DEPARTMENT OF ECONOMICS
UNIVERSITY OF STELLENBOSCH

The Link Between Market Correlation Structure and the Performance of Risk-Based Portfolios

by

[19959672]

Assignment presented in partial fulfilment of the requirements for the degree of Master's of Commerce at the University of Stellenbosch.

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March 2021

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The Link Between Market Correlation Structure and the Performance of Risk-Based Portfolios

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Abstract

This work uses Monte Carlo methods to design and simulate five distinctive market types, with each type possessing a unique correlation structure. Within each market type, the risk-based equal weight, minimum variance, inverse variance, equal risk contribution and maximum diversification portfolios are evaluated. The relative performance of each portfolio is compared within market types and the relationship between portfolio return characteristics and the market's correlation structure is investigated. The minimum variance portfolio was found to underperform in all market types. Overall, the equal weight, inverse variance and equal risk contribution portfolios performed above average. The equal weight and inverse variance consistently performed well across all the market types. The equal risk contribution only performed well in markets with significant correlation between assets. In these environments, its performance was akin to that of the inverse variance portfolio. The maximum diversification portfolio failed to maintain low volatility in all market types, but showed some potential in its ability to earn a high average return.

Keywords: Monte Carlo, Risk-based Portfolios, Portfolio Selection, Copula

JEL classification L250, L100

1. Introduction

Since Markowitz (1952) published his seminal work on mean-variance portfolios, scholars from around the globe have been striving to develop a robust algorithm capable of situating a portfolio on the efficient frontier *ex ante*. There are now a wide array of portfolio algorithms available, ranging from simple heuristic-based approaches to advanced mathematical algorithms based on quadratic optimization, random matrix theory and machine learning methods.

Unfortunately, the mean-variance portfolios of Markowitz (1952) suffer from severe sensitivity issues,

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Contributions:
Thank you to Nico Katzke for helping me puzzle and prod my way to the eventual completion of this research project. Thank you to Jessica Thibaud for always lending me an ear

where slight changes in their expected return input causes large changes in optimal portfolio weights. This is exacerbated by the fact that expected returns are notoriously difficult, if not impossible, to accurately forecast (De Prado, 2016). Due to this issue, this work focuses solely on so-called risk-based portfolios, defined by De Carvalho *et al.* (2012a) as "systemic quantitative approaches to portfolio allocation" that solely rely on views of risk when allocating capital (De Carvalho *et al.*, 2012a). These strategies do not require expected return forecasts, and are therefore, said to be more robust to estimation error. Despite their sole focus on risk mitigation, empirical back tests have shown that they often perform surprising well in terms of total return standpoint (Choueifaty *et al.*, 2013).

Rather than using the standard empirical approach to evaluate said strategies, this work attempts to use Monte Carlo simulation methods. This allows for an investigation of the link between the market's covariance structure and portfolio performance. Monte Carlo methods prove to be invaluable in this investigation, as they allow for the creation of *ad hoc* markets with predetermined risk return characteristics, and hence leave no uncertainty regarding the composition of the market. This creates an ideal environment for experimentation as the researcher has complete control over the market, and can therefore, adjust the independent variable and observe the response in the dependent variable. In this study the market correlation structure serves as the independent variable and the portfolio return series the dependent variable. Monte Carlo methods are also beneficial here because they enable the effective reduction of noise in the portfolio performance measures.

The risk-based portfolios evaluated in this work include the equal weight (EW), minimum variance (MV), inverse variance (IV), equal risk contribution (ERC) and maximum diversification (MD) portfolios. Section 2 below sets out the aims and objects. Section 3 provides a review of the relevant literature, this includes general issues plaguing the field of portfolio optimization, the rational and theoretical underpinnings behind the five risk-based portfolios, their relative performance in empirical back tests and the importance of using Monte Carlo methods in finance. Section 4 discusses the methodology, Section 5 provides and discusses the results, and Section 6 concludes.

2. Aims and Objectives

This work aims to use Monte Carlo Methods to uncover the relationship between a market's correlation structure and the risk return properties of various risk-based portfolio algorithms. This is achieved through the following objectives.

1. Design and create four distinctive *ad hoc* correlation matrices and estimate one empirical correlation matrix, each describing a market of 50 assets with a unique correlation structure. These matrices range from those possessing no clusters, to those exhibiting hierarchical clustering.
2. Use these five correlation matrices as the key input in their own Monte Carlo simulation, where each correlation matrix corresponds to a unique market type. All other risk characteristics remain equal between market types. The markets will each be built with a student t multivariate distribution with 4.5 degrees of freedom. The univariate distributions of each asset will each be normally distributed with means and standard deviations calibrated with S&P500 data. Each of the five market types will be simulated 10 000 times across 500 periods.
3. Calculate the returns obtained from each risk-based portfolio in each of the simulated markets. Use the first 250 periods to estimate an out-of-sample covariance matrix and calculate portfolio weights. Conduct periodic rebalancing every 50 periods, each time looking back 250 periods, recalculating the covariance matrix and the new portfolio weights. Repeat this process until all 500 simulated periods have been considered.
4. Calculate the average Sharp ratio, standard deviation, downside deviation, value at risk (VaR), inverse Herfindahl-Hirschman index and entropy for each portfolio in each market type.
5. Compare the relative performance of each portfolio within each of the market types and evaluate how the relative performance of each portfolio is effected when moving between market types.

3. Litrature Review

3.1. Review of Portfolio Optimisation Algorithms

3.1.1. Introduction

This literature review will cover common challenges reported in the literature regarding portfolio optimization, the five risk-based portfolios evaluated in this work, their respective performance in both empirical back tests and Monte Carlo studies and finally the importance of using Monte Carlo methods within the field of finance.

3.1.2. Issues with Portfolio optimization

When operating in-sample, portfolio optimization tends to be a perfect science, meanwhile, out-of-sample it becomes more of an art form where it is often preferable to use heuristics over advanced techniques. This section highlights some general issues that tend to worsen their performance out-of-sample.

Firstly, mean-variance optimizers, like those introduced by Markowitz (1952), rely heavily on the accuracy of their expected return forecasts. Small changes in their expected return input can lead to large changes in portfolio weights (De Prado, 2016). Since, in practice, expected returns are extremely difficult to estimate accurately, this issue serves as a hindrance to their wide spread use. Therefore, the so-called risk based portfolios that intentionally avoid using expected return forecasts have garnered a lot of attention (Maillard, 2010).

However, these risk-based portfolios are not void of issues. The quadratic programming methods used in many portfolio optimizers, including Markovitz's (1952) mean-variance and some risk-based portfolios, require the inversion of some positive-definite covariance matrix. This requirement for positive definiteness can cause problems, as covariance matrices estimated on empirical data are sometimes not positive definite. In which case, their inverse does not exist and these portfolios do not have solutions (De Prado, 2016). One approach to get around this issue is to simply compute the nearest positive definite matrix and use that instead (Bates & Maechler, 2019; Higham, 2002).

The covariance matrix estimation step is susceptible to measurement error, particularly if the underlying covariance matrix suffers from a high condition number (Zhou *et al.*, 2019). A condition number is defined as the absolute value of the ratio between a covariance matrix's largest and smallest eigenvalues (Bailey & Lopez De Prado, 2012; De Prado, 2016). The condition number is smallest in diagonal matrices (equal to 1) and increases as more correlated variables are added. When working with high condition number matrices, a small change in a single entry's estimated covariance can greatly alter its inverse. This, in turn, can effect the portfolio weights (De Prado, 2016). This is related to Markowitz's curse which De Prado (2016) summarized by stating that "the more correlated investments are, the greater is the need for a diversified portfolio—and yet the greater are that portfolio's estimation errors" (De Prado, 2016).

