GENERALIZED FUNCTIONS

Vol IV

Applications of Harmonic Analysis

I. M. Gel'fand
N. Ya. Vilenkin

GENERALIZED FUNCTIONS:

Volume 4

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Applications of Harmonic Analysis

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Translator's Note

This translation differs from the Russian edition of 1961 mainly in that the authors have revised the proof contained in Section 3.2 of Chapter 1, as well as a major portion of Section 3.6 of the same chapter (in particular, the proof of Theorem 7). The symbols ▼ and ▲ in the margin indicate the beginning and end of a few passages in which I have ventured to deviate (in a not completely trivial way) from the original, for the purpose of eliminating a gap or an obscurity, and in one case (in Section 5.1, Chapter 4) of reducing a ten-line proof to the obvious one-line proof. In addition, I have allowed myself the luxury of a number of remarks, which appear in footnotes marked by a dagger (†). Any comments regarding these or other portions of this volume will be welcome.

June, 1964 AMIEL FEINSTEIN

Foreword

This book is the fourth volume of a series of monographs on functional analysis appearing under the title "Generalized Functions." It should not, however, be considered a direct sequel to the preceding volumes. In writing this volume the authors have striven for the maximum independence from the preceding volumes. Only that material which is discussed in the first two chapters of Volume 1 must be considered as the indispensable minimum which the reader is required to know. In view of this, certain topics which were discussed in the preceding volumes are briefly repeated here.

This book is devoted to two general topics: recent developments in the theory of linear topological spaces and the construction of harmonic analysis in n-dimensional Euclidean and infinite-dimensional spaces.

After the appearance of a theory of topological spaces, the question arose of distinguishing a class of topological spaces, defined by rather simple axioms and including all (or nearly all) spaces which arise in applications. In the same way, after a theory of linear topological spaces was created, it became necessary to ascertain which class of spaces is most suitable for use in mathematical analysis. Such a class of linear topological spaces—nuclear spaces—was singled out by the French mathematician A. Grothendieck.

The class of nuclear spaces includes all or nearly all linear topological spaces which are presently used in analysis, and has a number of extremely important properties: the kernel theorem of L. Schwartz is valid in nuclear spaces, as is also the theorem on the spectral resolution of a self-adjoint operator. Furthermore, any measure on the cylinder sets in the conjugate space of a nuclear space is countably additive. The first and fourth chapters of this book are devoted to the discussion of these questions. In connection with spectral analysis, the concept of a rigged Hilbert space is introduced, which turns out, apparently, to be very useful also in many other questions in mathematics.

The second question which we study in this volume is the harmonic analysis of functions in various spaces. Harmonic analysis in Euclidean space (the Fourier integral) has already been discussed to some extent in previous volumes. We have given up the idea of repeating here the material in the preceding volumes which was devoted to the Fourier integral (possibly, had all of the volumes been written at the same time, many questions in the theory of the Fourier integral, for example the Paley–Wiener theorem for generalized functions, would have found their natural setting in this volume). We discuss here only questions of harmonic analysis in Euclidean space which were left unclarified in the previous volumes. Namely, we consider the Fourier transformation of measures having one or another order of growth (the theory of generalized positive definite functions) and its application in the theory of generalized random processes. The Fourier transformation of measures in linear topological spaces is considered at the same time.

In the following, fifth volume, we single out questions of harmonic analysis on homogeneous spaces (in particular, harmonic analysis on groups) and intimately related questions of integral geometry on certain spaces of constant curvature. This theory, which is very rich in the diversity of its results (connected, for example, with the theory of special functions, analytic functions of several complex variables, etc.) could not, of course, be discussed in its entirety within the confines of the fifth volume. We have restricted ourselves to discussing only questions of harmonic analysis on the Lorentz group. It should be remarked that harmonic analysis on the Lorentz group and the related homogeneous spaces is a considerably richer subject than harmonic analysis in the "degenerate" case of a Euclidean space. For example, in the case of a Euclidean space only the smoothness of the Fourier transform of a function is influenced by specifying one kind of behavior or another at infinity of the function itself. But in the case of the Lorentz group, specifying the behavior of the function at infinity leads to certain algebraic relations among the values of its Fourier transform at different points. However, at the present time these questions are only in the initial stages of investigation.

The material of this fourth volume represents a complete unit in itself, and, as we have said, the exposition is practically independent of the preceding volumes. In spite of the relation of one chapter to another, one can begin a reading of this book with the first chapter, which contains the general theory of nuclear and rigged Hilbert spaces, or with the second chapter, which discusses the more elementary theory of positive definite generalized functions.

We mention that certain chapters contain, together with general results, others of a more specialized nature; these can be passed over at the first reading.

The authors wish to express their deep gratitude to those who helped them in working on this book: F. V. Shirokov, whose contributions far exceeded the limits of ordinary editorial work, A. S. Dynin, B. S. Mityagin, and V. B. Lidskii, whose valuable advice the authors used in writing up various topics in the first chapter. They express their particular thanks to S. A. Vilenkin, who took upon herself all the work connected with preparing the manuscript for press.

I. M. Gel'fand
N. Ya. Vilenkin

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CHAPTER I

THE KERNEL THEOREM. NUCLEAR SPACES. RIGGED HILBERT SPACE

This chapter is devoted to the study of a class of countably normed spaces[1]—so-called *nuclear spaces*. These spaces first appeared in connection with the "kernel theorem," which will be used repeatedly in this book. Later it became evident that nuclear spaces also play an essential role in many other topics of functional analysis, namely, nuclear spaces turn out to be the most natural class of spaces for the study of the spectral decomposition of self-adjoint operators. These spaces were already introduced in Volume III (Chapter IV, Section 3.1), in connection with the consideration of spectral decompositions. However, the definition of nuclearity which was given there is not entirely suitable for the study of other questions. Therefore in this volume we will use another definition of nuclear space, which in the essential cases is equivalent to nuclearity in the sense discussed in Volume III. The discussion in this chapter does not depend upon that in Volume III. In order to attain a complete independence of these treatments, we will present in this chapter certain results on the spectral decomposition of self-adjoint operators. Here, however, our main attention will be given to the general aspects of theory, unlike the treatment in Volume III, where a not inconsiderable part was played by the applications of these results to specific differential operators.

An important role is played by the concept of a rigged Hilbert space.

[1] We assume that the reader is familiar with the concept of a countably normed space to the extent of Chapters I and II of Volume II of this series. Besides, a brief discussion of the basic facts relating to a special case of such spaces—countably Hilbert spaces—is given at the beginning of Section 3. Let us note that throughout this volume every countably normed space considered will be taken, without special mention, to be complete.

Moreover, we will as a rule assume that the compatible norms $\| \varphi \|_n$, $1 \leqslant n < \infty$, which define the topology in a countably normed space Φ are monotonically increasing, i.e., that for every element $\varphi \in \Phi$ the inequalities

$$\| \varphi \|_1 \leqslant \cdots \leqslant \| \varphi \|_n \leqslant \cdots$$

hold.

This concept arises in considering nuclear spaces in which an inner product is introduced in some way or another. The theory of rigged Hilbert spaces is discussed in Section 4, where applications of this theory to the spectral analysis of self-adjoint operators are presented.

Also related to the theory of nuclear spaces is the subject of measure theory in linear topological spaces, discussed in Chapter IV. We will show in that chapter that the nuclearity of a space Φ is a necessary and sufficient condition for every measure on the cylinder sets in the space Φ', conjugate to Φ, to be completely additive.

1. Bilinear Functionals on Countably Normed Spaces. The Kernel Theorem

In this section we will study the general form of bilinear functionals[2] on countably normed spaces. We will show that any bilinear functional $B(\varphi, \psi)$, continuous with respect to its arguments φ and ψ which range over countably normed spaces Φ and Ψ, is continuous with respect to certain norms $\| \varphi \|_m$ and $| \psi |_n$ in these spaces. In other words, we will show that

$$| B(\varphi, \psi) | \leqslant M \| \varphi \|_m | \psi |_n$$

for all elements φ and ψ of the spaces Φ and Ψ, where the numbers M, m, n do not depend upon φ and ψ.

Applying this result to various specific spaces, we obtain the general form of bilinear functionals on these spaces. One of the most important results thereby obtained is a description of the bilinear functionals on the space K of infinitely differentiable functions with bounded supports. It will be proved that these functionals have the form

$$B(\varphi, \psi) = (F, \varphi(x)\psi(y)),$$

where F is a linear functional on the space K_2 of infinitely differentiable functions of the variables $x = (x_1, ..., x_k)$ and $y = (y_1, ..., y_k)$, with

[2] A bilinear functional is a functional which is linear in both arguments φ and ψ. Further on we will encounter functionals $B(\varphi, \psi)$ which are linear in φ and antilinear in ψ, i.e.,

$$B(\alpha\varphi_1 + \beta\varphi_2, \psi) = \alpha B(\varphi_1, \psi) + \beta B(\varphi_2, \psi)$$

but

$$B(\varphi, \alpha\psi_1 + \beta\psi_2) = \bar{\alpha} B(\varphi, \psi_1) + \bar{\beta} B(\varphi, \psi_2) .$$

Such functionals are called Hermitean-bilinear, or simply Hermitean functionals.

bounded supports. This theorem is called "the kernel theorem"[3] and will be used frequently in this book.

1.1. Convex Functionals

The general properties of bilinear functionals are established by means of certain theorems concerning convex functionals on countably normed spaces.

A real functional $p(\varphi) \geqslant 0$, defined on a linear space Φ, is called *convex* if for any two elements φ and ψ of Φ the relation

$$p(\varphi + \psi) \leqslant p(\varphi) + p(\psi)$$

is satisfied, and for any element φ and complex number α the relation

$$p(\alpha\varphi) = |\alpha| p(\varphi)$$

is satisfied.

As an example of a convex functional one can take the norm $\| \varphi \|$ in a normed space, since in view of the definition of a norm $\| \varphi + \psi \| \leqslant \| \varphi \| + \| \psi \|$ and $\| \alpha\varphi \| = | \alpha | \| \varphi \|$.

We remark that finiteness of the value of the functional $p(\varphi)$ for every element φ of Φ is not assumed in our definition, i.e., $p(\varphi)$ may assume the value $+\infty$, in which case we take $a \cdot \infty = \infty$, $a \neq 0$, and $\infty + a = \infty + \infty = \infty$. Obviously the set of elements φ for which the functional $p(\varphi)$ takes on a finite value forms a linear subspace[4] in Φ. One could avoid using infinite values for $p(\varphi)$ by considering the functional $p(\varphi)$ only on this subspace.

The concept of a convex functional is related to the geometric concept of an absolutely convex set. A set A in a linear space Φ is called *absolutely convex* if it contains, together with any two elements φ and ψ, every element of the form $\alpha\varphi + \beta\psi$, where α and β are complex numbers such that $| \alpha | + | \beta | \leqslant 1$.[5]

In order to establish the connection between the concepts of a convex functional and an absolutely convex set, let us associate with the functional $p(\varphi)$ the set A consisting of every element $\varphi \in \Phi$ for which $p(\varphi) \leqslant 1$. We show that this set is absolutely convex. In fact, let φ

[3] More precisely, "the kernel theorem for the space K"; the general kernel theorem will be proved in Section 3.

[4] Nonclosed, generally speaking.

[5] In a real linear space the absolute convexity of a set is equivalent to its being convex and centrally symmetric.

and ψ be elements of the set A, i.e., $p(\varphi) \leqslant 1$ and $p(\psi) \leqslant 1$. Then if $|\alpha| + |\beta| \leqslant 1$, we have

$$p(\alpha\varphi + \beta\psi) \leqslant |\alpha| p(\varphi) + |\beta| p(\psi) \leqslant |\alpha| + |\beta| \leqslant 1$$

and so $\alpha\varphi + \beta\psi \in A$, which proves the absolute convexity of A.

Conversely, if A is any absolutely convex closed set (we are assuming now that Φ is a topological linear space), then there exists a convex functional $p(\varphi)$ such that A coincides with the set of elements φ for which $p(\varphi) \leqslant 1$. This functional is defined by

$$p(\varphi) = 1/q,$$

where q is the supremum of those values $\lambda > 0$ for which $\lambda\varphi \in A$. It is easily seen that $p(\varphi) \leqslant 1$ for every element φ of the set A and that $p(\varphi) > 1$ if φ does not belong to A. Moreover, from the absolute convexity of the set A it follows that $p(\varphi)$ is a convex functional.

We have thus established a correspondence between convex functionals $p(\varphi)$ and absolutely convex sets in the space Φ.

We remark that the functional $p(\varphi)$ is finite on the smallest linear subspace Ψ in Φ which contains the set A. In particular the functional $p(\varphi)$ is finite on the entire space Φ, if for any element φ of Φ one can find an element $\psi \neq 0$ in A such that φ lies on the line passing through ψ and the zero element (i.e., φ has the form $\varphi = \lambda\psi$). We will in this case call the set A *absorbing*. Observe that if A is an absorbing set, then every element φ of the space Φ belongs to one of the sets nA (we denote by nA the set of elements of the form $n\varphi$, $\varphi \in A$).

We call a functional $p(\varphi)$ *lower semicontinuous* if for any element φ_0 of the space Φ and any $\epsilon > 0$ there is a neighborhood U of φ_0 such that the inequality

$$p(\psi) \geqslant p(\varphi_0) - \epsilon$$

holds at every point ψ of this neighborhood.

Let us prove that if the functional $p(\varphi)$ is lower semicontinuous, then the set A, consisting of those elements φ for which $p(\varphi) \leqslant 1$, is closed. As a matter of fact, let φ_0 be a limit point of the set A. Then in any neighborhood of the point φ_0 there is a point ψ for which $p(\psi) \leqslant 1$. Consequently, in view of the lower semicontinuity of the functional $p(\varphi)$, $p(\varphi_0) \leqslant 1$, which proves that A is closed.

Let us now prove that if the absolutely convex set A is closed, then the corresponding convex functional $p(\varphi)$ is lower semicontinuous. In fact, the set A of points ψ for which $p(\psi) \leqslant 1$ being closed, the set of points ψ for which $p(\psi) > 1$ is open. But then any set which is defined by the inequality $p(\psi) > a$, where a is any real number, will be open. Setting $a = p(\varphi_0) - \epsilon$, we obtain that the set of points ψ for which $p(\psi) > p(\varphi_0) - \epsilon$ is open, and consequently contains a neighborhood of the point φ_0. This proves the lower semicontinuity of the functional $p(\varphi)$ at any point φ_0.

Thus we have proved that there is a one-to-one correspondence between lower semicontinuous convex functionals and absolutely convex closed sets.

Lemma 1. Let $\{p_\alpha(\varphi)\}$ be some collection of lower semicontinuous convex functionals on a linear topological space Φ. Then the functional $p(\varphi)$, defined by

$$p(\varphi) = \sup_\alpha p_\alpha(\varphi),$$

is also lower semicontinuous and convex.

Proof. Let us denote by A_α the absolutely convex closed set corresponding to the functional $p_\alpha(\varphi)$. Obviously, in order that the inequality

$$p(\varphi) = \sup_\alpha p_\alpha(\varphi) \leqslant 1$$

hold, it is necessary and sufficient that φ belong to the intersection of all of the sets A_α. But the intersection of closed absolutely convex sets is also closed and absolutely convex. Thus, to the functional $p(\varphi)$ corresponds the closed absolutely convex set $A = \bigcap_\alpha A_\alpha$, and therefore $p(\varphi)$ is a lower semicontinuous convex functional.

The following geometric theorem lies at the basis of the study of convex lower semicontinuous functionals.

Theorem 1. Let A be a closed absolutely convex set in a countably normed space Φ. If A is an absorbing set (i.e., if every element φ of the space Φ lies on the line passing through the zero element and some nonzero element of the set A), then A contains some neighborhood of the zero element of Φ.

Proof. Let us denote by nA the set of elements of the form na, $a \in A$. From the absolute convexity of A it follows that the set nA is nondecreasing in n, i.e.,

$$A \subset 2A \subset \ldots \subset nA \subset \ldots .$$

Since the set A is absorbing, any element φ of the space Φ belongs to the sets nA for all n sufficiently large. Therefore $\Phi = \bigcup_{n=1}^{\infty} nA$. But a countably normed space cannot be a countable sum of nowhere dense sets.[6] Consequently at least one of the sets nA is dense in some region of the space Φ, and, since it is closed, contains some ball $S(\varphi_0, r_0)$ defined by the inequality $\| \varphi - \varphi_0 \|_k \leqslant r_0$. But then the set A contains the ball S_0 defined by the inequality $\| \varphi - \varphi_1 \|_k \leqslant r_1$, where $\varphi_1 = \varphi_0/n$, $r_1 = r_0/n$. Since A is absolutely convex, it is easy to see that it therefore contains the ball $\| \varphi \|_k \leqslant r_1$. Thus we have proved that A contains a neighborhood of zero, which proves the theorem.

We remark that we have already encountered Theorem 1 in Volume II (cf. Chapter I, Section 3.4).

Theorem 1 is equivalent to the following theorem concerning convex functionals on countably normed spaces.

Theorem 1′. Let $p(\varphi)$ be a convex lower semicontinuous functional in a countably normed space Φ, which has a finite value at each point of this space. Then $p(\varphi)$ is bounded in some neighborhood of the zero element of Φ.

Proof. We consider the set A consisting of those points φ for which $p(\varphi) \leqslant 1$. In view of the convexity and lower semicontinuity of the functional $p(\varphi)$ this set is absolutely convex and closed. In as much as $p(\varphi)$ has finite values on the entire space Φ, the set A is absorbing (i.e., any element φ of Φ can be represented in the form $\lambda\psi$, where

[6] In fact, let $A_1 \subset A_2 \subset \ldots$ be an increasing sequence of nowhere dense sets in a countably normed space Φ. Since the set A_1 is nowhere dense, there is a ball $S_1(\varphi_1, r_1)$, defined by the inequality $\| \varphi - \varphi_1 \|_{p_1} \leqslant r_1$, $r_1 \leqslant 1$, which contains no points of the set A_1. Further, there is a ball $S_2(\varphi_2, r_2)$, defined by the inequality $\| \varphi - \varphi_2 \|_{p_2} \leqslant r_2$, $r_2 \leqslant \frac{1}{2}$, $p_2 > p_1$, which lies in the ball $S_1(\varphi_1, r_1)$ and contains no points of the set A_2. Continuing this process, we obtain a nested system of balls

$$S_1(\varphi_1, r_1) \supset S_2(\varphi_2, r_2) \supset \ldots \supset S_k(\varphi_k, r_k) \supset \ldots,$$

defined by the inequalities $\| \varphi - \varphi_k \|_{p_k} \leqslant r_k$, $r_k \leqslant 1/k$, $p_{k+1} > p_k$, and such that $S_k(\varphi_k, r_k)$ contains no points of the set A_k. It is easily seen that the centers $\varphi_1, \ldots, \varphi_k, \ldots$ of these balls form a fundamental sequence in Φ (i.e., that $\lim_{k,l\to\infty} \| \varphi_k - \varphi_l \|_n = 0$ for any n). Since we are considering only complete countably normed spaces, the sequence $\varphi_1, \ldots, \varphi_k, \ldots$ has a limit point φ. Obviously this point φ belongs to every ball $S_k(\varphi_k, r_k)$ and therefore does not belong to any one of the sets A_k. But this shows that we have found a point of the space Φ not belonging to the sum $\bigcup_{k=1}^{\infty} A_k$. Consequently $\Phi \neq \bigcup_{k=1}^{\infty} A_k$.

Let us remark that Theorem 1 and every result which we obtain from it is valid for every locally convex linear topological space which cannot be decomposed into a countable sum of nowhere dense sets (a linear topological space is called *locally convex* if it possesses a complete system of neighborhoods of zero, consisting of absolutely convex sets).

A linear topological space for which Theorem 1 holds is called a *barreled space*. Thus, this theorem can be stated as "every countably normed space is barreled."

$\psi \in A$). Consequently, by Theorem 1 A contains a neighborhood U of zero, in which, by the definition of A, we have $p(\varphi) \leqslant 1$, which proves the theorem.

In order to describe bilinear functionals we need the following generalization of Theorem 1′.

Theorem 2. Let $p_1(\varphi), ..., p_n(\varphi), ...$ be a sequence of convex lower semicontinuous functionals on a countably normed space Φ. Suppose further that

$$p_1(\varphi) \geqslant p_2(\varphi) \geqslant ...$$

where, for each point φ, $p_n(\varphi)$ is finite for all $n \geqslant n(\varphi)$. Then there are numbers n_0, m, M not depending upon φ, such that for $n \geqslant n_0$ the functional $p_n(\varphi)$ is finite on the entire space and, moreover, the inequality

$$p_n(\varphi) \leqslant M \| \varphi \|_m$$

holds for every point φ.

Proof. We denote by A_n the set of points of Φ for which the inequality $p_n(\varphi) \leqslant 1$ holds. Each of the sets A_n is absolutely convex and closed, and every element φ of Φ belongs to at least one set of the form kA_n; namely, if $p_n(\varphi)$ has a finite value and $p_n(\varphi) \leqslant k$, then φ belongs to the set kA_n. Thus $\Phi = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} kA_n$. Since the space Φ cannot be decomposed into a countable sum of nowhere dense sets, at least one of the sets kA_n, say $k_0A_{n_0}$, is dense in some region of Φ. But then the set A_{n_0} is dense in some region of Φ. Since A_{n_0} is closed, it follows that A_{n_0} contains some ball $S_m(\varphi_0, r)$ defined by the inequality $\| \varphi - \varphi_0 \|_m \leqslant r$. Therefore, in view of the absolute convexity of A_{n_0}, it also contains the ball $S_m(r)$: $\| \varphi \|_m \leqslant r$. In other words, we have shown that from the inequality $\| \varphi \|_m \leqslant r$ there follows the inequality $p_{n_0}(\varphi) \leqslant 1$. Since we have $p_n(\varphi) \leqslant p_{n_0}(\varphi)$ for $n \geqslant n_0$, then for every $n \geqslant n_0$ and every φ the inequality

$$p_n(\varphi) \leqslant M \| \varphi \|_m$$

holds, where $M = 1/r$, which proves the theorem.

1.2. Bilinear Functionals

We proceed now to the description of bilinear functionals on countably normed spaces. It is known that any linear functional F on a countably normed space has finite order, i.e., is continuous relative to one of the norms $\| \varphi \|_n$. In fact, in view of the continuity of the functional F there is a neighborhood U, defined by the inequality $\| \varphi \|_n \leqslant \delta$, in which the inequality $|(F, \varphi)| \leqslant 1$ holds. But then for $\| \varphi \|_n \leqslant \epsilon\delta$ the inequality $|(F, \varphi)| \leqslant \epsilon$ obtains, i.e., the functional F is continuous in the norm $\| \varphi \|_n$.

We shall now prove a similar result for bilinear functionals. Let us call a bilinear functional $B(\varphi, \psi)$, where φ and ψ range over the countably normed spaces Φ and Ψ, *continuous in each of its arguments*, if for fixed φ the functional $B(\varphi, \psi)$ is continuous in ψ, and for fixed ψ it is continuous in φ. The following theorem holds:

Theorem 3. Let $B(\varphi, \psi)$ be a bilinear functional in the countably normed spaces Φ and Ψ which is continuous in each of its arguments. φ and ψ. Then there are norms $\| \varphi \|_n$ and $| \psi |_m$ in the spaces Φ and Ψ relative to which the functional $B(\varphi, \psi)$ is jointly continuous in the variables φ and ψ. In other words, there exist norms $\| \varphi \|_n$ and $| \psi |_m$ such that[7]

$$| B(\varphi, \psi)| \leqslant M \| \varphi \|_n | \psi |_m, \tag{1}$$

where M does not depend upon φ and ψ.

Proof. For each n we introduce the functional $p_n(\psi)$ on the space Ψ, defined by

$$p_n(\psi) = \sup_{\|\varphi\|_n \leqslant 1} | B(\varphi, \psi)|. \tag{2}$$

This functional can assume both finite and infinite values. However, for each point ψ_0 there is an n (depending upon ψ_0) such that $p_n(\psi_0)$ is finite. In fact, $B(\varphi, \psi_0)$ is for fixed ψ_0 a continuous linear functional on the space Φ. But any continuous linear functional on a countably normed space is continuous with respect to some norm $\| \varphi \|_n$, and so there is an n such that $B(\varphi, \psi_0)$ is bounded in the ball $\| \varphi \|_n \leqslant 1$, i.e., in view of (2) the value $p_n(\psi_0)$ is finite.

Now we show that the functionals $p_n(\psi)$ are monotonically decreasing, convex, and lower semicontinuous. That the functionals p_n are monotonically decreasing

$$p_1(\psi) \geqslant \dots \geqslant p_n(\psi) \geqslant \dots$$

follows from the fact that the norms $\| \varphi \|_n$ satisfy the inequalities

$$\| \varphi \|_1 \leqslant \| \varphi \|_2 \leqslant \dots \leqslant \| \varphi \|_n \leqslant \dots$$

[7] We denote the norms in Φ by $\| \varphi \|_n$, and in Ψ by $| \psi |_m$.

Therefore the set $\| \varphi \|_n \leqslant 1$ contains the set $\| \varphi \|_{n+1} \leqslant 1$ and consequently

$$p_n(\psi) = \sup_{\|\varphi\|_n \leqslant 1} | B(\varphi, \psi)| \geqslant \sup_{\|\varphi\|_{n+1} \leqslant 1} | B(\varphi, \psi)| = p_{n+1}(\psi).$$

Further, the functionals $p_n(\psi)$ are convex. Indeed, for fixed φ the functional $| B(\varphi, \psi)|$ is convex, and so the functional

$$p_n(\psi) = \sup_{\|\varphi\|_n \leqslant 1} | B(\varphi, \psi)|$$

as the supremum of convex functionals, is likewise convex. In exactly the same way one proves that the functionals $p_n(\psi)$ are lower semicontinuous (the functional $|B(\varphi, \psi)|$, φ fixed, is continuous and *a fortiori* lower semicontinuous).

Now we apply Theorem 2 to the functionals $p_n(\psi)$. We find that $p_n(\psi) \leqslant M| \psi |_m$, where m and M do not depend upon ψ, and n is equal to or greater than some integer n_0. In view of the definition of the functional $p_n(\psi)$, this means that

$$\sup_{\|\varphi\|_n \leqslant 1} | B(\varphi, \psi)| \leqslant M| \psi |_m$$

and therefore

$$| B(\varphi, \psi)| \leqslant M\| \varphi \|_n | \psi |_m .$$

Thus our theorem is proved.

Let us now consider a linear operator which carries a countably normed space Φ into the conjugate space Ψ' of a countably normed space Ψ. With every such operator A can be associated a bilinear functional $B(\varphi, \psi)$, where $\varphi \in \Phi$ and $\psi \in \Psi$, setting

$$B(\varphi, \psi) = (A\varphi, \psi)$$

[we recall that $A\varphi \in \Psi'$ and so $(A\varphi, \psi)$ is defined]. Therefore, from Theorem 3 one can obtain a theorem concerning operators of this type. For this we assume that the operator A is continuous relative to the topology of the space Φ and the weak topology of the space Ψ'. In other words, we assume that for any elements $\psi_1, ..., \psi_n$ of Ψ there is a neighborhood U of zero in Φ such that $|(A\varphi, \psi_k)| \leqslant 1$, $1 \leqslant k \leqslant n$, for every element φ of U.[8] Then, as is easily seen, the bilinear functional

[8] A neighborhood of zero in the space Ψ', in the weak topology, is defined by the inequalities

$$| (F, \psi_k) | \leqslant 1, \qquad 1 \leqslant k \leqslant n,$$

where $\psi_1, ..., \psi_n$ are fixed elements of the space Ψ.

$B(\varphi, \psi)$ will be continuous with respect to each of its arguments φ and ψ. Applying Theorem 3 to $B(\varphi, \psi)$, we find that for every element $\varphi \in \Phi$ and $\psi \in \Psi$ an inequality of the form

$$|(A\varphi, \psi)| = |B(\varphi, \psi)| \leqslant M\|\varphi\|_n |\psi|_m$$

holds, where M, m, n are certain numbers which do not depend upon φ and ψ. Taking into account the definition of the norms in the space Ψ', we obtain

$$|A\varphi|_{-m} \leqslant M\|\varphi\|_n$$

(we have denoted[9] by $|F|_{-m}$ the norms in the space Ψ', defined by the equation $|F|_{-m} = \sup_{|\psi|_m \leqslant 1} |(F, \psi)|$). Thus we have proved that

$$\sup_{\|\varphi\|_n \leqslant 1} |A\varphi|_{-m} \leqslant M.$$

But this means that the operator A is continuous relative not only to the topology in the space Φ and the weak topology in the space Ψ', but also the topologies in these spaces defined by the norms $\|\varphi\|_n$ and $|F|_{-m}$. Thus the following holds:

Theorem 3′. Let Φ and Ψ be countably normed spaces and A a linear operator mapping Φ into the conjugate space Ψ' of Ψ. If A is continuous relative to the topology of the space Φ and the weak topology of the space Ψ', then it is continuous relative to certain norms $\|\varphi\|_n$ and $|F|_{-m}$ in these spaces.

We obtained Theorem 3′ as a corollary of Theorem 3. Conversely, Theorem 3 is a corollary of Theorem 3′. This is a consequence of the following easily proved assertion. *If $B(\varphi, \psi)$ is a bilinear functional on the spaces Φ and Ψ, continuous in each of its arguments φ and ψ, then the equation*

$$(A\varphi, \psi) = B(\varphi, \psi)$$

defines an operator A which maps Φ into Ψ' and is continuous relative to the topology of Φ and the weak topology of Ψ'. We remark that the bilinear functional $B(\varphi, \psi)$ defines, along with the operator A, also its *adjoint* A', defined by the equation $(A'\psi, \varphi) = B(\varphi, \psi)$.

[9] For reasons which will be clarified below, we have denoted the norms in a space, conjugate to a countably normed space Φ, by $\|F\|_{-m}$, i.e., we use negative indices.

1.3. The Structure of Bilinear Functionals on Specific Spaces (the Kernel Theorem)

The purpose of this section is to describe bilinear functionals on specific linear topological spaces, primarily on the space K of infinitely differentiable functions with bounded supports and the space S of infinitely differentiable functions which are rapidly decreasing for $|x| \to \infty$ together with their derivatives of all orders.[10] The determination of the structure of bilinear functionals on the space K represents the content of the so-called "kernel theorem" of L. Schwartz.

We will begin the solution of the problem at hand with the proof of a lemma on bilinear functionals on a Hilbert space of functions. For the sake of simplicity we carry out all considerations for functions of a single variable, stating here and there the changes which appear in the statements of the theorems upon passing to functions of several variables.

We denote by $H(a)$ the Hilbert space consisting of all functions $\varphi(x)$ defined on the interval $|x| \leqslant a$ and having square integrable moduli on this interval. The scalar product in the space $H(a)$ is defined by

$$(\varphi, \psi) = \int_{-a}^{a} \varphi(x)\overline{\psi(x)}\, dx.$$

We denote the norm in $H(a)$ by $\|\varphi\|$, i.e.,

$$\|\varphi\| = \left[\int_{-a}^{a} |\varphi(x)|^2\, dx\right]^{\frac{1}{2}}.$$

Lemma 1. Let $B(\varphi, \psi)$ be a continuous bilinear functional on the space $H(a)$, such that

$$
\begin{aligned}
|B(\varphi', \psi')| &\leqslant M\|\varphi\|\,\|\psi\|,\\
|B(\varphi, \psi')| &\leqslant M\|\varphi\|\,\|\psi\|, \qquad (3)\\
|B(\varphi', \psi)| &\leqslant M\|\varphi\|\,\|\psi\|,
\end{aligned}
$$

[10] The spaces K and S were introduced in Volume I. In the appendix to Section 1 we repeat the definition of these spaces and state their fundamental properties. Let us recall here only that a function $\varphi(x)$ is called rapidly decreasing if

$$\lim_{|x|\to\infty} |x^k\varphi(x)| = 0$$

for every k.

if the functions $\varphi(x)$ and $\psi(x)$ as well as their derivatives $\varphi'(x)$ and $\psi'(x)$ belong to the space $H(a)$. Then the functional $B(\varphi, \psi)$ has the form

$$B(\varphi, \psi) = \int_{-a}^{a} \int_{-a}^{a} F(x, y)\varphi(x)\psi(y)\, dx\, dy, \tag{4}$$

where $F(x, y)$ is a function with square integrable modulus.

Proof. We choose the orthonormal basis in $H(a)$ consisting of the functions

$$\chi_n(x) = \frac{1}{\sqrt{2a}} \exp\left(\frac{\pi i n x}{a}\right), \qquad -\infty < n < \infty.$$

The functions

$$\overline{\chi_m(x)}\,\overline{\chi_n(y)} \equiv \frac{1}{2a} \exp\left(-\frac{\pi i(mx + ny)}{a}\right)$$

form an orthonormal basis in the space $H_2(a)$ of functions $\varphi(x, y)$, $|x| \leqslant a$, $|y| \leqslant a$ with square integrable moduli. To construct the kernel $F(x, y)$ we consider the series

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} B(\chi_m, \chi_n)\overline{\chi_m(x)}\,\overline{\chi_n(y)}. \tag{5}$$

Let us show that this series converges in the mean. For this it suffices to show that the series

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} |B(\chi_m, \chi_n)|^2 \tag{6}$$

converges. But by inequalities (3) we have

$$|B(\chi'_m, \chi'_n)| \leqslant M\|\chi_m\|\,\|\chi_n\|$$

and

$$|B(\chi_m, \chi'_n)| \leqslant M\|\chi_m\|\,\|\chi_n\|.$$

Since

$$\chi'_n(x) = \frac{\pi i n}{a}\chi_n(x) \qquad \text{and} \qquad \|\chi_n\| = 1,$$

then

$$|B(\chi_m, \chi_n)| \leqslant \frac{M_1}{|m|\,|n|}, \qquad m \neq 0, \quad n \neq 0,$$

$$|B(1, \chi_n)| \leqslant \frac{M_1}{|n|}, \qquad n \neq 0,$$

$$|B(\chi_m, 1)| \leqslant \frac{M_1}{|m|}, \qquad m \neq 0,$$

where $M_1 = Ma/\pi$. Therefore the series (6) is majorized by the series

$$| B(1, 1)|^2 + 4M_1 \sum_{n=1}^{\infty} \frac{1}{n^2} + 4M_1 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^2n^2} . \tag{7}$$

Since the series (7) converges, so does the series (6), and therefore (5) converges in the mean.

Let us denote the sum of the series (5) by $F(x, y)$, i.e., we put

$$F(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} B(\chi_m, \chi_n)\overline{\chi_m(x)}\ \overline{\chi_n(y)}. \tag{8}$$

Since this function is the sum of a Fourier series which converges in the mean, it has square integrable modulus, i.e.,

$$\int_{-a}^{a} \int_{-a}^{a} | F(x, y)|^2 \, dx \, dy < \infty.$$

Consequently the expression

$$\int_{-a}^{a} \int_{-a}^{a} F(x, y)\varphi(x)\psi(y) \, dx \, dy$$

defines a continuous bilinear functional on the space $H(a)$. Let us show that this functional coincides with the functional $B(\varphi, \psi)$. Since these functionals are continuous, it is sufficient to show that they coincide for the elements $\chi_n(x)$ of the orthonormal basis in $H(a)$, i.e., that

$$B(\chi_m, \chi_n) = \int_{-a}^{a} \int_{-a}^{a} F(x, y)\chi_m(x)\chi_n(y) \, dx \, dy. \tag{9}$$

But Eq. (9) immediately follows from the fact that by formula (8) the numbers $B(\chi_m, \chi_n)$ are the Fourier coefficients of the function $F(x, y)$ relative to the orthonormal basis $\overline{\chi_m(x)}\ \overline{\chi_n(y)}$ in the space $H_2(a)$ of functions $\varphi(x, y)$, $| x | \leqslant a$, $| y | \leqslant a$ having square integrable moduli, which proves the lemma.

A similar assertion holds for functions of n variables, where, instead of inequalities (3), one has to assume that the inequalities

$$| B(\varphi^{(j)}, \psi^{(k)})| \leqslant M\| \varphi \| \| \psi \|, \qquad 0 \leqslant | j | \leqslant n, \quad 0 \leqslant | k | \leqslant n.$$

hold.

We now generalize the result obtained to bilinear functionals on the Hilbert space $H^{(m)}(a)$ consisting of all functions $\varphi(x)$ on the interval

$|x| \leqslant a$ which have $m-1$ continuous derivatives (one-sided at the end points $-a, a$), with the $(m-1)$st having the form

$$\varphi^{(m-1)}(x) = c + \int_{-a}^{x} \varphi^{(m)}(t)\, dt,$$

where $\varphi^{(m)}(t)$ is square integrable over $[-a, a]$. The scalar product in $H^{(m)}(a)$ is defined by

$$(\varphi, \psi)_m = \sum_{k=0}^{m} \int_{-a}^{a} \varphi^{(k)}(x)\overline{\psi^{(k)}(x)}\, dx.$$

We denote $\sqrt{(\varphi, \varphi)_m}$ by $\|\varphi\|_m$. The following lemma holds:

Lemma 1′. Let $B(\varphi, \psi)$ be a bilinear functional on the space $H^{(m)}(a)$, continuous with respect to each argument, and such that

$$|B(\varphi', \psi')| \leqslant M\|\varphi\|_m\|\psi\|_m,$$
$$|B(\varphi, \psi')| \leqslant M\|\varphi\|_m\|\psi\|_m,$$
$$|B(\varphi', \psi)| \leqslant M\|\varphi\|_m\|\psi\|_m,$$

if the functions $\varphi'(x)$ and $\psi'(x)$ belong to $H^{(m)}(a)$. Then the functional $B(\varphi, \psi)$ can be represented in the form

$$B(\varphi, \psi) = \sum_{k,l=0}^{m} \int_{-a}^{a}\int_{-a}^{a} \frac{\partial^{k+l}F(x, y)}{\partial x^k\, \partial y^l}\, \varphi^{(k)}(x)\psi^{(l)}(y)\, dx\, dy,$$

where $F(x, y)$ is a function whose derivatives

$$\frac{\partial^{k+l} F(x, y)}{\partial x^k\, \partial y^l}$$

$0 \leqslant k, l \leqslant m$ have square integrable moduli on the square $-a \leqslant x \leqslant a$, $-a \leqslant y \leqslant a$.

This lemma is proved in the same way as was Lemma 1. One has only to choose the kernel in the form

$$F(x, y) = \sum_{j,k=0}^{\infty} B(\chi_j, \chi_k)\, \frac{\overline{\chi_j(x)}\ \overline{\chi_k(y)}}{\|\chi_j\|_m^2\|\chi_k\|_m^2}.$$

An analogous lemma holds also for functions of several variables.

We pass now to the principal aim of this section—the description of every bilinear functional on the space $K(a)$ of infinitely differentiable functions which vanish outside the interval $|x| \leqslant a$. This space is countably normed, where the norms are defined by

$$\|\varphi\|_m = \sup_{0 \leqslant k \leqslant m} \max_{|x| \leqslant a} |\varphi^{(k)}(x)|.$$

It will be convenient to replace this system of norms by the system of norms $|\varphi|_m$, defined by the scalar products

$$(\varphi, \psi)_m = \sum_{k=0}^{m} \int_{-a}^{a} \varphi^{(k)}(x)\overline{\psi^{(k)}(x)}\, dx$$

This system of norms is equivalent to the preceding system. In fact, it is obvious that

$$(\varphi, \varphi)_m = \sum_{k=0}^{m} \int_{-a}^{a} |\varphi^{(k)}(x)|^2\, dx \leqslant 2a(m+1)\|\varphi\|_m^2.$$

On the other hand, for $|x| \leqslant a$ we have

$$|\varphi^{(k)}(x)| = |\varphi^{(k)}(x) - \varphi^{(k)}(-a)| = \left| \int_{-a}^{x} \varphi^{(k+1)}(\xi)\, d\xi \right| \leqslant \int_{-a}^{x} |\varphi^{(k+1)}(\xi)|\, d\xi$$

Applying the Bunyakovski–Schwartz inequality, we arrive at the relation

$$|\varphi^{(k)}(x)|^2 \leqslant 2a \int_{-a}^{x} |\varphi^{(k+1)}(\xi)|^2\, d\xi \leqslant 2a \int_{-a}^{a} |\varphi^{(k+1)}(\xi)|^2\, d\xi.$$

From this relation, which is valid for every x in the interval $|x| \leqslant a$ and every k, it follows that

$$\|\varphi\|_m^2 = \sup_{0 \leqslant k \leqslant m} \max_{|x| \leqslant a} |\varphi^{(k)}(x)| \leqslant 2a \sum_{k=0}^{m} \int_{-a}^{a} |\varphi^{(k+1)}(\xi)|^2\, d\xi \leqslant 2a(\varphi, \varphi)_{m+1},$$

i.e., $\|\varphi\|_m^2 \leqslant 2a(\varphi, \varphi)_{m+1}$.

Thus

$$(\varphi, \varphi)_m \leqslant 2a(m+1)\|\varphi\|_m^2$$

and

$$\|\varphi\|_m^2 \leqslant 2a(\varphi, \varphi)_{m+1}.$$

These inequalities show that the topologies defined by the system of norms $\|\varphi\|_m$ and $|\varphi|_m = \sqrt{(\varphi, \varphi)_m}$ coincide.

Now denote by $K^{(m)}(a)$ the completion of the space $K(a)$ with respect to the norm $| \varphi |_m = \sqrt{(\varphi, \varphi)_m}$. This completion is obviously a Hilbert space with scalar product

$$(\varphi, \psi)_m = \sum_{k=0}^{m} \int_{-a}^{a} \varphi^{(k)}(x)\overline{\psi^{(k)}(x)}\, dx,$$

and is in fact a subspace of $H^{(m)}(a)$. We also need the following result: If not only $\varphi(x)$ but also $\varphi'(x)$ belongs to the space $H^{(m)}(a)$, then the inequality

$$| \varphi' |_m \leqslant | \varphi |_{m+1} \tag{10}$$

holds. In fact

$$(\varphi', \varphi')_m = \sum_{k=0}^{m} \int_{-a}^{a} | \varphi^{(k+1)}(x)|^2\, dx \leqslant \sum_{k=0}^{m+1} \int_{-a}^{a} | \varphi^{(k)}(x)|^2\, dx = (\varphi, \varphi)_{m+1}$$

and therefore $| \varphi' |_m \leqslant | \varphi |_{m+1}$.

Let us now consider a bilinear function $B(\varphi, \psi)$ on the space $K(a)$. By Theorem 3, Section 1.2, this functional is continuous relative to certain norms $| \varphi |_m$ and $| \psi |_n$ in $K(a)$. Without loss of generality we can assume $m = n$. Therefore the functional $B(\varphi, \psi)$ satisfies the inequality

$$| B(\varphi, \psi)| \leqslant M| \varphi |_m| \psi |_m.$$

From this it follows that the functional $B(\varphi, \psi)$ can be extended by continuity to $K^{(m)}(a)$, and then arbitrarily to all of the Hilbert space $H^{(m)}(a)$. At the same time, if the functions $\varphi(x), \psi(x) \in H^{(m)}(a)$ are such that their derivatives $\varphi'(x)$ and $\psi'(x)$ also belong to $H^{(m)}(a)$, then by formula (10) the inequality

$$| B(\varphi', \psi')| \leqslant M| \varphi' |_m| \psi' |_m \leqslant M| \varphi |_{m+1}| \psi |_{m+1}$$

holds, as well as the inequalities

$$| B(\varphi', \psi)| \leqslant M| \varphi |_{m+1}| \psi |_{m+1}$$

and

$$| B(\varphi, \psi')| \leqslant M| \varphi |_{m+1}| \psi |_{m+1}.$$

We can therefore apply Lemma 1$'$. According to this lemma $B(\varphi, \psi)$ has the form

$$B(\varphi, \psi) = \sum_{k,l=0}^{m+1} \int_{-a}^{a} \int_{-a}^{a} \frac{\partial^{k+l}}{\partial x^k\, \partial y^l} F(x, y)\varphi^{(k)}(x)\psi^{(l)}(y)\, dx\, dy, \tag{11}$$

where $F(x, y)$ is a function whose derivatives $\frac{\partial^{k+l} F(x, y)}{\partial x^k \, \partial y^l}$, $0 \leqslant k, l \leqslant m + 1$, have square integrable moduli. Thus we have proved the following assertion.

Lemma 2. Any bilinear functional $B(\varphi, \psi)$, continuous in each argument, on the space $K(a)$ of infinitely differentiable functions which vanish outside the interval $|x| \leqslant a$ can be written in the form (11), where $F(x, y)$ is a function whose derivatives $\frac{\partial^{k+l} F(x, y)}{\partial x^k \, \partial y^l}$, $0 \leqslant k, l \leqslant m + 1$, have square integrable moduli.

We note now that if every derivative $\frac{\partial^{k+l} F(x, y)}{\partial x^k \, \partial y^l}, 0 \leqslant k, l \leqslant m + 1$, has square integrable modulus, then

$$(F, \varphi) = \sum_{k,l=0}^{m+1} \int_{-a}^{a} \int_{-a}^{a} \frac{\partial^{k+l}}{\partial x^k \, \partial y^l} F(x, y) \, \frac{\partial^{k+l}}{\partial x^k \, \partial y^l} \varphi(x, y) \, dx \, dy \qquad (11')$$

defines a continuous linear functional F on the space $K_2(a)$ of infinitely differentiable functions which vanish outside the square $|x| \leqslant a$, $|y| \leqslant a$. Therefore Lemma 2 may be formulated in the following way:

Theorem 4. Let $B(\varphi, \psi)$ be a bilinear functional, continuous in each argument, on the space $K(a)$ of infinitely differentiable functions which vanish outside of the interval $|x| \leqslant a$. Then $B(\varphi, \psi)$ can be written in the form

$$B(\varphi, \psi) = (F, \varphi(x) \, \psi(y))$$

where F is a linear functional on the space $K_2(a)$ of infinitely differentiable functions $\varphi(x, y)$ which vanish outside the square $|x| \leqslant a$, $|y| \leqslant a$.

It is now easy to obtain the general form of bilinear functionals on the space K of all infinitely differentiable functions having bounded supports. In fact, let $B(\varphi, \psi)$ be a bilinear functional on K. Then for any value a, $B(\varphi, \psi)$ defines a bilinear functional $B_a(\varphi, \psi)$ on the subspace $K(a)$ of functions which vanish outside the interval $|x| \leqslant a$. By Theorem 4 there is a linear functional F_a on the space $K_2(a)$ of infinitely differentiable functions $\varphi(x, y)$ which vanish outside the square $|x| \leqslant a$, $|y| \leqslant a$, such that

$$B(\varphi, \psi) = B_a(\varphi, \psi) = (F_a \, \varphi \, (x) \psi(y))$$

for $\varphi(x) \in K(a)$, $\psi(x) \in K(a)$. The functionals F_a are mutually compatible in the sense that

$$(F_b, \chi(x, y)) = (F_c, \chi(x, y)), \tag{12}$$

if the function $\chi(x, y)$ belongs to the space $K_2(a)$ and $b \geqslant a$, $c \geqslant a$. As a matter of fact, suppose that $b \geqslant a$ and $\varphi(x)$, $\psi(x)$ are functions from $K(a)$. Then

$$(F_b, \varphi(x)\psi(y)) = B_b(\varphi, \psi) = B(\varphi, \psi).$$

Since the right side of this equation does not depend upon b, we find that $(F_b, \varphi(x)\psi(y)) = (F_c, \varphi(x)\psi(y))$ for $b \geqslant a$, $c \geqslant a$. Thus Eq. (12) is proved for functions of the form $\chi(x, y) = \varphi(x)\psi(y)$. Since linear combinations of such functions are everywhere dense in the space $K_2(a)$, (12) holds for every function $\chi(x, y) \in K_2(a)$.

But to any compatible family of linear functionals F_a on the spaces $K_2(a)$ there corresponds a linear functional F on the space K_2, defined by the equation

$$(F, \varphi(x, y)) = (F_a, \varphi(x, y)),$$

where a is chosen so that $\varphi(x, y)$ vanishes outside the square $|x| \leqslant a$, $|y| \leqslant a$. From the compatibility of the functionals F_a it follows that the value of F does not depend upon a. Thus the functional F is uniquely defined for every function $\varphi(x, y)$ in K_2.

From the continuity and linearity of every functional F_a it follows that the functional F is continuous and linear. Moreover, for any functions $\varphi(x)$ and $\psi(x)$ in the space K we have

$$B(\varphi, \psi) = (F, \varphi(x)\psi(y)).$$

In fact, there is a number a such that $\varphi(x)$ and $\psi(x)$ vanish outside the interval $|x| \leqslant a$. Then

$$B(\varphi, \psi) = B_a(\varphi, \psi) = (F_a, \varphi(x)\psi(y)) = (F, \varphi(x)\psi(y)).$$

Thus we have proven the following theorem.

Theorem 5 (*The Kernel Theorem*). Every bilinear functional $B(\varphi, \psi)$, on the space K of all infinitely differentiable functions having bounded supports, which is continuous in each of the arguments φ and ψ has the form

$$B(\varphi, \psi) = (F, \varphi(x)\psi(y)),$$

where F is a continuous linear functional on the space K_2 of infinitely differentiable functions of two variables having bounded supports.

Of course, this theorem can without difficulty be generalized to the case where $\varphi(x)$ and $\psi(y)$ are functions of several variables.

A theorem analogous to Theorem 5 holds also for the space S of infinitely differentiable functions which, together with all their derivatives, are rapidly decreasing as $|x| \to \infty$. This theorem can be formulated in the following way.

Theorem 6. A bilinear functional $B(\varphi, \psi)$ on the space S, continuous in each argument, has the form

$$B(\varphi, \psi) = (F, \varphi(x)\psi(y)),$$

where F is a linear functional on the space S_2 of functions which, together with all of their derivatives, are rapidly decreasing as $x^2 + y^2 \to \infty$.

This theorem is proved almost exactly as was Theorem 5, with certain modifications in the construction of the kernel $F(x, y)$. Bearing in mind that a theorem will be proved in Section 3, of which Theorems 5 and 6 are particular cases, we omit the proof of Theorem 6.

We now indicate the general form of bilinear functionals on the space S. A description of linear functionals on the space S was given in Volume II (Chapter II, Section 4.3). Applying this to the functional F on S_2 which is defined by the bilinear functional $B(\varphi, \psi)$, we arrive at the following theorem.

Theorem 7. Any bilinear functional on the space S which is continuous in each argument has the form

$$B(\varphi, \psi) = \int F(x, y)\varphi^{(m)}(x)\psi^{(m)}(y)\, dx\, dy,$$

where $F(x, y)$ is a continuous function of power growth.[11]

One can obtain, for bilinear functionals on the space $K(a)$, a similar formula which is simpler than formula (11); namely, the following theorem holds.

[11] We say that a function $F(x, y)$ has power growth if there is a p such that

$$\lim_{x^2+y^2\to\infty} F(x, y)\,(x^2 + y^2)^{-p} = 0.$$

Theorem 8. Any bilinear functional on the space $K(a)$ which is continuous in each argument has the form

$$B(\varphi, \psi) = \int_{-a}^{a} \int_{-a}^{a} F(x, y)\varphi^{(m)}(x)\psi^{(m)}(y) \, dx \, dy,$$

where $F(x, y)$ is a continuous function defined on the square $|x| \leqslant a$, $|y| \leqslant a$.

In conclusion we remark that theorems analogous to the kernel theorem are valid also for polylinear functionals. For example, the corresponding theorem for the space K is formulated in the following way.

Theorem 5'. Let $B(\varphi_1, ..., \varphi_m)$ be a polylinear functional on the space K which is continuous in each argument. Then there is a linear functional F, on the space K_m of infinitely differentiable functions of m variables having bounded supports, such that

$$B(\varphi_1, ..., \varphi_m) = (F, \varphi_1(x_1) \dots \varphi_m(x_m)).$$

Appendix. The Spaces K, S, and Z

Throughout this volume we will, as a rule, be concerned only with the spaces K, S, and Z. Other spaces of test functions will be encountered only occasionally. In order to spare the reader the necessity of referring to earlier volumes of this series for the definition of each of these spaces, we give here a brief discussion of the basic results concerning the spaces K, S, and Z.

We denote by K the space of all functions $\varphi(x) = \varphi(x_1, ..., x_n)$ of n variables which are infinitely differentiable and have bounded supports. A sequence $\{\varphi_m(x)\}$ of functions in K is said to converge to zero if there exists a constant a such that every function $\varphi_m(x)$ vanishes for $|x| \geqslant a$,[12] and if for every q the sequence $\{\varphi_m^{(q)}(x)\}$ converges uniformly to zero.

This definition can be formulated in another way. Let us denote by $K(a)$ the subspace in K consisting of those functions $\varphi(x)$ which vanish for $|x| \geqslant a$. A sequence $\{\varphi_m(x)\}$ converges to zero in the space $K(a)$ if for every q the sequence $\{\varphi_m^{(q)}(x)\}$ converges uniformly to zero. If $a < b$ then $K(a) \subset K(b)$. The space K is the union of all the spaces

[12] We maintain the notation of the preceding volumes; for example, $|x|$ denotes the quantity $(x_1^2 + ... + x_n^2)^{\frac{1}{2}}$, and $|q|$, the sum $q_1 + ... + q_n$.

$K(a)$, and a sequence $\{\varphi_m(x)\}$ of functions in K converges to zero if and only if all the functions $\varphi_m(x)$ belong to the same subspace $K(a)$ and converge to zero relative to the topology of this subspace.

The topology in the space K may also be defined by giving a system of neighborhoods of zero in K. We note first that in the subspace $K(a)$ a topology can be defined by means of the system of norms

$$\|\varphi(x)\|_m = \sup_{|q|\leqslant m} \max_{|x|\leqslant a} |\varphi^{(q)}(x)|.$$

We call an absolutely convex set U in K a neighborhood of zero if for any a the set $U \cap K(a)$ is a neighborhood of zero in $K(a)$. This means the following: there exist numbers m and ϵ, depending upon a, such that $\varphi(x) \in U$ when $\varphi(x) \in K(a)$ and

$$\sup_{|q|\leqslant m} \max_{|x|\leqslant a} |\varphi^{(q)}(x)| < \epsilon.$$

It is not difficult to show that the topology defined by this (uncountable) system of neighborhoods of zero leads to precisely that notion of convergence which we introduced above.†

Another space which will be considered below is the space S. It consists of all infinitely differentiable functions $\varphi(x)$ which are rapidly decreasing, together with their derivatives of all orders, as $|x| \rightarrow \infty$. This means that for every $r \geqslant 0$ and q the relation

$$\lim_{|x|\rightarrow\infty} |(1 + |x|^2)^r \varphi^{(q)}(x)| = 0$$

holds. A sequence $\{\varphi_m(x)\}$ of functions in S is said to converge to zero if for every $r \geqslant 0$ and q

$$\lim_{m\rightarrow\infty} \max_{x} |(1 + |x|^2)^r \varphi_m^{(q)}(x)| = 0.$$

A topology in the space S can be defined by prescribing the following system of neighborhoods of zero. The neighborhood $U(r, k, \epsilon)$ is

† This definition of the topology in K appears to differ from that given in reference (69), Volume 1, Chapter III, §1, which is as follows: for each sequence of positive numbers $\epsilon_1 \geqslant \epsilon_2 \geqslant \cdots \rightarrow 0$ and non-negative integers $m_1 \leqslant m_2 \leqslant \cdots \rightarrow +\infty$ we define $V(\{\epsilon_n\}, \{m_n\})$ as the set of functions $\varphi \in K$ which satisfy, for each j, the inequalities

$$|\varphi^{(q)}(x)| \leqslant \epsilon_j \quad \text{for} \quad |x| \geqslant j \quad \text{and} \quad |q| \leqslant m_j.$$

The collection of all such $V(\{\epsilon_n\}, \{m_n\})$ is taken as a fundamental system of neighborhoods of zero in K. [Note: Footnotes cited by the dagger (†) are the translator's.]

defined by the positive integers r and k and a number $\epsilon > 0$, and consists of every function $\varphi(x)$ in the space S for which the inequality

$$\max_x |(1 + |x|^2)^r \varphi^{(q)}(x)| < \epsilon$$

holds when $|q| \leqslant k$.

Every function in the space K obviously belongs to the space S, and the functions in K form an everywhere dense set in S relative to the topology of S. Indeed, let $\varphi(x)$ be any function of the space S; we choose an arbitrary function $\alpha(x)$ in K such that $\alpha(0) = 1$. Then each function $\alpha(x/m)\,\varphi(x)$ belongs to K and, as is easily seen, as $m \to \infty$ the sequence of these functions converges to the function $\varphi(x)$ in the topology of S.

The imbedding of the space K into the space S is continuous, since the inequality

$$\max_x |(1 + |x|^2)^r \varphi^{(q)}(x)| < \epsilon, \qquad |q| < k,$$

which defines a neighborhood of zero in S, also defines a neighborhood of zero in K. Indeed, in each of the spaces $K(a)$ the topologies induced by the topologies of K and S coincide. From this it easily follows that $K(a)$ is a closed subspace not only of K but also of S.

Now we introduce the space Z. It consists of entire analytic functions $\varphi(z) = \varphi(z_1, \ldots, z_n)$ such that for any r the inequality[13]

$$|z^r \varphi(z)| \leqslant Ce^{a|y|}, \qquad z = x + iy, \tag{13}$$

holds, where the constant a depends upon $\varphi(z)$, and the constant C depends upon $\varphi(z)$ and r. From now on we will call an entire analytic function which satisfies an inequality of the form

$$|\varphi(z)| \leqslant Ce^{a|z|}$$

a function of exponential type. From inequality (13) it follows that a function in Z is a function of exponential type which is rapidly decreasing on the real axis. It can be shown that if a function $\varphi(z)$ belongs to Z, then for every r and q the relation

$$|z^r \varphi^{(q)}(z)| \leqslant C_1 e^{a|y|} \tag{14}$$

holds, where C_1 depends upon r and q.

[13] By z^r we understand, as usual, $z_1^{r_1} \cdot \ldots \cdot z_n^{r_n}$, and by $a|y|$ the sum $a_1|y_1| + \ldots + a_n|y_n|$.

A sequence $\{\varphi_m(z)\}$ of functions in Z is said to converge to zero if every function $\varphi_m(z)$ satisfies inequalities of the form

$$| z^r \varphi_m(z)| \leqslant Ce^{a|y|},$$

where the constant a does not depend upon m, and for every q and r the relation

$$\lim_{m\to\infty} \max_x |(1 + | x |^2)^r \varphi_m^{(q)}(x)| = 0$$

holds.

The family of functions $\varphi(z) \in Z$ which satisfy inequalities of the form (13) with a fixed value of a forms a closed linear subspace $Z(a)$ in Z. Thus the space Z is the union of the subspaces $Z(a)$, and a sequence $\{\varphi_m(z)\}$ converges to zero in Z if all the functions $\varphi_m(z)$ belong to the same subspace $Z(a)$ and converge to zero in this subspace.

We note now that considering a function $\varphi(z) \in Z$ for real values of its argument, we obtain an infinitely differentiable function which is rapidly decreasing as $| x | \to \infty$, together with its derivatives of all orders [see inequality (14)]. In this way there is defined a continuous imbedding of the space Z into the space S. This imbedding preserves the topology in each of the subspaces $Z(a)$. In other words, the topologies induced in the spaces $Z(a)$ by the topologies of Z and S coincide.

The space S is carried into itself by the Fourier transformation, which takes a function $\varphi(x)$ into the function $\tilde{\varphi}(\lambda)$:

$$\tilde{\varphi}(\lambda) = \int \varphi(x)e^{i(x,\lambda)}\,dx, \qquad (x, \lambda) = x_1\lambda_1 + \dots + x_n\lambda_n.$$

Under this transformation the space K is mapped onto the space Z, and the subspace $K(a)$ is mapped onto the subspace $Z(a)$. Since the functions in K are everywhere dense in S, the functions in Z are everywhere dense in S. Besides, this is easily verified directly. In fact, let $\alpha(z)$ be any function in Z such that $\int \alpha(x)\,dx = 1$. Then for any function $\varphi(z) \in S$ we have $\varphi(x) = \lim_{m\to\infty} \varphi_m(x)$, where

$$\varphi_m(x) = \varphi * \alpha_m(x), \qquad \alpha_m(x) = m^n\alpha\left(\frac{x}{m}\right).$$

But it is not hard to see that the functions $\varphi_m(x)$ belong to the space Z.[14]

[14] We denote by $\varphi * \alpha(x)$, where $\varphi(x) \in S$, $\alpha(x) \in S$, the convolution of the functions $\varphi(x)$ and $\alpha(x)$, defined by

$$\varphi * \alpha(x) = \int \varphi(y)\alpha(x - y)\,dy.$$

Since the functions in Z may also be considered as functions in S, the convolution is thus defined also for functions in Z. It can be shown that the convolution of two functions in Z belongs to Z. The assertion that the $\varphi_m(x)$ belong to Z means that there exist functions $\varphi_m(z)$ which coincide with the functions $\varphi_m(x)$ for real values of their argument z.

We now consider linear functionals on the spaces K, S, Z (generalized functions on these spaces). These linear functionals form linear spaces which are denoted by K', S', and Z', respectively. We shall regard K', S', Z' as linear topological spaces. A sequence $\{F_m\}$ of linear functionals is said to converge to zero if

$$\lim_{m\to\infty}(F_m, \varphi) = 0$$

for any test function φ.

Since the space K can be mapped continuously and one-to-one onto an everywhere dense subset of the space S, then each linear functional on S defines a linear functional on K, and to distinct functionals on S correspond distinct functionals on K. Thus there is defined a continuous imbedding of the space S' into the space K'. It can be shown that the elements of S' form an everywhere dense set in K'. In exactly the same way the elements of S' also form an everywhere dense set in Z'.

Examples of linear functionals on the space K are the functionals of the form

$$(f, \varphi) = \int \overline{f(x)}\varphi(x)\, dx,$$

where $f(x)$ is an arbitrary continuous function. Such a functional is called the *regular functional* corresponding to the function $f(x)$. In particular, to each function $\psi(x) \in K$ there corresponds a functional F_ψ on K having the form

$$(F_\psi, \varphi) = \int \overline{\psi(x)}\varphi(x)\, dx.$$

Let us show that the functionals of the form F_ψ, $\psi(x) \in K$, are everywhere dense in K'. To this effect, we choose positive functions $\alpha(x)$ and $\beta(x)$ in K such that $\int \alpha(x)\, dx = 1$ and $\beta(0) = 1$. We set

$$\alpha_m(x) = m^n \alpha\left(\frac{x}{m}\right) \qquad \text{and} \qquad \beta_m(x) = \beta(mx), \qquad x = (x_1, \ldots, x_n),$$

and associate with each linear functional F on K a sequence of linear functionals F_m, defined by

$$(F_m, \varphi) = \int \bar{\psi}_m(x)\varphi(x)\, dx,$$

where

$$f_m(x) = (F(y), \alpha_m(x - y)), \qquad \psi_m(x) = f_m(x)\beta_m(x).$$

Obviously the functions $f_m(x)$ are infinitely differentiable and therefore the functions $\psi_m(x) = f_m(x)\beta_m(x)$ belong to the space K. It is not hard to show that

$$\lim_{m\to\infty} (F_m, \varphi) = (F, \varphi)$$

for any function $\varphi(x) \in K$. Thus we have proved that the functionals of the form F_ψ, $\psi(x) \in K$, are everywhere dense in K'.

Further on we will need the Fourier transform not only for test functions but also for generalized functions. Let F be a generalized function (linear functional) on any space Φ of test functions and let $\tilde{\Phi}$ be the space consisting of the Fourier transforms of the functions in Φ (for example, $\tilde{K} = Z$, $\tilde{Z} = K$, $\tilde{S} = S$). The generalized function $\tilde{F}$ on $\tilde{\Phi}$, defined by[15]

$$(\tilde{F}, \tilde{\varphi}) = (2\pi)^n (F, \varphi)$$

is called the Fourier transform of the generalized function F. For regular generalized functions F_ψ, $\psi(x) \in \Phi$, the formula $\tilde{F}_\psi = F_{\tilde{\psi}}$ holds, which shows that our definition of the Fourier transform for generalized functions agrees with the definition of the Fourier transform for test functions.

In conclusion we indicate the general form of linear functionals on the spaces K, S, and Z.

Every linear functional on the space S has the form

$$(F, \varphi) = \int f(x)\varphi^{(m)}(x)\, dx,$$

where $f(x)$ is a continuous function such that for some $r > 0$

$$|(1 + |x|^2)^{-r} f(x)| \to 0 \qquad \text{as} \quad |x| \to \infty$$

(such functions are called functions of power growth).

A linear functional on any one of the spaces $K(a)$ is defined by an expression of the form

$$(F, \varphi) = \int_{|x| \leqslant a} f(x)\varphi^{(m)}(x)\, dx,$$

where $f(x)$ is a continuous function. Since the space K is the union of the subspaces $K(a)$, each linear functional F on K defines a family of

[15] n is the number of variables.

linear functionals F_a on the subspaces $K(a)$: $(F, \varphi) = (F_a, \varphi)$ if $\varphi(x) \in K(a)$ These functionals are mutually compatible in the sense that $(F_a, \varphi) = (F_b, \varphi)$ if $a < b$ and $\varphi(x) \in K(a)$. Conversely, any collection of compatible linear functionals F_a on the spaces $K(a)$ defines a linear functional F on K such that $(F, \varphi) = (F_a, \varphi)$ for $\varphi(x) \in K(a)$ (the continuity of the functional F follows directly from the definition of the topology in K). Every linear functional on the space Z is the Fourier transform of a linear functional on the space K.

2. Operators of Hilbert-Schmidt Type and Nuclear Operators

In the preceding section we proved the important kernel theorem for the space K. It was stated there that an analogous theorem holds also for the space S. One can without difficulty enlarge the number of analogous theorems.[1] In the following section we will present a wide class of spaces, called nuclear spaces, for which an analog of the kernel theorem holds. The definition of nuclear spaces is related to certain classes of operators in Hilbert space, which we consider in this section. Of these classes, two—completely continuous operators and operators of Hilbert–Schmidt type— are undoubtedly well known to the reader. However, for the sake of keeping the discussion more self-contained we will recall the basic properties of these operators.

The third class of operators—the class of nuclear operators—has in recent years gained importance in various connections. We define them in Section 2.3. Finally, in Section 2.4 we give an affine definition of the concept of a nuclear operator, which, in particular, makes it possible to generalize the definition to Banach spaces. We remark that in Banach spaces nuclear operators lose a number of important properties.

Every class of operators which is considered below arises upon completing the space of *degenerate operators*[2] with respect to one norm or another. Namely, as will be shown (cf. Theorem 1), every degenerate operator A can be written in the form $A = UT$, where U is an isometric operator and T is a positive-definite operator.[3] Let $\lambda_1, \ldots, \lambda_n$ be the

[1] A similar theorem holds, for example, in the space $\mathfrak{Z}$ of entire analytic functions, in which the topology is given by the norms

$$\| \varphi(z) \| = \sup_{|z|=n} | \varphi(z) |.$$

[2] An operator A is called *degenerate* if it maps the entire space onto a finite-dimensional subspace.

[3] We call an operator A *positive definite if* $(Af, f) \geqslant 0$, and *strictly positive definite* if $(Af, f) > 0, f \neq 0$.

eigenvalues of the operator T. The space of completely continuous operators is obtained from the space of degenerate operators by completing the latter in the norm $\| A \| = \max_k \lambda_k$; the space of Hilbert–Schmidt operators, by completion in the norm

$$\| A \|_2 = \sqrt{\sum_{k=1}^{n} \lambda_k^2};$$

and the space of nuclear operators, by completion in the norm $\| A \|_1 = \sum_{k=1}^{n} \lambda_k$.

The norms $\| A \|$, $\| A \|_1$, $\| A \|_2$ are isometrically invariant: the values of each coincide for the operators A, UA, and AU, where U is an isometric operator.

It is possible to describe every function $f(\lambda_1, \ldots, \lambda_n)$ of the eigenvalues of the operator T such that the equation

$$\| A \| = f(\lambda_1, \ldots, \lambda_n), \qquad A = UT$$

defines an isometrically invariant norm in the space of degenerate operators. Namely, $f(\lambda_1, \ldots, \lambda_n)$ must be a positive function, homogeneous of degree one, symmetric with respect to the variables $\lambda_1, \ldots, \lambda_n$, and such that after we extend its definition, as an even function, to all values of the variables, the result is a convex function in n-dimensional space. The functions $\max_k \lambda_k$, $\sqrt{\sum_{k=1}^{n} \lambda_k^2}$, and $\sum_{k=1}^{n} \lambda_k$ which we are considering satisfy these conditions.

2.1. Completely Continuous Operators

A linear operator A which maps a Hilbert space H_1 into a Hilbert space H_2 is called *completely continuous* if it carries any bounded set into a set whose closure is compact. This definition is equivalent to the following: the operator A is completely continuous if it takes every weakly convergent sequence of elements into a strongly convergent sequence.[4]

[4] A sequence of elements $h_1, h_2, \ldots$ of a Hilbert space H is said to converge strongly to the element h if

$$\lim_{n\to\infty} \| h_n - h \| = 0.$$

The sequence $h_1, h_2, \ldots$ is said to converge weakly to h if

$$\lim_{n\to\infty} (h_n, g) = (h, g)$$

for every element g of H.

We note the following properties of completely continuous operators:

(1) If A is a completely continuous operator which maps the space H_1 into the space H_2, then its adjoint[5] operator A^*, which maps H_2 into H_1, is likewise completely continuous.

(2) The product AB of a continuous linear operator A and a completely continuous operator B is completely continuous. The same assertion holds for the product BA.

(3) Any linear combination of completely continuous operators is a completely continuous operator.

Properties (2) and (3) are obvious. The proof of property (1) is carried out in the book of N. I. Akhiezer and I. M. Glazman, "Theory of Linear Operators in Hilbert Space," Chapter 2, §27. Moscow-Leningrad, 1950. (English translation by Ungar, New York, 1962).

If one considers linear operators acting in some Hilbert space H, then properties (1)–(3) can be formulated briefly by saying that the completely continuous operators form an ideal in the ring with an involution consisting of all continuous linear operators.

It can be shown that the set of completely continuous operators is closed in the set of all continuous linear operators relative to the (operator) norm

$$\| A \| = \sup_{\|f\|=1} \| Af \|$$

and, consequently, is complete relative to this norm.[6]

A self-adjoint completely continuous operator, i.e., a completely continuous operator A such that $(Af, g) = (f, Ag)$ for every $f, g \in H$, has a particularly simple structure. If A is a completely continuous self-adjoint operator, then one can choose an orthonormal basis $e_1, e_2, \ldots$ in H which consists of eigenvectors of A, $Ae_n = \lambda_n e_n$, and the eigenvalues $\lambda_1, \lambda_2, \ldots$ corresponding to the vectors $e_1, e_2, \ldots$ are real and converge to zero as $n \to \infty$, i.e., $\lim_{n\to\infty} \lambda_n = 0$.

Conversely, every operator A which is defined, in terms of some

[5] Let A be a linear operator which maps the Hilbert space H_1 into the Hilbert space H_2. The operator A^* mapping H_2 into H_1, and such that

$$(Af, g) = (f, A^*g)$$

for every element $f \in H_1$ and $g \in H_2$, is called the *adjoint* of A. More precisely, one should write $(Af, g)_2$ and $(f, A^*g)_1$, where $(Af, g)_2$ is the scalar product in H_2 and $(f, A^*g)_1$ is the scalar product in H_1. However, we are confident that the reader can in every case easily determine in which of the spaces the scalar product is taken.

[6] This assertion is proved in Akhiezer and Glazman, *loc. cit.*, Chapter 2, §28.

orthonormal basis $e_1, e_2, \ldots$, by $Ae_n = \lambda_n e_n$, where the λ_n are real numbers and $\lim_{n\to\infty} \lambda_n = 0$, is self-adjoint and completely continuous.[7]

If the operator A is positive definite [i.e., if $(Af, f) \geqslant 0$ for every vector $f \in H$], then its eigenvalues are either positive or equal to zero.

We now show that any (in general, nonself-adjoint) completely continuous operator differs from a positive-definite completely continuous operator only by an isometric factor, i.e., an operator U such that $\| Uf \| = \| f \|$. In other words, the following theorem holds:

Theorem 1. Let A be a completely continuous operator which maps H_1 into H_2. Then A has the form $A = UT$, where T is a positive-definite completely continuous operator in H_1, and U is an isometric operator† which maps the range of T into the space H_2.

Proof. Let us consider the operator $B = A^*A$. Since A maps H_1 into H_2, and A^* maps into H_2 into H_1, B takes H_1 into itself. As the product of two completely continuous operators A and A^*, B is completely continuous; further, B is positive definite. In fact, for any vector $f \in H_1$ one has the inequality

$$(Bf, f) = (A^*Af, f) = (Af, Af) \geqslant 0.$$

Consequently, as was stated above, the operator B has the form $Be_n = \lambda_n e_n$, where $e_1, e_2, \ldots$ is an orthonormal basis in H_1, $\lambda_n \geqslant 0$, and $\lim_{n\to\infty} \lambda_n = 0$. We now introduce a new operator $T = B^{\frac{1}{2}}$ defined by $Te_n = \sqrt{\lambda_n} e_n$. Obviously, $T^2 = B$. Moreover, it is clear that T is completely continuous and positive definite.

Let us compare $\| Af \|$ and $\| Tf \|$. We have

$$\| Af \|^2 = (Af, Af) = (A^*Af, f) = (T^2f, f).$$

But T is positive definite and consequently self-adjoint. Therefore

$$(T^2f, f) = (Tf, Tf) = \| Tf \|^2.$$

Thus the operators A and T are *metrically equal*, i.e.,

$$\| Af \| = \| Tf \|$$

[7] Cf. Akhiezer and Glazman, *loc. cit.*, Chapter 5, §55.

† The operator U can obviously be extended to the closure of the range of T by continuity. Often it is extended to all of H_1 by defining it to be zero on the orthogonal complement of the range of T, and then requiring linearity. An operator of this type is generally called *partially isometric*, and the decomposition $A = UT$ is called the *polar decomposition* of A.

for any element f of H_1. Now we define an operator U by the equation

$$Ug = Af$$

for every element g of the form $g = Tf$, $f \in H_1$. The operator U is isometric because $g = Tf$ and $\| Af \| = \| Tf \|$. Obviously, $Af = Ug = U(Tf)$ and therefore $A = UT$. Thus the theorem is proved.[8]

We remark that the operator U is defined on the set of elements of the form Tf, i.e., on the range of T. In view of its isometry we can extend it to the closure of its domain of definition. It is not hard to see that this closure is the subspace in H_1 spanned by the eigenvectors e_n corresponding to the nonzero eigenvalues λ_n of the operator T.

Theorem 1 enables us to give a geometric description of completely continuous operators (in general, nonself-adjoint). Let $A = UT$ be a completely continuous operator, and $e_1, e_2, \ldots$ orthogonal eigenvectors of the positive-definite operator T and $\lambda_n \geqslant 0$ the corresponding eigenvalues. Let us consider the sphere $\| x \| = 1$ in H_1. The operator T transforms this sphere into an ellipsoid, whose principal axes are directed along the vectors $e_1, e_2, \ldots$. The lengths of the semiaxes of this ellipsoid are equal to $\lambda_1, \lambda_2, \ldots$. Now the operator U isometrically maps this ellipsoid into the space H_2. As a result one obtains an ellipsoid in the space H_2, whose principal axes are directed along the vectors $h_n = Ue_n$ and whose semiaxes are of length λ_n. The lengths of the semiaxes of this ellipsoid tend to zero, since $\lim_{n\to\infty} \lambda_n = 0$.

Conversely, any operator A which transforms the sphere A into an ellipsoid whose principal semiaxes tend to zero is completely continuous.

The simplest example of a completely continuous operator is an operator P of the form

$$Pf = \lambda(f, e)h,$$

where e and h are fixed vectors of unit length, and λ is a fixed number. This operator maps all of H onto a one-dimensional space. We will now show that any completely continuous operator can be approximated by a sum of such operators. Specifically, we will show that a completely continuous operator A can be represented as the sum of a series

$$Af = \sum_{n=1}^{\infty} \lambda_n(f, e_n)h_n, \tag{1}$$

[8] An analogous assertion holds not only for completely continuous but also for arbitrary-bounded (and even for a wide class of unbounded) operators. However, the result stated in the text is sufficient for our purposes.

where the e_n (respectively, the h_n) are the elements of an orthonormal set in H_1 (respectively, in H_2), and $\lambda_1, \lambda_2, \ldots$ are positive numbers which tend to zero as $n \to \infty$. Conversely, every series of the form (1), in which e_n, h_n, λ_n have the aforementioned properties, defines a completely continuous operator.

The decomposition (1) may be obtained in the following way. We represent the operator A in the form $A = UT$ and denote by $\{e_n\}$ the set of vectors remaining after deleting, from an orthonormal basis for H_1 consisting of eigenvectors of T, those members with zero eigenvalues; let $\{\lambda_n\}$ be the eigenvalues of the e_n, and set $h_n = Ue_n$. Now for any $f \in H_1$ we have

$$Af = UTf = U\left(\sum_{n=1}^{\infty} \lambda_n(f, e_n)e_n\right) = \sum_{n=1}^{\infty} \lambda_n(f, e_n)h_n.$$

Now we show that the series (1) converges in operator norm, in other words, that the operators A_k, defined by

$$A_k f = \sum_{n=1}^{k} \lambda_k(f, e_k)h_k,$$

converge in the operator norm to the operator A. Suppose $\| f \| = 1$. Then, since the set $\{h_n\}$ is orthonormal,

$$\|(A - A_k)f\|^2 = \sum_{n=k+1}^{\infty} \lambda_n^2 |(f, e_n)|^2 \leqslant \Lambda_k^2 \sum_{n=k+1}^{\infty} |(f, e_n)|^2 \leqslant \Lambda_k^2 \|f\|^2 = \Lambda_k^2,$$

where Λ_k denotes the largest of the numbers $\lambda_{k+1}, \lambda_{k+2}, \ldots$. From this inequality it follows that

$$\| A - A_k \| = \sup_{\|f\|=1} \|(A - A_k)f\| \leqslant \Lambda_k$$

and, since $\lim_{k\to\infty} \Lambda_k = 0$, that $\lim_{k\to\infty} \| A - A_k \| = 0$. Therefore the operators A_k converge to the operator A in operator norm.

Now we show that the converse assertion holds, namely: any operator of the form

$$Af = \sum_{n=1}^{\infty} \lambda_n(f, e_n)h_n, \tag{1}$$

where $\{e_n\}$ and $\{h_n\}$ are orthonormal systems of vectors in the spaces H_1 and H_2, $\lambda_n > 0$, and $\lim_{n\to\infty} \lambda_n = 0$, is completely continuous. For the proof it suffices to remark that from $\lim_{n\to\infty} \lambda_n = 0$ follows $\lim_{k\to\infty} \| A - A_k \| = 0$, where

$$A_k f = \sum_{n=1}^{k} \lambda_n (f, e_n) h_n.$$

Since each of the operators A_k maps the space H_1 onto a finite-dimensional subspace in H_2, the A_k are completely continuous. Consequently the operator A, being the limit in operator norm of the A_k, is also completely continuous.

It is obvious that for an operator A of the form (1) the numbers λ_n are always eigenvalues of the positive-definite operator T which appears in the decomposition $A = UT$, the vectors e_n are eigenvectors of T, and the vectors h_n are of the form $h_n = Ue_n$.

We observe that at the same time we have proven the following assertion:

Any completely continuous operator A is the limit in operator norm of a sequence of degenerate operators A_k (i.e., operators which map the space H_1 onto a finite-dimensional subspace in H_2). We have thus shown that the space of completely continuous operators coincides with the completion of the set of degenerate operators in the norm $\| A \|$.

2.2. Hilbert-Schmidt Operators

For many questions of analysis the requirement that the eigenvalues λ_n (of the operator T appearing in the decomposition $A = UT$ of a completely continuous operator A) tend to zero is too weak. From now on we will consider operators on whose eigenvalues are imposed more restrictive requirements concerning their rate of decrease.

One of the most frequently used classes of such operators is the class of Hilbert-Schmidt operators.

A completely continuous operator $A = UT$ is said to be of *Hilbert–Schmidt type* if $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$, where the λ_n are the eigenvalues of the operator T.

Geometrically this says that an operator A of Hilbert–Schmidt type transforms the sphere $\| f \| = 1$ into an ellipsoid such that the series, consisting of the squares of the lengths of its semiaxes, converges.

Recalling that in the decomposition

$$Af = \sum_{n=1}^{\infty} \lambda_n (f, e_n) h_n, \tag{1}$$

which we established on page 31, λ_n is an eigenvalue of T, we can assert that an operator of Hilbert–Schmidt type admits a decomposition of the form (1), where $\{e_n\}$ and $\{h_n\}$ are orthonormal sets in the spaces H_1 and H_2, and the $\lambda_n > 0$ are such that the series $\Sigma_{n=1}^{\infty} \lambda_n^2$ converges.

Conversely, if $\{e_n\}$ and $\{h_n\}$ are orthonormal sets in Hilbert spaces H_1 and H_2, and $\lambda_n > 0$ are numbers such that the series $\Sigma_{n=1}^{\infty} \lambda_n^2$ converges, then formula (1) defines an operator of Hilbert–Schmidt type.

In fact, from the convergence of $\Sigma_{n=1}^{\infty} \lambda_n^2$ it follows that $\lim_{n\to\infty} \lambda_n = 0$. Therefore, as was shown in Section 2.1, the operator A is completely continuous and the λ_n are the nonzero eigenvalues of the operator T which appears in the decomposition $A = UT$. Consequently, the series $\Sigma_{n=1}^{\infty} \lambda_n^2$ consists of the squares of the eigenvalues of T, and so A is of Hilbert–Schmidt type.

We shall give a more convenient definition of an operator of Hilbert–Schmidt type. For this we need the following assertion.

Lemma 1. Let A be an operator, mapping the Hilbert space H_1 into a Hilbert space H_2, such that the series $\Sigma_{n=1}^{\infty} \| Af_n \|^2$ converges for some orthonormal basis $f_1, f_2, \ldots$ in H_1. Then $\Sigma_{n=1}^{\infty} \| Ag_n \|^2$ converges for any orthonormal basis $g_1, g_2, \ldots$ in H_1, and

$$\sum_{n=1}^{\infty} \| Af_n \|^2 = \sum_{n=1}^{\infty} \| Ag_n \|^2. \tag{2}$$

Proof. To prove this result, we choose some orthonormal basis $h_1, h_2, \ldots$ in H_2. Then

$$\| Af_n \|^2 = \sum_{k=1}^{\infty} |(Af_n, h_k)|^2 = \sum_{k=1}^{\infty} |(f_n, A^*h_k)|^2.$$

Therefore

$$\sum_{n=1}^{\infty} \| Af_n \|^2 = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |(f_n, A^*h_k)|^2.$$

But

$$\sum_{n=1}^{\infty} |(f_n, A^*h_k)|^2 = \| A^*h_k \|^2.$$

Since the right side of this equality does not depend upon the choice of the basis $\{f_n\}$ in H_1, Eq. (2) is proved. Moreover, we have shown that

$$\sum_{n=1}^{\infty} \| Af_n \|^2 = \sum_{n=1}^{\infty} \| A^*h_n \|^2, \tag{3}$$

where $h_1, h_2, \ldots$ is any orthonormal basis in H_2.

We now give another definition of an operator of Hilbert–Schmidt type. By definition, for a Hilbert–Schmidt operator A the series $\sum_{n=1}^{\infty} \lambda_n^2$ converges, where the λ_n are the nonzero eigenvalues of the positive-definite operator T appearing in the decomposition $A = UT$. Let $\{e_n\}$ be an orthonormal basis in H_1 consisting of eigenvectors of T. Since $\| Af \| = \| Tf \|$, $\lambda_n = \| Te_n \| = \| Ae_n \|$, and so the series $\sum_{n=1}^{\infty} \| Ae_n \|^2 = \sum_{n=1}^{\infty} \lambda_n^2$ converges. But then from (2) it follows that for a Hilbert–Schmidt operator A the series $\sum_{n=1}^{\infty} \| Af_n \|^2$ converges for any orthonormal basis $\{f_n\}$ in H_1.

We will show that the convergence of the series $\sum_{n=1}^{\infty} \| Af_n \|^2$ for some orthonormal basis in H_1 is not only a necessary but also a sufficient condition for the operator A to be of Hilbert–Schmidt type. In other words, we will prove the following theorem.

Theorem 2. In order that the operator A be of Hilbert–Schmidt type, it is necessary and sufficient that the series $\sum_{n=1}^{\infty} \| Af_n \|^2$ converge for at least one orthonormal basis $f_1, f_2, \ldots$ in H_1.

For the proof of this theorem we need the following lemma.

Lemma 2. Let the operator A be such that the series $\sum_{n=1}^{\infty} \| Af_n \|^2$ converges for some orthonormal basis $f_1, f_2, \ldots$ in H_1. Then $\| A \| \leqslant \| A \|_2$, where $\| A \|_2$ stands for the number $[\sum_{n=1}^{\infty} \| Af_n \|^2]^{\frac{1}{2}}$ (by (2) the value of $\| A \|_2$ depends only upon A and not upon the choice of the orthonormal basis $f_1, f_2, \ldots$ in H_1).

Proof. Let us choose any orthonormal basis $h_1, h_2, \ldots$ in H_2. Then

$$\begin{aligned} \| Af \|^2 &= \sum_{n=1}^{\infty} |(Af, h_n)|^2 \\ &= \sum_{n=1}^{\infty} |(f, A^*h_n)|^2 \leqslant \| f \|^2 \sum_{n=1}^{\infty} \| A^*h_n \|^2 = \| f \|^2 \sum_{n=1}^{\infty} \| Af_n \|^2. \end{aligned}$$

Thus

$$\| Af \|^2 \leqslant \| f \|^2 \sum_{n=1}^{\infty} \| Af_n \|^2 = \| f \|^2 \| A \|_2^2,$$

from which follows

$$\| A \| = \sup_{\| f \| = 1} \| Af \| \leqslant \| A \|_2.$$

Let us now prove Theorem 2. We need to prove only that the convergence of the series $\sum_{n=1}^{\infty} \| Af_n \|^2$ is a sufficient condition for A to be of Hilbert–Schmidt type. But for this it suffices to show that the convergence of this series implies that A is completely continuous. As a matter of fact, if A is completely continuous, then $A = UT$ and, in view of Lemma 1, the series $\sum_{n=1}^{\infty} \| Ae_n \|^2$ converges, where $e_1, e_2, \ldots$ is an orthonormal basis consisting of eigenvectors of T. Since $\lambda_n = \| Ae_n \|$, the convergence of the series $\sum_{n=1}^{\infty} \lambda_n^2$, consisting of the squares of the eigenvalues of T, is thereby proved.

We prove the complete continuity of the operator A. Let us denote by A_k the degenerate operator which takes the vector f_n into Af_n for $1 \leqslant n \leqslant k$ and into zero for $n > k$. Then

$$\| A - A_k \|^2 \leqslant \| A - A_k \|_2^2 = \sum_{n=1}^{\infty} \|(A - A_k)f_n \|^2 = \sum_{n=k+1}^{\infty} \| Af_n \|^2.$$

From the convergence of the series $\sum_{k=1}^{\infty} \| Af_k \|^2$ it follows that $\lim_{k\to\infty} \| A - A_k \| = 0$. Therefore A is the limit in operator norm $\| A \|$ of a sequence of degenerate operators. Since a degenerate operator is completely continuous, the operator A is also completely continuous. As we already mentioned, from the complete continuity of A and the convergence of the series $\sum_{n=1}^{\infty} \| Af_n \|^2$ it follows that A is of Hilbert–Schmidt type, which proves the theorem.

Henceforth we will call the number $\| A \|_2$ the *Hilbert–Schmidt norm* of A. Obviously the Hilbert–Schmidt norm is finite for Hilbert–Schmidt operators and only for such operators and satisfies, for these operators, the easily proved relations

$$\| A + B \|_2 \leqslant \| A \|_2 + \| B \|_2$$

and

$$\| \lambda A \|_2 = | \lambda | \| A \|_2.$$

From this it follows that the set of Hilbert–Schmidt operators forms a normed linear space $\mathfrak{h}$ relative to the norm $\| A \|_2$. Let us show that this space is a Hilbert space. In fact, a Hilbert–Schmidt operator A is defined by the numbers (Af_n, h_k), where $\{f_n\}$ is an orthonormal basis in H_1 and $\{h_k\}$ is an orthonormal basis in H_2, and

$$\| A \|_2 = \left[\sum_{n=1}^{\infty} \| Af_n \|^2\right]^{\frac{1}{2}} = \left[\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |(Af_n, h_k)|^2\right]^{\frac{1}{2}}.$$

From this it follows that the space $\mathfrak{h}$ of Hilbert–Schmidt operators is isomorphic to the space of infinite matrices $\| a_{nk} \|$ for which the series

$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} | a_{nk} |^2$ converges. But as is known, the space of such matrices is a Hilbert space; consequently $\mathfrak{h}$ is a Hilbert space.

Since the space of Hilbert–Schmidt operators is a Hilbert space, it is complete. We prove that this space is the completion of the set of degenerate operators in the norm $\| A \|_2$. In fact, in the proof of Theorem 2 it was shown that if A is of Hilbert–Schmidt type and $f_1, f_2, \ldots$ is an orthonormal basis in H_1, then $\lim_{k \to \infty} \| A - A_k \|_2 = 0$, where A_k is the operator which coincides with A on the elements $f_1, \ldots, f_k$ and which takes the elements $f_{k+1}, f_{k+2}, \ldots$ into zero. Thus every Hilbert–Schmidt operator is the limit in the norm $\| A \|_2$ of a sequence of degenerate operators $A_1, A_2, \ldots$. From this it follows that the set of degenerate operators is everywhere dense in the space of Hilbert–Schmidt operators with norm $\| A \|_2$. Since this space is complete, it is the completion of the set of degenerate operators in the norm $\| A \|_2$.

Let us pause to prove yet another property of Hilbert–Schmidt operators. Namely, we prove the following assertion.

Theorem 3. In order that the operator A, which takes a Hilbert space H_1 into a Hilbert space H_2, be of Hilbert–Schmidt type, it is necessary and sufficient that it admit a representation of the form

$$Af = \sum_{n=1}^{\infty} \lambda_n (f, e_n) h_n, \tag{4}$$

where $\{e_n\}$ and $\{h_n\}$ are orthonormal sets in H_1 and H_2 respectively, and the λ_n are positive numbers such that the series $\sum_{n=1}^{\infty} \lambda_n^2$ converges.

Proof. Let A be a Hilbert–Schmidt operator. Then A is completely continuous and, consequently, can be represented in the form of a series

$$Af = \sum_{n=1}^{\infty} \lambda_n (f, e_n) h_n,$$

where the λ_n are the nonzero eigenvalues of the positive-definite operator T appearing in the decomposition $A = UT$. Since A is a Hilbert–Schmidt operator, the series $\sum_{n=1}^{\infty} \lambda_n^2$ converges. This proves the necessity of the condition of the theorem.

We prove its sufficiency. Suppose that A admits a representation of the form (4), for which the series $\sum_{n=1}^{\infty} \lambda_n^2$ converges. Then $\lim_{n \to \infty} \lambda_n = 0$ and consequently A is completely continuous, and the λ_n are the nonzero eigenvalues of the operator T appearing in the decomposition $A = UT$.

Since by hypothesis $\Sigma_{n=1}^{\infty} \lambda_n^2$ converges, A is of Hilbert–Schmidt type, which proves the theorem.

In conclusion we note the following properties of Hilbert–Schmidt operators, which we will make use of further on.

(1) The adjoint A^* of a Hilbert–Schmidt operator is an operator of the same type.

In fact, if A is of Hilbert–Schmidt type, then the series $\Sigma_{n=1}^{\infty} \| Af_n \|^2$ converges for every orthonormal basis $\{f_1\}$ in H_1. But from this, by Lemma 1, follows the convergence of the series $\Sigma_{n=1}^{\infty} \| A^* h_n \|^2$ for any orthonormal basis $\{h_n\}$ in H_2. But this shows that A^* is likewise of Hilbert–Schmidt type.

(2) The product AB of a continuous linear operator A and a Hilbert–Schmidt operator B is of Hilbert–Schmidt type.

In fact, for any orthonormal basis $\{f_n\}$ in H_1 we have

$$\sum_{n=1}^{\infty} \| ABf_n \|^2 \leqslant \| A \|^2 \sum_{n=1}^{\infty} \| Bf_n \|^2. \tag{5}$$

But the series $\Sigma_{n=1}^{\infty} \| Bf_n \|^2$ converges, as B is a Hilbert–Schmidt operator. Therefore the series $\Sigma_{n=1}^{\infty} \| ABf_n \|^2$ converges, and consequently AB is likewise a Hilbert–Schmidt operator. We note that from this inequality there follows the useful relation

$$\| AB \|_2 \leqslant \| A \| \| B \|_2.$$

(3) The product BA, where A is a continuous linear operator and B is a Hilbert–Schmidt operator, is also a Hilbert–Schmidt operator.

In fact, $BA = (A^*B^*)^*$. But by property (1), B^* is a Hilbert–Schmidt operator, and by property (2) A^*B^* is of Hilbert–Schmidt type. A second application of property (1) shows that BA is a Hilbert–Schmidt operator.

2.3. Nuclear Operators

An even more restrictive requirement than that the operator A be of Hilbert–Schmidt type is that it be a nuclear operator.

A completely continuous operator is called *nuclear*† if $\Sigma_{n=1}^{\infty} \lambda_n < \infty$, where the λ_n are the eigenvalues of the operator T appearing in the

† Frequently, *operator of trace class*.

decomposition $A = UT$. Since the convergence of the series $\sum_{n=1}^{\infty} \lambda_n^2$ follows from the convergence of $\sum_{n=1}^{\infty} \lambda_n$, every nuclear operator is of Hilbert–Schmidt type.

Geometrically, the requirement of nuclearity says that the operator A maps the sphere $\| x \| = 1$ onto an ellipsoid in the space H_2 such that the series, consisting of the lengths of its principal semiaxes, converges.

We proved in Section 2.1 that every completely continuous operator which maps a Hilbert space H_1 into a Hilbert space H_2 can be represented in the form of a series

$$Af = \sum_{n=1}^{\infty} \lambda_n(f, e_n)h_n, \tag{1}$$

where $\{e_n\}$ and $\{h_n\}$ are orthonormal sets in H_1 and H_2, and $\lambda_n > 0$, $\lim_{n\to\infty} \lambda_n = 0$. From this it follows that every nuclear operator can be represented in the form of a series (1) in which $\lambda_n > 0$ and $\sum_{n=1}^{\infty} \lambda_n$ converges.

It was shown in Section 2.1 that every series of the form (1), where $\{e_n\}$ and $\{h_n\}$ are orthonormal sets in the spaces H_1 and H_2, and $\lambda_n > 0$ and $\lim_{n\to\infty} \lambda_n = 0$ defines a completely continuous operator A, whereby the λ_n are the eigenvalues of the positive-definite operator T appearing in the decomposition $A = UT$. Therefore any series of the form (1), for which $\lambda_n > 0$ and $\sum_{n=1}^{\infty} \lambda_n < +\infty$, defines a nuclear operator mapping the space H_1 into H_2.

For positive-definite operators the concept of a nuclear operator coincides with that of an operator having finite trace. A positive-definite operator A in a Hilbert space H is said to be an *operator with finite trace*,[9] if the series $\sum_{n=1}^{\infty} (Af_n, f_n)$ converges for any orthonormal basis $\{f_n\}$ in H.

The following assertion holds.

Lemma 3. In order that a completely continuous positive-definite operator T be nuclear, it is necessary and sufficient that it have finite trace.[10]

Proof. Let T be a positive-definite nuclear operator. We introduce the operator $T^{\frac{1}{2}}$, setting $T^{\frac{1}{2}}e_n = \lambda_n^{\frac{1}{2}}e_n$, where $\{e_n\}$ is an orthonormal

[9] If $\| a_{mn} \|$ is the matrix corresponding to the operator A with respect to the basis $\{f_n\}$, then $\sum_{n=1}^{\infty} (Af_n, f_n) = \sum_{n=1}^{\infty} a_{nn}$ and is thus the trace of the matrix $\| a_{mn} \|$.

[10] The analogous assertion, without the assumption of the complete continuity of T, is proved below in Theorem 7.

basis consisting of eigenvectors of T, and the λ_n are the corresponding eigenvalues. Since

$$\sum_{n=1}^{\infty} \| T^{\frac{1}{2}} e_n \|^2 = \sum_{n=1}^{\infty} \lambda_n < +\infty, \tag{6}$$

$T^{\frac{1}{2}}$ is a Hilbert–Schmidt operator. Therefore, for any orthonormal basis $f_1, f_2, \ldots$ we have

$$\sum_{n=1}^{\infty} \| T^{\frac{1}{2}} f_n \|^2 = \sum_{n=1}^{\infty} \| T^{\frac{1}{2}} e_n \|^2 = \sum_{n=1}^{\infty} \lambda_n.$$

In view of the self-adjointness of $T^{\frac{1}{2}}$,

$$\sum_{n=1}^{\infty} \| T^{\frac{1}{2}} f_n \|^2 = \sum_{n=1}^{\infty} (T^{\frac{1}{2}} f_n, T^{\frac{1}{2}} f_n) = \sum_{n=1}^{\infty} (T f_n, f_n).$$

Therefore for any orthonormal basis $f_1, f_2, \ldots$ in H

$$\sum_{n=1}^{\infty} (T f_n, f_n) = \sum_{n=1}^{\infty} \lambda_n < +\infty$$

from which it follows that the operator T has finite trace.

Conversely, let T be a completely continuous positive-definite operator having finite trace. We choose any orthonormal basis $e_1, e_2, \ldots$ in H consisting of eigenvectors of T, with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots$. Then

$$\sum_{n=1}^{\infty} \lambda_n = \sum_{n=1}^{\infty} (T e_n, e_n) < +\infty,$$

from which it follows that T is a nuclear operator.

From this it follows that any nuclear operator is the product $A = UT$ of an isometric operator U and a positive-definite operator T having finite trace.

Let us examine the relation of nuclear operators to Hilbert–Schmidt operators.

Theorem 4. The product AB of any two Hilbert–Schmidt operators is a nuclear operator. Conversely, every nuclear operator is the product of two Hilbert–Schmidt operators.

Proof. Suppose that B maps H_1 into H_2 and A maps H_2 into H_3, and let $AB = UT$ be the decomposition of the operator AB into the product of a positive-definite operator T, acting in the space H_1, and an isometric operator U which maps the range of T into the space H_3. We denote by $e_1, e_2, \ldots$ an orthonormal basis in H_1 consisting of eigenvectors of T, $Te_n = \lambda_n e_n$, and by $h_1, h_2, \ldots$ the orthonormal system in H_3 consisting of the vectors $h_n = Ue_n$, $\lambda_n \neq 0$. Then for $\lambda_n \neq 0$ we have

$$\begin{aligned} \lambda_n = (Te_n, e_n) &= (UTe_n, Ue_n) = (ABe_n, h_n) \\ &= (Be_n, A^*h_n) \leqslant \tfrac{1}{2}(\| Be_n \|^2 + \| A^*h_n \|^2). \end{aligned} \tag{7}$$

If A and B are Hilbert–Schmidt operators, then the series $\sum_{n=1}^{\infty} \| Be_n \|^2$ and $\sum_{n=1}^{\infty} \| A^*h_n \|^2$ converge, and so, in view of inequality (7), the series $\sum_{n=1}^{\infty} \lambda_n$ converges. Thus we have proved that the product AB of two Hilbert–Schmidt operators is a nuclear operator.

Let us now prove the converse assertion. Let A be a nuclear operator, and $A = UT$ its decomposition into the product of a positive-definite and an isometric operator. Then, as was shown above, the operator $T^{\frac{1}{2}}$ is of Hilbert–Schmidt type. Since[11]

$$\sum_{n=1}^{\infty} \| UT^{\frac{1}{2}}e_n \|^2 = \sum_{n=1}^{\infty} \| T^{\frac{1}{2}}e_n \|^2$$

$UT^{\frac{1}{2}}$ is likewise of Hilbert–Schmidt type. As $A = (UT^{\frac{1}{2}})T^{\frac{1}{2}}$, A is the product of two Hilbert–Schmidt operators, which proves the theorem.

The following properties of nuclear operators are a consequence of Theorem 4.

(1) The adjoint A^* of a nuclear operator A is a nuclear operator.

In fact, if $A = UT^{\frac{1}{2}}T^{\frac{1}{2}}$, then $A^* = T^{\frac{1}{2}}(UT^{\frac{1}{2}})^*$. The operator $(UT^{\frac{1}{2}})^*$, as the adjoint of a Hilbert–Schmidt operator, is itself Hilbert–Schmidt. Consequently A^* is a nuclear operator.

(2) The product AB of any bounded linear operator A with a nuclear operator B is a nuclear operator. The analogous assertion holds for the product BA.

In fact, if B is a nuclear operator, then $B = UT^{\frac{1}{2}}T^{\frac{1}{2}}$, where $UT^{\frac{1}{2}}$ and $T^{\frac{1}{2}}$ are Hilbert–Schmidt operators. Consequently $AB = (AUT^{\frac{1}{2}})T^{\frac{1}{2}}$ is the product of two Hilbert–Schmidt operators, i.e., a nuclear operator. The analogous assertion also holds for the product BA of a nuclear operator B with a bounded linear operator A, since $BA = (A^*B^*)^*$.

[11] The operator U is defined on the closure of the range of the operator T. It is easily seen that this closure coincides with the closure of the range of $T^{\frac{1}{2}}$. In both cases the closure in question is the subspace spanned by those vectors e_n corresponding to nonzero λ_n. Therefore $UT^{\frac{1}{2}}$ is an operator with domain H_1.

The properties of nuclear operators which have been proved suffice for the construction of the theory of nuclear spaces. However, in view of the importance of nuclear operators we will discuss some of their further properties.

Lemma 4. Let A and B be operators in a Hilbert space H_1, where B is a bounded linear operator and $A = UT$ is a nuclear operator. Then

$$\sum_{n=1}^{\infty} (ABe_n, e_n) = \sum_{n=1}^{\infty} (BAe_n, e_n), \tag{8}$$

where $\{e_n\}$ is an orthonormal basis in H consisting of eigenvectors of T.

Proof. Obviously

$$\sum_{n=1}^{\infty} (ABe_n, e_n) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (Be_n, e_m)(Ae_m, e_n) \tag{9}$$

and

$$\sum_{n=1}^{\infty} (BAe_n, e_n) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (Ae_n, e_m)(Be_m, e_n). \tag{9'}$$

The right sides of (9) and (9′) differ only in the order of the terms, and therefore, in order to prove (8), it suffices to show that the series (9) is absolutely convergent. But

$$(Ae_n, e_m) = (UTe_n, e_m) = \lambda_n(Ue_n, e_m),$$

where the λ_n denote the eigenvalues of T, corresponding to the eigenvectors e_n. Therefore

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |(Ae_n, e_m)| \cdot |(Be_m, e_n)| \leqslant \sum_{n=1}^{\infty} \lambda_n \sum_{m=1}^{\infty} |(Ue_n, e_m)| \cdot |(Be_m, e_n)|$$

$$\leqslant \frac{1}{2} \sum_{n=1}^{\infty} \lambda_n \sum_{m=1}^{\infty} [|(Ue_n, e_m)|^2 + |(Be_m, e_n)|^2].$$

But

$$\sum_{m=1}^{\infty} |(Ue_n, e_m)|^2 = \| Ue_n \|^2 \leqslant 1,$$

$$\sum_{m=1}^{\infty} |(Be_m, e_n)|^2 = \| B^* e_n \|^2 \leqslant \| B^* \|^2 = \| B \|^2$$

and therefore

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |(Ae_n, e_m)|\, |(Be_m, e_n)| \leqslant (\| B \|^2 + 1) \sum_{n=1}^{\infty} \lambda_n.$$

Since the series $\Sigma_{n=1}^{\infty} \lambda_n$ converges, the absolute convergence of the series (9) is proved. Hence (8) is also proved.

From Lemma 4 it follows that for any unitary operator V and any nuclear operator $A = UT$ we have

$$\sum_{n=1}^{\infty} (V^{-1}AVe_n, e_n) = \sum_{n=1}^{\infty} (Ae_n, e_n), \tag{10}$$

where $\{e_n\}$ is an orthonormal basis consisting of eigenvectors of T. For the proof it suffices to replace, in formula (8), B by V and A by $V^{-1}A$, and to note that the decomposition of the operator $V^{-1}A$ has the form $V^{-1}A = WT$, where $W = V^{-1}U$.

Equation (10) can be written in the form

$$\sum_{n=1}^{\infty} (Af_n, f_n) = \sum_{n=1}^{\infty} (Ae_n, e_n), \tag{11}$$

where $\{f_n\}$ is the orthonormal basis in H consisting of the vectors $f_n = Ve_n$. Thus, if A is a nuclear operator, then the series $\Sigma_{n=1}^{\infty} (Af_n, f_n)$ converges for any orthonormal basis $\{f_n\}$, and its sum does not depend upon the choice of basis, i.e., the operator A has finite trace (earlier this was proven only for positive-definite nuclear operators; cf. Lemma 3).

Equation (11) can be generalized by relinquishing the assumption of the orthogonality of the basis $\{f_n\}$. We call a set $\{f_n\}$ of vectors in H an *unconditional basis*[12] if $f_n = Bh_n$, where $\{h_n\}$ is an orthonormal basis in H and B is a bounded linear operator which has a bounded inverse

[12] If $\{f_n\}$ is an unconditional basis, then every basis $\{Cf_n\}$, where C is a bounded linear operator having a bounded inverse, is unconditional. Therefore the concept of an unconditional basis is an affine concept, depending only upon linear operators in H and the topology in H, but not depending upon a scalar product being given in H. In order that a basis $\{f_n\}$ be unconditional, it is necessary and sufficient that it remain a basis for any permutation of its vectors.†

† Presumably the condition that the set of norms $\{\| f_n \|\}$ be both bounded and bounded away from zero must be added; for it is clear that if $\{e_n\}$ is an orthonormal basis, then the set $\{a_ne_n\}$, where $a_n = n$ or n^{-1} as n is even or odd, is a basis for any permutation of its elements, but certainly not an unconditional basis in the authors' definition.

B^{-1}. If $\{f_n\}$, $f_n = Bh_n$, is an unconditional basis, then $\{g_n\}$, where $g_n = (B^{-1})^* h_n$, is also an unconditional basis, and

$$(f_m, g_n) = (Bh_m, (B^{-1})^* h_n) = (h_m, h_n) = \delta_{mn}.$$

The basis $\{g_n\}$ is said to be *biorthogonal* to the basis $\{f_n\}$.

We prove the following theorem.

Theorem 5. Let $\{f_n\}$ be an unconditional basis in the space H, and $\{g_n\}$ the basis which is biorthogonal to it. Then for any nuclear operator A the series

$$\sum_{n=1}^{\infty} (Af_n, g_n) \tag{12}$$

is absolutely convergent, and its sum does not depend upon the choice of basis $\{f_n\}$.

Proof. Let $f_n = Bh_n$, $g_n = (B^{-1})^* h_n$, where $\{h_n\}$ is an orthonormal basis in H. Then

$$\sum_{n=1}^{\infty} (Af_n, g_n) = \sum_{n=1}^{\infty} (ABh_n, (B^{-1})^* h_n) = \sum_{n=1}^{\infty} (B^{-1}ABh_n, h_n).$$

The operator $B^{-1}AB$ is nuclear, because A is a nuclear operator and B and B^{-1} are bounded. Therefore, as was proven above, the series $\sum_{n=1}^{\infty} (B^{-1}ABh_n, h_n)$ is absolutely convergent and hence the series $\sum_{n=1}^{\infty} (Af_n, g_n)$ is absolutely convergent. We prove the independence of the sum of this series upon the choice of basis $\{f_n\}$. In view of Eq. (11) the sum $\sum_{n=1}^{\infty} (B^{-1}ABh_n, h_n)$ does not depend upon the choice of orthonormal basis $\{h_n\}$. Therefore

$$\sum_{n=1}^{\infty} (B^{-1}ABh_n, h_n) = \sum_{n=1}^{\infty} (B^{-1}ABe_n, e_n),$$

where $\{e_n\}$ is an orthonormal basis consisting of eigenvectors of the positive-definite operator Q appearing in the decomposition $B^{-1}A = WQ$ (cf. Theorem 1). Applying Lemma 4 to the operators $B^{-1}A$ and B, we find that

$$\sum_{n=1}^{\infty} (B^{-1}ABe_n, e_n) = \sum_{n=1}^{\infty} (Ae_n, e_n).$$

Thus we have shown that

$$\sum_{n=1}^{\infty} (Af_n, g_n) = \sum_{n=1}^{\infty} (Ae_n, e_n).$$

Because the right side of the equation does not depend upon the choice of basis $\{f_n\}$, the value of $\Sigma_{n=1}^{\infty} (Af_n, g_n)$ does not depend upon the choice of unconditional basis $\{f_n\}$, which proves the theorem.

Somewhat more careful considerations show that the following theorem, proven by V. B. Lidskii, is valid.

Theorem 6 (*On the Trace*). If A is a nuclear operator, $\{f_n\}$ an unconditional basis, and $\{g_n\}$ the basis biorthogonal to it, then

$$\sum_{n=1}^{\infty} (Af_n, g_n) = \sum_{n=1}^{\infty} \mu_n, \tag{13}$$

where the μ_n are the eigenvalues of A.

We remark that if A is a nuclear operator, then the series $\Sigma_{n=1}^{\infty} (Af_n, g_n)$ is absolutely convergent not only when the bases $\{f_n\}$ and $\{g_n\}$ are biorthogonal, but also in the case where $f_n = B_1 e_n$, $g_n = B_2 e_n$, where B_1 and B_2 are bounded operators, and $\{e_n\}$ is an orthonormal basis in H. The proof of this assertion is analogous to that of Theorem 5.

From this result it follows that for any unconditional bases $\{f_n\}$ and $\{g_n\}$ (or parts of unconditional bases) and any nuclear operator A the series $\Sigma_{n=1}^{\infty} (Af_n, g_n)$ is absolutely convergent. The absolute convergence of such series is not only necessary, but also sufficient for the nuclearity of the operator A. In other words, the following theorem holds:

Theorem 7. In order that a bounded linear operator A in a Hilbert space H be nuclear, it is necessary and sufficient that the series $\Sigma_{n=1}^{\infty} (Af_n, g_n)$ converge for all systems of vectors $\{f_n\}$ and $\{g_n\}$ which are parts of unconditional bases in H.

Proof. It was shown above that if A is a nuclear operator, and $\{f_n\}$ and $\{g_n\}$ are parts of unconditional bases, then the series $\Sigma_{n=1}^{\infty} (Af_n, g_n)$ converges. Therefore the necessity of the condition of the theorem is proved, and we turn to the proof of its sufficiency.

Since A is bounded, it can be written in the form $A = UT$, where U is isometric and T is a positive-definite bounded operator (in Section 2.1 it was stated that such a decomposition exists for every bounded operator). We show that the operator T has a pure discrete spectrum, i.e., that one can choose an orthonormal basis in H, consisting of eigenvectors of T.

In fact, we show that if $\alpha > 0$ and $\beta > \| T \|$, then the range of $E(\Delta')$, where $\Delta' = [\alpha, \beta]$, is finite dimensional [$E(\Delta)$ is the resolution of unity of T; cf. the appendix to Section 4]. Clearly this implies that T has a pure discrete spectrum (and in addition that it is completely continuous). Suppose, to the contrary, that the range of $E(\Delta')$ is infinite dimensional. Then we can construct an infinite orthonormal set $f_1, f_2, \ldots$ in this range. All the f_n lie in the range of T, and $(Tf_n, f_n) \geqslant \alpha$. Set $g_n = Uf_n$. The systems $\{f_n\}$ and $\{g_n\}$ can be enlarged to orthonormal bases in H. Therefore the series $\sum_{n=1}^{\infty} (Af_n, g_n)$ is by hypothesis absolutely convergent. But

$$(Af_n, g_n) = (UTf_n, Uf_n) = (Tf_n, f_n) \geqslant \alpha > 0$$

and consequently

$$\sum_{n=1}^{\infty} (Af_n, g_n) = +\infty.$$

The contradiction shows that T has a discrete spectrum. We now show that T is a nuclear operator, i.e., that the series $\sum_{n=1}^{\infty} \lambda_n$ converges, where the λ_n are the eigenvalues of T. For this we note that

$$\lambda_n = (Te_n, e_n) = (UTe_n, Ue_n) = (Ae_n, g_n),$$

where the e_n denote the orthonormalized eigenvectors of T corresponding to the eigenvalues λ_n, and $g_n = Ue_n$. Therefore

$$\sum_{n=1}^{\infty} \lambda_n = \sum_{n=1}^{\infty} (Ae_n, g_n).$$

Since the systems $\{e_n\}$ and $\{g_n\}$ can be enlarged to orthonormal bases in H, the series $\sum_{n=1}^{\infty} (Ae_n, g_n)$ is convergent, and consequently the series $\sum_{n=1}^{\infty} \lambda_n$ converges, which proves the theorem.

It is useful to note that in the proof of the sufficiency of the condition of the theorem we used only the convergence of the series $\sum_{n=1}^{\infty} (Af_n, g_n)$ for any orthonormal systems $\{f_n\}$ and $\{g_n\}$. Therefore Theorem 7 can be strengthened in the following way.

Theorem 7′. In order that a bounded linear operator A in a Hilbert space H be nuclear, it is sufficient that the series $\sum_{n=1}^{\infty} (Af_n, g_n)$ converge for any orthonormal systems $\{f_n\}$ and $\{g_n\}$ of vectors in H.

This theorem is also valid for operators mapping one Hilbert space into another.

We also note the following necessary and sufficient criterion for the nuclearity of an operator.

Theorem 8. In order that an operator A be nuclear it is necessary and sufficient that the series $\sum_{n=1}^{\infty} \| Af_n \|$ converge for at least one orthonormal basis $f_1, f_2, \ldots$ in the space H_1.

Proof. Let $f_1, f_2, \ldots$ be an orthonormal basis in the space H_1 and suppose that the series $\sum_{n=1}^{\infty} \| Af_n \|$ converges. Then the series $\sum_{n=1}^{\infty} \| Af_n \|^2$ also converges. Therefore A is a Hilbert–Schmidt operator and *a fortiori* complete continuous. Consequently A can be represented in the form $A = UT$. Since $\| f_n \| = 1$ and the operator U is isometric on the range of T, we have

$$(Tf_n, f_n) \leqslant \| Tf_n \| = \| UTf_n \| = \| Af_n \|.$$

This inequality shows, in view of the convergence of $\sum_{n=1}^{\infty} \| Af_n \|$, that the series $\sum_{n=1}^{\infty} (Tf_n, f_n)$ converges. But then (by Lemma 3) the operator A is nuclear.

Conversely, let $A = UT$ be a nuclear operator. We choose an orthonormal basis $e_1, e_2, \ldots$ in H_1, consisting of eigenvectors of T. Then

$$\sum_{n=1}^{\infty} \| Ae_n \| = \sum_{n=1}^{\infty} \| Te_n \| = \sum_{n=1}^{\infty} \lambda_n,$$

which shows that the series $\sum_{n=1}^{\infty} \| Ae_n \|$ converges.

We remark that the nuclearity of an operator A does not imply the convergence of the series $\sum_{n=1}^{\infty} \| Af_n \|$ for every orthonormal basis in H_1. We construct a corresponding example.

