Undergraduate Topology

Robert H. Kasriel (Dover Publication)

Solutions to exercises

Part I

Chapters I to IV

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Status: DRAFT

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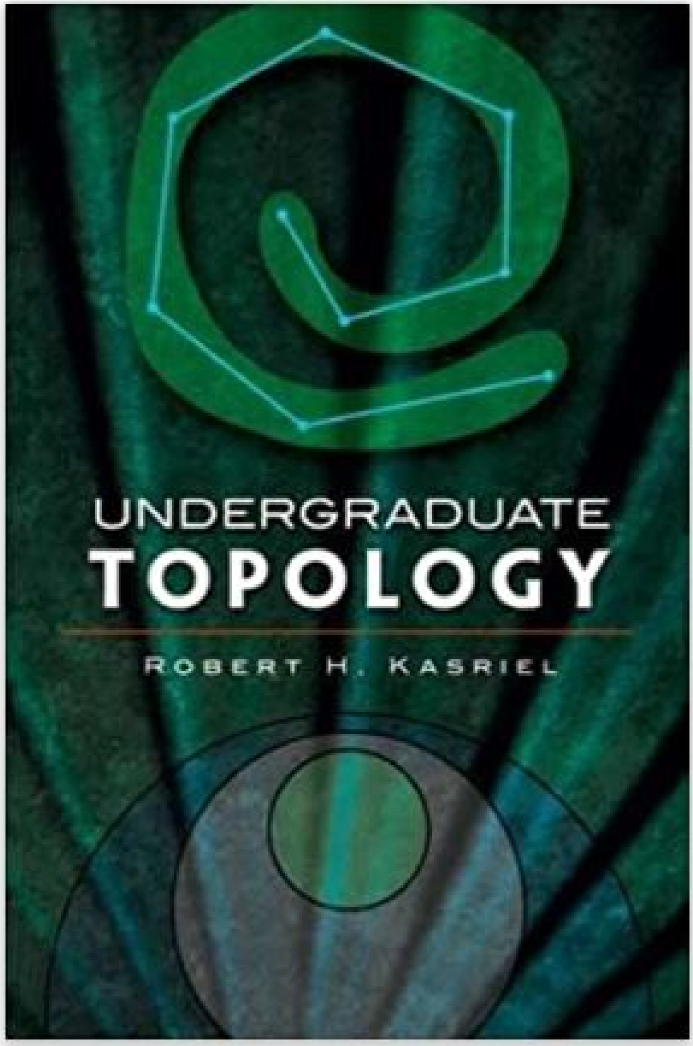


Figure 1

Remarks and warnings

You're welcome to use these notes, but they may contain errors, so proceed with caution : I graduated in 1979, went straight in the industry (where I didn't have to use fancy maths), and picked mathematics and physics again after I retired, so my mathematics got rusty for sure. If you do find an error, typo's , I'd be happy to receive bug reports, suggestions, and the like, through Github.

Contents

List of Figures

Sets, Functions, and Relations

1.1 Sets and Membership

1.1.1

List explicitly the elements of the set

$$\{x : x < 0 \text{ and } (x-1)(x+2)(x+3) = 0\}$$

$$\{-3, -2\}$$

♦

1.1.2

List the elements of the set

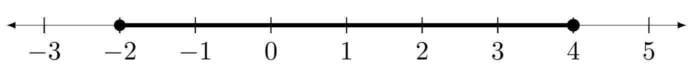
$$\{x : 3x - 1 \text{ is a multiple of } 3\}$$

$$\{x : x = k + \frac{1}{3}, k \in \mathbb{Z}\}$$

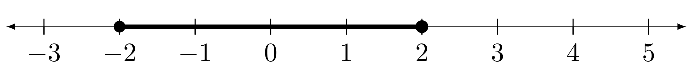
♦

1.1.3

Sketch on a number line each of the following sets.
(a) $\{x : |x-1| \leq 3\}$
(b) $\{x : |x-1| \leq 3 \text{ and } |x| \leq 2\}$
(c) $\{x : |x-1| \leq 3 \text{ or } |x| \leq 2\}$



(a)



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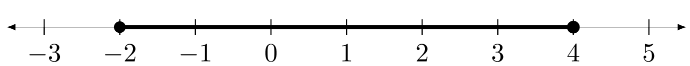


(c)

♦

1.2 Some remarks on the use of the connectives *and, or, implies*

1.2.1

Demonstrate by means of a table showing truth values that the following is a true statement for any choice of p and q. Thus show that it is a tautology.

$$(\neg q \Rightarrow \neg p) \Rightarrow (p \Rightarrow q)$$

p	q	$\neg q$	$\neg p$	$\neg q \Rightarrow \neg p$	$p \Rightarrow q$	$(\neg q \Rightarrow \neg p) \Rightarrow (p \Rightarrow q)$
T	T	F	F	T	T	T
T	F	T	F	F	F	T
F	T	F	T	T	T	T
F	F	T	T	T	T	T

♦

1.2.2

Show by means of a truth table that the statement

$$((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$$

is a tautology.

p	q	r	$p \Rightarrow q$	$q \Rightarrow r$	$(p \Rightarrow q) \wedge (q \Rightarrow r))$	$p \Rightarrow r$	$((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

♦

1.2.3

Show by means of a truth table that

$$(p \wedge q) \Rightarrow (p \vee q)$$

is a tautology.

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \Rightarrow (p \vee q)$
T	T	T	T	T
T	F	F	F	T
F	T	F	T	T
F	F	F	F	T

♦

1.2.4

Suppose that p and q are statements such that $(p \wedge q)$ is a false statement. Does it follow that the statement

$$(p \text{ is false}) \vee (q \text{ is false})$$

is a true statement?

p	q	$p \wedge q$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	F	F	F	T	T
F	T	F	T	F	T
F	F	F	T	T	T

The answer is Yes.

♦

1.2.5

Negate the following statement: *If two angles of a triangle have equal measure, then the length of two sides of that triangle are equal.*

First we note that $\neg(p \Rightarrow q) \Leftrightarrow (p \wedge \neg q)$. Indeed,

p	q	$p \Rightarrow q$	$\neg(p \Rightarrow q)$	$\neg q$	$p \wedge \neg q$	$\neg(p \Rightarrow q) \Leftrightarrow (p \wedge \neg q)$
T	T	T	F	F	F	T
T	F	F	T	T	T	T
F	T	T	F	F	F	T
F	F	T	F	T	F	T

Putting p as *two angles of a triangle have equal measure* and $\neg q$ as *no two sides of that triangle have equal length* we get the true 'false' statement:
Two angles of a triangle have equal measure $\wedge$ no two sides of that triangle have equal length.

♦

1.2.6

> Write the contrapositive of the statement in Exercise 5.

The contrapositive of $p \Rightarrow q$ is $\neg q \Rightarrow \neg p$. Putting $\neg p$ as *no two angles of a triangle have equal measure* and $\neg q$ as *no two sides of that triangle have equal length* we get
If no two sides of that triangle have equal length then no two angles of a triangle have equal measure.

♦

1.2.7

> Write the converse of the statement in Exercise 5.

The converse of $p \Rightarrow q$ is $q \Rightarrow p$, giving
If two sides of a triangle have equal length then two angles of a that triangle have equal measure.

♦

1.2.8

> Write the contrapositive of the following statement
> *If a person belongs to Committee A, then he must be a member of Committee B and he must be a member of Committee C.*

Lets put

$$p \equiv \text{a person belongs to Committee A}$$
$$q \equiv \text{a person belongs to Committee B}$$
$$r \equiv \text{a person belongs to Committee C}$$

then the given statement translates as

$$p \Rightarrow (q \wedge r)$$

and the contrapositive

$$\neg(q \wedge r) \Rightarrow \neg p$$

This last statement is equivalent with

$$(\neg q \vee \neg r) \Rightarrow \neg p$$

or in plain text:

If a person does not belong to Committee B or C , then he is not a member of Committee A.

♦

1.2.9

Write the contrapositive of the following statement

If $x \in A$ and $x \in B$, then $x \in C$

Lets put

$$p \equiv x \in A$$
$$q \equiv x \in B$$
$$r \equiv x \in C$$

then the given statement translates as

$$p \wedge r \Rightarrow r$$

and the contrapositive

$$\neg(r) \Rightarrow \neg(p \wedge q)$$

This last statement is equivalent with

$$\neg(r) \Rightarrow (\neg p \vee \neg q)$$

i.e:

$$x \notin C \Rightarrow (x \notin A \vee x \notin B)$$

♦

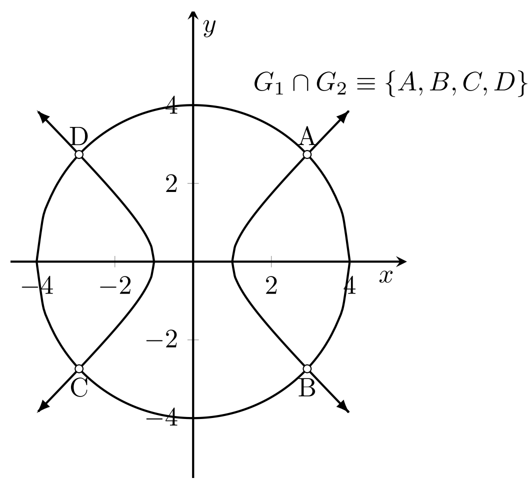
1.3 Subsets

No exercises!

1.4 Union and Intersection of sets

1.4.1

Let G_1 be the graph of the equation $x^2 + y^2 = 16$, and let G_2 be the graph of the equation $x^2 - y^2 = 1$. Sketch the sets $G_1 \cup G_2$ and $G_1 \cap G_2$.



$G_1 \cup G_2$ contains all the points defined by the graphs G_1 and G_2. $G_1 \cap G_2 \equiv \{A, B, C, D\}$ contains the 4 points at the intersection of the two graphs.

♦

1.4.2

We define the sets A, B, C as follows: $A = \{(x, y) : x^2 + y^2 \leq 9\}$, $B = \{(x, y) : x + y \geq 3\}$, $C = \{(x, y) : x \geq 0\}$.
Draw sketches of each of the following sets:

$$
\begin{array}{ll}
(a) & A \cup (B \cup C) \\
(b) & A \cap (B \cup C) \\
(c) & (A \cap B) \cup (A \cap C) \\
(d) & (A \cup B) \cup C \\
(e) & A \cup (B \cap C) \\
(f) & (A \cup B) \cap (A \cup C)
\end{array}
$$

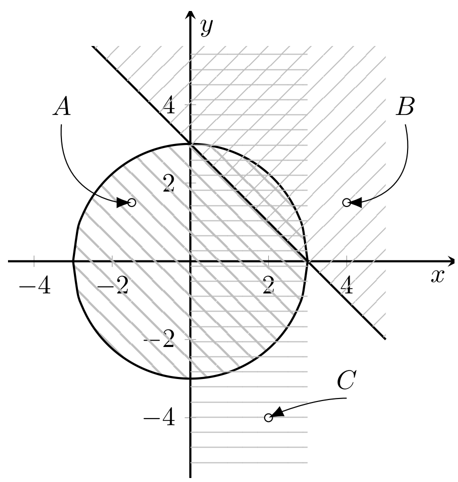
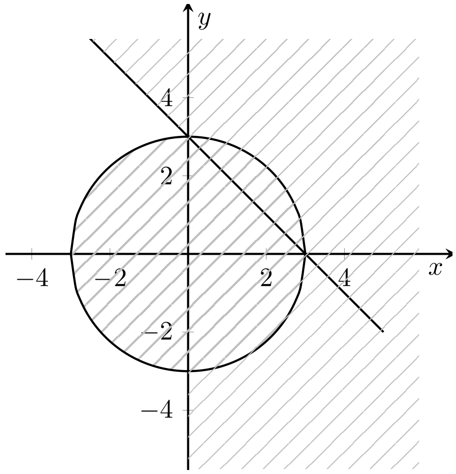
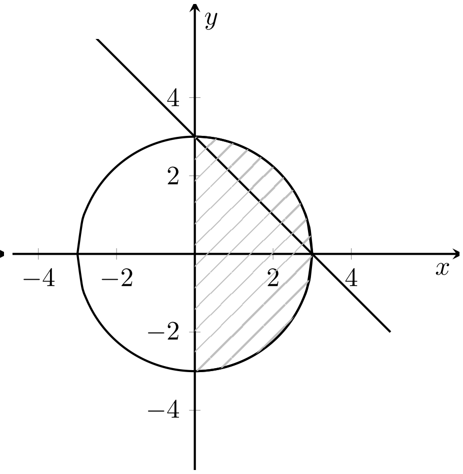


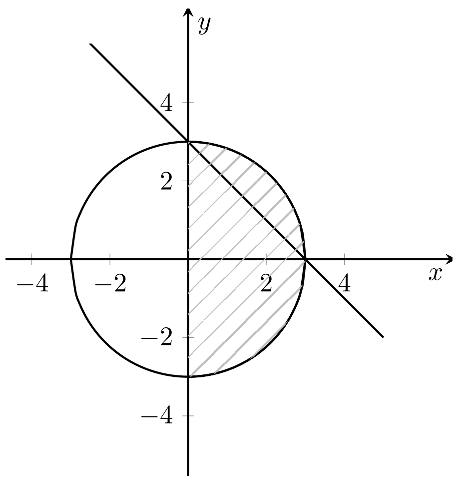
Figure 1.1: The 3 sets A, B, C



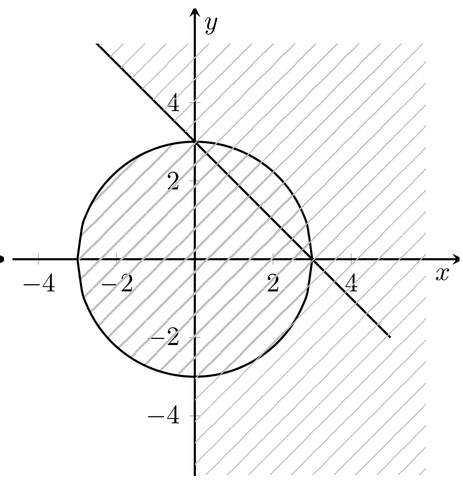
(a) $A \cup (B \cup C)$



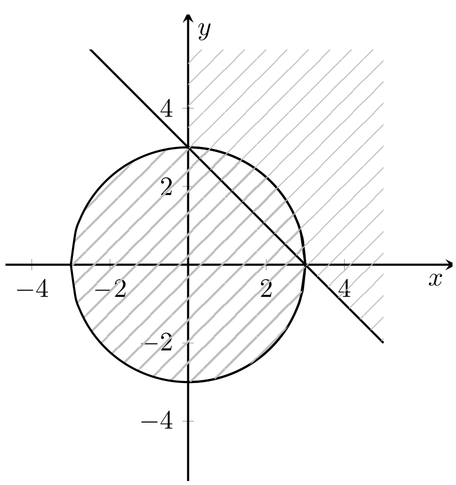
(b) $A \cap (B \cup C)$



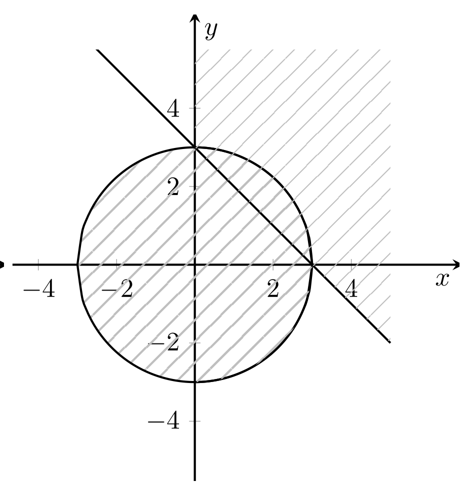
(c) $(A \cap B) \cup (A \cap C)$



(d) $(A \cup B) \cup C$



(e) $A \cup (B \cap C)$



(f) $(A \cup B) \cap (A \cup C)$

♦

1.4.3

Let A, B, C as follows: $A = \{(x, y) : x+y \leq 5\}$, $B = \{(x, y) : x+y \geq 3\}$, $C = \{(x, y) : x \geq 3\}$, and $D = \{(x, y) : y \geq 3\}$.
Draw a sketch for each of the following sets:

$$\begin{array}{ll} (a) & (A \cap B) \cap C \\ (b) & [(A \cap B) \cap C] \cap D \end{array}$$

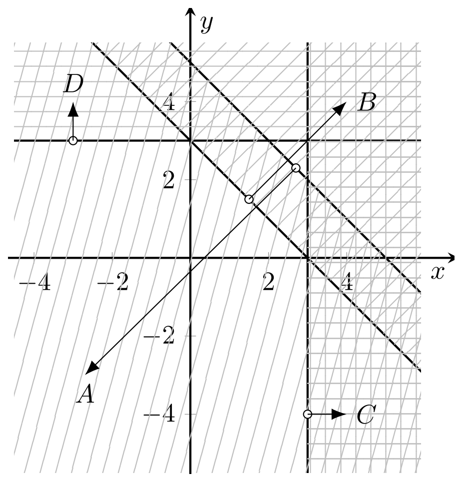
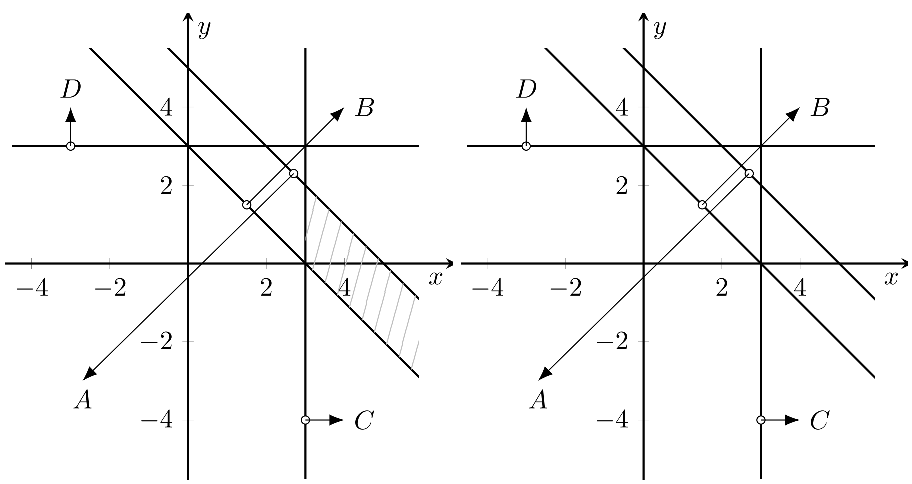


Figure 1.2: The 4 sets A, B, C, D



(a) $(A \cap B) \cap C$ (b) $((A \cap B) \cap C) \cap D = \varnothing$

♦

1.5 Complementation

1.5.1

Sketch each of the following sets: (the sets A, B, C are defined as in exercise 3page 8)

$$
\begin{array}{ll}
(a) & \sim (A \cap B) \\
(b) & (\sim A) \cup (B) \\
(c) & \sim (A \cup B) \\
(d) & (\sim A) \cap (B) \\
(e) & C - A \\
(f) & \sim (A \cap C) \\
(g) & (\sim A) \cup (\sim B) \\
(h) & (\sim A) \cap (A) \\
(i) & C - (A \cup B) \\
(j) & (C - A) \cap (C - B) \\
(k) & \sim (\sim A)
\end{array}
$$

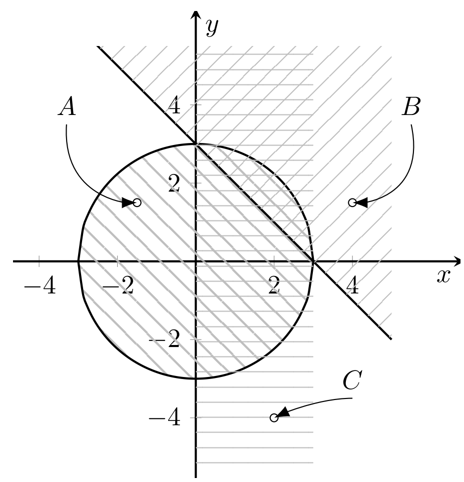
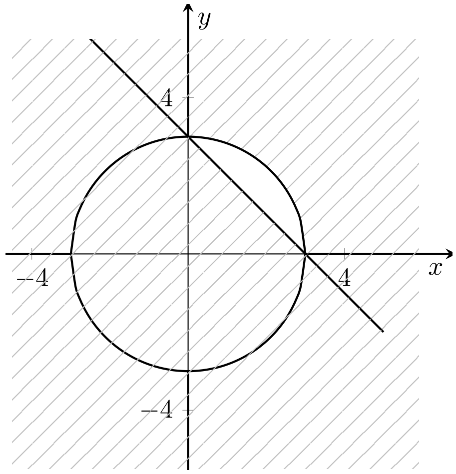
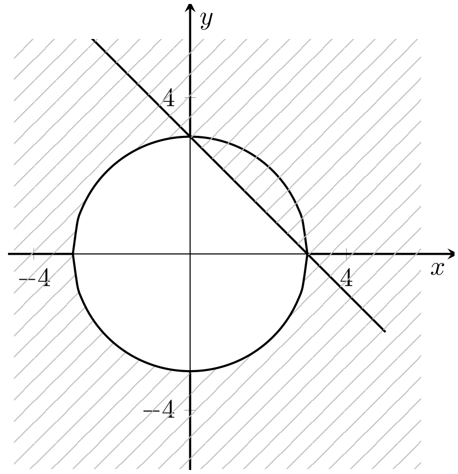


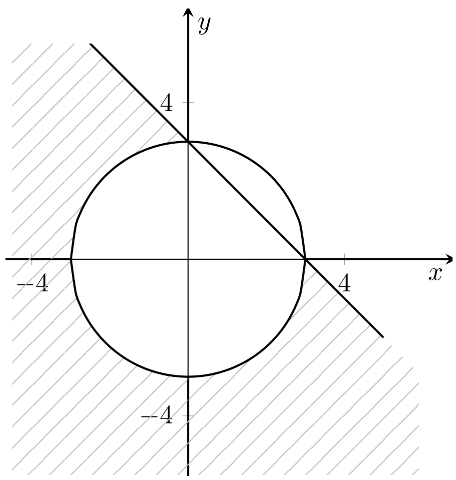
Figure 1.3: The 3 sets A, B, C



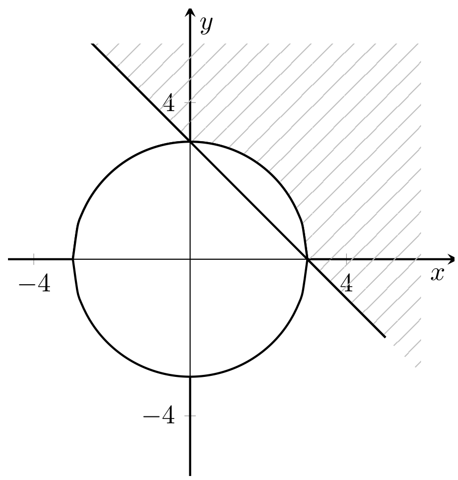
(a) $\sim (A \cap B)$



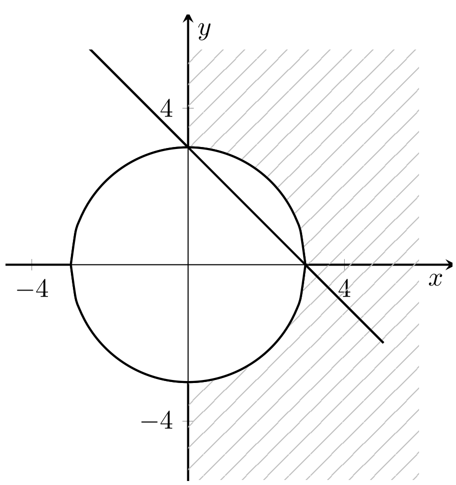
(b) $(\sim A) \cup (B)$



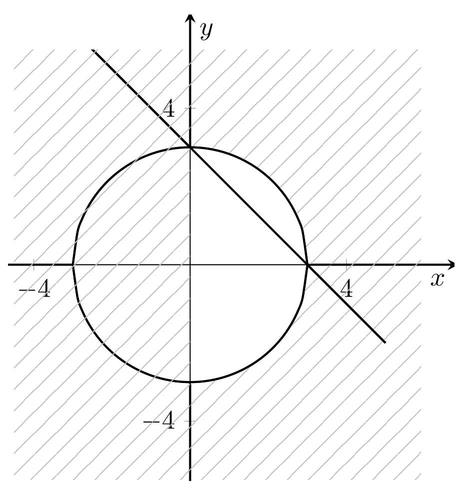
(c) $\sim (A \cup B)$



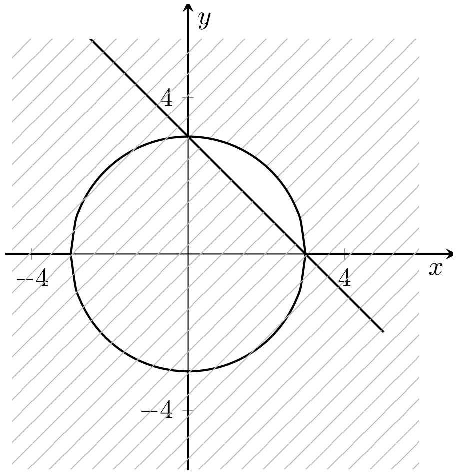
(d) $(\sim A) \cap (B)$



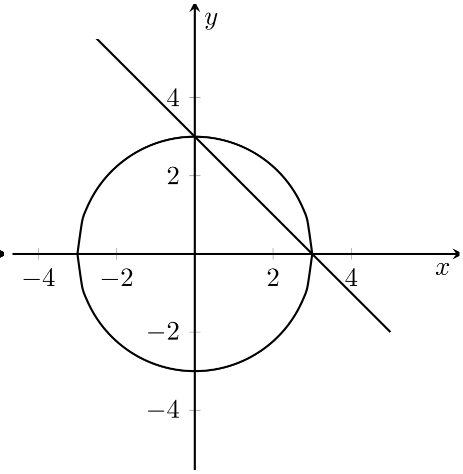
(e) $C - A$



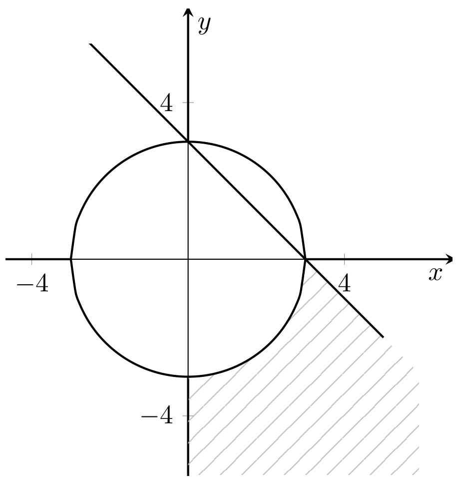
(f) $\sim (A \cap C)$



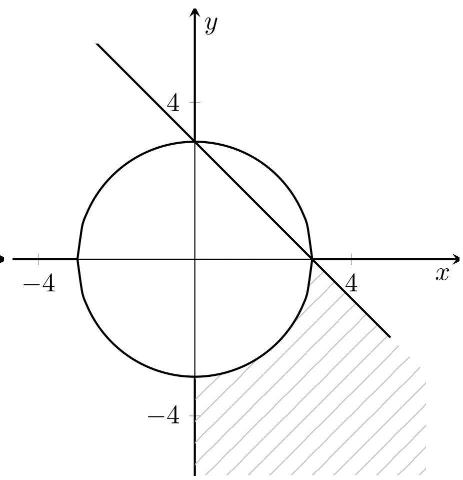
(g) $(\sim A) \cup (\sim B)$



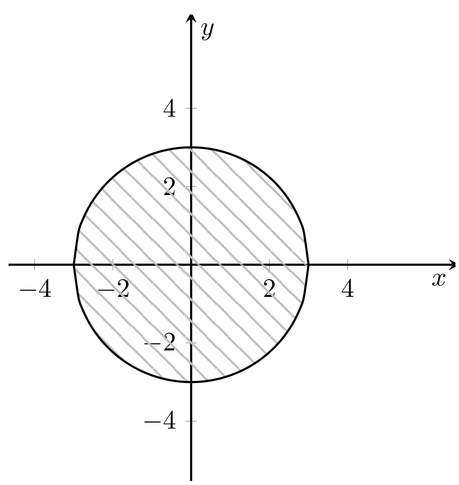
(h) $(\sim A) \cap (A) = \varnothing$



(i) $C - (A \cup B)$



(j) $(C - A) \cap (C - B)$



(k) $\sim (\sim A)$

♦

1.5.2

On the basis of the sketches made in the previous exeercise, formulate a proposition about relation that exist concerning complementation, union, and intersection. Try out your conjecture on other examples. In subsequent exercises you will be asked to try to prove such conjectures.

1.4.2 (a) and (d)	$A \cup (B \cup C) = (A \cup B) \cup C)$
1.4.2 (b) and (c)	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
1.4.2(e) and (f)	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
1.5.1(a) and (g)	$\sim (A \cap B) = (\sim A) \cup (\sim B)$
1.5.1(h)	$(\sim A) \cap A = \varnothing$
1.5.1(i) and (j)	$C - (A \cup B) = (C - A) \cap (C - B)$
1.5.1(k)	$\sim (\sim A) = A$

♦

1.6 Set identities and other set relations

1.6.1

Prove that if $A \subset B$, then:

$$\begin{array}{ll} (a) & A \cap C \subset B \cap C \\ (b) & \sim B \subset \sim A \\ (c) & A \cap B = A \\ (d) & A \cup C \subset B \cup C \end{array}$$

a) $A \cap C \subset B \cap C$

Given is $x \in B$ if $x \in A$. Suppose $x \in A \cap C$, then $x \in A$ (given) and $x \in C$ but $x \in B$ (given) and as $x \in C$ follows that $x \in B \cap C$. And we conclude that $A \cap C \subset B \cap C$.

◊

b) $\sim B \subset \sim A$

Given is $x \in B$ if $x \in A$. If $x \notin B$ then $x \in \sim B$. As $A \subset B$, x will not be in A but $x \in \sim A$. So $x \in \sim B \Rightarrow \quad x \in \sim A$ and thus $\sim B \subset \sim A$.

◊

c) $A \cap B = A$

Given is $x \in B$ if $x \in A$. Suppose $x \in A \cap B$, then $x \in A$ and thus $A \cap B \subset A$. Suppose $x \in A$, then $x \in B$ as $A \subset B$ and thus $x \in A \cap B$ from which we conclude $A \subset A \cap B$.

◊

d) $A \cup C \subset B \cup C$

Given is $x \in B$ if $x \in A$. Suppose $x \in A \cup C$, then $x \in A$ or $x \in C$. But $x \in B$ (given), so $x \in B$ or $x \in C$ and thus $x \in B \cup C$, from which we conclude $A \cup C \subset B \cup C$.

♦

1.6.2

Verify that each of the following is an an identity:

$$
\begin{aligned}
&(a) \quad A \cup \varnothing = A \\
&(b) \quad A \cap \varnothing = \varnothing \\
&(c) \quad A \cap A = A \\
&(d) \quad A \cup A = A \\
&(e) \quad (A \cup B) \cup C = A \cup (B \cup C) \\
&(f) \quad (A \cap B) \cap C = A \cap (B \cap C) \\
&(g) \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \\
&(h) \quad X - (A \cup B) = (X - A) \cap (X - B) \\
&(i) \quad A \cap \sim A = \varnothing \\
&(j) \quad A \cup \sim A = U
\end{aligned}
$$

a) $A \cup \varnothing = A$

This is a consequence of remark 3.3 page 7: the empty set $\varnothing$ is a subset of every set. So, $\varnothing \subset A$ giving the asked identity.

◊

b) $A \cap \varnothing = \varnothing$

If $x \in A \cap \varnothing$ then $x \in A$ and x must also be in $\varnothing$ which is impossible by definition. So there is no element $x \in \varnothing$ which can satisfy $x \in A \cap \varnothing$ giving the proposed identity.

◊

c) $A \cap A = A$

Suppose $x \in A \cap A$, then $x \in A$ and $x \in A$ and thus $x \in A$, giving $A \cap A \subset A$. Suppose $x \in A$, then obviously $x \in A$ and $x \in A$, giving $A \subset A \cap A$. Hence $A \cap A = A$

◊

d) $A \cup A = A$

Suppose $x \in A \cup A$, then $x \in A$ or $x \in A$ and thus $x \in A$, giving $A \cup A \subset A$. Suppose $x \in A$, then obviously $x \in A$ or $x \in A$, giving $A \subset A \cup A$. Hence $A \cup A = A$

◊

e) $(A \cup B) \cup C = A \cup (B \cup C)$

Suppose $x \in (A \cup B) \cup C$, then $x \in (A \cup B)$ or $x \in C$ and thus $x \in A$ or $x \in B$ or $x \in C$. So $x \in B$ or $x \in C$ can be written as $x \in (B \cup C)$. So $x \in A$ or $x \in (B \cup C)$, giving $(A \cup B) \cup C \subset A \cup (B \cup C)$. The same reasoning yields for $x \in A \cup (B \cup C)$ giving the identity.

◊

f) $(A \cap B) \cap C = A \cap (B \cap C)$

Suppose $x \in (A \cap B) \cap C$, then $x \in (A \cup B)$ and $x \in C$ and thus $x \in A$ and $x \in B$ and $x \in C$. So $x \in B$ and $x \in C$ can be written as $x \in (B \cap C)$. So $x \in A$ and $x \in (B \cup C)$, giving $(A \cap B) \cap C \subset A \cap (B \cap C)$. The same reasoning yields for $x \in A \cap (B \cap C)$ giving the identity.

◊

g) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Suppose $x \in A \cup (B \cap C)$, then $x \in A$ or $x \in (B \cap C)$. Take the case $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$ which implies $x \in (A \cup B) \cap (A \cup C)$, giving $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$. The other case: if $x \in B \cap C$ then $x \in B$ and $x \in C$. So, $x \in A \cup B$ and $x \in A \cup C$ giving also $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$.
On the other hand, be $x \in (A \cup B) \cap (A \cup C)$ then $x \in (A \cup B)$ and $x \in (A \cup C)$. Let's first take the case $x \in A$ then obviously $x \in A \cup (B \cap C)$ even if $x \notin B \cap C$. Alternatively, be $x \notin A$ then we must have $x \in B$ and $x \in C$ which implies $x \in B \cap C$, giving again $x \in A \cup (B \cap C)$.

◊

h) $X - (A \cup B) = (X - A) \cap (X - B)$

Suppose $x \in X - (A \cup B)$, then $x \notin A$ and $x \notin B$ which implies $x \in X - A$ and $x \in X - B$ and thus $x \in X - A \cap X - B$ giving $X - (A \cup B) \subset (X - A) \cap (X - B)$.
The other way around. Suppose $x \in (X - A) \cap (X - B)$. Then $x \in (X - A)$ and $x \in (X - B)$ which implies $x \notin A$ and $x \notin B$ giving $x \notin A \cup B$ which in turn implies $x \in X - (A \cup B)$ giving $(X - A) \cap (X - B) \subset X - (A \cup B)$.
Conclusion: $X - (A \cup B) = (X - A) \cap (X - B)$

◊

i) $A \cap \sim A = \varnothing$

Suppose $x \in A \cap \sim A$, then $x \in A$ and $x \notin A$ which is a contradiction, so the only element which is always an element of any set is the empty set, so $A \cap \sim A \subset \varnothing$. Suppose on the contrary that $x \in \varnothing$. This implies that x correspond to the empty set and as the empty set is an element of any set, we have $\varnothing \subset A \cap \sim A$

$\Diamond$

j) $A \cup \sim A = U$

Suppose $x \in A \cup \sim A$, then $x \in A$ or $x \notin A$. So, in any case $x \in U$ and thus $A \cup \sim A \subset U$.
On the opposite way suppose that $x \in U$. Then obviously $x \in A$ or $x \in \sim A$ and thus $U \subset A \cup \sim A$.

$\blacklozenge$

1.6.3

Prove that if $A \subset C$ and $B \subset C$, then $A \cup B \subset C$.

Given is $A \subset C$ and $B \subset C$. Take $x \in A$, then $x \in C$, so even if $x \notin B$, then $x \in A \cup B$ reduces to $x \in A$ and thus $x \in C$. The same reasoning yields for $x \in B$, giving $A \cup B \subset C$.

$\blacklozenge$

1.6.4

Prove that if $A \subset B$ and $A \subset C$, then $A \subset B \cap C$.

Given is $A \subset B$ and $A \subset C$. Take $x \in A$, then $x \in C$ and $x \in B$, which implies $x \in C \cap B$. giving indeed $A \subset B \cap C$.

$\blacklozenge$

1.7 Counterexamples

In each of the following exercises state whether the statement is necessarily true. Assume that A, B and C are subsets of a universal set U. Justify with a proof or a counterexample.

1.7.1

If $A \cup C = B \cup C$, then $A = B$

Not TRUE.

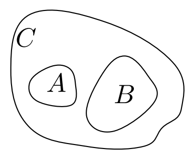


Figure 1.4: $A \cup C = B \cup C \not\Rightarrow A = B$

Be $A \subset C$ and $B \subset C$, then we have $A \cup C = B \cup C \equiv C = C$ even if $A \cap B = \varnothing$.

♦

1.7.2

$(A \cup B) - B = A$

Not TRUE.

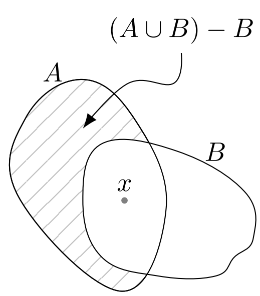


Figure 1.5: $(A \cup B) - B \neq A$

Be $A \cap B \neq \varnothing$, take $x \in A$ and $x \in B$, then x can't be $x \in (A \cup B) - B$ although it is an element of A.

♦

1.7.3

$(A - B) \cup B = A$

Not TRUE.

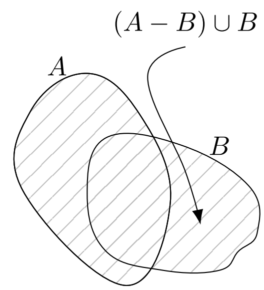


Figure 1.6: $(A - B) \cup B \neq A$

This is only true if $B \subset A$

♦

1.7.4

$\sim (A - B) = \sim (A \cap \sim B)$

TRUE.

Suppose first that A and B are disjoint, i.e. $A \cap B = \varnothing$, then $A - B = A$ and $\sim (A - B) = \sim A$. On the other hand $A \subset \sim B$, so $A \cap \sim B = A$, giving $\sim (A \cap \sim B) = \sim A$, giving indeed $\sim (A - B) = \sim (A \cap \sim B)$.

Suppose now that A and B are not disjoint, i.e. $A \cap B \neq \varnothing$. Be $x \in A - B \subset A$. This is equivalent with the statement $x \in A \wedge x \notin B$. Negating this statement: $\neg(x \in A \wedge x \notin B) \Leftrightarrow x \notin A \vee x \in B$. This give $\sim (A - B) \equiv x \notin A \vee x \in B$.
Be now $x \in A \cap \sim B$. This is equivalent with the statement $x \in A \wedge x \notin B$. Negating this statement: $\neg(x \in A \cap \sim B) \Leftrightarrow x \notin A \vee x \in B$. This give $\sim (A \cap \sim B) \equiv x \notin A \vee x \in B$, resulting in $\sim (A - B) = \sim (A \cap \sim B)$.

♦

1.7.5

$\sim (\sim (\sim A)) = \sim A$

TRUE.
Be $x \in \sim (\sim (\sim A))$. This is equivalent to $x \notin \sim (\sim A)$. Which on it's turn is equivalent with $x \in \sim A$. So, $\sim (\sim (\sim A)) \subset \sim A$.

Be $x \in \sim A$. This is equivalent to $x \notin \sim (\sim A)$. Which on it's turn is equivalent with $x \in \sim (\sim (\sim A))$. So, $\sim A \subset \sim (\sim (\sim A))$.
Both cases reduce to $\sim (\sim (\sim A)) = \sim A$.

♦

1.7.6

$A \cup (B - C) = (A \cup B) - C$

Not TRUE.
Be $x \in A \cup (B - C)$. This is equivalent to $x \in A \vee x \in (B - C)$. Suppose $x \in A$, then $x \in A \cup B$. Let's consider the set C so that $(A \cup B) \subset C$, then $(A \cup B) - C = \varnothing$. We get a contradiction and the proposed statement is not true.

♦

1.7.7

$\sim (A - B) = (\sim A) \cup B$

TRUE.
Be $x \in (A - B)$. This is equivalent to $x \in A \wedge x \notin B$. Negating this statement: $\neg(x \in A \wedge x \notin B) \Leftrightarrow x \notin A \vee x \in B$. This is equivalent to the statement $x \in (\sim A) \cup B$. So $\sim (A - B) \subset (\sim A) \cup B$.
Consider now $x \in (\sim A) \cup B$. So $x \notin A \vee x \in B$. If we have the case $x \notin A$ then also $x \notin (A - B)$ as x can not be one of the remaining elements of A after the complement of B relative to A. Also, if $x \in B$ then also $x \notin (A - B)$ as x is an element of B and thus can not be an element of $(A - B)$. Thus, in both cases we have, $x \notin (A - B)$ which implies $x \in \sim (A - B)$. So $(\sim A) \cup B \subset \sim (A - B)$.

♦

1.7.8

If $A - B = C - B$, then $A = C$.

Not TRUE.

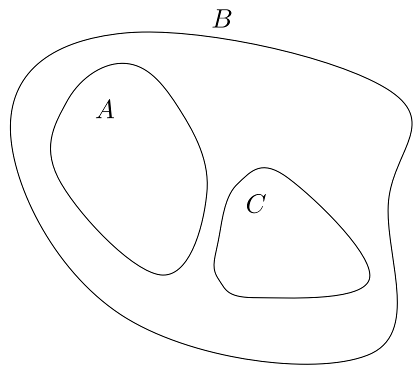


Figure 1.7: If $A - B = C - B \not\Rightarrow A = C$

Suppose $A \subset B$, then $A - B = \varnothing$. Choose a C such that $C \subset B$ and also $A \cap C = \varnothing$, then also $C - B = \varnothing$ and get $A - B = C - B$ although $A \neq C$.