For a sample with a given number of periods, larger dimension covariance matrices are prone to more noise in estimation. This is essentially due to a reduction in degrees of freedom as a sample of at least $1/2N(N+1)$ independent and identically distributed (iid) observations is required to estimate an $N \times N$ covariance matrix (De Prado, 2016: 60)]. Furthermore, financial market covariance structures tend to vary over time and have been known to change rapidly during so-called regime changes (De Prado, 2016). This exacerbates the issue of requiring a large number of observations when estimating the covariance matrix, since past data may not be a good refection of the future and looking back further increases this likelihood.

3.1.3. Risk Based Portfolios

This section reviews the intuition and technical underpinnings within the literature on risk-based portfolios. Those discussed here include the equal weight (EW), minimum variance (MV), inverse volatility (IV), equal risk contribution (ERC) and maximum diversification (MD) portfolios. The EW is a simple heuristic approach, the minimum variance is more akin to a Markowitz (1952) mean-variance portfolio, while the inverse-variance (IV), equal risk contrition (ERC) and maximum diversification (MD) are quite similar in that they all assume that adequate diversification can be obtained by allocating equal risk to each investible security.

3.1.3.1. Naive Equal Weight (EW). Perhaps the oldest and most simple approach to portfolio diversification involves holding a weight of $1/N$ of the N total available assets (DeMiguel *et al.*, 2009). In other words, this strategy can be described as putting an equal number of eggs into each available basket. It does not require any historical data when allocating capital and does not involve any form of optimization (DeMiguel *et al.*, 2009). This portfolio is commonly called the 'equal weight' or 1/N portfolio, however, its failure to recognize the importance of both asset variance and the covariance between assets has resulted in it also being referred to as the 'naive portfolio'. Meanwhile, its simplicity has resulted in it being used as a benchmark. Equal weighting is optimal from a mean-variance standpoint when there is no correlation between securities, and each possesses the same variance. In which case, the EW is theoretically equivalent to the MV portfolio. The EW's robustness to estimation errors and surprisingly good historical performance has lead to it being incorporated into hybrid portfolio strategies (Tu & Zhou, 2011). These strategies form a balance between the rules-based approach of the EW and the optimization approaches from more sophisticated portfolios and have been shown to outperform both the EW and its more sophisticated constituents.

3.1.3.2. Minimum Variance (MV). Portfolio optimizers designed to exhibit the minimum variance have, over the years, garnered a lot of attention. This can, in large, be attributed to them achieving surprisingly high returns and low variance in some historical back tests (Clarke *et al.*, 2011). Their excellent performance has been attributed to the empirical phenomena that low volatility stocks tend to earn returns in excess of the market, and high beta stocks tend not to be rewarded by higher returns (Clarke *et al.*, 2011; Fama & French, 1992). Interestingly, this latter finding contradicts traditional financial economic theory which predicts an asset's expected return to be proportional to its market beta (i.e. undiversifiable risk) (Perold, 2004).

The minimum variance portfolio selects security weights such that the resulting portfolio corresponds to that with the lowest possible in-sample volatility. Therefore, it has the lowest expected volatility and is, in theory, the safest and least risky portfolio (De Carvalho *et al.*, 2012b). Its primary input is a variance covariance matrix, which it uses to minimize aggregate portfolio volatility. This is accomplished by over-weighting low volatility and low correlation securities (De Carvalho *et al.*, 2012b). Notably, it is the only portfolio of the efficient frontier that does not depend on expected return forecasts (De Prado, 2016).

The following provides a technical explanation of how the MV portfolio weights are calculated. Let $\sum$ indicate the markets variance covariance matrix and $w = \{w_i, ..., w_N\}$ be a vector of length N containing individual security weights. The vector containing the MV portfolio weights can now be described as (De Carvalho *et al.*, 2012b):

$$w^* = arg\min(w' \sum w) \quad s.t. \ \sum_i^N w_i = 1$$

In some studies, the minimum variance (MV) portfolio has been found to earn cumulative returns equal to or slightly greater than market capitalization weighted portfolios, whilst maintaining a consistently lower variance and achieving a noticeable improvement in downside risk mitigation (Clarke *et al.*, 2011). The MV portfolio can, therefore, work well out-of-sample, however, if left unrestricted, tends to concentrate its holdings in a small number of assets (De Prado, 2016). Its sole objective to minimize portfolio volatility is likely the the primary reason for this. When near the trough of its objective function, it achieves minor reductions in *ex ante* volatility by greatly favoring a small number of low volatility/correlation securities (De Prado, 2016: 68)]. This tendency to produce highly concentrated portfolios can be costly out-of-sample since the portfolio remains exposed to the idiosyncratic risk of its major constituents. It puts too many eggs in too few baskets. In practice, this issue can be countered by applying clever maximum and minimum portfolio weight constraints.

3.1.3.3. Inverse-Varience (IV) Weighting. The IV portfolio, referred to as the equal-risk budget (ERB) portfolio in De Carvalho *et al.* (2012a), aims to allocate an equal risk budget to each investible security. The risk budget is defined as the the product of a security's weight and volatility (De Carvalho *et al.*, 2012a). If σ_i is defined as security i's volatility, then the portfolio risk budget can be equally distributed across N securities by setting security weights as:

$$w_{iv} = (\frac{1/\sigma_1}{\sum_{j=1}^N 1/\sigma}, ..., \frac{1/\sigma_N}{\sum_{j=1}^N 1/\sigma})$$

This indicates that each security'S weight is directly proportional to the inverse of its variance, thereby demonstrating why this is called the IV portfolio. The IV portfolio allocates capital based solely on security variance, and is therefore, oblivious to the covariance between its constitutes. De Carvalho *et al.* (2012a) found that if all securities posses the same Sharp ratio and their correlation coefficients are all equal, then the IV portfolio is efficient from a mean-variance stand point and obtains the highest possible Sharp ratio.

3.1.3.4. Equal Risk Contribution (ERC). The principle behind the ERC portfolio is similar to that of the IV, however, when balancing risk contributions the ERC does account for the covariance between securities (De Carvalho *et al.*, 2012a). The ERC allocates capital such that each security contributes equally to overall portfolio risk, which in theory, should maximize risk diversification (Maillard, 2010). According to Maillard (2010), in practice the ERC acts similar to a weight constrained MV portfolio, with constraints preventing high levels of portfolio concentration. Following the derivation and notation of Maillard (2010), the weights of an ERC portfolio $x = (x_1, x_2, ..., x_n)$ consisting of n assets can be calculated as follows.

Let σ_i^2 resemble asset i's variance, σ_{ij} the covariance between asset i and j and $\sum$ be the markets variance covariance matrix. Portfolio risk can now be written as $\sigma(x) = \sqrt{x^T \sum x} = \sum_i \sum_{j \neq i} x_i x_j \sigma_{ij}$ and the marginal risk contribution, $\partial_{x_i}\sigma(x)$, can then be defined as:

$$\partial_{x_i}\sigma(x) = \frac{\partial\sigma(x)}{\partial x_i} = \frac{x_i \sigma_i^2 + \sum_{j \neq i} x_j \sigma_{ij}}{\sigma(x)}$$

Therefore, $\partial_{x_i}\sigma(x)$ refers to the change in portfolio volatility resulting from a small change in asset i's weight (Maillard, 2010). ERC uses this definition to guide central objective to equate the risk contribution across each of the n assets. There is no closed-form solution describing the weights of the ERC portfolio, however, if we define $(\sum x)_i$ as the i^{th} row resulting from the product of $\sum$ with x and note that $\partial_{x_i}\sigma(x) = (\sum x)_i$, then optimal security weights, for the long-only ERC portfolio, can be described as those that satisfy the following statement (see Maillard (2010), p. 4-7 for more detail):

$$x^* = \{x \; \epsilon [0,1]^n : \sum x_i = 1, x_i \times (\sum x)_i = x_j \times (\sum x)_j \; \forall \; i, j\}$$

Maillard (2010) proved mathematically that the ERC portfolio's *ex ante* volatility is always somewhere between those of the EW and MV portfolios. De Carvalho *et al.* (2012a) reported that if all securities possess the same Sharp ratio, then the ERC and IV portfolios have identical weights. In addition to this, if the correlation coefficients between all securities are equal, then the ERC and IV merge into the EW portfolio, with each being mean-variance efficient with the maximum attainable Sharp ratio (De Carvalho *et al.*, 2012a).

