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# 2. Fuzzy Arithmetic

Recall that a fuzzy number is a normal, convex fuzzy set which is defined on the real line ***R***.

Thus such a fuzzy set consists of two parts: an interval on the real line defined in the ordinary sense and a *membership* function defined on the interval.

Now, some fundamental properties and operation rules pertaining to *intervals* are first studied, which will be needed in the sequel.

# 2.1 Some Fundamental Concepts

The concern here is the situation where the value of a member s is *uncertain* but bounded:

$$\underline{s} \leq s \leq \bar{s} \,,$$

where $[\, \underline{s} \,,\, \bar{s} \,] \subset R$ is called the *interval of confidence* about the values of $s$. Only closed intervals in the form of $[\, \underline{s} \,,\, \bar{s} \,]$ are considered in this course, not those like $(\, \underline{s} \,,\, \bar{s} \,]$, $[\, \underline{s} \,,\, \bar{s} \,)$, $(\, \underline{s} \,,\, \bar{s} \,)$, except perhaps $[\, \underline{s} \,, \infty)$ and $(-\infty \,,\, \bar{s} \,]$ in some special cases.

A special case is, when s = $\underline{s} = \bar{s}$, it becomes $[\, \underline{s} \,,\, \bar{s} \,] = [s \,,\, s\,] = s$.

- **Definition**

(a) ***Equality***: Two intervals $[\underline{s}_1, \overline{s}_1]$ and $[\underline{s}_2, \overline{s}_2]$ are said to be equal if and only if $\underline{s}_1 = \underline{s}_2$ and $\overline{s}_1 = \overline{s}_2$:

$$[\underline{s}_1, \overline{s}_1] = [\underline{s}_2, \overline{s}_2]$$

(b) ***Intersection***: The intersection of two intervals $[\underline{s}_1, \overline{s}_1]$ and $[\underline{s}_2, \overline{s}_2]$ is

$$[\underline{s}_1, \overline{s}_1] \cap [\underline{s}_2, \overline{s}_2] = [\max\{\underline{s}_1, \underline{s}_2\}, \min\{\overline{s}_1, \overline{s}_2\}]$$

Note: $[\underline{s}_1, \overline{s}_1] \cap [\underline{s}_2, \overline{s}_2] = \varnothing$ if and only if $\underline{s}_1 > \overline{s}_2$ or $\underline{s}_2 > \overline{s}_1$.

- **Definition**

(c) ***Union***: The *union* of two intervals $[\underline{s}_1, \bar{s}_1]$ and $[\underline{s}_2, \bar{s}_2]$ is

$$[\underline{s}_1, \bar{s}_1] \cup [\underline{s}_2, \bar{s}_2] = [\min\{\underline{s}_1, \underline{s}_2\}, \max\{\bar{s}_1, \bar{s}_2\}],$$

provided that $[\underline{s}_1, \bar{s}_1] \cap [\underline{s}_2, \bar{s}_2] \neq \varnothing$. Otherwise, it is undefined (since the result is not an interval).

(d) ***Inequality***: Interval $[\underline{s}_1, \bar{s}_1]$ is said to be *less than* (resp., *greater than*) interval $[\underline{s}_2, \bar{s}_2]$, denoted by

$$[\underline{s}_1, \bar{s}_1] < [\underline{s}_2, \bar{s}_2] \quad (\text{resp., } [\underline{s}_1, \bar{s}_1] > [\underline{s}_2, \bar{s}_2])$$

if and only if $\bar{s}_1 < \underline{s}_2$ (resp., $\underline{s}_1 > \bar{s}_2$). Otherwise, they cannot be compared. Note that the relations $\leq$ and $\geq$ are not defined for intervals.

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- **Definition**

  (e) ***Inclusion***: The interval $[\underline{s}_1, \overline{s}_1]$ is *being included* in the interval $[\underline{s}_2, \overline{s}_2]$ if and only if and $\underline{s}_2 \le \underline{s}_1$ and $\overline{s}_1 \le \overline{s}_2$:

$$[\underline{s}_1, \overline{s}_1] \subseteq [\underline{s}_2, \overline{s}_2]$$

- ***Example***

For 3 given intervals, $S_1 = [-1,0]$, $S_2 = [-1,2]$, and $S_3 = [2,10]$

$$S_1 \cap S_2 = [-1,0] \cap [-1,2] = [-1,0],$$

$$S_1 \cap S_3 = [-1,0] \cap [2,10] = \varnothing,$$

$$S_2 \cap S_3 = [-1,2] \cap [2,10] = [2,2] = 2,$$

$$S_1 \cup S_2 = [-1,0] \cup [-1,2] = [-1,2],$$

$$S_1 \cup S_3 = [-1,0] \cup [2,10] = \text{undefined},$$

$$S_2 \cup S_3 = [-1,2] \cup [2,10] = [-1,10],$$

$$S_1 = [-1,0] < [2,10] = S_3,$$

$$S_1 = [-1,0] \subset [-1,2] = S_2.$$

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# 2.2 Interval Arithmetic

Let $[\,\underline{s}\,,\ \overline{s}\,]$, $[\,\underline{s}_1, \overline{s}_1\,]$, and $[\,\underline{s}_2, \overline{s}_2\,]$ be intervals.

**Definition:** Interval Arithmetic

***(1) Addition***:

$$[\,\underline{s}_1, \overline{s}_1\,] + [\,\underline{s}_2, \overline{s}_2\,] = [\,\underline{s}_1 + \underline{s}_2,\ \overline{s}_1 + \overline{s}_2\,].$$

***(2) Subtraction*:**

$$[\,\underline{s}_1, \overline{s}_1\,] - [\,\underline{s}_2, \overline{s}_2\,] = [\,\underline{s}_1 - \overline{s}_2\,,\ \overline{s}_1 - \underline{s}_2\,].$$

***(3) Reciprocal:***

If $0 \notin [\,\underline{s}\,,\ \overline{s}\,]$ then $[\,\underline{s}\,,\ \overline{s}\,]^{-1} = [\,1/\overline{s}\,,\ 1/\underline{s}\,]$;

if $0 \in [\,\underline{s}\,,\ \overline{s}\,]$ then $[\,\underline{s}\,,\ \overline{s}\,]^{-1}$ is undefined.

