The Link Between Market Correlation Structure and the Performance of Risk-Based Portfolios

Nathan Potgieter[a]

[a] *Stellenbosch University, Stellenbosch, South Africa*

Abstract

This work uses Monte Carlo methods to design and simulate five distinctive markets types, with each market type possesing a unique correlation structure in its return matrix. The equal weight, minimum variance, inverse variance, equal risk contribution and maximum diversification risk-based portfolios are evaluated in each of the simulated markets. The relative performance of each portfolio is compaired within market types and the relationship between portfolio return characteristics and the market's correlation structure is investigated **FINDINGS**

Keywords: Monte Carlo, Risk-based Portfolios, Portfolio Selection, Copula

JEL classification L250, L100

1. Introduction

Since Markowitz (1952) published the seminal work on mean-variance portfolios, scholars from around the globe have been striving to develop a robust algorithm capable of situating a portfolio on the efficient frontier *ex ante*. There are now a wide array of available portfolio algorithms ranging from simple heuristic-based approaches to advanced mathematical algorithms based on quadratic optimization, random matrix theory and machine learning methods, with many more are still to come.

Unfortunately, the mean-variance portfolios of Markowitz (1952) suffer from sever sensitivity issues, where slight changes in their expected return input cause large changes in optimal portfolio weights. This is exacerbated by the fact that expected returns are notoriously difficult, if not impossible, to accurately forecast (De Prado, 2016). Due to this issue, this work focuses solely on so-called risk-based portfolios, defined by De Carvalho *et al.* (2012a) as "systemic quantitative approaches to portfolio allocation" that solely rely on views of risk when allocating capital. These strategies do not require expected return forecasts and, are therefore, said to be more robust to estimation error. Despite their

Email address: 19959672@sun.ac.za (Nathan Potgieter)

Contributions:
The author would like to thank Nico Katzke for helping me puzzle and prod my to to the eventual completion of this research project.

sole focus on risk mitigation, empirical back tests have shown that they often perform surprising well, from a total return standpoint (Choueifaty *et al.*, 2013).

Rather than using the standard empirical approach to evaluate said strategies, this work attempts to use Monte Carlo simulation methods. This allows for an investigation of the link between the markets covariance structure and portfolio performance. Monte Carlo methods prove to be invaluable in this investigation. They allow for the creation of *ad hoc* markets with predetermined risk return characteristics, and hence leave no uncertainty regarding the composition of the market. This creates an ideal environment for experimentation as the researcher has complete control over the market, and can therefore, adjust the independent variable and observe the response in the dependent variable. In this study the market correlation structure serves as the independent variable and the portfolio return series the dependent variable. Monte Carlo methods are also beneficial here because they enable the effective reduction of noise in portfolio performance measures.

The risk-based portfolios evaluated in this work include the naive equal weight (EW), minimum variance (MV), inverse variance (IV), equal risk contribution (ERC) and maximum diversification (MD) portfolios. Section 2 (below) sets out the aims and objects. Section 3 provides a review of the relevant literature; this includes general issues plaguing the field of portfolio optimization, the rational and theoretical underpinnings behind the five risk-based portfolios, their relative performance in empirical back tests and finally, the importance of using Monte Carlo methods in finance. Section 4 discusses the methodology, Section 5 provides and discusses the results, and Section 6 concludes.

2. Aims and Objectives

This work aims to use Monte Carlo Methods to uncover the relationship between a market's correlation structure and the risk return properties of various risk-based portfolio algorithms. This is achieved through the following objectives:

1. Design and create four distinctive *ad hoc* correlation matrices and estimate one empirical correlation matrix, each describing a market of 50 assets with a unique correlation structure. These matrices range from those possessing no clusters, to those exhibiting hierarchical clustering.
2. Use these five correlation matrices as the key input in their own Monte Carlo simulation, where each correlation matrix corresponds to a unique market type. All other risk characteristics remain equal between market types. The markets will each be built with a student t multivariate distribution with 4.5 degrees of freedom. The individual asset's univariate distributions will each be normally distributed with means and standard deviations calibrated with S&P500 data. Each of the five market types will be simulated 10 000 times across 500 periods, or approximately two years of trading daily return data.

3. Calculate the returns obtained from each risk-based portfolio in each of the simulated markets. Use the first 250 periods to estimate an out of sample covariance matrix and calculate portfolio weights. Conduct periodic rebalancing every 50 periods, each time looking back 250 periods, recalculating the covariance matrix and the new portfolio weights. Repeat this process until all 500 simulated periods have been considered.
4. Calculate the average Sharp ratio, standard deviation, downside deviation, value at risk (VaR), effective number of constitutes (ENC) and effective number of bets (ENB) for each portfolio across the 10 000 markets for each market type.
5. Compare the relative performance of each portfolio within each of the market types and evaluate how the relative performance of each portfolio is effected by the change in market correlation structure.

3. Litrature Review

3.1. Review of Portfolio Optimisation Algorithms

3.1.1. Introduction

This literature review will cover common issues reported in the literature of portfolio optimization, the five risk-based portfolios evaluated in this work, their respective performance in both empirical back tests and Monte Carlo studies and finally the importance of using Monte Carlo methods within the field of finance.

3.1.2. Issues with Portfolio optimization

When operating in sample, portfolio optimization tends to be a perfect science, out of sample it becomes more of an art form where it is often preferable to use simple heuristic over advanced techniques. This section highlights some general issues that tend to worsen their performance out of sample.

Firstly, mean-variance optimizers, like those introduced by Markowitz (1952), rely heavily on the accuracy of their expected return forecasts. Small changes in their expected return input can lead to large changes in portfolio weights (De Prado, 2016). Since, in practice, expected returns are extremely difficult to estimate accurately, this issue serves as a hindrance to their wide spread use. Therefore, the so-called risk based portfolios that intentionally avoid using expected return forecasts have garnered a lot of attention (Maillard, 2010).

However, these risk-based portfolios are not void of issues. The quadratic programming methods used in many portfolio optimizers, including Markovitz's (1952) mean-variance and some risk-based portfolios, require the inversion of some positive-definite covariance matrix. This requirement for positive definiteness can cause issues, as covariance matrices estimated on empirical data are sometimes not positive definite, in which case their inverse does not exist and these portfolios do not have solutions (De Prado, 2016). One approach that alleviates this issue is to simply compute the nearest positive definite matrix and use that instead (Bates & Maechler, 2019; Higham, 2002).

The covariance matrix estimation step is susceptible to measurement error, particularly if the underlying covariance matrix suffers from a high condition number (Zhou *et al.*, 2019). A condition number is defined as the absolute value of the ratio between a covariance matrix's largest and smallest eigenvalues (Bailey & Lopez De Prado, 2012; De Prado, 2016). The condition number is smallest in diagonal matrices (equal to 1) and increases as more correlated variables are added. When working with high condition number matrices, a small change in a single entry's estimated covariance can greatly alter its inverse, which in turn can effect the portfolio weights (De Prado, 2016). This is related to Markowitz's curse which De Prado (2016) summarized by stating that "the more correlated investments are, the greater is the need for a diversified portfolio—and yet the greater are that portfolio's estimation errors".

For a sample with a given number of periods, larger dimension covariance matrices are prone to more noise in estimation. This is essentially due to a reduction in degrees of freedom as a sample of at least $1/2N(N+1)$ independent and identically distributed (iid) observations is required to estimate an $N \times N$ covariance matrix (De Prado, 2016: 60)]. Furthermore, financial market covariance structures tend to vary over time and have been known to change rapidly during so-called regime changes (De Prado, 2016). This exacerbates the issue of requiring a large number of observations when estimating the covariance matrix, since passed data may not be a good refection of the future and looking further increases this likelihood.

