

**NANYANG**  
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# Fuzzy Systems



## *3.1 Introduction*

A **fuzzy system (FS)** is defined as a system with operating principles based on ***fuzzy information processing*** and ***decision making***. Namely the system makes use of fuzzy sets and of the corresponding mathematical framework.

Fuzzy sets can be involved in a system in a number of ways:

- 1) Systems with fuzzy parameters.

**Example**

$$y = \tilde{3}x_1 + \tilde{5}x_2$$

where  $\tilde{3}$  and  $\tilde{5}$  are fuzzy numbers “about three” and “about five”, respectively, defined by membership functions.

- 2) The input, output and state variables of a system may be fuzzy sets.

**Example**

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = \tilde{2}$$

where  $\tilde{2}$  is a fuzzy number

### 3) Rule-based systems

A fuzzy system in this case is a collection of fuzzy rules that converts inputs to outputs.

The basic configuration of a pure fuzzy system is shown in the Figure. 3.1

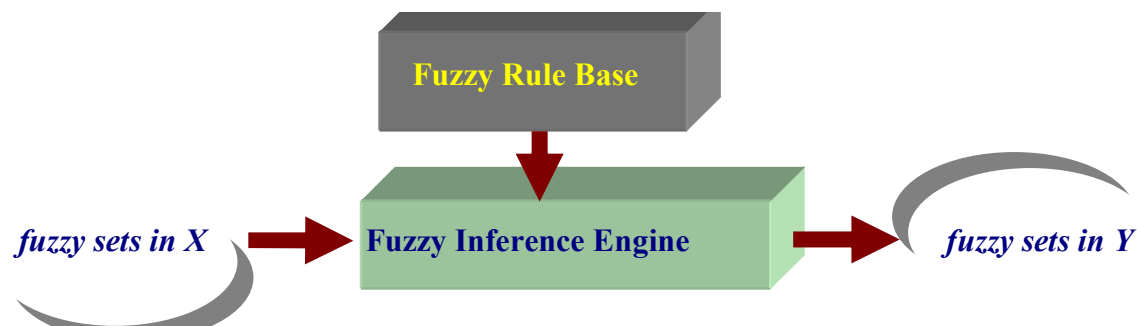


Figure. 3.1

The ***fuzzy inference engine*** (algorithm) combines fuzzy *IF-THEN* rules into a mapping from fuzzy sets in the ***input space X*** to fuzzy sets in the ***output space Y*** based on fuzzy logic principles.

## Rule-based Fuzzy Systems

- Linguistic (Mamdani) fuzzy system

**If  $x$  is  $A$  then  $y$  is  $B$**

### ***Example***

**If *the heating power is high*  
then *the temperature will increase fast*.**

- Takagi-Sugeno fuzzy system

### ***Example***

**If  $x$  is  $A$  then  $y = f(x)$**

Fuzzy systems are **NONLINEAR MAPPINGS** of inputs (stimuli) to outputs (responses).

The **main characteristics** of fuzzy systems are.

- Fuzzy systems are suitable for uncertain or approximate reasoning, especially for the system with a mathematical model that is difficult to derive;
- Fuzzy logic allows decision making with estimated values under incomplete or uncertain information.

## 3.2 Introduction to Fuzzy Rules

A fuzzy if-then rule (also known as fuzzy rule, fuzzy implication, or fuzzy conditional statement) assumes the form

IF  $x$  is  $A$  THEN  $y$  is  $B$

where  $x$  and  $y$  are linguistic variables,  $A$  and  $B$  are linguistic values determined by fuzzy sets on the universe of discourses  $X$  and  $Y$ , respectively.

“ $x$  is  $A$ ”-Antecedent or Premise

“ $y$  is  $B$ ”-Consequence or conclusion

The expression

If  $x$  is  $A$  then  $y$  is  $B$ , or  $A \rightarrow B$

describes a relation between variables  $x$  and  $y$ .

This suggests that a fuzzy rule be defined as a binary fuzzy relation  $R$  on the product space  $X \times Y$   
**(Mamdani'T-norm Rule)**

$$R = A \rightarrow B = A \times B = \int_{X \times Y} \mu_A(x) \wedge \mu_B(y) / (x, y)$$



### ***Example:***

**$X = \{1,2,3,4\}$  - uniqueness of invention**

**$Y = \{1, 2, 3, 4, 5, 6\}$  - market sizes**

**$A = \text{medium uniqueness} = \{0/1 + 0.6/2 + 1/3 + 0.2/4\}$**

**$B = \text{medium market size} = \{0/1 + 0.4/2 + 1/3 + 0.8/4 + 0.3/5 + 0.0/6\}$**

$$\mu_R(1, 2) = (0 \wedge 0.4) = 0$$

$$\mu_R(2, 2) = (0.6 \wedge 0.4) = 0.4$$

$$\mu_R(3, 2) = (1 \wedge 0.4) = 0.4$$

$$\mu_R(4, 2) = (0.2 \wedge 0.4) = 0.4$$

$\vdots$

$$A \rightarrow B = R = \begin{pmatrix} 0 \\ 0.6 \\ 1 \\ 0.2 \end{pmatrix} \wedge (0 \quad 0.4 \quad 1.0 \quad 0.8 \quad 0.3 \quad 0)$$

$$= \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & .4 & .6 & .6 & .3 & 0 \\ 0 & .4 & 1 & .8 & .3 & 0 \\ .0 & .2 & .2 & .2 & .2 & 0 \end{bmatrix} \end{matrix}, \subseteq X \times Y$$

## ***Firing Rules:***

In a fuzzy system, all rules fire to some extent, or in other words, some fire fully while others fire partially.

If the **antecedent** is true to **some degree** of membership, then the **consequent** is also true to the **same degree**.

