

Undergraduate Topology
Robert H. Kasriel (Dover Publication)
Solutions to exercises
Part I
Chapters I to IV

Bernard Carrette

May 1, 2023



Figure 1

Remarks and warnings

You're welcome to use these notes, but they may contain errors, so proceed with caution : I graduated in 1979, went straight in the industry (where I didn't have to use fancy maths), and picked mathematics and physics again after I retired, so my mathematics got rusty for sure. If you do find an error, typo's , I'd be happy to receive bug reports, suggestions, and the like, through Github.

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Sets, Functions, and Relations

1.1 Sets and Membership

1.1.1

List explicitly the elements of the set

$$\{x : x < 0 \text{ and } (x-1)(x+2)(x+3) = 0\}$$

$$\{-3, -2\}$$



1.1.2

List the elements of the set

$$\{x : 3x - 1 \text{ is a multiple of } 3\}$$

$$\{x : x = k + \frac{1}{3}, k \in \mathbb{Z}\}$$



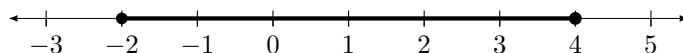
1.1.3

Sketch on a number line each of the following sets.

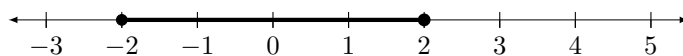
(a) $\{x : |x - 1| \leq 3\}$

(b) $\{x : |x - 1| \leq 3 \text{ and } |x| \leq 2\}$

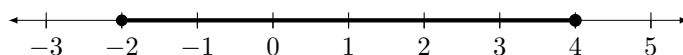
(c) $\{x : |x - 1| \leq 3 \text{ or } |x| \leq 2\}$



(a)



(b)



(c)



1.2 Some remarks on the use of the connectives *and*, *or*, *implies*

1.2.1

Demonstrate by means of a table showing truth values that the following is a true statement for any choice of p and q . Thus show that it is a tautology.

$$(\neg q \Rightarrow \neg p) \Rightarrow (p \Rightarrow q)$$

p	q	$\neg q$	$\neg p$	$\neg q \Rightarrow \neg p$	$p \Rightarrow q$	$(\neg q \Rightarrow \neg p) \Rightarrow (p \Rightarrow q)$
T	T	F	F	T	T	T
T	F	T	F	F	F	T
F	T	F	T	T	T	T
F	F	T	T	T	T	T



1.2.2

Show by means of a truth table that the statement

$$((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$$

is a tautology.

p	q	r	$p \Rightarrow q$	$q \Rightarrow r$	$(p \Rightarrow q) \wedge (q \Rightarrow r)$	$p \Rightarrow r$	$((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T



1.2.3

Show by means of a truth table that

$$(p \wedge q) \Rightarrow (p \vee q)$$

is a tautology.

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \Rightarrow (p \vee q)$
T	T	T	T	T
T	F	F	F	T
F	T	F	T	T
F	F	F	F	T



1.2.4

Suppose that p and q are statements such that $(p \wedge q)$ is a false statement. Does it follow that the statement

$$(p \text{ is false}) \vee (q \text{ is false})$$

is a true statement?

p	q	$p \wedge q$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	F	F	F	T	T
F	T	F	T	F	T
F	F	F	T	T	T

The answer is Yes.



1.2.5

Negate the following statement: *If two angles of a triangle have equal measure, then the length of two sides of that triangle are equal.*

First we note that $\neg(p \Rightarrow q) \Leftrightarrow (p \wedge \neg q)$. Indeed,

p	q	$p \Rightarrow q$	$\neg(p \Rightarrow q)$	$\neg q$	$p \wedge \neg q$	$\neg(p \Rightarrow q) \Leftrightarrow (p \wedge \neg q)$
T	T	T	F	F	F	T
T	F	F	T	T	T	T
F	T	T	F	F	F	T
F	F	T	F	T	F	T

Putting p as *two angles of a triangle have equal measure* and $\neg q$ as *no two sides of that triangle have equal length* we get the true 'false' statement:

Two angles of a triangle have equal measure \wedge no two sides of that triangle have equal length.



1.2.6

Write the contrapositive of the statement in Exercise 5.

The contrapositive of $p \Rightarrow q$ is $\neg q \Rightarrow \neg p$. Putting $\neg p$ as *no two angles of a triangle have equal measure* and $\neg q$ as *no two sides of that triangle have equal length* we get

If no two sides of that triangle have equal length then no two angles of a triangle have equal measure.



1.2.7

Write the converse of the statement in Exercise 5.

The converse of $p \Rightarrow q$ is $q \Rightarrow p$, giving

If two sides of a triangle have equal length then two angles of a that triangle have equal measure.



1.2.8

Write the contrapositive of the following statement

If a person belongs to Committee A, then he must be a member of Committee B and he must be a member of Committee C.