3.1.3. Risk Based Portfolios

This section reviews the intuition and technical underpinnings within the literature on risk-based portfolios. Those discussed here include the equal weight (EW), minimum variance (MV), inverse volatility (IV), equal risk contribution (ERC) and maximum diversification (MD) portfolios. The EW is a simple heuristic approach, the minimum variance is more akin to a Markowitz (1952) mean-variance portfolio, while the inverse-variance (IV), equal risk contrition (ERC) and maximum diversification (MD) are quite similar in that they all assume that adequate diversification can be obtained by allocating equal risk to each investible security.

3.1.3.1. Naive Equal Weight (EW). Perhaps the oldest and most simple approach to portfolio diversification involves holding a weight of $1/N$ of the N total available assets (DeMiguel *et al.*, 2009). In other words, this strategy can be described as putting an equal number of eggs into each available basket. It does not require any historical data when allocating capital and does not involve any form of optimization (DeMiguel *et al.*, 2009). This portfolio is commonly called the 'equal weight' or 1/N portfolio, however, its failure to recognize the importance of both asset variance and the covariance between assets has resulted in it also being referred to as the 'naive portfolio'. Meanwhile, its simplicity means that it has been widely used as a benchmark. Equal weighting is optimal from a mean-variance standpoint when there is no correlation between securities, and each possesses the same variance. In which case, the EW is theoretically equivalent to the MV portfolio. The EW's robustness to estimation errors and surprisingly good historical performance has lead to it being incorporated into hybrid portfolio strategies (Tu & Zhou, 2011). These strategies form a balance between the rules based approach of the EW and the optimization approaches from more sophisticated portfolios and have been shown to outperform both its EW and more sophisticated constituents.

3.1.3.2. Minimum Variance (MV). Portfolio optimizers designed to exhibit the minimum variance have over the years garnered a lot of attention. This can in large part be attributed to their tendency to achieve surprisingly high returns and low variance in some historical back tests (Clarke *et al.*, 2011). Their excellent performance has been attributed to the empirical phenomena that low volatility stocks tend to earn returns in excess of the market, and high beta stocks tend not to be rewarded by higher returns (Clarke *et al.*, 2011; Fama & French, 1992). Interesting, this latter finding contradicts traditional financial economic theory which predicts an asset's expected return to be proportional to its market beta (i.e. undiversifiable risk) (Perold, 2004).

The minimum variance portfolio selects security weights such that the resulting portfolio corresponds to that with the lowest possible in sample volatility. Therefore, it has the lowest expected volatility and is, in theory, the safest and least risky portfolio (De Carvalho *et al.*, 2012b). Its primary input is a variance covariance matrix, which it uses to minimize aggregate portfolio volatility. This is accomplished by over-weighting low volatility and low correlation securities (De Carvalho *et al.*, 2012b). Interestingly, it is the only portfolio of the efficient frontier that does not depend on expected return forecasts (De Prado, 2016).

The following provides a technical explanation of how the MV portfolio's weights are calculated. Let $\sum$ indicate the markets variance covariance matrix and $w = \{w_i, ..., w_N\}$ be a vector of length N containing individual security weights. The vector containing the MV portfolio's weights can now be described as (De Carvalho *et al.*, 2012b):

$$w^* = arg\min(w' \textstyle\sum w) \quad s.t. \ \textstyle\sum_i^N w_i = 1$$

In some studies, the minimum variance (MV) portfolio has been found by to earn cumulative returns equal to or slightly greater than market capitalization weighted portfolio's, whilst maintaining a consistently lower variance and achieving a noticeable improvement in downside risk mitigation (Clarke *et al.*, 2011). The MV portfolio can, therefore, work well out of sample, but if left unrestricted, tends to concentrate its holdings in a small number of assets (De Prado, 2016). Its sole objective to minimize portfolio volatility is likely the the primary reason for this. When near the trough of its objective function, it to achieves minor reductions in *ex ante* volatility by greatly favoring a small number of low volatility/correlation securities (De Prado, 2016: 68)]. This tendency to produce highly concentrated portfolio's can be costly out of sample since the portfolio exposed to the idiosyncratic risk of its major constituents. It puts too many eggs in too few baskets. In practice, this issue can be countered by applying cleaver maximum and minimum portfolio weight constraints.

3.1.3.3. Inverse-Varience (IV) Weighting. The IV portfolio, referred to as the equal-risk budget (ERB) portfolio in De Carvalho *et al.* (2012a), aims to allocate an equal risk budget to each investible security. Where the risk budget is defined as the the product of a security's weight and volatility (De Carvalho *et al.*, 2012a). If σ_i is defined as security i's volatility, then the portfolio risk budget can be equally distributed across N securities by setting security weights as:

$$w_{iv} = (\frac{1/\sigma_1}{\sum_{j=1}^{N} 1/\sigma}, ..., \frac{1/\sigma_N}{\sum_{j=1}^{N} 1/\sigma})$$

This indicates that each securities weight is directly proportional to the inverse of its variance, thereby demonstrating why this is called the IV portfolio. The IV portfolio allocates capital based solely on security variance and, is therefore, oblivious to the covariance between its constitutes. De Carvalho *et al.* (2012a) found that, if all securities posses the same sharp ratio and their correlation coefficients are all equal, then the IV portfolio is efficient from a mean-variance stand point and obtains the highest possible sharp ratio.

3.1.3.4. Equal Risk Contribution (ERC). The principle behind the ERC portfolio is similar to that of the IV, however, when balancing risk contributions the ERC does account for the covariance between securities (De Carvalho *et al.*, 2012a). The ERC allocates capital such that each security contributes equally to overall portfolio risk, which in theory should maximize risk diversification (Maillard, 2010). According to Maillard (2010), in practice the ERC acts similar to a weight constrained MV portfolio, with constraints preventing high levels of portfolio concentration. Following the derivation and notation of Maillard (2010), the weights of an ERC portfolio $x = (x_1, x_2, ..., x_n)$ consisting of n assets can be calculated as follows:

Let σ_i^2 resemble asset i's variance, σ_{ij} the covariance between asset i and j and $\sum$ be the markets variance covariance matrix. Portfolio risk can now be written as $\sigma(x) = \sqrt{x^T \sum x} = \sum_i \sum_{j \neq i} x_i x_j \sigma_{ij}$ and the marginal risk contribution, $\partial_{x_i} \sigma(x)$, can then be defined as:

$$\partial_{x_i}\sigma(x) = \frac{\partial\sigma(x)}{\partial x_i} = \frac{x_i\sigma_i^2+\sum_{j\neq i} x_j\sigma_{ij}}{\sigma(x)}$$

Therefore, $\partial_{x_i}\sigma(x)$ refers to the change in portfolio volatility resulting from a small change in asset i's weight (Maillard, 2010). ERC uses this definition to guide central objective to equate the risk contribution across each of the n assets. There is no closed-form solution describing the weights of the ERC portfolio, however, if we define $(\sum x)_i$ as the i^{th} row resulting from the product of $\sum$ with x and note that $\partial_{x_i}\sigma(x) = (\sum x)_i$, then optimal security weights, for the long only ERC portfolio, can be described as those that satisfy the following statement (see Maillard (2010), p. 4-7 for more detail):

$$x^* = \{x\ \epsilon[0,1]^n : \sum x_i = 1, x_i \times (\sum x)_i = x_j \times (\sum x)_j\ \forall\ i,j\}$$

Maillard (2010) proved mathematically that the ERC portfolio's *ex ante* volatility is always somewhere between those of the EW and MV portfolios. De Carvalho *et al.* (2012a) found that, if all securities posses the same sharp ratio, then the ERC and IV portfolios have identical weights. If, in addition, the correlation coefficients between all securities are equal, then the ERC and IV merge into the EW portfolio, with each being mean variance efficient with the maximum attainable sharp ratio (De Carvalho *et al.*, 2012a).