# Firing Fuzzy Rules - An Example

IF height is tall THEN weight is heavy

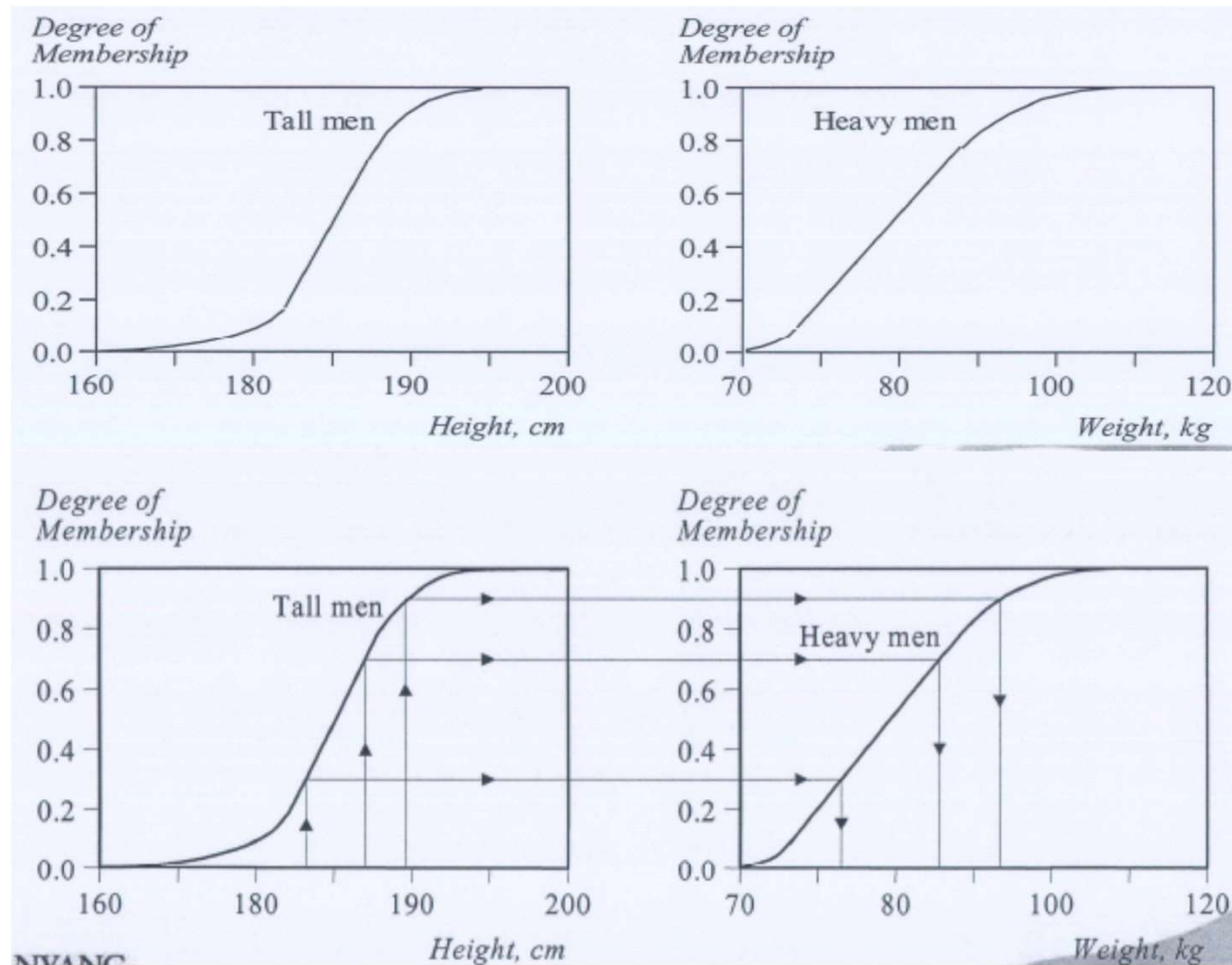


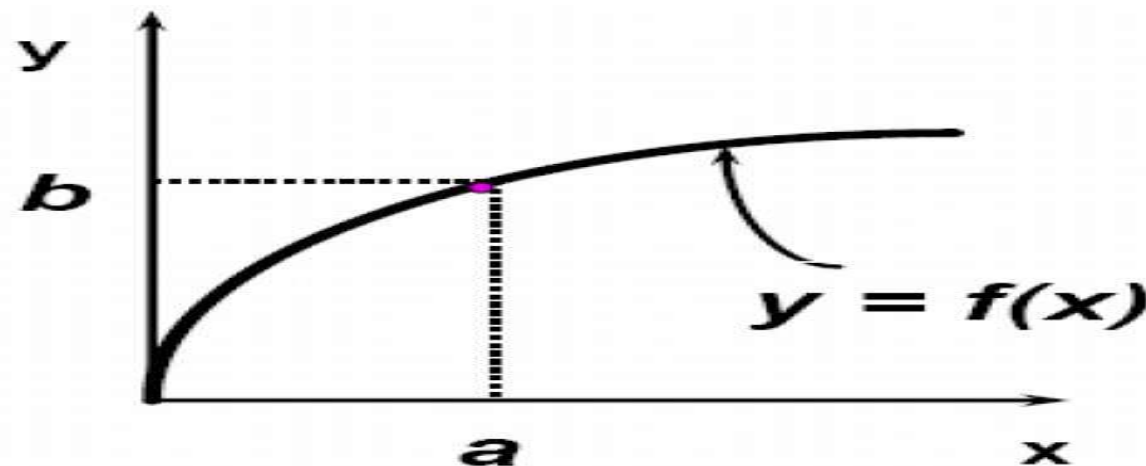
Figure. 3.2

### *3.3 FUZZY REASONING*

- Fuzzy reasoning is an inference procedure that derives conclusions from a set of fuzzy if-then rules and known facts.
- To better understand the reasoning process, we first consider some fundamentals.

## Compositional Rule of Inference

- The compositional rule of inference is a generalization of the following familiar notion:
- Suppose  $y=f(x)$ . If  $x = a$ , then from  $y=f(x)$  we can infer that  $y = b = f(a)$ .



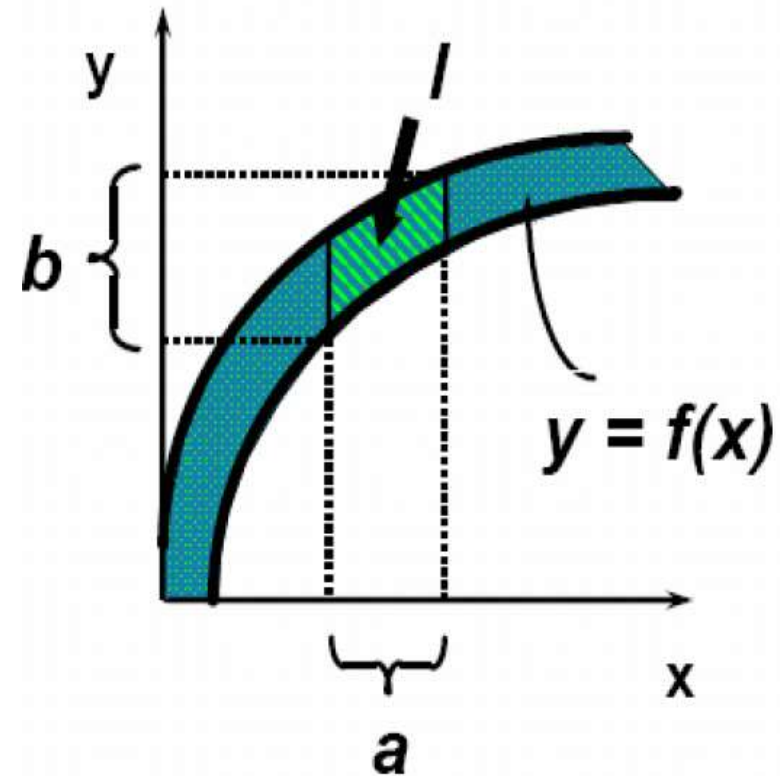
**$a$  and  $b$ : points**  
 **$y = f(x)$  : a curve**

Figure. 3.3

A generalization of the aforementioned process would allow  $a$  to be an interval and  $f(x)$  to be an interval valued function.

To find the resulting interval  $y = b$  corresponding to the interval  $x = a$ , we first construct a cylindrical extension of  $a$ .

Then find its intersection  $I$  with the interval-valued curve. The projection of  $I$  onto the  $y$ -axis yields the interval  $y=b$ .



**$a$  and  $b$ : intervals**

**$y = f(x)$  : an interval-valued function**

Figure. 3.4

Going one step further in our generalization, we may assume that  $R$  is a fuzzy relation on  $X \times Y$  and  $A$  is a fuzzy set of  $X$ , as shown in the Figure 3.5.

To find the resulting fuzzy set  $B$ , again we construct a **cylindrical extension**  $c(A)$  with base  $A$ . The **intersection** of  $c(A)$  and  $R$  forms the analog of the region of intersection  $I$ . By **projecting**  $c(A) \cap R$  on to the  **$y$ -axis**, we infer  $y$  as a **fuzzy set**  $B$  on the  $y$ -axis .