3.1.3.5. Maximum Diversification (MD). Choueifaty & Coignard (2008) originally designed the MD portfolio to maximize a diversification ratio (DR), which he defined as the sum of each security's risk bucket ($w'.V$) divided by portfolio volatility (De Carvalho *et al.*, 2012a). If we define V as a vector of asset volatilities, $\sum$ as the covariance matrix and w^* as the vector of MD portfolio weights. Then the the $w*$ can be expressed as:

$$w^* = arg\ max(DR) \quad with \quad DR = \frac{V'w}{\sqrt{w' \sum w}}$$

De Carvalho *et al.* (2012a) found that, in practice, the MV achieves a diversification ratio similar to that of the MD, and that the difference between the two is due to the MV's larger exposure to low residual volatility securities. Much like the IV and ERC portfolios, the MD portfolio attempts to diversify its portfolio by allocating equal risk to each security (Choueifaty & Coignard, 2008). The MD portfolio accomplishes this by over-weighting low volatility and low correlation securities (De Carvalho *et al.*, 2012a). For further detail regarding the theoretical results and properties of the MD portfolio see Choueifaty & Coignard (2008: 33–35).

3.2. Empirical Backtests and Monte Carlo Findings

Choueifaty *et al.* (2013) used empirical back-testing to compare the relative performance of numerous portfolio optimizers. They used historical data from the MSCI world index and considered the largest 50% of assets at each semi-annual rebalance date between 1999 and 2010. To reduce the noise in estimation at each rebalance date, covariance matrices were estimated using the previous years worth of data (Choueifaty *et al.*, 2013). These were then used as the primary inputs in estimating the long-only portfolio weights. The MV portfolio achieved an annual return of 6.7% and outperformed the ERC and EW portfolios which returned 6.3% and 5.8% respectively. The MV portfolio achieved the lowest daily volatility (10.0%) followed by the ERC and then the EW portfolio's (with 12.9% and 16.4% respectively). Accordingly, the MV portfolio had the highest Sharp ratio (0.36) followed by the ERC and EW portfolios of 0.24 and 0.16, respectively.

Despite the simplistic nature of the EW portfolio, empirical studies comparing it to the mean-variance, MV and Bayes-Stein portfolios often report statistically insignificant differences in Sharp ratio between the EW more the advanced portfolios (DeMiguel *et al.*, 2009). In addition, the EW performed surprisingly well from a total return perspective. In fact, many studies have found that the EW portfolio outperforms the mean variance and other sophisticated portfolios that are based on financial theory (Tu & Zhou, 2011).

Due to the aforementioned issues surrounding covariance matrix estimation error, Ardia *et al.* (2017) set out to evaluate the impact of covariance matrix misspecification on the properties of risk-based portfolios. The authors used Monte Carlo methods to build six distinctive investment universes, each with a unique covariance structure. Numerous covariance matrix estimation techniques were then used on the simulated data, one of which served as the benchmark. They then assessed the impact of alternative covariance specifications on the performance of the MV, IV, ERC and MD portfolios. The ERC and IV portfolios were found to be "relatively robust to covariance misspecification", the MV was found to be sensitive to misspecification in both the variance and covariance, and the MD portfolio was found to be robust to variance misspecification but sensitive to misspecification in the covariance (Ardia *et al.*, 2017: 1).

3.3. Monte Carlo Methods in Portfolio Optimization

Ever since the pioneering age of computers, people have shown a keen interest in leveraging their ability to perform rapid calculations to conduct randomized experiments. The core of Monte Carlo simulation is in the creation of random objects and/or processes using a computer (Kroese *et al.*, 2014: 1). There are a number of reasons for doing this, however, the primary one used in this work, and thereby discussed in this review, is of the sampling kind (Kroese *et al.*, 2014). This typically involves the modelling of some stochastic object or process, followed by sampling from some probability distribution and the manipulating of said sample through some deterministic process such that the result mimics the true underlying process. The primary idea behind Monte Carlo simulation is to repeat this simulation process many times so that interesting properties can be uncovered through the law of large numbers and central limit theorem (Glasserman, 2013).

A financial application of this can be found in Wang *et al.* (2012) who designed a Monte Carlo procedure that (1) models both the time-series and cross-section properties of financial market returns. This involves the use of extreme value theory to estimate a random term's probability distribution function (pdf). (2) Samples from the modelled process to produce an ensemble of market returns, with each exerting the same risk properties. The simulated data can then be used in risk management and/or the pricing of financial securities (Kroese *et al.*, 2014; Wang *et al.*, 2012). This unique ability to generate a large number of counterfactuals for an asset market with a known risk structure has made it a uniquely powerful tool in assessing the properties of portfolio optimization algorithms (Bailey & Lopez De Prado, 2012). Glasserman (2013) is a useful source for understanding the methods and applications of Monte Carlo methods in finance.

4. Methodology

This work used Monte Carlo simulation methods to investigate the link between a market's correlation structure and the relative performance of the EW, MV, IV, ERC and MD portfolios.

- The term market refers to a set of daily returns for a number of assets. For example, the daily returns for each of the JSE ALSI constituents between 1 January 2019 and 1 January 2020. Since this is a Monte Carlo study, thousands of markets are simulated, and can therefore, be thought of as a single observation from a population of markets.
- The term 'market type' refers to a population of markets, each with the same specified risk characteristics.

The R package *MCmarket* was used to simulate 10 000 markets from five separate market types, with the correlation structure being the only differentiating factor between them (Potgieter, 2020). Four of the correlation structures/matrices were designed *ad hoc*, while the fifth was estimated using S&P 500 data. These correlation matrices range from exhibiting no correlation (i.e. a diagonal matrix) to those will hierarchical clustering (see Section 4.1).

Thereafter, the long-only EW, MV, IV, ERC and MD portfolios were back-tested on the simulated markets (Section 4.3), and a number of portfolio analytics were calculated and aggregated across the 10 000 markets (Section 4.4). Finally, the portfolio metrics were compared within market types across portfolios and within portfolios, across market types.

4.1. Correlation Structures

This section discusses the five correlation matrices used in the Monte Carlo simulations. Section 4.1.1 describes the composition and attributes of the four *ad hoc* correlation matrices. While Section 4.1.2 describes the methodology behind the estimation of the empirical correlation matrix. The top 10 eigenvalues of each of the five matrices are listed in Table 4.1.

4.1.1. Ad Hoc

This section describes the four *ad hoc* 50 by 50 correlation matrices used as the key inputs in their respective Monte Carlo simulations. Figure 4.1 shows a graphical representation of each correlation matrix. The *gen_corr* function from the R package *MCmarket* was used in the construction of the four *ad hoc* matrices (Potgieter, 2020).

The first and most simplistic of the four matrices is a diagonal matrix (Diagonal Matrix in Figure 4.1). It describes a market with a zero correlation coefficient between each asset. Each of its 50 eigenvalues are equal to 1 (Table 4.1), it has no risk clusters and has plenty scope for diversification. This correlation matrix has the lowest possible condition number of 1. The Monte Carlo data set constructed using this matrix is referred to as Market 1.