♦

1.7.9

If $A - (B \cap C) = (A - B) \cap (A - C)$.

TRUE.

Suppose $x \in A - (B \cap C)$, then $x \in A \wedge x \notin B \cap C$. As x can not be simultaneously in B and C, then also x must be simultaneously in $A - B$ and $A - C$ as the "complementation of A with B and C will not "subtract" x out of A, and considering that $x \in A$ we have $A - (B \cap C) \subset (A - B) \cap (A - C)$ Suppose $x \in (A - B) \cap (A - C)$, then x must be an element of A but not an element of B and C. This means that $x \notin B \cap C$ and thus the complementation of A by $B \cap C$ has no effect on x. Thus, $\underbrace{(A - B) \cap (A - C)}_{=A} \subset A - (B \cap C)$.

♦

1.8 Collections of Sets

1.8.1

Suppose that A, B and C are the following subsets of the plane:
$A = \{(x,y) : x^2 + y^2 \leq 16\}$, $B = \{(x,y) : x \geq 0 \text{ and } y \leq 0\}$, $C = \{(x,y) : y \leq x\}$. If $\mathscr{K}$ is the collection of sets $\{A, B, C\}$, sketch each of the following sets:

$$
\begin{aligned}
&(a) \quad \bigcap \mathscr{K} \\
&(b) \quad \bigcup \mathscr{K} \\
&(c) \quad \bigcup \mathscr{K} - \bigcap \mathscr{K}
\end{aligned}
$$

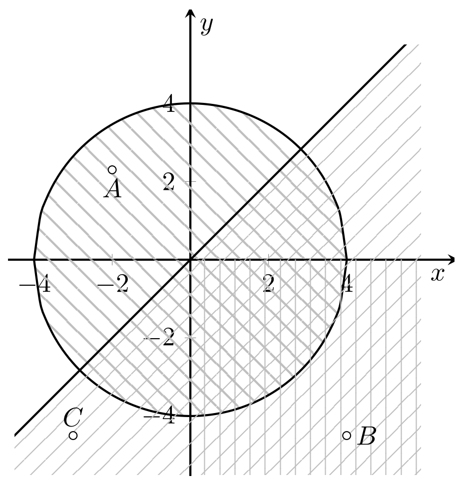


Figure 1.8: The sets A, B, C

a) $\bigcap \mathscr{K}$

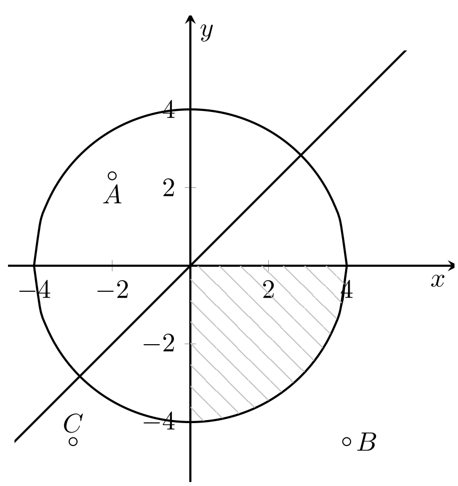


Figure 1.9: $\bigcap \mathscr{K}$

♢

b) $\bigcup \mathscr{K}$

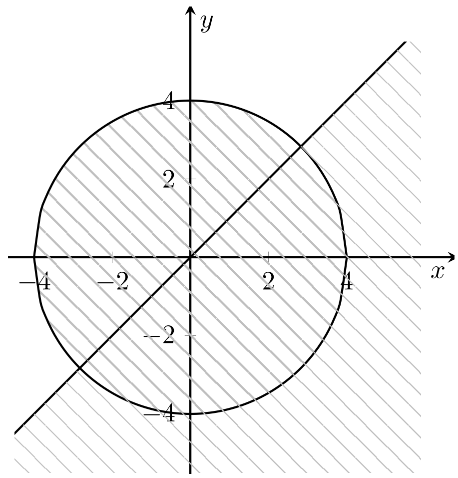


Figure 1.10: $\bigcup \mathscr{K}$

♢

c) $\bigcup \mathscr{K} - \bigcap \mathscr{K}$

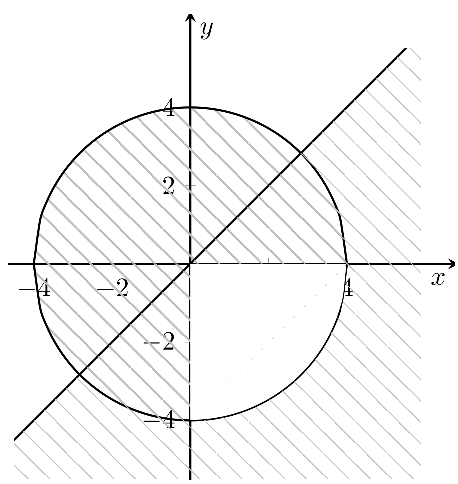


Figure 1.11: $\bigcup \mathscr{K} - \bigcap \mathscr{K}$

♦

1.8.2

Recall that $\mathbb{P}$ is the symbol for the set of positive integers. Suppose that for each $n \in \mathbb{P}$, we let $A_n = \{x \in \mathbb{R} : x \geq n\}$. Describe the sets $\bigcup\{A_n : n \in \mathbb{P}\}$ and $\bigcap\{A_n : n \in \mathbb{P}\}$.

$S = \bigcup\{A_n : n \in \mathbb{P}\}$

$$S = [1, +\infty)$$

$$\lozenge$$

$S = \bigcap\{A_n : n \in \mathbb{P}\}$

$$S = \varnothing$$

This can be understood by the fact that for every $x \in \mathbb{R}$, you can find a $n \in \mathbb{P}$ so that $x \notin A_n$. So, no x can be an element of S.

$$\blacklozenge$$

1.8.3

Suppose that for each $n \in \mathbb{P}$, K_n is a non-empty set such that $K_{n+1} \subset K_n$. Let $\mathscr{K} = \{K_n : n \in \mathbb{P}\}$.
In each of the following, if the statement is necessarily true, say so and justify your answer. If the statement is not necessarily true, give a counterexample to justify your answer.

(a) $\bigcup \mathscr{K} = K_1$
(b) $\bigcap\{K_i : i = 1, 2, \ldots, n\} = K_n$
(c) $\bigcap \mathscr{K} \neq \varnothing$

(a) $\bigcup \mathscr{K} = K_1$.
TRUE.
Be $x \in K_n$ for any arbitrary n. So, $x \in K_n \cup K_{n-1}$. But $K_n \cup K_{n-1} = K_{n-1}$, giving $x \in K_{n-1}$. Repeating that process with $K_{n-1} \subset K_{n-2} \subset \ldots K_2 \subset K_1$ we get $x \in K_n$.

$$\lozenge$$

(b) $\bigcap\{K_i : i = 1, 2, \ldots, n\} = K_n$.
TRUE.
Suppose first that for all n we have K_n is a *proper* subset of K_{n-1}. Then $K_n \cap K_{n-1} = K_n$. Be $x \in K_n$ but not in K_{n-1} for any arbitrary n. Then, $x \in K_n \cap K_{n-1}$ is equivalent to $x \in K_n$. Repeating that process with we have $K_n \cap K_{n-1} \cap K_{n-2} \cap \ldots K_2 \cap K_1 = K_n$ and get $x \in K_n$. Hence, $\bigcup \mathscr{K} = K_1$.
In the case that for some or all n we have $K_n = K_{n-1}$ we could also state that $\bigcap\{K_i : i = 1, 2, \ldots, n\} = K_{n-1}$ but as $K_n = K_{n-1}$ we can write $\bigcap\{K_i : i = 1, 2, \ldots, n\} = K_{n-1} = K_n$.

The same is true in the case that a sequence of the subsets are proper subset of each other i.e. $K_{n+p} = K_{n+p-1} = \ldots K_{n+1} = K_n = K_{n-1} = \cdots = K_{n-t}$. then one could write $\bigcap\{K_i : i = 1, 2, \ldots, n\} = K_{n+p}$ but as $K_{n+p} = K_n$, the original statement holds.

$\lozenge$

(c) $\bigcap \mathscr{K} \neq \varnothing$.
TRUE.
As no K_n is an empty set, K_n will always contain at least one element and due to (b) we get indeed $\bigcap \mathscr{K} \neq \varnothing$: suppose that for a given n, K_n contains only one element x, then all subsequent K_{n+p} must also have only one element i.e. x and we will get $\bigcap \mathscr{K} = \{x\}$

$\blacklozenge$

1.8.4

For each real number $r > 0$, let $L_r = \{x : x \geq r\}$. Sketch the set $\bigcup\{L_r : r > 0\}$ and $\bigcap\{L_r : r > 0\}$ on a number line. If a set happens to be empty, say so.

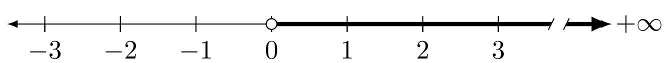


Figure 1.12: $\bigcup\{L_r : r > 0\}$

$\lozenge$

$\bigcap\{L_r : r > 0\} = \varnothing$.

Indeed, take an arbitrary r and be $\epsilon > 0$ then $\exists\, x \in L_r : x \notin L_{r+\epsilon}$. Then, $L_r \cap L_{r+\epsilon} = \varnothing$. So, whatever L_r we choose in the collection $\mathscr{L} = \{L_r : r \in \mathbb{R}^+\}$ there always be a $L_{r'}$ for which $L_r \cap L_{r'} = \varnothing$ and hence $\bigcap\{L_r : r > 0\} = \varnothing$.

$\blacklozenge$

1.8.5

Let U be a set and let $\mathscr{K}$ be a non-empty collection of subsets of U. $\sim$ will signify the complement with respect to U. Prove the following set identities. The identities are quite important and are known as De Morgan's Laws.

$$\begin{array}{ll} (a) & \sim(\bigcup\{K : K \in \mathscr{K}\}) = \bigcap\{\sim K : K \in \mathscr{K}\} \\ (b) & \sim(\bigcap\{K : K \in \mathscr{K}\}) = \bigcup\{\sim K : K \in \mathscr{K}\} \end{array}$$

(a) $\sim(\bigcup\{K : K \in \mathscr{K}\}) = \bigcap\{\sim K : K \in \mathscr{K}\}$
Suppose $x \in\sim(\bigcup\{K : K \in \mathscr{K}\})$, then $x \notin \bigcup\{K : K \in \mathscr{K}\}$. This means that x is not an element of any $K \in \mathscr{K}$ i.e. $\forall K \in \mathscr{K} : x \notin K$. This can also be expressed as $\forall K \in \mathscr{K} : x \in\sim K)$. This means that x is an element of all $\sim K$ giving $x \in \bigcap\{\sim K : K \in \mathscr{K}\}$ and thus $\sim(\bigcup\{K : K \in \mathscr{K}\}) \subset \bigcap\{\sim K : K \in \mathscr{K}\}$.
Suppose now that $x \in \bigcap\{\sim K : K \in \mathscr{K}\}$. This means that x is an element of $\{\sim K : K \in \mathscr{K}\}$ for all K i.e. $x \notin \{K : K \in \mathscr{K}\}$ for all K, (indeed if x would be an element of a $K \in \mathscr{K}$ then x would not be an element of its complement and so x could not be an element of $\bigcap\{\sim K : K \in \mathscr{K}\}$). The conclusion is that $x \notin \bigcup\{K : K \in \mathscr{K}\}$ and thus $x \in\sim \bigcup\{K : K \in \mathscr{K}\}$. Hence, $\bigcap\{\sim K : K \in \mathscr{K}\} \subset\sim(\bigcup\{K : K \in \mathscr{K}\})$.
Conclusion $\sim(\bigcup\{K : K \in \mathscr{K}\}) = \bigcap\{\sim K : K \in \mathscr{K}\}$.

$\lozenge$

(b) $\sim(\bigcap\{K : K \in \mathscr{K}\}) = \bigcup\{\sim K : K \in \mathscr{K}\}$
Suppose $x \in\sim(\bigcap\{K : K \in \mathscr{K}\})$, then $x \notin \bigcap\{K : K \in \mathscr{K}\}$. This means that there exists at least one $K \in \mathscr{K}$ so that x is not an element of this K i.e. $\exists K \in \mathscr{K} : x \notin K$. This can also be expressed as $\exists K \in \mathscr{K} : x \in\sim K$. This means that x is an element of $\bigcup\{\sim K : K \in \mathscr{K}\}$ and thus $\sim(\bigcap\{K : K \in \mathscr{K}\}) \subset \bigcup\{\sim K : K \in \mathscr{K}\}$.
Suppose now that $x \in \bigcup\{\sim K : K \in \mathscr{K}\}$. This means that x is an element of at least one $\sim K : K \in \mathscr{K}$. Stated differently, there exist at least one $K : K \in \mathscr{K}$ for which $x \notin K$. This means that x can not be an element of $\bigcap\{K : K \in \mathscr{K}\}$ and thus $x \in\sim \bigcap\{K : K \in \mathscr{K}\}$ which means $\bigcup\{\sim K : K \in \mathscr{K}\} \subset\sim(\bigcup\{K : K \in \mathscr{K}\})$
Conclusion $\sim(\bigcap\{K : K \in \mathscr{K}\}) = \bigcup\{\sim K : K \in \mathscr{K}\}$.

$\blacklozenge$

1.8.6

Let $S = \{1, 2, 3, 4, 5\}$ and let $\mathscr{P}(S)$ be the power set of S. List the elements in $\mathscr{P}(S)$.

We order them according to the number of elements in the subsets. We check the number of subsets by using the $\binom{5}{m}$ formula (i.e. combination without repetition).

$$\text{5 elements} \quad \binom{5}{5} = 1$$

$$\{1, 2, 3, 4, 5\}$$

$$\text{4 elements} \quad \binom{5}{4} = 5$$

$$\{1, 2, 3, 4\}$$

$$\{1, 2, 3, 5\}$$

$$\{1, 2, 4, 5\}$$

$$\{1, 3, 4, 5\}$$

$$\{2, 3, 4, 5\}$$

$$\text{3 elements} \quad \binom{5}{3} = 10$$

$$\{1, 2, 3\}$$

$$\{1, 2, 4\}$$

$$\{1, 2, 5\}$$

$$\{1, 3, 4\}$$

$$\{1, 3, 5\}$$

$$\{1, 4, 5\}$$

$$\{2, 3, 4\}$$

$$\{2, 3, 5\}$$

$$\{2, 4, 5\}$$

$$\{3, 4, 5\}$$

2 elements $\binom{5}{2} = 10$

$\{1,2\}$

$\{1,3\}$

$\{1,4\}$

$\{1,5\}$

$\{2,3\}$

$\{2,4\}$

$\{2,5\}$

$\{3,4\}$

$\{3,5\}$

$\{4,5\}$

1 element $\binom{5}{1} = 5$

$\{1\}$

$\{2\}$

$\{3\}$

$\{4\}$

$\{5\}$

0 elements $\binom{5}{0} = 1$

$\varnothing$

Note that the total number of subsets in $\mathscr{P}(S)$ is $1+5+10+10+5+1 = 32$ which corresponds to 2^5.

♦

1.9 Cartesian Product

1.9.1

Suppose that $A \subset B$ and C is a set. Prove that $A \times C \subset B \times C$.

Be $x \in A$ and $y \in C$. As $A \subset B$, then x is also in B. Thus $\underbrace{(x, y)}_{x \in A,\, y \in C} \in A \times C$ means also that $\underbrace{(x, y)}_{x \in B,\, y \in C} \in B \times C$

♦

1.9.2

Let $A = \{1, 2, 3\}$, $B = \{a, b\}$, and $C = \{\alpha, \beta\}$. List the elements of each of the following sets:

$$\begin{array}{ll} (a) & A \times (B \cup C) \\ (b) & (A \times B) \cup (A \times C) \\ (c) & (A \cup B) \times C \\ (d) & (A \times C) \cup (B \times C) \end{array}$$

(a) $A \times (B \cup C)$

$(1, a)$, $(1, b)$, $(2, a)$, $(2, b)$, $(3, a)$, $(3, b)$
$(1, \alpha)$, $(1, \beta)$, $(2, \alpha)$, $(2, \beta)$, $(3, \alpha)$, $(3, \beta)$

♢

(b) $(A \times B) \cup (A \times C)$

$(1, a)$, $(1, b)$, $(2, a)$, $(2, b)$, $(3, a)$, $(3, b)$
$(1, \alpha)$, $(1, \beta)$, $(2, \alpha)$, $(2, \beta)$, $(3, \alpha)$, $(3, \beta)$

♢

(c) $(A \cup B) \times C$

$(1, \alpha)$, $(1, \beta)$, $(2, \alpha)$, $(2, \beta)$, $(3, \alpha)$, $(3, \beta)$ $(1, \alpha)$, $(1, \beta)$, $(2, \alpha)$, $(2, \beta)$, $(3, \alpha)$, $(3, \beta)$
(a, α), (a, β), (b, α), (b, β)

♢

(d) $(A \times C) \cup (B \times C)$

$(1, \alpha)$, $(1, \beta)$, $(2, \alpha)$, $(2, \beta)$, $(3, \alpha)$, $(3, \beta)$ $(1, \alpha)$, $(1, \beta)$, $(2, \alpha)$, $(2, \beta)$, $(3, \alpha)$, $(3, \beta)$
(a, α), (a, β), (b, α), (b, β)

♦

1.9.3

Are any of the sets in Exercise 2 the same? If so write the set identities that are suggested by your observations. Try to prove your conjecture.

In exercise 2 we can see that the set (a) and (b) are the same. Also (c) and (d) are the same. This suggests the following identities $A\times(B\cup C) = (A\times B)\cup(A\times C)$ and $(A\cup B)\times C = (A\times C)\cup(B\times C)$.

$\mathbf{A} \times (\mathbf{B} \cup \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cup (\mathbf{A} \times \mathbf{C})$
Proof:
Be $x \in A$ and $y \in B \cup C$, so y is an element of B or C. Consider $(x, y) \in A \times (B \cup C)$. As the y can be an element of B or C follows immediately that $(x, y) \in (A \times B)$ or $(x, y) \in A \times C)$ and thus $(x, y) \in (A \times B) \cup (A \times C)$. And get $A \times (B \cup C) \subset (A \times B) \cup (A \times C)$
Suppose now that $(x, y) \in (A \times B) \cup (A \times C)$. The (x, y) is an element of $A \times B$ or $A \times C$. For the same $x \in A$ this implies that $y \in B$ or $y \in C$ and thus $(x, y) \in A \times (B \cup C)$, giving $(A \times B) \cup (A \times C) \subset A \times (B \cup C)$ leading with the previous $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

◊

$(\mathbf{A} \cup \mathbf{B}) \times \mathbf{C} = (\mathbf{A} \times \mathbf{C}) \cup (\mathbf{B} \times \mathbf{C})$
Proof:
Be $x \in A \cup B$ and $y \in C$, so x is an element of A or B. Consider $(x, y) \in (A \cup B) \times C$. As the x can be an element of A or B follows immediately that $(x, y) \in (A \times C)$ or $(x, y) \in (B \times C)$ and thus $(x, y) \in (A \times B) \cup (\mathbf{A} \times C)$. And get $(A \cup B) \times C \subset (A \times C) \cup (B \times C)$
Suppose now that $(x, y) \in (A \times B) \cup (\mathbf{A} \times C)$. The (x, y) is an element of $A \times C$ or $B \times C$. For the same $y \in C$ this implies that $x \in A$ or $x \in B$ and thus $(x, y) \in (A \times C) \cup (B \times C)$, giving $(A \times C) \cup (B \times C) \subset (A \cup B) \times C$ leading with the previous $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

⧫

1.9.4

Suppose that A is a set consisting of five elements and B is a set consisting of three elements. How many elements does the set $A \times B$ have? The set $B \times A$?

$A \times B$ has $5 \times 3 = 15$ elements. Indeed in the element $(x, y) \in A \times B$ we can choose for x out of the five elements of A and for each choice of x we are free to choose one element out of the 3 elements of B.
For $B \times A$, the reasoning is the same and get $3 \times 5 = 15$ elements.

⧫

1.9.5

> Suppose that A is a set consisting of m elements and B is a set consisting of n elements, where m and n are positive integers. How many elements are there in $A \times B$?

$A \times B$ has $m \times n$ elements. Indeed in the element $(x, y) \in A \times B$ we can choose for x out of the m elements of A and for each choice of x we are free to choose one element out of the n elements of B.

♦

1.9.6

> Suppose that A is a set consisting of three elements, B consists of four elements and C consists of two element. How many elements are there in the set $(A \times B) \times C$?

$(A \times B) \times C$ has $(3 \times 4) \times 2 = 24$ elements. Indeed in the element $((x, y), z) \in (A \times B) \times C$ we have for (x, y), $3 \times 4 = 12$ elements (see Exercise 1.9.5) and for each choice of this (x, y) we are free to choose one element out of the 2 elements of C.

♦

1.10 Functions

1.10.1

In each of the following, a set of ordered pairs Γ is given. In each case, determine whether Γ is a function and, if it is, determine if it is a one-to-one function.

(a) Let $\Gamma = \{(x, y) : -1 \leq x \leq 1 \text{ and } x^2 + y^2 = 1\}$.
(b) Let $\Gamma = \{(x, y) : -1 \leq x \leq 1,\ y \geq 0, \text{ and } x^2 + y^2 = 1\}$.
(c) Let $\Gamma = \{(x, y) : 0 \leq x \leq 1 \text{ and } x^2 + y^2 = 1\}$.
(d) Let $\mathscr{F}$ be the collection of all real-valued differentiable functions defined on the open interval (a, b).
Let $\Gamma = \{(f, f^{'}) : f \in \mathscr{F} \text{ and } f^{'} \text{ is the derivative of } f\}$.
(e) Let X be the collection of all continuous real-valued functions defined on the closed interval $[a, b]$.
Let $\Gamma = \left\{\left(f, \int_a^b f(x)dx\right) : f \in X\right\}$.

(a) Let $\Gamma = \{(x, y) : -1 \leq x \leq 1 \text{ and } x^2 + y^2 = 21\}$

Γ is not a function due to the ambiguity of the $\sqrt{\ }$ function. E.g. take $x = 0$ then $y = \pm 1$.

$\Diamond$

(b) Let $\Gamma = \{(x, y) : -1 \leq x \leq 1,\ y \geq 0, \text{ and } x^2 + y^2 = 1\}$.

This time, as the ambiguity on the range has been removed by the condition $y \geq 0$ Γ is a function. Yet, it is not one-to-one e.g. for $x = -1$ and $x = 1$ we get the same value for y.

$\Diamond$

(c) Let $\Gamma = \{(x, y) : 0 \leq x \leq 1 \text{ and } x^2 + y^2 = 2\}$.

This time, as the ambiguity on the range has been removed by the condition $y \geq 0$ Γ is a function. And, it is a one-to-one function as with the restriction on the domain $x \in [0, 1]$, y is well and uniquely defined.

$\Diamond$

(d) Let $\mathscr{F}$ be the collection of all real-valued differentiable functions defined on the open interval (a, b).Let $\Gamma = \{(f, f^{'}) : f \in \mathscr{F} \text{ and } f^{'} \text{ is the derivative of } f\}$.

Γ is a function as f is a real-valued differentiable function, meaning that $\forall f \in \mathscr{F}, \exists f^{'}$.
Yet, it is not one-to-one. E.g. take $f_1 = x + 1$ and $f_2 = x + 2$, both function give $f^{'} = 1$ meaning that Γ is not one-to-one.

$\Diamond$

(e) Let X be the collection of all continuous real-valued functions
defined on the closed interval $[a, b]$.
Let $\Gamma = \left\{ \left(f, \int_a^b f(x)dx \right) : f \in X \right\}$.

Γ is a function as f is a continuous real-valued function, and from calculus we know that every continuous is Riemann-integrable, meaning that for every f there exist a real number $\int_a^b f(x)dx$. Yet, Γ is not one-to-one as two different functions f_1 and f_2 could have the same value of their integral on the given domain e.g. take $f_1 = \frac{x-a}{b-a}$ and $f_2 = \frac{b-x}{b-a}$, both have the same value for the integral over $[a, b]$ namely $\frac{1}{2}(b-a)$.

♦

1.10.2

Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ be the function defined as follows:
For each $(x, y) \in \mathbb{R}$, let $f(x, y) = (a, b)$ where

$$a = x + 2y$$

and

$$b = 2x + 4y$$

Which of the following terms applies to $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$?
(a) surjective, (b) bijective, (c) injective.

f in not injective. Indeed the given function definition can be considered as a system of linear equations with x and y as unknowns and a, b as parameters. So for a given $(a, b) \in \mathbb{R} \times \mathbb{R}$ (the domain) the range will only span $\mathbb{R} \times \mathbb{R}$ only if the system of equations is not degenerated i.e. if the determinant of the system is not 0, but we have

$$det \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = 0$$

Hence, f is not surjective. It is however one-to-one (injective) as for a given (x, y), due to linear form of the function, there will be only one (a, b) on which (x, y) is mapped. As f is not surjective, f can not be bijective.

♦

1.10.3

Repeat the question in Ecercise 2 for the system

$$a = 3x + 2y$$

$$b = 6x - 2y$$

f is injective as we see that this time the determinant of the system is

$$det \begin{pmatrix} 3 & 2 \\ 6 & -2 \end{pmatrix} = -18$$

Hence, f is surjective. It is also one-to-one (injective) for the same reason mentioned in Exercise 2. . As f is surjective and injective , f is also bijective.

♦

1.10.4

Let f be a map from the set of all reals $\mathbb{R}$ into $\mathbb{R}$. Suppose furthermore that if x_1 and x_2 are in $\mathbb{R}$ and $x_1 < x_2$, then $f(x_1) < f(x_2)$. Is it necessarily true that f is one-to-one? Is it necessarily true that $f[\mathbb{R}] = \mathbb{R}$? Justify your answer.

It is necessarily true that f is one-to-one. (At each point x_1, the function for a given x_2 could be re-written as $f(x_2) = f(x_1) + \phi(x_1)(x_2 - x_1)$ with $\phi(x_1) > 0$. So $f(x_2)$ can not be equal to $f(x_1)$ unless $x_2 = x_1$.)
On the other hand $f[\mathbb{R}]$ is not necessarily equal to $\mathbb{R}$. As a counterexample, consider the function $f(x) = e^{-x}$, which is a monotone increasing function but the range is $(-\infty, 0) \neq \mathbb{R}$

♦

1.10.5

Consider the function $f : X \to Y$. Suppose that A and B are subsets of X. Decide which of the following statements are necessarily true. Justify your answers.

$$\begin{array}{ll}
(a) & \text{If } A \cap B = \varnothing, \text{ then } f[A] \cap f[B] = \varnothing. \\
(b) & \text{If } f[A] \cap f[B] = \varnothing, \text{ then } A \cap B = \varnothing. \\
(c) & \text{If } A \subset B, \text{ then } f[A] \subset f[B]. \\
(d) & f[A - B] = f[A] - f[B]. \\
(e) & f[A \cup B] = f[A] \cup f[B]. \\
(f) & f[A \cap B] \subset f[A] \cap f[B]. \\
(g) & f[A \cap B] = f[A] \cap f[B].
\end{array}$$

(a) If $A \cap B = \varnothing,$ then $f[A] \cap f[B] = \varnothing.$

This is not necessarily true. Take for example a non injective function like $f(x) = \sin(x)$ then $f[[0, \frac{\pi}{4}]] \cap f[\frac{3\pi}{4}, \pi] = [0, \frac{\sqrt{2}}{2}]$.

$\Diamond$

(b) If $f[A] \cap f[B] = \varnothing,$ then $A \cap B = \varnothing$

This is necessarily true as for f being a function we have $(x_2, f(x_2)) \in f$ and $(x_1, f(x_1)) \in f \Rightarrow f(x_1) = f(x_2)$ and $A \cap B \neq \varnothing$ would mean that $\exists\, x \in A \cap B$ for which x has two different images.

$\Diamond$

(c) If $A \subset B,$ then $f[A] \subset f[B].$

This is necessarily true as for the same reason as in (b).

$\Diamond$

(d) $f[A - B] = f[A] - f[B]$

This is not necessarily true. Let's take the same counterexample as in (a) i.e. $f(x) = \sin(x)$ and let's define $A = [0, 2\pi],\ B = [0, \frac{\pi}{4}]$, then $f[A] = [-1, 1]$ and $f[B] = [0, \frac{\sqrt{2}}{2}]$ and $f[A] - f[B] = [-1, 0) \cup (\frac{\sqrt{2}}{2}, 1]$ while $f[A - B] = [-1, 1]$.

$\Diamond$

(e) $f[A \cup B] = f[A] \cup f[B]$

This is true.

Suppose first that $A \cap B = \varnothing$ and take $x \in A$, then $f(x) \in f[A]$ and $x \notin f[B]$ giving $f(x) \in f[A] \cup f[B]$. On the other hand it is obvious that if $A \cap B \neq \varnothing$ then $f(x) \in f[A]$ and-or $f(x) inf[B]$ giving $f(x) \in f[A] \cup f[B]$. Hence, $f[A \cup B] \subset f[A] \cup f[B]$.

Suppose now that $f(x) \in f[A]$ this means that $x \in A$ regardless of $x \in B$ or not. So, $f[A] \cup f[B] \subset f[A \cup B]$ and with the previous we get $f[A \cup B] = f[A] \cup f[B]$.

$\Diamond$

(f) $f[A \cap B] \subset f[A] \cap f[B]$

True as if $f(x) \in f[A \cap B]$ means that $x \in A \cap B$ so x will be mapped in the image $f[A]$ and in the image $f[B]$ and thus $f[A \cap B] \subset f[A] \cap f[B]$.

$\Diamond$

(g) $f[A \cap B] = f[A] \cap f[B]$

Not true. Suppose $f(x) \in f[A] \cap f[B]$. But if f is not injective the possibility exists that for a given $x_a \in A$ and another $x_b \in B$ we have $f[x_a] = f[x_b]$ even if A and B are disjoint sets which would give $f[A \cap B] = f[\varnothing] = \varnothing$.

$\blacklozenge$

1.11 Relations

In Exercises 1 to 5, all relations are subsets of the plane. In each case, draw a sketch of R, and give Dom$R$, Range$R$, $R[0]$ and $R^{-1}[0]$.

1.11.1

Let $(x, y) \in R$ provided that (x, y) satisfies each of the following inequalities: $x + y \leq 3$, $y - x \geq 0$, $x \geq -3$.

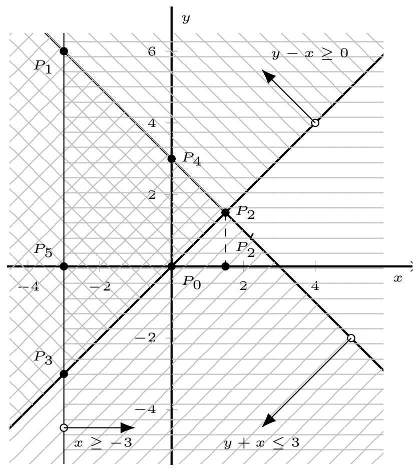


Figure 1.13: $x + y \leq 3$, $y - x \geq 0$, $x \geq -3$

Dom$R =$ segment $[P_5, P_2^{'}]$
Range$R =$ segment $[P_3, P_1]$
$R[0] :=$ segment $[P_0, P_4]$
$R^{-1}[0]: =$ segment $[P_5, P_0]$

♦

1.11.2

Let R be the set of all (x, y) that satisfy $x^2 - y^2 \leq 1$ and $y^2 - x^2 \leq 1$.

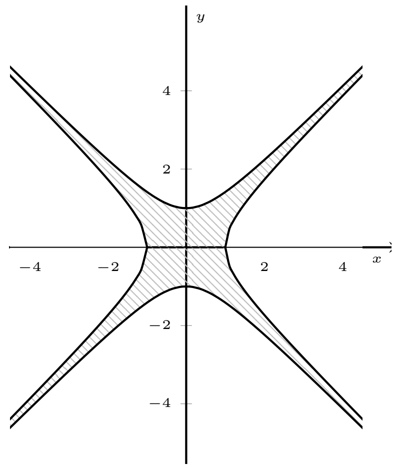


Figure 1.14: $x^2 - y^2 \leq 1$ and $y^2 - x^2 \leq 1$

Dom$R = (-\infty, +\infty)$
Range$R = (-\infty, +\infty)$
$R[0] := [-1, 1]$
$R^{-1}[0]: = [-1, 1]$

♦

1.11.3

Let R be the set of all (x, y) such that $x - y$ is a multiple of 3.

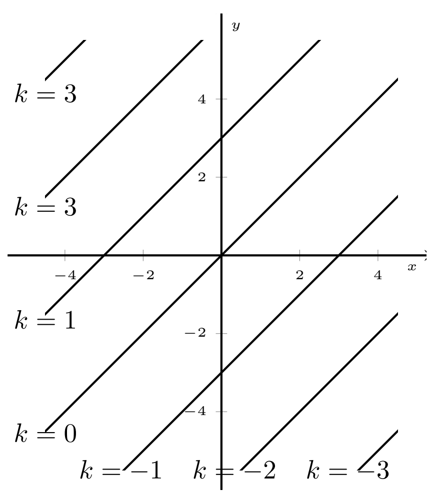


Figure 1.15: The relation $\{(x, y) : y = x - 3k, k \in \mathbb{Z}$

Dom$R = (-\infty, +\infty)$
Range$R = (-\infty, +\infty)$
$R[0] := \{y : y = 3k, k \in \mathbb{Z}\}$
$R^{-1}[0] := \{x : x = 3k, k \in \mathbb{Z}\}$

♦

1.11.4

Let R be a subset of the plane such that $(x, y) \in R$ provided that $x - y \leq \frac{1}{2}$

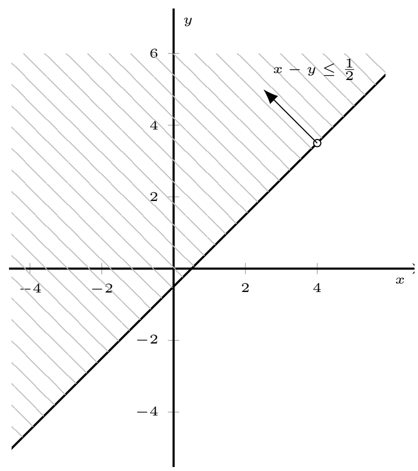


Figure 1.16: The relation $x - y \leq \frac{1}{2}$

Dom$R = (-\infty, +\infty)$
Range$R = (-\infty, +\infty)$
$R[0] := [-\frac{1}{2}, \infty)$
$R^{-1}[0] := (-\infty, \frac{1}{2}]$

♦

1.11.5

Let R be a subset of the plane such that $(x, y) \in R$ provided that $y = x^4$

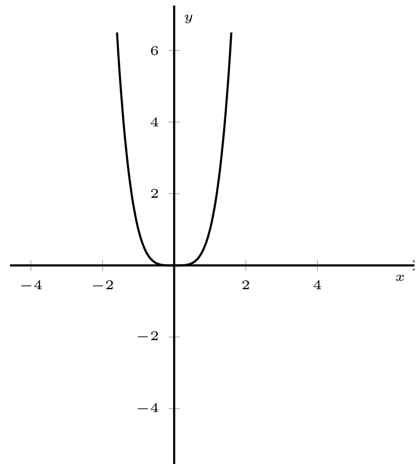


Figure 1.17: The relation $y = x^4$

$\mathbf{Dom}R = (-\infty, +\infty)$
$\mathbf{Range}R = [0, +\infty)$
$R[0] := \{0\}$
$R^{-1}[0] := \{0\}$

♦

1.11.6

Let $R = \{(x, y) : x \geq 0,\ x^2 + y^2 = 26\}$. Find $R[0]$, $R[5]$, and $R[I]$, where $I = \{r : 0 \leq r \leq 1\}$; $R^{-1}[J]$ where $J = \{r : -1 \leq r \leq 1\}$.

$R[0] := \{-\sqrt{26},\ \sqrt{26}\}$
$R[5] := \{-1,\ 1\}$
$R[I] := [-\sqrt{26}, -5] \cup [5, \sqrt{26}]$
$R^{-1}[J] := [5, \sqrt{26}]$

♦

1.11.7

Let $R = \{(x, y) : x \text{ is real and } y = x(x - 1)(x - 2)\}$. Find $R[0]$, $R[1]$, $R[2]$, $R^{-1}[0]$ and $R[I]$, where $I = \{x : 0 \leq x \leq 2\}$.

$R[0] := \{0\}$
$R[1] := \{0\}$
$R[2] := \{0\}$
$R^{-1}[0] := \{0, 1, 2\}$
$R[I] := [-1, 2]$

♦

1.11.8

Let R be a relation between sets X and Y, and suppose that A and B are subsets of X. In each of the following, tell wether the statement is necessarily true and give a justification of your answer.

$$
\begin{array}{ll}
(a) & R[A \cap B] = R[A] \cap R[B]. \\
(b) & R[A \cap B] \subset R[A] \cap R[B]. \\
(c) & R[A \cap B] \supset R[A] \cap R[B].
\end{array}
$$

(a) $R[A \cap B] = R[A] \cap R[B]$

This is not necessarily true. Take for example a non injective function as the relation R with $A \cap B = \varnothing$. This means that $R[A \cap B] = \varnothing$ but the relation being a non injective function it also possible that $R[A] \cap R[B] \neq \varnothing$. So, $R[A \cap B] \not\supset R[A] \cap R[B]$ and we can't have $R[A \cap B] = R[A] \cap R[B]$.

◊

(b) $R[A \cap B] \subset R[A] \cap R[B]$

This is necessarily true. Take $(x, y) : x \in A \cap B, y = R(x)$. Then we have obviously $y \in R[A]$ and also $y \in R[A \cap B]$ but as $x \in B$ (because $x \in A \cap B$) we have also $y \in R[B]$. So $y \in R[A]$ and $y \in R[B]$ and thus $y \in R[A] \cap R[B]$ giving $R[A \cap B] \subset R[A] \cap R[B]$.

◊

(c) $R[A \cap B] \supset R[A] \cap R[B]$

See (a).

♦

1.11.9

Let $\mathbb{Z}$ be the set of all integers. For each m and $n \in \mathbb{Z}$, let us write mRn if and only $m-n$ is an even integer. Thus this relation R is the set $\{(m,n) : m-n = 2k,\ k \in \mathbb{Z}\}$. Find $R[1]$ and $R[2]$. How many distinct sets of the form $R[i]$ are there?

$R[0] := \{n : n = 1-2k,\ k \in \mathbb{Z}\}$ i.e. the set of all odd integers.
$R[1] := \{n : n = 2-2k,\ k \in \mathbb{Z}\} \Leftrightarrow \{n : n = 2k,\ k \in \mathbb{Z}\}$ i.e. the set of all even integers.
There are 2 distinct sets in total.

♦

1.11.10

Let R be the relation defined as follows: For each ordered pair of integers (m,n), let mRn if and only $m-n$ is an integral multiple of 5 (including negative multiples of 5). Find $R[1]$, $R[2]$, and $R[6]$. How many distinct sets of the form $R[i]$ are there? Find $R^{-1}[1]$ and $R^{-1}[2]$. Is $R^{-1}[i] = R[i]$ for each i? For this relation R, if iRj and jRk, does it follow that iRk?

$R[1] := \{\dots, -9, -4, 1, 6, 11, \dots\}$
$R[2] := \{\dots, -8, -3, 2, 7, 12, \dots\}$
$R[6] := \{\dots, -9, -4, 1, 6, 11, \dots\}$
There are 5 distinct sets in total.
$R^{-1}[1] := \{\dots, -9, -4, 1, 6, 11, \dots\} = R[1]$
$R^{-1}[2] := \{\dots, -8, -3, 2, 7, 12, \dots\} = R[2]$
$R^{-1}[i] = R[i]$ for each i as the relation $n = m - 5k, \forall k \in \mathbb{Z}$ can be written as $m = n - 5p, \forall p \in \mathbb{Z}$. So the sets $R^{-1}[i]$ and $R[i]$ are not distinguishable.
If iRj and jRk, does it follow that iRk? Yes, as the composed relation $(jRk) \circ (iRj)$ has the relation $k = i - 5(p+q), p, q \in \mathbb{Z}$ and as $p+q \in \mathbb{Z}$ we can rewrite the relation $(jRk) \circ (iRj)$ as $j = i - 5p, p \in \mathbb{Z}$.

♦

1.12 Set inclusions for image and inverse image sets

1.12.1

Prove that
12.5 Suppose that R is a relation between X and Y . Then, if $\{A_\alpha : \alpha \in \Lambda\}$ is a non-empty collection of subsets of X, the following hold:

$$\mathbf{12.5(a)} \quad R\left[\bigcup\{A_\alpha : \alpha \in \Lambda\}\right] = \bigcup\{R[A_\alpha] : \alpha \in \Lambda\}.$$
$$\mathbf{12.5(b)} \quad R\left[\bigcap\{A_\alpha : \alpha \in \Lambda\}\right] \subset \bigcap\{R[A_\alpha] : \alpha \in \Lambda\}.$$

12.5(a) $\quad R\left[\bigcup\{A_\alpha : \alpha \in \Lambda\}\right] = \bigcup\{R[A_\alpha] : \alpha \in \Lambda\}$.
Be $y \in R\left[\bigcup\{A_\alpha : \alpha \in \Lambda\}\right]$, then there must be an x that is an element of at least one of the A_α and hence y must be in $R[A_\alpha]$, so y will also be in $\bigcup\{R[A_\alpha] : \alpha \in \Lambda\}$ and thus $R\left[\bigcup\{A_\alpha : \alpha \in \Lambda\}\right] \subset \bigcup\{R[A_\alpha] : \alpha \in \Lambda\}$.
Suppose now that $yin \bigcup\{R[A_\alpha] : \alpha \in \Lambda\}$. Then y must be an element of at least one of the $R[A_\alpha]$ and hence there mus be an x that is in a set A_α, so x will also be in $\bigcup\{A_\alpha : \alpha \in \Lambda\}$ and thus $\bigcup\{R[A_\alpha] : \alpha \in \Lambda\} \subset R\left[\bigcup\{A_\alpha : \alpha \in \Lambda\}\right]$.
From this and the previous conclusion follows $R\left[\bigcup\{A_\alpha : \alpha \in \Lambda\}\right] = \bigcup\{R[A_\alpha] : \alpha \in \Lambda\}$.