Lets put

$p \equiv$ a person belongs to Committee A

$q \equiv$ a person belongs to Committee B

$r \equiv$ a person belongs to Committee C

then the given statement translates as

$$p \Rightarrow (q \wedge r)$$

and the contrapositive

$$\neg(q \wedge r) \Rightarrow \neg p$$

This last statement is equivalent with

$$(\neg q \vee \neg r) \Rightarrow \neg p$$

or in plain text:

If a person does not belong to Committee B or C , then he is not a member of Committee A.



1.2.9

Write the contrapositive of the following statement

If $x \in A$ and $x \in B$, then $x \in C$

Lets put

$$p \equiv x \in A$$

$$q \equiv x \in B$$

$$r \equiv x \in C$$

then the given statement translates as

$$p \wedge q \Rightarrow r$$

and the contrapositive

$$\neg(r) \Rightarrow \neg(p \wedge q)$$

This last statement is equivalent with

$$\neg(r) \Rightarrow (\neg p \vee \neg q)$$

i.e:

$$x \notin C \Rightarrow (x \notin A \vee x \notin B)$$



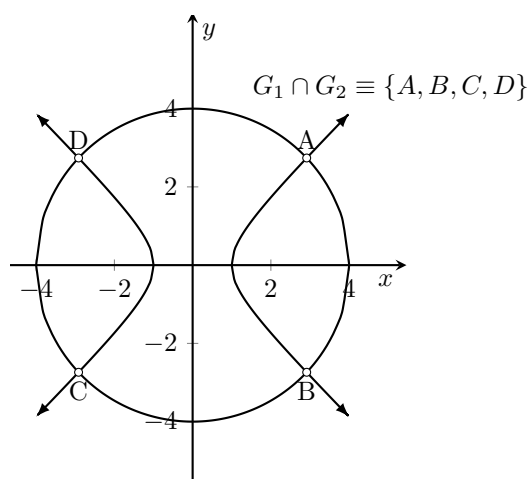
1.3 Subsets

No exercises!

1.4 Union and Intersection of sets

1.4.1

Let G_1 be the graph of the equation $x^2 + y^2 = 16$, and let G_2 be the graph of the equation $x^2 - y^2 = 1$. Sketch the sets $G_1 \cup G_2$ and $G_1 \cap G_2$.



$G_1 \cup G_2$ contains all the points defined by the graphs G_1 and G_2 . $G_1 \cap G_2 \equiv \{A, B, C, D\}$ contains the 4 points at the intersection of the two graphs.



1.4.2

We define the sets A , B , C as follows: $A = \{(x, y) : x^2 + y^2 \leq 9\}$, $B = \{(x, y) : x + y \geq 3\}$, $C = \{(x, y) : x \geq 0\}$.

Draw sketches of each of the following sets:

- (a) $A \cup (B \cup C)$
- (b) $A \cap (B \cup C)$
- (c) $(A \cap B) \cup (A \cap C)$
- (d) $(A \cup B) \cup C$
- (e) $A \cup (B \cap C)$
- (f) $(A \cup B) \cap (A \cup C)$