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***(4) Multiplication*:**

$$[\,\underline{s}_1, \overline{s}_1\,] \cdot [\,\underline{s}_2, \overline{s}_2\,] = [\,\underline{p},\ \overline{p}\,]$$

where

$$\underline{p} = \min\{\,\underline{s}_1\underline{s}_2,\ \underline{s}_1\overline{s}_2,\ \overline{s}_1\underline{s}_2,\ \overline{s}_1\overline{s}_2\,\},$$
$$\overline{p} = \max\{\,\underline{s}_1\underline{s}_2,\ \underline{s}_1\overline{s}_2,\ \overline{s}_1\underline{s}_2,\ \overline{s}_1\overline{s}_2\,\}.$$

***(5) Division*:**

$$[\,\underline{s}_1, \overline{s}_1\,] / [\,\underline{s}_2, \overline{s}_2\,] = [\,\underline{s}_1, \overline{s}_1\,] \cdot [\,\underline{s}_2, \overline{s}_2\,]^{-1},$$

provided that $0 \notin [\,\underline{s}_2, \overline{s}_2\,]$.

***(6) Maximum:***

$$\max\{[\underline{s}_1, \overline{s}_1], [\underline{s}_2, \overline{s}_2]\} = [\underline{p}, \overline{p}],$$

where

$$\underline{p} = \max\{\underline{s}_1, \underline{s}_2\},$$

$$\overline{p} = \max\{\overline{s}_1, \overline{s}_2\}.$$

***(5) Minimum*:**

$$\min\{[\underline{s}_1, \overline{s}_1], [\underline{s}_2, \overline{s}_2]\} = [\underline{p}, \overline{p}],$$

where

$$\underline{p} = \min\{\underline{s}_1, \underline{s}_2\},$$

$$\overline{p} = \min\{\overline{s}_1, \overline{s}_2\}.$$

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**Remarks:**

a) Interval arithmetic intends to obtain an interval as the result of an operation such that the resulting interval contains ***all possible solutions***. Therefore, these kind of operational rules are defined in a ***conservative way***, in the sense that it intends to make the resulting interval ***as large as necessary*** to ***avoid loosing any true solution***. For example, $[1,2]-[0,1]=[0,2]$ means that for any $a \in [1,2]$ and any $b \in [0,1]$, it is guaranteed that $a-b \in [0,2]$.

**Remarks:**

b) This conservatism may produce some unusual results that could seem to be inconsistent with the ordinary numerical solutions. For instance, according to the subtraction rule (2), one has $[1,2] - [1,2] = [-1,1] \neq [0,0] = 0.$

The result $[-1,1]$ here contains 0, but not only 0. The reason is that there can be other possible solutions: if one takes 1.5 from the first interval and 1.0 from the second, then the result is 0.5 rather than 0; and 0.5 is indeed in $[-1,1]$. Thus, an interval subtract itself is equal to zero (a point) only if this interval is itself a point (a trivial interval).

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# 2.3 Algebraic Properties of Interval Arithmetic

**Theorem** The addition and multiplication operations of intervals are *commutatitve* and *associative* but not *distributive*:

(1) $X + Y = Y + X$;
(2) $Z + ( X + Y ) = ( Z + X ) + Y$;
(3) $Z ( X Y ) = ( Z X ) Y$;
(4) $X Y = Y X$;
(5) $Z + \mathbf{0} = \mathbf{0} + Z = Z$ and $Z\mathbf{0} = \mathbf{0}Z = \mathbf{0}$, where $\mathbf{0} = [0,0]$;
(6) $Z\mathrm{I} = \mathrm{I}Z = Z$, where $\mathrm{I} = [1,1]$;
(7) $Z ( X + Y ) \neq ZX + ZY$, except when:
(a) $Z = [z,z]$ is a point; or
(b) $X = Y = \mathbf{0}$; or
(c) $xy \geq 0$ for all $x \in X$ and $y \in Y$.
In general, only the *subdistributive* law holds:

$$Z ( X + Y ) \subseteq ZX + ZY.$$

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***Example***

Let

$$Z = [1,2],\ X = \mathrm{I} = [1,1],\ Y = -\mathrm{I} = [-1,-1].$$

Then

$$Z\,(\,X + Y\,) = \ [1,2]\,(\,\mathrm{I} - \mathrm{I}\,) = [1,2]\cdot\mathbf{0} = \mathbf{0};$$

$$Z\,X + Z\,Y \ = \ [1,2]\cdot[1,1] + [1,2]\cdot[-1,-1] = [-1,1] \neq \mathbf{0}$$

A more general rule for interval arithmetic operations is the following fundamental law of *monotonic inclusion*.

**Theorem** Let $X_1, X_2, Y_1$, and $Y_2$ be intervals such that

$$X_1 \subseteq Y_1 \quad \text{and} \quad X_2 \subseteq Y_2.$$

Then, for all operations $* \in \{ +, -, \cdot, / \}$

$$X_1 * X_2 \subseteq Y_1 * Y_2.$$

**Corollary** Let X and Y be intervals with $x \in X$ and $y \in Y$ . Then, for any operation $* \in \{ +, -, \cdot, / \}$

$$x * y \in X * Y$$

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# 2.4 Interval Evaluation

An ordinary real-variable and real-valued functions $f: \mathrm{R} \rightarrow \mathrm{R}$ can easily be extended to an interval-variable and interval-valued function $f: \mathbf{I} \rightarrow \mathbf{I}$ , where **I** is the family of intervals defined on ***R***. Such extended functions include the following arithmetic functions:

$$Z = f(X, Y) = X * Y, \qquad * \in \{ +, -, \cdot, / \},$$

where $X, Y, Z \in \mathbf{I}$.

Note that for any ordinary continuous functions $f: \mathrm{R} \rightarrow \mathrm{R}$ and any interval $X \in \mathbf{I}$, the interval-variable and interval-valued function

$$f_{\mathbf{I}}(X) = \left[ \min_{x \in X} f(x), \quad \max_{x \in X} f(x) \right]$$

is also a continuous function.