$\lozenge$

12.5(b) $\quad R\left[\bigcap\{A_\alpha : \alpha \in \Lambda\}\right] \subset \bigcap\{R[A_\alpha] : \alpha \in \Lambda\}$.

We prove first that $R\left[\bigcap\{A_\alpha : \alpha \in \Lambda\}\right] \subset \bigcap\{R[A_\alpha] : \alpha \in \Lambda\}$
Take $y \in R\left[\bigcap\{A_\alpha : \alpha \in \Lambda\}\right]$. Then there must be an $x \in A_\alpha$, $\forall \alpha \in \Lambda$ and thus $y \in R[A_\alpha]$, $\forall \alpha \in \Lambda$ giving $y \in \bigcap\{R[A_\alpha] : \alpha \in \Lambda\}$ and thus $R\left[\bigcap\{A_\alpha : \alpha \in \Lambda\}\right] \subset \bigcap\{R[A_\alpha] : \alpha \in \Lambda\}$

We prove now that $R\left[\bigcap\{A_\alpha : \alpha \in \Lambda\}\right] \not\supset \bigcap\{R[A_\alpha] : \alpha \in \Lambda\}$
Be a non injective function as the relation R with $\bigcap\{R[A_\alpha] : \alpha \in \Lambda\} \neq \varnothing$. As R is a non injective function, we can have a $x \in \bigcap\{R[A_\alpha] : \alpha \in \Lambda\}$ but with $\bigcap\{A_\alpha : \alpha \in \Lambda\} = \varnothing$, which means that x can't be an element of $\bigcap\{A_\alpha : \alpha \in \Lambda\}$ and thus $R\left[\bigcap\{A_\alpha : \alpha \in \Lambda\}\right] \not\supset \bigcap\{R[A_\alpha] : \alpha \in \Lambda\}$.
(*Take for example the relation defined by* $R = \{(n, 1) : n \in \mathbb{N}\}$ *and the subsets of* $\mathbb{N}$, $A_\alpha = \{\alpha\} : \alpha \in \mathbb{N}\}$. *We have* $\bigcap\{R[A_\alpha] : \alpha \in \Lambda\} = \{1\}$ *with* $\bigcap\{A_\alpha : \alpha \in \Lambda\} = \varnothing$.)

$\blacklozenge$

1.12.2

Prove that

12.6 Let $f : X \to Y$ be a function. Let $\{A_\delta : \delta \in \Delta\}$ and $\{B_\lambda : \lambda \in \Lambda\}$ be non empty collections of subsets of X and Y respectively. Then,

12.6(a) $f\left[\bigcup\{A_\delta : \delta \in \Delta\}\right] = \bigcup\{f[A_\delta] : \delta \in \Delta\}$.
12.6(b) $f\left[\bigcap\{A_\delta : \delta \in \Delta\}\right] \subset \bigcap\{f[A_\delta] : \delta \in \Delta\}$.
12.6(c) $f^{-1}\left[\bigcup\{B_\lambda : \lambda \in \Lambda\}\right] = \bigcup\{f^{-1}[B_\lambda] : \lambda \in \Lambda\}$.
$f^{-1}\left[\bigcap\{B_\lambda : \lambda \in \Lambda\}\right] = \bigcap\{f^{-1}[B_\lambda] : \lambda \in \Lambda\}$.

12.6(a) $f\left[\bigcup\{A_\delta : \delta \in \Delta\}\right] = \bigcup\{f[A_\delta] : \delta \in \Delta\}$.

This a direct consequence of **12.6** with $R = f$.

♢

12.6(b) $f\left[\bigcap\{A_\delta : \delta \in \Delta\}\right] \subset \bigcap\{f[A_\delta] : \delta \in \Delta\}$.

This a direct consequence of **12.6** with $R = f$.

♢

12.6(c) $f^{-1}\left[\bigcup\{B_\lambda : \lambda \in \Lambda\}\right] = \bigcup\{f^{-1}[B_\lambda] : \lambda \in \Lambda\}$.

As f^{-1} is a relation and by **12.6** with $R = f^{-1}$ we get the asked identity.

♢

12.6(c′) $f^{-1}\left[\bigcap\{B_\lambda : \lambda \in \Lambda\}\right] = \bigcap\{f^{-1}[B_\lambda] : \lambda \in \Lambda\}$.

As f^{-1} is a relation and by **12.6** with $R = f^{-1}$ we get $f^{-1}\left[\bigcap\{B_\lambda : \lambda \in \Lambda\}\right] \subset \bigcap\{f^{-1}[B_\lambda] : \lambda \in \Lambda\}$.
We prove now that $f^{-1}\left[\bigcap\{B_\lambda : \lambda \in \Lambda\}\right] \supset \bigcap\{f^{-1}[B_\lambda] : \lambda \in \Lambda\}$.
Suppose $x \in \bigcap\{f^{-1}[B_\lambda] : \lambda \in \Lambda\}$ then $x \in f^{-1}[B_\lambda] : \forall\lambda \in \Lambda$. This means that there must be an unique $y = f(x)$ (f being a function) for which yields $y \in B_\lambda : \forall\lambda \in \Lambda$. Hence y must be in $\bigcap\{B_\lambda : \lambda \in \Lambda\}$ and thus $x \in f^{-1}\left[\bigcap\{B_\lambda : \lambda \in \Lambda\}\right]$ giving $f^{-1}\left[\bigcap\{B_\lambda : \lambda \in \Lambda\}\right] \supset \bigcap\{f^{-1}[B_\lambda] : \lambda \in \Lambda\}$.

♦

1.12.3

Prove that
12.7 Let $f : X \to Y$ be a function. Then, each of the following holds

$$\begin{array}{ll} \mathbf{12.7(a)} & \forall x \in X,\ x \in f^{-1}[f[x]]. \\ \mathbf{12.7(b)} & \forall A \subset X,\ A \subset f^{-1}[f[A]]. \\ \mathbf{12.7(c)} & \forall y \in \text{Range} f,\ f[f^{-1}[y]] = \{y\}. \end{array}$$

12.7(a) $\forall x \in X,\ x \in f^{-1}[f[x]]$.
Be $y = f(x)$, then obviously there is at least one x (there could be more if f is not injective), so that $x = f^{-1}[y]$. Hence, $\forall x \in X,\ x \in f^{-1}[f[x]]$.

♢

12.7(b) $A \subset X,\ A \subset f^{-1}[f[A]]$.
This is a consequence of the previous statement but with the remark that we could have (for a non injective function) a $x \in B$ with $A \cap B = \varnothing$ for which we have $f(x) \in f[A]$. So A is not always equal to $f^{-1}[f[A]]$ and get $A \subset X,\ A \subset f^{-1}[f[A]]$.

♢

12.7(c) $\forall y \in \text{Range} f,\ f[f^{-1}[y]] = \{y\}$.
This a direct consequence of f being a function. Indeed suppose for a given y we have the set $A = f^{-1}[\{y\}]$, so this set will contain all x as element which f maps (uniquely, f being a function) to y. So $f[A] = f[f^{-1}(y)] = \{y\}$.

♦

1.12.4

Prove that
Suppose that $f : X \to Y$ is a function and A and B are subsets of X. Suppose also that C and D are subsets of Y. For each of the following, determine whether the statement is necessarily true. In any case for which the statement is not necessarily true, determine whether it is under any of the following conditions: $f : X \to Y$ is a surjection,$f : X \to Y$ is a injection,$f : X \to Y$ is a bijection.

$$\begin{array}{ll} \mathbf{(a)} & f[A - B] = f[A] - f[B]. \\ \mathbf{(b)} & f^{-1}[D - C] = f^{-1}[D] - f^{-1}[C]. \\ \mathbf{(c)} & f^{-1}[f[A]] = A. \\ \mathbf{(b)} & f[f^{-1}[C]] = C. \end{array}$$

(a) $f[A - B] = f[A] - f[B]$.
This is not necessarily True.

Suppose, $x \in A$ but not in B and $y = f(x)$, so $y \in f[A - B]$ but if f is not a surjection then it is possible that a $x^{'} \in B$ exists which is mapped to y, meaning that y will not be an element of $f[A] - f[B]$ i.e. $y \notin f[A] - f[B]$ and thus $f[A - B] \not\subset f[A] - f[B]$ meaning that not always $f[A - B] = f[A] - f[B]$. So this identity can only be true if f is a surjection or a bijection as a bijection has to be an injection. Of course $f : X \to Y$ is a surjection, is not a sufficient condition for the identity to be true as a surjection is not necessarily an injection.

$\Diamond$

(**b**) $f^{-1}[D - C] = f^{-1}[D] - f^{-1}[C]$.
This is True if f is a surjection (or by extension a bijection).
Suppose, $x \in f^{-1}[D - C]$, so there is a $y \in D - C$ for which $x = f^{-1}(y)$. Also, $y \in D$ but not in C. Can there be $y^{'} \in C, \notin D$ for which $y^{'} = f(x)$? Obviously not, as f is a function meaning that $y^{'} = f(x)$ and $y = f(x) \Rightarrow y^{'} = y$. This means that x can't be an element of $f^{-1}[C]$ and thus that $x \in f^{-1}[D] - f^{-1}[C]$. Hence, $f^{-1}[D - C] \subset f^{-1}[D] - f^{-1}[C]$.
Suppose now that $x \in f^{-1}[D] - f^{-1}[C]$, so there is no $x \in f^{-1}[C]$ for which $y = f(x), y \in C$. This means that $y \notin C$ but y must be in D. i.e. $y \in D - C$ and thus $x \in f^{-1}[D - C]$ or $f^{-1}[D] - f^{-1}[C] \subset f^{-1}[D - C]$, leading to the identity.
Remark that the reasoning deployed implies that f is a surjection as if $y \in D - C$ has no inverse image in X, this would mean that $f^{-1}[D - C] = \varnothing$ and thus no x would exist for the given identity.

$\Diamond$

c) $f^{-1}[f[A]] = A$.
This is not necessarily True.
Be $y \in f[A]$, if f is not an injection then it is possible that there exist a $x^{'} \in B \not\subset A$ so that $f(x^{'}) = y$, so $x^{'}$ will be an element of $f^{-1}[f[A]]$ and as $x^{'} \notin A$ the identity can't be true. So, f needs to be an injection (and by extension a bijection) for the identity to be true.

$\Diamond$

(**d**) $f[f^{-1}[C]] = C$.
This is True if f is a surjection (or by extension a bijection).
Be $y \in C$ and $x \in f^{-1}[C]$, as f is a function then $f(x)$ will be in C and the set $f^{-1}[C]$ will contain all x for which $f(x) \in C$. But note that C may contain elements which are not mapped by f. In that case $f[f^{-1}[C]] \not\subset C$. So f needs to be a surjection (or by extension a bijection).
On the other hand, suppose $y \in C$, then $f^{-1}[C]$ will contain all $x \in X$ which are mapped to C and f being a function we will have $f[f^{-1}[C]] \subset C$.
Conclusion, the identity is true if f is a surjection (or by extension a bijection).
To illustrate this, take A as the subset $\mathbb{N} \subset \mathbb{R}$ and C as the subset of $\mathbb{R}$ with the even natural numbers as elements. Define now the function $f = \{(n, 4n) : n \in A\}$. Obviously $f[A] \neq C$ as $f[A] = \{4, 8, 12, \dots\} \neq \{2, 4, 6, 8, \dots\}$. Then as $f^{-1}[C] = A$, we have $f[f^{-1}[C]] \neq C$

$\blacklozenge$

1.12.5

Let $M : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be the map from $\mathbb{R}\times\mathbb{R}$ into $\mathbb{R}$ defined as follows: For each $(a, b) \in \mathbb{R}\times\mathbb{R}$, let $M((a, b)) = ab$. Is M a map from $\mathbb{R}\times\mathbb{R}$ onto $\mathbb{R}$? Representing $\mathbb{R}\times\mathbb{R}$ s a plane, draw a sketch for each of the following sets: $M^{-1}[0]$, $M^{-1}[1]$, $M^{-1}[I]$, where I is the closed interval $[0, 1]$.

Yes, M is a map from $\mathbb{R}\times\mathbb{R}$ onto $\mathbb{R}$ as every $x \in \mathbb{R}$ can be expressed as the product of two real numbers.

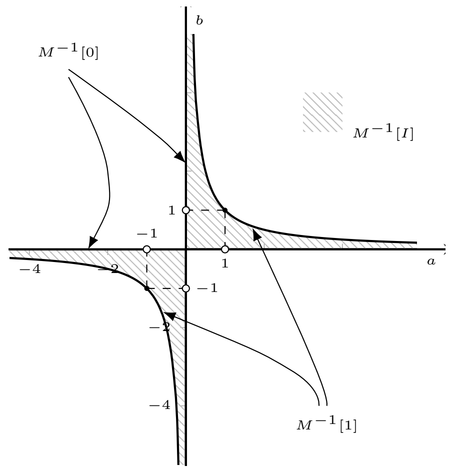


Figure 1.18: Inverse image of $M^{-1}(0)$, $M^{-1}(1)$, $M^{-1}(I)$

♦

1.12.6

Examine carefully the content of Theorem **12.6** and your answer to Exercise 4(a) and (b). Which seems to have a nicer behaviour on collections of sets, f or f^{-1}?

Putting aside the notion of 'nicer', we still could put forward that f^{-1} requires less restrictions in order to have certain identities.
Take first **12.6**(b) $\quad f\left[\bigcap\{A_\delta : \delta \in \Delta\}\right] \subset \bigcap\{f[A_\delta] : \delta \in \Delta\}$ compared to **12.6**(c') $\quad f^{-1}\left[\bigcap\{B_\lambda : \lambda \in \Lambda\}\right] = \bigcap\{f^{-1}[B_\lambda] : \lambda \in \Lambda\}$. f^{-1} requires no special condition (except for f being a function) in order to have an equality for the intersection of the sets in the collection.
Moreover in Exercise 4, for having the identity 4(a) $\quad f[A - B] = f[A] - f[B]$, we need f to be at least an injection while for 4(b) $\quad f^{-1}[D - C] = f^{-1}[D] - f^{-1}[C]$ we "only need af to be a surjection which can be achieved by restricting the target Y to $D \cup C$.

♦

1.13 The restriction of a function

1.13.1

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be mappings such that $f(x) = \sin x$ for each $x \in \mathbb{R}$ and $g(x) = \sqrt{1 - \cos^2 x}$ for each $x \in \mathbb{R}$. Find the largest interval of real numbers, I, whose left endpoint is 0 and which satisfies $f|I = g|I$.

As $g(x)$ can be expressed as $g(x) = |\sin x|$ the condition $f(x) = g(x)$ will only be met in the intervals $\{[2\pi k, \pi(2k+1)] : k \in \mathbb{P} \cup \{0\}\}$. Putting $k = 0$ we get $I = [0, \pi]$.

♦

1.13.2

Let $f : \mathbb{R} \to \mathbb{R}$ be defined as follows. For each $x \in \mathbb{R}$, let $f(x) = |x - 1|$. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by and $g(x) = x - 1$ for each $x \in \mathbb{R}$. Find the largest set $S \subset \mathbb{R}$ for which $f|I = g|I$.

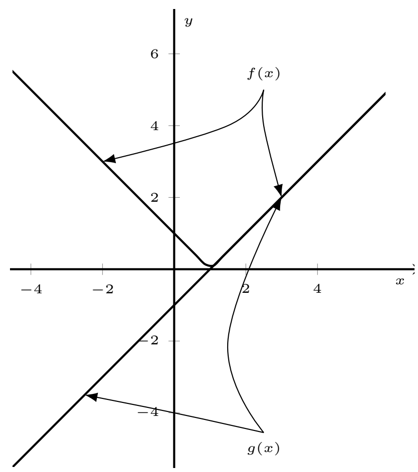


Figure 1.19: $f(x) = |x - 1|$ and $g(x) = x - 1$

From the figure we get $S = [1, +\infty)$.

♦

1.13.3

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) = \cos x$ and $f(x) = \sqrt{1 - \sin^2 x}$ for $x \in \mathbb{R}$. Find the largest set $S \subset \mathbb{R}$ for which $f|I = g|I$.

As $f(x) \equiv |\cos x|$, $f(x) = g(x)$ implies $S = \{[-\frac{\pi}{2}k, -\frac{\pi}{2}k + \pi] : k \in \mathbb{Z}_{/\{0\}}\}$.

1.14 Composition of functions

1.14.1

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \frac{x}{x+1}$ and $g(x) = x^2$. Give explicit formulas for $g \circ f(x)$ and $f \circ g(x)$. Determine the ranges of $g \circ f$ and $f \circ g$.

$g \circ f(x) = \frac{x^2}{x^2+2x+1}$
$f \circ g(x) = \frac{x^2}{x^2+1}$
$Range\, g \circ f = [0, +\infty)$
$Range\, f \circ g = [0, 1)$

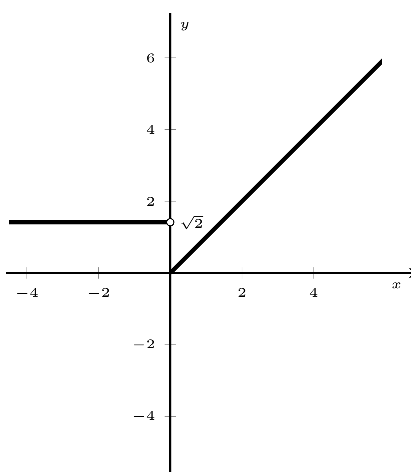
♦

1.14.2

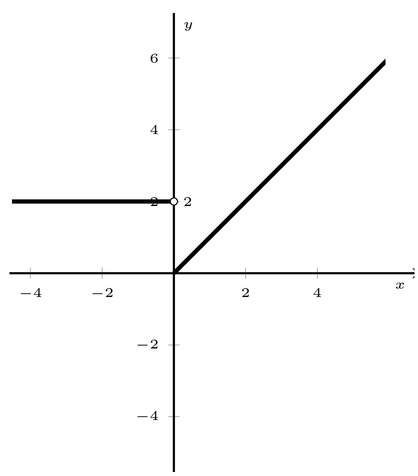
Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$\begin{aligned} f(x) &= x^2 && \text{for } x \geq 0 \\ &= 2 && \text{for } x < 0 \\ g(x) &= \sqrt{x} && \text{for } x \geq 0 \\ &= x && \text{for } x < 0 \end{aligned}$$

(a) Sketch the graph of $g \circ f$
(b) Sketch the graph of $f \circ g$
(c) Find $(f \circ g)^{-1}[x]$ for each $x \in \mathbb{R}$



(a) $g \circ f$



(b) $f \circ g$

(c) Find $(f \circ g)^{-1}[x]$ for each $x \in \mathbb{R}$

$(f \circ g)^{-1}[x] = \varnothing, \forall x < 0$

$(f \circ g)^{-1}[x] = \{x\}, \forall x \geq 0$

$(f \circ g)^{-1}[x] = (-\infty, 0) \cup \{2\}$ for $x = 2$

♦

1.14.3

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \sin x$ and $g(x) = |x|$. Write explicit expressions for $g \circ f(x)$ and $f \circ g(x)$ and find the range of each.

$g \circ f(x) = |\sin x|$
$f \circ g(x) = \sin |x|$
$\textit{Range}\, g \circ f = [0, 1]$
$\textit{Range}\, f \circ g = [-1, 1]$

♦

1.14.4

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2 + 2$ and $g(x) = x - 1$. Find expressions for $(g \circ f)(x)$ and $(f \circ g)(x)$ and note that $g \circ f \neq f \circ g$.

$(g \circ f)(x) = x^2 + 1$
$(f \circ g)(x) = x^2 - 2x + 3$

♦

1.14.5

Let $f : \mathbb{R} \to \mathbb{R}$,$g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2 + x$, $g(x) = (x-1)^2$ and $h(x) = x + 1$ for each $x \in \mathbb{R}$. Find an expression for $h \circ g \circ f(x)$ for $x \in \mathbb{R}$.

$$h \circ g \circ f(x) = \underbrace{\underbrace{\left((\underbrace{x^2 + x}_{f(x)}) - 1\right)^2}_{g(x)} + 1}_{h(x)} = x^4 + 2x^3 - x^2 + 2x + 2$$

♦

1.14.6

Suppose $f : X \to Y$ is a bijection. Show that $f^{-1} \circ f = i$ where $i : X \to X$ is the identity map on X and $f \circ f^{-1} = j$ where j is the identity map on Y.

$\mathbf{f^{-1} \circ f = i}$

Let $x \in X$. Then, $(x, f(x)) \in f$. Hence, $(f(x), x) \in f^{-1}$. thus, $f^{-1}(f(x)) = x$ and $\left(f^{-1} \circ f\right)(x) = x$ for each $x \in X$. hence, $f - 1 \circ f = i$, the identity map on X.

$\mathbf{f = i \circ f^{-1} = j}$

Let $y \in Y$. Then, $\left(y, f^{-1}(y)\right) \in f^{-1}$. Hence, $\left(f^{-1}(y), y\right) \in f$. thus, $f\left(f^{-1}(y)\right) = y$ and $\left(f \circ f^{-1}\right)(y) = y$ for each $y \in Y$. hence, $f \circ f^{-1} = j$, the identity map on Y.

Note that in the step $\left(y, f^{-1}(y)\right) \in f^{-1}$ we implicitly use the fact that f is a bijection.

♦

1.14.7

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be surjections. Suppose $g \circ f = i$ where $i : X \to X$ is the identity map on X. Show:

(a) f is one-to-one
(b) g is one-to-one
(c) $f \circ g = j$ where $j : X \to X$ is the identity map fromY onto Y
(d) $f = g^{-1}$
(d) $g = f^{-1}$

(a) **f** is one-to-one

Suppose that f is not one-to-one. This means that there exist 2 or more $x_1, x_2, \ldots$ which are mapped by f to an element $y^* \in Y$. But as g is a function, the image $g[\{y^*\}]$ can only have one element. Hence, x_1 or x_2 will not be included in this image. Suppose that x_2 is not an element of $g[\{y^*\}]$. As g is a surjection, there must be an $y^{'} \in Y$ so that $g(y^{'}) = x_2$. But also f is a surjection, so there must be an $x_3 \in X$ so that $f(x_3) = y^{'}$. So $g(y^{'}) = x_2$ implies $g(f(x_3)) = x_2$. But $g(f(x_3)) = g \circ f(x^3)$ and given that $g \circ f = i$ (the identity map) we get $g \circ f(x^3) = x_3$ and can conclude that $x_2 = x_3$ resulting in $f(x_2) = y^{'}$. But we started with the assumption that $f(x_2) = y^*$ and as f is a function, this means $y^{'} = y^*$, so there is no other $(y, x_2) \in g$ that maps y to x_2, meaning that only the element x_1 is mapped to the chosen y^*. Hence f is one-to-one.

$\Diamond$

(b) **g** is one-to-one

Suppose that g is not one-to-one. This means that for a given x^* we could have 2 (or more) $y_1, y_2, \ldots,$ so that $g(y_1) = x^*$ and $g(y_2) = x^*$. But f is on-to-one (see $1.14.7(a)$) so we must have two distinct x_1, x_2 for which we have $f(x_1) = y_1$ and $f(x_2) = y_2$, but $g(y_1) = x^*$ and $g(y_2) = x^*$, so $g \circ f(x_1) = x^*$ and $g \circ f(x_2) = x^*$ and given that $g \circ f = i$ (the identity map) we get $x_1 = x_2 = x^*$ and also $y_1 = y_2$ and conclude that g must be one-to-one.

$\Diamond$

(c) $\mathbf{f} \circ \mathbf{g} = \mathbf{j}$ where $\mathbf{j} : \mathbf{X} \to \mathbf{X}$ is the identity map from **Y** onto **Y**

We have $g \circ f = i$, so $g \circ f(x) = x$. As $x \in X$ we can apply f to x and this gives $(f \circ g \circ f)(x) = f(x)$ or $(f \circ g) \circ f(x) = f(x)$. Put $f(x) = y$, we can rewrite this as $(f \circ g)(y) = y$. Hence $f \circ g$ is the identity map $j : Y \to Y$.

$\Diamond$

(d) $\mathbf{f} = \mathbf{g}^{-1}$

Be $(y, x) \in g$. Then, $(x, y) \in g^{-1}$. As f is a bijection, we have one $x \in X$ and one $y \in Y$ so that $y = f(x)$. Then, $(x, y) \in g^{-1}$ is equivalent to $(x, f(x)) \in g^{-1}$. But, by definition , we have $(x, f(x)) \in f$, and conclude $f \subset g^{-1}$.

Be now, $(x, y) \in f$. As g is one-to-one (see above), there is just one (x, y) so that $x = g(y)$. Thus, $(g(y), y) \in f$. We notice that $(y, g(y)) \in g$, or $(g(y), y) \in g^{-1}$. This give with $(g(y), y) \in f$, $g^{-1} \subset f$. Combining with the first subset we get $f = g^{-1}$.

$\Diamond$

(e) $\mathbf{g} = \mathbf{f}^{-1}$

Be $(x, y) \in f$. Then, $(y, x) \in f^{-1}$. As g is a bijection, we have one $x \in X$ and one $y \in Y$ so that $x = g(y)$. Then, $(y, x) \in f^{-1}$is equivalent to $(y, g(y)) \in f^{-1}$. But, by definition , we have $(y, g(y)) \in g$, and conclude $g \subset f^{-1}$.

Be now, $(y, x) \in g$. As g is one-to-one (see above), there is just one (y, x) so that $x = g(y)$. Thus,

$(y, g(y)) \in g$. But $y = f(x)$. This gives $(f(x), \underbrace{f \circ g}_{=j}(y)) \in g$. But we notice that $(x, f(x)) \in f$ or $(f(x), x) \in f^{-1}$ and conclude with $(f(x), x) \in g$ that $f^{-1} \subset g$.
Combining with the first subset we get $g = f^{-1}$.

♦

1.14.8

Recall that $\mathbb{R}_+ = \{x : x \in \mathbb{R} \text{ and } x > 0\}$. Recall also that the natural logarithm function , ln, is defined on $\mathbb{R}_+$, with range $\mathbb{R}$; the exponential function, $\{(x, e^x) : x \in \mathbb{R}\}$, is the inverse of the of the ln function.; $\cosh x = \frac{e^x + e^{-x}}{2}$ for each $x \in \mathbb{R}$.
Sketch the cosh function and note that although cosh is not one-to-one, the restriction $\cosh|\mathbb{R}_+$ is one-to-one and hence, its inverse is a function. By using the results of Exercise 1.14.7, prove that $\cosh^{-1}(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$ for $x \geq 1$.

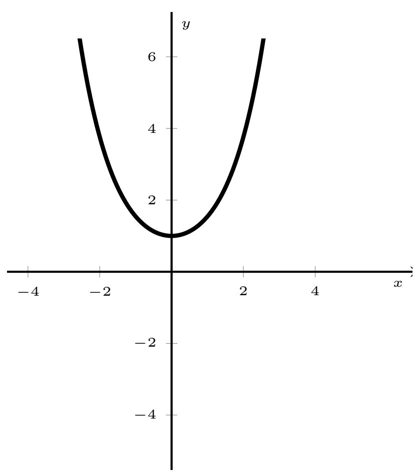


Figure 1.20: The function $\cosh x$

For two functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ for which yields $g \circ f = i$, we know from Exercise 1.14.7(c) that $f \circ g = j$ (the unit map). We put $f(x) = \cosh x = \frac{e^x + e^{-x}}{2}$ and try to find $g = f^{-1}$. From the sketch, we see that if we restrict x to $[0, +\infty)$ that f is bijective with $\mathit{Range}\, f = [1, +\infty)$. What we have to do, is to solve the functional equation $g\left(\frac{e^x + e^{-x}}{2}\right) = x$ (with $g = f^{-1}$) (this expression represent $g \circ f = i$).
We use the identity $a = e^{\ln a}$ and use the equivalence of $g \circ f = i$ and $f \circ g = j$, i.e.

$$\frac{e^{g(y(} + e^{-g(y)}}{2} = y$$

We put tentatively $g(y) = \ln p(y)$ with $p : [1, +\infty) \to \mathbb{R}$ and get from $a = e^{\ln a}$:

$$
\begin{aligned}
& \frac{e^{g(y)} + e^{-g(y)}}{2} = y \\
\Leftrightarrow \quad & \frac{e^{\ln p(y)} + e^{-g(y)}}{2} = y \\
\Leftrightarrow \quad & \frac{p(y) + \frac{1}{p(y)}}{2} = y \\
\Leftrightarrow \quad & p^2(y) - 2p(y)y = -1 \\
\Leftrightarrow \quad & p^2(y) - 2p(y)y + y^2 = y^2 - 1 \\
\Leftrightarrow \quad & \left(p(y) - y\right)^2 = y^2 - 1 \\
\Rightarrow \quad & p(y) = y \pm \sqrt{y^2 - 1}
\end{aligned}
$$

From our definition $g(y) = \ln p(y)$, we get $g(y) = \ln\left(y \pm \sqrt{y^2 - 1}\right)$. To get a real valued g we need obviously $y \geq 1$ but also need that $g(y) \geq 0$ as we restricted the function f to $\mathbb{R}_+$ which requires $y \pm \sqrt{y^2 - 1} \geq 1$ and thus only the solution $g(y) = \ln\left(y + \sqrt{y^2 - 1}\right), \quad y \geq 1$ can be retained.

♦

1.14.9

Consider the map $P : \mathbb{R}^2 \to \mathbb{R}$ such that for each $(x, y) \in \mathbb{R}^2$, $\alpha((x, y))$ is the element $(u, v) \in \mathbb{R}^2$ given by $u = 2x - y$, $v = 5x + y$. Recall that $\mathbb{R}^2$ denotes $\mathbb{R} \times \mathbb{R}$.

(a) Does $\alpha[\mathbb{R}^2] = \mathbb{R}^2$?.
(b) Is α one-to-one?.
(c) If α is one-to-one, find a rule for α^{-1} analogous to the rule given for α.

(a) Does $\alpha[\mathbb{R}^2] = \mathbb{R}^2$?
(b)Is α one-to-one?.
The answer is yes to both questions as we can consider the map as a system of linear equation with (x, y) as unknowns and (u, v) as parameters i.e.

$$
\begin{cases} 2x - y = u \\ 5x + y = v \end{cases}
$$

The determinant of this system is not zero (it is 7) and so for every (u, v) we have a unique (x, y). From the definition it is also clear that $u, v \in \mathbb{R}$ (it is in fact a map from one plane to another plane).

So, $\alpha[\mathbb{R}^2] = \mathbb{R}^2$ and α is a bijection.

$\diamondsuit$

(c) If α is one-to-one, find a rule for α^{-1} analogous to the rule given for α.

$$\begin{cases} x = & \frac{u+v}{7} \\ y = & \frac{2v-5u}{7} \end{cases}$$

$\blacklozenge$

1.14.10

Consider the map $P : \mathbb{R}^2 \to \mathbb{R}$ such that for each $(x, y) \in \mathbb{R}^2$, $P((x, y)) = x$. Note that P is not one-to-one. Find a subset S of $\mathbb{R}^2$ such that $P|S : S \to \mathbb{R}$ is a one-to-one map from S to $\mathbb{R}$.

We can find two types of restrictions:
A type with y a constant i.e. $P_r|S_r : S_r \to \mathbb{R}$ with $P_r = \{(x, r) : x \in \mathbb{R},\ r \in \mathbb{R} = \text{ constant}\}$.

And a type with $y = f(x)$ where $f : \mathbb{R} \to \mathbb{R}$ is a one-to-one function.

$\blacklozenge$

1.14.11

Suppose $f : A \to B$ is a one-to-one map from A to B and $g : B \to C$ is a one-to-one map from B into C. Prove that $g \circ f : A \to C$ is one-to-one map from A into C.

Suppose $g \circ f$ is not one-to-one. This means that for at least one $z \in C$ there exist two (or more) x_1, x_2 such that (x_1, z^*) and $(x_2, z^*) \in g \circ f$. But $f : A \to B$ and $g : B \to C$ are one-to-one maps which means that $x_1 \in A$ is mapped to a unique $y_1 \in B$ (the same for x_2 mapped to a unique y_2) and also $y_1 \in B$ is mapped to a unique $z_1 \in C$ (the same for y_2 mapped to a unique z_2). So the map $g \circ f$ will map x_1 into z_1 and x_2 into z_2. So supposition that there exist a z^* is in contradiction with our result that $z_1 \neq z_2$.
Is $g \circ f$ a surjection ?
As f and g are one-to-one, every $y \in B$ and every $z \in C$ are covered by f, g respectively, so every $z \in C$ is also covered by $g \circ f$. So, $g \circ f$ is a surjection and by the first argument also a bijection.

♦

1.14.12

Let $f : \mathbb{R} \to \mathbb{R}$ be a map such that for each pair of numbers x and y, $f(x+y) = f(x) + f(y)$.

(a) Show that $f(0) = 0$.
(b) Show that $f(-x) = -f(x), \quad \forall x \in \mathbb{R}$.
(c) Show that $f(mx) = mf(x), \quad \forall m \in \mathbb{Z}, x \in \mathbb{R}$.
(d) Show that $f(rx) = rf(x), \quad \forall r \in \mathbb{Q}, x \in \mathbb{R}$.

(a) Show that $f(0) = 0$.
Put $x = 0$ and $y = 0$. We have, $\underbrace{f(0+0)}_{=f(0)} = \underbrace{f(0) + f(0)}_{=2f(0)}$. This implies that $f(0) = 0$.

(b) Show that $f(-x) = -f(x), \quad \forall x \in \mathbb{R}$.
Put $y = -x$. We have, $\underbrace{f(x + (-x))}_{=f(0)=0} = f(x) + f(-x)$. This implies that $f(-x) = -f(x)$.

(c) Show that $f(mx) = mf(x), \quad \forall m \in \mathbb{Z}, x \in \mathbb{R}$.

$$\begin{aligned}
f(mx) &= f(x + (m-1)x) \\
&= f(x) + f((m-1)x) \\
&= f(x) + f(x + (m-2)x) \\
&= f(x) + f(x) + f((m-2)x) \\
&= \quad \vdots \\
&= \underbrace{f(x) + f(x) + \ldots f(x)}_{=mf(x)}
\end{aligned}$$

and get the required result.
(d) Show that $f(rx) = rf(x), \quad \forall r \in \mathbb{Q}, x \in \mathbb{R}$.

$$\begin{array}{ccc}
 & f(rx) & = f(\frac{m}{n}x) \\
\Leftrightarrow & f(rx) & = mf(\frac{1}{n}x) \\
\times n & nf(rx) & = mnf(\frac{1}{n}x) \\
\Leftrightarrow & nf(rx) & = mf(\frac{n}{n}x) \\
\Rightarrow & f(rx) & = \frac{m}{n}f(x) \\
\Rightarrow & f(rx) & = rf(x)
\end{array}$$

♦

1.15 Sequences

1.15.1

In each of the following find a formula for the nth term a_n of an infinite sequence whose first five terms are given.

$$\begin{array}{ll} (a) & a_l = 1, a_2 = \frac{1}{2}, a_3 = \frac{1}{4}, a_4 = \frac{1}{8}, a_5 = \frac{1}{16} \\ (b) & a_1 = 1, a_2 = 0, a_3 = 1, a_4 = 0, a_5 = 1. \\ (c) & a_1 = 1, a_2 = 0, a_3 = -1, a_4 = 0, a_5 = 1. \\ (d) & a_1 = 1, a_2 = 3, a_3 = 6, a_4 = 10, a_5 = 15. \end{array}$$

(a) $a_l = 1, a_2 = \frac{1}{2}, a_3 = \frac{1}{4}, a_4 = \frac{1}{8}, a_5 = \frac{1}{16}$

$$a_n = \frac{1}{2^{n-1}}$$

$\diamondsuit$

(b) $a_1 = 1, a_2 = 0, a_3 = 1, a_4 = 0, a_5 = 1$

$$a_n = n \pmod 2$$

$\diamondsuit$

(c) $a_1 = 1, a_2 = 0, a_3 = -1, a_4 = 0, a_5 = 1$

$$a_n = \left(\sin \frac{\pi}{2} n\right) (n \pmod 2)$$

$\diamondsuit$

(d) $a_1 = 1, a_2 = 3, a_3 = 6, a_4 = 10, a_5 = 15$

$$a_{n+1} = a_n + (n+1), \quad a_1 = 1$$

$\blacklozenge$

1.15.2

Let $f = (f_i)_{i=1}^{+\infty}$ g, be the sequence defined as follows: Let $g(x) = \sin x$. For each positive integer i, let $f_i = g^{(i)}(0)$, where $g^{(i)}$ is the i^{th} derivative of g. Write the terms off sufficiently far to see the pattern followed.

We have $g^{(1)}(x) = \cos x$, $g^{(2)}(x) = -\sin x$, $g^{(3)}(x) = -\cos x$, $g^{(4)}(x) = \sin x$, $g^{(5)}(x) = \cos x$, $\ldots$, so we get $f = (1, 0, -1, 0, 1, 0, -1, \ldots$, so from Exercise 1.15.1(c), we get

$$f_i = \left(\sin \frac{\pi}{2} i\right) (i \pmod 2)$$

♦

1.15.3

For each $n \in P$, let $a_n = \sum_{j=1}^{n} j^2$. Try to discover a formula for a_n.

We can suppose that when $n \to +\infty$ that $a_n = \sum_{j=1}^{n} j^2$ and $\int_0^n x^2 dx$ will tend to a common value. As $\int_0^n x^2 dx = \frac{1}{3}n^3$ we tentatively write a_n as a cubic expression $a_n = an^3 + bn^2 + cn + d$. As $a_{n+1} = a_n + (n+1)^2$ we try to find the parameters a, b, c, d by solving the expression

$$a(n+1)^3 + b(n+1)^2 + c(n+1) + d = an^3 + bn^2 + cn + d + (n+1)^2$$

for some values of n. Expanding gives

$$(3a-1)n^2 + (3a+2b-2)n + a + b + c - 1 = 0$$

For $n = 1, 2, 3$ we get the following system of linear equations

$$\begin{cases} 7a + 3b + c = 4 \\ 19a + 5b + c = 9 \\ 37a + 7b + c = 16 \end{cases}$$

and get $a_n = \frac{1}{3}$, $b = \frac{1}{2}$, $c = \frac{1}{6}$ giving $a_n = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n + d$ and for $n = 1$, $a_n = 1$ we get $1 = \underbrace{\frac{1}{3} + \frac{1}{2} + \frac{1}{6}}_{=1} + d$, giving $d = 0$. So, we get the expression $a_n = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$ which can be rewritten as

$$a_n = \frac{n(2n+1)(n+1)}{6}$$

Yet, we didn't prove formally that this equation works for every n:
We notice that this equation is exact for $n = 1$. Suppose that it is also correct for a n. Then:

$$\begin{aligned}
a_{n+1} &= a_n + (n+1)^2 \\
&= \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n + (n+1)^2 \\
&= \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n + n^2 + 2n + 1 \\
&= \frac{1}{3}n^3 + \frac{3}{2}n^2 + \frac{13}{6}n + 1
\end{aligned}$$

Using the proposed closed expression for $n+1$ gives

$$\begin{aligned}
a_{n+1} &= \frac{1}{3}(n+1)^3 + \frac{1}{2}(n+1)^2 + \frac{1}{6}(n+1) \\
&= \frac{1}{3}n^3 + n^2 + n + \frac{1}{3} + \frac{1}{2}n^2 + n + \frac{1}{2} + \frac{1}{6}n + \frac{1}{6} \\
&= \frac{1}{3}n^3 + \frac{3}{2}n^2 + \frac{13}{6}n + 1
\end{aligned}$$

So, both expression are the same and by the axiom of induction , we conclude that the expression is correct for every n.

♦

1.15.4

Suppose that a is a sequence such that $a_l = 1$, $a_2 = 3$, $a_3 = a_1 + a_2$, and for $j \geq 3$, $a_j = a_{j-1} + a_{j-2}$. Find a_4, a_5, a_6 and a_7.

$a_4 = 6$, $a_5 = 10$, $a_6 = 16$, $a_7 = 26$.

♦

1.15.5

Suppose a sequence a is given by $a_n = 2^n$ for each positive integer n. For which values of n is it true that $a_n \geq 10,000$?

We have to find a n such that $\log_2 2^n \geq \log_2 10,000$ (the logarithm function is a strict increasing function).

So, $n \geq \log_{10} 10,000 \log_2 10$ or $n \geq \underbrace{4 \times 3.3}_{=13.3}$ and hence $n \geq 14$.

♦

1.15.6

Let a be the sequence given by $a_n = \frac{n}{(n+1)}$. Find the smallest integer N such that for $n \geq N$, $a_n > \frac{9}{10}$.

We need $\frac{n}{(n+1)} > \frac{9}{10}$ or $n > 9$, hence $N = 10$.

♦

1.15.7

Let a be the sequence given by $a_n = \sqrt{n+1} - \sqrt{n}$. Find an integer N such that for $n \geq N$, $a_{n+1} < a_n$. Find an integer M such that for $n \geq M$, $a_n \leq \frac{1}{10}$.

$a_{n+1} < a_n$ gives $\sqrt{n+1} - \sqrt{n} < \sqrt{n} - \sqrt{n-1}$ or $\sqrt{n+1} + \sqrt{n-1} < 2\sqrt{n}$. As both sides are positive, we can take the power of thus inequality and get $2n + 2\sqrt{n^2-1} < 4n$ or $\sqrt{n^2-1} < n$ giving the trivial inequality $0 > -1$ meaning that the inequality yields for all $N > 0$.

$a_n \leq \frac{1}{10}$
We have

$$\begin{aligned} & \sqrt{n+1} - \sqrt{n} \leq \frac{1}{10} \\ \Rightarrow \quad & n+1 \leq \left(\sqrt{n} + \frac{1}{10}\right)^2 \\ \Rightarrow \quad & (4.95)^2 \leq n \end{aligned}$$

from which we conclude that n must be greater or equal to 25.