The second matrix (No Clusters in Figure 4.1) has no risk clusters but describes a market with significant correlation between its constituents. Each asset has a correlation of 0.9 with its closest neighbor (i.e. Asset 1 and 2, 5 and 6 and 11 and 12 each have a pairwise correlation coefficient of 0.9). Correlations then diminish exponentially by the absolute distance between the two assets (i.e. the correlation between Asset 1 and 5 is $0.9^{|1-5|} = 0.6561$). It has a large first eigenvalue of 15.93, however, they quickly diminish in size, such that its 9th largest eigenvalue is less than 1 at 0.79 (Table 4.1). This correlation matrix has the highest condition number of 302.4. The Monte Carlo data set constructed using this matrix is referred to as Market 2.

The third matrix (Five Clusters in Figure 4.1) contains five distinct non-overlapping risk clusters. Assets within the same cluster have a pairwise correlation coefficient of 0.6 while those that are not in the same cluster are uncorrelated. This correlation matrix has a condition number of 16.1. The Monte Carlo data set constructed using this matrix is referred to as Market 3.

The final *ad hoc* correlation matrix has three layers of overlapping risk clusters. The first layer has 10 distinctive clusters, within which, assets have a correlation coefficient of 0.7. The second layer has four clusters where assets that are not in same first layer cluster have a correlation coefficient of 0.5. Assets that are in the same third layer cluster but not clustered in layers one and two have a correlation coefficient of 0.3. Lastly, those that do not share any cluster have a correlation coefficient of 0.05. Its largest eigenvalue is 14.36, however, they diminish fairly quickly as its third largest is only 3.3 (Table 4.1). This correlation matrix has a condition number of 47.9. The Monte Carlo data set constructed using this matrix is referred to as Market 4.

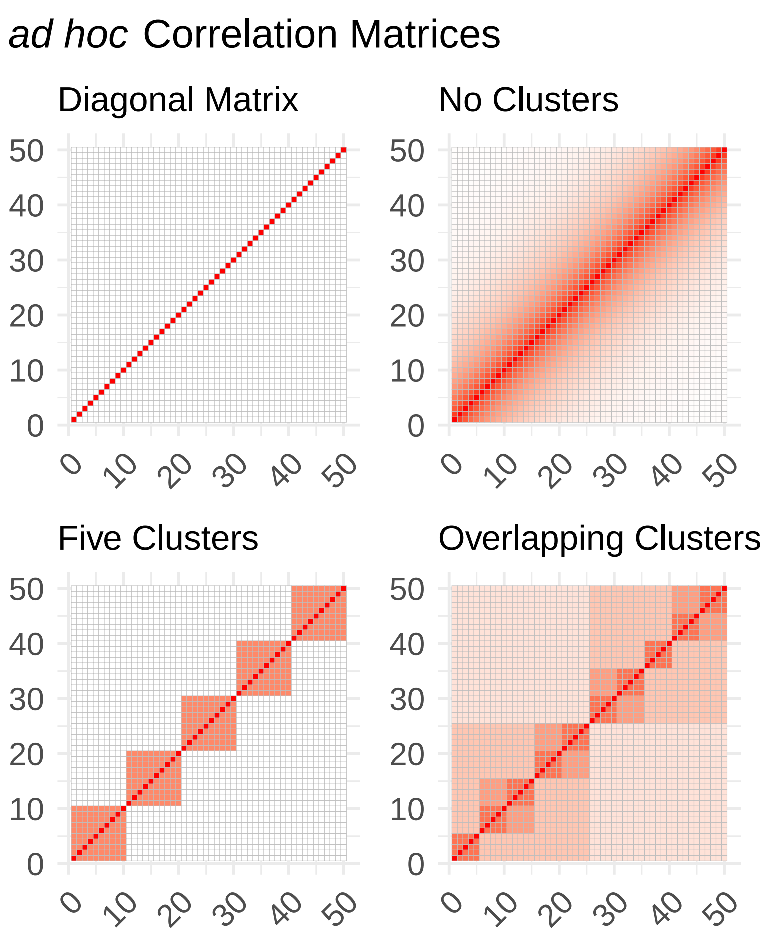

Figure 4.1: Correlation Matrices

4.1.2. Emperical

The empirical correlation matrix used in this study was estimated from the daily returns of a random subset of 50 of the largest 100 S&P 500 stocks, determined by market capitalization. The market capitalizations were measured on the 12 January 2020, while the S&P data were collected between 1 January 2016 and 1 January 2021.

The markets covariance matrix was then estimated using the *fit_mvt* function from the R package *fitHeavyTail* (Palomar & ZHOU, 2020). This covariance estimation method uses maximum likelihood estimation and generalized expectation maximization to fit a multivariate t-distribution to a matrix of asset returns (Liu & Rubin, 1995). The estimated multivariate t distribution was found to have 4.43 degrees of freedom and the correlation matrix shown in Figure 4.2.

In Figure 4.2 assets were ordered by hierarchical clustering so that the reader could more easily visualize the risk clusters. The correlation matrix's largest eigenvalue is 18.6 and they quickly diminish in size to below zero by its 8th largest eigenvalue (Table 4.1). This correlation matrix has a condition number of 211.1. The Monte Carlo data set constructed using this matrix is referred to as Market 5.

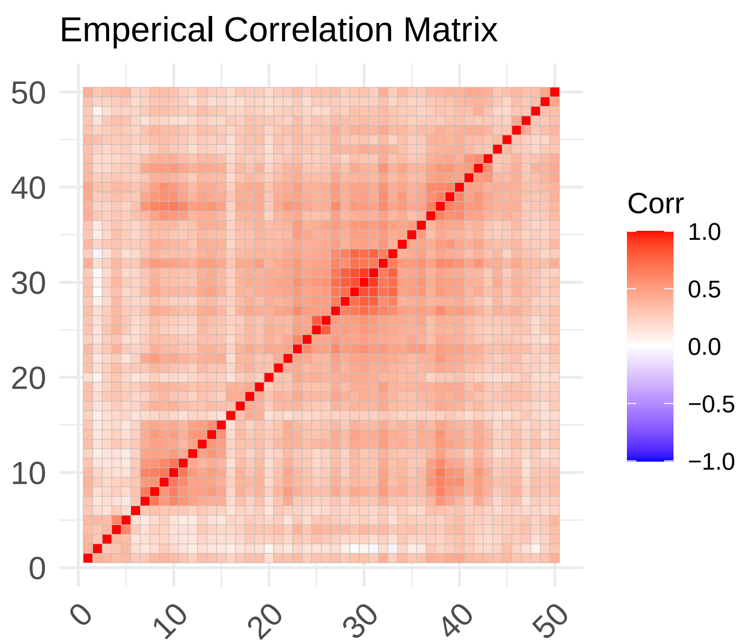

Figure 4.2: Emperical Correlation Matrix

Table 4.1: Eigenvalues

Diagonal	No Clusters	Five Clusters	Overlapping Clusters	Emperical
1	15.93	6.4	14.36	18.6
1	10.38	6.4	6.89	3.09
1	6.28	6.4	3.3	2.52
1	3.92	6.4	3.3	1.4
1	2.59	6.4	2.49	1.28
1	1.81	0.4	2.46	1.21
1	1.32	0.4	1.3	1.03
1	1.01	0.4	1.3	0.92
1	0.79	0.4	1.3	0.88
1	0.64	0.4	1.3	0.83

4.2. Monte Carlo

A generalized version of the Monte Carlo procedure developed in Wang *et al.* (2012) was used to simulate five distinctive market types, one type for each of the correlation matrices discussed in Section 4.1. This framework was built into the R package *MCmarket* which was used to carry out the Monte Carlo simulations in this work (Potgieter, 2020). The following briefly describes this process.