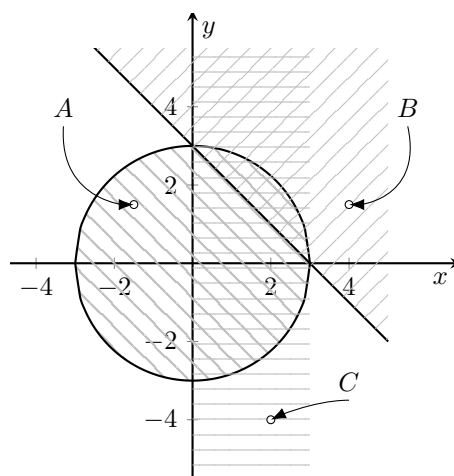
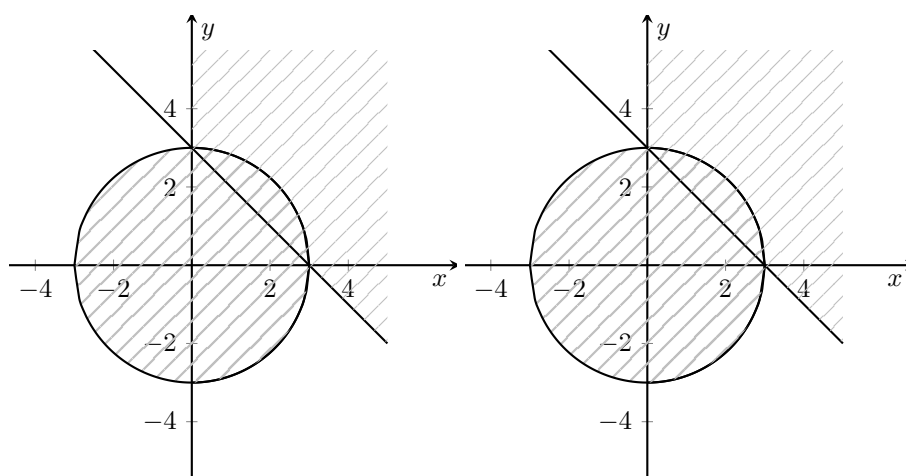
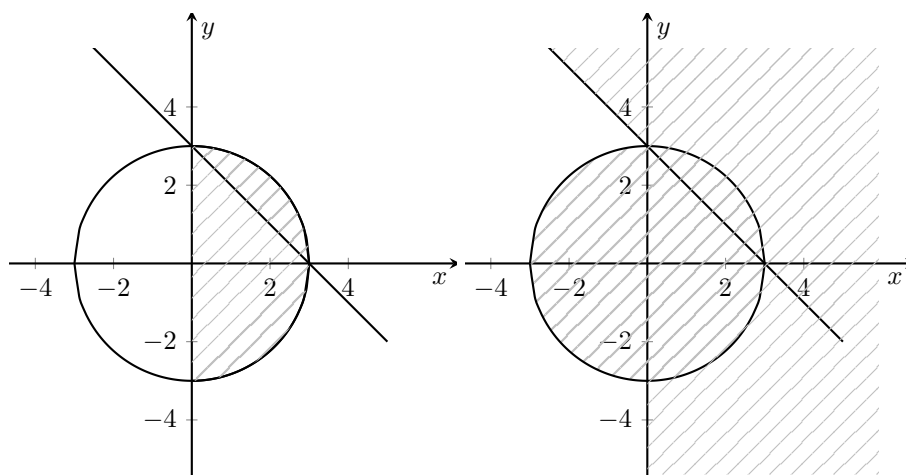
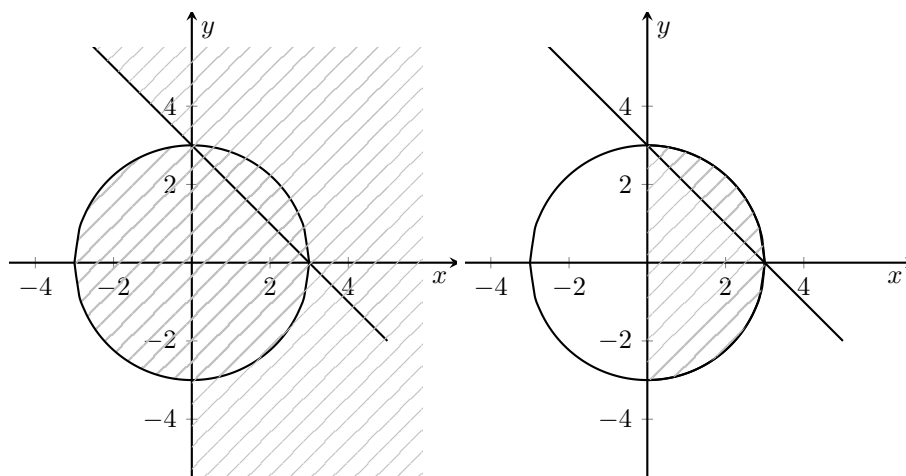


Figure 1.1: The 3 sets A , B , C



1.4.3

Let A , B , C as follows: $A = \{(x, y) : x + y \leq 5\}$, $B = \{(x, y) : x + y \geq 3\}$, $C = \{(x, y) : x \geq 3\}$, and $D = \{(x, y) : y \geq 3\}$.

Draw a sketch for each of the following sets:

- (a) $(A \cap B) \cap C$
 (b) $[(A \cap B) \cap C] \cap D$

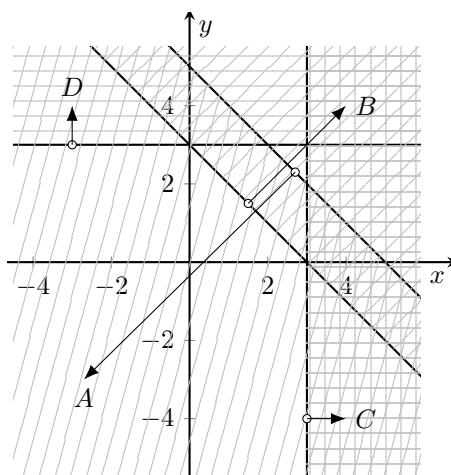
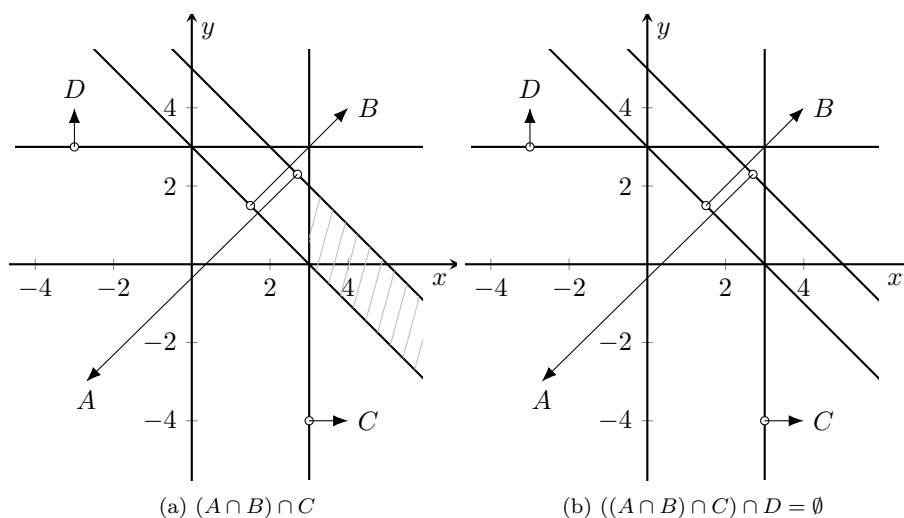