Now, let $A_1, \ldots, A_m$ be intervals in $\mathbf{I}$. For any interval $X \in \mathbf{I}$, one can further define a function, $f(x;a_1,\ldots,a_m)$, $x \in X$, which depends on $m$ parameters $a_k \in A_k,\ k = 1, 2,\ldots,m,$ by

$$f_{\mathbf{I}}(X;A_1,\ldots,A_m) = \{\ f(x;a_1,\ldots,a_m) \mid x \in X,\ a_k \in A_k,\ 1 \le k \le m\ \}$$

$$= \left[\ \min_{\substack{x\in X \\ a_k\in A_k,1\le k\le m}} f(x;a_1,\ldots,a_m),\ \max_{\substack{x\in X \\ a_k\in A_k,1\le k\le m}} f(x;a_1,\ldots,a_m)\ \right]$$

***Example***

Consider the real-variable and real-valued function

$$f(x;a) = \frac{ax}{1-x} \qquad x \neq 1, \qquad x \neq 0\ .$$

If $X$ = [2, 3] and $A$ = [0, 2] are intervals, with $x \in X$ and $a \in A$ , then the interval expression of $f$ is given by

$$f_{\mathbf{I}}(X;A) = \left\{ \frac{ax}{1-x} \;\middle|\; 2 \le x \le 3,\ 0 \le a \le 2 \right\}$$

$$= \left[ \min_{\substack{2\le x\le 3,\\ 0\le a\le 2}} \frac{ax}{1-x},\ \max_{\substack{2\le x\le 3,\\ 0\le a\le 2}} \frac{ax}{1-x} \right]$$

$$= [-4, 0].$$

The following result is important and useful. It states that all common interval arithmetic expressions have the ***inclusion monotonic*** *property.*

**Theorem** Let $f : \mathrm{R}^{n+m} \rightarrow \mathrm{R}$ be a real-variable and real-valued continuous function with an arithmetic interval expression $f_{\mathbf{I}}(X_1, \cdots, X_n, A_1, \cdots, A_m)$. Then, for all

$$X_k \subseteq Y_k, \quad k = 1,2,\ldots,n \quad \text{and} \quad A_l \subseteq B_l, \quad l = 1,2, \ldots, m,$$

one has

$$f_{\mathbf{I}}(X_1,\ldots,X_n,A_1,\ldots,A_m) \subseteq f_{\mathbf{I}}(Y_1,\ldots,Y_n,B_1,\ldots,B_m) .$$

***Example***

Let $X = [0.2, 0.4]$ and $Y = [0.1, 0.5]$ . Then $X \subset Y$.

(a) $X^{-1} = \dfrac{1}{[0.2, 0.4]} = [2.5, 5.0]$,

$Y^{-1} = \dfrac{1}{[0.1, 0.5]} = [2.0, 10.0]$,

$X^{-1} \subset Y^{-1}$.

(b) $1 - X = [1.0, 1.0] - [0.2, 0.4] = [0.6, 0.8]$,

$1 - Y = [1.0, 1.0] - [0.1, 0.5] = [0.5, 0.9]$,

$1 - X \subset 1 - Y$.

(c) $$\frac{1}{1-X} = \frac{1}{[0.6,0.8]} = [5/4,\ 5/3],$$

$$\frac{1}{1-Y} = \frac{1}{[0.5,0.9]} = [10/9,\ 2.0],$$

$$\frac{1}{1-X} \subset \frac{1}{1-Y}.$$

A very important issue in the evaluation of an interval expression is that the number of intervals involved in an interval expression should be reduced (whenever possible) before evaluating (i.e., computing) the expression, in order to obtain less-conservative lower and upper bounds of the resulting interval expression.

***Example: The difficulty for nonlinear functions***

Consider the function:

$$f(x;a) = \frac{ax}{1-x} \qquad x \neq 1, \quad x \neq 0 \,.$$

If one rewrites it as

$$\tilde{f}(x;a) = \frac{a}{\frac{1}{x}-1}, \quad x \neq 1, \; x \neq 0,$$

then, as a real-variable and real-valued function, $\tilde{f} \equiv f$ .

In the case of intervals, for x ∊ X = [2, 3] and a ∊ A = [0, 2], according to Example 1.6, one has

$$f_{\mathbf{I}}(X;A) = [-4, 0] \,.$$

On the other hand, one also has

$$\tilde{f}_{\mathbf{I}}(X;A) = \left\{ \frac{a}{1/x-1} \,\middle|\, 2 \le x \le 3, 0 \le a \le 2 \right\}$$

$$= \left[ \min_{\substack{2\le x\le 3\\ 0\le a\le 2}} \frac{a}{\frac{1}{x}-1}, \max_{\substack{2\le x\le 3\\ 0\le a\le 2}} \frac{a}{\frac{1}{x}-1} \right] = [-4,\ 0] = f_{\mathbf{I}}(X;A)$$

However, if one formally performs interval arithmetic (rather than numerical minimization and maximization, as just did), then one obtains

$$f_{\mathbf{I}}(X;A) = \frac{[0,2] \cdot [2,3]}{1-[2,3]} = \frac{[0,6]}{[-2,-1]} = [0,6] . [-1, -1/2] = [-6,\ 0]$$

and

$$\tilde{f}_{\mathbf{I}}(X;A) = \frac{[0,2]}{\dfrac{1}{[2,3]} - 1} = \frac{[0,2]}{\left[\dfrac{1}{3}, \dfrac{1}{2}\right] - 1} = \frac{[0,2]}{\left[-\dfrac{2}{3}, -\dfrac{1}{2}\right]}$$

$$= [0, 2] \cdot [-2, -3/2]$$

$$= [-4, 0].$$

Thus, $\tilde{f}_{\mathbf{I}}(X;A) \neq f_{\mathbf{I}}(X;A)$ but

$$[-4, 0] = \tilde{f}_{\mathbf{I}}(X;A) \subseteq f_{\mathbf{I}}(X;A) = [-6, 0].$$

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The reason is that formula $f_{\mathbf{I}}(X;A)$ has three intervals but $\tilde{f}_{\mathbf{I}}(X;A)$ has only two, so that the computational "errors" are accumulated more in the former therefore the result is more conservative.