3.1.3.5. Maximum Diversification (MD). Choueifaty & Coignard (2008) originally designed the MD portfolio to maximize some diversification ratio (DR), which he defined as the sum of each securities risk bucket ($w'.V$) divided by portfolio volatility (De Carvalho *et al.*, 2012a). If we define V as a vector of asset volatilities and $\sum$ as the covariance matrix and w^* as the vector of MD portfolio weights. Then the the $w*$ can be expressed as:

$$w^* = arg\ max(DR) \quad with \quad DR = \frac{w'.V}{\sqrt{w'Vw}}$$

De Carvalho *et al.* (2012a) found that, in practice, the MV achieves a diversification ratio similar to that of the MD, and that the difference between the two is due to the MV's larger exposure to low residual volatility securities. Much like the IV and ERC portfolios, the MD portfolio attempts to diversify its portfolio by allocating equal risk to each security (Choueifaty & Coignard, 2008). The MD portfolio accomplishes this by over-weighting low volatility and low correlation securities (De Carvalho *et al.*, 2012a). For further detail regarding the theoretical results and properties of the MD portfolio see Choueifaty & Coignard (2008: 33–35).

3.2. Empirical Backtests and Monte Carlo Findings

Choueifaty *et al.* (2013) used empirical back testing to compare the relative performance of numerous portfolio optimisers. They used historical data from the MSCI world index and considered the largest 50% of assets at each semi-annual rebalance date between 1999 and 2010. To reduce the noise in estimation, at each rebalance date, covariance matrices were estimated using the previous years worth of data (Choueifaty *et al.*, 2013). These were then used as the primary inputs in estimating the long-only portfolio weights. The MV portfolio achieved an annual return of 6.7% and outperformed the ERC and EW portfolios which returned 6.3% and 5.8% respectively. The MV portfolio achieved the lowest daily volatility (10%) followed by the ERC and then the EW portfolio's (with 12.9% and 16.4% respectively). Accordingly, the MV portfolio had the highest sharp ratio (0.36) followed by the ERC and EW portfolios (0.24 and 0.16, respectively).

Despite the simplistic nature of the EW portfolio, empirical studies comparing it to the mean-variance, MV and Bayes-Stein portfolios often report statistically insignificant differences in Sharp ratio between the EW more the advanced portfolios (DeMiguel *et al.*, 2009). In addition, the EW performed surprisingly well from a total return perspective. In fact, many studies have found that the EW portfolio outperforms the mean variance and other sophisticated portfolios that are based on financial theory (Tu & Zhou, 2011).

Due to the aforementioned issues surrounding covariance matrix estimation error, Ardia *et al.* (2017) set out to evaluate the impact of covariance matrix misspecification on the properties of risk-based portfolio's. The authors used Monte Carlo methods to build six distinctive investment universes, each with a unique covariance structure. Numerous covariance matrix estimation techniques were then used on the simulated data, one of which served as the benchmark. They then accessed the impact of alternative covariance specifications on the performance of the MV, IV, ERC and MD portfolio's. The ERC and IV portfolios were found to be "relatively robust to covariance misspecification", the MV was found to be sensitive to misspecification in both the variance and covariance and the MD portfolio was found to be robust to variances misspecification but sensitive to misspecification in the covariances (Ardia *et al.*, 2017: 1).

3.3. Monte Carlo Methods in Portfolio Optimisation

Ever since the pioneering age of computers people have shown a keen interest in leveraging their ability to perform rapid calculations to conduct randomized experiments. The core of Monte Carlo simulation is in the creation of random objects and/or processes using a computer (Kroese *et al.*, 2014: 1). There are a number of reasons for doing this, but the primary one used in this work, and thereby discussed in this review, is of the sampling kind (Kroese *et al.*, 2014). This typically involves the modeling of some stochastic object or process, followed by sampling from some probability distribution and the manipulating said sample through some deterministic process such that the result mimics the true underlying process. The primary idea behind Monte Carlo simulation is to repeat this simulation process many times so that interesting properties can be uncovered through the law of large numbers and central limit theorem (Glasserman, 2013).

A financial application of this can be found in Wang *et al.* (2012) who designed a Monte Carlo procedure that (1) models both the time-series and cross-section properties of financial market returns, involving the use of extreme value theory to estimates a random term's probability distribution function (pdf), and (2) sampling from the modeled process to produce an ensemble of market returns, with each exerting the same risk properties. The simulated data can then be used in risk management and/or the pricing of financial securities (Kroese *et al.*, 2014; Wang *et al.*, 2012). This unique ability to generate a large number of counterfactuals for an asset market with a known risk structure has made it a uniquely powerful tool in accessing the properties of portfolio optimization algorithms (Bailey & Lopez De Prado, 2012). See Glasserman (2013) as a useful source for understanding the methods and applications of Monte Carlo methods in finance.

4. Methodology

This work used Monte Carlo simulation methods to investigate the link between a markets correlation structure and the relative performance of the EW, MV, IV, ERC and MD portfolios.

- The term market refers to a set of daily returns for a number of assets. For example the daily returns for each of the JSE ALSI constitutes between 1 January 2019 and 1 January 2020. Since this is a Monte Carlo study, thousands of markets are simulated, they can therefore, be thought of as a single observation from a population of markets.
- The term 'market type' refers to a population of markets each with the same specified risk characteristics.

The R package MCmarket was used to simulate 10 000 markets from five separate market types, with the correlation structure being the only differentiating factor between between market types. (Potgieter, 2020). Four of the correlation structures/matrices were designed *ad hoc*, while the fifth was estimated using S&P 500 data. These correlation matrices range from one exhibiting no correlation (i.e. a diagonal matrix) to one with a hierarchical clustering structure (see Section 4.1).

Thereafter, the long only EW, MV, IV, ERC and MD portfolios were back tested on the simulated markets (Section 4.3) and portfolio analytics were calculated and aggregated across the 10 000 markets (Section 4.4). Finally, the portfolio metrics are compared within market types across portfolios and within portfolios, across market types.

4.1. Correlation Structures

This section discusses the five correlation matrices used in the Monte Carlo simulations. Section 4.1.1 describes the composition and attributes of the four *ad hoc* correlation matrices. While Section 4.1.2 describes the methodology behind the estimation of the empirical correlation matrix. Each of the five matrices top 10 eigenvalues are listed in Table 4.1.

4.1.1. Ad Hoc

This section describes the four *ad hoc* 50 by 50 correlation matrices used as the key inputs in their respective Monte Carlo simulations. See Figure 4.1 for a graphical representation of each correlation matrix. Note that the *gen_corr* function from the R package *MCmarket* was used in the construction of the four *ad hoc* matrices (Potgieter, 2020).

The first and most simplistic of the four matrices is a diagonal matrix (see Diagonal Matrix in Figure 4.1). It describes a market with a zero correlation coefficient between each asset. Each of its 50 eigenvalues are equal to 1 (Table 4.1), it has no risk clusters and has plenty scope for diversification. This correlation matrix has the lowest possible condition number of 1. The Monte Carlo data set constructed using this matrix is referred to as Market 1.