Example. Let us consider the fixed vector

$$f = (1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots)$$

in the Hilbert space H of sequences $x = (x_1, x_2, \ldots)$, the sum of whose square moduli converges. We denote by P the operator of orthogonal projection onto the subspace generated by the vector f. Since P maps the space H onto a one-dimensional subspace, it is a nuclear operator (its trace equals unity).

Now P transforms the vectors of the orthonormal basis $\{e_n\}$, where

$$e_n = (0, 0, \ldots, 1, 0, \ldots),$$

into the vectors

$$Pe_n = \frac{(e_n, f)f}{\|f\|} = \frac{f}{n\|f\|}.$$

But the series

$$\sum_{n=1}^{\infty} \| Pe_n \| = \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges.

Thus, if A is a nuclear operator, the series $\Sigma_{n=1}^{\infty} \| Ae_n \|$, where $\{e_n\}$ is an orthonormal basis in H, can diverge.

2.4. The Trace Norm

In this section it will be proved that the nuclear operators form a linear space and that this space is the completion of the space of degenerate operators relative to a certain norm, called the trace norm. First we prove the following theorem.

Theorem 9. If A is a nuclear operator mapping the Hilbert space H_1 into the Hilbert space H_2, then

$$\sup \sum_{n=1}^{\infty} |(Af_n, g_n)| = \sum_{n=1}^{\infty} \lambda_n, \tag{14}$$

where the λ_n are the eigenvalues of the positive-definite operator T appearing in the decomposition $A = UT$, and the supremum is taken over all orthonormal systems of vectors $\{f_n\}$ and $\{g_n\}$ in the spaces H_1 and H_2.

Proof. Let $\{e_n\}$ denote the set of vectors remaining after the deletion, from an orthonormal basis for H_1 consisting of eigenvectors of T, of those members with zero eigenvalue. Then

$$\lambda_n = (Te_n, e_n) = (UTe_n, Ue_n) = (Ae_n, h_n)$$

and therefore

$$\sum_{n=1}^{\infty} (Ae_n, h_n) = \sum_{n=1}^{\infty} \lambda_n.$$

Since $\{e_n\}$ and $\{h_n\}$ are orthonormal systems in H_1 and H_2,

$$\sum_{n=1}^{\infty} \lambda_n \leqslant \sup \sum_{n=1}^{\infty} |(Af_n, g_n)|. \tag{15}$$

Now we prove the reverse inequality. We take any orthonormal systems $\{f_n\}$ and $\{g_n\}$ in the spaces H_1 and H_2 and estimate the series $\sum_{n=1}^{\infty}|(Af_n, g_n)|$ Since those eigenvectors of T whose eigenvalue is zero are mapped by A into zero,

$$(Af_n, g_n) = \sum_{k=1}^{\infty} (f_n, e_k)(Ae_k, g_n) = \sum_{k=1}^{\infty} \lambda_k(f_n, e_k)(Ue_k, g_n),$$

where $\{e_k\}$ is as at the beginning of the proof. Therefore

$$\begin{aligned}\sum_{n=1}^{\infty} |(Af_n, g_n)| &\leqslant \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \lambda_k|(f_n, e_k)(Ue_k, g_n)| \\ &\leqslant \frac{1}{2}\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \lambda_k[|(f_n, e_k)|^2 + |(Ue_k, g_n)|^2].\end{aligned} \tag{16}$$

Since $\{f_n\}$ and $\{g_n\}$ are orthonormal systems in H_1 and H_2,

$$\sum_{n=1}^{\infty} |(f_n, e_k)|^2 \leqslant \| e_k \|^2 \leqslant 1$$

and

$$\sum_{n=1}^{\infty} |(Ue_k, g_n)|^2 \leqslant \| Ue_k \|^2 \leqslant 1.$$

From these inequalities and relation (16) it follows that

$$\sum_{n=1}^{\infty} |(Af_n, g_n)| \leqslant \sum_{k=1}^{\infty} \lambda_k.$$

Therefore

$$\sup \sum_{n=1}^{\infty} |(Af_n, g_n)| \leqslant \sum_{k=1}^{\infty} \lambda_k. \tag{17}$$

Comparing relations (15) and (17), we see that

$$\sup \sum_{n=1}^{\infty} |(Af_n, g_n)| = \sum_{k=1}^{\infty} \lambda_k,$$

which proves the theorem.

In Theorem 7′ it was proven that the convergence of the series $\sum_{n=1}^{\infty} |(Af_n, g_n)|$ for all orthonormal bases $\{f_n\}$ and $\{g_n\}$ in H_1 and H_2

implies that A is nuclear. Therefore nuclear operators can be characterized as operators for which the quantity

$$\sup \sum_{n=1}^{\infty} |(Af_n, g_n)|$$

is finite.

From this it follows at once that the nuclear operators form a linear space. For the proof we need only assert that the sum of two nuclear operators is a nuclear operator. But this assertion follows at once from

$$\sup \sum_{n=1}^{\infty} |((A+B)f_n, g_n)| \leqslant \sup \sum_{n=1}^{\infty} |(Af_n, g_n)| + \sup \sum_{n=1}^{\infty} |(Bf_n, g_n)|.$$

One can introduce a norm in the space of nuclear operators, setting

$$\| A \|_1 = \sup \sum_{n=1}^{\infty} |(Af_n, g_n)|, \tag{18}$$

where the supremum is taken over all orthonormal systems $\{f_n\}$ and $\{g_n\}$ in the spaces H_1 and H_2. It is not hard to verify that the norm $\| A \|_1$ satisfies the relations

$$\| A + B \|_1 \leqslant \| A \|_1 + \| B \|_1$$

and

$$\| \lambda A \|_1 = | \lambda | \, \| A \|_1.$$

From Theorem 9 it follows that the norm $\| A \|_1$ can also be defined as $\sum_{n=1}^{\infty} \lambda_n$, i.e., as the trace of the positive-definite operator T appearing in the decomposition $A = UT$ of the nuclear operator A [cf. Eq. (14)]. Therefore the norm $\| A \|_1$ is called the *trace norm*.

We recall that not only the trace norm but also the Hilbert–Schmidt norm $\| A \|_2$ and the operator norm $\| A \|$ can be expressed by means of the eigenvalues λ_n of the operator T appearing in the decomposition $A = UT$; namely, one has

$$\| A \| = \sup_n \lambda_n, \tag{19}$$

$$\| A \|_1 = \sum_{n=1}^{\infty} \lambda_n, \tag{20}$$

$$\| A \|_2 = \left[\sum_{n=1}^{\infty} \lambda_n^2 \right]^{\frac{1}{2}}. \tag{21}$$

We have already proven Eqs. (20) and (21); the proof of (19) is well known.

The norms $\| A \|$, $\| A \|_2$, $\| A \|_1$ are related by the inequalities

$$\| A \| \leqslant \| A \|_2 \leqslant \| A \|_1 \tag{22}$$

In fact, the inequality $\| A \| \leqslant \| A \|_2$ was proven above (cf. Lemma 2), and the inequality $\| A \|_2 \leqslant \| A \|_1$ follows from the fact that $\lambda_n \geqslant 0$ and therefore

$$\sum_{n=1}^{\infty} \lambda_n^2 \leqslant \left(\sum_{n=1}^{\infty} \lambda_n\right)^2.$$

At the beginning of Section 2.3 it was shown that any nuclear operator $A = UT$ can be written in the form

$$Af = \sum_{k=1}^{\infty} \lambda_k (f, e_k) h_k,$$

where $\{e_k\}$ is an orthonormal basis in H, consisting of eigenvectors of T, λ_k are the corresponding eigenvalues, $h_k = Ue_k$ for $\lambda_k \neq 0$, $\lambda_k \geqslant 0$, and $\sum_{k=1}^{\infty} \lambda_k$ converges. From Theorem 9 it follows that

$$\| A - A_n \|_1 = \sum_{k=n+1}^{\infty} \lambda_k,$$

where A_n denotes the operator defined by

$$A_n f = \sum_{k=1}^{n} \lambda_k (f, e_k) h_k.$$

Therefore

$$\lim_{n \to \infty} \| A - A_n \|_1 = 0.$$

Thus, any nuclear operator is the limit in the trace norm of a sequence $A_1, A_2, \ldots$ of degenerate operators.

Now we prove that the space of nuclear operators is the completion of the space of degenerate operators in the trace norm. For this it is sufficient to show that the space of nuclear operators is complete relative to the trace norm. First we shall prove the following theorem.

Theorem 10. Let $A_1, A_2, \ldots$ be a sequence of nuclear operators such that the set $\{\| A_n \|_1\}$ of trace norms is bounded. If the sequence $\{A_n\}$ converges weakly to an operator A, then A is a nuclear operator

Proof. From the boundedness of the set $\{\| A_n \|_1\}$ follows the existence of a number M such that

$$\sum_{k=1}^{\infty} |(A_n f_k, h_k)| \leqslant M$$

for all orthonormal systems $\{f_k\}$ and $\{h_k\}$ in H_1 and H_2 and every operator A_n. Now we let s be any positive integer, and pass to the limit $n \to \infty$ in the inequality

$$\sum_{k=1}^{s} |(A_n f_k, h_k)| \leqslant M.$$

Taking into account that, in view of the weak convergence of the sequence $\{A_n\}$, $\lim_{n\to\infty} (A_n f_k, h_k) = (Af_k, h_k)$, we obtain the inequality

$$\sum_{k=1}^{s} |(Af_k, h_k)| \leqslant M.$$

But then

$$\sum_{k=1}^{\infty} |(Af_k, h_k)| \leqslant M.$$

Since $\{f_k\}$ and $\{h_k\}$ are arbitrary orthonormal systems in H_1 and H_2, then

$$\sup \sum_{k=1}^{\infty} |(Af_k, h_k)| \leqslant M$$

and, consequently, A is nuclear.

Remark. From this theorem it follows, in particular, that $\| A \|_1 \leqslant \sup_n \| A_n \|_1$ if $\{A_n\}$ converges weakly to A.

It is now easy to establish that the space of all nuclear operators is complete relative to the trace norm. In fact, let $\{A_n\}$ be a sequence of nuclear operators which is fundamental relative to the norm $\| A \|_1$, i.e., suppose $\lim_{m,n\to\infty} \| A_m - A_n \|_1 = 0$. Then $\{A_n\}$ is also fundamental relative to the norm $\| A \|$. In view of the completeness of the space of all continuous linear operators relative to the norm $\| A \|$, there is an operator A such that $\lim_{n\to\infty} \| A - A_n \| = 0$. From this it follows that the sequence $\{A_n\}$ converges to A in the weak sense, and by Theorem 10 A is nuclear.

Now we show that the sequence $\{A_n\}$ converges to A in the sense of the norm $\| A \|_1$, i.e., that $\lim_{n\to\infty}\| A - A_n \|_1 = 0$. For this we note that since the sequence $\{A_n\}$ is fundamental relative to the norm $\| A \|_1$, for any $\epsilon > 0$ there is an N such that for $m, n \geqslant N$ and any orthonormal bases $\{f_n\}$ and $\{h_n\}$ in the spaces H_1 and H_2 the inequality

$$\sum_{k=1}^{\infty} |((A_m - A_n)f_k, g_k)| \leqslant \| A_m - A_n \|_1 \leqslant \epsilon$$

holds. But then, if $m \geqslant N$, $n \geqslant N$ we have

$$\sum_{k=1}^{s} |((A_m - A_n)f_k, g_k)| \leqslant \epsilon$$

for every value of s. From the weak convergence of the operators A_n to A it follows that the inequality

$$\sum_{k=1}^{s} |((A - A_n)f_k, g_k)| \leqslant \epsilon$$

holds for $n \geqslant N$ and every s, and therefore for $n \geqslant N$

$$\| A - A_n \|_1 = \sup \sum_{k=1}^{\infty} |((A - A_n)f_k, g_k)| \leqslant \epsilon.$$

Thus $\| A - A_n \|_1 \leqslant \epsilon$ for $n \geqslant N$, and, consequently, the operators A_n converge to A in the norm $\| A \|_1$. Therefore the space of nuclear operators is complete relative to the trace norm.

As we already remarked, it follows from this that the space of nuclear operators is the completion of the space of degenerate operators in the trace norm.

2.5. The Trace Norm and the Decomposition of an Operator into a Sum of Operators of Rank 1

We will give in this section another definition of the trace norm based upon the decomposition of an operator into a sum of operators of rank 1, i.e., operators which map the entire space H_1 onto a one-dimensional subspace of the space H_2.

Let us consider a degenerate operator A which maps the Hilbert space H_1 onto a finite-dimensional subspace G of the Hilbert space H_2.

We choose a basis in G, consisting of linearly independent vectors $g_1, \ldots, g_m$, and expand the vector Af in terms of this basis as

$$Af = \sum_{k=1}^{m} \alpha_k(f) g_k.$$

Obviously, for fixed k the coefficient $\alpha_k(f)$ is a linear continuous functional of f, and can therefore be represented in the form $\alpha_k(f) = (f, f_k)$, where f_k is a fixed element of H_1. Thus, a degenerate operator A can always be written in the form

$$Af = \sum_{k=1}^{m} (f, f_k) g_k. \tag{23}$$

Conversely, if $g_1, \ldots, g_m$ are any (possibly linearly dependent) vectors in H_2 and $f_1, \ldots, f_m$ are any vectors in H_1, then formula (23) defines a degenerate operator. Each term $(f, f_k)g_k$ in the formula is an operator which maps the entire space H_1 onto the one-dimensional subspace of H_2 generated by the vector g_k. Such an operator is called an operator of rank 1. Thus, formula (23) gives a decomposition of the degenerate operator A into a sum of operators of rank 1.

Let us introduce the operators $P_k f = (f, f_k) g_k$. Then Eq. (23) can be written in the form

$$Af = \sum_{k=1}^{m} P_k f. \tag{24}$$

We will call such a decomposition a *decomposition of the operator A into a sum of operators of rank* 1. We note that the norm $\| P_k \|$ is equal to $\| f_k \| \, \| g_k \|$. In fact,

$$\| P_k \| = \sup_{\|f\|=1} \|(f, f_k) g_k \| = \sup_{\|f\|=1} |(f, f_k)| \, \| g_k \| = \| f_k \| \, \| g_k \|.$$

Of course, every degenerate operator A can be decomposed into a sum of operators of rank 1 in various ways. We now prove that the trace norm of the operator A is the infimum of sums $\sum_{k=1}^{m} \| P_k \|$ taken over all decompositions of A into a sum of operators of rank 1, i.e.,

$$\| A \|_1 = \inf \sum_{k=1}^{m} \| P_k \|. \tag{25}$$

First we prove the inequality

$$\| A \|_1 \geqslant \inf \sum_{k=1}^{m} \| P_k \|. \tag{26}$$

Since the operator A is degenerate, the operator T appearing in the decomposition $A = UT$ has only a finite number of eigenvalues λ_k different from zero. Therefore A can be written in the form

$$Af = \sum_{k=1}^{m} \lambda_k(f, e_k)h_k,$$

where the e_k are eigenvectors of T and $h_k = Ue_k$. Since $\| \lambda_k e_k \| = \lambda_k$ and $\| h_k \| = 1$, for this decomposition the sum $\sum_{k=1}^{m} \| P_k \|$ equals $\sum_{k=1}^{m} \lambda_k$, i.e., the trace of the operator T, and so the infimum of sums of the form $\sum_{k=1}^{m} \| P_k \|$, taken over all the decompositions of the operator A into a sum of operators of rank 1, does not exceed[13] $\mathrm{Tr}(T)$, i.e.,

$$\inf \sum_{k=1}^{m} \| P_k \| \leqslant \mathrm{Tr}(T) = \| A \|_1.$$

Thus inequality (26) is proven.

Now we prove the reverse inequality. Let

$$A = \sum_{k=1}^{m} P_k$$

be a decomposition of A into a sum of operators of rank 1. Then by the properties of the trace norm we have

$$\| A \|_1 \leqslant \sum_{k=1}^{m} \| P_k \|_1. \tag{27}$$

But for an operator P of rank 1 the trace norm coincides with the ordinary norm, i.e., $\| P \|_1 = \| P \|$. In fact, let $Pf = (f, h)g$. Then for any orthonormal systems $\{f_n\}$ and $\{g_n\}$ in the spaces H_1 and H_2 we have

$$\sum_{k=1}^{\infty} |(Pf_k, g_k)| = \sum_{k=1}^{\infty} |(f_k, h)(g, g_k)|.$$

In view of the Bunyakovski–Schwartz inequality it follows from this that

$$\begin{aligned} \| P \|_1 &= \sup_{f,g} \sum_{k=1}^{\infty} |(Pf_k, g_k)| \\ &\leqslant \Big[\sum_{k=1}^{\infty} |(f_k, h)|^2 \sum_{k=1}^{\infty} |(g, g_k)|^2\Big]^{\frac{1}{2}} \leqslant \| h \| \| g \| = \| P \|. \end{aligned}$$

[13] $\mathrm{Tr}(T)$ denotes the trace of T.

Since the reverse inequality $\| P \| \leqslant \| P \|_1$ always holds, the equality $\| P \| = \| P \|_1$ is proven. Therefore inequality (27) may be rewritten in the form

$$\| A \|_1 \leqslant \sum_{k=1}^{m} \| P_k \|. \tag{28}$$

Since this inequality is valid for any decomposition of A into a sum of operators of rank 1, we have

$$\| A \|_1 \leqslant \inf \sum_{k=1}^{m} \| P_k \|. \tag{29}$$

From inequalities (26) and (29) it follows that

$$\| A \|_1 = \inf \sum_{k=1}^{m} \| P_k \|,$$

and since $\| P_k \| = \| f_k \| \| g_k \|$, that

$$\| A \|_1 = \inf \sum_{k=1}^{m} \| f_k \| \| g_k \|,$$

where the infimum is taken over all decompositions $Af = \sum_{k=1}^{m} (f, f_k) g_k$ of A into a sum of operators of rank 1. The infimum is achieved by any decomposition in which the f_k are eigenvectors (with nonzero eigenvalues) of the operator T appearing in the decomposition $A = UT$, and $g_k = Uf_k$.

We can now say that the nuclear operators are those obtained by completing the set of degenerate operators in the norm

$$\| A \|_1 = \inf \sum_{k=1}^{m} \| f_k \| \| g_k \|,$$

where the infimum is taken over all decompositions of the operator A into a sum of operators of rank 1.

In this form the definition of the nuclearity of an operator can be carried over to any Banach space. A degenerate operator which maps a Banach space E_1 into a Banach space E_2 has the form

$$Af = \sum_{k=1}^{m} (F_k, f) g_k,$$

where $F_1, \ldots, F_k$ are fixed linear functionals on the space E_1, and $g_1, \ldots, g_n$ are fixed elements of E_2. An operator B, mapping E_1 into E_2, is called nuclear if it belongs to the completion of the set of degenerate operators in the norm

$$\| A \|_1 = \inf \sum_{k=1}^{m} \| F_k \| \| g_k \|,$$

where the infimum is taken over all decompositions of A into a sum of operators of rank 1. We should remark, however, that we do not consider the concept of nuclearity in Banach spaces to be adequately justified, because such operators lack certain very important properties possessed by nuclear operators in Hilbert spaces. For example, the trace theorem (cf. Theorem 6) is not valid for nuclear operators in Banach spaces. In other words, if $\{f_n\}$ and $\{F_n\}$ are biorthogonal unconditional bases in the spaces E and E' (i.e., bases such that $(F_k, f_m) = \delta_{km}$), and μ_k are the eigenvalues of the nuclear operator A, then, generally speaking, the equality

$$\sum_{k=1}^{\infty} (F_k, Af_k) = \sum_{k=1}^{\infty} \mu_k$$

does not hold.

We remark that if an operator A, mapping a Banach space E_1 into a Banach space E_2, has the form

$$Af = \sum_{k=1}^{\infty} (F_k, f) g_k, \tag{30}$$

where $F_k \in E_1'$, $g_k \in E_2$, and the series

$$\sum_{k=1}^{\infty} \| F_k \| \| g_k \|$$

converges, then A is a nuclear operator. Conversely, if A is a nuclear operator which maps a Banach space E_1 into a Banach space E_2, then it can be written in the form of a series (30) for which the series $\sum_{k=1}^{\infty} \| F_k \| \| g_k \|$ converges.

We will not linger over the proofs of these assertions, since they have been proved in the case which is of interest to us, namely Hilbert space, and, as we said, the concept of nuclearity for operators in a Banach space is apparently not sufficiently worthwhile.

3. Nuclear Spaces. The Abstract Kernel Theorem

One of the basic problems which arises after setting up a general theory of linear topological spaces and, in particular, of countably normed spaces, is that of distinguishing a class of spaces which is defined by sufficiently simple requirements and is of service in analysis. We maintain that one such class of spaces is the class of nuclear spaces, which will be studied in this section.

Nuclear spaces were introduced in Volume III (Chapter IV, Section 3.1) in connection with the spectral analysis of self-adjoint operators. We give here another more natural definition of a nuclear space, which is equivalent to the previous definition for a wide class of linear topological spaces. Moreover, we prove an abstract version of the kernel theorem, i.e., a theorem on bilinear functionals on nuclear spaces, from which one can obtain the kernel theorem for the spaces K and S.

3.1. Countably Hilbert Spaces

We will call a strongly positive-definite Hermitean functional (φ, ψ), defined on a linear space Φ, i.e., a functional such that

(1) $(\varphi_1 + \varphi_2, \psi) = (\varphi_1, \psi) + (\varphi_2, \psi)$,

(2) $(\alpha\varphi, \psi) = \alpha(\varphi, \psi)$,

(3) $(\varphi, \psi) = \overline{(\psi, \varphi)}$,

(4) $(\varphi, \varphi) \geqslant 0$ and $(\varphi, \varphi) = 0$ if and only if $\varphi = 0$,

a scalar product in Φ.

With every scalar product (φ, ψ) in the space Φ one can associate a norm $\| \varphi \|$, setting $\| \varphi \| = \sqrt{(\varphi, \varphi)}$. Suppose now that we are given a countable system of scalar products $(\varphi, \psi)_n$ in the space Φ, which are *compatible* in the following sense: If a sequence $\{\varphi_k\}$ of elements of Φ converges to zero in the norm $\| \varphi \|_m = \sqrt{(\varphi, \varphi)_m}$ and is a fundamental sequence in the norm $\| \varphi \|_n$, then it also converges to zero in the norm $\| \varphi \|_n$.

We introduce a topology in Φ, taking as a complete neighborhood basis of zero in Φ the sets $U_{n,\epsilon}$, defined by the inequalities $\| \varphi \|_n \leqslant \epsilon$. We will say that a space Φ with a given countable collection of scalar products is *countably Hilbert,* if it is complete† relative to the stated

† Perhaps the simplest way to define completeness in the present context is to point out that the topology in Φ defined by the metric

$$\| \varphi \| = \sum_{n=1}^{\infty} 2^{-n} \frac{\| \varphi \|_n}{1 + \| \varphi \|_n}$$

is identical to that defined above. Completeness is then understood in its usual sense for metric spaces. It is not hard to see that completeness is equivalent to $\Phi = \bigcap_{n=1}^{\infty} \Phi_n$, where Φ_n is the completion of Φ relative to the norm $\| \varphi \|_n$.

topology. Thus, a countably Hilbert space is a complete linear topological space in which the topology is given by a countable set of compatible norms $\| \varphi \|_n$ having the form $\| \varphi \|_n = \sqrt{(\varphi, \varphi)_n}$.

We could also define a countable set of arbitrary (Banach) compatible norms $\| \varphi \|_n$. In this case the space is called countably normed.

At a first glance it might appear that the class of countably Hilbert spaces is essentially a narrow class of countably normed spaces, because the Hilbert norms $\| \varphi \|_n = \sqrt{(\varphi, \varphi)_n}$ are only special cases of general Banach norms. However, thanks to the fact that we are considering countable collections of norms, the differences between different Banach norms frequently fall away.[1] For example, the class of functions having continuous first derivatives is different from the class of functions whose first derivatives have square integrable moduli. However, it is obvious that the class of functions having continuous derivatives of all orders coincides with the class of functions having derivatives of all orders with square integrable moduli.

Frequently the system of norms $\| \varphi \|_n$ in a given countably normed space Φ can be replaced by a system of norms, defined by scalar products, without altering the topology of the space.

We will ordinarily consider a system of scalar products $(\varphi, \psi)_n$, $n = 1, 2, \ldots$, in the space Φ such that for any element $\varphi \in \Phi$ the inequalities

$$(\varphi, \varphi)_1 \leqslant (\varphi, \varphi)_2 \leqslant \cdots$$

hold. This condition does not restrict the class of spaces considered. If a given system of scalar products does not have this property, it can be replaced by a new system of scalar products $(\varphi, \psi)'_n$, setting

$$(\varphi, \psi)'_n = \sum_{k=1}^{n} (\varphi, \psi)_k.$$

As is easily seen, this does not alter the topology in Φ. At the same time the system of scalar products $(\varphi, \psi)'_n$ has the property that

$$(\varphi, \varphi)'_1 \leqslant (\varphi, \varphi)'_2 \leqslant \cdots$$

[1] The fact that differences between Banach spaces are either fictitious or too fine for many problems of functional analysis was made plain, for example, in the theory of representations of Lie groups.

Representations, given by the same formulas, but considered in different spaces, turned out, as a consequence, to be nonequivalent. Therefore it is most natural to consider group representations not in Banach spaces, but rather in linear topological spaces. Concerning details, cf. Volume V.

We denote by Φ_n the completion of the space Φ relative to the scalar product $(\varphi, \psi)_n$.[2] Obviously Φ_n is a Hilbert space. From the completeness of the space Φ it follows that Φ is the intersection of the spaces Φ_n, $n = 1, 2, \ldots$;

$$\Phi = \bigcap_{n=1}^{\infty} \Phi_n.$$

Conversely, if the topology in a linear topological space Φ is defined by means of a countable collection of scalar products $(\varphi, \psi)_n$ and Φ coincides with the intersection $\bigcap_{n=1}^{\infty} \Phi_n$ of its completions relative to these scalar products, then Φ is complete and therefore countably Hilbert. We will not linger over the proofs of these simple assertions. The reader who so desires can acquaint himself in Chapter I of Volume II with the proofs of these assertions for the more general case of countably normed spaces.

Further on we will need to consider certain elements of the space Φ as elements of the corresponding Hilbert spaces Φ_n. In such cases where this can lead to misunderstandings, we will write $\overset{(n)}{\varphi}$ instead of φ. Thus, the elements $\overset{(1)}{\varphi}, \overset{(2)}{\varphi}, \ldots$ are the same element φ of the space Φ, considered in the different spaces Φ_n.

Let us now clarify the structure of the adjoint space Φ' of a countably Hilbert space Φ. We show that Φ' is the union of the Hilbert spaces Φ'_n which are the adjoints of the Hilbert spaces Φ_n, $n = 1, 2, \ldots$. In fact, let F be an element of the space Φ'_n, i.e., a linear functional on the Hilbert space Φ_n. Then obviously F is continuous relative to the topology of Φ, i.e., F is an element of Φ'. From this it follows that $\bigcup_{n=1}^{\infty} \Phi'_n \subset \Phi'$. On the other hand, if F is a linear functional on Φ, then as was mentioned in Section 1.2, F is continuous relative to some one of the norms, say $\| \varphi \|_n = \sqrt{(\varphi, \varphi)_n}$, i.e., it belongs to the space Φ'_n. Thus $\Phi' = \bigcup_{n=1}^{\infty} \Phi'_n$. We note that the spaces Φ'_n form an increasing chain

$$\Phi'_1 \subset \Phi'_2 \subset \ldots .$$

In fact, since $(\varphi, \varphi)_m \leqslant (\varphi, \varphi)_n$ for $m \leqslant n$, then from the boundedness of a functional F in the ball $(\varphi, \varphi)_m \leqslant 1$ follows its boundedness in the ball $(\varphi, \varphi)_n \leqslant 1$, i.e., if F belongs to Φ'_m, then it belongs to Φ'_n.

The space Φ'_n is the adjoint of the Hilbert space Φ_n. Therefore there is defined in Φ'_n a norm which is conveniently denoted by $\| F \|_{-n}$, given by

$$\| F \|_{-n} = \sup_{\|\varphi\|_n = 1} (F, \varphi).$$

[2] That is, relative to the norm $\| \varphi \|_n = \sqrt{(\varphi, \varphi)_n}$ generated by the scalar product.

As is well known, the adjoint space of a Hilbert space is a Hilbert space and therefore the norm $\| F \|_{-n}$ is defined by a scalar product $(F, G)_{-n}$ in Φ'_n. In other words,

$$\| F \|_{-n} = \sqrt{(F, F)_{-n}}.$$

It should be kept in mind that the scalar products in the different spaces Φ'_n are different and that for $m \leqslant n$ the inequality $(F, F)_{-m} \geqslant (F, F)_{-n}$ holds.

Sometimes we will be led to consider a functional F from Φ' as an element of several of the spaces Φ'_n. In this case we will use the same notation $\overset{(1)}{F}, \overset{(2)}{F}, \dots$ as is used or elements of the space Φ. We note that if $m \leqslant n$ and $F \in \Phi'_m$, then

$$(\overset{(m)}{F}, \overset{(m)}{\varphi}) = (\overset{(n)}{F}, \overset{(n)}{\varphi})$$

for any element $\varphi \in \Phi$. In fact, both sides of this equality are the value of the functional F for the element $\varphi \in \Phi$.

One can introduce a topology in the space Φ' in various ways. For example, one can take as a complete system of neighborhoods of zero in Φ' the sets $U(\varphi_1, \dots, \varphi_m; \epsilon)$, consisting of those linear functionals F such that

$$|(F, \varphi_k)| \leqslant \epsilon, \qquad 1 \leqslant k \leqslant m.$$

Here $\varphi_1, \dots, \varphi_m$ are elements of Φ, and ϵ is an arbitrary positive number. This topology is called the *weak topology* in the space Φ'. Along with this topology we can consider the strong topology. In order to define the strong topology in Φ', we introduce the concept of a *bounded set* in Φ. A set A in Φ is said to be bounded, if for any k the set of numbers $(\varphi, \varphi)_k$, where $\varphi \in A$, is bounded. In this case, for any neighborhood U of zero in Φ there is a number n such that $A \subset nU$. A complete system of neighborhoods of zero which defines the strong topology in Φ' consists of all the sets $U(A, \epsilon)$, defined by the inequalities

$$\sup_{\varphi \in A} |(F, \varphi)| < \epsilon,$$

where A is any bounded set in Φ, and $\epsilon > 0$.

It can be shown that the elements of each of the subspaces Φ'_n form an everywhere dense set in the space Φ', relative to the weak topology.

One can also introduce the notion of bounded sets in Φ'. Here one must distinguish between a weakly and a strongly bounded set in Φ'. A set A in Φ' is called *strongly bounded*, if for any strong neighborhood U

of zero in Φ' there is an n such that $A \subset nU$. In the same way a set A is called *weakly bounded*, if for any weak neighborhood V of zero in Φ' there is an n such that $A \subset nV$. Since every weak neighborhood of zero in Φ' is also a strong neighborhood of zero, *every strongly bounded set in Φ' is weakly bounded.*

We consider, finally, the adjoint space Φ'' of Φ'. In this space also, one can consider different topologies, starting respectively from finite, strongly bounded, and weakly bounded sets in the space Φ'. We will construct a topology in Φ'', starting from the strongly bounded sets in Φ'. With each such set A and each number $\epsilon > 0$ we associate the set $U(A, \epsilon)$ in Φ'', consisting of those linear functionals $\hat{\varphi}$ on Φ' such that $\sup_{F \in A} |(\hat{\varphi}, F)| < \epsilon$. We take the collection of all the sets $U(A, \epsilon)$ for a complete system of neighborhoods of zero in Φ''.

With this topology the second adjoint space Φ'' is isomorphic to the original countably Hilbert space Φ, i.e., $\Phi = \Phi''$. A linear topological space Φ for which $\Phi = \Phi''$ is called a *reflexive* space. Thus *a countably Hilbert space is reflexive.*

The proof is carried out in the following way. To each element $\varphi \in \Phi$ there corresponds a linear functional $\hat{\varphi}$ on the space Φ', defined by the equation $(\hat{\varphi}, F) = (F, \varphi)$. The correspondence $\varphi \to \hat{\varphi}$ is a one-to-one imbedding of the space Φ into the space Φ''. We show that it is one-to-one onto, i.e., that each element $\hat{\varphi} \in \Phi''$ is the map of some element $\varphi \in \Phi$. In fact, a linear functional $\hat{\varphi}$ on the space $\Phi' = \bigcup_{k=1}^{\infty} \Phi_k'$ is at the same time a linear functional on each of the Hilbert spaces Φ_n'. But a Hilbert space is reflexive and therefore $\hat{\varphi}$ can be considered as an element of each of the Hilbert spaces which one obtains upon completing the space Φ relative to the scalar products $(\varphi, \psi)_k$. Since $\bigcap_{k=1}^{\infty} \Phi_k = \Phi$, we find that $\hat{\varphi} \in \Phi$. Thus we have proven that the spaces Φ and Φ'' coincide as sets of elements.

Now we show that the one-to-one correspondence which we have introduced between Φ and Φ'' is continuous in both directions. For this we make use of the description given in Volume II (Chapter I, Sections 5.2 and 5.3) of strongly bounded sets in a space Φ' which is the adjoint of a countably normed space $\Phi = \bigcap_{k=1}^{\infty} \Phi_k$. It was shown there that each such set A belongs to some one of the spaces Φ_k' and is bounded in it with respect to the corresponding norm $\| F \|_{-k}$.

Let us now consider some neighborhood $U(A, \epsilon)$ of zero in the space Φ''. By the remark just made, the bounded set A is bounded in one of the Hilbert spaces Φ_k', and therefore lies in some ball $\| F \|_{-k} < a$. We now consider the ball $\| \varphi \|_k \leqslant \epsilon/a$ in the space Φ. If an element φ belongs to this ball, then for any functional F from the ball $\| F \|_{-k} < a$, and *a fortiori* any functional F from the set A, the inequality $|(F, \varphi)| < \epsilon$ holds.

This means that the map of the ball $\| \varphi \|_k \leqslant \epsilon/a$ lies in the neighborhood $U(A, \epsilon)$ of zero in Φ'', i.e., the mapping $\varphi \to \hat{\varphi}$ is continuous.

The continuity of the inverse mapping is proven in the same way. Let $\| \varphi \|_k \leqslant \epsilon$ be a ball in the space Φ. We denote by A a strongly bounded set in Φ', consisting of those functionals F for which $|(F, \varphi)| \leqslant 1$ if $\| \varphi \|_k \leqslant \epsilon$. Obviously the map of the neighborhood $U(A, \epsilon)$ of zero in Φ'' is contained in the ball $\| \varphi \|_k \leqslant \epsilon$. This proves the assertion of the continuity of the inverse mapping.

Thus we have proven that the spaces Φ and Φ'' coincide not only as sets of elements but also according to their topologies, which proves that any countably Hilbert space is reflexive.

3.2. Nuclear Spaces

We now introduce the basic concept of this section—that of a nuclear space. Let Φ be a countably Hilbert space. We consider the Hilbert spaces Φ_n which are obtained by completing the space Φ in the norms $\| \varphi \|_n = \sqrt{(\varphi, \varphi)_n}$. In each of these spaces the set of elements of Φ is an everywhere dense set. By hypothesis, if $m \leqslant n$ then $(\varphi, \varphi)_m \leqslant (\varphi, \varphi)_n$. From this it follows that the mapping $\overset{(n)}{\varphi} \to \overset{(m)}{\varphi}$ is a continuous mapping of an everywhere dense set in Φ_n onto an everywhere dense set in Φ_m (we recall that $\overset{(n)}{\varphi}$ and $\overset{(m)}{\varphi}$ denote the same element $\varphi \in \Phi$, considered as an element of Φ_n and Φ_m). We can extend this mapping to a continuous linear operator T_m^n which maps the space Φ_n onto an everywhere dense subset of Φ_m, and note the obvious equality $T_m^p = T_m^n T_n^p$ if $m \leqslant n \leqslant p$.

We now introduce the following definition: A countably Hilbert space Φ is called *nuclear*, if for any m there is an n such that the mapping T_m^n of the space Φ_n into the space Φ_m is nuclear, i.e., has the form

$$T_m^n \varphi = \sum_{k=1}^{\infty} \lambda_k (\varphi, \varphi_k)_n \psi_k, \qquad \varphi \in \Phi_n,$$

where $\{\varphi_k\}$ and $\{\psi_k\}$ are orthonormal systems of vectors in the spaces Φ_n and Φ_m, respectively, $\lambda_k > 0$, and the series $\sum_{k=1}^{\infty} \lambda_k$ converges.

Geometrically this definition means the following: A countably Hilbert space Φ is nuclear, if for any m there is an n such that the set $(\varphi, \varphi)_n \leqslant 1$ is, relative to the scalar product $(\varphi, \varphi)_m$, an ellipsoid for which the series consisting of the lengths of its principal semiaxes converges.

We note further that instead of the nuclearity of the operator T_m^n one can require that it be of Hilbert–Schmidt type. In fact, it was shown in Section 2.3 that the product of any two Hilbert–Schmidt operators is a nuclear operator. Consequently, if the operators T_n^p and T_m^n are of Hilbert–Schmidt type, then $T_m^p = T_m^n T_n^p$ is a nuclear operator.

The concept of nuclearity can be generalized to any countably normed space. Namely, we call a countably normed space Φ nuclear, if for any m there is an n such that the operator T_m^n which imbeds the space Φ_n into the space Φ_m is nuclear, i.e., has the form

$$T_m^n\varphi = \sum_{k=1}^{\infty} \lambda_k(F_k, \varphi)\psi_k,$$

where $\{F_k\}$ and $\{\psi_k\}$ are bounded sequences of elements of the spaces Φ'_n and Φ_m, $\lambda_k > 0$, and the series $\Sigma_{k=1}^{\infty} \lambda_k$ converges.[3]

However, this generalization does not lead to an extension of the class of spaces considered—in any nuclear countably normed space a countable set of scalar products can be introduced in such a way that the space becomes a nuclear countably Hilbert space, without altering its topology.

These scalar products are constructed in the following way. We consider any value of m. There is an n such that the mapping T_m^n is nuclear, i.e., is given by a formula of the form

$$T_m^n\varphi = \sum_{k=1}^{\infty} \lambda_k(F_k, \varphi)\psi_k,$$

where $\{F_k\}$ is a bounded set in Φ'_n, $\{\psi_k\}$ is a bounded set in Φ_m, $\lambda_k > 0$, and the series $\Sigma_{k=1}^{\infty} \lambda_k$ converges.

We define $(\varphi, \varphi)_{mn}$ by

$$(\varphi, \varphi)_{mn} = \inf \sum_{k=1}^{\infty} \lambda_k | \alpha_k |^2,$$

where $\varphi \in \Phi$ and the infimum is taken over all sequences $\{\alpha_k\}$ such that $\Sigma_{k=1}^{\infty} \lambda_k \mid \alpha_k \mid^2 < +\infty$ and $T_m^n\varphi = \Sigma_{k=1}^{\infty}\lambda_k\alpha_k\psi_k$. Let us show that $(\varphi, \varphi)_{mn}$ is finite for all $\varphi \in \Phi$. Indeed, we can take $\alpha_k = (F_k, \varphi)$ and therefore

$$(\varphi, \varphi)_{mn} \leqslant \sum_{k=1}^{\infty} \lambda_k(F_k, \varphi)\overline{(F_k, \varphi)}.$$

[3] This can also be formulated as

$$T_m^n\varphi = \sum_{k=1}^{\infty} (F_k, \varphi)\psi_k,$$

where the series $\Sigma_{k=1}^{\infty} \| F_k \|_{-n} \| \psi_k \|_m$ converges.

This series converges for all $\varphi \in \Phi$, since it is majorized by the series

$$\sum_{k=1}^{\infty} \lambda_k \| F_k \|_{-n}^2 \| \varphi \|_n^2,$$

and by hypothesis the set of functionals $\{F_k\}$ is bounded in Φ_n' and the series $\sum_{k=1}^{\infty} \lambda_k$ converges.

It is not difficult to show that $(\varphi, \varphi)_{mn}$ is a Hermitean form in Φ and therefore defines a scalar product $(\varphi, \psi)_{mn}$ in Φ. We will now show that $(\varphi, \varphi)_{mn}$ satisfies inequalities of the form

(a) $\quad \| \varphi \|_m^2 \leqslant C_1(\varphi, \varphi)_{mn}$

and

(b) $\quad (\varphi, \varphi)_{mn} \leqslant C_2 \| \varphi \|_n^2,$

where C_1 and C_2 are positive and do not depend upon φ. In fact, from $T_m^n \varphi = \sum_{k=1}^{\infty} \lambda_k \alpha_k \varphi_k$ it follows that

$$\| \varphi \|_m^2 \leqslant \left[\sum_{k=1}^{\infty} \lambda_k | \alpha_k | \, \| \psi_k \|_m \right]^2.$$

Since the set $\{\psi_k\}$ in Φ_m is bounded,

$$\| \varphi \|_m^2 \leqslant C^2 \left(\sum_{k=1}^{\infty} \lambda_k | \alpha_k | \right)^2,$$

where $C = \sup_k \| \psi_k \|_m$. Further, in view of the Bunyakovski–Schwartz inequality

$$\left(\sum_{k=1}^{\infty} \lambda_k | \alpha_k | \right)^2 \leqslant \sum_{k=1}^{\infty} \lambda_k \sum_{j=1}^{\infty} \lambda_j | \alpha_j |^2,$$

and therefore

$$\| \varphi \|_m^2 \leqslant C_1 \sum_{k=1}^{\infty} \lambda_k | \alpha_k |^2,$$

where $C_1 = C^2 \sum_{k=1}^{\infty} \lambda_k$. Since this inequality holds for all sequences $\{\alpha_k\}$ such that $\sum_{k=1}^{\infty} \lambda_k \, | \alpha_k |^2 < \infty$ and $T_m^n \varphi = \sum_{k=1}^{\infty} \lambda_k \alpha_k \psi_k$, we have

$$\| \varphi \|_m^2 \leqslant C_1 \inf \sum_{k=1}^{\infty} \lambda_k | \alpha_k |^2 = C_1 (\varphi, \varphi)_{mn}.$$

On the other hand, since we can put $\alpha_k = (F_k, \varphi)$, then

$$(\varphi, \varphi)_{mn} \leqslant \sum_{k=1}^{\infty} \lambda_k |(F_k, \varphi)|^2$$

$$\leqslant \| \varphi \|_n^2 \sum_{k=1}^{\infty} \lambda_k \| F_k \|_{-n}^2 \leqslant C_2 \| \varphi \|_n^2,$$

where $C_2 = \sup_k \| F_k \|_{-n} \sum_{j=1}^{\infty} \lambda_j$. This proves the stated inequalities. Since m is arbitrary, it follows from these inequalities that the collection of scalar products $(\varphi, \psi)_{mn}$ defines the same topology in Φ as does the collection of norms $\| \varphi \|_n$.

▼ In order to prove that Φ is a countably Hilbert space, we have to further establish the compatibility of the norms $\sqrt{(\varphi, \varphi)_{mn}}$. For this it suffices, in view of inequality (a), to show that $\| \varphi \|_m$ and $\sqrt{(\varphi, \varphi)_{mn}}$ are compatible norms. To show this, we remark first that, as is well known, the set of all sequences $\alpha = \{\alpha_k\}$ such that $\sum_{k=1}^{\infty} \lambda_k \mid \alpha_k \mid^2 < \infty$ forms a Hilbert space L_λ^2, in which $\| \alpha \|^2 = \sum_{k=1}^{\infty} \lambda_k \mid \alpha_k \mid^2$. Now practically repeating the steps which led to inequality (a), we see that for each $\alpha \in L_\lambda^2$ the series $\sum_{k=1}^{\infty} \lambda_k \alpha_k \psi_k$ converges to some element $\varphi_\alpha \in \Phi_m$, and $\| \varphi_\alpha \|_m^2 \leqslant C_1 \| \alpha \|^2$. Now let $\mathfrak{M}$ be the set of all α such that $\varphi_\alpha = 0$. Clearly $\mathfrak{M}$ is linear, and the last inequality shows that it is a closed set in L_λ^2. We have seen that for any $\varphi \in \Phi \subset \Phi_m$ there is at least one $\alpha \in L_\lambda^2$ (namely, $\alpha = \{(F_k, \varphi)\}$) such that $\varphi = \varphi_\alpha$. Let P be the projection of L_λ^2 onto the orthogonal complement $\mathfrak{N}$ of $\mathfrak{M}$ in L_λ^2, and set $\alpha_\varphi = P\alpha$, where α is any element of L_λ^2 for which $\varphi = \varphi_\alpha$. Clearly α_φ is unique, and the set of all $\alpha \in L_\lambda^2$ for which $\varphi = \varphi_\alpha$ is just $\alpha_\varphi + \mathfrak{M}$. We observe further that $\| \alpha_\varphi \|^2 = (\varphi, \varphi)_{mn}$ (this shows, by the way, that $(\varphi, \varphi)_{mn}$ defines an inner product $(\varphi, \psi)_{mn}$). There is thus a mapping $\alpha \to \varphi_\alpha$ of $\mathfrak{N}$ into Φ_m such that the image of $\mathfrak{N}$ in Φ_m contains Φ, and if $\alpha_1 \neq \alpha_2$ are elements of $\mathfrak{N}$, then $\varphi_{\alpha_1} \neq \varphi_{\alpha_2}$ in Φ_m. Now if $\{\varphi_k\}$ is a sequence in Φ which is fundamental in the norm $\sqrt{(\varphi, \varphi)_{mn}}$, then there is a corresponding sequence $\{\alpha_k\}$ in $\mathfrak{N}$ which is fundamental in L_λ^2, and hence converges to some $\alpha_0 \in \mathfrak{N}$. Clearly (a) implies that $\{\varphi_k\}$ converges in Φ_m to some element $\varphi_0 \in \Phi_m$, which must be the element corresponding to α_0. Thus $\alpha_0 = 0$ if and only if $\varphi_0 = 0$. Hence $\| \varphi \|_m$ and $\sqrt{(\varphi, \varphi)_{mn}}$ are compatible. Thus the $\sqrt{(\varphi, \varphi)_{mn}}$ are compatible. We note also from (a), (b), and the compatibility of the $\| \varphi \|_m$ that any $\| \varphi \|_i$ and $\sqrt{(\varphi, \varphi)_{mn}}$ are compatible.

Now, recalling that $n = n(m)$, choose a sequence $m_1 < m_2 < \ldots$ such that $m_{i+1} > n(m_i)$, and set $(\varphi, \varphi)_i' = (\varphi, \varphi)_{m_i n(m_i)}$. From (a), (b), and $\| \varphi \|_1 \leqslant \| \varphi \|_2 \leqslant \ldots$ it follows that by multiplying each $(\varphi, \varphi)_i'$,

$i \geqslant 2$, by a suitable constant D_i, we have $(\varphi, \varphi)_1 \leqslant (\varphi, \varphi)_2 \leqslant \ldots$, where $(\varphi, \varphi)_1 = (\varphi, \varphi)_1'$ and $(\varphi, \varphi)_i = D_i(\varphi, \varphi)_i'$ for $i \geqslant 2$. It is evident, from (a) and (b), that the family of norms $\sqrt{(\varphi, \varphi)_i}$ defines the same topology in Φ as does the family $\| \varphi \|_m$. Thus Φ is a countably Hilbert space.

It remains to show that Φ is nuclear. Let $\Phi_{(i)}$ denote the completion of Φ with respect to $\sqrt{(\varphi, \varphi)_i}$. Since the $\sqrt{(\varphi, \varphi)_i}$ are compatible with the $\| \varphi \|_m$, then (a) and (b) show that $\Phi_{m_1} \subset \Phi_{(1)} \subset \Phi_{m_2} \subset \Phi_{(2)} \subset \ldots$. Now for any i there is $j > i$ such that the natural mapping $T^{m_j}_{m_{i+1}}$ of Φ_{m_j} into $\Phi_{m_{i+1}}$ is nuclear; and the natural mapping A of $\Phi_{(j)}$ into Φ_{m_j} is obviously bounded, as is the natural mapping B of $\Phi_{m_{i+1}}$ into $\Phi_{(i)}$. But then $BT^{m_j}_{m_{i+1}}A$, which is clearly the natural mapping of $\Phi_{(j)}$ into $\Phi_{(i)}$, is nuclear.[4] Thus Φ is a nuclear space. ▲

An important generalization of the concept of a nuclear countably normed space is that of a *nuclear linear topological space*. This concept is introduced in analogous fashion, the only difference being that the set of norms which define the topology is uncountable, and these norms can vanish on nonzero elements. We remark that the adjoint space of a nuclear countably Hilbert space is nuclear in this sense.

3.3. A Criterion for the Nuclearity of a Space

In this section we will formulate a necessary and sufficient condition for nuclearity which is convenient to use in various applications. To begin with, we introduce the following concepts.

A series $\sum_{k=1}^{\infty} F_k$ of functionals, defined on a linear topological space

[4] The assertion of the nuclearity of the product of a nuclear and a continuous operator is as valid for mappings in Banach spaces as it is for mappings in Hilbert space. In fact, if

$$T\varphi = \sum_{k=1}^{\infty} \lambda_k (F_k, \varphi) \psi_k$$

is a nuclear mapping of the space Φ into Ψ, and A is a continuous mapping of the space Ψ into X, then the mapping AT can be represented in the form

$$AT = \sum_{k=1}^{\infty} \lambda_k (F_k, \varphi)\, A\psi_k.$$

The set $\{A\psi_k\}$ is bounded in X in view of the continuity of the operator A. Therefore AT is a nuclear mapping. In just the same way one proves the nuclearity of the mapping TA, where A is a continuous mapping of X into Φ and T is a nuclear mapping of Φ into Ψ. Namely,

$$TA\chi = \sum_{k=1}^{\infty} \lambda_k (F_k, A\chi)\psi_k = \sum_{k=1}^{\infty} \lambda_k (A^* F_k, \chi)\psi_k,$$

and the set $\{A^* F_k\}$ is bounded in view of the continuity of the operator A^*.

Φ, is called *unconditionally convergent* if, for any element $\varphi \in \Phi$, the series $\sum_{k=1}^{\infty} |(F_k, \varphi)|$ converges. The series $\sum_{k=1}^{\infty} F_k$ is called *absolutely convergent* if one can find a neighborhood U of zero in Φ such that the series $\sum_{k=1}^{\infty} \|F_k\|_U$ converges, where $\|F_k\|_U = \sup_{\varphi \in U} |(F_k, \varphi)|$.

Obviously every absolutely convergent series is unconditionally convergent. In nuclear spaces the concepts of absolute and unconditional convergence are equivalent. In other words, the following theorem holds.

Theorem 1. Any unconditionally convergent series $\sum_{k=1}^{\infty} F_k$ of linear functionals on a nuclear space Φ is absolutely convergent.

First we prove the following lemma.

Lemma 1. If $\sum_{k=1}^{\infty} F_k$ is an unconditionally convergent series of linear functionals on a countably Hilbert space Φ, then there exist numbers M and m such that

$$\sum_{k=1}^{\infty} |(F_k, \varphi)| \leqslant M\|\varphi\|_m \tag{1}$$

for every element $\varphi \in \Phi$.

Proof. We define a functional $p(\varphi)$ on Φ, setting

$$p(\varphi) = \sum_{k=1}^{\infty} |(F_k, \varphi)|.$$

From the unconditional convergence of the series $\sum_{k=1}^{\infty} F_k$ it follows that $p(\varphi)$ is finite for every element $\varphi \in \Phi$. Let us show that $p(\varphi)$ is lower semicontinuous. In fact, we have

$$p(\varphi) = \sup_n p_n(\varphi) \tag{2}$$

where $p_n(\varphi) = \sum_{k=1}^{n} |(F_k, \varphi)|$. Since every one of the convex functionals $p_n(\varphi)$ is continuous, then, as was shown in Section 1.1, the functional $p(\varphi)$ is also convex and lower semicontinuous.

By Theorem 2 of Section 1 the functional $p(\varphi)$ is bounded in some neighborhood U of zero in Φ. From this follows the existence of numbers m and M such that

$$p(\varphi) \leqslant M\|\varphi\|_m$$

for every element $\varphi \in \Phi$, which proves the lemma.

Since, in view of the lemma just proven, the inequality

$$|(F_k, \varphi)| \leqslant \sum_{k=1}^{\infty} |(F_k, \varphi)| \leqslant M \| \varphi \|_m$$

holds for any k, then $\| F_k \|_{-m} \leqslant M$. Thus, if $\sum_{k=1}^{\infty} F_k$ is an unconditionally convergent series of functionals on the countably Hilbert space Φ, there exists an m such that the set of numbers $\| F_k \|_{-m}$, $k = 1, 2, \ldots$, is bounded. In particular, every functional F_k belongs to the space Φ'_m.

We now proceed to the proof of Theorem 1. Let the space Φ be nuclear, and $\sum_{k=1}^{\infty} F_k$ be an unconditionally convergent series of linear functionals on Φ. Then, as we have just proven, there exist M and m such that $F_k \in \Phi'_m$ and $\| F_k \|_{-m} \leqslant M$.

Since Φ is by hypothesis nuclear, there is an n such that the operator T^n_m, which takes the space Φ_n into Φ_m, is nuclear. We will show that the series $\sum_{k=1}^{\infty} \| F_k \|_{-n}$ converges.

Since the mapping $\Phi_n \to \Phi_m$ is nuclear, one can choose orthonormal bases $\{\varphi_k\}$ and $\{\psi_k\}$ in Φ_n and Φ_m such that for every element φ of Φ_n one has $\overset{(m)}{\varphi} = \sum_{k=1}^{\infty} \lambda_k (\overset{(n)}{\varphi}, \varphi_k)_n \psi_k$, where $\lambda_k > 0$ and the series $\sum_{k=1}^{\infty} \lambda_k$ converges. Thus, for any $\varphi \in \Phi$ and any linear functional F on Φ one has

$$(F, \varphi) = \sum_{k=1}^{\infty} \lambda_k (\varphi, \varphi_k)_n (F, \psi_k).$$

But since $\| \varphi_k \|_n = 1$, then $| (\varphi, \varphi_k)_n | \leqslant \| \varphi \|_n$, and therefore

$$|(F, \varphi)| \leqslant \sum_{k=1}^{\infty} \lambda_k |(F, \psi_k)| \, \| \varphi \|_n .$$

This means that

$$\| F \|_{-n} = \sup_{\| \varphi \|_n = 1} |(F, \varphi)| \leqslant \sum_{k=1}^{\infty} \lambda_k |(F, \psi_k)| .$$

Applying this inequality to the functionals $F_1, F_2, \ldots, F_j, \ldots$ and summing the resulting inequalities with respect to j, we find that

$$\sum_{j=1}^{\infty} \| F_j \|_{-n} \leqslant \sum_{k=1}^{\infty} \lambda_k \sum_{j=1}^{\infty} |(F_j, \psi_k)|,$$

and so, by inequality (1) we have

$$\sum_{j=1}^{\infty} \| F_j \|_{-n} \leqslant M \sum_{k=1}^{\infty} \lambda_k \| \psi_k \|_m = M \sum_{k=1}^{\infty} \lambda_k .$$

Thus the convergence of the series $\Sigma_{i=1}^{\infty} \| F_j \|_{-n}$ is proved; hence Theorem 1 is proved. Now we show that the converse theorem holds.

Theorem 2. If every unconditionally convergent series $\Sigma_{k=1}^{\infty} F_k$ of linear functionals on a countably Hilbert space Φ is absolutely convergent, then the space Φ is nuclear.

Proof. According to the definition of nuclearity we must show that for any m there is an n such that the mapping T_m^n of the space Φ_n into Φ_m is of Hilbert–Schmidt type. Since the adjoint of an operator of Hilbert–Schmidt type is also of that type, it suffices to exhibit an n for which the mapping $(T_m^n)'$ of the space Φ_m' into Φ_n' is of Hilbert–Schmidt type.

Let us select any orthonormal basis $F_1, F_2, \ldots$ in the Hilbert space Φ_m'. We will show that there is an $n > m$ for which the series $\Sigma_{k=1}^{\infty} \| F_k \|_{-n}^2$ converges; this implies that the operator $(T_m^n)'$, defined by $(T_m^n)' \overset{(m)}{F_k} = \overset{(n)}{F_k}$ is of Hilbert–Schmidt type. In the course of the proof we will make use of a well-known criterion for the convergence of the numerical series $\Sigma_1^{\infty} \mid a_k \mid^2$, namely, if the series $\Sigma_1^{\infty} x_k a_k$ converges for all numerical sequences $\{x_k\}$ satisfying the condition $\Sigma_1^{\infty} \mid x_k \mid^2 < \infty$, then the series $\Sigma_1^{\infty} \mid a_k \mid^2$ converges.

In view of the orthonormality of the basis $\{F_k\}$ in the space Φ_m', the series

$$\sum_{k=1}^{\infty} |(F_k, \varphi)|^2 \tag{3}$$

converges for any element $\varphi \in \Phi_m$. Let us consider any sequence $x = (x_1, x_2, \ldots)$ of numbers for which the series $\Sigma_{k=1}^{\infty} \mid x_k \mid^2$ converges. From the convergence of the series (3) it follows that the series of functionals $\Sigma_{k=1}^{\infty} x_k(F_k, \varphi)$ converges for any φ, i.e., the series $\Sigma_{k=1}^{\infty} x_k F_k$ is unconditionally convergent. By the conditions of the theorem this means that the series $\Sigma_{k=1}^{\infty} x_k F_k$ is absolutely convergent. In other words, there is an r [depending upon the choice of sequence $x = (x_1, x_2, \ldots)$] such that the series

$$\sum_{k=1}^{\infty} \mid x_k \mid \| F_k \|_{-r} \tag{4}$$

converges. Now we denote by H the Hilbert space consisting of all sequences $x = (x_1, x_2, \ldots)$ such that $\Sigma_{k=1}^{\infty} \mid x_k \mid^2$ converges. Setting

$$p_r(x) = \sum_{k=1}^{\infty} \mid x_k \mid \| F_k \|_{-r}$$

we obtain a sequence of functionals $p_1(x)$, $p_2(x)$, ... on H. Since the inequalities

$$\| F \|_{-1} \geqslant \| F \|_{-2} \geqslant \cdots$$

hold for any functional F on Φ, the functionals $p_r(x)$ satisfy the inequalities

$$p_1(x) \geqslant p_2(x) \geqslant \cdots$$

As was shown above, for any x there is an r such that the series (4) converges, i.e., $p_r(x)$ is finite. Since

$$p_r(x) = \sup_j p_{rj}(x),$$

where

$$p_{rj}(x) = \sum_{k=1}^{j} | x_k | \| F_k \|_{-r}$$

are convex and continuous functionals on the space H, then the functional $p_r(x)$ is convex and lower semicontinuous. We now apply Theorem 3 of Section 1 to the functionals $p_r(x)$, obtaining the existence of numbers n and M for which

$$p_n(x) \leqslant M \| x \|,$$

where

$$\| x \|^2 = \sum_{k=1}^{\infty} | x_k |^2.$$

This implies that the series

$$\sum_{k=1}^{\infty} | x_k | \| F_k \|_{-n}$$

converges for every sequence x in H. According to the criterion mentioned above for the convergence of a numerical series $\sum_{k=1}^{\infty} | a_k |^2$, we find that the series $\sum_{k=1}^{\infty} \| F_k \|_{-n}^2$ converges.

Therefore we see that for any orthonormal basis $F_1, F_2, \ldots$ in the space Φ'_m the series $\sum_{k=1}^{\infty} \| F_k \|_{-n}$ converges. In other words, the mapping $(T_m^n)'$ of the space Φ'_m into Φ'_n is of Hilbert–Schmidt type. This proves the nuclearity of the space Φ.

From Theorems 1 and 2 results the following assertion: In order that a countably Hilbert space Φ be nuclear, it is necessary and sufficient that every unconditionally convergent sequence of functionals on Φ be absolutely convergent.

In other words, a nuclear space can be defined as a countably Hilbert space in which every unconditionally convergent series of functionals is absolutely convergent. We remark that the absolute convergence of unconditionally convergent series of functionals holds for every nuclear linear topological space.

Remark. One can similarly prove a slightly different version of the preceding criterion for nuclearity, as follows: In order that a countably Hilbert space Φ be nuclear, it is necessary and sufficient that every one-parameter family f_λ $(a \leqslant \lambda \leqslant b)$ of linear functionals on Φ which satisfies the condition

$$\sup \sum_j |(f_{\lambda_{j+1}}, \varphi) - (f_{\lambda_j}, \varphi)| < C$$

for every $\varphi \in \Phi$, where C depends upon φ, also satisfies for some p and C_1 the condition

$$\sup \sum_j \| f_{\lambda_{j+1}} - f_{\lambda_j} \|_p < C_1.$$

Here the suprema are taken over all partitions $a = \lambda_0 < \lambda_1 < \dots < \lambda_n = b$ of the interval $[a, b]$. This definition of nuclearity played a fundamental role in Chapter IV of Volume III.

3.4. Properties of Nuclear Spaces

We proceed now to establish certain properties of nuclear spaces. The concept of nuclearity will be understood in the sense indicated in Section 3.2, i.e., only countably Hilbert nuclear spaces are considered.

First we show that any closed subspace Ψ of a nuclear countably Hilbert space Φ is a nuclear countably Hilbert space.[5] In fact, it is obvious that any scalar product $(\varphi, \psi)_n$ in the space Φ defines a scalar product in the space Ψ. Moreover, from the completeness of Φ and the assumption that Ψ is closed, it follows that Ψ is a complete space. Consequently, Ψ is a countably Hilbert space. Now we show that Ψ is nuclear. Given any m, there is an n such that the mapping T_m^n of the space Φ_n into Φ_m is nuclear. Then the mapping of Ψ_n [the completion of Ψ relative to the scalar product $(\varphi, \psi)_n$] into Ψ_m induced by the mapping T_m^n is also nuclear. This follows from the fact that for any orthonormal systems $\{\varphi_k\}$ and $\{\chi_k\}$ in the spaces Ψ_n and Ψ_m the series

$$\sum_{k=1}^{\infty} |(T_m^n \varphi_k, \chi_k)_n|$$

converges. This proves our assertion.

[5] This property is valid for any linear topological nuclear space.

Now we show that if Φ is a nuclear space, and Ψ is a closed subspace of Φ for which the factor space Φ/Ψ is complete, then Φ/Ψ is also a nuclear space.[6]

In fact, setting

$$(\varphi + \Psi, \varphi + \Psi)_n = \inf_{\psi \in \Psi} (\varphi + \psi, \varphi + \psi)_n,$$

Φ/Ψ becomes a countably Hilbert space (we leave the verification to the reader). Let us show that it is a nuclear space. To do this we consider an unconditionally convergent series of functionals $\sum_{k=1}^{\infty} f_k$ on the space Φ/Ψ. To each functional f_k there corresponds a functional F_k on the space Φ such that

$$(F_k, \varphi) = (f_k, \varphi + \Psi)$$

for every element $\varphi \in \Phi$. Obviously the series $\sum_{k=1}^{\infty} F_k$ is also unconditionally convergent. In view of the nuclearity of Φ it follows that there exists an m such that the series $\sum_{k=1}^{\infty} \| F_k \|_{-m}$ converges. But

$$\| F_k \|_{-m} = \sup_{\|\varphi\|_m \leqslant 1} |(F_k, \varphi)| = \sup_{\|\varphi\|_m \leqslant 1} |(f_k, \varphi + \Psi)| = \| f_k \|_{-m}.$$

Consequently, the series $\sum_{k=1}^{\infty} \| f_k \|_{-m}$ converges, and therefore $\sum_{k=1}^{\infty} f_k$ is an absolutely convergent series. This proves the nuclearity of Φ/Ψ.

We now show that *every nuclear space Φ is perfect*. In other words, every bounded closed set in a nuclear space Φ is compact. As a matter of fact, let A be a bounded[7] closed set in a nuclear space Φ. We denote by A_n the set A, considered as a set of elements in the Hilbert space Φ_n. In view of the boundedness of A, each set A_n is bounded. Moreover, for any m and n ($n > m$) the equation $T_m^n A_n = A_m$ holds, where T_m^n is the operator imbedding Φ_n into Φ_m. In view of the nuclearity of Φ, for every m there is an $n > m$ such that T_m^n is nuclear and *a fortiori*

[6] Let Ψ be a closed subspace of Φ. A set consisting of every element φ of the form $\varphi = \varphi_0 + \psi$, where φ_0 is a fixed element of Φ and ψ is an element of the subspace Ψ, is called a coset of Φ relative to Ψ. We will often denote this coset by $\varphi_0 + \Psi$. The space Φ/Ψ consisting of all cosets relative to Ψ is itself a linear space, namely, we take

$$(\varphi_1 + \Psi) + (\varphi_2 + \Psi) = \varphi_1 + \varphi_2 + \Psi$$

and

$$\lambda(\varphi_1 + \Psi) = \lambda\varphi_1 + \Psi.$$

If Φ is a normed space, then setting

$$\| \varphi_1 + \Psi \| = \inf_{\psi \in \Psi} \| \varphi_1 + \psi \|,$$

we make Φ/Ψ into a normed space. This space will be complete if Φ is complete.

[7] That is, a set such that for any neighborhood U of zero there is $\lambda > 0$ for which $\lambda A \subset U$.

completely continuous. But then the set A_m, as the image of the bounded set A_n by the operator T_m^n, has compact closure in Φ_m. Thus if $\{\varphi_i\}$ is a sequence of elements lying in A, then setting $m = 1$, we obtain the existence of a subsequence $\varphi_{i_1}, \varphi_{i_2}, \ldots$ which is fundamental in the norm $\| \varphi \|_1$. Taking $m = 2$, we can find a subsequence of $\{\varphi_{i_j}\}$ which is fundamental in the norm $\| \varphi \|_2$. Continuing in this way for all $m = 1, 2, \ldots$ and then taking the diagonal sequence of the resulting infinite family of sequences, we see that this diagonal sequence is fundamental with respect to every one of the norms $\| \varphi \|_n$, and hence with respect to the metric $\| \varphi \|$ defined in the footnote on page 57. Since a countably Hilbert space is by definition complete, the diagonal sequence converges, and its limit is in A since A was assumed closed. Thus A is compact.

From this result it follows that an infinite-dimensional Banach space is not nuclear, because a ball in such a space is bounded but not compact. At the same time any finite-dimensional space is nuclear, because the identity mapping in it is nuclear.

Since every nuclear space is perfect, all the theorems proven in Volume II for perfect spaces are valid for nuclear spaces, namely, the following assertions hold:

(1) Both in a nuclear space Φ and its adjoint space Φ', strong and weak convergence coincide.[8]

(2) If the space Φ is nuclear, then a closed bounded set A in its adjoint space Φ' is compact relative to weak and strong convergence.

(3) A nuclear space is separable (contains an everywhere dense countable set).

(4) A nuclear space is complete relative to weak convergence.

The reader will find the proofs of these assertions (for the more general class of perfect spaces) in Volume II (Chapter I, Section 6).

3.5. Bilinear Functionals on Nuclear Spaces

We have already indicated above that the kernel theorem is valid not only for the space K, but also for many other spaces, for example, the space S, the space $\mathfrak{Z}$ of entire analytic functions, and so forth. In this section we prove a theorem which embraces all of these particular cases. This theorem is formulated in the following way:

[8] A sequence $\{\varphi_k\}$ of elements in a countably normed space Φ is said to be weakly convergent to zero, if $\lim_{k\to\infty} (F, \varphi_k) = 0$ for every linear functional F on Φ. For linear functionals the concept of weak topology (and thus, of weak convergence) was defined on p. 60.

Theorem 3 (*Abstract Kernel Theorem*). Let Φ and Ψ be countably Hilbert spaces, one of which, say Φ, is nuclear, and $B(\varphi, \psi)$, $\varphi \in \Phi$, $\psi \in \Psi$, a bilinear functional, continuous in each of its arguments φ and ψ. Then there are values p and m such that $B(\varphi, \psi)$ can be represented in the form[9]

$$B(\varphi, \psi) = (A\varphi, \psi),$$

where A is a Hilbert–Schmidt operator which maps the Hilbert space Φ_p into the Hilbert space Ψ'_m (Φ_p is the completion of the space Φ in the norm $\| \varphi \|_p = \sqrt{(\varphi, \varphi)_p}$, and Ψ'_m is the adjoint space of Ψ_m).

Proof. Since the functional $B(\varphi, \psi)$ is continuous in each of its arguments φ and ψ, then by Theorem 3 of Section 1 there are norms $\| \varphi \|_n$ and $\| \psi \|_m$ in the spaces Φ and Ψ such that

$$| B(\varphi, \psi)| \leqslant M\| \varphi \|_n \| \psi \|_m, \tag{5}$$

where the value of M does not depend upon φ and ψ. We now introduce the linear operator A_1, mapping the space Φ_n into Ψ'_m and defined by the equation

$$(A_1\varphi, \psi) = B(\varphi, \psi).$$

From relation (5) it follows that

$$|(A_1\varphi, \psi)| = | B(\varphi, \psi)| \leqslant M\| \varphi \|_n \| \psi \|_m.$$

Therefore

$$\| A_1\varphi \|_{-m} = \sup_{\|\psi\|_m = 1} |(A_1\varphi, \psi)| \leqslant M\| \varphi \|_n,$$

and consequently the operator A_1 is bounded relative to the norms $\| \varphi \|_n$ and $\| F \|_{-m}$ in the spaces Φ_n and Ψ'_m.

Since the space Φ is by hypothesis nuclear, there exists a value p such that the operator T^p_n, mapping the space Φ_p into Φ_n and leaving the elements of Φ unchanged, is of Hilbert–Schmidt type. But then the operator $A = A_1 T^p_n$, mapping the space Φ_p into Ψ'_m, is also of that type. Since the operator T^p_n leaves the elements of Φ unchanged (more precisely, since $T^p_n \overset{(p)}{\varphi} = \overset{(n)}{\varphi}$), the bilinear functional $B(\varphi, \psi)$ can be written in the form

$$B(\varphi, \psi) = (A_1\varphi, \psi) = (A_1 T^p_n \varphi, \psi) = (A\varphi, \psi),$$

which proves our assertion.

[9] The symbol $(A\varphi, \psi)$ denotes the value of the functional $A\varphi$ on the element ψ (in distinction from the symbol $(\varphi, \psi)_n$, which denotes the scalar product in the space Φ_n).

We remark that any operator A of Hilbert–Schmidt type, mapping the space Φ_p into the space Ψ'_m, has, according to Theorem 3 of Section 2, the form

$$A\varphi = \sum_{k=1}^{\infty} \lambda_k(\varphi, \varphi_k)_p F_k,$$

where $\{\varphi_k\}$ and $\{F_k\}$ are orthonormal bases in Φ_p and Ψ'_m, $\lambda_k \geqslant 0$, and the series $\sum_{k=1}^{\infty} \lambda_k^2$ converges. Therefore we obtain from Theorem 3:

Corollary. Let Φ and Ψ be countably Hilbert spaces, where Φ is nuclear. Then for any bilinear functional $B(\varphi, \psi)$, $\varphi \in \Phi$, $\psi \in \Psi$, continuous in each argument, there exist m and p such that $B(\varphi, \psi)$ has the form

$$B(\varphi, \psi) = \sum_{k=1}^{\infty} \lambda_k(\varphi, \varphi_k)_p F_k(\psi),$$

where $\{\varphi_k\}$ and $\{F_k\}$ are orthonormal bases in the spaces Φ_p and Ψ'_m, $\lambda_k \geqslant 0$, and the series $\sum_{k=1}^{\infty} \lambda_k^2$ converges.

In certain cases a stronger version of Theorem 3, which says that, instead of being of Hilbert–Schmidt type, A is nuclear, turns out to be useful. Corresponding to this, the convergence of the series $\sum_{k=1}^{\infty} \lambda_k^2$ in the corollary can be replaced by the convergence of the series $\sum_{k=1}^{\infty} \lambda_k$. The proof of this stronger version requires no changes, since we can always take the operator T_n^p to be not only of Hilbert–Schmidt type, but also nuclear.

The kernel theorem can be given another formulation. We introduce the following notation. Let φ and ψ be elements belonging respectively to the Hilbert spaces H_1 and H_2. We associate with the element ψ the linear functional F_ψ on the space H_2, defined by $(F_\psi, \psi_1) = (\psi_1, \psi)_2$, where $\psi_1 \in H_2$ and $(\psi_1, \psi)_2$ is the scalar product in H_2. We denote by $\varphi \otimes \psi$ the linear operator mapping the space H_1 into H'_2, defined by

$$(\varphi \otimes \psi)\chi = (\chi, \varphi)_1 F_\psi,$$

where $(\chi, \varphi)_1$ is the scalar product in H_1. We will denote by $(\varphi \otimes \psi)^*$ the operator mapping H'_2 into H_1 and defined by

$$((\varphi \otimes \psi)^* F, \chi)_1 = (F, (\varphi \otimes \psi)\chi)_{-2},$$

where $F \in H'_2$, $\varphi \in H_1$, and $(F_1, F_2)_{-2}$ is the scalar product in H'_2. The operator $(\varphi \otimes \psi)^*$ can also be defined by the formula

$$((\varphi \otimes \psi)^* F, \chi)_1 = (F, (\chi, \varphi)_1 F_\psi)_{-2}.$$

Obviously the operators $\varphi \otimes \psi$ and $(\varphi \otimes \psi)^*$ are degenerate, and thus of Hilbert–Schmidt type. Therefore if A is any operator of Hilbert–Schmidt type, mapping the space H_1 into H_2', then the operator $(\varphi \otimes \psi)^* A$ is nuclear, and consequently its trace $\mathrm{Tr}[(\varphi \otimes \psi)^* A]$ exists. Setting

$$B(\varphi, \psi) = \mathrm{Tr}[(\varphi \otimes \psi)^* A],$$

we obtain a bilinear functional on the spaces H_1 and H_2. The Hilbert–Schmidt operator A is called the kernel of this bilinear functional.[10]

We shall now prove that in nuclear spaces any bilinear functional is defined by a kernel. In other words, the following theorem holds.

Theorem 4 (*Second Formulation of the Abstract Kernel Theorem*). Let Φ and Ψ be countably Hilbert spaces, where Φ is nuclear. For any bilinear functional $B(\varphi, \psi)$, $\varphi \in \Phi$, $\psi \in \Psi$, continuous in each of its arguments, there exist p, m, and a Hilbert–Schmidt operator A, mapping the space Φ_p into Ψ_m', such that

$$B(\varphi, \psi) = \mathrm{Tr}[(\varphi \otimes \psi)^* A].$$

Here $(\varphi \otimes \psi)^*$ denotes an operator, mapping the space Ψ_m' into Φ_p and defined by the formula

$$((\varphi \otimes \psi)^* F, \chi)_p = (F, (\chi, \varphi)_p F_\psi)_{-m},$$

where $F \in \Psi_m'$, $\chi \in \Phi_p$, $(\varphi, \chi)_p$ is the scalar product in Φ_p, $(F, F_\psi)_{-m}$ is the scalar product in Ψ_m', and F_ψ is a linear functional on Ψ_m defined by the formula $(F_\psi, \psi_1) = (\psi_1, \psi)_m$.

Proof. By Theorem 3 there are p and m such that

$$B(\varphi, \psi) = (A\varphi, \psi),$$

[10] If H_1 and H_2 are spaces of functions with square integrable moduli, then the operator $\varphi \otimes \psi$ is given by the formula

$$(\varphi \otimes \psi)\chi = \overline{\psi(y)} \int \chi(x) \overline{\varphi(x)} \, dx,$$

i.e., it is a degenerate integral operator with kernel $\overline{\varphi(x)\psi(y)}$. In this case the bilinear functional $B(\varphi, \psi)$ has the form

$$B(\varphi, \psi) = \int A(x, y)\varphi(x)\psi(y) \, dx \, dy,$$

where $A(x, y)$, a function having square integrable modulus, is the kernel of the functional $B(\varphi, \psi)$.

where A is a Hilbert–Schmidt operator, mapping the space Φ_p into Ψ'_m. We shall show that

$$B(\varphi, \psi) = \mathrm{Tr}[(\varphi \otimes \psi)^* A].$$

For this we fix $\varphi \in \Phi$ and choose an orthonormal basis $\{\varphi_k\}$ in Φ_p such that $\varphi_1 = \varphi/\| \varphi \|_p$. For brevity we denote the operator $\varphi \otimes \psi$ by T. Obviously

$$\mathrm{Tr}(T^* A) = \sum_{k=1}^{\infty} (T^* A \varphi_k, \varphi_k)_p = \sum_{k=1}^{\infty} (A\varphi_k, T\varphi_k)_{-m}.$$

But for $k \neq 1$

$$T\varphi_k = (\varphi \otimes \psi)\varphi_k = (\varphi_k, \varphi)_p F_\psi = \| \varphi \|_p (\varphi_k, \varphi_1)_p F_\psi = 0,$$

and

$$T\varphi_1 = \| \varphi \|_p F_\psi,$$

and therefore

$$\mathrm{Tr}(T^* A) = (A\varphi_1, \| \varphi \|_p F_\psi)_{-m} = (A\varphi, F_\psi)_{-m}.$$

Since the functional F_ψ is defined by $(F_\psi, \psi_1) = (\psi_1, \psi)_m$, then for any element F of Ψ'_m the relation $(F, F_\psi)_{-m} = (F, \psi)$ holds. Consequently

$$B(\varphi, \psi) = (A\varphi, \psi) = (A\varphi, F_\psi)_{-m} = \mathrm{Tr}(T^* A) = \mathrm{Tr}[(\varphi \otimes \psi)^* A],$$

which proves the theorem.

Applying Theorems 3 and 4 to specific spaces, we obtain for them the kernel theorem. For example, in Section 3.6 it will be proved that the space $K(a)$ of infinitely differentiable functions which vanish for $|x| \geqslant a$ is nuclear. The scalar products which define the topology in this space have the form

$$(\varphi, \psi)_m = \sum_{k=0}^{m} \int \varphi^{(m)}(x) \overline{\psi^{(m)}(x)}\, dx.$$

Applying the corollary of Theorem 3 to the space $K(a)$, we see that any bilinear functional $B(\varphi, \psi)$ on $K(a)$ has the form

$$B(\varphi, \psi) = \sum_{k=1}^{\infty} \lambda_k (\varphi, \varphi_k)_p (F_k, \psi), \tag{6}$$

where $\{\varphi_k\}$ is an orthonormal basis in the space $K_p(a)$,[11] $\{F_k\}$ is an orthonormal basis in the space $K'_m(a)$, $\lambda_k \geqslant 0$, and the series $\sum_{k=1}^{\infty} \lambda_k$ converges.

[11] We denote by $K_p(a)$ the completion of the space $K(a)$ in the norm $\| \varphi \|_p = \sqrt{(\varphi, \varphi)_p}$, and by $K'_m(a)$ the adjoint space of $K_m(a)$.

We shall now show that the bilinear functional $B(\varphi, \psi)$ can be written in the form

$$B(\varphi, \psi) = (F, \varphi(x)\psi(y)),$$

where F is a linear functional on the space $K_2(a)$ of infinitely differentiable functions which vanish outside the square $|x| \leqslant a$, $|y| \leqslant a$. To do this, we associate with the function $\varphi_k(x) \in K_p(a)$ and the functional $F_k \in K'_m(a)$ a functional on the space $K_2(a)$, which we denote by $F_k \times \varphi_k$ and define by the formula

$$(F_k \times \varphi_k, \varphi(x, y)) = (F_k, (\varphi(x, y), \varphi_k)_p).$$

This functional is defined for every function $\varphi(x, y) \in K_2(a)$. In fact, the function $\psi_k(y) = (\varphi(x, y), \varphi_k)_p$ has the form

$$\psi_k(y) = \sum_{j=0}^{p} \int \frac{\partial^j \varphi(x, y)}{\partial x^j} \overline{\varphi_k^{(j)}(x)}\, dx.$$

As $\varphi(x, y)$ is infinitely differentiable and $\varphi_k(x) \in K_p(a)$, it follows that $\psi_k(y)$ is defined for every value of y. Moreover, it is obvious that $\psi_k(y)$ is infinitely differentiable and vanishes for $|y| \geqslant a$. Therefore the expression

$$(F_k \times \varphi_k, \varphi(x, y)) = (F_k, \psi_k)$$

is meaningful. A simple estimate shows that

$$|(F_k \times \varphi_k, \varphi(x, y))| \leqslant \| F_k \|_{-m} \| \varphi_k \|_p \| \varphi(x, y) \|_{mp},$$

where $\| \varphi(x, y) \|_{mp}$ denotes the expression

$$\left[\sum_{i=0}^{m} \sum_{j=0}^{p} \int \left| \frac{\partial^{i+j} \varphi(x, y)}{\partial x^j\, \partial y^i} \right|^2 dx\, dy\right]^{\frac{1}{2}}.$$

Since by hypothesis $\| F_k \|_{-m} = \| \varphi_k \|_p = 1$, for every k we have the inequality

$$|(F_k \times \varphi_k, \varphi(x, y))| \leqslant \| \varphi(x, y) \|_{mp}.$$

From this it follows that for every function $\varphi(x, y) \in K_2(a)$ the series

$$\sum_{k=1}^{\infty} \lambda_k (F_k \times \varphi_k, \varphi(x, y))$$

converges, and it is easily seen that this series defines a continuous linear functional on the space $K_2(a)$. We denote this functional by F. Obviously formula (6) can be written, by means of F, in the form

$$B(\varphi, \psi) = (F, \varphi(x)\psi(y)). \tag{7}$$

But this means that any bilinear functional on the space $K(a)$, continuous in each argument, can be written in the form (7), where F is a continuous linear functional on the space $K_2(a)$. Thus we have obtained a new proof of the kernel theorem for the space $K(a)$.

The kernel theorem for the spaces S and $\mathfrak{Z}$ is proven in a completely analogous way.

In conclusion we mention yet another theorem, closely related to Theorem 3.

Theorem 5. Let A be a linear operator, mapping the nuclear space Φ into the adjoint space Ψ' of a countably Hilbert space Ψ. If A is continuous relative to the topology in Φ and the weak topology in Ψ', then there exist p and m such that A is of Hilbert–Schmidt type relative to the norms $\| \varphi \|_p$ and $\| F \|_{-m}$ in the spaces Φ and Ψ'.

Proof. By Theorem 3′ of Section 1 there exist n and m such that the operator A is continuous relative to the norms $\| \varphi \|_n$ and $\| F \|_{-m}$. In other words, the operator A_n, induced by A on the space Φ_n, is a continuous mapping of Φ_n into Ψ'_m. But in view of the nuclearity of Φ there exists p such that the mapping T_n^p of Φ_p into Φ_n is of Hilbert–Schmidt type. Obviously $A_p = A_n T_n^p$, where A_p is the operator induced by A on the space Φ_p. Since A_p is the product of a Hilbert–Schmidt operator T_n^p and a continuous operator A_n, it is itself of Hilbert–Schmidt type. This proves the theorem.

In the same way one proves the dual theorem.

Theorem 6. Let A be a continuous linear mapping of a countably Hilbert space Φ into the adjoint space Ψ' of a nuclear space Ψ (the space Ψ' is considered in the weak topology). Then there exist p and m such that A is of Hilbert–Schmidt type relative to the norms $\| \varphi \|_m$ and $\| F \|_{-p}$ in the spaces Φ and Ψ'.

3.6. Examples of Nuclear Spaces[12]

We have established various properties of nuclear spaces. The question naturally arises: To which specific spaces are our results applicable, i.e., which specific spaces have the property of nuclearity?

[12] Sections 3.6—3.8 can be omitted at the first reading.

First we show that the space $\hat{K}(a)$ of periodic infinitely differentiable functions $\varphi(x)$ of period $2a$ is nuclear.[13] We define the norms in this space by

$$\| \varphi \|_n^2 = \sum_{0 \leqslant q \leqslant n} \int_{-a}^{a} | \varphi^{(q)}(x)|^2 \, dx.$$

Obviously $\hat{K}(a)$ is a countably Hilbert space. To show that it is nuclear, we note that the functions $\psi_m(x) = e^{\pi i m x/a}$ form an orthogonal (but not normalized) basis in $\hat{K}(a)$ with respect to every one of the scalar products

$$(\varphi, \psi)_n = \sum_{0 \leqslant q \leqslant n} \int_{-a}^{a} \varphi^{(q)}(x) \overline{\psi^{(q)}(x)} \, dx,$$

and

$$\| \psi_m \|_n^2 = 2a \sum_{0 \leqslant q \leqslant n} \left(\frac{\pi m}{a} \right)^q$$

It follows at once that the imbedding operator of $\hat{K}_{n+2}(a)$ [the completion of $\hat{K}(a)$ with respect to the norm $\| \varphi \|_{n+2}$] into $\hat{K}_n(a)$ is positive definite and has finite trace, equal to

$$\sum_{m=-\infty}^{\infty} \frac{\| \psi_m \|_n}{\| \psi_m \|_{n+2}} .$$

But this means that the space $\hat{K}(a)$ is nuclear.

In view of the results proven in Section 3.4, it follows that the space $K(a)$, which is the closed subspace of $\hat{K}(a)$ consisting of all functions in $\hat{K}(a)$ which vanish, together with all their derivatives, for $| x | = a$, is nuclear. The space $\tilde{K}(a)$ of functions which are infinitely differentiable on the interval $[-a, a]$ (and which have one-sided derivatives of all orders at the end points) is also nuclear. Indeed, the completion of $\tilde{K}(a)$ relative to the norm $\| \varphi \|_n$ differs from the completion of $K(a)$ relative to the same norm only by a finite-dimensional direct component (i.e., it is isomorphic to the direct sum of the completion of $K(a)$ (relative to $\| \varphi \|_n$) and a finite-dimensional space).

We now describe a very general method of obtaining new nuclear spaces, starting from a given nuclear space. Let Φ be a countably Hilbert nuclear space, and $M = \| m_{np} \|$ an infinite matrix consisting of nonnegative numbers such that

(1) $0 < m_{np} \leqslant m_{n+1,p}$ and $m_{np} \leqslant m_{n,p+1}$.

[13] For the sake of simplicity we will carry out all considerations for functions of one variable. The passage to functions of several variables leads only to a somewhat more involved notation. The results, however, remain valid for functions of several variables.

(2) For any k there exists p such that the series $\sum_{n=1}^{\infty} m_{nk}/m_{np}$ converges, and its terms are monotonically decreasing.

We form a new space $\Phi(M)$ whose elements are sequences

$$\hat{\varphi} = (\varphi_1, \varphi_2, \ldots)$$

of elements of Φ such that for any p the series

$$\| \hat{\varphi} \|_p^2 = \sum_{n=1}^{\infty} m_{np} \| \varphi_n \|_p^2$$

converges. Let us show that $\Phi(M)$ is a nuclear space. This space is obviously countably Hilbert relative to the scalar products

$$(\hat{\varphi}, \hat{\psi})_p = \sum_{n=1}^{\infty} m_{np}(\varphi_n, \psi_n)_p.$$

Further, for any k there is p such that the imbedding operator of Φ_p into Φ_k [14] has finite Hilbert–Schmidt norm and the series $\sum_{n=1}^{\infty}(m_{nk}/m_{np})^2$ converges. But then it is easily seen that the imbedding operator of $\Phi(M)_p$ into $\Phi(M)_k$ is likewise an operator of Hilbert–Schmidt type. From this follows at once the nuclearity of $\Phi(M)$.

In view of the fact that the space $\tilde{K}(a)$ of infinitely differentiable functions on the interval $[-a, a]$ is nuclear, we obtain from the result just proven the following corollary:

If the matrix $M = \| m_{np} \|$ has the properties indicated above, then the space $\tilde{K}(M)$ of functions $\varphi(x)$, $-\infty < x < \infty$, infinitely differential on every interval $[n, n+1]$, having one-sided derivatives of all orders at every integer point, and such that for any p the series

$$\sum_{n=-\infty}^{\infty} m_{np} \int_n^{n+1} | \varphi^{(q)}(x)|^2 \, dx, \qquad 0 \leqslant q \leqslant p,$$

converge, is nuclear.

We can now deduce a criterion for the nuclearity of the spaces $K\{M_p\}$, which were introduced in Volume II (Chapter II, Section 1.2). Let us recall the definition of these spaces. Suppose that

$$1 \leqslant M_1(x) \leqslant M_2(x) \leqslant \ldots \tag{8}$$

[14] Φ_p is the completion of Φ relative to the norm $\| \varphi \|_p$.

is an increasing sequence of functions. Then $K\{M_p\}$ consists (by definition) of all infinitely differentiable functions $\varphi(x)$ for which every function of the form

$$|M_p(x)\varphi^{(k)}(x)|, \qquad 1 \leqslant p < \infty, \quad 0 \leqslant k \leqslant p$$

is bounded. The norms in $K\{M_p\}$ are defined by

$$\|\varphi\|_p = \max_{0\leqslant k\leqslant p} \sup_x |M_p(x)\varphi^{(k)}(x)|. \tag{9}$$

We will also assume that the functions $M_p(|x|)$ are monotonically increasing and that for any n and k there are p and C such that

$$|M_n^{(k)}(x)| \leqslant CM_p(x) \tag{10}$$

(these conditions are fulfilled in all cases of interest).

We will say that a space $K\{M_p\}$ satisfies condition (N), if for any n there is p such that the ratio

$$m_{np}(x) = \frac{M_n(x)}{M_p(x)}$$

goes to zero as $|x| \to \infty$ and is a summable function of x.

We prove the following theorem:

Theorem 7. If the space $K\{M_p\}$ satisfies condition (N), then it is nuclear.

Proof. First we prove that the space $K\{M_p\}$ is countably Hilbert. For this we introduce in $K\{M_p\}$ a countable collection of scalar products, setting

$$(\varphi, \psi)_p = \int [M_p(x)]^2 \sum_{0\leqslant q\leqslant p} \varphi^{(q)}(x)\overline{\psi^{(q)}(x)}\, dx. \tag{11}$$

We shall show that the topology in $K\{M_p\}$, defined by the norms $\|\varphi\|_p$ (see formula (9)), coincides with the topology defined by the norms $\|\varphi\|_p' = \sqrt{(\varphi, \varphi)_p}$. In fact, for $q \leqslant n < p$ we have

$$\int [M_n(x)]^2 |\varphi^{(q)}(x)|^2\, dx \leqslant \sup_x \{[M_p(x)]^2 |\varphi^{(q)}(x)|^2\} \int \left[\frac{M_n(x)}{M_p(x)}\right]^2 dx.$$

But by condition (N), p can be chosen so that the integral

$$\int \frac{M_n(x)}{M_p(x)}\, dx = \int m_{np}(x)\, dx$$

converges. Since $\lim_{|x|\to\infty} m_{np}(x) = 0$, the integral $\int m_{np}^2(x)\, dx$ converges. We denote its value by B_{np}^2. Then for $q \leqslant n < p$ we obtain

$$\int [M_n(x)]^2 |\varphi^{(q)}(x)|^2\, dx \leqslant B_{np}^2 \|\varphi\|_p^2.$$

From this it follows that

$$\|\varphi\|_n'^2 = \sum_{0\leqslant q\leqslant n} \int [M_n(x)]^2 |\varphi^{(q)}(x)|^2\, dx \leqslant C^2 \|\varphi\|_p^2,$$

i.e., $\|\varphi\|_n'^2 \leqslant C^2 \|\varphi\|_p^2$, where C is a constant not depending upon $\varphi(x)$.

We now estimate $\|\varphi\|_n$ in terms of $\|\varphi\|_p'$. Let $0 \leqslant q \leqslant n$. Then for any x we have†

$$|M_n(x)\varphi^{(q)}(x)| \leqslant \int_{-\infty}^{\infty} |(M_n(x)\varphi^{(q)}(x))'|\, dx.$$

In view of (10) there are p and C, not depending upon $\varphi(x)$, such that

$$\int_{-\infty}^{\infty} |(M_n(x)\varphi^{(q)}(x))'|\, dx \leqslant C \int_{-\infty}^{\infty} M_p(x)(|\varphi^{(q)}(x)| + |\varphi^{(q+1)}(x)|)\, dx.$$

It follows then from the Bunyakovski–Schwartz inequality that for any x we have

$$|M_n(x)\varphi^{(q)}(x)|^2 \leqslant 2C^2 \int_{-\infty}^{\infty} \left[\frac{M_p(x)}{M_s(x)}\right]^2 dx$$
$$\times \left(\int_{-\infty}^{\infty} |M_s(x)\varphi^{(q)}(x)|^2\, dx + \int_{-\infty}^{\infty} |M_s(x)\varphi^{(q+1)}(x)|^2\, dx\right).$$

But by condition (N) there is an s such that

$$\int_{-\infty}^{\infty} \left[\frac{M_p(x)}{M_s(x)}\right]^2 dx$$

converges, and we may certainly suppose that $s \geqslant q + 1$. Denoting the value of this integral by E^2, we obtain

$$\sup_x |M_n(x)\varphi^{(q)}(x)|^2 \leqslant 2C^2E^2(\|\varphi\|_s')^2.$$

† It is easily seen that condition (N) plus the requirement that every function $|M_p(x)\varphi^{(k)}(x)|$, $1 \leqslant p < \infty$, $0 \leqslant k \leqslant p$, be bounded implies that in fact these functions tend to zero as $|x| \to \infty$.

Since such inequalities hold for all $q \leqslant n$, we can find C_1 and s_1 such that

$$\| \varphi \|_n \leqslant C_1 \| \varphi \|'_{s_1}.$$

From this inequality and the previously proven inequality

$$\| \varphi \|'_n \leqslant C \| \varphi \|_p,$$

it follows that the system of norms $\| \varphi \|'_n$ defines the same topology in $K\{M_p\}$ as does the system of norms $\| \varphi \|_n$. Thus, if conditions (8), (10), and (N) are fulfilled, then $K\{M_p\}$ is a countably Hilbert space.

Let us now prove that $K\{M_p\}$ is nuclear. For this we note that the system of norms $\| \varphi \|'_p$ is equivalent to the system of norms†

$$||| \varphi |||_p^2 = \sum_{0 \leqslant q \leqslant p} \sum_{n=-\infty}^{\infty} m_{np} \int_n^{n+1} | \varphi^{(q)}(x)|^2 \, dx,$$

where we have put

$$m_{np} = \sup_{n \leqslant x \leqslant n+1} M_p(x).$$

It is not hard to see that in view of condition (N) and the conditions (8), (10) on the $M_p(x)$, the system of numbers m_{np} has the properties formulated earlier. Therefore the space $\tilde{K}(M)$ corresponding to the system $M = \| m_{np} \|$ is nuclear. But the space $K\{M_p\}$ is a closed subspace of $\tilde{K}(M)$ and therefore is also nuclear. This proves the theorem.

The class of spaces of type $K\{M_p\}$ and satisfying condition (N) is quite extensive. For example, the space S belongs to it. In this case $M_p(x) = (1 + x^2)^p$. Also belonging to this class is the space $S_{\alpha,A}$, consisting of those infinitely differentiable functions for which the integrals

$$\int \exp (a(1 - 1/p) | x |^{1/\alpha}) \varphi^{(q)}(x) \, dx, \qquad 0 \leqslant q \leqslant p,$$

$$p = 2, 3, \ldots; \qquad a = \frac{\alpha}{eA^{1/\alpha}}$$

converge. In this case $M_p(x) = \exp (a(1 - 1/p) \mid x \mid^{1/\alpha})$ and, as is easy to verify, condition (N) is fulfilled. Thus the kernel theorem holds for such spaces.

† This assertion is not apparent in the general case, but it is easy to prove for the particular spaces considered below. We mention that two systems of norms are said to be equivalent, if each member of each system is bounded by some multiple of a member of the other system, as was the case with the systems $\{\| \varphi \|_n\}$ and $\{\| \varphi \|'_n\}$.

Let us indicate a class of spaces of sequences, which are analogous to the space $K\{M_p\}$. Suppose that $\| m_{pq} \|$, $1 \leqslant p, q < \infty$, is a matrix consisting of elements m_{pq} such that

$$1 \leqslant m_{1q} \leqslant m_{2q} \leqslant \dots$$

for every q. We denote by $k\{m_{pq}\}$ the space of sequences $x = (x_1, x_2, \dots)$ such that for every p the sequence

$$(m_{p1}x_1, m_{p2}x_2, \dots)$$

is bounded. Obviously $k\{m_{pq})$ is a linear space. If we set

$$\| x \|_p = \sup_q | m_{pq}x_q |,$$

then $k\{m_{pq}\}$ is a countably normed space. We say that the space $k\{m_{pq}\}$ satisfies condition (N) if for each n there exists p such that the series

$$\sum_{q=1}^{\infty} \frac{m_{nq}}{m_{pq}}$$

converges.

One can show, by means of considerations analogous to those used above, that a space $k\{m_{pq}\}$ satisfying condition (N) is nuclear. In particular, the space s of rapidly decreasing sequences[15] is nuclear. For this space $m_{pq} = q^p$ and, as is easily verified, condition (N) is fulfilled.

From the nuclearity of the space s follows the nuclearity of the space 3 of entire analytic function, in which norms are defined by the formulas

$$\| \varphi \|_n = \sup_{|z|=n} | \varphi(z) |.$$

In fact, it can be established without difficulty that the coefficients $\varphi^{(n)}(0)/n!$ of the Taylor series of the entire analytic function $\varphi(z)$ form a rapidly decreasing sequence and, conversely, if the sequence of numbers $a_0, a_1, \dots$ is rapidly decreasing, then

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$$

[15] A sequence $x = (x_1, x_2, \dots)$ is called rapidly decreasing, if for every p the relation $\lim_{q\to\infty} | q^p x_q | = 0$ holds.

is an entire analytic function. Moreover, applying the well-known Cauchy inequalities for the coefficients of a Taylor series, we conclude that the correspondence

$$\varphi(z) \to \left\{\varphi(0), \ldots, \frac{\varphi^{(n)}(0)}{n!}, \ldots\right\}$$

is an isomorphism between the spaces $\mathfrak{Z}$ and s. Therefore, in view of the nuclearity of s, the space $\mathfrak{Z}$ is also nuclear.

We remark, in conclusion, that B. S. Mityagin proved the nuclearity of the spaces S_α^β, which were introduced in Volume II, Chapter IV.

3.7. The Metric Order of Sets in Nuclear Spaces[16]

Let A be a set in a metric space R. A set B in R is called an *ϵ-set* for the set A, if for each $x \in A$ there is a point $b \in B$ whose distance from x is less than ϵ, $\epsilon > 0$. Obviously, the balls of radius ϵ with centers at the points of an ϵ-set for A cover the set A.

If the set A is compact, then for any $\epsilon > 0$ there exists a finite ϵ-set for A. We denote by $N(\epsilon, A)$ the smallest number of elements in the ϵ-sets for the set A. Obviously, $N(\epsilon, A)$ is a nonincreasing function of ϵ, and if the set A is infinite, then $\lim_{\epsilon\to 0} N(\epsilon, A) = \infty$.

Many properties of the space R can be expressed in terms of the function $N(\epsilon, R)$. For example, a Banach space is finite dimensional if and only if the function $N(\epsilon, A)$ has power growth, as $\epsilon \to 0$, for any compact set A. In other words, a necessary and sufficient condition for the finite dimensionality of a Banach space R is that for any compact set A in R we have

$$N(\epsilon, A) < C\left(\frac{1}{\epsilon}\right)^n,$$

where C depends upon A. Here n is the dimension of R.

In this section we will characterize a nuclear countably Hilbert space by means of ϵ-sets.

First we introduce the concept of an ϵ-set for any linear topological space.

Let Φ be a linear topological space and U a neighborhood of zero in Φ. We call a set B an *ϵ-set for A relative to the neighborhood U of zero*, if for

[16] The results of Sections 3.7–3.8 have as their origin ideas of A. N. Kolmogorov on the ϵ-entropy and ϵ-capacity of compacta in functional spaces. Cf. references (33–35).

any element $\varphi \in A$ there is an element $\psi \in B$ such that $\varphi \in \psi + \epsilon U$ (ϵU denotes the collection of elements of the form $\epsilon\chi$, $\chi \in U$). We denote by $N(\epsilon, A, U)$ the smallest number of elements in the ϵ-sets for A relative to U. If the set A is compact (or has compact closure in Φ), then $N(\epsilon, A, U)$ is finite for every neighborhood U of zero and every $\epsilon > 0$.

We consider a perfect countably Hilbert space $\Phi = \bigcap_{k=1}^{\infty} \Phi_k$ and denote by U_m the unit ball in the space Φ_m. If $k < m$, then U_m has compact closure† relative to the norm $\| \varphi \|_k$ and therefore $N(\epsilon, U_m, U_k)$ is finite for any $\epsilon > 0$. For brevity of notation we denote $N(\epsilon, U_m, U_k)$ by $r_{km}(\epsilon)$.

Thus, with each perfect countably Hilbert space we have associated a family of functions $r_{km}(\epsilon)$, $1 \leqslant m < \infty$, $k < m$.

We will give a necessary and sufficient condition for the nuclearity of a space Φ, which is expressed in terms related to the rapidity of growth of the functions $r_{km}(\epsilon)$. Since a given topology in Φ can be defined by various collections of scalar products, other conditions expressed in different terms will be given on page 94.

Let us recall the following definition. Suppose $f(x)$ is a monotonically increasing function of x. The infimum ρ of all numbers μ, such that for some C (depending upon μ) the inequality

$$f(x) \leqslant C \exp (x^{\mu})$$

holds, is called the *order of growth* of the function $f(x)$. If $\rho = 0$, then

† This property, which is clearly equivalent to the assumption that any set $A \subset \Phi$ which is bounded relative to the norm $\| \varphi \|_m$ has compact closure in Φ_{m-1}, appears to be more restrictive than the assumption that Φ is perfect. The following remarks should clarify this assertion.

(1) If $p_1 < p_2 < \ldots$ is an increasing sequence of positive integers, then the topology in a countably normed space Φ is unaltered if we replace the family of norms $\{\| \varphi \|_i\}$ by the subfamily $\{\| \varphi \|_{p_i}\}$. This is obvious from the monotonicity $\| \varphi \|_1 \leqslant \| \varphi \|_2 \leqslant \ldots$ and the definition of the topology in Φ.

(2) If $p_1 < p_2 < \ldots$ is as above, and has the property that any set $A \subset \Phi$ which is bounded relative to the norm $\| \varphi \|_{p_{i+1}}$ has compact closure in Φ_{p_i}, then Φ is a perfect space. The proof of this is practically identical with the proof, on p. 72, that a nuclear space is perfect.

However, this property is entirely subsumed, in the proof in Theorem 8 of the necessity of (28), by the nuclearity of Φ. As for the proof of sufficiency, it appears to us that setting $\rho_{km} = \infty$ if U_m does not have compact closure relative to the norm $\| \varphi \|_k$ [which is just what (12) would give anyhow] permits it to be carried through with only a slight modification. Indeed, it suffices to observe, first, that $\rho_{km} < \infty$ implies $r_{km}(\epsilon) < \infty$ for all $\epsilon > 0$, hence that U_m has compact closure in Φ_k, so that T_k^m is completely continuous and one can speak about "the semiaxes of the ellipsoid U_m in Φ_k," and second, that by the monotonicity of the norms $\| \varphi \|_j$ one has $\rho_{km} \leqslant \rho_{j,j+1}$ if $k \leqslant j < k \leqslant j < m$.

$f(x)$ is called a *function of minimal order of growth.* In this case, for any $\mu > 0$ there exists C such that

$$f(x) \leqslant C \exp(x^{\mu})$$

From the definition of the order of growth it follows that

$$\rho = \overline{\lim_{x \to \infty}} \frac{\ln \ln f(x)}{\ln x}.$$

If Φ is a perfect countably Hilbert space, we will denote by ρ_{km} the order of growth of the functions $r_{km}(\epsilon)$ (considered as functions of ϵ^{-1}). In other words, we will set

$$\rho_{km} = \overline{\lim_{\epsilon \to 0}} \frac{\ln \ln N(\epsilon, U_m, U_k)}{\ln \epsilon^{-1}}. \tag{12}$$

Henceforth, we will call the number

$$\overline{\lim_{\epsilon \to 0}} \frac{\ln \ln N(\epsilon, A, U)}{\ln \epsilon^{-1}}$$

the *metric order of the set A relative to the neighborhood U of zero.* Thus ρ_{km} is the metric order of the ellipsoid U_m relative to the ball U_k, $k < m$.

The basic theorem of this section asserts the following: In order that a perfect countably Hilbert space $\Phi = \bigcap_{k=1}^{\infty} \Phi_k$ be nuclear, it is necessary and sufficient that

$$\sum_{j=p+1}^{\infty} \frac{1}{\rho_{j,j+1}} = \infty \tag{13}$$

for any $p > 0$ (if $\rho_{j,j+1} = 0$, then we set $1/\rho_{j,j+1} = \infty$).

For the proof of this theorem we need a certain new concept. Let $a_1, a_2, \ldots$ be a sequence of positive numbers converging to zero. We call the infimum λ, of those values μ for which the series $\sum_{n=1}^{\infty} a_n^{\mu}$ converges, the *exponent of convergence* of the sequence $\{1/a_n\}$. Thus, if the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lambda \leqslant 1$; if $\lambda < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.

To compute the exponent of convergence we can use the formula

$$\lambda = \overline{\lim_{\epsilon \to 0}} \frac{\ln n(\epsilon)}{\ln \epsilon^{-1}}, \tag{14}$$

where $n(\epsilon)$ denotes the number of terms of the sequence $\{a_n\}$ which are greater than ϵ.[17]

We return now to the consideration of perfect countably Hilbert spaces. Let U_m be the unit ball in the space Φ_m, the completion of the space Φ in the norm $\sqrt{(\varphi, \varphi)_m}$. Considered in the space Φ_k, where $k < m$, this ball is an ellipsoid. The sequence of lengths $a_1, a_2, \ldots$ of the semiaxes of this ellipsoid converges to zero, since the ellipsoid U_m is compact in Φ_k in view of the fact that Φ is perfect. We denote by λ_{km} the exponent of convergence of the sequence $\{1/a_n\}$. For brevity λ_{km} will be called the *exponent of convergence of the ellipsoid U_m in the space Φ_k.*

The following lemma lies at the basis of the proof of the theorem formulated above:

Lemma 2. The exponent of convergence λ_{km} of the ellipsoid U_m in the space Φ_k satisfies the inequality

$$\lambda_{km} \leqslant \rho_{km} \equiv \overline{\lim_{\epsilon \to 0}} \frac{\ln \ln N(\epsilon, U_m, U_k)}{\ln \epsilon^{-1}}, \tag{15}$$

where ρ_{km} is the order of growth of the function $r_{km}(\epsilon) \equiv N(\epsilon, U_m, U_k)$. If $\sum_{n=1}^{\infty} a_n < \infty$ (in particular, if $\lambda_{km} < 1$), then we also have the inequality

$$\rho_{km} \leqslant 2\lambda_{km}. \tag{16}$$

Proof. First we prove that

$$\ln n(e\epsilon) \leqslant \ln \ln r_{km}(\epsilon) \tag{17}$$

where $n(e\epsilon)$ denotes the number of terms in the sequence $\{a_n\}$ which are not less than $e\epsilon$. Without loss of generality, we may suppose that the sequence $\{a_n\}$ is nonincreasing, i.e., that $a_1 \geqslant a_2 \geqslant \ldots$.

We intersect the ellipsoid U_m with the finite-dimensional subspace spanned by the semiaxes $a_1, \ldots, a_n$, $n \equiv n(e\epsilon)$. The volume of the intersection equals $T_n \prod_{j=1}^{n} a_j$, where T_n is the volume of a unit n-dimensional ball.

Since the volume of an n-dimensional ball of radius ϵ equals $T_n \epsilon^n$,

[17] The proof of this assertion is carried out, for example, in the book of B. Ja. Levin, "Distribution of Zeroes of Entire Functions," Chapter 1, §4. Moscow, 1956. English translation by Amer. Math. Soc., Providence, Rhode Island, 1963.

in order to cover the ellipsoid U_m by translates of the ball ϵU_k we need at least $\prod_{j=1}^{n(e\epsilon)} (a_j/\epsilon)$ such translates. From this it follows that

$$r_{km}(\epsilon) \geqslant \prod_{j=1}^{n(e\epsilon)} \frac{a_j}{\epsilon}. \tag{18}$$

Since for $1 \leqslant j \leqslant n(e\epsilon)$ we have $a_j/\epsilon > e$, then from inequality (18) it follows that

$$r_{km}(\epsilon) \geqslant e^{n(e\epsilon)}.$$

Therefore

$$\ln \ln r_{km}(\epsilon) \geqslant \ln n(e\epsilon) \tag{19}$$

From this it follows that

$$\rho_{km} = \overline{\lim_{\epsilon \to 0}} \frac{\ln \ln r_{km}(\epsilon)}{\ln \epsilon^{-1}} \geqslant \overline{\lim_{\epsilon \to 0}} \frac{\ln n(e\epsilon)}{\ln \epsilon^{-1}} = \overline{\lim_{\epsilon \to 0}} \frac{\ln n(\epsilon)}{\ln \epsilon^{-1}} = \lambda_{km}.$$

Thus the relation $\rho_{km} \geqslant \lambda_{km}$ is proved.

We proceed to the proof of inequality (16). Let $\lambda_{km} < 1$. Then the series $\sum_{j=1}^{\infty} a_j$ converges. Denote its sum by a. If $n \geqslant n(\epsilon^2/4a)$, then we have

$$\sum_{j=n+1}^{\infty} a_j^2 \leqslant a_{n+1} \sum_{j=n+1}^{\infty} a_j \leqslant \tfrac{1}{4}\epsilon^2. \tag{20}$$

Now we will prove that

$$r_{km}(\epsilon) \leqslant \prod_{k=1}^{p} \frac{2[a_k\sqrt{p} + \epsilon]}{\epsilon},$$

where p denotes the number $n(\epsilon^2/4a)$. For this we consider the intersection $U_{m.p}$ of the ellipsoid U_m and the subspace H_p in Φ_k, spanned by the semiaxes $a_1, \ldots, a_p$. We construct, in the space H_p, a cubical lattice with mesh $\epsilon_1 = \epsilon/\sqrt{p}$, choosing as coordinate axes the axes of the ellipsoid $U_{m.p}$.

In view of the choice of ϵ_1, any point of the subspace H_p lies within a distance not exceeding $\frac{1}{2}\epsilon$ of the nearest point of this lattice. Similarly, any point of the ellipsoid $U_{m.p}$ lies at a distance not exceeding $\frac{1}{2}\epsilon$ from one of the lattice points lying in the parallelpiped T_p:

$$-\tfrac{1}{2}\epsilon_1 - a_k \leqslant x_k \leqslant \tfrac{1}{2}\epsilon_1 + a_k, \qquad 1 \leqslant k \leqslant p.$$

Obviously the number of lattice points in this parallelpiped is not greater than

$$\prod_{k=1}^{p} 2\left(\frac{a_k}{\epsilon_1}+1\right)=\prod_{k=1}^{p}\frac{2(a_k\sqrt{p}+\epsilon)}{\epsilon}$$

and, since $a_1 \geqslant a_k$, is not greater than

$$\left[\frac{2}{\epsilon}(a_1\sqrt{p}+1)\right]^p.$$

Let us show that the balls in Φ_k of radius ϵ with centers at those lattice points lying in the parallelpiped T_p cover the ellipsoid U_m. In fact, from inequality (20) it follows that each point A of the ellipsoid U_m is at a distance not exceeding $\frac{1}{2}\epsilon$ from the ellipsoid $U_{m,p}$. Thus $AB \leqslant \frac{1}{2}\epsilon$, where B is some point of $U_{m,p}$. But as was seen above, there is a lattice point C, belonging to the parallelpiped T_p, such that $BC < \frac{1}{2}\epsilon$. Consequently, $AC < \epsilon$ and the point A belongs to the ball of radius ϵ with center at the point C.

Therefore we have constructed $p_1 = [2\epsilon^{-1}(a_1\sqrt{p}+1)]^p$ points $x_1, \ldots, x_{p_1}$ such that

$$U_m \subset \bigcup_{j=1}^{p_1}(x_j+\epsilon U_k).$$

Therefore

$$r_{km}(\epsilon) = N(\epsilon, U_m, U_k) \leqslant \left[\frac{2}{\epsilon}(a_1\sqrt{p}+1)\right]^p.$$

Taking into consideration that $p = n(\epsilon^2/4a)$, we obtain from this the inequality

$$\ln\ln r_{km}(\epsilon) \leqslant \ln n\left(\frac{\epsilon^2}{4a}\right)+\ln\ln\frac{1}{\epsilon}+\ln\ln[2(a_1\sqrt{n(\epsilon^2/4a)}+1)]. \quad (21)$$

Since

$$\lim_{\epsilon\to 0}\frac{\ln(\epsilon^2/4a)}{\ln\epsilon}=2$$

we have

$$\rho_{km}=\overline{\lim_{\epsilon\to 0}}\frac{\ln\ln r_{km}(\epsilon)}{\ln\epsilon^{-1}} \leqslant \overline{\lim_{\epsilon\to 0}}\frac{\ln\ln n(\epsilon^2/4a)}{\ln\epsilon^{-1}}=2\lambda_{km}. \quad (22)$$

Thus inequality (16) is also proven.

We need yet the following lemma, relating the numbers ρ_{km}, ρ_{kl}, and ρ_{lm} for $k < l < m$.

Lemma 3. If $k < l < m$, then we have the inequality

$$\frac{1}{\rho_{km}} \geqslant \frac{1}{\rho_{kl}} + \frac{1}{\rho_{lm}}. \tag{23}$$

Proof. Obviously, for any $\delta > 0$ and $\epsilon > 0$ we have

$$N(\epsilon, \delta U_l, U_k) = N\left(\frac{\epsilon}{\delta}, U_l, U_k\right). \tag{24}$$

Moreover,

$$r_{km}(\epsilon) \equiv N(\epsilon, U_m, U_k) \leqslant N(\delta, U_m, U_l) N(\epsilon, \delta U_l, U_k), \tag{25}$$

since one can first cover the ellipsoid U_m with translates of the ball δU_l and then cover these translates with translates of the ball ϵU_k.

By definition of the numbers ρ_{kl} and ρ_{lm}, we have, for any $\gamma_1 > \rho_{kl}$ and $\gamma_2 > \rho_{lm}$:

$$N(\epsilon, U_l, U_k) \leqslant \exp\left(\frac{1}{\epsilon}\right)^{\gamma_1}$$

and

$$N(\epsilon, U_m, U_l) \leqslant \exp\left(\frac{1}{\epsilon}\right)^{\gamma_2}.$$

Therefore, in view of inequalities (24) and (25)

$$r_{km}(\epsilon) \leqslant \exp\left[\left(\frac{1}{\alpha}\right)^{\gamma_2} + \left(\frac{\alpha}{\epsilon}\right)^{\gamma_1}\right].$$

Setting $\alpha = \epsilon^{\gamma_1/(\gamma_1+\gamma_2)}$ in this inequality, we obtain

$$r_{km}(\epsilon) \leqslant \exp\left[2\left(\frac{1}{\epsilon}\right)^{\gamma_1\gamma_2/(\gamma_1+\gamma_2)}\right]. \tag{26}$$

Since $\gamma_1 - \rho_{kl}$ and $\gamma_2 - \rho_{lm}$ can be arbitrarily small, from inequality (15) follows the relation

$$\rho_{km} = \overline{\lim_{\epsilon \to 0}} \frac{\ln \ln r_{km}(\epsilon)}{\ln \epsilon^{-1}} \leqslant \frac{\rho_{kl}\rho_{lm}}{\rho_{kl} + \rho_{lm}}.$$

Consequently

$$\frac{1}{\rho_{km}} \geqslant \frac{1}{\rho_{kl}} + \frac{1}{\rho_{ml}},$$

which proves the lemma.