♦

1.15.8

For each $(x, y) \in \mathbb{R}^2$, let $f(x, y) = (x^2 - y^2, 2xy)$. Let the point p_1 in the plane be given by $f(\frac{1}{2}, \frac{1}{2})$, $p_2 = f(p_1)$, $p_3 = f(p_2)$, $p_4 = f(p_3)$, $p_5 = f(p_4)$. Calculate and plot the points $p_1, p_2, \ldots, p_5$.

$p(1) = (0, \frac{1}{2})$, $p(2) = (-\frac{1}{2^2}, 0)$, $p(3) = (\frac{1}{2^4}, 0)$, $p(4) = (\frac{1}{2^8}, 0)$, $p(5) = (\frac{1}{2^{16}}, 0)$

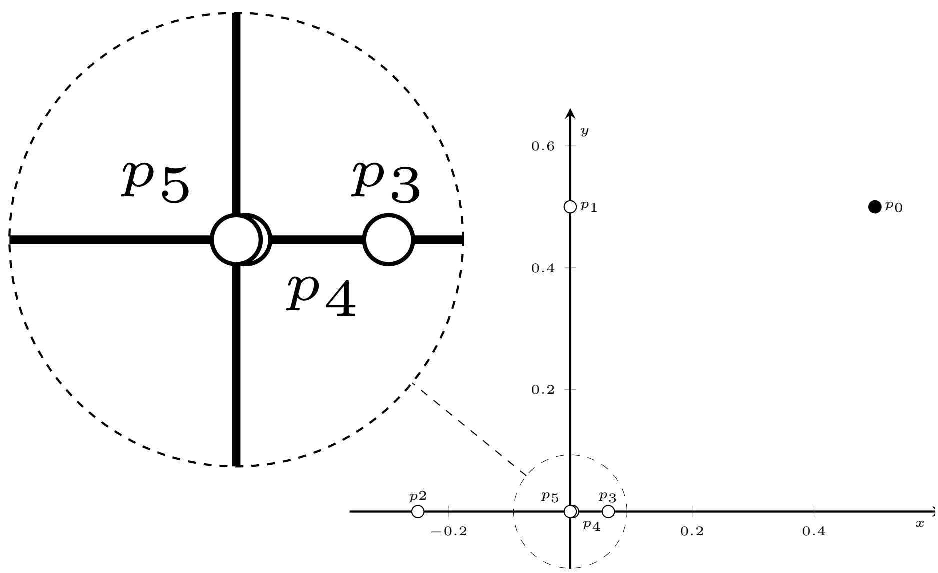


Figure 1.21: The dynamical system $f(x, y) = (x^2 - y^2, 2xy)$ with starting point $(\frac{1}{2}, \frac{1}{2})$

♦

1.15.9

Let $A_l = \{1, 2, 3\}, A_2 = \{1, 2\}, A_3 = \{a, b\}$. Write out the elements of $A_1 \times A_2 \times A_3$. Write out the elements of $(A_1 \times A_2) \times A_3$. Show that there is a "natural" one-to-one correspondence between the elements of these two sets.

$A_1 \times A_2 \times A_3$

$$\begin{Bmatrix} (1,1,a) & (1,1,b) & (1,2,a) & (1,2,b) \\ (2,1,a) & (2,1,b) & (2,2,a) & (2,2,b) \\ (3,1,a) & (3,1,b) & (3,2,a) & (3,2,b) \end{Bmatrix}$$

$(A_1 \times A_2) \times A_3$

$$\begin{Bmatrix} ((1,1),a) & ((1,1),b) & ((1,2),a) & ((1,2),b) \\ ((2,1),a) & ((2,1),b) & ((2,2),a) & ((2,2),b) \\ ((3,1),a) & ((3,1),b) & ((3,2),a) & ((3,2),b) \end{Bmatrix}$$

As the two sets have the same number of elements, we can define a bijection between the two sets. The most natural bijection is the one which maps an element $(x, y, z) \in A_1 \times A_2 \times A_3$ to $((x, y), z) \in (A_1 \times A_2) \times A_3$ according to the following scheme:

$$(x, y, z) \in A_1 \times A_2 \times A_3$$

$$((x, y), z) \in (A_1 \times A_2) \times A_3$$

Figure 1.22: "natural" one-to-one correspondence between the elements of $A_1 \times A_2 \times A_3$ and $(A_1 \times A_2) \times A_3$

♦

1.15.10

Suppose that for each $i \in \mathbb{P}$, $A_i = \{0,\ 1\}$. Describe in words the set $\bigtimes\{A_i : i \in \mathbb{P}\}$.

Each element of $\bigtimes\{A_i : i \in \mathbb{P}\}$ is of the form $(1, 0, 0, \ldots, 1, 1 \ldots 0, 1 \ldots)$ and can be interpreted as the binary representation of any $n \in \mathbb{P} \cup \{0\}$. (In the given example $n = 1 \times 2^0 + 0 \times 2^1 + 0 \times 2^2 + \cdots + 1 \times 2^k + 1 \times 2^{k+1} + \cdots + 0 \times 2^p + 1 \times 2^{p+1} + \ldots$).

♦

1.15.11

Let A_1 be the set of all real numbers $\mathbb{R}$. For each $i \in \mathbb{P}$ such that $i \geq 2$, let $A_i = \{0\}$. Describe in words the set $\bigtimes\{A_i : i \in \mathbb{P}\}$. Show that there exists a bijection from $\bigtimes\{A_i : i \in \mathbb{P}\}$ onto $\mathbb{R}$.

Each element of $\bigtimes\{A_i : i \in \mathbb{P}\}$ is of the form $(r, 0, 0, 0, \dots)$, $r \in \mathbb{R}$ and can be interpreted in different ways. One way is to consider an element of $\bigtimes\{A_i : i \in \mathbb{P}\}$ as the coordinates of a family of points in an infinite dimensional vector space where the points are strictly collinear with the first base vector. The bijection is trivial as for each point of this family, only one real number is assigned along the first base vector

♦

1.16 Sequences and Subsequences

1.16.1

Consider the sequence $S = (1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots)$. Find a map $N : \mathbb{P} \to \mathbb{P}$ such that $S \circ N$ is the sequence $(\frac{1}{3}, \frac{1}{6}, \ldots, \frac{1}{3n}, \ldots)$.

Define the map $N : \mathbb{P} \to \mathbb{P}$ as $N = \{n, 3n) : n \in \mathbb{P}\}$.

♦

1.16.2

Consider the sequence S such that $S(n) = (-1)^n \frac{1}{2^n}$. Find the nth term of the subsequence of S whose terms consist of all the positive terms of S and none of the negative terms of S.

Define the map $N : \mathbb{P} \to \mathbb{P}$ as $N = \{n, 2n) : n \in \mathbb{P}\}$ (each $n\mathbb{P}$ is mapped to an even element of $\mathbb{P}$). Hence $(S \circ N)_n = \frac{1}{4^n}$.

♦

1.16.3

For each positive integer n, let h_n be the function given by $h_n = \{(x, x^{n+1}) : 0 \leq x \leq 1\}$. Suppose that k is the sequence such that for each $n \in \mathbb{P}$, $k_n = \int_0^1 h_n dt$. Find the nth term of k.

$$\begin{aligned} k_n &= \int_0^1 t^{n+1} dt \\ &= \frac{1}{n+2} (x^{n+2})_0^1 \\ &= \frac{1}{n+2} \end{aligned}$$

♦

1.16.4

In each of the following determine whether or not the sequence is a strictly increasing sequence.

(a) $\left(\frac{n}{n+1}\right)_{n=1}^{+\infty}$
(b) $((n-\frac{1}{2})^2)_{n=1}^{+\infty}$
(c) $f=(50n-n^2)_{n=1}^{+\infty}$
(d) $g \circ f,$ where f is a strictly increasing sequence of positive integers and g is a strictly increasing sequence of real numbers.

(a) $\left(\frac{n}{n+1}\right)_{n=1}^{+\infty}$
We note that for $n>1$, $\Delta_n=\frac{n}{n+1}-\frac{n-1}{n}=\frac{n^2-(n+1)(n-1)}{(n+1)n}=\frac{1}{(n+1)n}$, so $\Delta_n>0, \forall n \in \mathbb{P}$ proving that the sequence is strictly increasing.

♢

(b) $((n-\frac{1}{2})^2)_{n=1}^{+\infty}$
We note that $\Delta_{n+1}=(n+1-\frac{1}{2})^2-(n-\frac{1}{2})^2=(n+\frac{1}{2})^2-(n-\frac{1}{2})^2=2n$, so $\Delta_n>0, \forall n>1 \in \mathbb{P}$ proving that the sequence is strictly increasing.

♢

(c) $f=(50n-n^2)_{n=1}^{+\infty}$
We note that $\Delta_{n+1}=(50(n+1)-(n+1)^2)-(50n-n^2)=(50n+50-n^2-2n-1-50n+n^2)=49-2n$, so $\Delta_n<0, \forall n \geq 25 \in \mathbb{P}$ proving that the sequence is not strictly increasing.

♢

(d) $g \circ f,$ where f is a strictly increasing sequence of positive integers
and g is a strictly increasing sequence of real numbers.
For f we have $f:\mathbb{P}\to\mathbb{P}$, such that $f=\{(n,i): n,i \in \mathbb{P} \text{ and } f(n+p)>f(n) \quad \forall n,p \in \mathbb{P}\}$. Analogously, for g we we have $g:\mathbb{P}\to\mathbb{R}$, such that $g=\{(k,x): k\in\mathbb{P}, x \in \mathbb{R} \text{ and } g(k+q)>g(k) \quad \forall k, q \in \mathbb{P}\}$.
Suppose that for a given n we have $f(n)=k$, then for $(g\circ f)(n)$ we have $(g\circ f)(n)=g(k)$. But we know that for a $n^{'}<n$ we will have $f(n^{'})=k^{'}<k$, so $(g\circ f)(n^{'})=g(k^{'})<g(k)$ and notice that $g\circ f$ is a strictly increasing sequence.

♦

1.16.5

Suppose that h is a subsequence of a sequence k and f is a subsequence of h. Is f a subsequence of k?

For h we have $\exists\, N : \mathbb{P} \to \mathbb{P}$, such that $h = k \circ N$. Analogously, for f we have $\exists\, M : \mathbb{P} \to \mathbb{P}$, such that $f = h \circ M$. Using the composition of functions, we have $f = k \circ N \circ M$. Be $Q = N \circ M$. Is Q a strictly increasing sequence? The answer is yes (see Exercise 1.16.4(d)). As Q is strictly increasing, f will be a subsequence of k.

♦

1.17 Finite induction and well-ordering for positive integers.

1.17.1

Prove that the following statement is equivalent to 17.1.
Suppose that h is an integer. Suppose further that $S(n)$ is a statement for each integer $n \geq h$, $S(h)$ is true, and $S(n)$ implies $S(n+1)$ for each integer $n \geq h$. Then $S(n)$ is true for each integer $n \geq h$.

Without loss of generality we take $h = 1$ (we could also define for a given h the map $N_h : \mathbb{P} \to \mathbb{Z}$ defined as $N = \{(p, h+p-1) : p \in \mathbb{P},\, h \in \mathbb{Z}\}$ and consider the statements $S \circ N_h(p)$.).
Let's define

$$M = \{m \in \mathbb{P} : S(m) \text{ is true}\}$$

Then the assumptions
i) $S(1)$ is true, is equivalent to $1 \in M$
ii) $S(n)$ implies $S(n+1)$ for each integer $n \geq 1$, is equivalent to $n \in M$ implies $n+1 \in M$ for each integer $n \geq 1$
iii) Then $S(n)$ is true for each integer $n \geq 1$, is equivalent to $m \in M$ every $m \geq 1$. This last statement implies that m contains all positive integers and thus $M = \mathbb{P}$.

♦

1.17.2

Prove that the sum of the first n positive integers is $\frac{1}{2}n(n+1)$.

Take $n = 1$, then $\frac{1}{2}n(n+1) = 1$ which corresponds to $\sum_{n=1}^{1} n = 1$.
Suppose that $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$ for a give n. Then $\sum_{k=1}^{n+1} k = \underbrace{\sum_{k=1}^{n} k}_{=\frac{1}{2}n(n+1)} + n + 1$, giving

$$\begin{aligned}\sum_{k=1}^{n+1} k &= \frac{1}{2}n(n+1) + n + 1 \\ &= \frac{1}{2}\left[n(n+1) + 2n + 2\right] \\ &= \frac{1}{2}\left[n^2 + n + 2n + 2\right] \\ &= \frac{1}{2}\left[(n^2 + 2n + 1) + (n+1)\right] \\ &= \frac{1}{2}\left[(n+1)^2 + (n+1)\right] \\ &= \frac{1}{2}(n+1)(n+2)\end{aligned}$$

confirming the expression.

♦

1.17.3

Prove that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$ for each $n \in \mathbb{P}$.

See Exercise **1.15.3**5

♦

1.17.4

Prove or disprove the following statement: For each $n \in \mathbb{P}$,

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$$

Take $n = 1$, then $\frac{1}{4}n^2(n+1)^2 = 1$ which corresponds to 1^3.

Suppose that $\sum_{k=1}^{n-1} k^3 = \frac{1}{4}(n-1)^2(n)^2$ for a give $n-1$. Then $\sum_{k=1}^{n} k^3 = \underbrace{\sum_{k=1}^{n-1} k}_{=\frac{1}{4}(n-1)^2(n)^2} + n^3$, giving

$$
\begin{aligned}
\sum_{k=1}^{n} k^3 &= \frac{1}{4}(n-1)^2 n^2 + n^3 \\
&= \frac{1}{4} n^2 \left[(n-1)^2 + 4n\right] \\
&= \frac{1}{4} n^2 \left[(n^2 + 2n + 1\right] \\
&= \frac{1}{4} n^2 (n+1)^2
\end{aligned}
$$

confirming the expression.

♦

1.17.5

> Is the following statement true? Justify your answer.
> For each positive integer n, $2^n - 1 \geq n$.

Take $n = 1$, then $2^n - 1 \geq 1$ which is a true statement.
Suppose now that for given $n-1$ we have $2^{n-1} - 1 \geq n-1$, then for n we have

$$
\begin{aligned}
2^n - 1 &= \left(2^{n-1}\right) 2 - 1 \\
&= \left(2^{n-1} - \frac{1}{2}\right) 2 \\
&= \left(\underbrace{2^{n-1} - 1}_{\geq n-1} + \frac{1}{2}\right) 2 \\
&\geq 2\left(n - \frac{1}{2}\right) \\
&\geq 2n - 1
\end{aligned}
$$

We prove that $2n - 1 \geq n$ for all positive integers. For $n = 1$ we have $2n - 1 \geq 1$.
Suppose now that for given n we have $2n - 1 \geq n$, then for $n+1$ we have $2(n+1) - 1 = 2n + 1$ which is $\geq 2n$ which is $\geq n$ as n is a positive integer, confirming the inequality stated.

♦

1.17.6

Either prove or disprove the following statement: For each positive integer n,

$$1.2 + 2.3 + 3.4 + \cdots + n(n+1) = \frac{1}{3}\left[n(n+1)(n+2) + 3\right]$$

.

Take $n = 1$, then $\frac{1}{3}\left[n(n+1)(n+2)+3\right] = 3$. But this should give $1.2 = 2$, so even for $n = 1$, the expression is wrong, hence can't be correct for all $n \in \mathbb{P}$.

♦

1.17.7

Either prove or disprove the following statement: For each $n \in \mathbb{P}$, $7^n - 3^n$ is divisible by 4.

For each $n \in \mathbb{P}$, $7^n - 3^n$ is divisible by 4 means that there exist a $k \in \mathbb{P}$ so that $7^n - 3^n = 4k$.
take $n = 1$, then $7^n - 3^n = 4$ and with $k = 1$ the statement is true .
Suppose now that for given $n-1$ we have $7^{n-1} - 3^{n-1} = 4k$, this means that $7^{n-1} = 3^{n-1} + 4k$. Then for n we get

$$\begin{aligned}
7^n - 3^n &= 7^{n-1}7 - 3^{n-1}3 \\
&= \left(3^{n-1} + 4k\right)7 - 3^{n-1}3 \\
&= 3^{n-1}7 + 28k - 3^{n-1}3 \\
&= 3^{n-1}(7-3) + 28k \\
&= 3^{n-1}4 + 28k \\
&= 4\left(3^{n-1} + 7k\right)
\end{aligned}$$

and get $7^n - 3^n = 4p^{'}$ with $p^{'} = 3^{n-1} + 7k$ where $p^{'} \in \mathbb{P}$. Hence, $7^n - 3^n$ is divisible by 4.

♦

1.17.8

Suppose that K is a nonempty collection of negative integers. Prove that there is a largest element in K.

Suppose K has no biggest element. Be $N = \mathbb{Z}_- - K$.
Note that $-1 \in N$ as otherwise -1 would be an element of K making this the largest element of K which we assumed had none. Consider now the set $N_n = \{n, n+1, \ldots, -2, -1\}$. Obviously $N_n \subset N$ as otherwise one of the element of N_n would be in K and thus K would have a largest element. Be $N_n \subset N$, then $N_{n-1} \subset N$ for otherwise $n-1$ would be the largest element of K. Continuing that process with $n-k$, $k \in \mathbb{P}$ we see that $N_{n-k} - > \mathbb{Z}_-$ meaning that $\mathbb{Z}_- \subset N$ and hence $N = \mathbb{Z}_-$. Thus K must be the empty-set which is a contradiction. Hence K must have a largest element.

♦

1.17.9

Is $3n^2 + n$ an even integer for each positive integer n? Justify your answer.

Take $n = 1$, then $3n^2 + n = 4$, an even integer.
Suppose that for a give n we have $3n^2 + n = 2p$ with $p \in \mathbb{P}$. Then for $n+1$ we have

$$\begin{aligned} 3(n+1)^2 + n + 1 &= 3n^2 + 6n + 3 + n + 1 \\ &= \underbrace{3n^2 + n}_{=2p} + 6n + 4 \\ &= 2p + 2(3n+2) \\ &= 2(p + 3n + 2) \end{aligned}$$

Hence, $3n^2 + n$ an even integer for each positive integer n.

♦

1.17.10

Try to discover a formula for the number of subsets (including the empty set) of a set of n objects. Then prove by induction that your conjecture is correct.

Be S the considered set with n objects, then the number of elements in the powerset $\mathscr{P}$ is, $\#\mathscr{P} = 2^n$. Be $n = 0$ ($S = \varnothing$), then $\#\mathscr{P} = 1$ which is correct as $\mathscr{P} = \{\varnothing\}$, containing only one subset.
Suppose that for a n we have $\#\mathscr{P}_n = 2^n$. Adding one object $x \notin S$ to the set S we get a new set $S^{'} = S \cup \{x\}$. Be $\mathscr{P}_n = \{P_1, P_2, \ldots P_{2^n}\}$, so the new powerset $\mathscr{P}_{n+1}$ can be expressed as $\mathscr{P}_{n+1} = \{P_1, P_2, \ldots P_{2^n}, P_1 \cup \{x\}, P_2 \cup \{x\}, \ldots P_{2_n} \cup \{x\}\} = \{P_1, P_2, \ldots P_{2_n}\} \cup \{P_1 \cup \{x\}, P_2 \cup \{x\}, \ldots P_{2^n} \cup \{x\}\}$ meaning that $\mathscr{P}_{n+1}$ will contain $2(2^n) = 2^{n+1}$ subsets.

♦

1.17.11

Is $n(n+1)(n+2)$ divisible by 3 for each positive integer n? Justify your answer.

We could perform the usual induction, but we can use another reasoning.
Suppose n is divisible by 3, then we are done. If n is not divisible by 3 then n will be of the form $n = 3k+1$ or $n = 3k+2$ for a given $k \in \mathbb{P}$. We don't need to consider the forms beyond $n = 3k+2$ because $n = 3k+3$, obviously is divisible by 3.
So, in the case n is not divisible by 3 , $n(n+1)(n+2)$ can be expressed as $(3k+1)(3k+2)(3k+3)$ or $(3k+2)(3k+3)(3k+4)$, so in both cases, one of the numbers in the product is divisble by 3, hence the total product also.

♦

1.17.12

Is $\frac{[n(n+1)(n+2)(n+3)]}{24}$ an integer for each positive integer n? Justify your answer.

We prove by induction that $\forall n \in \mathbb{P} : n(n+1)(n+2)(n+3) = 24m$, $m \in \mathbb{P}$
For $n = 1$, we have $n(n+1)(n+2)(n+3) = 24$.
Suppose $\exists\, n \in \mathbb{P} :\, n(n+1)(n+2)(n+3) = 24k$, $k \in \mathbb{P}$. The for $n+1$ have

$$(n+1)(n+2)(n+3)(n+4) = \underbrace{n(n+1)(n+2)(n+3)}_{=24k} + 4(n+1)(n+2)(n+3) \tag{1}$$

From **1.17.11** we know that $(n+1)(n+2)(n+3)$ is divisible by 3 for each positive integer n. Hence

(1) can be written as

$$(n+1)(n+2)(n+3)(n+4) = 24k + 4 \times 3 \times m \tag{2}$$

We claim that m is even. Indeed, for $n = 1$, $n(n+1)(n+2) = 2 \times 3$
Suppose $\exists\, n \in \mathbb{P} :\, n(n+1)(n+2) = 3 \times 2 \times q,\, q \in \mathbb{P}$. Then, for $n+1$ have

$$(n+1)(n+2)(n+3) = \underbrace{n(n+1)(n+2)}_{=3\times 2\times q} + 3(n+1)(n+2) \tag{3}$$

In the second term of (3), or $(n+1)$ or $(n+2)$ will be even, so (3) can be written as

$$\begin{aligned}(n+1)(n+2)(n+3) &= 3 \times 2 \times q + 3 \times 2 \times q^{'} \times n^* \quad (n^* = n+1 \text{ or } n+2)\\ &= 3 \times \underbrace{2 \times \left(q + q^{'} \times n^*\right)}_{=m}\end{aligned}$$

By induction, we conclude that m indeed, must be even. Hence (1) becomes

$$\begin{aligned}(n+1)(n+2)(n+3)(n+4) &= 24k + 4 \times 3 \times 2 \times \left(q + q^{'} \times n^*\right)\\ &= 24\left(k + q + q^{'} \times n^*\right)\end{aligned}$$

proving, by the induction principle (remember the initialisation with $n = 1$) that $\frac{n(n+1)(n+2)(n+3)}{24}$ is an integer for each positive integer n

♦

1.18 Sequences defined inductively

Clarification to example **18.3**

18.3. Example. Let $f(1) = 2$, $f(2) = 7$. Furthermore, let it be given that for each positive integer $n \geq 3$, $f(n) = \frac{1}{2}[(f(n-I) + f(n-2)]$. At this point it would be instructive for the reader to calculate a few terms off f. It is intuitively clear that there should exist a unique function satisfying the above properties. It furthermore seems reasonable that we should be able to prove by induction that such a function exists. This we can do, but not in as straightforward a manner as we might guess. To establish the existence of a function f satisfying the required properties, we proceed as follows: Let $f_2 = \{(1,2), (2,7)\}$. and for $n \geq 3$, let $S(n)$ be the following statement.
18.3(a). There exists a map $f_n : \mathbb{P}_n \to \mathbb{R}$ such that $f_n(1) = 2$ and $f_n(2) = 7$, and for $i \in \{3, 4, \ldots, n\}$,

$$f_n(i) = \frac{1}{2}[(f_n(i-1) + f_n(i-2)]$$

We see that $S(3)$ is true by considering the function $f_3 = \{(1,2), (2,7), (3, \frac{9}{2})\}$. Let $h \geq 3$ and assume that $S(h)$ is true. We show that $S(h+1)$ is true. To see this let

$$f_{h+1} = f_h \cup \{(h+1, \frac{1}{2}[(f_h(h) + f_h(h-1)]\}$$

It is easy to show that f_{h+1} satisfies the properties required of it so that $S(h+1)$ is true. Hence, by induction, $S(n)$ is true for each integer $n \geq 3$. Thus, there exists a collection of functions $\{f_n : n \geq 3\}$, each of which satisfies the conditions stated in **18.1(a)**. We next prove that for each f_n in the collection, $f_{n+1}|\mathbf{P}_n = f_n$. To see this, suppose that for some fixed integer $n \geq 3$, $f_{n+1}|\mathbf{P}_n \neq f_n$. Let j be the first positive integer for which $f_{n+1}(j) \neq f_n(j)$. We observe that $3 \leq j \leq n$. Then

$$f_{n+1}(j-1) = f_n(j-1) \text{ and } f_{n+1}(j-2) = f_n(j-2)$$

From this, we obtain

$$f_{n+1}(j) = \frac{1}{2}[(f_{n+1}(j-1) + f_{n+1}(j-2)] = \frac{1}{2}[(f_n(j-1) + f_n(j-2)] = f_n(j)$$

and we have arrived at a contradiction. For each $n \in \mathbb{P}$, let f_n be a function satisfying **18.3(a)**. We may apply **18.1** to $\{f_n : n \in \mathbb{P}\}$ and conclude that $f = \bigcup\{f_i = i \geq 3\}$ is a function. Next recall the remark in **18.2**, about the union of functions theorem. which states that $f(x) = f_n(x)$ for all x for which $x \in Dom f_n$. Thus, $f(1) = f_3(1) = 2$, $f(2) = f_3(2) = 7$, and $f(3) = f_3(3) = \frac{1}{2}[(f_3(2) + f_3(1)]$. Also, for $i \geq 3$,

$$f(i) = f_i(i) = \frac{1}{2}[(f_i(i-1) + f_i(i-2)] \frac{1}{2}[(f(i-1) + f(i-2)]$$

That f is a unique function that satisfies the given conditions is seen as follows: Suppose that there is another function f^* that satisfies the required conditions. Then the set $K = \{j : f^*(j) \neq f(j)\}$ would be a nonempty set of positive integers. Let m be the first element in K. Obviously $m \geq 4$. But then $f(i) = f^*(i)$ for $i < m$. However,

$$f(m) = \frac{1}{2}[(f(m-1) + f(m-2)] = \frac{1}{2}[(f^*(m-1) + f^*(m-2)] = f^*(m)$$

, and we have a contradiction.

1.18.1

Prove Theorem **18.4**, using the discussion in **18.3** as a hint.

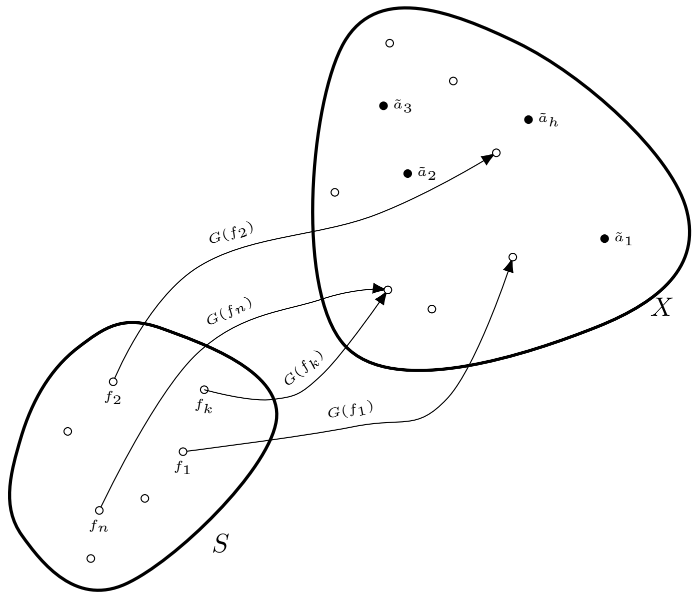


Figure 1.23: The map $G : S \to X$

Before attacking the proof, let's clarify some objects defined in the theorem.
First S is a (possibly infinite) set of finite sequences $f : \mathbb{P}_n \to X$. As X could be an infinite set and the f are finite sequences, S can be an infinite set.
Also, considering $S^{'} = \{f_k(n) : f_k \in S\}$ the collection of sets containing the terms $f_k(n)$ of a sequence f_k, we can state that

$$\bigcup S^{'} = X$$

What is this map G?
G will take an element (sequence) of S and map it to one element of X.
Note that G, by definition, is onto or or a bijection.
Example, if $X = \mathbb{Q}$, a sequence f will be mapped to a rational number.

$\Diamond$

Note that S is the set of all finite sequences with X as range. Hence, the following sequence will be in S:

$$\{(1, \tilde{a}_1), (2, \tilde{a}_2), (3, \tilde{a}_3), \ldots, (h, \tilde{a}_h), (h+1, a_{h+1}), \ldots, (n, a_n)\}, \quad (\tilde{a}_i, a_k \in X)$$

were the $\tilde{a}_k$ are the h first "given" terms and $a_k \in X$.

The given map G is a map from S into X. So, for the element (a sequence) of S of length h

$$\{(1, \tilde{a}_1),\, (2, \tilde{a}_2),\, (3, \tilde{a}a_3),\, \ldots,\, (h, \tilde{a}_h)\}$$

there exist an element $a_{h+1} = G\left(\{(1, \tilde{a}_1),\, (2, \tilde{a}_2),\, (3, \tilde{a}_3),\, \ldots,\, (h, \tilde{a}_h)\}\right) \in X$. As $a_{h+1} \in X$ the sequence

$$\{(1, \tilde{a}_1),\, (2, \tilde{a}_2),\, (3, \tilde{a}_3),\, \ldots,\, (h, \tilde{a}_h),\, (h+1, a_{h+1})\}$$

will be also be in S. Again this element (sequence of length $h+1$) will be mapped by G to an element $a_{h+2} \in X$. hence the sequence

$$\{(1, \tilde{a}_1),\, (2, \tilde{a}_2),\, (3, \tilde{a}_3),\, \ldots,\, (h, \tilde{a}_h),\, (h+1, a_{h+1}),\, (h+2, a_{h+2})\}$$

will be in S.
By induction, we conclude that there exists a sequence f such that

$$f_i = \begin{cases} \tilde{a}_i & \text{for } i \in \{1, 2, \ldots, h\} \\ G\left(\{\tilde{a}_1,\, \tilde{a}_2,\, \ldots,\, \tilde{a}_h,\, a_{h+1},\, \ldots,\, a_{i-1}\}\right) & \text{for } i \geq h+1 \end{cases}$$

Is this sequence, uniquely defined by $G : S \to X$? Yes, as G is a mapping (a function), so for a given element $\{a_1,\, a_2,\, \ldots,\, a_n\}$, there is a unique $G\left(\{a_1,\, a_2,\, \ldots,\, a_n\}\right) \in X$.

Note that the theorem does not say that the map G is "well-behaved" i.e. can be expressed by an algebraic or analytic function.

♦

1.18.2

Using Theorem **18.4**, prove that there exists a unique function f on the set of all nonnegative integers that satisfies the following conditions.
$f(0) = 1$ and $f(n) = nf(n-1)$ for each positive integer n. (Recall that common notation for $f(j)$ as defined inductively in this exercise is $j!$, read "j factorial.")

Be $X = \mathbb{N}$, $h = 1$, $a_1 = 1 \in \mathbb{N}$ and $G : S \to X$ with S a set of all finite-sequences with ranges in $\mathbb{N}$. From theorem **18.4** we know that theer is a unique sequence f such that

$$f_i = \begin{cases} a_i & \text{for } i \in \{1, 2, \ldots, h\} \\ G\left(\{f_1,\, f_2,\, \ldots,\, f_h,\, f_{h+1},\, \ldots,\, f_{i-1}\}\right) & \text{for } i \geq h+1 \end{cases}$$

As $h_1 = 1$ and $a_1 = 1$ and defining G as $G\left(\{f_1,\, f_2,\, \ldots f_{i-1}\}\right) = i\, f_{i-1}$, this simplifies to

$$f_i = \begin{cases} 1 & \text{for } i = 1 \\ i\, f(i-1) & \text{for } i \geq 2 \end{cases}$$

We can extend the function by adding one element $f_0 = 1$ to the sequence, giving as definition

$$f_i = \begin{cases} 1 & \text{for } i = 0 \\ 1 & \text{for } i = 1 \\ i\,f(i-1) & \text{for } i \geq 2 \end{cases}$$

♦

Following are some exercises concerning the factorial function that are useful in various branches of mathematics.

1.18.3

For each positive integer n and each nonnegative integer r such that $r \leq n$, define $\binom{n}{r} = \frac{n!}{r!(n-r)!}$. Verify each of the following:

(a) $\binom{n}{0} = 1$ for each positive integer n.

(b) $\binom{n}{n} = 1$ for each positive integer n.

(c) For each positive integer h and for each positive integer $j \leq h$

$$\binom{h}{j} + \binom{h}{j-1} = \binom{h+1}{j}$$

We can make use of Exercise 3 to prove the binomial expansion theorem in the next exercise.

(a) $\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{(n)!} = 1$

(b) $\binom{n}{n} = \frac{n!}{n!(n-n)!} = \frac{n!}{(n)!} = 1$

(c) $\binom{h}{j} + \binom{h}{j-1} = \binom{h+1}{j}$

$$\begin{aligned}
\binom{h}{j} + \binom{h}{j-1} &= \frac{h!}{j!(h-j)!} + \frac{h!}{(j-1)!(h-j+1)!} \\
&= \frac{h!}{(j-1)!}\left(\frac{1}{j(h-j)!} + \frac{1}{(h-j+1)!}\right) \\
&= \frac{h!}{(j-1)!}\left(\frac{(h-j+1)+j}{j(h-j+1)!}\right) \\
&= \frac{h!}{(j-1)!}\left(\frac{(h+1)}{j(h-j+1)!}\right) \\
&= \frac{(h+1)!}{j!(h+1-j)!} \\
&= \binom{h+1}{j}
\end{aligned}$$

♦

1.18.4

Prove that for each positive integer n,

$$(a+b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n-1} a^1 b^{n-1} + \binom{n}{n} a^0 b^n$$

Note that a short form for writing this, using summation notation, is

$$\sum_{i=0}^{n} = \binom{n}{i} a^{n-i} b^i$$

Initialize with $n = 1$, we have $(a+b)^1 = \underbrace{\binom{1}{0}}_{=1} a^1 b^0 + \underbrace{\binom{1}{1}}_{=1} a^{1-1} b^1 = a+b$. So, the proposed expression is true for $n = 1$.

Suppose now that it is alos correct for a certain n:

$$(a+b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n-1} a^1 b^{n-1} + \binom{n}{n} a^0 b^n$$

Then $(a+b)^{n+1} = (a+b)^n (a+b)$ giving

$$(a+b)^{n+1} = \begin{cases} \binom{n}{0} a^{n+1} b^0 + \binom{n}{1} a^n b^1 + \binom{n}{2} a^{n-1} b^2 + \cdots + \binom{n}{n-1} a^2 b^{n-1} + \binom{n}{n} a^1 b^n + \\ \\ \binom{n}{0} a^n b^1 + \binom{n}{1} a^{n-1} b^2 + \binom{n}{2} a^{n-2} b^3 + \cdots + \binom{n}{n-1} a^1 b^n + \binom{n}{n} a^0 b^{n+1} \end{cases}$$

regrouping as much as possible terms of the same power and using the results (a), (b), (c) of the previous exercise we get

$$(a+b)^{n+1} = \begin{cases} \binom{n}{0} a^{n+1}b^0 + \left[\binom{n}{1} + \binom{n}{0}\right] a^n b^1 + \left[\binom{n}{2} + \binom{n}{1}\right] a^{n-1}b^2 + \dots \\ \\ + \left[\binom{n}{n-2} + \binom{n}{n-1}\right] a^2 b^{n-1} + \left[\binom{n}{n-1} + \binom{n}{n}\right] a^1 b^n + \binom{n}{n} a^0 b^{n+1} \end{cases}$$

$$= \underbrace{\binom{n+1}{0}}_{=1=\binom{n}{0}} a^{n+1}b^0 + \binom{n+1}{1} a^n b^1 + \binom{n+1}{2} a^{n-1}b^2 + \dots + \binom{n+1}{n} a^1 b^n + \underbrace{\binom{n+1}{n+1}}_{=1=\binom{n}{n}} a^0 b^{n+1}$$

Hence, by the induction principle, the proposed expression is true.

♦

1.19 Some important properties of relations

1.19.1

In each of the following, classify the relation as to which of the properties discussed in Section 19 it possesses.

(a) Let S be the set of all triangles in the plane.
Let R be the relation in S defined as follows: for all a and b in S,
$a\,R\,b$ if and only if a is congruent tob.

(b) Let R be the set of all real numbers.
Let $S = \{(x, y) : (x, y) \in \mathbb{R} \times \mathbb{R} \text{ and } y \neq 0\}$.
For all (a, b) and $(c, d) \in S,$ let $(a, b)R(c, d)$ provided that $ad = be$.

(c) Suppose j is a fixed positive integer.
For each a and $b \in Z,$ let aRb if and only if $a - b = jk$ for some integer k.
(See Exercises 9 and 10, page 23 .)

(a)
- R is reflexive (a triangle is congruent to itself).
- R is symmetric (if a triangle A is congruent to a triangle B then the triangle B will also be congruent to A).
- R is transitive (if a triangle A is congruent to a triangle B and this triangle B is congruent to a triangle C then the triangle A will also be congruent to C).

(b) $(a, b)R(c, d) \Rightarrow ad = bc$
- R is reflexive $((a, b)R(a, b) \Rightarrow ad = ab)$.
- R is symmetric $((a, b)R(c, d) \Rightarrow ad = bc$ and $(c, d)R(a, b) \Rightarrow cb = ad,$).
- R is transitive ($[(a, b)R(c, d) \Rightarrow ad = bc\ ,\ (c, d)R(e, f) \Rightarrow cf = de] \Rightarrow af = be$ and $(a, b)R(e, f) \Rightarrow af = be$).

(c) $aRb \Rightarrow a - b = jk, k \in \mathbb{Z}$
- R is reflexive $(aRb \Rightarrow a - b = j \times 0, 0 \in \mathbb{Z})$.
- R is symmetric ($aRb \Rightarrow a - b = j \times k, k \in \mathbb{Z}$ and $bRa \Rightarrow b - a = j \times (-k), -k \in \mathbb{Z}$).
- R is transitive ($aRb \Rightarrow a - b = j \times k, k \in \mathbb{Z}$, $bRac \Rightarrow b - c = j \times (k^{'}), k^{'} \in \mathbb{Z} \Rightarrow a - b + b - c = a - c = j(k + k^{'}), \in \mathbb{Z}$ and $aRc \Rightarrow a - c = j \times k^{"}, k^{"} \in \mathbb{Z}$ with $k^{"} = k + k^{'}$).

♦

1.19.2

Suppose that R is a relation that is transitive in a set S. Let us define a new relation in S as follows: For each a and b in S, let aR^*b if and only if $a = b$ or aRb. Is R^* transitive in S? Is R^* reflexive in S? Illustrate with an R that is not reflexive.

(a) Is R^* transitive?
Be aR^*b and bR^*c, we can have 4 possibilities:
- $a = b$, $b = c$, then $a = c$ and by definition of the relation aR^*c.
- $a = b$, $b \neq c$, then $aR^*b \Leftrightarrow aR^*c$
- $a \neq b$, $b = c$, then $aR^*b \Leftrightarrow aR^*c$
- $a \neq b$, $b \neq c$, then aRb and bRc and as R is transitive, thus aRc and by definition of the relation we have aR^*c.

(b) Is R^* reflexive?
By the definition of R^* (case a=b) , we have indeed aR^*a.

◊

Consider the set $S = \mathbb{R}$ and the relation $aRb : a < b$. R is obviously not reflexive, yet is transitive ($a < b$, $b < c \Rightarrow a < c$), but $aR^*a \equiv (a, a) \in R^*$.

♦

1.19.3

Suppose that R is a relation in a set S. Let us define a new relation R^* as follows: For each a and b in S, let aR^*b if and only if aRb is true and bRa is false. Suppose R is transitive. Is R^* also transitive ? Is R^* necessarily antisymmetric?

a) Is R^* transitive?
We have

$$\begin{cases} aR^*b & \Leftrightarrow & aRb & \wedge & b\not R a \\ \\ bR^*c & \Leftrightarrow & bRc & \wedge & c\not R b \end{cases}$$

R is transitive, so aR^*c is transitive provided that $b\not R a \quad \wedge \quad c\not R b$. So, provided that R is antisymmetric and transitive , R^* will be transitive.

b) Is R^* necessarily antisymmetric?
Suppose

$$\begin{cases} aR^*b & \Leftrightarrow & aRb & \wedge & b\not{R}a \\ \\ bR^*a & \Leftrightarrow & bRa & \wedge & a\not{R}b \end{cases}$$

This is a contradiction, hence the answer is, yes, R^* is necessarily antisymmetric.

♦

1.19.4

Suppose that a relation R in a set S is transitive and antireflexive. Is it necessarily antisymmetric?

R transitive means aRb, $bRc \Rightarrow aRc$, R antireflexive means $(a, a) \notin R$ and antisymmetric means aRb, $bRa \Rightarrow a = b$.
Be $c = a$, then transitivity implies aRb, $bRa \Leftrightarrow aRa$, but as R is antireflexive we can't have aRa, meaning that $aRb \wedge bRa$ can not be true. So R needs definitely to be antysymmetric.

♦

1.19.5

Are the following propositions true?

(a) Suppose that R is a relation in a set S. Then R is symmetric if and only if $R \subset R^{-1}$.
(b) Suppose that R is a relation in a set S. Then R is symmetric if and only if $R = R^{-1}$.

(a)
Suppose R is symmetric. Then $aRb \Leftrightarrow bRa$ or stated differently $(a, b) \in R$ and $(b, a) \in R$ for all $a, b \in S$. For R^{-1} we have $(b, a) \in R^{-1}$ if $(a, b) \in R$. So we have $(b, a) \in R^{-1}$ and provided that R is symmetric $(b, a) \in R$, hence, $R \subset R^{-1}$ (starting from R is symmetric and noting that $(a, b) \in R \Rightarrow (b, a) \in R^{-1}$ we can also conclude that $R^{-1} \subset R$).
Suppose now that $R \subset R^{-1}$, this means that if $(b, a) \in R$ we also have also $(b, a) \in R^{-1}$, the latter implying $(a, b) \in R$, so we have $(a, b) \in R$ and $(b, a) \in R$, thus R is symmetric.