Therefore, this example has shown a fundamental principle of interval arithmetic: one should always try to reduce the number of intervals involved in evaluating an interval expression.

# 2.5 Operational on Fuzzy Sets

Arithmetic of fuzzy numbers (sets): the following is general rule.

**General Rule** Let $X$ and $Y$ be two fuzzy sets, with $Z \subseteq R$, and consider a two-variable function (e.g., $+$ , $-$ , $\cdot$ , $/$ , max, min):

$$\mathrm{F}: X \times Y \to Z.$$

Let $\mu(x)$, $\mu(y)$, and $\mu(z)$ be their associate membership functions. Given $\mu(x)$ and $\mu(y)$, define

$$\mu_Z(z) = \bigvee_{z=F(x,y)} \{ \mu_X(x) \wedge \mu_Y(y) \}.$$

where, for two real numbers $s_1$ and $s_2$ ,

$$s_1 \wedge s_2 = \min\{s_1, s_2\} \quad \text{and} \quad s_1 \vee s_2 = \max\{s_1, s_2\}.$$

Using the $\alpha$ -cut notation, this is equivalent to the following:

$$(Z)_\alpha = F(\ (X)_\alpha,\ (Y)_\alpha\ )$$
$$= \{\ z \in Z \mid z = F(x,y),\ x \in (X)_\alpha, y \in (Y)_\alpha\ \}.$$

The above computational rule and its $\alpha$-cut notation are now illustrated by several examples.

**<u>Addition (linear operator)</u>** Let $z = F(x,y) = x + y$. Then

$$Z = \{\ z \mid z = x + y,\ x \in \mathrm{X},\ y \in X\ \}$$

and

$$\mu_Z(z) = \bigvee_{z=x+y} \{\ \mu_X(x) \wedge \mu_Y(y)\ \}.$$

In the $\alpha$ -cut notation:

$$(Z)_\alpha = F((X)_\alpha, (Y)_\alpha) = (X)_\alpha + (Y)_\alpha .$$

***Example:***

Let $x$ and $y$ be such that

$$X = [-5,1], \qquad Y = [-5,12],$$

with associate membership functions shown in Figure 2.1:

$$\mu_X(x) = \begin{cases} \dfrac{x}{3} + \dfrac{5}{3}, & -5 \le x \le -2, \\ -\dfrac{x}{3} + \dfrac{1}{3}, & -2 \le x \le 1, \end{cases}$$

and

$$\mu_Y(y) = \begin{cases} 0 & -5 \le y \le -3, \\ \dfrac{y}{7} + \dfrac{3}{7} & -3 \le y \le 4, \\ -\dfrac{y}{8} + \dfrac{12}{8} & 4 \le y \le 12. \end{cases}$$

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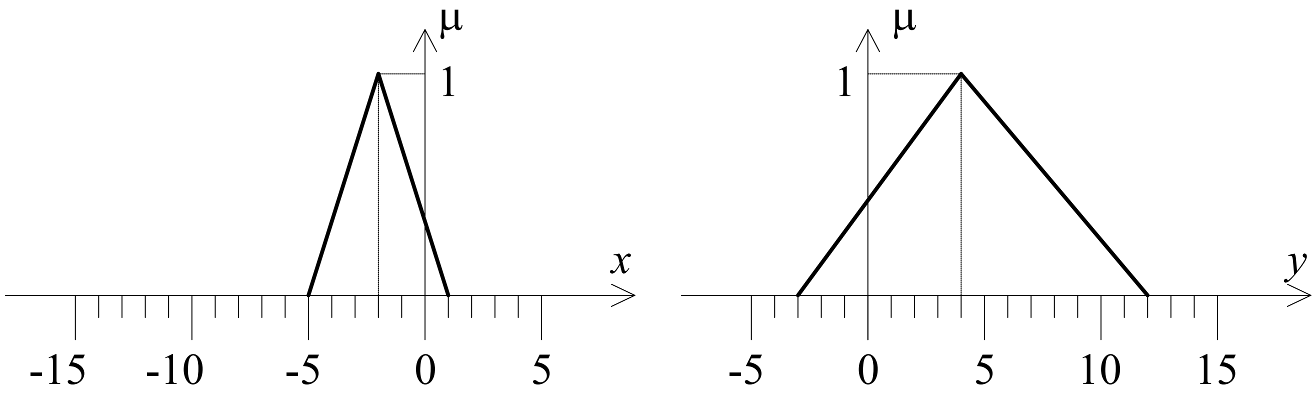


**Figure 2.1** Two membership functions for addition

Then, by the general operation rule, one has

$$Z = X + Y = [-5,1] + [-5,12] = [-10,13]$$

and, by comparing $\mu_X(x)$ and $\mu_Y(y)$ pointwise, one obtains

$$\mu_Z(z) = \bigvee_{z=x+y} \{ \mu_X(x) \wedge \mu_Y(y) \}$$

$$= \begin{cases} 0, & -10 \le z \le -8, \\ \dfrac{z}{10} + \dfrac{8}{10}, & -8 \le z \le 2, \\ -\dfrac{z}{11} + \dfrac{13}{11}, & 2 \le z \le 13. \end{cases}$$

Here, it is clear that the general “sup” rule does not yield the explicit formulas easily. On the contrary, an explicit formula for $\mu_Z(z)$ can be easily obtained by using the equivalent $\alpha$-cut operation as follows.

In the $\alpha$-cut notation, for any $\alpha$ value, the $\alpha$-cut of $X$ is obtained by letting

$$\alpha = x/3 + 5/3$$

and

$$\alpha = -x/3 + 1/3$$

which give

$$x_1 = 3\alpha - 5 \qquad \text{and} \qquad x_2 = -3\alpha + 1 ,$$

as shown in (the enlarged) Figure 2.2.