From Lemma 3 it follows that for any k and m

$$\frac{1}{\rho_{km}} \geqslant \sum_{j=k}^{m-1} \frac{1}{\rho_{j.j+1}}, \tag{27}$$

whereby if one of $\rho_{j.j+1}$, $k \leqslant j \leqslant m-1$, equals zero, then $\rho_{km} = 0$.

We now prove the theorem stated above, giving a necessary and sufficient condition for the nuclearity of a space.

Theorem 8. In order that a countably Hilbert perfect space Φ be nuclear, it is necessary and sufficient that

$$\sum_{j=p+1}^{\infty} \frac{1}{\rho_{j.j+1}} = \infty \tag{28}$$

hold for any p (where $1/\rho_{j.j+1} = \infty$ if $\rho_{j.j+1} = 0$).

Proof. First we prove the necessity of condition (28). Let $\Phi = \bigcap_{n=1}^{\infty} \Phi_n$ be a nuclear space. Then without loss of generality we may assume that every mapping T_j^{j+1} of the space Φ_{j+1} into Φ_j is nuclear. In other words, it can be assumed that each unit ball U_{j+1} is, relative to the norm $\| \varphi \|_j$, an ellipsoid such that the sum of the lengths of its semiaxes converges. Thus, if Φ is nuclear, then for any j the exponent of convergence $\lambda_{j,j+1}$ of the ellipsoid U_{j+1} in Φ_j does not exceed unity; $\lambda_{j.j+1} \leqslant 1$. In view of inequality (16), it follows from this that $\rho_{j.j+1} \leqslant 2$ for every j. But then for any p we have

$$\sum_{j=p}^{\infty} \frac{1}{\rho_{j.j+1}} = \infty.$$

This proves the necessity of the condition of the theorem.

Now we prove that the condition is sufficient for the nuclearity of Φ. We choose any index k. By hypothesis the series $\sum_{j=k}^{\infty} (1/\rho_{j.j+1})$ diverges. Therfore there is an m such that $\sum_{j=k}^{m-1} (1/\rho_{j.j+1}) > 1$. By inequality (27) it follows from this that $1/\rho_{km} > 1$. But as was shown in Lemma 2, $\lambda_{km} < \rho_{km}$ and therefore $\lambda_{km} < 1$.

Thus, for any k there is an m such that $\lambda_{km} < 1$, and, consequently, the series $\sum_{j=1}^{\infty} a_j$, consisting of the lengths of the semiaxes of the ellipsoid U_m in the space Φ_k, converges. Therefore the mapping T_k^m of the space Φ_m into Φ_k is nuclear. Since k was arbitrary, this shows that Φ is nuclear, which completes the proof.

As was already mentioned, the same topology in Φ may be defined by different collections of scalar products. We now give a necessary and sufficient condition for the nuclearity of a space Φ, in whose statement the scalar products $(\varphi, \psi)_n$ do not appear.

We will consider the order of growth of the function

$$N(\epsilon, A, U),$$

where A is a compact set [18] and U is a neighborhood of zero in the space Φ. In the case where Φ is n-dimensional, the function $N(\epsilon, A, U)$ grows as $(1/\epsilon)^n$, i.e., there is a constant C such that

$$N(\epsilon, A, U) \leqslant C\left(\frac{1}{\epsilon}\right)^n.$$

For a Hilbert space (or for Banach spaces) the function $N(\epsilon, A, U)$ can grow arbitrarily fast. This means that for any increasing function $f(x)$ one can construct a compact set A in a Hilbert space such that

$$N(\epsilon, A, U) > f\left(\frac{1}{\epsilon}\right).$$

We will see now that for nuclear spaces Φ, $N(\epsilon, A, U)$ is a function of minimal order of growth. In other words, if A is a compact set in a nuclear space and U is any neighborhood of zero in Φ, then for every $\alpha > 0$ there exists C such that

$$N(\epsilon, A, U) < C \exp \epsilon^{\alpha}. \tag{29}$$

The minimality of the order of growth of $N(\epsilon, A, U)$ is not only necessary, but also sufficient for the nuclearity of the space Φ. This assertion can be stated as the following theorem.

Theorem 9. Let $\Phi = \bigcap_{n=1}^{\infty} \Phi_n$ be a countably Hilbert perfect space. In order that Φ be nuclear, it is necessary and sufficient that for any compact set A in Φ and any neighborhood U of zero the condition

$$\overline{\lim_{\epsilon \to 0}} \frac{\ln \ln N(\epsilon, A, U)}{\ln \epsilon^{-1}} = 0 \tag{30}$$

be fulfilled.

Proof. We show the necessity of (30). In fact, let us assume that there

[18] As is often done in analysis, we call a set with compact closure compact.

exists in the nuclear space Φ a compact set A and a ball U_k,[19] such that

$$\overline{\lim_{\epsilon\to 0}} \frac{\ln \ln N(\epsilon, A, U_k)}{\ln \epsilon^{-1}} = C > 0. \tag{31}$$

Since the set A is compact, for any $m > k$ there is b_m such that A lies in the ball $b_m U_m$. But then from inequality (31) it follows that

$$\overline{\lim_{\epsilon\to 0}} \frac{\ln \ln N(\epsilon, b_m U_m, U_k)}{\ln \epsilon^{-1}} \geqslant C > 0$$

for any $m > k$, and therefore $\rho_{km} \geqslant C > 0$.

Therefore, if inequality (31) holds for some compact set A and some ball U_k, then for every $m > k$ we have $\rho_{km} \geqslant C > 0$. But then, by inequality (27)

$$\sum_{j=k}^{m-1} \frac{1}{\rho_{j,j+1}} \leqslant \frac{1}{\rho_{km}} < \frac{1}{C}$$

for every $m > k$, and therefore the series $\sum_{j=1}^{\infty} (1/\rho_{j,j+1})$ converges, which in view of Theorem 8 contradicts the nuclearity of the space Φ. This proves the necessity of condition (30).

Now we prove that this condition is sufficient. For this, it suffices to show the following: If condition (30) holds for all compact sets A and all neighborhoods U of zero, then for any k there is an m such that $\lambda_{km} < 1$.

We carry out the proof by contradiction. Let us assume that there exists a k such that for every $m > k$ the inequality $\lambda_{km} \geqslant 1$ holds. Then in view of relation (15), $\rho_{km} \geqslant 1$ also. Written out in detail, this inequality has the form

$$\overline{\lim_{\epsilon\to 0}} \frac{\ln \ln N(\epsilon, U_m, U_k)}{\ln \epsilon^{-1}} \geqslant 1. \tag{32}$$

Formally, this does not contradict (30), since the ellipsoid U_m, which is compact in the space Φ_k, is not in general a compact set in the space Φ. However we will now show that if inequality (32) is fulfilled for every $m > k$, then there exists a compact set F in Φ such that

$$\overline{\lim_{\epsilon\to 0}} \frac{\ln \ln N(\epsilon, F, U_k)}{\ln \epsilon^{-1}} \geqslant 1. \tag{33}$$

But this now contradicts Eq. (30).

[19] We can assume, without loss of generality, that U_k is a unit ball, since the metric order is not changed by replacing the sets A and U by sets which are similar to them.

The set F is constructed in the following manner. We consider, in the space Φ, the sets A_m defined by the inequalities $(\varphi, \varphi)_m \leqslant 1/m^2$. The closures of these sets in the spaces Φ_k, $k < m$, are compact sets† in Φ_k. These closures are simply the ellipsoids $m^{-1} U_m$ in the spaces Φ_k.

We assumed that inequality (32) holds for $m > k$. But then, obviously, the inequality

$$\overline{\lim_{\epsilon \to 0}} \frac{\ln \ln N(\epsilon, m^{-1} U_m, U_k)}{\ln \epsilon^{-1}} \geqslant 1 \tag{34}$$

holds (the value of the metric order is not changed by replacing sets with sets which are similar to them). Inequality (34) says that for any $m > k$ there is a sequence of numbers $\epsilon_{m1}, \epsilon_{m2}, \ldots$ such that

$$\ln N\left(\epsilon_{mn}, \frac{1}{m} U_m, U_k\right) \geqslant \frac{1}{\epsilon_{mn}}.$$

We choose a sequence of integers $n_k, n_{k+1}, \ldots$ such that $\lim_{m\to\infty} \epsilon_{mn_m} = 0$ and denote ϵ_{mn_m} by δ_m. Then we have

$$\ln N\left(\delta_m, \frac{1}{m} U_m, U_k\right) \geqslant \frac{1}{\delta_m}. \tag{35}$$

Now in each of the sets A_m we choose the largest subset $F_m = \{x_1, \ldots, x_n\}$ such that $\| x_i - x_j \|_k \geqslant \frac{2}{3}\delta_m$ for $i \neq j$. This subset is finite in view of the compactness of the closure $\bar{A}_m$ in Φ_k. In view of the maximality of the set F_m the balls $x_1 + \delta_m U_k, \ldots, x_n + \delta_m U_k$ cover the set $\bar{A}_m \equiv m^{-1} U_m$, i.e.,

$$\frac{1}{m} U_m \subset \bigcup_{j=1}^{n} (x_j + \delta_m U_k).$$

Therefore $N(\delta_m, m^{-1} U_m, U_k) \leqslant n$.

Since we have $\| x_i - x_j \|_k \geqslant \frac{2}{3}\delta_m$ for distinct x_i and x_j of the set F_m, then in order to cover F_m we need at least n balls of radius $\frac{1}{3}\delta_m$. Consequently, $n \leqslant N(\frac{1}{3}\delta_m, F_m, U_k)$. From the inequalities which have been obtained, it follows that

$$\ln N\left(\frac{\delta_m}{3}, F_m, U_k\right) \geqslant \ln n \geqslant \ln N\left(\delta_m, \frac{1}{m} U_m, U_k\right) \geqslant \frac{1}{\delta_m}.$$

Therefore

$$\ln N\left(\frac{\delta_m}{3}, F_m, U_k\right) \geqslant \frac{1}{\delta_m}. \tag{36}$$

† Cf. the footnote on p. 87. Here, however, this compactness appears to be essential to the argument.

Let us denote by F the union of all the F_m and the point $\varphi = 0$:

$$F = 0 \cup \bigcup_{m=k+1}^{\infty} F_m.$$

We prove that F is a compact set. For this it suffices to show that any neighborhood U of zero contains all but a finite number of the points of F. Let the neighborhood U of zero be defined by $\| \varphi \|_m \leqslant \rho$. We choose l so that $l \geqslant m$ and $\rho > l^{-1}$. Then it is obvious that $l^{-1} U_l \subset U$. Therefore for $p > l$ we have

$$F_p \subset A_p \subset \frac{1}{p} U_p \subset \frac{1}{l} U_l \subset U.$$

Thus only a finite number of points of F, namely those belonging to $F_1, \ldots, F_l$, can lie outside the neighborhood U.

From inequality (36) it follows that

$$\varlimsup_{m\to\infty} \frac{\ln \ln N(\frac{1}{3}\delta_m, F, U_k)}{\ln (1/\delta_m)} \geqslant 1$$

and therefore

$$\varlimsup_{\epsilon\to 0} \frac{\ln \ln N(\epsilon, F, U_k)}{\ln \epsilon^{-1}} \geqslant 1.$$

Thus we have constructed a compact set F for which inequality (33) holds. But the existence of such a set contradicts the hypothesis of the theorem. Therefore the assumption that $\rho_{km} \geqslant 1$ for every $m > k$ is false. Consequently, there is an $m > k$ such that $\rho_{km} < 1$. But then the mapping T_k^m is nuclear. In other words, for any k there is an m such that the mapping T_k^m is nuclear, and therefore Φ is a nuclear space.

We note a corollary of the assertions which we have proven.

Corollary. If, for any compact set A in a countably Hilbert perfect space Φ and any ball U_k, the inequality

$$\varlimsup_{\epsilon\to 0} \frac{\ln \ln N(\epsilon, A, U_k)}{\ln \epsilon^{-1}} \leqslant C \tag{37}$$

is satisfied, where C does not depend upon A and k, then the space Φ is nuclear.

In fact, using (37) we can construct, for any k, a sequence $k = k_0 < k_1 < \ldots$ such that $\rho_{k_n,k_{n+1}} < 2C$. Then for $n > 2C + 1$ we have

$$\frac{1}{\rho_{k,k_n}} \geqslant \sum_{j=1}^{n-1} \frac{1}{\rho_{k_j,k_{j+1}}} > 1$$

and therefore $\rho_{k,k_n} < 1$. But from this follows the nuclearity of Φ.

3.8. The Functional Dimension of Linear Topological Spaces

Let Φ be a linear topological space. We call the number df Φ, defined by

$$\mathrm{df}\,\Phi = \sup_U \inf_V \overline{\lim_{\epsilon\to 0}} \frac{\ln\ln N(\epsilon, V, U)}{\ln\ln \epsilon^{-1}}, \tag{38}$$

where U and V range over the neighborhoods of zero of Φ, the *functional dimension* of the space Φ. The basis for this terminology lies in the fact that for many linear topological spaces consisting of entire analytic functions, df Φ coincides with the number of variables on which the functions depend.

In this section we consider countably Hilbert spaces having finite functional dimension. For a countably Hilbert space $\Phi = \bigcap_{k=1}^{\infty} \Phi_k$ formula (38) can be rewritten in the following form:

$$\mathrm{df}\,\Phi = \sup_k \inf_m \overline{\lim_{\epsilon\to 0}} \frac{\ln\ln r_{km}(\epsilon)}{\ln\ln \epsilon^{-1}}, \tag{39}$$

where $r_{km}(\epsilon) = N(\epsilon, U_m, U_k)$. In other words, df $\Phi = \sup_k \sigma_k$, where $\sigma_k = \inf_m \sigma_{km}$, and

$$\sigma_{km} = \overline{\lim_{\epsilon\to 0}} \frac{\ln\ln r_{km}(\epsilon)}{\ln\ln \epsilon^{-1}}. \tag{40}$$

From this formula and Theorem 8 (Section 3.7) it follows that *every countably Hilbert space with finite functional dimension is nuclear.*

It can be shown that for countably Hilbert spaces the formula

$$\mathrm{df}\,\Phi = \sup_{U,A} \overline{\lim_{\epsilon\to 0}} \frac{\ln\ln N(\epsilon, A, U)}{\ln\ln \epsilon^{-1}} \tag{41}$$

is valid, where U ranges over all neighborhoods of zero in Φ, and A ranges over all compact sets in Φ. The proof of this assertion is analogous to the proof of Theorem 9 (Section 3.7) and we will not detain ourselves over it.

In this section we will introduce another formula for df Φ, giving it in terms of the lengths of the semiaxes of the ellipsoids U_m in the spaces Φ_k. For this we need the following definition. Let $a_1, a_2, \ldots$ be a sequence of positive numbers tending to zero. We call the exponent of convergence of the sequence $\ln(1/a_1), \ln(1/a_2), \ldots$ the *convergence type* of the sequence $\{a_n\}$. In other words, the convergence type τ is equal to the infimum of those numbers μ for which the series $\sum_{n=1}^{\infty} \ln^{-\mu}(1/a_n)$ converges. Obviously, for the finiteness of the convergence type of the sequence a_1,

a_2, ... it is necessary that the exponent of convergence of the sequence $(1/a_1)$, $(1/a_2)$, ... be equal to zero.

If $n(\epsilon)$ is the number of terms of the sequence a_1, a_2, ... which exceed ϵ, then one has

$$\tau = \overline{\lim_{\epsilon \to 0}} \frac{\ln n(\epsilon)}{\ln \ln \epsilon^{-1}} . \tag{42}$$

In fact, applying formula (14) of Section 3.7 to the sequence ln $(1/a_1)$, ln $(1/a_2)$, ... , we see that the exponent of convergence λ of $\{\ln (1/a_n)\}$ has the form

$$\overline{\lim_{\delta \to 0}} \frac{\ln m(\delta)}{\ln \delta^{-1}} ,$$

where $m(\delta)$ is the number of terms of $\{ln\,(1/a_n)\}$ greater than δ. But obviously $m(\delta) = n(e^{-1/\delta})$. Therefore

$$\tau = \overline{\lim_{\delta \to 0}} \frac{\ln n(e^{-1/\delta})}{\ln \delta^{-1}} .$$

Replacing $e^{-1/\delta}$ by ϵ, we arrive at formula (42).

We now consider a countably Hilbert space Φ and denote by τ_{km} the convergence type of the sequence a_1, a_2, ... consisting of the lengths of the semiaxes of the ellipsoid U_m in the space Φ_k. Thus

$$\tau_{km} = \overline{\lim_{\epsilon \to 0}} \frac{\ln n(\epsilon)}{\ln \ln \epsilon^{-1}} , \tag{43}$$

where $n(\epsilon)$ is the number of semiaxes of the ellipsoid U_m in the space Φ_k whose lengths exceed ϵ.

We now prove the following theorem.

Theorem 10. For a countably Hilbert space Φ one has the formula

$$\mathrm{df}\, \Phi = \tau_0 + 1, \tag{44}$$

where $\tau_0 = \sup_k \tau_k$, $\tau_k = \inf_m \tau_{km}$.

Proof. We show first that $\sup_k \tau_k + 1 \leqslant \mathrm{df}\, \Phi$. Let us choose an arbitrary k. Since $\sigma_k = \inf_m \sigma_{km}$, then for any $\alpha > 0$ there is an m such that $\sigma_{km} \leqslant \mathrm{df}\, \Phi + \alpha$. We denote by a_1, a_2, ... the lengths of the semiaxes of the ellipsoid U_m in the space Φ_k. Fix $\epsilon > 0$. Comparing the volume of an ellipsoid spanned by the axes $a_1, \ldots, a_{n(\sqrt{\epsilon})}$ with the volume of an $n(\sqrt{\epsilon})$-dimensional ball of radius ϵ, we obtain

$$\prod_{j=1}^{n(\sqrt{\epsilon})} \left(\frac{a_j}{\epsilon}\right) \leqslant r_{km}(\epsilon) \equiv N(\epsilon, U_m, U_k)$$

[see the derivation of formula (17), Section 3.7].

Since for $1 \leqslant j \leqslant n(\sqrt{\epsilon})$ we have $a_k > \sqrt{\epsilon}$, it follows from this inequality that

$$\left(\frac{1}{\epsilon}\right)^{\frac{1}{2}n(\sqrt{\epsilon})} \leqslant r_{km}(\epsilon),$$

and therefore

$$\ln \tfrac{1}{2} + \ln n(\sqrt{\epsilon}) + \ln \ln \frac{1}{\epsilon} \leqslant \ln \ln r_{km}(\epsilon).$$

From this follows the inequality

$$\overline{\lim_{\epsilon \to 0}} \frac{\ln n(\sqrt{\epsilon})}{\ln \ln \epsilon^{-1}} + 1 \leqslant \overline{\lim_{\epsilon \to 0}} \frac{\ln \ln r_{km}(\epsilon)}{\ln \ln \epsilon^{-1}} = \sigma_{km}.$$

Since

$$\lim_{\epsilon \to 0} \frac{\ln \ln (1/\sqrt{\epsilon})}{\ln \ln \epsilon^{-1}} = 1,$$

we find that

$$\tau_{km} + 1 = \overline{\lim_{\epsilon \to 0}} \frac{\ln n(\sqrt{\epsilon})}{\ln \ln \epsilon^{-1}} + 1 \leqslant \sigma_{km} \leqslant \mathrm{df}\, \Phi + \alpha.$$

From this it follows that $\tau_k + 1 = \inf_m \tau_{km} + 1 \leqslant \mathrm{df}\, \Phi + \alpha$. But then $\sup_k \tau_k + 1 \leqslant \mathrm{df}\, \Phi + \alpha$. In view of the arbitrariness of $\alpha > 0$ we see that $\sup_k \tau_k + 1 \leqslant \mathrm{df}\, \Phi$.

Therefore the inequality $\sup_k \tau_k + 1 \leqslant \mathrm{df}\, \Phi$ is proven. Now we prove the reverse inequality. Suppose that $\sup_k \tau_k$ is finite. Then for any k there is an m such that τ_{km} is finite, and therefore the exponent of convergence of the sequence $1/a_1, 1/a_2, \ldots$ is zero (as before, we denote by $a_1, a_2, \ldots$ the lengths of the semiaxes of the ellipsoid U_m in Φ_k). Therefore the series $\sum_{j=1}^{\infty} a_j$ converges. Without loss of generality we may suppose that the series $\sum_{j=1}^{\infty} a_j$ converges for any k and m; we denote its sum by a. In Section 3.7 it was shown that

$$\ln \ln r_{km}(\epsilon) \leqslant \ln n\left(\frac{\epsilon^2}{4a}\right) + \ln \ln \frac{1}{\epsilon} + \ln \ln \left[2\left(a_1 \sqrt{n\left(\frac{\epsilon^2}{4a}\right)} + 1\right)\right]$$

[cf. inequality (21)]. From this inequality it follows that

$$\sigma_{km} = \overline{\lim_{\epsilon \to 0}} \frac{\ln \ln r_{km}(\epsilon)}{\ln \ln \epsilon^{-1}}$$

$$\leqslant 1 + \overline{\lim_{\epsilon \to 0}} \frac{\ln n(\epsilon^2/4a) + \ln \ln[2(a_1 \sqrt{n(\epsilon^2/4a)} + 1)]}{\ln \ln \epsilon^{-1}} . \tag{45}$$

Let k be arbitrary. Since $\tau_0 = \sup_k \tau_k$ and $\tau_k = \inf_m \tau_{km}$, then for any $\alpha > 0$ there is an m such that

$$\tau_{km} = \overline{\lim_{\epsilon \to 0}} \frac{\ln n(\epsilon)}{\ln \ln \epsilon^{-1}} < \tau_0 + \alpha.$$

But then

$$\overline{\lim_{\epsilon \to 0}} \frac{\ln n(\epsilon^2/4a)}{\ln \ln \epsilon^{-1}} < \tau_0 + \alpha.$$

Therefore it follows from inequality (45) that $\sigma_{km} < 1 + \tau_0 + \alpha$. Since α is arbitrary we find that $\sigma_k = \inf_m \sigma_{km} < \tau_0 + 1$. But then $\operatorname{df} \Phi = \sup_k \sigma_k \leqslant \tau_0 + 1$. Thus we have proven that $\operatorname{df} \Phi \leqslant \tau_0 + 1$. Since we proved earlier that $\operatorname{df} \Phi \geqslant \tau_0 + 1$, then $\operatorname{df} \Phi = \tau_0 + 1$, which proves the theorem.

We now present examples of nuclear spaces having finite functional dimension.

Let $\mathfrak{Z}$ be the space of all entire analytic functions of s variables. We introduce a topology in $\mathfrak{Z}$ by means of the countable family of scalar products

$$(\varphi, \psi)_n = \int_{\Omega_n} \varphi(z_1, \ldots, z_s)\overline{\psi(z_1, \ldots, z_s)}\, dx_1 \ldots dx_s\, dy_1 \ldots dy_s,$$

where $z_k = x_k + iy_k$, and Ω_n is the region defined by the inequalities

$$0 \leqslant | z_k | \leqslant n, \qquad 1 \leqslant k \leqslant s.$$

The space $\mathfrak{Z}$ is a space of finite functional dimension. In order to prove this assertion, we consider the space $\mathfrak{Z}_n$, obtained by completing $\mathfrak{Z}$ relative to the norm $\| \varphi \|_n = \sqrt{(\varphi, \varphi)_n}$. The monomials

$$z_1^{p_1} \ldots z_s^{p_s}, \qquad 0 \leqslant p_k < \infty, \qquad 1 \leqslant k \leqslant s,$$

belong to each of the spaces $\mathfrak{Z}_n$. It is obvious that if the collections $p_1, \ldots, p_s$ and $q_1, \ldots, q_s$ do not coincide, then the functions $z_1^{p_1}, \ldots, z_s^{p_s}$ and $z_1^{q_1}, \ldots, z_s^{q_s}$ are orthogonal relative to each of the scalar products $(\varphi, \psi)_n$.

The norm of the monomial $z_1^{p_1}, \ldots, z_s^{p_s}$ relative to the scalar product $(\varphi, \psi)_n$ equals

$$\frac{n^{p_1+\ldots+p_s+s}\sqrt{\pi^{p_1+\ldots+p_s}}}{\sqrt{(p_1+1)\ldots(p_s+1)}}.$$

From this it follows that for any k and m, $k < m$, the lengths $a^{(k,m)}_{p_1,\ldots,p_s}$ of the semiaxes of the ellipsoid U_m in Φ_k are given by

$$a^{(k,m)}_{p_1,\ldots,p_s} = \left(\frac{k}{m}\right)^{p_1+\ldots+p_s+s}$$

It is not difficult to see that the exponent of convergence of the set of numbers $a^{(k,m)}_{p_1,\ldots,p_s}$ is equal to zero for any k and m, and the convergence type τ_{km} of this set equals s. Therefore $\tau_0 = \sup_k \inf_m \tau_{km} = s$. But then, by Theorem 10 we have $\mathrm{df}\, \Phi = s + 1$.

Another space of finite functional dimension is the space $\mathfrak{U}$ of all entire analytic functions $\varphi(z_1, \ldots, z_j)$, periodic in each variable with period 2π. Scalar products are defined in this space by

$$(\varphi, \psi)_n = \int_{P_n} \varphi(z_1, \ldots, z_s)\overline{\psi(z_1, \ldots, z_s)}\, dx_1 \ldots dx_s\, dy_1 \ldots dy_s,$$

where P_n denotes the region $0 \leqslant x_k \leqslant 2\pi$, $-n \leqslant y_k \leqslant n$, $1 \leqslant k \leqslant s$. The functions

$$\exp\left[i(p_1 z_1 + \ldots + p_s z_s)\right],$$

where $p_1, \ldots, p_s$ are any integers, are orthogonal relative to each of the scalar products $(\varphi, \psi)_n$. Proceeding in the same way as above, we conclude that $\mathrm{df}\, \mathfrak{U} = s + 1$.

It would be very interesting to consider, in the general case, the relation between the functional dimension of a space consisting of functions, and the number of variables on which these functions depend. Let us mention the following result. Let the countably Hilbert space Φ be the topologized tensor product[20] of countably Hilbert spaces $\Phi^{(1)}$ and $\Phi^{(2)}$, having finite functional dimension. Then the functional dimension of Φ is finite, and

$$\mathrm{df}\, \Phi = \mathrm{df}\, \Phi^{(1)} + \mathrm{df}\, \Phi^{(2)}.$$

[20] Concerning questions related to tensor products of linear topological spaces, see Grothendieck [reference (22)].

4. Rigged Hilbert Spaces. Spectral Analysis of Self-Adjoint and Unitary Operators

4.1. Generalized Eigenvectors

One of the basic results of linear algebra is the theorem on the existence of a complete system of eigenvectors for any self-adjoint linear operator A in an n-dimensional Euclidean space R_n. This theorem states that if A is a self-adjoint operator in an n-dimensional Euclidean space R_n, then an orthonormal basis $e_1, \ldots, e_n$ in R_n can be found, each vector of which is an eigenvector of the operator A: $Ae_k = \lambda_k e_k$, where λ_k is a real number. Expanding any vector f of the space R_n by means of the vectors $e_1, \ldots, e_n$:$f = a_1 e_1 + \ldots + a_n e_n$, $a_k = (f, e_k)$, we can write the operator A in the following form:

$$Af = \sum_{k=1}^{n} \lambda_k (f, e_k) e_k. \tag{1}$$

An analoguous statement is valid also for unitary operators, with the only difference that the λ_k are not necessarily real, but rather complex, numbers whose moduli equal unity.

The situation becomes complicated upon passing from the finite to the infinite-dimensional case. For example, in Hilbert spaces there exist unitary operators (i.e., operators U such that $\| Uf \| = \| f \| = \| U^{-1} f \|$ for any vector f of H), which do not have any eigenvectors different from zero.

The so-called abstract theorem on the spectral decomposition (see the appendix to this section) gives only a certain substitute for the expansion by means of eigenvectors. For any $\epsilon > 0$ and certain λ, $|\lambda| = 1$, it permits one to find a vector f_ϵ, $\| f_\epsilon \| = 1$, such that

$$\| Uf_\epsilon - \lambda f_\epsilon \| \leqslant \epsilon.$$

An example of such an operator is the operator of translation U_h in the Hilbert space L^2 of functions on the line, having square integrable moduli. As a matter of fact, let us suppose that the function $f(x)$ in L^2 is such that

$$U_h f(x) = f(x - h) = af(x). \tag{2}$$

Since the Fourier transform of the function $f(x - h)$ is $e^{i\lambda h} F(\lambda)$, where $F(\lambda) = \int f(x)\, e^{i\lambda x}\, dx$ is the Fourier transform of $f(x)$, then it follows from (2) that

$$e^{i\lambda h} F(\lambda) = aF(\lambda).$$

But this can hold only in the case where the function $F(\lambda)$ equals zero at every point for which $e^{i\lambda x} \neq a$, i.e., it differs from zero only at a countable set of points. Since $F(\lambda)$ has square integrable modulus, we find that $f(x) = 0$. Thus the operator U_h does not have eigenvectors in the space L^2. Nevertheless, it is easy to find functions not belonging to the space L^2 which are eigenvectors of the translation operator. For example, $U_h e^{-i\lambda x} = e^{i\lambda h} e^{-i\lambda x}$ i.e., $e^{-i\lambda x}$ is an eigenfunction of the operator U_h, corresponding to the eigenvalue $e^{i\lambda h}$. Now, as is well known, any function $f(x)$ from L^2 can be expanded in terms of the functions $e^{-i\lambda x}$:

$$f(x) = \frac{1}{2\pi} \int F(\lambda) e^{-i\lambda x} \, d\lambda,$$

where

$$F(\lambda) = \int f(x) e^{i\lambda x} \, dx,$$

and the action of the translation operator is given by the equation

$$\begin{aligned} U_h f(x) &= \frac{1}{2\pi} \int e^{i\lambda h} F(\lambda) e^{i\lambda x} \, d\lambda \\ &= \frac{1}{2\pi} \int e^{i\lambda h} \left[\int f(\xi) e^{i\lambda\xi} \, d\xi \right] e^{i\lambda x} \, d\lambda, \end{aligned}$$

analogous to (1). The system of eigenfunctions $e^{-i\lambda x}$ is complete in the sense that for any function $f(x) \in L^2$, the Plancherel equality

$$\int |f(x)|^2 \, dx = \frac{1}{2\pi} \int |F(\lambda)|^2 \, d\lambda \tag{3}$$

holds. Thus we see that although the operator U_h does not have eigenfunctions lying in L^2, it has a complete system of eigenfunctions lying outside this space. An analogous situation arises also for other operators [for example, for the operator of multiplication by a function, whose "eigenfunctions" are the functions of the form $\delta(x - h)$]. To interpret these eigenfunctions, drawing only upon concepts connected with the Hilbert space L^2 itself, turns out to be impossible. We shall show, however, that such an interpretation becomes possible, if together with the Hilbert space L^2 one considers a certain extension Φ' of it.

As a rule, Hilbert spaces arise in analysis upon considering a linear space Φ, in which is given a positive-definite Hermitean bilinear functional (φ, ψ). Taking (φ, ψ) as a scalar product in Φ and completing Φ with respect to it, one obtains a corresponding Hilbert space H. Following this one usually forgets about the space Φ and studies only the space H. However, it is just the simultaneous consideration of the space Φ and its completion H which enables one to interpret the "eigenfunctions" lying outside of the Hilbert space H.

For example, the functions $e^{-i\lambda x}$ may be considered as linear functionals on the linear space S of infinitely differentiable functions, rapidly decreasing on the real axis together with their derivatives of any order. The space L^2 is obtained from S by completion relative to the scalar product

$$(\varphi, \psi) = \int \varphi(x)\overline{\psi(x)}\, dx.$$

Thus, the eigenvectors of the operator U_h, which do not belong to the space L^2, turn out to be none other than linear functionals on the linear space S, embedded in L^2. In exactly the same way we can interpret an eigenfunction $\delta(x - h)$ of the operator of multiplication by a function as a linear functional on S.

We now introduce the following definition.

Let A be a linear operator in a linear topological space Φ. A linear functional F on Φ, such that

$$F(A\varphi) = \lambda F(\varphi) \tag{4}$$

for every element φ of Φ, is called a *generalized eigenvector of the operator A, corresponding to the eigenvalue* λ.[1]

We can now say that the functions $e^{-i\lambda x}$ are generalized eigenvectors for the translation operator considered in the space S. The Fourier transform $F(\lambda)$ of the function $\varphi(x)$ is none other than the value of the functional $(e^{-i\lambda x}, \varphi(x))$ for the function $\varphi(x)$. The Plancherel equality (3) shows that the set of generalized eigenfunctions $e^{-i\lambda x}$ is complete, i.e., that $F(\lambda) \equiv 0$ implies $\varphi(x) \equiv 0$.

In Section 4.5 we prove the existence of a complete system of generalized eigenvectors for any unitary and any self-adjoint operators, defined in nuclear spaces. For this we make use of the concept of a rigged Hilbert space,[2] which arises in the consideration of a nuclear space Φ in which is defined a scalar product (φ, ψ).

[1] We denote here the value of the functional F for the element φ by $F(\varphi)$, and not by (F, φ), in order to avoid the possibility of confusion with the scalar product.

[2] The concept of a rigged Hilbert space proves to be useful not only in the study of the spectral theory of linear operations, but also in a number of other topics of functional analysis (for example, in the theory of quasi-invariant measures; cf. Chapter IV). We believe that this concept is no less (if indeed not more) important than that of a Hilbert space.

4.2. Rigged Hilbert Spaces†

Suppose that in a countably Hilbert nuclear space Φ, defined by the scalar products $(\varphi, \psi)_n$, there is defined still another scalar product, i.e., a positive-definite nondegenerate Hermitean functional (φ, ψ), continuous in each variable φ and ψ. Thus, with each pair of elements $\varphi, \psi \in \Phi$ there is associated a complex number (φ, ψ) such that

(1) $(\varphi_1 + \varphi_2, \psi) = (\varphi_1, \psi) + (\varphi_2, \psi)$,

(2) $(\alpha\varphi, \psi) = \alpha(\varphi, \psi)$,

(3) $(\varphi, \psi) = \overline{(\psi, \varphi)}$,

(4) $(\varphi, \varphi) \geqslant 0$, and $(\varphi, \varphi) = 0$ only when $\varphi = 0$,

(5) if $\lim_{n\to\infty} \varphi_n = \varphi$, then $\lim_{n\to\infty} (\varphi_n, \psi) = (\varphi, \psi)$.

From conditions (3) and (5) it follows that $\lim_{n\to\infty}(\psi, \varphi_n) = (\psi, \varphi)$. Since Φ is countably Hilbert, it follows from condition (5) that (φ, ψ) is continuous relative to some norm $\| \varphi \|_m = \sqrt{(\varphi, \varphi)_m}$ in Φ, i.e., numbers M and m can be found such that

$$|(\varphi, \psi)| \leqslant M\| \varphi \|_m \| \psi \|_m. \tag{5}$$

We now construct a Hilbert space H, completing the space Φ relative to the norm $\| \varphi \| = \sqrt{(\varphi, \varphi)}$. The elements of Φ form an everywhere dense set in H, by which is therefore defined a continuous[3] linear operator T, mapping Φ into H. We will in the sequel frequently identify the space Φ with the corresponding subset of the Hilbert space H.

Together with the spaces Φ and H we will consider the adjoint space Φ' of Φ. The adjoint T' of T is an operator mapping H', the adjoint space of H, into Φ' and defined by the equation

$$(T'h')(\varphi) = h'(T\varphi)$$

for any elements $h' \in H'$ and $\varphi \in \Phi$. Since every linear functional h' on the Hilbert space H can be written in the form

$$h'(h) = (h, h_1),$$

where h_1 is some element of H, then T' can be considered as a mapping

† The Russian word here translated as "rigged" can be (and often is) translated as "equipped."

[3] The continuity of T follows from the continuity of the scalar product relative to the topology of Φ.

of H into Φ'. It should be kept in mind, however, that the mapping $h' \to h_1$ is antilinear, since if $h' \to h_1$, $h'' \to h_2$, then

$$(\alpha h' + \beta h'')(h) = \alpha h'(h) + \beta h''(h) = (h, \bar{\alpha} h_1 + \bar{\beta} h_2)$$

and therefore $\alpha h' + \beta h'' \to \bar{\alpha} h_1 + \bar{\beta} h_2$. Therefore, if T' is considered as a mapping of H into Φ', then T' is an antilinear operator, and $T'T$, mapping Φ into Φ', is also antilinear. If one considers only real spaces, then both T' and $T'T$ are linear.

We call a triple of spaces Φ, H, Φ', having the properties stated above (i.e., a nuclear countably Hilbert space Φ in which is defined a nondegenerate scalar product (φ, ψ), the completion H of Φ by this scalar product, and the space Φ' adjoint to Φ) a *rigged Hilbert space*. We showed that for a rigged Hilbert space there exists a continuous linear operator T which maps Φ one-to-one onto an everywhere dense subset in H, and its (antilinear) adjoint T' maps H one-to-one onto an everywhere dense subset in Φ'. Therefore we will denote a rigged Hilbert space by $\Phi \subset H \subset \Phi'$.

Let us note that T is continuous relative to one of the norms $\| \varphi \|_m$ defining the topology in Φ. As a matter of fact, in view of inequality (5) we have

$$\| T\varphi \| = \sqrt{(\varphi, \varphi)} \leqslant \sqrt{M} \| \varphi \|_m.$$

Therefore T can be extended onto the entire space Φ_n, $n \geqslant m$, obtained by completing Φ in the norm $\| \Phi \|_n = \sqrt{(\varphi, \varphi)_n}$. We denote the corresponding operator by T_n. From Theorem 5 of Section 3 it follows that there is a value n for which T_n, mapping the Hilbert space Φ_n into H, is a nuclear operator. The operator T_n', mapping H into Φ_n', is also nuclear.

We remark that in a number of cases it suffices to consider, instead of a rigged Hilbert space $\Phi \subset H \subset \Phi'$, the triple $G \subset H \subset G'$ of Hilbert spaces connected with the nuclear mappings T and T'. However, only for rigged Hilbert spaces does *any* continuous scalar product (φ, ψ) lead to such a triple of spaces.

We now indicate the form of the operator T. For this we apply to the operator T_n the description of nuclear operators given in Section 2.3. We obtain thereby the following result. There exist orthonormal bases $\{h_k\}$ and $\{\varphi_k\}$ in H and Φ_n such that for every element $\varphi \in \Phi_n$ one has

$$T_n \varphi = \sum_{k=1}^{\infty} \lambda_k (\varphi, \varphi_k)_n h_k, \tag{6}$$

where $\lambda_k \geqslant 0$ and the series $\Sigma_{k=1}^{\infty} \lambda_k$ converges.

In order to pass from T_n to T, we note that $T_n\varphi = T\varphi$ if φ belongs to Φ, and therefore for elements $\varphi \in \Phi$ formula (6) takes the form

$$T\varphi = \sum_{k=1}^{\infty} \lambda_k(\varphi, \varphi_k)_n h_k.$$

For fixed k the expression $(\varphi, \varphi_k)_n$ is a linear functional on Φ_n. We denote this functional by F_k: $F_k(\varphi) = (\varphi, \varphi_k)_n$. Since the correspondence $F \leftrightarrow \psi$, established by the formula $F(\varphi) = (\varphi, \psi)_n$ between the elements of Φ_n and certain linear functionals on Φ_n, is isometric, the functionals F_k form an orthonormal basis in Φ_n'. We have thus proved the following result.

Let $\Phi \subset H \subset \Phi'$ be a rigged Hilbert space, and let T be the natural imbedding operator of Φ into H. Then there is an n, and also orthonormal bases $\{h_k\}$ and $\{F_k\}$ in H and Φ_n', such that for every element $\varphi \in \Phi$ one has

$$T\varphi = \sum_{k=1}^{\infty} \lambda_k F_k(\varphi) h_k, \tag{7}$$

where $\lambda_k \geqslant 0$ and the series $\Sigma_{k=1}^{\infty} \lambda_k$ converges.

We remark that one can associate with a rigged Hilbert space a two-sided infinite decreasing chain of Hilbert spaces

$$\ldots \supset \Phi_{-k} \supset \ldots \supset \Phi_0 \supset \ldots \supset \Phi_k \supset \ldots, \tag{8}$$

such that for any k, $-\infty < k < \infty$, there exists a nuclear mapping T_k^{k+1} of the space Φ_{k+1} onto an everywhere dense subset of Φ_k.

In order to construct such a chain, we take into account the fact that a nuclear space Φ is the intersection of a decreasing chain of Hilbert spaces: $\Phi = \bigcap_{k=1}^{\infty} \Phi_k$,

$$\Phi_1 \supset \Phi_2 \supset \ldots \supset \Phi_k \supset \ldots, \tag{9}$$

and for every k the natural mapping T_k^{k+1} is nuclear.† Now the space Φ' is the union of an increasing chain of Hilbert spaces: $\Phi' = \bigcup_{k=1}^{\infty} \Phi_{-k}$,

$$\Phi_{-1} \subset \Phi_{-2} \subset \ldots \subset \Phi_{-k} \subset \ldots, \tag{10}$$

where Φ_{-k} denotes the space Φ_k' adjoint to Φ_k. We denote by T_k^{k+1}, for $k < -1$, the operator adjoint to T_{-k-1}^{-k}. This operator is also nuclear. In order to connect the chains (9) and (10), we note the following. We saw

† See remark (1) in the footnote on p. 87.

above that there is a value n for which the operator T_n, mapping Φ_n into H, is nuclear. Then the mapping T_{-n} of H into Φ_{-n} is nuclear (this mapping, as was shown above, is antilinear). Without loss of generality we may suppose that $n = 1$. We now denote H by Φ_0, and the mappings T_1 and T_{-1} by T_0^1 and T_{-1}^0, respectively. We thereby obtain the desired sequence of spaces (8).

Chains of this sort appear in the consideration of symmetric positive-definite differential operators. Let A be such an operator. We associate with this operator a family of scalar products in the space K of infinitely differentiable functions with bounded supports

$$(\varphi, \psi)_n = \sum_{k=0}^{n} \int A^k\varphi(x)\overline{\psi(x)}\, dx, \qquad n = 0, 1, \ldots \tag{11}$$

Obviously one has the inequality $(\varphi, \varphi)_{n+1} \geqslant (\varphi, \varphi)_n$ for any $\varphi \in K$. Therefore, completing K relative to these scalar products, we obtain a decreasing chain of Hilbert spaces

$$\Phi_0 \supset \Phi_1 \supset \ldots \supset \Phi_n \supset \ldots$$

Let us denote the intersection of these spaces by Φ. Obviously the operator A carries Φ into itself. As a matter of fact, in view of the symmetry of A we have

$$(A\varphi, A\varphi)_n = \sum_{k=0}^{n} \int A^{k+1}\varphi(x)\overline{A\varphi(x)}\, dx = \sum_{k=0}^{n} \int A^{k+2}\varphi(x)\overline{\varphi(x)}\, dx \leqslant (\varphi, \varphi)_{n+2}.$$

Consequently A maps the Hilbert space Φ_{n+2} into Φ_n in a continuous manner, and therefore carries the space Φ into itself.

Now we consider the spaces Φ_{-n} adjoint to the Φ_n. These spaces form an increasing chain

$$\Phi_0 \subset \Phi_{-1} \subset \ldots \subset \Phi_{-n} \subset \ldots .$$

We identified the space Φ_{-0} with Φ_0, associating with a function $\psi(x) \in \Phi_0$ the functional F_ψ such that

$$F_\psi(\varphi) = \int \varphi(x)\overline{\psi(x)}\, dx.$$

Thus we have obtained a chain of spaces of the form

$$\Phi' \supset \ldots \supset \Phi_{-n} \supset \ldots \supset \Phi_0 \supset \ldots \supset \Phi_n \supset \ldots \supset \Phi.$$

For any two elements φ and ψ of the space $\Phi = \bigcap_{n=1}^{\infty} \Phi_n$ there is thus defined, together with the scalar products $(\varphi, \psi)_n$, $n \geq 0$, also the scalar products $(\varphi, \psi)_{-n}$, $n \geq 0$, which appear if φ and ψ are considered as elements of the space Φ_{-n}. The scalar products of the form $(\varphi, \psi)_{-n}$ prove to be useful in certain questions in the theory of partial differential equations.

4.3. The Realization of a Hilbert Space as a Space of Functions, and Rigged Hilbert Spaces

As is well known, a Hilbert space admits various realizations as a space of functions. These realizations are constructed in the following way. We choose a positive measure σ on some set X (for example, on the real line) and denote by L_σ^2 the space of all functions $\varphi(x)$ for which the integral

$$\int_X |\varphi(x)|^2 \, d\sigma(x)$$

converges. Introducing in the space L_σ^2 a scalar product by the formula[4]

$$(\varphi, \psi) = \int \varphi(x)\overline{\psi(x)} \, d\sigma(x), \tag{12}$$

we obtain a Hilbert space. Precisely speaking, the elements of L_σ^2 are not separate functions $\varphi(x)$, but rather classes of functions which differ from each other only on a set of σ-measure zero.

A drawback of this realization is the circumstance that, associating with the functions $\varphi(x)$ of L_σ^2 their value at some point x_0, generally speaking we do not obtain a continuous linear functional (the exceptions to this are those points whose individual measure is nonzero). Moreover, since the functions $\varphi(x) \in L_\sigma^2$ are only defined up to a set of σ-measure zero, we lose the possibility of speaking of their values at a fixed point x_0. However, in many questions, in particular in questions connected with the spectral decomposition of self-adjoint operators, it is desirable to consider the value of a function at a point as a linear functional. Just as in many similar cases, this turns out to be possible if we pass from the consideration of a Hilbert space to the consideration of a rigged Hilbert space.

Let $\Phi \subset H \subset \Phi'$ be a rigged Hilbert space. We consider a realization

[4] Henceforth in integrating over the set X we will omit any indication of the region of integration.

$h \to h(x)$ of the space H as a space of functions with scalar product of the form (12). Then to each element φ of the nuclear space Φ there corresponds a function $\varphi(x)$, associated by this realization with the element $T\varphi$ of H (we denote by T the natural imbedding of Φ into its completion H). Thus we obtain a realization $\varphi \to \varphi(x)$ of the space Φ, induced by the realization $h \to h(x)$ of the space H. In this section the following theorem concerning such a realization will be proved.

Theorem 1. Let $\Phi \subset H \subset \Phi'$ be a rigged Hilbert space and $\varphi \to \varphi(x)$ be the realization of Φ, as a space of functions, induced by the realization of the Hilbert space H as a space L_σ^2. Then to each value x one can associate a linear functional F_x on the space Φ such that for any function $\varphi(x) \in \Phi$ the equation

$$\varphi(x_0) = F_{x_0}(\varphi) \tag{13}$$

holds for almost every x_0 (relative to the measure σ).

For the proof of this theorem we begin with the following lemma on orthogonal series.

Lemma 1. Suppose that the functions $\{h_k(x)\}$ form an orthonormal system relative to a positive measure σ. Then for any sequence of positive numbers λ_k such that the series $\sum_{k=1}^\infty \lambda_k$ converges, the series

$$\sum_{k=1}^{\infty} \lambda_k |\, h_k(x)| \tag{14}$$

converges almost everywhere (relative to σ).

Proof. We apply the following well-known criterion for the convergence almost everywhere of a series of nonnegative functions.

If the series $\sum_{k=1}^\infty \int g_k(x)\, d\sigma(x)$ converges, where $g_k(x) \geqslant 0$ and σ is a positive measure, then the series $\sum_{k=1}^\infty g_k(x)$ converges almost everywhere relative to σ.

In view of the Cauchy–Bunyakovski inequality we have

$$\left(\sum_{k=1}^{\infty} \lambda_k |\, h_k(x)|\right)^2 \leqslant \sum_{k=1}^{\infty} \lambda_k \sum_{k=1}^{\infty} \lambda_k |\, h_k(x)|^2.$$

Since by hypothesis the series $\sum_{k=1}^\infty \lambda_k$ converges, it suffices for us to show the convergence almost everywhere of the series

$$\sum_{k=1}^{\infty} \lambda_k |\, h_k(x)|^2. \tag{15}$$

Let us consider the integral

$$\int \left[\sum_{k=1}^{\infty} \lambda_k | h_k(x)|^2\right] d\sigma(x). \tag{16}$$

Since the functions $h_k(x)$ are normalized, this integral equals $\Sigma_{k=1}^{\infty} \lambda_k$. As the series $\Sigma_{k=1}^{\infty} \lambda_k$ converges, by hypothesis, the integral (16) has a finite value, and therefore the series (15) converges for almost every x (relative to σ). From this, as we saw, also follows the convergence almost everywhere of the series (14), which proves the lemma.

We now turn to the proof of Theorem 1. Thus, we are given that $\varphi \to \varphi(x)$ is a realization of the nuclear space Φ, arising from the realization of H as L^2_σ. We have to construct functionals F_x on Φ such that $F_x(\varphi) = \varphi(x)$ for every $\varphi \in \Phi$. We define these functionals in the following way. Let us consider T, the natural imbedding operator of Φ into H. We saw in Section 4.2 that T can be written in the form

$$T\varphi = \sum_{k=0}^{\infty} \lambda_k F_k(\varphi) h_k, \tag{17}$$

where $\{F_k\}$ is an orthonormal basis in one of the spaces Φ'_n adjoint to Φ_n, $\{h_k\}$ is an orthonormal basis in H, $\lambda_k \geqslant 0$, and the series $\Sigma_{k=1}^{\infty} \lambda_k$ converges. Since by the hypothesis of the theorem H is realized as a space of functions, then to each element h_k there corresponds a function $h_k(x)$.[5] Let us consider the series

$$F_x = \sum_{k=1}^{\infty} \lambda_k h_k(x) F_k, \tag{18}$$

and prove that it converges for almost every x (relative to σ) in the norm of the space Φ_n. To do this we apply Lemma 1 to the series $\Sigma_{k=1}^{\infty} \lambda_k | h_k(x) |$. Since $\Sigma_{k=1}^{\infty} \lambda_k$ converges, and the $h_k(x)$ are orthonormal relative to σ,[6] then by this lemma the series $\Sigma_{k=1}^{\infty} \lambda_k \mid h_k(x) \mid$ converges almost everywhere (relative to σ).

Now we take into account the fact that the functionals $\{F_k\}$ form an orthonormal basis in Φ'_n, and therefore $\| F_k \|_{-n} = 1$ ($\| F \|_{-n}$ is the norm

[5] Actually, an element of L^2_σ, which is a class of functions. We assume that $h_k(x)$ is some arbitrary but fixed element of this class, and therefore $h_k(x)$ is meaningful for every value of x.

[6] These functions correspond, under the realization of H, to the elements h_k of the orthonormal basis in H.

of a functional $F \in \Phi'_n$). Consequently, the series $\sum_{k=1}^{\infty} \lambda_k \mid h_k(x) \mid$ can be rewritten in the form

$$\sum_{k=1}^{\infty} \lambda_k | h_k(x)| \, \| F_k \|_{-n}. \tag{19}$$

Thus the series (19) having positive terms converges for almost every x. Since

$$\| \lambda_k h_k(x) F_k \|_{-n} = \lambda_k | h_k(x)| \, \| F_k \|_{-n},$$

the series of functionals (18) converges in the norm of Φ'_n for almost every x.

Thus the convergence of the series of functionals (17) relative to the topology of Φ'_n for almost every x is proven. We take for the functional F_x the sum of the series (18) (at those points for which it converges), setting, for example, $F_x = 0$ at the remaining points.

It remains for us to show that if $\varphi(x)$ is any function in Φ, then for almost every x (relative to σ) the equality $F_x(\varphi) = \varphi(x)$ holds. But from (18) it follows that for almost every x we have

$$F_x(\varphi) = \sum_{k=1}^{\infty} \lambda_k h_k(x) F_k(\varphi). \tag{20}$$

On the other hand, from (17) it follows that

$$\varphi(x) = T\varphi = \sum_{k=1}^{\infty} \lambda_k F_k(\varphi) h_k(x), \tag{21}$$

where $\varphi(x)$ and $h_k(x)$ are considered as functions in L^2_σ. From this it follows that the series (21) is the expansion of $\varphi(x)$ in terms of the orthonormal system of functions $\{h_k(x)\}$. Since $\varphi(x)$ has square integrable modulus, the series converges to it in the mean. But if a series converges almost everywhere to the function $F_x(\varphi)$ and converges in the mean to the function $\varphi(x)$, then $F_x(\varphi) = \varphi(x)$ for almost every x, which proves the theorem.

We remark now that each function $\varphi(x)$ is defined only up to a set of σ-measure zero. Therefore we can change the value of each of these functions on a set of σ-measure zero (different for different functions) without changing the realization of the space. Henceforth we will always have in mind a realization of H for which $\varphi(x) = F_x(\varphi)$ for every x and every element $\varphi \in \Phi$.

It is useful to note that every functional F_x in Theorem 1 belongs to the same space Φ'_n. We also note that the condition of nuclearity of the space Φ can be weakened. The theorem remains valid for any pair $\Phi \subset H$ consisting of a locally convex linear topological space Φ and a Hilbert space H, if there exists a nuclear mapping T of Φ into H.[7]

4.4. Direct Integrals of Hilbert Spaces, and Rigged Hilbert Spaces

The theorem proved in the preceding section is a particular case of a more general theorem, connected with the representation of a Hilbert space as a direct integral of Hilbert spaces.

The concept of a direct integral of Hilbert spaces is a generalization of the concept of the orthogonal direct sum of a countable family of Hilbert spaces H_1, H_2, ...

We recall that one calls "orthogonal direct sum of the Hilbert spaces H_1, H_2, ..." the Hilbert space

$$\mathfrak{h} = \sum_{n=1}^{\infty} \oplus H_n,$$

whose elements are sequences

$$\xi = (h_1, h_2, \ldots), \qquad h_n \in H_n,$$

such that the series $\sum_{n=1}^{\infty} \| h_n \|_n^2$, where $\| h_n \|_n$ is the norm in H_n, converges. The linear operations in $\mathfrak{h}$ are defined coordinate-wise: If $\xi = (h_1, h_2, \ldots)$, $\eta = (g_1, g_2, \ldots)$, then $\xi + \eta = (h_1 + g_1, h_2 + g_2, \ldots)$ and $a\xi = (ah_1, ah_2, \ldots)$, and the scalar product (ξ, η) is defined by

$$(\xi, \eta) = \sum_{n=1}^{\infty} (h_n, g_n)_n,$$

where $(h_n, g_n)_n$ is the scalar product in H_n. One can consider the somewhat more general concept of the orthogonal direct sum of the Hilbert spaces H_1, H_2, ..., taken with the positive weights $\mu_1, \mu_2, \ldots, \mu_n > 0$. In this case the scalar product is defined by the formula

$$(\xi, \eta) = \sum_{n=1}^{\infty} \mu_n (h_n, g_n)_n.$$

[7] A linear topological space is called locally convex if each neighborhood of any element contains a convex neighborhood of the same element. A mapping of Φ into H is called nuclear if it is the product of a continuous linear mapping of Φ into some Hilbert space H_1 and a nuclear mapping of H_1 into H.

This equation can be written in the form

$$(\xi, \eta) = \int_X (h(x), g(x))_x \, d\mu(x),$$

where we denote by X the set consisting of the points $x = 1, 2, \ldots$, and by $\mu(x)$ the measure on X which equals μ_n at the point $x = n$. Corresponding to this, the orthogonal direct sum of the Hilbert spaces H_1, $H_2, \ldots$, taken with weights $\mu_1, \mu_2, \ldots$, can be written in the form

$$\mathfrak{h} = \int_X \oplus \, H(x) \, d\mu(x).$$

We now generalize the concept of the orthogonal direct sum of Hilbert spaces, taken with given weights, by relinquishing the assumption of a countable number of terms. In other words, we consider some set X on which is defined a positive measure μ. Suppose that with each point $x \in X$ there is associated a separable Hilbert space $H(x)$ of dimension $n(x)$, where $n(x)$ may assume the values $1, 2, \ldots$ or ∞ and is measurable with respect to μ. We consider first the case where all the spaces $H(x)$ have the same dimension n (n is an integer or ∞). In this case we identify each of the Hilbert spaces $H(x)$ with some one Hilbert space H of dimension n.

We construct the space $\mathfrak{h}$, consisting of those vector functions† $\xi = h(x)$ on the set X, taking values in H, such that

(1) for any $h \in H$ the numerical function $(h(x), h)$ is measurable with respect to μ,

(2) the numerical function $\| h(x) \|$ is square integrable with respect to μ;

$$\int_X \| h(x) \|^2 \, d\mu(x) < \infty.$$

We define the linear operations in $\mathfrak{h}$, setting for the vector functions $\xi = h(x)$ and $\eta = g(x)$

$$\xi + \eta = h(x) + g(x),$$
$$a\xi = ah(x),$$

and we introduce a scalar product, setting

$$(\xi, \eta) = \int_X (h(x), g(x)) \, d\mu(x). \tag{22}$$

† Rather, equivalence classes of such vector functions, two vector functions being considered equivalent if they differ only on a subset of X having μ-measure zero.

This scalar product is defined for all elements $\xi = h(x)$ and $\eta = g(x)$ of $\mathfrak{h}$. As a matter of fact, in view of condition (1), for any orthogonal basis $h_1, h_2, \ldots$ in H the functions $(h(x), h_n)$ and $(h_n, h(x))$ are measurable, as is therefore the function

$$(h(x), g(x)) = \sum_{n=1}^{\infty} (h(x), h_n)(h_n, g(x)).$$

But

$$\int_X |(h(x), g(x))|\, d\mu(x) \leqslant \tfrac{1}{2}\left[\int_X \| h(x)\|^2\, d\mu(x) + \int_X \| g(x)\|^2\, d\mu(x)\right],$$

and therefore in view of condition (2) the integral (22) converges.

It is easy to verify that the space $\mathfrak{h}$ satisfies all the axioms of a Hilbert space. In particular, it is complete. The proof of the completeness of $\mathfrak{h}$ is well known in the case where $n(x) = 1$, i.e., when $\mathfrak{h}$ is the space L^2_μ of scalar functions $f(x)$ having square integrable moduli with respect to μ. In the general case the proof is carried out analogously.

We will call the Hilbert space $\mathfrak{h}$ the *direct integral of the Hilbert spaces* $H(x)$ *with respect to the measure* μ and denote it by the same symbol

$$\mathfrak{h} = \int_X \oplus\, H(x)\, d\mu(x)$$

that was used for the orthogonal direct sum.

In the case where the spaces $H(x)$ have different dimensions, we proceed in the following way. We divide the set X, on which the measure μ is defined, into measurable subsets $X_1, X_2, \ldots$, on each of which $n(x) = n$. We already know the definition of the Hilbert space

$$\mathfrak{h}_n = \int_{X_n} \oplus\, H(x)\, d\mu(x).$$

We now denote by $\mathfrak{h}$ the orthogonal direct sum of the spaces $\mathfrak{h}_1, \mathfrak{h}_2, \ldots$,

$$\mathfrak{h} = \sum_{n=1}^{\infty} \oplus\, \mathfrak{h}_n.$$

We will also call this space $\mathfrak{h}$ the *direct integral of the spaces* $H(x)$ *with respect to the measure* μ and denote it by

$$\mathfrak{h} = \int_X \oplus\, H(x)\, d\mu(x).$$

The concept of an orthogonal direct sum of Hilbert spaces is a special case of that of a direct integral of Hilbert spaces. It corresponds to the case where the measure μ is defined on a countable set. The Hilbert space L^2_μ of functions with square integrable moduli relative to the measure μ is a direct integral of one-dimensional spaces relative to μ.

We proceed now to the consideration of results, analogous to those of Section 4.3, for direct integrals.

We note the following lemma, analogous to Lemma 1 of Section 4.3.

Lemma 1′. Let $\mathfrak{h}$ be the direct integral of the Hilbert spaces $H(x)$ relative to the measure μ;

$$\mathfrak{h} = \int_X \oplus\, H(x)\, d\mu(x).$$

If $\{\xi_n\} = \{h_n(x)\}$ is any orthonormal system in $\mathfrak{h}$ and the series $\sum_{n=1}^{\infty} \lambda_n$, where $\lambda_n > 0$, converges, then the series

$$\sum_{n=1}^{\infty} \lambda_n h_n(x)$$

converges for almost every x (with respect to μ) in the norm of $H(x)$.

Since the proof of this lemma proceeds word for word like that of Lemma 1, we omit it.

Let us consider now a rigged Hilbert space $\Phi \subset H \subset \Phi'$. Suppose that H is represented as a direct integral

$$\mathfrak{h} = \int_X \oplus\, H(x)\, d\mu(x)$$

of Hilbert spaces $H(x)$ relative to the measure μ. Then to each element $h \in H$ there corresponds a vector function $\xi = h(x)$ whose value for each x is a vector $h(x) \in H(x)$, and

$$(\xi, \eta) = \int_X (h(x), g(x))\, d\mu(x).$$

Since to each element $\varphi \in \Phi$ there corresponds the element $T\varphi \in H$ (T is the natural imbedding of Φ into H), then we can associate with an element $\varphi \in \Phi$ a function $\varphi(x) \in \mathfrak{h}$. We now prove the following theorem concerning these functions, generalizing Theorem 1 of Section 4.3.

Theorem 1′. Let $\Phi \subset H \subset \Phi'$ be a rigged Hilbert space and $h \leftrightarrow \xi \equiv h(x) \in H(x)$ be a representation of H as a direct integral

$$\mathfrak{h} = \int_X \oplus\, H(x)\, d\mu(x).$$

Then for any x there exists a nuclear operator T_x, mapping Φ into $H(x)$, such that for $\varphi \in \Phi$ the functions $\varphi(x)$ and $T_x(\varphi)$ differ only on a set of μ-measure zero.

Proof. Since Φ is nuclear, there is an n such that the natural imbedding T of Φ into H can be written in the form

$$T\varphi = \sum_{k=1}^{\infty} \lambda_k F_k(\varphi) h_k, \tag{23}$$

where $\{h_k\}$ is an orthonormal basis in H, $\{F_k\}$ is an orthonormal basis in Φ_n', $\lambda_k \geqslant 0$, and $\sum_{k=1}^{\infty} \lambda_k$ converges. We associate with each k and each $x \in X$ the operator $\lambda_k F_k h_k(x)$ of rank 1, taking the element $\varphi \in \Phi_n$ into the element $\lambda_k F_k(\varphi) h_k(x)$ of $H(x)$. We will now prove that the series

$$\sum_{k=1}^{\infty} \lambda_k F_k h_k(x), \tag{24}$$

consisting of these operators, converges in norm for almost every x (with respect to μ), and for almost every x the sum T_x of this series is a nuclear operator. For this we note that from the definition of the operator $\lambda_k F_k h_k(x)$ follows the equality

$$\| \lambda_k F_k h_k(x) \| = \lambda_k \| F_k \|_{-n} \| h_k(x) \|,$$

where $\| F_k \|_{-n}$ is the norm of the linear functional F_k in the space Φ_n', and $\| h_k(x) \|$ is the norm of the element $h_k(x)$ in the space $H(x)$. Therefore, according to Section 2.5, both the convergence almost everywhere of the series (24) and the nuclearity almost everywhere of the operator T_x will be proven, if we prove that the series

$$\sum_{k=1}^{\infty} \lambda_k \| F_k \|_{-n} \| h_k(x) \|,$$

consisting of the norms of operators of rank 1, converges almost everywhere. But this follows at once from Lemma 1′, taking into account that $\| F_k \|_{-n} = 1$ and the vector functions $h_k(x)$ correspond to the elements of an orthonormal basis $\{h_k\}$ in H and therefore form an orthonormal basis of functions in $\mathfrak{h}$.

We now set

$$T_x = \sum_{k=1}^{\infty} \lambda_k F_k h_k(x) \tag{25}$$

for each point x at which the sum of the series (24) is a nuclear operator, and $T_x = 0$ at the remaining points of X. Let us show that for each element $\varphi \in \Phi$ the equation $T_x(\varphi) = \varphi(x)$ holds for almost every x. In fact, it follows from (25) that almost everywhere

$$T_x(\varphi) = \sum_{k=1}^{\infty} \lambda_k F_k(\varphi) h_k(x). \tag{26}$$

But the function $\varphi(x)$ corresponds to the element $T\varphi \in H$ and therefore we have, by (25), that

$$\varphi(x) = \sum_{k=1}^{\infty} \lambda_k F_k(\varphi) h_k(x), \tag{27}$$

where the series converges in the mean (with respect to μ). Comparing (26) and (27), we conclude that the vector functions $T_x(\varphi)$ and $\varphi(x)$ are equal almost everywhere. The theorem is proved.

We note, as in Theorem 1, that the operators T_x can be chosen so that they will all be nuclear relative to the same scalar product $(\varphi, \psi)_n$ in Φ. Moreover, we remark that instead of the requirement that Φ be nuclear, it is sufficient to require that Φ be a locally convex linear topological space which admits a nuclear imbedding T into the space H.

Henceforth, in speaking of a realization $h \leftrightarrow h(x)$ of H, we will have in mind a realization for which the relation $T_x(\varphi) = \varphi(x)$ holds for all x and all $\varphi \in \Phi$.

4.5. The Spectral Analysis of Operators in Rigged Hilbert Spaces

We proceed now to the basic topic of this section—the spectral analysis of operators in rigged Hilbert spaces. We first recall the concept, introduced in Section 4.1, of a generalized eigenvector. Let A be an operator in a linear topological space Φ. A linear functional $F \in \Phi'$, such that

$$F(A\varphi) = \lambda F(\varphi)$$

for every $\varphi \in \Phi$, is called a *generalized eigenvector of A corresponding to the eigenvalue* λ. This equality can be written in the form

$$A'F = \lambda F,$$

where A' is an operator in Φ' such that

$$A'F(\varphi) = F(A\varphi)$$

for all $\varphi \in \Phi$ and $F \in \Phi'$.

To each value λ there corresponds the *eigenspace* Φ'_λ of A, consisting of all the generalized eigenvectors F whose eigenvalue is λ. We now introduce the notion of a spectral decomposition for elements of the space Φ. Associating with each element $\varphi \in \Phi$ and each number λ a linear functional $\tilde{\varphi}_\lambda$ on Φ'_λ, taking the value $F_\lambda(\varphi)$ on the element F_λ of Φ'_λ, we obtain vector functions $\tilde{\varphi}_\lambda$, whose values are linear functionals defined on the subspaces Φ'_λ. We call the correspondence $\varphi \to \tilde{\varphi}_\lambda$ the *spectral decomposition of the element* φ *corresponding to the operator* A. It is obvious that if $\varphi \to \tilde{\varphi}_\lambda$ is the spectral decomposition of the element φ, then the spectral dècomposition of the element $\psi = A\varphi$ is the vector function $\tilde{\psi}_\lambda \equiv \lambda\tilde{\varphi}_\lambda$. In fact, for any functional $F_\lambda \in \Phi'_\lambda$ we have

$$F_\lambda(\psi) = F_\lambda(A\varphi) = \lambda F_\lambda(\varphi),$$

and so by definition of $\tilde{\varphi}_\lambda$ and $\tilde{\psi}_\lambda$ we have

$$\tilde{\psi}_\lambda = \lambda\tilde{\varphi}_\lambda.$$

If the subspaces Φ'_λ are one dimensional (or, as we will say, the operator A has a simple spectrum), then the functions $\tilde{\varphi}_\lambda$ are scalars.

An example of a spectral decomposition with simple spectrum is the correspondence by which to a function $\varphi(x)$ of the space S there corresponds its Fourier transform $\tilde{\varphi}(\lambda)$,

$$\tilde{\varphi}(\lambda) = \int \varphi(x)e^{i\lambda x}\,dx.$$

This decomposition corresponds to the translation operator U_h: $\varphi(x) \to \varphi(x-h)$, since the functions $e^{i\lambda x}$ are the generalized eigenvectors of this operator.

If the operator A has "few" generalized eigenvectors, it can happen that $\tilde{\varphi}_\lambda \equiv 0$ while at the same time $\varphi \neq 0$. In this case the same vector function will correspond to various elements of the space Φ. We will say that the operator A has a *sufficient family of generalized eigenvectors*, or else that *the set of generalized eigenvectors of the operator A is complete*, if $\tilde{\varphi}_\lambda \equiv 0$ implies $\varphi = 0$. If the set of generalized eigenvectors of A is complete, then distinct elements φ of Φ give rise to distinct vector functions $\tilde{\varphi}_\lambda$.

We will show that if we have a rigged Hilbert space $\Phi \subset H \subset \Phi'$ and the operator A, acting in Φ, can be extended to a unitary or self-adjoint operator in H, then the system of generalized eigenvectors of A is complete.

For the proofs of these results we will make use of certain theorems of the spectral theory of linear operators in Hilbert space. In order not to interrupt the discussion, we will only quote here the statements of these theorems (some of them will be proven in the appendix to this section).

We recall the following definitions. An operator U in a Hilbert space H is called *unitary* if for any vectors $f, g \in H$ we have $(f, g) = (Uf, Ug) = (U^{-1}f, U^{-1}g)$. A unitary operator U is called *cyclic* if there exists a vector $f \in H$ such that the vectors $U^n f$, $-\infty < n < \infty$, where n is an integer, generate the entire space H.

Let us give an example of a unitary cyclic operator. Suppose that L_σ^2 is the space of functions $\varphi(\lambda)$, $|\lambda| = 1$, on the unit circle, having square integrable moduli with respect to a positive finite measure σ on this circle. Then the operator U which takes a function $\varphi(\lambda)$ into the function $\lambda\varphi(\lambda)$ is unitary. In fact

$$(U\varphi, U\psi) = \int_{|\lambda|=1} \lambda\varphi(\lambda)\overline{\lambda\psi(\lambda)}\, d\sigma(\lambda) = \int \varphi(\lambda)\overline{\psi(\lambda)}\, d\sigma(\lambda) = (\varphi, \psi).$$

It is not difficult to see that U is a cyclic operator, for which the function $\varphi_0(\lambda) \equiv 1$ is a generating vector. It turns out that any unitary cyclic operator has this form. In other words, the following theorem holds.

Theorem 2. Let U be a unitary cyclic operator in a Hilbert space H. Then the space H can be realized as a space L_σ^2 of functions $\varphi(\lambda)$, $|\lambda| = 1$, on the unit circle, having square integrable moduli with respect to a positive finite measure σ, in such a way that to the operator U corresponds the operator of multiplication by λ: if $h \to h(\lambda)$, then $Uh \to \lambda h(\lambda)$.

Let us now consider a rigged Hilbert space $\Phi \subset H \subset \Phi'$. An operator U, mapping Φ onto itself, is called *unitary* if

$$(U\varphi, U\psi) = (\varphi, \psi)$$

for any elements $\varphi, \psi \in \Phi$, where (φ, ψ) is the scalar product by which Φ is completed to yield H.

As Φ is dense in H and U is unitary, U can be extended to a unitary operator in H. We will denote this operator by the same letter U. If the extended operator U is cyclic in H, then we will also call the operator U in Φ *cyclic.*

We now prove the following theorem on the completeness of the system of generalized eigenvectors of a unitary cyclic operator U in a rigged Hilbert space $\Phi \subset H \subset \Phi'$

Theorem 3. Let U be a cyclic unitary operator in a rigged Hilbert space. Then the set of generalized eigenvectors of U is complete, i.e. from the vanishing of all components $\tilde{\varphi}_\lambda$ of the spectral decomposition of φ corresponding to the operator U, it follows that $\varphi = 0$.

Proof. Since Φ is everywhere dense in H, then U can be extended to a unitary operator in H. We apply Theorem 2 to it, obtaining a realization $h \to h(\lambda)$ of H as a space of functions on the unit circle, having square integrable moduli with respect to a positive measure σ. To the operator U corresponds, by this realization, the operator of multiplication by λ, $|\lambda| = 1$, i.e. if $h \to h(\lambda)$, then $Uh \to \lambda h(\lambda)$.

By the realization $h \to h(\lambda)$ there corresponds to each element $\varphi \in \Phi$ a function $\varphi(\lambda)$. By Theorem 1 of Section 4.3 the functions $\varphi(\lambda)$ can be chosen so that

$$\varphi(\lambda) = F_\lambda(\varphi)$$

for any λ, where F_λ is a continuous linear functional on Φ. We show that the F_λ are generalized eigenvectors of U. In fact, let φ be any element of Φ and let $U\varphi = \psi$. Then for any λ we have† $\psi(\lambda) = \lambda\varphi(\lambda)$. But

$$\psi(\lambda) = F_\lambda(\psi) = F_\lambda(U\varphi),$$

and

$$\varphi(\lambda) = F_\lambda(\varphi),$$

and therefore

$$F_\lambda(U\varphi) = \lambda F_\lambda(\varphi).$$

But this means that F_λ is a generalized eigenvector of U, corresponding to the eigenvalue λ.

Now we note that

$$\varphi(\lambda) = F_\lambda(\varphi) = \tilde{\varphi}_\lambda.$$

This means that the function $\varphi(\lambda)$ coincides, on the space Φ'_λ of the generalized eigenvectors F_λ, with $\tilde{\varphi}_\lambda$. Hence it follows that if $\tilde{\varphi}_\lambda \equiv 0$, then $\varphi(\lambda) \equiv 0$.

In order to prove the completeness of the system of generalized eigenvectors of U, it remains for us to show that if $\varphi(\lambda) = 0$ for all λ, $|\lambda| = 1$,

† Since the representatives $\varphi(\lambda)$ have already been chosen so that $\varphi(\lambda) = F_\lambda(\varphi)$ for *all* λ and all $\varphi \in \Phi$, it is not clear why $\psi(\lambda) = \lambda\varphi(\lambda)$ should hold for *all* λ and all $\varphi \in \Phi$. A similar question also arises in the proof of Theorem 5 below.

then $\varphi = 0$. But this assertion follows at once from the fact that the mapping $h \to h(\lambda)$ of H into L_σ^2 is isometric, and therefore

$$\| \varphi \|^2 = (\varphi, \varphi) = \int_{|\lambda|=1} | \varphi(\lambda)|^2 \, d\sigma(\lambda). \tag{28}$$

Consequently, if $\varphi(\lambda) \equiv 0$, $| \lambda | = 1$, then $\varphi = 0$. Thus Theorem 3 is proven.

We remark that (28) can also be written in the form

$$\| \varphi \|^2 = \int | F_\lambda(\varphi)|^2 \, d\sigma(\lambda),$$

which is a generalization of the Plancherel equality for the ordinary Fourier transform.

Now let us consider self-adjoint operators. A linear operator A, acting in a Hilbert space H and defined on an everywhere dense linear subset Ω_A in H, is called *self-adjoint* if:

(1) for any vectors $f, g \in \Omega_A$

$$(Af, g) = (f, Ag),$$

(2) for no vector $g \notin \Omega_A$ can there be found a vector g_1 such that $(Af, g) = (f, g_1)$ for all $f \in \Omega_A$.

A self-adjoint operator A is called *cyclic* if there is a vector f such that the vectors $A^n f$, $n = 0, 1, \ldots$ generate the entire space H.

For self-adjoint operators a theorem analogous to Theorem 2 is valid.

Theorem 2′. Let A be a self-adjoint cyclic operator in a Hilbert space H. Then there is a realization $h \to h(\lambda)$ of H as a space L_σ^2 of functions on the real line, having square integrable moduli with respect to a positive measure σ, such that

(1) the domain of definition of A is carried by this realization into the set of all functions $f(\lambda) \in L_\sigma^2$ for which the integral

$$\int | \lambda f(\lambda)|^2 \, d\sigma(\lambda)$$

converges,

(2) if to the element f corresponds the function $f(\lambda)$, then to the element Af corresponds the function $\lambda f(\lambda)$.[8]

We call an operator A, acting in a nuclear space Φ, self-*adjoint* relative to a scalar product (φ, ψ), if its closure in the completion H of φ relative to the norm $\sqrt{(\varphi, \varphi)}$ is self-adjoint. In this case we will say that A is self-adjoint in the rigged Hilbert space $\Phi \subset H \subset \Phi'$. If the operator in H thereby obtained turns out to be cyclic, then we will also call A *cyclic.*

Using Theorem 2′, the following theorem can be proved.

Theorem 3′. Let A be a self-adjoint cyclic operator in a rigged Hilbert space $\Phi \subset H \subset \Phi'$. Then the set of generalized eigenvectors of A, corresponding to real eigenvalues, is complete.

We remark that the conditions of Theorems 3 and 3′ can be weakened, by relinquishing the requirement that U (or A) map the space Φ into itself (this condition is not fulfilled, for example, if Φ is a space of infinitely differentiable functions, and A is a linear differential operator whose coefficients are functions having only a finite number of continuous derivatives). Theorems 3 and 3′ remain valid also in the case where U (or A) carries into itself a completion Φ_n of Φ such that the natural imbedding T_n of Φ_n into H is nuclear. This follows from the validity of Theorem 1 in the case where Φ is not nuclear, but the imbedding T of Φ into H is nuclear.

Let us now consider the case in which the operator is not cyclic. In this case Theorems 2 and 2′ are replaced by the following more general theorems.

Theorem 4. Let U be a unitary operator in a Hilbert space H. Then there exists a positive measure σ on the unit circle and a representation of H as a direct integral

$$\mathfrak{h} = \int_{|\lambda|=1} \oplus\, H(\lambda)\, d\sigma(\lambda)$$

[8] Theorems 2 and 2′ are closely related to each other. If U is a unitary operator, then the operator A defined by

$$A = \frac{U - iE}{U + iE}$$

is self-adjoint. Any self-adjoint operator A can be written in the form indicated, where U is a unitary operator. Using these remarks, one can obtain Theorem 2′ from Theorem 2 and vice versa.

of Hilbert spaces $H(\lambda)$ with respect to the measure σ, such that to the operator U there corresponds by this realization the operator of multiplication by λ. In other words, if to the element $h \in H$ there corresponds the vector function $h(\lambda)$, then to the element Uh there corresponds the vector function $\lambda h(\lambda)$.

Theorem 4′. Let A be a self-adjoint operator in a Hilbert space H. Then there exists a positive measure $\sigma(\lambda)$ on the real line and a representation of H as a direct integral $\mathfrak{h}$ of spaces $H(\lambda)$ with respect to σ:

$$\mathfrak{h} = \int_{-\infty}^{\infty} \oplus H(\lambda)\, d\sigma(\lambda),$$

such that to A corresponds the operator of multiplication by λ.

We shall apply these theorems to proving the completeness of the system of generalized eigenvectors of unitary and self-adjoint operators in a rigged Hilbert space $\Phi \subset H \subset \Phi'$. We will consider in detail only the case of unitary operators, since the proof for self-adjoint operators is entirely analogous.

Thus, let U be a unitary operator in a rigged Hilbert space $\Phi \subset H \subset \Phi'$. By Theorem 4 there exists a representation of H as a direct integral

$$\mathfrak{h} = \int_{|\lambda|=1} \oplus H(\lambda)\, d\sigma(\lambda),$$

such that to the operator U corresponds the operator of multiplication by λ. Applying Theorem 1′ to this representation, we find that

$$\varphi(\lambda) = T_\lambda(\varphi)$$

for all elements $\varphi \in \Phi$ and all λ, where T_λ is a nuclear operator mapping Φ into $H(\lambda)$.

From this it follows that to each element $\xi \in H(\lambda)$ there corresponds a linear functional $\tilde{\xi}$ on Φ, defined by

$$\tilde{\xi}(\varphi) = (\varphi(\lambda), \xi)_\lambda \equiv (T_\lambda \varphi, \xi)_\lambda$$

(the scalar products are taken in $H(\lambda)$). Now in view of the fact that to U there corresponds in $\mathfrak{h}$ the operator of multiplication by λ, we find that the functional $\tilde{\xi}$, corresponding to the element $\xi \in H(\lambda)$, is a generalized eigenvector of U, i.e., that $U'\tilde{\xi} = \lambda\tilde{\xi}$; if $\xi \neq 0$, then $\tilde{\xi} \neq 0$.

Thus we have constructed an imbedding of each space $H(\lambda)$ into the space Φ'_λ consisting of linear functionals F on Φ for which $U'F = \lambda F$.

It is not hard to see that this imbedding is continuous: if $\lim_{n\to\infty} \xi_n = \xi$, then $\lim_{n\to\infty} \tilde{\xi}_n = \tilde{\xi}$.

Now let φ be an element of Φ such that $\tilde{\varphi}_\lambda \equiv 0$. Then for any λ and any $\xi \in H(\lambda)$ we have

$$0 = \tilde{\varphi}_\lambda(\xi) = \tilde{\xi}(\varphi) = (\varphi(\lambda), \xi)_\lambda.$$

Hence it follows that $\varphi(\lambda) \equiv 0$. Since

$$\| \varphi \|^2 = (\varphi, \varphi) = \int \| \varphi(\lambda)\|_\lambda^2 \, d\sigma(\lambda),$$

we obtain $\varphi = 0$.

Thus, we have proven that if $\tilde{\varphi}_\lambda \equiv 0$ for all λ, then $\varphi = 0$. In other words, we have proven the completeness of the system of generalized eigenvectors of U.

Theorem 5. A unitary operator in a rigged Hilbert space has a complete system of generalized eigenvectors, corresponding to eigenvalues λ having modulus one.

The following theorem is proven in exactly the same way.

Theorem 5′. A self-adjoint operator in a rigged Hilbert space has a complete system of generalized eigenvectors, corresponding to real eigenvalues.

In certain cases, analogous theorems, regarding commutative systems of unitary or self-adjoint operators, are useful.

Let $\{A_k\}$, $k = 1, \ldots, n$, be a system of commuting self-adjoint operators in a rigged Hilbert space. This means that the operators $E_k(\Delta)$, $k = 1, \ldots, n$, appearing in the resolutions of unity of the self-adjoint closures of the A_k in H, commute with each other. We call a linear functional F on Φ a *generalized eigenvector for the system* $\{A_k\}$, if for any k, $1 \leqslant k \leqslant n$,

$$A_k' F = \lambda_k F.$$

We shall call the set of numbers $\lambda = (\lambda_1, \ldots, \lambda_n)$ the *eigenvalues* corresponding to the eigenvector F.

The following result holds.

Theorem 6. If $\{A_k\}$, $1 \leqslant k \leqslant n$, is a system of commuting self-adjoint operators in a rigged Hilbert space, then the set of generalized eigenvectors of this system is complete.

An analogous theorem is valid also for commutative systems of unitary operators in a rigged Hilbert space.

Appendix. The Spectral Analysis of Self-Adjoint and Unitary Operators in Hilbert Space

1. The Abstract Theorem on Spectral Decomposition

In this section use was made of certain results of the spectral theory of operators. Since not all of these results can be considered as generally known, we give here a discussion of them based upon the theorem on the abstract spectral decomposition of a self-adjoint operator (concerning the definition of a self-adjoint operator, cf. Section 4.5).

In order to formulate this theorem, we introduce the concept of a resolution of unity. Suppose that to every interval $\Delta = [a, b)$ on the real line there corresponds a bounded self-adjoint operator $E(\Delta)$ in a Hilbert space H, whereby the following properties are satisfied:

(1) for any two intervals Δ_1 and Δ_2

$$E(\Delta_1)E(\Delta_2) = E(\Delta_1 \cap \Delta_2); \tag{1}$$

(2)

$$\lim_{x \to +\infty} E(x) = E, \qquad \lim_{x \to -\infty} E(x) = 0, \tag{2}$$

where we have put $E(x) = E(\Delta_x)$ [Δ_x is the interval $(-\infty, x)$], E denotes the identity operator, and 0 the null operator[9];

(3) if the interval Δ is a countable union of nonintersecting intervals Δ_n, $\Delta = \bigcup_{n=1}^{\infty} \Delta_n$, then $E(\Delta) = \sum_{n=1}^{\infty} E(\Delta_n)$.

Such a system of operators $E(\Delta)$ is called a *resolution of unity.*

From (1) it follows that for any interval Δ one has $E^2(\Delta) = E(\Delta)$. This means that $E(\Delta)$ is a projection operator, projecting the space H onto the subspace $H_\Delta = E(\Delta)H$. The operators $E(\Delta)$ are positive definite, i.e., $(E(\Delta)f, f) \geqslant 0$ for any $f \in H$. In fact,

$$(E(\Delta)f, f) = (E^2(\Delta)f, f) = (E(\Delta)f, E(\Delta)f) \geqslant 0.$$

[9] Here and further on the convergence of operators is understood in the weak sense; $\lim_{x\to+\infty} E(x) = E$ means that

$$\lim_{x \to +\infty} (E(x)f, g) = (f, g)$$

for any two elements $f, g \in H$.

We set, for any interval Δ and any element $f \in H$,

$$\mu_f(\Delta) = (E(\Delta)f, f).$$

From the discussion above it follows that $\mu_f(\Delta)$ is a countably additive positive measure defined on the intervals Δ. This measure can be extended to every Borel set. We will call $\mu_f(\Delta)$ the *spectral measure* corresponding, by the resolution of unity $E(\Delta)$, to the element f.

The theorem on the spectral decomposition of a self-adjoint operator is formulated in the following way.

Theorem 1. Let A be a self-adjoint operator in a Hilbert space H. Then there exists a resolution of unity $E(\Delta)$ such that the operator A is defined on the set Ω_A of those elements $f \in H$ for which the integral

$$\int_{-\infty}^{\infty} x^2 \, d\mu_f(x)$$

converges, where $\mu_f(x) = (E(\Delta)f, f)$. The operator A is given, for these elements f, by the formula[10]

$$Af = \int_{-\infty}^{\infty} x \, d(E(x)f), \tag{3}$$

where $E(x) = E(-\infty, x)$.

From Theorem 1 it follows that if Δ is any interval, then

$$E(\Delta)A = AE(\Delta) = \int_{\Delta} x \, dE(x). \tag{4}$$

In fact,

$$E(\Delta)A = E(\Delta) \int_{-\infty}^{\infty} x \, dE(x) = \int_{-\infty}^{\infty} xE(\Delta) \, dE(x).$$

But by Eq. (1), $E(\Delta)\, dE(x) = 0$ if x does not belong to Δ, and $E(\Delta)$ $dE(x) = dE(x)$ if $x \in \Delta$. Therefore

$$E(\Delta)A = \int_{\Delta} x \, dE(x).$$

[10] Formula (3) is also understood in the weak sense; for any two elements $f, g \in \Omega_A$ one has

$$(Af, g) = \int_{-\infty}^{\infty} x \, d(E(x) f, g).$$

Similarly one can prove that

$$AE(\Delta) = \int_{\Delta} x \, dE(x).$$

From (4) it follows that for any vector f in the subspace $H_\Delta = E(\Delta)H$, $\Delta = [a, b)$, one has

$$\| Af - af \| \leqslant (b - a)\| f \|.$$

Therefore if $b - a$ is small, then f is "almost an eigenvector" of A. If $\Delta_1, \Delta_2, \ldots$ are nonintersecting intervals which cover the real line, then H is the orthogonal direct sum of the subspaces H_{Δ_n}, in each of which A "almost coincides with a similarity operator."

An analogous theorem holds also for unitary operators, the only difference being that the intervals Δ lie not on the real line, but rather on the unit circle.

2. Cyclic Operators

Cyclic operators have a particularly simple structure. A self-adjoint operator A is called *cyclic*, if there exists a vector f such that the linear combinations of the vectors $E(\Delta)f$ are everywhere dense in H. The vector f is called a *cyclic vector*.

If A is a cyclic operator, then the Hilbert space H can be realized as a space L_f^2 of functions $\varphi(x)$, having square integrable moduli with respect to the measure $\mu_f(\Delta)$, whereby to the operator A corresponds the operator of multiplication of the functions $\varphi(x)$ by x.

Thus, the domain of definition of A by this realization consists of those functions $\varphi(x) \in L_f^2$ for which the integral

$$\int_{-\infty}^{\infty} x^2 | \varphi(x)|^2 \, d\mu_f(x)$$

converges.

This realization can be accomplished in the following way. With each vector of the form $E(\Delta)f$ we associate the characteristic function $\chi_\Delta(x)$ of the interval Δ. In particular, we associate with f the function identically equal to unity on the real line. Let us show that this correspondence is isometric in the sense that

$$\| E(\Delta)f \|^2 = \int_{-\infty}^{\infty} | \chi_\Delta(x)|^2 \, d\mu_f(x).$$

In fact, it follows from (1) that

$$\| E(\Delta)f\|^2 = (E(\Delta)f, E(\Delta)f) = (E(\Delta)f, f)$$

$$= \mu_f(\Delta) = \int_\Delta d\mu_f(x) = \int_{-\infty}^{\infty} |\chi_\Delta(x)|^2 \, d\mu_f(x).$$

We now extend the isometric correspondence $E(\Delta)f \to \chi_\Delta(x)$, using linear combinations and passing to the limit. Since the linear combinations of the vectors $E(\Delta)f$ are everywhere dense in H, and the linear combinations of the characteristic functions $\chi_\Delta(x)$ are everywhere dense in L_f^2, we obtain an isometry between H and L_f^2.

It is obvious that

$$(AE(\Delta)f, g) = \int_\Delta x \, d(E(x)f, g)$$

for any interval Δ. Therefore

$$(AE(\Delta)f, f) = \int_\Delta x \, d(E(x)f, f) = \int_\Delta x \, d\mu_f(x) = \int_{-\infty}^{\infty} x\chi_\Delta(x) \, d\mu_f(x).$$

This means that to the operator A there corresponds in L_f^2 the operator carrying the characteristic functions $\chi_\Delta(x)$ into the functions $x\chi_\Delta(x)$. Since the functions $\chi_\Delta(x)$ form an everywhere dense set in L_f^2, we find that for the realization under discussion there corresponds to A the operator of multiplication by x of the functions $\varphi(x) \in L_f^2$.

3. The Decomposition of a Hilbert Space into a Direct Integral Corresponding to a Given Self-Adjoint Operator

We will show: If A is a self-adjoint operator in a Hilbert space H, then there exists a representation of H as a direct integral

$$\mathfrak{h} = \int \oplus H(x) \, d\mu(x)$$

of Hilbert spaces $H(x)$, for which A is given in each of the spaces $H(x)$ by the operator of multiplication by x.

Let A be a self-adjoint operator in H and f a vector in H. The smallest closed subspace H_f in H containing every vector $E(\Delta)f$, where $E(\Delta)$ is the resolution of unity corresponding to A, we call the *cyclic subspace* in H generated by the vector f. If a vector h is orthogonal to a cyclic subspace H_f, then every vector $E(\Delta)h$ is orthogonal to H_f. In fact, since $E(\Delta)$ is self-adjoint, then for $g \in H_f$ we have

$$(E(\Delta)h, g) = (h, E(\Delta)g).$$

Since the subspace H_f contains, along with the vector g, every vector $E(\Delta)g$, then $(E(\Delta)h, g) = 0$ for $g \in H_f$, i.e., the vectors $E(\Delta)h$ are orthogonal to H_f. From this it follows that if a vector h is orthogonal to a cyclic subspace H_f, then the cyclic subspace H_h, generated by h, is orthogonal to H_f.

We proceed now to construct the representation $\mathfrak{h}$ of H corresponding to the operator A. Choose a countable dense set $f_1, f_2, \ldots$ in H and denote by H_1 the cyclic subspace generated by f_1. Suppose that we have already constructed pairwise orthogonal cyclic subspaces $H_1, \ldots, H_n$ in H. We choose the first among the f_k, $1 \leqslant k < \infty$, not belonging to the direct sum $H^n = H_1 + \ldots + H_n$. Let this be f_{k_n}. In the subspace G spanned by H^n and f_{k_n} we choose an element h_{n+1} ($\| h_{n+1} \| = 1$) orthogonal to H^n, and denote by H_{n+1} the cyclic subspace generated by h_{n+1}. Obviously $f_{k_n} \in H_1 + \ldots + H_{n+1}$. Since the set of elements $f_1, f_2, \ldots$ is everywhere dense in H, then continuing the process described, we obtain a decomposition

$$H = \sum_{n=1}^{\infty} \oplus H_n$$

of H into an orthogonal direct sum of cyclic subspaces $H_1, H_2, \ldots$.

It was shown above that each of the cyclic subspaces H_n can be realized as a space of functions $h_n(x)$, in which the scalar product is given by

$$(f_n(x), g_n(x)) = \int f_n(x)\overline{g_n(x)}\, d\mu_n(x),$$

where $\mu_n(\Delta) = (E(\Delta)h_n, h_n)$ is a positive measure. From this it follows that each element $f \in H$ is given in the form of a sequence of functions,

$$f = (f_1(x), f_2(x), \ldots),$$

and the scalar product in H has the form

$$(f, g) = \sum_{n=1}^{\infty} \int f_n(x)\overline{g_n(x)}\, d\mu_n(x).$$

The operator A takes each of the functions $f_n(x)$ into the function $xf_n(x)$, and consequently

$$Af = (xf_1(x), xf_2(x), \ldots).$$

The space H has been realized as a direct sum of spaces of functions, in each of which the scalar product is defined by means of some positive measure. Now we show that one can obtain a realization of H as a direct sum of spaces of functions such that the scalar product in each of these spaces H_n is defined by the same measure μ. This measure is defined by

$$\mu(\Delta) = \sum_{n=1}^{\infty} \frac{1}{2^n} \mu_n(\Delta) \tag{5}$$

(since for any n we have

$$\mu_n(\Delta) = (E(\Delta)h_n, h_n) \leqslant \| h_n \| = 1,$$

the series (5) converges). The measure μ has the following property, which follows at once from (5): if $\mu(\Delta) = 0$ for some set Δ, then $\mu_n(\Delta) = 0$ for every n.

By the Radon–Nikodym theorem[11] it follows that each of the measures μ_n can be written in the form

$$\mu_n(\Delta) = \int_\Delta \varphi_n(x)\, d\mu(x),$$

where $\varphi_n(x)$ is a nonnegative function. We denote by L^2_μ the Hilbert space consisting of all functions $\psi(x)$ for which the integral

$$\int |\psi(x)|^2\, d\mu(x)$$

converges. Obviously if $h_n(x)$ is a function in H_n, then the function $\psi_n(x) = \sqrt{\varphi_n(x)}\, h_n(x)$ belongs to L^2_μ, and

$$\| \psi_n(x) \|^2_\mu = \int |\psi_n(x)|^2\, d\mu(x) = \int |h_n(x)|^2 \varphi_n(x)\, d\mu(x)$$

$$= \int |h_n(x)|^2\, d\mu_n(x) = \| h_n(x) \|^2.$$

In other words, $h_n(x) \to \psi_n(x)$ is an isometric mapping of H_n into L^2_μ. ▼ Now let A_n be the set of x for which $\varphi_n(x) > 0$. It is clear that the mapping $h_n(x) \to \psi_n(x)$ is an isometric mapping of H_n *onto* $L^2_\mu(A_n)$, the set of all functions defined on A_n and having square integrable moduli relative to μ. We thus have an isomorphism

$$H \leftrightarrow L^2_\mu(A_1) \oplus L^2_\mu(A_2) \oplus \ldots$$

[11] Concerning the Radon–Nikodym theorem, see the footnote on p. 351.

Now define $n(x)$ as the number of values of m, $m = 1, 2, \ldots$ for which $x \in A_m$. The values of $n(x)$ are $0, 1, \ldots, \infty$ and $n(x)$ is clearly measurable; further, $\mu(\{n(x) = 0\}) = 0$. Let $B_n = \{n(x) = n\}$. For each element

$$\{\psi_1(x), \psi_2(x), \ldots\} \in \sum_{i=1}^{\infty} \oplus L^2_\mu(A_i)$$

we construct, for each $n = 1, 2, \ldots, \infty$, a set of functions $\varphi_1^{(n)}(x), \ldots, \varphi_n^{(n)}(x)$ defined on B_n, in the following way. Fix n and $x \in B_n$; let $m_1(x) < m_2(x) < \ldots < m_n(x)$ be those values of m for which $x \in A_m$. We define

$$\varphi_1^{(n)}(x) = \psi_{m_1}(x), \ldots, \varphi_n^{(n)}(x) = \psi_{m_n}(x).$$

It is not hard to see that this mapping (i.e., of sequences $\{\psi_i(x)\}$ into infinite sets $\{\varphi_j^{(n)}(x)\}$) is one-to-one, that the $\varphi_j^{(n)}(x)$ are measurable, and that

$$\sum_{n=1}^{\infty} \int_{B_n} \sum_{i=1}^{n} |\varphi_i^{(n)}(x)|^2 \, d\mu(x) = \int \sum_{n=1}^{\infty} |\psi_n(x)|^2 \, d\mu(x).$$

Now for each $x \in B_n$ we choose, in an n-dimensional Hilbert space $H(x)$, an orthonormal basis $e_1(x), \ldots, e_n(x)$, and with each $f \in H$ we associate the vector

$$\varphi_1^{(n)}(x) e_1(x) + \ldots + \varphi_n^{(n)}(x) e_n(x) \in H(x),$$

where the $\varphi_j^{(n)}(x)$ are as above. We see thus that H is represented as a direct integral

$$\mathfrak{h} = \int \oplus \, H(x) \, d\mu(x)$$

of the Hilbert spaces $H(x)$ with respect to the measure μ. Since the correspondence $f \leftrightarrow \{\psi_1(x), \psi_2(x), \ldots\}$ implies $Af \leftrightarrow \{x\psi_1(x), x\psi_2(x), \ldots\}$, it follows from the definition of the $\varphi_j^{(n)}(x)$ that if $f \leftrightarrow \{f(x)\}$, where ▲ $f(x) \in H(x)$, then $Af \leftrightarrow \{xf(x)\}$. The operator A (considered in $\mathfrak{h}$) is defined on the set of those vector functions $\{f(x)\} \in \mathfrak{h}$ for which the integral

$$\int |x|^2 \|f(x)\|^2 \, d\mu(x)$$

converges.

We have thus proven the following theorem.

Theorem 2. Let A be a self-adjoint operator in a Hilbert space H. Then H can be represented as a direct integral

$$\mathfrak{h} = \int \oplus H(x)\, d\mu(x)$$

of Hilbert spaces $H(x)$ relative to a positive measure μ in such a way that to the operator A there corresponds in $\mathfrak{h}$ the operator of multiplication by x.

CHAPTER II

Positive and Positive-Definite Generalized Functions

1. Introduction

In this chapter we will discuss a number of results of the theory of generalized functions, related in one way or another to the concepts of positivity and positive definiteness. At the focus of our attention will be the question of defining such generalized functions by means of positive measures on various sets. For continuous functions the classical example of such a means of definition is given by Bochner's theorem, which says that every continuous positive-definite function is the Fourier transform of a positive measure. Here we will deal with various generalizations of this theorem. In particular, we will consider conditionally positive-definite generalized functions, which have useful applications in the theory of random processes (cf. Section 4).

The subsequent part of this chapter is devoted to the theory of evenly positive-definite generalized functions. This theory gives an example of how the uniqueness or nonuniqueness of the positive measure defining a generalized function depends upon the *a priori* estimates imposed upon the function. A typical theorem in this range of topics is a theorem concerning even functions $f(x)$, for which the kernel

$$K(x, y) = f(x + y) + f(x - y)$$

is positive-definite. As M. G. Krein showed [reference (39); see also A. Ya. Povzner, reference (54)], such functions are Fourier transforms of positive measures, concentrated on the real and imaginary axes. Here, in distinction from Bochner's theorem, the measure μ which gives the function $f(x)$ is not always uniquely defined, but only under certain assumptions concerning the growth of the function at infinity. *We consider the cases of uniqueness and nonuniqueness of the measure μ to be fundamentally different.* Within the class of uniqueness, the proof of the existence theorem can be carried out by general methods, and it is possible to carry over the theorem from functions of one variable to functions of several variables. At the same time examples are known to us which show that outside the class of uniqueness, the existence theorem is valid only for functions of one variable. A similar situation also occurs in the theory of moments.

The methods used in this chapter show that the general plan, which consists in constructing for each problem an appropriate space of test or generalized functions, furnishes the key to the solution of the question in the present case. For example, in the theorem of M. G. Krein it turned out to be expedient to consider functions which grow more slowly than every one of the functions exp (ax^2), $a > 0$ (functions which are linear functionals on the space $S^{\frac{1}{2}}_{\frac{1}{2}}$; cf. p. 198).

The general methods which can be applied for the study of the indicated range of questions break up into spectral methods and methods connected with the use of normed rings. With the help of spectral methods, A. G. Kostyuchenko and B. S. Mityagin recently proved a number of theorems on positive-definite generalized functions. This method borders upon M. G. Krein's method and is a synthesis of the methods of M. G. Krein [reference (39)] and I. M. Gel'fand and A. G. Kostyuchenko [reference (18)]. The main difficulty in the use of spectral methods is the proof of the self-adjointness of the differential or difference operators which appear. However, this difficulty can be successfully overcome by considering the Cauchy problem associated with these operators and using the results of Volume III. The other general method —that of normed rings—has been applied to the range of questions under consideration in a paper of I. M. Gel'fand and M. A. Naimark (19).

The problems considered here concerning positive-definite functions are interesting in that they show the close connection among uniqueness in the Cauchy problem, quasi-analytic functions, self-adjoint operators, moment problems, and normed rings. As was already stated, in the class of uniqueness a general method is possible. This method ought presumably to be a first step toward a properly constructed theory of linear topological rings, which would enable one to combine the various approaches indicated above. The problems considered in this chapter should also serve as a beginning in the creation of such a theory. Since this theory has not yet been constructed, and in order not to introduce unnecessary complications, we have discussed these problems by elementary methods, not based on any general theory.

1.1. Positivity and Positive Definiteness

We will call a generalized function F *positive*, and write $F \geqslant 0$, if $(F, \varphi) \geqslant 0$ for every positive[1] test function. For example, $\delta(x) \geqslant 0$, and

[1] A test function $\varphi(x)$ is called *positive* if $\varphi(x) \geqslant 0$ for every $x = \{x_1, ..., x_n\}$. If $\varphi(x) > 0$, the function is called *strictly positive*.

$\delta'(x)$ and $\delta''(x)$ do not possess this property. In many cases positive generalized functions can be defined by means of positive measures μ, i.e., they have the form

$$(F, \varphi) = \int \varphi(x)\, d\mu(x),$$

where the positive measure μ satisfies some growth condition at infinity (these growth conditions depend upon which space of test functions one considers the generalized functions to be defined on).

Let us consider the space C_A of real functions $\varphi(x)$, continuous on a bounded closed set A of n-dimensional space (for example, A can be the ball $|x| \leqslant a$.[2]

According to a theorem of F. Riesz,[3] any positive linear functional F on the space C_A is given by a uniquely defined finite positive measure μ on the Borel subsets of A, i.e.,

$$(F, \varphi) = \int \varphi(x)\, d\mu(x).$$

An arbitrary linear functional F can be represented in the same way, but only with a measure μ which takes positive and negative values. Finally, if the space C_A consists of complex functions, and the functional F is also complex, then one has the same representation with a complex measure.

The theorem of Riesz also remains valid for certain other spaces of continuous functions. Thus, any linear positive functional on the space $C(a)$ of functions $\varphi(x)$ which are continuous in the ball $|x| \leqslant a$ and vanish for $|x| \geqslant a$ is also given by a positive finite measure.

Another concept which is studied in this chapter and is connected with the concept of positivity is that of *positive definiteness*. Positive-definite functions arise in considering the Fourier transforms of positive summable functions, and appear in such different topics as probability theory (cf. Chapter III), the theory of group representations (cf. Volume V), and many other areas of mathematics. To better orient the reader we will now formulate certain results regarding positive-definite functions.[4] We first consider functions of one variable.

[2] Let us recall notation:

$$x = \{x_1, \ldots, x_n\} \qquad \text{and} \qquad |x| = \sqrt{x_1^2 + \ldots + x_n^2}\,.$$

[3] Cf. S. L. Lusternik and V. I. Sobolev, "Elements of Functional Analysis," Chapter III, Section 22. Ungar, New York, 1962.

[4] We will not formally make use of these results. Moreover, they will be derived in Section 3 from more general theorems which we shall prove.

Let $f(x)$ be the Fourier transform of a positive summable function $F(\lambda)$,

$$f(x) = \int_{-\infty}^{\infty} F(\lambda) e^{i\lambda x} \, d\lambda.$$

In this case $f(x)$ has the following properties: It is continuous, and for any real numbers $x_1, \ldots, x_m$ and complex numbers $\xi_1, \ldots, \xi_m$ one has

$$\sum_{k,j=1}^{m} f(x_k - x_j) \xi_k \bar{\xi}_j \geqslant 0. \tag{1}$$

In fact, substituting for $f(x)$, in the left side of this inequality, its expression in terms of $F(\lambda)$, we obtain

$$\sum_{k,j=1}^{m} f(x_k - x_j) \xi_k \bar{\xi}_j = \sum_{k,j=1}^{m} \xi_k \bar{\xi}_j \int_{-\infty}^{\infty} \exp\,[i\lambda(x_k - x_j)] F(\lambda) \, d\lambda$$

$$= \int_{-\infty}^{\infty} \left| \sum_{j=1}^{m} \exp\,(i\lambda x_j) \xi_j \right|^2 F(\lambda) \, d\lambda.$$

Since $F(\lambda)$ is positive, the expression standing on the right side is either positive or zero; hence inequality (1) follows.

A function $f(x)$, satisfying inequality (1) for any real numbers $x_1, \ldots, x_m$ and complex numbers $\xi_1, \ldots, \xi_m$, is called *positive-definite*. We have therefore proven that the Fourier transform of any positive summable function is positive-definite. In precisely the same way one proves the continuity and positive definiteness of every function $f(x)$ having the form

$$f(x) = \int_{-\infty}^{\infty} e^{i\lambda x} \, d\mu(\lambda), \tag{2}$$

where μ is a positive finite measure on the line.[5]

Now consider a positive finite measure μ in n-dimensional space. The expression

$$f(x) = \int e^{i(\lambda, x)} \, d\mu(\lambda) \tag{2'}$$

(where, as usual, $x = \{x_1, \ldots, x_n\}$, $\lambda = \{\lambda_1, \ldots, \lambda_n\}$, and $(\lambda, x) = \lambda_1 x_1$

[5] This assertion is a generalization of that proven earlier, since to every positive summable function $F(\lambda)$ there corresponds a positive finite measure μ defined by

$$\mu(A) = \int_A F(\lambda) \, d\lambda.$$

$+ \ldots + \lambda_n x_n)$ defines a continuous function $f(x)$, positive-definite in the following sense.

For any points $x_1, \ldots, x_m$ of the n-dimensional space R_n and any complex numbers $\xi_1, \ldots, \xi_m$ one has

$$\sum_{k,j=1}^{m} f(x_k - x_j)\xi_k\bar{\xi}_j \geqslant 0$$

($x_k - x_j$ denotes the point with coordinates $x_{k1} - x_{j1}, \ldots, x_{kn} - x_{jn}$, where $x_k = \{x_{k1}, \ldots, x_{kn}\}$, $x_j = \{x_{j1}, \ldots, x_{jn}\}$).

As S. Bochner showed, the functions of the form (2′) exhaust the class of continuous positive-definite functions, namely, every positive-definite function is the Fourier transform[6] of some positive finite measure μ.

We have considered the Fourier transforms of positive summable functions (or, in formulas (2) and (2′), of positive finite measures). A substantial generalization of Bochner's theorem is obtained if one considers the Fourier transforms of positive functions which are not summable over the entire space. It is natural to introduce the definition of positive definiteness in such a way that the Fourier transforms of such functions (as well as the Fourier transforms of infinite positive measures) also will be positive-definite. We know that the Fourier transforms of such functions are generalized functions. Obviously the definition of positive definiteness expressed by inequality (1) does not carry over to generalized functions, since for generalized functions the notion of their value at a point is lacking and therefore the expression $f(x_k - x_j)$ is meaningless. However, it can be shown that the definition of positive definiteness which we gave for continuous functions is equivalent to the following:

[6] By the Fourier transform $\tilde{\mu}$ of a measure μ (in general, complex) we understand the Fourier transform of the corresponding generalized function (μ, φ), defined for test functions $\varphi(\lambda)$ by

$$(\mu, \varphi) = \int \varphi(\lambda)\, \overline{d\mu(\lambda)}.$$

In other words, by definition we set

$$(\tilde{\mu}, \tilde{\varphi}) = (2\pi)^n \int \varphi(\lambda)\, \overline{d\mu(\lambda)},$$

where

$$\tilde{\varphi}(x) = \int e^{i(\lambda, x)}\varphi(\lambda)\, d\lambda.$$

It is not hard to verify that if the measure μ is finite, then the generalized function $\tilde{\mu}$ is given by a continuous function $f(x)$, where

$$f(x) = \int e^{i(\lambda, x)}\, d\mu(\lambda).$$

a continuous function $f(x)$ is called *positive-definite*, if for any infinitely differentiable function $\varphi(x)$ with bounded support one has

$$\iint \overline{f(t-y)}\varphi(t)\overline{\varphi(y)}\,dt\,dy \geqslant 0 \tag{3}$$

(the proof of equivalence is carried out in Section 3.2).

Let us rewrite this definition so that it can be carried over to generalized functions. For this we make the substitution $t - y = x$ in the integral in (3). Then (3) becomes

$$\int \overline{f(x)} \int \varphi(t)\overline{\varphi(t-x)}\,dt\,dx \geqslant 0. \tag{3'}$$

But the integral $\int \varphi(t)\overline{\varphi(t-x)}\,dt$ represents simply the convolution[7] of the functions $\varphi(x)$ and $\varphi^*(x) = \overline{\varphi(-x)}$

$$\int \varphi(t)\overline{\varphi(t-x)}\,dt = \varphi * \varphi^*(x).$$

Thus, we find that a continuous function $f(x)$ is positive definite if the functional

$$(f, \psi) = \int \overline{f(x)}\psi(x)\,dx$$

corresponding to it assumes nonnegative values for all functions of the form $\psi(x) = \varphi * \varphi^*(x)$, i.e., if $(f, \varphi * \varphi^*) \geqslant 0$ for every infinitely differentiable function φ with bounded support. In this form the definition of positive definiteness can be extended to generalized functions. Namely, let F be a generalized function on the space K of infinitely differentiable functions with bounded supports. We will call F *positive-definite* if for every test function $\varphi(x)$ the inequality $(F, \varphi * \varphi^*) \geqslant 0$ holds, where $\varphi^*(x) = \overline{\varphi(-x)}$. Further on, we will extend this definition to generalized functions on other spaces of test functions.

L. Schwartz has generalized Bochner's theorem to positive-definite generalized functions on the space K. In order to formulate the theorem which he obtained, we introduce the notion of a tempered measure. We call a positive measure μ *tempered* if the integral $\int (1 + |\lambda|^2)^{-p}\, d\mu(\lambda)$ converges[†] for some $p \geqslant 0$. The theorem of Bochner–Schwartz

[7] We recall that the *convolution* of the test functions $\varphi(x)$ and $\psi(x)$ is the function

$$\varphi * \psi(x) = \int \varphi(t)\psi(x-t)\,dt.$$

[†] In certain cases (for instance, Theorem 1 of Section 4), where μ is not necessarily finite in any neighborhood of zero, the region of integration in this definition is presumably taken to be of the form $|\lambda| \geqslant a > 0$.

asserts that *the class of positive-definite generalized functions* (i.e., functionals) *on the space* K *coincides with the class of Fourier transforms of positive tempered measures.* In other words, every such generalized function F can be written in the form

$$(F, \varphi) = \int \tilde{\varphi}(\lambda)\, d\mu(\lambda), \tag{4}$$

where $\tilde{\varphi}(\lambda)$ is the Fourier transform of the test function $\varphi(x)$, and μ is a positive tempered measure. Conversely, every generalized function of the form (4) is positive definite.

As examples of positive-definite generalized functions, one can take the Fourier transforms of the functions $|x|^\lambda$, x_+^λ, x_-^λ, where $\lambda \geqslant 0$, etc. (concerning the definition of these Fourier transforms, cf. Vol. I, Chapter I, Sections 2 and 3). In particular, the positive-definite generalized functions $\delta(x)$ and $-\delta''(x)$ are respectively the Fourier transforms of the positive functions 1 and x^2. However, we can ascertain the positive definiteness of these generalized functions directly. In fact, for any infinitely differentiable function $\varphi(x)$ with bounded support we have

$$\int \delta(t-y)\varphi(t)\overline{\varphi(y)}\, dy\, dt = \int |\varphi(t)|^2\, dt \geqslant 0$$

and

$$-\int \delta''(t-y)\varphi(t)\overline{\varphi(y)}\, dy\, dt = -\int \varphi(t)\overline{\varphi''(t)}\, dt = \int |\varphi'(t)|^2\, dt \geqslant 0.$$

The Bochner–Schwartz theorem is connected with the Fourier transforms of positive functions having power growth (and also tempered measures). One can also consider the Fourier transforms of rapidly growing functions, which are usually linear functionals on spaces of analytic functions. Let us consider, for example, the positive function e^{cx}, where c is a real number. Its Fourier transform is the generalized function $F = 2\pi\delta(\lambda - ic)$ on the space Z of entire analytic functions of exponential type, rapidly decreasing on the real axis together with all of their derivatives (cf. Volume I, Chapter II, Section 2.2). This generalized function is positive-definite in the following sense. Let $\varphi(z)$ be a function in Z. We denote by $\varphi^*(z)$ the function in Z defined by $\varphi^*(z) = \overline{\varphi(-\bar{z})}$. If z is real, $z = x$, then $\varphi^*(x) = \overline{\varphi(-x)}$, so that this notation agrees with that introduced earlier.

A generalized function F on Z is called positive-definite, if $(F, \varphi * \varphi^*) \geqslant 0$ for every $\varphi(z) \in Z$.[8]

[8] Since every function $\varphi(z) \in Z$, when considered for real values of z, belongs to the space S, the notion of convolution is defined for these functions; the convolution of two functions from Z belongs to Z.

We shall also study other generalizations of the notion of positive definiteness in this chapter.

2. Positive Generalized Functions

We have already mentioned, in the introduction to this chapter, the connection between positive measures and positive-definite generalized functions. This connection consists in the following. The Fourier transformation takes positive-definite functions F into generalized functions $\tilde{F}$ having the following property, which we will call the property of *multiplicative positivity*: For any test function $\tilde{\varphi}(\lambda)$ of the dual space the inequality $(\tilde{F}, \tilde{\varphi}\bar{\tilde{\varphi}}) \geqslant 0$ holds.

Multiplicative positivity is, in general, a weaker requirement than positivity for generalized functions. However, as we shall see below, for many spaces of test functions the class of multiplicatively positive linear functionals coincides with the class of positive functionals. These, as a rule, are given by positive measures. In the present section we will also study positive linear functionals on certain spaces of test functions and establish the connection between the concepts of positivity and multiplicative positivity.

2.1. Positive Generalized Functions on the Space of Infinitely Differentiable Functions Having Bounded Supports

A generalized function F is called *positive*, if

$$(F, \varphi) \geqslant 0$$

for any positive test function φ.

In this paragraph we will study positive generalized functions on the space K.

Theorem 1. Every generalized function F such that $(F, \varphi) \geqslant 0$ for every infinitely differentiable function $\varphi(x)$ with bounded support has the form

$$(F, \varphi) = \int \varphi(x)\, d\mu(x),$$

where μ is some positive measure (not necessarily finite).

Conversely, every positive measure μ defines a positive linear function F on K by the above formula.