(b)

Suppose R is symmetric. From (a) we have $R = R^{-1}$.
We must prove that also $R^{-1} \subset R$.
Suppose now that $R = R^{-1}$, this means that if $(b, a) \in R$ we also have also $(b, a) \in R^{-1}$, the latter implying $(a, b) \in R$, so we have $(a, b) \in R$ and $(b, a) \in R$, thus R is symmetric.

♦

1.19.6

Suppose that R is a relation defined in S. Is $R \cap R^{-1}$ a symmetric relation in S?

Suppose, $R \cap R^{-1}$ is symmetric, then $\forall a, b \in S : aR \cap R^{-1}b \Rightarrow bR \cap R^{-1}a$ which means $(a, b) \in R \cap R^{-1} \wedge (b, a) \in R \cap R^{-1}$. This means that (b, a) also must be an element of R or bRa. If R is not symmetric then the statement is not true and $R \cap R^{-1}$ can not be symmetric in a general case.

♦

1.19.7

Suppose that R is a transitive relation in S. Is R^{-1} transitive in S?

We have $(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R$. But, $(a, c) \in R$ implies that $(c, a) \in R^{-1}$. For R^{-1} to be transitive, we need $(b, a) \in R^{-1}, (c, b) \in R^{-1} \Rightarrow (c, a) \in R^{-1}$.
So, yes indeed, R^{-1} is transitive in S

♦

1.19.8

Suppose that R is symmetric in S. Is R^{-1} symmetric in S?

We have $(a, b) \in R \Rightarrow (b, a) \in R$. But, $(b, a) \in R$ implies that $(a, b) \in R^{-1}$. For R^{-1} to be symmetric, we need $(b, a) \in R^{-1} \Rightarrow (a, b) \in R^{-1}$.
So, yes indeed, R^{-1} is symmetric in S

♦

1.19.9

Suppose that R is reflexive in S. Is R^{-1} reflexive in S?

$(a, a) \in R \Rightarrow (a, a) \in R^{-1}$. Hence, R^{-1} is reflexive in S

♦

1.19.10

Does there exist a nonempty set S and a relation R in S such that R is both symmetric and antisymmetric in S?

Be $S = \{a\}$ an arbitrary non-empty set with 1 element. Let's define R as the identity map i.e. $R = \{(a, a) : a \in S\}$.
Symmetry means

$$\forall x, y \in S : (x, y) \in R \Rightarrow (y, x) \in R$$

which is a true statement as $(x = a, y = x)$, $(y = x, x = a) \in R$ because R is the identity map.

Anti-symmetry means

$$\forall x, y \in S : (x, y) \in R \wedge (y, x) \in R \Rightarrow x = y$$

which is a true statement as $x = a,\ y = a \quad \forall x, y \in S$ and R is the identity map.

♦

1.20 Decomposition of a set

1.20.1

> For each real number r, let $F_r = \{(r, y) : y \in \mathbb{R}\}$. Is $\{F_r : r \in \mathbb{R}\}$ a partition of $\mathbb{R} \times \mathbb{R}$?

First note that $F_r \cap F_{r'} = \varnothing$ if $r \neq r^{'}$.
Also $\bigcup\{(r, y) : y \in \mathbb{R}\}$ will cover, for a given r, the whole $\mathbb{R}$-axis. Also, as F_r is defined for each real number, for each of this number will be associated the whole real number axis, and hence $\{F_r : r \in \mathbb{R}\}$ a partition of $\mathbb{R} \times \mathbb{R}$.

♦

1.20.2

> Let $A_0 = \{x : -1 \leq x \leq 1\}$. For each $x \in \mathbb{R} - A_0$ let $A_x = \{x\}$. Is the following collection $\mathscr{K}$ a decomposition of the real line $\mathbb{R}$?
>
> $$\mathscr{K} = \{A_x : x = 0 \text{ or } |x| > 1\}$$

- $\bigcup\{A_x : x = 0 \text{ or } |x| > 1\} = \mathbb{R}$. This can be seen as A_0 is the closed interval $[-1, 1]$ and for all other $x \in \mathbb{R} - A_0$ i.e. $x \in (-\infty, -1) \cup (1, +\infty)$, each A_x, $x \geq 1$ is $\{x\}$ itself. So, $\bigcup\{A_x : x = 0 \text{ or } |x| > 1\} = (-\infty, 1) \cup [-1, 1] \cup (1, +\infty) = \mathbb{R}$
- $A_0 \cap A_x = \varnothing, \forall |x| > 1$.
- $A_x \cap A_{x'} = \{x\} \cap \{x^{'}\} = \varnothing, \forall |x|, |x^{'}| > 1$.

Conclusion: $\mathscr{K}$ a decomposition of the real line $\mathbb{R}$

♦

1.20.3

> Suppose X is a nonempty set and $f : X \to Y$ is a surjection. Is $\{f^{-1}[y] : y \in Y\}$ a decomposition of X?

f is a surjection, so every $y \in Y$ has at least one element of $x \in X$ such that $f(x) = y$. But for two different y_1, $y_2 \in Y$, as f is a function, we must have $f^{-1}[y_1] \cap f^{-1}[y_2] = \varnothing$ as otherwise we

would have a x^*, such that $f(x^*) = y_1$ and $f(x^*) = y_2$, which is excluded as f is a function. So, one condition for $\{f^{-1}[y] : y \in Y\}$ a being a decomposition of X is fulfilled
What about $\bigcup\{f^{-1}[y] : y \in Y\}$? We notice that $\{f^{-1}[y] : y \in Y\}$ corresponds to *dom* f which is not necessarily equal to X (*dom* $f \subset X$, see definition **10.3** page 16). So, $\{f^{-1}[y] : y \in Y\}$ a not always a decomposition of X.

♦

1.21 Equivalence classes

1.21.1

For each ordered pair of real numbers (a, b) such that $a \neq 0$ and $b \neq 0$, let $E(a, b)$ be the equation $ax + by = 0$. Let $\xi = \{E(a, b) : a \neq 0 \text{ and } b \neq 0\}$. For $E(a, b)$ and $E(c, d)$ in ξ, let us define a relation as follows. $E(a, b) \simeq E(c, d)$ if and only if every solution (x, y) of $ax + by = 0$ is a solution of $cx + dy = 0$, and every solution of $cx + dy = 0$ is a solution of $ax + by = 0$. Note that $\simeq$ is an equivalence relation. Is it true that $E(2, 3) \simeq E(4, 6)$? Try to discover an equation relating $a, b,\ c,$ and d so that $E(a, b) \simeq E(c, d)$ provided that $a, b,\ c,$ and d satisfy your equation. Justify your conjecture. Write an equation that is in the same equivalence class as is $E(3, 2)$ but which is not the same as $E(3, 2)$.

We first check that $\simeq$ is an equivalence relation.
i) reflexivity: $E(a, b) \simeq E(a, b)$ is obvious as (x, y) is of course a solution of the same equation $ax + by = 0$ on both sides of the relation $\simeq$.
ii) symmetric: $E(a, b) \simeq E(c, d) \Leftrightarrow E(c, d) \simeq E(a, b)$ is obvious as a solution (x, y) of $E(a, b)$ will be a solution of $E(c, d)$ and vice-versa, a solution of $E(c, d)$ will also be a solution of $E(a, b)$.
iii) transitivity $E(a, b) \simeq E(c, d)$, $E(c, d) \simeq E(e, f) \Rightarrow E(a, b) \simeq E(e, f)$: also obvious as a solution of $ax + by = 0$ will be a solution of $cx + dy = 0$ but then also a solution of $ex + fy = 0$.

a) $E(2, 3) \simeq E(4, 6)$

$$\begin{cases} 2x + 3y = 0 \\ 4x + 6y = 0 \Leftrightarrow 2x + 3y = 0 \end{cases}$$

so, yes, $E(2, 3) \simeq E(4, 6)$

b) Try to discover an equation relating $a, b,\ c,$ and d so that $E(a, b) \simeq E(c, d)$

Suppose (x, y) is a solution of $ax + by = 0$ and a solution of $cx + dy = 0$.

$$\begin{cases} ax + by = 0 \\ cx + dy = 0 \end{cases}$$
$$\Rightarrow \begin{cases} cax + cby = 0 \\ acx + ady = 0 \end{cases}$$
$$\Rightarrow \begin{cases} cax + cby = 0 \\ (ad - bc)y = 0 \end{cases}$$

If we exclude the trivial solution $(x, y) = (0, 0)$ we see that $ad - bc$ must be zero, hence the condition needed is

$$ad - bc = 0$$

c)Write an equation that is in the same equivalence class as is $E(3, 2)$ but which is not the same as $E(3, 2)$.

Using the previous result we need $3d - 2c = 0$ or $c = \frac{3}{2}d$. Hence another equation is with $d = 1$ $\frac{3}{2}x + y = 0$

♦

1.21.2

> Let $\mathbb{R}_+$, be the collection of all positive real numbers. For $a \in \mathbb{R}_+$, and $b \in \mathbb{R}_+$,, let aRb if and only if $\frac{a}{b}$ is a rational number. Is R an equivalence relation on $\mathbb{R}_+$? Justify your answer. What is the form of all the numbers b such that $b \in R[\sqrt{2}]$? If a is an irrational positive number and bRa, is b necessarily an irrational number?

a) Is R an equivalence relation on $\mathbb{R}_+$?

i) reflexivity: aRa implies $\frac{a}{a} = 1 \in \mathbb{Q}$.

ii) symmetric : aRb implies $\frac{a}{b} \in \mathbb{Q}$ We can then express $\frac{a}{b}$ as $\frac{a}{b} = \frac{p}{q}$ with $p,\, q \in \mathbb{Z}_+$. Obviously, $\frac{b}{a} = \frac{q}{p} \in \mathbb{Q}$ and thus bRa.

iii) transitivity : We need $aRb, bRc \Rightarrow aRc$.

aRb implies $\frac{a}{b} =\in \mathbb{Q}$ and cRd implies $\frac{c}{d} =\in \mathbb{Q}$. We can then express $\frac{a}{b}$ as $\frac{a}{b} = \frac{p}{q}$ with $p,\, q \in \mathbb{Z}_+$.
Also $\frac{c}{d} = \frac{r}{s}$ with $r,\, s \in \mathbb{Z}_+$.
Hence, we have $\frac{a}{b}\frac{b}{c} = \frac{pr}{qs}$ or $\frac{a}{c} = \frac{u}{v}$ with $(u = pr, v = qs) \in \mathbb{Q}$ and thus aRc.

b) What is the form of all the numbers b such that $b \in R[\sqrt{2}]$?

First remember (see section 11) that $R[a] = \{b : aRb\}$, so $R[\sqrt{2}] = \{b : \sqrt{2}Rb\}$ and from the definition of R we must have $\frac{\sqrt{2}}{b} \in \mathbb{Q}$. As $\sqrt{2}$ is irrational we need b to be of the form $q\sqrt{2}$ with $q \in \mathbb{Q}$.

c) If a is an irrational positive number and bRa, is b necessarily an irrational number?

Suppose $\frac{b}{a} \in \mathbb{Q}$. Then $\frac{b}{a} = \frac{p}{q}$ with $p, q \in \mathbb{Z}_+$. Thus $b = a\frac{p}{q}$. Suppose b is rational then $b = \frac{u}{v}$,

$u, v \in \mathbb{Z}_+$, giving $\frac{u}{v} = a\frac{p}{q}$. Hence $\underbrace{uq}_{=s\in\mathbb{Z}_+} = a \underbrace{pv}_{=t\in\mathbb{Z}_+}$ from which follows that $a = \frac{s}{t}$ and thus rational.

We get a contradiction and hence b is necessarily irrational.

♦

1.21.3

Let $\mathbb{R}$ be the set of all real numbers and let $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. Let m be a fixed real number. For each (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$ let $(x_1, y_1)R(x_2, y_2)$ provided that $y_1 - mx_1 = y_2 - mx_2$. Is R an equivalence relation ? Let $m = 3$. Sketch $R[(1, 2)]$.

a) Is R an equivalence relation?

i) reflexive: obviously $y_1 - mx_1 = y_1 - mx_1$ hence R is a reflexive relation.

ii) symmetric:

$$\underbrace{(x_1, y_1)R(x_2, y_2)}_{y_1-mx_1=y_2-mx_2} \Rightarrow \underbrace{(x_2, y_2)R(x_1, y_1)}_{y_2-mx_2=y_1-mx_1}$$

hence R is a symmetric relation.

iii) transitive Obviously from $y_1 - mx_1 = y_2 - mx_2$ and $y_2 - mx_2 = y_3 - mx_3$ follows $y_1 - mx_1 = y_3 - mx_3$ and thus $(x_1, y_1)R(x_3, y_3)$ hence R is a transitive relation.

Conclusion: R is an equivalence relation.

b) The set generated by $R^[(1, 2)]$ with $m = 3$ is defined by $y_1 - 3x_1 = y_2 - 3x_2$ and $R^[(1, 2)] = \{(x, y) : (1, 2)R(x, y)\} = \{(x, y) : y - 3x = -1\}$ giving the following sketch:

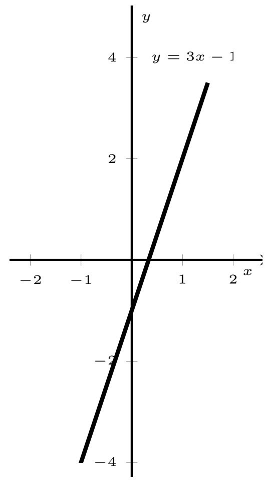


Figure 1.24: Sketch of the set generated by $R[(1,2)]$ with $m=3$

♦

1.21.4

Let $\mathscr{D}$ be the set of all real-valued functions which are defined and have derivatives on the open interval (a,b). For $f \in \mathscr{D}$ and $g \in \mathscr{D}$, let fRg provided that $f^{'} = g^{'}$. Is R an equivalence relation in $\mathscr{D}$? Let $f(x) = x^2$ for $x \in (a,b)$. Find $R[f]$.

a) Is R an equivalence relation?

i) reflexive: obviously $fRf \Rightarrow f^{'} = f^{'}$ hence R is a reflexive relation.

ii) symmetric:

$$\underbrace{fRg}_{f'=g'} \Rightarrow \underbrace{gRf}_{g'=f'}$$

hence R is a symmetric relation.

iii) transitive Obviously from fRg and gRh follows $f^{'} = g^{'}$ and $g^{'} = h^{'}$ and thus $f^{'} = h^{'}$ or fRh, hence R is a transitive relation.

Conclusion: R is an equivalence relation.

b) Let $f(x) = x^2$ for $x \in (a,b)$. Find $R[f]$.

$f(x) = x^2 \Rightarrow f^{'}(x) = 2x$.
From $R[f] = \{h : fRh\}$ follows $2x = h^{'}(x) \Rightarrow h = x^2 + C$ with C a constant in $\mathbb{R}$.

♦

1.23 Partially ordered and totally ordered sets

Some clarifications:

Although evident, let's recall some basic remarks:

- **Upper and lower bounds** of a subset $A \subset S$ are **not** necessarily elements of A (well of S of course).

- On the other hand, **Least and Greatest elements** of a subset $A \subset S$ are **necessarily** elements of A.

- Remark on the definition **23.6**:

23.6. Definition. Least upper bound and greatest lower bound.
Let $(S, \leq)$ be a partially ordered set. Suppose that A is a subset of S and A has an upper bound in S. If the set of *upper bounds of* S has a least element l, then l is called the least upper bound of $A\,(l.u.b.(A))$. If A has a lower bound and the set of lower bounds of S has a greatest element g, then g is called the greatest lower bound of $A\,(g.l.b.(A))$.
The text in italic is a little bit confusing for me and would restate the definition as :

23.6. Definition. Least upper bound and greatest lower bound.
Let $(S, \leq)$ be a partially ordered set. Suppose that A is a subset of S and A has an upper bound in S. **If the set P of upper bounds of** A has a least element l, then $l \in P$ is called the least upper bound of $A\,(l.u.b.(A))$. If A has a lower bound and **the set P of lower bounds** of A has a greatest element g, then $\mathbf{g} \in \mathbf{P}$ is called the greatest lower bound of $A\,(g.l.b.(A))$.
Note that $l \in P$ or $g \in P$ are not necessarily an element of A.

- **Maximal and minimal elements vs. Greatest and Least elements**

The distinction between those two elements are quite confusing at first sight. Herunder is a clarification borrowed from a Reddit feed.

The notion that's important in understanding the distinction is that of a comparability, and more importantly, a chain. Two elements, x and y, or a set ordered by $\leq$ (where $\leq$ is just some arbitrary ordering) are said to be comparable if $x \leq y$ or $y \leq x$. A chain is a subset of A such that any two elements in the chain are comparable.

The usual relation $\leq$ in $\mathbb{R}$ is what's called a linear ordering, meaning every subset is a chain. However, the relation $|$ (where $x|y$ means x divides y) is not a linear ordering, so it allows for the existence of sets that are not chains.

Now a maximal element b in a subset B of an ordered set A is one such that there is no x in B such that $b \leq x$ and x is distinct from b. A greatest element c is one such that, if x is in B, $\leq c$.

The important part: The greatest and maximal elements in a set are only ever distinct if the set is not a chain. Consider the set $\{2, 4, 8, 16, 3, 9, 27\}$ *ordered under the relation* $|$*. Then it has two maximal elements,* 16 *and* 27*, and no greatest element. However, the subsets* $\{2, 4, 8, 16\}$ *and* $\{3, 9, 27\}$ *are both chains, so the maximal elements* 16 *and* 27 *are both greatest elements, too.*

Another important relation where it's easy to find subsets which are not chains is the subset relation: say, a relation R *on* $\mathscr{P}(\mathbb{N})$ *such that* ARB *if* A *is a subset of* B*. Then* $\{\{1\}, \{2\}, \{1, 2\}\}$ *has the greatest element* $\{1, 2\}$*, but* $\{\{2\}, \{2, 3\}, \{4\}, \{4, 5\}\}$ *has two maximal elements,* $\{2, 3\}$ *and* $\{4, 5\}$*, but no greatest element.*

The notion of a maximal [element] is weaker than that of the greatest element and least element (which are also known, respectively, as maximum and minimum); indeed a partially ordered set may have multiple maximal and minimal elements. As an example, in the collection

$$S = \{\{d, o\}, \{d, o, g\}, \{g, o, a, d\}, \{o, a, f\}\}$$

ordered by containment, the element $\{d, o\}$ *is minimal, the element* $\{g, o, a, d\}$ *is maximal, the element* $\{d, o, g\}$ *is neither, and the element* $\{o, a, f\}$ *is both minimal and maximal. By contrast, neither a maximum nor a minimum exists for* S*.*

1.23.1

Consider the system $(\mathbb{R}, \leq)$ of the real line $\mathbb{R}$ together with the usual ordering $\leq$.
(a) Give an example of subset A of $\mathbb{R}$ that is bounded below but not above. Similarly, give an example of a subset B of $\mathbb{R}$ that is bounded above but not below.
(b) Give an example of a subset S of $\mathbb{R}$ that has a least upper bound but whose least upper bound does not belong to S.

(a) Consider $A \subset \mathbb{R}$ defined by $A = \{x^2 : x \in \mathbb{R}\}$. Then A is bounded below by the element 0, but A is not bounded above.

(a') Consider $A \subset \mathbb{R}$ defined by $A^{'} = \{e^{-x} : x \in \mathbb{R}\}$ Then $A^{'}$ in bounded above by the element 0, but $A^{'}$ is not bounded below.

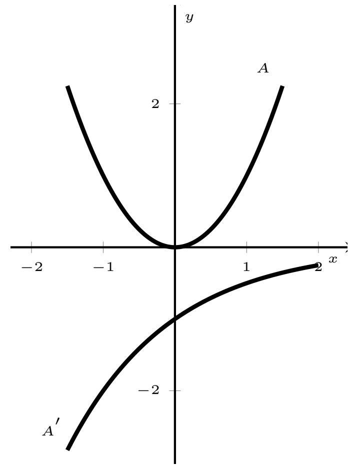


Figure 1.25: The sets $A = \{x^2 : x \in \mathbb{R}\}$ and $A^{'} = \{-e^{-x} : x \in \mathbb{R}\}$

(b) Consider the set $A^{'}$ above. $l.u.b.(A^{'}) = 0$ but $0 \notin A^{'}$.

♦

1.23.2

Give an example of a collection of sets $\mathscr{K}$ such that the partially ordered set $(\mathscr{K}, \subset)$ satisfies the following two conditions:
(a) $\mathscr{K}$ is not linearly ordered.
(b) Every linearly ordered subset of $\mathscr{K}$ has an upper bound in In your example, does $\mathscr{K}$ have a maximal element ?

(a)Consider the collection of sets $\mathscr{K}_+ = \{[0, e^{-p^2}] : p \in \mathbb{Z}_+\}$ and $\mathscr{K}_- = \{[0, -e^{-p^2}] : p \in \mathbb{Z}_-\}$ and the union of them $\mathscr{K} = \mathscr{K}_+ \cup \mathscr{K}_-$

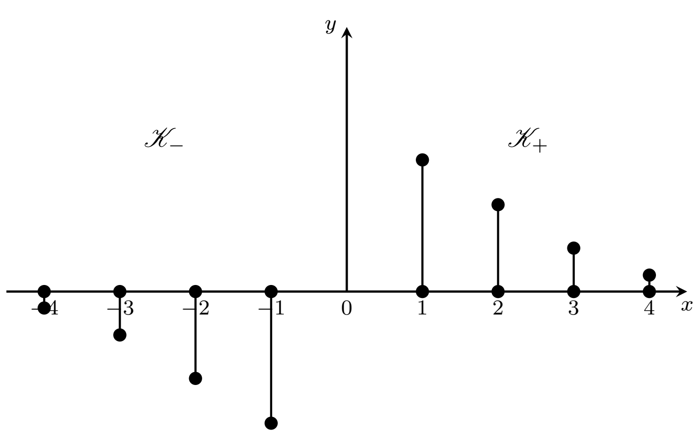


Figure 1.26: The collection $\mathscr{K} = \{[0, e^{-p^2}] : p \in \mathbb{Z}_+\} \cup \{[0, -e^{-p^2}] : p \in \mathbb{Z}_-\}$

We see that $\mathscr{K}_+ \cap \mathscr{K}_- = \varnothing$. Hence for any arbitrary element K_n of $\mathscr{K}_+$ and any arbitrary element K_m of $\mathscr{K}_-$we can't have $K_n \subset K_m$ or $K_n \supset K_m$. Hence, $\mathscr{K}$ is not linearly ordered.

$\Diamond$

(b)For $\mathscr{K}_-$ we have $[-e, 0]$ as upper bound and for $\mathscr{K}_+$ we have $[0, e]$ as upper bound.
Does $\mathscr{K}$ has a maximal element? For $\mathscr{K}$ having a maximal element we need

$$\exists M \in \mathscr{K}, \forall K \in \mathscr{K} : M \subset K \Rightarrow M = K$$

It is clear that $\mathscr{K}$ does not have a maximal element as any candidate M will be either an element of $\mathscr{K}_+$ or an element of $\mathscr{K}_-$. Suppose $M \in \mathscr{K}_+$ than for any element $K \in \mathscr{K}_-$ which is also an element of $\mathscr{K}$ we statement $M \subset K$ is always false, giving that the necessary condition for a maximal element is never true, hence no maximal element can be defined.

$\blacklozenge$

1.23.3

> Suppose that S is a set and R is a relation in S that is transitive and antireflexive (19.2). Define $\underset{=}{R}$ in S as follows: For all x and y in S, $x\underset{=}{R}y$ if and only if xRy or $x = y$.
> Is $\underset{=}{R}$ a partial ordering in S?

We check whether $\underset{=}{R}$ is transitive, anti-symmetric and reflexive.

i) transitive: $x\underset{=}{R}y$, $y\underset{=}{R}z \Rightarrow x\underset{=}{R}z$
We look at three possible cases:
• $x = z$: then, although $x\not{R}z$, we have by definition $x\underset{=}{R}z$.
• $x \neq y$ and $y \neq z$: we have in that case xRy and yRz, so by definition of $\underset{=}{R}$ we have $x\underset{=}{R}y$ and $y\underset{=}{R}z$. as R is transitive we have $xRyz$, so by definition of $\underset{=}{R}$ we have also $x\underset{=}{R}z$.
• $(x \neq y$ and $y = z)$ or $(x = y$ and $y \neq z)$: Let's take the case $x = y$ and $y \neq z$.
xRy is not defined as R is anti-reflexive. But, by definition, $x\underset{=}{R}y$ is defined. Also $y\underset{=}{R}z$ is defined because yRz is defined. Then obviously $y\underset{=}{R}z \Leftrightarrow x\underset{=}{R}z$, because $x = y$.

So, $\underset{=}{R}$ is transitive.

ii) reflexive: $\forall x \in S : x\underset{=}{R}x$
This is true by the definition of $\underset{=}{R}$.

So, $\underset{=}{R}$ is reflexive.

iii) anti-symmetric : $\forall x,\, y \in S : x\underset{=}{R}y$ and $y\underset{=}{R}x \Rightarrow x = y$

Recall: ($x\underset{=}{R}y$ then or xRy or $x = y$) and also ($y\underset{=}{R}x$ then or yRx or $y = x$).

Suppose we have $x\underset{=}{R}y$ and $y\underset{=}{R}x$. Then, by definition of $\underset{=}{R}$ we would have xRy and yRx if both exist. Yet, nothing is said about the symmetrical properties of R. If R is symmetric then xRy and yRx is possible, even if $x \neq y$. Hence, $\underset{=}{R}$ is not necessarily anti-symmetric.

Conclusion: No, $\underset{=}{R}$ is not necessarily a partial ordering of S.

♦

1.23.4

4. Let (A, R) and (B, Γ) be partially ordered sets. Define a relation $\leq$ in $A \times B$,

$$(a_l, b_1) \leq (a_2, b_2) \text{ if and only if } a_l R a_2 \text{ and } b_1 \Gamma b_2.$$

(a) Show that $\leq$ is a partial ordering for $A \times B$.
(b) Suppose that R and Γ are total orderings for A and B, respectively. Is $\leq$ necessarily a total ordering for $A \times B$?

(a) $\leq$ is a partial ordering of $A \times B$.

i) transitive: To prove $(a_1, b_1) \leq (a_2, b_2)$, $(a_2, b_2) \leq (a_3, b_3) \Rightarrow (a_1, b_1) \leq (a_3, b_3)$.

We have $(a_1, b_1) \leq (a_2, b_2) \Leftrightarrow a_1Ra_2$ and $b_1\Gamma b_2$, and also $(a_2, b_2) \leq (a_3, b_3) \Leftrightarrow a_2Ra_3$ and $b_2\Gamma b_3$.
We know that R and Γ are transitive relations so a_1Ra_2 and $a_2Ra_3 \Rightarrow a_1Ra_3$, and also $b_1\Gamma b_2$ and $b_2\Gamma b_3 \Rightarrow b_1\Gamma b_3$. So we have indeed $(a_1, b_1) \leq (a_2, b_2)$, $(a_2, b_2) \leq (a_3, b_3) \Rightarrow a_1Ra_3$ and $b_1\Gamma b_3$ and thus $(a_1, b_1) \leq (a_2, b_2)$, $(a_2, b_2) \leq (a_3, b_3) \Rightarrow (a_1, b_1) \leq (a_3, b_3)$ Conclusion : $\leq$ is a transitive relation on $A \times B$.

ii) reflexive: To prove $\forall (a, b) \in A \times B : (a, b) \leq (a, b)$.

Both R and Γ are reflexive, so $\forall a \in A : aRa$ and $\forall b \in B : b\Gamma b$. By definition $\leq$ we need $(a, b) \leq (a, b) \Rightarrow aRa$ and $b\Gamma b$. Hence, the right part of the statement is true, so $\leq$ is a reflexive relation.

iii) anti-symmetric: To prove $\forall (a_1,\, b_1), (a_2,\, b_2) \in A \times B : (a_1,\, b_1) \leq (a_2,\, b_2)$ and $(a_2,\, b_2) \leq (a_1,\, b_1) \Leftrightarrow a_1 = a_2$ and $b_1 = b_2$.

• $\Rightarrow$:
We know that R and Γ are anti-symmetric in A respectively B. By definition we have:
○ $(a_1, b_1) \leq (a_2, b_2) \Leftrightarrow a_1 R a_2$ and $b_1 \Gamma b_2$.
○ $(a_2, b_2) \leq (a_1, b_1) \Leftrightarrow a_2 R a_1$ and $b_2 \Gamma b_1$.
So, if $(a_1, b_1) \leq (a_2, b_2)$ and $(a_2, b_2) \leq (a_12, b_1)$ we have

$$(a_1 R a_2 \text{ and } b_1 \Gamma b_2) \text{ and } (a_2 R a_1 \text{ and } b_2 \Gamma b_1)$$

This implies (R and Γ being anti-symmetric)

$$a_1 = a_2 \text{ and } b_1 = b_2$$

an thus we conclude

$$\forall (a_1, b_1), (a_2, b_2) \in A \times B : (a_1, b_1) \leq (a_2, b_2) \text{ and } (a_2, b_2) \leq (a_1, b_1) \Rightarrow a_1 = a_2 \text{ and } b_1 = b_2$$

• $\Leftarrow$: This requires

$$\forall a_1 \text{ and } b_1 \Rightarrow (a_1, b_1), (a_1, b_1) \in A \times B : (a_1, b_1) \leq (a_1, b_1) \text{ and } (a_1, b_1) \leq (a_1, b_1)$$

As A and Γ are reflexive $(a_1, b_1) \leq (a_1, b_1)$ is a true statement and conclude

$$\forall (a_1, b_1), (a_2, b_2) \in A \times B : (a_1, b_1) \leq (a_2, b_2) \text{ and } (a_2, b_2) \leq (a_1, b_1) \Leftarrow a_1 = a_2 \text{ and } b_1 = b_2$$

Conclusion: From i), ii) and ii) $\leq$ is an partial ordering in $A \times B$.

◊

(b) Suppose that R and Γ are total orderings for A and B, respectively. Is $\leq$ necessarily a total ordering for $A \times B$?

The answer is yes as $aRa^{'}$ and $b\Gamma b^{'}$ are defined for all elements in A and B, hence $\leq$ is defined for all elements in $A \times B$.

♦

1.23.5

Let $\mathscr{L}$ be the relation defined in $\mathbb{R} \times \mathbb{R}$ as follows. For all (a_1, a_2) and (b_1, b_2) in $\mathbb{R} \times \mathbb{R}$, let $(a_1, a_2) \mathscr{L} (b_1, b_2)$ if and only if

$$a_1 \leq b_1, \text{ and if } a_l = b_1, \text{ then } a_2 \leq b_2$$

For obvious reasons this relation $\mathscr{L}$, is called a dictionary or lexicographical order for $\mathbb{R} \times \mathbb{R}$.
(a) Is $\mathscr{L}$ a partial ordering for $\mathbb{R} \times \mathbb{R}$?
(b) If the answer to (a) is yes, is $\mathscr{L}$ a total ordering for $\mathbb{R} \times \mathbb{R}$?

(a) $\mathscr{L}$ being a partial ordering of $\mathbb{R}\times\mathbb{R}$ requires that $\mathscr{L}$ is transitive, reflexive and anti-symmetric.

i) transitivity: $(a_1,\, a_2)\mathscr{L}(b_1, b_2),\, (b_1,\, b_2)\mathscr{L}(c_1, c_2) \Rightarrow (a_1,\, a_2)\mathscr{L}(c_1, c_2)$

From the definition of $\mathscr{L}$ we have:

$$\begin{cases} (a_1,\, a_2)\mathscr{L}(b_1, b_2) \Leftrightarrow a_1 \leq b_1, \text{ and if } a_1 = b_1, \text{ then } a_2 \leq b_2 \\ \\ (b_1,\, b_2)\mathscr{L}(c_1, c_2) \Leftrightarrow b_1 \leq c_1, \text{ and if } b_1 = c_1, \text{ then } b_2 \leq c_2 \end{cases}$$

We consider three cases:

1. $a_1 \neq b_1,\, b_1 \neq c_1$

Then, we have $a_1 \leq b_1$ and $b_1 \leq c_1 \Rightarrow a_1 \leq c_1$ which satisfies the definition of $\mathscr{L}$ (note that we can't have the case $a_1 = c_1$).

2. $a_1 \neq b_1,\, b_1 = c_1$

Then, we have $a_1 \leq b_1$ and $b_1 = c_1 \Rightarrow a_1 \leq c_1$ which satisfies the definition of $\mathscr{L}$ for $(a_1, a_2)\mathscr{L}(c_1, c_2)$ as $a_1 \neq c_1$.

3. $a_1 = b_1,\, b_1 \neq c_1$

Then, we have $a_1 = b_1$ and $b_1 \leq c_1 \Rightarrow a_1 \leq c_1$ which satisfies the definition of $\mathscr{L}$ for $(a_1, a_2)\mathscr{L}(c_1, c_2)$ as $a_1 \neq c_1$.

4. $a_1 = b_1,\, b_1 = c_1$

Then, we have $a_1 = b_1$ and $b_1 = c_1 \Rightarrow (a_1 = b_1$ and $a_2 \leq b_2)$ and $(b_1 = c_1$ and $b_2 \leq c_2)$ which implies $a_1 = c_1$ and $a_2 \leq c_2)$ which satisfies the definition of $\mathscr{L}$ for $(a_1, a_2)\mathscr{L}(c_1, c_2)$.

Conclusion: $\mathscr{L}$ is a transitive relation.

$\Diamond$

ii) reflexivity $(a_1,\, a_2)\mathscr{L}(a_1, a_2)$ is true.

From the definition of $\mathscr{L}$, we should have

$$(a_1,\, a_2)\mathscr{L}(a_1, a_2) \Leftrightarrow a_1 \leq a_1, \text{ and if } a_1 = a_1, \text{ then } a_2 \leq a_2$$

$\Rightarrow$: $(a_1,\, a_2)\mathscr{L}(a_1, a_2) \Rightarrow a_1 \leq a_1,$ and if $a_1 = a_1,$ then $a_2 \leq a_2$.

Obviously, we have the condition $a_1 \leq a_1$ and as $a_1 = a_1$ we should have $a_2 \leq a_2$ which is true as

$a_2 = a_2$.

$\Leftarrow$: $(a_1,\, a_2)\mathscr{L}(a_1, a_2) \Leftarrow a_1 \leq a_1,\;$ and if $a_l = a_1,\;$ then $a_2 \leq a_2$.

We have by assumption,as $a_1 = a_1$, $a_1 \leq a_1$ and as as $a_2 = a_2$we have also $a_2 \leq a_2$. By definition, we have $(a_1,\, a_2)\mathscr{L}(a_1, a_2)$.

Conclusion $\mathscr{L}$ is a reflexive relation.

iii) anti-symmetric $\forall(a_1,\, a_2), (b_1,\, b_2)\mathscr{L}(b_1, b_2)$ and $(b_1,\, b_2)\mathscr{L}(a_1, a_2) \Rightarrow (a_1,\, a_2) = (b_1,\, b_2)$.

From the definition of $\mathscr{L}$ we have:

$$\begin{cases} (a_1,\, a_2)\mathscr{L}(b_1, b_2) \Leftrightarrow a_1 \leq b_1,\; \text{ and if } a_1 = b_1,\; \text{then } a_2 \leq b_2 \\ \\ (b_1,\, b_2)\mathscr{L}(a_1, a_2) \Leftrightarrow b_1 \leq a_1,\; \text{ and if } b_1 = a_1,\; \text{then } b_2 \leq a_2 \end{cases} \tag{1}$$

If $(a_1,\, a_2)\mathscr{L}(b_1, b_2)$ and $(b_1,\, b_2)\mathscr{L}(a_1, a_2)$ we see from the definition of $\mathscr{L}$ that we have $a_1 \leq b_1$ and $b_1 \leq a_1$, hence we will have $a_1 = b_1$. Then, the definition implies $a_2 \leq b_2$ and $b_2 \leq a_2$, and thus $a_2 = b_2$. Hence we have indeed $(a_1,\, a_2) = (b_1,\, b_2)$ and thus $\mathscr{L}$ is anti-symmetric.

Conclusion: from i), ii) and iii) we see that $\mathscr{L}$ is a partial ordering of $\mathbb{R} \times \mathbb{R}$.

$\lozenge$

(b) If the answer to (a) is yes, is $\mathscr{L}$ a total ordering for $\mathbb{R} \times \mathbb{R}$?

Yes, as the usual relation $\leq$ in $\mathbb{R}$ is a total ordering of $\mathbb{R}$. Hence for all $(a,\, b) \in \mathbb{R} \times \mathbb{R}$ the $\leq$ relation of the two entries are defined for all a and b and thus from the definition also for all $(a,\, b) \in \mathbb{R} \times \mathbb{R}$.

$\blacklozenge$

1.23.6

Prove that if $(S, \leq)$ is a well-ordered set, then it is a linearly ordered set.

Let's remind some definitions:

$a)$ A well ordered set means: If $(S, \leq)$ is a partially ordered set such that every nonempty subset of S has a first element, then $\leq$ is said to be a well ordering for S and $(S, \leq)$ is said to be a well-ordered set.

$b)$ A first element m of a partially ordered set $(T, \leq)$ implies that $\exists\, m \in T, \forall s \in T : m \leq s$.

c) A linearly ordered set means that $\forall x, y \in S : x \leq y$ or $y \leq x$.

Now, suppose that $(S, \leq)$ is not linearly ordered, then:

$$\exists\, x \in S, \forall y \in S : x \nleq y \text{ and } y \nleq x \tag{1}$$

Take a subset $S_x \subset S$ such that $x \in S_x$. As $(S, \leq)$ is a well ordered set, S_x has a first element m_x. This means that for the element $x \in S_x$ we have $m_x \leq x$.
But we get a contradiction as $m_x \leq x$ is defined but we supposed that $x \nleq m_x$ and $m_x \nleq x$ (substitute y with m_x in (1)). Hence our supposition that $(S, \leq)$ is not linearly ordered is wrong and conclude $(S, \leq)$ is linearly ordered.

♦

1.24 Axiom of choice and Zorn's lemma

1.24.1

Show that the two forms of Zorn's lemma given in **24.3** are equivalent.

24.3 gives only one version of Zorn's lemma. So it is not clear what the author wants. Does he want that, starting with Hausdorff maximal principle we can prove Zorn's lemma? If yes we have to prove that:

24.4. Hausdorff Maximality Principle.
Let $(S, \leq)$ be a partially ordered set. Then S contains a maximal linearly ordered subset.

$$\Downarrow$$

24.3. Zorn's Lemma.
Suppose that $(S, \leq)$ is a nonempty partially ordered set such that every linearly ordered subset in S has an upper (lower) bound in S.
Then S has a maximal (minimal) element in S

$\circ$

First,note that in this book the notion of *maximal linearly ordered subset* is not defined explicitly. So, we put as definition: a linearly ordered subset $P \subset S$ with $(S, \leq)$ a partially ordered set, is a maximal linearly ordered subset of S if $\forall X^* \subset S : X^* \subset P$ where X^* are linearly ordered subsets of S.
Let's show the asked equivalence.

Consider a partially ordered set S, where every linearly ordered subset has an upper bound.
Note: This is possible because the considered subset is linearly order, so the relation $a \leq b$ exists for all a, b in this subset. So, we will certainly have an upper bound (in fact , a largest element) in this subset (although other upper bounds may exist which are not in this subset).

According to the Hausdorff maximum principle there exists a maximal linearly ordered subset $P \subset S$ and P has an upper bound , m. Suppose m is not the largest element in P then $\{m\} \cup P$ is also a linearly ordered set[1] in which P would be properly contained i.e. $P \subset \{m\} \cup P$, contradicting the definition. Thus m is a maximal element in S.

$\blacklozenge$

[1]This follows form the fact that we supposed that m is not the largest element. This assumption on itself, means that $m \leq x$ or $x \leq m, x \in P$ is defined.

1.24.2

Prove the following variation of **24.4**. Let $(S, \leq)$ be a partially ordered set. If A is a linearly ordered subset of S, then there exists a maximal linearly ordered subset M of S such that $A \subset M$.

Following the line of proof of **24.4**. **Hausdorff Maximality Principle**, we know that there exists a set M such that M is a maximally linearly ordered subset of S.
Now, suppose we have a linearly ordered subset $A^{'}$ of S for which $A^{'} \supset M$.
This means that M is contained in $A^{'}$, meaning that there exists an upper bound $A^{'}$ in S for which $M \subset A^{'}$ with $A^{'}$ a linearly ordered subset of S, meaning that M would not be maximal, a contradiction. Hence, $A^{'} \subset M$.

♦

1.24.3 †

Let $\mathscr{K}$ be a collection of sets. Prove that there exists a maximal nested subcollection $\mathscr{K}$.

Can we simply use Hausdorff maximality principle on $(\mathscr{K}, \subset)$?
Then, we know that $(\mathscr{K}, \subset)$ contains a maximal linearly ordered subset $(\mathscr{K}^*, \subset)$. As $(\mathscr{K}^*, \subset)$ is a linearly ordered subset we have, $\forall a, b \in \mathscr{K}^* : a \subset b$ or $b \subset a$ which is a definition of a nested set.
Conclusion: $\mathscr{K}^*$ is maximal and nested.

♦

1.25 Cardinality of sets (Introduction)

1.25.1 †

Prove Theorem **25.3**.

Be $\mathscr{K} = \{A, B, C, \dots\}$ a collection of sets. If $\sim$ is defined in $\mathscr{K}$, this means that for some sets in the collection we have a bijection between these sets.
An equivalence relation requires 3 constraints:

$$\begin{cases} \forall a \in A : (a,a) \\ \\ \forall a \in A, b \in B : (a,b) \Rightarrow (b,a) \\ \\ \forall a \in A, b \in B, c \in C : (a,b),(b,c) \Rightarrow (a,c) \end{cases}$$

Recall that $\sim$ means that there exists a bijection f between the two involved sets.

i) Suppose $A \sim A$, obviously we can take the identity map as trivial bijection which can be expressed as (a,a), $\forall a \in A$.// So, the first prerequisite for an equivalence relation is fulfilled.

ii) Suppose $A \sim B$. This means that for every element a in A there exists an element b in B such that (a,b) exists but as f is a bijection, the map f^{-1} exists which maps b again to a. Hence, (b,a) is a valid set and exists.
So, the second prerequisite for an equivalence relation is fulfilled.

iii) Suppose $A \sim B$, $B \sim C$ this means that there exist bijection f and g such that $\forall a \in A$, $\exists! b \in B : b = f(a)$ and $\forall b \in B$, $\exists! c \in C : c = g(b)$, hence, $(g \circ f)(a) = c$.
So, the third prerequisite for an equivalence relation is fulfilled.