Figure 1.2: The 4 sets A , B , C , D



1.5 Complementation

1.5.1

Sketch each of the following sets: (the sets A , B , C are defined as in exercise 3page 8)

- (a) $\sim (A \cap B)$
- (b) $(\sim A) \cup (B)$
- (c) $\sim (A \cup B)$
- (d) $(\sim A) \cap (B)$
- (e) $C - A$
- (f) $\sim (A \cap C)$
- (g) $(\sim A) \cup (\sim B)$
- (h) $(\sim A) \cap (A)$
- (i) $C - (A \cup B)$
- (j) $(C - A) \cap (C - B)$
- (k) $\sim (\sim A)$

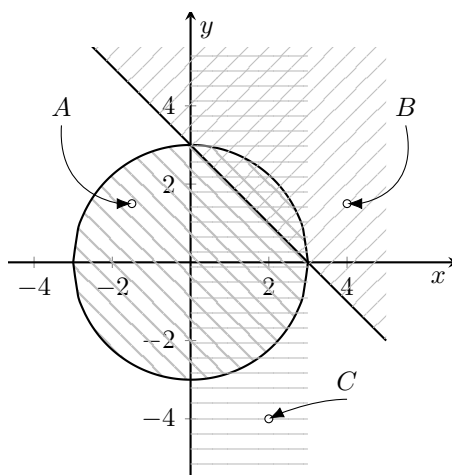
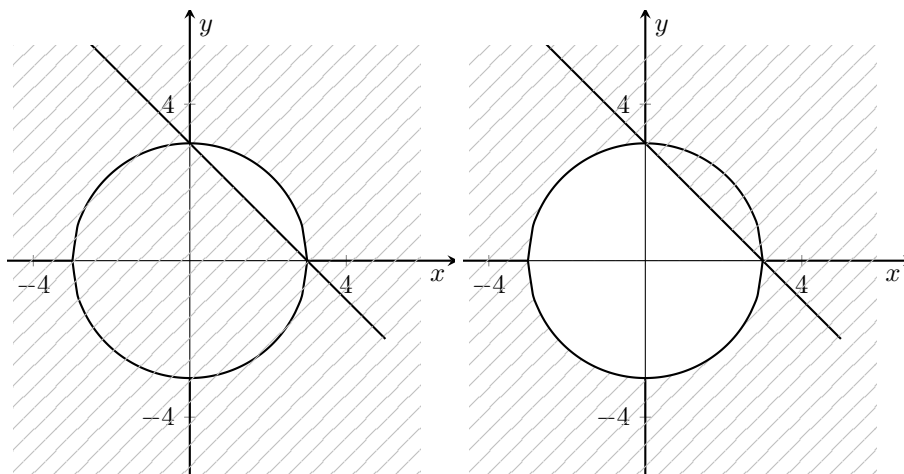
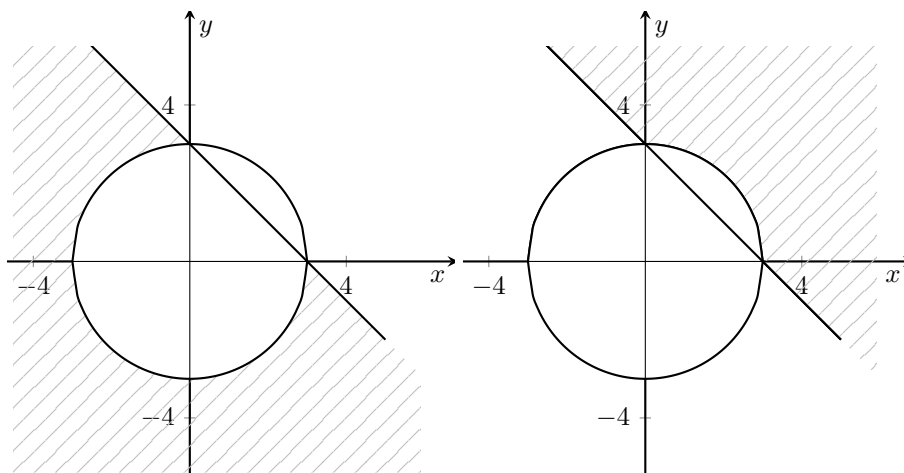