An Elliptical t copula with 4.5 degrees of freedom was used in conjunction with a 50 by 50 correlation matrix to simulate 500 random uniformly distributed draws (corresponding to 500 trading days or approximately two years worth of daily trading data) across 50 assets. The uniformly distributed observations were then transformed via the inverted normal cumulative distribution function into normally distributed observations (Potgieter, 2020; Wang *et al.*, 2012: 3). This process was repeated 10 000 times for each of the five correlation matrices/ market types set out in Section 4.1. The correlation matrix is the only distinguishing factor between market types as all other factors remain equal. Overall, this process created 5 data sets each containing 10 000 markets with 50 assets and 500 periods.

The expected returns and standard deviation of the simulated assets were calibrated using a random subset of 50 of the largest 100 S&P500 stocks between 1 January 2020 and 1 January 2021. Maximum likelihood estimation was used to fit the multivariate t distribution to the return series using the method developed by Liu & Rubin (1995). This produced a series of estimated means and variances which were then used to calibrate the expected returns and standard deviations of the simulated variables (see Table 6.1).

4.3. Back Tests

To remain consistent with the literature, as well as the mandate for the majority of portfolio managers, a long-only weight constraint was applied to all portfolios. In addition, a constraint limiting the maximum weight of a single security to 10.0% is also applied. This prevents some portfolios from building unreasonably highly concentrated holdings, while remaining flexible enough to punish those that do so. These constraints, therefore, act to provide a fair playing ground for the portfolios to compete. The back testing procedure works as follows.

The first 250 periods are used to fit a multivariate t distribution via maximum likelihood and estimate a covariance matrix (Liu & Rubin, 1995). Interestingly, the identifying assumption used in this covariance matrix estimation method, i.e. that the data comes from a multivariate t distribution, is correct by definition since this is the distribution used to simulate the data set. The estimated covariance matrix is then used as the only input when calculating the weights for each of the respective risk-based portfolios. Each portfolio holds these weights over the next 50 periods, they are then rebalanced by looking back 250 periods, calculating a covariance matrix and new weights. This process is repeated until all periods in the data set are exhausted. Since there are 500 periods in each market, each portfolio is weighted 5 times and 250 periods of daily returns are calculated for each portfolio.

4.4. Portfolio Analytics

This section describes the portfolio performance and concentration metrics used to evaluate and compare portfolios. The Sharp ratio is used to evaluate the risk adjusted return, while the standard deviation (SD), downside deviation (DD) and value at risk (VaR) are used to assess portfolio risk. Finally, the effective number of constituents estimated as the inverse of the Herfindahl-Hirschman index (HHI), and portfolio entropy calculated following Meucci (2010), are used to compare diversification between portfolios (Rhoades, 1993).

Since Markowitz (1952), the variance of asset returns has been the standard measure for risk in the financial industry (Meucci, 2010). SD is simply the square-root of variance, and is therefore, also widely used as a measure of risk. SD is beneficial due to its relative ease in interpretation. It is also key in calculating the next two portfolio performance metrics described in this study, namely the Sharp ratio and value at risk (VaR).

The Sharp ratio is a measure of a portfolio's risk adjusted returns. Generally speaking, the Sharp ratio is calculated by dividing the portfolio return by some measure of portfolio risk. It is therefore, interpreted as the return per unit of risk. In this work, the Sharp ratio is calculated by dividing portfolio return by its standard deviation.

The 95% VaR is another risk metric used to evaluate portfolio risk performance in this study. It is one of the financial industry's standard measurements for downside risk and can be interpreted as the maximum return expected in the worst 5% of scenarios (Peterson & Carl, 2020). That is, in the worst 5% of scenarios, one should expect to loose at least this amount. The particular version of VaR used here is the Gaussian VaR, which is calculated by assuming that returns are normally distributed $N(\mu, \sigma)$, where μ and σ are estimated using historical data. The probability distribution assumptions enable one to attach probability values to possible future portfolio returns. This assumption can be dangerous in practice, however, in this study it is correct by construction, as the return series were each simulated to be normally distributed. It should, therefore, result in fairly accurate estimates of downside risk. In this study, a higher VaR is viewed as a good thing as this implies that a smaller amount is at risk (i.e. a VaR of -0.01 is preferred to -0.02).

The HHI estimates portfolio concentration and is calculated as the sum of squared portfolio weights (Rhoades, 1993). A portfolio with capital allocated evenly across a large number of securities will have an HHI of approximately zero, while a portfolio with all its capital invested in a single security will have the maximum HHI of 10000. The effective number of constitutes (ENC) can then be approximated as the inverse of the HHI, where an equally weighted portfolio has an inverse HHI equal to the number of securities, and more concentrated portfolios will have an inverse HHI of less than the number of securities. Weight-based measures like the HHI are limited in that they are oblivious to covariation between portfolio components. The inverse HHI can therefore, be misleading in financial applications where portfolio components are known to exhibit significant dependence.

Meucci (2010) attempted to rectify this issue when he introduced a new method to evaluate portfolio diversification that considers portfolio risk structure. He used a principle component (PC) approach to estimate the total number of orthogonal bets within a portfolio, which he referred to as the principle portfolios. With this, he estimated a portfolio diversification distribution using the percentage of total portfolio variation attributed to each principle portfolio. Portfolio entropy can then be approximated as the dispersion of the diversification distribution (Meucci, 2010: 10). In this context, entropy can be interpreted as the effective number of orthogonal bets within a portfolio.

5. Results and Discussion

This section has two main parts, the first (Section 5.1), compares the relative performance of the five risk-based portfolios in each of the five market types, and the second (Section 5.2), investigates how each portfolio's relative performance changes across the market types. However, some interesting trends and caveats regarding the portfolio concentration measures are first discussed.

5.0.0.1. A Note on the Potfolio Concerntration Metrics . The average inverse HHI and entropy for each portfolio across the five market types can be found in Tables 5.1, 5.2, 5.3, 5.4 and 5.5. Throughout the five market types, the inverse HHI and entropy measures are typically at odds. In other words, portfolios rated as highly diversified by the inverse HHI are rated as relatively undiversified in entropy and *vice verse*. An interesting observation is that the inverse HHI measure coincides better with the portfolio risk measures.

Furthermore, when ranked in order of concentration score, the results tend to be consistent across the five market types. The EW's inverse HHI is the highest in all five market types and its entropy is the lowest. Conversely, the MV portfolio's inverse HHI is the lowest in all Market types, while its entropy appears to be one of the highest. The IV portfolio typically ranks second for the highest inverse HHI, and lowest for entropy. The ERC has the third highest inverse HHI and third highest entropy. The MD has the second lowest HHI, while its entropy consistently ranks as on of the highest. Overall, this indicates that the EW, IV and ERC portfolios diversify well across assets and poorly across risk sources, while the MD and MV portfolios diversify poorly across assets and well across sources of risk.

5.1. Comparing Portfolios Within Market Types

5.1.1. Market 1

The portfolios compared in Market 1 were evaluated within the markets simulated using the 'Diagonal Matrix' (Figure 4.1). Due to the lack of dependence between its constituents, this market type is arguably the least realistic. According to the average Sharp ratio, SD, DD and VaR measures indicate that the EW portfolio performed the best overall (Table 5.1). The IV portfolio was a close second, it has the second highest Sharp ratio, tied the lowest SD, and is ranked second in the DD and VaR metrics. The ERC ranked the third best, followed by the MD and finally the MV.