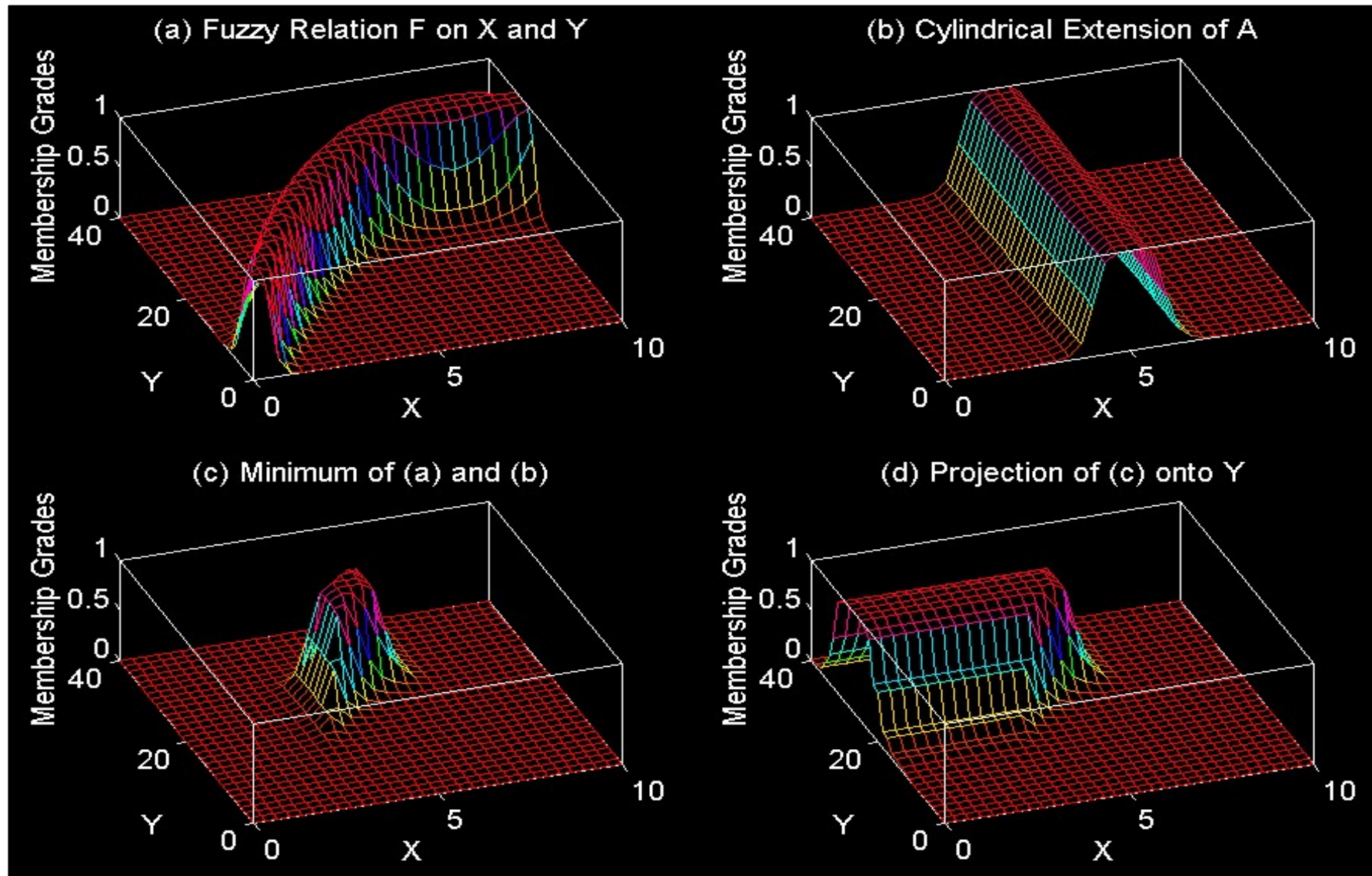


Figure. 3.5

Specifically, let  $\mu_A, \mu_{c(A)}, \mu_B$  and  $\mu_R$  be MFs of  $A, c(A), B$  and  $R$ , where  $\mu_{c(A)}$  is related to  $\mu_A$  through  $\mu_{c(A)}(x,y) = \mu_A(x)$ .

Then

$$\begin{aligned}\mu_{c(A) \cap R} &= \min\{ \mu_{c(A)}(x,y), \mu_R(x,y) \} \\ &= \min\{ \mu_A(x), \mu_R(x,y) \}\end{aligned}$$

By projecting  $c(A) \cap R$  on to the  $y$ -axis, we have

$$\begin{aligned}\mu_B(y) &= \max_x \min\{ \mu_A(x,y), \mu_R(x,y) \} \\ &= V_x [\mu_A(x) \wedge \mu_R(x,y)]\end{aligned}$$

This formula reduces to the max-min composition and is represented as:

$$B = A \circ R$$

The above is called **fuzzy or approximate reasoning**  
(also called generalized modus ponens(GMP))  
which is formally summarized as follows:

<b>Premise 1 (fact):</b>	<b><math>x</math> is <math>A'</math>,</b>
<b>Premise 2 (rule):</b>	<b>if <math>x</math> is <math>A</math> then <math>y</math> is <math>B</math></b>
<hr/>	
<b>consequence (conclusion)</b>	<b><math>y</math> is <math>B'</math></b>

where  $A$ ,  $A'$ , and  $B$  are fuzzy sets of  $X$ ,  $X$ , and  $Y$ .

***Example:***

$$\underline{X = \{20 \ 25 \ 30 \ 35 \ 40 \ 45 \ 50\}; Y = \{10 \ 30 \ 50 \ 70 \ 80 \ 100\}}$$

$$\underline{A = 1/20 + 0.8/25 + 0.4/30, B = 0.6/70 + 0.8/80 + 1.0/100}$$

If x is A Then y is B

If x is A' what is y ?,

$$\underline{\text{where } A' = 0.4/30 + 0.6/35 + 1.0/40 + 0.2/45 + 0/50}$$

$$\mu_R(x, y) = \mu_A(x) \wedge \mu_B(y)$$

$$= \begin{pmatrix} 1 \\ .8 \\ .4 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \wedge (0 \quad 0 \quad 0 \quad .6 \quad .8 \quad 1)$$

$$= \begin{pmatrix} 0 & 0 & 0 & .6 & .8 & 1 \\ 0 & 0 & 0 & .6 & .8 & .4 \\ 0 & 0 & 0 & .4 & .4 & .4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B' = A' \circ R = (0 \quad 0 \quad .4 \quad .6 \quad 1 \quad .2 \quad 0) \circ \begin{pmatrix} 0 & 0 & 0 & .6 & .8 & 1 \\ 0 & 0 & 0 & .6 & .8 & .4 \\ 0 & 0 & 0 & .4 & .4 & .4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$=(0 \ 0 \ 0 \ 0.4 \ 0.4 \ 0.4)$$

**Further simplification of the rule gives:**

$$\begin{aligned}\mu_{B'}(y) &= [\vee_x (\mu_{A'}(x) \wedge \mu_A(x))] \wedge \mu_B(y) \\ &= w \wedge \mu_B(y)\end{aligned}$$

**In other words, first we find the degree of match  $w$  as the maximum of  $\mu_{A'}(x) \wedge \mu_A(x)$  then the MF of the resulting  $B'$  is equal to the MF of clipped by  $w$ .**