Figure 1.3: The 3 sets A , B , C



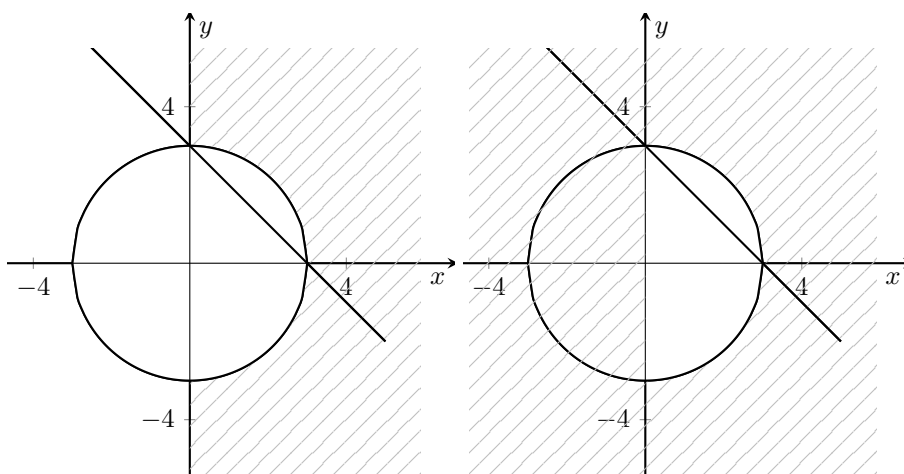
(a) $\sim (A \cap B)$

(b) $(\sim A) \cup (B)$



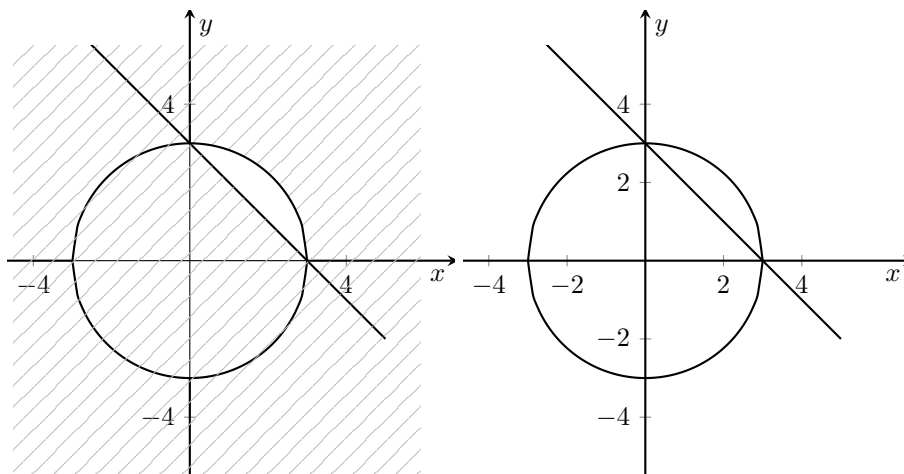
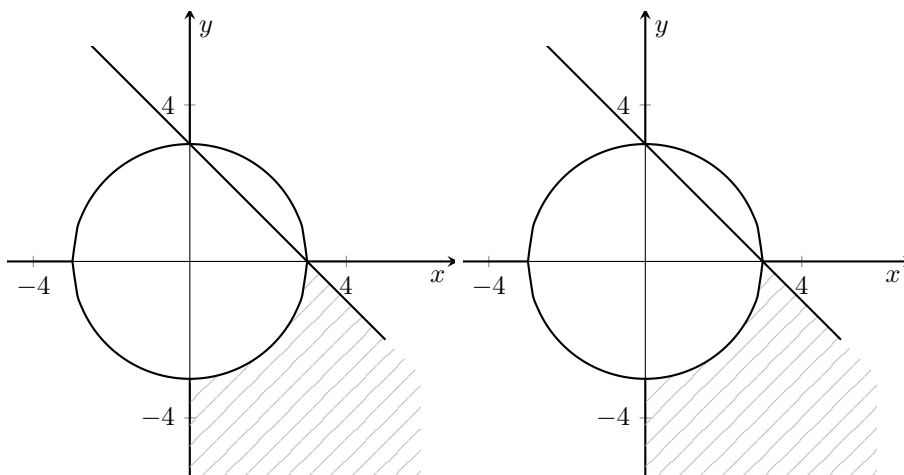
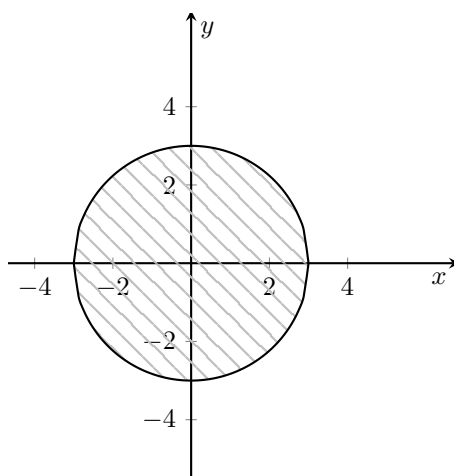
(c) $\sim (A \cup B)$