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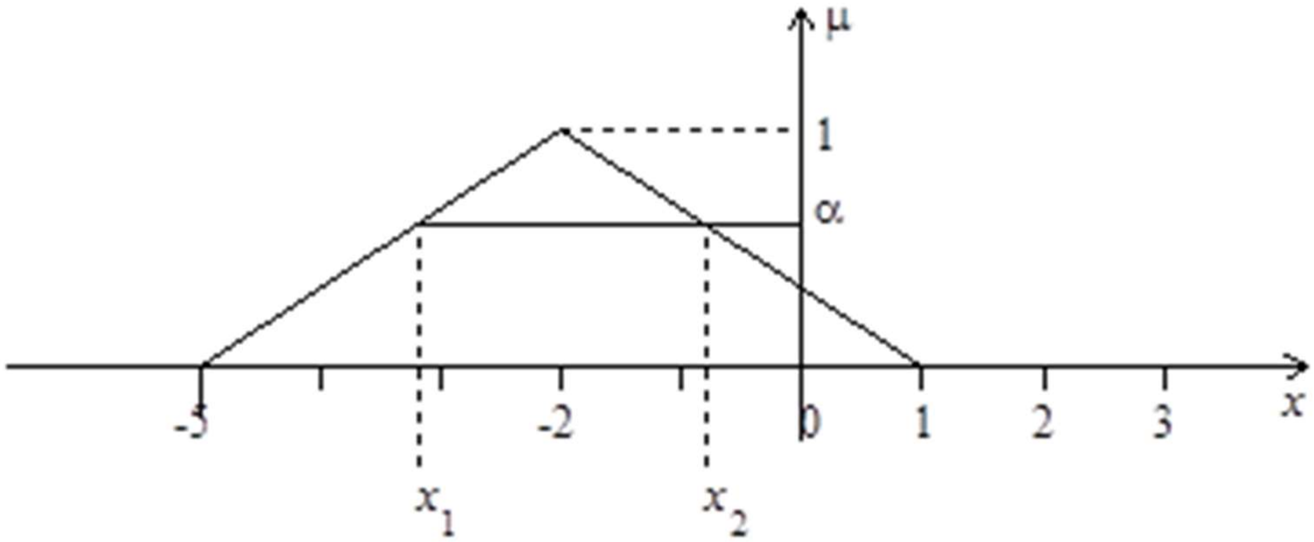


**Figure 2.2** The $\alpha$-cut of the membership function $\mu_X(x)$

Hence, the projection interval is

$$(X)_\alpha = [\, x_1, x_2\,] = [3\alpha - 5, -3\alpha + 1].$$

Similarly,

$$(Y)_\alpha = [7\alpha - 3, -8\alpha + 12],$$

so that

$$(Z)_\alpha = (X)_\alpha + (Y)_\alpha = [10\alpha - 8, -11\alpha + 13].$$

Setting $z_1 = 10\alpha - 8$ and $z_2 = -11\alpha + 13$ gives $\alpha = z_1/10 + 8/10$ and $\alpha = -z_2/11 + 12/11$, which yield the membership function shown in Figure 2.3.

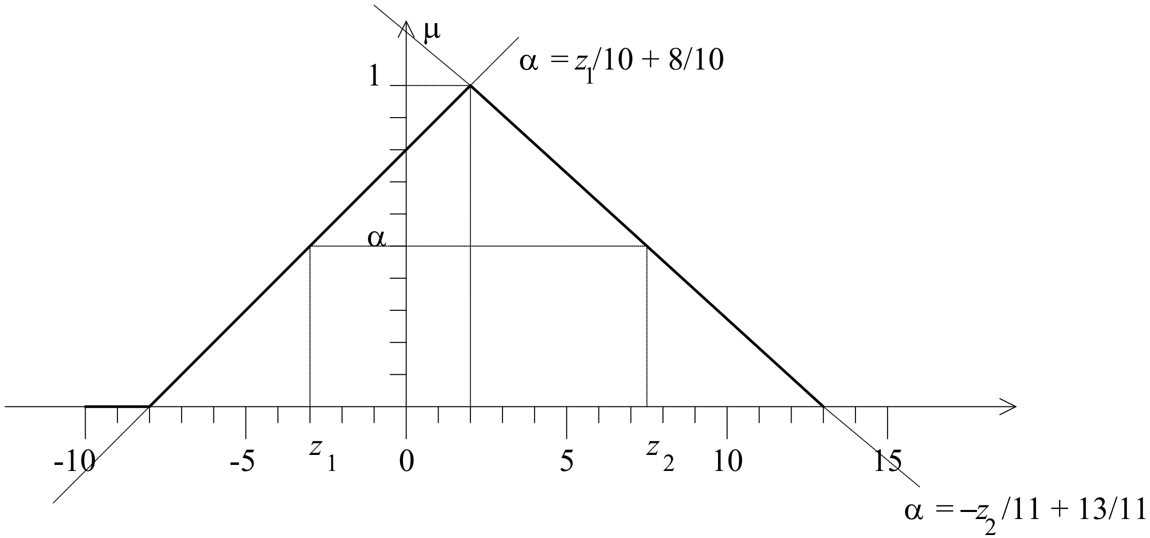


**Figure 2.3** The resulting membership function

Taking $Z = [-10,13]$ into account, one finally arrives at

$$\mu_Z(z) = \begin{cases} 0, & -10 \le z \le -8, \\ \dfrac{z}{10} + \dfrac{8}{10}, & -8 \le z \le 2, \\ -\dfrac{z}{11} + \dfrac{13}{11}, & 2 \le z \le 13. \end{cases}$$