The second matrix (labeled No Clusters in Figure 4.1) has no risk clusters but describes a market with significant correlation between its constituents. Each asset has a correlation of 0.9 with its closest neighbor (i.e. Asset 1 and 2, 5 and 6 and 11 and 12 each have a pairwise correlation coefficient of 0.9). Correlations then diminish exponentially by the absolute distance between the two assets (i.e. the correlation between Asset 1 and 5 is $0.9^{|1-5|} = 0.6561$). It has a large first eigenvalue of 15.93, but they quickly diminish in size, such that, its 9th largest eigenvalue is less than 1 at 0.79 (Table 4.1). This correlation matrix has the highest condition number of 302.4. The Monte Carlo data set constructed using this matrix is referred to as Market 2.

The third matrix (labeled Five Clusters in Figure 4.1) contains five distinct non-overlapping risk clusters. Assets within the same cluster have a pairwise correlation coefficient of 0.6 while those that are not in the same cluster are uncorrelated. This correlation matrix has a condition number of 16.1. The Monte Carlo data set constructed using this matrix is referred to as Market 3.

The final *ad hoc* correlation matrix has three layers of overlapping risk clusters. The first layer has 10 distinctive clusters, within which, assets have a correlation coefficient of 0.7. The second layer has four clusters where assets that are not in same first layer cluster have a correlation coefficient of 0.5. Assets that are in the same third layer cluster but not clustered in layers one and two have a correlation coefficient of 0.3. Finally, those that do not share any cluster have a correlation coefficient of 0.05. Its largest eigenvalue is 14.36, but they diminish fairly quickly as its third largest is only 3.3 (Table 4.1).

This correlation matrix has a condition number of 47.9. The Monte Carlo data set constructed using this matrix is referred to as Market 4.

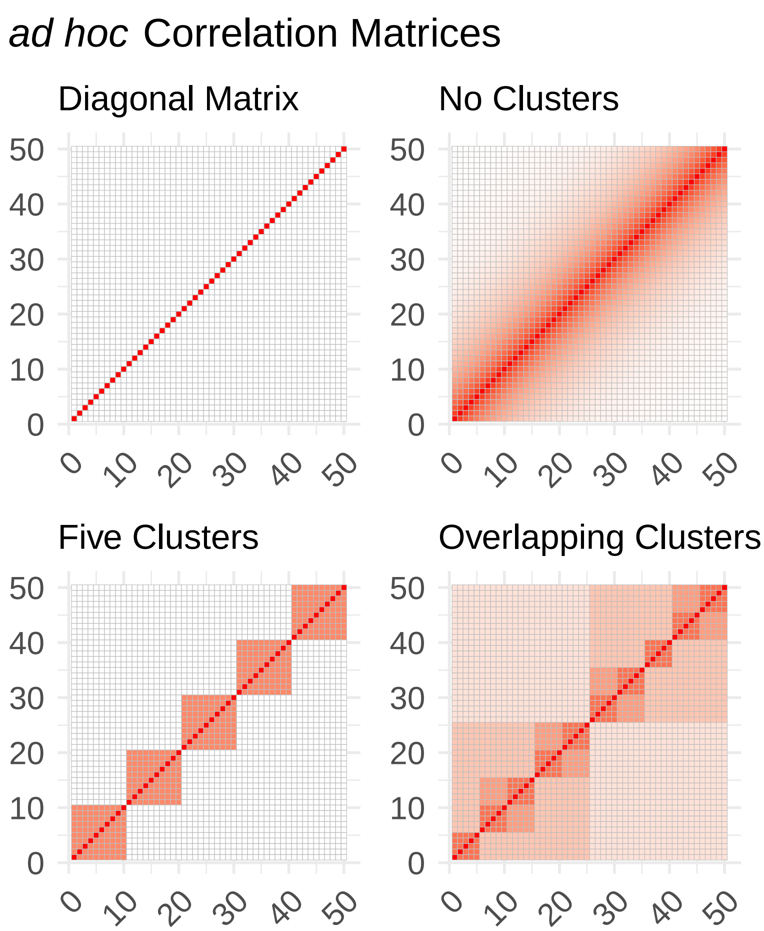

Figure 4.1: Correlation Matrices

4.1.2. Emperical

The empirical correlation matrix used in this study was estimated from the daily returns of a random subset of 50 of the largest 100 S&P 500 stocks, determined by market capitalization, between 1 January 2016 and 1 January 2021. The market capitalizations were measured as of 12 January 2020.

The markets covariance matrix was then estimated using the *fit_mvt* function from the R package *fitHeavyTail* (Palomar & ZHOU, 2020). This covariance estimation method uses maximum likelihood estimation and generalized expectation maximization to fit a multivariate t-distribution to a matrix of asset returns (Liu & Rubin, 1995). The estimated multivariate t distribution was found to have 4.43 degrees of freedom and a correlation matrix shown in Figure 4.2.

Note that in Figure 4.2 assets were ordered by hierarchical clustering so that the reader could more easily visualize the risk clusters. The correlation matrix's largest eigenvalue is 18.6 and they quickly diminish to below zero by its 8th largest eigenvalue (Table 4.1). This correlation matrix has a condition number of 211.1. The Monte Carlo data set constructed using this matrix is referred to as Market 5.

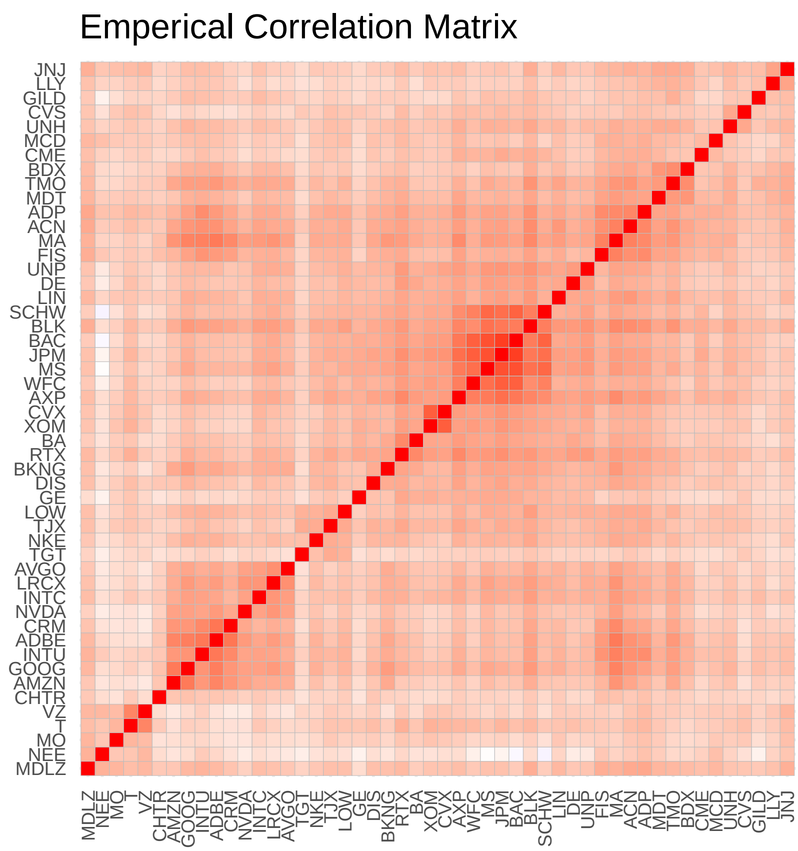

Figure 4.2: Emperical Correlation Matrix

Table 4.1: Eigenvalues

Diagonal	No Clusters	Five Clusters	Overlapping Clusters	Emperical
1	15.93	6.4	14.36	18.6
1	10.38	6.4	6.89	3.09
1	6.28	6.4	3.3	2.52
1	3.92	6.4	3.3	1.4
1	2.59	6.4	2.49	1.28
1	1.81	0.4	2.46	1.21
1	1.32	0.4	1.3	1.03
1	1.01	0.4	1.3	0.92
1	0.79	0.4	1.3	0.88
1	0.64	0.4	1.3	0.83