(d) $(\sim A) \cap (B)$



(e) $C - A$

(f) $\sim (A \cap C)$

(g) $(\sim A) \cup (\sim B)$ (h) $(\sim A) \cap (A) = \emptyset$ (i) $C - (A \cup B)$ (j) $(C - A) \cap (C - B)$ (k) $\sim(\sim A)$ 

1.5.2

On the basis of the sketches made in the previous exercise, formulate a proposition about relation that exist concerning complementation, union, and intersection. Try out your conjecture on other examples. In subsequent exercises you will be asked to try to prove such conjectures.

$$1.4.2(a) \text{ and } (d) \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$1.4.2(b) \text{ and } (c) \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$1.4.2(e) \text{ and } (f) \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$1.5.1(a) \text{ and } (g) \quad \sim (A \cap B) = (\sim A) \cup (\sim B)$$

$$1.5.1(h) \quad (\sim A) \cap A = \emptyset$$

$$1.5.1(i) \text{ and } (j) \quad C - (A \cup B) = (C - A) \cap (C - B)$$

$$1.5.1(k) \quad \sim (\sim A) = A$$



1.6 Set identities and other set relations

1.6.1

Prove that if $A \subset B$, then:

$$(a) \quad A \cap C \subset B \cap C$$

$$(b) \quad \sim B \subset \sim A$$

$$(c) \quad A \cap B = A$$

$$(d) \quad A \cup C \subset B \cup C$$

a) $A \cap C \subset B \cap C$

Given is $x \in B$ if $x \in A$. Suppose $x \in A \cap C$, then $x \in A$ (given) and $x \in C$ but $x \in B$ (given) and as $x \in C$ follows that $x \in B \cap C$. And we conclude that $A \cap C \subset B \cap C$.

◇

b) $\sim B \subset \sim A$

Given is $x \in B$ if $x \in A$. If $x \notin B$ then $x \in \sim B$. As $A \subset B$, x will not be in A but $x \in \sim A$. So $x \in \sim B \Rightarrow x \in \sim A$ and thus $\sim B \subset \sim A$.

◇

c) $A \cap B = A$

Given is $x \in B$ if $x \in A$. Suppose $x \in A \cap B$, then $x \in A$ and thus $A \cap B \subset A$. Suppose $x \in A$, then $x \in B$ as $A \subset B$ and thus $x \in A \cap B$ from which we conclude $A \subset A \cap B$.

◇

d) $A \cup C \subset B \cup C$

Given is $x \in B$ if $x \in A$. Suppose $x \in A \cup C$, then $x \in A$ or $x \in C$. But $x \in B$ (given), so $x \in B$ or $x \in C$ and thus $x \in B \cup C$, from which we conclude $A \cup C \subset B \cup C$.

◆

1.6.2

Verify that each of the following is an identity:

- (a) $A \cup \emptyset = A$
- (b) $A \cap \emptyset = \emptyset$
- (c) $A \cap A = A$
- (d) $A \cup A = A$
- (e) $(A \cup B) \cup C = A \cup (B \cup C)$
- (f) $(A \cap B) \cap C = A \cap (B \cap C)$
- (g) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- (h) $X - (A \cup B) = (X - A) \cap (X - B)$
- (i) $A \cap \sim A = \emptyset$
- (j) $A \cup \sim A = U$

a) $A \cup \emptyset = A$

This is a consequence of remark 3.3 page 7: the empty set \emptyset is a subset of every set. So, $\emptyset \subset A$ giving the asked identity.

◇

b) $A \cap \emptyset = \emptyset$

If $x \in A \cap \emptyset$ then $x \in A$ and x must also be in \emptyset which is impossible by definition. So there is no element $x \in \emptyset$ which can satisfy $x \in A \cap \emptyset$ giving the proposed identity.

◇

c) $A \cap A = A$

Suppose $x \in A \cap A$, then $x \in A$ and $x \in A$ and thus $x \in A$, giving $A \cap A \subset A$. Suppose $x \in A$, then obviously $x \in A$ and $x \in A$, giving $A \subset A \cap A$. Hence $A \cap A = A$

◇

d) $A \cup A = A$

Suppose $x \in A \cup A$, then $x \in A$ or $x \in A$ and thus $x \in A$, giving $A \cup A \subset A$. Suppose $x \in A$, then obviously $x \in A$ or $x \in A$, giving $A \subset A \cup A$. Hence $A \cup A = A$

◇

e) $(A \cup B) \cup C = A \cup (B \cup C)$

Suppose $x \in (A \cup B) \cup C$, then $x \in (A \cup B)$ or $x \in C$ and thus $x \in A$ or $x \in B$ or $x \in C$. So $x \in B$ or $x \in C$ can be written as $x \in (B \cup C)$. So $x \in A$ or $x \in (B \cup C)$, giving $(A \cup B) \cup C \subset A \cup (B \cup C)$. The same reasoning yields for $x \in A \cup (B \cup C)$ giving the identity.