To prove this theorem we will first show that any positive linear functional on K can be extended, preserving its positivity, to the space C of all continuous functions having bounded supports. The topology in C is defined in the following manner: A sequence $\{\varphi_m(x)\}$ of functions in C converges to zero if every one of the functions $\varphi_m(x)$ vanishes outside some fixed ball $|x| \leqslant a$ and the sequence $\{\varphi_m(x)\}$ converges uniformly to zero.

Lemma 1. Every positive linear functional on the space K is continuous on K relative to the topology of C.

Proof. Let $\{\varphi_m(x)\}$ be a sequence of functions in K, converging to zero in the sense of the topology of C. In other words, suppose that the $\varphi_m(x)$ vanish outside the ball $|x| \leqslant a$ and converge uniformly to zero. Then for any $\epsilon > 0$ there is an n such that

$$-\epsilon \leqslant \varphi_m(x) \leqslant \epsilon \tag{1}$$

for $m \geqslant n$. We multiply every term of (1) by a positive function $\alpha(x) \in K$ such that $\alpha(x) = 1$ for $|x| \leqslant a$. Since $\varphi_m(x) = 0$ for $|x| \geqslant a$, $\alpha(x)\varphi_m(x) = \varphi_m(x)$. Therefore we obtain

$$-\epsilon\alpha(x) \leqslant \varphi_m(x) \leqslant \epsilon\alpha(x).$$

Applying the positive functional F to this inequality, we obtain

$$-\epsilon(F, \alpha) \leqslant (F, \varphi_m) \leqslant \epsilon(F, \alpha).$$

Since ϵ is arbitrary, it follows that $\lim_{m\to\infty}(F, \varphi_m) = 0$.[1] This proves the continuity of F relative to the topology of C.

Similar considerations show that if a sequence of functions $\{\varphi_m(x)\}$ in K is fundamental in the sense of the topology of C, and F is a positive linear functional, then the numerical sequence $\{(F, \varphi_m)\}$ is also fundamental. From this it follows that every positive linear functional on K can be extended by continuity to the space C of continuous functions having bounded supports.

In fact, if $\varphi(x) \in C$, then there exists a sequence $\{\varphi_m(x)\}$ in K which converges to $\varphi(x)$ in the topology of C. One can set, for example,

[1] The functional F cannot be applied directly to the terms of inequality (1), since a constant does not have bounded support. We will frequently apply the device of multiplying by a function $\alpha(x)$ in similar cases further on.

$\varphi_m(x) = \varphi * \alpha_m(x)$, where $\{\alpha_m(x)\}$ is some δ-sequence,[2] consisting of functions in K. But then the functions $\{\varphi_m(x)\}$ form a fundamental sequence in the sense of the topology of C. By an earlier remark the numerical sequence $\{(F, \varphi_m)\}$ also will be fundamental. Setting $(F, \varphi) = \lim_{m\to\infty}(F, \varphi_m)$, we extend F to the function φ. The reader can convince himself without difficulty that the linear functional thus obtained is continuous on all of C. It is also positive on C, since if $\varphi(x) \in C$ is positive, the approximating sequence $\{\varphi_m(x)\}$ can likewise be chosen positive (choosing, for example, the functions of the δ-sequence to be positive). But then we obtain

$$(F, \varphi) = \lim_{m\to\infty} (F, \varphi_m) \geqslant 0.$$

Thus, we have proven that every positive linear functional on K can be extended in a unique way, preserving its positivity, to the space C of continuous functions having bounded supports. We now give a description of positive linear functionals on C.

Lemma 2. Every positive linear functional on C is given by

$$(F, \varphi) = \int \varphi(x)\, d\mu(x),$$

where μ is some positive measure (generally speaking, not finite).

Proof. The functional F is defined on every continuous function having bounded support. Therefore it is defined also on every space $C(a)$ consisting of those continuous functions $\varphi(x)$ such that $\varphi(x) = 0$ for $|x| \geqslant a$. By a theorem of F. Riesz, on each of the spaces $C(a)$ the functional F has the form $(F, \varphi) = \int \varphi(x)\, d\mu_a(x)$, where μ_a is a positive measure defined in the ball $|x| \leqslant a$. Since the measures μ_a are uniquely defined by the values of F, then for $a < b$ the measures μ_a and μ_b coincide in the ball $|x| \leqslant a$. Therefore there exists a measure μ which coincides in each ball $|x| \leqslant a$ with the measure μ_a. But then

$$(F, \varphi) = \int \varphi(x)\, d\mu(x)$$

for any function $\varphi(x) \in C$.

[2] By a δ-*sequence* or a δ-*convergent sequence* we mean a sequence of functions $\{\varphi_m(x)\}$ such that

$$\lim_{m\to\infty} \int f(x)\varphi_m(x)\, dx = (\delta, f) = f(0)$$

for any bounded continuous function $f(x)$. If $\{\varphi_m(x)\}$ is a δ-sequence, then $\{\varphi_m^*(x)\}$ is also a δ-sequence. The termwise convolution $\{\varphi_m * \psi_m(x)\}$ of two δ-sequences is a δ-sequence.

From Lemmas 1 and 2 it follows at once that every positive linear functional on K is given by a positive measure μ. This proves the first part of Theorem 1. The converse (second part of Theorem 1)—every positive measure μ defines a positive linear functional

$$(F, \varphi) = \int \varphi(x)\, d\mu(x)$$

on K—is obvious. This completes the proof of Theorem 1.

2.2. The General Form of Positive Generalized Functions on the Space S

Now we consider the space S of infinitely differentiable functions, rapidly decreasing as $|x| \to \infty$ together with their derivatives of all orders (we recall that $\varphi(x)$ is called rapidly decreasing if $\lim_{|x|\to\infty} | x^k \varphi(x)| = 0$ for every k).

Suppose that we are given a positive linear functional F on S. Then we are also given a positive linear functional on K (since K is contained in S, and convergence in K implies convergence in S). In view of the results of Section 2.1, there exists a positive measure μ such that F can be represented in the form

$$(F, \varphi) = \int \varphi(x)\, d\mu(x) \tag{2}$$

for every function $\varphi(x) \in K$.

In order that (2) be meaningful for every function $\varphi(x) \in S$, the measure μ has to satisfy certain growth conditions at infinity. Namely, we will show that the continuity of the functional F relative to the topology of S implies the convergence of the integral

$$\int (1 + |x|^2)^{-p}\, d\mu(x)$$

for some $p > 0$ (or, as we will usually say, the measure μ is tempered).

In order to prove the necessity of this condition, we use the following lemma of Fatou.[3]

Let μ be a positive measure, and $\{\varphi_m(x)\}$ a sequence of positive functions such that $\int \varphi_m(x)\, d\mu(x) \leqslant A$ for every m. If $\lim_{m\to\infty} \varphi_m(x) = \varphi(x)$ at every point x, then

$$\int \varphi(x)\, d\mu(x) \leqslant A.$$

[3] Cf. I. P. Natanson, "Theory of Functions of a Real Variable," Volume I, p. 160. Ungar, New York, 1955.

In view of this lemma, to prove that μ is tempered it suffices to construct a sequence $\{\varphi_m(x)\}$ of positive functions in K such that $\int \varphi_m(x)\, d\mu(x) \leqslant 1$ and

$$\lim_{m\to\infty} \varphi_m(x) = A(1 + |x|^2)^{-p}$$

for some $A > 0$ and $p > 0$. This sequence is constructed in the following way.

From the continuity of F relative to the topology of S follows the existence of a neighborhood U of zero in S such that $|(F, \varphi)| \leqslant 1$ for every $\varphi \in U$. In view of the definition of the topology in S this neighborhood is defined by positive integers p and k and a number η, and consists of those functions $\varphi(x) \in S$ satisfying, for $|q| \leqslant k$, the inequality

$$\sup_x |(1 + |x|^2)^p \varphi^{(q)}(x)| \leqslant \eta.$$

The desired functions $\varphi_m(x)$ are defined by

$$\varphi_m(x) = A\alpha\left(\frac{x}{m}\right)(1 + |x|^2)^{-p},$$

where $\alpha(x)$ is any infinitely differentiable function with bounded support such that $\alpha(x) = 1$ for $|x| \leqslant 1$. For sufficiently small values of A each of the $\varphi_m(x)$ belongs to the neighborhood U of zero[4] and therefore $(F, \varphi_m) \leqslant 1$. But since the $\varphi_m(x)$ have bounded supports, (F, φ_m) is given by formula (2). Consequently the $\varphi_m(x)$ satisfy the inequality

$$\int \varphi_m(x)\, d\mu(x) \leqslant 1. \tag{3}$$

Moreover, by the construction of the $\varphi_m(x)$ we have

$$\lim_{m\to\infty} \varphi_m(x) = A(1 + |x|^2)^{-p}$$

for every x. Passing to the limit in (3), we obtain, by Fatou's lemma,

$$A \int (1 + |x|^2)^{-p}\, d\mu(x) \leqslant 1.$$

But this shows that μ is tempered.

Thus, if a functional F, defined by (2), is continuous relative to the topology of S, then the measure μ corresponding to it is tempered.

[4] This follows easily from inequalities of the form

$$[(1 + |x|^2)^{-p}]^{(q)} \leqslant C_q(1 + |x|^2)^{-p}.$$

Conversely, if μ is a positive tempered measure, then the integral $\int \varphi(x)\, d\mu(x)$ converges for every $\varphi(x) \in S$ and defines a linear functional on S. The continuity of this functional easily follows from the definition of the topology in S.

We are now in a position to prove that formula (2) is valid for every $\varphi(x) \in S$. In fact, we have seen that the linear functionals (F, φ) and $\int \varphi(x)\, d\mu(x)$ coincide on the everywhere dense set in S consisting of functions having bounded supports. Moreover, both functionals are continuous relative to the topology of S. From this it follows that these functionals coincide over all of S.

Thus, we have proven the following theorem.

Theorem 2. Every positive generalized function F on the space S is given by a tempered measure μ;

$$(F, \varphi) = \int \varphi(x)\, d\mu(x). \tag{2}$$

Conversely, if μ is a positive tempered measure, then (2) defines a positive generalized function on S. The definition of a tempered measure is given on p. 140.

2.3. Positive Generalized Functions on Some Other Spaces[5]

We now apply the method, used in order to describe positive generalized functions on S, to a substantially wider class of linear topological spaces, in particular to spaces of type $K\{M_p\}$ and their unions.

Concerning the definition of spaces of type $K\{M_p\}$, cf. Chapter I, Section 3.6. We will restrict ourselves here to considering spaces $K\{M_p\}$ for which the following conditions are satisfied:

(a) The $M_p(x)$ are infinitely differentiable outside some neighborhood of zero (which is the same for all p) and are nowhere infinite;

(b) for any p there are numbers q, a, and C such that if $|x| \geqslant a$ and $1 \leqslant |k| \leqslant p$, then

$$\left[\frac{1}{M_q(x)}\right]^{(k)} \leqslant \frac{C}{M_p(x)}.$$

For such spaces $K\{M_p\}$ the positive generalized functions are described by the following theorem.

[5] This section can be omitted at the first reading.

Theorem 3. If the sequence of functions $M_p(x)$ satisfies conditions (a) and (b), then a positive generalized function F on $K\{M_p\}$ is given by a positive measure μ such that the integral $\int [M_p(x)]^{-1}\, d\mu(x)$ converges for some p. Conversely, every positive measure μ such that the integral $\int [M_p(x)]^{-1}\, d\mu(x)$ converges for some p defines a positive generalized function on $K\{M_p\}$.

The proof of this theorem proceeds in a manner entirely analogous to that of the corresponding theorem for the space S, which is the particular case of the space $K\{M_p\}$ corresponding to $M_p(x) = (1 + |x|^2)^p$. We leave it to the reader to carry through the details of the corresponding proofs.

We now consider a union of spaces of type $K\{M_p\}$. Such a union is given by a doubly indexed sequence of functions $\{M_{rp}(x)\}$. We require that for each fixed r the functions $M_{rp}(x)$ satisfy conditions (a) and (b) and the condition $1 \leqslant M_{r1}(x) \leqslant M_{r2}(x) \leqslant \dots$. We will denote the space corresponding to the sequence of functions $\{M_{rp}(x)\}$ (for fixed r) by $K_r\{M_{rp}\}$. Suppose now that for each p one has $M_{r+1,p} \leqslant M_{rp}$. Then if a function $\varphi(x)$ belongs to the space $K_r\{M_{rp}\}$, it also belongs to the space $K_{r+1}\{M_{r+1,p}\}$. Thus, we obtain an increasing chain of spaces

$$K_1\{M_{1p}\} \subset K_2\{M_{2p}\} \subset \dots .$$

The union of these spaces, with the corresponding topology, will also be called a space of type $K\{M_{rp}\}$ (a sequence $\{\varphi_m(x)\}$ in $K\{M_{rp}\}$ converges to zero if all the $\varphi_m(x)$ belong to some one space $K_r\{M_{rp}\}$ and converge to zero in this space).

Positive generalized functions on spaces of type $K\{M_{rp}\}$ are described by the following theorem.

Theorem 4. Suppose that we are given a doubly indexed sequence $\{M_{rp}(x)\}$ of functions satisfying conditions (a) and (b) above, and also the inequalities

$$1 \leqslant M_{r1}(x) \leqslant M_{r2}(x) \leqslant \dots ,$$

$$M_{rp}(x) \geqslant M_{r+1,p}(x).$$

Every positive generalized function on $K\{M_{rp}\}$ is given by a positive measure μ such that for any r the integral $\int [M_{rp}(x)]^{-1}\, d\mu(x)$ converges for some p. Conversely, if μ is a positive measure such that for any r the preceding integral converges for some p, then $(F, \varphi) = \int \varphi(x)\, d\mu(x)$ defines a positive generalized function on $K\{M_{rp}\}$.

Let us consider, for example, the space S_α, consisting of infinitely differentiable functions $\varphi(x)$ satisfying the inequalities

$$| x^k \varphi^{(q)}(x)| \leqslant C_q A^k k^{k\alpha},$$

where the constants C_q and A depend upon $\varphi(x)$.

As was shown in Volume II (Chapter IV, Section 3.1), this space is a space of type $K\{M_{rp}\}$, where

$$M_{rp}(x) = \exp [r^{-1}(1 - p^{-1})| x |^{1/\alpha}].$$

We have, from Theorem 4, the following description of positive linear functionals on S_α.

Theorem 5. Every positive linear functional on S_α is given by a positive measure μ such that for any $r > 0$ the integral

$$\int \exp[-r^{-1}(1 - p^{-1})| x |^{1/\alpha}] \, d\mu(x) \tag{3}$$

converges for some $p > 0$. Conversely, if a measure μ is such that for each $r > 0$ the integral (3) converges for some $p > 0$, then $(F, \varphi) = \int \varphi(x) \, d\mu(x)$ defines a positive linear functional on S_α.

We remark that the condition that for any $r > 0$ the integral (3) converge for some $p > 0$ is equivalent to the condition that the integral $\int \exp (-a \mid x |^{1/\alpha}) \, d\mu(x)$ converge for any $a > 0$.

2.4. Multiplicatively Positive Generalized Functions

Many of the spaces which we are considering (in particular, the spaces K and S) are linear algebras, i.e., the products of functions belonging to one of these spaces also belong to this same space. In such spaces there exists, together with the concept of positivity which was investigated above, another concept—that of multiplicative positivity. It will be used further on in an essential way for the study of positive-definite generalized functions.

A generalized function F is called *multiplicatively positive* if $(F, \varphi\bar{\varphi}) \geqslant 0$ for every test function $\varphi(x)$.[6]

[6] The spaces considered contain, together with every function $\varphi(x)$, the function $\overline{\varphi(x)}$. The passage from $\varphi(x)$ to $\overline{\varphi(x)}$ has the usual properties of an involution.

Every positive generalized function is multiplicatively positive. In fact, since $\varphi(x)\overline{\varphi(x)} \geqslant 0$, the inequality $(F, \psi) \geqslant 0$ for all positive test functions implies the inequality $(F, \varphi\bar{\varphi}) \geqslant 0$ for all test functions $\varphi(x)$.

The converse statement is in general false—there exist linear spaces (for example, the space of all real polynomials in two variables) in which the validity of the inequality $(F, \varphi\bar{\varphi}) \geqslant 0$ for all test functions φ belonging to the space does not by itself imply the inequality $(F, \psi) \geqslant 0$ for all positive test functions $\psi(x)$ in the space (cf. Section 7). However, *for the spaces K and S every multiplicatively positive generalized function is positive.*

We will first carry out the proof of this assertion for the space K. Let F be a generalized multiplicatively positive function on K. Since F is continuous, it suffices, for the proof of its positivity, to show that the functions of the form $\varphi(x)\overline{\varphi(x)}$ are everywhere dense in the set of positive functions in K.[7]

Let us consider a positive function $\psi(x) \in K$. This function vanishes for $|x| \geqslant a$ for $a > 0$ sufficiently large. We denote by $\alpha(x)$ a positive function in K such that $\alpha(x) = 1$ for $|x| \leqslant a$, and we set $\varphi_m(x) = \alpha(x) \sqrt{\psi(x) + 1/m}$. Then obviously $\varphi_m(x)$ belongs to K,[8] and the sequence $\varphi_m^2(x) \equiv \varphi_m(x)\overline{\varphi_m(x)}$ converges to $\psi(x)$ in the topology of K.

Thus, we have proven that the set of all functions of the form $\varphi(x)\overline{\varphi(x)}$ is everywhere dense in the set of positive functions in K. Therefore the validity of the inequality $(F, \varphi\bar{\varphi}) \geqslant 0$ for all functions $\varphi(x) \in K$ implies that of the inequality $(F, \psi) \geqslant 0$ for all positive $\psi(x) \in K$, i.e., the positivity of F.

Let us now prove that the concepts of positivity and multiplicative positivity coincide for generalized functions on the space S.

Suppose that $\psi(x)$ is a positive function in S. There exists a sequence of positive functions $\varphi_m(x) \in K$ converging to $\psi(x)$ in the topology of S.[9] Therefore the set of positive functions in K is everywhere dense in the set of positive functions in S. But as we have seen above, the set of functions of the form $\varphi(x)\overline{\varphi(x)}$, where $\varphi(x) \in K$, is everywhere dense in the set of positive functions in K. Since the imbedding of K into S is

[7] We note that the set of functions of the form $\varphi(x)\overline{\varphi(x)}$, $\varphi(x) \in K$, does not coincide with the set of all positive functions in K. This is connected with the fact that the function $\sqrt{\psi(x)}$ can have a break at those points where $\psi(x) = 0$.

[8] Since $\psi(x) + 1/m$ does not vanish, the function $\sqrt{\psi(x) + 1/m}$ is infinitely differentiable.

[9] For example, we set $\varphi_m(x) = \alpha(x/m)\psi(x)$, where $\alpha(x) \in K$, $0 \leqslant \alpha(x) \leqslant 1$, and $\alpha(x) = 1$ in some neighborhood of $x = 0$. Then the sequence $\{\varphi_m(x)\}$ has the required properties.

continuous, the set of functions of the form $\varphi(x)\overline{\varphi(x)}$ is also everywhere dense in the set of positive functions in S. But then the positivity of a generalized function F follows from its multiplicative positivity. Thus, we have proven that the concepts of positivity and multiplicative positivity also coincide in the space S.[10]

Since we know the general form of positive generalized functions on the spaces K and S, the assertions just proven imply the following theorems.

Theorem 6. Every multiplicatively positive generalized function on the space K of all infinitely differentiable functions having bounded supports is given by a positive measure.

Theorem 7. Every multiplicatively positive generalized function, on the space S of infinitely differentiable functions which are rapidly decreasing as $|x| \to \infty$ together with their derivatives of all orders, is given by a positive tempered measure.

3. Positive-Definite Generalized Functions. Bochner's Theorem

In this section we find the general form of positive-definite generalized functions on the spaces K and S. We will show that the positive-definite generalized functions on these spaces are Fourier transforms of positive tempered measures. We begin our consideration of positive-definite generalized functions with the space S, as this case is simpler.

3.1. Positive-Definite Generalized Functions on S

A generalized function F on the space S is called *positive-definite*, if $(F, \varphi * \varphi^*) \geqslant 0$ for every $\varphi(x) \in S$, where $\varphi * \varphi^*(x)$ denotes the convolution of the functions $\varphi(x)$ and $\varphi^*(x) = \overline{\varphi(-x)}$;

$$\varphi * \varphi^*(x) = \int \varphi(t)\overline{\varphi(t-x)}\, dt.$$

We show: The Fourier transformation carries positive-definite generalized functions on S into multiplicatively positive generalized functions

[10] One can similarly prove that the concepts of positivity and multiplicative positivity coincide in any space Φ under the condition that the positive functions in K are everywhere dense in the positive functions in Φ and that the mapping of K into Φ is continuous. This condition is fulfilled, for example, for all spaces $K\{M_{rp}\}$.

on S,[1] and every multiplicatively positive generalized function on S can be obtained in this manner.

In fact, the Fourier transformation takes the convolution of the functions $\varphi_1(x)$ and $\varphi_2(x)$ into the product of their Fourier transforms $\psi_1(\lambda)$ and $\psi_2(\lambda)$, and takes $\varphi^*(x)$ into $\overline{\psi(\lambda)}$, where $\psi(\lambda)$ is the Fourier transform of $\varphi(x)$. Therefore the Fourier transform of $\varphi * \varphi^*(x)$ is $\psi(\lambda)\overline{\psi(\lambda)}$, where $\psi(\lambda)$ is the Fourier transform of $\varphi(x)$.

Now since S is self-dual with respect to the Fourier transformation, any function of the form $\psi(\lambda)\overline{\psi(\lambda)}$, where $\psi(\lambda) \in S$, is the Fourier transform of a function of the form $\varphi * \varphi^*(x)$, $\varphi(x) \in S$. But by definition, one calls "the Fourier transformation of the generalized function F on S" that generalized function $\tilde{F}$ on S for which $(\tilde{F}, \tilde{\varphi}) = (2\pi)^n(F, \varphi)$ for every $\varphi(x) \in S$. Consequently, the validity of the inequality $(F, \varphi * \varphi^*) \geqslant 0$ for all $\varphi(x) \in S$ is equivalent to that of the inequality $(\tilde{F}, \psi\bar{\psi}) \geqslant 0$ for all $\psi(\lambda) \in S$. Therefore the positive definiteness of a generalized function F on S is equivalent to the multiplicative positivity of its Fourier transform $\tilde{F}$.

Since multiplicatively positive generalized functions on S are given by positive tempered measures (cf. Section 2.2), one has the following theorem.

Theorem 1. In order that a generalized function F on S be positive-definite, it is necessary and sufficient that it be the Fourier transform of a positive tempered measure.

3.2. Continuous Positive-Definite Functions

In this section we consider a special case of positive-definite generalized functions on the space S—that of continuous positive-definite functions. A continuous function $f(x)$ is called *positive-definite*, if for any real numbers $x_1, \ldots, x_m$ and complex numbers $\xi_1, \ldots, \xi_m$ one has

$$\sum_{j,k=1}^{m} f(x_k - x_j)\xi_k\bar{\xi}_j \geqslant 0. \tag{1}$$

We will show that a continuous positive-definite function $f(x)$ is a positive-definite generalized function on S. Before proving this, let us

[1] We recall that the space S is self-dual with respect to the Fourier transformation.

establish some simple properties of continuous positive-definite functions.

In the first place we note that if $f(x)$ is positive-definite, then $\overline{f(x)}$ is also, since

$$\sum_{j,k=1}^{m} \overline{f(x_k - x_j)}\xi_k\bar{\xi}_j = \overline{\sum_{j,k=1}^{m} f(x_k - x_j)\bar{\xi}_k\xi_j} \geqslant 0.$$

We further note that inequality (1) denotes the positive definiteness of the matrix with elements $f(x_k - x_j)$. It is known that the conditions for the positive definiteness of a matrix are that it be Hermitean and that its principal minors be nonnegative. From the first condition on the matrix $\| f(x_k - x_j) \|$ we obtain

$$f(-x) = \overline{f(x)},$$

and from the positivity of its diagonal elements it follows that

$$f(0) \geqslant 0.$$

Finally, since the numbers $x_k - x_j$ are arbitrary, the nonnegativity of the second-order principal minors implies

$$\begin{vmatrix} f(0) & f(x) \\ f(-x) & f(0) \end{vmatrix} \geqslant 0,$$

from which it follows that

$$|f(x)| \leqslant f(0),$$

i.e., $f(x)$ is bounded.

Thus we see that a positive-definite continuous function is bounded. Since every continuous bounded function $f(x)$ defines a generalized function

$$(f, \varphi) = \int \overline{f(x)}\varphi(x)\,dx$$

on S, we obtain the following assertion: Every positive-definite continuous function defines a generalized function on S.

We now show that this generalized function is positive-definite, i.e., that $(f, \varphi * \varphi^*) \geqslant 0$ for every $\varphi(x) \in S$. To this end, we note that the expression $(f, \varphi * \varphi^*)$ can be represented as an integral

$$\int \overline{f(x - y)}\varphi(y)\overline{\varphi(x)}\,dx\,dy.$$

This integral is the limit as $T \to \infty$ of the integral

$$\int_{-T}^{T}\int_{-T}^{T} \overline{f(x-y)}\varphi(y)\overline{\varphi(x)}\, dx\, dy \tag{2}$$

(since $\varphi(x)$ is summable and $f(x)$ is bounded). But for each T the integral (2) is the limit of sums

$$\sum_{j,k=1}^{m} \overline{f(x_k - x_j)}\varphi(x_k)\overline{\varphi(x_j)}\, \Delta x_k\, \Delta x_j,$$

which, in view of (1), are nonnegative. From this it follows that the inequality $(f, \varphi * \varphi^*) \geqslant 0$ holds for all $\varphi(x) \in S$, i.e., the generalized function (f, φ) is positive definite.

This proves our assertion—the validity of inequality (1) for a continuous function $f(x)$ implies the inequality $(f, \varphi * \varphi^*) \geqslant 0$ for all $\varphi(x) \in S$. In particular, this last inequality holds for every $\varphi(x) \in K$.

The converse assertion is also valid—if a continuous function $f(x)$ is such that

$$(f, \varphi * \varphi^*) = \int \overline{f(x-y)}\varphi(x)\overline{\varphi(y)}\, dx\, dy \geqslant 0 \tag{3}$$

for every $\varphi(x) \in S$, then $f(x)$ is positive definite, i.e., inequality (1) holds for all real numbers $x_1, \ldots, x_m$ and complex numbers $\xi_1, \ldots, \xi_m$. We will actually prove the more general inequality

$$\int \overline{f(x-y)}\, d\mu(x)\, \overline{d\mu(y)} \geqslant 0, \tag{4}$$

where μ is any finite measure which is concentrated on a bounded set (inequality (1) is a special case of inequality (4), in which the points $x_1, \ldots, x_m$ have μ-measure $\xi_1, \ldots, \xi_m$ respectively, and $f(x)$ is replaced by $\overline{f(x)}$).

In fact, suppose that μ is a finite measure with bounded support. Let $\{\varphi_m(x)\}$ be a δ-sequence of functions in the space K. Then the functions

$$\mu_m(x) = \int \varphi_m(x-y)\, \overline{d\mu(y)}$$

also belong to K. In fact, there is an A such that μ and $\varphi_m(x)$ vanish for $|x| \geqslant A$. But then $\mu_m(x) = 0$ for $|x| \geqslant 2A$ and therefore the functions

$\mu_m(x)$ have bounded supports. Since the $\varphi_m(x)$ are infinitely differentiable, so are the $\mu_m(x)$, and therefore the $\mu_m(x)$ belong to K.

The functions $\mu_m(x)$ are such that for any bounded continuous function $f(x)$ one has

$$\lim_{m\to\infty} (f, \mu_m * \mu_m^*) = \int f(x-y)\, d\mu(x)\, \overline{d\mu(y)}. \tag{5}$$

In fact,

$$\begin{aligned}(f, \mu_m * \mu_m^*) &= \int \overline{f(u-v)}\mu_m(u)\overline{\mu_m(v)}\, du\, dv \\ &= \int \overline{f(u-v)}\varphi_m(u-x)\overline{\varphi_m(v-y)}\, du\, dv\, d\mu(x)\, \overline{d\mu(y)}.\end{aligned} \tag{6}$$

Since $\{\varphi_m(x)\}$ is a δ-sequence,

$$\lim_{m\to\infty} \int \overline{f(u-v)}\varphi_m(u-x)\varphi_m(v-y)\, du\, dv = \overline{f(x-y)}.$$

Therefore, passing to the limit $m \to \infty$ in (6), we obtain (5).

Now let $f(x)$ be such that $(f, \varphi * \varphi^*) \geqslant 0$ for any $\varphi(x) \in K$. Then we have

$$(f, \mu_m * \mu_m^*) \geqslant 0.$$

Passing to the limit $m \to \infty$ and taking (5) into account, we obtain

$$\int f(x-y)\, d\mu(x)\, \overline{d\mu(y)} \geqslant 0.$$

We have therefore proven that the validity of inequality (3) for all $\varphi(x) \in K$ implies the validity of (4) for any finite measure with bounded support. In particular, from this it follows also that inequality (1) is satisfied.

Thus we see that for continuous functions $f(x)$ inequalities (1), (3), and (4) lead to mutually equivalent definitions of the notion of positive definiteness.

We have carried out the arguments for functions of one variable; however, the assertion is also valid in the case of several variables.

Now we proceed to the description of all continuous positive-definite functions (of several variables).

Theorem 2. (*Bochner*). Every continuous positive-definite function $f(x)$ is the Fourier transform of a finite positive measure μ.

Proof. We have already seen that a continuous positive-definite function defines a positive-definite generalized function on the space S. By Theorem 1 (Section 3.1) there exists a positive tempered measure μ such that the generalized function (f, φ) is the Fourier transform of μ. This means that

$$(f, \varphi) = (2\pi)^{-n} \int \tilde{\varphi}(\lambda)\, d\mu(\lambda) \tag{7}$$

for all functions $\varphi(x) \in S[\tilde{\varphi}(\lambda)$ is the Fourier transform of $\varphi(x)]$. It remains for us to show that the measure μ is finite (i.e., that $\int d\mu(\lambda) < +\infty$). For this we apply (7) to the functions $\varphi_m(x) = \alpha_m * \alpha_m^*(x)$, where $\{\alpha_m(x)\}$ is a δ-sequence in S. We obtain

$$(f, \varphi_m) = (2\pi)^{-n} \int \tilde{\varphi}_m(\lambda)\, d\mu(\lambda). \tag{8}$$

Now we pass to the limit $m \to \infty$. Since the termwise convolution of two δ-sequences is a δ-sequence, $\{\varphi_m(x)\}$ is a δ-sequence. From this it follows that the left side of (8) tends to $f(0)$ as $m \to \infty$. Now consider the right side of (8). From the relation

$$\lim_{m\to\infty} \tilde{\varphi}_m(\lambda) = \lim_{m\to\infty} \int e^{i(\lambda, x)} \varphi_m(x)\, dx = (\delta, e^{i(\lambda, x)}) = 1$$

it follows that $\tilde{\varphi}_m(\lambda) \to 1$ for any λ as $m \to \infty$. Moreover, these functions are positive in view of the relation $\tilde{\varphi}_m(\lambda) = |\tilde{\alpha}_m(\lambda)|^2$. Hence it follows that

$$\lim_{m\to\infty} \int \tilde{\varphi}_m(\lambda)\, d\mu(\lambda) \geqslant \int d\mu(\lambda)$$

and therefore

$$\int d\mu(\lambda) \leqslant (2\pi)^{-n} f(0).^2$$

This proves the finiteness of μ.

The converse assertion is also true: *if μ is any finite positive measure, then its Fourier transform*

$$f(x) = \int e^{i(\lambda, x)}\, d\mu(\lambda)$$

is a continuous positive-definite function.

[2] In fact, we have here equality.

We first prove that $f(x)$ is continuous. Since μ is finite, for any $\epsilon > 0$ there is an $N > 0$ such that

$$\int_{|\lambda| > N} d\mu(\lambda) < \tfrac{1}{4}\epsilon.$$

Let $| x_1 - x_2 | < \epsilon/2NM$, where M is the μ-measure of the ball $| \lambda | \leqslant N$. Then it is easily seen that

$$|f(x_1) - f(x_2)| \leqslant \tfrac{1}{2}\epsilon + \int_{|\lambda| \leqslant N} | \exp[i(\lambda, x_1)] - \exp[i(\lambda, x_2)] | \, d\mu(\lambda) \leqslant \epsilon.$$

This proves the continuity of $f(x)$. To prove its positive definiteness, we note that

$$\sum_{j,k=1}^{m} f(x_k - x_j)\xi_k\bar{\xi}_j = \sum_{j,k=1}^{m} \xi_k\bar{\xi}_j \int \exp[i(\lambda, x_k - x_j)] \, d\mu(\lambda)$$

$$= \int \left| \sum_{j=1}^{m} \exp[i(\lambda, x_j)]\xi_j \right|^2 d\mu(\lambda).$$

Since μ is positive, we find that

$$\sum_{j,k=1}^{m} f(x_k - x_j)\xi_k\bar{\xi}_j \geqslant 0,$$

i.e., that $f(x)$ is positive definite.

3.3. Positive-Definite Generalized Functions on K

In Section 3.1 we described the totality of positive-definite generalized functions in S'. The space K' is substantially richer in generalized functions than S'; nevertheless it turns out that in passing from S' to K' the totality of positive-definite generalized functions is not enlarged. In other words, although $S' \subset K'$, the class of positive-definite functions in K' is the same as that in S'. This class is described by the following theorem.

Theorem 3. (*Bochner–Schwartz*). Every positive-definite generalized function F on the space K of infinitely differentiable functions having bounded supports is the Fourier transform of a positive tempered measure μ, i.e., can be written as

$$(F, \varphi) = \int \tilde{\varphi}(\lambda)\, d\mu(\lambda). \tag{9}$$

Conversely, the Fourier transform of any positive tempered measure defines a positive-definite generalized function on K.

The first step in the proof of this theorem will be to consider certain continuous functions $F_\alpha(x)$ connected with the generalized function F. We prove the following assertion:

Lemma 1. If F is a positive-definite generalized function on K, and $\alpha(x)$ is any function in K, then the generalized function $F_\alpha = \alpha * \alpha^* * F$, defined for all $\varphi \in K$ by

$$(F_\alpha, \varphi) = (F, \alpha * \alpha^* * \varphi),$$

is given by a continuous positive-definite function.

Proof. To show that F_α is positive definite, we observe that since F is positive definite,

$$(F_\alpha, \varphi * \varphi^*) = (F, \alpha * \alpha^* * \varphi * \varphi^*) = (F, (\alpha * \varphi) * (\alpha * \varphi)^*) \geqslant 0$$

for any $\varphi \in K$, i.e., F_α is positive definite. As $\theta(x) \equiv \alpha * \alpha^*(x)$ is infinitely differentiable and has bounded support, it is easy to verify that

$$\begin{aligned}(F_\alpha, \varphi) = (F, \theta * \varphi) &= (F, \int \theta(y - x)\varphi(y)\, dy) \\ &= \int (F, \theta(y - x))\varphi(y)\, dy\end{aligned}$$

for all $\varphi \in K$, and that $(F, \theta(y - x))$ is a continuous (in fact, infinitely differentiable) function of y (it is understood that $\theta(y - x)$ is considered as a function of x in the expression $(F, \theta(y - x))$). In other words, setting $F_\alpha(y) = \overline{(F, \theta(y - x))}$, we have

$$(F_\alpha, \varphi) = \int \overline{F_\alpha(x)}\varphi(x)\, dx.$$

As the generalized function F_α has already been shown to be positive definite, this proves the lemma.

From this lemma and Bochner's theorem, we have the following assertion.

Lemma 2. If F is a positive-definite generalized function on K, and $\alpha(x)$ is any function in K, then the generalized function $F_\alpha = \alpha * \alpha^* * F$ has the form

$$(F_\alpha, \varphi) = \int \tilde{\varphi}(\lambda)\, d\mu_\alpha(\lambda), \tag{10}$$

where $\tilde{\varphi}(\lambda)$ is the Fourier transform of $\varphi(x)$, and μ_α is a positive finite measure.

Let us now consider the connection between measures μ_α and μ_β corresponding to functions $\alpha(x)$, $\beta(x) \in K$. In view of the fact that the Fourier transform of $\beta * \beta^* * \varphi(x)$ is the function $\tilde{\beta}(\lambda)\bar{\tilde{\beta}}(\lambda)\tilde{\varphi}(\lambda)$, it follows from (10) that

$$(F, (\alpha * \alpha^*) * (\beta * \beta^*) * \varphi) = \int \tilde{\varphi}(\lambda) |\tilde{\beta}(\lambda)|^2\, d\mu_\alpha(\lambda) \tag{10'}$$

and

$$(F, (\alpha * \alpha^*) * (\beta * \beta^*) * \varphi) = \int \tilde{\varphi}(\lambda) |\tilde{\alpha}(\lambda)|^2\, d\mu_\beta(\lambda). \tag{10''}$$

Since the left sides coincide, for every $\varphi(x) \in K$ we have

$$\int \tilde{\varphi}(\lambda) |\tilde{\beta}(\lambda)|^2\, d\mu_\alpha(\lambda) = \int \tilde{\varphi}(\lambda) |\tilde{\alpha}(\lambda)|^2\, d\mu_\beta(\lambda),$$

where $\tilde{\varphi}(\lambda)$, the Fourier transform of $\varphi(x)$, belongs to the space Z. Since the space Z is sufficiently rich in functions,[3] it follows from this that

$$|\tilde{\beta}(\lambda)|^2\, d\mu_\alpha(\lambda) = |\tilde{\alpha}(\lambda)|^2\, d\mu_\beta(\lambda). \tag{11}$$

We can now rewrite equation (10), eliminating $\alpha(x)$ from it. To do this we introduce a positive measure μ, setting

$$\int \tilde{\varphi}(\lambda)\, d\mu(\lambda) = \int \tilde{\varphi}(\lambda) \frac{d\mu_\alpha(\lambda)}{|\tilde{\alpha}(\lambda)|^2}$$

[3] It follows from a lemma proven in Volume II (Chapter IV, Section 8.4) that

$$\int f(x)\varphi(x)\, dx = 0$$

for all $\varphi(x) \in Z$ implies the vanishing of the continuous function $f(x)$. This assertion carries over without difficulty to any finite positive measure (for example, using the convolution of the measure with a δ-sequence).

for all $\varphi(x) \in K$. From (11) it follows that μ does not depend upon the choice of $\alpha(x) \in K$. By means of μ, Eq. (10) can be written in the form

$$(F, \alpha * \alpha^* * \varphi) = \int |\tilde{\alpha}(\lambda)|^2 \tilde{\varphi}(\lambda) \frac{d\mu_\alpha(\lambda)}{|\tilde{\alpha}(\lambda)|^2} = \int |\tilde{\alpha}(\lambda)|^2 \tilde{\varphi}(\lambda)\, d\mu(\lambda). \tag{12}$$

Denote the function $\alpha * \alpha^* * \varphi(x)$ by $\psi(x)$. Then (12) becomes

$$(F, \psi) = \int \tilde{\psi}(\lambda)\, d\mu(\lambda). \tag{13}$$

This formula is valid for every function $\psi(x)$ which is representable in the form

$$\psi(x) = \alpha * \alpha^* * \varphi(x), \tag{14}$$

where $\alpha(x)$ and $\varphi(x)$ are any functions in K.

Thus, (13) has been proved for functions $\psi(x)$ having the form (14). We now have to prove that (13) remains valid for every $\psi(x) \in K$ and that μ is tempered.

The first of these assertions is easily concluded from the second. Indeed, if μ is tempered, then the functional $\int \tilde{\psi}(\lambda)\, d\mu(\lambda)$ is continuous in the topology of S and *a fortiori* in that of K. But this functional coincides with the functional F on the set of all functions representable in the form (14). Since the set of such functions is everywhere dense in K, the functionals F and $\int \tilde{\psi}(\lambda)\, d\mu(\lambda)$ coincide on K, i.e., (13) is valid for every function in K.

Thus, it remains for us to prove the following lemma.

Lemma 3. Let the positive measure μ be such that for all functions $\psi(x)$ of the form $\psi(x) = \alpha * \alpha^* * \varphi(x)$, where $\alpha(x), \varphi(x) \in K$, one has

$$\int \tilde{\psi}(\lambda)\, d\mu(\lambda) = (F, \psi),$$

where F is a generalized function on K, and $\tilde{\psi}(\lambda)$ is the Fourier transform of $\psi(x)$. Then μ is tempered.

Proof. Since F is continuous on K, it is continuous on all of the subspaces $K(a)$ of K. Fix $a > 0$. Then there is a neighborhood U of zero in $K(a)$ such that $|(F, \varphi)| \leqslant 1$ for all $\varphi(x) \in U$.

To prove that μ is tempered, it is sufficient to construct a sequence of functions $\psi_m(x)$, belonging to the previously given neighborhood U of zero in $K(a)$ and having the following properties:

(1) The $\psi_m(x)$ are of the form (14), for which the representation (13) is already proven, i.e.,

$$\psi_m(x) = \alpha_m * \alpha_m^* * \varphi_m(x), \tag{15}$$

where $\alpha_m, \varphi_m \in K$;

(2) $\tilde{\psi}_m(\lambda) \geqslant 0$, where $\tilde{\psi}_m(\lambda)$ is the Fourier transform of $\psi_m(x)$;

(3) $\lim_{m\to\infty} \tilde{\psi}_m(\lambda) = \tilde{\omega}(\lambda)$ exists for every λ, and

$$\tilde{\omega}(\lambda) \geqslant \frac{A}{(1 + |\lambda|^2)^{p+n+1}}, \tag{16}$$

where $A > 0$, $p > 0$ are certain numbers, and n is the number of variables.

We call this sequence of functions and the corresponding limit function $\tilde{\omega}(\lambda)$ a *barrier sequence,* since it makes it possible to estimate the growth of the measure μ (such a barrier sequence was also constructed in studying positive generalized functions on S).

Assume that we have constructed such a sequence. Since all the $\psi_m(x)$ belong to the neighborhood U of zero, one has $|(F, \psi_m)| \leqslant 1$. But since they are of the form (15), then

$$(F, \psi_m) = \int \tilde{\psi}_m(\lambda)\, d\mu(\lambda).$$

Therefore $|\int \tilde{\psi}_m(\lambda)\, d\mu(\lambda)| \leqslant 1$, or since $\mu \geqslant 0$, $\tilde{\psi}_m(\lambda) \geqslant 0$, $0 \leqslant \int \tilde{\psi}_m(\lambda)\, d\mu(\lambda) \leqslant 1$. Passing to the limit $m \to \infty$ and using Fatou's lemma (cf. p. 145), we find that $\int \tilde{\omega}(\lambda)\, d\mu(\lambda)$ converges. In view of (16), it follows from this that

$$\int \frac{d\mu(\lambda)}{(1 + |\lambda|^2)^{p+n+1}} \leqslant \frac{1}{A},$$

i.e., μ is tempered.

Thus, Lemma 3 will be proven if we construct a barrier sequence. We proceed to construct such a sequence, belonging to a given neighborhood U of zero in K.

Choose any $a > 0$; then $U \cap K(a)$ is a neighborhood of zero in $K(a)$. By definition of the topology in $K(a)$ (cf. p. 20) one can find numbers p and η such that from $|\varphi^{(q)}(x)| \leqslant \eta$ for all q, $0 \leqslant |q| \leqslant 2p$, it follows that $\varphi(x) \in U \cap K(a)$ (we assume $\varphi \in K(a)$). This number p is at the basis of the construction of the barrier sequence $\psi_m(x)$. First we construct the limit function $\tilde{\omega}(\lambda)$. To do this we denote by $\gamma(x)$ the function whose Fourier transform is

$$\tilde{\gamma}(\lambda) = (1 + |\lambda|^2)^{-p-n-1}.$$

Now we choose an infinitely differentiable function $\chi(x)$ which vanishes for $|x| > \frac{1}{4}a$, and set $\chi_0(x) = \chi(x) * \chi^*(x)$,

$$\omega(x) = \gamma(x)\chi_0(x) = \gamma(x)[\chi(x) * \chi^*(x)].$$

Lemma 4. The function $\omega(x)$ has the following properties:

(1) $\omega(x) = 0$ for $|x| > \frac{1}{2}a$,

(2) $\tilde{\omega}(\lambda) > A(1 + |\lambda|^2)^{-p-n-1}$,

(3) all the derivatives of $\omega(x)$ up to order $2p$ are bounded.[4]

Proof. Since $\chi(x) = 0$ for $|x| > \frac{1}{4}a$, then $\chi(x) * \chi^*(x) = 0$ for $|x| > \frac{1}{2}a$. Consequently $\omega(x) = \gamma(x)[\chi(x) * \chi^*(x)] = 0$ for $|x| > \frac{1}{2}a$. Further, by definition

$$\gamma(x) = (2\pi)^{-n} \int \frac{e^{-i(x,\lambda)}}{(1 + |\lambda|^2)^{p+n+1}}\, d\lambda,$$

and so

$$|\gamma^{(q)}(x)| < (2\pi)^{-n} \int |\lambda|^{|q|}(1 + |\lambda|^2)^{-n-p-1}\, d\lambda.$$

Since the integral converges for $|q| \leqslant 2p$, the derivatives $\gamma^{(q)}(x)$ are bounded for $|q| \leqslant 2p$.

Now we estimate $\tilde{\omega}(\lambda)$. Since the Fourier transformation takes a product into a convolution, and a convolution into a product, it follows from

$$\omega(x) = \gamma(x)[\chi(x) * \chi^*(x)]$$

that

$$\tilde{\omega}(\lambda) = \tilde{\gamma}(\lambda) * |\tilde{\chi}(\lambda)|^2 = (1 + |\lambda|^2)^{-n-p-1} * |\tilde{\chi}(\lambda)|^2,$$

i.e.,

$$\tilde{\omega}(\lambda) = \int |\tilde{\chi}(u)|^2(1 + |\lambda - u|^2)^{-n-p-1}\, du.$$

Choosing any number $\rho > 0$, we have

$$\tilde{\omega}(\lambda) \geqslant \int_{|u|<\rho} |\tilde{\chi}(u)|^2(1 + |\lambda - u|^2)^{-n-p-1}\, du$$

and for $|\lambda| > \rho$ we have

$$\tilde{\omega}(\lambda) \geqslant (1 + |\lambda + \rho|^2)^{-n-p-1} \int_{|u|<\rho} |\tilde{\chi}(u)|^2\, du = C(1 + |\lambda + \rho|^2)^{-n-p-1}.$$

[4] The derivatives of order $2p + 1$ have a discontinuity at $x = 0$.

Setting

$$A = C \sup_{\lambda}[(1 + |\lambda + \rho|^2)^{p+n+1}(1 + |\lambda|^2)^{-p-n-1}],$$

we arrive at inequality (16). This proves the lemma.

The function $\omega(x)$ which has been constructed has continuous derivatives only up to order $2p$. However, if we convolute it with an infinitely differentiable function having bounded support, we obtain an infinitely differentiable function. We therefore construct the $\psi_m(x)$ from $\omega(x)$ in the following way. Choose a function $\alpha(x)$ vanishing, say, for $|x| > \frac{1}{8}a$, such that $\int \alpha(x)\,dx = 1$, and set $\alpha_m(x) = m^n\alpha(mx)$. We denote by $\beta_m(x)$ the function $\alpha_m(x) * \alpha_m^*(x)$ and construct $\psi_m(x)$ according to

$$\psi_m(x) = C\omega(x) * \beta_m(x) * \beta_m^*(x).$$

Lemma 5. The sequence $\{\psi_m(x)\}$ is a barrier sequence, i.e., it satisfies the following conditions:

(1) The ψ_m are of the form

$$\psi_m = \alpha_m * \alpha_m^* * \varphi_m(x),$$

where $\alpha_m, \varphi_m \in K(a)$;

(2) $\tilde{\psi}_m(\lambda) \geqslant 0$;

(3) $\lim_{m\to\infty} \tilde{\psi}_m(\lambda) = \tilde{\omega}(\lambda)$, where

$$\tilde{\omega}(\lambda) \geqslant \frac{A}{(1 + |\lambda|^2)^{p+n+1}}.$$

Moreover, for sufficiently small C the functions $\psi_m(x)$ belong to the given neighborhood U of zero in $K(a)$.

Proof. Set

$$\varphi_m(x) = C\alpha_m * \alpha_m^* * \omega(x).$$

Since $\psi_m(x) = C\omega * \beta_m * \beta_m^*(x)$, where $\beta_m(x) = \alpha_m * \alpha_m^*(x)$, then $\psi_m(x) = \alpha_m * \alpha_m^* * \varphi_m(x)$. From the very construction of $\varphi_m(x)$ it is clear that $\varphi_m \in K(a)$, and therefore condition 1 is fulfilled.

Further, we have $\tilde{\psi}_m(\lambda) = \tilde{\omega}(\lambda)\,|\tilde{\beta}_m(\lambda)|^2 \geqslant 0$. Let us now find $\lim_{m\to\infty} \tilde{\psi}_m(\lambda)$. Since

$$\tilde{\psi}_m(\lambda) = \tilde{\omega}(\lambda)|\tilde{\beta}_m(\lambda)|^2 = \tilde{\omega}(\lambda)|\tilde{\alpha}_m(\lambda)|^4,$$

it suffices to find $\lim_{m\to\infty} \tilde{\alpha}_m(\lambda)$;

$$\lim_{m\to\infty} \tilde{\alpha}_m(\lambda) = \lim_{m\to\infty} m^{-n} \int \alpha(mx)e^{i(\lambda,x)}\, dx$$

$$= \lim_{m\to\infty} \int \alpha(x)e^{(i/m)(\lambda,x)}\, dx = \int \alpha(x)\, dx = 1,$$

and so $\lim_{m\to\infty} \tilde{\psi}_m(\lambda) = \tilde{\omega}(\lambda)$, i.e., condition 3 is also fulfilled.

It remains to show that for sufficiently small values of C the $\psi_m(x)$ all belong to the neighborhood U of zero. For this we note that for any functions $\varphi(x)$, $\psi(x)$ one has

$$[\varphi * \psi(x)]^{(q)} = \varphi^{(q)} * \psi(x)$$

and therefore

$$\sup_x | [\varphi * \varphi(x)]^{(q)}| \leqslant \sup_x | \varphi^{(q)}(x)| \int | \psi(x)|\, dx.$$

Apply this inequality to the $\psi_m(x)$. For every m one has

$$\int \beta_m * \beta_m^*(x)\, dx = \left| \int \beta_m(x)\, dx \right|^2 = \left| \int \alpha_m * \alpha_m^*(x)\, dx \right|^2$$

$$= \left| \int \alpha_m(x)\, dx \right|^4 = \left| m^n \int \alpha(mx)\, dx \right|^4 = \left| \int \alpha(x)\, dx \right|^4 = 1.$$

Since $\omega(x)$ has bounded derivatives of order q, $0 \leqslant | q | \leqslant 2p$, there is an M such that

$$\sup_x | \omega^{(q)}(x)| \leqslant M, \qquad 0 \leqslant | q | \leqslant 2p.$$

Consequently, for $C = \eta/M$ the functions $\psi_m(x)$ satisfy the inequalities

$$\sup_x | \psi_m^{(q)}(x) | \leqslant \eta, \qquad 0 \leqslant | q | \leqslant 2p$$

and therefore belong to the neighborhood U of zero. This proves the lemma.

Thus, the existence of barrier sequences is proved, and from this, as was mentioned, follows Lemma 3.

We have already shown that Lemma 3 implies the validity of formula (13) for all functions in K. In other words, we have proven that *any positive-definite generalized function on* K is given by

$$(F, \psi) = \int \tilde{\psi}(\lambda)\, d\mu(\lambda), \tag{17}$$

where $\tilde{\psi}(\lambda)$ is the Fourier transform of $\psi(x)$ and μ is a positive tempered measure. But this assertion is simply the Bochner–Schwartz theorem.

We mention that the Bochner–Schwartz theorem can also be formulated in the following way.

Theorem 3′. Any multiplicatively positive generalized function F on Z has the form

$$(F, \varphi) = \int \varphi(\lambda)\, d\mu(\lambda),$$

where μ is a positive tempered measure.

The next result follows without difficulty from the Bochner–Schwartz theorem.

Theorem 4. Every positive-definite generalized function F on K can be represented in the form

$$F = (1 - \Delta)^p f(x), \tag{18}$$

where $f(x)$ is a continuous positive-definite function, and Δ is the Laplace operator

$$\Delta = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}.$$

Indeed, let $\tilde{F}$ be the Fourier transform of the generalized function F. By the Bochner–Schwartz theorem, $\tilde{F}$ is given by

$$(\tilde{F}, \tilde{\varphi}) = \int \tilde{\varphi}(\lambda)\, d\mu(\lambda),$$

where μ is a positive tempered measure. Therefore there is a p such that the integral $\int (1 + |\lambda|^2)^{-p}\, d\mu(\lambda)$ converges. Set

$$d\nu(\lambda) = (1 + |\lambda|^2)^{-p}\, d\mu(\lambda).$$

The measure ν is positive and finite. Therefore its Fourier transform is a continuous positive-definite function $f_1(x)$:

$$f_1(x) = \int e^{i(\lambda, x)}\, d\nu(\lambda).$$

The function $f(x) = \overline{f_1(x)}$ is also positive definite. Now from $d\mu(\lambda) = (1 + |\lambda|^2)^p \, d\nu(\lambda)$ it follows that

$$
\begin{aligned}
(F, \varphi) &= (2\pi)^{-n}(\tilde{F}, \tilde{\varphi}) = \int \tilde{\varphi}(\lambda) \, d\mu(\lambda) \\
&= \int \varphi(x) e^{i(\lambda,x)} (1 + |\lambda|^2)^p \, d\nu(\lambda) \, dx \\
&= \int [(1 - \Delta)^p \varphi(x)] e^{i(\lambda,x)} \, d\nu(\lambda) \, dx \\
&= \int [(1 - \Delta)^p \varphi(x)] f_1(x) \, dx \\
&= \int [(1 - \Delta)^p \varphi(x)] \overline{f(x)} \, dx = (f, (1 - \Delta)^p \varphi)
\end{aligned}
$$

for every $\varphi \in K$. But this means that $F = (1 - \Delta)^p f$.

The converse is also true: Every generalized function of the form $(1 - \Delta)^p f$, where $f(x)$ is a continuous positive-definite function, is positive definite.

3.4. Positive-Definite Generalized Functions on Z

It is easier to find the positive-definite generalized functions on Z than on K and S. Let $\varphi(z)$ be some function from Z. We will denote by $\varphi^*(z)$ the function $\overline{\varphi(-\bar{z})}$. Obviously $\varphi^*(z) \in Z$. The notation $\varphi^*(z)$ is justified by the fact that for real values of z, $\varphi^*(z) = \overline{\varphi(-z)}$

We will call a generalized function F on Z positive-definite if, for every $\varphi \in Z$,

$$(F, \varphi * \varphi^*) \geqslant 0.$$[5]

But the Fourier transformation carries the function $\varphi * \varphi^*(z)$ into the function $|\psi(\lambda)|^2$, where $\psi(\lambda)$ is the Fourier transform of $\varphi(z)$ and lies in K. Also, every function of the form $|\psi(\lambda)|^2$, $\psi(\lambda) \in K$, can be obtained in this way. From this it follows that the Fourier transform of a positive-definite generalized function F on Z is a multiplicatively positive generalized function $\tilde{F}$ on K. Since by Theorem 6 of Section 2 all multiplicatively

[5] If we consider functions in Z for real values of their argument, we obtain functions belonging to S. These functions can be convoluted. Now it is easily seen that convoluting the functions $\varphi(x)$ and $\psi(x)$ corresponding to the functions $\varphi(z)$ and $\psi(z)$ from Z, we obtain a function $\varphi * \psi(x)$ which also corresponds to some function from Z (this can be seen more simply by taking Fourier transforms). We denote this function from Z by $\varphi * \psi(z)$.

positive generalized functions on K are defined by positive measures, and any positive measure defines a multiplicatively positive generalized function on K, we arrive at the following theorem.

Theorem 5. Positive-definite generalized functions on Z are Fourier transforms of positive measures. Conversely, the Fourier transform of any positive measure is a positive-definite generalized function on Z.

3.5. Translation-Invariant Positive-Definite Hermitean Bilinear Functionals

Positive-definite generalized functions most frequently turn up in connection with translation-invariant positive-definite bilinear functionals. We will now examine this connection in detail. For simplicity we will consider bilinear functionals on the space K, although the results to be obtained can without particular difficulty be extended to other spaces of test functions.

A functional $B(\varphi, \psi)$, where φ, ψ range over the space K, is called a *Hermitean bilinear functional,* if

(1) for fixed $\psi(x)$, $B(\varphi, \psi)$ is a linear functional in $\varphi(x)$, continuous in the topology of K,

(2) for fixed $\varphi(x)$, $\overline{B(\varphi, \psi)}$ is a linear functional in ψ, continuous in the topology of K.

A bilinear functional $B(\varphi, \psi)$ is called *translation-invariant* if its value does not change under simultaneous translation of $\varphi(x)$ and $\psi(x)$ by the same vector h;

$$B[\varphi(x), \psi(x)] = B[\varphi(x + h), \psi(x + h)]. \tag{19}$$

Let us exhibit the general form of such functionals. To do this, we note that the value of a convolution $\varphi * \psi^*$ remains unchanged under simultaneous translation of $\varphi(x)$ and $\psi(x)$ by the same vector h. Indeed, let $\varphi_1(x) = \varphi(x + h)$, $\psi_1(x) = \psi(x + h)$. Then

$$\begin{aligned}\varphi_1 * \psi_1^*(x) &= \int \varphi_1(y)\overline{\psi_1(y - x)}\, dy \\ &= \int \varphi(y + h)\overline{\psi(y + h - x)}\, dy \\ &= \int \varphi(y)\overline{\psi(y - x)}\, dy = \varphi * \psi^*(x).\end{aligned}$$

Set

$$B(\varphi, \psi) = (F, \varphi * \psi^*), \tag{20}$$

where F is any generalized function on K. An elementary calculation shows that $B(\varphi, \psi)$ is a Hermitean bilinear functional on K. From the invariance of $\varphi * \psi^*$ relative to simultaneous translation of $\varphi(x)$ and $\psi(x)$ by the vector h it follows that $B(\varphi, \psi)$ is translation-invariant. We will show that any translation-invariant Hermitean bilinear functional on K can be written in the form $(F, \varphi * \varphi^*)$, where F is a generalized function on K.

This assertion is based upon the kernel theorem for the space K, discussed in Chapter I. It follows from this theorem that every Hermitean bilinear functional $B(\varphi, \psi)$ on K can be written in the form

$$B(\varphi, \psi) = (F_1, \varphi(x)\overline{\psi(y)}),$$

where F_1 is a linear functional on the space K_2 of infinitely differentiable functions of x and y (i.e., of $2n$ variables $x_1, \ldots, x_n; y_1, \ldots, y_n$) with bounded supports.

Since the linear combinations of functions of the form $\varphi(x)\overline{\psi(y)}$, where $\varphi(x)$ and $\psi(y)$ range over K, are everywhere dense in K_2, it follows from (19) that

$$(F_1, \varphi(x, y)) = (F_1, \varphi(x + h, y + h)). \tag{21}$$

Thus, the generalized function F_1 is invariant under simultaneous translation of the arguments x and y by a vector h.

But every generalized function F_1, satisfying (21), has the form

$$(F_1, \varphi(x, y)) = (F, \chi(y)), \tag{22}$$

where F is some generalized function on K, and $\chi(y)$ is defined by

$$\chi(y) = \int \varphi(x, x - y)\, dx.$$

Indeed, let us introduce a generalized function F_2 on K_2 by setting

$$(F_2, \psi(x, y)) = (F_1, \psi(x + y, x - y)). \tag{23}$$

It follows from (21) that F_2 is invariant under translation of x by h. It was shown in Volume I (Chapter I, Section 4.1) that such a generalized function is given by

$$(F_2, \psi(x, y)) = (F(y), \int \psi(x, y)\, dx), \tag{24}$$

where F is a generalized function on K. Setting

$$\psi(x, y) = \varphi(\tfrac{1}{2}x + \tfrac{1}{2}y, \tfrac{1}{2}x - \tfrac{1}{2}y),$$

we find from (23) and (24) that

$$\begin{aligned}(F_1, \varphi(x, y)) &= (F_2, \psi(x, y)) = \left(F, \int \psi(x, y)\, dx\right) \\ &= \left(F, \int \varphi(\tfrac{1}{2}x + \tfrac{1}{2}y, \tfrac{1}{2}x - \tfrac{1}{2}y)\, dx\right) = 2\left(F, \int \varphi(x, x - y)\, dx\right)\end{aligned}$$

This proves (22) (the factor 2 can be absorbed into F).

From (22) it follows that for functions of the form $\varphi(x, y) = \varphi(x)\overline{\psi(y)}$ the generalized function F_1 is given by

$$(F_1(y), \varphi(x)\overline{\psi(y)}) = \left(F, \int \varphi(x)\overline{\psi(x - y)}\, dx\right) = (F, \varphi * \psi^*). \tag{25}$$

But

$$(F_1, \varphi(x)\overline{\psi(y)}) = B(\varphi, \psi).$$

We thus obtain the following result: Every translation-invariant Hermitean functional $B(\varphi, \psi)$ on K has the form $B(\varphi, \psi) = (F, \varphi * \psi^*)$, where F is some generalized function on K.

A bilinear functional $B(\varphi, \psi)$ on K is called positive-definite, if $B(\varphi, \varphi) \geqslant 0$ for every $\varphi \in K$. If $B(\varphi, \psi)$ is moreover translation-invariant, then the generalized function F corresponding to it satisfies $(F, \varphi * \varphi^*) \geqslant 0$, i.e., F is also positive definite.

Availing ourselves of the Bochner–Schwartz theorem (cf. Section 3.3), which gives a representation of positive-definite generalized functions by means of tempered measures, and taking into consideration that $\widetilde{\varphi * \varphi^*} = \tilde{\varphi}(\lambda)\overline{\tilde{\varphi}(\lambda)}$, we obtain the following result.

Theorem 6. Every translation-invariant positive-definite Hermitean bilinear functional $B(\varphi, \psi)$ on K has the form

$$B(\varphi, \psi) = \int \tilde{\varphi}(\lambda)\overline{\tilde{\psi}(\lambda)}\, d\mu(\lambda),$$

where μ is some positive tempered measure and $\tilde{\varphi}(\lambda)$, $\tilde{\psi}(\lambda)$ are the Fourier transforms, respectively, of $\varphi(x)$ and $\psi(x)$.

3.6. Examples of Positive and Positive-Definite Generalized Functions

In this section we present various examples of positive and positive-definite generalized functions on S and on other spaces.

We will begin with functions of one variable. The simplest positive functions are the monomials x^{2m}. Since the Fourier transform of a positive generalized function is positive definite, and the Fourier transform of x^{2m} is[6] $(-1)^m 2\pi\delta^{(2m)}(x)$, we obtain the following result: The generalized function $(-1)^m\delta^{(2m)}(x)$ is positive definite. This result, however, is easily obtained directly by computing the integral

$$(-1)^m \int \delta^{(2m)}(x-y)\varphi(x)\overline{\varphi(y)}\, dx\, dy = (-1)^m \int \varphi^{(2m)}(x)\overline{\psi(x)}\, dx.$$

Integrating by parts m times, we find that this integral is equal to

$$\int |\varphi^{(m)}(x)|^2\, dx \geqslant 0.$$

This result is connected with the following general fact: *If F is positive-definite, then $(-1)^m\ (d^{2m}F/dx^{2m})$ is also positive-definite.* Indeed,

$$\left((-1)^m \frac{d^{2m}F}{dx^{2m}}, \varphi * \varphi^*\right) = \left(F, (-1)^m \frac{d^{2m}}{dx^{2m}}\varphi * \varphi^*\right)$$
$$= (F, \varphi^{(m)} * (\varphi^{(m)}(x))^*) \geqslant 0.$$

If $\lambda > -1$, then the generalized function $|x|^\lambda$ is positive. In fact, it is defined by

$$(|x|^\lambda, \varphi(x)) = \int |x|^\lambda\varphi(x)\, dx$$

and it is clear that $(|x|^\lambda, \varphi(x)) \geqslant 0$ for $\varphi(x) \geqslant 0$. The Fourier transform of the generalized function $|x|^\lambda$ is

$$-2\sin(\tfrac{1}{2}\lambda\pi)\Gamma(\lambda+1)|x|^{-\lambda-1}$$

(cf. Volume I, Chapter II, Section 2.3). Thus, for $\lambda > -1$ the generalized function $-2\sin(\frac{1}{2}\lambda\pi)\,\Gamma(\lambda+1)\,|x|^{-\lambda-1}$ is positive definite. One can similarly show the positive definiteness, for $\lambda > -1$, of the generalized functions

$$ie^{\frac{1}{2}\pi\lambda i}\Gamma(\lambda+1)(x+i0)^{-\lambda-1}$$

and

$$-ie^{-\frac{1}{2}\pi\lambda i}\Gamma(\lambda+1)(x-i0)^{-\lambda-1}$$

[6] We will find it convenient, throughout this paragraph, to use the same symbol for the argument of a function and that of its Fourier transform.

(they are the Fourier transforms of the positive generalized functions x_+^λ and x_-^λ).

The situation is otherwise for $\lambda < -1$. Thus, for example, for $\lambda > -m-1$ the generalized function x_+^λ is defined by

$$(x_+^\lambda, \varphi) = \int_0^1 x^\lambda \left[\varphi(x) - \sum_{k=0}^{m-1} \frac{x^k}{k!} \varphi^{(k)}(0)\right] dx$$

$$+ \int_1^\infty x^\lambda \varphi(x)\, dx + \sum_{k=1}^{m} \frac{\varphi^{(k-1)}(0)}{(k-1)!\,(\lambda+k)}. \tag{26}$$

This expression can take on negative values for positive test functions $\varphi(x)$. However, if $\varphi(x)$ is such that $\varphi(0) = \varphi'(0) = \ldots = \varphi^{(m-1)}(0) = 0$, then (26) becomes

$$(x_+^\lambda, \varphi) = \int_0^\infty x^\lambda \varphi(x)\, dx.$$

Therefore for $\lambda > -m-1$ the generalized function x_+^λ is not positive on the entire space of test functions, but only on its subspace consisting of those functions φ for which $\varphi(0) = \varphi'(0) = \ldots = \varphi^{(m-1)}(0) = 0$. An analogous remark applies also to the generalized function x_-^λ.

It follows that for $-2m-1 < \lambda$ the inequalities $(x_+^\lambda, \varphi\bar{\varphi}) \geqslant 0$ and $(x_-^\lambda, \varphi\bar{\varphi}) \geqslant 0$ are fulfilled if

$$\varphi(0) = \varphi'(0) = \ldots = \varphi^{(m-1)}(0) = 0. \tag{27}$$

Indeed, in this case every derivative of the function $\varphi(x)\,\overline{\varphi(x)}$ up to and including the $(2m-1)$st vanishes for $x = 0$, and therefore one has

$$(x_+^\lambda, \varphi\tilde{\varphi}) = \int_0^\infty x^\lambda\, |\varphi(x)|^2\, dx \geqslant 0$$

and a similar inequality for x_-^λ. Considering the Fourier transforms of the generalized functions x_+^λ and x_-^λ, we find that the generalized functions

$$ie^{\frac{1}{2}\pi\lambda i}\Gamma(\lambda+1)(x+i0)^{-\lambda-1}$$

and

$$-ie^{-\frac{1}{2}\pi\lambda i}\Gamma(\lambda+1)(x-i0)^{-\lambda-1}$$

are not positive definite, for $-2m-1 < \lambda$, on the entire space S, but only on a certain subspace of S. It can be shown that this subspace consists of those functions for which every moment $m_k = \int_{-\infty}^{\infty} x^k \varphi(x)\, dx$ of order up to and including the $(m-1)$st vanishes.†

† Presumably m is the smallest integer satisfying $-2m-1 < \lambda$.

In other words,

$$(ie^{\frac{1}{2}\pi\lambda i}\Gamma(\lambda + 1)(x + i0)^{-\lambda-1}, \varphi * \varphi^*(x)) \geqslant 0$$

for every $\varphi(x) \in S$ such that

$$\int_{-\infty}^{\infty} x^k\varphi(x)\, dx = 0, \qquad 0 \leqslant k \leqslant m - 1.$$

Let us now consider the generalized functions of the form $(ax^2 + bx + c)^\lambda$. These functions were studied in Volume I (Chapter II, Section 2.6), where their Fourier transforms were obtained. The generalized function $(1 + x^2)^\lambda$ is positive for any real λ. Therefore its Fourier transform

$$\frac{2\sqrt{\pi}}{\Gamma(-\lambda)} (\tfrac{1}{2}| x |)^{-\lambda-\frac{1}{2}}K_{-\lambda-\frac{1}{2}}(x)$$

is positive definite for any real λ.

The generalized functions $(x^2 - 1)_+^\lambda$ and $(x^2 - 1)_-^\lambda$ are positive for $\lambda > - 1$. Therefore for $\lambda > - 1$ their Fourier transforms

$$\Gamma(\lambda + 1) \sqrt{\pi}(\tfrac{1}{2}| x |)^{-\lambda-\frac{1}{2}}N_{-\lambda-\frac{1}{2}}(| x |)$$

and

$$\Gamma(\lambda + 1) \sqrt{\pi}(\tfrac{1}{2}| x |)^{-\lambda-\frac{1}{2}}J_{-\lambda-\frac{1}{2}}(| x |)$$

are positive definite.

For positive integer values m, the generalized functions

$$\left(1 - \frac{d^2}{dx^2}\right)^m \delta(x)$$

and

$$2\pi(-1)^m \left(1 + \frac{d^2}{dx^2}\right)^m \delta(x) + (-1)^{m+1} \sqrt{\pi}(\tfrac{1}{2}| x |)^{-m-\frac{1}{2}}J_{m+\frac{1}{2}}(x),$$

which are the Fourier transforms respectively of the positive generalized functions $(x^2 + 1)^m$ and $(x^2 - 1)^m$, are positive definite.

Let us now consider functions of several variables. For $\lambda > - n$ the generalized function r^λ is defined by

$$(r^\lambda, \varphi) = \Omega_n \int_0^{\infty} r^{\lambda+n-1}S_\varphi(r)\, dr,$$

where Ω_n denotes the "surface area" of the n-dimensional unit sphere $[\Omega_n = 2\pi^{\frac{1}{2}n}/\Gamma(\frac{1}{2}n)]$, and $S_\varphi(r)$ denotes the average of the function

$\varphi(x)$ over the sphere of radius r. From this formula one sees that the generalized function r^λ is positive for $\lambda > -n$. Consequently, its Fourier transform

$$\widetilde{r^\lambda} = 2^{\lambda+n}\pi^{\frac{1}{2}n}\frac{\Gamma(\frac{1}{2}\lambda + \frac{1}{2}n)}{\Gamma(-\frac{1}{2}\lambda)}r^{-\lambda-n}$$

is positive definite for $\lambda > -n$.

If $-n - 2m < \lambda < -n - 2m + 2$, the generalized function r^λ is defined by

$$(r^\lambda, \varphi) = \Omega_n \int_0^\infty r^{\lambda+n-1}\left[S_\varphi(r) - \sum_{k=0}^{m-1}\frac{r^{2k}}{(2k)!}S_\varphi^{(2k)}(0)\right]dr.$$

In this case r^λ is positive on the subspace consisting of those functions $\varphi(x)$ for which

$$S_\varphi(0) = S_\varphi''(0) = \dots = S_\varphi^{(2m-2)}(0) = 0. \tag{28}$$

According to formula (6) of Volume I (Chapter I, Section 3.9) condition (28) can be represented in the form

$$\varphi(0) = \Delta\varphi(0) = \dots = \Delta^{m-1}\varphi(0) = 0, \qquad \Delta = \sum_{k=1}^{n}\frac{\partial^2}{\partial x_k^2}. \tag{29}$$

Thus, we have proven that the generalized function (r^λ, φ), where

$$r^\lambda = (x_1^2 + \dots + x_n^2)^{\frac{1}{2}\lambda}, \qquad -n - 2m < \lambda < -n - 2m + 2,$$

is positive on the subspace consisting of those test functions satisfying condition (29).

More complicated examples of positive and positive-definite generalized functions of several variables arise in considering generalized functions connected with quadratic forms (cf. Volume I, Chapter III, Section 2).

For example, let $P = \sum_{j,k=1}^n g_{jk}x_jx_k$ be a quadratic form in n variables. Then the generalized functions P_+^λ and P_-^λ are positive for $\lambda > -1$. Therefore their Fourier transforms are positive-definite.

These Fourier transforms are given by (Volume *I*, Chapter III, Section 2.6)

$$\widetilde{P_+^\lambda} = 2^{n+2\lambda}\pi^{\frac{1}{2}n-1}\Gamma(\lambda + \tfrac{1}{2}n)\frac{1}{2i\sqrt{|D|}}$$
$$\times\left[e^{i(\frac{1}{2}q+\lambda)\pi}(Q - i0)^{-\lambda-\frac{1}{2}n} - e^{-i(\frac{1}{2}q+\lambda)\pi}(Q + i0)^{-\lambda-\frac{1}{2}n}\right] \tag{30}$$

and

$$\widetilde{P_-^\lambda} = -2^{n+2\lambda}\pi^{\frac{1}{2}n-1}\Gamma(\lambda+1)\Gamma(\lambda+\tfrac{1}{2}n)\frac{1}{2i\sqrt{|D|}}$$

$$\times\, [e^{-\frac{1}{2}\pi qi}(Q-i0)^{-\lambda-\frac{1}{2}n} - e^{\frac{1}{2}\pi qi}(Q+i0)^{-\lambda-\frac{1}{2}n}], \tag{31}$$

where q is the number of negative terms in the canonical representation of P; $Q = \sum_{j,k=1}^n g^{jk}x_jx_k$ is the quadratic form adjoint to P (i.e., for which $\sum_{k=1}^n g_{jk}g^{km} = \delta_j^m$); D is the discriminant of P; and $(Q+i0)^\lambda$, $(Q-i0)^\lambda$ denote the generalized functions $Q_+^\lambda + e^{\pi\lambda i}Q_-^\lambda$ and $Q_+^\lambda + e^{-\pi\lambda i}$ Q_-^λ respectively.

Let us point out that for $s \geqslant 0$ and even, the generalized function

$$(2\pi)^n \sum_{k=0}^{s} \frac{(-1)^n(\frac{1}{2}c)^{2s-2k}}{4^k k!(s-k)!} L^k\delta(x),$$

where

$$L = \sum_{j,k=1}^n g^{jk}\frac{\partial^2}{\partial x_j\,\partial x_k},$$

is positive definite, since it is the Fourier transform of the positive generalized function $(c^2+P)^s/\Gamma(s+1)$. We will not give here the more complicated examples connected with the positive generalized functions $(c^2+P)_+^\lambda$ and $(c^2+P)_-^\lambda$, since the expressions for the Fourier transforms of these generalized functions are rather complicated.

Up to this point we have considered examples of positive-definite generalized functions on the space K (or, what is the same, on the space S). Let us give examples of positive-definite generalized functions on the space Z. These generalized functions are Fourier transforms of positive generalized functions on K. Thanks to the fact that test functions in K have bounded supports, generalized functions on K can have any behavior at infinity.

Consider, for example, the positive function e^{ax}, a real. Its Fourier transform is the positive-definite generalized function $2\pi\delta(z-ia)$,

$$(2\pi\delta(z-ia), \varphi(z)) = 2\pi\varphi(ia),$$

on Z.

In the same way, the Fourier transform of the positive generalized function $\exp(\frac{1}{2}x^2)$ on K is the positive-definite generalized function

$$(F,\varphi) = -i\sqrt{2\pi}\int_{-i\infty}^{i\infty} e^{z^2}\varphi(z)\,dz$$

on the space Z.

4. Conditionally Positive-Definite Generalized Functions[1]

4.1. Basic Definitions

One can obtain new positive-definite generalized functions from given positive-definite generalized functions, by applying to them differential operators of the form $D\bar{D}$, where $D = \sum_{|k|=s} a_k(d^k/dx^k)$ is a linear homogeneous constant-coefficient differential operator of order s,[2] and $\bar{D}$ denotes the operator $(-1)^s \sum_{|k|=s} \overline{a_k}(d^k/dx^k)$.

Indeed, from the easily proven relation

$$D\bar{D}(\varphi * \varphi^*) = D\varphi * (D\varphi)^* \tag{1}$$

it follows that $(D\bar{D}F, \varphi * \varphi^*) = (F, D\varphi * (D\varphi)^*)$. Therefore the inequality $(D\bar{D}F, \varphi * \varphi^*) \geqslant 0$ holds for any positive-definite generalized function F.

The converse assertion is not true—from the positive definiteness of the generalized function $D\bar{D}F$ it does not in general follow that F is itself positive-definite. For example, the function $-x^2$ is not positive definite. However, applying $D\bar{D}$ to it, where $D = d/dx$, we obtain the positive-definite function $D\bar{D}(-x^2) = 2$.

We will call a generalized function F a *conditionally positive-definite generalized function of order* s if the inequality $(D\bar{D}F, \varphi * \varphi^*) \geqslant 0$ holds for all test functions $\varphi(x)$ and all linear *homogeneous* constant-coefficient differential operators D of order s. Such generalized functions arise, for example, in the theory of generalized random processes in Chapter III.

Using (1), we can formulate the definition of conditional positive definiteness in another way. Namely, a generalized function F is conditionally positive-definite if the inequality $(F, \varphi * \varphi^*) \geqslant 0$ holds for all test functions of the form $\varphi(x) = D\psi(x)$, where D is a linear homogeneous constant-coefficient differential operator of order s, and $\psi(x)$ is a test function.

The study of conditionally positive-definite generalized functions is more conveniently carried out by replacing them by their Fourier transforms.

[1] This section can be omitted at a first reading. It should be studied after reading Chapter III.

[2] As usual, d^k/dx^k denotes the operator

$$\frac{\partial^{k_1+\cdots+k_n}}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}}, \qquad |k| = k_1 + \ldots + k_n.$$

Let us associate with each differential operator

$$D = \sum_{|k|=s} a_k \frac{d^k}{dx^k}$$

the polynomial

$$P(\lambda) = \frac{1}{(2\pi)^n} \sum_{|k|=s} a_k(-i\lambda)^k.$$

Since the relation $\widetilde{d^kF/dx^k} = [(-i\lambda)^k/(2\pi)^n]\tilde{F}$ is satisfied by any generalized function, $\widetilde{DF} = P(\lambda)\tilde{F}$. It is easily shown that to the operator $\bar{D}$ corresponds the polynomial

$$\bar{P}(\lambda) = \frac{1}{(2\pi)^n} \sum_{|k|=s} \overline{a_k}(-i\lambda)^k.$$

Therefore the Fourier transform of the generalized function $D\bar{D}F$ is the generalized function $P\bar{P}\tilde{F}$. Since the Fourier transformation takes the function $\varphi * \varphi^*(x)$ into the function $\tilde{\varphi}(\lambda)\overline{\tilde{\varphi}}(\lambda)$, the conditional positive definiteness of order s of a generalized function F is equivalent to its satisfying the inequality

$$(P\bar{P}\tilde{F}, \psi\bar{\psi}) \geqslant 0$$

for all *homogeneous* polynomials P of degree s and all functions $\psi(\lambda)$ of the dual space.

In accordance with this we will call a generalized function F *conditionally positive of order* s, if the inequality $(P\bar{P}F, \varphi\bar{\varphi}) \geqslant 0$ holds for all *homogeneous polynomials of degree* s and all test functions $\varphi(x)$ (it would be more correct to call such generalized functions *conditionally multiplicatively positive*, but for brevity we omit the word "multiplicatively").

Since we will be interested in conditionally positive-definite generalized functions on the space K, we will consider conditionally positive generalized functions on its dual space Z.

4.2. Conditionally Positive Generalized Functions (Case of One Variable)[3]

Here we shall ascertain the general form of conditionally positive generalized functions of order s, for functions of one variable. Since, for functions of one variable, x^s is the only homogeneous polynomial of

[3] This case is the most important for the theory of random processes. The reader who is interested in the theory of random fields should also familiarize himself with Section 4.3, in which conditionally positive generalized functions of several variables are considered.

degree s, the conditional positivity of a generalized function F is equivalent to the multiplicative positivity of the generalized function $x^{2s}F$. But by the Bochner–Schwartz theorem given in Section 3, every multiplicatively positive generalized function on Z is given by a positive tempered measure. Thus, if F is a conditionally positive generalized function of order s, there exists a positive tempered measure ν such that

$$(F, z^{2s}\varphi(z)) = \int \varphi(x)\, d\nu(x)$$

for all $\varphi(z) \in Z$. But any function of one variable $\psi(z) \in Z$, having a zero of order $2s$ at $z = 0$, can be represented in the form $\psi(z) = z^{2s}\varphi(z)$, where $\varphi(z)$ also belongs to Z. Therefore the result which we have proven shows that

$$(F, \psi) = \int \psi(x) \frac{d\nu(x)}{x^{2s}} \tag{2}$$

for any function $\psi(z) \in Z$ having a zero of order $2s$ at $z = 0$.

It will be convenient for us to formulate this result in another way. To do this, we separate out the point $x = 0$ from the integral in (2). In view of the relation

$$\lim_{x\to 0} \frac{\psi(x)}{x^{2s}} = \frac{\psi^{(2s)}(0)}{(2s)!},$$

we obtain

$$(F, \psi) = \int_{\Omega_0} \psi(x) \frac{d\nu(x)}{x^{2s}} + a\, \frac{\psi^{(2s)}(0)}{(2s)!}, \tag{3}$$

where Ω_0 is the region complementary to the point $x = 0$, and the coefficient a is the ν-measure of this point. Introducing a new measure $d\mu(x) = d\nu(x)/x^{2s}$ in Ω_0, we can rewrite (3) in the simpler form

$$(F, \psi) = \int_{\Omega_0} \psi(x)\, d\mu(x) + a\, \frac{\psi^{(2s)}(0)}{(2s)!}. \tag{4}$$

We have thus found the general form of the functional F on functions $\psi(z)$ having a zero of order $2s$ at $z = 0$. Let us now find the general form of F on an arbitrary function $\psi(z) \in Z$. Choose any function $\alpha(z) \in Z$ such that $\alpha(z) - 1$ has a zero of order[4] $2s + 1$ at $z = 0$, and

[4] Such a function $\alpha(z)$ is easily constructed by considering a function of the form $p(z)\beta(z)$, where $\beta(z) \in Z$ and $p(z)$ is a polynomial of degree $2s + 1$, and suitably choosing the coefficients of $p(z)$.

associate with every function $\psi(z) \in Z$ the function

$$\theta(z) = \psi(z) - \alpha(z) \sum_{k=0}^{2s-1} \frac{\psi^{(k)}(0) z^k}{k!} .$$

This function has a zero of order $2s$ at $z = 0$, and therefore (4) is applicable to it[5];

$$(F, \theta) = \int_{\Omega_0} \theta(x)\, d\mu(x) + a \frac{\theta^{(2s)}(0)}{2s!} .$$

But

$$(F, \psi) = (F, \theta) + \sum_{k=0}^{2s-1} \frac{\psi^{(k)}(0)}{k!} (F, \alpha(z) z^k),$$

and $\theta^{(2s)}(0) = \psi^{(2s)}(0)$; therefore

$$(F, \psi) = \int_{\Omega_0} \left[\psi(x) - \alpha(x) \sum_{k=0}^{2s-1} \frac{\psi^{(k)}(0)}{k!} x^k\right] d\mu(x) + \sum_{k=0}^{2s} a_k \frac{\psi_k(0)}{k!} ,$$

where for brevity we have set $a_k = (F, \alpha(z) z^k)$ for $0 \leqslant k \leqslant 2s - 1$ and $a_{2s} = a = \nu(0)$.

Since the measure ν is finite on all bounded sets, $\int_{0<|x|<1} d\nu(x) < +\infty$ and therefore $\int_{0<|x|<1} x^{2s}\, d\mu(x) < +\infty$.

Moreover, we note that $a_{2s} \geqslant 0$ since a_{2s} is the ν-measure of the point $x = 0$.

Thus, we have proven the following theorem:

Theorem 1. Every conditionally positive generalized function F of order s of one variable, on the space Z, has the form

$$(F, \psi) = \int_{\Omega_0} \left[\psi(x) - \alpha(x) \sum_{k=0}^{2s-1} \frac{\psi^{(k)}(0)}{k!} x^k\right] d\mu(x) + \sum_{k=0}^{2s} a_k \frac{\psi^{(k)}(0)}{k!} . \tag{5}$$

Here μ is a positive tempered measure such that the integral $\int_{0<|x|<1} x^{2s}\, d\mu(x)$ converges, $\alpha(z)$ is a function in Z such that $\alpha(z) - 1$ has a zero of

[5] We were not able to set simply

$$\theta(z) = \psi(z) - \sum_{k=0}^{2s-1} \frac{\psi^{(k)}(0)}{k!} z^k,$$

since then the function $\theta(z)$ would not tend to zero along the real axis and would not belong to Z.

We note that the function $\alpha(z)$ is not uniquely defined. The final expression for (F, ψ) depends upon the choice of $\alpha(z)$ and is therefore also nonunique.

order $2s + 1$ at $z = 0$, $a_{2s} \geqslant 0$, and a_k, $0 \leqslant k \leqslant 2s - 1$ are certain numbers.

We mention that the converse assertion is also true—*any generalized function F of the form* (5), *where μ, $\alpha(z)$, and the a_k satisfy the conditions stated, is a conditionally positive function of order s.* Indeed, if $\varphi(z)$ is any function in Z, then the function $z^{2s}\varphi(z)\bar{\varphi}(z)$ has a zero of order at least $2s$ at $z = 0$. Therefore all of its derivatives of order up to the $(2s - 1)$st inclusive vanish at $z = 0$. The derivative of order $2s$ equals $(2s)! |\varphi(0)|^2$ for $z = 0$.

Therefore

$$(F, z^{2s}\varphi(z)\bar{\varphi}(z)) = \int_{\Omega_0} x^{2s} |\varphi(x)|^2 \, d\mu(x) + a_{2s} |\varphi(0)|^2. \tag{6}$$

The integral in (6) converges at infinity, since μ is tempered and the function $x^{2s} |\varphi(x)|^2$ is rapidly decreasing. It also converges at zero, since by assumption the integral $\int_{0<|x|<1} x^{2s} \, d\mu(x)$ converges. But since μ is positive and $a_{2s} \geqslant 0$, it follows from (6) that $(F, z^{2s}\varphi(z)\bar{\varphi}(z)) \geqslant 0$. Thus we have proven that if the measure μ, the function $\alpha(z)$, and the numbers a_k satisfy the conditions of Theorem 1, then formula (5) defines a conditionally positive generalized function of order s.

4.3. Conditionally Positive Generalized Functions (Case of Several Variables)

In the case of several variables, a theorem analogous to Theorem 1 holds.

Theorem 1′. Any conditionally positive generalized function F of order s on the space Z has the form

$$(F, \varphi) = \int_{\Omega_0} \left[\varphi(x) - \alpha(x) \sum_{|k|=0}^{2s-1} \frac{\varphi^{(k)}(0)}{k!} x^k\right] d\mu(x) + \sum_{|k|=0}^{2s} a_k \frac{\varphi^{(k)}(0)}{k!}. \tag{7}$$

Here μ is a positive tempered measure, defined in the complement Ω_0 of the point $x = 0$, such that the integral $\int_{0<|x|<1} |x|^{2s} \, d\mu(x)$ converges, $\alpha(z)$ is a function in Z such that $\alpha(z) - 1$ has a zero of order $2s + 1$ at $z = 0$[6]; the a_k, $|k| = 2s$, are numbers such that the Hermitean form $\sum_{|i|=|j|=s} a_{i+j}\xi_i\bar{\xi}_j$ is positive-definite, and the a_k, for $|k| \leqslant 2s - 1$, are certain fixed numbers.[7]

[6] In other words, $\alpha(0) = 1$ and $\alpha^{(q)}(0) = 0$ for $1 \leqslant |q| \leqslant 2s$.

[7] We set here, as everywhere in this book, $|x| = (x_1^2 + \ldots + x_n^2)^{\frac{1}{2}}$; $k!$ denotes $k_1! \ldots k_n!$; z^k denotes $z_1^{k_1} \ldots z_n^{k_n}$, and $|k|$ denotes the sum $k_1 + \ldots + k_v$.

We first prove a special case of this theorem; namely, we indicate the general form of (F, φ) for functions $\varphi(z) \in Z$ of the form

$$\varphi(z) = \sum_{|k|=2s} z^k \varphi_k(z), \qquad \varphi_k(z) \in Z. \tag{8}$$

Lemma 1. Let F be a conditionally positive generalized function of order s. Then for any function $\varphi(z) \in Z$ which is a linear combination of functions $z^k\varphi_k(z)$, where $|k| = 2s$ and $\varphi_k(z) \in Z$, F is given by

$$(F, \varphi) = \int_{\Omega_0} \varphi(x)\, d\mu(x) + \sum_{|k|=2s} a_k \frac{\varphi^{(k)}(0)}{k!}. \tag{9}$$

Here μ is a positive tempered measure such that the integral $\int_{0<|x|<1} |x|^{2s}\, d\mu(x)$ converges, and the a_k, $|k| = 2s$, are numbers such that the Hermitean form $\sum_{|i|=|j|=s} a_{i+j}\xi_i\bar{\xi}_j$ is positive-definite.

First we find the general form of F for functions of the form $\varphi(z) = z^k\psi(z)$, where $\psi(z) \in Z$ and $|k| = 2s$. Since for $|k| = 2s$ a monomial z^k can be written in the form $z^k = z^i z^j$, where $|i| = |j| = s$, the function $z^k\psi(z)$ can be represented in the form

$$z^k\psi(z) = (\tfrac{1}{2}z^i + \tfrac{1}{2}z^j)^2\psi(z) - (\tfrac{1}{2}z^i - \tfrac{1}{2}z^j)^2\psi(z).$$

But for any homogeneous polynomial $P(z)$ of degree s one can find a positive tempered measure ν_p such that the relation

$$(F, P\bar{P}\psi) = \int \psi(x)\, d\nu_p(x) \tag{10}$$

holds for all $\psi(z) \in Z$. In fact, by the definition of the conditional positivity of a generalized function F the generalized function $P\bar{P}F$ is multiplicatively positive. According to Theorem 3′ of Section 3, a multiplicatively positive generalized function on Z is given by a positive tempered measure ν_p. Therefore

$$(F, P\bar{P}\psi) = (P\bar{P}F, \psi) = \int \psi(x)\, d\nu_p(x).$$

Applying this result to the polynomials $P_1(z) = (z^i + z^j)/2$ and $P_2 = (z^i - z^j)/2$, we conclude that

$$\begin{aligned}(F, z^k\psi(z)) &= (F, P_1\bar{P}_1\psi) - (F, P_2\bar{P}_2\psi)\\ &= \int \psi(x)\, d\nu_{p_1}(x) - \int \psi(x)\, d\nu_{p_2}(x) = \int \psi(x)\, d\nu_k(x),\end{aligned}$$

where we have set

$$d\nu_k(x) = d\nu_{p_1}(x) - d\nu_{p_2}(x).$$

At first glance the measure ν_k depends not only upon the value k, but also upon the method of decomposition $z^k = z^i z^j$ of the monomial z^k into a product of monomials $z^i = z_1^{i_1} \dots z_n^{i_n}$ and $z^j = z_1^{j_1} \dots z_n^{j_n}$ of degree s. As a matter of fact, ν_k depends only upon k. Indeed, suppose that the measure σ_k corresponds to a different decomposition $z^k = z^p z^q$, $|p| = |q| = s$. Then for any $\psi(z) \in Z$ we have

$$(F, z^k\psi(z)) = \int \psi(x)\, d\nu_k(x) = \int \psi(x)\, d\sigma_k(x).$$

But Z is everywhere dense in S, and therefore

$$\int \psi(x)\, d\nu_k(x) = \int \psi(x)\, d\sigma_k(x)$$

for all $\psi(x) \in S$. This can hold only if $\nu_k = \sigma_k$.

Thus, we have proven that for any k, $|k| = 2s$, there exists a unique measure ν_k (in general, not positive) such that for every function of the form $z^k\, \psi(z)$, $\psi(z) \in Z$, the generalized function F is given by

$$(F, z^k\psi(z)) = \int \psi(x)\, d\nu_k(x). \tag{11}$$

This formula still does not enable one to find the general form of (F, φ) for all functions of the form $\sum_{|k|} z^k\varphi_k(z)$, since the measure ν_k in (11) depends upon the value k (i.e., on the collection of numbers $k_1, \dots, k_n$). We would like, therefore, to rewrite this formula in a form not depending upon the value k. To do this we divide the entire space of the independent variables into two parts—the manifold L_k, where $x^k = 0$,[8] and the complementary region Ω_k. In each region Ω_k we introduce, in place of ν_k, a new measure μ_k, setting[9] $d\mu_k(x) = d\nu_k(x)/x^k$.

Let us show that these measures have the following *compatibility* property: $\mu_j = \mu_k$ in the region $\Omega_j \cap \Omega_k$. Let x^j and x^k be two monomials of order $2s$. Then for any $\psi(z) \in Z$ one has

$$(F, z^{j+k}\psi(z)) = \int x^k\psi(x)\, d\nu_j(x)$$

[8] Recall that the equation $x^k = 0$ is a short notation for the equation $x_1^{k_1} \dots x_n^{k_n} = 0$. Therefore the manifold L_k consists of all hyperplanes $x_i = 0$ for which k_i is different from zero.

[9] This is possible because the denominator x^k does not vanish in Ω_k.

and

$$(F, z^{j+k}\psi(z)) = \int x^j\psi(x)\, d\nu_k(x)$$

and so, for any $\psi(z) \in Z$,

$$\int x^k\psi(x)\, d\nu_j(x) = \int x^j\psi(x)\, d\nu_k(x),$$

i.e.,

$$x^k\, d\nu_j(x) = x^j\, d\nu_k(x). \tag{12}$$

But then we have $\mu_j = \mu_k$ in $\Omega_j \cap \Omega_k$. It follows that in the union Ω_0 of the Ω_k there exists a measure μ which coincides, in each of the regions Ω_k, with the corresponding measure μ_k. It is not hard to see that Ω_0 is the complement of the point $x = 0$. We can now write (11) in the form

$$\begin{aligned}(F, z^k\psi(z)) &= \int_{\Omega_k} \psi(x)\, d\nu_k(x) + \int_{L_k} \psi(x)\, d\nu_k(x)\\ &= \int_{\Omega_k} x^k\psi(x)\, d\mu_k(x) + \int_{L_k} \psi(x)\, d\nu_k(x)\\ &= \int_{\Omega_k} x^k\psi(x)\, d\mu(x) + \int_{L_k} \psi(x)\, d\nu_k(x).\end{aligned} \tag{13}$$

Since the function $x^x\psi(x)$ vanishes outside the set Ω_k, the first term in (13) is not changed if we replace the region of integration Ω_k by Ω_0;

$$\int_{\Omega_k} x^k\psi(x)\, d\mu(x) = \int_{\Omega_0} x^k\psi(x)\, d\mu(x).$$

Let us now consider the second term. From (12) we have $\nu_k(L_k') = 0$ where L_k' denotes the set of points x for which $x^k = 0$ but at least one of the monomials x^j does not vanish. Since the manifold L_k can be divided into the point $x = 0$ and a finite number of sets, in each of which at least one of the x^j is different from zero, the measure ν_k (in L_k) is concentrated at the point $x = 0$. Therefore

$$\int_{L_k} \psi(x)\, d\nu_k(x) = a_k\psi(0),$$

where a_k is the ν_k-measure of the point $x = 0$.

Substituting this expression into (13), we obtain

$$(F, z^k\psi(z)) = \int_{\Omega_0} x^k\psi(x)\, d\mu(x) + a_k\psi(0). \tag{14}$$

Setting $z^k\psi(z) = \varphi(z)$ and taking into consideration the equality $\psi(0) = \varphi^{(k)}(0)/k!$, we can write (14) in the form

$$(F, \varphi) = \int_{\Omega_0} \varphi(x)\, d\mu(x) + a_k \frac{\varphi^{(k)}(0)}{k!}. \tag{15}$$

Now (15) can be represented in the following form, which does not depend upon k:

$$(F, \varphi) = \int_{\Omega_0} \varphi(x)\, d\mu(x) + \sum_{|k|=2s} a_k \frac{\varphi^{(k)}(0)}{k!}. \tag{16}$$

Indeed, each derivative of $\varphi(z) = z^k\psi(z)$, of order j, $|j| = 2s$, vanishes at $z = 0$ if $j \neq k$. Therefore (15) is equivalent to (16).

Thus, we have found the general form of the functional (F, φ) for functions of the form $\varphi(z) = z^k\psi(z)$, $|k| = 2s$, $\psi(z) \in Z$. But the right side of (16) does not depend upon k, and therefore (16) holds for linear combinations of functions $z^k\psi(z)$. This proves that the value of (F, φ), for all functions of the form

$$\varphi(z) = \sum_{|k|=2s} z^k\varphi_k(z), \qquad \varphi_k(z) \in Z,$$

is given by (16).

In order to complete the proof of the lemma, it remains for us to show that the measure μ and the numbers a_k have the properties stated in the lemma. In other words, we must show that μ is positive and tempered, that the integral[10] $\int_{0<|x|<1} |x|^{2s}\, d\mu(x)$ converges, and that the a_k ($|k| = 2s$) are such that the Hermitean form $\sum_{|i|=|j|=s} a_{i+j}\xi_i\bar{\xi}_j$ is positive definite.

To prove the positivity of μ we use the relation $x^k\, d\mu(x) = d\nu_k(x)$ which holds for μ. We choose $k = 2j$ (i.e., $x^k = x_1^{2j_1} \dots x_n^{2j_n}$). This is possible since k can be any vector with integer components and length $|k| = 2s$. In view of the conditional positivity of the generalized function F, the generalized function $z^{2j}F$ is multiplicatively positive. By Theorem 3′ of Section 3, the measure ν_{2j} which defines it is positive and tempered. But then μ, which is related to ν_{2j} by $x^{2j}\, d\mu(x) = d\nu_{2j}(x)$, is also positive and tempered. Furthermore, it follows from $\int_{0<|x|<1} d\nu_{2j}(x) < +\infty$ that

$$\int_{0<|x|<1} x^{2j}\, d\mu(x) < +\infty. \tag{17}$$

[10] $|x|^{2s} = (x_1^2 + \dots + x_n^2)^s$.

Since the integral $\int_{0<|x|<1} |x|^{2s}\, d\mu(x)$ is a linear combination of integrals of the form (17), it also converges. This proves our assertions regarding the measure μ.

In order to prove the assertion regarding the numbers a_k, recall that these numbers were defined by $a_k = \nu_k(0)$, the ν_k-measure of the point $x = 0$. Therefore we must prove that for any complex numbers $\xi_1, \ldots, \xi_n$

$$\sum_{|i|=|j|=s} \nu_{i+j}(0)\xi_i\bar{\xi}_j \geqslant 0. \tag{18}$$

To prove (18), we form the homogeneous polynomial $P(z) = \sum_{|j|=s} \xi_j z^j$ of degree s. As was remarked at the beginning of the proof, there corresponds to this polynomial a positive measure $d\nu_p$ such that

$$(F, P\bar{P}\varphi) = \int \varphi(x)\, d\nu_p(x) \tag{19}$$

for all $\varphi(z) \in Z$. But, on the other hand,

$$\begin{aligned} (F, P\bar{P}\varphi) &= \sum_{|i|=|j|=s} (F, z^{i+j}\varphi)\xi_i\bar{\xi}_j \\ &= \sum_{|i|=|j|=s} \xi_i\bar{\xi}_j \int \varphi(x)\, d\nu_{i+j}(x). \end{aligned} \tag{20}$$

Since (19) and (20) are valid for any $\varphi(z) \in Z$, then

$$\nu_p(A) = \sum_{|i|=|j|=s} \nu_{i+j}(A)\xi_i\bar{\xi}_j$$

for any set A. Choose the point $x = 0$ as the set A. Since $\nu_{i+j}(0) = a_{i+j}$, we obtain

$$\nu_p(0) = \sum_{|i|=|j|=s} a_{i+j}\xi_i\bar{\xi}_j.$$

Since $\nu_p(0) \geqslant 0$, inequality (18) follows. This proves the positive definiteness of $\sum_{|i|=|j|=s} a_{i+j}\xi_i\bar{\xi}_j$.

Lemma 1 gives the general form of the functional (F, φ) for functions φ which can be written as in (8). Now we show that (16) holds for all functions $\varphi(z)$ having a zero of order $2s$ at $z = 0$. To prove this, we take into account that the functions of the form (8) are everywhere dense in the set of functions in Z having a zero of order $2s$ at $z = 0$ (the proof of this statement is carried out in the Appendix to this section, p. 194).

Therefore it suffices, for the proof of our assertion, to show that the functional

$$(F_1, \varphi) = \int_{\Omega_0} \varphi(x)\, d\mu(x) + \sum_{|k|=2s} a_k \frac{\varphi^{(k)}(0)}{k!}, \tag{21}$$

considered on functions of the form

$$\varphi(z) = \sum_{|k|=2s} z^k \varphi_k(z), \qquad \varphi_k(z) \in Z, \tag{22}$$

is continuous in the topology of Z.

Let $\varphi(z)$ be a function in Z having the form indicated. Then the function $\varphi(x)/|x|^{2s}$ is bounded for $0 < |x| < 1$ and the convergence of $\int_{0<|x|<1} |x|^{2s}\, d\mu(x)$ implies the convergence of $\int_{0<|x|<1} \varphi(x)\, d\mu(x)$. The convergence of $\int_{|x|\geqslant 1} \varphi(x)\, d\mu(x)$ follows from the fact that $\varphi(x)$ is rapidly decreasing and that μ is tempered. From this it follows that (F_1, φ) is defined for all functions $\varphi(z) \in Z$ of the form (22).

Let us prove the continuity of the functional F_1 in the topology of Z. Suppose that a sequence of functions $\varphi_m(z) \in Z$, having the form

$$\varphi_m(z) = \sum_{|k|=2s} z^k \varphi_{mk}(z), \qquad \varphi_{mk}(z) \in Z, \tag{23}$$

converges to zero (in the topology of Z). From the properties of the measure μ it follows that

$$\int \frac{|x|^{2s}\, d\mu(x)}{(1+|x|^2)^p}$$

converges for some $p > 0$.

Since the $\varphi_m(x)$ converge to zero in the topology of Z and have the form (23), for any $\epsilon > 0$ there is an N such that $|\varphi_m(z)| < (\epsilon\, |x|^{2s})/[(1+|x|^2)^p]$ and $|\varphi_m^{(k)}(0)| < \epsilon$, $|k| = 2s$, for $m \geqslant N$. But then

$$|(F_1, \varphi_m)| \leqslant \epsilon \left[\int \frac{|x|^{2s}\, d\mu(x)}{(1+|x|^2)^p} + \sum_{|k|=2s} \frac{|a_k|}{k!}\right]$$

for $m \geqslant N$. In view of the arbitrariness of ϵ, it follows from this that F_1 is continuous. Since F_1 coincides with F on functions of the form (22), and these functions are everywhere dense in the subspace of functions $\varphi(z) \in Z$ having a zero of order $2s$ at $z = 0$, the equality

$$(F, \varphi) = \int \varphi(x)\, d\mu(x) + \sum_{|k|=2s} a_k \frac{\varphi^{(k)}(0)}{k!}$$

is valid for all functions $\varphi(z)$ of this subspace. Thus Lemma 1 is proved.

It follows from these considerations that if the measure μ is tempered and $\int_{0<|x|<1} |x|^{2s} \, d\mu(x)$ converges, then $\int \varphi(x) \, d\mu(x)$ converges for all functions $\varphi(z) \in Z$ having a zero of order $2s$ at $z = 0$, and defines a continuous linear functional on the subspace of such functions.

We proceed now to the proof of Theorem 1′, i.e., to the establishing of the form of the generalized function F for all $\varphi(z) \in Z$. Just as for functions of one variable, we introduce a new function $\theta(z)$, setting

$$\theta(z) = \varphi(z) - \alpha(z) \sum_{|k|=0}^{2s-1} \frac{\varphi^{(k)}(0)}{k!} z^k,$$

where $\alpha(z)$ is any function in Z such that $\alpha(z) - 1$ has a zero of order $2s + 1$ at $z = 0$. The function $\theta(z)$ has a zero of order $2s$ at $z = 0$.

We have already shown that for such functions the value of the generalized function F is given by

$$(F, \theta) = \int_{\Omega_0} \theta(x) \, d\mu(x) + \sum_{|k|=2s} a_k \frac{\theta^{(k)}(0)}{k!}.$$

Since the derivatives of order $2s$ of $\varphi(z)$ and $\theta(z)$ coincide, this formula can be written in the form

$$(F, \theta) = \int_{\Omega_0} \theta(x) \, d\mu(x) + \sum_{|k|=2s} a_k \frac{\varphi^{(k)}(0)}{k!}.$$

But

$$(F, \varphi) = (F, \theta) + \sum_{|k|=0}^{2s-1} \frac{\varphi^{(k)}(0)}{k!} (F, \alpha(z) z^k).$$

Denoting $(F, \alpha(z) \, z^k)$ by a_k, we obtain

$$(F, \varphi) = \int_{\Omega_0} \left[\varphi(x) - \alpha(x) \sum_{|k|=0}^{2s-1} \frac{\varphi^{(k)}(0)}{k!} x^k\right] d\mu(x) + \sum_{|k|=0}^{2s} a_k \frac{\varphi^{(k)}(0)}{k!}.$$

Thus, the general form of F has been found for any $\varphi(z) \in Z$; Theorem 1′ is proved.

The theorem converse to Theorem 1′ is also valid.

Theorem 2. Let μ be a positive tempered measure such that $\int_{0<|x|<1} |x|^{2s} \, d\mu(x)$ converges; a_k, $|k| = 2s$,—numbers such that the Hermitean form $\sum_{|i|=|j|=s} a_{i+j} \xi_i \bar{\xi}_j$ is positive definite; a_k, $0 \leqslant |k|$

$\leqslant 2s - 1$—arbitrary numbers, and $\alpha(z)$, a function in Z such that $\alpha(z) - 1$ has a zero of order $2s + 1$ at $z = 0$. Then the generalized function F, defined by

$$(F, \varphi) = \int_{\Omega_0} \left[\varphi(x) - \alpha(x) \sum_{|k|=0}^{2s-1} \frac{\varphi^{(k)}(0)}{k!} x^k\right] d\mu(x) + \sum_{|k|=0}^{2s} a_k \frac{\varphi^{(k)}(0)}{k!}, \quad (24)$$

is conditionally positive of order s.

Proof. First we prove that (24) defines a functional on Z. Let $\varphi(z) \in Z$ be arbitrary. Then the function

$$\theta(z) = \varphi(z) - \alpha(z) \sum_{|k|=0}^{2s-1} \frac{\varphi^{(k)}(0)}{k!} z^k$$

has a zero of order $2s$ at $z = 0$.

We showed on p. 186 that from this follows the convergence of $\int_{\Omega_0} \theta(x)\, d\mu(x)$ and the continuity of its dependence upon $\theta(x)$. Therefore (24) defines a continuous linear functional on Z.

Now let us show that F is conditionally positive of order s, i.e., that $(F, P\bar{P}\varphi\,\bar{\varphi}) \geqslant 0$ for any homogeneous polynomial $P(z)$ of degree s and any $\varphi(z) \in Z$. Indeed, the function $\psi(z) = P(z)\bar{P}(z)\varphi(z)\bar{\varphi}(z)$ has a zero of order $2s$ at $z = 0$, i.e., all of its derivatives up to order $2s - 1$ inclusive vanish at $z = 0$.

By Leibnitz's formula, the derivatives $\psi^{(k)}(0)$, $|k| = 2s$ are given by[11]

$$\psi^{(k)}(0) = \sum_{|i|=|j|=s} (i + j)!\, \frac{P^{(i)}(0)\varphi(0)}{i!}\, \frac{\overline{P^{(j)}(0)\varphi(0)}}{j!}.^{12}$$

Therefore the generalized function F is given, for $\psi(z) = P(z)\bar{P}(z)\varphi(z)\bar{\varphi}(z)$, by

$$(F, P\bar{P}\varphi\bar{\varphi}) = \int_{\Omega_0} |P(x)\varphi(x)|^2\, d\mu(x) + \sum_{|i|=|j|=s} a_{i+j} \frac{P^{(i)}(0)\varphi(0)}{i!}\, \frac{\overline{P^{(j)}(0)\varphi(0)}}{j!}.$$

[11] We denote by $(i + j)!/i!j!$ the expression

$$\frac{(i_1 + j_1)! \ldots (i_n + j_n)!}{i_1! \ldots i_n!\, j_1! \ldots j_n!}.$$

[12] The remaining terms in Leibnitz's formula vanish, since $P^{(i)}(0) = 0$ if $|i| \leqslant s - 1$. If $|i| \geqslant s + 1$, then $|j| \leqslant s - 1$ and $P^{(j)}(0) = 0$.

Since both terms on the right side are positive (in view of the positivity of the measure μ and the positive definiteness of the form $\sum_{|i|=|j|=s} a_{i+j}\xi_i\bar{\xi}_j$), we obtain $(F, P\bar{P}\varphi\bar{\varphi}) \geqslant 0$. This proves the conditional positivity of F.

We remark that μ and the numbers a_k, $|k| = 2s$, are *uniquely* defined by the conditionally positive generalized function F. At the same time, the function $\alpha(z)$ and the numbers a_k, $|k| \leqslant 2s - 1$, are not uniquely defined. To prove the uniqueness of μ and the a_k, $|k| = 2s$, we note the following.

By the Bochner–Schwartz theorem (Theorem 3 of Section 3), for every polynomial $P(z)$ the functional F *uniquely* defines a measure ν_p such that $(F, P\bar{P}\varphi) = \int \varphi(x)\, d\nu_p(x)$. The measures ν_p uniquely define measures ν_k, $|k| = 2s$, such that $(F, z^k\varphi(z)) = \int \varphi(x)\, \nu_k(x)$. Lastly, the ν_k uniquely define a measure μ and numbers a_k, $|k| = 2s$, since $d\mu(x) = d\nu_k(x)/x^k$ and $a_k = \nu_k(0)$. Thus the uniqueness of μ and the a_k, $|k| = 2s$, is proved.

4.4. Conditionally Positive-Definite Generalized Functions on K

We have seen in Section 3 that the Fourier transformation takes conditionally positive-definite generalized functions into conditionally multiplicatively positive generalized functions on the dual space of test functions. Therefore the results of the preceding section permit us to give a description of conditionally positive-definite generalized functions on K. We recall that a generalized function F on K is called conditionally positive-definite of order s if $(D\bar{D}F, \varphi * \varphi^*) \geqslant 0$ for all $\varphi(x) \in K$ and all linear homogeneous constant-coefficient differential operators D of order s.

From Theorem 1′ of the preceding section follows:

Theorem 3. Let F be a conditionally positive-definite generalized function of order s on the space K. Then F has the form

$$(F, \varphi) = \int_{\Omega_0} \left[\tilde{\varphi}(\lambda) - \alpha(\lambda) \sum_{|k|=0}^{2s-1} \frac{\tilde{\varphi}^{(k)}(0)}{k!} \lambda^k\right] d\mu(\lambda) + \sum_{|k|=0}^{2s} a_k \frac{\tilde{\varphi}^{(k)}(0)}{k!} \qquad (25)$$

Here $\tilde{\varphi}(\lambda)$ is the Fourier transform of $\varphi(x)$, μ is a positive tempered measure such that $\int_{0<|\lambda|<1} |\lambda|^{2s}\, d\mu(\lambda)$ converges, the a_k, $|k| = 2s$, are numbers such that the Hermitean form $\sum_{|i|=|j|=s} a_{i+j}\xi_i\bar{\xi}_j$ is positive-definite, the a_k for $0 \leqslant |k| \leqslant 2s - 1$ are numbers depending upon F, and $\alpha(\lambda)$ is a function in Z such that $\alpha(\lambda) - 1$ has a zero of order $2s + 1$ at $\lambda = 0$.

Formula (25) can be written in another way, introducing the moments of the function $\varphi(x)$. For this we note the following. From

$$\tilde{\varphi}(\lambda) = \int e^{i(\lambda,x)}\varphi(x)\,dx$$

it follows that

$$\tilde{\varphi}^{(k)}(\lambda) = i^{|k|}\int x^k e^{i(\lambda,x)}\varphi(x)\,dx.$$

Setting $\lambda = 0$, we obtain

$$\tilde{\varphi}^{(k)}(0) = i^{|k|}\int x^k\varphi(x)\,dx = i^{|k|}b_k,$$

where b_k denotes the kth moment of $\varphi(x)$.

Using this equality, we can write (25) in the form

$$(F,\varphi) = \int_{\Omega_0}\left[\tilde{\varphi}(\lambda) - \alpha(\lambda)\sum_{|k|=0}^{2s-1}\frac{i^{|k|}\lambda^k b_k}{k!}\right]d\mu(\lambda) + \sum_{|k|=0}^{2s} a_k\frac{i^{|k|}b_k}{k!}\,. \tag{25'}$$

In particular, if the moments of $\varphi(x)$ vanish for $|k| < 2s$, then (25′) becomes

$$\begin{aligned}(F,\varphi) &= \int_{\Omega_0}\tilde{\varphi}(\lambda)\,d\mu(\lambda) + (-1)^s\sum_{|k|=2s} a_k\frac{b_k}{k!}\\ &= \int_{\Omega_0}\tilde{\varphi}(\lambda)\,d\mu(\lambda) + \sum_{|k|=2s} a_k\frac{\tilde{\varphi}^{(k)}(0)}{k!}\,.\end{aligned} \tag{25''}$$

In conclusion we remark that if a conditionally positive-definite generalized function F has the form

$$(F,\varphi) = \int \overline{F(x)}\varphi(x)\,dx, \tag{26}$$

where $F(x)$ is a continuous function, then the positive measure μ corresponding to F is finite in any region of the form $|\lambda| \geqslant a > 0$, and $\int_{0<|\lambda|<1}|\lambda|^{2s}\,d\mu(\lambda)$ converges. The converse is also true.

4.5. Bilinear Functionals Connected with Conditionally Positive-Definite Generalized Functions

In this paragraph we discuss Hermitean bilinear functionals $B(\varphi, \psi)$ on K, having the following properties: For any linear homogeneous constant-coefficient differential operator $D = \sum_{|k|=s} a_k\,(d^k/dx^k)$ the Hermitean bilinear functional $B_D(\varphi, \psi)$, defined by

$$B_D(\varphi, \psi) = B(D\varphi, D\psi),$$

is translation-invariant and positive-definite.

For the description of such functionals we use the theorem of Section 3, according to which $B_D(\varphi, \psi)$ can be represented in the form

$$B_D(\varphi, \psi) = (F_D, \varphi * \psi^*),$$

where F_D is a positive-definite generalized function. Thus, for any two functions $\varphi(x)$, $\psi(x) \in K$

$$B(D\varphi, D\psi) = (F_D, \varphi * \psi^*).$$

We introduce a new generalized function F, setting $(F, D\bar{D}\theta) = (F_D, \theta)$ for any $\theta(x) \in K$. This defines F only on functions of the form $D\bar{D}\theta(x)$. The functional F can be extended to all of K in an arbitrary manner.[13]

The bilinear functional $B(\varphi, \psi)$ is related to F by the equation

$$\begin{aligned} B(D\varphi, D\psi) &= B_D(\varphi, \psi) = (F_D, \varphi * \psi^*) \\ &= (F, D\bar{D}(\varphi * \psi^*)) = (D\bar{D}F, \varphi * \psi^*). \end{aligned}$$

This equation holds for any $\varphi(x)$, $\psi(x) \in K$, and any homogeneous constant-coefficient linear differential operator D of order s.