4.2. Monte Carlo

A generalized version of the Monte Carlo procedure developed in Wang *et al.* (2012) was used to simulate five distinctive market types, one type for each of the correlation matrices discussed in Section 4.1. This framework was build into the R package *MCmarket* which was used to carry out this project's Monte Carlo simulations (Potgieter, 2020). The following briefly describes the process:

An Elliptical t copula with 4.5 degrees of freedom was used, in conjunction with a 50 by 50 correlation matrix, to simulate 500 random uniformly distributed draws (corresponding to 500 trading days or approximately two years worth of trading days) across the 50 assets. The uniformly distributed observations were then transformed, via the inverted normal cumulative distribution function, into normally distributed observations (Potgieter, 2020; Wang *et al.*, 2012: 3). This process was repeated 10 000 times for each of the five correlation matrices/ market types set out in Section 4.1. The correlation matrix is the only distinguishing factor between market types as all other factors remain equal. Overall, this process created 5 data sets each containing 10 000 markets with 50 assets and 500 periods.

The expected returns and standard deviation of these simulated variables were calibrated using a random subset of 50 of the largest 100 S&P500 stocks between 1 January 2020 and 1 January 2021. Maximum likelihood estimation was used to fit the multivariate t distribution to the return series using the method developed by Liu & Rubin (1995). This produced a series of estimated means and variances which were used to calibrate the expected returns and standard deviations of the simulated variables (see Table 6.1).

4.3. *Back Tests*

To remain consistent with the literature, as well as, the mandate for the majority of portfolio managers, a long-only weight constraint was applied to all portfolios. In addition, a constraint limiting the maximum weight of a single security to 10% is also applied. This prevents some portfolios from building unreasonably highly concentrated holdings, while remaining flexible enough to punish those that do so. These constraints therefore act to provide a fair playing ground for the portfolio's to compete. The back testing procedure works as follows.

The first 250 periods, or approximately one years worth of daily return data, are used to fit a multivariate t distribution via maximum likelihood and estimate a covariance matrix (Liu & Rubin, 1995). Interestingly, the identifying assumption used in this covariance matrix estimation method, i.e. that the data comes from a multivariate t distribution, is correct by definition since this is the distribution used to simulate the data set. The estimated covariance matrix is then used as the sole input when calculating the weights for each of the respective risk-based portfolios. Each portfolio holds these weights over the next 50 periods, when they are rebalanced by looking back 250 periods, calculating a covariance matrix and new weights. This process is repeated until all periods in the data set are exhausted. Since there are 500 periods in each market, each portfolio is weighted 5 times and 250 periods of daily returns are calculated for each portfolio.

4.4. *Portfolio Analytics*

This section describes the portfolio performance and concentration metrics used to evaluate and compare portfolios. The Sharp ratio is used to evaluate the risk adjusted return, while the standard deviation (SD), downside deviation (DD) and value at risk (VaR) are used to access portfolio risk. Finally, the effective number of constituents, calculated as the inverse of the Herfindahl-Hirschman index (HHI), and the effective number of bets, calculated following Meucci (2010), are used to compare diversification between portfolios (Rhoades, 1993).

Since Markowitz (1952), the variance of asset returns has been the standard measure for risk in the financial industry (Meucci, 2010). SD is simply the square-root of variance, and too is widely used as a measure of risk. SD benefits due to its relative ease in interpretation. It is also key in calculating the next two portfolio performance metrics described in this study, namely the Sharp ratio and value at risk (VaR).

The Sharp ratio is a measure of a portfolio's risk adjusted returns. Generally speaking, the Sharp ratio is calculated by dividing the portfolio return by some measure of portfolio risk, it is therefore, interpreted as the return per unit of risk. In this Sharp ratio is calculated by dividing portfolio return by its standard deviation.

The 95% VaR is another risk metric used to evaluate portfolio risk performance in this study. It is one of the financial industry standard measurements for downside risk and can be interpreted as the maximum return expected from in the worst 5% of scenarios (Peterson & Carl, 2020). That is, in the worst 5% of scenarios, one should expect to loose at least this amount. The particular version of VaR used here is the Gaussian VaR, this is calculated by assuming that returns are normally distributed $N(\mu, \sigma)$, where μ and σ are estimated using historical data. The probability distribution assumptions enable one to attach probability values to possible future portfolio returns. This assumption can be dangerous in practice, however, in this study it correct by definition, as the return series were each simulated to be normally distributed. It should, therefore, result in an accurate estimate of downside risk. In this study, a higher VaR is viewed as a good thing as this implies that a smaller amount is at risk (i.e. a VaR of -0.01 is preferred to -0.02, where -0.01 is interpreted as being the higher measure for VaR).

The HHI estimates portfolio concentration, and is calculated as the sum of squared portfolio weights (Rhoades, 1993). A portfolio with capital allocated evenly across a large number of securities will have an HHI of approximately zero, while a portfolio with all its capital invested in a single security will have the maximum HHI of 10000. The effective number of constitutes (ENC) can then be approximated as the inverse of the HHI, where an equally weighted portfolio with have an inverse HHI equal to the number of securities, and more concentrated portfolios will have an inverse HHI of less than the number of securities. Weight based measures like the HHI are limited, in that, they are oblivious to covariation between portfolio components. The inverse HHI can therefore, be misleading in financial applications where portfolio components are known to exhibit significant dependence.

Meucci (2010) attempted to rectify this issue when he introduced a new method to evaluate portfolio diversification that considers portfolio risk structure. He used a principle component (PC) approach to estimate the total number of orthogonal bets within a portfolio, which he referred to as the principle portfolios. With this, he estimated a portfolio diversification distribution using the percentage of total portfolio variation attributed to each principle portfolio. Portfolio entropy can then be approximated as the dispersion of the diversification distribution (Meucci, 2010: 10). In this context entropy can be interpreted as the effective number of orthogonal bets within a portfolio.

5. Results and Discussion

This section has tow main parts, the first (Section 5.1), compares the five risk-based portfolios relative performance in each of the five market types, and the second (Section 5.2), investigates how each portfolio's performance changes across the market types. However, a short section discussing some interesting trends and caveats regarding the portfolio concentration measures is included.

5.0.0.1. A Note on the Potfolio Concerntration Metrics. Each portfolios average inverse HHI and entropy across the five market types can be found in Tables 5.1, 5.2, 5.3, 5.4 and 5.5. Since Market 1's underlying correlation matrix expresses no dependence between assets (Diagonal matrix in Figure 4.1), its assets return returns series are orthogonal. Therefore, in theory the EW portfolio should have an entropy of 50, however, reported entropy is 17.5 (see Table 5.1). The reported figure falls well short of its theoretical value, this can not simply be due to noisy measurement. The entropy figures reported are an average across the 10 000 simulated markets, and hence, the majority of noise in estimation should be canceled out. Rather, it is likely due to a persistent bias that arises then rotating the returns matrix based on noise rather than signal. Since the underlying correlation structure implies that the returns are orthogonal, but finite sample estimates of the correlation matrix will inevitably contain some noise, it is likely that entropy is estimated with spurious principle components. Due to the presence of significant correlation between assets, the entropy estimates in Table 5.2 are all significantly lower compared to Market 1. The significant correlation structure in the underlying correlation matrix makes the entropy a better measure of portfolio diversification than what it was in Market 1.