◇

f) $(A \cap B) \cap C = A \cap (B \cap C)$

Suppose $x \in (A \cap B) \cap C$, then $x \in (A \cap B)$ and $x \in C$ and thus $x \in A$ and $x \in B$ and $x \in C$. So $x \in B$ and $x \in C$ can be written as $x \in (B \cap C)$. So $x \in A$ and $x \in (B \cap C)$, giving $(A \cap B) \cap C \subset A \cap (B \cap C)$. The same reasoning yields for $x \in A \cap (B \cap C)$ giving the identity.

◇

g) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Suppose $x \in A \cup (B \cap C)$, then $x \in A$ or $x \in (B \cap C)$. Take the case $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$ which implies $x \in (A \cup B) \cap (A \cup C)$, giving $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$. The other case: if $x \in B \cap C$ then $x \in B$ and $x \in C$. So, $x \in A \cup B$ and $x \in A \cup C$ giving also $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$.

On the other hand, be $x \in (A \cup B) \cap (A \cup C)$ then $x \in (A \cup B)$ and $x \in (A \cup C)$. Let's first take the case $x \in A$ then obviously $x \in A \cup (B \cap C)$ even if $x \notin B \cap C$. Alternatively, be $x \notin A$ then we must have $x \in B$ and $x \in C$ which implies $x \in B \cap C$, giving again $x \in A \cup (B \cap C)$.

◇

h) $X - (A \cup B) = (X - A) \cap (X - B)$

Suppose $x \in X - (A \cup B)$, then $x \notin A$ and $x \notin B$ which implies $x \in X - A$ and $x \in X - B$ and thus $x \in X - A \cap X - B$ giving $X - (A \cup B) \subset (X - A) \cap (X - B)$.

The other way around. Suppose $x \in (X - A) \cap (X - B)$. Then $x \in (X - A)$ and $x \in (X - B)$ which implies $x \notin A$ and $x \notin B$ giving $x \notin A \cup B$ which in turn implies $x \in X - (A \cup B)$ giving $(X - A) \cap (X - B) \subset X - (A \cup B)$.

Conclusion: $X - (A \cup B) = (X - A) \cap (X - B)$

◇

i) $A \cap \sim A = \emptyset$

Suppose $x \in A \cap \sim A$, then $x \in A$ and $x \notin A$ which is a contradiction, so the only element which is always an element of any set is the empty set, so $A \cap \sim A \subset \emptyset$. Suppose on the contrary that $x \in \emptyset$. This implies that x correspond to the empty set and as the empty set is an element of

any set, we have $\emptyset \subset A \cap \sim A$

◇

j) $A \cup \sim A = U$

Suppose $x \in A \cup \sim A$, then $x \in A$ or $x \notin A$. So, in any case $x \in U$ and thus $A \cup \sim A \subset U$.

On the opposite way suppose that $x \in U$. Then obviously $x \in A$ or $x \in \sim A$ and thus $U \subset A \cup \sim A$.

◆

1.6.3

Prove that if $A \subset C$ and $B \subset C$, then $A \cup B \subset C$.

Given is $A \subset C$ and $B \subset C$. Take $x \in A$, then $x \in C$, so even if $x \notin B$, then $x \in A \cup B$ reduces to $x \in A$ and thus $x \in C$. The same reasoning yields for $x \in B$, giving $A \cup B \subset C$.

◆

1.6.4

Prove that if $A \subset B$ and $A \subset C$, then $A \subset B \cap C$.

Given is $A \subset B$ and $A \subset C$. Take $x \in A$, then $x \in C$ and $x \in B$, which implies $x \in C \cap B$. giving indeed $A \subset B \cap C$.

◆

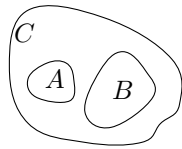
1.7 Counterexamples

In each of the following exercises state whether the statement is necessarily true. Assume that A , B and C are subsets of a universal set U . Justify with a proof or a counterexample.

1.7.1

If $A \cup C = B \cup C$, then $A = B$

Not TRUE.



(I) $A \cup C = B \cup C \not\Rightarrow A = B$

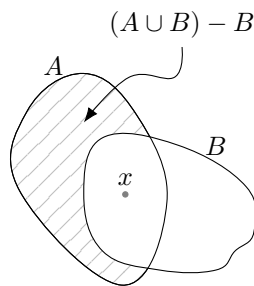
Be $A \subset C$ and $B \subset C$, then we have $A \cup C = B \cup C \equiv C = C$ even if $A \cap B = \emptyset$.



1.7.2

$(A \cup B) - B = A$

Not TRUE.



(m) $(A \cup B) - B \neq A$

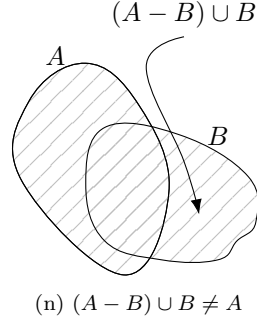
Be $A \cap B \neq \emptyset$, take $x \in A$ and $x \in B$, then x can't be $x \in (A \cup B) - B$ although it is an element of A .