**Graphic Representation:**

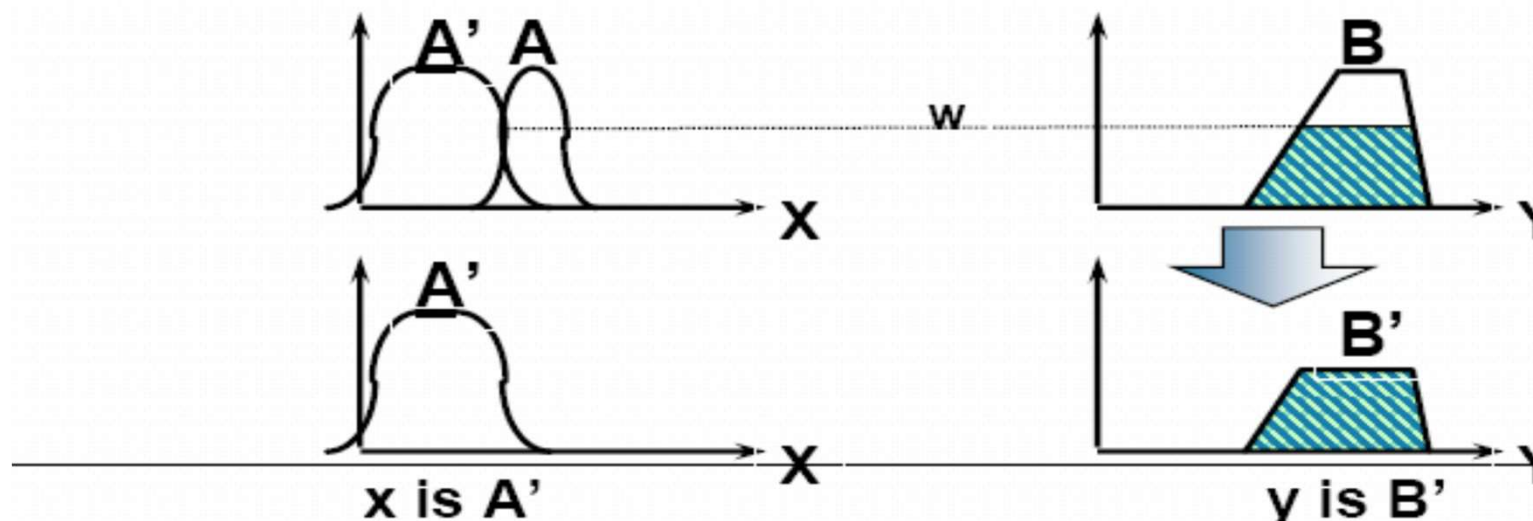


Figure. 3.6

More on Clipping will be discussed later.

## ***Single rule with multiple antecedent***

A fuzzy if-then rule with two antecedents is usually written as

“if  $x$  is  $A$  and  $y$  is  $B$  then  $z$  is  $C$ ”.

premise 1 (fact)	$x$ is $A'$ and $y$ is $B'$
premise 2(rule)	if $x$ is $A$ and $y$ is $B$ then $z$ is $C$
<hr/>	
consequence (conclusion)	$z$ is $C'$

The fuzzy rule in premise 2 can be put into a simpler form “ $A \times B \rightarrow C$ ”.

Based on Mamdani's fuzzy implication function,

$$C' = (A' \times B') \circ (A \times B \rightarrow C)$$

$$\begin{aligned}\mu_{C'}(z) &= \bigvee_{x,y} [\mu_{A'}(x) \wedge \mu_{B'}(y)] \wedge [\mu_A(x) \wedge \mu_B(y) \wedge \mu_C(z)] \\ &= \bigvee_{x,y} \{[\mu_{A'}(x) \wedge \mu_{B'}(y) \wedge \mu_A(x) \wedge \mu_B(y)]\} \wedge \mu_C(z) \\ &= \underbrace{\{\bigvee_x [\mu_{A'}(x) \wedge \mu_A(x)]\}}_{w_1} \wedge \underbrace{\{\bigvee_y [\mu_{B'}(y) \wedge \mu_B(y)]\}}_{w_2} \wedge \mu_C(z) \\ &= \underbrace{(w_1 \wedge w_2)}_{\text{firing strength}} \wedge \mu_C(z)\end{aligned}$$

$w_1$  and  $w_2$  are maxima of the MFs of  $A \cap A'$  and  $B \cap B'$

$w_1$  denote the degree of compatibility between A and A' and similarly for  $w_2$ .  $w_1 \wedge w_2$  is called the firing strength or degree of fulfillment of the fuzzy rule.



## Graphic Representation:

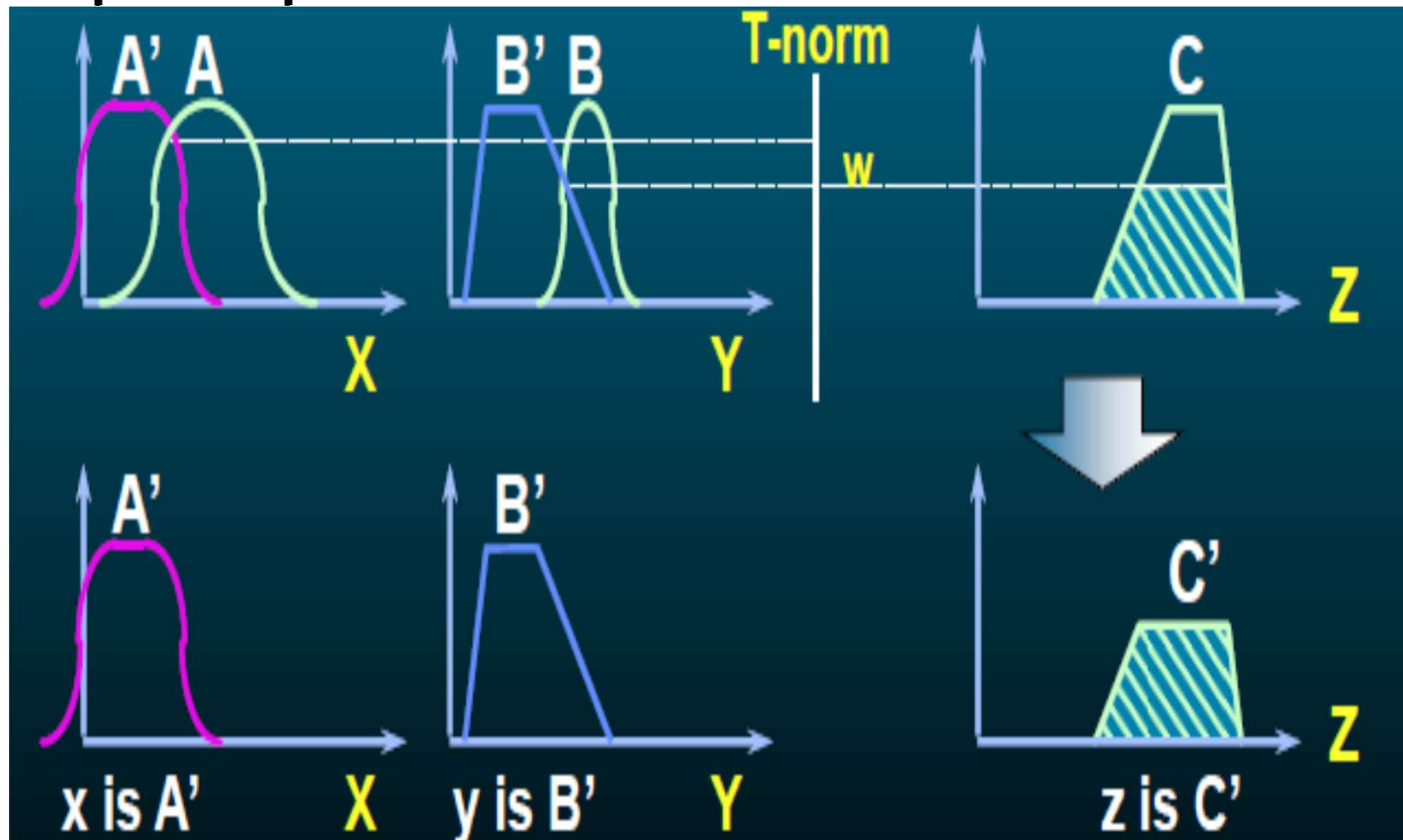


Figure. 3.7

### *3.4 Fuzzy Inference Systems*

As mentioned, a **FUZZY SYSTEM** is a collection of fuzzy rules that converts inputs to outputs.