Since by hypothesis $B(D\varphi, D\varphi) \geqslant 0$, then $(D\bar{D}F, \varphi * \varphi^*) \geqslant 0$, and therefore the generalized function $D\bar{D}F$ is positive-definite for any D of the form considered. But this means that F is a conditionally positive-definite generalized function of order s.

We can now prove the following theorem, describing bilinear functionals of the type considered.

Theorem 4. Suppose that the Hermitean functional $B(\varphi, \psi)$ is such that for any homogeneous linear constant-coefficient differential operator D of order s the functional $B_D(\varphi, \psi) = B(D\varphi, D\psi)$ is translation-invariant and positive-definite. Then for any infinitely differentiable functions

[13] The functions of the form $D\bar{D}\theta(x)$, $\theta(x) \in K$, form a subspace K_D in K such that the factor space K/K_D is finite dimensional. Choosing a linearly independent basis $\varphi_1 + K_D, \ldots, \varphi_m + K_D$ in this factor space and taking any values for $(F, \varphi_1), \ldots, (F, \varphi_m)$, we obtain an extension of F to all of K. † It is not evident that F is independent of D.

$\varphi(x)$ and $\psi(x)$ having bounded supports, whose moments up to order $s-1$ inclusive vanish, one has

$$B(\varphi, \psi) = \int_{\Omega_0} \tilde{\varphi}(\lambda)\overline{\tilde{\psi}(\lambda)}\, d\mu(\lambda) + \sum_{|i|=|j|=s} a_{i+j} \frac{\tilde{\varphi}^{(i)}(0)}{i!} \frac{\overline{\tilde{\psi}^{(j)}(0)}}{j!}.$$

Here μ is a positive tempered measure such that

$$\int_{0<|\lambda|<1} |\lambda|^{2s}\, d\mu(\lambda)$$

converges, and the a_k, $|k| = 2s$, are numbers such that the form $\sum_{|i|=|j|=s} a_{i+j}\xi_i\bar{\xi}_j$ is positive-definite.

Proof. First we show that the vanishing of the moments b_k and c_k of $\varphi(x)$ and $\psi(x)$ for $|k| \leqslant s-1$ implies the vanishing of the moments of $\varphi * \psi^*(x)$ up to order $2s-1$ inclusive. Indeed, suppose $|k| \leqslant 2s-1$; then

$$\int x^k[\varphi * \psi^*(x)]\, dx = \iint x^k \varphi(y)\overline{\psi(y-x)}\, dx\, dy.$$

Setting $x = y + t$ in this integral, we obtain[14]

$$\begin{aligned}\int x^k[\varphi * \psi^*(x)]\, dx &= \iint (y+t)^k \varphi(y)\overline{\psi(t)}\, dy\, dt \\ &= \sum_{i+j=k} C_{i+j}^{j} \int y^j\varphi(y)\, dy \overline{\int t^i\psi(t)\, dt} \\ &= \sum_{i+j=k} C_{i+j}^{j} b_j \bar{c}_i. \end{aligned} \tag{27}$$

But this sum equals zero, since from $|i+j| = |k| \leqslant 2s-1$ it follows that in every term either $|i| \leqslant s-1$ or $|j| \leqslant s-1$, and therefore every term in (27) equals zero.

We proceed now to the description of the bilinear functional $B(\varphi, \psi)$. We have seen above that for functions of the form $\varphi = D\varphi_1$, $\psi = D\psi_1$,

$$B(\varphi, \psi) = (F, \varphi * \psi^*), \tag{28}$$

where F is a conditionally positive-definite generalized function of order s. Since the functions of the form $D\varphi_1$ are everywhere dense in the set of functions having zero moments up to order $s-1$ inclusive

[14] We denote by C_{i+j}^{j} the expression

$$\frac{(i+j)!}{i!j!} = \frac{(i_1+j_1)! \ldots (i_n+j_n)!}{i_1! \ldots i_n! j_1! \ldots j_n!}.$$

(see Appendix to this section), (28) holds for all such functions. To compute $(F, \varphi * \psi^*)$ we can apply (25''), since it was shown above that all the moments of $\varphi * \psi^*(x)$, up to order $2s - 1$ inclusive, equal zero.

This proves that

$$B(\varphi, \psi) = \int_{\Omega_0} \tilde{\theta}(\lambda)\, d\mu(\lambda) + \sum_{|k|=2s} a_k \frac{\tilde{\theta}^{(k)}(0)}{k!}, \tag{29}$$

where $\tilde{\theta}(\lambda)$ is the Fourier transform of $\theta(x) = \varphi * \psi^*(x)$, and μ and the a_k, $|k| = 2s$, are as in (25'').

In view of the fact that $\tilde{\theta}(\lambda) = \tilde{\varphi}(\lambda)\overline{\tilde{\psi}(\lambda)}$, where $\tilde{\varphi}(\lambda)$ and $\tilde{\psi}(\lambda)$ are the Fourier transforms of $\varphi(x)$ and $\psi(x)$, it follows that[15]

$$\tilde{\theta}^{(k)}(0) = \sum_{i+j=k} C^j_{i+j}\tilde{\varphi}^{(i)}(0)\overline{\tilde{\psi}^{(j)}(0)} = \sum_{\substack{|i|=|j|=s\\ i+j=k}} C^j_{i+j}\tilde{\varphi}^{(i)}(0)\overline{\tilde{\psi}^{(j)}(0)}.$$

Substituting these values for $\tilde{\theta}(\lambda)$ and $\tilde{\theta}^{(k)}(0)$ into (29), we obtain

$$B(\varphi, \psi) = \int_{\Omega_0} \tilde{\varphi}(\lambda)\overline{\tilde{\psi}(\lambda)}\, d\mu(\lambda) + \sum_{|i|=|j|=s} a_{i+j} \frac{\tilde{\varphi}^{(i)}(0)}{i!}\frac{\overline{\tilde{\psi}^{(j)}(0)}}{j!}, \tag{29'}$$

which proves the theorem.

Since for $|i| = |j|$ we have $\tilde{\varphi}^{(i)}(0)\overline{\tilde{\psi}^{(j)}(0)} = b_i\overline{c_j}$, (29') can be rewritten in the form

$$B(\varphi, \psi) = \int_{\Omega_0} \tilde{\varphi}(\lambda)\overline{\tilde{\psi}(\lambda)}\, d\mu(\lambda) + \sum_{|i|=|j|=s} a_{i+j} \frac{b_i\overline{c_j}}{i!\,j!}. \tag{29''}$$

Let us now clarify the form of the Hermitean bilinear functional $B(\varphi, \psi)$ for any $\varphi(x)$, $\psi(x) \in K$. For this we introduce functions $\theta_j(x) \in K$, $|j| \leqslant s - 1$, such that

$$\int x^i\theta_j(x)\, dx = \delta_{ij}, \qquad |i| \leqslant s - 1, \quad |j| \leqslant s - 1, \tag{30}$$

where δ_{ij} is the multidimensional Kronecker symbol: $\delta_{ij} = \delta_{i_1j_1} \cdots \delta_{i_nj_n}$.

The existence of such functions is proven without difficulty. Let $\theta(x)$ be a function such that $\int \theta(x)\, dx = 1$. Consider the functions $\theta^{(j)}(x)$. It is obvious that $\int x^k\theta^{(j)}(x)\, dx = 0$ if $|k| < |j|$, or if $|k| = |j|$ but $k \neq j$, and also that

$$\int x^j\theta^{(j)}(x)\, dx = j!.$$

[15] Recall that $\tilde{\varphi}^{(i)}(0) = 0$ if $|i| \leqslant s - 1$.

It then follows from these relations that one can form linear combinations $\theta_i(x)$ of the $\theta^{(j)}(x)$, $|j| \leqslant s-1$, which will satisfy the relations (30).

With each $\varphi(x) \in K$ we associate the function

$$\varphi_0(x) = \varphi(x) - \sum_{|j| \leqslant s-1} b_j \theta_j(x),$$

where the b_j, $|j| \leqslant s-1$, are the moments of $\varphi(x)$. It is obvious that the moments of $\varphi_0(x)$ up to order $s-1$ inclusive equal zero, and its moments of order s coincide with those of $\varphi(x)$. Therefore if $\varphi(x)$ and $\psi(x)$ are any two functions in K, we find from (29″) that

$$B(\varphi_0, \psi_0) = \int_{\Omega_0} \tilde{\varphi}_0(\lambda)\overline{\tilde{\psi}_0(\lambda)}\, d\mu(\lambda) + \sum_{|i|=|j|=s} a_{i+j} \frac{b_i \overline{c_j}}{i!j!}\,. \tag{31}$$

Here μ, a_k ($|k| = 2s$), b_j, and c_j have the same meaning as earlier, and $\tilde{\varphi}_0(\lambda)$, $\tilde{\psi}_0(\lambda)$ are the Fourier transforms of $\varphi_0(x)$ and $\psi_0(x)$.

From the hermiticity of $B(\varphi, \psi)$ it follows that

$$B(\varphi, \psi) = B(\varphi_0, \psi_0) + \sum_{|i| \leqslant s-1} b_i B(\theta_i, \psi_0) + \sum_{|j| \leqslant s-1} \overline{c_j} B(\varphi_0, \theta_j) + \sum_{|i|, |j| \leqslant s-1} b_i \overline{c_j} B(\theta_i, \theta_j).$$

But $B(\varphi_0, \theta_j)$ is, for fixed j, a linear functional on K; $B(\varphi_0, \theta_j) = L_j(\varphi)$, and $B(\theta, \psi_0) = \overline{L_j(\psi)}$.

Thus, we have proven the following theorem.

Theorem 5. Let $B(\varphi, \psi)$ be a Hermitean bilinear functional, and suppose that for any homogeneous constant-coefficient linear differential operator D of order s the functional $B_D(\varphi, \psi) = B(D\varphi, D\psi)$ is translation-invariant and positive-definite. Then for any functions $\varphi(x)$, $\psi(x) \in K$, $B(\varphi, \psi)$ is given by

$$B(\varphi, \psi) = \int_{\Omega_0} \tilde{\varphi}_0(\lambda)\overline{\tilde{\psi}_0(\lambda)}\, d\mu(\lambda) + \sum_{|i|=|j|=s} a_{i+j} \frac{b_i \overline{c_j}}{i!j!} + \sum_{|i| \leqslant s-1} b_i \overline{L_i(\psi)} + \sum_{|j| \leqslant s-1} \overline{c_j} L_j(\varphi) + \sum_{|i|, |j| \leqslant s-1} A_{ij} b_i \overline{c_j}.$$

Here μ is a positive tempered measure such that $\int_{0<|\lambda|<1} |\lambda|^{2s}\, d\mu(\lambda)$ converges; the a_k, $|k| = 2s$, are numbers such that the Hermitean form $\sum_{|i|=|j|=s} a_{i+j}\xi_i\bar{\xi}_j$ is positive-definite; the L_i are linear functionals on K, $A_{ij} = \bar{A}_{ji}$ are certain numbers; b_i and c_i are the moments of $\varphi(x)$ and $\psi(x)$, and $\tilde{\varphi}_0(\lambda)$, $\tilde{\psi}_0(\lambda)$ are the Fourier transforms of

$$\varphi_0(x) = \varphi(x) - \sum_{|i| \leqslant s-1} b_i \theta_i(x)$$

and

$$\psi_0(x) = \psi(x) - \sum_{|i| \leqslant s-1} c_i \theta_i(x).$$

Appendix

In the proof of Theorem 1 we made use of the following assertion.

Theorem 6. Every function $\varphi(z) \in Z$ having a zero of order m at $z = 0$ is the limit (in the sense of convergence in Z) of a sequence of functions $\varphi_k(z) \in Z$ having the form

$$\varphi_k(z) = \sum_{|r|=m} z^r \varphi_{kr}(z),$$

where $\varphi_{kr}(z) \in Z$.

Let us prove this theorem. To begin with, we show that $\varphi(z)$ can be represented in the form

$$\varphi(z) = \sum_{|r|=m} z^r \psi_r(z),$$

where the $\psi_r(z)$ are entire analytic functions of exponential type, having power growth for real values of z. This assertion is obvious if there is only one independent variable, since in this case $\varphi(z) = z^m \psi(z)$, where $\psi(z) \in Z$. Suppose now that the assertion is already proven for the case where the number of variables is less than n. Expand $\varphi(z)$ as a Taylor series in powers of the variable z_n. This expansion can be written in the form

$$\varphi(z) = \sum_{s=0}^{m-1} \frac{\varphi^{(s)}(z^*)}{s!} z_n^s + z_n^m \hat{\varphi}(z), \tag{32}$$

where z^* denotes the collection of variables $z_1, \ldots, z_{n-1}$, the $\varphi^{(s)}(z^*)$ are the values of the partial derivatives $\partial^s \varphi / \partial z_n^s$ for $z_n = 0$, and $\hat{\varphi}(z) = \sum_{s=m}^{\infty} [(z_n^{s-m} \varphi^{(s)}(z^*))/s!]$. Each of the functions $\varphi^{(s)}(z^*)$, $0 \leqslant s \leqslant m-1$, belongs to the space Z^* of functions of $n-1$ variables $z_1, \ldots, z_{n-1}$ and has a zero of order $m - s$ for $z^* = 0$. Therefore, by the induction hypothesis these functions can be written in the form

$$\varphi^{(s)}(z^*) = \sum_{|l|=m-s} (z^*)^l \psi_{sl}(z^*),$$

where the $\psi_{sl}(z^*)$ are entire analytic functions of exponential type having power growth for real values of z^*. Substituting these values into (32), we obtain

$$\varphi(z) = \sum_{s=0}^{m-1} \sum_{|l|=m-s} \frac{z_n^s (z^*)^l \psi_{sl}(z^*)}{s!} + z_n^m \hat{\varphi}(z). \tag{33}$$

The function $\hat{\varphi}(z)$ is also an entire function. Since all the remaining terms in the right side of (33) are of exponential type and have power growth for real values of z, and $\varphi(z)$ also has these properties, then $z_n^m \hat{\varphi}(z)$ also has these properties. But then $\hat{\varphi}(z)$ is also a function of exponential type having power growth for real values of z.

Since we have $s + |l| = m$ in (33), then the latter can be rewritten in the form

$$\varphi(z) = \sum_{|r|=m} z^r \psi_r(z),$$

where $\psi_r(z) = \psi_{sl}(z^*)$ if $z^r = z_n^s (z^*)^l$, $l > 0$, and $\psi_r(z) = \hat{\varphi}(z)$ if $z^r = z_n^m$. As we have proven, all of the functions $\psi_r(z)$ are of exponential type and have power growth for real values of z. This proves our auxiliary statement.

Now take any function $\alpha(z) \in Z$ such that $\alpha(0) = 1$. The sequence of functions $\alpha(z/k)\, \varphi(z)$ converges to $\varphi(z)$ in the topology of Z. But

$$\alpha\left(\frac{z}{k}\right) \varphi(z) = \sum_{|r|=m} z^r \alpha\left(\frac{z}{k}\right) \psi_r(z),$$

and the functions $\varphi_{kr}(z) = \alpha(z/k)\, \psi_r(z)$, as is easily seen, belong to Z. This proves the theorem.

Using the Fourier transformation, we obtain the following corollary of Theorem 6.

Corollary. Every function $\varphi(x) \in K$ whose moments up to order m inclusive equal zero is the limit (in the sense of the topology in K) of a sequence of functions $\varphi_k(x)$ having the form

$$\varphi_k(x) = \sum_{|r|=m} \varphi_{kr}^{(r)}(x),$$

where $\varphi_{kr}(x) \in K$.

5. Evenly Positive-Definite Generalized Functions

We consider here one of the most typical examples of theorems analogous to the Bochner–Schwartz theorem in which, as distinguished from the cases considered in Section 3, the positive measure is not defined uniquely. Such theorems appear rather frequently (for example, in the moment problem).

5.1. Preliminary Remarks

A generalized function F is called *even with respect to each argument*, if

$$F(\pm x_1, \dots, \pm x_n) = F(x_1, \dots, x_n)$$

for any combination of signs. Henceforth the words "with respect to each argument" will, for brevity, be omitted.

If F is an even generalized function, and the test function $\varphi(x)$ is odd with respect to at least one argument, then $(F, \varphi) = 0$.

Many questions lead to the consideration of even generalized functions such that $(F, \varphi * \varphi^*) \geqslant 0$ only for even test functions $\varphi(x)$. We will call such generalized functions *evenly positive-definite*. Of course, the class of evenly positive-definite generalized functions is broader than the class of even positive-definite generalized functions since, if F is an even positive-definite generalized function, the inequality $(F, \varphi * \varphi^*) \geqslant 0$ holds not only for even test functions, but for all test functions.

The definition of an evenly positive-definite generalized function can be given another form. Assume that the space Φ of test functions is such that together with any two functions $\varphi(x)$ and $\psi(x)$, $x = (x_1, \dots, x_n)$, it contains also the function

$$\theta(x) = \sum \int \varphi(y_1, \dots, y_n)\psi(x_1 \pm y_1, \dots, x_n \pm y_n)\, dy_1 \dots dy_n,$$

where the summation is taken over all combinations of signs. It is easy to show that if F is an evenly positive-definite generalized function, then the bilinear functional on Φ defined by $B(\varphi, \psi) = (F, \theta)$ is positive-definite, i.e., $B(\varphi, \varphi) \geqslant 0$. Conversely, if the bilinear functional defined by $B(\varphi, \psi) = (F, \theta)$ is positive definite, then F is evenly positive-definite. We will not stop to prove these simple assertions.

M. G. Krein has obtained a description of all *continuous evenly positive-definite functions* $f(x)$ *of one variable*, i.e., even functions $f(x)$ such that

$$\int f(x)\varphi(y)\overline{\varphi(y - x)}\, dx\, dy \geqslant 0 \tag{1}$$

for all even functions $\varphi(x) \in K$. He proved that every such function has the form

$$f(x) = \int_0^\infty \cos \lambda x \, d\mu_1(\lambda) + \int_0^\infty \cosh \lambda x \, d\mu_2(\lambda), \tag{2}$$

where μ_1 and μ_2 are positive measures, μ_1 is finite, and μ_2 is such that the integral $\int_0^\infty \cosh \lambda x \, d\mu_2(\lambda)$ converges for all $x \geqslant 0$.

A similar result, under restrictions on the growth of $f(x)$, has been obtained by A. Ya. Povzner [reference (54)].

If the measure μ_2 equals zero, then $f(x)$ assumes the form

$$f(x) = \int_0^\infty \cos \lambda x \, d\mu_1(\lambda), \tag{3}$$

and is consequently an even positive-definite function.

Krein's theorem is obviously an analog of Bochner's theorem. In the case of Krein's theorem the measures μ_1 and μ_2 are not uniquely defined by $f(x)$. A corresponding example will be given in Section 6. However, if $f(x)$ satisfies certain restrictions on its growth for $|x| \to \infty$, then its integral representation (2) is unique. It is sufficient, for example, that $\int_0^\infty \exp(-cx^2) f(x) \, dx$ converge for all $c > 0$.

We will obtain this result further on as a corollary of more general results connected with evenly positive-definite generalized functions. In order to establish the connection of this circle of topics with the theory of generalized functions, we remark the following. If the integral $\int_0^\infty \exp(-cx^2) f(x) \, dx$ converges for all $c > 0$, then

$$(f, \varphi) = \int \overline{f(x)} \varphi(x) \, dx \tag{4}$$

defines a generalized function not only on the space K of infinitely differentiable functions having bounded supports, but also on larger spaces of test functions. For example, (4) defines a generalized function on the space[1] $S_{\frac{1}{2}}^{\frac{1}{2}}$ consisting of entire analytic functions $\varphi(z)$ satisfying inequalities of the form

$$|\varphi(x + iy)| \leqslant C \exp(-ax^2 + by^2), \qquad 0 < a \leqslant b.$$

It turns out that the possibility of extending the functional (f, φ) to $S_{\frac{1}{2}}^{\frac{1}{2}}$ is sufficient for the uniqueness of μ_1 and μ_2. In other words, in order

[1] Concerning the definition of the space $S_{\frac{1}{2}}^{\frac{1}{2}}$, cf. Volume II, Chapter IV, Section 2.3. A brief definition is given on p. 198.

that the measures μ_1 and μ_2 in (2) be uniquely defined by the evenly positive-definite continuous function $f(x)$, it is sufficient that (4) define a continuous linear functional on the space $S_{\frac{1}{2}}^{\frac{1}{2}}$.

We will see further on that in this form the corresponding assertion carries over to evenly positive-definite generalized functions of several variables. Moreover, for evenly positive-definite generalized functions on $S_{\frac{1}{2}}^{\frac{1}{2}}$ we will prove a theorem on the existence of an integral representation not only for functions of one variable, but also for functions of several variables.

Thus we will see that for a suitable choice of the space of test functions one can achieve the result that every evenly positive-definite generalized function on the space is defined by means of a unique positive measure.

We remark that for certain spaces (for example, K) one can prove an existence theorem for the corresponding measures, in which the measures are not uniquely defined. However, in these cases the existence theorem is proven only for functions of one variable. At the same time, in the class of uniqueness the results obtained are true also for functions of several variables.

Apparently this is not accidental, and outside of the class of uniqueness the existence theorem for functions of several variables is, as a rule, not true. That is, one is unable to deduce the positivity of a generalized function from its multiplicative positivity. It would be interesting to construct corresponding examples (at present such examples are known for the multidimensional moment problem; cf. Section 7.2).

5.2. Evenly Positive-Definite Generalized Functions on $S_{\frac{1}{2}}^{\frac{1}{2}}$ [2]

In this paragraph we study evenly positive-definite generalized functions on the space $S_{\frac{1}{2}}^{\frac{1}{2}}$. In other words, we will consider even generalized functions on $S_{\frac{1}{2}}^{\frac{1}{2}}$ such that $(F, \varphi * \varphi^*) \geqslant 0$ for all even functions $\varphi(z) \in S_{\frac{1}{2}}^{\frac{1}{2}}$.

[2] In the case of several variables, $S_{\frac{1}{2}}^{\frac{1}{2}}$ denotes the space of entire analytic functions $\varphi(z) = \varphi(z_1, \ldots, z_n)$ satisfying inequalities of the form

$$|\varphi(x + iy)| \leqslant K \exp(-ax^2 + by^2), \qquad 0 < a \leqslant b. \tag{5}$$

Here ax^2 denotes the expression $\Sigma_{k=1}^n a_k x_k^2$; by^2, the expression $\Sigma_{k=1}^n b_k y_k^2$; and the inequality $0 < a \leqslant b$ means that $0 < a_k \leqslant b_k$ for all k. The topology in $S_{\frac{1}{2}}^{\frac{1}{2}}$ is defined in the following manner: a sequence $\{\varphi_m(z)\}$ of functions in $S_{\frac{1}{2}}^{\frac{1}{2}}$ is said to converge to zero, if the functions $\varphi_m(z)$ converge to zero uniformly in every finite region of z-space and satisfy the inequality

$$|\varphi_m(x + iy)| \leqslant K \exp(-ax^2 + by^2), \qquad 0 < a \leqslant b$$

with constants K, a, b which do not depend upon m.

Let us denote by $\mathfrak{M}$ the set of points $z = (z_1, \ldots, z_n)$ in n-dimensional complex space, each of whose coordinates is either real or pure imaginary. The following theorem, generalizing the assertions formulated earlier for functions of one variable, holds.

Theorem 1. Let F be an evenly positive-definite generalized function on the space $S_{\frac{1}{2}}^{\frac{1}{2}}$. Then F is the Fourier transform of a uniquely defined even positive measure μ, concentrated on the set $\mathfrak{M}$ of points $\lambda = (\lambda_1, \ldots, \lambda_n)$, each of whose coordinates λ_k is either real or pure imaginary, and such that the integral[3] $\int_{\mathfrak{M}} \exp(-c\lambda^2)\, d\mu(\lambda)$ converges for every $c > 0$.

In other words, for any even function[4] $\varphi(z) \in S_{\frac{1}{2}}^{\frac{1}{2}}$

$$(F, \varphi) = \int_{\mathfrak{M}} \tilde{\varphi}(\lambda)\, d\mu(\lambda),$$

where μ is a measure with the properties indicated above, and $\tilde{\varphi}$ is the Fourier transform of $\varphi(z)$.

If F is a generalized function of one variable, then the condition that the integral $\int_{\mathfrak{M}} \exp(-c\lambda^2)\, d\mu(\lambda)$ converge means that

$$\int_0^\infty \exp(-c\lambda^2)\, d\mu_1(\lambda) + \int_0^\infty \exp(c\lambda^2)\, d\mu_2(\lambda) < +\infty$$

for any $c > 0$ (here μ_1 is the restriction of the even measure μ to the real axis, and μ_2 is its restriction to the imaginary axis). The generalized function F is given, for $n = 1$, by

$$(F, \varphi) = \int_{-\infty}^{\infty} \tilde{\varphi}(\lambda)\, d\mu_1(\lambda) + \int_{-\infty}^{\infty} \tilde{\varphi}(i\lambda)\, d\mu_2(\lambda).$$

This equation can be written in the form

$$F = 2\left[\int_0^\infty \cos \lambda x\, d\mu_1(\lambda) + \int_0^\infty \cosh \lambda x\, d\mu_2(\lambda)\right].$$

Thus, for $n = 1$ we obtain a generalization of the results indicated earlier for evenly positive-definite continuous functions of one variable.

We will, as usual, prove the theorem which is dual to Theorem 1

[3] We denote by $c\lambda^2$ the expression $\sum_{k=1}^n c_k\lambda_k^2$. The inequality $c > 0$ signifies that $c_k > 0$, $1 \leqslant k \leqslant n$.

[4] It is sufficient to indicate the form of F for even test functions only: every test function $\varphi(z)$ is the sum of an even test function and functions which are odd in at least one variables. For functions which are odd in at least one variable, in view of the evenness of F, $(F, \varphi) = 0$.

relative to the Fourier transformation. Since this transformation carries $S_{\frac{1}{2}}^{\frac{1}{2}}$ into itself, $\tilde{S}_{\frac{1}{2}}^{\frac{1}{2}} = S_{\frac{1}{2}}^{\frac{1}{2}}$, the dual theorem reads as follows.

Theorem 1′. Let F be an even generalized function on $S_{\frac{1}{2}}^{\frac{1}{2}}$ such that $(F, \varphi\bar{\varphi}) \geqslant 0$ for all even functions $\varphi(z) \in S_{\frac{1}{2}}^{\frac{1}{2}}$. Then[5] F is given by an even positive measure μ concentrated on the set $\mathfrak{M}$ of points, each of whose coordinated is either real or pure imaginary, and such that the integral $\int_{\mathfrak{M}} \exp(-cz^2)\, d\mu(z)$ converges for all $c > 0$.[6] In other words,

$$(F, \varphi) = \int_{\mathfrak{M}} \varphi(z)\, d\mu(z).$$

For the proof of Theorem 1′ we will find it convenient to pass from functions in $S_{\frac{1}{2}}^{\frac{1}{2}}$ to functions of exponential type, associating with every even function $\varphi(z_1, \ldots, z_n) \in S_{\frac{1}{2}}^{\frac{1}{2}}$ the function $\psi(z_1, \ldots, z_n)$, defined by

$$\psi(z_1, \ldots, z_n) = \varphi(\sqrt{z_1}, \ldots, \sqrt{z_n}).$$

In view of the evenness of $\varphi(z)$, the function $\psi(z)$ is well defined. The space of all the functions $\psi(z)$ obtained by this mapping will be denoted by Q. Obviously, every function in Q is an entire analytic function.

From the inequalities

$$|\varphi(x + iy)| < K \exp(-ax^2 + by^2), \qquad 0 < a \leqslant b, \tag{5}$$

which the functions in $S_{\frac{1}{2}}^{\frac{1}{2}}$ satisfy, it follows that the functions $\psi(z) \in Q$ satisfy the inequalities

$$|\psi(x + iy)| \leqslant K \exp(-cx + d\|z\|), \qquad 0 \leqslant d < c, \tag{6}$$

where

$$cx = \sum_{k=1}^{n} c_k x_k, \qquad d\|z\| = \sum_{k=1}^{n} d_k |z_k|.\text{[7]}$$

[5] For the sake of convenience in writing, we denote the dual generalized function by F and not by $\tilde{F}$; moreover, we denote the argument by z and not by λ, as was done in Theorem 1. We trust that these changes will not cause the reader trouble.

[6] That is, $c_1 > 0, \ldots, c_n > 0$. By cz^2 we understand $\Sigma_{k=1}^{n} c_k z_k^2$.

[7] Indeed, it follows from inequality (5) that

$$|\psi(x + iy)| = |\varphi(\sqrt{x_1 + iy_1}, \ldots, \sqrt{x_n + iy_n})|$$

$$= |\varphi(\sqrt{\tfrac{1}{2}x_1 + \tfrac{1}{2}|z_1|} + i\sqrt{-\tfrac{1}{2}x_1 + \tfrac{1}{2}|z_1|}, \ldots, \sqrt{\tfrac{1}{2}x_n + \tfrac{1}{2}|z_n|} + i\sqrt{-\tfrac{1}{2}x_n + \tfrac{1}{2}|z_n|})|$$
$$\leqslant K \exp(-cx + d\|z\|),$$

where $c = \frac{1}{2}a + \frac{1}{2}b$, $d = \frac{1}{2}b - \frac{1}{2}a$, and cx and $d\|z\|$ have the meaning indicated above. It is obvious here that $0 \leqslant d < c$ (i.e., that $0 \leqslant d_k < c_k$, $1 \leqslant k \leqslant n$).

Conversely, if $\psi(z)$ is an entire analytic function satisfying inequality (6), then the function $\varphi(z) = \psi(z^2)$ satisfies an inequality of the form (5), i.e., belongs to $S_{\frac{1}{2}}^{\frac{1}{2}}$. Thus, the space Q can be defined as the space of entire analytic functions which satisfy inequalities of the form

$$|\psi(x + iy)| \leqslant K \exp(-cx + d\|z\|), \tag{6}$$

where $0 \leqslant d < c$. In expanded form, inequality (6) is just

$$|\psi(x_1 + iy_1, \ldots, x_n + iy_n)| \leqslant K \exp\left[-\sum_{k=1}^{n} c_k x_k + \sum_{k=1}^{n} d_k |z_k|\right],$$

where $0 \leqslant d_k < c_k$, $1 \leqslant k \leqslant n$.

The topology in $S_{\frac{1}{2}}^{\frac{1}{2}}$ induces a topology in Q. From the definition of the topology in $S_{\frac{1}{2}}^{\frac{1}{2}}$ it follows that a sequence $\{\psi_m(z)\}$ in Q converges to zero if and only if the functions $\psi_m(z)$ converge to zero uniformly in every finite region of z-space and satisfy an inequality

$$|\psi_m(x + iy)| \leqslant K \exp(-cx + d\|z\|)$$

where K, c, and d do not depend upon m.

With every generalized function F on $S_{\frac{1}{2}}^{\frac{1}{2}}$ we associate a generalized function Φ on the space Q, defined by

$$(\Phi, \psi(z)) = (F, \psi(z^2)).$$

It is obvious that if the inequality $(F, \varphi\bar{\varphi}) \geqslant 0$ holds for all even functions $\varphi(z) \in S_{\frac{1}{2}}^{\frac{1}{2}}$, then $(\Phi, \psi\bar{\psi}) \geqslant 0$ for all $\psi(z) \in Q$, and conversely.

Since the set $\mathfrak{M}$ of points with real or pure imaginary coordinates goes over, under the transformation $z \to z^2$, into the set $\mathfrak{R}$ of points with real coordinates, Theorem 1′ is equivalent to the following assertion.

Theorem 1″. Every mutiplicatively positive generalized function Φ on the space Q has the form

$$(\Phi, \psi) = \int_{\mathfrak{R}} \psi(x)\, d\nu(x),$$

where ν is a uniquely defined positive measure on the set $\mathfrak{R}$ of points with real coordinates, such that the integral $\int e^{-cx}\, d\nu(x)$ converges for all $c > 0$.[8]

[8] As above, we denote by cx the expression $c_1x_1 + \ldots + c_nx_n$; $c > 0$ denotes $c_1 > 0, \ldots, c_n > 0$.

There is obviously a similarity between Theorem 1″ and Theorem 3′ of Section 3. We will reduce Theorem 1″ to Theorem 3′ of Section 3.

The plan of the proof consists in the following. First of all, we introduce, by means of the equation

$$(\Phi_c, \theta(z)) = (\Phi, e^{-cz}\theta(z)),$$

a multiplicatively positive generalized function Φ_c on the space Z. By Theorem 3′ of Section 3, Φ_c is given by a positive tempered measure σ_c i.e.,

$$(\Phi_c, \theta) = \int \theta(x)\, d\sigma_c(x).$$

Setting $d\nu_c(x) = e^{cx}\, d\sigma_c(x)$, we obtain for every $c > 0$ a positive measure ν_c such that

$$(\Phi, \psi) = \int \psi(x)\, d\nu_c(x)$$

for all functions $\psi(z) \in Q$ having the form $\psi(z) = e^{-cz}\theta(z)$, where $\theta(z) \in Z$.

Using the continuity of Φ relative to the topology in Q, one is able to show that the measures ν_c do not depend upon the choice of $c > 0$. Following this, one proves that the equality

$$(\Phi, \psi) = \int \psi(x)\, d\nu(x) \qquad (\nu(x) \equiv \nu_c(x))$$

holds not only for functions ψ of the form $\psi(z) = e^{-cz}\theta(z)$, where $\theta(z) \in Z$, but also for all functions $\psi(z) \in S_{\frac{1}{2}}^{\frac{1}{2}}$.

Let us now proceed to carry out this plane. First we prove the following lemma.

Lemma 1. For any $c > 0$, all functions of the form $\psi(z) = e^{-cz}\theta(z)$, where $\theta(z) \in Z$, belong to Q, and the mapping $\theta(z) \to e^{-cz}\theta(z)$ of Z into Q is continuous for any fixed c.

Proof. Since $\theta(z) \in Z$, then by the definition of the space Z we have $|\theta(x + iy)| \leqslant Ce^{a\|y\|}$,[9] and therefore

$$|e^{-cz}\theta(z)| \leqslant Ce^{-cx+a\|y\|}. \tag{7}$$

[9] By $a\|y\|$ we denote $\sum_{k=1}^{n} a_k |y_k|$.

For any a and c there is an $r = (r_1, \ldots, r_n)$, $r_k > 0$, such that $c_k + r_k > \sqrt{a_k^2 + r_k^2}$, $1 \leqslant k \leqslant n$. We set $a' = (\sqrt{a_1^2 + r_1^2}, \ldots, \sqrt{a_n^2 + r_n^2})$ and $c' = (c_1 + r_1, \ldots, c_n + r_n)$. Since

$$r\| x \| + a\| y \| = (r_1 | x_1 | + \ldots + r_n | x_n |) + (a_1 | y_1 | + \ldots + a_n | y_1 |)$$

$$\leqslant \sum_{k=1}^{n} \sqrt{r_k^2 + a_k^2}\, \sqrt{| x_k |^2 + | y_k |^2} = a' \| z \|,$$

then

$$\exp(-cx + a\| y \|) = \exp(-c'x + rx + a\| y \|) \leqslant \exp(-c'x + a'\| z \|)$$

and therefore

$$| \exp(-cz)\theta(z)| \leqslant C \exp(-cx + a\| y \|) \leqslant C \exp(-c'x + a'\| z \|). \tag{8}$$

Since, in view of the choice of r, $c' > a'$, the function $e^{-cz}\theta(z)$ belongs to $\underset{\sim}{Q}$.

Thus, the functions of the form $e^{-cz}\theta(z)$, $\theta(z) \in Z$, belong to the space $\underset{\sim}{Q}$ for any c. It is easy to ascertain the continuity of the mapping $\theta(z) \to e^{-cz}\theta(z)$ of Z into $\underset{\sim}{Q}$, starting from the definition of convergence in these spaces.

From Lemma 1 it follows that

$$(\Phi_c, \theta(z)) = (\Phi, e^{-cz}\theta(z)) \tag{9}$$

defines a continuous linear functional Φ_c on the space Z. By the hypothesis of Theorem 1″ the inequality $(\Phi, \psi\bar{\psi}) \geqslant 0$ holds for all $\psi(z) \in \underset{\sim}{Q}$. Therefore $(\Phi_c, \theta\bar{\theta}) \geqslant 0$ for all $\theta(z) \in Z$. Indeed, we have

$$(\Phi_c, \theta\bar{\theta}) = (\Phi, e^{-\frac{1}{2}cz}\theta(z)\overline{e^{-\frac{1}{2}c\bar{z}}\theta(\bar{z})}) \geqslant 0.$$

Since $(\Phi_c, \theta\bar{\theta}) \geqslant 0$ for all $\theta(z) \in Z$, then by Theorem 3 of Section 3, the generalized function Φ_c is given by a uniquely defined positive tempered measure σ_c on the set $\Re$ of points with real coordinates. Therefore, if the generalized function Φ satisfies the conditions of Theorem 1″, then for any $c > 0$ there is a positive tempered measure σ_c such that

$$(\Phi_c, \theta) = \int_{\Re} \theta(x)\, d\sigma_c(x) \tag{10}$$

for all $\theta(x) \in Z$.

Equation (10) can also be written in the form

$$(\Phi, \psi) = \int_{\Re} \psi(x)\, d\nu_c(x), \tag{10'}$$

if we set $\psi(z) = e^{-cz}\theta(z)$, $d\nu_c(x) = e^{cx}\, d\sigma_c(x)$.

Thus, we have proven that (10′) holds for all $\psi(z) \in Q$ representable in the form $e^{-cz}\theta(z)$, where $\theta(z) \in Z$. The set of functions $\psi(z) \in Q$ having the form $e^{-cz}\theta(z)$, $\theta(z) \in Z$, for fixed c will be denoted by Q_c. Thus the generalized function Φ is defined on the subspace Q_c by a positive measure ν_c. We proceed now to the central point of the proof, namely, we show that ν_c does not depend upon the choice of $c > 0$, i.e., that $\nu_a = \nu_b = \nu$ for any b, $c > 0$. This will show that Φ is defined by the same positive measure on each of the subspaces Q_c, i.e., that

$$(\Phi, \psi) = \int_{\Re} \psi(x)\, d\nu(x)$$

holds for all functions $\psi(x)$ which belong to at least one of the Q_c. Theorem 1″ is easily obtained from this by a limit passage.

The following two lemmas lie at the basis of the subsequent reasoning. Although their proofs are not hard, they require a certain amount of computation. Therefore, in order not to interrupt the discussion, we will present their proofs at the conclusion of the proof of the theorem.

Lemma 2. If $0 < b < 2c$, i.e., $0 < b_k < 2c_k$ for all k, $1 \leqslant k \leqslant n$, then there exists a sequence of functions $\theta_m(z) \in Z$ such that

(1) the sequence $\{e^{-cz}\theta_m(z)\}$ converges to e^{-bz} in the topology of Q,

(2) the functions $\theta_m(z)$ assume positive values for real values of z (i.e., $\theta_m(x) \geqslant 0$),

(3) for real values x we have

$$|\theta_m(x)| \leqslant K_1 e^{\|(b-c)x\|} \tag{11}$$

with the constant K_1 not depending upon m.[10]

Lemma 3. Every function $\psi(z) \in Q$ belongs to the closure in Q of at least one of the sets Q_c.

Lemma 2 enables us to establish the independence of the measures

[10] We denote by $\|(b-c)x\|$ the sum $\sum_{k=1}^{n} |(b_k - c_k)x_k|$.

ν_c upon the value of $c > 0$. To do this we first show that for $0 < b < 2c$ the integral

$$\int e^{-bx}\, d\nu_c(x) \tag{12}$$

converges. Indeed, since the functions $\theta_m(z)$ in Lemma 2 belong to Z, the functions $e^{-cz}\theta_m(z)$ lie in Q_c, and so

$$(\Phi, e^{-cz}\theta_m(z)) = \int e^{-cx}\theta_m(x)\, d\nu_c(x). \tag{12$'$}$$

Since $e^{-cz}\theta_m(z) \to e^{-bz}$ in the topology of Q, it follows from the continuity of Φ that the left side of (12′) is bounded. In other words, for all m we have

$$\left| \int e^{-cx}\theta_m(x)\, d\nu_c(x) \right| < A.$$

Further, in as much as $\theta_m(x)e^{-cx} \geqslant 0$ and $\theta_m(x)e^{-cx} \to e^{-bx}$ uniformly in every finite region, it follows that

$$\int e^{-bx}\, d\nu_c(x) < A.$$

Therefore the integral (12) converges. Let us now show that

$$(\Phi, e^{-bz}\theta(z)) = \int e^{-bx}\theta(x)\, d\nu_c(x) \tag{13}$$

for any $\theta(z) \in Z$, if $0 < b < 2c$.

In fact,

$$\lim_{m\to\infty} e^{-cz}\theta_m(z)\theta(z) = e^{-bz}\theta(z)$$

in the topology of Q. Hence

$$(\Phi, e^{-bz}\theta(z)) = \lim_{m\to\infty} (\Phi, e^{-cz}\theta_m(z)\theta(z)),$$

i.e., by definition of the measures ν_c,

$$(\Phi, e^{-bz}\theta(z)) = \lim_{m\to\infty} \int e^{-cx}\theta_m(x)\theta(x)\, d\nu_c(x). \tag{13$'$}$$

For the proof of (13) it suffices to prove that one can pass to the limit under the integral in (13′). To do this we note that in view of inequality (11) and the boundedness of $\theta(x)$, one has

$$| e^{-cx}\theta(x)\theta_m(x)| \leqslant Ke^{-cx+|i(b-c)x|^{2}}, \tag{14}$$

where the constant K does not depend upon m. In expanded form the expression $-cx + \| (b - c)x \|$ is just

$$- \sum_{k=1}^{n} [c_k x_k + |(b_k - c_k)x_k|] = - \sum_{k=1}^{n} x_k[c_k - | b_k - c_k | \operatorname{sign} x_k].$$

But in view of the inequalities $0 < b_k < 2c_k$, $1 \leqslant k \leqslant n$, we have $0 < c_k - | b_k - c_k | \operatorname{sign} x_k < 2c_k$. Let us denote the expression $c_k - | b_k - c_k | \operatorname{sign} x_k$ by h_k. We saw that $0 < h_k < 2c_k$. Therefore it follows from inequality (14) that

$$| e^{-cx}\theta(x)\theta_m(x)| < Ke^{-hx}, \qquad h = (h_1, \ldots, h_n),$$

where $0 < h < 2c$. But, as was shown above, the integral $\int e^{-hx}\, d\nu_c(x)$ converges for $0 < h < 2c$. Thus, each of the functions $e^{-cx}\theta(x)\theta_m(x)$ is bounded by the function e^{-hx}, which is summable with respect to ν_c. As is well known, this permits passing to the limit under the integral sign, which proves (13).

Now we are able to prove the independence of the measures ν_c upon the choice of $c > 0$. In fact, by definition of ν_b we have, for every $\theta(z) \in Z$,

$$(\Phi, e^{-bz}\theta(z)) = \int e^{-bx}\theta(x)\, d\nu_b(x).$$

Comparing this relation with (10′), we see that

$$\int e^{-bx}\theta(x)\, d\nu_c(x) = \int e^{-bx}\theta(x)\, d\nu_b(x)$$

for every $\theta \in Z$. But this can be so only if $\nu_c = \nu_b$.

Thus we have proven that $\nu_c = \nu_b$ if $0 < b < 2c$. But then this equality holds for any positive values of b and c. Denote the common value of the measures ν_c by ν. From the properties of ν_c it follows that the integral

$$\int e^{-cx}\, d\nu(x) \tag{15}$$

converges for all $c > 0$.

As was remarked above (cf. p. 204), it follows from this that the equality

$$(\Phi, \psi) = \int \psi(x)\, d\nu(x)$$

holds for every function $\psi(x)$ which belongs to at least one of the sets Q_c. In order to prove its validity for every function $\psi(x) \in Q$, we use Lemma

3. According to this lemma, for any function $\psi(z) \in Q$ there exists a sequence $\{\psi_m(z)\}$ which converges to $\psi(z)$ in the topology of Q, and all the $\psi_m(z)$ belong to the same space Q_c. The generalized function Φ is defined, for each of these functions, by

$$(\Phi, \psi_m) = \int \psi_m(x)\, d\nu(x)$$

and we therefore find, in view of the continuity of Φ, that

$$(\Phi, \psi) = \lim_{m\to\infty} \int \psi_m(x)\, d\nu(x). \tag{16}$$

In view of the earlier proven convergence of the integral (15) and the estimate

$$|\psi_m(x)| \leqslant Ke^{-cx}$$

which the $\psi_m(x)$ satisfy (by definition of convergence in Q), we can pass to the limit under the integral sign in (16). It follows that

$$(\Phi, \psi) = \int \psi(x)\, d\nu(x).$$

This proves Theorem 1″. But, as we have seen above, Theorems 1, 1′, and 1″ are mutually equivalent. Therefore the proof of Theorem 1 is complete.

The converse of Theorem 1 also holds.

Theorem 2. Suppose that the positive even measure μ, defined on the set $\mathfrak{M}$ of points, each of whose coordinates is either real or pure imaginary, is such that the integral

$$\int_{\mathfrak{M}} \exp(-cz^2)\, d\mu(z)$$

converges for all $c > 0$. Then

$$(F, \varphi) = \int_{\mathfrak{M}} \varphi(z)\, d\mu(z)$$

defines an even generalized function F on the space $S_{\frac{1}{2}}^{\frac{1}{2}}$ such that $(F, \varphi * \varphi^*) \geqslant 0$ for all even $\varphi(z) \in S_{\frac{1}{2}}^{\frac{1}{2}}$.

The proof of this assertion is trivial.

In the course of proving Theorem 1 we omitted the proofs of Lemmas 2 and 3. Let us fill this gap.

First we prove Lemma 2, i.e., we show that for $0 < b < 2c$ the function e^{-bz} can be approximated in Q by a sequence of functions of the form $e^{-cz}\theta_m(z)$, where the $\theta_m(z) \in Z$ are such that $\theta_m(x) \geqslant 0$ and $|\theta_m(x)| \leqslant K_1 e^{\|(b-c)x\|}$, with the constant K_1 not depending upon m. For the construction of the $\theta_m(x)$ we take any function $\alpha(z) \in Z$ such that $\alpha(0) = 1$ and

$$|\alpha(x + iy)| \leqslant Ce^{r\|y\|},$$

where r satisfies the inequality[11] $0 < r < \frac{1}{2}(c - \|b - c\|)$ (since by hypothesis $0 < b < 2c$, such an r exists).

We set

$$\theta_m(z) = \alpha\left(\frac{z}{m}\right)\bar{\alpha}\left(\frac{z}{m}\right)\Big[\sum_{|k|=0}^{m} \frac{(c-b)^k z^k}{2^k k!}\Big]^2 \tag{17}$$

and show that the sequence $\{\theta_m(z)\}$ satisfies all the conditions of the lemma. Note that each of the $\theta_m(z)$ belongs to Z, being the product of the function $\alpha(z/m)\bar{\alpha}(z/m) \in Z$ and a polynomial. Further, the expression appearing within the square brackets is the partial sum of the Taylor series for $e^{\frac{1}{2}(c-b)z}$ and therefore converges to $e^{\frac{1}{2}(c-b)z}$ uniformly in every bounded region as $m \to \infty$. At the same time, the functions $\alpha(z/m)\bar{\alpha}(z/m)$ converge to $\alpha(0) = 1$ uniformly in every bounded region as $m \to \infty$. Therefore, $e^{-cz}\theta_m(z) \to e^{-bz}$ uniformly in every bounded region.

Moreover, these functions satisfy the inequalities

$$|e^{-cz}\theta_m(z)| < Le^{-cx+s\|z\|}, \qquad 0 < s < c$$

with constants L, c, s not depending upon m. Indeed,

$$\Big|\sum_{|k|=0}^{m} \frac{(c-b)^k z^k}{2^k k!}\Big| \leqslant e^{\frac{1}{2}\|(c-b)z\|}. \tag{18}$$

and therefore

$$\begin{aligned} |e^{-cz}\theta_m(z)| &\leqslant \Big|e^{-cz}\alpha\left(\frac{z}{m}\right)\bar{\alpha}\left(\frac{z}{m}\right)\Big| e^{\|(c-b)z\|} \\ &\leqslant C^2 \exp\left(-cx + \frac{2r\|y\|}{m} + \|(c-b)z\|\right). \end{aligned} \tag{19}$$

But since $\|y\| \leqslant \|z\|$, it follows from (19) that

$$|e^{-cz}\theta_m(z)| \leqslant Le^{-cx+s\|z\|}, \tag{20}$$

[11] This inequality means that

$$0 < r_k < \tfrac{1}{2}(c_k - |b_k - c_k|).$$

for all k, $1 \leqslant k \leqslant n$.

where we have put $L = C^2$, $s = \| c - b \| + 2r$. Here $0 < s < c$ in view of the choice of r. From the uniformity, in every bounded region, of the convergence of $\{e^{-cz}\theta_m(z)\}$ to e^{-bz} and from (20) it follows also that the sequence $\{e^{-cz}\theta_m(z)\}$ converges to e^{-bz} in the topology of Q. Further, the $\theta_m(z)$ assume positive values on the set $\mathfrak{R}$ of points having real coordinates and satisfy on $\mathfrak{R}$ the inequality

$$| e^{-cx}\theta_m(x)| \leqslant Le^{-cx+s\|x\|}, \qquad 0 < s < c.$$

Lastly, from (17) and the fact that the functions $\alpha(z/m)$ have a common bound on the real line, we have

$$| \theta_m(x)| < K_1 e^{\|(c-b)x\|}$$

with K_1 independent of m. This proves Lemma 2.

Now we prove Lemma 3, i.e., we show that every function $\psi(z) \in Q$ belongs to the closure in Q of some one of the Q_c. To do this, we have to find for every $\psi(z) \in Q$ a $c > 0$ and a sequence of functions $\psi_m(z) \in Q$, having the form $\psi_m(z) = e^{-cz}\varphi_m(z)$, $\varphi_m(z) \in Z$, such that $\lim_{m\to\infty} \psi_m(z) = \psi(z)$ (the limit is understood to be in the topology of Q).

Thus, let $\psi(z)$ be any function in Q. By the definition of Q, $\psi(z)$ satisfies an inequality of the form

$$| \psi(x + iy)| < Ce^{-ax+b\|z\|}, \qquad 0 < a < b. \tag{21}$$

We take any $c > a$ and expand the entire function $e^{cz}\psi(z)$ in a Taylor series, and denote by $\rho_m(z)$ the mth partial sum of this series. We choose any $\alpha(z) \in Z$ such that $\alpha(0) = 1$ and $| \alpha(x + iy) | < Le^{r|y|}$, where $0 < r < \frac{1}{4}(a - b)$, and set

$$\psi_m(z) = e^{-cz}\alpha\left(\frac{z}{m}\right)\rho_m(z).$$

Since $\alpha(z/m) \in Z$ and the $\rho_m(z)$ are polynomials, the $\psi_m(z)$ belong to the set Q_c. Let us show that $\lim_{m\to\infty}\psi_m(z) = \psi(z)$, where the limit is understood in the sense of the topology of Q.

First of all, it is obvious that the $\rho_m(z)$ converge to the entire function $e^{cz}\psi(z)$ uniformly in every bounded region. Further, $\{\alpha(z/m)\} \to 1$ uniformly in every bounded region. Therefore $\{\psi_m(z)\} \to \psi(z)$ uniformly in every bounded region.

Now we show that the $\psi_m(z)$ satisfy the inequalities

$$| \psi_m(x + iy)| < Ne^{-cx+s\|z\|}, \qquad 0 \leqslant s < c, \tag{22}$$

with constants N, c, s not depending upon m. For this, we note that in view of (21)

$$|e^{cz}\psi(z)| < Ce^{(c-a)x+b\|z\|} \leqslant K\exp(b_1\|z\|), \tag{23}$$

where $b_1 = b + c - a$. But then for any $b_2 > b_1$ one has[12]

$$|\rho_m(z)| \leqslant K_1 \exp(b_2\|z\|),$$

where K_1 does not depend upon m. Choose $b_2 = c - \frac{1}{2}(a - b)$.
For this choice of b_2 one has the estimate

$$|\psi_m(x + iy)| = e^{-cx}\left|\alpha\left(\frac{z}{m}\right)\right| |\rho_m(z)|$$
$$\leqslant LK_1 \exp\left(-cx + \frac{r\|y\|}{m} + \tfrac{1}{2}[2c - a - b]\|z\|\right) \leqslant Ne^{-cx+s\|z\|},$$

where $s = c - \frac{1}{4}(a - b)$. Obviously $0 < s < c$.

Thus we have proven that (22) holds with constants N, c, s not depending upon m, and, consequently, that $\{\psi_m(z)\}$ converges to $\psi(z)$ in the topology of Q.

This proves Lemma 3, and with it Theorem 1″ (and so also Theorem 1′ and 1) is completely proved.

With the help of Theorem 1″ it is easy to obtain a description of evenly positive-definite continuous functions $f(x) = f(x_1, ..., x_n)$ such that the integral $\int \exp(-cx^2) f(x)\, dx$ converges for all $c > 0$. Such a function defines an evenly positive-definite generalized function

$$(f, \varphi) = \int \overline{f(x)}\varphi(x)\, dx$$

on the space $S_{\frac{1}{2}}^{\frac{1}{2}}$. By Theorem 1 this function has the form

$$f(x) = \int_{\mathfrak{M}} e^{i(x,z)}\, d\mu(z), \tag{24}$$

where μ is a uniquely defined positive even measure on the set $\mathfrak{M}$, such that $\int_{\mathfrak{M}} \exp(-cz^2)\, d\mu(z)$ converges for all $c > 0$. However, the continuity of $f(x)$ imposes additional conditions upon μ. It can be shown that μ must be such that not only does $\int_{\mathfrak{M}} \exp(-cz^2)\, d\mu(z)$ converge for

[12] Indeed, since $e^{cz}\psi(z)$ is an entire analytic function satisfying (23), its Taylor coefficients satisfy the inequalities $|d_k| < M(b_1e/k)^k$. But then $k!|d_k| < M_1\sqrt{k}\, b_1^k$. If $b_2 > b_1$, then there is a K_1 such that $|k!d_k| < K_1 b_2^k$, where K_1 does not depend upon k. Obviously,

$$|\rho_m(z)| \leqslant K_1 \sum_{|k|=0}^{m} \left|\frac{b_2^k z^k}{k!}\right| \leqslant K_1 \exp(b_2\|z\|).$$

all $c > 0$, but also $\int_{\mathfrak{M}} \exp(cy^2)\, d\mu(z)$, $z = x + iy$. Conversely, if μ is such that

$$\int_{\mathfrak{M}} \exp(cy^2)\, d\mu(z) \tag{25}$$

converges for all $c > 0$, then the function $f(x)$ defined by (24) is continuous.

Thus, the following theorem holds.

Theorem 3. Let $f(x)$ be a continuous evenly positive-definite function such that $\int \exp(-cx^2) f(x)\, dx$ converges for all $c > 0$. Then $f(x)$ is the Fourier transform of an even positive measure μ on the set $\mathfrak{M}$ of points, each of whose coordinates is either real or pure imaginary. The measure μ is such that the integral (25) converges for all $c > 0$. Conversely, if the positive even measure μ is such that the integral (25) converges for all $c > 0$, then (24) defines a continuous evenly positive-definite function $f(x)$ such that the integral $\int \exp(-cx^2) f(x)\, dx$ converges for all $c > 0$. We omit the proof of Theorem 3.

Theorem 1 also enables us to clarify under what conditions the equality of the Fourier transforms of two positive even measures μ_1 and μ_2 implies the equality of the measures themselves. Namely, the following assertion holds.

Theorem 4. Let μ_1 and μ_2 be positive even measures, on the set $\mathfrak{M}$ of points, each of whose coordinates is either real or pure imaginary, which define generalized functions on the space Z. If the Fourier transforms μ_1 and μ_2 of these measures coincide and if the integrals $\int_{\mathfrak{M}} \exp(-cz^2)\, d\mu_1(z)$ and $\int_{\mathfrak{M}} \exp(-cz^2)\, d\mu_2(z)$ converge for all $c > 0$, then μ_1 and μ_2 coincide.

Indeed, under the conditions of the theorem the measures μ_1 and μ_2 define generalized functions on $S_{\frac{1}{2}}^{\frac{1}{2}}$. But then their Fourier transforms are also generalized functions on $S_{\frac{1}{2}}^{\frac{1}{2}}$, i.e., $\mu_1 = \mu_2 = F$, where F is an evenly positive-definite generalized function on $S_{\frac{1}{2}}^{\frac{1}{2}}$.

Since for evenly positive-definite generalized functions on $S_{\frac{1}{2}}^{\frac{1}{2}}$ the corresponding positive measures are uniquely defined, we find that $\mu_1 = \mu_2$.

5.3. Evenly Positive-Definite Generalized Functions on $S_{\frac{1}{2}}$

Results analogous to those proven in Section 5.2 hold for generalized functions on the space[13] $S_{\frac{1}{2}}$.

[13] The space $S_{\frac{1}{2}}$ consists of infinitely differentiable functions $\varphi(x)$, satisfying inequalities of the form

$$|\varphi^{(q)}(x)| \leqslant C_q \exp(-bx^2),$$

The theorem regarding evenly positive-definite generalized functions on $S_{\frac{1}{2}}$ is formulated in the following way.

Theorem 5. Let F be an even generalized function on $S_{\frac{1}{2}}$ such that $(F, \varphi * \varphi^*) \geqslant 0$ for all even functions $\varphi \in S_{\frac{1}{2}}$. Then F can be represented in the form

$$(F, \varphi) = \int_{\mathfrak{M}} \tilde{\varphi}(z)\, d\mu(z), \tag{26}$$

where $\tilde{\varphi}(z)$ is the Fourier transform of $\varphi(x)$, and μ is a uniquely defined positive even measure on the set $\mathfrak{M}$ of points each of whose coordinates is either real or pure imaginary. The measure μ has the following properties: For any $c > 0$ the integral

$$\int_{\mathfrak{M}} (1 + |x|^2)^{-p} \exp(c|y|^2)\, d\mu(z) \tag{27}$$

converges for some p. Conversely, if μ is an even positive measure on the set $\mathfrak{M}$ having the properties indicated, then (26) defines an even generalized function F on $S_{\frac{1}{2}}$ such that $(F, \varphi * \varphi^*) \geqslant 0$ for all even functions $\varphi(x) \in S_{\frac{1}{2}}$.

We will not carry out a detailed proof of this theorem, but simply indicate the idea of the proof. As usual, one can pass from Theorem 5 to a theorem which is dual to it relative to the Fourier transformation, regarding multiplicatively positive generalized functions on $S^{\frac{1}{2}}$ (the reader can formulate this theorem without difficulty). Since $S_{\frac{1}{2}}^{\frac{1}{2}}$ is a subspace of $S^{\frac{1}{2}}$, such a multiplicatively positive generalized function induces a generalized function with similar properties on the space $S_{\frac{1}{2}}^{\frac{1}{2}}$. By Theorem 1′ this generalized function is given (for functions

where the constants C_q and b depend upon $\varphi(x)$ (cf. Volume II, Chapter IV, Section 2.2), and bx^2 denotes the sum $\sum_{k=1}^{n} b_k x_k^2$. A topology in $S_{\frac{1}{2}}$ is introduced as follows. A sequence $\{\varphi_m(x)\}$ in this space is said to converge to zero, if for any q the sequence $\{\varphi_m^{(q)}(x)\}$ converges to zero uniformly in every bounded region, and

$$|\varphi_m^{(q)}(x)| \leqslant C_q \exp(-bx^2)$$

for some constants C_q and b not depending upon m.

We remark that the convolution of two functions from $S_{\frac{1}{2}}$ belongs to $S_{\frac{1}{2}}$. Moreover, together with any function $\varphi(x)$, $S_{\frac{1}{2}}$ also contains the function $\varphi^*(x) = \overline{\varphi(-x)}$.

The space $S_{\frac{1}{2}}$ is dual, with respect to the Fourier transformation, to the space $S^{\frac{1}{2}}$ of entire analytic functions $\varphi(z)$ satisfying inequalities of the form

$$|x^k \varphi(x + iy)| \leqslant C_k \exp(by^2)$$

(cf. Volume II, Chapter IV, Section 6.2).

from $S_{\frac{1}{2}}^{\frac{1}{2}}$) by an even positive measure μ on the set $\mathfrak{M}$, such that the integral

$$\int_{\mathfrak{M}} \exp(-cz^2)\, d\mu(z)$$

converges for all $c > 0$. Following this we have to prove only that μ defines a generalized function F on all of $S^{\frac{1}{2}}$ and has the properties indicated in the statement of the theorem. The proof of this can be carried out by approximating functions $\varphi(z) \in S^{\frac{1}{2}}$ by functions of the form $\varphi_m(z) = \exp(-z^2/m)\,\varphi(z)$ from $S_{\frac{1}{2}}^{\frac{1}{2}}$ and using Fatou's lemma. We omit the details of the proof.

5.4. Positive-Definite Generalized Functions and Groups of Linear Transformations

The concept of an evenly positive-definite generalized function which we have studied is a special case of a more general concept, connected with groups of linear transformations.

Let G be some group of linear transformations of n-dimensional space.

A function $f(x) = f(x_1, \ldots, x_n)$ is said to be *symmetric* (or invariant) *relative to the group of transformations* G, if $f(gx) = f(x)$ for every element $g \in G$. For example, if the group G consists of all transformations in which some variables change sign, then the functions which are symmetric relative to G are the even functions.

If a continuous function $f(x)$ is symmetric relative to some group G, then for all functions $\varphi(x) \in K$ one has

$$(f, \varphi(g^{-1}x)) = (\det g)(f, \varphi(x)). \tag{28}$$

Indeed,

$$(f, \varphi(g^{-1}x)) = \int \overline{f(x)}\varphi(g^{-1}x)\, dx = (\det g) \int \overline{f(gx)}\varphi(x)\, dx$$

$$= (\det g) \int \overline{f(x)}\varphi(x)\, dx = (\det g)(f, \varphi(x)).$$

In accordance with this we call a generalized function F *symmetric relative to the group* G, if

$$(F, \varphi(g^{-1}x)) = (\det g)(F, \varphi(x))$$

for all test functions $\varphi(x)$ and all elements $g \in G$.

Lastly, we call a test function $\varphi(x)$ a *symmetric function of the second kind relative to the group* G, if

$$\varphi(g^{-1}x) = (\det g)\varphi(x) \tag{29}$$

for all $g \in G$. Obviously, if $\varphi_1(x)$ and $\varphi_2(x)$ are symmetric functions of the second kind, then their convolution $\varphi(x) = \varphi_1 * \varphi_2(x)$ is also a symmetric function of the second kind. Indeed,

$$\begin{aligned}\varphi(g^{-1}x) &= \int \varphi_1(y)\varphi_2(g^{-1}x - y)\, dy \\ &= (\det g)^{-1} \int \varphi_1(g^{-1}y)\varphi_2(g^{-1}(x - y))\, dy \\ &= (\det g) \int \varphi_1(y)\varphi_2(x - y)\, dy = (\det g)\varphi(x).\end{aligned}$$

Let us now introduce the concept of a generalized function F, *positive-definite relative to a group G*. We will apply this name to any generalized function F which is symmetric relative to G and such that

$$(F, \varphi * \varphi^*) \geqslant 0 \tag{30}$$

for any test function $\varphi(x)$ which is a symmetric function of the second kind relative to G. For example, if G is the group of all changes of sign of the arguments, then those generalized functions which are positive-definite relative to G are just the evenly positive-definite ones.

It is easy to describe a method which enables one to construct an entire class of generalized functions which are symmetric relative to a given group G. For this we consider, along with the group G, the group G^*, consisting of the transformations g^* which are adjoint to the transformations $g \in G$, i.e., such that

$$(\lambda, gx) = (g^*\lambda, x)$$

for all vectors $x = (x_1, ..., x_n)$ and $\lambda = (\lambda_1, ..., \lambda_n)$. Let us denote by $\mathfrak{M}$ the subset of n-dimensional complex space consisting of all points $\lambda = (\lambda_1, ..., \lambda_n)$ such that $g^*\lambda = \bar{\lambda}$ for all elements $g^* \in G^*$. Then every positive measure μ, defined on $\mathfrak{M}$ and symmetric[14] relative to the group G^*, defines a generalized function

$$(F, \varphi) = \int_{\mathfrak{M}} \tilde{\varphi}(\lambda)\, d\mu(\lambda) \tag{31}$$

[14] A measure μ is said to be *symmetric relative to the group* G^*, if

$$\int_{\mathfrak{M}} \tilde{\varphi}(g^*\lambda)\, d\mu(\lambda) = \int_{\mathfrak{M}} \tilde{\varphi}(\lambda)\, d\mu(\lambda)$$

for any function $\tilde{\varphi}(\lambda)$. We remark that the set $\mathfrak{M}$ itself is invariant relative to all transformations of the group G^*.

which is positive-definite relative to G (as usual, $\tilde{\varphi}(\lambda)$ denotes the Fourier transform of $\varphi(x)$).

In order to prove this assertion, we show first that the Fourier transform of the test function $\varphi(g^{-1}x)$ is the function $(\det g)\,\tilde{\varphi}(g^*\lambda)$. We have

$$\int \varphi(g^{-1}x)e^{i(\lambda,x)}\,dx = (\det g)\int \varphi(x)e^{i(\lambda,gx)}\,dx$$

$$= (\det g)\int \varphi(x)e^{i(g^*\lambda,x)}\,dx = (\det g)\tilde{\varphi}(g^*\lambda). \tag{32}$$

From this it follows that if μ is symmetric relative to G^*, then F is symmetric relative to G. Indeed,

$$(F, \varphi(g^{-1}x)) = (\det g)\int_{\mathfrak{M}} \tilde{\varphi}(g^*\lambda)\,d\mu(\lambda)$$

$$= (\det g)\int_{\mathfrak{M}} \tilde{\varphi}(\lambda)\,d\mu(\lambda) = (\det g)(F, \varphi(x))$$

for any test function $\varphi(x)$.

It remains for us to show that if $\varphi(x)$ is a symmetric test function of the second kind relative to G, then

$$(F, \varphi * \varphi^*) \geqslant 0. \tag{33}$$

To do this, we observe that in view of (32)

$$\tilde{\varphi}(g^*\lambda) = (\det g)^{-1}\int \varphi(g^{-1}\lambda)e^{i(\lambda,x)}\,dx.$$

Since $\varphi(x)$ is a symmetric function of the second kind, we have $\varphi(g^{-1}x) = (\det g)\varphi(x)$, and therefore

$$\tilde{\varphi}(g^*\lambda) = \int \varphi(x)e^{i(\lambda,x)}\,dx = \tilde{\varphi}(\lambda).$$

From $\tilde{\varphi}(g^*\lambda) = \tilde{\varphi}(\lambda)$ and the definition of the set $\mathfrak{M}$ it follows that $\bar{\tilde{\varphi}}(\lambda) = \overline{\tilde{\varphi}(\lambda)}$ for all points $\lambda \in \mathfrak{M}$ (here $\bar{\tilde{\varphi}}(\lambda)$ denotes the function $\overline{\tilde{\varphi}(\bar{\lambda})}$). Indeed,

$$\bar{\tilde{\varphi}}(\lambda) = \overline{\tilde{\varphi}(\bar{\lambda})} = \overline{\tilde{\varphi}(g^*\lambda)} = \overline{\tilde{\varphi}(\lambda)}.$$

But the Fourier transform of $\varphi * \varphi^*(x)$ is $\tilde{\varphi}(\lambda)\bar{\tilde{\varphi}}(\lambda)$. Therefore we have

$$(F, \varphi * \varphi^*) = \int_{\mathfrak{M}} \tilde{\varphi}(\lambda)\bar{\tilde{\varphi}}(\lambda)\,d\mu(\lambda) = \int_{\mathfrak{M}} |\tilde{\varphi}(\lambda)|^2\,d\mu(\lambda).$$

Since μ is positive, the integral is positive, which proves (33).

We have described a method which enables us to construct an entire class of generalized functions which are positive-definite relative to a group G of transformations. It would be interesting to know for which groups G and spaces Φ one can obtain by this method all generalized functions which are positive-definite relative to G, and also to clarify the question of conditions for the uniqueness of the measure μ. In this section we have solved these problems for the group G of all changes of sign of the arguments and the space $S_{\frac{1}{2}}^{\frac{1}{2}}$ of test functions.

6. Evenly Positive-Definite Generalized Functions on the Space of Functions of One Variable with Bounded Supports

6.1. Positive and Multiplicatively Positive Generalized Functions

In this section we consider evenly positive-definite generalized functions on the space K, restricting ourselves to the case of functions of one variable. In other words, we consider even generalized functions F on K such that $(F, \varphi * \varphi^*) \geqslant 0$ for any even infinitely differentiable function φ of one variable having bounded support. We will show that such a generalized function is the Fourier transform of a positive measure concentrated on the real and imaginary axes, which measure is not, however, uniquely defined.

Let us consider the Fourier transform $\tilde{F}$ of an evenly positive-definite generalized function F on K. In view of the duality, relative to the Fourier transformation, between convolution and multiplication of functions, $\tilde{F}$ has the following property: $(F, \varphi\bar{\varphi}) \geqslant 0$ for any even function $\varphi(z) \in Z$.

Thus, our problem reduces to that of describing all even generalized functions[1] F on Z such that $(F, \varphi\bar{\varphi}) \geqslant 0$ for all even functions in Z. In the case of functions of one variable which we are considering, we will succeed in reducing this problem to the simpler problem of describing those generalized functions F such that $(F, \varphi) \geqslant 0$ for all functions $\varphi(z)$ which assume positive values on the real and imaginary axes. For this we need the following auxiliary theorem.

Theorem 1. Let $\psi(z)$ be an entire analytic function of one variable of order $\frac{1}{2}$ and finite type (i.e., it satisfies an inequality of the form $|\psi(z)| < C \exp(a|z|^{\frac{1}{2}})$) which assumes positive values on the real axis. Then $\psi(z)$ has the form $\psi(z) = \varphi(z)\bar{\varphi}(z)$,[2] where $\varphi(z)$ is an entire analytic function of order $\frac{1}{2}$ and finite type.

[1] Henceforth we will denote a generalized function on Z by the letter F, and not by $\tilde{F}$.

[2] As above, $\bar{\varphi}(z)$ denotes the function $\overline{\varphi(\bar{z})}$.

Proof. Since $\psi(z)$ is of order $\frac{1}{2}$, it can be expanded in an infinite product of the form[3]

$$\psi(z) = Az^m \prod_{k=0}^{\infty} \left(1 - \frac{z}{a_k}\right), \tag{1}$$

where the a_k are the roots of $\psi(z)$ and m is the order of the root $z = 0$. Since $\psi(z)$ is real on the real axis, the coefficients in its power series expansion are real, and therefore the complex roots of $\psi(z)$ occur in conjugate pairs. Since, moreover, $\psi(z)$ is positive on the real axis, its real roots have even order (in particular, m is even), and the coefficient A is positive. We number the roots of $\psi(z)$ so that for any k one has $a_{2k} = \overline{a_{2k+1}}$, and introduce a new function

$$\varphi(z) = \sqrt{A} z^{\frac{1}{2}m} \prod_{k=0}^{\infty} \left(1 - \frac{z}{a_{2k}}\right). \tag{2}$$

Then it is obvious that $\psi(z) = \varphi(z)\bar{\varphi}(z)$. It remains for us to show only that $\varphi(z)$ has order $\frac{1}{2}$ and finite type. In other words, we have to show that $\varphi(z)$ satisfies an inequality

$$|\varphi(z)| \leqslant C_1 \exp(a_1 |z|^{\frac{1}{2}}).$$

For this we make use of the following relation between the growth of an entire function and the density of its zeroes (cf. B. Ja. Levin, "Distribution of Zeroes of Entire Functions," Chapter I, Theorem 14. Amer. Math. Soc., Providence, Rhode Island, 1964).

If an entire analytic function $\varphi(z)$ has order ρ and type a, and ρ is not an integer, then

$$\overline{\lim_{r\to\infty}} \frac{\ln n(r)}{\ln r} = \rho$$

and

$$\overline{\lim_{r\to\infty}} \frac{n(r)}{r^\rho} = a,$$

where $n(r)$ is the number of roots of $\varphi(z)$ in the disk $|z| < r$. Conversely, if these relations are satisfied, then $\varphi(z)$ has order ρ and type a.

The number $n_1(r)$ of roots of $\varphi(z)$ in the disk $|z| < r$ is equal to one half the number $n(r)$ of roots of $\psi(z)$ in the same disk. Since the order

[3] Cf. E. C. Titchmarsh, "Theory of Functions," 2nd ed., p. 250, Oxford Univ. Press, New York and London, 1939.

of $\psi(z)$ is not an integer, then in view of the theorem quoted $\varphi(z)$ has order $\frac{1}{2}$ and finite type. This proves the theorem.

We can now prove the following assertion, which enables us to reduce the study of even generalized functions on Z, satisfying the inequality $(F, \varphi\bar{\varphi}) \geqslant 0$ for any even function $\varphi(z) \in Z$, to the study of functions F for which $(F, \varphi) \geqslant 0$ if $\varphi(z)$ is positive on the real and imaginary axes.

Theorem 2. Any even function $\theta(z) \in Z$, which assumes positive values on the set $\mathfrak{M}$ consisting of the real and imaginary axes, can be represented in the form $\theta(z) = \chi(z)\bar{\chi}(z)$, where $\chi(z)$ is some even function in Z.

Proof. We associate with the function $\theta(z)$ the function $\psi(z) = \theta(\sqrt{z})$ (this function is well defined in view of the evenness of $\theta(z)$). Obviously $\psi(z)$ is an entire analytic function which assumes positive values on the real axis and has order $\frac{1}{2}$. By Theorem 1 we can write $\psi(z)$ in the form $\psi(z) = \varphi(z)\bar{\varphi}(z)$, where $\varphi(z)$ is an entire analytic function of order $\frac{1}{2}$ and finite type. Set $\chi(z) = \varphi(z^2)$. Since

$$\theta(z) = \psi(z^2) = \varphi(z^2)\bar{\varphi}(z^2) = \chi(z)\bar{\chi}(z),$$

then for the proof of our assertion it suffices to show that $\chi(z) \in Z$ (the evenness of $\chi(z)$ follows from its definition).

In other words, we have to prove that for any k, $\chi(z)$ satisfies an inequality of the form

$$|z^{2k}\chi(z)| < C_k e^{a|y|}.$$

For this we note that by construction $\varphi(z)$ has order $\frac{1}{2}$ and finite type, and therefore the function $\chi(z) = \varphi(z^2)$ has order 1 and finite type, i.e., $|\chi(z)| < Ce^{a|z|}$. The same is true of the function $z^{2k}\chi(z)$. But for real values of z this last function is bounded, since

$$|x^{2k}\chi(x)|^2 = |x^{4k}\theta(x)|,$$

and $z^{4k}\theta(z)$ is bounded on the real axis in view of $\theta(z) \in Z$. But if

$$\sup_x |x^k\chi(x)| = M,$$

then it follows from a known inequality of S. N. Bernstein[4] that

$$|z^k\chi(z)| \leqslant Me^{a|y|}.$$

Thus $\chi(z) \in Z$, and Theorem 2 is proved.

[4] Cf. N. I. Achieser, "Theory of Approximation," pp. 137–139, Ungar, New York, 1956.

We have thus proven that any even function of one variable $\theta(z) \in Z$, assuming positive values on the real and imaginary axes, can be represented in the form $\theta(z) = \chi(z)\bar{\chi}(z)$, where $\chi(z)$ is an even function in Z (for functions of several variables this theorem is apparently not true). Consequently, if the inequality $(F, \chi\bar{\chi}) \geqslant 0$ is valid for all even functions in Z, then $(F, \theta) \geqslant 0$ for all functions $\theta(z) \in Z$ which assume positive values on the real and imaginary axes.

Thus, the problem of describing the even generalized functions F of one variable, for which $(F, \chi\bar{\chi}) \geqslant 0$ for all even functions $\chi(z) \in Z$, is equivalent to the problem of describing the even generalized functions F such that $(F, \theta) \geqslant 0$ for all even functions $\theta \in Z$ which assume positive values on the real and imaginary axes.

This problem will be solved in Section 6.3 with the help of a theorem on the extension of positive linear functionals.

6.2. A Theorem on the Extension of Positive Linear Functionals[5]

We will consider a linear space L whose elements are functions $\varphi(x)$, defined on some set $\mathfrak{M}$. A function $\varphi(x)$ is called *positive*, if $\varphi(x) \geqslant 0$ for all $x \in \mathfrak{M}$. An additive homogeneous functional F defined on L is called *positive*, if $(F, \varphi) \geqslant 0$ for all positive functions $\varphi(x) \in L$. We will say that a function $\varphi(x)$ defined on $\mathfrak{M}$ is *subordinate* to the space L, if there exists a function $\psi(x) \in L$ such that $-\psi(x) \leqslant \varphi(x) \leqslant \psi(x)$. The following theorem holds.

Theorem 3. Let L be a linear space whose elements are functions defined on some set $\mathfrak{M}$, and let F be a positive additive homogeneous functional on L. Then the functional F can be extended, without losing its positivity, to any linear space M consisting of functions which are subordinate to L.

Proof. Take any function $\varphi_1(x) \in M$ which does not belong to L, and denote by L_1 the linear space generated by $\varphi_1(x)$ and L. In order to extend the functional F to L_1, consider all the functions $\psi(x)$ and $\chi(x)$ in L such that $\psi(x) \leqslant \varphi_1(x) \leqslant \chi(x)$ (the existence of at least one such

[5] In point of fact, a theory of the extension of positive additive homogeneous functionals can be constructed for any linear space in which a partial ordering of elements is defined, having the usual properties:

(a) If $x \leqslant y$, $y \leqslant z$, then $x \leqslant z$,
(b) if $x \leqslant y$, $y \leqslant x$, then $x = y$,
(c) $x \leqslant x$.

pair of functions follows from the fact that $\varphi_1(x)$ is subordinate to L). In view of the positivity of F, we have $(F, \psi) \leqslant (F, \chi)$. But then sup $(F, \psi) \leqslant \inf(F, \chi)$, where $\sup(F, \psi)$ is taken over all functions $\psi(x)$ such that $\psi(x) \leqslant \varphi_1(x)$, and $\inf(F, \chi)$ is taken over all functions $\chi(x)$ such that $\varphi_1(x) \leqslant \chi(x)$ (it is understood that $\psi(x), \chi(x) \in L$). We choose a number c lying between $\sup(F, \psi)$ and $\inf(F, \chi)$, and set $(F, \varphi) = (F, \theta) + \lambda c$ for all functions of the form $\varphi(x) = \theta(x) + \lambda\varphi_1(x)$, where $\theta(x) \in L$. Obviously this defines an additive homogeneous functional on L_1. We show that this functional is positive. Suppose that the function $\varphi(x) = \theta(x) + \lambda\varphi_1(x)$ is positive. If $\lambda > 0$, then $-\lambda^{-1}\theta(x) < \varphi_1(x)$ and consequently, in view of $\sup(F, \psi) \leqslant c$, we have $-\lambda^{-1}(F, \theta) \leqslant c$. But then $(F, \varphi) = (F, \theta) + \lambda c \geqslant 0$. Similarly, one shows that $(F, \varphi) \geqslant 0$ if $\lambda < 0$. In the case $\lambda = 0$ the inequality $(F, \varphi) \geqslant 0$ follows from the assumed positivity of F on L.

Thus, we have extended F to L_1 while preserving its positivity. Choosing after this another function $\varphi_2(x) \in M$ not belonging to L_1, we extend F, preserving its positivity, to the space generated by $\varphi_2(x)$ and L_1, and so on. Continuing this process and using transfinite induction, we can extend F to all of M while preserving its positivity.

We remark that the extension of F which we have carried out is, in general, not unique. This is the essential difference between the construction carried out here and those carried out in previous sections, where the functionals were extended by continuity and the extensions were therefore unique. We will see below that the positive measures, the proofs of whose existence will be based upon Theorem 3, are in certain cases defined in a nonunique manner.

6.3. Even Positive Generalized Functions on Z

In this paragraph we describe the even generalized functions F on Z such that $(F, \theta) \geqslant 0$ for all even functions $\theta(z) \in Z$ which assume positive values on the real and imaginary axes.

Theorem 4. Let F be an even generalized function of one variable on Z such that $(F, \theta) \geqslant 0$ for all even functions $\theta(z) \in Z$ which assume positive values on the real and imaginary axes. Then there exist positive even measures μ_1 and μ_2 such that

$$(F, \varphi) = \int_{-\infty}^{\infty} \varphi(x)\, d\mu_1(x) + \int_{-\infty}^{\infty} \varphi(iy)\, d\mu_2(y) \tag{3}$$

for all even functions $\varphi(z) \in Z$. These measures are such that the integral

$$\int_{-\infty}^{\infty} (1 + x^2)^{-p} \, d\mu_1(x) \tag{4}$$

converges for some $p > 0$, and the integral

$$\int_{-\infty}^{\infty} e^{a|y|} \, d\mu_2(y) \tag{5}$$

converges for all $a > 0$.

Conversely, if two positive even measures μ_1 and μ_2 satisfy the latter conditions, then (3) defines a generalized function F on Z such that $(F, \theta) \geqslant 0$ for all even functions $\theta(z) \in Z$ which assume positive values on the real and imaginary axes.

Proof. We start with the second (converse) part of the theorem. We have to prove only that, if the conditions on μ_1 and μ_2 are satisfied, then the functional F is continuous on Z. Since Z is the union of its subspaces $Z(a)$, it suffices to show the continuity of F in the topology of $Z(a)$ for all $a > 0$. From the definition of the topology in Z (cf. the appendix to Section 1 of Chapter I, p. 23) it follows that a neighborhood of zero in $Z(a)$ is defined by an inequality of the form

$$\sup_x (1 + |x|^2)^r |\varphi(x)| \leqslant \eta. \tag{6}$$

Moreover, if $\varphi(z) \in Z(a)$, then

$$\sup_x |\varphi(x + iy)| \leqslant Ae^{a|y|}, \tag{7}$$

where

$$A = \sup_x |\varphi(x)|.$$

From this and the conditions upon the measures μ_1 and μ_2 it follows that the integrals

$$\int_{-\infty}^{\infty} \varphi(x) \, d\mu_1(x) \qquad \text{and} \qquad \int_{-\infty}^{\infty} \varphi(iy) \, d\mu_2(y)$$

converge for all functions $\varphi(z)$ in the neighborhood U of zero defined by an inequality of the form (6), if we set $r = p$. In addition, the values of these integrals will remain bounded as $\varphi(z)$ ranges over the neighborhood U. We have therefore proven that F is bounded on U. Consequently, it is continuous on the subspace $Z(a)$. From this follows its continuity on all of Z.

Now we proceed to the proof of the direct assertion of the theorem. We will denote the space of even functions in Z by Z_+ Functions in Z_+ will be considered only on the set $\mathfrak{M}$ consisting of the real and imaginary axes. Correspondingly, these functions will be called *positive* if they assume positive values on the real and imaginary axes. From the conditions of the theorem it follows that F is positive (on Z_+). Therefore, by Theorem 3 it can be extended, while preserving its positivity, to all functions $\psi(z)$ defined on $\mathfrak{M}$ and subordinate to the space Z_+ (i.e., such that the inequality $-\theta(z) \leqslant \psi(z) \leqslant \theta(z)$ holds on $\mathfrak{M}$ for some function $\theta(z) \in Z_+$).

Using these remarks, we extend the functional F, preserving its positivity, to all functions of the form $\varphi(z)f(z)$, where $\varphi(z)$ is a positive function in Z_+, and $f(z)$ is a function from the space C_0 of continuous functions on $\mathfrak{M}$ which tend to zero as $|z| \to \infty$ (C_0 is taken with the usual topology); it is easily seen that all functions of this form are subordinate to Z_+. We may suppose, without loss of generality, that $(F, \varphi(z)f(z)) = 0$ if $f(z)$ is an odd function.

We now associate with every positive function $\varphi(z) \in Z_+$ a functional F_φ on C_0, defined by

$$(F_\varphi, f) = (F, \varphi f)$$

(the right side of this equation is meaningful for any $f(z) \in C_0$).

The functional F was extended without losing its positivity. Therefore $(F_\varphi, f) \geqslant 0$ for all positive functions $f(z) \in C_0$(i.e., functions which assume positive values on the set $\mathfrak{M}$). Moreover, F_φ is continuous relative to the topology of C_0, because

$$-\sup_{z\in\mathfrak{M}} |f(z)|(F, \varphi) \leqslant (F_\varphi, f) \leqslant \sup_{z\in\mathfrak{M}} |f(z)|(F, \varphi).$$

By a theorem of F. Riesz, there exists a measure ν_φ on $\mathfrak{M}$ such that

$$(F_\varphi, f) = \int_{\mathfrak{M}} f(z)\, d\nu_\varphi(z) \tag{8}$$

for all functions $f(z) \in C_0$. Since $(F_\varphi, f) = 0$ for odd functions $f(z)$, the measure ν_φ is even.

Equation (8) can be rewritten in the form

$$(F, \varphi f) = (F_\varphi, f) = \int_{\mathfrak{M}} \varphi(z) f(z) \frac{d\nu_\varphi(z)}{\varphi(z)}.$$

Setting $\varphi(z)f(z) = \theta(z)$ and $d\nu_\varphi(z)/\varphi(z) = d\mu_\varphi(z)$, we obtain

$$(F, \theta) = \int_{\mathfrak{M}} \theta(z)\, d\mu_\varphi(z). \tag{9}$$

By considering the expression $(F, \varphi_1\varphi_2 f)$, we observe that the measure μ_φ does not depend upon the choice of $\varphi(z)$. Therefore we will denote it simply by μ. Let μ_1 and μ_2 be the restrictions of μ to the real and imaginary axis, respectively. Then (9) becomes

$$(F, \theta) = \int_{-\infty}^{\infty} \theta(x)\, d\mu_1(x) + \int_{-\infty}^{\infty} \theta(iy)\, d\mu_2(y). \tag{10}$$

We have obtained this equality for functions of the form $\theta(z) = \varphi(z)f(z)$, where $\varphi(z)$ is a positive function in Z_+, and $f(z) \in C_0$. But any positive function $\theta(z) \in Z_+$ can be represented in this form, setting, for example,

$$\theta(z) = \theta(z)(1 + z^4)\frac{1}{(1 + z^4)}$$

(it is obvious that $\theta(z)(1 + z^4)$ belongs to Z_+ and is positive on $\mathfrak{M}$, and $1/(1 + z^4)$ belongs to C_0). Therefore (10) is proven for all positive functions in Z_+.

Using the validity of (10) for positive functions in Z_+, let us prove that μ_1 and μ_2 satisfy the conditions of the theorem. Indeed, the functional F is continuous relative to the topology of Z. Therefore for any $a > 0$ there is a neighborhood U of zero in the space $Z(2a)$ such that $|(F, \varphi)| \leqslant 1$ for all $\varphi(z) \in U$. This neighborhood U is defined by an inequality of the form

$$\sup_x |(1 + x^2)^p \varphi(x)| \leqslant \eta.$$

Let us now choose a sequence of positive even functions[6] $\varphi_n(z) \in U$

[6] Such a sequence can be constructed in the form

$$\varphi_n(z) = \epsilon\beta(z)\alpha\left(\frac{z}{n}\right)\bar{\alpha}\left(\frac{z}{n}\right),$$

where $\alpha(z)$ is an even function in the space $Z(\frac{1}{2}a)$ such that $\alpha(0) = 1$, and $\beta(z)$ is a function of the form

$$\beta(z) = \int_{-\infty}^{\infty} \frac{\gamma(z - x)}{(1 + x^2)^p}\, dx,$$

where $\gamma(z)$ is an even function in $Z(a)$ such that $\int_{-\infty}^{\infty} \gamma(x)\, dx \neq 0$ and the Fourier transform of $\gamma(x)$ is positive. We omit the details of the corresponding estimates, because similar techniques appeared in Lemma 3 of Section 3.

such that $\lim_{n\to\infty} \varphi_n(z) = \beta_0(z)$ exists for any $z \in \mathfrak{M}$ and $\beta_0(z)$ satisfies the inequalities

$$\beta_0(x) \geqslant \frac{A}{(1+x^2)^p} \tag{11}$$

and

$$\beta_0(iy) \geqslant Be^{a|y|}. \tag{12}$$

From the inequalities

$$(F, \varphi_n) = \int_{-\infty}^{\infty} \varphi_n(x)\, d\mu_1(x) + \int_{-\infty}^{\infty} \varphi_n(iy)\, d\mu_2(y) \leqslant 1$$

and the estimates (11) and (12) it follows, in view of Fatou's lemma, that the integrals

$$\int_{-\infty}^{\infty} \frac{d\mu_1(x)}{(1+x^2)^p}$$

and

$$\int_{-\infty}^{\infty} e^{a|y|}\, d\mu_2(y)$$

converge. Since a was arbitrary, this proves that μ_1 and μ_2 satisfy the conditions of the theorem.

We have already seen that the fulfillment of these conditions implies the continuity of the functional

$$\int_{-\infty}^{\infty} \varphi(x)\, d\mu_1(x) + \int_{-\infty}^{\infty} \varphi(iy)\, d\mu_2(y)$$

relative to the topology of Z. But this functional coincides with the functional (F, φ) on those functions in Z_+ which are representable in the form $\theta(z)f(z)$, where $\theta(z) \in Z_+$ is positive and $f(z) \in C_0$. Therefore, to prove that (10) holds for all functions in Z_+ it remains for us to show that the set of functions of the form $\theta(z)f(z)$ is everywhere dense in Z_+.

Let $\varphi(z)$ be some function in Z_+. Then there is a real number c such that

$$-A(2+\cos cz) < \varphi(z) < A(2+\cos cz)$$

on the real and imaginary axes. Now take any function $\alpha(z) \in Z_+$ such that $\alpha(0) = 1$, and set

$$\varphi_n(z) = \varphi(z)\alpha\left(\frac{z}{n}\right)\bar{\alpha}\left(\frac{z}{n}\right).$$

Then the sequence $\{\varphi_n(z)\}$ converges to $\varphi(z)$ in the topology of Z, and the $\varphi_n(z)$ can be represented in the form

$$\varphi_n(z) = A(2 + \cos cz)(1 + z^4)\alpha\left(\frac{z}{n}\right)\bar{\alpha}\left(\frac{z}{n}\right)\frac{\theta(z)}{(2 + \cos cz)(1 + z^4)}.$$

But

$$\theta_n(z) = A(2 + \cos cz)(1 + z^4)\alpha\left(\frac{z}{n}\right)\bar{\alpha}\left(\frac{z}{n}\right)$$

is obviously a positive function in Z_+, and $f(z) = [\theta(z)/(2 + \cos cz)(1 + z^4)]$ tends to zero as $|z| \to \infty$. This proves that the functions of the form $\theta(z)f(z)$ constitute an everywhere dense set in Z_+, and therefore (10) holds for all functions in Z_+ Theorem 4 is thus proved.

We remark that this theorem is also valid for functions of several variables. Namely, the following assertion holds.

Theorem 4′. Let F be an even generalized function on the space Z of several variables such that $(F, \theta) \geqslant 0$ for all even functions $\theta(z) \in Z$ which assume positive values on the set $\mathfrak{M}$ consisting of all points, each of whose coordinates is either real or pure imaginary. Then F has the form

$$(F, \varphi) = \int_{\mathfrak{M}} \varphi(z)\, d\mu(z),$$

where μ is an even positive measure on the set $\mathfrak{M}$ such that for any $a > 0$ the integral

$$\int_{\mathfrak{M}} (1 + |x|^2)^{-p} e^{a|y|}\, d\mu(z)$$

converges for some $p > 0$.

The proof of this theorem repeats almost verbatim that of Theorem 4. We have not stopped to prove Theorem 4′ because we do not know, for functions of several variables, whether or not the concepts of positivity and multiplicative positivity are equivalent.

For functions of one variable, as has been shown in Section 6.1, this equivalence holds. Thus, (3) describes not only positive, but also multiplicatively positive generalized functions on Z_+ (i.e., even generalized functions on Z such that $(F, \varphi\bar{\varphi}) \geqslant 0$ for all even functions $\varphi(z) \in Z$). By means of the Fourier transformation we obtain the following theorem, which describes evenly positive-definite generalized functions[7] on the

[7] That is, even generalized functions F on K such that $(F, \varphi * \varphi^*) \geqslant 0$ for all even functions $\varphi(x) \in K$.

space K of infinitely differentiable functions of one variable, having bounded supports.

Theorem 5. Let F be an evenly positive-definite generalized function on the space K of one variable. Then there exist even positive measures μ_1 and μ_2 such that for all $\varphi \in K$ one has

$$(F, \varphi) = \int \tilde{\varphi}(x)\, d\mu_1(x) + \int \tilde{\varphi}(iy)\, d\mu_2(y), \tag{13}$$

where $\tilde{\varphi}(z)$ is the Fourier transform of $\varphi(x)$. The measures μ_1 and μ_2 are such that $\int e^{a|y|}\, d\mu_2(y)$ converges for all $a > 0$, and $\int (1 + x^2)^{-p}\, d\mu_1(x)$ converges for some $p > 0$.

If an evenly positive-definite generalized function F has the form

$$(F, \varphi) = \int \overline{f(x)}\varphi(x)\, dx,$$

where $f(x)$ is a continuous function, then the measure μ_1 in (13) is finite. Conversely, if μ_1 is finite, and $\int e^{a|y|}\, d\mu_2(y)$ converges for all $a > 0$, then F is given by a continuous function. Hence we obtain a description of continuous evenly positive-definite functions of one variable.

Theorem 6. Let the continuous function $f(x)$ be evenly positive-definite. Then $f(x)$ has the form

$$f(x) = \int e^{i\lambda x}\, d\mu_1(\lambda) + \int e^{\lambda x}\, d\mu_2(\lambda),$$

where μ_1 and μ_2 are even positive measures such that μ_1 is finite, and $\int e^{a\lambda}\, d\mu_2(\lambda)$ converges for all $a > 0$.

6.4. An Example of the Nonuniqueness of the Positive Measure Corresponding to a Positive Functional on Z_+

We have already pointed out in Section 6.2 that the extension of a positive functional may be nonunique. Therefore the positive measures μ_1 and μ_2 defined, according to Theorem 4, by a positive functional on Z_+ are, generally speaking, not uniquely defined by this functional. We present here an example of this nonuniqueness.

Choose any a, $1 < a < 2$. Set $b = \frac{1}{4}\pi a$, and let $\exp(-z^a e^{-ib}) = \psi_1(z) + i\psi_2(z)$, where ψ_1 and ψ_2 are real functions.

Let $\psi_1^+(x)$ denote the function which coincides with $\psi_1(x)$ where $\psi_1(x) \geqslant 0$ and is zero otherwise. Similarly, let $\psi_1^-(x)$ coincide with $-\psi_1(x)$ where $\psi_1(x) \leqslant 0$ and be zero otherwise. We define $\psi_2^+(x)$ and $\psi_2^-(x)$ in the same way. We take the functions ψ_1^+, ψ_1^-, ψ_2^+, and ψ_2^- as the densities of measures μ_1^+, μ_1^-, μ_2^+ and μ_2^-. It is easily seen that these measures are defined by

$$\mu_1^+(x) = \int_0^x \max[0, \exp(-t^a \cos b)\cos(t^a \sin b)]\, dt,$$

$$\mu_1^-(x) = -\int_0^x \min[0, \exp(-t^a \cos b)\cos(t^a \sin b)]\, dt,$$

$$\mu_2^+(x) = \int_0^x \max[0, \exp(-t^a \cos b)\sin(t^a \sin b)]\, dt,$$

$$\mu_2^-(x) = -\int_0^x \min[0, \exp(-t^a \cos b)\sin(t^a \sin b)]\, dt.$$

It is obvious that $\mu_1^+(x)$, $\mu_1^-(x)$, $\mu_2^+(x)$, $\mu_2^-(x)$ are increasing functions of x, and

$$\mu_1^+(x) \neq \mu_1^-(x), \qquad \mu_2^+(x) \neq \mu_2^-(x).$$

We now prove that

$$\int_{-\infty}^{\infty} \varphi(x)\, d\mu_1^+(x) + \int_{-\infty}^{\infty} \varphi(iy)\, d\mu_2^-(y) = \int_{-\infty}^{\infty} \varphi(x)\, d\mu_1^-(x) + \int_{-\infty}^{\infty} \varphi(iy)\, d\mu_2^+(y) \tag{14}$$

for any even function $\varphi(z) \in Z$. This will show that the distinct pairs μ_1^+, μ_2^- and μ_1^-, μ_2^+ define the same positive functional on Z_+.

To prove (14), we note that

$$\int_0^\infty \exp[-x^a e^{-ib}]\varphi(x)\, dx = i\int_0^\infty \exp[-y^a e^{-ib}]\varphi(iy)\, dy \tag{15}$$

for any function $\varphi(z) \in Z$, where $b = \frac{1}{4}\pi a$.

Indeed, the function $\exp[-z^a e^{-ib}]\,\varphi(z)$ is analytic in the region bounded by intervals on the real and imaginary axes and the circles $|z| = \rho$ and $|z| = R$ (Fig. 1). Therefore its integral along the contour of this region equals zero. But

$$|\exp[-z^a e^{-ib}]| = \exp(-|z|^a)\cos a(\alpha - \tfrac{1}{4}\pi),$$

where $\alpha = \arg z$. In view of $1 < a < 2$ and $0 \leqslant \alpha \leqslant \frac{1}{2}\pi$, the integrals along the arcs of the circles $|z| = \rho$ and $|z| = R$ tend to zero as $\rho \to 0$ and $R \to \infty$ (recall that $\varphi(z) \in Z$ and therefore satisfies an inequality of the form $|\varphi(z)| \leqslant Ce^{b|y|} \leqslant Ce^{b|z|}$). From this it follows that the integrals of the function $\exp[-z^a e^{-ib}]\varphi(z)$ along the real axis and the imaginary axis are equal. This proves (15).

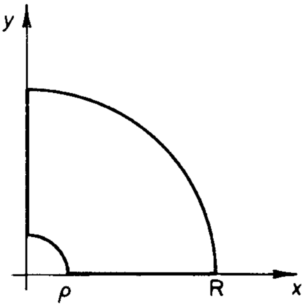

FIG. 1

Now let $\varphi(z)$ be an even function which assumes real values on the real axis. Then it also assumes real values on the imaginary axis. Equating the real terms in (15), we obtain

$$\int_0^\infty \exp[-x^a \cos b] \cos(x^a \sin b)\varphi(x)\, dx = \int_0^\infty \exp[-y^a \cos b] \sin(y^a \sin b)\varphi(iy)\, dy. \quad (16)$$

But this is just another way of writing (14), which one can easily see by breaking up the integral on the left side into integrals over the regions where $\cos(x^a \sin b) > 0$ and $\cos(x^a \sin b) < 0$, and the integral on the right side into integrals over the regions where $\sin(x^a \sin b) > 0$ and $\sin(x^a \sin b) < 0$.

This proves (14) for any even function $\varphi(z)$ which assumes real values on the real axis.

Now take any even function $\varphi(z) \in Z$ and represent it in the form

$$\varphi(z) = \varphi_1(z) + i\varphi_2(z),$$

where $\varphi_1(z) = \frac{1}{2}\varphi(z) + \frac{1}{2}\overline{\varphi(\bar{z})}$ and $\varphi_2(z) = [\varphi(z) - \overline{\varphi(\bar{z})}]/2i$. The functions $\varphi_1(z)$ and $\varphi_2(z)$ are even functions in Z which assume real values on the real axis. Formula (14) holds for these functions, and therefore it holds also for $\varphi(z)$.

We have therefore constructed the sought for example.

7. Multiplicatively Positive Linear Functionals on Topological Algebras with Involutions

7.1. Topological Algebras with Involutions

As we have already said, the basic concepts introduced and studied in Sections 2–6 (positivity, multiplicative positivity, positive definiteness) pertain in essence to the theory of topological algebras with involutions. Let us give the basic definitions concerning these algebras.

A linear space L is called an *algebra with an involution*, if an operation of multiplication of elements and an operation of passing to the adjoint element x^* (involution) is defined in it, satisfying the following axioms:

(1) $(xy)z = x(yz)$;

(2) $x(y + z) = xy + xz$; $(y + z)x = yx + zx$;

(3) $\lambda(xy) = (\lambda x)y = x(\lambda y)$;

(4) $(x^*)^* = x$;

(5) $(\lambda x + \mu y)^* = \bar{\lambda}x^* + \bar{\mu}y^*$,

(6) $(xy)^* = y^*x^*$.

An algebra L is called *commutative* if the multiplication of elements is commutative, i.e., $xy = yx$. If L contains an element e (the unit of the algebra) such that $xe = ex = x$ for all $x \in L$, then L is called an *algebra with a unit.*

As a rule, one considers algebras which are linear topological spaces. In this case the algebraic operations (including the involution) are required to be continuous in the topology of the space.

If L is a normed space, then the algebra is also called *normed*, and one requires that the norm introduced into the algebra L satisfy the following conditions:

(1) $\| xy \| \leqslant \| x \| \| y \|$,

(2) $\| x^* \| = \| x \|$,

(3) the algebra L is complete relative to the norm $\| x \|$, *i.e.*, $\lim_{m.n\to\infty} \| x_n - x_m \| = 0$ implies the existence of an element $x \in L$ for which

$$\lim_{n\to\infty} \| x - x_n \| = 0.$$

The spaces K and S become topological algebras with involutions if for multiplication we take, for example, the ordinary multiplication of functions, and for involution we take the operation of passing from $\varphi(x)$ to $\overline{\varphi(x)}$. The space Z is also a topological algebra with involution. Multiplication in Z is defined as the ordinary multiplication of functions, and for involution is taken the operation $\varphi(z) \rightarrow \bar{\varphi}(z) = \overline{\varphi(\bar{z})}$ (it is easily seen that $\bar{\varphi}(z) \in Z$).

For algebras with an involution, the concept of a multiplicatively positive linear function is defined in the following way.

A linear functional F on a topological algebra with involution L is called *multiplicatively positive* if $(F, xx^*) \geqslant 0$ for all $x \in L$.

It is easy to see that multiplicatively positive generalized functions on the spaces K and S are simply multiplicatively positive linear functionals for the corresponding algebras.

The concept of positive definiteness can also be considered as a special case of multiplicative positivity for linear functionals, for a suitable definition of the multiplication of elements in the algebra.

Namely, if a new product is defined in S, taking the convolution of two functions as their product, and the involution is defined as the transformation $\varphi(x) \rightarrow \varphi^*(x) = \overline{\varphi(-x)}$, then multiplicatively positive linear functionals on this algebra are positive-definite generalized functions.

The concept of even positive definiteness also lies within this scheme. Namely, we define the product of the functions $\varphi(x)$ and $\psi(x)$ as the function

$$\chi(x) = \sum \int \varphi(y_1, \ldots, y_n)\psi(x_1 \pm y_1, \ldots, x_n \pm y_n)\, dy_1, \ldots, dy_n, \tag{1}$$

and involution is defined as the transformation $\varphi(x) \rightarrow \varphi^*(x) = \overline{\varphi(-x)}$. Then multiplicatively positive functionals are evenly positive-definite generalized functions.

One can obtain a complete description of multiplicatively positive functionals on a normed algebra with involution. In order to do this, we introduce the concept of a symmetric homomorphism of an algebra with an involution into the field of complex numbers.

A *symmetric homomorphism* of an algebra L with involution into the field of complex numbers is a linear functional M on L such that

$$(M, xy) = (M, x)(M, y) \qquad \text{and} \qquad (M, x^*) = \overline{(M, x)}.$$

For algebras consisting of functions of a real variable, in which the multiplication of elements is defined as the multiplication of functions, and the involution as $\varphi(x) \rightarrow \overline{\varphi(x)}$, an example of a symmetric homomorphism is the correspondence $\varphi(x) \to \varphi(x_0)$, where x_0 is some real number. However, certain algebras have other symmetric homomorphisms. For example, if we introduce a multiplication into the space of even functions in Z or in S_1^1 by means of (1), and the involution is taken as $\varphi(z) \to \overline{\varphi(-\bar{z})}$, then every correspondence

$$\varphi(z_1, \ldots, z_n) \to \varphi(z_1^0, \ldots, z_n^0),$$

where $z_1^0, \ldots, z_n^0$ are real or pure imaginary numbers, is a symmetric homomorphism.

We will denote the set of all symmetric homomorphisms of an algebra L by $\mathfrak{M}$. Obviously, to each element $x \in L$ there corresponds a function $x(M) = (M, x)$, defined on the set $\mathfrak{M}$. If the algebra L is commutative and normed, then one can introduce a topology in the set $\mathfrak{M}$ such that every function $x(M)$ will be continuous, and $\mathfrak{M}$ will be compact.

To every positive finite measure $\sigma(M)$ defined on $\mathfrak{M}$ there corresponds a linear functional on the algebra L, defined by

$$(F_1, x) = \int x(M)\, d\sigma(M).$$

For any $x \in L$ we have

$$(F_1, xx^*) = \int xx^*(M)\, d\sigma(M) = \int |\, x(M)|^2\, d\sigma(M) > 0.$$

Therefore the functional F_1 is multiplicatively positive. It can be shown that these functionals exhaust the set of multiplicatively positive linear functionals on commutative normed algebras with involutions. Precisely speaking, the following theorem holds.

Theorem 1. Every multiplicatively positive linear functional F on a commutative normed algebra L with involution can be represented, in a unique way, in the form

$$(F, x) = \int_{\mathfrak{M}} x(M)\, d\sigma(M),$$

where $\sigma(M)$ is a positive measure on the set $\mathfrak{M}$ of symmetric homomorphisms of the algebra L into the field of complex numbers.

For the proof of this theorem the reader can consult, for example, M. A. Naimark, "Normed Rings." Nordhoff, Groningen, 1959.

7.2. The Algebra of Polynomials in Two Variables

Using Theorem 1, one can obtain a description of evenly positive-definite continuous functions which do not grow faster than $e^{a|x|}$, $a > 0$. The methods which were used in Section 5 enable one to obtain a corresponding description for functions which grow more slowly than all of the functions $\exp(\epsilon\,|\,x\,|^2)$, $\epsilon > 0$, and, in particular, for all functions which do not grow faster than $\exp(a\,|\,x\,|)$. Presumably the results of Section 5 could also be obtained if one succeeded in constructing a sufficiently developed theory of topological rings with involutions.

However, for arbitrary commutative algebras with involutions which are not normed, a theorem analogous to Theorem 1 is not valid, generally speaking. There exist topological algebras with involutions for which some multiplicatively positive linear functionals are not positive. An example of such an algebra is the algebra P of polynomials in two variables. This algebra consists of all polynomials of the form

$$\varphi(x, y) = \sum_{k,l} a_{kl} x^k y^l.$$

Multiplication is defined as the usual multiplication of polynomials; and involution, as passing from the polynomial $\varphi(x, y)$ to the polynomial

$$\bar{\varphi}(x, y) = \sum_{k,l} \bar{a}_{kl} x^k y^l.$$

A sequence

$$\varphi_n(x, y) = \sum_{k,l} a_{kl}^{(n)} x^k y^l$$

of polynomials in P is said to converge to the polynomial $\varphi(x, y) = \sum_{k,l} a_{kl} x^k y^l$, if the degree of all of the $\varphi_n(x, y)$ is the same as that of $\varphi(x, y)$, and

$$\lim_{n\to\infty} a_{kl}^{(n)} = a_{kl}$$

for all k and l.

To define a homomorphism of the algebra P into the field of complex numbers, it is sufficient to specify the values x_0 and y_0 which the monomials x and y assume under the homomorphism (x_0 and y_0 may be complex numbers). Then the homomorphism is given by

$$f(x, y) \to \sum_{k,l=0} a_{kl} x_0^k y_0^l.$$

Obviously, such a homomorphism will be symmetric if x_0 and y_0 are real numbers. Therefore the positive elements in the algebra P are those for which $\varphi(x, y) \geqslant 0$ for all real values of x and y. Hilbert has constructed an example of a positive polynomial of two variables which is not a linear combination of polynomials of the form $\psi(x, y)\overline{\psi(x, y)}$.

This example is constructed in the following way. We consider any eight points $M_1, \ldots, M_8$ in the plane such that no three of them are collinear and no six of them lie on a curve of the second order. As is proved in algebraic geometry, an infinite set of third-order curves can be passed through these points, and all of these curves intersect in some ninth point M_9 (cf. R. J. Walker, "Algebraic Curves," Theorem 6.2, p. 70. Princeton Univ. Press, Princeton, New Jersey, 1950). Let us show that any polynomial $f(x, y)$ of the sixth degree which is a sum of squares of polynomials $\varphi_k(x, y)$, and vanishes at the points $M_1, \ldots, M_8$, must also vanish at M_9. Indeed, suppose that

$$f(x, y) = \sum_{k=1}^{n} \varphi_k^2(x, y) \qquad \text{and} \qquad f(M_1) = \ldots = f(M_8) = 0.$$

Then each of the $\varphi_k(x, y)$ vanishes at the points $M_1, \ldots, M_8$. But since these points do not lie on any one second-order curve, all of the $\varphi_k(x, y)$ are polynomials of the third degree. Now the curves $\varphi_k(x, y) = 0$ pass through the points $M_1, \ldots, M_8$. But then, as was remarked above, they also pass through the point M_9, i.e., $\varphi_k(M_9) = 0$. Consequently $f(M_9) = 0$.

It follows that any polynomial $f(x, y)$ of the sixth degree which equals zero at the points $M_1, \ldots, M_8$, and is different from zero at M_9, cannot be represented as a sum of squares of polynomials. Let us now show that there exists a sixth-degree polynomial $f(x, y)$ which is nonnegative for all x, y, vanishes at the points $M_1, \ldots, M_8$, and is different from zero at M_9. This will prove the existence of a positive polynomial which cannot be represented as a sum of squares of polynomials.

The polynomial $f(x, y)$ is constructed in the following way. Let $\varphi(x, y) = 0$ and $\psi(x, y) = 0$ be two fixed curves of the third order which pass through the points $M_1, \ldots, M_9$. We pass a second-order curve $\alpha(x, y) = 0$ through $M_1, \ldots, M_5$, and a fourth-order curve $\beta(x, y) = 0$ through $M_1, \ldots, M_8$, for which M_6, M_7, M_8 are double points. Let us show that these curves do not pass through M_9. Indeed, suppose that $\alpha(M_9) = 0$. Choose any point N on the curve $\alpha(x, y) = 0$, different from the M_k, and set $\lambda = -\varphi(N)/\psi(N)$. The third-order curve $\theta(x, y) = \varphi(x, y) + \lambda\psi(x, y) = 0$ passes through the points $M_1, \ldots, M_9$ and N, and thus has seven points in common with the second-order curve $\alpha(x, y) = 0$. But if curves of order m and n have more than mn common points, then they have a common component (cf. R. J. Walker, "Algebraic

Curves," Theorem 3.1, p. 59. Princeton Univ. Press, Princeton, New Jersey, 1950). This component can only be the curve $\alpha(x, y) = 0$. Thus, the curve $\theta(x, y) = 0$ splits up into a curve of the second order and a straight line. Since the curve $\alpha(x, y) = 0$ does not pass through the points M_6, M_7, M_8 (otherwise six of the points M_k, $1 \leqslant k \leqslant 8$, would lie on a second-order curve), the points M_6, M_7, M_8 must lie on the second component of the curve $\theta(x, y) = 0$, i.e., on some straight line. But this contradicts the choice of the points $M_1, \ldots, M_8$.

Thus, we have proven that the curve $\alpha(x, y) = 0$ does not pass through M_9. One proves in the same way that the curve $\beta(x, y) = 0$ does not pass through M_9. Without loss of generality we may suppose that $\alpha(M_9)\beta(M_9) > 0$.

Now consider the curve (with $p > 0$)

$$\chi(x, y) = \varphi^2(x, y) + \psi^2(x, y) + p\alpha(x, y)\beta(x, y) = 0.$$

The polynomial $\chi(x, y)$ is a polynomial of the sixth degree which vanishes at the points M_k, $1 \leqslant k \leqslant 8$, and is positive at M_9. Therefore it cannot be represented as a sum of squares of polynomials. It remains to show that for some choice of $p > 0$ this polynomial will be positive. For this we note that for $p = 0$ $\chi(x, y)$ becomes $\varphi^2(x, y) + \psi^2(x, y)$, for which the points M_k, $1 \leqslant k \leqslant 9$ are minimum points, and $\varphi^2(x, y) + \psi^2(x, y)$ vanishes at these points. The points M_k, $1 \leqslant k \leqslant 8$, are also stationary points of the polynomial $\alpha(x, y)\beta(x, y)$, since its partial derivatives vanish at these points (at the points $M_1, \ldots, M_5$ both $\alpha(x, y)$ and $\beta(x, y)$ vanish, and M_6, M_7, M_8 are double points for the curve $\beta(x, y) = 0$). Therefore there is a $p_0 > 0$ such that for $0 < p \leqslant p_0$ the polynomial $\varphi^2(x, y) + \psi^2(x, y) + p\alpha(x, y)\beta(x, y)$ has zero minima at the points M_k, $1 \leqslant k \leqslant 8$, and is positive at M_9.

We can find disks Ω_k with centers at M_k, $1 \leqslant k \leqslant 9$, such that in the region made up of these disks the polynomial $\chi(x, y)$ will be nonnegative for $0 < p \leqslant p_0$, and vanishes only at M_k, $1 \leqslant k \leqslant 8$. Outside of these disks the function $|(\varphi^2 + \psi^2)/\alpha\beta|$ has a nonzero minimum A. Therefore for $p \leqslant A$ one has $\varphi^2(x, y) + \psi^2(x, y) \geqslant p \mid \alpha(x, y)\beta(x, y) \mid$ outside these disks. But then the polynomial $\chi(x, y)$ is positive for $0 \leqslant p \leqslant \min (p_0, A)$.

This proves the existence of a positive polynomial in two variables which cannot be represented as a sum of squares of polynomials.

It can be proven that Hilbert's polynomial not only is not a sum of squares of polynomials, but also cannot be approximated in the space of polynomials of the sixth degree by sums of squares of polynomials.

Denote now by T_6 the cone, in the linear space P_6 of polynomials of the sixth degree, consisting of all polynomials of the form

$$\sum_{k=1}^{n} \varphi_k^2(x, y),$$

where the $\varphi_k(x, y)$ are polynomials with real coefficients. Hilbert's polynomial $f(x, y)$ does not belong to the closure of this cone. Therefore one can construct a linear functional on the space P_6 which is positive on the cone T_6 and assumes a negative value at $f(x, y)$. This functional can be extended to the entire algebra of polynomials in such a way that it assumes positive values on all polynomials which are representable as sums of squares of polynomials. As a result one obtains a linear functional on the algebra of all polynomials in two variables which is multiplicatively positive but not positive.

The example which has been considered shows that for topological rings with involutions the concepts of positivity and multiplicative positivity of linear functionals do not, generally speaking, coincide. It would be very important to distinguish the class of topological rings in which these concepts coincide. As we have seen, the rings K, S, Z, $S_{\frac{1}{2}}^{\frac{1}{2}}$, and others belong to this class.

