform of a Dirac Delta Comb or Impulse Train $\nabla_T(t)$, also known as the "Shah" function, which is defined by the Dirac Delta function repeated infinitely with period T. Because the Dirac comb function is periodic, it can be represented as a Fourier series as follows, given that $F[\delta(t)] = 1$.

$$\begin{aligned} g(t) = \nabla_T(t) &= \sum_{k=-\infty}^{\infty} \delta(t - kT) \\ &= \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{-i2\pi t \frac{k}{T}} \end{aligned} \tag{38}$$

Now we can compute the fourier transform of Dirac Delta Comb as follows.

$$G(f) = \frac{1}{T} \sum_{k=-\infty}^{\infty} d_0(f - \frac{k}{T}) \tag{39}$$

where $d_0(f) = \lim_{B\to\infty} \frac{1}{\pi f} \sin(Bf)$ is the Fourier Transform of 1.

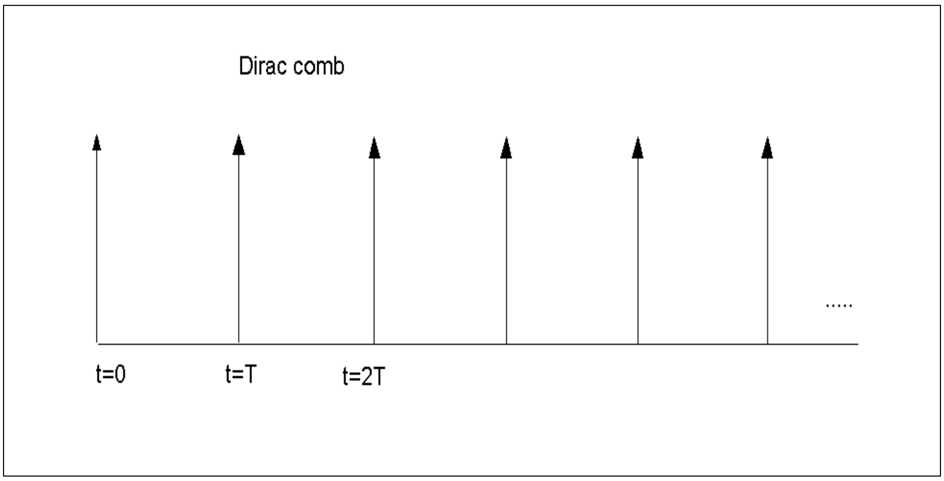

Figure 2:

5. Related results

We have shown that

$$\begin{aligned} d_0(t) = F^{-1}[1] &= \lim_{B\to\infty} \frac{1}{\pi t}\sin(Bt) \\ d_N(t) = \frac{d^N}{dt^N} d_0(t) &= \frac{d^N}{dt^N}[\lim_{B\to\infty} \frac{1}{\pi t}\sin(Bt)] \\ d_N(t) &\longleftrightarrow (i\omega)^N \end{aligned} \tag{40}$$

We have shown that $d_0(t)$ can be viewed as a sinc function whose height tends to ∞ at t=0 and whose zero crossings become more and more closer as $B \to \infty$. We have also derived related results.

Let us compare this specific version of delta function with another interpretations of Dirac Delta function $\delta_a(t)$.

$$\delta_a(t) = \lim_{T\to 0} \frac{1}{T} rect(\frac{t}{T}) \quad F[\delta_a(t)] = \lim_{T\to 0} sinc(fT) = 1 \tag{41}$$

We can see that $\delta_a(t)$ is similar to the definition of Dirac Delta function $\delta(t)$ in the sense that for $t \neq 0$, $\delta_a(t) = 0$ and it goes to infinity at $t = 0$.