As all prerequisites are fulfilled we are able to define a equivalence relation between some sets in $\mathscr{K}$.

♦

1.25.2

Show that the set $\mathbb{P}$ of all positive integers is equivalent to the set of all positive even integers.

Define the one-to-one map $f : \mathbb{P} \to \mathbb{P}_{2n}$ with $f = \{(n, 2n) : n \in \mathbb{P}\}$ and $\mathbb{P}_{2n}$ being the set of positive even numbers.

Obviously, f is one-to-one and thus the function defines the equivalence of the two sets.

♦

1.25.3

Show that the set $\mathbb{P}$ of all positive integers is equivalent to the set $\mathbb{P} - \{1\}$.

Define the one-to-one map $f : \mathbb{P} \to \mathbb{P} - \{1\}$ with $f = \{(n, n+1) : n \in \mathbb{P}\}$.

Obviously, f is one-to-one and $1 \notin \mathbb{P} - \{1\}$.
Thus the function defines the equivalence of the two sets.

♦

1.25.4

Recall that $\mathbb{P}_n = \{1, 2, ..., n\}$. Show that $\mathbb{P} \sim (\mathbb{P} - \mathbb{P}_n)$.

Define the one-to-one map $f : \mathbb{P} \to \mathbb{P} - \mathbb{P}_n$ with $f = \{(p, p + n : n \in \mathbb{P}\}$.

Obviously, f is one-to-one and the image of f is $\{n + 1, n + 2, n + 3, \dots\} = \mathbb{P} - \mathbb{P}_n$.
Thus the function defines the equivalence of the two sets.

♦

1.26 Countable sets

1.26.1

Prove that the set of all rational real numbers is a countably infinite set.

Every $q \in \mathbb{Q}$ is of the form $\frac{m}{n}$, $m \in \mathbb{Z}$, $n \in \mathbb{P}$ and can be represented by the ordered set (m, n).

Define the relation

$$f = \begin{cases} \frac{p}{2} & \text{for } p \in \mathbb{P},\ p \text{ is even} \\ (-1)^p \frac{p-1}{2} & \text{for } p \in \mathbb{P},\ p \text{ is odd} \end{cases}$$

This relation maps the even elements of $\mathbb{P}$ onto the positive elements of $\mathbb{Z}$ and the odd elements of $\mathbb{P}$ onto the negative elements of $\mathbb{Z}$ and the zero element. Hence f is a surjection.
By **26.6**, as $\mathbb{P}$ is countable, then $\mathbb{Z}$ is also countable.

By theorem **26.7** we know that $\mathbb{P} \times \mathbb{P}$ is countably infinite. Considering that (m, n) as a representation of a rational number can be expressed by $(f(p), n)$ and hence we can associate with every rational number an ordered set $(p, n) \in \mathbb{P} \times \mathbb{P}$ we conclude that $\mathbb{Q}$ is countably infinite.

♦

1.26.2

Prove that if X is a set that is equivalent to a countable set, then X is also a countable set.

Be C a countable set. As $X \sim C$ there is a bijection f from C into X. The same yields for C with a bijection g from P^* into C with $\mathbb{P}^* = \mathbb{P}_n$ or $\mathbb{P}$.
$h = g \circ f : \mathbb{P}^* \to X$ is also a bijection.
Hence, X is countable.

♦

1.26.3

Suppose that $f : X \to Y$ is a map from a set X onto a countable set Y. Suppose that for each $y \in Y$, $f^{-1}[y]$ is a countable set. Is X necessarily a countable set?

As Y is countable, there exists a bijection $f : \mathbb{P}^* \to Y$ with $\mathbb{P}^* = \mathbb{P}_n$ or $\mathbb{P}$.

Be $X_y = f^{-1}[y]$ for a certain $y \in Y$. By assumption, X_y is a countable set. Be $\mathscr{K}$ the collection of the sets X_y for all $y \in Y$. As Y is countable, $\mathscr{K}$ will also be countable. From theorem **26.8** we know that $\bigcup \mathscr{K}$ is countable. As $f : X \to Y$ is a map from a set X onto a countable set Y, we know that $\cup X_y : \forall y \in Y$ will correspond to X, hence, $X = \bigcup \mathscr{K}$ and as $\bigcup \mathscr{K}$ is countable, X will also be countable.

♦

1.26.4

Prove that if A and B are countable sets, then so is $A \times B$.

Be $\mathbb{P}^*$ equal to $\mathbb{P}_n$ or $\mathbb{P}$. A and B are countable sets implies that $A \overset{f}{\sim} \mathbb{P}_{n_A}$ and $B \overset{g}{\sim} \mathbb{P}_{n_B}$, with f, g the bijective maps associated with the equivalence. .
So, for each $(a, b) \in A \times B$ let's define a map $f : A \times B \to \mathbb{P}^*$ with $\mathbb{P}^* = \mathbb{P}_{n_A \times b_B}$ or $\mathbb{P}$

$$f = \{2^{f(a)} 3^{g(b)} : a \in A,\, b \in B\}$$

We note that $2^{f(a)} 3^{g(b)} : a \in A,\, b \in B$ is indeed unique (i.e. f is really a map).
$2^{f(a)} 3^{g(b)} : a \in A,\, b \in B$ can be written as $2^m 3^n : m \in \mathbb{P}^*_A,\, n \in \mathbb{P}^*_B$
Suppose we have the more general expression

$$2^{m'} 3^{n'} = 2^m 3^n : m,\, m',\, n,\, n' \in \mathbb{P}$$

Be $m' = m + p$, $n' = n + q$ with $p, q \in \mathbb{Z}$. Then, $2^m 3^n 2^p 3^q = 2^m 3^n$. This implies $2^p 3^q = 1$ or $p = 0$, $q = 0$. So, indeed, $2^{f(a)} 3^{g(b)} : a \in A,\, b \in B$ is unique (i.e. f is really a map).
Hence, we can write $f(a, b) = h(m, n)$ where (m, n) are unique for a given (a, b).
Thus,

$$A \times B \sim \mathbb{P}^*_A \times \mathbb{P}^*_B$$

and hence $A \times B$ is countable.

♦

1.26.5

Let $f : X \to Y$ be a surjection. Show that there is a subset of X that is equivalent to Y.

Be the relation f^{-1} the inverse of the function f. Let, $X_y = f^{-1}[y]$. This exists for all $y \in Y$ (f is a surjection).
By the axiom of choice, for each X_y we can choose a single element $x_y \in X_y$. As each $x \in X$ will be an element of a certain X_y we choose the x_y such that $x_y \neq x^{'}_{y}$, $\forall y$, $y^{'} \in Y$.
Be $S = \bigcup\{x_y\} \subset X$. We know that the $\{x_y\}$ are disjoint. So for each element of S there is a unique $y \in Y$ and for each $y \in Y$ we have one and only one $x \in S$. Moreover $S \subset X$. We conclude that the constructed relation is a bijection and hence

$$S \in Y$$

.

♦

1.27 Uncountable sets

1.27.1 †

It is known that every real number between 0 and 1 inclusive has a (binary) representation in the form $.a_1a_2a_3\dots a_n\dots$ where each a_i, is either 0 or 1. However, as in the decimal system, the representation is not unique. For example, .01100 (remaining terms 0) represents the same number as .0101111 (remaining terms 1). But each real number between 0 and 1 has at least one and no more than two representations. Use this information to prove that the reals are uncountable. Point out why this implies that the set of all irrational numbers is uncountable.

♦

1.27.2

Suppose that A is an uncountable set and C is a countable subset of A. Show that $A - C$ is an uncountable set.

Suppose $A - C$ is countable.
Consider $\mathscr{K} = \{A - C, C\}$. Obviously $\mathscr{K}$ is countable and it's elements are countable. By **26.8** we know that the union of a countable collection of countable set, is also countable. But

$$\bigcup \mathscr{K} = (A - C) \cup C = A$$

We have a contradiction as A was not countable.
Conclusion: $A - C$ is not countable

♦

1.27.3

Suppose that A, B, C, and D are sets such that $A \cap C = B \cap D = \varnothing$. Suppose further that $A \sim B$ and $C \sim D$. Is $(A \cup C) \sim (B \cup D)$?

As $A \sim B$ and $C \sim D$ there exist two bijection $f : A \to B$ and $g : C \to D$.
Let's define $h : (A \cup C) \to (B \cup D)$ as

$$h(x) = \begin{cases} f(x) & \text{for } x \in A \\ g(x) & \text{for } x \in C \end{cases}$$

As $A \cap C = \varnothing$ we will have $h(x) = f(x)$ or $h(x) = g(x)$ for a certain $x \in A \cup C$ and as f and g have different images B and D with $B \cap D = \varnothing$ we will have $h(x) \in B$ or $h(x) \in D$. Hence the function $h : (A \cup C) \to (B \cup D)$ is a bijection as f and g are bijections.
Conclusion: $(A \cup C) \sim (B \cup D)$

♦

1.27.4

Prove that every infinite set contains a countably infinite subset.

Be X the considered infinite set.
Lets define recursively, the following sets

$$\begin{array}{ll} S_1 = \{x_1\} & x_1 \in X \\ S_2 = \{x_2\} \cup S_1 & x_2 \in X - S_1 \\ S_3 = \{x_3\} \cup S_2 & x_3 \in X - S_2 \\ S_4 = \{x_4\} \cup S_3 & x_4 \in X - S_3 \\ & \vdots \end{array}$$

where the x_i are chosen randomly in the relevant set (this is possible by the axiom of choice).
Each S_k is countable. Also be $\mathscr{K} = \{S_1, S_2, S_3, \dots\}$. $\mathscr{K}$ is countable. By **26.8** we know that $\bigcup \mathscr{K}$ is countable and as $\bigcup \mathscr{K} \subset X$ we find that X contains a countably infinite set.

♦

1.27.5

Prove that if S is an infinite set and $x \in S$, then $S - \{x\} \sim S$.

By the previous exercise 4 we know that every infinite set contains a countably infinite set. So, as $S - \{x\}$ and S are infinite sets they will both contain a countably infinite set.
Let's denote C and $C^{'}$ these sets for S and $S - \{x\}$ respectively. This means that we can express S as $S = S^{'} \cup C$, where $S^{'} = S - C$. Also, we can express $S - \{x\}$ as $S - \{x\} = S^{"} \cup C^{'}$, where $S^{"} = (S - \{x\}) - C^{'}$.
Notice that nor $S^{"}$ nor $C^{'}$ can contain x. On the other hand or $S^{'}$ or C will contain x. We know that C is countably infinite. If $x \in S^{'}$ we can 'extract' x out of $S^{'}$ and make it an element of C. C will still be infinite and countable. Hence, $S^{'}$ will be $S^{"}$. Doing so, we get the following expressions

$$\begin{cases} S = S^{"} \cup C & \text{with} \quad S^{"} \cap C = \varnothing \\ S - \{x\} = S^{"} \cup C^{'} & \text{with} \quad S^{"} \cap C^{'} = \varnothing \end{cases}$$

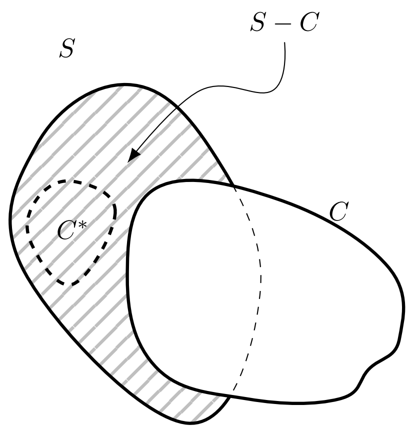
Obviously, $S^{"} \sim S^{"}$ (the identity map is a bijection).
By exercise 3 of this section we know that if $A \cap C = B \cap D = \varnothing$ and further that $A \sim B$ and $C \sim D$, that we have $(A \cup C) \sim (B \cup D)$.
So, provided that $C \sim C^{'}$ we can apply this last theorem and would indeed get that $S - \{x\} \sim S$.
Question is $C \sim C^{'}$?
The answer is yes by applying again theorem **26.8** noting that we can write $C = \bigcup\{\{x\}, C^{'}\}$. As both $\{x\}$ and $C^{'}$ are countable and the collection $\{\{x\}, C^{'}\}$ is countable, C will also be countable and thus we can find a bijection between C and $C^{'}$.
Conclusion: $S - \{x\} \sim S$

♦

1.27.6

Suppose S is an uncountable set and C is a countable set. Show that $S - C \sim S$.

The proof is analogue to the previous problem. We investigate the more general case where C is countably infinite.
First, note that S is necessarily infinite as it is non countable. Then, $S - C$ will be an infinite uncountable set (this is a consequence of the theorem stated in exercise 2 of this section) .
As $S - C$ is an infinite set, it will contain a countably infinite set, which we will denote by C^*.
Be $\hat{S} = S - C - C^*$. We can express S as $S = \hat{S} \cup C^{'} \cup C^*$, where $C^{'} = S \cap C$.



(a) $\hat{S} = S - C - C^*$

$$\begin{tikzcd}
C^* \arrow[rr, shift right, "g \circ h \circ f^{-1}"'] \arrow[d, shift right, "f^{-1}"'] & & C^* \cup C^{'} \arrow[ll, shift right, "f \circ h^{-1} \circ g^{-1}"'] \arrow[d, shift left, "g^{-1}"] \\
\mathbb{P} \arrow[u, shift right, "f"'] \arrow[rr, shift right, "h"'] & & \mathbb{P} \arrow[ll, shift right, "h^{-1}"'] \arrow[u, shift left, "g"]
\end{tikzcd}$$

(b) f, g, h are bijections, hence $g \circ h \circ f^{-1}$ is also a bijection.

Figure 1.27: $S - C \sim S$ with S an uncountable set and C a countable set

We use the result of exercise 3 of this section:
If $A \cap E = B \cap D = \varnothing$ and further that $A \sim B$ and $E \sim D$, then we have $(A \cup E) \sim (B \cup D)$.

Put, $A = B = \hat{S}$, $E = C^*$ and $D = (C^{'} \cup C^*)$. We have indeed $\hat{S} \cap C^* = \hat{S} \cap (C^{'} \cup C^*) = \varnothing$.
Obviously $A \sim B$ and also we have $E \sim D$ (see figure 1.26. (b)) as we have bijections f and g between C^*, $C^{'} \cup C^*$ (countable sets) and $\mathbb{P}$ and always can define a bijection $h : \mathbb{P} \to \mathbb{P}$.

Hence, we get $(A \cup E) \sim (B \cup D)$ or

$$(\hat{S} \cup C^*) \sim \hat{S} \cup (C^{'} \cup C^*)$$

and thus $S - C^{'} \sim S$. Here, we can replace $C^{'}$ by C as $C^{'} = S \cap C$ and get

$$S - C \sim S$$

♦

1.29 Review exercises

1.29.I

Let p and q, r, and s be statements. Consider the following compound statement: If (p and q), then (r or s). Choose the statement or statements below which would be the correct way to state the contrapositive of the given compound statement.

1.29.I.1

1. If (r is false and s is false), then (p is false or q is false).
2. If r is false or s is false, then p is false and q is false.
3. If (p and q) is a false statement, then (r or s) is a false statement.
4. If (r or s), then (p and q).
5. None of the previous choices is correct, but a correct one is ...

$$
\begin{aligned}
& (p \wedge q) \Rightarrow (r \vee s) \\
\Leftrightarrow \quad & \neg(r \vee s) \Rightarrow \neg(p \wedge q) \\
\Leftrightarrow \quad & (\neg r \wedge \neg s) \Rightarrow (\neg p \vee \neg q)
\end{aligned}
$$

Hence, (1) is the correct answer.

♦

1.29.II

1.29.II.1

Let A and B and C be nonempty sets. Either prove the following statement or give a counterexample.

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$$

.

Be $(a,b) \in A \times B$ and $(c,d) \in C \times D$. Then $(A \times B) \cap (C \times D) \neq \varnothing$ implies that there exist (a,b) and (c,d) such that $a = c$ and $b = d$. As in that case a and $c \in A \cap C$ and b and $d \in B \cap D$ implies that (a,c) and $(b,d) \in (A \cap C) \times (B \cap D)$ and thus

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$$

♦

1.29.III

1.29.III.1

Give the names of the three properties that a relation must possess in order to be called an equivalence relation.

Associative, reflexive and symmetric.

♦

1.29.III.2

Suppose that R is a relation defined on the set of real numbers as follows: xRy if and only there exists an integer k such that $x - y = k$. Is R an equivalence relation ?
Justify your answers.

We check the three required properties: Associative, reflexive and symmetric.

i) associative : aRb and bRc implies aRc.
Be $x - y = k$ and $y - z = m$. Adding those two expressions gives $x - z = m + k$.
The relation is associative.

ii) reflexive : means that aRa is defined.
Obviously $x - x = 0$ with $0 \in \mathbb{Z}$ and thus the relation is reflexive.

iii)symmetric : aRb implies bRa is also defined.
Be$x - y = k$. Obviously $-x + y = -k$ or $y - x = k^{'}$ with $k^{'} = -k \in \mathbb{Z}$ and thus the relation is symmetric.

♦

1.29.III.3

Suppose that R is an equivalence relation defined in a set A. Which of the following is necessarily true? Justify your answers.
(a) $R = A \times A$
(b) $R \subset A \times A$
(c) $\{(x, x) : x \in A\} \subset R$
(d) If a and b are distinct elements in A, then $R[a] \cap R[b] = \varnothing$
(e) If $R[a] \cap R[b] \neq \varnothing$, , then $R[a] = R[b]$
(f) $R = R^{-1}$

(a) is necessarily true as by definition of an equivalence relation (see page 39), xRx is defined for all $x \in A$.

(b) is true by the definition of a relation.

(c) is true as an equivalence relation must be reflexive.

(d) Suppose $R[a] \cap R[b] \neq \varnothing$ with an element $y \in R[a] \cap R[b]$. This means that we have $(a, y) \in R$ and $(b, y) \in R$ for $a \neq b$. We know that R is symmetric, hence $(y, b) \in R$ and by the transitivity we have aRb. Hence, $b \in R[a]$. Also, by the symmetry we have $a \in R[b]$. By the reflexivity of R we have also $a \in R[a]$ and $b \in R[b]$. Thus, $\{a,\ b\} \in R[a]$ and $\{a,\ b\} \in R[b]$ and finally $R[a] \cap R[b] \neq \varnothing$. This reasoning suggest the following counterexample to show that (d) is not necessarily true:

Be $A = \{a,\ b,\ c\}$ and $R = \{(a, a),\ (b, b),\ (c, c),\ (a, b),\ (b, a),\ (a, c),\ (c, a),\ (b, c),\ (c, b)\}$.We have $R[a] = \{a,\ b,\ c\}$ and $R[b] = \{a,\ b,\ c\}$ giving $R[a] \cap R[b] = \{a,\ b,\ c\} \neq \varnothing$.

(e) We know by assumption $\exists y \in R[a] \cap R[b]$. So, $(a, y) \in R$ and $(b, y) \in R$. By symmetry and transitivity we have $(a, b) \in R$. Suppose $\exists y^{'} \in R[a]$ and $y^{'} \notin R[b]$. This means that $(a, y^{'}) \in R$ and by symmetry $(y^{'}, a) \in R$.
This combined with $(a, b) \in R$ gives by transitivity $(y^{'}, b) \in R$ and by symmetry $(b, y^{'}) \in R$. Hence $y^{'} \in R[b]$. We get a contradiction and conclude $\nexists y^{'} \in R[a]$ and $y^{'} \notin R[b]$ and get

$$R[a] = R[b]$$

(f) By symmetry of an equivalence relation we have $(a, b) \in R$ and $(b, a) \in R$ which is the definition of the inverse of a relation.

♦

1.29.IV

Let R be the relation defined as follows:

$$R = \{(x, y) : x \text{ is real}, y \text{ is real, and } |x - y| = 5\}$$

1.29.IV.1

Is R a symmetric relation?

Obviously as $|x - y| = |y - x|$.

♦

1.29.IV.2

2. Is R a transitive relation?

No, take as counterexample $(2, -3) \in R$ and $(-3, 2) \in R$, we would get $(2, 2) \notin R$.

♦

1.29.IV.3

Determine $R[2]$.

$R[2] = \{-3,\ 7\}$

♦

1.29.IV.4

Is R a function?

No, as for $x = 0$ we have $0R5$ and $0R(-5)$.

♦

1.29.IV.5

Find the domain of R^{-1}

$Dom\, R^{-1} = \mathbb{R}$.

♦

1.29.V

Let $f : X \to Y$ be a bijection. Let A and B be subsets of X.

1.29.V.1

Prove that $f[A \cap B] = f[A] \cap f[B]$.

Be $x \in A \cap B$. Then, as f is a bijection, hence "onto", so there $\exists y \in Y : y = f(x)$. Hence $y = f(x) \in f[A] \cap f[B]$. Hence we have $f[A \cap B] \subset f[A] \cap f[B]$

Suppose now we have $y \in f[A]$ and $f[B]$. This means that we must have an $x^{'} \in A$ and a $x" \in B$, for which we have $y = f(x^{'}) = f(x")$. Could it be possible that $x^{'} \neq x"$? The answer is no, as f is surjective, so once an element of A is mapped to a certain element of Y no other element of $X \supset A$, B will be mapped to that element (the same reasoning yields for an element in B). So,$x^{'} = x"$, which means $x^{'} \in f[A \cap B]$ and thus $f[A \cap B] \supset f[A] \cap f[B]$

Ans conclude $f[A \cap B] = f[A] \cap f[B]$.

♦

1.29.V.2

Give a counterexample to show that the preceding is not true if f is not a one-to-one function but simply a function.

Take $f : \mathbb{R} \to \mathbb{R} : y = x^2$ and $A = [-2, 0]$ and $B = [0, 2]$. We get $f[A \cap B] = \{f(0)\} = \{0\}$ and $f[A] = f[B] = [0, 4]$ and get $f[A] \cap f[B] = [0, 4]$.

♦

1.29.V.3

Explain what step in your proof of the first part breaks down if the hypothesis does not say "one-to-one."

See $1.29.V.1$ second part where we treat the case $f[A \cap B] \supset f[A] \cap f[B]$.

♦

1.29.VI

Consider the map $f : \mathbb{R} \to \mathbb{R}$ given by the following: $f(x) = x(x-2)$ for each $x \in \mathbb{R}$.

1.29.VI.1

Find the range of f.

Using calculus, the extrema are determined by $f^{'} = 2(x-1) = 0$, or $x = 1$ and thus $y_{\min} = -1$ and get

$$\text{Range } f = [-1, +\infty)$$

♦

1.29.VI.2

Determine the set $f^{-1}[\{y : -1 \leq y \leq 0\}]$.

We know from $1.29.VI.1$ that for $y = -1$ we have $x = 1$. And for $y = 0$ we have $x(x-2) = 0$ giving $f^{-1}[0] = \{0,\, 2\}$. Hence

$$f^{-1}[\{y : -1 \leq y \leq 0\}] = [0, 2]$$

♦

1.29.VI.3

Find the largest number z such that f restricted to the set $\{x : 0 \leq x \leq z\}$ is a one-to-one function.

Using calculus again, the slope of f is $f^{'} = 2(x-1)$. At $x = 0$ the slope is negative but the sign of the slope changes at $x = 1$, hence f is monotone decreasing between $[0,\, 1]$ and we conclude that z must be 1.

♦

1.29.VII

1.29.VII.1

Give a precise statement of the principle of finite induction.

Suppose that M is a subset of $\mathbb{P}$ such that $1 \in M$, and $h \in M$ implies that $h+1 \in M$. Then $M = \mathbb{P}$.

♦

1.29.VII.2

Give a precise statement of the well-ordering principle for integers.

Let K be a nonempty set of positive integers. Then there is a first (smallest) element in K.

♦

1.29.VII.3

Prove that the well-ordering principle implies the principle of finite induction.

Let M be a set of positive integers such that $1 \in M$, and $h \in M$ implies that $h+1 \in M$. We wish to show that $M = \mathbb{P}$. If $M \neq \mathbb{P}$, then $\mathbb{P} - M$ is a nonempty set of positive integers. By the well-ordering principle, there is a smallest integer k in $\mathbb{P} - M$, and thus $k - 1 \notin \mathbb{P} - M$. Since $k \neq 1$, $k - 1 \in \mathbb{P}$ and, thus, $k - 1 \in M$. However, by our assumption for the set M, it now follows that $k \in M$ and we have a contradiction.

♦

1.29.VII.4

Prove that $9^n - 8n - 1$ is divisible by 64, if n is any positive integer.

We prove by induction.
Take $n = 1$. We have $9^1 - 8 \times 1 - 1 = 0$ and is divisible by 64.
Suppose, for a certain n we have $9^n - 8n - 1 = 64k$ with $k \in \mathbb{P} \cup \{0\}$. Then, for $n+1$ we have

$$\begin{aligned} 9^{(n+1)} - 8(n+1) - 1 &= 9^n 9 - 8n - 8 - 1 \\ &= 9(9^n - 8n - 1) + 64n \\ &= 9 \times 64k + 64n \\ &= 64m \quad \text{ with } m = 9k + n \in \mathbb{P} \end{aligned}$$

By induction, the expression is proved.

♦

1.29.VIII

1.29.VIII.1

> Define what is meant by an infinite sequence.

A infinite sequence is a function defined on the set of the positive integers.

♦

1.29.VIII.2

> Define what is meant by a subsequence of an infinite sequence.

Let $f : \mathbb{P} \to X$ be a sequence with functional values in a set X. Let $N : \mathbb{P} \to \mathbb{P}$ be a strictly increasing sequence from $\mathbb{P}$ into $\mathbb{P}$ (i.e. if $i > j$ then $N(i) > N(j)$). Then, the composition $h = f \circ N$ is said to be a subsequence of f.

♦

1.29.VIII.3

> Define what is meant by a decomposition of a set.

If S is a set, form the collection $\mathscr{K}$ with subsets of S as elements but in such a way that all elements in $\mathscr{K}$ are disjoint (i.e. $\forall A \in \mathscr{K}, B \in \mathscr{K}, A \neq B : A \cap B = \varnothing$. Moreover if we have that $\bigcup \mathscr{K} = S$ then $\mathscr{K}$ is a decomposition of S.

♦

1.29.VIII.4

> Define what is meant by a function that is one-to-one.

Synonym is *injection*. This means that for a function $f : X \to Y$ if $f(x) = f(x^{'})$ then we have $x = x^{'}$.

♦

1.29.VIII.5

> Suppose that $f : X \to Y$ is a one-to-one function, and $g : Y \to Z$ is also a one-to-one function. Prove that the composition $g \circ f$ is a one-to-one function.

We have for, $f : X \to Y$: if $f(x) = f(x^{'})$ then we have $x = x^{'}$ and for $g : Y \to Z$: if $g(y) = f(y^{'})$ then we have $y = y^{'}$.
Suppose for $h : X \to Z : h = g \circ f$ we could have $x, x^{'} \in X$ such that $h(x) = h(x^{'})$ with $x \neq x^{'}$. This means $g(f(x)) = g(f(x^{'}))$ with $x \neq x^{'}$. But f is an injection so $f(x) \neq f(x^{'})$. Moreover g is also an injection. This means $g(f(x)) \neq g(f(x^{'}))$ as $f(x) \neq f(x^{'})$. So we can't have $h(x) = h(x^{'})$ with $x \neq x^{'}$ and thus

$$(g \circ f)(x) = (g \circ f)(x^{'}) \Rightarrow x = x^{'}$$

.

♦

1.29.IX

Is the following statement necessarily true? Justify your answer.
If R is a relation and $R^{-1} \subset R$, then R is a symmetric relation.

Yes as, if R^{-1} is a subset of R we must have that $(a, b) \in R$ implies that $(b, a) \in R^{-1}$ and thus $(b, a) \in R$ as $R^{-1} \subset R$. Conclusion (a, b) and (b, a) are elements of R and thus R is a symmetric relation.

♦

1.29.X

1.29.X.1

Define what is meant by a partially ordered set.

Be S a set and define a relation $\leq$ from S into S, such that this relation is transitive, reflexive but anti-symmetric i.e. $\forall a, b \in S : a \leq b$ and $b \leq a \Rightarrow a = b$. Note that it is not necessary that the relation is defined for all $b \in S$. This means that it is possible that for some elements $x \in S$, we can't find an element $a \in S$ such that $a \leq x$.

♦

1.29.X.2

Define what is meant by a totally ordered set.

This means that any arbitrary element x in S, we will find an element $a \in S$ such that $a \leq x$.

♦

1.29.X.3

Give an example of a partially ordered set that is not totally ordered.

See Exercise **1.23.2(a)**.

♦

1.29.X.4

Give an example of a partially ordered set that has a maximal element but no greatest element.

Consider the poset $S = (\mathbb{C}, \leq)$ with $\leq$ the usual ordering in the real numbers. Then

$$A \subset S = \{1, i, 2, 2i, 3, 3i\}$$

has no Greatest element but has 3 as Maximal element. Indeed, recall the definitions.

g is said to be the greatest element of S provided that

$$g \in A \text{ and } x \leq g, \forall x \in A$$

The important thing here is the "$\forall x \in A$". Obviously we can not compare let's say 3 with $3i$ with the given ordering relation, so can not find a $g \in A$.

On the other hand from the definition of a Maximal element m in S:

$$\forall s \in S : m \leq s \Rightarrow s = m$$

This means that if we can compare certain elements in S then we can perhaps find a Maximal element. This is the case in the example given as we can say

$$1 \leq 1,\ 1 \leq 2,\ 1 \leq 3,\ 2 \leq 2,\ 2 \leq 3,\ 3 \leq 3$$

♦

1.29.X.5

Prove that a set has at most one greatest lower bound.

The ordering relation is anti-symmetric. So, from the definition of a poset $(S, \leq)$:
Suppose that a subset A of S has an lower bound in S. If the set of lower bounds of S has a greatest element g, then g is called the greatest lower bound of A (g.l.b.(A)).
Suppose that $L = \{l_1, l_1, \dots\}$ is the set of lower bounds of S. Be l_g a greatest lower bound. That means that $l_g \in L$ and $x \leq l_g \forall x \in L$. Suppose now that there is another greatest lower bound $l_g^{'}$. Then we would have

$$l_g^{'} \in L \text{ and } x \leq l_g^{'} \forall x \in L$$

As this must be true for all $x \in L$ and $l_g \in L$ we would have

$$l_g \leq l_g^{'}$$

But as l_g was supposed to be the greatest lower bound and $l_g^{'} \in L$ we would also have

$$l_g^{'} \leq l_g$$

As the ordering relation is anti-symmetric we must conclude

$$l_g^{'} = l_g$$

♦

1.29.XI

1.29.XI.1

Use the axiom of choice to give an alternate proof of Theorem **26.6**.

Let's recall the theorem and it's proof:

26.6. Theorem. If $f : X \to Y$ is a surjection and X is countable, then Y is countable.
PROOF. Let $\mathbb{P}^*$ be $\mathbb{P}_n$, or $\mathbb{P}$ according to whether X is finite with cardinality $n \geq 1$ or is infinite. Then there exists a bijection $\alpha : \mathbb{P}^* \to X$. Using the well-ordering principle for positive integers, define $\beta : Y \to \mathbb{P}^*$ by $\beta(y) =$ least element of $\alpha^{-1}\left[f^{-1}[y]\right]$. The mapping $\beta : Y \to \mathbb{P}^*$ is an injection (one-to-one) and $\beta[Y]$ is a countable set by **26.4**.. The fact that $\beta : Y \to \beta[Y]$ is a bijection shows, by **26.5**., that Y is countable.

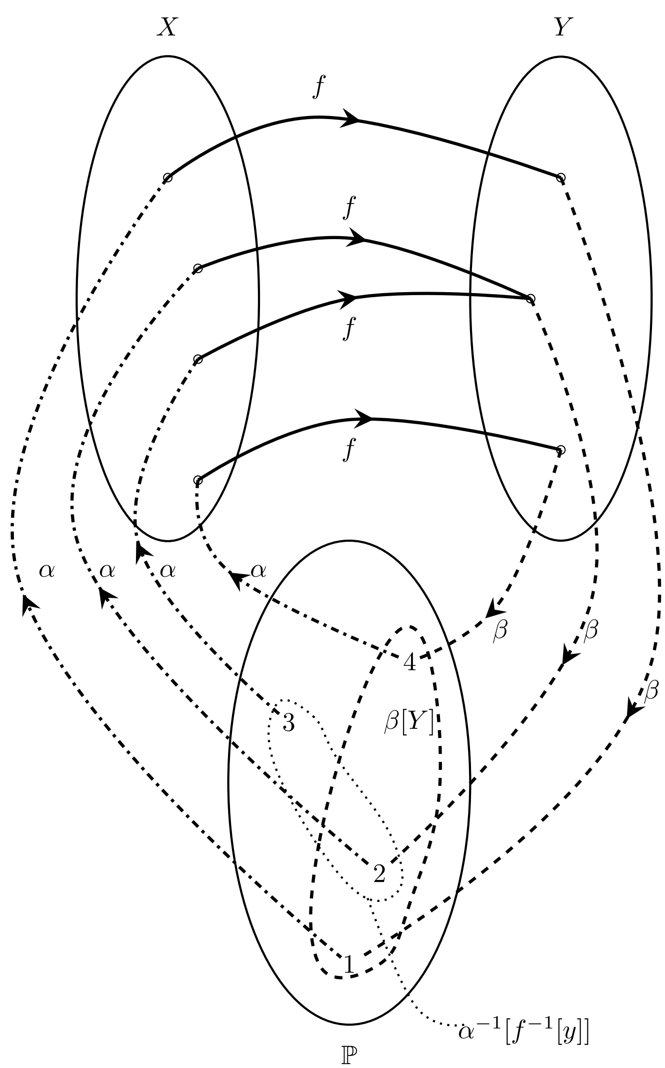


Figure 1.28: Scheme of proof that if $f : X \to Y$ is a surjection and X is countable, then Y is countable.

$\lozenge$

The axiom of choice as stated in the book is:

Suppose $\mathscr{K}$ is a nonempty collection of nonempty sets. Then there is a function s defined on $\mathscr{K}$ such that $s(K) \in K$ for each $K \in \mathscr{K}$.

The alternate proof is very similar of course with the proof given above. The only thing to change is the argument

Using the well-ordering principle for positive integers, define $\beta : Y \to \mathbb{P}^*$ by $\beta(y) =$ least element of $\alpha^{-1}\left[f^{-1}[y]\right]$.

We now change the definition of $\beta : Y \to \mathbb{P}^*$ to $\beta = \alpha^{-1} \circ f^{-1}$. For a given $y \in Y$, $\beta[y] = \alpha^{-1}\left[f^{-1}[y]\right]$ will be a set consisting of n elements with $n \in \mathbb{P}$ (as f is a surjection). Obviously $\beta[y]$ is countable. We now construct the collection of sets $\mathscr{K} = \{\beta[y], y \in Y\}$. We now apply the axiom of choice with the "choice function" s, and form the set $S = \{s(K) \in K : K \in \mathscr{K}\}$. As $S \subset \mathbb{P}$ and as $\mathbb{P}$ is countable, S is also countable. Hence with every element $y \in Y$ we can associate one and only one element of S and thus Y is countable.

$\blacklozenge$

1.29.XI.2

Let $\mathscr{S}$ be the collection of all finite-sequences of integers. Is $\mathscr{S}$ a countable set?

We first have to check whether or not $\mathscr{S}$ is infinite or infinite set, as if it is finite, then by definition, it will be countable.

Lemma: $\mathscr{S}$ is an infinite set.

The elements S_i [2] of $\mathscr{S}$ are all finite sequences.
Be the set $S = \bigcup \mathscr{S}$. We will "expand" $\mathscr{S}$ recursively, with the following two procedure.
- If S is not in $\mathscr{S}$ then we "expand" $\mathscr{S}$ and form $\mathscr{S} \cup S$.
- If S is in $\mathscr{S}$, we know that S is a finite set of integers, hence will contain a greatest element g. We "expand" S and form $S \cup \{g+1\}$. We "expand" again $\mathscr{S}$ and form $\mathscr{S} \cup S$.

We can perform this procedure indefinitely and conclude that $\mathscr{S}$ is an infinite set.

$\lozenge$

$\blacklozenge$

[2] The index is in fact an abuse of notation, as this would suggest that $\mathscr{S}$ is countable, which still has to be proven. The index is to be considered here as a place holder for an identification between sets

1.29.XII

Let $\mathscr{F}$ be the set of all functions that map the closed interval $[0,1]$ into $[0,1]$. Prove that $[0,1] < \mathscr{F}$ (Hint: Imitate somewhat the proof of **27.1**.)

As preliminary, we first note that $\mathscr{F}$ "is onto " $\mathbb{R}_{[0,1]}$ (e.g. the identity function is an element of $\mathscr{F}$).

We first prove that $\exists \mathscr{K} \subset \mathscr{F} : \mathbb{R}_{[0,1]} \sim \mathscr{F}$.

$\mathbb{R}_{[0,1]} \sim \mathscr{K}$ means that there exists a bijection between these two sets.

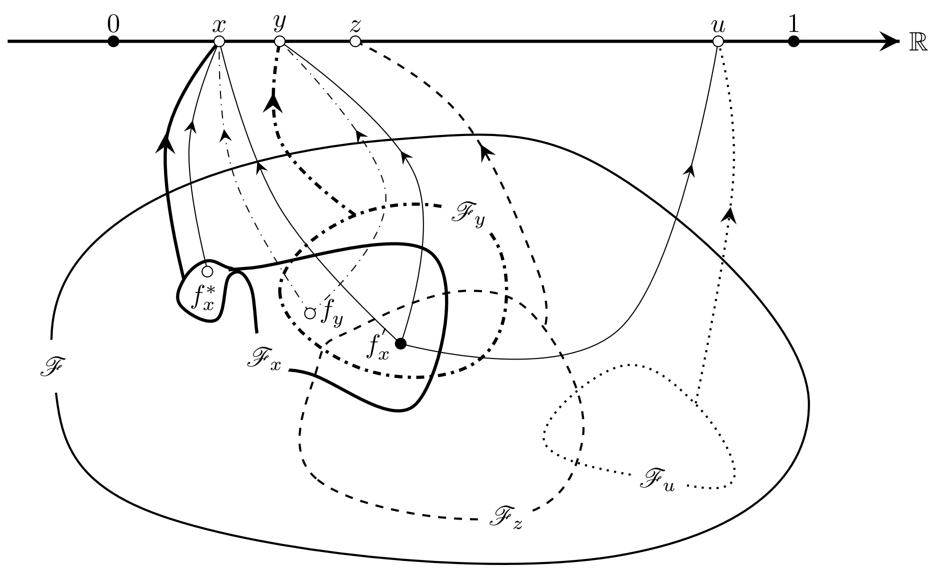


Figure 1.29: Relations from $\mathscr{F} \to \mathbb{R}_{[0,1]}$.

For each $x \in \mathbb{R}_{[0,1]}$ we can associate (see figure above) the subset $\mathscr{F}_x \subset \mathscr{F}$ such that

$$\mathscr{F}_x = \{f \in \mathscr{F} : \exists\, a \in \mathbb{R}_{[0,1]}, x = f(a) \in \mathbb{R}_{[0,1]}\}$$

Obviously, with every $x \in \mathbb{R}_{[0,1]}$ there is only one $\mathscr{F}_x$ associated and conclude that there is a bijection between $\mathbb{R}_{[0,1]}$ and the collection of sets $\{\mathscr{F}_x : x \in \mathbb{R}_{[0,1]}\}$.
Suppose that indeed, $\mathbb{R}_{[0,1]} \sim \mathscr{K}$, then for every $x \in \mathbb{R}_{[0,1]}$ there must exist a unique $f_x^* \in \mathscr{F}_x$ which is not an element of any element in $\{\mathscr{F}_y : y \in \mathbb{R}_{[0,1]} - \{x\}\}$ (recall that an element $f_x \in \mathscr{F}_x$ could also be in another set $\mathscr{F}_y$ as we could have $x = f_x(a) = f_y(b)$.
Consider the associated (to $\mathscr{F}_x$) set , $\mathscr{F}^x = \bigcup\{\mathscr{F}_y : y \in \mathbb{R}_{[0,1]} - \{x\}\}$. Then we have $\mathscr{F}_x - \mathscr{F}x = \{f_x^*\}$, where f_x^* is the constant function $f_x^* : \mathbb{R}_{[0,1]} \to \mathbb{R}_{[0,1]}$ defined as $f_x^*(a) = x, \forall a \in \mathbb{R}_{[0,1]}$ (obviously, there is only one and unique constant function[3] for a given x). So for every $x \in \mathbb{R}_{[0,1]}$ we can

[3] Be $f[\mathbb{R}_{[0,1]}] = \{x\}$ and $f'[\mathbb{R}_{[0,1]}] = \{x\}$, then, $f = f'$

associate a unique function $f^*_x : \mathbb{R}_{[0,1]} \to \mathbb{R}_{[0,1]}$, hence we have $\mathscr{K} = \{f^*_x; x \in \mathbb{R}_{[0,1]}\} \subset \mathscr{F}$ with a bijection between $\mathbb{R}_{[0,1]}$ and $\mathscr{K}$ and thus $\mathbb{R}_{[0,1]} \sim \mathscr{K}$.

We still have to prove that there is no bijection between $\mathbb{R}_{[0,1]}$ and $\mathscr{F}$ (i.e $\mathbb{R}_{[0,1]} \not\sim \mathscr{F}$).