Hilbert's example is closely connected with the moment problem for functions of two variables. This problem consists in the following. Given numbers μ_{jk}, $0 \leqslant j, k < \infty$, it is required to find a positive measure σ such that

$$\mu_{jk} = \iint x^j y^k \, d\sigma(x, y).$$

If the moment problem has a solution, then for any polynomial

$$\varphi(x, y) = \sum_{j,k=0}^{n} a_{jk} x^j y^k$$

which is positive for all real values of x and y, one has

$$\sum_{j,k=0}^{n} a_{jk}\mu_{jk} = \sum_{j,k=0}^{n} a_{jk} \iint x^j y^k \, d\sigma(x, y) = \iint \varphi(x, y) \, d\sigma(x, y) \geqslant 0.$$

In other words, the linear functional F on the space P of polynomials in two variables, defined by

$$\left(F, \sum_{j,k=0}^{n} a_{jk} x^j y^k\right) = \sum_{j,k=0}^{n} a_{jk}\mu_{jk}, \tag{2}$$

must be positive. It can be shown, using the theorem on the extension of positive linear functionals (cf. Theorem 3 of Section 6), that this condition is also sufficient.

In the case of functions of one variable the weaker requirement of multiplicative positivity for F turns out to be sufficient for the moment problem to be solvable. As Hilbert's example shows, for functions of two variables there exist multiplicatively positive linear functionals which are not positive. Therefore the condition that

$$(F, \varphi^2(x, y) \geqslant 0$$

for all polynomials $\varphi(x, y)$, where F is the linear functional defined by (2), is not sufficient for the solvability of the moment problem. It should be remarked, however, that if one imposes upon the growth of the moments μ_{jk} certain known restrictions which guarantee the possibility of extending the functional (2) from the space of polynomials to one or another space of entire functions, then the condition of multiplicative positivity becomes not only necessary but also sufficient for the solvability of the moment problem. In this case the solution of the moment problem will be unique.

CHAPTER III

Generalized Random Processes

1. Basic Concepts Connected with Generalized Random Processes

1.1. Random Variables

We assume that the reader is familiar with the basic concepts of probability theory. However, since the concept of a random variable is fundamental for this chapter and since the definition of this concept which will be used in this book differs outwardly from that generally used, we begin with the definition of a random variable.

We say that a *random variable* ξ is defined, if for each real x a number $\mathbf{P}(x)$ is given, called the *probability of the event* $\xi < x$, with the following properties:

(1) $\mathbf{P}(x_1) \leqslant \mathbf{P}(x_2)$, if $x_1 \leqslant x_2$,

(2) $\lim_{x \to -\infty} \mathbf{P}(x) = 0$, $\lim_{x \to \infty} \mathbf{P}(x) = 1$,

(3) $\lim_{x \to a-0} \mathbf{P}(x) = \mathbf{P}(a)$.

Defining the function $\mathbf{P}(x)$ uniquely defines the probability $\mathbf{P}(X)$ that the value of the random variable ξ belong to the Borel set X.

If we consider several random variables $\xi_1, \ldots, \xi_n$, then giving the probability distribution of each of the ξ_k is no longer sufficient. It is necessary to know also the probability $\mathbf{P}(x_1, \ldots, x_n)$ that one have simultaneously $\xi_1 < x_1, \ldots, \xi_n < x_n$. The probability $\mathbf{P}(X)$ that the point $\xi = (\xi_1, \ldots, \xi_n)$ belongs to a Borel set X in n-dimensional space R_n is uniquely defined by these probabilities. We will call the collection $\xi = (\xi_1, \ldots, \xi_n)$ an n-dimensional random variable.

Of course, giving the probability distribution $\mathbf{P}(x_1, \ldots, x_n)$ uniquely defines the probability distributions of the random variables ξ_k. These are given by

$$\mathbf{P}(x_k) = \mathbf{P}(\infty, \ldots, x_k, \ldots, \infty).$$

One can similarly find the probability distribution for any subset

$\xi_{k_1}, \ldots, \xi_{k_r}$ of the ξ_k. To do this, one replaces those x_k in $\mathbf{P}(x_1, \ldots, x_n)$ for which k is not one of $k_1, \ldots, k_r$ by ∞.

Let us now consider an infinite set $\{\xi_\alpha\}$ of random variables. In this case we define, for any finite collection $\xi_1, \ldots, \xi_n$, the probability $\mathbf{P}(x_1. \ldots, x_n)$ that $\xi_1 < x_1, \ldots, \xi_n < x_n$.[1] These probabilities must be compatible in the sense that for any random variables $\xi_1, \ldots, \xi_{n+1}$ the probability $\mathbf{P}(x_1, \ldots, x_n)$ that

$$\xi_k < x_k, \qquad 1 \leqslant k \leqslant n \tag{1}$$

must coincide with the probability $\mathbf{P}(x_1, \ldots, x_n, \infty)$ that

$$\xi_k < x_k, \qquad 1 \leqslant k \leqslant n, \qquad \xi_{n+1} < \infty.$$

In other words, imposing a condition of the form $\xi_{n+1} < \infty$ does not change the probability of the event (1).

Thus, *a set $\{\xi_\alpha\}$ of random variables is considered to be defined* if for any finite collection of random variables $\xi_1, \ldots, \xi_n$ we specify the joint probability distribution $\mathbf{P}(x_1, \ldots, x_n)$, and these distributions are compatible.

Now let us introduce the concept of the *equality of two random variables* ξ and η. We will say that $\xi = \eta$, if for any real numbers $a < b$ the probability that simultaneously $a \leqslant \xi < b$ and $a \leqslant \eta < b$ is the same as the probability that $a \leqslant \xi < b$, and is also the same as the probability that $a \leqslant \eta < b$. Thus, if $\mathbf{P}(X, Y)$ denotes the probability that $\xi \in X$, $\eta \in Y$, and $P_1(X)$ denotes the probability that $\xi \in X$, then $\xi = \eta$ means that $\mathbf{P}(X, X) = \mathbf{P}_1(X)$. Further, if $X \cap Y = 0$ and $\xi = \eta$, then $\mathbf{P}(X, Y) = 0$. Indeed, by the compatibility condition

$$\mathbf{P}_1(X) = \mathbf{P}(X, R_1),$$

where R_1 is the real line. But

$$\mathbf{P}(X, R_1) = \mathbf{P}(X, X) + \mathbf{P}(X, R_1 - X),$$

and by definition of the equality of two random variables, $\mathbf{P}_1(X) = \mathbf{P}(X, X)$. From this it follows that $\mathbf{P}(X, R_1 - X) = 0$. Since $Y \subset R_1 - X$, *a fortiori* $\mathbf{P}(X, Y) = 0$.

In order to show that our definition of the equality of random variables is a natural one, we prove the following theorem.

[1] Of course the probabilities $\mathbf{P}(x_1, \ldots, x_n)$ depend upon which random variables $\xi_1, \ldots, \xi_n$ from $\{\xi_\alpha\}$ we are considering.

Theorem 1. Let $\xi = \eta$. Then for any random variables $\xi_1, ..., \xi_n$, the probability $\mathbf{P}(x_1, ..., x_n, x)$ that

$$\xi_1 < x_1, ..., \xi_n < x_n, \qquad \xi < x \tag{2}$$

coincides with the probability $\mathbf{P}_1(x_1, ..., x_n, x)$ that

$$\xi_1 < x_1, ..., \xi_n < x_n, \qquad \eta < x. \tag{3}$$

Proof. We prove that both $\mathbf{P}(x_1, ..., x_n, x)$ and $\mathbf{P}_1(x_1, ..., x_n, x)$ coincide with the probability $\mathbf{P}(x_1, ..., x_n, x, x)$ that $\xi_1 < x_1, ..., \xi_n < x_n, \xi < x$, $\eta < x$. Indeed, we have shown that $\xi = \eta$ implies the vanishing of the probability of having simultaneously $\xi < x$ and $\eta \geqslant x$. Certainly, then, the probability $\bar{\mathbf{P}}(x_1, ..., x_n, x, x)$ that

$$\xi_1 < x_1, ..., \xi_n < x_n, \qquad \xi < x, \qquad \eta \geqslant x$$

vanishes. But it follows from the compatibility condition that

$$\mathbf{P}(x_1, ..., x_n, x) = \mathbf{P}(x_1, ..., x_n, x, \infty),$$

where $\mathbf{P}(x_1, ..., x_n, x, \infty)$ denotes the probability that

$$\xi_1 < x_1, ..., \xi_n < x_n, \qquad \xi < x, \qquad \eta < \infty.$$

Since

$$\mathbf{P}(x_1, ..., x_n, x, \infty) = \mathbf{P}(x_1, ..., x_n, x, x) + \bar{\mathbf{P}}(x_1, ..., x_n, x, x)$$

and

$$\bar{\mathbf{P}}(x_1, ..., x_n, x, x) = 0,$$

then

$$\mathbf{P}(x_1, ..., x_n, x) = \mathbf{P}(x_1, ..., x_n, x, x).$$

In exactly the same way one proves that the probability $\mathbf{P}_1(x_1 , ..., x_n, x)$ of (3) being fulfilled coincides with $\mathbf{P}(x_1, ..., x_n, x, x)$, which proves that

$$\mathbf{P}(x_1, ..., x_n, x) = \mathbf{P}_1(x_1, ..., x_n, x).$$

One can similarly prove that if $\xi_1 = \eta_1, ..., \xi_n = \eta_n$, then the probability $\mathbf{P}(X)$ that $(\xi_1, ..., \xi_n) \in X$ coincides with the probability $\mathbf{P}_1(X)$ that $(\eta_1, ..., \eta_n) \in X$.

Henceforth we will not distinguish between equal random variables.

Let $y = f(x_1, ..., x_n)$ be some measurable function of the arguments $x_1, ..., x_n$, and let $\xi_1, ..., \xi_n$ be random variables. We will define a random variable $\eta = f(\xi_1, ..., \xi_n)$, i.e., we will specify, for any random variables

$\zeta_1, \ldots, \zeta_m$, the probability of the point $(\zeta_1, \ldots, \zeta_m, \eta)$ belonging to a given set $\tilde{Z}$ in $(n + 1)$-dimensional space R_{n+1}. Of course, it is sufficient to specify this probability only for sets of the form $\tilde{Z} = Z \times Y$, where Z is some set in m-dimensional space, Y is a set on the real line, and $Z \times Y$ denotes the set of those points $(z_1, \ldots, z_m, y)$ such that $(z_1, \ldots, z_m) \in Z$ and $y \in Y$.

We consider $(n + m)$-dimensional space R_{n+m} and denote by W the set of all points $(z_1, \ldots, z_m; x_1, \ldots, x_n)$ in R_{n+m} such that $(z_1, \ldots, z_m) \in Z$ and $f(x_1, \ldots, x_n) \in Y$.

Now set

$$\mathbf{P}_1(\tilde{Z}) = \mathbf{P}_1(Z \times Y) = \mathbf{P}(W),$$

where $\mathbf{P}(W)$ denotes the probability that

$$(z_1, \ldots, z_m; x_1, \ldots, x_n) \in W.$$

We take $\mathbf{P}_1(\tilde{Z})$ for the probability distribution of the random variable $(\zeta_1, \ldots, \zeta_m, \eta)$. We have thus defined the joint probability distribution of the random variable $\eta = f(\xi_1, \ldots, \xi_n)$ with any random variables $\zeta_1, \ldots, \zeta_m$, i.e., the random variable $\eta = f(\xi_1, \ldots, \xi_n)$ is well defined.

If

$$y_1 = f_1(x_1, \ldots, x_n),$$
$$\cdots\cdots\cdots\cdots$$
$$y_m = f_m(x_1, \ldots, x_n)$$

are functions of the variables $x_1, \ldots, x_n$, then with the help of the definition which we have given it is easy to find the probability distribution of the random variable $(\eta_1, \ldots, \eta_m)$, where we have put

$$\eta_k = f_k(\xi_1, \ldots, \xi_n).$$

This probability distribution is given by

$$\mathbf{P}_1(Y) = \mathbf{P}(X),$$

where X denotes the set of those points $(x_1, \ldots, x_n)$ for which $(y_1, \ldots, y_m) \in Y$, and $\mathbf{P}(X)$ denotes the probability that $(\xi_1, \ldots, \xi_n) \in X$.

We indicate the formula for the mean value of a random variable $\eta = f(\xi_1, \ldots, \xi_n)$. If $\mathbf{P}_1(y)$ denotes the probability that $\eta < y$, then by definition the mean value of η is given by[2]

$$\mathbf{E}\eta = \int y \, d\mathbf{P}_1(y). \tag{4}$$

[2] We denote the mean value of the random variable η by $\mathbf{E}\eta$.

But by our definition of a function of random arguments, the probability $\mathbf{P}_1(y)$ coincides with the probability that $f(x_1, \ldots, x_n) < y$. Therefore

$$\mathbf{E}\eta = \int f(x_1, \ldots, x_n)\, d\mathbf{P}(x_1, \ldots, x_n),$$

where $\mathbf{P}(X)$ is the probability distribution of the random variable $(\xi_1, \ldots, \xi_n)$. For example, the mean value of the product of the random variables ξ_1 and ξ_2 is given by

$$\mathbf{E}(\xi_1 \xi_2) = \int x_1 x_2\, d\mathbf{P}(x_1, x_2),$$

where $\mathbf{P}(x_1, x_2)$ is the probability that $\xi_1 < x_1$, $\xi_2 < x_2$. The mean value of the random variable $e^{i\lambda\xi}$ is given by

$$\mathbf{E}(e^{i\lambda\xi}) = \int e^{i\lambda x}\, d\mathbf{P}(x).$$

The expression $\mathbf{E}^{i\lambda\xi}(e)$ is called the *characteristic function* of the random variable ξ.

We now proceed to the definition of the limit of a sequence of random variables. Suppose that we are given a sequence $\{\xi_{k1}, \ldots, \xi_{kn}\}$ of n-dimensional random variables. We will say that this sequence converges to the n-dimensional random variable $(\xi_1, \ldots, \xi_n)$, if for any random variables $\eta_1, \ldots, \eta_m$ and any bounded continuous function $f(x_1, \ldots, x_n; y_1, \ldots, y_m)$ one has

$$\lim_{k\to\infty} \int f(x_1, \ldots, x_n; y_1, \ldots, y_m)\, d\mathbf{P}_k(x; y) = \int f(x_1, \ldots, x_n; y_1, \ldots, y_m)\, d\mathbf{P}(x, y),$$

where $\mathbf{P}_k(x, y)$ denotes the measure in R_{n+m} corresponding to the random variable $(\xi_{k1}, \ldots, \xi_{kn}; \eta_1, \ldots, \eta_m)$, and $\mathbf{P}(x, y)$ denotes the measure in R_{n+m} corresponding to the random variable $(\xi_1, \ldots, \xi_n; \eta_1, \ldots, \eta_m)$.[3]

We further introduce the concept of a random function, also called a random process. Suppose that with every real number t there is associated a random variable $\xi(t)$ (i.e., in other words, suppose that for any n real numbers $t_1, \ldots, t_n$ there is given a joint distribution function for the random variables $\xi(t_1), \ldots, \xi(t_n)$, and these distribution functions are compatible). In this case we will say that we are given a *random function* $\xi(t)$. One can introduce the notions of continuity, integrals,

[3] The probability that $(\xi_1, \ldots, \xi_n) \in X$ is called the measure of the set X corresponding to the random variable $(\xi_1, \ldots, \xi_n)$.

and derivatives of a random function similarly to the way in which this is done for ordinary functions. For example, a random function $\xi(t)$ is called continuous if $\lim_{k\to\infty} t_{kj} = t_j$, $1 \leqslant j \leqslant n$, implies

$$\lim_{k\to\infty} (\xi(t_{k1}), \ldots, \xi(t_{kn})) = (\xi(t_1), \ldots, \xi(t_n)).$$

If $\xi(t)$ is a continuous random function, and $\varphi(t)$ is a continuous scalar function with bounded support, then the integral

$$\int \varphi(t)\xi(t)\, dt$$

exists, where this integral is understood as the limit of sums:

$$\int \varphi(t)\xi(t)\, dt = \lim_{\max \Delta t \to 0} \sum_{k=1}^{n} \varphi(t_k)\xi(t_k)\, \Delta t_k.$$

We will not dwell on the details of the theory of random functions, since the basic results of this theory will be obtained below as special cases of a more general theory.

In conclusion we point out that along with real random variables, one can consider random variables which assume complex values. In this case one gives the probability that the point $(\xi_1, \ldots, \xi_n)$ belong to a given set in n-dimensional complex space.

1.2. Generalized Random Processes

We proceed now to the definition of the basic concept of this chapter—the concept of a generalized random process. Consider the space K of infinitely differentiable functions $\varphi(t)$ having bounded supports. We will say that a *random functional* Φ is defined on K, if with every element $\varphi(t) \in K$ there is associated a random variable $\Phi(\varphi)$. In accordance with the discussion of Section 1.1, this means that for every n elements $\varphi_1(t), \ldots, \varphi_n(t)$ in K one specifies the probability that

$$a_k \leqslant \Phi(\varphi_k) < b_k, \qquad 1 \leqslant k \leqslant n,$$

and these probability distributions are compatible in the sense indicated above.

The random functional $\Phi(\varphi)$ is called *linear*, if for any elements $\varphi, \psi \in K$ and any numbers α and β one has

$$\Phi(\alpha\varphi + \beta\psi) = \alpha\Phi(\varphi) + \beta\Phi(\psi)$$

(concerning the definition of equality of random variables, see Section 1.1). Finally, the random functional $\Phi(\varphi)$ is called *continuous*, if the convergence in K of the functions $\varphi_{kj}(t)$ to $\varphi_j(t)$, $1 \leqslant j \leqslant n$, implies

$$\lim_{k\to\infty} (\Phi(\varphi_{k1}), \ldots, \Phi(\varphi_{kn})) = (\Phi(\varphi_1), \ldots, \Phi(\varphi_n)).$$

As we do not consider any random variables other than those of the form $\Phi(\varphi)$, the continuity of Φ means the following.

If $\lim_{k\to\infty} \varphi_{kj}(t) = \varphi_j(t)$, $1 \leqslant j \leqslant n$ (in the space K), then for any continuous bounded function $f(x_1, \ldots, x_n)$ one has

$$\lim_{k\to\infty} \int f(x_1, \ldots, x_n)\, d\mathbf{P}_k(x) = \int f(x_1, \ldots, x_n)\, d\mathbf{P}(x),$$

where $\mathbf{P}(x)$ denotes the measure corresponding to the random variable $(\Phi(\varphi_1), \ldots, \Phi(\varphi_n))$, and $\mathbf{P}_k(x)$ denotes the measure corresponding to the random variable $(\Phi(\varphi_{k1}), \ldots, \Phi(\varphi_{kn}))$.

Just as continuous linear functionals on the space K are called generalized functions, we will call a continuous linear random functional on K a *generalized random function*. In the case where K consists of functions of one variable, the corresponding random function will be called a *generalized random process*. In the case where K is a space of functions of several variables, Φ is called a *generalized random field*.

Let us pause to consider the physical motivation for the concept of a generalized random function. The usual concept of a random function, which we gave in Section 1.1, is based upon the assumption that it is possible to measure the value of the random function at every moment of time t without calculating the value of the function at other moments of time. However, every actual measurement is accomplished by means of an apparatus which has a certain inertia. Therefore the reading which the apparatus gives is not the value of the random variable $\xi(t)$ at the instant t, but rather a certain averaged value $\Phi(\varphi) = \int \varphi(t)\xi(t)\, dt$, where $\varphi(t)$ is a function characterizing the apparatus. These quantities are compatible and depend linearly upon φ. Moreover, small changes of the function $\varphi(t)$ cause small changes in the random variable $\Phi(\varphi)$ (apparatuses which differ only slightly give close readings). Thus, as a consequence of measuring the value of a random function by means of apparatuses, we obtain a continuous linear random functional, i.e., a generalized random process.

As a result of the smoothing action of apparatuses, one can thus obtain a probability distribution not only for processes which exist at each instant of time t, but also for "generalized" processes, for which there do not exist probability distributions at isolated instants of time. Typical examples of such processes (the velocity of a Brownian particle which does not have inertia) will be considered in Section 1.3. This is analogous to the fact that the values of a generalized function F at separate points may not exist, while, however, integrals of the form $(F, \varphi) = \int F(t)\varphi(t)\, dt$ do exist.

1.3. Examples of Generalized Random Processes

Let us give examples of generalized random processes. These examples will be studied at greater length in Section 2.5. We associate with the linearly independent functions $\varphi_1(x), \ldots, \varphi_n(x)$ in K the random variable $(\Phi(\varphi_1), \ldots, \Phi(\varphi_n))$ having the probability distribution

$$\mathbf{P}_\varphi(X) = \frac{\sqrt{\det \Lambda_\varphi}}{(2\pi)^{\frac{1}{2}n}} \int_X \exp[-\tfrac{1}{2}(\Delta_\varphi x, x)]\, dx, \tag{5}$$

where Λ_φ is the inverse of the matrix $\| b_{jk} \|$, whose elements are

$$b_{jk} = \int \varphi_j(t)\varphi_k(t)\, dt.$$

It can be shown that these random variables are compatible, and are continuous and linear in φ. The generalized random process, defined by the probability distribution (5), is called the *unit* process. This process can be interpreted as the result of measuring, by means of some apparatus, the velocity of a particle which is undergoing one-dimensional Brownian motion and has no inertia. The unit random process is not an ordinary random process, because the velocity of a Brownian particle at a given moment of time does not have a probability distribution. Therefore there does not exist a continuous random function $\xi(t)$ such that $\Phi(\varphi) = \int \varphi(t)\xi(t)\, dt$.

We remark that the path traced out by a Brownian particle is also a random function of time. In this instance, however, for any n moments of time $0 \leqslant t_1 < \ldots < t_n$ one can specify the probability distribution of the n-dimensional random variable $(\xi(t_1), \ldots, \xi(t_n))$, where $\xi(t)$ is the coordinate of the particle at time t. Namely, if $\xi(0) = 0$, then for appropriate choice of the unit of time the probability that $(\xi(t_1), \ldots, \xi(t_n)) \in X$ is expressed by

$$\mathbf{P}(X) = \frac{1}{(2\pi)^{\frac{1}{2}n} \sqrt{t_1(t_2 - t_1) \ldots (t_n - t_{n-1})}} \times \int_X \exp\left\{-\frac{1}{2}\left[\frac{x_1^2}{t_1} + \frac{(x_2 - x_1)^2}{t_2 - t_1} + \ldots + \frac{(x_n - x_{n-1})^2}{t_n - t_{n-1}}\right]\right\} dx_1 \ldots dx_n. \tag{6}$$

It can be shown that the generalized random process corresponding to the random function $\xi(t)$, with probability distribution (6), associates with the functions $\varphi_1(t), \ldots, \varphi_n(t)$ the random variable $(\Phi(\varphi_1), \ldots, \Phi(\varphi_n))$ with probability distribution

$$\mathbf{P}(X) = \frac{\sqrt{\det \Lambda_\varphi}}{(2\pi)^{\frac{1}{2}n}} \int_X \exp[-\tfrac{1}{2}(\Lambda_\varphi x, x)]\, dx.$$

Here Λ_φ is the inverse of the matrix $B = \| b_{jk} \|$ consisting of the elements $b_{jk} = B(\varphi_j, \varphi_k)$, where

$$B(\varphi, \psi) = \int [\hat{\varphi}(t) - \hat{\varphi}(\infty)][\hat{\psi}(t) - \hat{\psi}(\infty)]\, dt,$$

$$\hat{\varphi}(t) = \int_0^t \varphi(t)\, dt, \qquad \hat{\psi}(t) = \int_0^t \psi(t)\, dt.$$

This random process is called the *Wiener* process.

One can construct examples of generalized random fields in the same way. For example, the analog of the unit generalized random process is the *unit random field* Φ. If $\varphi_k(x) = \varphi_k(x_1, \ldots, x_m)$, $1 \leqslant k \leqslant n$, are functions in K, then the unit random field associates with them the random variable $(\Phi(\varphi_1), \ldots, \Phi(\varphi_n))$ with probability distribution given by the same formula (5) as in the case of the unit random process. The sole difference consists in the fact that the numbers b_{jk} are defined by

$$b_{jk} = \int \varphi_j(x_1, \ldots, x_n)\varphi_k(x_1, \ldots, x_n)\, dx_1 \ldots dx_n.$$

1.4. Operations on Generalized Random Processes

The operations which can be performed on generalized random processes are defined in a manner analogous to that by which they are defined for generalized functions. For example, by a *linear combination* $\alpha\Phi_1 + \beta\Phi_2$ of the generalized random processes Φ_1 and Φ_2 is understood the generalized random process Φ which associates with every function $\varphi(t) \in K$ the random variable† $\alpha\Phi_1(\varphi) + \beta\Phi_2(\varphi)$. Thus, the set of all generalized random processes forms a linear space.

The ordinary operations on generalized random processes are defined by means of the corresponding operations on the test functions $\varphi(t)$. Thus, the product $f(t)\Phi$ of an infinitely differentiable function $f(t)$ and

† Of course, a correlation between Φ_1 and Φ_2 must be specified.

a generalized random process Φ is defined as the process for which there corresponds to the function $\varphi(t) \in K$ the random variable $\Phi(f\varphi)$. In the same way, the *derivative* Φ' of a generalized random process Φ is defined as the process for which there corresponds to the function $\varphi(t)$ the random variable $-\Phi(\varphi')$.

We note that while the derivative of an ordinary random process may no longer be a process of the same type, the derivative of a generalized random process always exists and is a generalized random process. In particular, although the derivative of the Wiener process is not an ordinary random process, it is a generalized random process.

We consider, lastly, the notion of a *translation* of a generalized random process. If Φ is a generalized random process, by its translation by a number h we mean the random process Φ_h which associates with each function $\varphi(t)$ the random variable $\Phi_h(\varphi) = \Phi[\varphi(t - h)]$. It is not difficult to show that

$$\Phi' = \lim_{h\to\infty} \frac{\Phi_n - \Phi}{h},$$

but we will not detain ourselves over this.

2. Moments of Generalized Random Processes. Gaussian Processes. Characteristic Functionals

2.1. The Mean of a Generalized Random Process

If Φ is a generalized random process, then to every function $\varphi(t) \in K$ there corresponds a random variable $\Phi(\varphi)$. Let us assume that every one of the random variables $\Phi(\varphi)$ has a mean $m(\varphi)$, which is continuous in φ. Then $m(\varphi)$ is a continuous functional on K. We will call this functional the *mean of the generalized random process* Φ. Thus, the mean of the generalized random process Φ is defined by

$$m(\varphi) = \mathbf{E}[\Phi(\varphi)] = \int x \, d\mathbf{P}(x)$$

($\mathbf{P}(x)$ denotes the probability that $\Phi(\varphi) < x$).

The functional $m(\varphi)$ is obviously linear, as the random variable $\Phi(a\varphi_1 + b\varphi_2)$ is by definition equal to the random variable $a\Phi(\varphi_1) + b\Phi(\varphi_2)$, and therefore

$$\begin{aligned} m(a\varphi_1 + b\varphi_2) &= \mathbf{E}[\Phi(a\varphi_1 + b\varphi_2)] = \mathbf{E}[a\Phi(\varphi_1) + b\Phi(\varphi_2)] \\ &= a\mathbf{E}[\Phi(\varphi_1)] + b\mathbf{E}[\Phi(\varphi_2)] = am(\varphi_1) + bm(\varphi_2). \end{aligned}$$

Thus, $m(\varphi)$ is a linear functional on K, i.e., a generalized function.

The generalized random process $\Phi(\varphi) - m(\varphi)$ has mean zero. Every generalized random process is the sum of a linear functional $m(\varphi)$ and a process having mean zero (when $m(\varphi)$ exists).

If the mean of the random variable $\Phi(\varphi)\Phi(\psi)$ exists for all φ and ψ and is continuous in each of the arguments φ and ψ, we call it the *correlation functional* of Φ. Thus, the correlation functional $B(\varphi, \psi)$ of the process Φ is given by

$$B(\varphi, \psi) = \mathbf{E}[\Phi(\varphi)\Phi(\psi)].$$

This equation can be rewritten in the form

$$B(\varphi, \psi) = \int x_1 x_2 \, d\mathbf{P}(x_1, x_2),$$

where $\mathbf{P}(x_1, x_2)$ denotes the joint distribution function of the random variables $\Phi(\varphi)$ and $\Phi(\psi)$.

From the linearity of the random functional $\Phi(\varphi)$, it follows that $B(\varphi, \psi)$ is a bilinear functional.

The functional $B(\varphi, \psi)$ gives the connection between readings of the apparatuses characterized by the functions $\varphi(t)$ and $\psi(t)$.

Since the random variable $[\Phi(\varphi)]^2$ is positive, its mean $B(\varphi, \varphi) = \mathbf{E}[\Phi(\varphi)\Phi(\varphi)]$ is also positive, i.e., $B(\varphi, \varphi) \geqslant 0$. Therefore the correlation functional $B(\varphi, \psi)$ is positive-definite.[1] Moreover, the functional $C(\varphi, \psi)$, defined by

$$C(\varphi, \psi) = B(\varphi, \psi) - m(\varphi)m(\psi),$$

is positive-definite, where $m(\varphi)$ is the mean of the process Φ.

Indeed,

$$\begin{aligned} C(\varphi, \varphi) = B(\varphi, \varphi) - m(\varphi)m(\varphi) &= \mathbf{E}[\Phi(\varphi)\Phi(\varphi)] \\ &\quad -2\mathbf{E}[\Phi(\varphi)]m(\varphi) + m^2(\varphi) = \mathbf{E}[|\Phi(\varphi) - m(\varphi)|^2] \geqslant 0. \end{aligned}$$

We define the *n*th-*order moment of the generalized random process* Φ as the polylinear functional $m[\Phi(\varphi_1) \cdot \ldots \cdot \Phi(\varphi_n)]$, i.e., the mean of the random variable $\Phi(\varphi_1) \cdot \ldots \cdot \Phi(\varphi_n)$. It is given by

$$m[\Phi(\varphi_1) \ldots \Phi(\varphi_n)] = \int x_1 \cdot \ldots \cdot x_n \, d\mathbf{P}(x_1, \ldots, x_n),$$

[1] For complex generalized random processes the correlation functional is defined by

$$B(\varphi, \psi) = \mathbf{E}[\Phi(\varphi)\overline{\Phi(\psi)}]$$

and is a positive-definite Hermitean functional.

where $\mathbf{P}$ is the measure corresponding to the n-dimensional random variable

$$(\Phi(\varphi_1), \ldots, \Phi(\varphi_n)).^2$$

By Theorem 5′ of Chapter I, Section 1.3, every polylinear functional F on K can be written in the form

$$F(\varphi_1, \ldots, \varphi_n) = (F_0, \varphi_1(t_1) \cdot \ldots \cdot \varphi_n(t_n)),$$

where F_0 is a linear functional (generalized function) on the space K_n of infinitely differentiable functions of n variables, having bounded supports. Thus, the nth moment of a generalized random process Φ is given by a generalized function F_0 of n variables. In particular, the correlation functional $B(\varphi, \psi)$ is given by a generalized function $B(t_1, t_2)$ of two variables.

2.2. Gaussian Processes

One of the important classes of generalized random processes is the class of Gaussian random processes. We first consider real Gaussian processes. A generalized random process is called a *proper Gaussian process*, if for any linearly independent functions $\varphi_1, \ldots, \varphi_n$ in K the random variable $(\Phi(\varphi_1), \ldots, \Phi(\varphi_n))$ is Gaussianly distributed. This means that the probability that $(\Phi(\varphi_1), \ldots, \Phi(\varphi_n)) \in X$ is expressed by the formula

$$\mathbf{P}(X) = \frac{\sqrt{\det \Lambda}}{(2\pi)^{\frac{1}{2}n}} \int_X \exp[-\tfrac{1}{2}(\Lambda x, x)]\, dx, \tag{1}$$

where $\Lambda = \| \lambda_{ij} \|$ is a nondegenerate positive-definite matrix, and $(\Lambda x, x)$ denotes the quadratic form

$$(\Lambda x, x) = \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_{ij} x_i x_j.$$

A simple computation shows that $\mathbf{P}(R_n) = 1$. Indeed, since the matrix Λ is positive definite, it can be written in the form $\Lambda = C'C$, where C is a square matrix and C' is its transpose. Therefore $(\Lambda x, x) = (Cx, Cx)$. Let us make the substitution $Cx = y$ in the integral

$$\mathbf{P}(R_n) = \frac{\sqrt{\det \Lambda}}{(2\pi)^{\frac{1}{2}n}} \int \exp[-\tfrac{1}{2}(\Lambda x, x)]\, dx.$$

[2] The mean of the process Φ is the first-order moment, and the correlation functional $B(\varphi, \psi)$ coincides with the second-order moment.

Since $\det C = \sqrt{\det \Lambda}$, we find that

$$\mathbf{P}(R_n) = \frac{1}{(2\pi)^{\frac{1}{2}n}} \int \exp[-\tfrac{1}{2}(y, y)]\, dy$$

$$= \frac{1}{(2\pi)^{\frac{1}{2}n}} \int \exp[-\tfrac{1}{2}(y_1^2 + \dots + y_n^2)]\, dy_1 \dots dy_n.$$

As $\int_{-\infty}^{\infty} \exp(-\tfrac{1}{2} y_k^2)\, dy_k = \sqrt{2\pi}$, we arrive at $\mathbf{P}(R_n) = 1$.

Now we prove that the probability distribution $\mathbf{P}(X)$ is uniquely defined by the correlation functional $B(\varphi, \psi)$ of the process Φ. Namely, *if Φ is a proper Gaussian process, then for any linearly independent functions $\varphi_1, \dots, \varphi_n$ in K one has $\Lambda = \| B(\varphi_i, \varphi_j) \|^{-1}$.* To prove this assertion, we will calculate the value of $\mathbf{E}[\Phi(\varphi_i)\Phi(\varphi_j)]$ in two ways.

By definition of the correlation functional we have

$$\mathbf{E}[\Phi(\varphi_i)\Phi(\varphi_j)] = B(\varphi_i, \varphi_j). \tag{2}$$

But the random variable $\Phi(\varphi_i)\Phi(\varphi_j)$ can also be considered as a function of the n-dimensional random variable whose distribution function is given by (1). Therefore

$$\mathbf{E}[\Phi(\varphi_i)\Phi(\varphi_j)] = \frac{\sqrt{\det \Lambda}}{(2\pi)^{\frac{1}{2}n}} \int x_i x_j \exp[-\tfrac{1}{2}(\Lambda x, x)]\, dx. \tag{3}$$

To compute the integral (3) we use the formula

$$\frac{\sqrt{\det C}}{(2\pi)^{\frac{1}{2}n}} \int (Ax, x) \exp[-\tfrac{1}{2}(Cx, x)]\, dx = \mathrm{Tr}(AC^{-1}),^{3} \tag{4}$$

which is valid for any strictly positive-definite matrix C and any matrix A (for the proof of (4), see below). Since $x_i x_j = (A_{ij}x, x)$, where A_{ij} is the matrix, all of whose elements vanish with the exception of a_{ij}, which equals 1, the integral (3) is equal to $\mathrm{Tr}(A_{ij}\Lambda^{-1})$.

But $A_{ij}\Lambda^{-1}$ is the matrix, all of whose rows vanish with the exception of the ith row, which coincides with the jth row of Λ^{-1}. Therefore $\mathrm{Tr}(A_{ij}\Lambda^{-1}) = \mu_{ij}$, where the μ_{ij} are the elements of Λ^{-1}.

Thus, we have proven that

$$\mathbf{E}[\Phi(\varphi_i)\Phi(\varphi_j)] = \mathrm{Tr}(A_{ij}\Lambda^{-1}) = \mu_{ij}.$$

[3] Tr B denotes the trace of the matrix B, i.e., the sum of its diagonal elements.

Comparing this with (2), we conclude that

$$\| B(\varphi_i, \varphi_j)\| = \| \mu_{ij} \| = \Lambda^{-1}.$$

This proves our assertion.

It remains for us to prove formula (4). To this end we note that if C is a positive-definite matrix of order n, and A is any real matrix of the same order, then for sufficiently small real ϵ we have

$$\frac{1}{(2\pi)^{\frac{1}{2}n}} \int \exp[-\tfrac{1}{2}((C + \epsilon A)x, x)]\, dx = [\det(C + \epsilon A)]^{-\frac{1}{2}}$$

(the proof of this identity coincides with the proof of $\mathbf{P}(R_n) = 1$ which was carried out above). Since

$$\det[(C + \epsilon A)]^{-\frac{1}{2}} = (\det C)^{-\frac{1}{2}}[\det(E + \epsilon AC^{-1})]^{-\frac{1}{2}},$$

this identity can also be written in the form

$$\frac{(\det C)^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}n}} \int \exp[-\tfrac{1}{2}((C + \epsilon A)x, x)]\, dx = [\det(E + \epsilon AC^{-1})]^{-\frac{1}{2}}. \tag{5}$$

Expand both sides of (5) in a power series in ϵ, and equate coefficients. Since

$$\det(E + \epsilon AC^{-1}) = 1 + \epsilon \operatorname{Tr}(AC^{-1}) + \dots,$$

we obtain

$$\frac{(\det C)^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}n}} \int (Ax, x) e^{-\frac{1}{2}(Cx, x)}\, dx = \operatorname{Tr}(AC^{-1}).$$

This proves formula (4).

We remark that by similar means one can also compute integrals such as

$$\int (Ax, x)^2 e^{-\frac{1}{2}(Cx, x)}\, dx$$

and so on.

Thus, we have proven the following theorem.

Theorem 1. Let Φ be a proper Gaussian generalized random process. Then for any linearly independent functions $\varphi_1, \dots, \varphi_n$ in K the probability distribution of the random variable $(\Phi(\varphi_1), \dots, \Phi(\varphi_n))$ has the form

$$\mathbf{P}(X) = \frac{\sqrt{\det \Lambda}}{(2\pi)^{\frac{1}{2}n}} \int_X e^{-\frac{1}{2}(\Lambda x, x)}\, dx,$$

where $\Lambda = \| \lambda_{ij} \|$ is the inverse of the matrix $\| B(\varphi_i, \varphi_j) \|$, and $(\Lambda x, x)$ denotes the quadratic form

$$(\Lambda x, x) = \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_{ij} x_i x_j.$$

If the functions $\varphi_1, \ldots, \varphi_n$ are linearly dependent, then the probability distribution of the random variable $(\Phi(\varphi_1), \ldots, \Phi(\varphi_n))$ is concentrated on a subspace R'_m of R_n, whose dimension is equal to the dimension of the linear space spanned by $\varphi_1, \ldots, \varphi_n$. It consists of those points $(x_1, \ldots, x_n)$ whose coordinates satisfy the same linear relations as do the functions $\varphi_1, \ldots, \varphi_n$. The probability distribution in question is given, on R'_m, by a formula similar to (1).

We have obtained the probability distribution for proper Gaussian processes. Sometimes improper Gaussian processes are considered, for which the correlation functional $B(\varphi, \varphi)$ can vanish for functions other than $\varphi(t) \equiv 0$. For such processes the random variable $(\Phi(\varphi_1), \ldots, \Phi(\varphi_n))$ can be distributed according to an improper Gaussian law, i.e., a Gaussian distribution which is concentrated on some subspace of n-dimensional space. We will not dwell further on this question.

We further remark that if the mean $m(\varphi)$ of the Gaussian process Φ is different from zero, then formula (1) is replaced by a somewhat more complicated formula

$$\mathbf{P}(X) = \frac{\sqrt{\det \Lambda}}{(2\pi)^{\frac{1}{2}n}} \int_X \exp[-\tfrac{1}{2}(\Lambda(x - x^0), x - x^0)]\, dx, \tag{1'}$$

where x^0 is the vector with coordinates $m(\varphi_1), \ldots, m(\varphi_n)$.

Along with real Gaussian processes, one can introduce complex Gaussian processes. For such processes the probability that

$$(\Phi(\varphi_1), \ldots, \Phi(\varphi_n)) \in X$$

(X is a set in n-dimensional complex space) is expressed by the formula

$$\mathbf{P}(X) = \frac{\det \Lambda}{(2\pi)^n} \int_X \exp[-\tfrac{1}{2}(\Lambda z, z)]\, dz, \tag{1''}$$

where $\Lambda = \| \lambda_{ij} \|$ is the inverse of the matrix $\| B(\varphi_i, \varphi_j) \|$,

$$(\Lambda z, z) = \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_{ij} z_i \bar{z}_j$$

is the Hermitean form corresponding to the matrix Λ, and dz denotes the measure $dx_1\, dy_1 \ldots dx_n\, dy_n$, $z_k = x_k + iy_k$.

2.3. The Existence of Gaussian Processes with Given Means and Correlation Functionals

We have seen in the previous paragraph that the probability distribution of the random variable $(\Phi(\varphi_1), \ldots, \Phi(\varphi_n))$, where Φ is a generalized Gaussian process, is uniquely defined by the correlation functional $B(\varphi, \psi)$ and the mean $m(\varphi)$ of Φ. As was proven in Section 2.1, $B(\varphi, \psi)$ and $m(\varphi)$ must be such that the bilinear functional

$$C(\varphi, \psi) = B(\varphi, \psi) - m(\varphi)m(\psi)$$

is positive-definite.

Now we show: If a continuous bilinear functional $B(\varphi, \psi)$ on K and a continuous linear functional $m(\varphi)$ on K are such that the functional $B(\varphi, \psi) - m(\varphi)m(\psi)$ is positive-definite, then there exists a generalized Gaussian process Φ of which $B(\varphi, \psi)$ is the correlation functional and $m(\varphi)$ is the mean.

Obviously it is sufficient to prove the theorem in the case where $m(\varphi) = 0$ and $B(\varphi, \psi)$ is positive-definite.

We first consider the case where $B(\varphi, \psi)$ is strictly positive-definite, i.e., where $B(\varphi, \varphi) > 0$ for all functions $\varphi(t) \in K$ which are not identically zero. In this case the matrix $\| B(\varphi_i, \varphi_j) \|$ is nondegenerate for any linearly independent functions $\varphi_1, \ldots, \varphi_n$ in K. Indeed, if $\varphi_1, \ldots, \varphi_n$ are linearly independent and the numbers $a_1, \ldots, a_n$ are not simultaneously zero, then the function

$$\psi(t) = \sum_{i=1}^{n} a_i \varphi_i(t)$$

is not identically zero, and therefore

$$\sum_{i=1}^{n} \sum_{j=1}^{n} B(\varphi_i, \varphi_j) a_i a_j = B(\psi, \psi) > 0.$$

From this follows the nondegeneracy of the matrix $\| B(\varphi_i, \varphi_j) \|$.

Since $B = \| B(\varphi_i, \varphi_j) \|$ is nondegenerate, it has an inverse A. We now associate with the functions $\varphi_1, \ldots, \varphi_n$ the random variable $(\Phi(\varphi_1), \ldots, \Phi(\varphi_n))$ with probability distribution

$$\mathbf{P}(X) = \frac{\sqrt{\det A}}{(2\pi)^{\frac{1}{2}n}} \int_X e^{-\frac{1}{2}(Ax,x)}\, dx, \tag{6}$$

where

$$(\Lambda x, x) = \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_{ij} x_i x_j. \tag{7}$$

The random variable $(\Phi(\varphi_1), ..., \Phi(\varphi_n))$ is thus defined for linearly independent functions $\varphi_1, ..., \varphi_n$.

Suppose now that $\varphi_1, ..., \varphi_n$ are linearly dependent. Choose a linearly independent basis $\psi_1, ..., \psi_m$ in the linear space spanned by $\varphi_1, ..., \varphi_n$. Since the functions $\psi_1, ..., \psi_m$ are linearly independent, the random variable $(\Phi(\psi_1), ..., \Phi(\psi_m))$ is defined by a probability distribution $\mathbf{P}_1(X)$ having the form (6). Since the functions $\varphi_1, ..., \varphi_n$ are linear combinations of $\psi_1, ..., \psi_m$,

$$\varphi_k(t) = \sum_{j=1}^{m} b_{kj} \psi_j(t),$$

we define the probability distribution of the random variable $(\Phi(\varphi_1), ..., \Phi(\varphi_n))$ by

$$\mathbf{P}(X) = \mathbf{P}_1(\tilde{X});$$

here $\tilde{X}$ denotes the set of points $x = (x_1, ..., x_m)$ in m-dimensional space R_m such that $(y_1, ..., y_n) \in X$, where $y_k = \sum_{j=1}^{m} b_{kj} x_j$.

Thus, we have defined the probability distribution $\mathbf{P}(X)$ of the n-dimensional random variable $(\Phi(\varphi_1), ..., \Phi(\varphi_n))$ for any functions $\varphi_1, ..., \varphi_n$ in K. It can be shown that these probability distributions define a continuous linear random functional on K, i.e., that the following conditions are fulfilled:

(1) For any functions $\varphi_1, ..., \varphi_n, \varphi$ and numbers $x_1, ..., x_n$ the probability of the event

$$\Phi(\varphi_k) \leqslant x_k, \qquad 1 \leqslant k \leqslant n$$

coincides with the probability of the event

$$\Phi(\varphi_k) \leqslant x_k, \qquad 1 \leqslant k \leqslant n, \qquad \Phi(\varphi) < \infty$$

(the condition of compatibility);

(2) $$\Phi(\alpha\varphi + \beta\psi) = \alpha\Phi(\varphi) + \beta\Phi(\psi),$$

where equality of random variables is understood in the sense established in Section 1;

(3) the dependence of the random variable $(\Phi(\varphi_1), ..., \Phi(\varphi_n))$ upon the functions $\varphi_1, ..., \varphi_n$ is continuous.

The proof of assertions (1) and (2) is based upon the following lemma.

Lemma 1. Suppose that $\varphi_k(t)$, $1 \leqslant k \leqslant n$, is a system of linearly independent functions in K, and $\psi_k(t)$, $1 \leqslant k \leqslant m$, are linearly independent functions which belong to the linear space spanned by the functions $\varphi_1, ..., \varphi_n$,

$$\psi_k(t) = \sum_{j=1}^{n} a_{kj}\varphi_j(t). \tag{8}$$

Let $\mathbf{P}(X)$ denote the probability distribution of the random variable $(\Phi(\varphi_1), ..., \Phi(\varphi_n))$, and $\mathbf{P}_1(Y)$, that of the random variable $(\Phi(\psi_1), ..., \Phi(\psi_m))$. Then

$$\mathbf{P}(\tilde{Y}) = \mathbf{P}_1(Y),$$

where Y is any set in the space R_m, and $\tilde{Y}$ is the set of those points $x = (x_1, ..., x_n)$ in R_n such that the point $y = (y_1, ..., y_m)$ with coordinates $y_k = \sum_{j=1}^{n} a_{kj}x_j$ belongs to Y.

Proof. The relation $\mathbf{P}(\tilde{Y}) = \mathbf{P}_1(Y)$ can be written in the form

$$\frac{\sqrt{\det \Lambda_\varphi}}{(2\pi)^{\frac{1}{2}n}} \int_{\tilde{Y}} \exp[-\tfrac{1}{2}(\Lambda_\varphi x, x)]\, dx = \frac{\sqrt{\det \Lambda_\psi}}{(2\pi)^{\frac{1}{2}m}} \int_{Y} \exp[-\tfrac{1}{2}(\Lambda_\psi y, y)]\, dy, \tag{9}$$

where Λ_ψ denotes the matrix $\| B(\psi_i, \psi_j) \|^{-1}$, and Λ_φ denotes the matrix $\| B(\varphi_i, \varphi_j) \|^{-1}$.

To prove relation (9), note that in view of (8) we have

$$B(\psi_k, \psi_l) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ki}a_{lj}B(\varphi_i, \varphi_j),$$

and therefore

$$\| B(\psi_k, \psi_l)\| = A\| B(\varphi_i, \varphi_j)\|A', \tag{10}$$

where A denotes the matrix $\| a_{ki} \|$, and A' is the transpose of A.

Consider first the case where $m = n$. In this case the matrix A is nondegenerate, because each of the collections $\{\psi_k\}$ and $\{\varphi_j\}$ is linearly independent. Consequently, it follows from (10) that

$$\Lambda_\varphi = (A')^{-1}\Lambda_\psi A^{-1},$$

i.e., that

$$\Lambda_\varphi = A'\Lambda_\psi A. \tag{11}$$

Therefore

$$(\Lambda_\varphi x, x) = (A'\Lambda_\psi Ax, x) = (\Lambda_\psi Ax, Ax). \tag{12}$$

Moreover, it follows from (11) that

$$\det \Lambda_\varphi = (\det A)^2 \det \Lambda_\psi. \tag{13}$$

Substituting the expressions (12) and (13) for $(\Lambda_\varphi x, x)$ and $\det \Lambda_\varphi$ into the left side of (9), we obtain

$$\frac{\sqrt{\det \Lambda_\varphi}}{(2\pi)^{\frac{1}{2}n}} \int_{\tilde{Y}} \exp[-\tfrac{1}{2}(\Lambda_\varphi x, x)]\, dx$$
$$= \frac{|\det A| \sqrt{\det \Lambda_\psi}}{(2\pi)^{\frac{1}{2}n}} \int_{\tilde{Y}} \exp[-\tfrac{1}{2}(\Lambda_\psi Ax, Ax)]\, dx.$$

We make the change of variables $Ax = y$ in the right side of this equality. Taking into account that $dx = |\det A|^{-1}\, dy$ and that $\tilde{Y}$ is the inverse image of the set Y under the transformation $Ax = y$, we arrive at (9).

Let us now consider the general case. Any linear transformation A which carries the functions $\varphi_1, \ldots, \varphi_n$ into the functions $\psi_1, \ldots, \psi_m$ can be represented in the form

$$\{\varphi_1, \ldots, \varphi_n\} \xrightarrow{A_1} \{\hat{\varphi}_1, \ldots, \hat{\varphi}_n\} \xrightarrow{A_2} \{\hat{\varphi}_1, \ldots, \hat{\varphi}_m\} \xrightarrow{A_3} \{\psi_1, \ldots, \psi_m\},$$

where A_1 and A_3 are nondegenerate linear transformations, and $\hat{\varphi}_1, \ldots, \hat{\varphi}_n$ are functions such that $B(\hat{\varphi}_i, \hat{\varphi}_j) = \delta_{ij}$ (δ_{ij} is the Kronecker symbol). For the transformations A_1 and A_3 the compatibility relation has been proven. Therefore it is sufficient to prove the equality $\mathbf{P}_1(Y) = \mathbf{P}(\tilde{Y})$ for mappings of the form $(\varphi_1, \ldots, \varphi_n) \rightarrow (\varphi_1, \ldots, \varphi_m)$, where $\varphi_1, \ldots, \varphi_n$ are functions such that $B(\varphi_i, \varphi_j) = \delta_{ij}$. For such functions, $\| B(\varphi_i, \varphi_j) \|$, Λ_φ, and $\| B(\psi_i, \psi_j) \|$, Λ_ψ are the identity matrix. Consequently, the result to be proved assumes the form

$$\frac{1}{(2\pi)^{\frac{1}{2}m}} \int_Y \exp[-\tfrac{1}{2}(y_1^2 + \cdots + y_m^2)]dy_1 \ldots dy_m$$
$$= \frac{1}{(2\pi)^{\frac{1}{2}n}} \int_{\tilde{Y}} \exp[-\tfrac{1}{2}(x_1^2 + \cdots + x_n^2)]\, dx_1 \ldots dx_n. \tag{14}$$

Since the set $\tilde{Y}$ consists of those points $x = (x_1, \ldots, x_n)$ for which $(x_1, \ldots, x_m) \in Y$, (14) follows at once from the relation

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-\tfrac{1}{2}x^2)\, dx = 1.$$

Thus, Lemma 1 is also proven in the case where $m \neq n$. Applying Lemma 1 to a mapping of the form

$$(\varphi_1, \ldots, \varphi_n, \varphi) \to (\varphi_1, \ldots, \varphi_n)$$

we conclude that the probability distributions of the random variables $(\Phi(\varphi_1), \ldots, \Phi(\varphi_n), \Phi(\varphi))$ and $(\Phi(\varphi_1), \ldots, \Phi(\varphi_n))$ are compatible if the functions $\varphi_1, \ldots, \varphi_n, \varphi$ are linearly independent. If they are not linearly independent, the compatibility follows easily from the definition of the probability distribution for linearly dependent functions.

Further, applying Lemma 1 to the mapping

$$(\varphi_1, \varphi_2) \to \alpha\varphi_1 + \beta\varphi_2,$$

we observe that the random functional $\Phi(\varphi)$ is linear.

Finally, the continuity of the dependence of the random variable $(\Phi(\varphi_1), \ldots, \Phi(\varphi_n))$ upon $\varphi_1, \ldots, \varphi_n$ follows from the fact that the bilinear functional $B(\varphi, \psi)$ is continuous in φ and ψ.

From the results of Section 2.2 it follows that the correlation functional of the process which we have constructed is equal to $B(\varphi, \psi)$.

This completes the discussion of the case in which $B(\varphi, \psi)$ is strongly positive. If $B(\varphi, \psi)$ is degenerate on some subspace in K, then the proof is carried out in a similar manner. Here it may turn out that the probability distribution of the random variable $(\Phi(\varphi_1), \ldots, \Phi(\varphi_n))$ is concentrated on some subspace of R_n even in the case where the functions $\varphi_1, \ldots, \varphi_n$ are linearly independent.

We have therefore proven the following theorem.

Theorem 2. In order that a continuous linear functional $m(\varphi)$ on K and a continuous bilinear functional $B(\varphi, \psi)$ on K be respectively the mean and the correlation functional of a generalized random process Φ, it is necessary and sufficient that the bilinear functional

$$C(\varphi, \psi) = B(\varphi, \psi) - m(\varphi)m(\psi)$$

be positive-definite, in which case the process Φ can be chosen to be Gaussian.

From this theorem follows:

Corollary. Let Φ be any generalized random process with mean $m(\varphi)$ and correlation functional $B(\varphi, \psi)$. Then there exists a Gaussian generalized random process having the same mean and correlation functional as Φ.

Indeed, if $m(\varphi)$ is the mean, and $B(\varphi, \psi)$ is the correlation functional of Φ, then the bilinear functional $B(\varphi, \psi) - m(\varphi)m(\psi)$ is positive-definite. Then in view of Theorem 2 there exists a Gaussian generalized random process of which $m(\varphi)$ is the mean and $B(\varphi, \psi)$, the correlation functional.

2.4. Derivatives of Generalized Gaussian Processes

We now prove that the derivative of a generalized Gaussian random process is itself a Gaussian random process. Indeed, suppose that the probability distribution of the random variable $(\Phi(\varphi_1), \ldots, \Phi(\varphi_n))$ is given by

$$\mathbf{P}_n(X) = \frac{(\det A)^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}n}} \int_X e^{-\frac{1}{2}(Ax,x)}\, dx,$$

where A is the inverse of the matrix $B = \| B(\varphi_i, \varphi_j) \|$. By definition of the derivative of a random process, the random variable $(\Phi'(\varphi_1), \ldots, \Phi'(\varphi_n))$ has the same probability distribution as the random variable $(-\Phi(\varphi_1'), \ldots, -\Phi(\varphi_n'))$. In other words, the probability $\mathbf{P}_n'(X)$ of the event $(\Phi'(\varphi_1), \ldots, \Phi'(\varphi_n)) \in X$ is given by

$$\mathbf{P}_n'(X) = \frac{(\det A')^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}n}} \int_{-X} \exp[-\tfrac{1}{2}(A'x, x)]\, dx, \tag{15}$$

where A' is the inverse of the matrix $B' = \| B(\varphi_i', \varphi_j') \|$, and $-X$ is the reflection of X through the origin of coordinates.

As $(A'x, x) = (-A'x, -x)$, we can replace $-X$ by X in (15). We find that $(\Phi'(\varphi_1), \ldots, \Phi'(\varphi_n))$ is a Gaussian random variable whose matrix of second moments is $\| B(\varphi_i', \varphi_j') \|$.

We have thus proven that *the derivative of a Gaussian random process with correlation functional $B(\varphi, \psi)$ is a Gaussian random process with correlation functional $B(\varphi', \psi')$.*

2.5. Examples of Gaussian Generalized Random Processes

A Gaussian random process for which the correlation functional $B(\varphi, \psi)$ is given by

$$B(\varphi, \psi) = \int B(t, s)\varphi(t)\psi(s)\, dt\, ds, \tag{16}$$

where $B(t, s)$ is a positive-definite continuous kernel,[4] is called a continuous (or classical) Gaussian random process. For such processes a probability distribution exists for any moments of time $t_1, \ldots, t_n$. This distribution is given by

$$\mathbf{P}_n(X) = \frac{\sqrt{\det \Lambda}}{(2\pi)^{\frac{1}{2}n}} \int_X e^{-\frac{1}{2}(\Lambda x, x)}\, dx, \tag{17}$$

where Λ is the inverse of the matrix $B = \| B(t_i, t_j) \|$. Conversely, if Φ is a continuous random process and if for any moments of time $t_1, \ldots, t_n$ the probability distribution is given by (17), where $\Lambda = \| B(t_i, t_j) \|^{-1}$, then the correlation functional of Φ is given by (16). We omit the proofs of these statements.

As an example of a continuous Gaussian process we consider the Wiener process, namely, that process $\Phi(t)$ for which the probability of the event $(\Phi(t_1), \ldots, \Phi(t_n)) \in X$, $0 < t_1 < \ldots < t_n$, is given by[5]

$$\mathbf{P}_n(X) = [(2\pi)^n t_1(t_2 - t_1) \ldots (t_n - t_{n-1})]^{-\frac{1}{2}}$$

$$\times \int_X \exp\left\{-\frac{1}{2}\left[\frac{x_1^2}{t_1^2} + \ldots + \frac{(x_n - x_{n-1})^2}{t_n - t_{n-1}}\right]\right\} dx_1 \ldots dx_n.$$

It can be shown that the Wiener process describes the coordinate of a particle lacking inertia, undergoing one-dimensional Brownian motion, and starting from the point $x = 0$ at $t = 0$.

Let us compute the correlation matrix for the Wiener process. Suppose $t < s$. Then we have

$$(\Lambda x, x) = \frac{x_1^2}{t} + \frac{(x_2 - x_1)^2}{s - t}$$

and therefore the matrix Λ has the form

$$\Lambda = \left\| \begin{matrix} \dfrac{s}{(s-t)t} & -\dfrac{1}{s-t} \\ -\dfrac{1}{s-t} & \dfrac{1}{s-t} \end{matrix} \right\| = \frac{1}{s-t} \left\| \begin{matrix} \dfrac{s}{t} & -1 \\ -1 & 1 \end{matrix} \right\|$$

[4] That is, a kernel such that

$$\int B(t, s)\varphi(t)\varphi(s)\, dt\, ds \geqslant 0$$

for any $\varphi(t) \in K$.

[5] To times $t < 0$ there corresponds the probability distribution which is concentrated at the point $x = 0$.

But this means that

$$B = A^{-1} = \begin{Vmatrix} t & t \\ t & s \end{Vmatrix}.$$

Since the matrix B has the form

$$B = \begin{Vmatrix} B(t, t) & B(t, s) \\ B(s, t) & B(s, s) \end{Vmatrix},$$

we find that for $t < s$

$$B(t, t) = t; \quad B(t, s) = t; \quad B(s, t) = t; \quad B(s, s) = s.$$

These relations can be written as one formula, namely

$$B(t, s) = \min(t, s),$$

where $t, s > 0$. If $t < 0$ or $s < 0$, then $B(t, s) = 0$.

Let us find the form of the correlation functional of the Wiener process. By formula (16) we have

$$\begin{aligned} B(\varphi, \psi) &= \int_0^\infty \int_0^\infty \varphi(t)\psi(s) \min(t, s)\, dt\, ds \\ &= \int_0^\infty \varphi(t) \int_0^t s\psi(s)\, ds\, dt + \int_0^\infty \psi(s) \int_0^s t\varphi(t)\, dt\, ds. \end{aligned} \tag{18}$$

Set

$$\hat{\varphi}(t) = \int_0^t \varphi(t)\, dt, \qquad \hat{\psi}(t) = \int_0^t \psi(t)\, dt.$$

Integrating the right side of (18) by parts, we have

$$B(\varphi, \psi) = \int_0^\infty [\hat{\varphi}(\infty) - \hat{\varphi}(t)]t\psi(t)\, dt + \int_0^\infty [\hat{\psi}(\infty) - \hat{\psi}(s)]s\varphi(s)\, ds.$$

Integrating the first term by parts once again and taking into account that

$$\int_0^\infty s\varphi(s)\, ds = \int_0^\infty [\hat{\varphi}(\infty) - \hat{\varphi}(s)]\, ds,$$

we find after simple manipulations that

$$B(\varphi, \psi) = \int_0^\infty [\hat{\varphi}(t) - \hat{\varphi}(\infty)][\hat{\psi}(t) - \hat{\psi}(\infty)]\, dt. \tag{19}$$

Thus, we have found the correlation functional of the Wiener process.

Let us now consider the derivative of the Wiener process, i.e., the probability distribution of the velocity of a Brownian particle. It can be shown that this derivative is not a continuous random process. However, the derivative of the Wiener process exists as a generalized random process. The correlation functional of the derivative of the Wiener process is given by $B'(\varphi, \psi) = B(\varphi', \psi')$, where $B(\varphi, \psi)$ is defined by (19). Since

$$[\varphi'(t)]^\wedge = \int_0^t \varphi'(s)\, ds = \varphi(s) - \varphi(0), \qquad [\psi'(t)]^\wedge = \psi(s) - \psi(0),$$

and, as $\varphi(s)$ and $\psi(s)$ have bounded supports, $\varphi(\infty) = \psi(\infty) = 0$, then

$$B'(\varphi, \psi) = \int_0^\infty \varphi(t)\psi(t)\, dt.$$

This formula can be written in the form

$$B'(\varphi, \psi) = \int_0^\infty \int_0^\infty \delta(t - s)\varphi(t)\psi(s)\, ds.$$

Thus, the correlation functional of the derivative of the Wiener process is the generalized function $B(t, s) = \delta(t - s)$. The derivative of the Wiener process is the simplest generalized process of Gaussian type. It plays a role analogous to that of the δ-function in the theory of generalized functions, and is called the *unit generalized random process.*

The case of the complex Wiener process can be treated in similar fashion; here the correlation functional has the form

$$B'(\varphi, \psi) = \int_0^\infty \varphi(t)\overline{\psi(t)}\, dt.$$

2.6. The Characteristic Functional of a Generalized Random Process

We now introduce the notion of the characteristic functional of a generalized random process, which generalizes the notion of the characteristic function of a probability distribution. Let Φ be a generalized random process. The mean of the random variable $e^{i\Phi(\varphi)}$ is called the *characteristic functional* of Φ. In other words, the characteristic functional $L(\varphi)$ is defined by

$$L(\varphi) = \mathbf{E}[e^{i\Phi(\varphi)}] = \int e^{ix}\, d\mathbf{P}(x), \tag{20}$$

where $\mathbf{P}(x)$ denotes the probability that $\Phi(\varphi) < x$.

As an example, let us compute the characteristic functional of a Gaussian random process. Let $B(\varphi, \psi)$ be the correlation functional of the Gaussian random process. According to Section 2.2, the distribution function of $\Phi(\varphi)$ is given by

$$\mathbf{P}(x) = \frac{1}{\sqrt{2\pi B(\varphi, \varphi)}} \int_{-\infty}^{x} \exp\left(-\frac{x^2}{2B(\varphi, \varphi)}\right) dx.$$

Therefore

$$L(\varphi) = \frac{1}{\sqrt{2\pi B(\varphi, \varphi)}} \int \exp\left(ix - \frac{x^2}{2B(\varphi, \varphi)}\right) dx = \exp[-\tfrac{1}{2}B(\varphi, \varphi)].$$

Thus, the characteristic functional of a Gaussian generalized random process having correlation functional $B(\varphi, \psi)$ is given by

$$L(\varphi) = \exp[-\tfrac{1}{2}B(\varphi, \varphi)]. \tag{21}$$

In particular, for the unit Gaussian process we have $B(\varphi, \varphi) = \int \varphi^2(t)\, dt$, and therefore

$$L(\varphi) = \exp\left[-\frac{1}{2}\int \varphi^2(t)\, dt\right]. \tag{22}$$

The properties of characteristic functionals are similar to those of characteristic functions. Namely, we have the following assertion:

The characteristic functional $L(\varphi)$ of a generalized random process Φ is positive-definite, i.e., for any functions $\varphi_1(t), \ldots, \varphi_n(t)$ in K and any complex numbers $\alpha_1, \ldots, \alpha_n$ one has

$$\sum_{j=1}^{n}\sum_{k=1}^{n} L(\varphi_j - \varphi_k)\alpha_j\overline{\alpha_k} \geqslant 0,$$

In fact, we have

$$\sum_{j=1}^{n}\sum_{k=1}^{n} L(\varphi_j - \varphi_k)\alpha_j\bar{\alpha}_k = \sum_{j=1}^{n}\sum_{k=1}^{n} \mathbf{E}[\alpha_j\bar{\alpha}_k \exp[i\Phi(\varphi_j - \varphi_k)]]$$

$$= \mathbf{E}\left|\sum_{j=1}^{n} \alpha_j \exp[i\Phi(\varphi_j)]\right|^2 \geqslant 0.$$

Further, *the characteristic functional $L(\varphi)$ is continuous*. Indeed, if $\lim_{k\to\infty} \varphi_k(t) = \varphi(t)$, then

$$\lim_{k\to\infty} \int f(x)\, d\mathbf{P}_k(x) = \int f(x)\, d\mathbf{P}(x)$$

for any continuous bounded function $f(x)$, where $\mathbf{P}_k(x)$ is the distribution function of the random variable $\Phi(\varphi_k)$, and $\mathbf{P}(x)$ is the distribution function of $\Phi(\varphi)$. Setting $f(x) = e^{ix}$, we obtain

$$\lim_{k\to\infty} L(\varphi_k) = L(\varphi).$$

This proves the continuity of $L(\varphi)$. Lastly, one has

$$L(0) = \int dP(x) = 1.$$

These properties are not only necessary, but also sufficient for a functional $L(\varphi)$ to be the characteristic functional of some generalized random process Φ. In other words, the following theorem holds.

Theorem 3. Let $L(\varphi)$ be a positive-definite continuous functional on the space K such that $L(0) = 1$. Then there exists a generalized random process Φ whose characteristic functional is $L(\varphi)$.

A detailed proof of this theorem will be carried out in Chapter IV, Section 4, in connection with Fourier transforms of measures in linear topological spaces. Here we will indicate only the idea of the proof.

Let $\varphi_1(t), \ldots, \varphi_n(t)$ be functions in K. We associate with them a function $\psi(y_1, \ldots, y_n)$ of n variables, setting

$$\psi(y_1, \ldots, y_n) = L\left[\sum_{k=1}^{n} y_k \varphi_k(t)\right].$$

Simple verification shows that this function is continuous and positive-definite. Consequently, by Bochner's theorem $\psi(y_1, \ldots, y_n)$ is the Fourier transform of some positive measure $\mathbf{P}(x_1, \ldots, x_n)$ in n-dimensional space R_n. We take $\mathbf{P}(x_1, \ldots, x_n)$ as the distribution function of the random variable $(\Phi(\varphi_1), \ldots, \Phi(\varphi_n))$. It can be proven that these random variables are compatible and that $\Phi(\varphi)$ is linear and continuous in φ. Thus a generalized random process Φ has been constructed, of which $L(\varphi)$ is the characteristic functional.

3. Stationary Generalized Random Processes. Generalized Random Processes with Stationary nth-Order Increments

3.1. Stationary Processes

A generalized random process Φ is called *stationary*, if for any functions $\varphi_1(t), \ldots, \varphi_n(t)$ in K and any number h the random variables

$(\Phi[\varphi_1(t+h)], \ldots, \Phi[\varphi_n(t+h)])$ and $(\Phi[\varphi_1(t)], \ldots, \Phi[\varphi_n(t)])$ are identically distributed. In other words, the process Φ is stationary if the result of measurements on it by apparatuses characterized by the functions $\varphi_1(t), \ldots, \varphi_n(t)$ is not changed by the simultaneous translation of all the measurements by the same time interval h.

If Φ is stationary, then its mean is invariant under translation. Thus, for any function $\varphi(t) \in K$ and any number h,

$$m[\varphi(t)] = m[\varphi(t+h)].$$

But the only linear functionals on K which are invariant under translation are those of the form

$$m(\varphi) = a \int \varphi(t)\, dt, \tag{1}$$

where a is some number.[1] Therefore the mean of a stationary generalized random process has the form (1). Since, for stationary random processes, the mean $m(\varphi)$ is uniquely defined by the number a, we will also call a the mean of the stationary process Φ.

3.2. The Correlation Functional of a Stationary Process

Let us now find the general form of the correlation functional of a (complex) stationary generalized random process. From the stationarity of the process Φ it follows that

$$B[\varphi(t), \psi(t)] = B[\varphi(t+h), \psi(t+h)] \tag{2}$$

for any two functions $\varphi(t), \psi(t) \in K$. Thus, the correlation functional of a stationary generalized random process is a positive-definite bilinear Hermitean functional which is translation-invariant. The general form

[1] Indeed, by the results of Volume II (Chapter II, Section 4.3), the functional m has the form

$$m(\varphi) = \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} f_k(t) \varphi^{(k)}(t)\, dt,$$

where the $f_k(t)$ are continuous functions, only a finite number of which are different from zero on any given finite interval. But it follows from the condition of stationarity that the $f_k(t)$ are constants. For $k \geqslant 1$ we have

$$\int_{-\infty}^{\infty} \varphi^{(k)}(t)\, dt = \varphi^{(k-1)}(t) \Big|_{-\infty}^{\infty} = 0$$

and therefore $m(\varphi)$ has the form $m(\varphi) = a \int \varphi(t)\, dt$.

of such functionals was found in Chapter II, Section 3.5. In view of the results proven there, such functionals have the form

$$B(\varphi, \psi) = (B_0, \varphi * \psi^*), \tag{3}$$

where B_0 is a generalized function of one variable which is the Fourier transform of some positive tempered measure.

Thus, the following theorem holds.

Theorem 1. The correlation functional $B(\varphi, \psi)$ of a stationary generalized random process Φ has the form $B(\varphi, \psi) = (B_0, \varphi * \psi^*)$, where B_0 is the Fourier transform of some positive tempered measure σ.

Since the Fourier transform of the function $\varphi * \psi^*(x)$ is $\tilde{\varphi}(\lambda)\bar{\tilde{\psi}}(\lambda)$, where $\tilde{\varphi}(\lambda)$ and $\tilde{\psi}(\lambda)$ are the Fourier transforms of the functions $\varphi(x)$ and $\psi(x)$,[2] then Theorem 1 can also be formulated in the following way.

Theorem 1′. The correlation functional $B(\varphi, \psi)$ of a stationary generalized random process Φ has the form

$$B(\varphi, \psi) = \int \tilde{\varphi}(\lambda)\bar{\tilde{\psi}}(\lambda)\, d\sigma(\lambda), \tag{4}$$

where σ is some positive tempered measure.

The measure σ is called the *spectral measure* of the process Φ. We remark that the spectral measure σ, as well as the mean a of a stationary generalized random process Φ, is uniquely defined by the process.

As an example, let us consider the unit random process, i.e., the derivative of the Wiener process. As was shown in Section 2.5, the correlation functional of this process is given by

$$\begin{aligned} B(\varphi, \psi) &= \int_{-\infty}^{\infty} \varphi(t)\overline{\psi(t)}\, dt \\ &= \int_{-\infty}^{\infty} \delta(t) \left[\int_{-\infty}^{\infty} \varphi(s)\overline{\psi(s-t)}\, ds\right] dt = (\delta, \varphi * \psi^*). \end{aligned} \tag{5}$$

Therefore $B_0(t) = \delta(t)$. But the function $\delta(t)$ is the Fourier transform of Lebesgue measure. Consequently, the spectral measure of the unit generalized random process (the derivative of the Wiener process) is Lebesgue measure, i.e., $d\sigma(\lambda) = d\lambda$.

[2] Recall that $\bar{\varphi}(\lambda) = \overline{\varphi(\bar{\lambda})}$.

3.3. Processes with Stationary Increments

We proceed now to the study of generalized random processes with stationary nth-order increments.

A generalized random process Φ is called a *process with stationary* nth-*order increments*, if its nth derivative is a stationary generalized random process. Thus, for a generalized random process with stationary nth-order increments, the (k-dimensional) random variables

$$(\Phi^{(n)}[\varphi_1(t+h)], \ldots, \Phi^{(n)}[\varphi_k(t+h)])$$

and

$$(\Phi^{(n)}[\varphi_1(t)], \ldots, \Phi^{(n)}[\varphi_k(t)])$$

are identically distributed.[3] In view of the definition of the derivative of a generalized random process, this means that the random variables

$$(\Phi[\varphi_1^{(n)}(t+h)], \ldots, \Phi[\varphi_k^{(n)}(t+h)])$$

and

$$(\Phi[\varphi_1^{(n)}(t)], \ldots, \Phi[\varphi_k^{(n)}(t)])$$

are identically distributed.

Let us now find the general form of the mean $m(\varphi)$ of a generalized random process Φ with stationary nth-order increments. Denote by $m_n(\varphi)$ the mean of the process $\Phi^{(n)}(\varphi) = (-1)^n\Phi(\varphi^{(n)})$. From the relation

$$m_n(\varphi) = \mathbf{E}[\Phi^{(n)}(\varphi)] = (-1)^n\mathbf{E}\Phi[\varphi^{(n)}] = (-1)^n m(\varphi^{(n)}) = m^{(n)}(\varphi)$$

it follows that the generalized function $m_n(\varphi)$ is the nth derivative of the generalized function $m(\varphi)$.

Since the process $\Phi^{(n)}$ is stationary, according to Section 3.1 its mean $m_n(\varphi)$ has the form

$$m_n(\varphi) = a\int \varphi(t)\,dt,$$

where a is some constant. Thus, the mean $m(\varphi)$ of a generalized random process Φ with stationary nth-order increments satisfies the differential equation

$$m^{(n)} = a,$$

[3] It can be shown that this definition is equivalent to the following: for any h the process $\Delta_h^n\Phi$ is stationary. Here $\Delta_h\Phi$ denotes the process defined by

$$\Delta_h\Phi(\varphi) = \Phi[\varphi(t+h) - \varphi(t)],$$

and $\Delta_h^n\Phi$, the process $\Delta_h[\Delta_h^{n-1}\Phi]$. This definition also justifies the name "process with stationary nth-order increments."

where a is some constant. From this it follows that the generalized function $m(\varphi)$ has the form

$$m(\varphi) = \left(\sum_{k=0}^{n} a_k t^k, \varphi \right) = \sum_{k=0}^{n} a_k \int_{-\infty}^{\infty} t^k \varphi(t)\, dt = \sum_{k=0}^{n} a_k (t^k, \varphi). \tag{6}$$

Let us now consider the correlation functional $B(\varphi, \psi)$ of a generalized random process Φ with stationary nth-order increments.

This correlation functional is related to the correlation functional $B_n(\varphi, \psi)$ of the process $\Phi^{(n)}$ according to

$$B(\varphi^{(n)}, \psi^{(n)}) = B_n(\varphi, \psi). \tag{7}$$

Indeed,

$$\begin{aligned} B(\varphi^{(n)}, \psi^{(n)}) &= \mathbf{E}[\Phi(\varphi^{(n)})\overline{\Phi(\psi^{(n)})}] \\ &= \mathbf{E}[\Phi^{(n)}(\varphi)\overline{\Phi^{(n)}(\psi)}] = B_n(\varphi, \psi), \end{aligned}$$

which proves (7).

Since the process $\Phi^{(n)}$ is by definition a stationary generalized random process, its correlation functional $B_n(\varphi, \psi)$ is translation-invariant. We clarified the structure of Hermitean functionals $B(\varphi, \psi)$, for which $B(\varphi^{(n)}, \psi^{(n)})$ is a positive-definite translation-invariant Hermitean functional, in Chapter II (Section 4.5). Applying the result obtained there, we conclude the validity of the following theorem.

Theorem 2. The correlation functional $B(\varphi, \psi)$ of a generalized random process with stationary nth-order increments has the form

$$\begin{aligned} B(\varphi, \psi) = \int_{\Omega_0} &\left[\tilde{\varphi}(\lambda) - \alpha(\lambda) \sum_{j=0}^{n-1} \tilde{\varphi}^{(j)}(0) \frac{\lambda^j}{j!}\right] \\ &\times \overline{\left[\tilde{\psi}(\lambda) - \alpha(\lambda) \sum_{j=0}^{n-1} \tilde{\psi}^{(j)}(0) \frac{\lambda^j}{j!}\right]} d\sigma(\lambda) \\ &+ a_{2n} \alpha_n \overline{\beta_n} + \sum_{j=0}^{n-1} \alpha_j \overline{L_j(\psi)} + \sum_{j=0}^{n-1} \overline{\beta_j} L_j(\varphi) + \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} c_{jk} \alpha_j \overline{\beta_k}. \end{aligned} \tag{8}$$

Here $\tilde{\varphi}(\lambda)$ and $\tilde{\psi}(\lambda)$ are the Fourier transforms of $\varphi(x)$ and $\psi(x)$, σ is a positive tempered measure for which

$$\int_{0<\lambda<1} |\lambda|^{2n}\, d\sigma(\lambda) < +\infty,$$

$\alpha(\lambda)$ is an entire analytic function in the space Z such that $\alpha(\lambda) - 1$ has a zero of order n at $\lambda = 0$, α_j and β_j are the moments of the functions $\varphi(t)$ and $\psi(t)$, i.e.,

$$\alpha_j = (t^j, \varphi); \qquad \beta_j = (t^j, \psi),$$

a_{2n} is a positive number, the c_{jk}, $0 \leqslant j, k \leqslant n - 1$, are certain numbers, the L_j are linear functionals on the space K, and Ω_0 is the complement of the point $\lambda = 0$.

The moments α_j and β_j in (8) can be replaced by the expressions $i^{-j}\tilde{\varphi}^{(j)}(0)$ and $i^{-j}\tilde{\psi}^{(j)}(0)$. Indeed, differentiating the equation

$$\tilde{\varphi}(\lambda) = \int \varphi(x) e^{i\lambda x}\, dx$$

j times with respect to λ and setting $\lambda = 0$, we see that $\alpha_j = i^{-j}\tilde{\varphi}^{(j)}(0)$. In the same way one proves that $\beta_j = i^{-j}\tilde{\psi}^{(j)}(0)$.

The correlation functional $B(\varphi, \psi)$ assumes a particularly simple form if all the moments of the functions $\varphi(x)$ and $\psi(x)$ up to order $n - 1$ inclusive equal zero. Namely, the following theorem holds.

Theorem 2′. If $B(\varphi, \psi)$ is the correlation functional of a generalized random process with stationary nth-order increments, and the moments of the functions $\varphi(x)$ and $\psi(x)$ up to order $n - 1$ inclusive equal zero, then

$$B(\varphi, \psi) = \int_{\Omega_0} \tilde{\varphi}(\lambda)\overline{\tilde{\psi}(\lambda)}\, d\sigma(\lambda) + a_{2n}\alpha_n\overline{\beta}_n,$$

where $\tilde{\varphi}(\lambda)$, $\tilde{\psi}(\lambda)$, σ and a_{2n} have the same meaning as in Theorem 2.

Suppose that Φ is a generalized random process with stationary nth-order increments, and $c_0, \ldots, c_{n-1}$ are arbitrary random variables such that $\mathbf{E}(c_j\bar{c}_k)$ exists for all j and k. Then the process Φ_1, defined by

$$\Phi_1(\varphi) = \Phi(\varphi) + \sum_{k=0}^{n-1} c_k\alpha_k, \qquad \alpha_k = \int \varphi(t) t^k\, dt,$$

is also a process with stationary nth-order increments, because $\Phi_1{}^{(n)} = \Phi^{(n)}$ is a stationary process. The correlation functional $B_1(\varphi, \psi)$ of Φ_1 is expressed in terms of the correlation functional $B(\varphi, \psi)$ of Φ according to

$$B_1(\varphi, \psi) = B(\varphi, \psi) + \sum_{k=0}^{n-1} \overline{\beta}_k L_k(\varphi) + \sum_{k=0}^{n-1} \alpha_k\overline{L_k(\psi)} + \sum_{j,k=0}^{n-1} a_{jk}\alpha_j\overline{\beta}_k,$$

where $\alpha_k = \int \varphi(t)t^k\,dt$, $\beta_k = \int \psi(t)t^k\,dt$, $L_j(\varphi)$ denotes the linear functional $\mathbf{E}[\bar{c}_j\Phi(\varphi)]$, and $a_{jk} = \mathbf{E}(c_j\bar{c}_k)$.

Thus, the presence in (8) of the terms $\alpha_j\overline{L_j(\psi)}$, $\bar{\beta}_jL_j(\varphi)$, $a_{jk}\alpha_j\bar{\beta}_k$ is related to the fact that by adding polynomials of order $n - 1$ having random coefficients to a process Φ with stationary nth-order increments, we obtain a process with stationary nth-order increments.

We remark that in setting up the theory developed in this section we made use only of the fact that neither the mean nor the correlation functional of the process Φ or the process $\Phi^{(n)}$, respectively, changes under translation of the functions $\varphi(x)$ and $\psi(x)$. Therefore all of the results which we have proven are applicable to processes which are stationary or have stationary nth-order increments, respectively, in the wide sense. A generalized random process Φ is called stationary in the wide sense, if its mean $m(\varphi)$ and correlation functional $B(\varphi, \psi)$ are translation-invariant. Processes with wide-sense stationary nth-order increments are defined in analogous fashion.

Since a Gaussian process is uniquely defined by a knowledge of its mean $m(\varphi)$ and correlation functional $B(\varphi, \psi)$, for such processes stationarity in the wide sense coincides with stationarity in the ordinary sense, and the stationarity in the wide sense of the nth-order increments coincides with the ordinary stationarity of the nth-order increments.

3.4. The Fourier Transform of a Stationary Generalized Random Process

Formula (4) of Section 3.2 gives an expression for the correlation functional of a stationary generalized random process in the form of the Fourier transform of some positive measure. This representation of the correlation functional suggests the construction of the Fourier transform of a stationary generalized process Φ itself. In order to render this idea more precise, we introduce the concept of a random measure. Suppose that with every† Borel set Δ on the real line there is associated a random variable $Z(\Delta)$. We will say that $Z(\Delta)$ is a *random measure*, if

(1) $Z(\Delta)$ is a completely additive random function of sets, i.e., for any decomposition $\Delta = \bigcup_{n-1}^{\infty} \Delta_n$ of a set Δ into the countable union of nonintersecting sets Δ_n, the equality $Z(\Delta) = \sum_{n=1}^{\infty} Z(\Delta_n)$ holds in the sense of convergence in the mean[4];

† It is usually assumed, and presumably should be here also, that $Z(\Delta)$ is defined only for Borel sets Δ satisfying $\sigma(\Delta) < \infty$, where σ is introduced in (9) below, and that σ is finite on bounded intervals.

[4] A sequence of random variables ξ_n is said to converge in the mean to a random variable ξ, if $\mathbf{E}[|\xi_n - \xi|^2] \to 0$ as $n \to \infty$.

(2) there exists a positive measure σ such that

$$E(Z(\Delta_1)\overline{Z(\Delta_2)}) = \sigma(\Delta_1 \cap \Delta_2) \tag{9}$$

for any sets Δ_1 and Δ_2;

(3) the mean of the random variable $Z(\Delta)$ equals zero for any set Δ.

Note that in view of (9) we have $E(Z(\Delta_1)\overline{Z(\Delta_2)}) = 0$ for any two nonintersecting sets Δ_1 and Δ_2. In other words, the random variables $Z(\Delta_1)$ and $Z(\Delta_2)$ corresponding to nonintersecting sets Δ_1 and Δ_2 are uncorrelated.

It would be more natural to consider measures for which the random variables $Z(\Delta_1)$ and $Z(\Delta_2)$ corresponding to nonintersecting sets Δ_1 and Δ_2 are not only uncorrelated but also independent. If the $Z(\Delta)$ form a Gaussian family of random variables (i.e., the joint distribution of any finite collection of them is Gaussian), then the uncorrelatedness of $Z(\Delta_1)$ and $Z(\Delta_2)$ implies their independence.[5] Therefore it is more natural, in our opinion, to consider random measures only in the case where the $Z(\Delta)$ form a Gaussian family. However, the consideration of random measures, without imposing upon them the requirement that the $Z(\Delta)$ form a Gaussian family, is accepted in probability theory.

[5] Indeed, let ξ and η be real random variables such that $E(\xi) = E(\eta) = 0$, and suppose that the distribution of the two-dimensional random variable $\zeta = (\xi, \eta)$ has the form

$$\mathbf{P}_\zeta(a, b) = \frac{(\det \Lambda)^{\frac{1}{2}}}{2\pi} \int_{-\infty}^{a} \int_{-\infty}^{b} \exp[-\tfrac{1}{2}(\lambda_{11}x^2 + 2\lambda_{12}xy + \lambda_{22}y^2)]\, dx\, dy, \tag{10}$$

where $\mathbf{P}_\zeta(a, b)$ denotes the probability that $\xi < a$, $\eta < b$, and $\Lambda = \begin{Vmatrix} \lambda_{11} & \lambda_{21} \\ \lambda_{12} & \lambda_{22} \end{Vmatrix}$ is the inverse of the matrix of second moments, i.e., of the matrix

$$B = \begin{Vmatrix} \mathbf{E}(\xi^2) & \mathbf{E}(\xi\eta) \\ \mathbf{E}(\xi\eta) & \mathbf{E}(\eta^2) \end{Vmatrix}.$$

If $\mathbf{E}(\xi\eta) = 0$, then $\lambda_{12} = 0$ and (10) assumes the form

$$\mathbf{P}_\zeta(a, b) = \frac{\lambda_{11}^{\frac{1}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{a} \exp(-\tfrac{1}{2}\lambda_{11}x^2)\, dx \frac{\lambda_{22}^{\frac{1}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{b} \exp(-\tfrac{1}{2}\lambda_{22}y^2)\, dy = \mathbf{P}_\xi(a)\mathbf{P}_\eta(b),$$

where $\mathbf{P}_\xi(a)$ is the distribution function of ξ, and $\mathbf{P}_\eta(b)$ is the distribution function of η. Therefore $\mathbf{E}(\xi\eta) = 0$ implies that $\mathbf{P}_\zeta(a, b) = \mathbf{P}_\xi(a)\mathbf{P}_\eta(b)$, i.e., ξ and η are independent.

If ζ has a degenerate Gaussian distribution, then one of ξ, η is a multiple of the other, and so it follows from $E(\xi\eta) = 0$ that one of ξ, η is identically zero, hence independent of the other.

We now introduce the concept of the Fourier transform of a random measure, which is defined as the random process Φ given by[6,†]

$$\Phi = \int e^{i\lambda t}\, dZ(\lambda).$$

This equation means that with a function $\varphi(t) \in K$ is associated the random variable

$$\Phi(\varphi) = \int \varphi(t) e^{i\lambda t}\, dZ(\lambda)\, dt = \int \tilde{\varphi}(\lambda)\, dZ(\lambda). \tag{11}$$

It is not difficult to show that if the measure $\sigma(\Delta) = \mathbf{E}[|Z(\Delta)|^2]$ is tempered, then (11) defines a continuous linear random functional on K, i.e., a generalized random process.

Let us show that this process is stationary in the wide sense. Indeed, since the mean of every one of the random variables $Z(\Delta)$ equals zero, then for any $\varphi(t) \in K$ we have $\mathbf{E}[\Phi(\varphi)] = 0$. Consequently,

$$m[\varphi(t)] = m[\varphi(t+h)] = 0.$$

Further,

$$\begin{aligned} B(\varphi, \psi) &= \mathbf{E}[\Phi(\varphi)\overline{\Phi(\psi)}] \\ &= \mathbf{E}\left[\int \tilde{\varphi}(\lambda)\, dZ(\lambda) \int \overline{\tilde{\psi}(\mu)\, dZ(\mu)}\right] \\ &= \iint \tilde{\varphi}(\lambda)\overline{\tilde{\psi}(\mu)}\mathbf{E}[dZ(\lambda)\, \overline{dZ(\mu)}]. \end{aligned}$$

In view of (9) we can rewrite this formula in the form

$$B(\varphi, \psi) = \int \tilde{\varphi}(\lambda)\overline{\tilde{\psi}(\lambda)}\, d\sigma(\lambda).$$

Under translation of the functions $\varphi(t)$ and $\psi(t)$ by h, their Fourier transforms are multiplied by $e^{-i\lambda h}$, and therefore

$$\begin{aligned} B[\varphi(t+h), \psi(t+h)] &= \int e^{-i\lambda h}\tilde{\varphi}(\lambda)\overline{e^{-i\lambda h}\tilde{\psi}(\lambda)}\, d\sigma(\lambda) \\ &= \int \tilde{\varphi}(\lambda)\overline{\tilde{\psi}(\lambda)}\, d\sigma(\lambda) = B[\varphi(t), \psi(t)]. \end{aligned}$$

[6] The integral of a function $\varphi(\lambda)$ with respect to the measure $Z(\lambda)$ is understood as the limit of the corresponding sums

$$\sum_{k=0}^{n-1} \varphi(\lambda_k)\, \Delta Z(\lambda_k).$$

† For further details, see J. L. Doob, "Stochastic Processes," Chapter IX, Section 2. Wiley, New York, 1953.

This proves that the process Φ, defined by (11), is stationary in the wide sense.

Now we prove that the converse is true.

Theorem 3. Let Φ be a generalized random process which is stationary in the wide sense, such that $\mathbf{E}[|\Phi(\varphi)|^2]$ is finite for all φ, and let σ be the corresponding spectral measure. Then there exists a random measure $Z(\Delta)$ such that

$$\Phi = \int e^{it\lambda}\, dZ(\lambda), \tag{12}$$

and

$$\mathbf{E}[Z(\Delta_1)\overline{Z(\Delta_2)}] = \sigma(\Delta_1 \cap \Delta_2). \tag{13}$$

Proof. We introduce a scalar product in K by means of the positive-definite functional $B(\varphi, \psi)$, the correlation functional of Φ, i.e., we set

$$(\varphi, \psi) = B(\varphi, \psi). \tag{14}$$

We further consider the linear space $\mathfrak{R}$ consisting of all the random variables $\Phi(\varphi)$ (this space is linear in view of the linearity of the random functional Φ), and introduce a scalar product in $\mathfrak{R}$, setting

$$(\Phi(\varphi), \Phi(\psi)) = B(\varphi, \psi) = \mathbf{E}[\Phi(\varphi)\overline{\Phi(\psi)}]. \tag{15}$$

Since $(\Phi(\varphi),\ \Phi(\psi)) = B(\varphi, \psi) = (\varphi, \psi)$, then the mapping $\varphi \to \Phi(\varphi)$ is an isometric mapping of K onto $\mathfrak{R}$. This mapping can be extended onto the Hilbert spaces H and $\mathfrak{h}$ which are obtained by completing K and $\mathfrak{R}$ with respect to the scalar products (14) and (15) (if the bilinear functional $B(\varphi, \psi)$ is degenerate, then it is first necessary to take the factor spaces of K and $\mathfrak{R}$ relative to the subspaces in K and $\mathfrak{R}$ on which $B(\varphi, \psi)$ and the inner product (15), respectively, are degenerate).

We thus obtain an isometry between the spaces H and $\mathfrak{h}$. Since Φ is stationary in the wide sense, for any two functions $\varphi(t), \psi(t) \in K$ one has

$$(\varphi, \psi) = B(\varphi, \psi) = \int \tilde{\varphi}(\lambda)\bar{\tilde{\psi}}(\lambda)\, d\sigma(\lambda),$$

where $\sigma(\Delta)$ is the spectral measure of Φ. Therefore the correspondence $\varphi(t) \to \tilde{\varphi}(\lambda)$, $\varphi \in K$, can be extended to an isometry† between H and the

† Indeed, the Fourier transformation is a homeomorphism of S onto S. Since K is dense in S, the set $\mathfrak{F}K = \{\tilde{\varphi} \mid \varphi \in K\}$ is dense in S in the topology of S. Now since σ is tempered, $S \subset L^2_\sigma$, and convergence in S implies convergence in L^2_σ; hence $\mathfrak{F}K$ is dense in S in the topology of L^2_σ. But K and, consequently, S is dense in L^2_σ. Thus $\mathfrak{F}K$ is dense in L^2_σ.

space L_σ^2 of functions $\tilde{\varphi}(\lambda)$ having square integrable moduli with respect to the measure σ.

In particular, L_σ^2 contains the characteristic functions $\chi_\Delta(\lambda)$ of all bounded Borel sets Δ. Denote by $Z(\Delta)$ the element of the space $\mathfrak{h}$ corresponding to the function $\chi_\Delta(\lambda)$. If $\Delta = \bigcup_{n=1}^\infty \Delta_n$ is a decomposition of Δ into a countable union of nonintersecting sets, then obviously $\chi_\Delta(\lambda) = \sum_{n=1}^\infty \chi_{\Delta_n}(\lambda)$. Since the mapping of L_σ^2 onto $\mathfrak{h}$ is linear, it follows that $Z(\Delta) = \sum_{n=1}^\infty Z(\Delta_n)$ in the sense of mean square convergence. In other words, we have proven that $Z(\Delta)$ is a random measure. In view of the isometry of the mapping of L_σ^2 onto $\mathfrak{h}$ and the equation $\chi_{\Delta_1}(\lambda)\overline{\chi_{\Delta^2}(\lambda)} = \chi_{\Delta_1 \cap \Delta_2}(\lambda)$, we have

$$\mathbf{E}(Z(\Delta_1)\overline{Z(\Delta_2)}) = (Z(\Delta_1), Z(\Delta_2))$$
$$= \int \chi_{\Delta_1}(\lambda)\overline{\chi_{\Delta_2}(\lambda)}\, d\sigma(\lambda) = \int \chi_{\Delta_1 \cap \Delta_2}(\lambda)\, d\sigma(\lambda) = \sigma(\Delta_1 \cap \Delta_2).$$

This proves (13).

We now prove (12), i.e., we show that for any $\varphi(t) \in K$ one has

$$\Phi(\varphi) = \int \tilde{\varphi}(\lambda)\, dZ(\lambda).$$

Indeed, to the random variable $\Phi(\varphi) \in \mathfrak{R}$ corresponds the function $\tilde{\varphi}(\lambda) \in L_\sigma^2$. But $\tilde{\varphi}(\lambda)$ can be approximated by sums of the form

$$\sum_{k=1}^n \tilde{\varphi}(\lambda_k)\chi_{\Delta_k}(\lambda),$$

where λ_k is a point in the set Δ_k, and so the random variable $\Phi(\varphi)$ is the limit of sums of the form

$$\sum_{k=1}^n \tilde{\varphi}(\lambda_k)Z(\Delta_k).$$

But this means that

$$\Phi(\varphi) = \int \tilde{\varphi}(\lambda)\, dZ(\lambda),$$

which proves Theorem 3.

We have therefore obtained a representation of the wide-sense stationary process Φ as the Fourier transform of a random measure $Z(\Delta)$. In line with remarks made earlier, this representation can be considered valuable only for Gaussian random processes—for these processes the random variables $Z(\Delta_1)$ and $Z(\Delta_2)$, corresponding to nonintersecting sets Δ_1 and Δ_2, will be mutually independent.

A similar representation can also be obtained for processes with stationary nth-order increments, but we will not stop to consider this question.

4. Generalized Random Processes with Independent Values at Every Point

4.1. Processes with Independent Values

It is not possible, in the framework of the ordinary theory of random processes, to introduce processes with continuously varying time whose values at distinct times are independent random variables.† Applying the theory of generalized random processes, however, enables one to consider such processes. In doing this we will establish a connection between processes with independent values at every point and infinitely divisible random variables.

We will say that a *generalized random process Φ has independent values at every point*, if the random variables $\Phi(\varphi_1)$ and $\Phi(\varphi_2)$ [1] are mutually independent whenever $\varphi_1(t)\varphi_2(t) = 0$. Physically, this means that the results of measuring the random quantity Φ in nonintersecting time intervals are mutually independent. An example of a process with independent values at every point is the velocity of a particle undergoing Brownian motion.

It is always convenient to carry out the study of processes with independent values at every point with the help of their characteristic functionals. We now give a necessary and sufficient condition for a functional to be the characteristic functional of a process with independent values at every point.

Theorem 1. In order that a continuous functional $L(\varphi) \not\equiv 0$ be the characteristic functional of a generalized random process with independent values at every point, it is necessary and sufficient that it be positive-definite and that for any functions $\varphi_1(t)$ and $\varphi_2(t)$ whose product vanishes, one have

$$L(\varphi_1 + \varphi_2) = L(\varphi_1)L(\varphi_2). \tag{1}$$

† Strictly speaking, this assertion is not true. However, there is little one can do with such processes.

[1] We will consider only real functions $\varphi(t)$ in this section.

Proof. Suppose that the functional $L(\varphi)$ is the characteristic functional of a generalized random process Φ with independent values at every point. As was shown in Section 2.6, $L(\varphi)$ is continuous and positive-definite.

Take two functions $\varphi_1(t)$ and $\varphi_2(t)$ from K. Obviously

$$L(\varphi_1 + \varphi_2) = \mathbf{E}[\exp(i\Phi(\varphi_1 + \varphi_2))] = \mathbf{E}[\exp(i\Phi(\varphi_1)) \exp(i\Phi(\varphi_2))].$$

If φ_1 and φ_2 are such that $\varphi_1(t)\varphi_2(t) = 0$, then the random variables $\Phi(\varphi_1)$ and $\Phi(\varphi_2)$ are mutually independent by the definition of a process with independent values at every point. Then the random variables $\exp(i\Phi(\varphi_1))$ and $\exp(i\Phi(\varphi_2))$ are independent. Since the mean of the product of independent random variables is equal to the product of their means, for $\varphi_1(t)\varphi_2(t) = 0$ one has

$$L(\varphi_1 + \varphi_2) = \mathbf{E}[\exp(i\Phi(\varphi_1)]\mathbf{E}[\exp(i\Phi(\varphi_2))] = L(\varphi_1)L(\varphi_2).$$

This proves the necessity of the condition of the theorem. We proceed to the proof of its sufficiency. Setting $\varphi_2(t) \equiv 0$ in (1) and choosing φ_1 so that $L(\varphi_1) \neq 0$, we obtain $L(0) = 1$. Further, since $L(\varphi)$ is continuous and positive-definite, by Theorem 3 of Section 2 it is the characteristic functional of some generalized random process Φ. We have to show only that the values of Φ are independent at every point, i.e., that the random variables $\Phi(\varphi_1)$ and $\Phi(\varphi_2)$ are independent if $\varphi_1(t)\varphi_2(t) = 0$.

For this we note that $\varphi_1(t)\varphi_2(t) = 0$ implies, by the conditions of the theorem, that

$$L(s\varphi_1 + s\varphi_2) = L(s\varphi_1)L(s\varphi_2) \tag{2}$$

for all values of s. Since the characteristic function of the random variable $\Phi(\varphi)$ equals $L(s\varphi)$,

$$f(s) = f_1(s)f_2(s), \tag{2'}$$

where $f_1(s)$, $f_2(s)$, and $f(s)$ denote the characteristic functions of the random variables $\Phi(\varphi_1)$, $\Phi(\varphi_2)$, and $\Phi(\varphi_1 + \varphi_2) = \Phi(\varphi_1) + \Phi(\varphi_2)$, respectively.

But (2') implies that the random variables $\Phi(\varphi_1)$ and $\Phi(\varphi_2)$ are mutually independent, which completes the proof.

An example of a functional $L(\varphi)$ which satisfies (1) is any functional of the form

$$L(\varphi) = e^{M(\varphi)},$$

$$M(\varphi) = \int f[\varphi(t), \varphi'(t), \ldots, \varphi^{(n)}(t), t]\, dt, \tag{3}$$

where $f(x_0, ..., x_n, t)$ is a continuous function of $n + 2$ variables, such that $f(0, ..., 0, t) = 0$. Indeed, if $\varphi_1(t)\varphi_2(t) = 0$, then the integral which expresses $M(\varphi_1 + \varphi_2)$ is the sum of two integrals, taken respectively over the sets $\mathfrak{M}_1$ and $\mathfrak{M}_2$ on which $\varphi_1(t)$ and $\varphi_2(t)$ are different from zero (the integral over the rest of the line equals zero in view of the condition $f(0, ..., 0, t) = 0$). But on $\mathfrak{M}_1$ the sum $\varphi_1(t) + \varphi_2(t)$ coincides with $\varphi_1(t)$, and on $\mathfrak{M}_2$ it coincides with $\varphi_2(t)$; therefore

$$M(\varphi_1 + \varphi_2) = \int_{\mathfrak{M}_1} f[\varphi_1(t), \varphi_1'(t), ..., \varphi_1^{(n)}(t), t]\, dt$$

$$+ \int_{\mathfrak{M}_2} f[\varphi_2(t), \varphi_2'(t), ..., \varphi_2^{(n)}(t), t]\, dt.$$

Since, by definition of the sets $\mathfrak{M}_1$ and $\mathfrak{M}_2$, the right side does not change if the integrals over $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are replaced by integrals over the entire real line, we obtain[2]

$$M(\varphi_1 + \varphi_2) = M(\varphi_1) + M(\varphi_2)$$

and consequently

$$L(\varphi_1 + \varphi_2) = L(\varphi_1)L(\varphi_2).$$

This proves our assertion. The number n in formula (3) is called the *order* of the functional $L(\varphi)$.

4.2. A Condition for the Positive Definiteness of the Functional $\exp(\int f[\varphi(t)]\, dt)$

Let us now clarify the conditions which are imposed upon the continuous function $f(x_0, ..., x_n, t)$ by the requirement that the functional $L(\varphi)$, defined by formula (3), be positive-definite. First we consider the case where $L(\varphi)$ is given by

$$L(\varphi) = \exp\left(\int f[\varphi(t)]\, dt\right). \tag{4}$$

A necessary and sufficient condition for the positive definiteness of such a functional is given by the following theorem.

Theorem 2. In order that the functional $L(\varphi)$, defined by (4), be positive-definite, it is necessary and sufficient that the function $e^{sf(x)}$ be positive-definite for all positive values of the parameter s.

[2] A functional $M(\varphi)$ such that $M(\varphi_1 + \varphi_2) = M(\varphi_1) + M(\varphi_2)$, if $\varphi_1(t)\varphi_2(t) = 0$, is called *local*. It would be interesting to find tbe general form of local functionals.

Proof. We first prove the necessity of the condition. If $L(\varphi)$ is given by (4), then it can be extended to all piecewise-continuous functions with bounded supports, following which it will still satisfy the condition

$$\sum_{j,k=1}^{m} L(\varphi_j - \varphi_k)\xi_j\bar{\xi}_k \geqslant 0 \tag{5}$$

of positive definiteness. Denote by $\varphi_j(t)$ the function which is equal to a constant x_j in the interval $0 \leqslant t \leqslant s$ and zero elsewhere. For these functions $\varphi_j(t)$, $1 \leqslant j \leqslant m$, (5) becomes

$$\sum_{j,k=1}^{m} \exp(sf(x_j - x_k))\xi_j\bar{\xi}_k \geqslant 0.$$

This proves the positive definiteness of the functions $e^{sf(x)}$.

Now we prove the sufficiency of the condition of the theorem, i.e., that the positive definiteness of the functions $e^{sf(x)}$ for all $s > 0$ implies the positive definiteness of the functional

$$L(\varphi) = \exp\left(\int f[\varphi(t)]\, dt\right).$$

In other words, we must prove that if, for all $s > 0$, the function $e^{sf(x)}$ is positive-definite, then for any functions $\varphi_1(t), \ldots, \varphi_n(t)$ in K the matrix $A = \| a_{ij} \|$, with elements

$$a_{ij} = \exp\left(\int_{-\infty}^{\infty} f[\varphi_i(t) - \varphi_j(t)]\, dt\right),$$

is positive-definite. Denote by $[-b, b]$ an interval outside of which all of the $\varphi_i(t)$, $1 \leqslant i \leqslant m$, vanish. Then the elements of A can be represented in the form

$$a_{ij} = \exp\left(\int_{-b}^{b} f[\varphi_i(t) - \varphi_j(t)]\, dt\right).$$

Since $f(x)$ is a continuous function,

$$\int_{-b}^{b} f[\varphi_i(t) - \varphi_j(t)]\, dt = \lim_{k\to\infty} \frac{b}{k} \sum_{q=-k}^{k-1} \alpha_q,$$

where

$$\alpha_q = f\left[\varphi_i\left(\frac{qb}{k}\right) - \varphi_j\left(\frac{qb}{k}\right)\right].$$

Therefore it suffices to prove the positive definiteness of the matrix A_k with elements

$$a_{ij} = \exp\left(\frac{b}{k}\sum_{q=-k}^{k-1}\alpha_q\right) = \prod_{q=-k}^{k-1}\exp\left(\frac{b}{k}\alpha_q\right).$$

But a matrix whose elements are the products of the corresponding elements of positive-definite matrices is itself positive-definite (Schur's theorem; for proof, see below). Therefore the positive definiteness of the matrix A_k (and thus also of A) follows from the positive definiteness of the matrices A_{kq} with elements

$$a_{ij}^{(kq)} = \exp\left(\frac{b}{k}\alpha_q\right) = \exp\left\{\frac{b}{k}f\left[\varphi_i\left(\frac{qb}{k}\right) - \varphi_j\left(\frac{qb}{k}\right)\right]\right\}.$$

The positive definiteness of the matrix A_{kq} follows from the fact that the function $\exp(b/k)f(t)$ is by hypothesis positive-definite. This proves the theorem.

For completeness, we present a proof of the theorem of Schur referred to.

Theorem. If the Hermitean matrices $\| a_{ij} \|$ and $\| b_{ij} \|$ are positive-definite, then the matrix $\| a_{ij}b_{ij} \|$ is also positive-definite.

To prove this theorem, we need the following lemma.

Lemma 1. Every positive-definite matrix $\| a_{ij} \|$ can be represented as a sum of matrices of the form $\| \alpha_i\bar{\alpha}_j \|$, i.e.,

$$a_{ij} = \sum_{p=1}^{m} \alpha_i^{(p)}\bar{\alpha}_j^{(p)}. \tag{6}$$

Proof. Any positive-definite Hermitean form $\sum_{i,j=1}^{n} a_{ij}x_i\bar{x}_j$ can be reduced to a sum of squares, i.e., can be written in the form

$$\sum_{i,j=1}^{n} a_{ij}x_i\bar{x}_j = \sum_{p=1}^{m} x'_p\bar{x}'_p, \tag{7}$$

where the x'_p are linear combinations of the variables x_j;

$$x'_p = \sum_{i=1}^{n} \alpha_i^{(p)}x_i, \qquad 1 \leqslant p \leqslant m. \tag{8}$$

Substituting into (7) the expression (8) for x'_p and equating the coefficients of $x_i\bar{x}_j$, we obtain (6).

The converse is also true: any matrix whose elements are of the form (6) is positive-definite. Indeed, for any $\xi_1, \ldots, \xi_n$ we have

$$\sum_{i,j=1}^{n} \alpha_i \bar{\alpha}_j \xi_i \bar{\xi}_j = \left| \sum_{i=1}^{n} \alpha_i \xi_i \right|^2 \geqslant 0,$$

and therefore the matrix $\| \alpha_i \bar{\alpha}_j \|$ is positive-definite. But then a sum of matrices of this form is positive-definite.

We now turn to the proof of the theorem. Suppose that $\| a_{ij} \|$ and $\| b_{ij} \|$ are positive-definite. By the lemma, their elements can be represented in the form

$$a_{ij} = \sum_{p=1}^{m} \alpha_i^{(p)} \bar{\alpha}_j^{(p)}$$

and

$$b_{ij} = \sum_{q=1}^{l} \beta_i^{(q)} \bar{\beta}_j^{(q)}.$$

From this it follows that the matrix $\| a_{ij} b_{ij} \|$ is a sum of matrices having elements of the form

$$\alpha_i^{(p)} \beta_i^{(q)} \overline{\alpha_j^{(p)} \beta_j^{(q)}}$$

and is therefore positive-definite.

Theorems 1 and 2 imply the following assertion.

Theorem 3. In order that the functional $L(\varphi)$, defined by

$$L(\varphi) = \exp\left(\int_{-\infty}^{\infty} f[\varphi(t)]\, dt \right),$$

where $f(x)$ is a continuous function such that $f(0) = 0$, be the characteristic functional of some generalized random process with independent values at every point, it is necessary and sufficient that the function $e^{sf(x)}$ be positive-definite for every positive value of s.

From the properties of positive-definite functions it follows, in this case, that

$$| e^{sf(x)} | \leqslant e^{sf(0)} = 1$$

for any x.

4.3. Processes with Independent Values and Conditionally Positive-Definite Functions

For a definitive description of those characteristic functionals, of generalized random processes with independent values at every point, which are defined by a formula of the form (4), it remains to describe those functions $f(x)$ for which $e^{sf(x)}$ is positive-definite for all positive values of s. We note first that if $e^{sf(x)}$ is positive-definite and $s > 0$, then $e^{sf(-x)} = \overline{e^{sf(x)}} = e^{s\overline{f(x)}}$ and therefore $f(-x) = \overline{f(x)}$. From this it follows that the expression

$$\sum_{j,k=1}^{n} f(x_j - x_k)\xi_j\bar{\xi}_k$$

assumes real values for any real numbers $x_1, \ldots, x_n$ and complex numbers $\xi_1, \ldots, \xi_n$.

We now prove the following theorem.

Theorem 4. In order that the function $e^{sf(x)}$ be positive-definite for all positive values of s, it is necessary and sufficient that the inequality

$$\sum_{j=1}^{n}\sum_{k=1}^{n} f(x_j - x_k)\xi_j\bar{\xi}_k \geqslant 0 \tag{9}$$

hold for all real values $x_1, \ldots, x_n$ and any complex values $\xi_1, \ldots, \xi_n$ such that $\Sigma_{k=1}^{n} \xi_k = 0$.

Proof. First we prove that inequality (9), under the condition $\Sigma_{k=1}^{n} \xi_k = 0$, is necessary for the positive definiteness of $e^{sf(x)}$ for all $s \geqslant 0$. If, in fact, $e^{sf(x)}$ is positive-definite for all $s \geqslant 0$, then for any $\xi_1, \ldots, \xi_n$ and any real $x_1, \ldots, x_n$ we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \exp[sf(x_i - x_j)]\xi_i\bar{\xi}_j \geqslant 0.$$

Expanding $e^{sf(x)}$ according to Taylor's formula, we obtain

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n} \exp[sf(x_i - x_j)]\xi_i\bar{\xi}_j &= \left|\sum_{k=1}^{n} \xi_k\right|^2 + s\sum_{i=1}^{n}\sum_{j=1}^{n} f(x_i - x_j)\xi_i\bar{\xi}_j \\ &\quad + \frac{s^2}{2!}\sum_{i=1}^{n}\sum_{j=1}^{n} \exp[s\theta_{ij}f(x_i - x_j)]f^2(x_i - x_j)\xi_i\bar{\xi}_j,\end{aligned} \tag{10}$$

where the θ_{ij} are some numbers lying between 0 and 1.

Let us now assume that $\Sigma_{k=1}^{n} \xi_k = 0$, and inequality (9) does not hold, i.e., that $\Sigma_{i,j=1}^{n} f(x_i - x_j)\ \xi_i\bar{\xi}_j < 0$. Then, choosing s sufficiently small, we find from (10) that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \exp[sf(x_i - x_j)]\xi_i\bar{\xi}_j < 0,$$

contrary to the assumption that $e^{sf(x)}$ is positive-definite.

Let us now show the sufficiency of condition (9). First we prove that if (9) holds, then for any fixed numbers $x_1, \ldots, x_n$ one can find s_0 such that for $0 \leqslant s < s_0$ the matrix with elements

$$a_{ij} = 1 + sf(x_i - x_j)$$

is positive-definite. Suppose that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} f(x_i - x_j)\xi_i\bar{\xi}_j \geqslant 0,$$

when $\Sigma_{i=1}^{n} \xi_i = 0$. Denote by A the smallest value which the Hermitean form

$$\sum_{i=1}^{n}\sum_{j=1}^{n} f(x_i - x_j)\xi_i\bar{\xi}_j \tag{11}$$

assumes on the hyperplane $\Sigma_{i=1}^{n} \xi_i = 1$. This minimum exists, because the form (11) has, by virtue of (9), a minimum on the hyperplane $\Sigma_{k=1}^{n} \xi_k = 0$, and therefore assumes a minimum on any hyperplane parallel to it.

It is easy to see that in this case

$$\left|\sum_{i=1}^{n} \xi_i\right|^2 + \sum_{i,j=1}^{n} f(x_i - x_j)\xi_i\bar{\xi}_j \geqslant (A + 1)\left|\sum_{i=1}^{n} \xi_i\right|^2 \tag{12}$$

for any values $\xi_1, \ldots, \xi_n$. We have to show the positive definiteness of the matrix having elements $1 + sf(x_i - x_j)$ for sufficiently small s, i.e., we have to verify the positive definiteness of the form

$$\left|\sum_{i=1}^{n} \xi_i\right|^2 + s\sum_{i,j=1}^{n} f(x_i - x_j)\xi_i\bar{\xi}_j.$$

But in view of (12) we have

$$\left|\sum_{i=1}^{n} \xi_i\right|^2 + s \sum_{i,j=1}^{n} f(x_i - x_j)\xi_i\bar{\xi}_j$$

$$= (1 - s)\left|\sum_{i=1}^{n} \xi_i\right|^2 + s\left[\left|\sum_{i=1}^{n} \xi_i\right|^2 + \sum_{i,j=1}^{n} f(x_i - x_j)\xi_i\bar{\xi}_j\right]$$

$$\geqslant (1 - s)\left|\sum_{i=1}^{n} \xi_i\right|^2 + s(A + 1)\left|\sum_{i=1}^{n} \xi_i\right|^2 = (1 + sA)\left|\sum_{i=1}^{n} \alpha_i\right|^2.$$

Therefore if $A > 0$, then the inequality

$$\left|\sum_{i=1}^{n} \xi_i\right|^2 + s \sum_{i,j=1}^{n} f(x_i - x_j)\xi_i\bar{\xi}_j \geqslant 0$$

holds for all $s > 0$. If $A < 0$, the inequality holds for $s < -A^{-1}$. By Schur's theorem it follows that for any $s > 0$ the matrix having elements $[1 + (s/n)f(x_i - x_j)]^n$ will be positive-definite for n sufficiently large. Then the limit of this matrix, i.e., the matrix having elements

$$b_{ij} = \exp[sf(x_i - x_j)] = \lim_{n\to\infty}\left[1 + \frac{sf(x_i - x_j)}{n}\right]^n,$$

will also be positive-definite. But this means that the function $e^{sf(x)}$ is positive-definite for all $s > 0$, which proves Theorem 4.

Let us proceed now to find all continuous functions $f(x)$ such that

$$\sum_{i,j=1}^{n} f(x_i - x_j)\xi_i\bar{\xi}_j \geqslant 0 \tag{13}$$

under the condition $\sum_{i=1}^{n} \xi_i = 0$. It is easy to show, passing from sums to integrals, that all such functions satisfy the inequality $(f, \varphi * \varphi^*) \geqslant 0$, if $\int \varphi(x)\, dx = 0$.

But $\int \psi'(x)\, dx = 0$ for all functions $\psi \in K$. Therefore the inequality

$$(f, \psi' * (\psi')^*) \geqslant 0$$

holds for all functions $f(x)$ which satisfy inequality (13) for $\sum_{i=1}^{n} \xi_i = 0$. The converse is also true.