It is also interesting it see that, throughout the five market types, the inverse HHI and entropy measures are typically at odds. Portfolio's rated as highly diversified by the inverse HHI are are rated as relatively undiversified in entropy and *vice verse*. It is also interesting to see that the inverse HHI measure better coincides with the other portfolio risk measures.

5.1. Comparing Portfolios Within Market Types

5.1.1. Market 1

The portfolios compared here were evaluated within the markets simulated using the 'Diagonal Matrix" (Figure 4.1). Due to the lack of dependence between its constituents, this market type is arguable the least realistic. According to the portfolio average Sharp ratio, SD, DD and VaR measures the EW portfolio performed the best overall (Table 5.1). The IV portfolio was a close second, it has the second highest Sharp ratio, tied the lowest SD, and is ranked second in the DD and VaR metrics. The ERC ranked the third best overall, followed by the MD and finally the MV.

Table 5.1: Market 1 - Portfolio Risk Metrics

Metric	EW	MV	IV	ERC	MD
Sharp	0.19353	0.13501	0.18341	0.15876	0.14284
SD	0.00386	0.005	0.00386	0.00447	0.00494
Downside Deviation	0.00235	0.00318	0.00236	0.00279	0.00312
VaR	-0.00545	-0.00735	-0.00548	-0.00645	-0.00722
Inv HHI	50	27.2	47.3	35.6	28.2
Entropy	17.5	22.8	20.1	18.9	21

Since this market type has real no correlation structure it is unsurprising that the two portfolios that neglect using covariance information have performed relatively well. The EW and IV portfolios effectively assume that there is no correlation between assets. In Market 1 this assumption is correct by construction. However, since the variance does differ between assets it would have been reasonable to assume that the IV portfolio would be the least volatile. The fact that it is not may be because asset return variances are not sufficiently different to punish the EW ignorance and/or the IV portfolio's out of sample variance forecast is not accurate enough to effectively reduce risk out of sample.

The MV, ERC and MD portfolios performed poorly compared to the EW and IV portfolios. This is likely be due to there being no real dependence in the underlying market's correlation structure. Therefore, these portfolios are more noisy in comparison, and likely use spurious covariance information when allocating capital.

5.1.2. Market 2

The portfolios compared here were evaluated within the markets simulated using the 'No Clusters' correlation matrix (Figure 4.1). Unlike in Market 1 the portfolio risk metrics in Table 5.2 do not indicate a clear winner. Despite having the highest average SD, the MD portfolio achieved the highest Sharp ratio. The ERC performed best in the SD, DD and VaR measures while obtaining the second highest Sharp ratio. The EW and IV portfolios were similar in their performance and the MV performed poorly.

Table 5.2: Market 2 - Portfolio Risk Metrics

Metric	EW	MV	IV	ERC	MD
Sharp	0.05158	0.03432	0.04948	0.053	0.06484
SD	0.01454	0.01446	0.01438	0.01431	0.01508
Downside Deviation	0.00985	0.00993	0.00976	0.00969	0.01011
VaR	-0.02276	-0.02289	-0.02255	-0.02238	-0.02338
Inv HHI	50	12.2	47.5	47.4	13.9
Entropy	1.1	1.8	1.1	1.2	2

In Market 2, the MV portfolio has the lowest overall Sharp ratio, the third highest SD, the second highest DD and the worst VaR. Despite it managing to attain a fairly low average SD, its poor performance across the other risk measures demonstrates a crucial failure to mitigate downside risk exposure (measured by DD and VaR). The mitigation of downside risk is arguably far more important than reducing volatility in general. Additionally, its relatively low sharp ratio indicates that it was not well compensated for for holding its risk. In fact, it seems to have performed performed exceedingly poorly from a total return perspective. Its entropy score indicates that it diversifies relatively well across risk sources. However, its inverse HHI suggests that it is highly concentrated in a few number of assets (Table 5.2). Despite its interesting concentration results, MV is considered the worst performing portfolio in Market 2.

The EW and IV portfolios performed very similarly in Market 2. This is a recurring theme thought the market types (for example see Tables 5.3 and 5.4). The EW has the higher Sharp ratio of the two, while the IV performs better in the SD, DD and VaR measures. Therefore, it seems as if the the EW portfolio finds itself sightly closer to the top right quadrant of the efficient frontier, where it earns higher returns with greater risk compared to the IV portfolio. They each have a high inverse HHI, indicating that they were well diversified across assets. In comparison to the other portfolios, they have a low entropy measure and are therefore, are not well diversified across risk sources. This has however, not translated into higher average portfolio volatility.

The MD portfolio acts as Market 2's wild card. It has the highest average Sharp ratio by a significant margin, but performs the worst in the SD and DD risk metrics. Despite it having the largest SD, the MD is on par with the best VaR. This indicates that, compared to the other portfolios, the MD earned exceptionally high returns. Interestingly, it has the highest entropy score and is therefore well diversified across risk sources. Comparatively, it has a low inverse HHI and in therefore highly concentrated in a few number of assets.

In comparison, the ERC seems to be the safer and more consistent option. It scored the best overall across the three risk metrics and achieved the second highest Sharp ratio. Therefore, from a risk mitigation perspective the ERC is the clear winner. However, the high returns obtained by the MD portfolio may entice less risk averse investors. Compared to the MD, the ERC is well diversified across assets and poorly diversified across risk sources.

5.1.3. Market 3

The portfolios compared here were evaluated within the markets simulated using the 'Five Clusters' correlation matrix (Figure 4.1). This market structure has not produced a clear winner. On average the EW, IV, ERC and MD portfolios performed quite similarly, while, the MV portfolio seems to be the clear looser. The MV is the worst performer across the Sharp, SD, DD and VaR measures (Table 5.3). The IV and ERC portfolios performed very similarly. The EW portfolio scored the highest average Sharp ratio while maintaining relatively low risk. The MD portfolio performed poorly compared to the EW, IV and ERC portfolios, but still outperformed the MV by a comfortable margin.

Table 5.3: Market 3 - Portfolio Risk Metrics

Metric	EW	MV	IV	ERC	MD
Sharp	0.07949	0.05165	0.07608	0.07602	0.07098
SD	0.00941	0.01034	0.00932	0.00933	0.01011
Downside Deviation	0.00625	0.00701	0.0062	0.00621	0.00675
VaR	-0.01439	-0.01613	-0.01428	-0.0143	-0.01555
Inv HHI	50	14.6	47.6	46	17.8
Entropy	1.3	3.7	1.4	1.5	2.2

The EW portfolio has the third lowest SD, DD and VaR across Market 3. Therefore, despite there being significant and distinct risk clusters in the the underlying market structure the EW portfolio achieved relatively high level of diversification. It has the highest average Sharp ratio, suggesting that it performed well from a total return stand point. According to its entropy measure, it is the least diversified across risk sources. This finding holds across Markets 3, 4 and 5 and will therefore not be repeated (see Tables 5.4 and 5.5).