Usually we start with **crisp inputs and find** crisp outputs.

With such considerations, we have the following four steps of fuzzy inference process:

- Fuzzification of the input variables
- Evaluation of output for each rule
- Aggregation of the rules' outputs
- Defuzzification

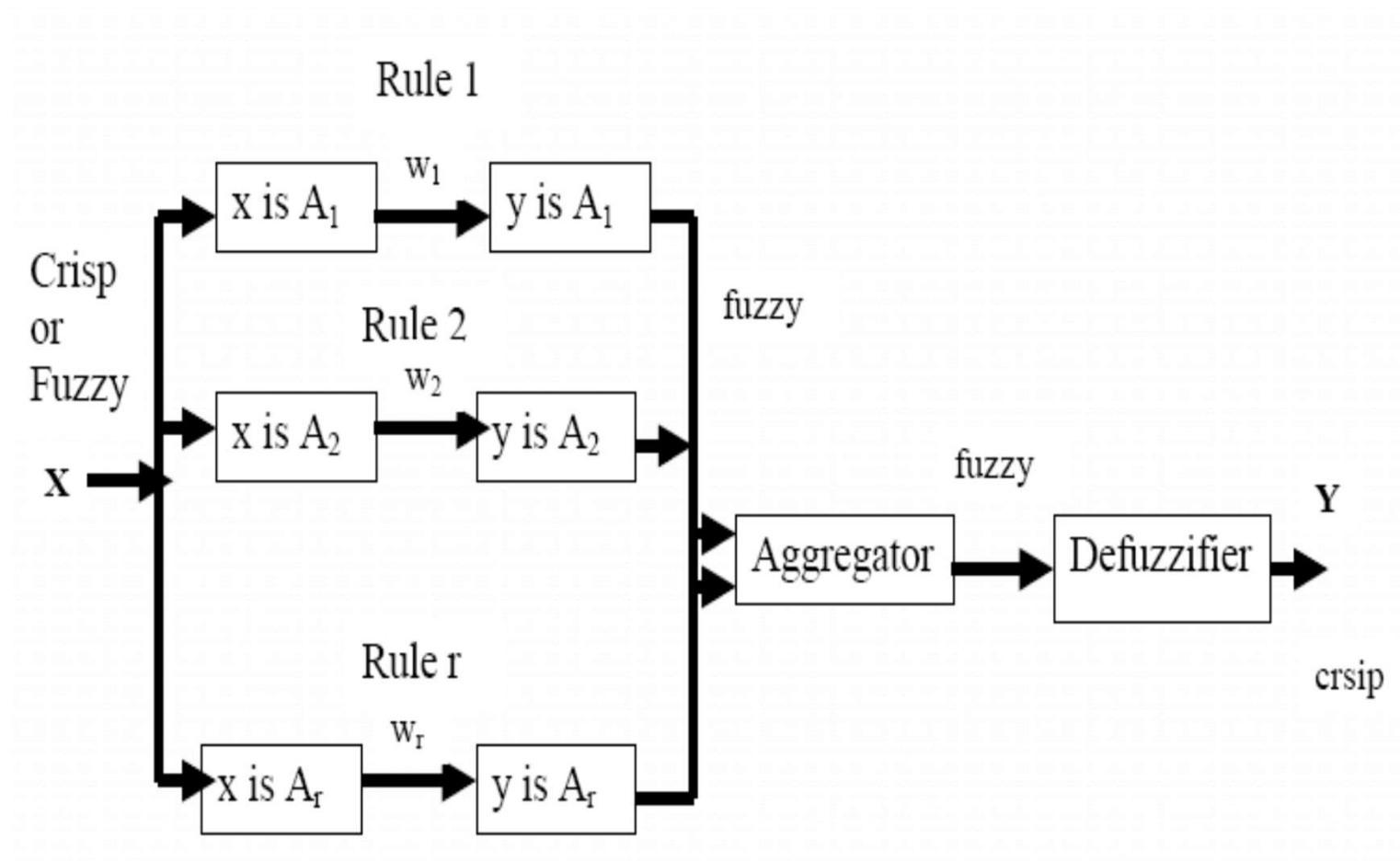


Figure. 3.8

We illustrate the above steps with *examples*.

## Step 1: Fuzzification

The first step is to take the crisp inputs  $x_1$  and  $y_1$ , and determine the degree to which these inputs belong to each of the appropriate fuzzy sets.

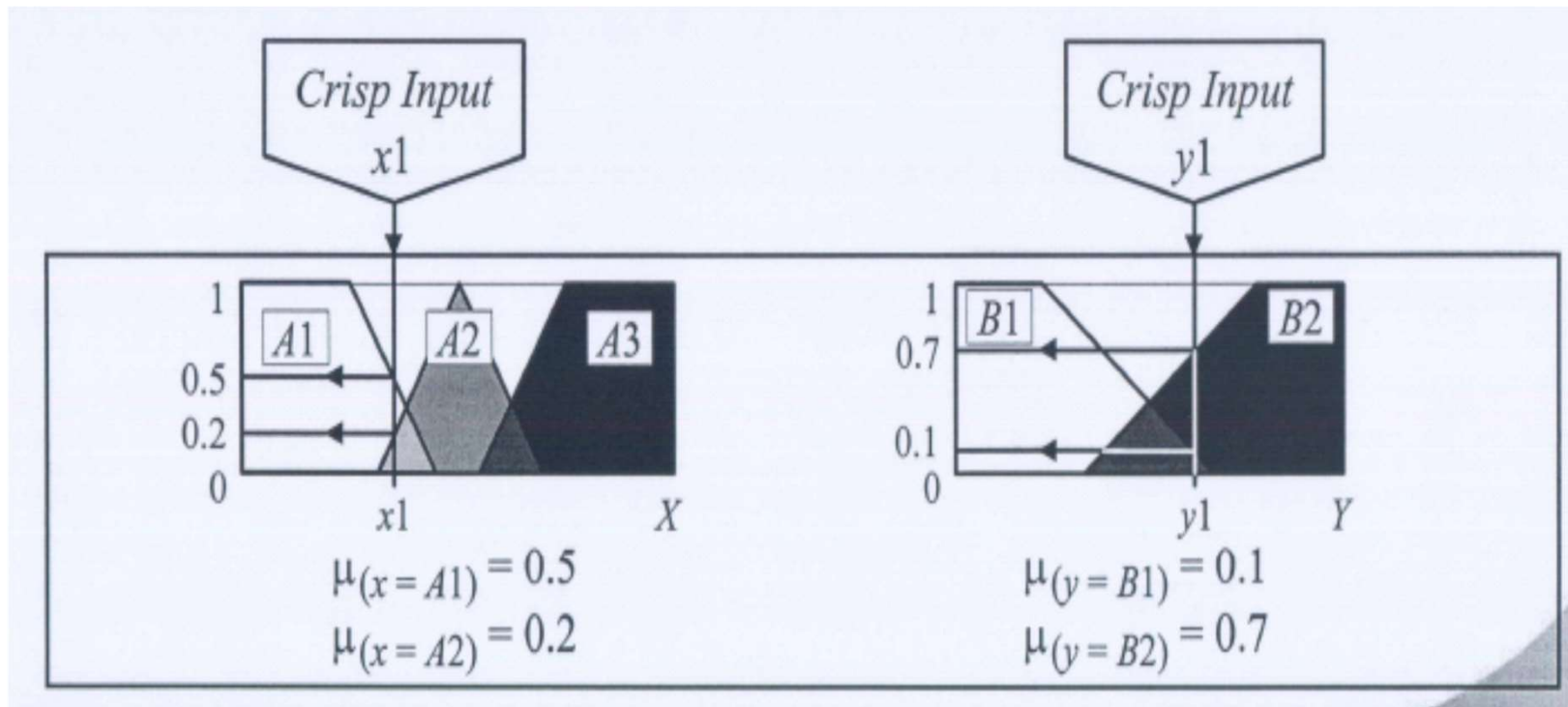


Figure. 3.9

## Step 2: Rule Evaluation

- The second step is to take the fuzzified inputs,  $\mu_{(x=A1)} = 0.5$ ,  $\mu_{(x=A2)} = 0.2$ ,  $\mu_{(y=B1)} = 0.1$  and  $\mu_{(y=B2)} = 0.7$ , and apply them to the antecedents of the fuzzy rules.
- If a given fuzzy rule has multiple antecedents, the fuzzy operator (AND or OR) is used to obtain a single number that represents the result of the antecedent evaluation.
- This number (the truth value) is then applied to the consequent membership function.