Table 5.1: Market 1 - Portfolio Risk Metrics

Metric	EW	MV	IV	ERC	MD
Sharp	0.19353	0.13501	0.18341	0.15876	0.14284
SD	0.00386	0.005	0.00386	0.00447	0.00494
Downside Deviation	0.00235	0.00318	0.00236	0.00279	0.00312
VaR	-0.00545	-0.00735	-0.00548	-0.00645	-0.00722
Inv HHI	50	27.2	47.3	35.6	28.2
Entropy	17.5	22.8	20.1	18.9	21

Since Market 1 has no real correlation structure, it is unsurprising that the two portfolios that do not use covariance information have performed relatively well. The EW and IV portfolios effectively assume that there is no correlation between assets. In Market 1 this assumption is correct by construction. However, since the variance does differ between assets, it would have been reasonable to assume that the IV portfolio would be the least volatile. The fact that it is not may be because asset return variances are not sufficiently different to punish the EW's ignorance. Alternatively, the IV portfolio's out-of-sample variance forecast is not accurate enough to effectively reduce risk out-of-sample.

The MV, ERC and MD portfolios performed poorly compared to the EW and IV portfolios. This is likely due to there being no real dependence in the underlying market's correlation structure. Therefore, these portfolios are more noisy in comparison, and likely use spurious covariance information when allocating capital. Out of the three, the ERC portfolio performs significantly better than the rest.

5.1.2. Market 2

The portfolios compared in Market 2 were evaluated within the markets simulated using the 'No Clusters' correlation matrix (Figure 4.1). Unlike in Market 1, the portfolio risk metrics in Table 5.2 do not rank the portfolios in a definitive order. Despite having the highest average SD, the MD portfolio achieved the highest Sharp ratio. The ERC performed best in the SD, DD and VaR measures while obtaining the second highest Sharp ratio. The EW and IV portfolios were similar in their performance and the MV performed poorly.

Table 5.2: Market 2 - Portfolio Risk Metrics

Metric	EW	MV	IV	ERC	MD
Sharp	0.05158	0.03432	0.04948	0.053	0.06484
SD	0.01454	0.01446	0.01438	0.01431	0.01508
Downside Deviation	0.00985	0.00993	0.00976	0.00969	0.01011
VaR	-0.02276	-0.02289	-0.02255	-0.02238	-0.02338
Inv HHI	50	12.2	47.5	47.4	13.9
Entropy	1.1	1.8	1.1	1.2	2

In Market 2, the MV portfolio has the lowest overall Sharp ratio, the third highest SD, the second highest DD and the worst VaR. Despite it managing to attain a fairly low average SD, its poor performance across the other risk measures (i.e. DD and VaR) demonstrates a crucial failure to mitigate downside risk. Arguably, the mitigation of downside risk is more important than simply reducing volatility. Its low Sharp ratio indicates that it was not well compensated for holding its risk, and performed exceedingly poorly from a total return perspective. Its entropy score indicates that it is diversified relatively well across risk sources. However, its inverse HHI suggests that it is highly concentrated in a few number of assets (Table 5.2).

The EW and IV portfolios performed very similarly in Market 2. This is a recurring theme thought the market types (for example see Tables 5.3 and 5.4). The EW has the higher Sharp ratio of the two, while the IV performs better in the SD, DD and VaR measures. Therefore, it seems that the the EW portfolio finds itself sightly closer to the top right quadrant of the efficient frontier where it earns higher returns with greater risk compared to the IV portfolio. Both portfolios have a high inverse HHI, indicating that they were well diversified across assets. In comparison to the other portfolios, they have a low entropy measure, and are therefore, not well diversified across risk sources. This has, however, not translated into higher average portfolio volatility.

The MD portfolio acts as a wild card in Market 2. It has the highest average Sharp ratio by a significant margin, however, it performs the worst in the SD and DD risk metrics. Despite it having the largest SD, the MD portfolio is on par with the best in VaR. This indicates that, compared to the other portfolios, the MD earned exceptionally high returns. Interestingly, it has the highest entropy score and is therefore, well diversified across risk sources. Comparatively, it has a low inverse HHI indicating that it is highly concentrated in a small number of assets.

Compared to the MD portfolio, the ERC seems to be safer and more consistent across metrics. The ERC scored the best overall across the three risk metrics and achieved the second highest Sharp ratio. Therefore, from a risk mitigation perspective the ERC is the top portfolio in Market 2. However, the high returns obtained by the MD portfolio may entice less risk averse investors. Compared to the MD, the ERC is well diversified across assets and poorly diversified across risk sources.

5.1.3. Market 3

The portfolios compared in Market 3 were evaluated within the markets simulated using the 'Five Clusters' correlation matrix (Figure 4.1). This market structure has not produced an obvious winner. On average, the EW, IV, ERC and MD portfolios performed relatively similarly. The MV portfolio performed the worst across the Sharp, SD, DD and VaR measures (Table 5.3). The IV and ERC portfolios performed very similarly. The EW portfolio obtained the highest average Sharp ratio and maintained relatively low risk. The MD portfolio performed poorly compared to the EW, IV and ERC portfolios, but still outperformed the MV by a comfortable margin.

Table 5.3: Market 3 - Portfolio Risk Metrics

Metric	EW	MV	IV	ERC	MD
Sharp	0.07949	0.05165	0.07608	0.07602	0.07098
SD	0.00941	0.01034	0.00932	0.00933	0.01011
Downside Deviation	0.00625	0.00701	0.0062	0.00621	0.00675
VaR	-0.01439	-0.01613	-0.01428	-0.0143	-0.01555
Inv HHI	50	14.6	47.6	46	17.8
Entropy	1.3	3.7	1.4	1.5	2.2

In Market 3, the EW portfolio has the third lowest SD, DD and VaR. Therefore, despite there being significant and distinct risk clusters in the underlying market structure, the EW portfolio achieved a relatively high level of diversification. It has the highest average Sharp ratio, suggesting that it performed well from a total return stand point. The above findings hold in Markets 3, 4 and 5 and will therefore, not be repeated (see Tables 5.4 and 5.5).

On average, the MV portfolio performed exceedingly poorly. It obtained the lowest Sharp ratio by far and is the worst performer across the SD, DD and VaR measures. Its inability to maintain a low average out-of-sample SD indicates that it suffers from estimation error. The inverse HHI and entropy metrics in Table 5.3 indicate that the MV portfolio is the least diversified across assets and the most diversified across sources of risk. These findings hold in Market 4 and are not repeated (see 5.4).

The IV and ERC portfolios were strong contenders for first place. Their scores were close across all 6 measures (Table 5.3). They were approximately equal in their ability to mitigate risk. The IV has a slightly lower SD, the ERC has a slightly lower DD and the IV has a slightly higher VaR. Their Sharp ratios also seem to suggest that they were fairly well compensated for holding risk. In addition to this, their respective inverse HHI and entropy scores indicate that they are well diversified across assets and poorly diversified across risk sources.

The MD portfolio performed worse than the EW, IV and ERC portfolios. It has a lower Sharp ratio, higher SD, DD and a lower VaR. However, it underperformed these portfolios by a fairly small margin, and did substantially better than the MV portfolio in this regard. Its inverse HHI and entropy indicate it is relatively highly concentrated across assets and well diversified across risk sources.

5.1.4. Market 4

The portfolios compared in Market 4 were evaluated within the markets simulated using the 'Overlapping Clusters' correlation matrix (Figure 4.1). There is therefore, a positive correlation coefficient between all assets in the underlying correlation structure. Despite this, the EW and IV portfolios which are oblivious to asset covariance managed significant risk reduction. The IV and ERC portfolios performed approximately equally well across all the metrics (Table 5.4). The MV performed the worst, with the MD narrowly outperforming it in the SD, DD and VaR measures.

Table 5.4: Market 4 - Portfolio Risk Metrics

Metric	EW	MV	IV	ERC	MD
Sharp	0.05368	0.03225	0.05155	0.05154	0.05006
SD	0.01397	0.01447	0.0138	0.0138	0.01443
Downside Deviation	0.00946	0.00996	0.00935	0.00936	0.00979
VaR	-0.02184	-0.02294	-0.02159	-0.0216	-0.0226
Inv HHI	50	13	47.6	46.7	13.7
Entropy	1.1	2	1.1	1.1	1.6

The IV and ERC performed in a very similar manner in this market. They can both be viewed as the best performing portfolios across the SD, DD and VaR measures. They are also approximately equal with the second highest Sharp ratio. They are both well diversified across assets, but relatively highly concentrated across risk sources (Table 5.4).