On average the MV portfolio performed exceedingly poorly, its has by far the lowest Sharp ratio and is the worst performer across the SD, DD and VaR measures. Its inability to to maintain a low average out of sample SD indicates that it suffers from estimation error. The inverse HHI and entropy metrics in Table 5.3 indicate that the MV portfolio is the least diversified across assets and the most diversified across sources of risk. These findings hold in Market 4 and are not repeated (see 5.4).

The IV and ERC portfolios were strong contenders for first place. Their scores were close across all 6 measures in Table 5.3, and therefore, were about equal in their ability to mitigate risk. The IV has a slightly lower SD, the ERC a slightly lower DD and the IV has a slightly higher VaR. Their Sharp ratios also seem to suggest that they were fairly well compensated per unit of risk. Their respective inverse HHI and entropy scores indicate that they are well diversified across asset and poorly diversified across risk sources.

The MD portfolio performed worse than the EW, IV and ERC portfolios. It has a lower Sharp ratio, higher SD, DD and a low VaR. However, it underperformed these portfolios by a fairly small margin, and did substantially better than the MV portfolio. It is relatively highly concentrated across assets and fairly well diversified across risk sources.

5.1.4. Market 4

The portfolios compared here were evaluated within the markets simulated using the 'Overlapping Clusters' correlation matrix (Figure 4.1). There is therefore, a positive correlation coefficient between all assets in the underlying correlation structure. Despite this the EW and IV portfolios, which are oblivious to asset covariance, managed significant risk reduction. The IV and ERC portfolios performed about equally well across all the metrics in Table 5.4. The MV and MD performed worst, however, the MD was the better of the two.

Table 5.4: Market 4 - Portfolio Risk Metrics

Metric	EW	MV	IV	ERC	MD
Sharp	0.05368	0.03225	0.05155	0.05154	0.05006
SD	0.01397	0.01447	0.0138	0.0138	0.01443
Downside Deviation	0.00946	0.00996	0.00935	0.00936	0.00979
VaR	-0.02184	-0.02294	-0.02159	-0.0216	-0.0226
Inv HHI	50	13	47.6	46.7	13.7
Entropy	1.1	2	1.1	1.1	1.6

The IV and ERC performed very similarly in this market. They are approximately tied as the best performing assets across the SD, DD and VaR measures. They are also approximately tie with the second highest Sharp ratio. They are both well diversified across assets, but relatively highly concentrated across risk sources (Table 5.4).

The MD portfolio is the second worst performing portfolio in Market 2. It ranks second to last across Sharp ratio, SD, DD and VaR metrics (Table 5.4). Its Sharp ratio is not substantially lower than that of the IV and ERC, thereby indicating that the MD was compensated fairly for holding risk. The MD has a fairly low inverse HHI and has the the highest entropy.

5.1.5. Market 5

The portfolios compared here were evaluated within the markets simulated using the empirical correlation matrix (Figure 4.2). This is therefore the most realistic of the five correlation matrices. It exhibits significant correlation between assets, with some noticeable risk clusters. The IV and ERC portfolios performed similarly across the metrics in Table 5.5. The MV and MD portfolios performed the worst. The EW has the highest sharp ratio, but was out performed by the IV and ERC in terms of risk mitigation.

Table 5.5: Market 5 - Portfolio Risk Metrics

Metric	EW	MV	IV	ERC	MD
Sharp	0.04708	0.03473	0.04526	0.04511	0.04556
SD	0.01585	0.01684	0.01564	0.01579	0.0171
Downside Deviation	0.01078	0.01157	0.01065	0.01076	0.01164
VaR	-0.02497	-0.02676	-0.02466	-0.0249	-0.02697
Inv HHI	50	13	47.5	43.6	12.5
Entropy	1.1	1.6	1.2	1.2	1.5

In Market 5 the MV portfolio has the lowest Sharp ratio and ranked second worst in the SD, DD and VaR measures. The fact that the relatively more simple EW and IV portfolios achieved a lower average SD again suggests that its out of sample performance is suffering due to estimation error. Its inverse HHI indicates that in is highly concentrated in a relatively small number of assets. Conversely, its entropy suggests it is the most successful portfolio in diversifying across risk sources.

The IV portfolio has the third highest Sharp ratio, the lowest SD, the second lowest DD and is ranked best in VaR. The IV narrowly outperformed the ERC in all the aforementioned metrics. Furthermore, the IV portfolio is well diversified across assets, as shown by its inverse HHI, and comparatively its entropy score suggests it is comparatively less diversified across risk sources.

The MD portfolio received the second highest Sharp ratio, the highest SD and DD, and performed the worst in VaR. Its inverse HHI indicates that its holding are highly concentrated in a small number of assets while its entropy suggests that it has done relatively well in diversifying across risk sources.

5.2. Comparing Portfolios Across Market Types

This section compares the relative portfolio performance across market types, thereby investigating the link between correlation structure and portfolio performance. Tables 5.6, 5.7, 5.8, 5.9 and 5.10 provide normalized Sharp ratio, SD, DD, and VaR performance metrics, as well as the concentration scores, for the each portfolio across the five market types. Values were normalized by subtracting the mean portfolio score within each market, and dividing by the standard deviation.

Table 5.6: Equal Weight

Metric	Market 1	Market 2	Market 3	Market 4	Market 5
Sharp	1.2192	0.0858	0.7753	0.6666	0.7076
SD	-1.0181	-0.0457	-0.6014	-0.3728	-0.5851
Downside Deviation	-1.029	-0.1105	-0.6266	-0.4497	-0.6218
VaR	1.0309	0.0837	0.6304	0.4427	0.6113
Inv HHI	50	50	50	50	50
Entropy	20.362	1.3586	3.2539	1.101	1.1167

Table 5.7: Minimum Variance

Metric	Market 1	Market 2	Market 3	Market 4	Market 5
Sharp	-1.0958	-1.4968	-1.7213	-1.7695	-1.7667
SD	1.0325	-0.3068	1.3139	1.1305	0.885
Downside Deviation	1.0541	0.3807	1.4084	1.3636	1.0157
VaR	-1.0528	-0.2563	-1.4008	-1.3346	-0.9931
Inv HHI	27.4705	11.4806	14.0377	12.2394	12.1014
Entropy	21.9552	2.0646	5.3911	2.2467	1.7717

Table 5.8: Inverse Volatility

Metric	Market 1	Market 2	Market 3	Market 4	Market 5
Sharp	0.8189	-0.1067	0.4696	0.4245	0.343
SD	-1.0181	-0.5679	-0.7867	-0.8839	-0.8969
Downside Deviation	-1.0039	-0.6632	-0.7604	-0.8486	-0.8913
VaR	0.998	0.633	0.7588	0.8466	0.8891
Inv HHI	47.2773	47.3	47.2838	47.287	47.2623
Entropy	21.2181	1.3726	3.3521	1.1399	1.1482

Table 5.9: Equal Risk Contribution

Metric	Market 1	Market 2	Market 3	Market 4	Market 5
Sharp	-0.1563	0.216	0.4642	0.4233	0.3129
SD	0.0791	-0.7963	-0.7661	-0.8839	-0.6741
Downside Deviation	0.0753	-1.093	-0.7337	-0.8124	-0.6633
VaR	-0.0658	1.0776	0.7354	0.8305	0.674
Inv HHI	35.9119	46.3298	44.9382	46.1096	42.7687
Entropy	21.1598	1.4843	3.3722	1.1407	1.1584

Table 5.10: Maximum Diversification

Metric	Market 1	Market 2	Market 3	Market 4	Market 5
Sharp	-0.786	1.3016	0.0122	0.2551	0.4031
SD	0.9245	1.7167	0.8403	1.0102	1.2711
Downside Deviation	0.9035	1.486	0.7122	0.7471	1.1608
VaR	-0.9102	-1.538	-0.7237	-0.7852	-1.1813
Inv HHI	29.1668	12.8975	16.211	14.2876	13.2099
Entropy	21.1969	2.3394	4.5327	1.5828	1.6707

Since this market type exhibits significant correlation between its constituents it is unsurprising to see the EW and IV portfolios fall out, in favor of the more sophisticated ERC and MD portfolios.