In Chapter II, Section 4.1, we called such functions conditionally positive-definite functions of the first order.

We see, thus, that a function $f(x)$ for which (13) is satisfied whenever $\sum_{i=1}^{n} \xi_i = 0$, is a conditionally positive-definite function of the first order.

The form of such functions was established in Chapter II, Section 4.4. Using this result, we obtain the following theorem.

Theorem 5. In order that the functional $L(\varphi)$, defined by

$$L(\varphi) = \exp\left(\int f[\varphi(t)]\, dt\right)$$

be the characteristic functional of a generalized random process with independent values at every point, it is necessary and sufficient that the continuous function $f(x)$ have the form

$$f(x) = \int_{|\lambda|>0} [e^{i\lambda x} - \alpha(\lambda)(1 + i\lambda x)]\, d\sigma(\lambda) + a_0 + ia_1 x - \frac{a_2 x^2}{2!}. \tag{14}$$

Here σ is a positive measure such that

$$\int_{|\lambda|>1} d\sigma(\lambda) + \int_{0<|\lambda|\leqslant 1} \lambda^2\, d\sigma(\lambda) < \infty,$$

a_2 is a positive number, $\alpha(\lambda)$ is a function in the space Z such that $\alpha(\lambda) - 1$ has a zero of the third order at $\lambda = 0$, and a_0, a_1 are any numbers.

It is necessary here that

$$\int_{|\lambda|>0} [1 - \alpha(\lambda)]\, d\sigma(\lambda) + a_0 = 0,$$

since the relation $f(0) = 0$ must hold.

We remark that among the functions which can be represented by formula (14) are the functions

$$f(x) = c(e^{ihx} - 1).$$

These functions appear when σ is concentrated at the point $\lambda = h$, for suitable choice of the function $\alpha(\lambda)$ and the constants a_0, a_1, and a_2. In this case the characteristic functional $L(\varphi)$ is given by

$$L(\varphi) = \exp\left[c \int [\exp ih\varphi(t) - 1]\, dt\right]. \tag{15}$$

A generalized random process having such a characteristic functional is called a *Poisson process*.

4.4. A Connection between Processes with Independent Values at Every Point and Infinitely Divisible Distribution Laws

Theorem 2 implies a connection between processes with independent values at every point and infinitely divisible random variables. A random variable ξ is called *infinitely divisible*, if for any n it can be represented in the form

$$\xi = \xi_1 + \dots + \xi_n,$$

where $\xi_1, \dots, \xi_n$ are independent and identically distributed random variables. Let $\chi(x)$ be the characteristic function of an infinitely divisible random variable. Since the characteristic function of a sum of independent random variables is the product of their individual characteristic functions, for any positive integer n the function $\chi(x)$ can be represented as

$$\chi(x) = [\chi_n(x)]^n, \tag{16}$$

where $\chi_n(x)$ is the characteristic function of a random variable ξ_n.

The above mentioned connection between processes with independent values at every point and infinitely divisible random variables is established by the following theorem.

Theorem 6. In order that the functional defined by

$$L(\varphi) = \exp\left(\int_{-\infty}^{\infty} f[\varphi(t)]\, dt\right),$$

where $f(x)$ is a continuous function such that $f(0) = 0$, be the characteristic functional of some generalized random process with independent values at every point, it is necessary and sufficient that the function $e^{f(x)}$ be the characteristic function of some infinitely divisible random variable.

Proof. Suppose that $L(\varphi)$ is the characteristic functional of a process with independent values at every point. Then in view of Theorem 2, the function $\exp[n^{-1}f(x)]$ is positive-definite for any n and is therefore, by Bochner's theorem, the characteristic function of some random variable $\xi^{(n)}$(i.e., the Fourier transform of some positive normalized measure). Since

$$\exp[f(x)] = \exp[n^{-1}f(x)]^n,$$

$e^{f(x)}$ is the characteristic function of the random variable

$$\xi = \xi_1 + \dots + \xi_n,$$

where the ξ_i are independent and have the same distribution as the random variable $\xi^{(n)}$. It follows that ξ is an infinitely divisible random variable.

Conversely, suppose that $e^{f(x)}$ is the characteristic function of an infinitely divisible random variable. Then for any n the function $\exp[n^{-1}f(x)]$ is the characteristic function of some random variable and is therefore positive-definite. But then, for any m and n the function $\exp[(m/n)f(x)]$, as a product of positive-definite functions, is positive-definite. Finally, $e^{sf(x)}$, as a limit of positive-definite functions, is positive-definite for any $s > 0$. But then, in view of Theorem 2 the functional $L(\varphi) = \exp(\int f[\varphi(t)]\, dt)$ is the characteristic functional of some random process with independent values at every point.

It follows from this theorem that the characteristic function of an infinitely divisible random variable has the form $e^{f(x)}$, where $f(x)$ is given by an expression of the form (14).

As an example of a process with independent values at every point we can take the derivative of the Wiener process (i.e., the unit process). We saw in Section 2.6 that the characteristic functional of this process has the form

$$L(\varphi) = \exp\left[-\frac{1}{2}\int \varphi^2(t)\, dt\right]. \tag{17}$$

But this functional is a particular case of (4), corresponding to $f(x) = -\frac{1}{2}x^2$.

4.5. Processes Connected with Functionals of the nth order

Up to now, we have considered processes with independent values at every point whose characteristic functionals have the form $L(\varphi) = e^{M(\varphi)}$, where $M(\varphi) = \int f[\varphi(t)]\, dt$. We now consider processes whose characteristic functionals have the more general form $L(\varphi) = e^{M(\varphi)}$, where

$$M(\varphi) = \int f[\varphi, \varphi', \ldots, \varphi^{(n)}]\, dt.$$

With almost no change in the preceding argument, we can state a condition which is *sufficient* for the functional $L(\varphi) = e^{M(\varphi)}$ to be positive definite, in other words, for $L(\varphi)$ to be the characteristic functional of a generalized random process with independent values at every point. This condition is given by the following theorem.

Theorem 7. In order that the functional $L(\varphi) = e^{M(\varphi)}$, where

$$M(\varphi) = \int f[\varphi, \varphi', \ldots, \varphi^{(n)}]\, dt, \tag{18}$$

be positive-definite, it is sufficient that for any $s > 0$ the function exp $[sf(x_0, \ldots, x_n)]$ be a positive-definite function of the variables $x_0, \ldots, x_n$.

The proof of this theorem proceeds in entirely the same way as the proof of the corresponding part of Theorem 2, and we omit it. It is not known whether the above condition is also *necessary* for the positive definiteness of $L(\varphi)$.

The description of those functions $f(x_0, \ldots, x_n)$ for which $\exp[sf(x_0, \ldots, x_n)]$ is positive-definite for $s > 0$ proceeds word for word as in the case of a single variable. Without carrying out its detailed proof, we state the following theorem.

Theorem 8. In order that the function $\exp[sf(x)] \equiv \exp[sf(x_0, \ldots, x_n)]$ be positive-definite for all $s > 0$, it is necessary and sufficient that the inequality $(f, \varphi * \varphi^*) \geqslant 0$ hold for all functions $\varphi(x) = \varphi(x_0, \ldots, x_n) \in K$ such that $\int \varphi(x)\, dx = 0$.

According to Theorem 3 of Chapter II, Section 4.4, a function $f(x)$ having this property has the form

$$f(x) = \int_{|\lambda|>0} [e^{i(\lambda, x)} - \alpha(\lambda)(1 + i(\lambda, x))]\, d\sigma(\lambda) + \sum_{|k|=0}^{2} a_k \frac{(ix)^k}{k!}. \tag{19}$$

Here σ is a positive measure such that the integrals

$$\int_{0<|\lambda|\leqslant 1} |\lambda|^2\, d\sigma(\lambda) \qquad \text{and} \qquad \int_{|\lambda|>1} d\sigma(\lambda)$$

converge; $\alpha(\lambda)$ is a function in the space Z such that $\alpha(\lambda) - 1$ has a zero of the third order at $\lambda = 0$; the a_k, $|k| < 2$, are certain numbers depending upon $f(x)$; and the a_k, $|k| = 2$, are numbers such that the quadratic form $\sum_{|r|=|s|=1} a_{r+s} \xi_r \bar{\xi}_s$ is positive-definite. Of course, here also one must have

$$\int_{|\lambda|>0} [1 - \alpha(\lambda)]\, d\sigma(\lambda) + a_0 = 0.$$

4.6. Processes of Generalized Poisson Type

Suppose that the measure σ in formula (19) is concentrated at the point $\lambda = h$. For appropriate choice of the function $\alpha(\lambda)$ and the numbers a_k, $|k| = 0, 1, 2$, $f(x)$ will have the form

$$f(x) = C(\exp[i(h, x)] - 1) = C(\exp[i(h_0 x_0 + \ldots + h_n x_n)] - 1).$$

In this case the characteristic functional is defined by the formula

$$L(\varphi) = \exp\left[C \int \exp[ih_1\varphi(t) + \dots + ih_n\varphi^{(n)}(t)]\right] dt.$$

Processes having this type of characteristic functional may be considered as generalizations of the Poisson processes considered earlier. These processes have random jumps whose distribution is Poisson, random changes of velocity whose distribution is also Poisson, and so on up to random changes of the nth derivative, which will have a Poisson distribution.

4.7. Correlation Functionals and Moments of Processes with Independent Values at Every Point

Let us now find the general form of the correlation functional of a process Φ with independent values at every point. Without restriction of generality, we may suppose that the mean of this process equals zero.

According to the kernel theorem (Chapter I, Section 1.3, Theorem 5) the correlation functional of any real generalized random process has the form

$$B(\varphi; \psi) = (F, \varphi(x)\psi(y)),$$

where F is a generalized function of two variables. Let us show that if the process Φ has independent values at every point, then F is concentrated on the diagonal $x = y$(i.e., that $(F, \theta(x, y)) = 0$ if the function $\theta(x, y)$ equals zero in some neighborhood of this diagonal).

Indeed, let $\varphi(x)$ and $\psi(x)$ be functions in the space K such that $\varphi(x)\psi(x) = 0$. Then, by the definition of a process with independent values at every point (cf. Section 4.1), the random variables $\Phi(\varphi)$ and $\Phi(\psi)$ are independent. Since the mean of the product of independent random variables is equal to the product of their means, then

$$B(\varphi, \psi) = \mathbf{E}[\Phi(\varphi)\Phi(\psi)] = \mathbf{E}[\Phi(\varphi)]\mathbf{E}[\Phi(\psi)] = m(\varphi)m(\psi) = 0$$

(recall that we are considering processes with mean zero).

Thus we have proven that $B(\varphi, \psi) = 0$ for any two functions $\varphi(x)$ and $\psi(x)$ whose product vanishes in some neighborhood of the diagonal $x = y$. But any function $\theta(x, y)$ in K_2 which vanishes in some neighborhood of the diagonal can be approximated by linear combinations of functions of the form $\varphi(x)\psi(y)$ which vanish in some neighborhood of the diagonal. We have therefore proven that

$$(F, \theta(x, y)) = 0$$

if $\theta(x, y)$ vanishes in some neighborhood of the diagonal $x = y$. But this means that F is concentrated on the diagonal.

The general form of a generalized function on the space $K_2(a)$ was obtained in Volume II (Chapter II, Section 4.3). It follows from this result that the generalized function F has the form

$$(F, \theta) = \int \sum_{j,k} Q_{jk}(x, y) \frac{\partial^{j+k}\theta(x, y)}{\partial x^j \, \partial y^k} \, dx \, dy,$$

where the $Q_{jk}(x, y)$ are continuous functions, only a finite number of which are different from zero on any given bounded set. Since F is concentrated on the diagonal $x = y$, we obtain

$$(F, \theta) = \int \sum_{j,k} R_{jk}(x) \left. \frac{\partial^{j+k}\theta(x, y)}{\partial x^j \, \partial y^k} \right|_{x=y} dx,$$

where we have put $R_{jk}(x) = Q_{jk}(x, x)$.

Hence the following theorem results.

Theorem 9. The correlation functional $B(\varphi, \psi)$ of a process with independent values at every point is given by

$$B(\varphi, \psi) = \int \sum_{j,k} R_{jk}(x) \varphi^{(j)}(x) \psi^{(k)}(x) \, dx, \tag{20}$$

where only a finite number of the functions $R_{jk}(x)$ are different from zero on any given bounded set.

Since the functional $B(\varphi, \psi)$ is positive-definite, then for any function $\varphi(x) \in K$, necessarily

$$\int \sum_{j,k} R_{jk}(x) \varphi^{(j)}(x) \varphi^{(k)}(x) \, dx \geqslant 0. \tag{21}$$

The following general theorem is proven in a completely analogous way.

Theorem 9′. The nth-order moment of a generalized random process with independent values at every point is given by

$$m_n(\varphi_1, \ldots, \varphi_n) = \int \sum_{j_1, \ldots, j_n} R_{j_1, \ldots, j_n}(x) \varphi_1^{(j_1)}(x) \ldots \varphi_n^{(j_n)}(x) \, dx, \tag{22}$$

where the $R_{j_1,\ldots,j_n}(x)$ are continuous functions, only a finite number of which are different from zero on any given bounded set.

4.8. Gaussian Processes with Independent Values at Every Point

The results obtained in the previous paragraph enable us to show the general form of Gaussian processes with independent values at every point. We know that a Gaussian process Φ is completely defined by its correlation functional $B(\varphi, \psi)$ (we suppose here that the mean $m(\varphi)$ of Φ equals zero). The probability distributions for a Gaussian process with correlation functional $B(\varphi, \psi)$ have the form

$$P_n(X) = \frac{|\det A|^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}n}} \int_X e^{-\frac{1}{2}(Ax,x)}\, dx, \tag{23}$$

where A is the inverse of the matrix $\| B(\varphi_k, \varphi_r) \|$. But we already know the general form of the correlation functional for processes with independent values at every point. It follows that any Gaussian process Φ with independent values at every point is defined in the following manner. Consider a bilinear functional

$$B(\varphi, \psi) = \int \sum_{j,k} R_{jk}(x)\varphi^{(j)}(x)\psi^{(k)}(x)\, dx, \tag{24}$$

such that only a finite number of the functions $R_{jk}(x)$ are different from zero on any given finite interval, and such that $B(\varphi, \varphi) \geqslant 0$ for any function $\varphi(x) \in K$. With any functions $\varphi_1(t), \ldots, \varphi_n(t) \in K$ we associate the (n-dimensional) random variable $\Phi(\varphi_1, \ldots, \varphi_n)$ with probability distribution given by (23), where A is the inverse of the matrix $\| B(\varphi_k, \varphi_r) \|$. Then this family of probability distributions defines a Gaussian process with independent values at every point. Conversely, every such process can be obtained in this way.

Thus, to every Gaussian process with independent values at every point there corresponds a uniquely defined bilinear functional (24) with the properties indicated above, and every such functional defines a Gaussian process with independent values at every point.

Example. Let Φ_0 be the unit process (cf. Section 2.5), and T be any differential operator of finite order. Then the process $\Phi = T\Phi_0$ will be a Gaussian process with independent values at every point. In fact, in the real case the correlation functional $B(\varphi, \psi)$ of Φ has the form

$$\begin{aligned} B(\varphi, \psi) &= \mathbf{E}[\Phi(\varphi)\Phi(\psi)] = \mathbf{E}[T\Phi_0(\varphi)T\Phi_0(\psi)] \\ &= \mathbf{E}[\Phi_0(T\varphi)\Phi_0(T\psi)] = B_0(T\varphi, T\psi), \end{aligned}$$

where B_0 is the correlation functional of Φ_0. But the correlation functional of the unit process has the form

$$B_0(\varphi, \psi) = \int \varphi(x)\psi(x)\, dx.$$

Therefore

$$B(\varphi, \psi) = \int T\varphi(x)T\psi(x)\, dx. \tag{25}$$

Since a functional of the form (25) is of the form (24), the process $T\Phi_0$ is a Gaussian process with independent values at every point.

Of course, not every such Gaussian process has the form $T\Phi_0$, since not every positive-definite functional of the form (24) can be represented in the form (25).

5. Generalized Random Fields

5.1. Basic Definitions

Up to now, we have considered generalized random processes, i.e., generalized random functions of one variable. In this section we consider generalized random functions of several variable. In order to distinguish them from functions of one variable, we will call such functions *generalized random fields*.

Thus, we will say that a generalized random field Φ depending upon n variables is defined, if to each collection

$$\{\varphi_1(x), \ldots, \varphi_m(x)\}, \qquad x = (x_1, \ldots, x_n)$$

of functions of n variables which are infinitely differentiable and have bounded supports there corresponds an m-dimensional random variable $(\Phi(\varphi_1), \ldots, \Phi(\varphi_m))$, and the probability distributions of these random variables are mutually compatible and continuous (since these conditions are formulated in the same way as for functions of one variable, we refer the reader to Section 1 for their precise statement).

A substantial portion of the theory of generalized random fields is analogous to the corresponding portion of the theory of generalized random processes. In these cases we will restrict ourselves to the statements only of the corresponding results (for example, in the theory of homogeneous fields, which is similar to the theory of stationary processes). The only essentially new results which appear in the theory of random fields are those regarding the behavior of fields under rotations and reflections of the space R_n on which the functions $\varphi(x)$ are defined. Here we will not restrict ourselves to the consideration of scalar fields, in which one random variable $\Phi(\varphi)$ is associated with each function $\varphi(x) \in K$, but will in fact consider also multidimensional fields, i.e., we will associate with each $\varphi(x) \in K$ a random vector

$$(\Phi_1(\varphi), ..., \Phi_N(\varphi)).$$

Of course, with each collection of functions $\varphi_1, ..., \varphi_m \in K$ there is associated, accordingly, a random matrix with elements $\Phi_i(\varphi_j)$. We will also consider the question of the transformation of these matrices under rotations and reflections of the space R_n.

5.2. Homogeneous Random Fields and Fields with Homogeneous sth-Order Increments

In this paragraph we will formulate definitions and theorems which are analogous to the results obtained in Section 3. The analog of the notion of a stationary generalized random process is that of a *homogeneous generalized random field.*

A generalized random field Φ is called homogeneous, if for any functions $\varphi_1(x), ..., \varphi_m(x)$ in K and any vector $h = (h_1, ..., h_n)$, the m-dimensional random variables

$$(\Phi(\varphi_1(x)), ..., \Phi(\varphi_m(x)))$$

and

$$(\Phi(\varphi_1(x + h), ..., \Phi(\varphi_m(x + h)))$$

have identical distributions.

Just as for stationary processes, one can show that the correlation functional $B(\varphi, \psi)$ of a homogeneous generalized random field has the form

$$B(\varphi, \psi) = \int \tilde{\varphi}(\lambda)\bar{\tilde{\psi}}(\lambda)\, d\sigma(\lambda), \tag{1}$$

where $\tilde{\varphi}(\lambda)$ and $\tilde{\psi}(\lambda)$ are the Fourier transforms of $\varphi(x)$ and $\psi(x)$, and σ is a positive tempered measure, which is called the *spectral measure* of the field.

Further, we say that a generalized random field Φ has *homogeneous* sth-*order increments*, if the following condition is satisfied:

▼ For each $j = (j_1, ..., j_n)$, $|j| = s$, let $\varphi_{j1}, ..., \varphi_{jm_j}$ be any finite set of functions in K, and let $\Phi^{(j)}$ be the corresponding partial derivative[1] of Φ. Then the joint probability distribution of the $N = \sum_{|j|=s} m_j$ random variables $\Phi^{(j)}(\varphi_{ji})$ is invariant† under the simultaneous translation of the φ_{ji} by the same vector $h = (h_1, ..., h_n)$ in R_n. ▲

From this definition it is clear that if Φ is a generalized random field with homogeneous sth-order increments, and D is a linear homogeneous sth-order differential operator with constant coefficients, then the field $D\Phi$ is homogeneous.

The correlation functional $B(\varphi, \psi)$ of a generalized random field with homogeneous sth-order increments has almost the same form as that of a generalized process with stationary sth-order increments. This can be established in a way similar to that used in the case of random processes, by using the results of Chapter II, Section 4. We exhibit the formula for $B(\varphi, \psi)$ in the case where all the moments of the functions $\varphi(x)$ and $\psi(x)$ up to and including order $s - 1$ vanish; in other words, when

$$\alpha_k = \int x^k \varphi(x)\, dx = 0; \qquad \beta_k = \int x^k \psi(x)\, dx = 0$$

for $|k| \leqslant s - 1$. In this case one has

$$B(\varphi, \psi) = \int_{\Omega_0} \tilde{\varphi}(\lambda)\bar{\tilde{\psi}}(\lambda)\, d\sigma(\lambda) + \sum_{|j|=|k|=s} a_{jk}\alpha_j\bar{\beta}_k, \tag{2}$$

where $\tilde{\varphi}(\lambda)$ and $\tilde{\psi}(\lambda)$ are the Fourier transforms of $\varphi(x)$ and $\psi(x)$, σ is a positive tempered measure such that $\int_{0<|\lambda|<1} |\lambda|^{2s}\, d\sigma(\lambda)$ converges, $\Omega_0 = R_n - \{0\}$, α_j and β_k are the moments of $\varphi(x)$ and $\psi(x)$ respectively, and the a_{jk}, $|j| = |k| = s$, are numbers such that the form

$$\sum_{|j|=|k|=s} a_{jk}\xi_j\bar{\xi}_k \tag{3}$$

is positive-definite.

The converse is also true. That is, *any bilinear functional* $B(\varphi, \psi)$ *of the form* (1) *is the correlation functional of some homogeneous generalized random field*. Similarly, *any bilinear functional, defined for functions* $\varphi(x)$ *and* $\psi(x)$ *whose moments up to and including order* $s - 1$ *vanish and having the form* (2), *coincides on such functions with the correlation functional of*

[1] Differentiation is defined for generalized random fields in the same way as for generalized random functions.

† Note that this is a stronger requirement than that each $\Phi^{(j)}$ separately be homogeneous.

some generalized random field with homogeneous sth-*order increments.* Moreover, both of the random fields mentioned may be chosen Gaussian, i.e., such that for any functions $\varphi_1, \ldots, \varphi_m \in K$ the random variable $(\Phi(\varphi_1), \ldots, \Phi(\varphi_m))$ has a Gaussian distribution.

5.3. Isotropic Homogeneous Generalized Random Fields

As we already said in Section 5.1, the most interesting question in the study of generalized random fields is the consideration of their behavior under rotations and reflections of the space R_n on which the functions $\varphi(x)$ are defined. We begin by considering fields which are invariant with respect to these transformations. Such fields are called *isotropic.* Thus, a generalized random field Φ is called isotropic, if for any functions $\varphi_1, \ldots, \varphi_m \in K$ and any rotation or reflection g of the space R_n on which the φ_i are defined, the m-dimensional random variables

$$(\Phi(\varphi_1(x)), \ldots, \Phi(\varphi_m(x)))$$

and

$$(\Phi(\varphi_1(g^{-1}x)), \ldots, \Phi(\varphi_m(g^{-1}x)))$$

are identically distributed. Here $g^{-1}x$ denotes the point into which the point x is transformed by the transformation g^{-1}. For brevity the function $\varphi(g^{-1}x)$ will be denoted by $\varphi_g(x)$.

We will usually consider fields which are simultaneously homogeneous and isotropic. The condition of homogeneity permits us to apply formula (1) to these fields, and the condition of isotropy imposes certain restrictions on the spectral measure σ appearing in (1).

Namely, the following assertion holds.

Lemma 1. If a generalized random field Φ is homogeneous and isotropic, then its spectral measure σ is invariant with respect to rotation and reflection.

Proof. From the isotropy of Φ it follows that

$$B(\varphi, \psi) = B(\varphi_g, \psi_g)$$

for all elements g of the group G of rotations and reflections of the space R_n. It is easy to show that the Fourier transform of the function $\varphi_g(x) = \varphi(g^{-1}x)$ is $\tilde{\varphi}(g^{-1}\lambda)$, where $\tilde{\varphi}(\lambda)$ is the Fourier transform of $\varphi(x)$. From this it follows that the correlation functional $B(\varphi_g, \psi_g)$ is given by

$$B(\varphi_g, \psi_g) = \int \tilde{\varphi}(g^{-1}\lambda)\bar{\tilde{\psi}}(g^{-1}\lambda)\, d\sigma(\lambda)\,,$$

and therefore

$$\int \tilde{\varphi}(\lambda)\bar{\tilde{\psi}}(\lambda)\, d\sigma(\lambda) = \int \tilde{\varphi}(g^{-1}\lambda)\bar{\tilde{\psi}}(g^{-1}\lambda)\, d\sigma(\lambda).$$

Making the substitution $g^{-1}\lambda = \lambda_1$ in the right side of this equation, we obtain

$$\int \tilde{\varphi}(\lambda)\bar{\tilde{\psi}}(\lambda)\, d\sigma(\lambda) = \int \tilde{\varphi}(\lambda)\bar{\tilde{\psi}}(\lambda)\, d\sigma(g\lambda).$$

Since the spectral measure σ is uniquely defined by the field Φ, it follows from this that $\sigma(A) = \sigma(gA)$ for any Borel set A in R_n, i.e., σ is invariant with respect to rotation and reflection.

Lemma 1 makes it possible to simplify the expression for the correlation function, in the case where the field Φ is homogeneous and isotropic, by replacing the integral over n-dimensional space by a double integral. Denote by $\hat{\theta}_0(r)$ the average of the function $\hat{\theta}(\lambda) = \tilde{\varphi}(\lambda)\bar{\tilde{\psi}}(\lambda)$ over the hypersphere $S(r)$ of radius r,[2] and by $\sigma(r)$ the σ-measure of the ball with center at the origin and radius r.

Then

$$B(\varphi, \psi) = \int \hat{\theta}_0(r)\, d\sigma(r), \tag{4}$$

which is easily obtained from (1) by changing to polar coordinates.

But $\hat{\theta}(\lambda)$ is the Fourier transform of the function $\varphi * \psi^*(x) = \theta(x)$. In order to simplify (4), we will express $\hat{\theta}_0(r)$ in terms of the average of $\theta(x)$ over $S(R)$. To do this we use the following lemma.

Lemma 2. Let $\tilde{f}(\lambda)$ be the Fourier transform of $f(x)$. Then the average $\tilde{f}_0(r)$ of $\tilde{f}(\lambda)$ over the hypersphere $S(r)$ can be expressed in terms of the average $f_0(R)$ of $f(x)$ over the hypersphere $S(R)$ according to

$$\tilde{f}_0(r) = \frac{(2\pi)^{p+1}}{r^p} \int_0^\infty R^{p+1} f_0(R) J_p(rR)\, dR, \tag{5}$$

where $J_p(R)$ is the Bessel function of order $p = \frac{1}{2}(n-2)$;

$$J_p(R) = (\tfrac{1}{2}R)^p \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+p+1)} (\tfrac{1}{2}R)^{2n}.$$

[2] In other words, we set

$$\hat{\theta}_0(r) = \int_{S(r)} \tilde{\theta}_0(\lambda)\, d\tau(\lambda),$$

where $\tau(\lambda)$ is the rotationally invariant measure on the sphere $S(r)$, normalized by the condition $\tau[S(r)] = 1$.

This lemma is also very easy to prove by changing to polar coordinates in the Fourier integral. One has only to bear in mind that[3]

$$\int_0^\pi e^{iR\cos\theta} \sin^{2p}\theta\, d\theta = \frac{\Gamma(p+\frac{1}{2})\sqrt{\pi}}{(\frac{1}{2}R)^p} J_p(R).$$

Applying this theorem to the integral in (4), and taking into account that $\tilde{\theta}(\lambda) = \tilde{\varphi}(\lambda)\bar{\tilde{\psi}}(\lambda)$ is the Fourier transform of $\theta(x) = \varphi * \psi^*(x)$, we arrive at the following result.

Theorem 1. The correlation functional of a homogeneous and isotropic field Φ is given by

$$B(\varphi, \psi) = (2\pi)^{p+1} \int_0^\infty \int_0^\infty \theta_0(R) \left(\frac{R}{r}\right)^p J_p(Rr) R\, dR\, d\sigma(r), \tag{6}$$

where σ is a positive tempered measure on the half-line $r \geqslant 0$, $\theta_0(R)$ is the average of $\theta(x) = \varphi * \psi^*(x)$ on the hypersphere with center at the origin and radius R, $p = \frac{1}{2}(n-2)$, and $J_p(R)$ is the Bessel function of order p.

5.4. Generalized Random Fields with Homogeneous and Isotropic sth-Order Increments

Let us introduce fields with homogeneous and isotropic sth-order increments. One says that a field Φ has *homogeneous and isotropic* s*th-order increments*, if the following condition is satisfied:

For each $j = (j_1, \ldots, j_n)$, $|j| = s$, let $\varphi_{j1}, \ldots, \varphi_{jm_j}$ be any finite set of functions in K, and let $\Phi^{(j)}$ be the corresponding partial derivative of Φ. Then the joint probability distribution of the $N = \sum_{|j|=s} m_j$ random variables $\Phi^{(j)}(\varphi_{ji})$ is invariant under the simultaneous translation, or simultaneous rotation or reflection, of all the φ_{ij}.

We will exhibit the form of the correlation functional $B(\varphi, \psi)$ of such a field for functions $\varphi(x)$ and $\psi(x)$, whose moments up to and including order $s-1$ vanish. In this case†

$$B(\varphi, \psi) = B(\varphi_g, \psi_g), \tag{7}$$

[3] See, e.g., I. M. Ryzhik and I. S. Gradshtein, "Tables of Integrals, Sums, Series, and Products" (in Russian) 6, 412 (6), p. 345. Moscow, 1951. German translation: I. M. Ryshik and I. S. Gradstein, "Summen-, Produkt- und Integraltafeln," 6, 412(6), p. 312, Berlin, 1957.

† It follows easily from the definition of a process with homogeneous and isotropic sth-order increments that (7) holds when φ and ψ are sums of the form $\sum_{|j|=s} \varphi_j^{(j)}$ and

where, we recall, $\varphi_g(x) = \varphi(g^{-1}x)$, $\psi_g(x) = \psi(g^{-1}x)$. Therefore the spectral measure σ, appearing in the formula

$$B(\varphi, \psi) = \int_{\Omega_0} \tilde{\varphi}(\lambda)\bar{\tilde{\psi}}(\lambda)\, d\sigma(\lambda) + \sum_{|j|=|k|=s} a_{jk}\alpha_j\overline{\beta_k} \tag{8}$$

for $B(\varphi, \psi)$,[4] is invariant relative to rotation and reflection in R_n, and the bilinear form

$$\sum_{|j|=|k|=s} a_{jk}\alpha_j\overline{\beta_k} \tag{9}$$

satisfies the relation

$$\sum_{|j|=|k|=s} a_{jk}\alpha_j\overline{\beta_k} = \sum_{|j|=|k|=s} a_{jk}\alpha_j^{(g)}\overline{\beta_k^{(g)}}, \tag{10}$$

where $\alpha_k^{(g)}$ and $\beta_k^{(g)}$ denote the moments of the functions $\varphi(g^{-1}x)$ and $\psi(g^{-1}x)$.

Using the invariance of the measure σ relative to rotations and reflections, we prove, just as for isotropic processes, that the first term in formula (8) can be written in a form analogous to formula (6), with the sole difference that the measure $\sigma(r)$ must not only be tempered, but also must be such that the integral $\int_{0<r<1} r^{2s}\, d\sigma(r)$ converges (this last assertion follows from the fact that the integral $\int_{0<|\lambda|<1} |\lambda|^{2s}\, d\sigma(\lambda)$ converges; cf. Section 5.2).

Let us now clarify the restrictions on the Hermitean form (9) which are imposed by relation (10). For this we make use of the fact that α_j and β_k are the moments of $\varphi(x)$ and $\psi(y)$. Therefore the form (9) can be written as

$$\sum_{|j|=|k|=s} a_{jk}\alpha_j\overline{\beta_k} = \int P(x, y)\varphi(x)\overline{\psi(y)}\, dx\, dy,$$

where $P(x, y)$ denotes the polynomial $\sum_{|j|=|k|=s} a_{jk}x^j y^k$. It follows

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$\sum_{|j|=s} \psi_j^{(j)}$, $\varphi_j, \psi_j \in K$. But by the corollary in the appendix to Chapter II, Section 4, any function in K, whose moments up to and including order $s - 1$ equal zero, is the limit in K of a sequence of such sums. Since the definition of the correlation functional included its continuity in each variable separately, and therefore (by Theorem 3 of Chapter I, Section 1.2) its joint continuity in every space $K(a)$, hence in all of K, the validity of (7) extends by continuity to all $\varphi, \psi \in K$ whose moments up to and including order $s - 1$ equal zero.

[4] Since Φ has homogeneous sth-order increments, according to Section 5.2 its correlation functional has the form (8).

from (10) that $P(x, y)$ must be invariant relative to all rotations and reflections in n-dimensional space:

$$P(x, y) = P(gx, gy).$$

But any polynomial which is invariant under rotation and reflection in R_n can be represented in the form of a polynomial in the expressions

$$(x, x) = \sum_{k=1}^{n} x_k^2, \qquad (x, y) = \sum_{k=1}^{n} x_k y_k, \qquad (y, y) = \sum_{k=1}^{n} y_k^2$$

(a proof of this fact is given, for example, in H. Weyl, "The Classical Groups," pp. 31-32. Princeton Univ. Press, Princeton, New Jersey, 1946), namely,

$$P(x, y) = \sum_{i,j,k} b_{ijk}(x, x)^i (x, y)^j (y, y)^k.$$

Since every term of the polynomial $P(x, y)$ is of degree $2s$, then necessarily $i + j + k = s$. Since, moreover, every term of $P(x, y)$ has the same degree s in x and y, then necessarily $i = k$. This means that $P(x, y)$ has the following form:

$$P(x, y) = \sum_{k=0}^{[\frac{1}{2}s]} b_k(x, x)^k (x, y)^{s-2k} (y, y)^k.$$

We have thus proven that the Hermitean form (9) is given by the following formula:

$$\sum_{j+k=s} a_{jk}\alpha_j \overline{\beta_k} = \sum_{k=0}^{[\frac{1}{2}s]} b_k \int (x, x)^k (x, y)^{s-2k} (y, y)^k \varphi(x)\overline{\psi(y)}\, dx\, dy. \tag{11}$$

Let us now show that every coefficient b_k in (11) is nonnegative. Indeed, (9) is positive-definite. Therefore the inequality

$$\sum_{k=0}^{[\frac{1}{2}s]} b_k \int (x, x)^k (x, y)^{s-2k} (y, y)^k \varphi(x)\overline{\varphi(y)}\, dx\, dy \geqslant 0 \tag{12}$$

must hold for any function $\varphi(x) \in K$. Multiplying out the expressions

$$(x, y)^{s-2k} = (x_1 y_1 + \dots + x_n y_n)^{s-2k},$$

we verify without difficulty that every integral occurring in (12) is the product of two integrals of complex conjugate functions, and is therefore positive. By appropriately choosing the function $\varphi(x)$, these integrals can be given any positive values. But from this it follows that inequality (12) can hold only if every coefficient b_k is nonnegative.

Thus, we have proven the following theorem.

Theorem 2. If Φ is a generalized random field with homogeneous and isotropic sth-order increments and if the moments of the functions $\varphi(x)$ and $\psi(x)$ up to and including order $s - 1$ vanish, then the correlation functional $B(\varphi, \psi)$ of Φ has the form

$$B(\varphi, \psi) = (2\pi)^{p+1} \int_0^\infty \int_0^\infty \theta_0(R) \left(\frac{R}{r}\right)^p J_p(Rr) R \, dR \, d\sigma(r) + \sum_{k=0}^{[\frac{1}{2}s]} b_k \int (x, x)^k (x, y)^{s-2k} (y, y)^k \varphi(x) \overline{\psi(y)} \, dx \, dy. \tag{13}$$

Here $\theta_0(R)$ is the average of the function $\theta(x) = \varphi * \psi^*(x)$ over the sphere with center at the origin and radius R, σ is a positive tempered measure on the half-line $0 < r < \infty$, for which the integral $\int_{0<r<1} |r|^{2s} d\sigma(r)$ converges, $J_p(R)$ is the Bessel function of order $p = \frac{1}{2}(n - 2)$, and the b_k are nonnegative numbers.

From this theorem it is easy to obtain the form of $B(\varphi, \psi)$ for any functions $\varphi(x)$ and $\psi(x)$ in K, replacing a function $\varphi(x) \in K$ by a function of the form

$$\varphi(x) - \sum_{|k|=0}^{s-1} \alpha_k \theta_k(x).$$

Here the α_k denote the moments of $\varphi(x)$, and the $\theta_k(x)$ are functions in K such that

$$\int x^j \theta_k(x) \, dx = \delta_{jk}$$

(δ_{jk} is the Kronecker symbol). We omit the exact statement of the result thereby obtained, which is rather cumbersome.

5.5. Multidimensional Generalized Random Fields

In certain applications of the theory of random fields, it is not sufficient to consider only scalar fields. For example, the velocity of particles in a turbulent flow can be considered as a random quantity. However, since velocity is a vector quantity, we obtain a vector rather than a scalar random field. In this section we will give the basic definitions relating to multidimensional generalized random fields.

Let R_{Nm} denote the linear space consisting of matrices with N rows and m columns.

We will say that an *N-dimensional generalized random field Φ* is defined, if with every m functions $\varphi_j \in K$, $1 \leqslant j \leqslant m$, there is associated a probability distribution in the space R_{Nm}, these probability distributions are compatible, and their dependence on the φ_j is continuous. In other words, an N-dimensional generalized random field associates with a vector function $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_m)$ the random matrix $\| \Phi(\boldsymbol{\varphi}) \|$ with elements $\Phi_i(\varphi_j)$;

$$\| \Phi(\boldsymbol{\varphi}) \| = \left\| \begin{matrix} \Phi_1(\varphi_1) & \Phi_1(\varphi_2) \ldots & \Phi_1(\varphi_m) \\ \ldots & \ldots & \ldots \\ \Phi_N(\varphi_1) & \Phi_N(\varphi_2) \ldots & \Phi_N(\varphi_m) \end{matrix} \right\|.$$

In particular, with every function $\varphi \in K$ is associated a random column vector with components $\Phi_1(\varphi), \ldots, \Phi_N(\varphi)$.

We now introduce the notions of the mean and the correlation matrix of a multidimensional generalized random field. Let φ be some function in K, and $\Phi(\varphi)$ the random vector corresponding to φ. We assume that all of the random variables $\Phi_k(\varphi)$ have means whose dependence upon φ is continuous. Then, setting

$$\mathfrak{M}(\varphi) = \left\| \begin{matrix} m_1(\varphi) \\ \vdots \\ m_N(\varphi) \end{matrix} \right\|,$$

where $m_k(\varphi) = \mathbf{E}[\Phi_k(\varphi)]$, we obtain a vector whose coordinates $m_k(\varphi)$ are generalized functions on the space K. We call this vector the *vector mean value* of the field Φ.

In place of the the correlation functional $B(\varphi, \psi)$, we introduce for multidimensional generalized random fields the correlation matrix $\mathfrak{B}(\varphi, \psi)$. Namely, suppose that for any functions $\varphi, \psi \in K$ and any i, j, $0 \leqslant i, j \leqslant N$, the mean value

$$B_{ij}(\varphi, \psi) = \mathbf{E}[\Phi_i(\varphi)\overline{\Phi_j(\psi)}]$$

exists and has continuous dependence upon φ and ψ. We call the matrix consisting of the functionals $B_{ij}(\varphi, \psi)$ the *correlation matrix* of the field Φ, and denote it by $\mathfrak{B}(\varphi, \psi)$.

The correlation matrix $\mathfrak{B}(\varphi, \psi) = \| B_{ij}(\varphi, \psi \|$ of a multidimensional generalized random field Φ has the following property of strong positive definiteness: For any complex numbers α_{ir}, $1 \leqslant i \leqslant N$, $1 \leqslant r \leqslant m$, and any functions $\varphi_1, \ldots, \varphi_m \in K$, one has

$$\sum_{i,j=1}^{N} \sum_{r,s=1}^{m} B_{ij}(\varphi_r, \varphi_s)\alpha_{ir}\overline{\alpha_{js}} \geqslant 0. \tag{14}$$

To prove this inequality, it suffices to note that the left side can be written in the form

$$\mathbf{E}\left[\left|\sum_{i=1}^{N}\sum_{r=1}^{m} \alpha_{ir}\Phi_i(\varphi_r)\right|^2\right]$$

and is therefore, as the mean of a nonnegative random variable, nonnegative.

From the strong positive definiteness of the matrix $\mathfrak{B}(\varphi, \psi)$ it follows that for any numbers $\alpha_1, \ldots, \alpha_N$ the bilinear functional

$$B_\alpha(\varphi, \psi) = \sum_{i,j=1}^{N} \alpha_i\overline{\alpha_j}B_{ij}(\varphi, \psi)$$

is positive-definite. Indeed, if $\varphi_1, \ldots, \varphi_m$ are any functions in K, and $\xi_1, \ldots, \xi_m$ are any complex numbers, then

$$\sum_{r,s=1}^{m} B_\alpha(\varphi_r, \varphi_s)\xi_r\overline{\xi_s} = \sum_{i,j=1}^{N}\sum_{r,s=1}^{m} B_{ij}(\varphi_r, \varphi_s)\alpha_i\xi_r\overline{\alpha_j\xi_s}.$$

But the right side of this equation is nonnegative, which one observes by setting $\alpha_{ir} = \alpha_i\xi_r$ in (14). Therefore

$$\sum_{r,s=1}^{m} B_\alpha(\varphi_r, \varphi_s) \geqslant 0,$$

which proves the positive definiteness of $B_\alpha(\varphi, \psi)$.

We remark that the converse does not hold in general: the positive definiteness of the functional $B_\alpha(\varphi, \psi)$ for all $\alpha_1, \ldots, \alpha_N$ does not imply the strong positive definiteness of the matrix $\mathfrak{B}(\varphi, \psi)$.

We proceed now to the consideration of homogeneous multidimensional fields. A multidimensional generalized random field Φ is called *homogeneous*, if the probability distribution of the random matrix $\| \Phi_i(\varphi_j) \|$ does not change under the simultaneous translation of each of the functions

$\varphi_1(x), \ldots, \varphi_m(x)$ by the same vector $h = (h_1, \ldots, h_n)$. For homogeneous fields the vector mean value has the form

$$\mathfrak{M}(\varphi) = a \int \varphi(x)\, dx,$$

where $a = (a_1, \ldots, a_N)$ is some N-dimensional vector.

The correlation matrix for such fields can be described in the following way: *The correlation matrix $\mathfrak{B}(\varphi, \psi)$ of a homogeneous N-dimensional generalized random field has the form*

$$\mathfrak{B}(\varphi, \psi) = \int \tilde{\varphi}(\lambda)\bar{\tilde{\psi}}(\lambda)\, d\overline{\mathfrak{F}(\lambda)}. \tag{15}$$

Here $\tilde{\varphi}(\lambda)$ and $\tilde{\psi}(\lambda)$ are the Fourier transforms of $\varphi(x)$ and $\psi(x)$, and

$$\mathfrak{F}(X) = \| \mathfrak{F}_{ij}(X)\|, \qquad 1 \leqslant i \leqslant N, \quad 1 \leqslant j \leqslant N,$$

is a matrix consisting of complex tempered measures in the space R_n such that the matrix $\mathfrak{F}(X)$ is positive-definite for all sets X (we will call the matrix $\mathfrak{F}(X)$ the *spectral matrix* of the field Φ).

The proof of this assertion is carried out by considering the homogeneous random field

$$\Psi_\xi = \xi_1\Phi_1 + \ldots + \xi_N\Phi_N, \tag{16}$$

which is linear in the complex parameters $\xi_1, \ldots, \xi_N$.

Let us now turn to the consideration of *multidimensional random fields having homogeneous* sth-*order increments*, i.e., such that the joint probability distribution of any collection of random variables $\Phi_k^{(j)}(\varphi_{kji})$, $\varphi_{kji} \in K$, $|j| = s$, $k = 1, \ldots, N$, $i = 1, \ldots, m_{kj}$, is invariant under the simultaneous translation of the φ_{kji} by the same vector $h = (h_1, \ldots, h_n)$ in R_n.

Here also, a description of the correlation matrix can be obtained by passing to the one-dimensional field

$$\Psi_\xi = \xi_1\Phi_1 + \ldots + \xi_N\Phi_N.$$

The correlation matrix $\mathfrak{B}(\varphi, \psi)$ has a particularly simple form if every moment of the functions $\varphi(x)$ and $\psi(x)$ up to order $s-1$ inclusive equals zero. In this case the elements $B_{ij}(\varphi, \psi)$ of $\mathfrak{B}(\varphi, \psi)$ are of the form

$$B_{ij}(\varphi, \psi) = \int_{\Omega_0} \tilde{\varphi}(\lambda)\bar{\tilde{\psi}}(\lambda)\, dF_{ij}(\lambda) + \sum_{|p|=|q|=s} a_{pq}^{ij}\alpha_p\bar{\beta}_q, \tag{17}$$

where Ω_0 is the set obtained by deleting the point $\lambda = 0$ from the entire space, F_{ij} is a complex tempered measure such that the integral

$$\int_{0<|\lambda|<1} |\lambda|^{2s} \, dF_{ij}(\lambda)$$

converges, and the matrix $\mathfrak{F}(X) = \| F_{ij}(X) \|$ is positive-definite for every set X. The α_p in (17) denote the moments

$$\int x^p \varphi(x) \, dx$$

of $\varphi(x)$, and the β_q, the moments of $\psi(x)$. Finally, the numbers a_{pq}^{ij}, $|p| = |q| = s$, are such that the inequality

$$\sum_{i,j=1}^{N} \sum_{|p|=|q|=s} a_{pq}^{ij} \alpha_{ip} \overline{\alpha_{jq}} \geqslant 0$$

holds for all complex numbers α_{ip}, $1 \leqslant i \leqslant N$, $|p| = s$.

5.6. Isotropic and Vectorial Multidimensional Random Fields

A multidimensional field Φ is called *isotropic*, if for any rotation or reflection g in R_n the random matrix $\Phi_g(\boldsymbol{\varphi})$, defined by

$$\Phi_g(\boldsymbol{\varphi}) = \| \Phi_i[\varphi_j(g^{-1}x)] \|,$$

has the same probability distribution as the random matrix $\Phi(\boldsymbol{\varphi})$.

One can prove the following result: The correlation matrix $\mathfrak{B}(\varphi, \psi)$ of a homogeneous isotropic N-dimensional generalized random field Φ has the form

$$\mathfrak{B}(\varphi, \psi) = (2\pi)^{\nu+1} \int_0^\infty \int_0^\infty \theta_0(R) \left(\frac{R}{r}\right)^\nu J_\nu(Rr) R \, dR \, d\sigma(r),$$

where $\theta_0(R)$ is the average of the function $\theta(x) = \varphi * \psi^*(x)$ over the sphere with center at the origin and radius R, and σ is a matrix whose components are complex tempered measures σ_{ij} defined on the half-line $0 \leqslant r < \infty$, and such that for any set X the matrix $\sigma(X)$ is positive-definite.

We now introduce the concept of a vectorial field. A generalized n-dimensional random field Φ is called *vectorial* if, for any rotation or reflection g in R_n, the random matrices $\| \Phi_i[\varphi_j(g^{-1}x)] \|$ and $\| g \| \cdot \| \Phi_i[\varphi_j(x)] \|$ are identically distributed ($\| g \|$ denotes the matrix of the transformation g).

The following result holds:

Theorem 3. Let Φ be a homogeneous generalized random vectorial field. Then its mean equals zero. The correlation matrix $\mathfrak{B}(\varphi, \psi)$ of Φ consists of elements $B_{ij}(\varphi, \psi)$, $1 \leqslant i, j \leqslant n$, given by

$$B_{ij}(\varphi, \psi) = \int \theta_{ij}(R) J_p(R_r) \left(\frac{R}{r}\right)^p \frac{d\sigma(r)}{r^2} R\, dR, \qquad i \neq j,$$

and

$$B_{ii}(\varphi, \psi) = \int \theta_0(R) J_p(Rr) \left(\frac{R}{r}\right)^p d\sigma_2(r) R\, dR$$

$$+ \int \theta_{ii}(R) J_p(Rr) \left(\frac{R}{r}\right)^p \frac{d\sigma(r)}{r^2} R\, dR.$$

Here $p = \frac{1}{2}(n - 2)$, $\theta_0(R)$ denotes the average of the function $\theta(x) = \varphi * \psi^*(x)$ over the sphere $|x| = R$, $\theta_{ij}(R)$ denotes the average of the function $\dfrac{\partial^2 \theta(x)}{\partial x_i\, \partial x_j}$ over the same sphere, and $\sigma = \sigma_1 - \sigma_2$, where σ_1 and σ_2 are positive tempered measures on the half-line $0 \leqslant r < \infty$ such that $\sigma_1(\{0\}) = \sigma_2(\{0\})$; finally, J_p is the Bessel function of order p.

We omit the proof of this result, which is connected with the representation theory of the group of rotations and reflections in Euclidean space.

CHAPTER IV

MEASURES IN LINEAR TOPOLOGICAL SPACES

1. Basic Definitions

1.1. Cylinder Sets

In this chapter we study measures in linear topological spaces. We will restrict ourselves to considering measures in spaces Φ' which are adjoint to some linear topological space Φ. We will first study measures on the simplest sets in Φ'—the cylinder sets. Following this, measures on sets of a more general form will be considered. Let us define the notion of a cylinder set in the space Φ'. We choose any fixed elements $\varphi_1, ..., \varphi_n$ in Φ. To each element $F \in \Phi'$ there corresponds the point $((F, \varphi_1), ..., (F, \varphi_n))$ in n-dimensional space R_n. Thus the elements $\varphi_1, ..., \varphi_n$ in Φ define a mapping

$$F \to ((F, \varphi_1), ..., (F, \varphi_n)) \tag{1}$$

of Φ' into R_n.

Let A be a given set in R_n, and consider the set Z of all linear functionals F such that

$$((F, \varphi_1), ..., (F, \varphi_n)) \in A.$$

We call this set Z the *cylinder set* defined by the elements $\varphi_1, ..., \varphi_n$ and the set A in R_n.

As examples of cylinder sets we may consider the half-spaces in Φ' defined by an inequality of the form $(F. \varphi) \leqslant a$, and also sets of a more general type—strips, defined by systems of inequalities

$$a_k \leqslant (F, \varphi_k) \leqslant b_k, \qquad 1 \leqslant k \leqslant n.$$

One can give another definition of a cylinder set. Let $\varphi_1, ..., \varphi_n$ be elements of Φ. We decompose the space Φ' into cosets, regarding all linear functionals which are carried into the same point in R_n by the mapping (1) as constituting one coset. In other words, two functionals F_1 and F_2 belong to the same coset if and only if

$$(F_1, \varphi_k) = (F_2, \varphi_k), \qquad 1 \leqslant k \leqslant n.$$

Obviously, the cylinder set Z is the union of those cosets corresponding to the points of the set A. Conversely, any union of cosets is a cylinder set in Φ'.

The decomposition of Φ' into cosets is uniquely defined by specifying the linear subspace Ψ^0 in Φ', consisting of those functionals which are carried into zero by the mapping (1). Indeed, the condition

$$(F_1, \varphi_k) = (F_2, \varphi_k), \qquad 1 \leqslant k \leqslant n,$$

is equivalent to the condition

$$(F_1 - F_2, \varphi_k) = 0, \qquad 1 \leqslant k \leqslant n.$$

Therefore two functionals belong to the same coset if and only if their difference belongs to the subspace Ψ^0.

Note that the equations $(F, \varphi_k) = 0$, $1 \leqslant k \leqslant n$, imply that $(F, \psi) = 0$ for any element of the form

$$\psi = a_1\varphi_1 + \dots + a_n\varphi_n.$$

Therefore the subspace Ψ^0 in Φ' can be defined as the subspace of those linear functionals F for which $(F, \psi) = 0$ for any element $\psi \in \Psi$, where Ψ is the subspace in Φ generated by the elements $\varphi_1, \dots, \varphi_n$.

Thus we arrive at the following definition of a cylinder set in Φ'. Let Ψ be a finite-dimensional subspace in Φ, and let Ψ^0 denote the linear subspace in Φ' consisting of those elements F for which

$$(F, \psi) = 0 \qquad \text{for} \quad \psi \in \Psi.$$

This subspace $\Psi^0 \subset \Phi'$ is called the *annihilator* of the subspace Ψ. We decompose Φ' into cosets, putting into the same cosets all functionals which take the same values on Ψ (or, what is the same, all functionals whose differences lie in the subspace Ψ^0). We thus obtain the factor space Φ'/Ψ^0, whose elements are cosets. Associating with every functional $F \in \Phi'$ the coset which contains it, we obtain a linear mapping of Φ' into Φ'/Ψ^0. Now choose any subset $A \subset \Phi'/\Psi^0$. The collection of all elements

$F \in \Phi'$ which are carried into elements of A by the mapping $\Phi' \to \Phi'/\Psi^0$ is called the *cylinder set* Z *with base* A *and generating subspace* Ψ^0.[1]

This definition is more convenient to use than that given above, because it does not require that a basis $\varphi_1, \ldots, \varphi_n$ be given in the subspace Ψ.

1.2. Simplest Properties of Cylinder Sets

Before studying the properties of cylinder sets, we stop to consider some simple assertions concerning linear topological spaces. We will consider only *locally convex* linear topological spaces, i.e., spaces in which every neighborhood of zero contains an absolutely convex neighborhood of zero. The class of locally convex spaces is adequately broad; in particular, it contains all countably normed spaces.

The following theorem on the extension of linear functionals holds for these spaces.

Any linear functional F which is defined on a subspace Ψ of a locally convex linear topological space Φ can be extended to a linear functional on all of Φ.

Indeed, the continuity of F implies that there exists a neighborhood U of zero in Φ such that $|(F, \varphi)| \leqslant 1$ for $\varphi \in U \cap \Psi$. Choosing an absolutely convex neighborhood of zero $V \subset U$, we take V as the unit sphere in Φ of a seminorm $\|\varphi\|$ (i.e., we set $\|\varphi\| = 1/\sup|\lambda|$, where $\lambda\varphi \in V$, for all $\varphi \in \Phi$). Clearly $|(F, \varphi)| \leqslant \|\varphi\|$ for all $\varphi \in \Psi$. By the Hahn–Banach theorem† the functional F has an extension $\bar{F}$, defined on all of Φ, which is additive and homogeneous and satisfies $|(\bar{F}, \varphi)| \leqslant \|\varphi\|$ for all $\varphi \in \Phi$. It follows that $|(\bar{F}, \varphi)| \leqslant 1$ for $\varphi \in V$, which means that $\bar{F}$ is continuous relative to the topology of Φ.

Next we show that *if Φ is a locally convex linear topological space and Ψ is a subspace of Φ, then the space Φ'/Ψ^0 is the adjoint space of Ψ.*

Indeed, any element $F \in \Phi'$ is a linear functional on Φ, and consequently on Ψ. Now two functionals F_1 and F_2 coincide on Ψ if and only if they belong to the same coset relative to Ψ^0, i.e., if they correspond to the same element in the factor space Φ'/Ψ^0. Thus, to every element

[1] The notion of a cylinder set can be introduced for any linear topological space Φ. Namely, let Ψ be some closed linear subspace in Φ, and A some set in the factor space Φ/Ψ. Then it is natural to call the collection of elements $\varphi \in \Phi$ such that the coset which contains φ belongs to A a cylinder set. However, all that we need are cylinder sets in Φ' corresponding to annihilators of finite-dimensional subspaces.

† The usual proofs of the Hahn–Banach theorem are entirely valid for a space which has a seminorm, with respect to which the space need not be complete.

$\tilde{F} \in \Phi'/\Psi^0$ there corresponds a linear functional on Ψ, and to distinct elements of Φ'/Ψ^0 there correspond distinct functionals on Ψ. Now let us show that every linear functional on Ψ can be obtained in this way. Let F_0 be a linear functional on Ψ. Then, as we saw above, F_0 can be extended to a linear functional on all of Φ. The various possible extensions of F_0, since they all coincide on Ψ, belong to the same coset relative to Ψ^0. Thus, every linear functional on Ψ corresponds to some element of Φ'/Ψ^0, which completes the proof.

It follows from this result that *if a subspace $\Psi \subset \Phi$ is n-dimensional, then the factor space Φ'/Ψ^0 is also n-dimensional.*

Now let us consider cylinder sets. A given cylinder set may be defined by various generating subsets and bases. For example, if

$$\psi = a_1\varphi_1 + \dots + a_n\varphi_n,$$

then the inequalities $(F, \psi) \leqslant a$ and

$$a_1(F, \varphi_1) + \dots + a_n(F, \varphi_n) \leqslant a$$

define the same half-space in Φ'.

Let us now clarify the conditions under which a cylinder set Z_1, having generating subspace Ψ_1^0 and base A_1, coincides with the cylinder set Z_2 having generating subspace Ψ_2^0 and base A_2. First we note that the cylinder sets Z_1 and Z_2 can be given by the same generating subspace Ψ_3^0. This subspace is the annihilator of the subspace Ψ_3 in Φ generated by the subspaces Ψ_1 and Ψ_2, and coincides with the intersection $\Psi_1^0 \cap \Psi_2^0$. Since $\Psi_3^0 \subset \Psi_1^0$, any coset with respect to Ψ_3^0 belongs to some coset with respect to Ψ_1^0. Associating with every coset with respect to Ψ_3^0 that coset with respect to Ψ_1^0 which contains it, we obtain a linear mapping T_1 of the factor space Φ'/Ψ_3^0 onto the factor space Φ'/Ψ_1^0. If we denote the inverse image of the set A_1 under the mapping T_1 by $T_1^{-1}(A_1)$, then it is obvious that the cylinder set Z_1 can be defined by the generating subspace Ψ_3^0 and the base $T_1^{-1}(A_1)$.

It follows in the same way that the cylinder set Z_2 can be defined by the generating subspace Ψ_3^0 and the base $T_2^{-1}(A_2)$ (T_2 denotes the linear mapping from Φ'/Ψ_3^0 onto Φ'/Ψ_2^0, by which every coset with respect to Ψ_3^0 is carried into that coset with respect to Ψ_2^0 which contains it).

Since evidently two cylinder sets with the same generating subspace coincide if and only if their bases coincide, we obtain the following result:

Suppose that the cylinder sets Z_1, Z_2 are defined respectively by the generating subsets Ψ_1^0 and Ψ_2^0 and the bases A_1 and A_2. Set $\Psi_3^0 = \Psi^0$

$\cap \Psi_2^0$. In order that Z_1 and Z_2 coincide, it is necessary and sufficient that

$$T_1^{-1}(A_1) = T_2^{-1}(A_2), \tag{2}$$

where T_1 denotes the natural linear mapping of Φ'/Ψ_3^0 onto Φ'/Ψ_1^0, and T_2 denotes the natural linear mapping of Φ'/Ψ_3^0 onto Φ'/Ψ_2^0.[2]

We note also the following properties of cylinder sets.

(1) The complement of any cylinder set is a cylinder set. Indeed, if the cylinder set Z is defined by the generating subspace Ψ^0 and the base A, then $\Phi' - Z$ has the same generating subspace, and its base is the complement of A in the factor space Φ'/Ψ^0.

(2) The intersection of any two cylinder sets is a cylinder set. Indeed, we have seen that Z_1 and Z_2 can be defined by the same generating subspace Ψ^0 in Φ'. Suppose that their bases are accordingly A_1 and A_2. Then $Z_1 \cap Z_2$ is the cylinder set with generating subspace Ψ^0 and base $A_1 \cap A_2$.

The following property is proven in entirely the same way.

(3) The sum of any two cylinder sets is a cylinder set.

We see, thus, that the cylinder sets form an algebra of sets.[3]

1.3. Cylinder Set Measures

We will henceforth consider only cylinder sets Z whose bases are Borel sets in Φ'/Ψ^0 (recall that we are considering only generating subspaces Ψ^0 such that Φ'/Ψ^0 is finite dimensional). If $Z_1, Z_2, \ldots$ are cylinder sets with Borel bases, having the same generating subspace Ψ^0, then their union $\bigcup_{n=1}^{\infty} Z_n$ and intersection $\bigcap_{n=1}^{\infty} Z_n$ are also cylinder sets with Borel bases.

By a *cylinder set measure* in the space Φ' we will mean a numerical valued function $\mu(Z)$, defined on the family of all cylinder sets with Borel bases, which has the following properties:

(1) $0 \leqslant \mu(Z) \leqslant 1$ for all Z,

(2) $\mu(\Phi') = 1$,

(3) if the set Z is the union of a sequence $Z_1, Z_2, \ldots$ of nonintersecting

[2] Obviously, if the factor spaces Φ'/Ψ_1^0 and Φ'/Ψ_3^0 are finite dimensional, then Φ'/Ψ_2^0 will be also.

[3] A system of sets is called an algebra if it contains, along with any two sets, their union and their complements.

cylinder sets having Borel bases and a common generating subspace Ψ^0, then

$$\mu(Z) = \sum_{n=1}^{\infty} \mu(Z_n),$$

(4) for any cylinder set Z (with Borel base)†

$$\mu(Z) = \inf \mu(U),$$

where U runs through all open cylinder sets containing Z.

A cylinder set measure $\mu(Z)$ defines a measure on the Borel sets in every factor space Φ'/Ψ^0. Namely, if A is some Borel set in Φ'/Ψ^0, and Z is a cylinder set with base A and generating subspace Ψ^0, then we set

$$\nu_{\Psi}(A) = \mu(Z). \tag{3}$$

Obviously ν_{Ψ} is a positive normalized measure in Φ'/Ψ^0 which is regular in the sense of Caratheodory.[4]

The measures induced by μ in different factor spaces Φ'/Ψ^0 are not independent. If a given cylinder set Z can be defined by the generating subspace Ψ_1^0 and base A_1 as well as the generating subspace Ψ_2^0 and base A_2, then it is necessary that

$$\nu_{\Psi_1}(A_1) = \nu_{\Psi_2}(A_2),$$

because both sides coincide with $\mu(Z)$.

Taking into account the condition indicated in Section 1.2 for two cylinder sets, defined by different generating subspaces and bases, to

† The following remarks may be useful. The authors are considering the weak (often called "weak dual" or "weak *") topology on Φ'. Since any annihilator Ψ^0 is a closed set in Φ' in this topology, elementary facts concerning the definition of a topology in a factor space Φ'/Ψ^0, plus the fact that there is only one topology on a finite-dimensional vector space (in our case, Φ'/Ψ^0) which makes it a linear topological space and separates points, imply that a cylinder set is open in the weak topology of Φ' if and only if its base is an open set in Φ'/Ψ^0. We observe also that condition 4 is superfluous. Indeed, as is pointed out in Section 2.1 below, μ is finitely additive; hence $Z \subset U$ implies $\mu(Z) \leqslant \mu(U)$. But it is a standard result of measure theory that any finite (or even Borel) measure on the Borel sets in R_n is regular. Applying this to the measures ν_{Ψ} (see text further ahead), we see that condition 4 is satisfied when U runs over all cylinder sets which have the same generating subspace as Z and whose bases are open sets containing the base of Z.

[4] A measure ν is called regular in the sense of Caratheodory, if for any Borel set A one has

$$\nu(A) = \inf_{U} \nu(U),$$

where U runs through all open sets containing A.

coincide, we can formulate the preceding equality in the following way. If $\Psi_1 \subset \Psi_2$, then for any Borel set A in the factor space Φ'/Ψ_1^0 one has

$$\nu_{\Psi_1}(A) = \nu_{\Psi_2}[T^{-1}(A)], \tag{4}$$

where $T^{-1}(A)$ denotes the inverse image of A with respect to the natural mapping T of Φ'/Ψ_2^0 onto Φ'/Ψ_1^0 (T carries every coset with respect to Ψ_2^0 into that coset with respect to Ψ_1^0 which contains it).

Thus, we have found a necessary condition for a system of measures ν_Ψ in the factor spaces Φ'/Ψ^0 to be induced by a cylinder set measure. This condition is also sufficient. In other words, the following assertion holds.

Suppose that $\{\nu_\Psi(A)\}$ *is a system of normalized positive measures, regular in the sense of Caratheodory, in the factor spaces* Φ'/Ψ^0. *If Eq.* (4) *holds for every Borel set* A *in* Φ'/Ψ_1^0 *whenever* $\Psi_1 \subset \Psi_2$, *then the measures* ν_Ψ *are induced by a cylinder set measure* $\mu(Z)$ *in* Φ'.

Indeed, for any cylinder set Z with generating subspace Ψ^0 and base A we set

$$\mu(Z) = \nu_\Psi(A).$$

From (4) it follows that $\mu(Z)$ does not depend upon the manner in which Z is defined. Obviously $\mu(Z)$ is a cylinder set measure in Φ', and all the measures ν_Ψ are induced by μ.

From now on we will call (4) the *compatibility condition* for the measures ν_Ψ. It can be shown that it is sufficient to verify (4) only for half-spaces in Φ'. This assertion easily results from the following lemma:

If the values of two positive normalized measures ν_1 and ν_2 in a finite-dimensional space R coincide for all half-spaces in R, then ν_1 and ν_2 are identical.

For the proof of this lemma cf. reference (29).

1.4. The Continuity Condition for Cylinder Set Measures

We will henceforth consider only measures which satisfy a certain continuity condition. This condition is formulated in the following way:

▼ A cylinder set measure μ is said to be *continuous*, if for any bounded continuous function $f(x_1, \ldots, x_m)$ of m variables, the function

$$I(\varphi_1, \ldots, \varphi_m) = \int_{\Phi'} f((F, \varphi_1), \ldots, (F, \varphi_m))\, d\mu(F)$$

is sequentially continuous† in the variables $\varphi_1, \ldots, \varphi_m \in \Phi$. In other words, if $\lim_{i\to\infty} \varphi_{ij} = \varphi_j$, $j = 1, \ldots, m$, where the convergence is in Φ, then

$$\lim_{i\to\infty} I(\varphi_{i1}, \ldots, \varphi_{im}) = I(\varphi_1, \ldots, \varphi_m).$$

We point out that the integral is well defined, because if Ψ denotes the linear subspace generated by $\varphi_1, \ldots, \varphi_m$, then $f((F, \varphi_1), \ldots, (F, \varphi_m))$, as a function of $F \in \Phi'$, is clearly measurable with respect to the σ-algebra of all cylinder sets with generating subspace Ψ^0, on which μ is by definition countably additive.

We note the following result: If the cylinder set measure μ is continuous, then for any $A > 0$ and any sequence $\{\varphi_i\}$ such that $\lim_{i\to\infty} \varphi_i = 0$, we have $\lim_{i\to\infty} \mu\{|(F, \varphi_i)| \geqslant A\} = 0$.

In fact, let $f(x)$ be a continuous bounded nonnegative function such that $f(0) = 0$ and $f(x) = 1$ for $|x| \geqslant A$. By the continuity of μ we have

$$\lim_{i\to\infty} \int f((F, \varphi_i))\, d\mu(F) = \int f((F, 0))\, d\mu(F) = 0 \cdot d\mu(F) = 0$$

But

$$\int f((F, \varphi_i))\, d\mu(F) \geqslant \mu\{|(F, \varphi_i)| \geqslant A\},$$

from which the assertion follows.

The converse is also true: If, for any $A > 0$ and any sequence $\{\varphi_i\}$ such that $\lim_{i\to\infty} \varphi_i = 0$, one has

$$\lim_{i\to\infty} \mu\{|(F, \varphi_i)| \geqslant A\} = 0,$$

then μ is continuous. For the proof, the reader is referred to a paper by Minlos [reference (49)].

This enables us to use the following sufficient condition for the continuity of μ:

Let μ_φ be the measure on the real line defined by $\mu_\varphi(-\infty, a) = \mu\{(F, \varphi) < a\}$. If, for any sequence $\{\varphi_i\}$ for which $\lim_{i\to\infty} \varphi_i = 0$, and any bounded continuous function $f(x)$, one has

$$\lim_{i\to\infty} \int f(x)\, d\mu_{\varphi_i}(x) = f(0),$$

then μ is continuous.

† In countably normed spaces, which have a metric topology, sequential and ordinary continuity are of course equivalent. The only place where sequential continuity intervenes is in Section 4.2, where the use of sequences appears to be unavoidable.

Indeed, by the result mentioned above it suffices to show that

$$\lim_{i\to\infty} \mu\{|(F, \varphi_i)| \geq A\} = 0$$

for any $A > 0$. But if $f(x)$ is the function defined earlier, then

▲ $$\lim_{i\to\infty} \mu\{|(F, \varphi_i)| \geq A\} = \lim_{i\to\infty} \mu_{\varphi_i}((-A, A)^c) \leq \lim_{i\to\infty} \int f(x)\, d\mu_{\varphi_i}(x) = 0.$$

1.5. Induced Cylinder Set Measures

Let T be a continuous linear mapping of the linear topological space Φ_2 into the linear topological space Φ_1. Denote by T' the mapping of Φ_1' into Φ_2' which is adjoint to T, i.e., the mapping such that $(T'F, \varphi) = (F, T\varphi)$ for any $\varphi \in \Phi_2$ and $F \in \Phi_1'$. Obviously, if T carries the finite-dimensional subspace $\Psi_2 \subset \Phi_2$ into the finite-dimensional subspace $\Psi_1 \subset \Phi_1$, then T' carries the subspace Ψ_1^0 into the subspace Ψ_2^0. Indeed, suppose that $F \in \Psi_1^0$. Then $T\psi \in \Psi_1$ for any $\psi \in \Psi_2$, and therefore $(F, T\psi) = 0$. But this means that $(T'F, \psi) = 0$ for all $\psi \in \Psi_2$, i.e., that $T'F \in \Psi_2^0$.

Thus we have proven that $T'\Psi_1^0 \subset \Psi_2^0$. From this it follows that the mapping T' induces a mapping T_1' of the factor space Φ_1'/Ψ_1^0 into the factor space Φ_2'/Ψ_2^0, by which the coset $F + \Psi_1^0$ is carried into the coset $T'F + \Psi_2^0$ (in view of the inclusion $T'\Psi_1^0 \subset \Psi_2^0$, the correspondence $F + \Psi_1^0 \to T'F + \Psi_2^0$ does not depend upon the choice of representative F in the coset $F + \Psi_1^0$).

Thus we have proven: If T is a continuous linear mapping of Φ_2 into Φ_1, then for every finite-dimensional subspace $\Psi_1 \subset \Phi_1$ there exsist a linear mapping T_1' of the factor space Φ_1'/Ψ_1^0 into the factor space Φ_2'/Ψ_2^0, where Ψ_1^0 denotes the annihilator of the subspace $\Psi_1 = T\Psi_2$ in Φ_1. The mapping T_1' takes the coset $F + \Psi_1^0$ into the coset $T'F + \Psi_2^0$.

Suppose now that a cylinder set measure μ_1 is given in Φ_1'. We introduce a measure on the cylinder sets in Φ_2' in the following way. Suppose that the cylinder set Z_2 in Φ_2' is defined by the subspace Ψ_2 (in Φ_2) and the base A_2. Let $\Psi_1 = T\Psi_2$ and $A_1 = (T_1')^{-1}(A_2) \subset \Phi_1'/\Psi_1^0$. We set

$$\mu_2(Z_2) = \mu_1(Z_1),$$

where Z_1 is the cylinder set in Φ_1' with generating subspace Ψ_1 and base A_1. It is easy to see that μ_2 is a cylinder set measure in Φ_2', and that in view of the continuity of T, μ_2 satisfies the continuity condition. We will call μ_2 *the measure induced from* μ_1 *by the mapping* T.

For example, if Φ_m is the completion of the countably Hilbert space Φ in the norm $\| \varphi \|_m$, then to every measure μ_m in Φ'_m there corresponds a measure μ in Φ' (and a measure μ_n in any space Φ'_n, where $n > m$). We will call a measure in Φ' m-continuous if it is induced by a continuous measure in Φ'_m.

2. The Countable Additivity of Cylinder Set Measures in Spaces Adjoint to Nuclear Spaces

2.1. The Additivity of Cylinder Set Measures

Cylinder set measures have the following property of finite additivity: If $Z_1, \ldots, Z_n$ is a finite system of disjoint cylinder sets in Φ', then

$$\mu\left(\bigcup_{k=1}^{n} Z_k\right) = \sum_{k=1}^{n} \mu(Z_k).$$

In fact, since, as was shown in Section 1.1, we can find a common generating subspace for any *finite* system of cylinder sets, this assertion follows from the additivity of the measure ν_Ψ in the factor space Φ'/Ψ^0.

However, the measure μ does not by any means always have the property of *countable additivity*: it does not follow, generally speaking, that if a cylinder set Z is the union of a countable family $Z_1, Z_2, \ldots$ of nonintersecting cylinder sets, then

$$\mu(Z) = \sum_{k=1}^{\infty} \mu(Z_k)$$

(of course, the equality does hold if all of the Z_k are defined by the same generating subspace).

For us, however, it is essential that μ be countably additive. This is connected with the fact that the class of cylinder sets in Φ' is rather narrow.[1] Therefore it is natural to want to extend μ to a wider class of sets. This class is the σ-algebra generated by the (Borel) cylinder sets. As usual, by the σ-algebra of sets generated by the cylinder sets we mean the smallest class of sets which contains the cylinder sets and is closed under the operations of countable union and complementation. We will call the members of this σ-algebra the *Borel sets* in Φ'.

[1] For example, the polar of a set $A \subset \Phi$, generally speaking, is not a cylinder set in Φ' (the *polar* of a set A is the set of all functionals F such that $|(F, \varphi)| \leqslant 1$ for all $\varphi \in A$).

The class of Borel sets is adequately broad; for example, if Φ contains a countable everywhere dense set of elements, then the polar of every set $A \subset \Phi$ is a Borel set in Φ'.

In the case where the measure μ, defined on the cylinder sets, is completely additive, it can be extended to all the Borel sets.

This extension can be carried out in the following way. We call the cylinder sets Borel sets of the zeroth class. Suppose that Borel sets of class β have already been defined, where β is any transfinite number less than α. We call "Borel sets of class α" all countable unions of nonintersecting sets of class less than α and all complements of such unions. Thus, Borel sets are defined for all transfinite numbers of the first and second classes. If

$$B = \bigcup_{k=1}^{\infty} B_k$$

is a decomposition of a Borel set of class α into nonintersecting Borel sets of lower classes, then we set

$$\mu(B) = \sum_{k=1}^{\infty} \mu(B_k)$$

and

$$\mu(\Phi' - B) = 1 - \mu(B).$$

Using the completely additivity of μ for cylinder sets, we now show that starting from two decompositions

$$B = \bigcup_{k=1}^{\infty} B_k$$

and

$$B = \bigcup_{k=1}^{\infty} B'_k \qquad \left(\text{or} \quad B = \Phi' - \bigcup_{k=1}^{\infty} B'_k\right)$$

of the Borel set B into nonintersecting Borel sets of lower classes, we always obtain the same value for $\mu(B)$. This is easy to prove for sets of the first class: If

$$B = \bigcup_{k=1}^{\infty} Z_k = \bigcup_{k=1}^{\infty} Z'_k$$

are two decompositions of a set B of the first class into nonintersecting cylinder sets, then

$$\sum_{k=1}^{\infty} \mu(Z_k) = \sum_{k=1}^{\infty} \mu(Z'_k).$$

Indeed,

$$\sum_{k=1}^{\infty} \mu(Z_k) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \mu(Z_k \cap Z'_j) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \mu(Z_k \cap Z'_j) = \sum_{j=1}^{\infty} \mu(Z'_j).$$

If now

$$B = \Phi' - \bigcup_{k=1}^{\infty} Z'_k = \bigcup_{k=1}^{\infty} Z_k,$$

then

$$\left(\bigcup_{k=1}^{\infty} Z'_k\right) \cup \left(\bigcup_{k=1}^{\infty} Z_k\right) = \Phi'$$

is a decomposition of Φ' into nonintersecting cylinder sets, and therefore

$$\sum_{k=1}^{\infty} \mu(Z_k) + \sum_{k=1}^{\infty} \mu(Z'_k) = 1,$$

i.e.,

$$\sum_{k=1}^{\infty} \mu(Z_k) = 1 - \sum_{k=1}^{\infty} \mu(Z'_k).$$

This proves that μ is unambiguously defined on Borel sets of class 1. It can be shown that μ remains countably additive following this extension. For sets of higher classes the proof is carried out by means of transfinite induction.

We remark that the extension of μ to the Borel sets in Φ' has the following property (regularity in the sense of Caratheodory):

For any Borel set $B \subset \Phi'$,

$$\mu(B) = \inf_Z \mu(Z),$$

where Z runs through all countable unions of open cylinder sets Z_k such that $B \subset \bigcup_{k=1}^{\infty} Z_k$.

The proof of this assertion is easily carried out by means of transfinite induction.

We will see further on that there exist spaces for which every positive normalized cylinder set measure which has the continuity property is countably additive, and can therefore be extended to all the Borel sets. At the same time, there exist spaces in which not every measure can be extended to the Borel sets, but only measures satisfying certain additional conditions.

The class of spaces for which any positive normalized cylinder set measure satisfying the continuity condition can be extended to the Borel sets is the class of spaces which are adjoint to nuclear spaces. This result will be proven in Section 2.4. For the proof of this basic result we need certain results of measure theory. First of all we indicate the following simple criterion for the countable additivity of a measure.

Theorem 1. In order that a measure μ on the cylinder sets in Φ' be countably additive, it is necessary and sufficient that

$$\sum_{k=1}^{\infty} \mu(Z_k) = 1$$

for any decomposition $\Phi' = \bigcup_{k=1}^{\infty} Z_k$ of Φ' into nonintersecting cylinder sets.

Proof. The necessity of the condition follows directly from the definition of countable additivity. As for the sufficiency, suppose that $Z = \bigcup_{k=1}^{\infty} Z_k$ is a decomposition of some cylinder set Z into nonintersecting cylinder sets $Z_1, Z_2, \ldots$. Then the space Φ' can be decomposed into the nonintersecting cylinder sets $\Phi' - Z, Z_1, Z_2, \ldots$, and therefore by the hypothesis of the theorem

$$\mu(\Phi' - Z) + \sum_{k=1}^{\infty} \mu(Z_k) = 1. \tag{1}$$

From the finite additivity of μ it follows that

$$\mu(\Phi' - Z) + \mu(Z) = 1. \tag{2}$$

Comparing (1) and (2), we obtain

$$\mu(Z) = \sum_{k=1}^{\infty} \mu(Z_k),$$

which proves the countable additivity of μ.

This theorem can be stated in another, equivalent, way.

Theorem 1′. In order that a measure μ on cylinder sets be countably additive, it is necessary and sufficient that

$$\lim_{n\to\infty} \mu(Z_n') = 0 \tag{3}$$

for any decreasing sequence $Z_1 \supset Z_2 \supset \ldots$ of cylinder sets whose intersection is empty.

Proof. Only the sufficiency of the condition needs to be proven. Let

$$\Phi' = \bigcup_{k=1}^{\infty} Z_k$$

be a decomposition of Φ' into nonintersecting cylinder sets. Then the cylinder sets

$$Z'_n = \Phi' - \bigcup_{k=1}^{n} Z_k$$

form a decreasing sequence with empty intersection, and so by hypothesis

$$\lim_{n\to\infty} \mu(Z'_n) = 0.$$

In view of the finite additivity of μ, this means that

$$\lim_{n\to\infty} \left[1 - \sum_{k=1}^{n} \mu(Z_k)\right] = 0$$

or, that $\sum_{k=1}^{\infty} \mu(Z_k) = 1$. Consequently, μ is countably additive by Theorem 1.

Theorem 1″. In order that the measure μ be countably additive, it is necessary and sufficient that for any sequence $\{Z_k\}$ of (not necessarily disjoint) cylinder sets whose union is Φ',

$$\sum_{k=1}^{\infty} \mu(Z_k) \geqslant 1. \tag{4}$$

To prove the sufficiency of this condition, we note that if the sets Z_k whose union is Φ' are nonintersecting, then in view of the finite additivity of μ one has

$$\sum_{k=1}^{\infty} \mu(Z_k) \leqslant 1. \tag{5}$$

On the other hand, inequality (4) is satisfied. Inequalities (4) and (5) imply

$$\sum_{k=1}^{\infty} \mu(Z_k) = 1,$$

and therefore μ is countably additive by Theorem 1. The necessity of the condition is obvious.

Finally, we note that it is sufficient to require only that inequality

(4) hold for all sequences of open cylinder sets whose union is Φ'. This follows at once from the fact that in view of the regularity of μ, for any cylinder set Z we can find some open cylinder set whose measure exceeds that of Z by as little as desired.

2.2. A Condition for the Countable Additivity of Cylinder Set Measures in Spaces Adjoint to Countably Hilbert Spaces

The conditions for countable additivity given in the preceding section are inconvenient to apply. Here we introduce a condition for the countable additivity of measures on the cylinder sets in spaces adjoint to countably Hilbert spaces, which is more convenient to use.

Suppose that the cylinder set measure μ in the space Φ' adjoint to a countably Hilbert space Φ is countably additive. Then, as we have seen above, it can be extended to all the Borel sets in Φ'. In particular, μ can be extended to all balls $S_n(R)$ defined by inequalities of the form $\| F \|_{-n} \leqslant R$. Indeed, $S_n(R)$ consists of all continuous linear functionals on Φ such that $|(F, \varphi)| \leqslant R$ if $\| \varphi \|_n \leqslant 1$. Choose a countable set $\{\varphi_k\}$ of elements which are everywhere dense in the unit ball $S_n = \{\| \varphi \|_n \leqslant 1\}$ of the Hilbert space Φ_n and which lie in $S_n \cap \Phi$. If we denote the strips $|(F, \varphi_k)| \leqslant R$ in Φ' by A_k, it is obvious that

$$S_n(R) = \bigcap_{k=1}^{\infty} A_k,$$

i.e., $S_n(R)$ is a Borel set in Φ' (moreover, it is a Borel set of class 1). Therefore μ can be extended to every ball.

Now let us show that for any $\epsilon > 0$ there is a ball $S_n(R)$, defined by an inequality of the form $\| F \|_{-n} \leqslant R$, such that the μ measure of the complement of $S_n(R)$ is less than ϵ (assuming μ to be countably additive). Indeed, every element $F \in \Phi'$ belongs to one of the spaces Φ_n' and therefore satisfies some inequality of the form $\| F \|_{-n} \leqslant R$. Therefore Φ' is a countable union of balls,

$$\Phi' = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} S_n(k).$$

Since $m \leqslant n$ implies that $\| F \|_{-m} \geqslant \| F \|_{-n}$ for any element $F \in \Phi'$, then $S_n(n) \subset S_{n+1}(n+1)$. Consequently, Φ' is the union of an increasing sequence of balls $S_n(n)$, i.e.,

$$\Phi' = \bigcup_{n=1}^{\infty} S_n(n),$$

where

$$S_1(1) \subset S_1(2) \subset \dots .$$

Since $\mu(\Phi') = 1$, we have

$$\lim_{n\to\infty} \mu[S_n(n)] = 1.$$

But this shows that for any $\epsilon > 0$ there is an n such that the complement of $S_n(n)$ has measure less than ϵ.

We have therefore proven the following assertion.

Theorem 2. If μ is a positive normalized countably additive cylinder set measure in the adjoint space Φ' of a countably Hilbert space Φ, then for any $\epsilon > 0$ there is a ball $S_n(R)$ such that the μ-measure of any cylinder set Z lying outside $S_n(R)$ is less than ϵ.

Now we prove that the converse also holds.

Theorem 2′. Suppose that μ is a positive normalized cylinder set measure on the adjoint space Φ' of a countably Hilbert space Φ. If for any $\epsilon > 0$ there is a ball $S_n(R)$ in Φ' such that the measure of any cylinder set lying outside $S_n(R)$ is less than ϵ, then μ is countably additive.

For the proof of Theorem 2′ we need the following lemma.

Lemma 1. From any covering of a ball $S(R) = \{\| \varphi \| \leqslant R\}$ in a Hilbert space H by open cylinder sets, one can extract a finite subcovering.

A cylinder set in a Hilbert space H is defined by the condition

$$\{(\varphi, \varphi_1), \dots, (\varphi, \varphi_n)\} \in A,$$

where $\varphi_1, \dots, \varphi_n$ are fixed elements in H, and A is some set in n-dimensional space R_n. Since (φ, φ_k) is a linear functional on H, this definition agrees with that of Section 1.

Since an open cylinder set in a Hilbert space is an open set in the weak topology of H, Lemma 1 can be expressed briefly by saying that a *ball in a Hilbert space is weakly compact*[2] (i.e., compact in the weak topology).

[2] A set A lying in a topological space X is said to be *compact*, if from any covering of A by open sets one can extract a finite subcovering. We are considering here the weak topology in the Hilbert space H, in which the neighborhoods of zero are defined by inequalities of the form

$$|(\varphi, \varphi_k)| < \epsilon, \qquad 1 \leqslant k \leqslant n,$$

i.e., they are open cylinder sets in H. The compactness of a set A in this topology means that from any covering of A by open cylinder sets one can extract a finite subcovering.

The proof of Lemma 1 is as follows. With each element φ, $\| \varphi \| \leqslant 1$, we associate the interval $-R \leqslant x \leqslant R$ of the real line, and denote these intervals by I_φ. Let I be the Tikhonov product of all of these intervals (see, for example, M. A. Naimark, "Normed Rings," Chapter 1, Section 2.12. Nordhoff, Groningen, 1959). Since the Tikhonov product of compact sets is compact, I is a compact set. Now with each point $\varphi_0 \in S(R)$we associate the point (φ_0, φ) on the interval I_φ. We thus obtain a correspondence, to each $\varphi_0 \in S(R)$, of a point in the direct product of all the I_φ, i.e., of a point in I. A simple examination shows that the mapping $\varphi_0 \to (\varphi_0, \varphi)$ is a homeomorphism of $S(R)$ (with the topology induced in it by the weak topology in H) onto a closed subset $\alpha(S)$ of I. Since a closed subset of a compact space is compact, and α is a homeomorphism, then $S(R)$ is itself compact. In other words, from any covering of $S(R)$ by open cylinder sets one can extract a finite subcovering, which completes the proof.

Let us now prove Theorem 2'. To see that μ is countably additive, we have to prove, by the remark following Theorem 1'', that if

$$\Phi' = \bigcup_{k=1}^{\infty} Z_k$$

is any covering of Φ' by open cylinder sets, then

$$\sum_{k=1}^{\infty} \mu(Z_k) \geqslant 1.$$

Given $\epsilon > 0$, there exists by hypothesis a ball $S_n(R) = \{\| F \|_{-n} \leqslant R\}$ in Φ' such that the measure of any cylinder set lying outside $S_n(R)$ is less than ϵ. Let Z_{nk} denote the intersection of the set Z_k with the Hilbert space Φ_n' in Φ'. It is obvious that the sets Z_{nk} are open cylinder sets in Φ_n' which cover Φ_n' and, consequently, cover the ball $S_n(R)$. By Lemma 1 we can choose a finite number of the Z_{nk}, say $Z_{n1}, \ldots, Z_{nj}$, which cover $S_n(R)$. Therefore we have

$$S_n(R) \subset \bigcup_{k=1}^{j} Z_{nk} \subset \bigcup_{k=1}^{j} Z_k. \tag{6}$$

Let Z denote the cylinder set $\Phi' - \bigcup_{k=1}^{j} Z_k$. It follows from (6) that Z lies outside $S_n(R)$ and its measure is therefore less than ϵ. Then

$$\epsilon > \mu(Z) = \mu\left(\Phi' - \bigcup_{k=1}^{j} Z_k\right) \geqslant 1 - \sum_{k=1}^{j} \mu(Z_k)$$

(recall that the Z_k may intersect) and therefore

$$\sum_{k=1}^{\infty} \mu(Z_k) \geqslant \sum_{k=1}^{j} \mu(Z_k) \geqslant 1 - \epsilon.$$

Since ϵ is arbitrary, it follows from this that

$$\sum_{k=1}^{\infty} \mu(Z_k) \geqslant 1,$$

which shows that μ is countably additive, and completes the proof of Theorem 2′.

Theorems 2 and 2′ give a necessary and sufficient condition for a cylinder set measure in the adjoint space Φ' of a countably Hilbert space Φ to be countably additive. This condition is that for any $\epsilon > 0$ it is possible to find a ball $S_n(R)$ in Φ' such that the measure of any cylinder set lying outside $S_n(R)$ is less than ϵ.

2.3. Cylinder Sets Measures in the Adjoint Spaces of Nuclear Countably Hilbert Spaces

In this paragraph we shall prove a basic result concerning measures in the adjoint space of a nuclear countably Hilbert space. The statement of this result is as follows.

Theorem 3. Suppose that Φ' is the adjoint space of a countably Hilbert nuclear space Φ. Then any positive normalized cylinder set measure μ in Φ', satisfying the continuity condition, is countably additive.

We precede the proof of this theorem by certain lemmas on the connection between the measures of half-spaces and balls in n-dimensional space. Let μ be a positive normalized measure in an n-dimensional Euclidean space. We denote by $\mu(r, \omega)$ the measure of the half space $(x, \omega) \geqslant r$, which is bounded by the plane perpendicular to the unit vector ω and situated a distance r from the origin ((x, ω) denotes the scalar product in the Euclidean space being considered). Further, we denote by $\mu(R)$ the measure of a ball of radius R and center at the origin.

In order to establish a connection between $\mu(r, \omega)$ and $\mu(R)$, we will consider not the measure $\mu(r, \omega)$ but rather its average over ω. This average is defined in the following way. A unit vector ω can be considered as the radius vector of a point on the unit sphere. We introduce on the space Ω of these unit vectors the measure τ, defined naturally as normalized (surface) measure on the unit sphere.

In the spherical coordinates

$$\begin{aligned} x_1 &= \rho \cos \varphi_1, \\ x_2 &= \rho \sin \varphi_1 \cos \varphi_2, \\ &\cdots\cdots\cdots \\ x_n &= \rho \sin \varphi_1 \dots \sin \varphi_{n-1}, \end{aligned} \qquad \begin{aligned} &0 \leqslant \rho < \infty, \\ &0 \leqslant \varphi_k \leqslant \pi, \quad 1 \leqslant k \leqslant n-2, \\ &0 \leqslant \varphi_{n-1} < 2\pi, \end{aligned}$$

this measure is given by

$$d\tau(\omega) = \frac{\Gamma(\frac{1}{2}n)}{2\pi^{\frac{1}{2}n}} \sin^{n-2} \varphi_1 \dots \sin \varphi_{n-2}\, d\varphi_1 \dots d\varphi_{n-1}. \tag{7}$$

If $f(\omega)$ is any function on the sphere Ω (or what is the same, a function of unit vectors ω), then by its average value we mean the integral $\int f(\omega)\, d\tau(\omega)$, which we will denote by $\langle f(\omega) \rangle$. Thus

$$\langle f(\omega) \rangle = \int_\Omega f(\omega)\, d\tau(\omega). \tag{8}$$

Let us now express the average $\langle \mu(r, \omega) \rangle$ of the measure of the half-space $(x, \omega) \geqslant r$ in terms of $\mu(R)$. By definition we have

$$\langle \mu(r, \omega) \rangle = \int_\Omega \mu(r, \omega)\, d\tau(\omega). \tag{9}$$

But

$$\mu(r, \omega) = \int f(x, r, \omega)\, d\mu(x),$$

where $f(x, r, \omega)$ denotes the characteristic function of the half-space $(x, \omega) \geqslant r$, i.e., the function which equals unity for $(x, \omega) \geqslant r$ and zero for $(x, \omega) < r$. Substituting this expression for $\mu(r, \omega)$ into (9), we obtain

$$\langle \mu(r, \omega) \rangle = \int f(x, r, \omega)\, d\mu(x)\, d\tau(\omega).$$

From this it follows that

$$\langle \mu(r, \omega) \rangle = \int \varphi(x, r)\, d\mu(x),$$

where

$$\varphi(x, r) = \int_\Omega f(x, r, \omega)\, d\tau(\omega).$$

From this formula one sees that $\varphi(x, r)$ equals the τ-measure of the set of those vectors ω for which $(x, \omega) \geqslant r$. Since the ends of these vectors form a segment on the unit sphere which is cut off from the sphere by the plane $(x, \omega) = r/|x|$, situated a distance $r/|x|$ from the origin, then

$$\varphi(x, r) = \frac{\Gamma(\frac{1}{2}n)}{\Gamma(\frac{1}{2}n - \frac{1}{2})\sqrt{\pi}} \int_{r/|x|}^{1} (1 - y^2)^{\frac{1}{2}n-\frac{3}{2}}\, dy, \qquad \text{if} \quad r < |x|,$$

and

$$\varphi(x, r) = 0 \qquad \text{if} \quad r \geqslant |x|.$$

From this it follows that

$$\langle \mu(r, \omega) \rangle = \frac{\Gamma(\frac{1}{2}n)}{\Gamma(\frac{1}{2}n - \frac{1}{2})\sqrt{\pi}} \int_{r \leqslant |x|} d\mu(x) \int_{r/|k|}^{1} (1 - y^2)^{\frac{1}{2}n-\frac{3}{2}}\, dy.$$

Since the expression

$$\int_{r/|x|}^{1} (1 - y^2)^{\frac{1}{2}n-\frac{3}{2}}\, dy$$

depends only upon $|x|$, then changing to spherical coordinates ρ, $\varphi_1, \dots, \varphi_{n-1}$, we obtain the formula

$$\langle \mu(r, \omega) \rangle = \frac{\Gamma(\frac{1}{2}n)}{\Gamma(\frac{1}{2}n - \frac{1}{2})\sqrt{\pi}} \int_{r}^{\infty} \int_{r/\rho}^{1} (1 - y^2)^{\frac{1}{2}n-\frac{3}{2}}\, dy\, d\mu(\rho), \tag{10}$$

where, as stated above, $\mu(\rho)$ denotes the μ-measure of the ball of radius ρ and center at the origin.

Thus, we have established a connection between the average $\langle \mu(r, \omega) \rangle$ of the measures of the half-spaces $(x, \omega) \geqslant r$ and the measures $\mu(\rho)$ of balls.

Since the function

$$\psi(\rho) = \int_{r/\rho}^{1} (1 - y^2)^{\frac{1}{2}n-\frac{3}{2}}\, dy$$

is a monotone increasing function of ρ, we find from (10) that

$$\langle \mu(r, \omega) \rangle \geqslant \frac{\Gamma(\frac{1}{2}n)}{\Gamma(\frac{1}{2}n - \frac{1}{2})\sqrt{\pi}} \int_{r/R}^{1} (1 - y^2)^{\frac{1}{2}n-\frac{3}{2}}\, dy[1 - \mu(R)]$$

for any r and $R \geqslant r$. It follows from this inequality that

$$1 - \mu(R) \leqslant C\left(n, \frac{r}{R}\right) \cdot \langle \mu(r, \omega) \rangle, \tag{11}$$

where $C(n, r/R)$ denotes the ratio

$$\frac{\Gamma(\frac{1}{2}n)}{\Gamma(\frac{1}{2}n-\frac{1}{2})\sqrt{\pi}\int_{r/R}^{1}(1-y^2)^{\frac{1}{2}n-\frac{3}{2}}\,dy}=\frac{2\int_0^1(1-y^2)^{\frac{1}{2}n-\frac{3}{2}}\,dy}{\int_{r/R}^{1}(1-y^2)^{\frac{1}{2}n-\frac{3}{2}}\,dy}.$$

Now we show that if we set $r/R = 1/\sqrt{n}$ in $C(n, r/R)$, then the set of numbers $C_n = C(n, \sqrt{n})$ will be bounded. To show this, we observe that

$$C_n=\frac{2\int_0^1(1-y^2)^{\frac{1}{2}n-\frac{3}{2}}\,dy}{\int_{1/\sqrt{n}}^{1}(1-y^2)^{\frac{1}{2}n-\frac{3}{2}}\,dy}=\frac{2\int_0^{\sqrt{n}}[1-(y^2/n)]^{\frac{1}{2}n-\frac{3}{2}}\,dy}{\int_1^{\sqrt{n}}[1-(y^2/n)]^{\frac{1}{2}n-\frac{3}{2}}\,dy}.$$

Therefore

$$\lim_{n\to\infty}C_n=\frac{2\int_0^\infty\exp(-\frac{1}{2}y^2)\,dy}{\int_1^\infty\exp(-\frac{1}{2}y^2)\,dy}<+\infty.$$

Since all of the C_n, $n = 1, 2, \ldots$, are positive and $\lim_{n\to\infty} C_n < +\infty$, the set $\{C_n\}$ is bounded. Let $C = \sup_n C_n$. Then from (11) there follows the inequality

$$1-\mu(R)\leqslant C\left\langle\mu\left(\frac{R}{\sqrt{n}},\omega\right)\right\rangle.$$

We have thus proven the following lemma, which enables us to estimate $\mu(R)$ by means of $\mu(r, \omega)$.

Lemma 2. Let μ be a positive normalized measure in an n-dimensional Euclidean space, let $\mu(R)$ be the measure of the ball of radius R and center at the origin, and let $\langle\mu(r, \omega)\rangle$ be defined by (9). Then

$$1-\mu(R)\leqslant C\cdot\left\langle\mu\left(\frac{R}{\sqrt{n}},\omega\right)\right\rangle, \tag{12}$$

where C is a constant not depending upon either n or R.

The half-spaces which are involved in Lemma 2 are bounded by the planes $(x, \omega) = R/\sqrt{n}$ which are tangent to the sphere with radius $R/\sqrt{n}$ and center at the origin. To prove Theorem 3, we need a lemma, similar to Lemma 2, in which the sphere of radius $R/\sqrt{n}$ is replaced by an ellipsoid. We precede this lemma by the following assertion concerning the average value of the square of the distance from the origin to the tangent planes of an ellipsoid.

Lemma 3. Let $r(\omega)$ be the (perpendicular) distance from the origin to the tangent plane of the ellipsoid

$$\frac{x_1^2}{\lambda_1^2} + \dots + \frac{x_n^2}{\lambda_n^2} = 1$$

which is perpendicular to the unit vector ω. Then

$$\langle r^2(\omega)\rangle = n^{-1}(\lambda_1^2 + \dots + \lambda_n^2).$$

Proof. A simple calculation shows that

$$r^2(\omega) = \lambda_1^2\omega_1^2 + \dots + \lambda_n^2\omega_n^2,$$

where $\omega_1, \dots, \omega_n$ are the coordinates of the vector.[3]

From this it follows that

$$\langle r^2(\omega)\rangle = \lambda_1^2\langle\omega_1^2\rangle + \dots + \lambda_n^2\langle\omega_n^2\rangle.$$

But since $\omega_1^2 + \dots + \omega_n^2 = 1$, and the coordinates $\omega_1, \dots, \omega_n$ are equally distributed, then

$$\langle\omega_k^2\rangle = n^{-1}\langle\omega_1^2 + \dots + \omega_n^2\rangle = n^{-1}$$

and therefore

$$\langle r^2(\omega)\rangle = n^{-1}(\lambda_1^2 + \dots + \lambda_n^2),$$

which proves the lemma.

We now turn to the proof of the following lemma, which is (together with Lemma 2) the central point in the proof of Theorem 3.

[3] Indeed, if the coordinates of the point of tangency are $x_1^{(0)}, \dots, x_n^{(0)}$, then the equation of the tangent plane to the ellipsoid has the form

$$\frac{x_1x_1^{(0)}}{\lambda_1^2} + \dots + \frac{x_nx_n^{(0)}}{\lambda_n^2} = 1.$$

Bringing this equation into normal form, we find that $\omega_k = r(\omega)x_k^{(0)}/\lambda_k^2$ and therefore $x_k^{(0)} = \omega_k\lambda_k^2/r(\omega)$. But

$$\frac{x_1^{(0)2}}{\lambda_1^2} + \dots + \frac{x_n^{(0)2}}{\lambda_n^2} = 1$$

and consequently $r^2(\omega) = \lambda_1^2\omega_1^2 + \dots + \lambda_n^2\omega_n^2$.

Lemma 4. Let μ be a positive normalized measure in n-dimensional. Euclidean space R_n, and let Q be an ellipsoid such that for each of its tangent planes, the measure of the half-space not containing the ellipsoid is less than ϵ. Then the measure of the region outside any given sphere of radius R and center at origin which contains Q does not exceed $C(\epsilon + H^2/R^2)$, where H^2 is the sum of the squares of the semiaxes of Q, and C is a constant which depends neither upon n nor upon the choice of sphere or ellipsoid Q.

Proof. Construct, for each unit vector ω, the plane $(x, \omega) = R/\sqrt{n}$ perpendicular to it which is tangent to the sphere of radius $R/\sqrt{n}$. Parallel to each of these planes there is a plane $(x, \omega) = r(\omega)$ which is tangent to the ellipsoid $Q(r(\omega)$ is the distance from this plane to the origin). Certain of the vectors ω satisfy the inequality $R/\sqrt{n} \leqslant r(\omega)$; let Ω_1 be the set of these vectors. Let Ω_2 denote the sets of those vectors remaining, i.e., those for which $R/\sqrt{n} > r(\omega)$.

We now show that the measure $\tau(\Omega_1)$ does not exceed H^2/R^2 (recall that τ is the measure on the set of unit vectors which corresponds to the ordinary normalized measure on the surface of the unit sphere in R_n). Indeed, it follows from Lemma 3 that

$$\frac{H^2}{n} = \langle r^2(\omega)\rangle = \int_\Omega r^2(\omega)\, d\tau \geqslant \int_{\Omega_1} r^2(\omega)\, d\tau.$$

But for $\omega \in \Omega_1$ we have $r^2(\omega) \geqslant R^2/n$, and therefore $H^2/n \geqslant \tau(\Omega_1)\, R^2/n$, which means that

$$\tau(\Omega_1) \leqslant \frac{H^2}{R^2}\,. \tag{13}$$

Let, as usual, $\mu(R/\sqrt{n}, \omega)$ denote the measure of the half-space $(x, \omega) \geqslant R/\sqrt{n}$. We wish to estimate its average value $\langle\mu(R/\sqrt{n}, \omega)\rangle$. Obviously

$$\begin{aligned}\left\langle \mu\left(\frac{R}{\sqrt{n}}, \omega\right)\right\rangle &= \int_\Omega \mu\left(\frac{R}{\sqrt{n}}, \omega\right) d\tau \\ &= \int_{\Omega_1} \mu\left(\frac{R}{\sqrt{n}}, \omega\right) d\tau + \int_{\Omega_2} \mu\left(\frac{R}{\sqrt{n}}, \omega\right) d\tau. \end{aligned}\tag{14}$$

From inequality (13) and the trivial estimate $\mu(R/\sqrt{n}, \omega) \leqslant 1$ it follows that

$$\int_{\Omega_1} \mu\left(\frac{R}{\sqrt{n}}, \omega\right) d\tau \leqslant \frac{H^2}{R^2}\,. \tag{15}$$

On the other hand, for $\omega \in \Omega_2$ the plane $(x, \omega) = R/\sqrt{n}$ is further from the origin than is the tangent plane to the ellipsoid Q which is parallel to it. Therefore for $\omega \in \Omega_2$ the half-space $(x, \omega) \geqslant R/\sqrt{n}$ lies in the half-space $(x, \omega) \geqslant r(\omega)$. But by the hypothesis of the lemma the measure of the half-space defined by a tangent plane to the ellipsoid and not containing the ellipsoid does not exceed ϵ. *A fortiori* we have $\mu(R/\sqrt{n}, \omega) \leqslant \epsilon$ for $\omega \in \Omega_2$. Taking into account the trivial estimate $\tau(\Omega_2) \leqslant 1$, we obtain

$$\int_{\Omega_2} \mu(R/\sqrt{n}, \omega)\, d\tau \leqslant \epsilon. \tag{16}$$

From (14), (15), and (16) follows the estimate

$$\langle \mu(R/\sqrt{n}, \omega) \rangle \leqslant \frac{H^2}{R^2} + \epsilon. \tag{17}$$

By Lemma 3, this implies that

$$1 - \mu(R) \leqslant C\left(\epsilon + \frac{H^2}{R^2}\right), \tag{18}$$

where C is a constant not depending either upon n, H, or R, and $1-\mu(R)$ is the measure of the region outside the sphere of radius R and center at the origin. This proves the lemma.

We will show that a lemma similar to Lemma 4 holds for spaces adjoint to a countably Hilbert space. We precede this lemma by the following remarks. Let $\Phi' = \bigcup_{n=1}^{\infty} \Phi_n'$ be the adjoint space of the countably Hilbert space $\Phi = \bigcap_{n=1}^{\infty} \Phi_n$; let Ψ be a finite-dimensional subspace of Φ and Ψ^0 its annihilator, and fix n. The natural mapping T of Φ' onto the factor space Φ'/Ψ^0 induces a mapping T_n of the subspace Φ_n' into Φ'/Ψ^0. But every Φ_n' is everywhere dense in Φ' (in the weak topology; cf. Chapter I, Section 3.1) and therefore the image of Φ_n' in Φ'/Ψ^0 is everywhere dense in Φ'/Ψ^0. Since Φ'/Ψ^0 is finite-dimensional and the image of Φ_n' is a linear subspace in Φ'/Ψ^0, it follows that T_n carries Φ_n' onto Φ'/Ψ^0. Therefore

$$\Phi'/\Psi^0 = \Phi_n'/\Psi^0 \cap \Phi_n'.$$

Now denote by Ψ^* the orthogonal complement of $\Psi^0 \cap \Phi_n'$ in the Hilbert space Φ_n' (i.e., the collection of those $F \in \Phi_n'$ such that $(F, F_1)_{-n} = 0$ for all $F_1 \in \Psi^0 \cap \Phi_n'$). Obviously the natural mapping of Φ_n' onto Φ'/Ψ^0 maps the subspace Ψ^* one-one onto Φ'/Ψ^0. Therefore any $F \in \Phi'$

can be written in unique fashion as $F = F^0 + F^*$, where $F^0 \in \Psi^0$, $F^* \in \Psi^*$. We will call Ψ^* the *orthogonal complement of* Ψ^0 *in* Φ'_n.†

We now proceed to the statement and proof of the analog of Lemma 4.

Lemma 4′. Let μ be a positive normalized cylinder set measure on the adjoint space

$$\Phi' = \bigcup_{n=1}^{\infty} \Phi'_n$$

of a countably Hilbert space $\Phi = \bigcap_{n=1}^{\infty} \Phi_n$. Let Q be an ellipsoid in the Hilbert space Φ'_n,[4] such that the sum of the squares of its principal semiaxes is equal to H^2, and the measure of any half-space in Φ', not containing Q, is less than ϵ. If $S_n(R) = \{\| F \|_{-n} \leqslant R\}$ is any ball in Φ'_n containing Q, then the measure of any cylinder set Z, lying outside $S_n(R)$, is less than $C(\epsilon + H^2/R^2)$, where C is the same constant as in Lemma 4.

Proof. Let Z be any cylinder set lying outside $S_n(R)$. Let Ψ^0 be its generating subspace, $A \subset \Phi'/\Psi^0$ its base, and denote by Ψ^* the orthogonal complement of $\Psi^0 \cap \Phi'_n$ in Φ'_n.

If F is any element in Φ', then, as we have seen, it can be written in unique fashion as $F = F^0 + F^*$, where $F^0 \in \Psi^0$, $F^* \in \Psi^*$. Denote by P the mapping which takes $F = F^0 + F^*$ into F^*, i.e., $P(F) = F^*$. Since by construction Ψ^* is orthogonal to $\Psi^0 \cap \Phi'_n$, then for elements $F \in \Phi'_n$, P is the orthogonal projection of Φ'_n onto Ψ^*. This orthogonal projection takes $S_n(R)$ into a ball $S_n^*(R)$ in the subspace Ψ^* and Q into an ellipsoid Q^* lying in Ψ^*. Since the cylinder set Z lies outside $S_n(R)$, its image Z^* in Ψ^* lies outside $S_n^*(R)$.

We introduce a measure μ^* in Ψ^*, defined by

$$\mu^*(X) = \mu[P^{-1}(X)] \equiv \mu(X + \Psi^0).$$

and apply Lemma 4 to $S_n^*(R)$, Q^*, and μ^* in the finite-dimensional space Ψ^*. For this we note that the sum of the squares of the principal semiaxes of Q^* does not exceed H^2. Indeed, the ellipsoid Q can be considered as the image of the unit sphere in Φ'_n under a mapping T

† For the case at hand, the foregoing seems unnecessarily involved. Indeed, given $n > 0$ and $F \in \Phi'$, there is a unique $\psi_F \in \Psi$ such that $(F, \varphi) = (\varphi, \psi_F)_n$ for all $\varphi \in \Psi$. Define $F^* \in \Phi'$ by $(F^*, \varphi) = (\varphi, \psi_F)_n$; then $F^0 \equiv F - F^* \in \Psi^0$, and $F = F^0 + F^*$ is the desired decomposition.

[4] By an ellipsoid in the Hilbert space Φ'_n we mean the image of the unit sphere $\| F \|_{-n} \leqslant 1$ by some linear mapping T. The condition that the series consisting of the sum of the squares of the principal semiaxes of this ellipsoid converge means that T is a Hilbert–Schmidt operator.

whose Hilbert–Schmidt norm equals H. The ellipsoid Q^* is the image of the unit sphere in Φ_n' under the mapping PT. But the Hilbert–Schmidt norm of PT does not exceed H.[5] Consequently, the sum of the squares of the semiaxes of Q^* does not exceed H^2. Let us now show that the measure of any half-space in Ψ^* which does not intersect Q^* is less than ϵ. Indeed, if C^* is a half-space in Ψ^* which does not intersect Q^*, then $C = C^* + \Psi^0$ is a half-space in Φ' which does not intersect the ellipsoid Q, and therefore

$$\mu^*(C^*) = \mu(C^* + \Psi^0) = \mu(C) \leqslant \epsilon.$$

Finally, since $Q \subset S_n(R)$, then $Q^* \subset S_n^*(R)$. Therefore, in view of Lemma 4, the μ^*-measure of the region outside $S_n^*(R)$ does not exceed $C(\epsilon + H^2/R^2)$. But the base A^* of Z lies† outside $S_n^*(R)$, and therefore

$$\mu(Z) = \mu^*(A^*) \leqslant C(\epsilon + H^2/R^2).$$

Let us now proceed to our main goal—the proof of Theorem 3. In other words, we wish to prove that a positive normalized cylinder set measure in the adjoint space of a nuclear space is countably additive.

Proof of Theorem 3. As has been shown in Theorem 2′, to prove the countable additivity of μ it suffices to show that for any $\epsilon > 0$ one can find n and R such that the measure of any cylinder set lying outside the ball $S_n(R) = \{\| F \|_{-n} \leqslant R\}$ is less than ϵ. First we use the continuity condition imposed upon μ. It follows from this condition that there exists a ball $S_m(\rho) = \{\| F \|_{-m} \leqslant \rho\}$ such that the measure of any half-space in Φ' which does not intersect $S_m(\rho)$ has measure less than $\epsilon/2C$, where C is the constant in Lemma 4′.††

Since the space Φ is nuclear, there is an n such that the ball $S_m(\rho)$, considered in the Hilbert space Φ_n', is an ellipsoid, the sum of whose

[5] In fact, if $f_1, f_2, \ldots$ is an orthonormal basis in Φ_n', then

$$\| PT \|_2 = \left[\sum_{k=1}^{\infty} \| PTf_k \|^2\right]^{\frac{1}{2}} \leqslant \| P \| \left[\sum_{k=1}^{\infty} \| Tf^k \|^2\right]^{\frac{1}{2}} = \| P \| \, \| T \|_2 = H \| P \|.$$

Since P is a projection operator, $\| P \| = 1$ and therefore $\| PT \|_2 \leqslant H$.

† A^* is the image of A under the one-one correspondence between Ψ^* and Φ'/Ψ^0 mentioned just preceding Lemma 4′.

†† Indeed, since sequential and ordinary continuity are equivalent in a nuclear space, for any δ, $A > 0$ there is a neighborhood U of zero in Φ (say, $U = \{\| \varphi \|_m < a)$ such that $\mu\{|(F, \varphi)| \geqslant A\} < \delta$ for $\varphi \in U$. Choose any $A > 0$ and take $\delta = \epsilon/2C$; then $\{\| F \|_{-m} \leqslant A/a\}$ is the desired ball, which follows from the observation that any half-space which does not contain the zero functional can be written as $\{(F, \psi) \geqslant A\}$, and from the existence of nonzero functionals F for which $(F, \psi) = \| F \|_{-m} \| \psi \|_m$.

principal semiaxes is finite (here n must be chosen so that the mapping of Φ'_m into Φ'_n is Hilbert–Schmidt). Let H^2 denote the sum of the squares of the principal semiaxes of the ellipsoid $S_m(\rho)$ in Φ'_n, and choose R so large that the ball $S_n(R)$ in Φ'_n contains the ellipsoid $S_m(\rho)$, and also

$$\frac{H^2}{R^2} \leqslant \frac{\epsilon}{2C} \,. \tag{19}$$

By Lemma 4′, for any cylinder set Z in Φ', lying outside $S_n(R)$, one has the estimate

$$\mu(Z) \leqslant C\left(\frac{\epsilon}{2C} + \frac{H^2}{R^2}\right).$$

It follows from (19) that $\mu(Z) \leqslant \epsilon$.

Thus, we have found a ball $S_n(R)$ such that the measure of any cylinder set Z which lies outside $S_n(R)$ has μ-measure not exceeding the given value $\epsilon > 0$. Hence Theorem 2′ implies that the measure μ is countably additive.

This concludes the proof of Theorem 3.

It can be shown that the nuclearity of the space Φ is not only a sufficient, but also a necessary condition for *every* cylinder set measure in the adjoint space Φ' to be countably additive. This assertion is proven by constructing, for any nonnuclear countably Hilbert space Φ, a positive normalized cylinder set measure in Φ' which is not countably additive. Moreover, this measure can be chosen to be a so-called Gaussian measure (cf. Theorem 2 of Section 3).

We have already mentioned that in certain questions one can consider, instead of nuclear spaces, two Hilbert spaces which are connected by a nuclear mapping. Let us indicate here the analog of Theorem 3 which is obtained by this replacement.

Theorem 4. Let H_1 and H_2 be Hilbert spaces, and let T be a nuclear mapping of H_2 into H_1. Suppose that μ is a positive normalized cylinder set measure, satisfying the continuity condition, in the adjoint space H_1' of H_1. Then the measure in H_2' induced[6] by T and μ is countably additive.

The proof of this theorem is a word-for-word repetition of the proof of Theorem 3. The point is that in proving Theorem 3 all that was used was that μ is continuous in one of the Hilbert spaces Φ'_m and that there exists a nuclear mapping of Φ'_m into one of the spaces Φ'_n.

We note that while Theorem 3 talks about *all* measures in Φ', Theorem

[6] Concerning a measure induced by a mapping, cf. Section 1.4.

4 concerns only measures in H'_2 which are induced by some measure in H'_1.

2.4. The Countable Additivity of Cylinder Set Measures in Spaces Adjoint to Union Spaces of Nuclear Spaces

The theorem proven in the preceding section is not adequate for certain applications. From example, this theorem does not apply to measures on the cylinder sets in the conjugate space K' of the space K of all infinitely differentiable functions having bounded supports. This is due to the fact that K is not a countably Hilbert space. but rather the union space of its nuclear subspaces $K(a)$, consisting of those functions in K which vanish for $|x| \leqslant a$. Now every subspace $K(a)$ is closed in K, and a sequence $\{\varphi_m(x)\}$ of functions in K converges to zero if and only if all of the $\varphi_m(x)$ belong to some one subspace $K(a)$ and converge to zero in the topology of $K(a)$. Moreover, from the general form of linear functionals on $K(a)$ it follows that every linear functional on $K(a)$ can be extended to all of K.