**Subtraction (linear operator)** Let $z = F(x,y) = x - y$. Then

$$Z = \{ z \mid z = x - y ,\ x \in \mathrm{X},\ y \in X \}$$

and

$$\mu_Z(z) = \bigvee_{z=x-y} \{ \mu_X(x) \wedge \mu_Y(y) \}.$$

In the $\alpha$ -cut notation:

$$(Z)_\alpha = F((X)_\alpha, (Y)_\alpha) = (X)_\alpha - (Y)_\alpha .$$

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***Example***

Let $X$ and $Y$ be such that

$$X = [0,20], \qquad Y = [0,10],$$

with the membership functions

$$\mu_X(x) = \begin{cases} 0, & 0 \le x \le 7, \\ \dfrac{x}{7} - 1, & 7 \le x \le 14, \\ -\dfrac{x}{5} + \dfrac{19}{5}, & 14 \le x \le 19, \\ 0, & 19 \le x \le 20, \end{cases}$$

and

$$\mu_Y(y) = \begin{cases} 0, & 0 \le y \le 3, \\ \dfrac{y}{2} - \dfrac{3}{2}, & 3 \le y \le 5, \\ -\dfrac{y}{5} + 2, & 5 \le y \le 10. \end{cases}$$

Then, one obtains, via the interval arithmetic, that

$$Z = X - Y = [-10,20],$$

with

$$\mu_Z(z) = \begin{cases} 0, & -10 \le z \le -3, \\ \dfrac{z}{12} + \dfrac{3}{12} & -3 \le z \le 9, \\ -\dfrac{z}{7} + \dfrac{16}{7} & 9 \le z \le 16, \\ 0, & 16 \le z \le 20. \end{cases}$$

In the $\alpha$-cut notation, for any value, the $\alpha$ -cut of $X$ is obtained by letting $\alpha = x/7 - 1$ and $\alpha = -x/5 + 19/5$ , respectively, which give $x_1 = 7\alpha + 7$ and $x_2 = -5\alpha + 19$. Hence, the projection interval is

$$(X)_\alpha = [\, x_1, x_2 \,] = [7\alpha + 7, -5\alpha + 19].$$

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Similarly,

$$(Y)_\alpha = [2\alpha + 3, -5\alpha + 10],$$

so that

$$(Z)_\alpha = (X)_\alpha - (Y)_\alpha = [12\alpha - 3, -7\alpha + 16].$$

Setting $z_1 = 12\alpha - 3$ and $z_2 = -7\alpha + 16$ gives $\alpha = z_1/12 + 1/4$ and $\alpha = -z_2/7 + 16/7$, which yield the membership function shown in Figure 2.4.

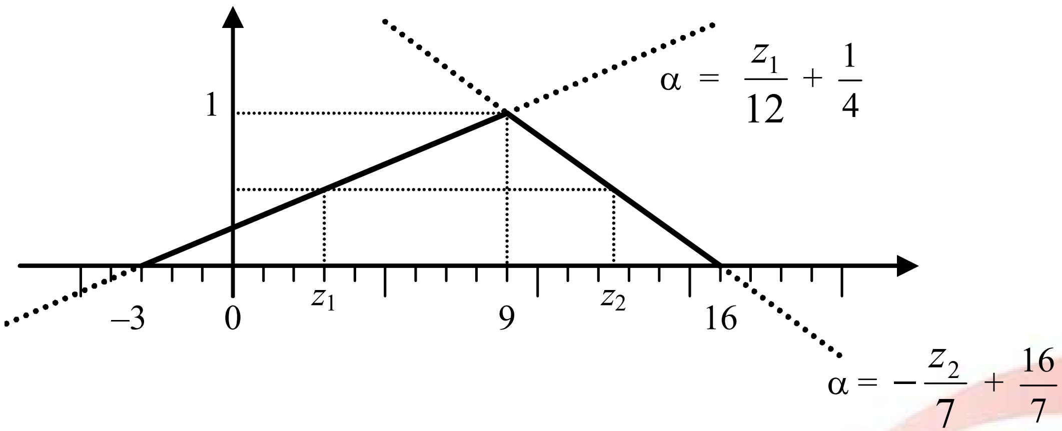


**Figure 2.4** The resulting membership function of Example 1.12

**Multiplication (nonlinear operator)** Let $z = F(x,y) = x \cdot y$.

Then

$$Z = \{ z \mid z = x\,y,\ x \in X,\ y \in Y \}$$

and

$$\mu_Z(z) = \bigvee_{z = x \cdot y} \{ \mu_X(x) \wedge \mu_Y(y) \}.$$

In the $\alpha$-cut notation:

$$(Z)_\alpha = F(\ (X)_\alpha,\ (Y)_\alpha\ ) = (X)_\alpha \cdot (Y)_\alpha\ .$$

***Example***

Let $X$ and $Y$ be such that

$$X = [2,5], \qquad Y = [3,6],$$

with the membership functions as shown in Figure 2.5:

$$\mu_X(x) = \begin{cases} x-2, & 2 \le x \le 3, \\ -\dfrac{x}{2}+\dfrac{5}{2}, & 3 \le x \le 5, \end{cases}$$

and

$$\mu_Y(y) = \begin{cases} \dfrac{y}{2}-\dfrac{3}{2}, & 3 \le y \le 5, \\ -y+6, & 5 \le y \le 6. \end{cases}$$

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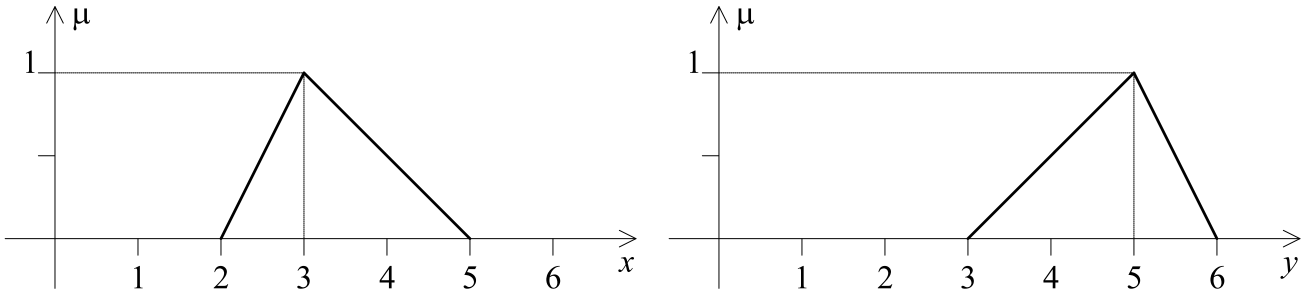