5.3. Discussion

MD seems to perform well from a return when risk structures are not clean. At the same time obtains higher SD.

6. Conclusion

I hope you find this template useful. Remember, stackoverflow is your friend - use it to find answers to questions. Feel free to write me a mail if you have any questions regarding the use of this package. To cite this package, simply type citation("Texevier") in Rstudio to get the citation for Katzke (2017) (Note that united references in your bibtex file will not be included in References).

References

Ardia, D., Bolliger, G., Boudt, K. & Gagnon-Fleury, J.-P. 2017. The impact of covariance misspecification in risk-based portfolios. *Annals of Operations Research*. 254(1-2):1–16.

Bailey, D.H. & Lopez De Prado, M. 2012. Balanced baskets: A new approach to trading and hedging risks. *Journal of Investment Strategies (Risk Journals)*. 1(4).

Bates, D. & Maechler, M. 2019. *Matrix: Sparse and dense matrix classes and methods*. ed. [Online], Available: https://CRAN.R-project.org/package=Matrix.

Choueifaty, Y. & Coignard, Y. 2008. Toward maximum diversification. *The Journal of Portfolio Management*. 35(1):40–51.

Choueifaty, Y., Froidure, T. & Reynier, J. 2013. Properties of the most diversified portfolio. *Journal of investment strategies*. 2(2):49–70.

Clarke, R., De Silva, H. & Thorley, S. 2011. Minimum-variance portfolio composition. *The Journal of Portfolio Management*. 37(2):31–45.

De Carvalho, R.L., Lu, X. & Moulin, P. 2012a. Demystifying equity risk–based strategies: A simple alpha plus beta description. *The Journal of Portfolio Management*. 38(3):56–70.

De Carvalho, R.L., Lu, X. & Moulin, P. 2012b. Demystifying equity risk–based strategies: A simple alpha plus beta description. *The Journal of Portfolio Management*. 38(3):56–70.

DeMiguel, V., Garlappi, L. & Uppal, R. 2009. Optimal versus naive diversification: How inefficient is the 1/n portfolio strategy? *The review of Financial studies*. 22(5):1915–1953.

De Prado, M.L. 2016. Building diversified portfolios that outperform out of sample. *The Journal of Portfolio Management*. 42(4):59–69.

Fama, E.F. & French, K.R. 1992. The cross-section of expected stock returns. *the Journal of Finance*. 47(2):427–465.

Glasserman, P. 2013. *Monte carlo methods in financial engineering*. ed. Vol. 53. Springer Science & Business Media.

Higham, N.J. 2002. Computing the nearest correlation matrix—a problem from finance. *IMA journal of Numerical Analysis*. 22(3):329–343.

Katzke, N.F. 2017. *Texevier: Package to create elsevier templates for rmarkdown*. ed. Stellenbosch, South Africa: Bureau for Economic Research.

Kroese, D.P., Brereton, T., Taimre, T. & Botev, Z.I. 2014. Why the monte carlo method is so important today. *Wiley Interdisciplinary Reviews: Computational Statistics*. 6(6):386–392.

Liu, C. & Rubin, D.B. 1995. ML estimation of the t distribution using em and its extensions, ecm and ecme. *Statistica Sinica*. 19–39.

Maillard, T., Roncalli. 2010. The properties of equally weighted risk contribution portfolios. *The Journal of Portfolio Management*. 36(4):60–70.

Markowitz, H. 1952. Portfolio selection. *The Journal of Finance*. 7(1):77–91.

Meucci, A. 2010. Managing diversification.

Palomar, D.P. & ZHOU, R. 2020. *fitHeavyTail: Mean and Covariance Matrix Estimation under Heavy Tails*. ed. [Online], Available: https://CRAN.R-project.org/package=fitHeavyTail.

Perold, A.F. 2004. The capital asset pricing model. *Journal of economic perspectives*. 18(3):3–24.

Peterson, B.G. & Carl, P. 2020. *PerformanceAnalytics: Econometric tools for performance and risk analysis*. ed. [Online], Available: https://CRAN.R-project.org/package=PerformanceAnalytics.

Potgieter, N. 2020. *MCmarket: Package to simplify the monte carlo simulation of financial markets*. ed. Stellenbosch, South Africa: Stellenbosch University.

Rhoades, S.A. 1993. The herfindahl-hirschman index. *Fed. Res. Bull.* 79:188.

Tu, J. & Zhou, G. 2011. Markowitz meets talmud: A combination of sophisticated and naive diversification strategies. *Journal of Financial Economics*. 99(1):204–215.

Wang, P., Sullivan, R.N. & Ge, Y. 2012. Risk-based dynamic asset allocation withExtreme tails and correlations. *The Journal of Portfolio Management*. 38(4):26–42.

Zhou, R., Liu, J., Kumar, S. & Palomar, D.P. 2019. Robust factor analysis parameter estimation. In ed. Springer *International conference on computer aided systems theory*. 3–11.

Appendix

Table 6.1: Asset Means and Sd's

Asset	Mean	Sd	Asset...	Mean...	Sd...
Asset_1	-0.00011	0.03277	Asset_26	0.00247	0.02671
Asset_2	0.00023	0.02314	Asset_27	-0.00132	0.04888
Asset_3	-3e-05	0.02359	Asset_28	0.00085	0.02387
Asset_4	-0.0018	0.03654	Asset_29	0.00024	0.01613
Asset_5	-0.00036	0.02925	Asset_30	0.00086	0.01884
Asset_6	0.00052	0.02396	Asset_31	0.00118	0.03147
Asset_7	0.00058	0.02451	Asset_32	0.00053	0.02882
Asset_8	0.00099	0.02287	Asset_33	0.00166	0.02253
Asset_9	0.00057	0.02929	Asset_34	0.00179	0.03002
Asset_10	0.00049	0.01736	Asset_35	0.00235	0.02574
Asset_11	-0.00088	0.01907	Asset_36	0.00157	0.02307
Asset_12	5e-05	0.0301	Asset_37	0.0017	0.02566
Asset_13	-0.00095	0.03185	Asset_38	0.00157	0.02262
Asset_14	-1e-04	0.02111	Asset_39	0.00054	0.03116
Asset_15	0.00102	0.02555	Asset_40	0.00242	0.02494
Asset_16	0.00174	0.02582	Asset_41	0.00246	0.02186
Asset_17	0.00241	0.03639	Asset_42	0.00113	0.01951
Asset_18	0.00033	0.03967	Asset_43	0.00181	0.02099
Asset_19	0.00102	0.02636	Asset_44	-0.00045	0.02297
Asset_20	0.00173	0.02697	Asset_45	4e-05	0.02354
Asset_21	-0.00034	0.01405	Asset_46	0.00485	0.03553
Asset_22	0.0011	0.0233	Asset_47	0.00064	0.02275
Asset_23	0.00236	0.02663	Asset_48	1e-04	0.02713
Asset_24	-0.00055	0.03195	Asset_49	0.00071	0.02186
Asset_25	-0.00283	0.03227	Asset_50	-0.00203	0.03131

Appendix A

Some appendix information here

Appendix B