### ***Example***

R1: If  $x$  is  $A_1$  AND  $y$  is  $B_1$ , THEN  $z$  is  $C_1$

R2: If  $x$  is  $A_2$  AND  $y$  is  $B_2$ , THEN  $z$  is  $C_2$

## ***Mamdani's minimum fuzzy implication rule:***

The firing levels of the rules, denoted by  $\alpha_i$ ,  $i = 1, 2$ , are computed by

$$\alpha_1 = A_1(x_0) \wedge B_1(y_0),$$

$$\alpha_2 = A_2(x_0) \wedge B_2(y_0)$$

The individual rule outputs are obtained by

$$\mu_{c'_1} = \alpha_1 \wedge \mu_{c_1}(Z)$$

$$\mu_{c'_2} = \alpha_2 \wedge \mu_{c_2}(Z)$$

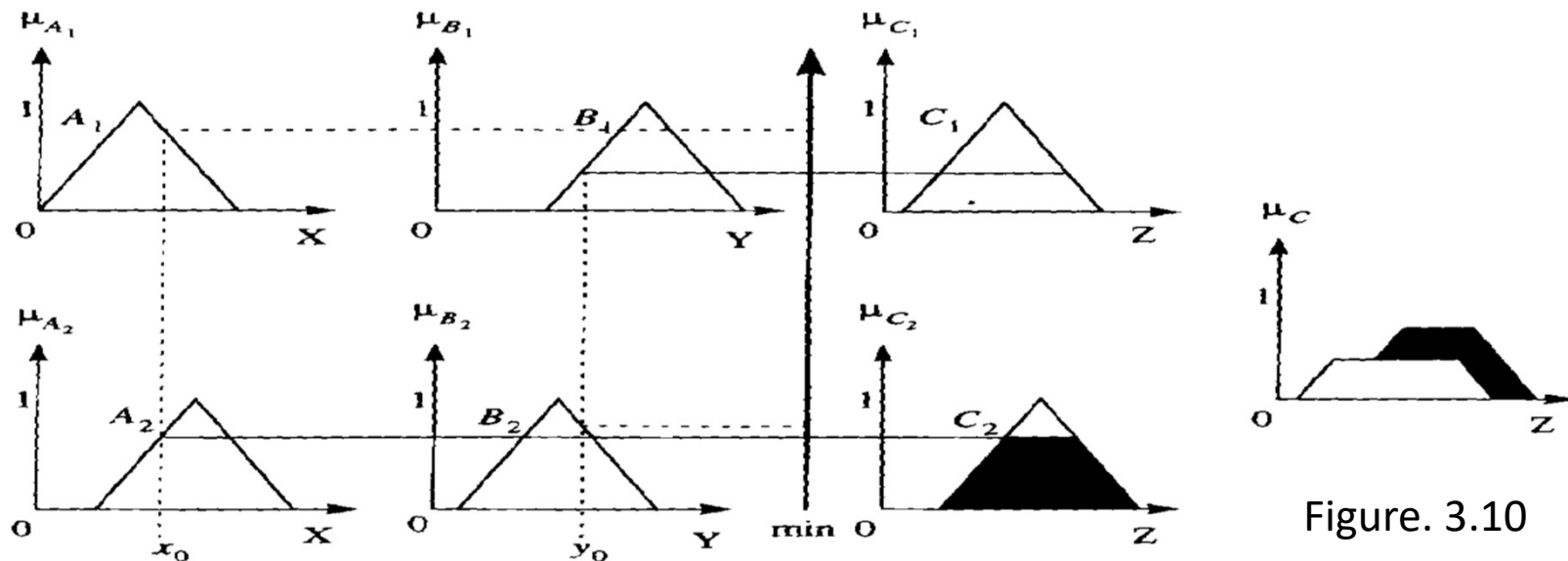


Figure. 3.10

### ***Larsen's product fuzzy implication rule:***

The firing levels of the rules, denoted by  $\alpha_i$ ,  $i = 1, 2$ , are computed by

$$\alpha_1 = A_1(x_0) \wedge B_1(y_0),$$

$$\alpha_2 = A_2(x_0) \wedge B_2(y_0)$$

The individual rule outputs are obtained by

$$\mu_{c'_1} = \alpha_1 \mu_{c_1}(Z)$$

$$\mu_{c'_2} = \alpha_2 \mu_{c_2}(Z)$$

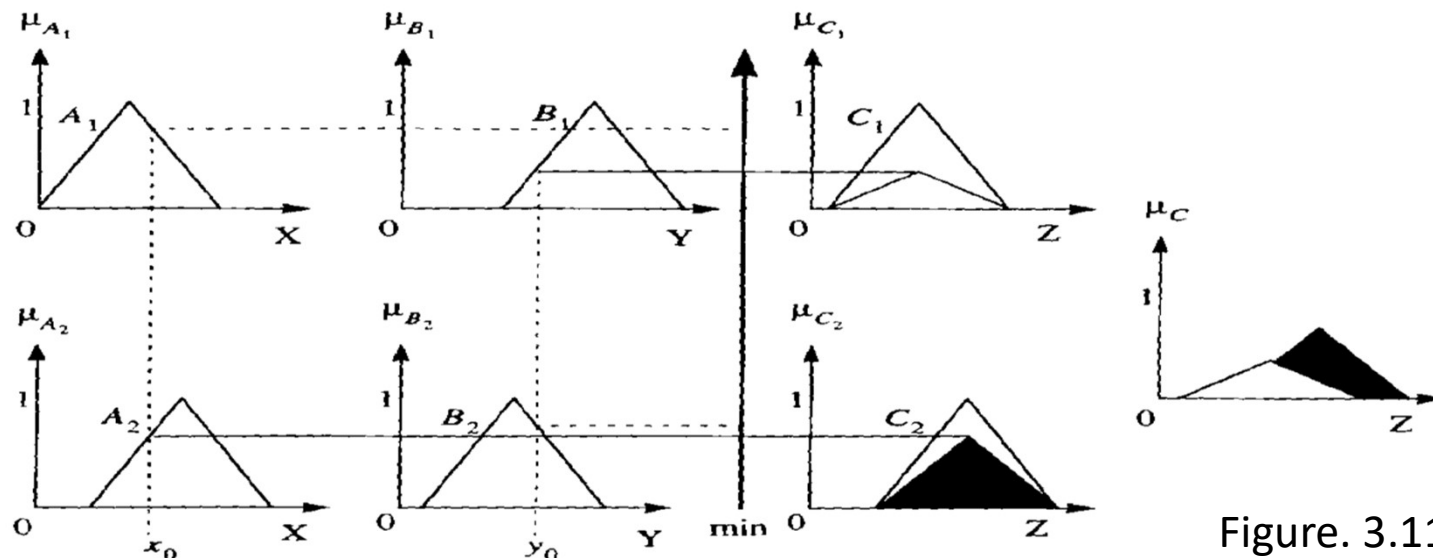


Figure. 3.11

From the above, there are two main methods for applying the result of the antecedent evaluation to the membership function of the consequent

- Clipping
- Scaling

The most common method of correlating the rule consequent with the truth value of the rule antecedent is to cut the consequent membership function at the level of the antecedent truth. This method is called clipping.

