9. HEAVISIDE'S EXPANSION FORMULA

Theorem 10. (Heaviside's Expansion Formula) Let $f(s)$ and $g(s)$ be two polynomials in s with $\deg f(s) < \deg g(s)$. If $g(s)$ has n distinct roots $\alpha_1, \alpha_2, ..., \alpha_n$, then

$$L^{-1}\left\{\frac{f(s)}{g(s)}\right\} = \sum_{i=1}^{n} \frac{f(\alpha_i)}{g'(\alpha_i)} e^{\alpha_i t}.$$

Proof. Let $\dfrac{f(s)}{g(s)} = \dfrac{A_1}{s-\alpha_1} + \dfrac{A_2}{s-\alpha_2} + ... + \dfrac{A_s}{s-\alpha_n}$. ...(1)

Multiplying both sides by $(s-\alpha_i)$, where $1 \le i \le n$, and taking limit as $s \to \alpha_i$, we get

$$\lim_{s\to\alpha_i} \frac{f(s)(s-\alpha_i)}{g(s)} = A_i$$

$$\Rightarrow \quad A_i = f(\alpha_i) \lim_{s\to\alpha_i} \frac{s-\alpha_i}{g(s)} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= f(\alpha_i) \lim_{s\to\alpha_i} \frac{1}{g'(s)}$$

[Using L'Hospital Rule]

$$= \frac{f(\alpha_i)}{g'(\alpha_i)}.$$

Putting these values of A_i in (1), we get

$$\frac{f(s)}{g(s)} = \sum_{i=1}^{n} \left(\frac{f(\alpha_i)}{g'(\alpha_i)} \cdot \frac{1}{s-\alpha_i}\right)$$

$$\therefore \quad L^{-1}\left\{\frac{f(s)}{g(s)}\right\} = \sum_{i=1}^{n} \frac{f(\alpha_i)}{g'(\alpha_i)} e^{\alpha_i t}.$$

ILLUSTRATIVE EXAMPLES

Example 1. Using Heaviside's expansion formula, evaluate :

(i) $L^{-1}\left\{\dfrac{2s^2+5s-4}{s^3+s^2-2s}\right\}$

(ii) $L^{-1}\left\{\dfrac{s^2-6}{s^3+4s^2+3s}\right\}$

(iii) $L^{-1}\left\{\dfrac{3s+1}{(s-1)(s^2+1)}\right\}$.

Sol. (i) Let $f(s) = 2s^2+5s-4$, $g(s) = s^3+s^2-2s$.

Clearly, $g(s) = s(s^2+s-2) = s(s+2)(s-1)$

and $g'(s) = 3s^2+2s-2$.

$\therefore$ $g(s)$ has three distinct roots namely

$\alpha_1 = 0,\ \alpha_2 = 1,\ \alpha_3 = -2.$

$\therefore$ By Heaviside's expansion formula,

$$L^{-1}\left\{\frac{f(s)}{g(s)}\right\} = \sum_{i=1}^{n} \frac{f(\alpha_i)}{g'(\alpha_i)} e^{\alpha_i t}$$

$$= \frac{f(\alpha_1)}{g'(\alpha_1)} e^{\alpha_1 t} + \frac{f(\alpha_2)}{g'(\alpha_2)} e^{\alpha_2 t} + \frac{f(\alpha_3)}{g'(\alpha_3)} e^{\alpha_3 t}$$

$$= \frac{f(0)}{g'(0)} e^{0t} + \frac{f(1)}{g'(1)} e^{t} + \frac{f(-2)}{g'(-2)} e^{-2t}$$

$$= \frac{-4}{-2} + \frac{3}{3} e^{t} + \frac{-6}{6} e^{-2t}$$

$$= 2 + e^{t} - e^{-2t}.$$

(*ii*) Let $f(s) = s^2 - 6,\ g(s) = s^3 + 4s^2 + 3s$.

Then $g(s) = s(s^2 + 4s + 3) = s(s+3)(s+1)$

and $g'(s) = 3s^2 + 8s + 3$.