Consider the subset $\mathscr{F}_{\{0,1\}} \subset \mathscr{F}$ such that $f[\mathbb{R}_{[0,1]}] = \{0, 1\}$ for $f \in \mathscr{F}_{\{0,1\}}$.
The functions f (represented as a collection of tuples) will have a form of $\{(0, 0), \dots (x, 1) \dots (y, 0) \dots (z, 1) \dots\}$, $\{(0, 1), \dots (x, 1) \dots (y, 1) \dots (z, 1) \dots\}$, $\{(0, 0), \dots (x, 0) \dots (y, 0) \dots (z, 0) \dots\}$, etc.

We are now ready to apply the scheme used in **27.1**. Suppose that $\mathbb{R}_{[0,1]} \sim \mathscr{F}_{\{0,1\}}$. Then, with every $f \in \mathscr{F}_{\{0,1\}}$ we can associate a unique $x \in \mathbb{R}_{[0,1]}$ as we have a bijection between $\mathbb{R}_{[0,1]}$ and $\mathscr{F}_{\{0,1\}}$.
Let's denote f_x the function associated with x.
Let's define the function $\alpha : \mathbb{R}_{[0,1]} \to \{0,\, 1\}$, such that $\alpha(x) = 1 - f_x(x)$ i.e. this function is of the form

$$\alpha = \{(0,\, 1 - f_0(0))\,, \dots, (x,\, 1 - f_x(x))\,, \dots, (y,\, 1 - f_y(y))\,, \dots, (z,\, 1 - f_z(z))\,, \dots\}$$

and $\alpha(x) \in \mathscr{F}_{\{0,1\}}$ but there is no $x^{'} \in \mathbb{R}_{[0,1]}$ left, which can be mapped by the bijection, to $\alpha(x)$.
Indeed, suppose $\alpha = f_{x'}$, then

$$\alpha = \{(0,\, f_{x'}(0))\,, \dots, (x,\, f_{x'}(x))\,, \dots, (y,\, f_{x'}(y))\,, \dots, \underbrace{\left(x^{'},\, f_{x'}(x^{'})\right)}_{\leftrightarrow\left(x',\, 1 - f_{x'}(x')\right)}, \dots, (z,\, f_{x'}(z))\,, \dots\}$$

We have a contradiction in the position $(x^{'}, \circ)$ as $f_{x'}(x^{'})$ is either 0 or 1 and so $\mathbb{R}_{[0,1]} \not\sim \mathscr{F}_{\{0,1\}}$ and as $\mathscr{F}_{\{0,1\}} \subset \mathscr{F}$ we can't have $\mathbb{R} \sim \mathscr{F}_{\{0,1\}}$.

♦

1.29.XIII

1.29.XIII.1

Give an example of a set A and a relation R in A that is symmetric and transitive in A but is not reflexive.

Be $A = \mathbb{R}$ and let xRy if and only if the product xy is strictly positive, in other words

$$\forall x,\, y \in \mathbb{R} : xRy \Leftrightarrow xy > 0$$

This is not reflexive because the number zero does not produce a strictly positive product with itself. R is symmetric and also transitive as, if $xy > 0$ and $yz > 0$, then $xz > 0$.

♦

1.29.XIII.2

Tell what is wrong with the following argument, which claims that to show that if a relation R in a set A is transitive and symmetric, then it is reflexive.
Let $a \in A$. Choose an element $b \in A$ such that aRb. Since R is symmetric, it then follows that bRa. Since R is transitive, aRb and bRa imply that aRa. Hence, we have shown that R is reflexive.

Consider the relation R on a set $A = \{a, b, c\}$ defined as

$$R = \{(a, b),\, (b, a),\, (a, c),\, (c, a),\, (b, c),\, (c, b),\, (b, b),\, (c, c)\}$$

Clearly, R is transitive and symmetric, but not reflexive as $(a, a) \notin R$.
Recall the exact definition of transitivity as a statement:

R is transitive in A if and only if for all $x,\, y,\, z \in A$

$$xRy \text{ and } yRz \Rightarrow xRz$$

In order to have that statement to be true we need that both sides of the $\Rightarrow$ connective, have to be true. In the example given, the right side is true for every element, giving the wrong conclusion as stated in the assignment.

♦

1.29.XIV

Let $(S, \leq)$ be a nonempty partially ordered set such that every linearly ordered subset A has an upper bound. Show that if $a \in A$, then there is a maximal element $m \in S$ such that $a \leq m$.

Let's recall some definitions:

• $(S, \leq)$ means that for some (not necessarily all) x, $y \in S$ that $x \leq$ or $y \leq x$ is defined.
• Linearly or totally ordered means that the relation $x \leq y$ or $y \leq x$ is defined for every x, $y \in S$.
• Upper bound of a subset $A \subset S$: means that there exists a $u \in S$ (not necessarily an element of A) such that $x \leq u$, $\forall x \in A$.
• Maximal element $m \in S$ means that m is not **followed** by any element in S or stated in a different way: $\forall \in S : m \leq s \Rightarrow m = s$.

We know that A has an upper bound $u \in S$.
Suppose first that $u \in A$. This mean that $\forall x \in A : a \leq u$. It is clear that u is not followed by any element in A, which implies that u is a maximal element of A.

Suppose now that $u \notin A$.
If there are no other element $p \in S$ such that $u \leq p$, then u is a maximal element and we can put $m = u$.
If there is another element $p \in S$ such that $u \leq p$, then by transitivity, p is also an upper bound for A and can put $u = p$. We repeat the reasoning until we find no other element in S such that $u \leq p$. Then, u is a maximal element and we can put $m = u$.

♦

Structure of $\mathbb{R}$ and $\mathbb{R}^n$

2.30 Algebraic structure of $\mathbb{R}$

Remark on the Archimedean principle

The Archimedean principle is sometimes stated as :
If x, $y \in \mathbb{R}$ with $y > 0$, then $\exists\, n \in \mathbb{Z} : (n-1)y \leq x < ny$.

On page 25, we have

30.6 The least upper bound axiom for the real number system $\mathbb{R}$.
If S is a nonempty subset of $\mathbb{R}$ and S has an upper bound, then S has a least upper bound in $\mathbb{R}$.

It is strange that the author first gives the Archimedean principle as a principle (aka Axiom ?) and afterwards **30.6** as an axiom as the Archimedean principle can be proved with this axiom.

Proof (by contradiction):
Be arbitrary x, $y \in \mathbb{R}$ with $y > 0$. Suppose that we have $py \leq x$ is true for all $p \in \mathbb{Z}$. Be A the subset of $\mathbb{R}$ with all the real numbers py as elements. From axiom **30.6** we know that A has a least upper bound with x as an upper bound. Be $\xi = lub(A)$. As we have $y > 0$, then $\xi - y$ is not an upper bound of A. This mean that there is a p such that $py > \xi - y$. Hence, $(p+1)y > \xi$. Thus ξ can not be a upper bound of A, and we have a contradiction. Hence, there must be a $p \in \mathbb{Z}$ for which $py > x$.

2.30.1

> Explain by example why it is that the system of all rational numbers does not satisfy the least upper bound axiom.

Be $A \subset \mathbb{Q}$ such that $A = \{x^2 \leq 2,\, x \in \mathbb{Q}\}$. A has upper bounds (e.g. $sup(A) = 2$). This set can be re-expressed as $A = (-\sqrt{2}, \sqrt{2})$ and has no $l.u.b.(A)$ as $\sqrt{2} \notin \mathbb{Q}$.

♦

2.30.2

> Prove that the least upper bound property implies the following: If S is a nonempty subset of real numbers that has a lower bound, then S has a greatest lower bound.

Suppose S is a nonempty subset of real numbers that has a lower bound $\boldsymbol{s}$. For all $x \in S$ we have $\boldsymbol{s} \leq x$. Let's define a new set $\hat{S}$ with $\hat{S} = \{2\boldsymbol{s} - x : x \in S\}$. Then, $\boldsymbol{s}$ is an upper bound for $\hat{S}$. By the upper bound axiom for real number, we have a $\hat{\boldsymbol{s}} = l.u.b.(\hat{S})$ and we have $2\boldsymbol{s} - x \leq \hat{\boldsymbol{s}}$ for all $x \in S$. And thus,

$$2\boldsymbol{s} - \hat{\boldsymbol{s}} \leq x$$

for all $x \in S$.
$2\boldsymbol{s} - \hat{\boldsymbol{s}}$ is a $g.l.b.(A)$ as, suppose we would have a $\epsilon > 0$ such that $2\boldsymbol{s} - \hat{\boldsymbol{s}} + \epsilon \leq x$ and thus $\underbrace{2\boldsymbol{s} - x}_{\in \hat{S}} \leq \hat{\boldsymbol{s}} - \epsilon$ for all $\hat{x} \in \hat{S}$. We get a contradiction as we would have $\hat{\boldsymbol{s}} - \epsilon$ as an upper bound which is smaller than $l.u.b.(\hat{S})$.

♦

2.30.3

> Let $S = \left\{x : x = 1 - \frac{1}{n},\, n \in \mathbb{P}\right\}$. Find $l.u.b.(S)$ and $g.l.b.(S)$ if they exist.

For $n = 1$ we have $x = 0$ and thus $g.l.b(S) = 0$.

For $n \to +\infty$ we have $x \to 1$ and thus $l.u.b(S) = 1$.

♦

2.30.4

Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3$. Find the $l.u.b.(f[\{x : 0 < x < 1\}])$.
Find $l.u.b.(f[\{x : 0 \le x \le 1\})$.

$l.u.b.(f[\{x : 0 < x < 1\}]) = 1.$

$l.u.b.(f[\{x : 0 \le x \le 1\}) = 1$

$\blacklozenge$

2.30.5

Suppose that $f : \{x : 0 < x\} \to \mathbb{R}$ is given by $f(x) = \frac{1}{x}$ for $0 < x$. Does $f[\{x : 0 < x\}]$ have a $l.u.b.$? Does it have a $g.l.b.$?

$l.u.b.(f[\{x : 0 < x\}])$ does not exist.

$g.l.b.(f[\{x : 0 < x\}]) = 0.$

$\blacklozenge$

2.30.6

Give an example of a function f defined on a closed interval S such that $l.u.b.(f[S])$ exists but f does not attain a maximum value on S.

Consider $f : \mathbb{R}_{[0,1]} \to \mathbb{R}$ such that $f = \left\{ \frac{1-a^{-\frac{x}{1-x}}}{1+a^{-\frac{x}{1-x}}} : x \in \mathbb{R}_{[0,1]},\ a > 1 \right\}$ with $l.u.b.(f[S]) = 1 \notin f[S]$.

$\blacklozenge$

2.30.7

Prove the following statement: If $a = l.u.b.(A)$, then for each $\epsilon > 0$, there is an $x \in A$ such that $a - \epsilon < x \le a$. State and prove an analogous proposition for $g.l.b.(A)$.

Suppose for a given ϵ, $\nexists\ x \in A :\ a - \epsilon < x$, then, $x \le a - \epsilon$. Thus $a - \epsilon$ is a lower bound which is smaller than $l.u.b.(S)$. We have a contradiction and thus $\forall \epsilon \in \mathbb{R}, \exists\ x \in A :\ a - \epsilon < x \le a$.

For $b = g.l.b.(A)$, the statement becomes

$$\forall \epsilon \in \mathbb{R}, \exists\, x \in A : \, b \leq x < b + \epsilon$$

Suppose for a given ϵ, $\nexists\, x \in A : \, x < b + \epsilon$, then, $x \leq b + \epsilon$. Thus $b + \epsilon$ is a upper bound which is greater than $g.l.b.(S)$. We have a contradiction and thus $\forall \epsilon \in \mathbb{R}, \exists\, x \in A : \, b \leq x < b + \epsilon$.

♦

2.30.8

Prove that if A is a nonempty bounded set of real numbers then $g.l.b.(A) \leq l.u.b.(A)$. For what kind of set A is $g.l.b.(A) = l.u.b.(A)$?

A is bounded, hence by the upper bound property and its corollary (see Exercise 2.30.2) then A has both a $l.u.b.(A)$ and a $g.l.b.(A)$. This mean that for every $x \in A$ we have $l.u.b.(A) \leq x \leq g.l.b.(A)$ and thus $l.u.b.(A) \leq g.l.b.(A)$

We can have $g.l.b.(A) = l.u.b.(A)$ when A is a singleton, i.e. $A = \{x\}$. E.g. $A = f[\mathbb{R}]$ with $f = \{c\}$ with c a constant, has $g.l.b.(A) = l.u.b.(A) = c$.

♦

2.30.9

Prove that if A and B are nonempty subsets of $\mathbb{R}$ and $A \subset B$, then $g.l.b.(B) \leq g.l.b.(A) \leq l.u.b.(A) \leq l.u.b.(B)$.

From **2.30.8** we know that $l.u.b.(A) \leq g.l.b.(A)$ and $l.u.b.(B) \leq g.l.b.(B)$.

Suppose we have $g.l.b.(A) < g.l.b.(B)$. This implies that $\exists x \in A : x \notin B$. We have a contradiction as $A \subset B$ and thus we must have $g.l.b.(B) \leq g.l.b.(A)$.
In the same vein, suppose we have $l.u.b.(B) < l.u.b.(A)$. This implies that $\exists x \in A : x \notin B$. We have a contradiction as $A \subset B$ and thus we must have $l.u.b.(A) \leq l.u.b.(B)$.
Putting this all together we get indeed

$$g.l.b.(B) \leq g.l.b.(A) \leq l.u.b.(A) \leq l.u.b.(B)$$

♦

2.31 Distance between two points in $\mathbb{R}$

2.31.1

Verify the properties stated in **31.2**

31.2(a) and **31.2(b)** are trivial as $d(x, y) = |x - y|$.

31.2(c):
We consider 4 possibilities (the cases $a = b$ or $b = c$ are excluded as in that case $d(a, b)$ or $d(b, c)$ are zero).
i) $a < b$ and $b < c$. Then,

$$d(a,c) = |a - b + b - c| = |-|a-b| - |b-c|| = ||a-b| + |b-c|| = d(a,b) + d(b,c)$$

ii) $a < b$ and $b > c$. Then,

$$d(a,c) = |a - b + b - c| = |-|a-b| + |b-c|| < ||a-b| + |b-c|| = d(a,b) + d(b,c)$$

ii) $a > b$ and $b < c$. Then,

$$d(a,c) = |a - b + b - c| = ||a-b| - |b-c|| < ||a-b| + |b-c|| = d(a,b) + d(b,c)$$

iv) $a > b$ and $b > c$. Then,

$$d(a,c) = |a - b + b - c| = ||a-b| + |b-c|| = ||a-b| + |b-c|| = d(a,b) + d(b,c)$$

and conclude

$$d(a,b) + d(b,c) \geq d(a,c)$$

♦

2.31.2

Let $p \in \mathbb{R}$. Give an example of a collection $\mathscr{K}$ of neighborhoods of p such that $\bigcap \mathscr{K}$ is not a neighborhood. Show that if $\mathscr{K}$ is a nonempty finite collection of neighborhoods of p, then $\bigcup \mathscr{K}$ and $\bigcap \mathscr{K}$ are neighborhoods of p.

Consider the infinite collection $\mathscr{K} = \{N(p; \frac{1}{n}) : n \in \mathbb{P}\}$, then $\bigcap \mathscr{K} = \{p\}$ and no $\epsilon > 0$ can be found for which $N(p; \epsilon)$ exists.

If $\mathscr{K}$ is a finite collection then $\mathscr{K}$ is countable with element $N(p; \epsilon_i)$ and we can construct a finite set

$$\mathscr{E} = \{\epsilon_i : i = 1, 2, \ldots, n\}$$

As $\mathscr{E} \subset \mathbb{R}$ and by the least upper bound principle, $\mathscr{E}$ has both a $\epsilon_u = l.u.b.(\mathscr{E})$ and a $\epsilon_l = g.l.b(\mathscr{E})$. Moreover, $N(p;\epsilon_u)$ and $N(p;\epsilon_l)$ are elements of $\mathscr{K}$ and hence we get

$$\bigcup \mathscr{K} = N(p;\epsilon_u)$$

and

$$\bigcap \mathscr{K} = N(p;\epsilon_l)$$

.

♦

2.32 Limit of a sequence in $\mathbb{R}$

2.32.1

Suppose that (a_n) is a sequence such that $a_n \leq a_{n+1}$, $(a_{n+1} \leq a_n)$ for each positive integer n. Suppose further that the sequence (a_n) is bounded above (below). Then $\lim(a_n)$ exists.

$\mathbf{a_n \leq a_{n+1}}$

As (a_n) is bounded above and is a subset of $\mathbb{R}$, then by the upper bound property of $\mathbb{R}$, has a lower upper bound, $A = l.u.b.$.
We first notice that for any arbitrary $\epsilon \in \mathbb{R}$, there must be a a_N such that $A - \epsilon < a_N \leq A$ (as otherwise $A - \epsilon$ would be an upper bound, giving a contradiction).
So we have $A - \epsilon < a_N$ and thus $A - a_N < \epsilon$. Furthermore as $a_{N+k} \geq a_N$ for all k we will have $A - a_{N+k} \leq A - a_N < \epsilon$. As $\forall n, A \geq a_n$, $A - a_n$ will stay positive for all n. Thus, the last result can be written as $|A - a_{N+k}| < \epsilon$ for a chosen ϵ. From the definition of the limit, we conclude that the $\lim(a_n) = l.u.b.(a_n)$ exists.

$\mathbf{a_{n+1} \leq a_n}$

As (a_n) is bounded below and is a subset of $\mathbb{R}$, then by the corollary of the upper bound property of $\mathbb{R}$, has a greatest lower bound, $L = g.l.b.$.
We first notice that for any arbitrary $\epsilon \in \mathbb{R}$, there must be a a_N such that $L \leq a_N < L + \epsilon$ (as otherwise $L + \epsilon$ would be a lower bound, giving a contradiction).
So we have $a_N < L + \epsilon$ and thus $a_N - L < \epsilon$. Furthermore as $a_{N+k} \leq a_N$ for all k we will have $a_{N+k} - L \leq a_N - L < \epsilon$. As $\forall n, L \leq a_n$, $a_n - L$ will stay positive for all n. Thus, the last result can be written as $|L - a_{N+k}| < \epsilon$ for a chosen ϵ. From the definition of the limit, we conclude that the $\lim(a_n) = g.l.b.(a_n)$ exists.

♦

2.32.2

Let the sequence (c_n) in $\mathbb{R}$ be given by $c_n = a_n + b_n$, where $\lim(a_n) = A$ and $\lim(b_n) = B$. Then, $\lim(c_n) = A + B$.

From the definition of the limit of a sequence, for a given $\frac{\epsilon}{2}$ there will be elements a_N and $b_{N'}$ such that for all $n \geq N$ and $n^{'} \geq N^{'}$ we will have

$$|A - a_n| < \frac{\epsilon}{2} \text{ and } |B - b_{n'}| < \frac{\epsilon}{2}$$

Let's put $M = sup(N, N^{'})$. Adding the two inequalities gives

$$|A - a_m| + |B - b_m| < \epsilon$$

for any $m \geq M$. The triangle inequality of the distance in $\mathbb{R}$ states

$$d(x, z) \leq d(x, y) + d(y, z)$$

Put $x = a_m - B$, $z = A - b_m$ and $y = A - B$. The triangle inequality gives $|a_m - B - A + b_m| \leq |a_m - B - A + B| + |A - b_m - A + B|$ or

$$|a_m + b_m - (A + B)| \leq |A - a_m| + |B - b_m|$$

But $|A - a_m| + |B - b_m| < \epsilon$ and get

$$|a_m + b_m - (A + B)| < \epsilon$$

giving $\lim(a_n + b_n) = A + B$

♦

2.32.3

Let the sequence (c_n) in $\mathbb{R}$ be given by $c_n = ka_n$, where k is a constant and $\lim(a_n) = A$. Then, $\lim(c_n) = kA$.

Be given k. Let's chose a $\frac{\epsilon}{|k|}$ (we suppose $k \neq 0$, this case being trivial).
We have $|a_n - A| < \frac{\epsilon}{|k|}$ for all $n \geq$ then certain N. Multiplying by $|k|$ gives $|k||a_n - A| < \frac{|k|\epsilon}{|k|}$ and thus

$$|ka_n - kA| < \epsilon$$

and conclude $\lim ka_n = kA$.

♦

2.32.4

Let the sequence (c_n) in $\mathbb{R}$ be given by $c_n = a_n b_n$, where $\lim(a_n) = A$ and $\lim(b_n) = B$. Then, $\lim(c_n) = AB$.

Choose arbitrary ϵ_a, $\epsilon_b > 0$. Then,
There exists a $n_a \in \mathbb{P}$, such that

$$|a_n - A| < \epsilon_a \quad \forall n \in \mathbb{P}, n < n_a \tag{1}$$

there also exists a $n_b \in \mathbb{P}$, such that

$$|b_n - B| < \epsilon_b \quad \forall n \in \mathbb{P}, n < n_b \tag{2}$$

Also, there exists a $\hat{n}$ such that

$$|a_n| - |A| \leq ||a_n| - |A|| \leq |a_n - A| < 1$$

for all $n > \hat{n}$, so that $|a_n| < |A| + 1$, i.e. $\frac{|a_n|}{|A|+1} < 1$.
Hence, for all $n > \sup\{n_a, n_b, \hat{n}\}$ we have

$$|a_n b_n - AB| = |a_n b_n - a_n B + a_n B - AB| = |a_n(b_n - B) + B(a_n - A)|$$

from which follows, using (1) and (2)

$$|a_n b_n - AB| \leq |a_n(b_n - B)| + |B(a_n - A)| < |a_n|\epsilon_b + |B|\epsilon_a \tag{3}$$

Let's put $\epsilon_a = \frac{\epsilon}{2(|B|+1)}$ and $\epsilon_b = \frac{\epsilon}{2(|A|+1)}$ such that we have

$$|a_n - A| < \frac{\epsilon}{2(|B|+1)} \quad \forall n \in \mathbb{P}, n < n_a \tag{4}$$

$$|b_n - B| < \frac{\epsilon}{2(|A|+1)} \quad \forall n \in \mathbb{P}, n < n_b \tag{5}$$

As $\frac{|a_n|}{|A|+1} < 1$ (see above) and $\frac{|B|}{|B|+1} \leq 1$, we get for (3), using (4) and (5)

$$|a_n b_n - AB| < \underbrace{\frac{|a_n|}{(|A|+1)}}_{<1} \frac{\epsilon}{2} + \underbrace{\frac{|B|}{(|B|+1)}}_{<1} \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

for any arbitrary ϵ.

♦

2.32.5

Let the sequence (c_n) in $\mathbb{R}$ be given by $c_n = \frac{a_n}{b_n}$, where $b_n \neq 0$, $\lim(a_n) = A$ and $\lim(b_n) = B \neq 0$.
Then, $\lim(c_n) = \frac{A}{B}$.

Choose arbitrary ϵ_a, $\epsilon_b > 0$. Then,
There exists a $n_a \in \mathbb{P}$, such that

$$|a_n - A| < \epsilon_a \quad \forall n \in \mathbb{P}, n < n_a \tag{1}$$

there also exists a $n_b \in \mathbb{P}$, such that

$$|b_n - B| < \epsilon_b \quad \forall n \in \mathbb{P}, n < n_b \tag{2}$$

Also, there exists a $\hat{n}$ such that

$$|b_n| > \frac{|B|}{2}$$

for all $n > \hat{n}$ (as otherwise we would have $|b_n| \leq \frac{|B|}{2}$ for all $n > \hat{n}$, meaning that $|b_n|$ will not be able to get arbitrarily close to $|B|$.)
Thus we can write

$$\frac{1}{|b_n|} < \frac{2}{|B|} \quad \forall n > \hat{n} \tag{3}$$

.

Hence, for all $n > \sup\{n_a, n_b, \hat{n}\}$ we have

$$\left|\frac{a_n}{b_n} - \frac{A}{B}\right| = \left|\frac{a_n}{b_n} - \frac{A}{b_n} + \frac{A}{b_n} - \frac{A}{B}\right| \leq \frac{|a_n - A|}{|b_n|} + |A|\left|\frac{1}{b_n} - \frac{1}{B}\right| = \frac{|a_n - A|}{|b_n|} + |A|\frac{|B - b_n|}{|B||b_n|}$$

Using (1), (2) and (3) we get

$$\begin{aligned} \left|\frac{a_n}{b_n} - \frac{A}{B}\right| &< \frac{1}{|b_n|}\left(\epsilon_a + \frac{|A|}{|B|}\epsilon_b\right) \qquad (4) \\ &< \frac{2}{|B|}\left(\epsilon_a + \frac{|A|}{|B|}\epsilon_b\right) \qquad (5) \end{aligned}$$

Let's put $\epsilon_a = \frac{|B|}{4}\epsilon$ and $\epsilon_b = \frac{|B|^2}{4|A|}\epsilon$ we get

$$\begin{aligned} \left|\frac{a_n}{b_n} - \frac{A}{B}\right| &< \frac{1}{2}\epsilon + \frac{1}{2}\epsilon \\ &= \epsilon \end{aligned}$$

for any arbitrary ϵ.

♦

2.32.6

If a sequence (a_i) has a limit, it is unique.

Suppose we have for an arbitrary ϵ

$$\exists\, n_1,\ |a_n - A| < \epsilon, \forall n > n_1$$
$$\exists\, n_2,\ |a_n - A^{'}| < \epsilon, \forall n > n_2$$

Suppose that $A \neq A^{'}$. Let $\epsilon = \frac{|A-A^{'}|}{2}$. By hypothesis there exists a $n_1 \in \mathbb{P}$ such that

$$|a_n - A| < \frac{|A - A^{'}|}{2} \quad \text{if} \quad n \geq n_1$$

By hypothesis, there exists $n_2 \in \mathbb{P}$ such that

$$|a_n - A^{'}| < \frac{|A - A^{'}|}{2} \quad \text{if} \quad n \geq n_2$$

Let $\hat{n} = sup\{n_1, n_2\}$. If $n \geq \hat{n}$, then by the triangle inequality

$$|A - A^{'}| = |(a_n - A) - (a_n - a^{'})| < |a_n - A| + |a_n - A^{'}| < 2\frac{|A - A^{'}|}{2} = |A - A^{'}|$$

And we have a contradiction as we get $|A - A^{'}| < |A - A^{'}|$.

♦

2.33 The nested interval theorem for $\mathbb{R}$

2.33.1

Give the details of the proof of Theorem **33.1**.

First we note that we have

$$a_1 \leq \cdots \leq a_n \leq a_{n+1} \leq \cdots \leq b_1 \quad (1)$$

$$a_1 \leq \cdots \leq b_{n+1} \leq b_n \leq \cdots \leq b_1 \quad (2)$$

We see that the subset $\{a_n\} \subset \mathbb{R}$ is bounded above, and the subset $\{b_n\} \subset \mathbb{R}$ is bounded below, hence, by the upper bound principle (and its corollary) in $\mathbb{R}$, $\{a_n\}$ has a *l.u.b.* and $\{b_n\}$ has a *g.l.b.* Let's denote them A and B respectively.
Be $\epsilon > 0$, then there exists a $n_1 \in \mathbb{P}$ such that $\forall n \geq n_1$ we have $A - \epsilon < a_n \leq A$ (as otherwise $A - \epsilon$ would be an upperbound, which is contradictory with the fact that A is a *l.u.b.* Then, we have as $\epsilon > 0$,

$$A - \epsilon < a_n < A + \epsilon$$

and thus

$$|a_n - A| < \epsilon \quad (3)$$

Giving $A = \lim\{a_n\}$

The same reasoning on the lower bound of b_n gives $B \leq b_n < B + \epsilon$ and thus

$$|b_n - B| < \epsilon \quad (4)$$

and $B = \lim\{b_n\}$

Then we have

$$a_1 \leq \cdots \leq a_n \leq a_{n+1} \leq \cdots \leq A \leq b_1 \quad (5)$$

$$a_1 \leq B \leq \cdots \leq b_{n+1} \leq b_n \leq \cdots \leq b_1 \quad (6)$$

From this we conclude that $A \leq B$. Indeed, suppose we would have $A > B$.
From (5) and (6) we can write (3) and (4) as

$$|a_n - A| = A - a_n < \epsilon \quad \forall n \geq n_1 \quad (7)$$

$$|b_n - B| = b_n - B < \epsilon \quad \forall n \geq n_2 \quad (8)$$

Choose $\epsilon = \frac{A-B}{2}$, (> 0 as we supposed $A > B$) and $\hat{n} = \max(n_1,\, n_2)$. Adding (7) and (8) gives

$$\begin{aligned} & A - B + b_n - a_n < \epsilon + \epsilon && (9) \\ \Rightarrow\quad & A - B + b_n - a_n < A - B && (10) \\ \Rightarrow\quad & b_n < a_n && (11) \end{aligned}$$

which is impossible as we have $a_n \leq b_n$, $\forall n \in \mathbb{P}$.

Till now, we proved that $A = \lim\{a_n\}$ and $B = \lim\{b_n\}$ exist and that $A \leq B$.

Consider $[A,\, B]$ and an element $x \in [A,\, B]$. We have

$$A \leq x \leq B$$

But we know $a_n \leq A$ and $b_n \geq B$, $\forall n \in \mathbb{P}$ and so we get

$$a_n \leq A \leq x \leq B \leq b_n$$

and thus $x \in [a_n, b_n]$ and conclude

$$[A, B] \subset [a_n, b_n],\, \forall n \in \mathbb{P}$$

and we can state

$$\exists\, x \in \bigcap_{i=1}^{\infty} [a_i, b_i]$$

and thus

$$\bigcap_{i=1}^{\infty} [a_i, b_i] \neq \varnothing$$

We now prove that if $\lim\left(|b_i - a_i|\right) = 0$ then $\bigcap_{i=1}^{\infty}[a_i, b_i]$ has exactly one element.
Suppose $x,\, y \in \bigcap_{i=1}^{\infty}[a_i, b_i]$, two different elements. Then $|b_i - a_i| \geq |x - y| > 0$, $\forall i \in \mathbb{P}$ as $x \neq y$. Choose $\epsilon = \frac{|x-y|}{2}$, then $|b_i - a_i| > \epsilon$, $\forall i \in \mathbb{P}$ and thus $|b_i - a_i|$ doe not have a limit. We get a contradiction as we supposed $\lim\left(|b_i - a_i|\right) = 0$.

♦

2.33.2

Give an example of nonempty intervals I_i (see **33.2**) such that

$$I_1 \supset I_2 \supset \cdots \supset I_n \supset \ldots \text{ and } \bigcap_{i=1}^{\infty} I_i = \varnothing$$

Choose $I_i = (0,\, 2^{-i})$. Suppose there exists $x \in \bigcap_{i=1}^{\infty} I_i$, then for any $i \in \mathbb{P}$, there exists a $n > i$ such that $x \geq 2^{-n}$ and thus $x \notin I_n$ and hence $x \notin \bigcap_{i=1}^{\infty} I_i$.

Thus

$$\bigcap_{i=1}^{\infty} I_i = \varnothing$$

♦

2.33.3

Is the following statement true? Given an interval I and a point $p \in I$, there exists a countable collection of closed intervals $\{I_i\}$ such that $p \in I_1 \subset I_2 \subset I_3 \cdots \subset I_n \ldots$ (possibly only a finite number needed) and such that $I = \bigcup_{i=1}^{\infty} I_i$.

Without loss of generalization, put $I = (0, 1)$ and $p \in I$. Define, the following collection of closed sets

$$I_n = [a_n,\, b_n] = \left[\frac{p}{2^n},\, \frac{p + 2^n - 1}{2^n}\right]$$

We have $I_n \subset I_{n+1}$. Indeed,

$$\begin{aligned} b_{n+1} = \frac{p + 2^{n+1} - 1}{2^{n+1}} &= \frac{1}{2}\frac{p + 2.2^n - 1}{2^n} \\ &= \frac{1}{2}\frac{2p + 2.2^n - 2 + 1 - p}{2^n} \\ &= \frac{p + 2^n - 1}{2^n} + \frac{1}{2}\underbrace{\frac{1-p}{2^n}}_{>0} \\ &> \frac{p + 2^n - 1}{2^n} = b_n \end{aligned}$$

and also $a_{n+1} = \frac{p}{2^{n+1}} < \frac{p}{2^n} = a_n$ giving $I_n \subset I_{n+1}$.

Also, for every $n \in \mathbb{P}$ we have $a_n = \frac{p}{2^n} > 0$ and

$$b_n = \frac{p + 2^n - 1}{2^n} = \underbrace{\frac{p-1}{2^n}}_{<0} + 1$$
$$< 1$$

Moreover $a_n < p$, $\forall p \in \mathbb{P}$ and

$$\begin{aligned}
b_n - p &= \frac{p + 2^n - 1}{2^n} - p \\
&= \frac{p + 2^n - 1 - 2^n p}{2^n} \\
&= \frac{(2^n - 1) - (2^n - 1)p}{2^n} \\
&= \underbrace{\frac{(2^n - 1)}{2^n}}_{>0} \underbrace{(1 - p)}_{>0} \\
&> 0 \\
\Rightarrow \quad b_n &> p
\end{aligned}$$

So I_n are closed intervals containing p and for which we have,

$$p \in I_1 \subset I_2 \subset I_3 \cdots \subset I_n \ldots$$

and

$$I_n \subset I, \, \forall n \in \mathbb{P}$$

Obviously, the collection of I_n is countable as $n \in \mathbb{P}$ and $\mathbb{P}$ is countable.

We now prove that

$$I = \bigcup_{i=1}^{\infty} I_i$$

Suppose we have $x \in I$, then,

$$\exists \, \hat{n} \in \mathbb{P}, \, \frac{p}{2^{\hat{n}}} \leq x \leq \frac{p + 2^{\hat{n}} - 1}{2^{\hat{n}}}$$

Indeed, it suffice to take $\hat{n} = max(\log_2 \frac{p}{x}, \log_2 \frac{1-p}{1-x})$ with $\hat{n} \in \mathbb{P}$.
So, for every $x \in I$ we can find a closed interval containing x and thus

$$I \subset \bigcup_{i=1}^{\infty} I_i$$

and as $I_i \subset I$, $\forall i \in \mathbb{P}$ we have also

$$\bigcup_{i=1}^{\infty} I_i \subset I$$

giving

$$I = \bigcup_{i=1}^{\infty} I_i$$

♦

2.33.4

Suppose that $\mathscr{K}$ is a collection of intervals such that $\bigcap \mathscr{K} \neq \varnothing$. Is $\bigcup \mathscr{K}$ necessarily an interval?

Be $a \in \bigcup \mathscr{K}$ and $b \in \bigcup \mathscr{K}$. Consider the closed interval $[a, b]$. Is it possible that $[a, b] \not\subset \bigcup \mathscr{K}$?

$[a, b] \not\subset \bigcup \mathscr{K}$ would mean that $\exists\, x \in [a, b]$ such that $x \notin \bigcup \mathscr{K}$.
As $a \in \bigcup \mathscr{K}$ there exists at least one interval $K_a \in \mathscr{K}$ such that $a \in K_a$. In the same way $\exists\, K_b \in \mathscr{K}$ for which we have $b \in K_b$.

If we have $a \in K_a$ and also $b \in K_a$, then we can put $K_a = K_b$ with the consequence that $x \in K_a$ as K_a is an interval and hence $\forall a, b \in K_a, [a, b] \subset K_a$.
Hence, x will also be an element of $\bigcup \mathscr{K}$ and we get $[a, b] \subset \bigcup \mathscr{K}$.

Suppose now, $K_a \neq K_b$. As we suppose $x \in [a, b]$, but $x \notin K_a$ and $x \notin K_b$ we will have

$$a < x < b \quad (\text{compared to } a \leq x \leq b)$$

As a and b are arbitrary (insofar $a \in K_a$ and $b \in K_b$) and $x \notin K_a \cup K_b$, we get

$$K_a \cap K_b = \varnothing$$

This is not possible as we have $\bigcap \mathscr{K} \neq \varnothing$ and thus x must be an element of $\bigcup \mathscr{K}$ implying $[a, b] \subset \bigcup \mathscr{K}$.

Conclusion: Provided that we have $\bigcap \mathscr{K} \neq \varnothing$, $\bigcup \mathscr{K}$ is indeed an interval

♦

2.34 Algebraic structure for $\mathbb{R}^n$

2.34.1

Verify the properties stated in **34.2** and **34.4**.

34.2(a). For vector addition: For all x, y, and z in $\mathbb{R}^n$,

(i) $x + y = y + x$

$$\begin{aligned} x + y &= (x_1 + y_1,\, x_2 + y_2,\, \ldots,\, x_n + y_n) \\ &= (y_1 + x_1,\, y_2 + x_2,\, \ldots,\, y_n + x_n) = y + x \end{aligned}$$

(ii) $x + (y + z) = (x + y) + z$

$$\begin{aligned} x + (y + z) &= (x_1 + (y_1 + z_1),\, x_2 + (y_2 + z_2),\, \ldots,\, x_n + (y_n + z_n)) \\ &= ((x_1 + y_1) + z_1,\, (x_2 + y_2) + z_2,\, \ldots,\, (x_n + y_n) + z_n = (x + y) + z \end{aligned}$$

(iii) $\theta + x = x$

$$\begin{aligned} \theta + x &= (0 + x_1,\, 0 + x_2,\, \ldots,\, 0 + x_n) \\ &= (x_1,\, x_2,\, \ldots,\, x_n) = x \end{aligned}$$

(iv) $x - x = \theta$ (by $x - y$ we shall mean $x + (-y)$)

$$\begin{aligned} x - x = x + (-x) &= (x_1 + (-x_1),\, x_2 + (-x_2),\, \ldots,\, x_n + (-x_n)) \\ &= (0,\, 0,\, \ldots,\, 0) = \theta \end{aligned}$$

$\Diamond$

34.2(b). For scalar multiplication: For scalars α, β and vectors x and y,

(i) $\alpha(\beta x) = \alpha\beta(x)$

$$\begin{aligned}
\alpha(\beta x) &= \alpha(\beta x_1,\, \beta x_2,\, \ldots,\, \beta x_n)\\
&= ((\alpha\beta)x_1,\, (\alpha\beta)x_2,\, \ldots,\, (\alpha\beta)x_n)\\
&= (\alpha\beta)(x_1,\, x_2,\, \ldots,\, x_n) = (\alpha\beta)(x)
\end{aligned}$$

(ii) $(\alpha+\beta)x = \alpha x + \beta x$

$$\begin{aligned}
(\alpha+\beta)x &= ((\alpha+\beta)x_1,\, (\alpha+\beta)x_2,\, \ldots,\, (\alpha+\beta)x_n)\\
&= (\alpha x_1 + \beta x_1,\, \alpha x_2 + \beta x_2,\, \ldots,\, \alpha x_n + \beta x_n)\\
&= (\alpha x_1,\, \alpha x_2,\, \ldots,\, \alpha x_n +) + (\beta x_1,\, \beta x_2,\, \ldots,\, \beta x_n)\\
&= \alpha x + \beta x
\end{aligned}$$

(iii) $\alpha(x+y) = \alpha x + \alpha y$

$$\begin{aligned}
\alpha(x+y) &= \alpha(x_1+y_1,\, x_2+y_2,\, \ldots,\, x_n+y_n)\\
&= (\alpha x_1 + \alpha y_1,\, \alpha x_2 + \alpha y_2,\, \ldots,\, \alpha x_n + \alpha y_n)\\
&= (\alpha x_1,\, \alpha x_2,\, \ldots,\, \alpha x_n +) + (\alpha y_1,\, \alpha y_2,\, \ldots,\, \alpha y_n +)\\
&= \alpha x + \alpha y
\end{aligned}$$

(iv) $1x = x$

$$\begin{aligned}
1x &= (1.x_1,\, 1.x_2,\, \ldots,\, 1.x_n)\\
&= (x_1,\, x_2,\, \ldots,\, x_n)\\
&= x
\end{aligned}$$

$\Diamond$

34.4. Properties of the inner product. For x, and z in $\mathbb{R}^n$ and scalars α and β

(i) $x.y = y.x$

$$\begin{aligned}
x.y &= x_1y_1 + x_2y_2 + \cdots + x_ny_n\\
&= y_1x_1 + y_2x_2 + \cdots + y_nx_n\\
&= y.x
\end{aligned}$$

(ii) $x.(\alpha y + \beta z) = \alpha(x.y) + \beta(y.z)$

$$\begin{aligned}
x.(\alpha y + \beta z) &= x_1(\alpha y_1 + \beta z_1) + x_2(\alpha y_2 + \beta z_2) + \cdots + x_n(\alpha y_n + \beta z_n) \\
&= \alpha x_1 y_1 + \beta x_1 z_1 + \alpha x_2 y_2 + \beta x_2 z_2 + \cdots + \alpha x_n y_n + \beta x_n z_n \\
&= (\alpha x_1 y_1 + \alpha x_2 y_2 + \cdots + \alpha x_n y_n) + (\beta x_1 z_1 + \beta x_2 z_2 + \cdots + \beta x_n z_n) \\
&= \alpha(x.y) + \beta(y.z)
\end{aligned}$$

(iii) $(\alpha x + \beta y).z = \alpha(x.z) + \beta(y.z)$

$$\begin{aligned}
(\alpha x + \beta y).z &= (\alpha x_1 + \beta y_1)z_1 + (\alpha x_2 + \beta y_2)z_2 + \cdots + (\alpha x_n + \beta y_n)z_n \\
&= \alpha x_1 z_1 + \beta y_1 z_1 + \alpha x_2 z_2 + \beta y_2 z_2 + \cdots + \alpha x_n z_n + \beta y_n z_n \\
&= (\alpha x_1 z_1 + \alpha x_2 z_2 + \cdots + \alpha x_n z_n) + (\beta y_1 z_1 + \beta y_2 z_2 + \cdots + \beta y_n z_n) \\
&= \alpha(x.z) + \beta(y.z)
\end{aligned}$$

(iv) $x.x > 0$ if $x \neq \theta$ and $\theta.\theta = 0$

$$x.x = x_1 x_1 + x_2 x_2 + \cdots + x_n x_n$$

is a sum of positive numbers and possibly zero's of $x \neq \theta$ and thus $x > 0$.