1.7.3

$$(A - B) \cup B = A$$

Not TRUE.



This is only true if $B \subset A$



1.7.4

$$\sim (A - B) = \sim (A \cap \sim B)$$

TRUE.

Suppose first that A and B are disjoint, i.e. $A \cap B = \emptyset$, then $A - B = A$ and $\sim (A - B) = \sim A$. On the other hand $A \subset \sim B$, so $A \cap \sim B = A$, giving $\sim (A \cap \sim B) = \sim A$, giving indeed $\sim (A - B) = \sim (A \cap \sim B)$.

Suppose now that A and B are not disjoint, i.e. $A \cap B \neq \emptyset$. Be $x \in A - B \subset A$. This is equivalent with the statement $x \in A \wedge x \notin B$. Negating this statement: $\neg(x \in A \wedge x \notin B) \Leftrightarrow x \notin A \vee x \in B$. This give $\sim (A - B) \equiv x \notin A \vee x \in B$.

Be now $x \in A \cap \sim B$. This is equivalent with the statement $x \in A \wedge x \notin B$. Negating this statement: $\neg(x \in A \cap \sim B) \Leftrightarrow x \notin A \vee x \in B$. This give $\sim (A \cap \sim B) \equiv x \notin A \vee x \in B$, resulting in $\sim (A - B) = \sim (A \cap \sim B)$.



1.7.5

$$\sim (\sim (\sim A)) = \sim A$$

TRUE.

Be $x \in \sim (\sim (\sim A))$. This is equivalent to $x \notin \sim (\sim A)$. Which on it's turn is equivalent with $x \in \sim A$. So, $\sim (\sim (\sim A)) \subset \sim A$.

Be $x \in \sim A$. This is equivalent to $x \notin \sim (\sim A)$. Which on it's turn is equivalent with $x \in \sim (\sim (\sim A))$. So, $\sim A \subset \sim (\sim (\sim A))$.

Both cases reduce to $\sim (\sim (\sim A)) = \sim A$.



1.7.6

$$A \cup (B - C) = (A \cup B) - C$$

Not TRUE.

Be $x \in A \cup (B - C)$. This is equivalent to $x \in A \vee x \in (B - C)$. Suppose $x \in A$, then $x \in A \cup B$. Let's consider the set C so that $(A \cup B) \subset C$, then $(A \cup B) - C = \emptyset$. We get a contradiction and the proposed statement is not true.



1.7.7

$$\sim (A - B) = (\sim A) \cup B$$

TRUE.

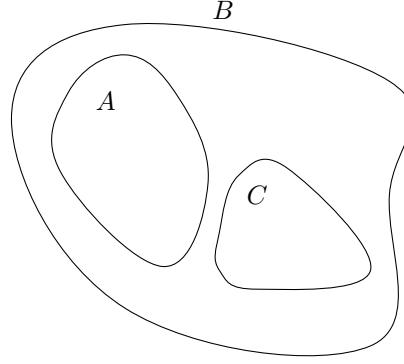
Be $x \in (A - B)$. This is equivalent to $x \in A \wedge x \notin B$. Negating this statement: $\neg(x \in A \wedge x \notin B) \Leftrightarrow x \notin A \vee x \in B$. This is equivalent to the statement $x \in (\sim A) \cup B$. So $\sim (A - B) \subset (\sim A) \cup B$. Consider now $x \in (\sim A) \cup B$. So $x \notin A \vee x \in B$. If we have the case $x \notin A$ then also $x \notin (A - B)$ as x can not be one of the remaining elements of A after the complement of B relative to A . Also, if $x \in B$ then also $x \notin (A - B)$ as x is an element of B and thus can not be an element of $(A - B)$. Thus, in both cases we have, $x \notin (A - B)$ which implies $x \in \sim (A - B)$. So $(\sim A) \cup B \subset \sim (A - B)$.



1.7.8

$$\text{If } A - B = C - B, \text{ then } A = C.$$

Not TRUE.



(o) If $A - B = C - B \not\Rightarrow A = C$

Suppose $A \subset B$, then $A - B = \emptyset$. Choose a C such that $C \subset B$ and also $A \cap C = \emptyset$, then also $C - B = \emptyset$ and get $A - B = C - B$ although $A \neq C$.



1.7.9

If $A - (B \cap C) = (A - B) \cap (A - C)$.

TRUE.

Suppose $x \in A - (B \cap C)$, then $x \in A \wedge x \notin B \cap C$. As x can not be simultaneously in B and C , then also x must be simultaneously in $A - B$ and $A - C$ as the "complementation of A with B and C will not "subtract" x out of A , and considering that $x \in A$ we have $A - (B \cap C) \subset (A - B) \cap (A - C)$. Suppose $x \in (A - B) \cap (A - C)$, then x must be an element of A but not an element of B and C . This means that $x \notin B \cap C$ and thus the complementation of A by $B \cap C$ has no effect on x . Thus, $(A - B) \cap (A - C) = A \subset A - (B \cap C)$.