**Figure 2.5** Two membership functions for Example 1.13

In the $\alpha$-cut notation, for any $\alpha$ value, letting

$$\alpha = x - 2 \qquad \text{and} \qquad \alpha = -\frac{x}{2} + \frac{5}{2}$$

gives

$$x_1 = \alpha + 2 \qquad \text{and} \qquad x_2 = -2\alpha + 5,$$

so that

$$(X)_\alpha = [\ \alpha + 2, -2\alpha + 5\ ].$$

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Similarly,

$$(Y)_\alpha = [\,2\alpha + 3, -\alpha + 6\,].$$

It then follows that

$$Z = [6,30]$$

and

$$\begin{aligned}(Z)_\alpha &= (X)_\alpha \cdot (Y)_\alpha \\ &= [\alpha + 2, -2\alpha + 5\,] \cdot [\,2\alpha + 3, -\alpha + 6\,] \\ &= [\underline{p}(\alpha), \overline{p}(\alpha)\,],\end{aligned}$$

where

$$\underline{p}(\alpha) = min\{2\alpha^2+7\alpha+6,-\alpha^2+4\alpha+12,-4\alpha^2+4\alpha+15,2\alpha^2-17\alpha+30\},$$

$$\overline{p}(\alpha) = max\{2\alpha^2+7\alpha+6,-\alpha^2+4\alpha+12,-4\alpha^2+4\alpha+15,2\alpha^2-17\alpha+30\},$$

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with the curves shown in Figure 2.6.

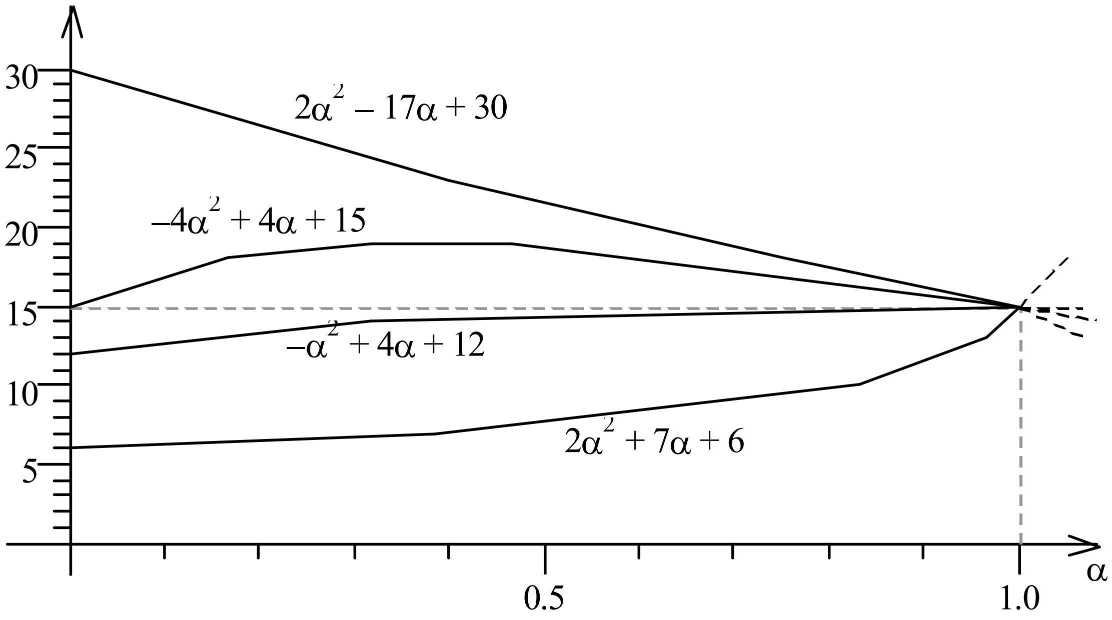


**Figure 2.6** The intermediate membership functions of Example 1.13

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Hence, one has

$$\underline{p}(\alpha) = 2\alpha^2+7\alpha+6 \quad \text{and} \quad \overline{p}(\alpha) = 2\alpha^2-17\alpha+30,$$

so that

$$(Z)_\alpha = [\underline{p}(\alpha), \overline{p}(\alpha)] = [\, 2\alpha^2+7\alpha+6,\ 2\alpha^2-17\alpha+30 \,].$$

Let, moreover,

$$z_1 = 2\alpha^2+7\alpha+6 \quad \text{and} \quad z_2 = 2\alpha^2-17\alpha+30.$$

Then, one can solve them for α, subject to $0 \le \alpha \le 1$ , and obtain

$$\alpha = \frac{-7+\sqrt{1+8z_1}}{4} \quad \text{or} \quad \alpha = \frac{17-\sqrt{49+8z_2}}{4}.$$

Consequently,

$$\mu_Z(z) = \begin{cases} \dfrac{-7+\sqrt{1+8z}}{4}, & 6 \le z \le 15, \\ \dfrac{17-\sqrt{49+8z}}{4}, & 15 \le z \le 30, \end{cases}$$

as shown in Figure 2.7.

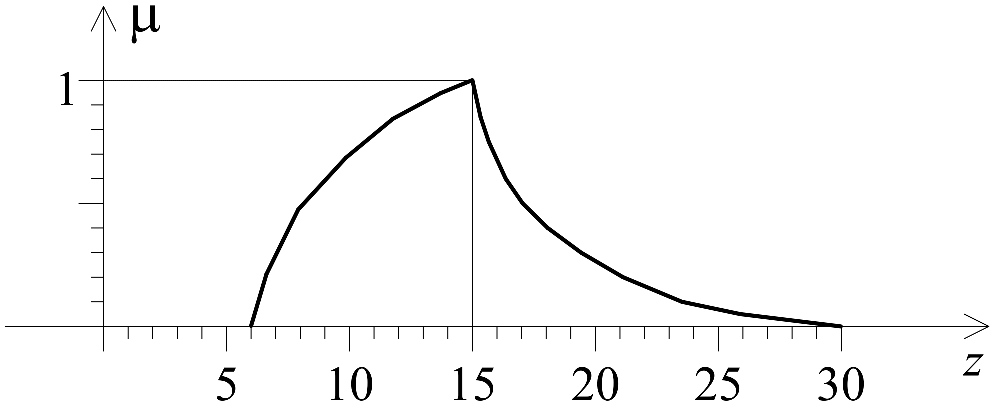


**Figure 2.7** The resulting membership function of Example 1.13

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**Division (nonlinear operator)** Let $z = F(x,y) = \dfrac{x}{y}$ . Then

$$Z = \{ z \mid z = x / y,\ x \in X ,\ y \in Y \}$$

and

$$\mu_Z(z) = \bigvee_{z=x/y} \{ \mu_X(x) \wedge \mu_Y(y) \}.$$

In the $\alpha$-cut notation:

$$(Z)_\alpha = F(\ (X)_\alpha ,\ (Y)_\alpha\ ) = \frac{(X)_\alpha}{(Y)_\alpha} .$$