Since the top of the membership function is sliced, the clipped fuzzy set loses some information.

However, clipping is still often preferred because it involves less complex and faster mathematics, and generates an aggregated output surface that is easier to defuzzify.



While clipping is a frequently used method, scaling offers a better approach for preserving the original shape of the fuzzy set.

The original membership function of the rule consequent is adjusted by multiplying all its membership degrees by the truth value of the rule antecedent.

This method, which generally loses less information, can be very useful in fuzzy expert systems.

### Step 3: Aggregation of Rule Outputs

Aggregation is the process of unification of the outputs of all rules.

We take the membership functions of all rule consequents previously clipped or scaled and combine them into a single fuzzy set.

$$C(w) = C'_1(w) \vee C'_2(w)$$

#### *Example*

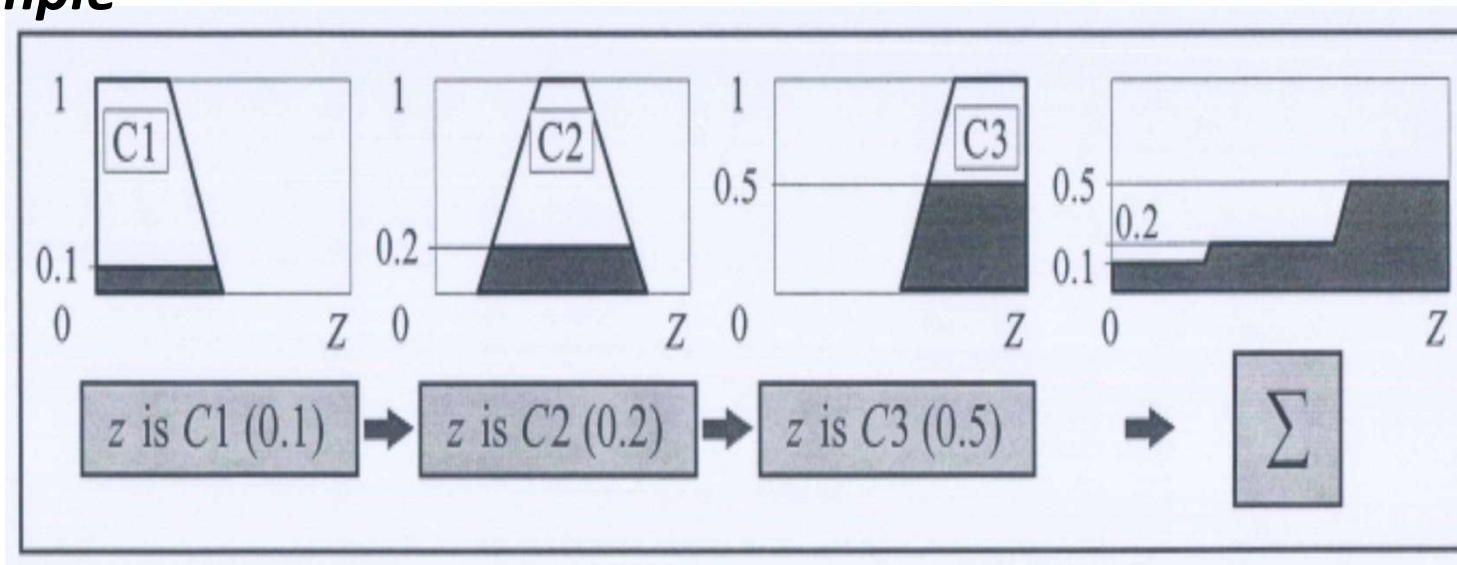


Figure. 3.12

The above 3 steps can be summarized by the following illustration with ***Mamdani's implication rule***

### Graphics representation:

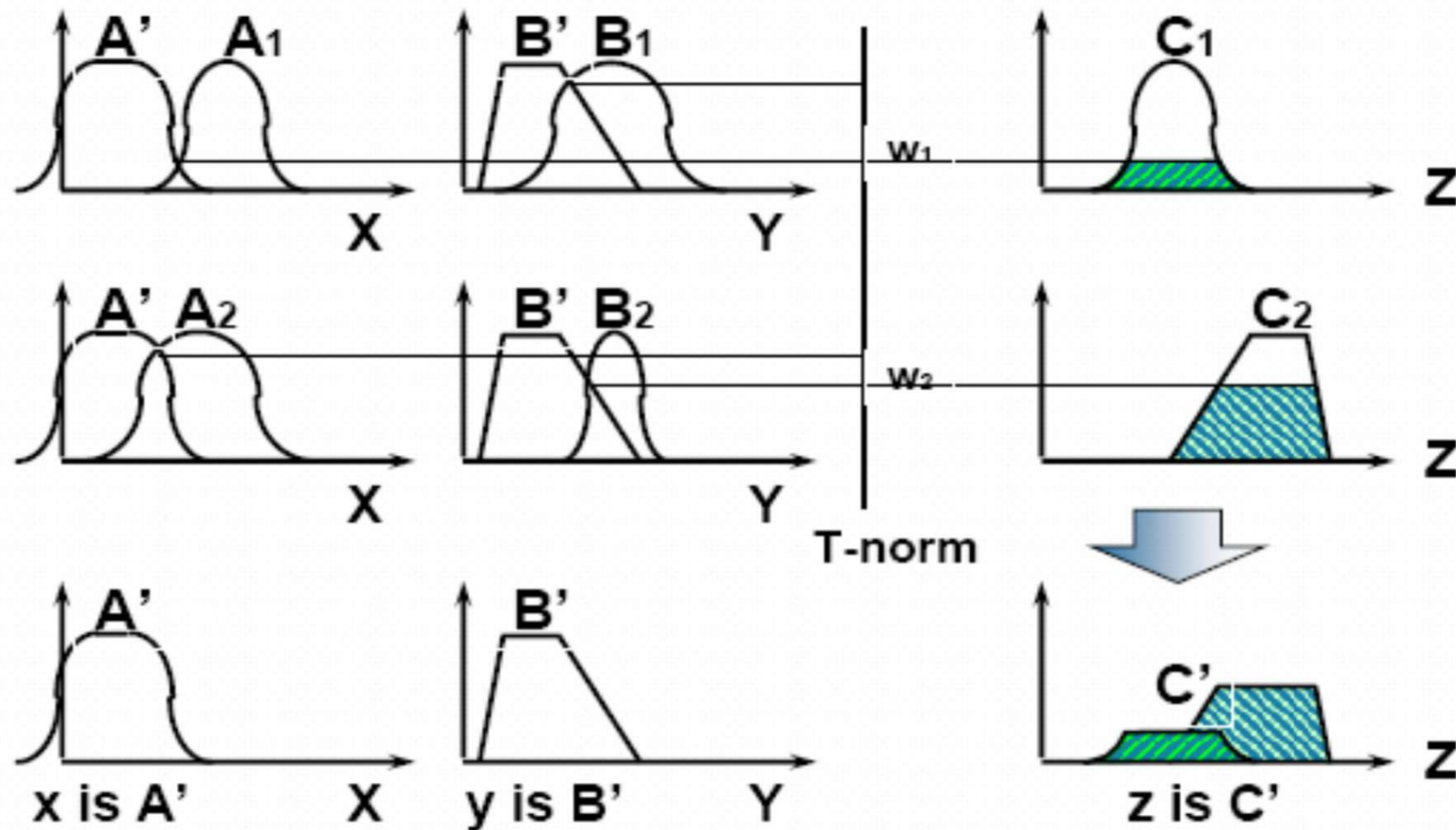


Figure. 3.13

## Step 4: Defuzzification

Fuzziness helps us to evaluate the rules, but the final output of a fuzzy system has to be a crisp number.