The MD portfolio is the second worst performing portfolio in Market 2. It ranks second to last across the Sharp ratio, SD, DD and VaR metrics (Table 5.4). Its Sharp ratio is not substantially lower than that of the IV and ERC, thereby indicating that the MD was compensated fairly for holding risk. The MD has a relatively low inverse HHI with the highest entropy.

5.1.5. Market 5

The portfolios compared in Market 5 were evaluated within the markets simulated using the 'Empirical Correlation Matrix' (Figure 4.2). This is therefore, the most realistic of the five correlation matrices. It exhibits significant correlation between assets, with some noticeable risk clustering. The IV and ERC portfolios performed similarly across all metrics (Table 5.5), while the MV and MD portfolios performed the worst. The EW has the highest Sharp ratio, but was out performed by the IV and ERC in terms of risk mitigation.

Table 5.5: Market 5 - Portfolio Risk Metrics

Metric	EW	MV	IV	ERC	MD
Sharp	0.04708	0.03473	0.04526	0.04511	0.04556
SD	0.01585	0.01684	0.01564	0.01579	0.0171
Downside Deviation	0.01078	0.01157	0.01065	0.01076	0.01164
VaR	-0.02497	-0.02676	-0.02466	-0.0249	-0.02697
Inv HHI	50	13	47.5	43.6	12.5
Entropy	1.1	1.6	1.2	1.2	1.5

In Market 5, the MV portfolio has the lowest Sharp ratio and ranked second worst in the SD, DD and VaR measures. It therefore, failed in its objective to minimize portfolio volatility. The relatively simple EW and IV portfolios outperformed it in this regard. The MV portfolio's out-of-sample performance is, therefore, suffering due to estimation error. Its inverse HHI indicates that it is highly concentrated in a relatively small number of assets. Conversely, its entropy suggests it is the most successful portfolio in diversifying across risk sources. As discussed previously (Section 5.0.0.1) this is a relatively common observation since the inverse HHI and entropy measures are typically at odds.

The IV portfolio has the third highest Sharp ratio, the lowest SD, the second lowest DD and the highest VaR. It narrowly outperformed the ERC in all of these metrics. Furthermore, as shown by its concentration scores, the IV portfolio is well diversified across assets and poorly diversified across risk sources. On the other hand, the MD portfolio received the second highest Sharp ratio, the highest SD and DD, and performed the worst in VaR. Its inverse HHI indicates that it is highly concentrated in a small number of assets, while its entropy suggests that it has done relatively well in diversifying across risk sources.

5.2. Comparing Portfolios Across Market Types

This section compares the relative portfolio performance across the five market types, thereby investigating the link between correlation structure and portfolio performance. Each portfolio's standardized Sharp ratio, SD, DD, and VaR performance metrics, as well as their inverse HHI and entropy scores across the five market types are provided in Tables 5.6, 5.7, 5.8, 5.9 and 5.10. Values were standardized via the commonly-used z-score transformation and can therefore, be interpreted as standard deviations. Section 5.2.1 discusses how each portfolio's entropy changes between the five markets and Section 5.2.2 describes the relative portfolio performance across market types.

5.2.1. Entropy Across Market Types

All portfolios had their highest entropy score in Market 1 and second highest in Market 3. These markets simply had the most scope for diversification. Therefore, the interesting changes in entropy occur between markets 2, 4 and 5. When moving from Market 2 to 4, the EW, IV, ERC and MD portfolios experienced a decline in their entropy score, while the MV portfolio's entropy improved. Observing the change in entropy when moving from Market 2 to Market 5, the IV and ERC experience improvements and the EW, MV, and MD declines. Furthermore, when moving from Market 4 to 5, the EW, IV, ERC and MD all experience improvements, while the the MV suffers a decline.

5.2.2. Performance Metrics Across Market Types

Table 5.6: Equal Weight

Metric	Market 1	Market 2	Market 3	Market 4	Market 5
Sharp	1.2192	0.0858	0.7753	0.6666	0.7076
SD	-1.0181	-0.0457	-0.6014	-0.3728	-0.5851
Downside Deviation	-1.029	-0.1105	-0.6266	-0.4497	-0.6218
VaR	1.0309	0.0837	0.6304	0.4427	0.6113
Inv HHI	50	50	50	50	50
Entropy	20.362	1.3586	3.2539	1.101	1.1167

The EW portfolio performed better than average across all five market types in the Sharp ratio, SD, DD and VaR metrics. According to the standardized scores in Table 5.6, the EW portfolio performed best in Market 1, followed by Market 3, Market 5 and lastly, Market 2. As previously mentioned, its excellent Market 1 performance is unsurprising due to there being no correlation within the underlying market structure. However, its ability to achieve substantial risk reduction in markets characterized by significant dependence, demonstrates the effectiveness of this simple strategy. Table 5.6 indicates that it exceeds in markets with little or no correlation between assets and maintains its success through a variety of market structures. The EW's inverse HHI is by construction equal to 50 in all markets, while its entropy declined with the number of correlated assets. The decline in entropy is often simply due to a decrease in the scope for diversification and is therefore, experienced across all portfolios.

Table 5.7: Minimum Variance

Metric	Market 1	Market 2	Market 3	Market 4	Market 5
Sharp	-1.0958	-1.4968	-1.7213	-1.7695	-1.7667
SD	1.0325	-0.3068	1.3139	1.1305	0.885
Downside Deviation	1.0541	0.3807	1.4084	1.3636	1.0157
VaR	-1.0528	-0.2563	-1.4008	-1.3346	-0.9931
Inv HHI	27.4705	11.4806	14.0377	12.2394	12.1014
Entropy	21.9552	2.0646	5.3911	2.2467	1.7717

The MV portfolio performs below average across all five markets. In Markets 1, 3 and 4 it scores approximately one standard deviation below average in the Sharp ratio, SD, DD and VaR metrics. Meanwhile, its Sharp ratio is at least one standard deviation below average in all five markets. It performed best in Market 2, however, even then it performed below average in Sharp ratio, DD and VaR. Therefore, across all of the market types the MV portfolio's out-of-sample performance suffered due to estimation error.

Table 5.8: Inverse Volatility

Metric	Market 1	Market 2	Market 3	Market 4	Market 5
Sharp	0.8189	-0.1067	0.4696	0.4245	0.343
SD	-1.0181	-0.5679	-0.7867	-0.8839	-0.8969
Downside Deviation	-1.0039	-0.6632	-0.7604	-0.8486	-0.8913
VaR	0.998	0.633	0.7588	0.8466	0.8891
Inv HHI	47.2773	47.3	47.2838	47.287	47.2623
Entropy	21.2181	1.3726	3.3521	1.1399	1.1482

Across the five markets, the IV portfolio performed above average in SD, DD and VaR. The IV portfolio performed best in Market 1, and worst in Market 2. Its Sharp ratio in Market 2 is the only occasion it performed below average. Overall, the IV portfolio proved to be successful and robust across market types. In general, it seems to strive in markets with little or no correlation between assets, but maintains its success through a variety of market structures.