In order to obtain a theorem on the complete additivity of measures on cylinder sets which is also useful for the space K', we introduce the concept of the union space of linear topological spaces.

Let $\Phi^{(1)} \subset \Phi^{(2)} \subset \ldots$ be an increasing sequence of linear topological spaces such that the topology in each $\Phi^{(m)}$ coincides with that induced in it by $\Phi^{(m+1)}$. We call the linear space $\Phi = \bigcup_{m=1}^{\infty} \Phi^{(m)}$ the *union space* of the $\Phi^{(m)}$. The space Φ is not necessarily a linear topological space; however, we will say that a sequence $\{\varphi_n\}$ of elements in Φ converges to zero, if all of the φ_n belong to some one $\Phi^{(m)}$, and $\{\varphi_n\}$ converges to zero in $\Phi^{(m)}$.

An additive homogeneous functional on Φ will be called *linear*, if its restriction to each $\Phi^{(m)}$ is a linear functional on $\Phi^{(m)}$. Conversely, we will *assume* that any linear functional $F^{(m)}$ on $\Phi^{(m)}$ can be extended (not necessarily uniquely) to a linear functional F on Φ. Thus, every linear functional $F^{(m)}$ on $\Phi^{(m)}$ can be considered as the restriction to $\Phi^{(m)}$ of some linear functional F on Φ. The functionals $F^{(1)}$, $F^{(2)}$, ... corresponding to a given functional F on Φ are compatible in the following sense: if $m \leqslant n$, then for any element $\varphi \in \Phi^{(m)}$ we have $(F^{(m)}, \varphi) = (F^{(n)}, \varphi)$, since both sides coincide with (F, φ).

Thus we see that an element F of the conjugate space Φ' can be written in the form

$$F = \{F^{(1)}, F^{(2)}, \ldots\},$$

where $F^{(m)} \in \Phi^{(m)\prime}$, and the functionals $F^{(1)}$, $F^{(2)}$, ... are compatible.

We now consider cylinder sets in Φ'. It is easy to see that any such set is defined by a cylinder set $Z^{(m)}$ in one of the spaces $\Phi^{(m)\prime}$ and consists of all elements $F = \{F^{(1)}, F^{(2)}, \ldots\}$ of the space Φ' for which $F^{(m)} \in Z^{(m)}$. We will denote the cylinder set in Φ', defined by a cylinder set $Z^{(m)}$ in $\Phi^{(m)\prime}$, by $Z(Z^{(m)})$.

It is easily verified that the weak neighborhoods in Φ' coincide with the cylinder sets in Φ' defined by the weak neighborhoods in the spaces $\Phi^{(m)\prime}$.

Further on we will use the mappings T^m of Φ' into $\Phi^{(m)\prime}$, which carry an element $F = \{F^{(1)}, F^{(2)}, \ldots\}$ of Φ' into its mth coordinate $F^{(m)}$. From the compatibility of the $F^{(m)}$ it follows that for $m \leqslant n$ we have $T^m(T^n)^{-1}F^{(n)} = F^{(m)}$. Therefore, setting $T_m^n = T^m(T^n)^{-1}$, we obtain a mapping of the space $\Phi^{(n)\prime}$ onto $\Phi^{(m)\prime}$, by which every functional $F^{(n)}$ is carried into a compatible functional $F^{(m)}$.

Let us now consider a measure μ on the cylinder sets in Φ'. This measure defines a measure on the cylinder sets in every one of the $\Phi^{(m)\prime}$. Indeed, we set

$$\mu_m(Z^{(m)}) = \mu[Z(Z^{(m)})]$$

for any cylinder set $Z^{(m)}$ in $\Phi^{(m)\prime}$, where $Z(Z^{(m)})$ is the cylinder set in Φ' defined by $Z^{(m)}$. Obviously μ_m is a measure on the cylinder sets in $\Phi^{(m)\prime}$. The measures $\mu_1, \mu_2, \ldots$ thus obtained are compatible in the sense that for $m \leqslant n$ and any $Z^{(m)}$ in $\Phi^{(m)\prime}$ we have

$$\mu_m(Z^{(m)}) = \mu_n[(T_m^n)^{-1}Z^{(m)}]. \tag{20}$$

Indeed, it is easy to see that the sets $Z^{(m)}$ and $(T_m^n)^{-1}Z^{(m)}$ define the same cylinder set Z in Φ', and therefore both sides of (20) coincide with $\mu(Z)$.

The converse is also true: if $\mu_1, \mu_2, \ldots$, is any sequence of compatible measures on the cylinder sets in the spaces $\Phi^{(1)\prime}, \Phi^{(2)\prime}, \ldots$, then there exists a measure μ on Φ' such that

$$\mu[Z(Z^{(m)})] = \mu_m(Z^{(m)})$$

for all m and all cylinder sets $Z^{(m)}$ in $\Phi^{(m)\prime}$.

Of course, since the measures μ_m are finitely additive, μ will also be finitely additive. With somewhat greater effort, one can show that the countable additivity of each of the μ_m implies, under certain assumptions regarding μ, the countable additivity of μ. That is, the following theorem holds.

Theorem 5. Let Φ be the union space of the countably normed spaces $\Phi^{(1)}, \Phi^{(2)}, \ldots$. Suppose further that μ is a positive normalized measure on the conjugate space Φ', such that for every m the measure μ_m induced by μ on $\Phi^{(m)\prime}$ is countably additive and regular in the sense of Caratheodory. Then μ is also countably additive.

Proof. According to Theorem 1′, we have to show that any decreasing sequence $Z_1 \supset Z_2 \supset \ldots$ of cylinder sets Z_k in Φ', such that $\lim_{k\to\infty} \mu(Z_k) = \epsilon > 0$, has nonempty intersection. Now each of the Z_k is defined by a cylinder set $Z^{(m_k)}$ in $\Phi^{(m_k)\prime}$. If the set of all m_k is bounded, then without loss of generality we may suppose that $m_1 = m_2 = \ldots = m$. In this case the nonemptiness of $\bigcap_{k=1}^{\infty} Z_k$ follows from the countable additivity of μ_m.

Now assume that the m_k are unbounded. In this case we may without loss of generality suppose that $m_1 < m_2 < \ldots$. By hypothesis each of the μ_{m_k} is regular in the sense of Caratheodory. Therefore in each of the spaces $\Phi^{(m_k)\prime}$ there is a weakly closed set B_k such that $B_k \subset Z^{(m_k)}$ and

$$\mu_{m_k}[Z^{(m_k)} - B_k] \leqslant \frac{\epsilon}{2^{k+2}} .$$

Further, by Theorem 2 the countable additivity of μ_{m_k} implies the existence in each $\Phi^{(m_k)\prime}$ of a weakly compact set (a ball) C_k such that $\mu_{m_k}(C_k) \geqslant 1 - \epsilon/2^{-(k+2)}$. Obviously $D_k = B_k \cap C_k$ is also weakly compact and

$$\mu_{m_k}[Z^{(m_k)} - D_k] \leqslant \frac{\epsilon}{2^{k+1}} .$$

Set

$$E_k = \bigcap_{j\leqslant k} (T^{m_k}_{m_j})^{-1} D_j .$$

It is easy to see that $E_k \subset Z^{(m_k)}$, and

$$\mu_{m_k}[Z^{(m_k)} - E_k] \leqslant \sum_{j=1}^{k} \frac{\epsilon}{2^{j+1}} \leqslant \tfrac{1}{2}\epsilon.$$

Therefore $\mu_{m_k}(E_k) \geqslant \frac{1}{2}\epsilon$. It is further obvious that each of the E_k is weakly compact in $\Phi^{(m_k)\prime}$, and $T^{m_k}_{m_j} E_k \subset E_j$ for $j < k$.

We now choose an element $F^{(m_k)}$ from each of the E_k (the E_k are nonempty, as $\mu_{m_k}(E_k) \geqslant \frac{1}{2}\epsilon$). For $j \leqslant m_k$ set

$$F_k^{(j)} = T^{m_k}_{m_j} F^{(m_k)}.$$

Since $T^{m_k}_{m_1}(E_k) \subset E_1$, the elements $F_1^{(1)}$, $F_2^{(1)}$, ... belong to the weakly compact set E_1. One can therefore choose a subsequence

$$F^{(1)}_{i_1}, \quad F^{(1)}_{i_2}, \ldots$$

which converges weakly to some element F_{m_1} in $\Phi^{(m_1)'}$. Now we consider the elements $F_{i_1}^{(2)}$, $F_{i_2}^{(2)}$, ... (we may clearly assume that $i_1 \geqslant 2$) which lie in E_2 (since $T_{m_2}^{m_k} E_k \subset E_2$). Since E_2 is weakly compact, we can find a weakly convergent subsequence of $\{F_{i_n}^{(2)}\}$ which has limit F_{m_2} in $\Phi^{(m_2)'}$. From the weak continuity of $T_{m_1}^{m_2}$ (which is easy to show) it follows that $T_{m_1}^{m_2} F_{m_2} = F_{m_1}$. In similar fashion we construct elements F_{m_3}, F_{m_4}, ... in $\Phi^{(m_3)'}$, $\Phi^{(m_4)'}$, ..., such that $F_{m_j} = T_{m_j}^{m_k} F_{m_k}$ for $j \leqslant k$, and we set $F_m = T_m^{m_k} F_{m_k}$ for $m < m_k$. Then $F = \{F_1, F_2, ...\}$ is an element in Φ', and moreover belongs to each of the Z_k, since $F_{m_k} \in E_k \subset Z^{(m_k)}$. This proves that the sequence Z_1, Z_2, ... of cylinder sets in Φ' has nonempty intersection, and therefore that μ is countably additive, which completes the proof.

Since, according to Theorem 3, any positive normalized cylinder set measure, in the conjugate space of a nuclear space, which satisfies the continuity condition is countably additive, we obtain from Theorem 5 the following result.

Theorem 6. Suppose that Φ is the union space of an increasing sequence $\Phi^{(1)} \subset \Phi^{(2)} \subset ...$ of nuclear spaces. Then any positive normalized measure, on the cylinder sets of the conjugate space Φ', which satisfies the continuity condition is countably additive.

Since the space K is the union space of its nuclear subspaces $K(n)$, any measure on the cylinder sets of K' which has the properties indicated above is countably additive.†

2.5. A Condition for the Countable Additivity of Measures on the Cylinder Sets in a Hilbert Space

Theorem 3 gives a condition for the countable additivity of all cylinder set measures in the conjugate space of a countably Hilbert space. In this paragraph we indicate a condition for the countable additivity of a specified measure μ defined on a Hilbert space H.

Let B_1, B_2, ... be a sequence of positive-definite operators in H, with domain H. By means of these operators we introduce a new topology in H, taking as a base of neighborhoods of zero the system of sets $U_n(R)$ in H, where $U_n(R)$ consists of all $\varphi \in H$ satisfying $(B_n\varphi, \varphi) \leqslant R^2$. We will call this topology the *topology defined by the operators* B_1, B_2, ... The following theorem holds.

† It is perhaps worth mentioning explicitly that this result plays the same role in the theory of generalized random processes as does the Kolmogorov extension theorem in the theory of ordinary random processes.

Theorem 7. In order that a positive normalized measure μ on the cylinder sets[7] in the Hilbert space H be countably additive, it is necessary and sufficient that μ be continuous relative to the topology in H defined by some sequence $B_1, B_2, \ldots$ of positive-definite nuclear operators.

The continuity of μ means the following: For any $\epsilon > 0$ there exists a $\delta > 0$ and n such that the inequality $(B_n\varphi, \varphi) \leqslant \delta$ implies that $\mu(\Gamma_\varphi) \leqslant \epsilon$, where Γ_φ denotes the strip defined by $|(\varphi, \psi)| \geqslant 1$.

Proof. First we prove the necessity of the condition. Suppose that μ is countably additive. We construct for any $n > 0$ a positive-definite nuclear operator B_n such that the inequality $(B_n\varphi, \varphi) \leqslant 1/2n$ implies that $\mu(\Gamma_\varphi) < 1/n$. This operator is constructed in the following way. Since μ is countable additive, it can be extended to all balls in H. But H is the union of a countable family of increasing balls, and therefore there is a ball $S(R) = \{\| \varphi \| \leqslant R\}$ such that the measure of its complement is less than $1/2n$. We define B_n by setting

$$(B_n\varphi, \varphi) = \int_{S(R)} |(\varphi, \psi)|^2 \, d\mu(\psi).$$

Obviously B_n is positive-definite. To show that it is nuclear, we note that for any orthonormal basis $\{\varphi_k\}$ in H one has

$$\begin{aligned}\sum_{k=1}^{\infty} (B_n\varphi_k, \varphi_k) &= \int_{S(R)} \sum_{k=1}^{\infty} |(\varphi_k, \psi)|^2 \, d\mu(\psi) \\ &= \int_{S(R)} \| \psi \|^2 \, d\mu(\psi) \leqslant R^2 \int_{S(R)} d\mu(\psi) \leqslant R^2.\end{aligned}$$

In other words, the series $\sum_{k=1}^{\infty} (B_n\varphi_k, \varphi_k)$ converges for any orthonormal basis $\{\varphi_k\}$. As was shown in Chapter I, Section 2.3, it follows that B_n is a nuclear operator.

Now consider any element φ such that $(B_n\varphi, \varphi) \leqslant 1/2n$, and let us estimate the measure of the strip Γ_φ defined by $|(\varphi, \psi)| \geqslant 1$. Obviously

$$\mu(\Gamma_\varphi) = \mu(\Gamma'_\varphi) + \mu(\Gamma''_\varphi),$$

where Γ'_φ is that part of Γ_φ contained in the ball $S(R)$, and Γ''_φ is that part lying outside $S(R)$. In view of the choice of $S(R)$ we have $\mu(\Gamma''_\varphi) \leqslant 1/2n$.

[7] By a cylinder set in H we mean the collection of those elements φ for which

$$((\varphi, \varphi_1), \ldots, (\varphi, \varphi_k)) \in A$$

where $\varphi_1, \ldots, \varphi_k$ are elements of H, and A is a Borel set in k-dimensional Euclidean space. Since (φ, φ_j) is a linear functional on H, this definition agrees with that given in Section 1.

On the other hand, from the inequality $|(\varphi, \psi)| \geqslant 1$, which holds for all $\psi \in \Gamma_\varphi$ and therefore for all $\psi \in \Gamma'_\varphi$, it follows that

$$\mu(\Gamma'_\varphi) = \int_{\Gamma'_\varphi} d\mu(\psi) \leqslant \int_{\Gamma'_\varphi} |(\varphi, \psi)|^2 \, d\mu(\psi)$$

$$\leqslant \int_{S(R)} |(\varphi, \psi)|^2 \, d\mu(\psi) = (B_n\varphi, \varphi) \leqslant \frac{1}{2n} \, .$$

Hence $\mu(\Gamma_\varphi) \leqslant 1/n$.

Thus we have constructed a sequence B_1, B_2, ... of positive-definite nuclear operators such that the inequality $(B_n\varphi, \varphi) \leqslant 1/2n$ implies $\mu(\Gamma_\varphi) \leqslant 1/n$, where Γ_φ is the strip defined by $|(\varphi, \psi)| \geqslant 1$. This means that μ is continuous relative to the topology in H defined by the B_k, and proves the necessity of the condition of the theorem.

The sufficiency of the condition can be proven by using Lemma 4. We omit the details of this proof.

3. Gaussian Measures in Linear Topological Spaces

3.1. Definition of Gaussian Measures

We will consider here Gaussian measures in linear topological spaces. First we describe Gaussian measures in the finite-dimensional case. Let R_n be an n-dimensional linear space, in which is defined a scalar product (x, y). This scalar product defines a metric in R_n and, in particular, defines Lebesgue measure in R_n. We introduce the Gaussian measure in R_n corresponding to the scalar product (x, y), setting

$$\mu(X) = \frac{1}{(2\pi)^{\frac{1}{2}n}} \int_X e^{-\frac{1}{2}(x,x)} \, dx. \tag{1}$$

In other words, Gaussian measures in an n-dimensional linear space are always defined by means of scalar products.

If A is a non-degenerate linear transformation in R_n and (x, y) is a scalar product in R_n, then $(x, y)_1 = (Ax, Ay)$ is also a scalar product. To this scalar product there corresponds the Gaussian measure

$$\mu_1(X) = \frac{1}{(2\pi)^{\frac{1}{2}n}} \int_X e^{-\frac{1}{2}(Ax,Ax)} \, d_1x,$$

where d_1x denotes the Lebesgue measure corresponding to the scalar product $(x, y)_1$. It is easy to see that $d_1x = |\det A|\, dx$. Therefore the measure μ_1 can be written in the form

$$\mu_1(x) = \frac{|\det A|}{(2\pi)^{\frac{1}{2}n}} \int_X e^{-\frac{1}{2}(Ax,Ax)}\, dx.$$

We note the following lemma concerning Gaussian measures in finite-dimensional spaces, which we will use further on.

Lemma 1. Let R_n be an n-dimensional Euclidean space with scalar product (x, y), and R_m an m-dimensional subspace in R_n. Let μ_n be the Gaussian measure in R_n corresponding to the scalar product (x, y), and denote by μ_m the Gaussian measure in R_m corresponding to the same scalar product. Then for any subset X of R_m we have the following compatibility condition between μ_n and μ_m:

$$\mu_m(x) = \mu_n[Q^{-1}(x)], \tag{2}$$

where Q denotes the operator of orthogonal projection of R_n onto R_m.

Proof. Let R_{n-m} denote the inverse image of the origin with respect to Q. Obviously R_n is the orthogonal sum of the subspaces R_m and R_{n-m}, and therefore any $x \in R_n$ can be written in unique fashion as $x = x' + x''$, where $x' \in R_m$ and $x'' \in R_{n-m}$. It is also obvious that

$$(x, y) = (x', y') + (x'', y''),$$

and that the Lebesgue measure dx in R_n is the product of the Lebesgue measures dx' in R_m and dx'' in R_{n-m}, defined by the scalar product (x, y). Since the set $Q^{-1}(X)$ is the orthogonal sum of X and R_{n-m}, we have

$$\mu_n[Q^{-1}(X)] = \frac{1}{(2\pi)^{\frac{1}{2}n}} \int_{Q^{-1}(X)} e^{-\frac{1}{2}(x,x)}\, dx$$
$$= \frac{1}{(2\pi)^{\frac{1}{2}n}} \int_X e^{-\frac{1}{2}(x',x')}\, dx' \int_{R_{n-m}} e^{-\frac{1}{2}(x'',x'')}\, dx''.$$

Since

$$\mu_m(X) = \frac{1}{(2\pi)^{\frac{1}{2}m}} \int_X e^{-\frac{1}{2}(x',x')}\, dx',$$

to prove (2) it suffices to verify that

$$\frac{1}{(2\pi)^{\frac{1}{2}(n-m)}} \int_{R_{n-m}} e^{-\frac{1}{2}(x'',x'')}\, dx'' = 1. \tag{3}$$

But in coordinate form, the integral in (3) is

$$\frac{1}{(2\pi)^{\frac{1}{2}(n-m)}} \int \exp\{-\tfrac{1}{2}[(x_1'')^2 + \dots + (x_{n-m}'')^2]\} \, dx_1'' \dots dx_{n-m}''. \tag{3'}$$

Since

$$\int_{-\infty}^{\infty} \exp(-\tfrac{1}{2}x^2) \, dx = \sqrt{2\pi},$$

the integral (3′) is equal to 1, which proves (2).

Along with the Gaussian measures just considered, so-called improper (or degenerate) Gaussian measures can also be considered. These are defined by a formula of the form

$$\mu(X) = \frac{1}{(2\pi)^{\frac{1}{2}m}} \int_{X \cap R_m} e^{-\frac{1}{2}(x,x)} \, dx, \tag{4}$$

where R_m is an m-dimensional linear subspace of R_n, (x, y) is a scalar product in R_m, and dx is the Lebesgue measure in R_m defined by this scalar product.

Let us now proceed to the construction of Gaussian measures in the infinite-dimensional case. These measures will be constructed in the conjugate space Φ' of a locally convex linear topological space Φ. As we saw in Section 1.1, the local convexity of Φ guarantees that any linear functional on any subspace Ψ of Φ can be extended to a linear functional on all of Φ. In addition, the conjugate space Ψ' of any finite-dimensional subspace Ψ of Φ is isomorphic to the factor space Φ'/Ψ^0, where Ψ^0 denotes the annihilator of Ψ (i.e., the collection of all linear functionals on Φ such that $(F, \psi) = 0$ for $\psi \in \Psi$).

Gaussian measures in an infinite-dimensional space Φ' are defined by means of scalar products $B(\varphi, \psi)$ defined in Φ.

Thus, suppose that $B(\varphi, \psi)$ is a nondegenerate scalar product in a real locally convex linear topological space Φ, continuous in the topology of Φ. First we define a Gaussian measure in every finite-dimensional subspace Ψ of Φ by means of the scalar product $B(\varphi, \psi)$, and then we carry over these measures to the factor spaces Φ'/Ψ^0.

We define a measure τ_Ψ in each n-dimensional subspace Ψ of Φ by

$$\tau_\Psi(Y) = \frac{1}{(2\pi)^{\frac{1}{2}n}} \int_Y e^{-\frac{1}{2}B(\psi,\psi)} \, d\psi, \tag{5}$$

where $d\psi$ is the Lebesgue measure in Ψ corresponding to the scalar product $B(\varphi, \psi)$. In view of Lemma 1 these measures are compatible in the following sense: If $\Psi_1 \subset \Psi_2$, then for any $Y \subset \Psi_1$ we have

$$\tau_1(Y) = \tau_2[Q_1^{-1}(Y)], \tag{6}$$

where Q_1 denotes the operator of orthogonal projection of Ψ_2 onto Ψ_1 (relative to the scalar product $B(\varphi, \psi)$), and τ_1, τ_2 are the measures τ_{Ψ_1} and τ_{Ψ_2}.

We remark that the finite-dimensional Euclidean space Ψ (with scalar product $B(\varphi, \psi)$) is isomorphic to its conjugate space Ψ'.[1] But, as was pointed out above, the space Ψ' is isomorphic to the factor space Φ'/Ψ^0. We have thus established a natural isomorphism A_Ψ between Ψ and Φ'/Ψ^0. In view of this isomorphism there corresponds to the measure τ_Ψ in Ψ a measure ν_Ψ in Φ'/Ψ^0, defined by

$$\nu_\Psi(X) = \tau_\Psi[A_\Psi^{-1}(X)], \qquad X \subset \Phi'/\Psi^0. \tag{7}$$

Let us show that the measures ν_Ψ define a measure on the cylinder sets in Φ', i.e., that they are compatible. For this, according to Section 1.3, it suffices to show that if $\Psi_1 \subset \Psi_2$, then for any set $X \subset \Phi'/\Psi_1^0$ one has[2]

$$\nu_1(X) = \nu_2[Q^{-1}(X)], \tag{8}$$

where Q denotes the natural mapping of Φ'/Ψ_2^0 into Φ'/Ψ_1^0.

In view of (7), we can write (8), which is to be proven, in the following form:

$$\tau_1[A_1^{-1}(X)] = \tau_2[A_2^{-1}Q^{-1}(X)]$$

(we have denoted A_{Ψ_1} and A_{Ψ_2} by A_1 and A_2 respectively). If we set $A_1^{-1}(X) = Y$, then this equation becomes

$$\tau_1(Y) = \tau_2[A_2^{-1}Q^{-1}A_1(Y)]. \tag{9}$$

In view of relation (6) it remains to show that the mapping $Q_1 = A_1^{-1}QA_2$ is the orthogonal projection of Ψ_2 onto Ψ_1. But this follows directly from the fact that $A_1^{-1}QA_2$ acts on elements $\psi \in \Psi_1$ according to the scheme

$$\psi \xrightarrow[A_2]{} A_2\psi + \Psi_2^0 \xrightarrow[Q]{} A_2\psi + \Psi_1^0 \xrightarrow[A_1^{-1}]{} \psi,$$

and on elements φ in the orthogonal complement of Ψ_1 in Ψ_2 according to[3]

$$\varphi \xrightarrow[A_2]{} A_2\varphi + \Psi_2^{(0)} \xrightarrow[Q]{} \Psi_1^0 \xrightarrow[A_1^{-1}]{} 0.$$

This proves that the measures ν_Ψ are compatible.

[1] With every element $\psi \in \Psi$ we associate the linear functional F_ψ, defined by $F_\psi(\varphi) = (B\,\varphi, \psi)$. From the nondegeneracy of $B\,(\varphi, \psi)$ it follows that the image of Ψ is all of Ψ'.

[2] We denote here ν_{Ψ_1} by ν_1, ν_{Ψ_2} by ν_2, and further τ_{Ψ_1} by τ_1 and τ_{Ψ_2} by τ_2.

[3] Since $B(\varphi, \psi) = 0$, $\psi \in \Psi_1$, for such φ, then $A_2\varphi \in \Psi_1^0$ and therefore $A_2\varphi + \Psi_2^0 \subset \Psi_1^0$.

We denote by μ the measure defined on the cylinder sets of Φ' by the ν_{Ψ}.

From the fact that the scalar product $B(\varphi, \psi)$ was assumed to be continuous in the topology of Φ, it follows easily that the measure μ satisfies the continuity condition. We omit the simple proof of this statement, which involves writing down the measures in coordinate form.

One can remove the condition of nondegeneracy that was imposed upon $B(\varphi, \psi)$. Assume that there are nonzero elements φ for which $B(\varphi, \varphi) = 0$, and let X be the totality of all such φ. Then X is a linear subspace of Φ which is closed in view of the assumed continuity of $B(\varphi, \psi)$. Let X^0 be the subspace in Φ' consisting of all continuous linear functionals F which vanish on X. Then X^0 is the conjugate space of the factor space Φ/X, and $B(\varphi, \psi)$ induces a nondegenerate scalar product $B_1(\varphi, \psi)$ on the latter. We can therefore construct the Gaussian measure μ_1 in X^0 defined by the scalar product $B_1(\varphi, \psi)$. The measure μ in Φ', defined for any cylinder set Y in Φ' by

$$\mu(Y) = \mu_1(Y \cap X^0),$$

is called a *Gaussian measure in Φ'*. Such Gaussian measures, which are concentrated on subspaces of Φ', are called *degenerate* or *improper.*

Let us now stop to consider the case where Φ is a Hilbert space, and the Gaussian measure μ is defined by the scalar product (φ, ψ) in Φ. In this case the spaces Φ and Φ' can be identified, and we can suppose that μ is defined in the space Φ itself. The cylinder sets Z in Φ are the orthogonal sums of subspaces A^0 of Φ, having finite-dimensional orthogonal complements A, with (Borel) sets X lying in A. The measure of such a cylinder set is given by

$$\mu(Z) = \frac{1}{(2\pi)^{\frac{1}{2}n}} \int_X e^{-\frac{1}{2}(\varphi, \varphi)} \, d\varphi,$$

where (φ, ψ) is the scalar product induced in the finite dimensional subspace A by the scalar product in Φ, and $d\varphi$ is the corresponding Lebesgue measure in A.

3.2. A Condition for the Countable Additivity of Gaussian Measures in the Conjugate Spaces of Countably Hilbert Spaces

In Theorem 4 of Section 2, it was shown that if a measure μ_1 in a Hilbert space H_1 satisfies the continuity condition and if T is a Hilbert–

Schmidt mapping of H_1 into a Hilbert space H_2, then the measure μ_2 in H_2, induced by T and μ_1, is countably additive. With the help of this theorem it is easy to show a sufficient condition for a Gaussian measure μ, defined in the conjugate space Φ' of a countably Hilbert space Φ by a scalar product $B(\varphi, \psi)$ in Φ, to be countably additive.

Let $B(\varphi, \psi)$ be a scalar product (nondegenerate) in a countably Hilbert space Φ, which is continuous jointly in both arguments relative to the topology of Φ.† Completing Φ relative to this scalar product, we obtain a Hilbert space Φ_B. From the continuity of $B(\varphi, \psi)$ it follows that the natural imbedding of Φ into Φ_B is continuous. Therefore there is an m such that for $n \geqslant m$ the imbedding T_n of the Hilbert space Φ_n into Φ_B is continuous (Φ_n is the completion of Φ relative to the scalar product $(\varphi, \psi)_n$). A sufficient condition for the countable additivity of the Gaussian measure on Φ' defined by $B(\varphi, \psi)$ is given by the following theorem.

Theorem 1. In order that the measure μ, defined in the conjugate space Φ' of a countably Hilbert space Φ by a continuous nondegenerate scalar product $B(\varphi, \psi)$, be countably additive, it is sufficient that for some n the mapping T_n of Φ_n into Φ_B be of Hilbert–Schmidt class.

Proof. The scalar product $B(\varphi, \psi)$ in the Hilbert space Φ_B defines a Gaussian measure μ_1 in the conjugate space Φ_B' which satisfies the continuity condition. The mapping T_n' adjoint to T_n maps Φ_B' into Φ_n' and is also Hilbert–Schmidt. Now apply Theorem 4 of Section 2 to the Hilbert spaces Φ_B' and Φ_n' and to the measure μ_1 in Φ_B'. We find that μ_1 induces a countably additive measure μ_n in Φ_n', which in turn induces a countably additive measure in Φ'. Obviously this last measure coincides with the measure μ defined in Φ' by the scalar product $B(\varphi, \psi)$, which is thus countably additive.

The proof of Theorem 1 was based upon Theorem 4 of Section 2. The central and most difficult point in the proof of the latter theorem was to establish the inequality

$$1 - \mu(R) \leqslant C(\epsilon + H^2/R^2)$$

(cf. Lemma 4 of Section 2). For Gaussian measures one can avoid this inequality by using the more simply proven inequality

$$\mu(\Omega) = \frac{\sqrt{\det C}}{(2\pi)^{\frac{1}{2}n}} \int_\Omega e^{-\frac{1}{2}C(x,x)}\, dx \leqslant \frac{\operatorname{Tr}(C^{-1})}{r^2} . \tag{10}$$

† By Theorem 3 of Chapter I, Section 1.2, it is sufficient that $B(\varphi, \psi)$ be continuous in each argument separately.

Here $C(x, x)$ denotes a strictly positive-definite quadratic form in the space R_n, $\mathrm{Tr}(C^{-1})$ denotes the trace of the matrix C^{-1}, and Ω denotes the region outside the sphere of radius r and center at the origin of R_n.

In order to prove inequality (10), we note that

$$\mu(\Omega) = \frac{\sqrt{\det C}}{(2\pi)^{\frac{1}{2}n}} \int_{R_n} \chi(x) e^{-\frac{1}{2}(Cx,x)}\, dx,$$

where $\chi(x)$ denotes the characteristic function of the region Ω. Since $\chi(x) = 1$ for those x satisfying $(x, x) > r^2$ and vanishes for those x satisfying $(x, x) \leqslant r^2$, then the inequality $\chi(x) \leqslant (x, x)/r^2$ holds for all $x \in R_n$. It follows that

$$\mu(\Omega) \leqslant \frac{\sqrt{\det C}}{(2\pi)^{\frac{1}{2}n} r^2} \int_{R_n} (x, x) e^{-\frac{1}{2}(Cx,x)}\, dx.$$

Applying formula (4) of Section 2.2, Chapter III, to the right side of this inequality, we obtain

$$\mu(\Omega) \leqslant \frac{\mathrm{Tr}(C^{-1})}{r^2},$$

which proves inequality (10).

Now we prove the following lemma.

Lemma 2. If μ is the Gaussian measure in a Hilbert space H which is defined by the scalar product (φ, ψ) in H, and T is a Hilbert–Schmidt mapping of H into a Hilbert space H_1, then the measure μ_1 in H_1, induced by T and μ, is countably additive.

Proof. According to Theorem 2 of Section 2, it suffices, for the proof of the countable additivity of μ_1, to show that for any $\epsilon > 0$ there is a ball $S_1(r)$ with radius r and center at the origin in H_1 such that the measure of any cylinder set lying outside $S_1(r)$ is less than ϵ. One constructs $S_1(r)$ in the following way. Let T' be the mapping of H_1 into H which is adjoint to T, and consider the operator $Q = T'T$. Since T is of Hilbert–Schmidt class, then by Theorem 4 of Chapter I, Section 2.3, Q is a nuclear operator. Let r be any number such that $\mathrm{Tr}\, Q < \epsilon r^2$, where $\mathrm{Tr}\, Q$ is the trace of Q. Then r is the desired radius.

To prove this assertion it suffices to write out the explicit expression for μ_1 and to apply inequality (10). We omit the details of the argument.

Theorem 1 follows directly from Lemma 2. Indeed, the scalar product $B(\varphi, \psi)$ defines a Gaussian measure μ_B in the conjugate space Φ_B' of Φ_B. But the mapping T_n of Φ_n into Φ_B is, by hypothesis, of Hilbert–Schmidt class. Therefore its adjoint T_n', which maps Φ_B' into Φ_n', is of

Hilbert–Schmidt class. But then the measure μ_n, induced in Φ_n' by μ_B and T_n', is countably additive by Lemma 2, as is therefore the measure μ induced in Φ' by μ_n. But μ is none other than the measure defined in Φ' by $B(\varphi, \psi)$. This proves Theorem 1.

In view of Theorem 3 of Section 2, *any* Gaussian measure in the conjugate space Φ' of a nuclear space Φ is countably additive (we remark that this assertion can also be proven more simply than the general case, by using inequality (10)).

Now we show that the requirement of nuclearity of the space is not only a sufficient, but also a necessary condition for every Gaussian measure in the conjugate space Φ' of the countably Hilbert space Φ to be countably additive. To do this, we need the following estimate for Gaussian measures in a Euclidean space R_n.

Lemma 3. Let μ be the Gaussian measure in n-dimensional Euclidean space R_n defined by the scalar product (x, y) in R_n, and let Ω denote the region defined by the inequalities

$$\operatorname{Tr} C - 2\sqrt{\operatorname{Tr} C} \leqslant C(x, x) \leqslant \operatorname{Tr} C + 2\sqrt{\operatorname{Tr} C},$$

where $C(x, x)$ is a positive-definite quadratic form in R_n, and $\operatorname{Tr} C$ is the trace of the matrix C consisting of the coefficients of the quadratic form. Then

$$\mu(\Omega) \geqslant 1 - \frac{\lambda_1^2 + \dots + \lambda_n^2}{2 \operatorname{Tr} C}, \tag{11}$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of C.[4]

Proof. Let $\chi(x)$ denote the characteristic function of the region Ω. Obviously the inequality

$$\chi(x) \geqslant 1 - \frac{[C(x, x) - \operatorname{Tr} C]^2}{4 \operatorname{Tr} C}$$

is satisfied for all points $x \in R_n$. Therefore

$$\begin{aligned} \mu(\Omega) &= \int_{R_n} \chi(x)\, d\mu(x) \geqslant \int_{R_n} \left\{ 1 - \frac{[C(x, x) - \operatorname{Tr} C]^2}{4 \operatorname{Tr} C} \right\} d\mu(x) \\ &= 1 - \int_{R_n} \frac{(C(x, x))^2 - 2 \operatorname{Tr} C C(x, x) + (\operatorname{Tr} C)^2}{4 \operatorname{Tr} C}\, d\mu(x). \end{aligned} \tag{12}$$

[4] The reader can without difficulty establish the connection of this lemma with the well-known Chebyshev inequality of probability theory.

From formula (4) of Section 2.2, Chapter III, it follows that

$$\int_{R_n} C(x, x)\, d\mu(x) = \frac{1}{(2\pi)^{\frac{1}{2}n}} \int_{R_n} C(x, x) e^{-\frac{1}{2}(x,x)}\, dx = \operatorname{Tr} C,$$

and therefore

$$\mu(\Omega) \geqslant 1 - \frac{1}{4 \operatorname{Tr} C} \int_{R_n} [(C(x, x))^2 - (\operatorname{Tr} C)^2]\, d\mu(x).$$

In order to estimate the integral

$$\int_{R_n} (C(x, x))^2\, d\mu(x),$$

we choose a Cartesian coordinate system in R_n in which the form $C(x, x)$ reduces to a sum of squares

$$C(x, x) = \lambda_1 x_1^2 + \dots + \lambda_n x_n^2.$$

Then the above integral assumes the form

$$\frac{1}{(2\pi)^{\frac{1}{2}n}} \int (\lambda_1 x_1^2 + \dots + \lambda_n x_n^2)^2 \exp[-\tfrac{1}{2}(x_1^2 + \dots + x_n^2)]\, dx$$

$$= \frac{1}{(2\pi)^{\frac{1}{2}n}} \sum_{j,k=1}^{n} \lambda_j \lambda_k \int x_j^2 x_k^2 \exp[-\tfrac{1}{2}(x_1^2 + \dots + x_n^2)]\, dx.$$

But for $j \neq k$

$$\frac{1}{(2\pi)^{\frac{1}{2}n}} \int x_j^2 x_k^2 \exp[-\tfrac{1}{2}(x_1^2 + \dots + x_n^2)]\, dx = 1,$$

and

$$\frac{1}{(2\pi)^{\frac{1}{2}n}} \int x_j^4 \exp[-\tfrac{1}{2}(x_1^2 + \dots + x_n^2)]\, dx = 3,$$

and therefore the integral under consideration is equal to

$$3 \sum_{j=1}^{n} \lambda_j^2 + \sum_{\substack{j,k=1\\ j \neq k}}^{n} \lambda_j \lambda_k.$$

Consequently, in view of inequality (11)

$$\mu(\Omega) \geqslant 1 - \frac{1}{4 \operatorname{Tr} C} \Big[3 \sum_{j=1}^{n} \lambda_j^2 + \sum_{\substack{j,k=1\\ j \neq k}}^{n} \lambda_j \lambda_k - \sum_{j=1}^{n} \lambda_j^2 - \sum_{\substack{j,k=1\\ j \neq k}}^{n} \lambda_j \lambda_k\Big]$$

$$= 1 - \frac{\lambda_1^2 + \dots + \lambda_n^2}{2 \operatorname{Tr} C}.$$

This proves Lemma 3.

Consider the Gaussian measure μ in a (real) Hilbert space H, defined by the scalar product (φ, ψ) in H. Let T be an operator which maps H into another Hilbert space H_1, and denote by μ_1 the measure induced in H_1 by μ and T. We prove the following lemma.

Lemma 4. If T is not of Hilbert–Schmidt class, and $\| T \| \leqslant 1$, then for any $r > 0$ there is a cylinder set Z in H_1, lying outside the ball $S_1(r)$ in H_1 with radius r and center at zero, whose measure is greater than $1/2$.

Proof. Consider first the case where the positive-definite operator $Q = T'T$ has a pure discrete spectrum. Let $\lambda_1, \lambda_2, \ldots$ be the eigenvalues of Q, and $h_1, h_2, \ldots$ the corresponding normalized eigenvectors. Obviously the inverse image of the ball $S_1(r)$ under the mapping T is the set Ω in H defined by the inequality $(T\varphi, T\varphi) \leqslant r^2$, or equivalently $(Q\varphi, \varphi) \leqslant r^2$. In coordinate form the set Ω is defined by the inequality

$$\sum_{k=1}^{\infty} \lambda_k(\varphi, h_k)^2 \leqslant r^2. \tag{13}$$

We note that the series $\sum_{k=1}^{\infty} \lambda_k$ diverges, because T is by hypothesis not of Hilbert–Schmidt class, and so Q is not a nuclear operator. Consequently, there are m and n such that

$$\sum_{k=m+1}^{n} \lambda_k - 2\sqrt{\sum_{k=m+1}^{n} \lambda_k} \geqslant r^2. \tag{14}$$

We now assert that the desired cylinder set Z in H is the cylinder set defined in coordinate form by

$$\sum_{k=m+1}^{n} \lambda_k(\varphi, h_k)^2 \geqslant r^2. \tag{15}$$

Indeed, comparing this inequality with (13), we conclude that Z lies outside the region Ω. Let us now estimate the measure of Z. In view of inequality (14), the set Z contains the set Z_1, defined by the inequalities

$$\Lambda - 2\sqrt{\Lambda} \leqslant \sum_{k=m+1}^{n} \lambda_k(\varphi, h_k)^2 \leqslant \Lambda + 2\sqrt{\Lambda},$$

where $\Lambda = \sum_{k=m+1}^{n} \lambda_k$. Therefore $\mu(Z) \geqslant \mu(Z_1)$. But according to Lemma 3 we have the estimate

$$\mu(Z_1) \geqslant 1 - \frac{\lambda_{m+1}^2 + \ldots + \lambda_n^2}{2(\lambda_{m+1} + \ldots + \lambda_n)}.$$

Since by hypothesis $\| T \| \leqslant 1$, it follows that $\lambda_k \leqslant 1$ for all k. Therefore we obtain $\mu(Z) \geqslant \mu(Z_1) \geqslant \frac{1}{2}$.

Thus we have proven that $\mu(Z) \geqslant \frac{1}{2}$. Mapping Z into H_1, we obtain a cylinder set in H_1, lying outside $S_1(r)$ and having measure at least $\frac{1}{2}$. This proves Lemma 4 when $T'T$ has a discrete spectrum.

When the spectrum of $T'T$ is not purely discrete, the proof is carried out in similar fashion, replacing the vectors $h_1, h_2, \ldots$ by orthonormal vectors $\varphi_1, \varphi_2, \ldots$ such that $(T'T\varphi_k, \varphi_k) \geqslant C > 0$ (the existence of these vectors follows directly from the fact that the spectrum of $T'T$ is not purely discrete). We omit the details.

We can now prove that if a countably Hilbert space Φ is not nuclear, then there exists a Gaussian measure μ in the conjugate space Φ' which is not countably additive.

Indeed, since Φ is not nuclear, there exists an m such that the mapping T_m^n of Φ_n into Φ_m is not of Hilbert–Schmidt class for any $n \geqslant m$. Then the adjoint operators $(T_m^n)'$ are also not of Hilbert–Schmidt class. Now consider the Gaussian measure μ_m in Φ_m' defined by the scalar product $(F, G)_{-m}$. This measure induces a measure μ_n in each of the Φ_n' and a measure μ in Φ'. Let us show that μ is not countably additive. Indeed, since $(T_m^n)'$ is not Hilbert–Schmidt for any $n \geqslant m$, then by Lemma 4 there exists, for any n and r, a cylinder set Z in Φ_n' lying outside the ball $(F, F)_{-n} \leqslant r^2$, whose measure is at least $\frac{1}{2}$ (Lemma 4 applies since it is clear, from the monotonicity of the inner products in a countably Hilbert space, that $\| T_m^n \| \leqslant 1$). But then Theorem 2 of Section 2 implies that μ is not countably additive.

We have thus proven the following theorem.

Theorem 2. In order that every Gaussian measure in the conjugate space Φ' of a given countably Hilbert space Φ be countably additive, it is necessary and sufficient that Φ be a nuclear space.

Obviously the nuclearity of Φ is *a fortiori* necessary for the countable additivity of all (not only Gaussian) measures in Φ'.

4. Fourier Transforms of Measures in Linear Topological Spaces

4.1. Definition of the Fourier Transform of a Measure

The Fourier transform of a nonnegative measure μ in n-dimensional Euclidean space R_n is defined as the function $f(x)$ given by

$$f(x) = \int e^{i(x,y)} \, d\mu(y). \tag{1}$$

Let us carry over this definition to a linear topological space.

Let Φ be a linear topological space and μ a cylinder set measure in the conjugate space Φ'. We define the *Fourier transform of* μ as the (non-linear) functional $L(\varphi)$ defined on Φ by

$$L(\varphi) = \int e^{i(F,\varphi)}\, d\mu(F). \tag{2}$$

We remark that to compute $L(\varphi)$ it suffices to know the measures of half-spaces in Φ'. Indeed, if $\varphi \in \Phi$, then the inequality $(F, \varphi) \leqslant x$ defines a half-space in Φ', whose measure we denote by $\mu_\varphi(x)$. Then $L(\varphi)$ can be written in the form

$$L(\varphi) = \int e^{ix}\, d\mu_\varphi(x). \tag{3}$$

We note that for positive λ the half-space $(F, \lambda\varphi) \leqslant x$ coincides with the half-space $(F, \varphi) \leqslant x/\lambda$. Therefore for all positive λ we have

$$L(\lambda\varphi) = \int e^{ix}\, d\mu_{\lambda\varphi}(x) = \int e^{ix}\, d\mu_\varphi\left(\frac{x}{\lambda}\right) = \int e^{i\lambda x}\, d\mu_\varphi(x). \tag{4}$$

Now if $\lambda < 0$, then the half-space $(F, \lambda\varphi) \leqslant x$ coincides with the half-space $(F, \varphi) \geqslant x/\lambda$, and therefore at the points of continuity of the function $\mu_{\lambda\varphi}(x)$ we have

$$\mu_{\lambda\varphi}(x) = 1 - \mu_\varphi\left(\frac{x}{\lambda}\right).$$

Therefore

$$L(\lambda\varphi) = \int e^{ix}\, d\mu_{\lambda\varphi}(x) = -\int e^{ix}\, d\mu_\varphi\left(\frac{x}{\lambda}\right) = \int e^{i\lambda x}\, d\mu_\varphi(x)$$

also holds for $\lambda < 0$.

It is easy to show that if Ψ is a finite-dimensional subspace in Φ and μ_Ψ is the measure in the factor space Φ'/Ψ^0 corresponding to μ, then for any $\varphi \in \Psi$ one has

$$L(\varphi) = \int_{\Phi'/\Psi^0} e^{i(F,\varphi)}\, d\mu_\Psi(F). \tag{5}$$

Indeed, if $\varphi \in \Psi$, then the half-space $(F, \varphi) \leqslant x$ consists of cosets with respect to the subspace Ψ^0 in Φ'. Therefore the μ-measure of this half-space in Φ' coincides with the μ_Ψ-measure of the half-space $(F, \varphi) \leqslant x$ in Φ'/Ψ^0. Since the Fourier transform of a measure is uniquely defined by the measures of half-spaces, this proves (5).

As an example, let us calculate the Fourier transform of the Gaussian measure μ defined by a functional $B(\varphi, \psi)$ on Φ.

Since the Gaussian measure of the half-space $(F, \varphi) \leqslant a$ is equal to

$$\frac{1}{\sqrt{2\pi B(\varphi, \psi)}} \int_{-\infty}^{a} \exp\left(\frac{-x^2}{2B(\varphi, \varphi)}\right) dx,$$

the functional $L(\varphi)$ is given by

$$L(\varphi) = \frac{1}{\sqrt{2\pi B(\varphi, \varphi)}} \int_{-\infty}^{\infty} \exp(ix) \exp\left(-\frac{x^2}{2B(\varphi, \varphi)}\right) dx.$$

But this integral equals $e^{-\frac{1}{2}B(\varphi,\varphi)}$. It follows that

$$L(\varphi) = e^{-\frac{1}{2}B(\varphi,\varphi)}. \tag{6}$$

4.2. Positive-Definite Functionals on Linear Topological Spaces

Let $L(\varphi)$ be a functional on a linear topological space Φ. This functional is called *positive-definite* if

$$\sum_{j,k=1}^{m} L(\varphi_j - \varphi_k)\xi_j\bar{\xi}_k \geqslant 0 \tag{7}$$

for any elements $\varphi_1, \ldots, \varphi_m$ in Φ and any complex numbers $\xi_1, \ldots, \xi_m$.

An example of a positive-definite functional is furnished by any functional $L(\varphi)$ which is the Fourier transform of a cylinder set measure in the conjugate space Φ' of Φ (recall that we are considering only positive normalized measures). Indeed, suppose that $L(\varphi)$ is the Fourier transform of a measure μ. Let Ψ be the finite-dimensional subspace spanned by the elements $\varphi_1, \ldots, \varphi_m$, and let μ_Ψ be the measure in the factor space Φ'/Ψ^0 corresponding to the measure μ. Then for $\varphi \in \Psi$, $L(\varphi)$ is given by formula (5). But then we have

$$\sum_{j,k=1}^{m} L(\varphi_j - \varphi_k)\xi_j\bar{\xi}_k = \sum_{j,k=1}^{m} \xi_j\bar{\xi}_k \int_{\Phi'/\Psi^0} e^{i(F,\varphi_j-\varphi_k)}\, d\mu_\Psi(F)$$

$$\int_{\Phi'/\Psi^0} \left| \sum_{j=1}^{m} \xi_j e^{i(F,\varphi_j)} \right|^2 d\mu_\Psi(F) \geqslant 0,$$

from which it is evident that $L(\varphi)$ is positive-definite.

Let us remark that if μ satisfies the continuity condition, then its Fourier transform is continuous. In fact, suppose that the sequence $\{\varphi_n\}$ converges to the element $\varphi_0 \in \Phi$. Let μ_n be the measure on the line corresponding to the element φ_n, and μ_0 the measure corresponding to φ_0. Then

$$L(\varphi_n) = \int e^{ix}\, d\mu_n(x), \qquad n = 0, 1, \dots .$$

But the continuity condition says that

$$\lim_{n\to\infty} \int f(x)\, d\mu_n(x) = \int f(x)\, d\mu_0(x)$$

for any bounded continuous function $f(x)$. Letting $f(x) = e^{ix}$, we obtain $\lim_{n\to\infty} L(\varphi_n) = L(\varphi_0)$, which shows that $L(\varphi)$ is continuous. Lastly, we note that $L(0) = 1$, since the measure μ_0 corresponding to the zero element is concentrated at the point $x = 0$, and therefore

$$\int e^{ix}\, d\mu_0(x) = 1.$$

Thus we see that the Fourier transform $L(\varphi)$ of any measure on the cylinder sets in Φ' is positive-definite and continuous (in the sequential sense) and $L(0) = 1$. Now we show that these conditions are not only necessary, but also sufficient for a functional $L(\varphi)$ to be the Fourier transform of some cylinder set measure in Φ'.

Theorem 1. In order that a functional $L(\varphi)$ on a linear topological space Φ be the Fourier transform of some cylinder set measure in the conjugate space Φ', it is necessary and sufficient that $L(\varphi)$ be positive-definite and continuous (in the sequential sense) and that $L(0) = 1$.

Proof. The necessity of the conditions was proven above. To show their sufficiency, let $L(\varphi)$ be a functional satisfying the conditions of the theorem. Considering $L(\varphi)$ on a finite-dimensional subspace Ψ of Φ, we obtain a positive-definite continuous function $L_\Psi(\varphi)$ on Ψ. By Bochner's theorem (cf. Chapter II, Section 3.2) this function is the Fourier transform of a positive measure μ_Ψ defined in the conjugate space Ψ' of Ψ. But we have seen in Section 3.1 that Ψ' can be identified in a natural way with the factor space Φ'/Ψ^0, where Ψ^0 consists of all linear functionals F which vanish on Ψ. It follows that in each of the factor spaces Φ'/Ψ^0, where Ψ is finite dimensional, there is defined a measure μ_Ψ. It remains for us to show that these measures are compatible and satisfy the continuity condition.

To prove compatibility, consider two finite-dimensional subspaces Ψ_1, Ψ_2 with $\Psi_1 \subset \Psi_2$ in Φ, and let μ_1 and μ_2 be the measures corresponding to them. We have to prove that μ_1 coincides with the measure ν induced in the factor space Φ'/Ψ_1^0 by μ_2. In other words, we must prove that

$$\mu_1(A) = \nu(A) = \mu_2[Q^{-1}(A)] \tag{8}$$

for any set A in Φ'/Ψ_1^0, where Q denotes the mapping of Φ'/Ψ_2^0 onto Φ'/Ψ_1^0 by which the coset $F + \Psi_2^0$ is carried into the coset $F + \Psi_1^0$. We prove (8) by showing that the Fourier transforms of the measures μ_1 and ν coincide.

The Fourier transform of μ_1 is by definition of μ_1 the function $L_1(\varphi)$ defined on Ψ_1 and coinciding there with $L(\varphi)$. The Fourier transform of ν is also defined on Ψ_1, and is given by

$$\int_{\Phi'/\Psi_1^0} e^{i(F,\varphi)}\, d\nu(F). \tag{9}$$

If $\varphi \in \Psi_1$, then the value of (F, φ) is the same for all functionals F belonging to the same coset $F_0 + \Psi_1^0$. Since, moreover, ν and μ_2 are related by (8), we can rewrite (9) in the form

$$\int_{\Phi'/\Psi_2^0} e^{i(F,\varphi)}\, d\mu_2(F).$$

Thus (9) is the Fourier transform of μ_2 for all elements $\varphi \in \Psi_1$. But by definition of μ_2 this Fourier transform is that function $L_2(\varphi)$ on Ψ_2 which coincides there with $L(\varphi)$. But $L_1(\varphi)$ and $L_2(\varphi)$ coincide on Ψ_1, since $L_1(\varphi) = L(\varphi) = L_2(\varphi)$ for $\varphi \in \Psi_1$. Thus the Fourier transforms of μ_1 and ν coincide. But then μ_1 and ν coincide, which proves that the measures μ_Ψ are compatible.

We can thus associate with the functional $L(\varphi)$ a cylinder set measure μ in Φ'. It remains for us to show that μ satisfies the continuity condition. For this we use the following theorem from the theory of Fourier integrals:

If a sequence $\{\mu_n\}$ of positive normalized measures is such that

$$\lim_{n\to\infty} \int e^{i\lambda x}\, d\mu_n(x) = \int e^{i\lambda x}\, d\mu_0(x),$$

for any value of λ, then the measures μ_n converge weakly to μ_0.[1]

[1] A sequence $\{\mu_n\}$ of measures is said to converge weakly to a measure μ_0, if

$$\lim_{n\to\infty} \int f(x)\, d\mu_n(x) = \int f(x)\, d\mu_0(x)$$

for any bounded continuous function $f(x)$.

To prove, now, that μ satisfies the continuity condition, let $\{\varphi_n\}$ be a sequence of elements in Φ which converges to an element φ_0, and let μ_n be the measure corresponding to φ_n, $n = 0, 1, \ldots$. Then for any real value of λ we have

$$L(\lambda\varphi_n) = \int e^{i\lambda x}\, d\mu_n(x), \qquad n = 0, 1, \ldots$$

(cf. formula (4)). Since in view of the continuity of $L(\varphi)$ we have $\lim_{n\to\infty} L(\lambda\varphi_n) = L(\lambda\varphi_0)$, then for any λ

$$\lim_{n\to\infty} \int e^{i\lambda x}\, d\mu_n(x) = \int e^{i\lambda x}\, d\mu_0(x).$$

But this, as we said, implies that the measures μ_n, $n = 1, 2, \ldots$ converge weakly to μ_0. Thus the measure μ which we have constructed on the cylinder sets of Φ' satisfies the continuity condition (cf. Section 1.4), which concludes the proof of the theorem.

If Φ is a nuclear space, then by Theorem 3 of Section 2, any cylinder set measure in Φ' which satisfies the continuity condition is countably additive. We therefore have the following assertion.

Theorem 2. Any continuous† positive-definite functional $L(\varphi)$ on a nuclear space Φ, such that $L(0) = 1$, is the Fourier transform of a countably additive positive normalized measure in Φ'.

This theorem is simply Bochner's theorem for nuclear spaces.

5. Quasi-Invariant Measures in Linear Topological Spaces

5.1. Invariant and Quasi-Invariant Measures in Finite-Dimensional Spaces

In this section we will consider questions connected with the transformation of measures in linear topological spaces by parallel displacement. By the term "measure" we mean a positive countably additive measure $\mu(X)$ on the Borel sets in a conjugate space Φ', which is regular in the sense of Caratheodory, and such that the entire space is a countable union of sets of finite measure (this last property is called the σ-finiteness of μ).

† Since a nuclear space has a metric topology (cf. footnote on p. 57), continuity and sequential continuity are equivalent.

We start by considering measures in finite-dimensional linear spaces. In a finite-dimensional linear space R_n there exists Lebesgue measure $\mu_0(X)$ which is *invariant* under any parallel displacement in R_n. In other words, the measure $\mu_0(X)$ is such that $\mu_0(X) = \mu_0(y + X)$ for all vectors y and measurable sets X in R_n. The property of invariance under parallel displacement is characteristic for Lebesgue measures—any two measures which are invariant under all parallel displacements are identical up to a constant factor.

Let us now consider measures which are equivalent to Lebesgue measure. We say that two measures μ and ν are *equivalent*, if they have the same family of null sets (i.e., if $\mu(X) = 0$ implies $\nu(X) = 0$ and conversely). A description of all measures equivalent to Lebesgue measure is given by the following theorem.

Theorem 1. Any measure μ in R_n which is equivalent to Lebesgue measure has the form

$$\mu(X) = \int_X f(x)\, dx,$$

where $f(x)$ is a strictly positive function which is summable over every bounded set in R_n.

Proof. Since $\mu_0(X) = 0$ implies $\mu(X) = 0$, by the Radon–Nikodym theorem[1] there exists a finite-valued nonnegative measurable function $f(x)$ such that

$$\mu(X) = \int_X f(x)\, dx \tag{1}$$

for all measurable sets X in R_n. Let X_0 denote the set of points at which $f(x) = 0$. Obviously

$$\mu(X_0) = \int_{X_0} f(x)\, dx = 0;$$

since μ and μ_0 are by hypothesis equivalent, $\mu_0(X) = 0$. Consequently $f(x)$ is almost everywhere positive. Since $f(x)$ can be altered on a set of

[1] The Radon–Nikodym theorem says the following:

Suppose that μ and ν are measures such that $\mu(X) = 0$ for all sets X of ν-measure zero. Then there exists a finite-valued nonnegative measurable function $f(x)$ such that

$$\mu(X) = \int_X f(x)\, d\nu(x)$$

for all measurable sets X. The function $f(x)$ is defined up to a set of ν-measure zero.

Lebesgue measure zero without affecting (1), we can suppose that it is positive everywhere.

Lastly, we show that $f(x)$ is summable over any bounded set. Since μ and μ_0 are equivalent and the Lebesgue measure of a point $x \in R_n$ is zero, then $\mu(\{x\}) = 0$ for all $x \in R_n$. But since we assumed that μ is regular in the sense of Caratheodory, for every $x \in R_n$ there is an open set $V(x)$ containing x whose μ-measure is finite. Since any closed bounded set X in R_n can be covered by a finite number of the $V(x)$ (and hence any bounded set can), the μ-measure of any bounded set is finite.

We remark that the Lebesgue measure μ_0 is expressible in terms of μ by the formula

$$\mu_0(X) = \int_X \frac{d\mu(x)}{f(x)} .$$

As a matter of fact, if μ and ν are measures equivalent to Lebesgue measure, then they are mutually equivalent, and considerations similar to those used in the proof of the preceding theorem show that we have

$$\nu(X) = \int_X f_{\mu\nu}(x)\, d\mu(x),$$

where $f_{\mu\nu}(x)$ is summable (relative to μ) over every bounded set and is positive for all x.

Measures μ which are equivalent to Lebesgue measure have the following weakened property of invariance under parallel displacement.

If a set X has μ-measure zero, then every translate of X has μ-measure zero. Indeed, from the invariance of Lebesgue measure and the definition of equivalence we have the chain of implications

$$\mu(X) = 0 \Rightarrow \mu_0(X) = 0 \Rightarrow \mu_0(y + X) = 0 \Rightarrow \mu(y + X) = 0.$$

A measure which has the property that $\mu(X) = 0$ implies $\mu(y + X) = 0$ for all y will be called quasi-invariant (relative to parallel displacement = translation). We have therefore proven that all measures which are equivalent to Lebesgue measure are quasi-invariant.

The converse is also true.

Theorem 2. If a measure μ is quasi-invariant, then it is equivalent to Lebesgue measure.

First we prove the following lemma.

Lemma 1. If a measure μ is quasi-invariant, then the μ-measure of any bounded set is finite.

Proof. The quasi-invariance of μ implies that the μ-measure of each point $x \in R_n$ is zero. Indeed, if for some x_0 one had $\mu(\{x_0\}) > 0$, then this would imply $\mu(\{x\}) > 0$ for all x. Thus any set containing a non-denumerable number of points would contain an infinite number of points whose measures all exceed some fixed positive constant, and therefore the set would have infinite measure. But this clearly contradicts the σ-finiteness of μ. Now the regularity of μ implies that for every $x \in R_n$ there is an open set $V(x)$ containing x which has finite μ-measure. Since any closed bounded set (and therefore any bounded set) can be covered by finitely many of the $V(x)$, the μ-measure of such a set is finite.

Proof of Theorem. Suppose that μ is quasi-invariant. Clearly if X is bounded, then $\mu(y + X)$ is finite for all y. As for its measurability as a function of y, this follows at once from

$$\mu(y + X) = \int_{-\infty}^{\infty} \chi(x - y)\, d\mu(x), \tag{2}$$

where χ is the characteristic function of X. Further, if Y is a bounded set, then we can obviously find a bounded measurable set Z such that $(y + X) \subset Z$ for all $y \in Y$; hence $\mu(y + X) \leqslant \mu(Z) < \infty$ for all $y \in Y$. Now suppose that $\mu(X) > 0$, $\mu_0(X) = 0$, and $\mu_0(Y) > 0$. Then by the quasi-invariance of μ we have $\mu(y + X) > 0$ for all y. Then

$$\begin{aligned} 0 < \int_Y \mu(y + X)\, dy &= \int_Y \left[\int_{-\infty}^{\infty} \chi(x - y)\, d\mu(x)\right] dy \\ &= \int_{-\infty}^{\infty} \left[\int_Y \chi(x - y)\, dy\right] d\mu(x), \end{aligned} \tag{3}$$

where Fubini's theorem is applicable because our definition of quasi-invariance includes the assumption of σ-finiteness. But

$$\int_Y \chi(x - y)\, dy = \mu_0(Y \cap (x - X)) \leqslant \mu_0(x - X) = 0 \tag{4}$$

for every x, since $\mu_0(x - X) = \mu_0(X) = 0$. But then the right side of (3) vanishes, which is a contradiction.

Thus, we have shown that any bounded set† of Lebesgue measure

† The use of bounded sets and Lemma 1 in order to avoid the consideration of sets having (possibly) infinite μ-measure appears unnecessary, as Fubini's theorem for non-negative functions is valid without any summability restrictions.

zero has μ-measure zero. Since any set of Lebesgue measure zero can be written as a countable union of bounded sets of Lebesgue measure zero, then $\mu(X) = 0$ for all sets X of Lebesgue measure zero. To show the converse, suppose that $\mu_0(X_0) > 0$ but $\mu(X_0) = 0$. Let $\{x_k\}$ be an everywhere dense sequence of points in R_n, and define Z as the union

$$Z = \bigcup_{k=1}^{\infty} (x_k + X_0). \tag{5}$$

It is not hard to show† that $\mu_0(R_n - Z) \equiv \mu_0(Z_1) = 0$. But

$$\mu(R_n) = \mu(Z) + \mu(Z_1) \leqslant \mu(Z_1) + \sum_{k=1}^{\infty} \mu(x_k + X_0). \tag{6}$$

Since $\mu(X_0) = 0$ and μ is quasi-invariant, then $\mu(x_k + X_0) = 0$ for all k. Furthermore, as we saw above, $\mu_0(Z_1) = 0$ implies $\mu(Z_1) = 0$. Therefore $\mu(R_n) = 0$. Thus μ is either equivalent to Lebesgue measure or else vanishes identically.

We have thus proven that the class of quasi-invariant measures in R_n coincides with the class of measures equivalent to Lebesgue measure. Therefore, all quasi-invariant measures in R_n are equivalent to one another.

5.2. Quasi-Invariant Measures in Linear Topological Spaces

Let us now consider measures in infinite-dimensional linear topological spaces. The definition of a quasi-invariant measure can be carried over formally to this case, by calling a measure in a linear topological space quasi-invariant if parallel displacement takes sets of measure zero into sets of measure zero. However, this formal extension is unsuccessful, owing to the fact that for the most important classes of infinite-dimensional spaces there are no nonzero measures which are quasi-invariant in the sense indicated.

† Indeed, if $\mu_0(X_0) > 0$, then by the theory of differentiation we can find, for any $\epsilon > 0$, a sequence $\{J_k\}$ of cubes whose diameters tend to zero and such that

$$\mu_0(J_k \cap X_0) \geqslant (1 - \epsilon)\, \mu_0(J_k).$$

Let I and I' be fixed cubes such that I lies in the interior of I'. Clearly the family $\mathfrak{F} = \{x_i + J_k\}$ of cubes covers I in the sense of Vitali, and so by Vitali's covering theorem there exists a sequence $\{I_k\}$ of disjoint cubes from $\mathfrak{F}$ such that $\mu_0(I - \bigcup_{k=1}^{\infty} I_k) = 0$, and in addition the I_k can be chosen so that they lie in I'. But this implies that $\mu_0(I - Z) \leqslant \epsilon\mu_0(I')$, where Z is defined by (5). Since ϵ is arbitrary, $\mu_0(I - Z) = 0$, and since I is an arbitrary cube, it follows that $\mu_0(R_n - Z) = 0$.

Let us consider measures μ in the conjugate space Φ' of a countably normed space Φ. We show that if the spaces Φ_n and Φ_{n+1} are different for every n, where $\Phi = \bigcap_{n=1}^{\infty} \Phi_n$, then there exists no quasi-invariant measure in Φ'. Indeed, Φ' is the union of the subspaces Φ_n' conjugate to the Φ_n, $\Phi' = \bigcup_{n=1}^{\infty} \Phi_n'$, where $\Phi_1' \subset \Phi_2' \subset \ldots$. Therefore

$$\mu(\Phi') = \lim_{n \to \infty} \mu(\Phi_n'),$$

as the Φ_n' are Borel sets in Φ' for which μ is therefore defined (cf. the opening remarks in Section 2.2). Since by hypothesis $\mu(\Phi') \neq 0$, there is an n such that $\mu(\Phi_n') \neq 0$. Since Φ_n and Φ_{n+1} are different from one another, then Φ_n' and Φ' differ from each other. We decompose Φ' into cosets with respect to Φ_n'. Each of these cosets is obtained by a parallel displacement of Φ_n', which has nonzero measure, and so by the quasi-invariance of μ these cosets have nonzero measures. Since the family of all these cosets has the power of the continuum, we arrive at a contradiction with the σ-finiteness of μ.

This assertion holds also for the conjugate space Φ' of any normed space having a countable everywhere dense set. In particular, there exists no quasi-invariant measure in a Hilbert space (cf. Section 5.3).

As in many similar cases, the difficulties which arise are successfully overcome by considering rigged Hilbert spaces. Thus, let $\Phi \subset H \subset \Phi'$ be a rigged Hilbert space, i.e., a nuclear space Φ in which there is given a scalar product (φ, ψ) (as in Section 4 of Chapter 1, H denotes the completion of Φ in the norm $\| \varphi \| = \sqrt{(\varphi, \varphi)}$). With each element $\psi \in \Phi$ we associate a functional F_ψ on Φ, defined by $(F_\psi, \varphi) = (\varphi, \psi)$. We obtain thereby an (antilinear) imbedding $\psi \to F_\psi$ of Φ into Φ'. It is obvious that the functionals of the form F_ψ, $\psi \in \Phi$, are everywhere dense in Φ'.

We will say that a measure μ in Φ' is *quasi-invariant*, if $\mu(F_\psi + X) = 0$ for every element $\psi \in \Phi$ and every set X such that $\mu(X) = 0$. Thus, we eliminate the requirement that *every* translation carries sets of measure zero into sets of measure zero, requiring that this be true only for translations by the elements F_ψ, $\psi \in \Phi$.

We remark that since the elements of the form F_ψ are everywhere dense in Φ', then the translations by elements F are "sufficiently numerous." This means that if $\Phi'/\Psi^0 = R$ is a finite-dimensional factor space, then any translation in R can be induced by a translation in Φ' corresponding to some element F_ψ, where $\psi \in \Phi$. This assertion follows from the fact that the map of Φ into R is everywhere dense in R, which, in view of the finite dimensionality of R, means that it coincides with R.

We will prove that there exist quasi-invariant measures in the conjugate spaces of nuclear spaces (here, of course, quasi-invariance is understood in the sense just indicated). That is, we will show that if an imbedding of Φ into Φ' is defined by a continuous Hermitean functional $B(\varphi, \psi)$, then the Gaussian measure defined by this functional is quasi-invariant.

In other words, we will prove the following theorem.

Theorem 3. Let $B(\varphi, \psi)$ be a nondegenerate positive-definite Hermitean functional, continuous in each argument, on a nuclear space Φ, and let μ be the Gaussian measure in Φ' defined by $B(\varphi, \psi)$. If X is a set in Φ' such that $\mu(X) = 0$, and ψ is any element of Φ, then $\mu(F_\psi + X) = 0$, where F_ψ denotes the functional on Φ defined by $(F_\psi, \varphi) = B(\varphi, \psi)$.

The proof of this theorem is based upon the following lemma.

Lemma 2. Let μ be the Gaussian measure in the conjugate space Φ' of a nuclear (complex) countably Hilbert space defined by a continuous (in each variable) nondegenerate Hermitean functional $B(\varphi, \psi)$. If the base A of a cylinder set Z in Φ' lies in the projection of the ball $\| F \|_{-n} \leqslant R$ in the factor space Φ'/Ψ^0, where Ψ^0 is the generating subspace of Z, then for any $\psi \in \Psi$ we have

$$\frac{\mu(F_\psi + Z)}{\mu(Z)} \leqslant e^{R\|\psi\|_n}. \tag{7}$$

Proof. By the definition of a Gaussian measure we have[2]

$$\mu(Z) = \frac{1}{(2\pi)^n} \int_A \exp[-\tfrac{1}{2} B_\Psi(\tilde{F}, \tilde{F})] \, d_\Psi \tilde{F}$$

and

$$\mu(F_\psi + Z) = \frac{1}{(2\pi)^n} \int_{\tilde{F}_\psi + A} \exp[-\tfrac{1}{2} B_\Psi(\tilde{F}, \tilde{F})] \, d_\Psi \tilde{F}$$

$$= \frac{1}{(2\pi)^n} \int_A \exp[-\tfrac{1}{2} B_\Psi(\tilde{F} - \tilde{F}_\psi, \tilde{F} - \tilde{F}_\psi)] \, d_\Psi \tilde{F},$$

where $B_\Psi(\tilde{F}_1, \tilde{F}_2)$ is the functional on Φ'/Ψ^0 defined by $B(\varphi, \psi)$, and

[2] For brevity of notation we denote by $\tilde{F}$ the coset with respect to Ψ^0 containing the element $F \in \Phi'$, and by $\tilde{F}_\psi + A$ the projection into Φ'/Ψ^0 of the cylinder set $F_\psi + Z$.

$d_\Psi \tilde{F}$ is the Lebesgue measure in Φ'/Ψ^0 corresponding to the scalar product $B_\Psi(\tilde{F}_1, \tilde{F}_2)$. Therefore

$$\frac{\mu(F_\psi + Z)}{\mu(Z)} \leqslant \sup_F \frac{\exp[-\frac{1}{2}B_\Psi(\tilde{F} - \tilde{F}_\psi, \tilde{F} - \tilde{F}_\psi)]}{\exp[-\frac{1}{2}B_\Psi(\tilde{F}, \tilde{F})]}$$

$$= \sup_F \exp[B_\Psi(\tilde{F}, \tilde{F}_\psi) - \tfrac{1}{2}B_\Psi(\tilde{F}_\psi, \tilde{F}_\psi)] \leqslant \sup_F \exp[B_\Psi(\tilde{F}, \tilde{F}_\psi)]$$

From the definition of the functional $B_\Psi(\tilde{F}_1, \tilde{F}_2)$ (cf. Section 3.2) it follows that

$$B_\Psi(\tilde{F}, \tilde{F}_\psi) = (F, \psi)$$

for any $\tilde{F}$ in Φ'/Ψ^0, where (F, ψ) denotes the value of the functional F for the element ψ. If $\tilde{F} \in A$, then by hypothesis one can choose F in $\tilde{F}$ so that $\| F \|_{-n} \leqslant R$. Then

$$| B_\Psi(\tilde{F}, \tilde{F}_\psi)| \leqslant |(F, \psi)| \leqslant \| F \|_{-n} \| \psi \|_n \leqslant R \| \psi \|_n.$$

Therefore

$$\frac{\mu(F_\psi + Z)}{\mu(Z)} \leqslant e^{R\|\psi\|_n}$$

which proves the lemma.

Proof of the Theorem. Let X be a set of μ-measure zero, and let ψ be any element in Φ. We have to show that the set $F_\psi + X$, the translate of X by the element F_ψ, also has zero μ-measure. To do this, it suffices to show that $\mu(F_\psi + X) < \epsilon$ for any $\epsilon > 0$.

Thus, let $\epsilon > 0$. Since μ is countably additive and Φ' is the union of balls $\| F \|_{-n} \leqslant R$, one can find n and R such that $\mu(\Phi' - S) < \frac{1}{2}\epsilon$, where S is the ball $\| F \|_{-n} \leqslant R$, and $F_\psi \in \Phi'_n$. Let $X_1 = X \cap S_1$, where S_1 is the ball $\| F \|_{-n} \leqslant R + \| F_\psi \|_{-n}$, and $X_2 = X - X_1$. Obviously the set $F_\psi + X_2$ lies in the complement of S, and therefore

$$\mu(F_\psi + X_2) < \tfrac{1}{2}\epsilon.$$

Let us show that also $\mu(F_\psi + X_1) < \frac{1}{2}\epsilon$. Indeed, since $\mu(X_1) = 0$, it can be covered by a countable union $\bigcup_{k=1}^\infty Z_k$ of cylinder sets such that

$$\sum_{k=1}^\infty \mu(Z_k) < \tfrac{1}{2}\epsilon \exp[-(R + \| F_\psi \|_{-n}) \| \psi \|_n].$$

Since $X_1 \subset S$, the cylinder sets Z_k, $1 \leqslant k < \infty$, can be chosen so that their bases A_k lie respectively in the projection of the ball S_1 in Φ'/Ψ_k^0

(Ψ_k^0 denotes the generating subspace of Z_k). We may suppose without loss of generality that $\psi \in \Psi_k$ for all k. Since the A_k lie in the projection of the ball $S_1 = \{\| F \|_{-n} \leqslant R + \| F_\varphi \|_{-n}\}$, then by Lemma 2

$$\mu(F_\varphi + Z_k) \leqslant \exp[(R + \| F_\varphi \|_{-n})\| \psi \|_n]\mu(Z_k)$$

and therefore

$$\mu(F_\varphi + X_1) \leqslant \sum_{k=1}^{\infty} \mu(F_\varphi + Z_k)$$

$$\leqslant \exp[(R + \| F_\varphi \|_{-n})\| \psi \|_n] \sum_{k=1}^{\infty} \mu(Z_k) < \tfrac{1}{2}\epsilon.$$

Thus $\mu(F_\varphi + X_1) < \frac{1}{2}\epsilon$. But this means that

$$\mu(F_\varphi + X) = \mu(F_\varphi + X_1) + \mu(F_\varphi + X_2) < \epsilon,$$

which proves the theorem.

Thus, Gaussian measures in the conjugate space Φ' of a nuclear space Φ are quasi-invariant. One can say that to every continuous positive-definite nondegenerate Hermitean functional on Φ there corresponds a quasi-invariant measure. If Φ is infinite dimensional, then there exist infinitely many pairwise inequivalent quasi-invariant measures in Φ' (recall that in a finite-dimensional space all quasi-invariant measures are equivalent to one another).

In fact, let $B_1(\varphi, \psi)$ and $B_2(\varphi, \psi)$ be positive-definite continuous non-degenerate Hermitean functionals on a nuclear space Φ, and let μ_1 and μ_2 be the Gaussian measures in Φ' defined by these functionals. Assume that $B_2(\varphi, \psi)$ is bounded relative to the scalar product defined by $B_1(\varphi,\psi)$, i.e., that there exists $M > 0$ such that the inequality

$$| B_2(\varphi, \varphi)| \leqslant M| B_1(\varphi, \varphi)|$$

holds for all $\varphi \in \Phi$. Then $B_2(\varphi, \psi)$ defines a positive-definite bounded linear operator A in the space H (the completion of Φ relative to the norm $\| \varphi \| = \sqrt{B_1(\varphi, \varphi)}$) defined by

$$B_1(A\varphi, \psi) = B_2(\varphi, \psi). \tag{8}$$

The following assertion holds: If the operator A defined by (8) has a discrete spectrum, and the series $\sum_{k=1}^{\infty} \lambda_k$ consisting of the eigenvalues of A converges, then μ_1 and μ_2 are inequivalent.

The proof of this assertion is based upon Lemmas 2 and 3 of Section 3. Namely, consider the ball S in Φ defined by $B_1(\varphi, \varphi) \leqslant R^2$ and denote the map of S in Φ' by S'. It is easy to show, using Lemma 3 of Section 3, that $\mu_1(S') = 0$. At the same time, if $\sum_{k=1}^{\infty} \lambda_k$ converges, then using the estimates from Lemma 2 of Section 3, one can show that for R sufficiently large $\mu_2(S') \neq 0$. This shows that μ_1 and μ_2 are inequivalent.[3]

Using these statements, it is not difficult to construct an infinite set of pairwise inequivalent quasi-invariant measures in Φ'. To do this, it suffices to consider an infinite sequence $B_1(\varphi, \psi)$, $B_2(\varphi, \psi)$, ... of positive-definite Hermitean functionals on Φ such that the operator A_n, defined by

$$B_n(A_n\varphi, \psi) = B_{n+1}(\varphi, \psi),$$

has a discrete spectrum, and the series consisting of its eigenvalues converges.

It would be very interesting to give a complete description of all quasi-invariant measures in nuclear spaces.

5.3. Quasi-Invariant Measures in Complete Metric Spaces

In the previous paragraph it was shown that under very general conditions there exist no measures in the conjugate spaces of countably normed spaces which are quasi-invariant relative to all translations. We now prove a similar result for complete linear metric spaces.

Theorem 4. Let A be a complete metric linear space containing a countable everywhere dense set, and such that the absolutely convex hull of any compact set[4] X in A is nowhere dense in A. Then the only quasi-invariant (under all translations) measure on A is the identically zero measure.

Proof. First we show that if there is no normalized quasi-invariant measure (i.e., a quasi-invariant measure such that $\mu(A) = 1$), then the only quasi-invariant measure in A is identically zero. Indeed, let μ be a quasi-invariant measure in A. Since μ is σ-finite (recall that we are only considering such measures), A can be written at a countable union of disjoint sets A_1, A_2, ... having finite positive μ-measure. Let $f(x)$ be defined on A by $f(x) = 1/2^k\mu(A_k)$ if $x \in A_k$, and set

$$\nu(X) = \int_X f(x)\, d\mu(x). \tag{9}$$

[3] One can show that if the product $\Pi_{k=1}^{\infty} \lambda_k$ converges, then μ_1 and μ_2 are equivalent. In this case A is the sum of the identity operator and a nuclear operator.

[4] By the absolutely convex hull of a set X we mean the set $\tilde{X}$ consisting of all linear combinations of the form $a_1x_1 + ... + a_nx_n$, where $x_k \in X$, $1 \leqslant k \leqslant n$, and

$$|a_1| + ... + |a_n| \leqslant 1.$$

Since

$$\nu(\Lambda) = \sum_{k=1}^{\infty} \nu(\Lambda_k) = \sum_{k=1}^{\infty} \frac{1}{2^k} = 1,$$

then ν is a normalized measure in Λ. From the quasi-invariance of μ it follows easily that ν is also quasi-invariant. But by hypothesis there is no quasi-invariant normalized measure on Λ. Consequently $\mu(\Lambda) = 0$.

Thus, our problem has been reduced to proving that there are no normalized quasi-invariant measures in Λ. Suppose that μ is a countably additive measure in Λ such that $\mu(\Lambda) = 1$. We show that for any n there is a compact set X_n in Λ such that $\mu(X_n) \geqslant 1 - n^{-1}$. Indeed, choose a countable everywhere dense set $x_1, x_2, \ldots$ in Λ and consider the closed balls S_{kp} with centers at x_k and radii p^{-1}. Since for any fixed p the balls $S_{1p}, S_{2p}, \ldots$ cover Λ (since the set $\{x_k\}$ is everywhere dense), in view of the countable additivity of μ there is a number $k(p)$ such that the measure of the set

$$X_{np} = \bigcup_{j=1}^{k(p)} S_{jp}$$

is not less than $1 - (1/2^p n)$. Let $X_n = \bigcap_{p=1}^{\infty} X_{np}$; we show that X_n is the desired set. Indeed, it is obvious that

$$\mu(\Lambda - X_n) \leqslant \sum_{p=1}^{\infty} \mu(\Lambda - X_{np}) \leqslant \sum_{p=1}^{\infty} \frac{1}{2^p n} = \frac{1}{n}, \tag{10}$$

from which it follows that $\mu(X_n) \geqslant 1 - n^{-1}$. Further, for any p the set X_n is covered by the finite set of balls $S_{1p}, \ldots, S_{k(p)p}$ of radius p^{-1}. Finally, X_n is closed because each of the S_{kp} is closed and consequently X_{np}, as the union of finitely many of the S_{kp}, is closed. But in a complete metric space any closed set Z which can be covered by a finite number of balls of any preassigned radius is compact.[5] Consequently X_n is compact.

Now let $X = \bigcup_{n=1}^{\infty} X_n$. From the relation $\mu(X_n) \geqslant 1 - n^{-1}$ it follows that $\mu(X) = 1$. But then the (nonclosed) linear span $\hat{X}$ of X has measure 1.[6,†] Let us show that the set $\hat{X}$ does not coincide with the

[5] The proof of this assertion is carried out in the same way as the proof of the compactness of a closed bounded set in a finite-dimensional space, with the sole difference that the coverings of Z by balls of arbitrarily small radius play the role of the partitions.

[6] By the linear span $\hat{X}$ of a set X we mean the set consisting of all finite linear combinations $\lambda_1 x_1 + \ldots + \lambda_n x_n$ of elements of X.

† We remark that the question of the measurability or nonmeasurability of $\hat{X}$ is inconsequential to the proof, and can be avoided in various ways, of which perhaps the simplest is to observe that in the very last step of the proof $\hat{X}$ can be replaced by X.

entire space Λ. To do this, consider the compact sets $Y_n = \bigcup_{k=1}^{n} X_k$, and let $\tilde{Y}_n$ denote the absolutely convex hull of Y_n. By the conditions of the theorem, the sets $\tilde{Y}_n$ are nowhere dense in Λ. Therefore the sets $k\tilde{Y}_n$ consisting of all elements of the form ky, $y \in \tilde{Y}_n$, are also nowhere dense in Λ. Since a complete metric space cannot be written as a countable union of nowhere dense sets (cf. Chapter I, Section 1.1), the union $Y = \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} k\tilde{Y}_n$ does not coincide with Λ. But $\hat{X} \subset Y$, since if $x \in \hat{X}$, then $x = \lambda_1 x_1 + \dots + \lambda_p x_p$, where $x_j \in X_{n(j)}$, and therefore $x \in \tilde{Y}_n$, where $n = \max_{1 \leq j \leq p} n(j)$ and $k \geqslant \sum_{j=1}^{n} |\lambda_j|$. This proves that the set $\hat{X}$, having measure 1, does not coincide with Λ.

Now let y be any element in Λ which does not lie in $\hat{X}$. Since $\hat{X}$ is linear, then $\hat{X}$ and $y + \hat{X}$ are disjoint. Since therefore $y + \hat{X}$ lies in the complement of $\hat{X}$ and $\mu(\hat{X}) = 1$, then $\mu(y + \hat{X}) = 0$. It follows that μ is not quasi-invariant.

We now show that it follows from this theorem that there exist no quasi-invariant measures in infinite-dimensional complete normed spaces having countable everywhere dense sets. In fact, we have to show only that in such spaces the absolutely convex hull of any compact set is nowhere dense. Let X be a compact set in Λ, and $S(x_0, r)$ the ball in Λ with radius r and center at x_0. Since X is compact, it can be covered by a finite number of balls $S(x_k, \frac{1}{2}r)$, $1 \leqslant k \leqslant n$, of radius $\frac{1}{2}r$. Therefore its absolutely convex hull $\tilde{X}$ lies in the absolutely convex hull $\tilde{S}$ of the set $S = \bigcup_{k=1}^{n} S(x_k, \frac{1}{2}r)$. Any element $y \in \tilde{S}$ can be represented in the form

$$y = \lambda_1(x_1 + y_1) + \dots + \lambda_n(x_n + y_n),$$

where $|\lambda_1| + \dots + |\lambda_n| \leqslant 1$ and $\|y_i\| \leqslant \frac{1}{2}r$, $i = 1, \dots, n$. But all elements $\lambda_1 x_1 + \dots + \lambda_n x_n$ lie in the subspace V spanned by $x_1, \dots, x_n$, and

$$\|\lambda_1 y_1 + \dots + \lambda_n y_n\| \leqslant \tfrac{1}{2}r(|\lambda_1| + \dots + |\lambda_n|) \leqslant \tfrac{1}{2}r.$$

Therefore every element in $\tilde{S}$ can be written in the form $y = v + z$, where $v \in V$ and $\|z\| \leqslant \frac{1}{2}r$.

To show that the closure of $\tilde{S}$ does not contain the ball $S(x_0, r)$, it suffices to find an element $y_1 \in S(0, r)$ which cannot be represented in the form $y_1 = v - x_0 + z$, where $v \in V$ and $\|z\| \leqslant \frac{1}{2}r$. But the existence of such an element follows directly from the finite dimensionality of V and the infinite dimensionality of Λ.† By Theorem 4 this implies the

† This result is usually stated as a theorem for the case $r = 1$, from which the result for arbitrary $r > 0$ is a trivial consequence. The statement is the following. If Y is a closed subspace of a normed linear space Λ and $Y \neq \Lambda$, then for any $\epsilon > 0$ there exists an element $x \in \Lambda$ such that $\|x\| = 1$ and $\|y - x\| > 1 - \epsilon$ for all $y \in Y$. In our case Y is the linear space spanned by $x_0, \dots, x_n$.

nonexistence in Λ of quasi-invariant measures. In the same way one can prove the nonexistence of quasi-invariant measures in any complete countably normed space having a countable everywhere dense set.

5.4. Nuclear Lie Groups and Their Unitary Representations. The Commutation Relations of the Quantum Theory of Fields

The quasi-invariant measures which we have constructed in Section 5.2 find applications in the theory of infinite-dimensional Lie groups. Let G be some (topological) group. We will call G a *nuclear Lie group*, if there exists a neighborhood of the unit element in G which is homeomorphic to a neighborhood of zero in a countably Hilbert nuclear space Φ. As a rule, nuclear Lie groups are considered for which Φ is a rigged Hilbert space, i.e., such that a scalar product (φ, ψ) is defined in Φ. Every nuclear space Φ can be looked upon as a commutative nuclear Lie group.

Let us present a somewhat more complicated example of a nuclear Lie group. Suppose that $\Phi \subset H \subset \Phi'$ is a rigged Hilbert space. The elements of the group G_0 will be all triples $g = (\varphi, \psi; \alpha)$, where φ and ψ are elements of Φ, and α is a complex number of unit modulus. We introduce a multiplication in G_0, setting

$$\begin{aligned} g_1 g_2 &= (\varphi_1, \psi_1; \alpha_1)(\varphi_2, \psi_2; \alpha_2) \\ &= (\varphi_1 + \varphi_2, \psi_1 + \psi_2; e^{i(\varphi_2, \psi_1)}\alpha_1\alpha_2) \end{aligned} \tag{11}$$

((φ, ψ) is the scalar product in Φ).

This group is connected with the commutation relations of the quantum theory of fields. In quantum mechanics a system having one degree of freedom is studied by means of operators p and q which are connected by the commutation relation

$$pq - qp = 1.$$

This commutation relation is the commutation relation for the operators of the Lie group G whose elements are triples of numbers (x, y, α), $\alpha \neq 0$, and multiplication is defined by

$$(x_1, y_1, \alpha_1)(x_2, y_2, \alpha_2) = (x_1 + x_2, y_1 + y_2, e^{ix_2 y_1}\alpha_1\alpha_2). \tag{12}$$

In the same way, the consideration of a system with n degrees of freedom leads to the system of commutation relations

$$p_j q_j - q_j p_j = 1, \qquad 1 \leqslant j \leqslant n. \tag{13}$$

These are the commutation relations for the operators of the Lie group G whose elements have the form (x, y, α), where x and y are vectors in n-dimensional space, and multiplication is defined by (12), the sole difference being that instead of $x_2 y_1$ one has to take the scalar product (x_2, y_1). Finally, the consideration of quantized fields (systems with an infinite number of degrees of freedom) leads to an infinite system of commutation relations of the form (13). It is natural to regard these relations as the commutation relations of the nuclear Lie group G_0.

We will consider here unitary representations of the groups Φ and G_0. By a *unitary representation* of any group G we mean a continuous operator-valued function $U(g)$ defined on G, whose values are unitary operators in a Hilbert space $\mathfrak{h}$, such that

$$U(g_1 g_2) = U(g_1) U(g_2)$$

for any two elements $g_1, g_2 \in G$. A unitary representation $U(g)$ is called *cyclic*, if there exists a vector $h \in \mathfrak{h}$ such that the smallest closed subspace in $\mathfrak{h}$ which contains all vectors $U(g)h$, $g \in G$, coincides with $\mathfrak{h}$. Without loss of generality, we may suppose that $\| h \| = 1$. The vector h is called a cyclic vector for the representation $U(g)$.

We begin by considering cyclic representations of the group Φ. In other words, we consider continuous operator-valued functions $U(\varphi)$, whose values are unitary operators in a Hilbert space $\mathfrak{h}$, and $U(\varphi_1 + \varphi_2) = U(\varphi_1)\, U(\varphi_2)$. The fact that the group Φ is commutative implies that $U(\varphi_1)\, U(\varphi_2) = U(\varphi_2) U(\varphi_1)$ for any elements $\varphi_1, \varphi_2 \in \Phi$. Since $U(0) = I$, it follows from the group property and the properties of unitary operators that $U(-\varphi) = U^{-1}(\varphi) = U^*(\varphi)$.

With every cyclic unitary representation $U(\varphi)$ of Φ we associate a functional $L(\varphi)$ on Φ, setting

$$L(\varphi) = (U(\varphi)h, h)_1,$$

where h is some fixed cyclic vector of the representation $U(\varphi)$, and $(\, , \,)_1$ denotes the scalar product in $\mathfrak{h}$. This functional is positive-definite. Indeed, for any elements $\varphi_1, \ldots, \varphi_n$ in Φ and any complex numbers $\alpha_1, \ldots, \alpha_n$ we have

$$\sum_{j=1}^{n} \sum_{k=1}^{n} L(\varphi_j - \varphi_k)\alpha_j \bar{\alpha}_k = \sum_{j=1}^{n} \sum_{k=1}^{n} (U(\varphi_j - \varphi_k)h, h)_1 \alpha_j \bar{\alpha}_k.$$

But

$$(U(\varphi_j - \varphi_k)h, h)_1 = (U^*(\varphi_k) U(\varphi_j)h, h)_1 = (U(\varphi_j)h, U(\varphi_k)h)_1 \, ,$$

and therefore

$$\sum_{j=1}^{n}\sum_{k=1}^{n} L(\varphi_j - \varphi_k)\alpha_j\bar{\alpha}_k = \sum_{j=1}^{n}\sum_{k=1}^{n} (U(\varphi_j)h,\ U(\varphi_k)h)_1\alpha_j\bar{\alpha}_k$$

$$= \left\| \sum_{j=1}^{n} \alpha_j U(\varphi_j)h \right\|_1^2 \geqslant 0$$

($\| h \|_1$ is the norm in $\mathfrak{h}$).

This proves the positive definiteness of $L(\varphi)$. It is further obvious that $L(0) = (h, h)_1 = 1$, and that in view of the continuity of the representation $U(\varphi), L(\varphi)$ is continuous. Applying Bochner's theorem in nuclear spaces (cf. Theorem 1 of Section 4) to $L(\varphi)$, there is a normalized countably additive positive measure μ in Φ' such that $L(\varphi)$ is its Fourier transform, i.e.,

$$L(\varphi) = \int e^{i(F,\varphi)}\, d\mu(F). \tag{14}$$

Just as in the spectral analysis of operators (cf. the appendix to Chapter I, Section 4), one can prove that $\mathfrak{h}$ can be realized as the space L^2_μ of functions $f(F)$, defined on Φ' and having square integrable moduli with respect to μ, in such a way that the operator which corresponds by this realization to the operator $U(\varphi)$ is the operator of multiplication by $e^{i(F,\varphi)}$.

This realization consists in associating with the vector

$$h_1 = \sum_{k=1}^{n} \lambda_k U(\varphi_k)h \tag{15}$$

in $\mathfrak{h}$ the function

$$f_1(F) = \sum_{k=1}^{n} \lambda_k e^{i(F,\varphi_k)} \tag{16}$$

on Φ'. It follows from (14) that this correspondence is isometric. Since h is a cyclic vector, the vectors of the form (15) are everywhere dense in $\mathfrak{h}$, and therefore this correspondence can be extended to all vectors of $\mathfrak{h}$. Now the operator $U(\varphi)$ takes a vector of the form (15) into the vector

$$U(\varphi)h_1 = \sum_{k=1}^{n} \lambda_k U(\varphi_k + \varphi)h,$$

to which corresponds the function

$$\sum_{k=1}^{n} \lambda_k e^{i(F,\varphi_k+\varphi)} = e^{i(F,\varphi)} \sum_{k=1}^{n} \lambda_k e^{i(F,\varphi_k)}.$$

Consequently, on functions of the form (16) the operator corresponding to $U(\varphi)$ is the operator of multiplication by $e^{i(F,\varphi)}$. But it is easy to prove that these functions are everywhere dense in L^2_μ. Therefore the operator in L^2_μ corresponding to $U(\varphi)$ is the operator of multiplication by $e^{i(F,\varphi)}$.

Thus we have proven the following theorem.

Theorem 5. Let $U(\varphi)$ be a unitary cyclic representation of the group Φ, and let $h \in \mathfrak{h}$ be a cyclic vector for this representation. Then there exists a normalized countably additive positive measure μ in Φ' such that

$$L(\varphi) \equiv (U(\varphi)h, h) = \int e^{i(F,\varphi)}\, d\mu(F). \tag{17}$$

The space $\mathfrak{h}$ can be realized as the space L^2_μ of functions $f(F)$ on Φ' having square integrable moduli with respect to μ, in such a way that the operator corresponding by this realization to $U(\varphi)$ is the operator of multiplication by $e^{i(F,\varphi)}$.

If one chooses a vector $h_1 \in \mathfrak{h}$ different from h (in general, h_1 will not be cyclic), then to h_1 also there corresponds a positive-definite continuous functional $L_1(\varphi)$, defined by

$$L_1(\varphi) = (U(\varphi)h_1, h_1)_1.$$

The functional $L_1(\varphi)$ is the Fourier transform of a positive measure μ_1 in Φ':

$$L_1(\varphi) = \int e^{i(F,\varphi)}\, d\mu_1(F). \tag{18}$$

The measures μ_1 and μ are connected by the relation

$$d\mu_1(F) = |f(F)|^2\, d\mu(F), \tag{19}$$

where $f(F)$ is the function which corresponds by Theorem 5 to the vector h_1. Indeed, since to the operator $U(\varphi)$ there corresponds the operator of multiplication by $e^{i(F,\varphi)}$ in L^2_μ, and the correspondence between $\mathfrak{h}$ and L^2_μ is isometric, then

$$L_1(\varphi) = (U(\varphi)h_1, h_1)_1 = \int e^{i(F,\varphi)} |f(F)|^2\, d\mu(F).$$

Comparing this with (18), we conclude that (19) holds.

It follows from (19) that if $\mu(X) = 0$, then $\mu_1(X) = 0$. If h_1 is also a cyclic vector, then the converse is true. Thus the measures corresponding to different cyclic vectors in $\mathfrak{h}$ are equivalent to one another.

Finally, we remark that given a normalized measure μ in Φ', there exists a unitary representation $U(\varphi)$ of Φ and a vector h in the space of the representation such that

$$(U(\varphi)h, h) = \int e^{i(F,\varphi)}\, d\mu(F).$$

Indeed, denote by L_μ^2 the space of all functions $f(F)$ on Φ' having square integrable moduli with respect to μ, and associate with each $\varphi \in \Phi$ the operator $U(\varphi)$ in L_μ^2 which takes any function $f(F)$ into the function $e^{i(F,\varphi)} f(F)$. Obviously $U(\varphi)$ is the desired representation.

Let us consider unitary representations of Φ which are not cyclic. In this case there is a finite or countable set $\{\mu_n\}$ of measures in Φ' such that $\mathfrak{h}$ is the direct orthogonal sum of the spaces $L_{\mu_n}^2$, and to the operator $U(\varphi)$ there corresponds in each of the $L_{\mu_n}^2$ the operator of multiplication by $e^{i(F,\varphi)}$. From this it follows that $\mathfrak{h}$ can be realized as a direct integral of Hilbert spaces

$$\mathfrak{h} = \int_{\Phi'} \oplus\, H(F)\, d\mu(F)$$

in such a way that to the operator $U(\varphi)$ there corresponds the operator of multiplication by $e^{i(F,\varphi)}$. We shall not carry out the details of the corresponding arguments.

We now turn to unitary representations of the group G_0. Recall that this group consists of elements of the form $(\varphi, \psi; \alpha)$, where φ and ψ are vectors in a nuclear space Φ in which is defined a scalar product (φ, ψ), and α is a complex number of modulus 1. Multiplication in G_0 is defined by

$$(\varphi_1, \psi_1; \alpha_1)(\varphi_2, \psi_2; \alpha_2) = (\varphi_1 + \varphi_2, \psi_1 + \psi_2; e^{i(\varphi_2,\psi_1)}\alpha_1\alpha_2). \tag{20}$$

Consider the set Φ_1 in G_0 consisting of all elements of the form $(\varphi, 0; 1)$. Since

$$(\varphi_1, 0; 1)(\varphi_2, 0; 1) = (\varphi_1 + \varphi_2, 0; 1),$$

then this set of elements forms a subgroup in G_0 which is isomorphic to the group Φ. In the same way, the elements of the form $(0, \psi; 1)$ form a subgroup Ψ_1 in G_0 which also is isomorphic to Φ. Finally, the elements of the form $(0, 0; \alpha)$ form a subgroup A in G_0 which is isomorphic to the multiplicative group T of complex numbers of modulus 1.

Let $U(g)$ be a unitary representation of the group G_0. Restricting $U(g)$ to the subgroup Φ_1, we obtain a unitary representation $U(\varphi)$ of the group Φ. In the same way, the restriction of $U(g)$ to Ψ_1 yields another unitary representation $V(\psi)$ of Φ (we denote by $V(\psi)$ the operator $U(g)$

corresponding to g of the form $(0, \psi; 1)$). Finally, the restriction of $U(g)$ to the subgroup A is a unitary representation $W(\alpha)$ of the group T. Since any element $g = (\varphi, \psi; \alpha)$ in G_0 can be written as a product

$$(\varphi, \psi; \alpha) = (\varphi, 0; 1)(0, \psi; 1)(0, 0; \alpha) \tag{21}$$

of elements of the subgroups Φ_1, Ψ_1, and A, then the operator $U(g)$ corresponding to g can be written in the form

$$U(g) = U(\varphi)V(\psi)W(\alpha).$$

Therefore to define $U(g)$ it suffices to specify the representations $U(\varphi)$, $V(\psi)$, and $W(\alpha)$.

For simplicity we restrict ourselves to the case where the representation $W(\alpha)$ of T has the form $W(\alpha) = \alpha$, and the representation $U(\varphi)$ is cyclic (the general case can easily be reduced to this case). We show that in this case a complete description of all representations of G_0 reduces to the description of all pairs $(U(\varphi), V(\psi))$ of representations of the group Φ, satisfying the commutation relations

$$V(\psi)U(\varphi) = e^{i(\varphi,\psi)}U(\varphi)V(\psi). \tag{22}$$

Indeed, let $U(g)$ be a unitary representation of G_0. It follows from (20) that

$$(0, \psi; 1)(\varphi, 0; 1) = (\varphi, 0; 1)(0, \psi; 1)(0, 0; e^{i(\varphi,\psi)}). \tag{23}$$

Since we have assumed that $W(\alpha) = \alpha$, the proof of (22) follows directly from (23).

Conversely, if the unitary representations $U(\varphi)$ and $V(\psi)$ satisfy the commutation relations (22), then, taking

$$U(g) = \alpha U(\varphi)V(\psi)$$

for $g = (\varphi, \psi; \alpha)$, we obtain a unitary representation of G_0. Indeed, if $g_1 = (\varphi_1, \psi_1; \alpha_1)$ and $g_2 = (\varphi_2, \psi_2; \alpha_2)$, then

$$\begin{aligned} U(g_1)U(g_2) &= \alpha_1\alpha_2 U(\varphi_1)V(\psi_1)U(\varphi_2)V(\psi_2) \\ &= e^{i(\varphi_2,\psi_1)}\alpha_1\alpha_2 U(\varphi_1)U(\varphi_2)V(\psi_1)V(\psi_2) \\ &= e^{i(\varphi_2,\psi_1)}\alpha_1\alpha_2 U(\varphi_1 + \varphi_2)V(\psi_1 + \psi_2) = U(g_1g_2). \end{aligned}$$

We now turn to the description of all pairs $(U(\varphi), V(\psi))$ of representations of the group Φ which satisfy the commutation relations (22).

Since we are considering only the case in which $U(\varphi)$ is cyclic, by Theorem 5 there exists a normalized measure μ in Φ' such that

$$(U(\varphi)h, h) = \int e^{i(F,\varphi)}\, d\mu(F)$$

for all $\varphi \in \Phi$ (h is a fixed cyclic vector of the representation $U(\varphi)$). The space $\mathfrak{h}$ of the representation can be realized as the space L^2_μ of functions $f(F)$ on Φ' having square integrable moduli with respect to μ, and to the operator $U(\varphi)$ there corresponds in L^2_μ the operator of multiplication by $e^{i(F,\varphi)}$.

We now prove that for this representation $U(\varphi)$ the operators $V(\psi)$ are given (in the space L^2_μ) by

$$V(\psi)f(F) = a_\psi(F)f(F + F_\psi), \tag{24}$$

where $a_\psi(F) = V(\psi)f_0(F), f_0(F) \equiv 1$, and F_ψ is the linear functional on Φ defined by $(F_\psi, \varphi) = (\varphi, \psi)$. Indeed, since the operator in L^2_μ corresponding to $U(\varphi)$ is that of multiplication by $e^{i(F,\varphi)}$, any function of the form

$$f(F) = \sum_{k=1}^{n} \lambda_k e^{i(F,\varphi_k)}. \tag{25}$$

can be written in the form

$$f(F) = \sum_{k=1}^{n} \lambda_k U(\varphi_k) f_0(F).$$

Consequently, in view of the commutation relations we have

$$\begin{aligned} V(\psi)f(F) &= \sum_{k=1}^{n} \lambda_k V(\psi) U(\varphi_k) f_0(F) \\ &= \sum_{k=1}^{n} \lambda_k \exp[i(\varphi_k, \psi)]\, U(\varphi_k) V(\psi) f_0(F) \\ &= \sum_{k=1}^{n} \lambda_k \exp[i(\varphi_k, \psi)] \exp[i(F, \varphi_k)] a_\psi(F). \end{aligned}$$

Since $\exp[i(\varphi_k, \psi)] = \exp[i(F_\psi, \varphi_k)]$, this relation can be written in the form

$$V(\psi)f(F) = a_\psi(F) \sum_{k=1}^{n} \lambda_k \exp[i(F + F_\psi, \varphi_k)] = a_\psi(F) f(F + F_\psi).$$

As the functions of the form (25) are everywhere dense in L^2_μ, the relation

$$V(\psi)f(F) = a_\psi(F)f(F + F_\psi) \tag{26}$$

holds for all functions $f(F)$ in L^2_μ.

The functions $a_\psi(F)$ satisfy the functional equation

$$a_{\psi_1+\psi_2}(F) = a_{\psi_1}(F)a_{\psi_2}(F + F_{\psi_1}). \tag{27}$$

Indeed, in view of (26)

$$\begin{aligned} a_{\psi_1+\psi_2}(F) &= V(\psi_1 + \psi_2)f_0(F) = V(\psi_1)V(\psi_2)f_0(F) \\ &= V(\psi_1)a_{\psi_2}(F) = a_{\psi_1}(F)a_{\psi_2}(F + F_{\psi_1}). \end{aligned}$$

We have found the realizations of $U(\varphi)$ and $V(\psi)$ in L^2_μ. Let us now show that the measure μ, corresponding to the representation $U(\varphi)$, is quasi-invariant (in Φ'). To do this, we note that from the unitarity of $V(\psi)$ it follows that

$$(f, f)_1 = (V(\psi)f, V(\psi)f)_1$$

for every $f \in L^2_\mu$. This can be written as

$$\int |f(F)|^2 \, d\mu(F) = \int |f(F + F_\psi)|^2 | a_\psi(F)|^2 \, d\mu(F).$$

Replacing the variable F by $F + F_\psi$ in the left side, we obtain

$$\int |f(F + F_\psi)|^2 \, d\mu(F + F_\psi) = \int |f(F + F_\psi)|^2 | a_\psi(F)|^2 \, d\mu(F). \tag{28}$$

Since (28) holds for all functions $f(F) \in L^2_\mu$, then

$$d\mu_\psi(F) \equiv d\mu(F + F_\psi) = | a_\psi(F)|^2 \, d\mu(F). \tag{29}$$

Thus, under translation by the vector F_ψ corresponding to the element $\psi \in \Phi$, the measure μ is taken into the measure μ_ψ defined by

$$\mu_\psi(X) = \int_X | a_\psi(F)|^2 \, d\mu(F).$$

Obviously $\mu_\psi(X) = 0$ if $\mu(X) = 0$. But this means that μ is quasi-invariant.

We have thus proven the following theorem.

Theorem 6. Suppose that $U(g)$ is a unitary representation of the group G_0, in a Hilbert space H, which induces a cyclic (unitary) representation of the subgroup Φ_1 of G_0 consisting of all elements of the form $(\varphi, 0; 1)$, and the representation $W(\alpha) = \alpha$ of the subgroup of G_0 consisting of all elements of the form $(0, 0; \alpha)$. Then there exists a quasi-invariant measure (in the sense of the definition on p. 355) μ in Φ' such that

$$(U(\varphi)h, h) = \int e^{i(F,\varphi)}\, d\mu(F),$$

where φ is an element of the subgroup Φ_1, and h is a cyclic vector of the representation $U(\varphi)$. The Hilbert space H can be realized as the space L^2_μ of functions on Φ' which have square integrable moduli with respect to μ, in such a way that $U(\varphi)$ is the operator of multiplication by $e^{i(F,\varphi)}$, and $V(\psi)$ is given by

$$V(\psi) f(F) = a_\psi(F) f(F + F_\psi),$$

where the $a_\psi(F)$ are functions on Φ' satisfying the functional equation

$$a_{\psi_1+\psi_2}(F) = a_{\psi_1}(F) a_{\psi_2}(F + F_{\psi_1}),$$

and F_ψ is the element of Φ' such that $(F_\psi, \varphi) = (\varphi, \psi)$.

Under translation by the vector $F_\psi \in \Phi'$, the measure μ transforms according to

$$\mu_\psi(X) = \int_X |\, a_\psi(F)|^2\, d\mu(F).$$

NOTES AND REFERENCES TO THE LITERATURE

Chapter I, Section 1

Theorem 1 is due to I. M. Gel'fand (15), who also proved Theorem 2, which is published here for the first time. The kernel theorem for the spaces K and S was proven by L. Schwartz (68), who showed the importance of this theorem in analysis; for other proofs see A. Grothendieck (22) and L. Ehrenpreis (11).

Chapter I, Section 2

The general form of a norm in a space of matrices was considered by J. von Neumann (76). The trace norm for operators in Banach spaces has been studied by R. Schatten (62, 63), and also by Schatten and von Neumann (64, 65). See also the work of A. F. Ruston (60). The definition of a nuclear mapping was given by L. Schwartz (69) and A. Grothendieck (22). Theorem 6 concerning the trace is due to V. B. Lidskii (45).

Chapter I, Section 3

The general definition of nuclear spaces was given by Grothendieck (22). A definition of nuclearity, which coincides with that of Grothendieck in the case of countably normed spaces, was given in connection with the theory of eigenelements of self-adjoint operators by I.M. Gel'fand and A. G. Kostyuchenko (18). Most of the results of this section are due to Grothendieck. Theorem 1 was proven by D. A. Raikov (56). A. S. Dynin (10), using a result of Grothendieck, proved Theorem 2. The proof given in the text, which does not rest upon Grothendieck's results, is due to N. Ya. Vilenkin. The nuclearity of the space S_α^β was proven by B. S. Mityagin (50). The results of Section 3.7 are due to B. S. Mityagin (51). Estimates, similar to those used in Section 3.7, were obtained by V. I. Arnol'd [see reference (35), Section 6). The material in Section 3.8, which ties in with Mityagin's results, is due to N. Ya. Vilenkin. The basic ideas, connected with the study of infinite-dimensional spaces by means of estimates of the smallest number of elements in the ϵ-sets of compacta, were given by A. N. Kolmogorov (33, 34). These estimates are computed for many concrete spaces in the paper of A.N. Kolmogorov and V. M. Tikhomirov (35) [see also A. G. Vitushkin (85)]. Some important results concerning nuclear spaces are due to Ch. Bessaga and A. Pilchinskii, who proved, in particular, that any nuclear space can be imbedded in the space of all infinitely differentiable functions (6).

Chapter I, Section 4

The concept of a rigged Hilbert space was essentially introduced by I. M. Gel'fand and A. G. Kostyuchenko in reference (18) in connection with the spectral theory of self-adjoint operators. They proved the theorem on the existence of a complete system of generalized eigenvectors for a self-adjoint operator. Yu. M. Berezanskii (1–5) simplified the proof somewhat and indicated further applications to the theory of partial differential equations and the theory of positive-definite functions. See also papers of G. I. Kats (27, 28), L. Gårding (13), and K. Maurin (46–48). The reader will find further references in Volume III of this series [see also the books of A. Friedman (82) and L. Hörmander (84)]. In writing Sections 4.3 and 4.4, a refinement of the method of proof of I. M. Gel'fand and A. G. Kostyuchenko, due to Maurin and Gårding (48), was used. Direct integrals of Hilbert spaces were first studied by J. von Neumann (77).

Chapter II, Section 2

The theorem giving the general form of a linear functional on a space of continuous functions is due to F. Riesz (57). The theorem on the general form of positive generalized functions on the space K was shown by L. Schwartz (67).

Chapter II, Section 3

The basic theorem of the theory of continuous positive-definite functions (Theorem 2) was proven by S. Bochner (9). Positive-definite generalized functions were introduced by L. Schwartz, who proved Theorem 3 in reference (67). The proof given here is a new version, obtained by simplifying and systematizing Schwartz's proof.

Chapter II, Section 4

Conditionally positive-definite functions of one variable have been considered in connection with infinitely divisible distribution laws and the theory of random processes with stationary increments [see B. V. Gnedenko (21) and A. M. Yaglom and M. S. Pinsker (80)]. A number of interesting results, relating to this circle of topics, were presented in a note of M. G. Krein (40). Conditionally positive-definite functions of the first order, of several variables, were considered by A. M. Yaglom (79). The general theorem was proven by N. Ya. Vilenkin (cf. Theorem 2), and is published here for the first time.

Chapter II, Section 5

Continuous evenly positive-definite functions of one variable were studied by M. G. Krein (39). Conditions for uniqueness in Krein's theorem were obtained by B. M. Levitan and N. N. Meiman (42, 43). See also the work of E. B. Vul (78) and Yu. M. Berezanskii (3, 4).

For generalized functions, even positive definiteness was studied in the paper of I. M. Gel'fand and Sya-do-shin (20) for the case of one variable, and by N. Ya. Vilenkin (75) for the case of several variables. A substantial generalization of these results was given by A. G. Kostyuchenko and B. S. Mityagin (36).

Chapter II, Section 6

The theorem on the extensions of positive functionals on partially ordered linear spaces is due to M. G. Krein (37), who developed an idea of M. Riesz (58). A description of evenly positive-definite generalized functions of one variable was given in reference (20) by Sya-do-shin, to whom is due also the example of the nonuniqueness of the measure.

Chapter II, Section 7

The theorem on positive-definite functionals on commutative normed rings was proven by I. M. Gel'fand and M. A. Naimark (19). Positive-definite linear functionals on the ring of polynomials of two variables have been considered by R. B. Zarkhina (81). For an example of a positive polynomial which is not a sum of squares of polynomials, cf. D. Hilbert (24).

The theorem on the decomposition of entire analytic functions of exponential type, which are positive on the real axis, was proven by M. G. Krein (38); cf. also B. Ja. Levin (41).

Chapter III

The concept of a generalized random process, as well as the basic results of this chapter (in particular, the theory of processes with independent values at every point), is due to I. M. Gel'fand (16). The correlation theory of generalized random processes was constructed by K. Ito (25); cf. also the interesting book of S. Bochner (8). Another definition of a generalized random process is due to K. Urbanik (73); cf. also L. Schwartz (70). The characteristic functional was introduced by A. N. Kolmogorov (31), and has been studied by S. Bochner (7).

Probability distributions in normed spaces have been considered by A. N. Kolmogorov (31), Yu. V. Prokhorov (55), and E. Mourier (53).

Concerning the considerations in Section 4 on infinitely divisible random variables, cf. A. Ya. Khinchin (30). Theorem 4 was proven by Schoenberg in reference (66) in connection with questions on the imbedding of metric spaces in a Hilbert space.

The theory of generalized random fields, studied in Section 5, was constructed by A. M. Yaglom (79). The results on fields with homogeneous nth-order increments are due to N. Ya. Vilenkin; cf. also Ito (26).

Chapter IV, Section 1

The definition of a measure by means of the measures of cylinder sets was introduced by A. N. Kolmogorov (32).

Chapter IV, Section 2

The main result of this section, Theorem 3, was obtained by R. A. Minlos (49), proving the validity of an earlier conjecture of I. M. Gel'fand (17). Similar questions have been taken up by Yu. V. Prokhorov (55) and V. Sazonov (61), who studied probability distributions in linear normed spaces. As A. N. Kolmogorov showed, Lemmas 4 and 4', which are at the heart of Minlos' proof, follow from certain estimates obtained by Yu. V. Prokhorov. The proof presented in the text is a simplification of the proof given by R. A. Minlos. Theorems 2 and 2' are due to V. D. Erokhin (unpublished). Theorems 5 and 6 were proven by N. Ya. Vilenkin. Theorem 7 is due to V. Sazanov (61).

Chapter IV, Section 3

The question of conditions for the countable additivity of Gaussian measures was considered by V. A. Golubev (unpublished) before the general theorem of R. A. Minlos. The necessity of the condition of nuclearity of the space for the countable additivity of any cylinder set measure in the conjugate space was proven by R. A. Minlos (49).

Chapter IV, Section 4

The results of this section follow directly from those of Section 2. Generalizations of Bochner's theorem to Banach space have been treated by Yu. V. Prokhorov (55) and V. Sazanov (61). See also the monograph of L. Gross (83).

Chapter IV, Section 5

A study of the commutation relations in the quantum theory of fields was carried out by Gårding and Wightman (14). Similar results were obtained in another form by Segal (71). In the form presented in the text, these results are due to I. M. Gel'fand and are published here for the first time.

The first series of nontrivial representations of the commutation relations were obtained by Friedrichs (12), Van Hove (74), and Haag (23). An interesting form of representation of the commutation relations is contained in the detailed Princeton dissertation of J. S. Lew (44).

Some results on quasi-invariant measures are contained in papers of V. N. Sudakov (72) and B. S. Mityagin (52).

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† Added by translator.

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