The input for the defuzzification process is the aggregate output fuzzy set  $C$  and the output is a single crisp number.

There are several defuzzification methods, but probably the most popular one is the centroid technique.

Centroid defuzzification method finds a point representing the centre of gravity of the fuzzy set,  $A$ , on the interval,  $[a \ b]$ .

A reasonable estimate can be obtained by calculating it over a sample of points.

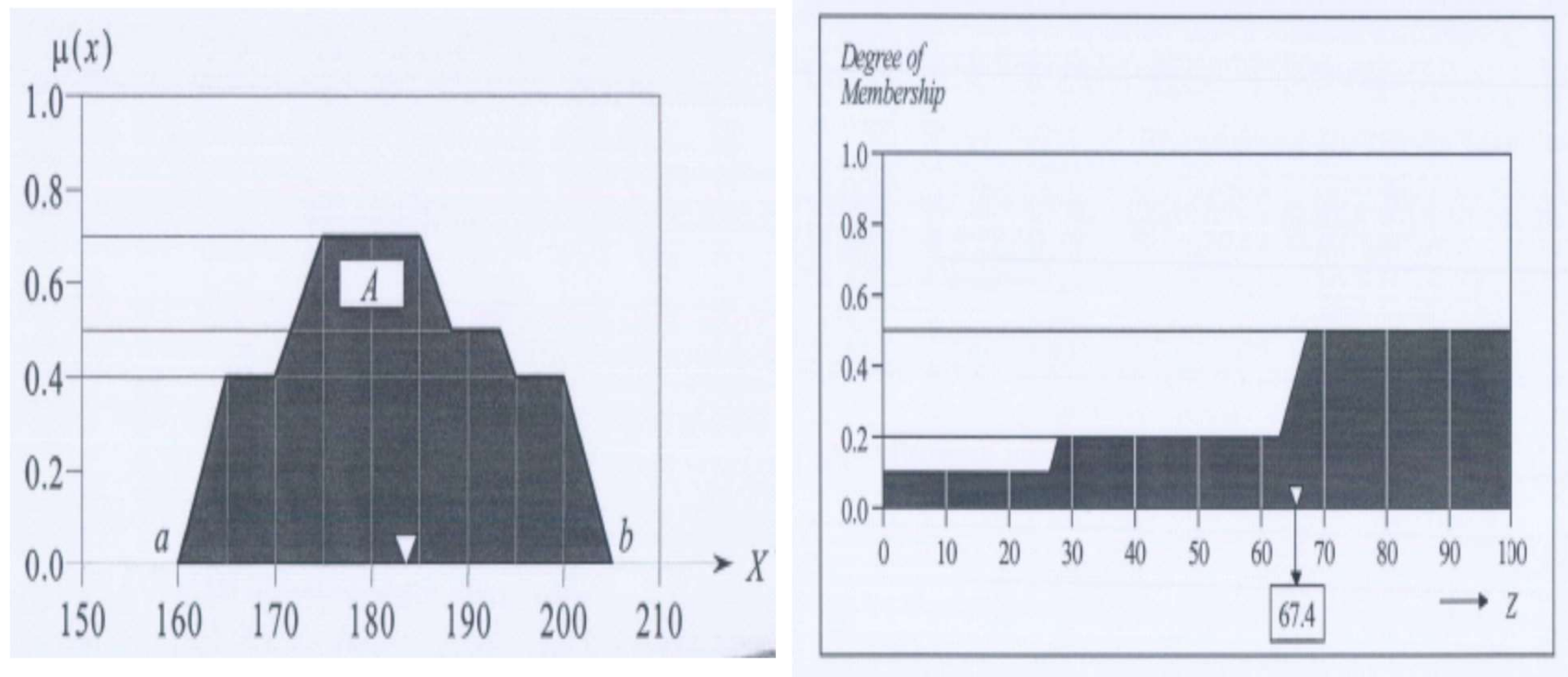


Figure. 3.14

Mathematically this centre of gravity (COG) can be expressed as:

$$z_{COG} = \frac{\int_Z \mu_A(z)zdz}{\int_Z \mu_A(z)dz}$$

$$z_{COG} = \frac{\sum \mu_{ci}z_i}{\sum \mu_{ci}}$$

The advantage of the COG defuzzifier lies in its intuitive plausibility. The disadvantage is that it is computationally intensive. In fact, the membership function is usually irregular and therefore the integrations in general are difficult to compute.

***Example: Given a fuzzy control system:***

Rule 1: IF  $x$  is  $A_1$  AND  $y$  is  $B_1$ , THEN  $z$  is  $C_1$ .

Rule 2: IF  $x$  is  $A_2$  AND  $y$  is  $B_2$ , THEN  $z$  is  $C_2$ .

Suppose  $x_0$  and  $y_0$  are the sensor readings for linguistic input variables  $x$  and  $y$  and the following membership functions for fuzzy predicates  $A_1, A_2, B_1, B_2, C_1$ , and  $C_2$  are given:

$$\mu_{A_1}(x) = \begin{cases} \frac{x-2}{3} & 2 \leq x \leq 5 \\ \frac{8-x}{3} & 5 < x \leq 8, \end{cases}$$

$$\mu_{A_2}(x) = \begin{cases} \frac{x-3}{3} & 3 \leq x \leq 6 \\ \frac{9-x}{3} & 6 < x \leq 9, \end{cases}$$

$$\mu_{B_1}(y) = \begin{cases} \frac{y-5}{3} & 5 \leq y \leq 8 \\ \frac{11-y}{3} & 8 < y \leq 11, \end{cases}$$

$$\mu_{B_2}(y) = \begin{cases} \frac{y-4}{3} & 4 \leq y \leq 7 \\ \frac{10-y}{3} & 7 < y < 10, \end{cases}$$

$$\mu_{C_1}(z) = \begin{cases} \frac{z-1}{3} & 1 \leq z \leq 4 \\ \frac{7-z}{3} & 4 < z \leq 7, \end{cases}$$

$$\mu_{C_2}(z) = \begin{cases} \frac{z-3}{3} & 3 \leq z \leq 6 \\ \frac{9-z}{3} & 6 < z \leq 9. \end{cases}$$

Further assume that at time  $t_1$  we are reading the sensor values at  $x_0(t_1) = 4$  and  $y_0(t_1) = 8$ . Let us illustrate how the final control output is computed.

First, the sensor readings  $x_0(t_1)$  and  $y_0(t_1)$  have to be matched against the preconditions  $A_1$  and  $B_1$  of rule 1, respectively. This will produce  $\mu_{A_1}(x_0 = 4) = \frac{2}{3}$  and  $\mu_{B_1}(y_0 = 8) = 1$ . Similarly, for rule 2, we have  $\mu_{A_2}(x_0) = \frac{1}{3}$  and  $\mu_{B_2}(y_0) = \frac{2}{3}$ . The firing strength of rule 1 is calculated by

$$\alpha_1 = \min(\mu_{A_1}(x_0), \mu_{B_1}(y_0)) = \min\left(\frac{2}{3}, 1\right) = \frac{2}{3},$$

and similarly for rule 2, the firing strength is

$$\alpha_2 = \min(\mu_{A_2}(x_0), \mu_{B_2}(y_0)) = \min\left(\frac{1}{3}, \frac{2}{3}\right) = \frac{1}{3}.$$