Table 5.9: Equal Risk Contribution

Metric	Market 1	Market 2	Market 3	Market 4	Market 5
Sharp	-0.1563	0.216	0.4642	0.4233	0.3129
SD	0.0791	-0.7963	-0.7661	-0.8839	-0.6741
Downside Deviation	0.0753	-1.093	-0.7337	-0.8124	-0.6633
VaR	-0.0658	1.0776	0.7354	0.8305	0.674
Inv HHI	35.9119	46.3298	44.9382	46.1096	42.7687
Entropy	21.1598	1.4843	3.3722	1.1407	1.1584

In Markets 2, 3, 4 and 5 the ERC portfolio performed above average in Sharp ratio, SD, DD and VaR. In Market 1 its performance was mediocre, scoring below average in Sharp ratio and VaR and above average in SD and DD. It performed the best from a downside risk mitigation perspective in Market 2 (see DD and VaR). Overall, it seems like Market 1 is not a good match for the ERC, however, across Markets 2, 3, 4 and 5 its relative performance was fairly consistent. Furthermore, its performance seems to improve in markets with high condition numbers. This suggests that the ERC is able to perform well across a variety of markets with significant correlation between their assets.

Table 5.10: Maximum Diversification

Metric	Market 1	Market 2	Market 3	Market 4	Market 5
Sharp	-0.786	1.3016	0.0122	0.2551	0.4031
SD	0.9245	1.7167	0.8403	1.0102	1.2711
Downside Deviation	0.9035	1.486	0.7122	0.7471	1.1608
VaR	-0.9102	-1.538	-0.7237	-0.7852	-1.1813
Inv HHI	29.1668	12.8975	16.211	14.2876	13.2099
Entropy	21.1969	2.3394	4.5327	1.5828	1.6707

From a risk mitigation perspective, the MD portfolio consistently performed well below average. It has a standardized score of greater than 0.7 for SD and DD, and less that -0.7 for VaR. Meanwhile, it managed to achieve above average Sharp ratios in Markets 2, 3, 4 and 5, suggesting that it made returns well above average in these markets. In Markets 1, 3 and 4, the MD portfolio performed quite poorly across all measures. In Market 2, it received the highest standardized Sharp ratio while performing its worst across the SD, DD and VaR measures. In Market 5 it obtained a fairly good Sharp ratio, however, showed poor risk mitigation (see SD, DD and VaR). Overall, it does not seem as though any of the market structures were a particularly good match for the MD portfolio, but it does show potential to achieve high returns.

5.3. Discussion

The MV portfolio continued to perform below average in all market types (Table 5.7). It failed to maintain low out-of-sample volatility and seemed to be particularly bad at mitigating downside risk. It consistently received the lowest Sharp ratio, indicating that, on average, it returned less than the other portfolios. Its low average inverse HHI and high average entropy indicate that it diversifies well across sources of risk by concentrating its holdings in a small number of assets.

The EW and IV portfolios performed above average in all measures and across all market types. Both portfolios performed their best in Market 1 and worst in Market 2. In Markets 3, 4 an 5 the

IV portfolio outperformed the EW from a risk mitigation perspective, however, the EW achieved a higher Sharp ratio (Tables 5.6 and 5.8). Both the EW and IV portfolios exhibit high diversification across assets, however, they fail to diversify across sources of risk. These results indicate that the EW and IV portfolios are successful across a variety of correlation structures.

The ERC portfolio showed above average performance in Markets 2, 3, 4 and 5. It performed its worst in Market 1 and seemed to improve in markets with higher condition numbers. Its performance was very similar to the IV portfolio in Markets 3, 4 and 5, however, it outperformed the IV portfolio in Market 2. The ERC portfolio concentration scores indicate that it diversifies across assets better than what it does across risk sources. It does, however, show improvement over the EW and IV portfolios in its diversification across risk sources.

The MD portfolio performance showed mixed results within markets and was clearly at its worst in Market 1. In Markets 2 and 5, however, it showed great promise in its ability to earn a high Sharp ratio. At the same time it struggled to keep its SD and DD down. This characteristic proved to quite interesting in that it often scored exceptionally well in its Sharp ratio, whilst performing exceedingly poorly in risk mitigation (for example see Market 2 in Table 5.10). It would be interesting to investigate if this high risk, high reward attribute holds in other Monte Carlo environments. The MD portfolio concentration scores suggest that it diversifies well across risk sources and poorly across assets.

Interestingly, these findings seem to be related to those in Ardia *et al.* (2017). Who reported that the MV is sensitive to misspecification in both the variance and covariance, the MD portfolio is sensitive to misspecification in the covariance and the IV and ERC are robust to mispecifcation in both the variance and covariance.

6. Conclusion

This work used Monte Carlo methods to calculate the returns of various risk-based portfolios across a set of five unique market types, thereby enabling the successful and comprehensive comparison of the relative performance of said portfolios across each market type. Overall, the EW and IV portfolios performed well across all market types, while the ERC portfolio also preformed well, but only in markets with significant correlation between constituents. In all market types, the MV portfolio proved unable to maintain low out-of-sample volatility and performed poorly from a total return perspective. This demonstrated its sensitivity to covariance matrix estimation error. The EW and IV portfolios were found to perform above average across all market types. Notably, they performed their best in the markets simulated with the “Diagonal Matrix” correlation structure, and their worst in the markets simulated with the “No Clusters” correlation matrix. Overall, these simple portfolios proved to be the most robust to changes in market type. The ERC portfolio performed poorly in the market with the “Diagonal Matrix” correlation structure and its performance improved in markets with higher condition number correlation matrices. In these markets, the ERC and IV portfolios performed similarly. In all market types, the MD portfolio proved to be unsuccessful in reducing volatility, however, it showed potential to achieve high returns in some market structures. The average inverse HHI and entropy measures showed that the EW, IV and ERC portfolios diversify well across assets and poorly across risk sources. Conversely, the MV and MD portfolios diversify well across risk sources and poorly across assets. These results indicate the potential for incorporating simple heuristic approaches, like those of the EW and IV portfolios, in the development of future risk-based portfolio algorithms.

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Appendix

Table 6.1: Asset Means and Sd's

Asset	Mean	Sd	Asset...	Mean...	Sd...
Asset_1	-0.00011	0.03277	Asset_26	0.00247	0.02671
Asset_2	0.00023	0.02314	Asset_27	-0.00132	0.04888
Asset_3	-3e-05	0.02359	Asset_28	0.00085	0.02387
Asset_4	-0.0018	0.03654	Asset_29	0.00024	0.01613
Asset_5	-0.00036	0.02925	Asset_30	0.00086	0.01884
Asset_6	0.00052	0.02396	Asset_31	0.00118	0.03147
Asset_7	0.00058	0.02451	Asset_32	0.00053	0.02882
Asset_8	0.00099	0.02287	Asset_33	0.00166	0.02253
Asset_9	0.00057	0.02929	Asset_34	0.00179	0.03002
Asset_10	0.00049	0.01736	Asset_35	0.00235	0.02574
Asset_11	-0.00088	0.01907	Asset_36	0.00157	0.02307
Asset_12	5e-05	0.0301	Asset_37	0.0017	0.02566
Asset_13	-0.00095	0.03185	Asset_38	0.00157	0.02262
Asset_14	-1e-04	0.02111	Asset_39	0.00054	0.03116
Asset_15	0.00102	0.02555	Asset_40	0.00242	0.02494
Asset_16	0.00174	0.02582	Asset_41	0.00246	0.02186
Asset_17	0.00241	0.03639	Asset_42	0.00113	0.01951
Asset_18	0.00033	0.03967	Asset_43	0.00181	0.02099
Asset_19	0.00102	0.02636	Asset_44	-0.00045	0.02297
Asset_20	0.00173	0.02697	Asset_45	4e-05	0.02354
Asset_21	-0.00034	0.01405	Asset_46	0.00485	0.03553
Asset_22	0.0011	0.0233	Asset_47	0.00064	0.02275
Asset_23	0.00236	0.02663	Asset_48	1e-04	0.02713
Asset_24	-0.00055	0.03195	Asset_49	0.00071	0.02186
Asset_25	-0.00283	0.03227	Asset_50	-0.00203	0.03131