$\therefore$ $g(s)$ has three distinct roots namely

$\alpha_1 = 0,\ \alpha_2 = -1,\ \alpha_3 = -3.$

$\therefore$ By Heaviside's expansion formula,

$$L^{-1}\left\{\frac{f(s)}{g(s)}\right\} = \sum_{i=1}^{n} \frac{f(\alpha_i)}{g'(\alpha_i)} e^{\alpha_i t}$$

$$= \frac{f(\alpha_1)}{g'(\alpha_1)} e^{\alpha_1 t} + \frac{f(\alpha_2)}{g'(\alpha_2)} e^{\alpha_2 t} + \frac{f(\alpha_3)}{g'(\alpha_3)} e^{\alpha_3 t}$$

$$= \frac{f(0)}{g'(0)} e^{0t} + \frac{f(-1)}{g'(-1)} e^{-t} + \frac{f(-3)}{g'(-3)} e^{-3t}$$

$$= -\frac{6}{3} + \frac{-5}{-2} e^{-t} + \frac{3}{6} e^{-3t}$$

$$= -2 + \frac{5}{2} e^{-t} + \frac{1}{2} e^{-3t}.$$

(*iii*) Let $f(s) = 3s + 1,\ g(s) = (s-1)(s^2+1)$.

Then $g(s) = (s-1)(s-i)(s+i)$

and $g'(s) = (s^2+1) + (s-1)(2s) = 3s^2 - 2s + 1$.

$\therefore$ $g(s)$ has three distinct roots namely

$\alpha_1 = 1,\ \alpha_2 = i,\ \alpha_3 = -i.$

$\therefore$ By Heaviside's expansion formula,

$$L^{-1}\left\{\frac{f(s)}{g(s)}\right\} = \sum_{i=1}^{n} \frac{f(\alpha_i)}{g'(\alpha_i)} e^{\alpha_i t}$$

$$= \frac{f(1)}{g'(1)} e^{t} + \frac{f(i)}{g'(i)} e^{it} + \frac{f(-i)}{g'(-i)} e^{-it}$$

$$= \frac{4}{2} e^{t} + \frac{3i+1}{-2(i+1)} e^{it} + \frac{1-3i}{-2(1-i)} e^{-it}$$

$$= 2e^{t} - \frac{1}{2}\frac{(3i+1)(1-i)}{(1+i)(1-i)} e^{it} - \frac{1}{2}\frac{(1-3i)(1+i)}{(1-i)(1+i)} e^{-it}$$

$$= 2e^{t} - \frac{1}{4}(4+2i)e^{it} - \frac{1}{4}(4-2i)e^{-t}$$

$$= 2e^{t} - (e^{it} + e^{-it}) - i\frac{(e^{it} - e^{-it})}{2}$$

$$= 2e^{t} - 2\cos t + \sin t.$$

EXERCISE–4(*e*)

Use Heavinside's expansion formula to find inverse Laplace transforms of the following :

1. $\dfrac{4s+3}{9s^2-16}$
2. $\dfrac{s+1}{s^2+2s-8}$
3. $\dfrac{s^2+s-2}{s(s-2)(s+3)}$
4. $\dfrac{s^2-10s+13}{(s-7)(s^2+5s+6)}$
5. $\dfrac{19s+37}{(s-2)(s+1)(s+3)}$
6. $\dfrac{6s^2+22s+18}{s^3+6s^2+11s+6}$
7. $\dfrac{s+5}{(s+1)(s^2+1)}$
8. $\dfrac{1}{s^3+1}$
9. $\dfrac{1}{s^3-1}$
10. $\dfrac{1}{s(s^2+9)}$ **(Pbi. U. 2015)**
11. $\dfrac{1}{(s+4)(s^2+16)}$
12. $\dfrac{s}{(s^2+a^2)(s^2+b^2)}$.

ANSWERS

1. $\frac{4}{9}\cosh\frac{4t}{3}+\frac{1}{4}\sinh\frac{4t}{3}$

2. $e^{-t}\cosh 3t$

3. $\frac{1}{3}+\frac{2}{5}e^{2t}+\frac{4}{15}e^{-3t}$

4. $-\frac{3}{5}e^{2t}+2e^{3t}-\frac{2}{5}e^{7t}$

5. $-2e^{-3t}-3e^{-t}+5e^{2t}$

6. $3e^{-3t}+2e^{-2t}+e^{-t}$

7. $2e^{-t}-2\cos t+3\sin t$

8. $\frac{1}{3}\left[e^{-t}-e^{\frac{t}{2}}\left(\cos\frac{\sqrt{3}\,t}{2}-\sqrt{3}\sin\frac{\sqrt{3}\,t}{2}\right)\right]$

9. $\frac{1}{3}\left[e^{-t}-e^{-\frac{t}{2}}\left(\cos\frac{\sqrt{3}\,t}{2}+\sqrt{3}\sin\frac{\sqrt{3}\,t}{2}\right)\right]$

10. $\frac{1}{9}(1-\cos 3t)$

11. $\frac{1}{32}(e^{-4t}-\cos 4t+\sin 4t)$

12. $\frac{1}{a^2-b^2}(\cos bt-\cos at)$