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***Example***

Let $X$ and $Y$ be such that

$$X = [18,33], \qquad Y = [5,8],$$

with the membership functions:

$$\mu_X(x) = \begin{cases} \dfrac{x}{4} - \dfrac{18}{4}, & 18 \le x \le 22, \\ -\dfrac{x}{11} + 3, & 22 \le x \le 33, \end{cases}$$

and

$$\mu_Y(y) = \begin{cases} y - 5, & 5 \le y \le 6, \\ -\dfrac{y}{2} + 4, & 6 \le y \le 8, \end{cases}$$

as shown in Figure 2.8.

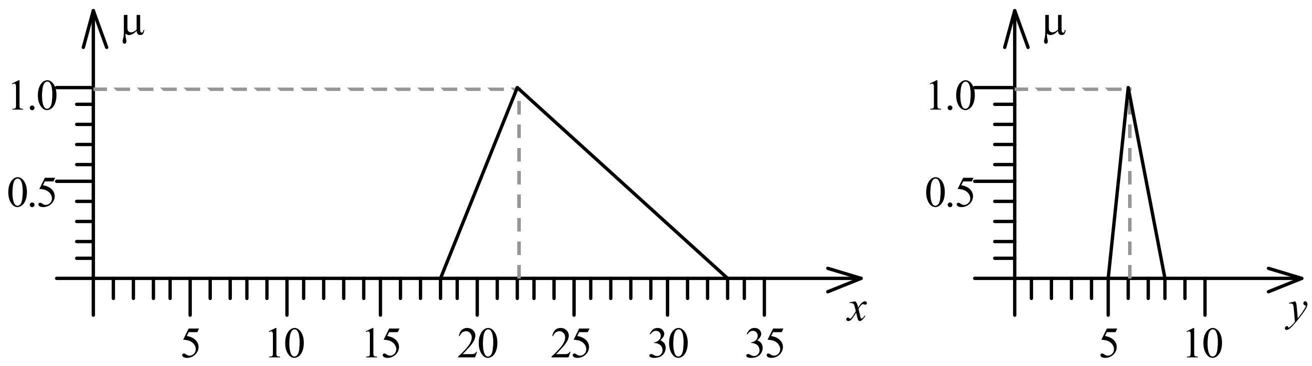


**Figure 2.8** The two membership functions for Example 1.14

In the $\alpha$-cut notation, let

$$\alpha = \frac{x}{4} - \frac{18}{4} \quad \text{and} \quad \alpha = -\frac{x}{11} + 3.$$

One has

$$x_1 = 4\alpha + 18 \quad \text{and} \quad x_2 = -11\alpha + 33,$$

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so that

$$(X)_\alpha = [\,4\alpha + 18, -11\alpha + 33\,].$$

Similarly,

$$(Y)_\alpha = [\,\alpha + 5, -2\alpha + 8\,].$$

Hence,

$$(Z)_\alpha = \frac{(X)_\alpha}{(Y)_\alpha} = \frac{[4\alpha + 18, -11\alpha + 33]}{[\alpha + 5, -2\alpha + 8]}$$

$$= \left[\frac{4\alpha + 18}{-2\alpha + 8}, \frac{-11\alpha + 33}{\alpha + 5}\right].$$

Next, letting

$$z_1 = \frac{4\alpha + 18}{-2\alpha + 8} \quad \text{and} \quad z_2 = \frac{-11\alpha + 33}{\alpha + 5},$$

gives

$$\alpha = \frac{8z_1 - 18}{2z_1 + 4} \quad \text{and} \quad \alpha = \frac{-5z_2 + 33}{z_2 + 11},$$

so that

$$\mu_Z(z) = \begin{cases} \dfrac{8z-18}{2z+4}, & \dfrac{9}{4} \le z \le \dfrac{11}{3}, \\ \dfrac{-5z+33}{z+11}, & \dfrac{11}{3} \le z \le \dfrac{33}{5}, \end{cases}$$

as shown in Figure 2.9.

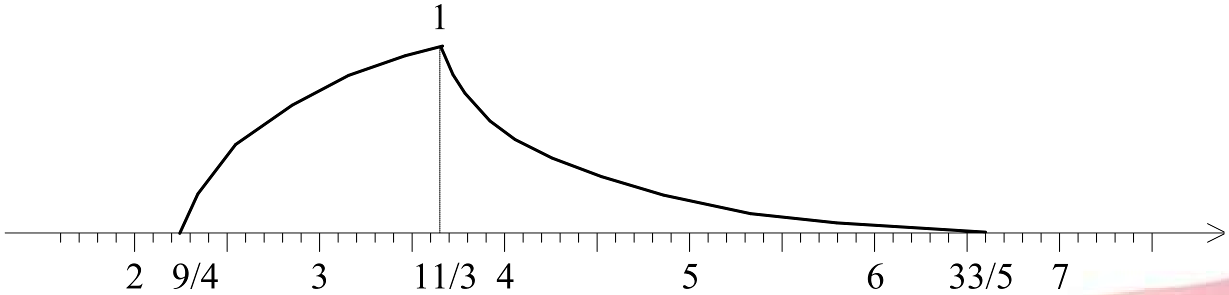


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**Figure 2.9** The resulting membership function of Example 1.14