♦

2.34.2

Show that for points in $\mathbb{R}^2$, $|x - y|$ gives the usual distance formula with which the reader is familiar from analytic geometry.

$$\begin{aligned}
|x - y|^2 = ((x - y).(x - y)) &= (x_1 - y_1)^2 + (x_2 - y_2)^2 \\
\Rightarrow \qquad |x - y| &= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}
\end{aligned}$$

♦

2.34.3

Let x, y, and z be distinct points in R^2. Let L_1 be the line segment with endpoints x and z. Let L_2 be the line segment with endpoints y and z. Let α be the smaller angle (or one of the angles if equal) formed by L_1 and L_2 at z. Show that

$$\cos\alpha = \frac{(x-z)(y-z)}{|x-z||(y-z|}$$

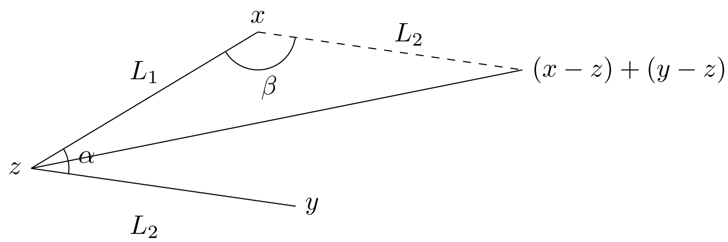


Figure 2.1: Proof that the scalar product determine the angle of two segments

Consider the triangle $\triangle(z, x, (x-z)+(y-z))$. From trigonometry we have

$$|x-z|^2 + |y-z|^2 = |x-z+y-z|^2 + 2|x-z||y-z|\cos\beta \tag{1}$$

where β is the inner angle at x. We have

$$|x-z+y-z|^2 = |x-z|^2 + |y-z|^2 + 2(x-z)(y-z)$$

and can write (1) as

$$|x-z|^2 + |y-z|^2 = |x-z|^2 + |y-z|^2 + 2(x-z)(y-z) + 2|x-z||y-z|\cos\beta$$
$$\Rightarrow \quad \cos\beta = -\frac{(x-z)(y-z)}{|x-z||y-z|}$$

From geometry we have $2\alpha + 2\beta = 2\pi$ and thus $\cos\beta = \cos(\pi-\alpha) = -\cos\alpha$, giving

$$\cos\alpha = \frac{(x-z)(y-z)}{|x-z||y-z|}$$

♦

2.34.4

Let x and y be two elements in R^2. Show that $|x.y| \leq |x||y|$ and $|x+y| \leq |x| + |y|$

We suppose $y \neq 0$, then for any $p \in \mathbb{R}$ we have

$$\begin{aligned} 0 &\leq (x+py).(x+py) && (1)\\ &= (x.x) + p^2(y.y) + 2p(x.y) && (2)\\ &= |x|^2 + p^2|y|^2 + 2p(x.y) && (3) \end{aligned}$$

As p is arbitrary, choose $p = -\frac{(x.y)}{|y|^2}$ and substitute in (3), we get

$$\begin{aligned} 0 &\leq |x|^2 - \frac{(x.y)^2}{|y|^2} - \frac{(x.y)^2}{|y|^4}|y|^2\\ &= |x|^2|y|^2 - (x.y)^2 \end{aligned}$$

from which we get

$$(x.y)^2 \leq |x|^2|y|^2$$

Taking the positive square root:

$$|x.y| \leq |x||y|$$

Note that we have also $(x.y) \leq |x.y| \leq |x||y|$ *as* $(x.y)$ *can be negative.*

◊

$$\begin{aligned} |x+y|^2 &= (x+y).(x+y)\\ &= (x.x) + (y.y) + 2(x.y)\\ &= |x|^2 + |y|^2 + 2(x.y)\\ &\leq |x|^2 + |y|^2 + 2|x||y| \quad \text{(by the previous inequality)}\\ &= (|x|+|y|)^2 \end{aligned}$$

Taking again the positive square root, we get

$$|x+y| \leq |x| + |y|$$

Note: this last inequality is called the ***Minkowski inequality***.

♦

2.34.5

The results of this exercise will be needed in the next section. Consider the function f given by $f(x) = Ax^2 + 2Bx + C$, where $A > 0$, and which further satisfies $f(x) \geq 0$ for all real x. Prove that $B^2 - AC \leq 0$.

$f(x) \geq 0$, $\forall x \in \mathbb{R}$, then $f(-\frac{B}{A}) \geq 0$ which gives

$$\begin{aligned} f(-\frac{B}{A}) &= \frac{B^2}{A} - 2\frac{B^2}{A} + C \geq 0 \\ \Rightarrow \quad & B^2 - AC \leq 0 \end{aligned}$$

♦

2.35 The Cauchy-Schwartz inequality

2.35.1

Verify **35.3** .

35.3. Theorem. For all x, y and z in $\mathbb{R}^n$ and real numbers α,

35.3(a). $|x| > 0$ if $x \neq 0$ and $|\theta| = 0$
This is direct consequence of **34.4.(iv)** and the definition **34.5** of the magnitude of a vector in $\mathbb{R}^n$.

$\Diamond$

35.3(b). $|\alpha x| = |\alpha||x|$
$|\alpha x|^2 = (\alpha x).(\alpha x) = \alpha^2(x.x)$. Hence $|\alpha x|^2 = \sqrt{\alpha^2(x.x)} = |\alpha||x|$ (as the magnitude of a vector is positive or zero by definition positive).

$\Diamond$

35.3(c). $|x| = |-x|$
$|x| = |(-1)(-1)x| = |-1||(-1)x| = |-x|$ by **35.3(b)**

$\Diamond$

35.3(d). $|x-y| + |y-z| \geq |x-z|$
$|x-z| = |x-y+y-z| \leq |x-y| + |y-z|$ by the triangle equality.

$\blacklozenge$

2.35.2

Suppose tha $a \in \mathbb{R}^n$. Consider the collection $S = \{\alpha a : -\infty < \alpha < \infty\}$. Show that S is a line.

The definition of a line is

$$S = \{x : x = (1-t)u + tv,\, t \in \mathbb{R}\} \text{ and } u \neq v$$

This can be written as

$$S = \{x : x = u + (v-u)t,\, t \in \mathbb{R}\} \text{ and } u \neq v$$

put $u = \theta$, $v = a$ and $t = \alpha$ an we get

$$S = \{x : x = \alpha a,\ \alpha \in \mathbb{R}\}$$

♦

2.36 The distance formula in $\mathbb{R}^n$

2.36.1

Prove theorem **36.1**.

36.1. Theorem. Let $d(x, y) = |x - y|$ for x and y in $\mathbb{R}^n$. Then for every x, y, and z in $\mathbb{R}^n$

36.1(a). $d(x, y) \geq 0$
This follows immediately from the definition of $|x - y|$ as a sum of squared reals, hence positive or zero.

$\Diamond$

36.1(b). $d(x, y) = 0 \Leftrightarrow x = y$
$\Leftarrow$: follows from the definition $|x - x| = \sqrt{0^2 + 0^2 + \cdots + 0^2}$

$\Rightarrow$: from **34.4(iv)** we know that $z.z > 0$ if $z \neq \theta$, hence $z \neq \theta$, we have $|z| > 0$ (by the definition of the norm). And conclude since $d(x, y) = |x - y| = |z| = 0$ (put $z = x - y$) that z must be equal to θ and thus $x = y$.

$\Diamond$

36.1(c). $d(x, y) = d(y, x)$
$d(x, y) = |x - y| = |y - x| = d(y, x)$

$\Diamond$

36.1(d). $d(x, y) + d(y, z) \geq d(x, z)$
This follows immediately from the triangle equality **35.3.(d)**.

$\blacklozenge$

2.36.2

Verify that if properties (b), (c), and (d) of **36.1** are assumed for a real-valued function defined on $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, then (a) follows automatically.

We have (with $d(x, y)$ replaced by a real-valued function $f(x, y) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

36.1(b). $f(x, y) = 0 \Leftrightarrow x = y$
36.1(c). $f(x, y) = f(y, x)$
36.1(d). $f(x, y) + f(y, z) \geq f(x, z)$

In **36.1(d)** put $z = x$ then we get

$$\underbrace{f(x,y) + f(y,x)}_{=2f(x,y)} \geq \underbrace{f(x,x)}_{=0}$$

giving

$$f(x,y) \geq 0$$

♦

2.36.3

Prove remark **36.3**.

Remark: Let $x = \{x_1,\, x_2,\, \ldots x_n\}$ and $y = \{y_1,\, y_2,\, \ldots y_n\}$ be points in $\mathbb{R}^n$. Then for each $i \in \mathbb{P}_n$:

$$|x_i - y_i| \leq d(x,y)$$

and

$$d(x,y) \leq \sqrt{n} \max\{|x_i - y_i| : i \in \mathbb{P}_n\}$$

As $d^2(x,y) = (x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2$ it is clear that $d^2(x,y) \geq (x_i - y_i)^2$ for any arbitray $i \in \mathbb{P}_n$ and thus

$$|x_i - y_i| \leq d(x,y)$$

Also

$$\begin{aligned} d^2(x,y) &= (x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2 \\ &\leq (x_m - y_m)^2 + (x_m - y_m)^2 + \cdots + (x_m - y_m)^2 \quad \text{with } (x_m - y_m)^2 = \max\{(x_i - y_i)^2 : i \in \mathbb{P}_n\} \\ &= n(x_m - y_m)^2 \end{aligned}$$

giving

$$d(x,y) \leq \sqrt{n} \max\{|x_i - y_i| : i \in \mathbb{P}_n\}$$

♦

2.37 Open subsets of $\mathbb{R}^n$

2.37.1

Verify Examples 37.2, 37.3, 37.4, and 37.5. Let $p \in \mathbb{R}^n$ and let $\epsilon > 0$. Prove that the set $N(p;\epsilon)$ is an open subset of $\mathbb{R}^n$.

37.2

i) $\varnothing$

As there is no p such that there exist a $N(p;\epsilon)$, we have $N(p;\epsilon) = \varnothing \subset \varnothing$.

ii) $\mathbb{R}^n$

For every $p \in \mathbb{R}^n$ there exist a $\epsilon > 0$ such that $\exists\, q \in \mathbb{R}^n \in N(p;\epsilon)$ and thus $N(p;\epsilon) \subset \mathbb{R}^n$.

37.3 $\mathbb{R}^n \times \{0\}$

$\mathbb{R}^n \times \{0\}$ is a subset of $\mathbb{R}^{n+1}$ and any ϵ-neighbourhood of a point $p \in \mathbb{R}^n \times \{0\}$ will contain points of the form $(x_1,\ x_2,\ \ldots x_n,\ x_{n+1})$ with x_{n+1} not necessarily 0, hence the neighbourhood will not be a subset of $\mathbb{R}^n \times \{0\}$.

37.4 $S = \{(x,\, y) : x + y < 2\}$

As shown in the figure 2.2 it suffice to take $\epsilon < |p - q|$ where q determines the largest 2-ball for which the line $y = 2 - x$ is tangent to the 2-surface with center p.

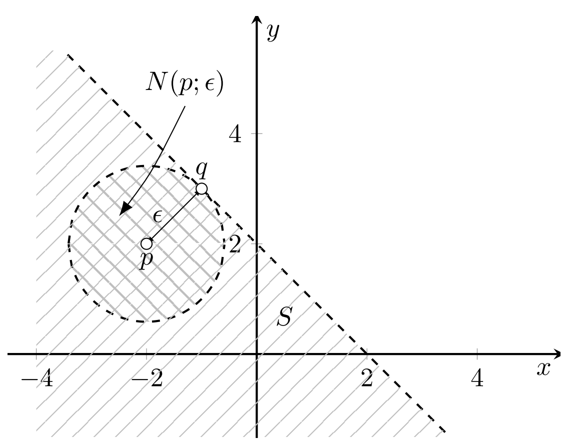


Figure 2.2: ϵ-neighbourhood for the subset $S = \{(x, y) : x + y < 2\}$

37.4 $A = \{(x, y) : x^2 + y^2 < 1\}$, $B = \{(x, y) : (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 < 1\}$

It is obvious that A and B are open subsets (they are both 2-balls in $\mathbb{R}^2$). Then $A \cap B$ and $A \cup B$ are open subsets (see Exercise hereunder).

♦

2.37.2

Prove that if A and B are open subsets of $\mathbb{R}^n$, then $A \cup B$ and $A \cap B$ are open subsets of $\mathbb{R}^n$.

$\mathbf{A \cap B}$ is an open subset:
Even if $A \cap B = \varnothing$ as $\varnothing$ is a open set (see **37.2**).
If $A \cap B \neq \varnothing$, then for a $p \in A \cap B$, we have $p \in A$ and $p \in B$ and as A and B are open sets, there exist a ϵ_a-neighbourhood of p which is a subset of A and a ϵ_b-neighbourhood of p which is a subset of B. Take $\epsilon = \min(\epsilon_a, \epsilon_b)$ and we have $N(p; \epsilon) \subset A \cap B$.

$\mathbf{A \cup B}$ is an open subset:
As A and B are open sets, there exist a ϵ_a-neighbourhood of p which is a subset of A and a ϵ_b-neighbourhood of p which is a subset of B and thus we have a $\epsilon > 0$ so that $\forall p \in A : N(p; \epsilon) \subset A$ or $\forall p \in B : N(p; \epsilon) \subset B$ as $p \in A \cup B$ ans thus $p \in A$ or $p \in B$.

♦

2.37.3

Let $\mathscr{K}$ be a collection of open subsets of $\mathbb{R}^n$. Prove that $\bigcup \mathscr{K}$ is an open subset of $\mathbb{R}^n$. Prove that if $\mathscr{K}$ is a nonempty finite collection of open subsets of $\mathbb{R}^n$, then $\bigcap \mathscr{K}$ is open. The reader, by virtue of having proved the previous exercise has proved the following very important theorem.
The union of an arbitrary collection of open subsets of $\mathbb{R}^n$ is open. The intersection of a finite collection of open subsets of $\mathbb{R}^n$ is open.

The proof is nearly similar to the previous exercise.

$\bigcup \mathscr{K}$ is an open subset:
As every $K \in \mathscr{K}$ is an open set, there exist a ϵ_K-neighbourhood ,for every $p \in K$, which is a subset of K. Thus $\forall p \in \bigcup \mathscr{K}, \exists \epsilon : N(p; \epsilon) \subset \bigcup \mathscr{K}$ proving that $\bigcup \mathscr{K}$ is an open set.

$\bigcap \mathscr{K}$ is an open subset provided that $\mathscr{K}$ is a finite set:

i) Even if $\bigcap \mathscr{K} = \varnothing$, $\bigcap \mathscr{K} = \varnothing$ is an open set as $\varnothing$ is a open set (see **37.2**).

ii) If $\bigcap \mathscr{K} \neq \varnothing$, then for a $p \in \bigcap \mathscr{K}$, p is an element of all $K_n \in \mathscr{K}$ and as all K_n are open sets, there exist for every K_n an ϵ_n-neighbourhood of p which is a subset of K_n. As $\mathscr{K}$ is finite, it is countable and can form the set $E = \{\epsilon_1, \epsilon_2, \ldots \epsilon_N\}$. Take $\epsilon = \min(E)$ and we have $N(p;\epsilon) \subset A \cap B$.

♦

2.38 Limit points in $\mathbb{R}^n$

2.38.1

1. Let F be a finite subset of $\mathbb{R}^n$. Can F have any limit points?

Be $F = \{p_1, p_2, \ldots, p_N\}$ the finite set. Define

$$E = \{|p_i - p_j|, \, i, \, j \in \mathbb{P}_N, \, i \neq j\}$$

Then we can find $\epsilon = \min E$ and therefore $N(p_i; \frac{\epsilon}{2}) - \{p_i\}$ will not be a subset of F and thus

$$\exists \, \epsilon^{'} : \Big(N(p_i; \epsilon^{'}) - \{p_i\}\Big) \cap F = \varnothing$$

♦

2.38.2

Give an example of a subset S of $\mathbb{R}^2$ such that every point of S is a limit point of S.

$$S = \{(x, y) : |x - y| < r\} \quad \text{a 2-ball in } \mathbb{R}^2$$

♦

2.38.3

Give an example of a subset S of $\mathbb{R}^2$ that is infinite and has no limit points.

$$Z = \{(p, \, q) : p, \, q \in \mathbb{Z}\}$$

♦

2.38.4

Suppose S is a nonempty open subset of $\mathbb{R}^n$. Is every point of S a limit point of S? Give an example of an open nonempty subset of $\mathbb{R}^n$ that contains all of its limit points.

S is a nonempty open subset of $\mathbb{R}^n$ so we have for any $p \in S$,

$$\exists\ \epsilon > 0 : N(p;\epsilon) \subset S$$

We prove that

$$\forall p \in S, \exists\ \epsilon > 0 : N(p;\epsilon) \subset S \Rightarrow \forall p \in S, \forall \epsilon^{'} > 0 : \Big(N(p;\epsilon^{'}) - \{p\}\Big) \cap S \neq \varnothing$$

Be an arbitrary $\epsilon^{'} > 0$. Besides the obvious case $\epsilon^{'} = \epsilon$, we examine the two cases $\epsilon^{'} > \epsilon$ and $\epsilon^{'} < \epsilon$.

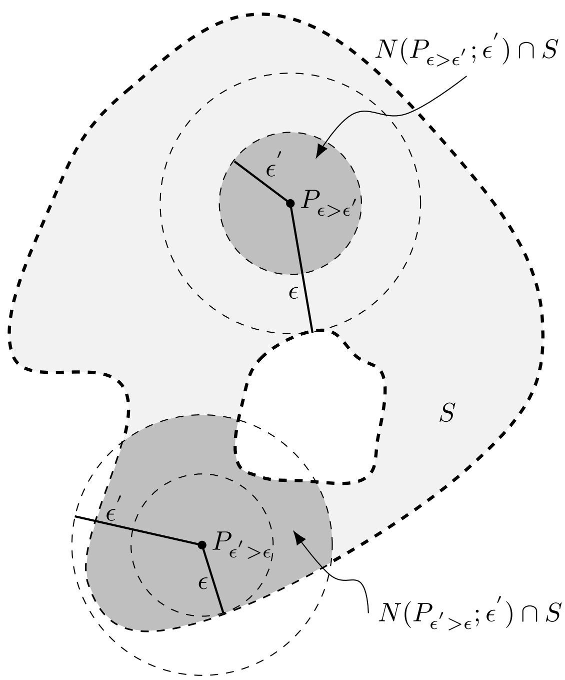


Figure 2.3: ϵ-neighbourhood for the open non empty set S

i) $\epsilon^{'} > \epsilon$

In that case we will have $N(p;\epsilon) \subset N(p;\epsilon^{'})$. This means that $N(p;\epsilon) \cap S \subset N(p;\epsilon^{'}) \cap S$ and as $N(p;\epsilon) \cap S = N(p;\epsilon) \neq \varnothing$ we conclude $N(p;\epsilon^{'}) \cap S \neq \varnothing$.

ii) $\epsilon > \epsilon^{'}$

In that case we will have $N(p;\epsilon^{'}) \subset N(p;\epsilon)$. As $N(p;\epsilon) \subset S$ we also have $N(p;\epsilon^{'}) \subset S$ and conclude $N(p;\epsilon^{'}) \cap S \neq \varnothing$.

So, for all cases of $\epsilon^{'}$ we get the conclusion

$$\forall p \in S, \forall \epsilon^{'} > 0 : N(p;\epsilon^{'}) \cap S \neq \varnothing$$

Can we prove that also

$$\forall p \in S, \forall \epsilon^{'} > 0 : \Big(N(p;\epsilon^{'}) - \{p\}\Big) \cap S \neq \varnothing \tag{1}$$

Suppose,

$$\forall p \in S, \forall \epsilon^{'} > 0 : \Big(N(p;\epsilon^{'}) - \{p\}\Big) \cap S = \varnothing$$

This means that p would be the only element common to $N(p;\epsilon^{'})$ and S. This is not possible as $\epsilon^{'}$ being greater than 0 means that $N(p;\epsilon^{'})$ contains other elements than p and as $N(p;\epsilon^{'}) \subset S$ these other elements must also belong to S, hence $\Big(N(p;\epsilon^{'}) - \{p\}\Big) \cap S \neq \varnothing$ and conclude that (1) is a true statement and thus p is a limit point of S. As p was arbitrary chosen, this yields for all elements of S and thus every point of S is a limit point of S.

$\lozenge$

Example of an open nonempty subset of $\mathbb{R}^n$ that contains all of its limit points.

$$S = \mathbb{R}^n$$

$\blacklozenge$

2.38.5

Suppose that $\mathscr{K}$ is a collection of subsets of $\mathbb{R}^n$. Suppose p is a limit point of $\bigcup \mathscr{K}$. Is p necessarily a limit point of at least one $A \in \mathscr{K}$?

We give a counterexample to prove this is not true.

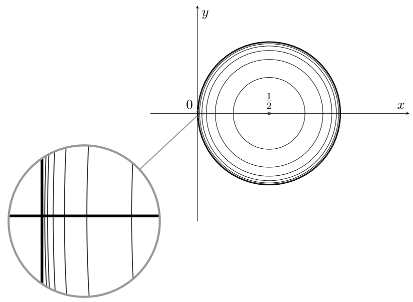


Figure 2.4: Open subsets $K_n = \{x : |x - (\frac{1}{2}, 0)| < \frac{2^n-1}{2^{n+1}},\ n \in \mathbb{P}\}$ in $\mathbb{R}^2$.

In $\mathbb{R}^2$, take the open subsets $K_n = \{x : |x - (\frac{1}{2}, 0)| < \frac{2^n-1}{2^{n+1}},\ n \in \mathbb{P}\}$ and $\mathscr{K} = \{K_n,\ n \in \mathbb{P}\}$. This set for $\mathbb{R}^2$ is illustrated in the figure above.

$\hat{O} = (0, 0)$ is a limit point for $\bigcup \mathscr{K}$.
Indeed, choose an arbitrary $\epsilon > 0$. If $\hat{O}$ is a limit point of $\bigcup \mathscr{K}$, then we must have

$$\forall \epsilon : \Big(N(\hat{O}; \epsilon) - \{\hat{O}\}\Big) \cap \bigcup \mathscr{K} \neq \varnothing$$

Consider a point $(x, 0) \in K_n \subset \bigcup \mathscr{K}$. We have $|x - \frac{1}{2}| < \frac{2^n-1}{2^{n+1}}$ or

$$-\frac{2^n - 1}{2^{n+1}} < x - \frac{1}{2} < \frac{2^n - 1}{2^{n+1}} \tag{1}$$

$$\Leftrightarrow \quad \frac{1}{2^{n+1}} < x < 1 - \frac{1}{2^{n+1}} \tag{2}$$

Consider the points $(x^{'}, 0) \in N(\hat{O}; \epsilon)$. We have $|x^{'}| < \epsilon$ or

$$-\epsilon < x^{'} < \epsilon$$

and for $\Big(N(\hat{O}; \epsilon) - \{\hat{O}\}\Big)$, considering only the positive $x^{'}$'s

$$0 < x^{'} < \epsilon \tag{3}$$

Considering (2) and (3), we can always find a n such that $\frac{1}{2^{n+1}} < \epsilon$. Hence $\Big(N(\hat{O}; \epsilon) - \{\hat{O}\}\Big) \cap K_n = \Big(\frac{1}{2^{n+1}}, \epsilon\Big) \neq \varnothing$ and thus as $K_n \subset \bigcup \mathscr{K}$ we have

$$\forall \epsilon : \Big(N(\hat{O}; \epsilon) - \{\hat{O}\}\Big) \cap \bigcup \mathscr{K} \neq \varnothing$$

.

On the other hand, for a given $K_n \in \mathscr{K}$ it is clear that $\hat{O}$ is not a limit point as we can always find an $\epsilon < \frac{1}{2^{n+1}}$, meaning that K_n and $\Big(N(\hat{O}; \epsilon) - \{\hat{O}\}\Big)$ have no common points, hence, $\hat{O}$ is not a limit point of K_n.

♦

2.38.6

If your answer to Exercise 5 is no, prove the following: Suppose that $\mathscr{K}$ is a finite collection of subsets of $\mathbb{R}^n$. If p is a limit point of $\bigcup \mathscr{K}$, then p is a limit point of at least one $A \in \mathscr{K}$.

Be p a limit point of $\bigcup \mathscr{K}$. This means

$$\forall \epsilon > 0, (N(p;\epsilon) - \{p\}) \cap \bigcup \mathscr{K} \neq \varnothing$$

Be $S = (N(p;\epsilon) - \{p\}) \cap \bigcup \mathscr{K} \neq \varnothing$. This means $\exists\, x \in \mathbb{R}^n$ such that $x \in S$ and $x \in \bigcup \mathscr{K}$, the latter means that x must be an element of at least one $A^* \in \mathscr{K}$.
But x must also be an element of $(N(p;\epsilon) - \{p\})$ so we have

$$\forall \epsilon > 0, (N(p;\epsilon) - \{p\}) \cap A^* \neq \varnothing$$

Hence, there exists at least one A^* for which p is a limit point.

♦

2.38.7

Let $S \subset \mathbb{R}^n$ and let z be a limit point of S. Show that for every $\epsilon > 0$, $N(z;\epsilon) \cap S$ is an infinite set.

We have

$$\forall \epsilon > 0, (N(z;\epsilon) - \{z\}) \cap S \neq \varnothing \tag{1}$$
$$\Rightarrow \quad \forall \epsilon > 0, N(z;\epsilon) \cap S \neq \varnothing \tag{2}$$

Suppose $K = N(z;\epsilon) \cap S$ is a finite set. Be $K = \{x_1, x_2, \ldots, x_N\}$ and $E = \{|x_1 - z|, |x_2 - z|, \ldots, |x_N - z|\}$. Put $\delta = \min E$. Then for any $\epsilon^{'} < \delta$ we have

$$N(z;\epsilon^{'}) \cap S = \varnothing \quad , \epsilon^{'} < \delta \tag{3}$$

We get a contradiction as (3) puts a constraint on the requirement ($\forall$) in (2) and thus we conclude

$$N(z;\epsilon^{'}) \cap S \text{ is a infinite set.}$$

♦

2.39 Closed subsets of $\mathbb{R}^n$

Clarifications on Theorem 39.5

For each subset of $\mathbb{R}^n$, $cl(S)$ is a closed subset of $\mathbb{R}^n$.
(Each step in the book is labelled, so we can refer to it in the following steps).

Proof.
(a) Let x be a limit point of $cl(S)$.
(b) We wish to show that $x \in cl(S)$.
(c) If $x \notin cl(S)$, then $x \notin S$ and x is not a limit point of S.
- *(This is because $cl(S)$ by definition must contain all limit point of S).*

(d) Hence, there is an $\epsilon > 0$ such that $N(x;\epsilon) \subset \mathbb{R}^n - S$.
- *(This is by the definition of a limit point i.e. $\forall \epsilon > 0, (N(x;\epsilon) - \{x\}) \cap S \neq \varnothing$. As x is not a limit point of S (see (b)) $(N(x;\epsilon) - \{x\}) \cap S = \varnothing$ and thus all elements of $N(x;\epsilon)$ must be an element of the complement of S i.e. $\mathbb{R}^n - S$).*

(e) However, since x is a limit point of $cl(S)$, there is a point $z \in cl(S)$ such that $0 < d(x,z) < \epsilon$.
- *(This follows from the definition of a limit point $\forall \epsilon > 0, (N(x;\epsilon) - \{x\}) \cap S \neq \varnothing$ and thus there must be at least one $z \in (N(x;\epsilon) - \{x\}) \cap S$).*

(f) Then since $z \notin S$, z is a limit point of S.
- *(This is because $cl(S)$ by definition must contain all limit point of S).*

(g) Since $\epsilon - d(x,z) > 0$ and z is a limit point of S, there is a point $y \in S$ such that $d(y,z) \leq \epsilon - d(x,z)$.
(h) However, $d(x,y) \leq d(y,z) + d(z,x) < \epsilon - d(x,z) + d(z,x) = \epsilon$ (see accompanying figure).
(i) Hence, $y \in S \cap N(x;\epsilon)$, contrary to the way in which ϵ was chosen.
- *(This follows from (d) $N(x;\epsilon) \subset \mathbb{R}^n - S$),* meaning that S and $N(x;\epsilon$ can't have common elements)

2.39.1

Suppose that F is a finite subset of $\mathbb{R}^n$. Is F necessarily closed in $\mathbb{R}^n$?

Consider $\sim F$. Then, all $p \in F$ are limit points of $\sim F$. Indeed, we have, $\forall \epsilon > 0 : (N(p;\epsilon) - \{p\}) \subset\sim F$ and thus $\forall \epsilon > 0 : (N(p;\epsilon) - \{p\}) \cap \sim F \neq \varnothing$.
Conclusion p is a limit point of $\sim F$.
And as $p \notin\sim F$ we can rewrite this as $\forall \epsilon > 0 : N(p;\epsilon) \cap \sim F \neq \varnothing$. Also, consider an arbitrary point $z \in\sim F$. Be $E = \{|p_1 - z|, |p_2 - z|, , \ldots, |p_n - z|\}$ where the p_i are the elements of F. Be $\delta = \min E$ and choose $\epsilon^{'} < \delta$, the we will have that $N(z;\epsilon^{'}) \subset\sim F$ and thus $\sim F$ is an open subset and by theorem **39.2** F is a closed subset.

♦

2.39.2

Prove Theorems **39.2** and **39.3**.

39.2 Let $S \subset \mathbb{R}^n$. Then S is closed if and only if its complement is open in $\mathbb{R}^n$.

$$\Rightarrow$$

S is closed.
Be $p \in\sim S$. Then, $p \notin S$ and thus p can not be a limit point of S. Hence, the statement $\forall \epsilon > 0, (N(p;\epsilon) - \{p\}) \cap S \neq \varnothing$ is false. We have,

$$\begin{aligned} & \neg\left(\forall \epsilon > 0, (N(p;\epsilon) - \{p\}) \cap S \neq \varnothing\right) \\ \Leftrightarrow\quad & \exists\ \epsilon > 0, (N(p;\epsilon) - \{p\}) \cap S = \varnothing \\ \Leftrightarrow\quad & \exists\ \epsilon > 0, N(p;\epsilon) \cap S = \varnothing \end{aligned}$$

The last statement follows from the fact that $p \notin S$, hence the presence of p in $N(p;\epsilon)$ is of no object for the statement. By the last statement , $N(p;\epsilon) \not\subset S$ and thus $N(p;\epsilon)$ lies completely in the complement of S and we get for any arbitrary $p \in (\sim S)$:

$$\exists\ \epsilon > 0, N(p;\epsilon) \subset (\sim S)$$

proving that $\sim S$ is an open set.

◊

$\Leftarrow$

$\sim$ S is open.
Be q a limit point of S but not in S, then $q \in (\sim S)$. As $\sim S$ is an open set we have $\exists\ \epsilon > 0, N(q;\epsilon) \subset (\sim S)$ and thus $\exists\ \epsilon > 0, N(q;\epsilon) \not\subset S$ giving $\exists\ \epsilon > 0, N(q;\epsilon) \cap S = \varnothing$. The presence of q in $N(q;\epsilon)$ does not change the last statement and we have

$$\exists\ \epsilon > 0, (N(q;\epsilon) - \{q\}) \cap S = \varnothing$$

We have a contradiction as we supposed that q was a limit point of S, hence the supposition that q is not in S is wrong. So a limit point of S must be in S and thus S is closed.

$\Diamond$

39.3
(a) **The empty set and $\mathbb{R}^n$ are closed subsets of $\mathbb{R}^n$.**

We use in both case theorem **39.2** *(Let $S \subset \mathbb{R}^n$. Then S is closed if and only if its complement is open in $\mathbb{R}^n$).*

• As $\varnothing = \sim \mathbb{R}^n$ and from **37.2** we know that $\mathbb{R}^n$ is an open set. By theorem **39.2** we then know that it's complement is closed. Hence, $\varnothing$ is closed.

• As $\mathbb{R}^n = \sim \varnothing$ and from **37.2** we know that $\varnothing$ is an open set. By theorem **39.2** we then know that it's complement is closed. Hence, $\mathbb{R}^n$ is closed.

$\Diamond$

For the following statements we use the De Morgan's laws

$$\bigcup\{\sim K : K \in \mathscr{K}\} = \sim \left(\bigcap\{K : K \in \mathscr{K}\}\right) \tag{1}$$
$$\bigcap\{\sim K : K \in \mathscr{K}\} = \sim \left(\bigcup\{K : K \in \mathscr{K}\}\right) \tag{2}$$

and also the results of exercise **2.37.2**

The union of an arbitray colection of open subsets of $\mathbb{R}^n$ is open. (3)

The intersection of a finite colection of open subsets of $\mathbb{R}^n$ is open. (4)

(b) **The intersection of an arbitrary nonempty collection of closed sets is closed**

We know that K is closed and thus $\sim K$ is open. By (3) we have

$$\begin{aligned} & \bigcup\{\sim K : K \in \mathscr{K}\} \text{ is open} \\ \text{and by (1)} \quad & \bigcup\{\sim K : K \in \mathscr{K}\} = \sim \left(\bigcap\{K : K \in \mathscr{K}\}\right) \\ \Rightarrow \quad & \sim \left(\bigcap\{K : K \in \mathscr{K}\}\right) \text{ is open} \end{aligned}$$

By theorem **39.2** we conclude

$$\bigcap\{K : K \in \mathscr{K}\} \text{ is closed.}$$

(c) **The union of a finite collection of closed sets is closed.**

We know that K is closed and thus $\sim K$ is open. By (4) we have (as $\mathscr{K}$ is finite)

$$\begin{aligned} & \bigcap\{\sim K : K \in \mathscr{K}\} \text{ is open} \\ \text{and by (2)} \quad & \bigcap\{\sim K : K \in \mathscr{K}\} = \sim \left(\bigcup\{K : K \in \mathscr{K}\}\right) \\ \Rightarrow \quad & \sim \left(\bigcup\{K : K \in \mathscr{K}\}\right) \text{ is open} \end{aligned}$$

By theorem **39.2** we conclude

$$\bigcup\{K : K \in \mathscr{K}\} \text{ is closed.}$$

♦

2.39.3

Is the set in Example **37.3** a closed set?

Consider $S = \mathbb{R} \times \{0\}$. Then, $\sim S = \varnothing \times \mathbb{R}_0$. We know that $\varnothing$ is open. Also $\mathbb{R}_0$ is open (choose for any $p \in \mathbb{R}_0$ an $\epsilon < \frac{|p|}{2}$, giving $N(p;\epsilon) \subset \mathbb{R}_0$.)
So $\sim S$ is open and by theorem **39.2** we conclude that S is a closed subset of $\mathbb{R}^2$.

♦

2.39.4

Let a and b be real numbers and let S be the following subset of $\mathbb{R}^2$. $S = \{(x, y) : ax+by \leq 1\}$. Is S a closed set?

$\sim S = \{(x, y) : ax + by > 1\}$ is open and by by theorem **39.2** we conclude that S is a closed subset of $\mathbb{R}^2$.

♦

2.39.5

Let S be a subset of $\mathbb{R}^n$. Let $S^{'} = \{x : x \in \mathbb{R}^n$ and x is a limit point of $S\}$. Is $S^{'}$ necessarily a closed set?

The proof is analogue to the proof of Theorem **39.5**.
Let x be a limit point of $S^{'}$. We wish to show that $x \in S^{'}$.
If $x \notin S^{'}$, then x is not a limit point of S *(This is because $S^{'}$, by definition, must contain all limit point of S)*.
Hence, there is an $\epsilon > 0$ such that $N(x; \epsilon) \subset \mathbb{R}^n - S$ *(This is by the definition of a limit point i.e. $\forall \epsilon > 0, (N(x; \epsilon) - \{x\}) \cap S \neq \varnothing$. As x is not a limit point of S , there exists an ϵ such that $(N(x; \epsilon) - \{x\}) \cap S = \varnothing$ and thus all elements of $N(x; \epsilon)$ must be an element of the complement of S i.e. $\mathbb{R}^n - S$)*.
However, since x is a limit point of $S^{'}$, there is a point $z \in S^{'}$ such that $0 < d(x, z) < \epsilon$ *(This follows from the definition of a limit point $\forall \epsilon > 0, (N(x; \epsilon) - \{x\}) \cap S \neq \varnothing$ and thus there must be at least one $z \in (N(x; \epsilon) - \{x\}) \cap S$)*.
Then since $z \in S^{'}$, z is a limit point of S *(This is because $S^{'}$ by definition must contain all limit point of S)*.
Since $\epsilon - d(x, z) > 0$ and z is a limit point of S, there is a point $y \in S$ such that $d(y, z) \leq \epsilon - d(x, z)$.
However, $d(x, y) \leq d(y, z) + d(z, x) < \epsilon - d(x, z) + d(z, x) = \epsilon$.
Hence, $y \in S \cap N(x; \epsilon)$, and we get a contradiction as $N(x; \epsilon) \subset \mathbb{R}^n - S$, meaning that S and $N(x; \epsilon)$ can't have common elements.
Conclusion x must be an element of $S^{'}$. So, $S^{'}$ is a closed subset of $\mathbb{R}^n$.

♦

2.39.6

Give an example of a countable collection $\mathscr{K}$ of closed subsets of $\mathbb{R}^2$ such that $\bigcup \mathscr{K}$ is not closed.

See Exercise **2.38.2** where the subset $\{\hat{O}\} = \{(0,0)\}$ is a limit point of $\mathscr{K}$ but no element of it.

♦

2.39.7

Let S be a subset of $\mathbb{R}^n$. Let F consist of all points p in $\mathbb{R}^n$ such that for each $\epsilon > 0$, $N(p;\epsilon) \cap S \neq \varnothing$ and $N(p;\epsilon) \cap (\sim S) \neq \varnothing$. Is F necessarily a closed set?

Consider $S = \{p\}$ a set in $\mathbb{R}^n$ consisting of a single point. Then, $\forall \epsilon > 0$, $N(p;\epsilon) \cap S = \{p\} (\neq \varnothing)$, which implies that $p \in F$. Also, $N(p;\epsilon) \cap (\sim S) \neq \varnothing$ as $N(p;\epsilon) - \{p\}$ contains only elements of $\sim S$. Also, no other element q of $\sim S$ can comply with the condition $\forall \epsilon > 0$, $N(q;\epsilon) \cap S \neq \varnothing$ and hence $F = S$. So F has no limit point (and thus does not contain any), hence F is closed [1]

♦

2.39.8

Show that lines and polygons are closed subsets of $\mathbb{R}^n$ (see **35.5** and **35.6**).

First note that, for polygons, by theorem **39.3(c)** (*The union of a finite collection of closed sets is closed*) it suffice to prove that a line is a closed subset as a polygon is the union of a finite number of line segments and the union of a finite collection of closed sets is closed.
Recall the definition of a line segment $L(a,b)$ determined by a and b:

$$L = \{x : x = (1-t)a + tb,\, t \in \mathbb{R}_{[0,1]}\}$$

Be a p a limit point of the line $L(a,b)$. Then, $\forall \epsilon : (N(p;\epsilon) - \{p\}) \cap L(a,b) \neq \varnothing$. This means that $\exists\, x \in L(a,b)$ which is also an element of $N(p;\epsilon)$. This implies

$$d(p;x) < \epsilon$$

[1] Because $\forall x \in \varnothing : P(x)$ is always true regardless of the proposition $P(x)$.

Suppose that $p \notin L(a, b)$. This implies

$$\nexists\, t \in \mathbb{R} : p = (1 - t)a + bt$$

Consider the set

$$D = \{d = d(p; x) : x = (1 - t)a + bt, \forall t \in \mathbb{R}\}$$

As $p \notin L(a, b)$, the ordered set $(D, \leq) \subset (\mathbb{R}, \leq)$ does not contain the element $d = 0$, hence D has a lower bound $= 0 \notin D$. By the least upper bound axiom on page **61** and it's corollary (exercise 2 page 62) D has a greatest lower bound. Be $g.l.b\, D = \delta$. We have

$$0 < \delta$$

and thus

$$\forall \epsilon > 0 : 0 < \delta \leq d(p; x) < \epsilon$$

But this inequality must be true for any $\epsilon > 0$. We get a contradiction as this inequality can't be satisfied for $0 < \epsilon < \delta$ and thus our assumption the $p \notin L(a, b)$ is false.

♦

2.40 Bounded subsets of $\mathbb{R}^n$

Remark on 40.4 - The Bolzano-Weierstrass Theorem

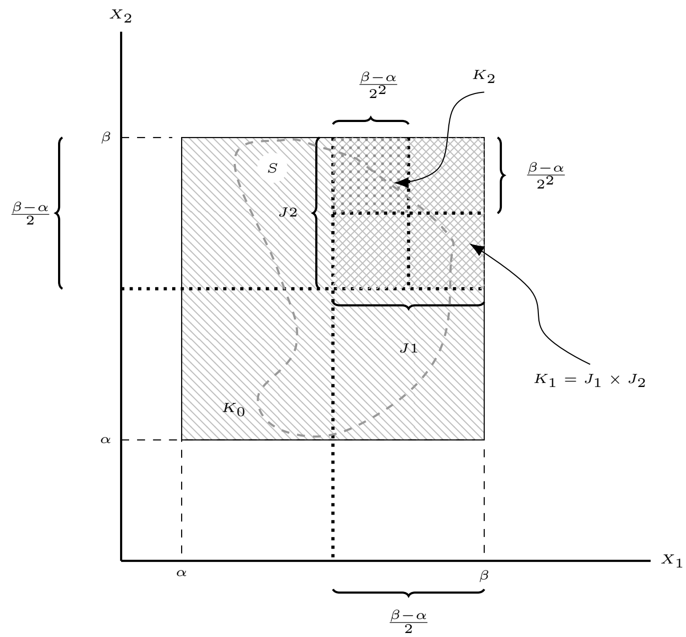


Figure 2.5: The Bolzano-Weierstrass Theorem

The succcesive partition of the close interval $K_i^{(p)}$ in equal n^2 closed intervals $K_{i+1}^{(q)}(q = 1, 2, \ldots 2^n)$ makes that we can always choose one of this intervals such that $S \cap K_{i+1}^{(q')} \neq \varnothing$

2.41 Convergent sequences in $\mathbb{R}^n$

Remark on 41.3 on the "boundness" of a convergent sequence

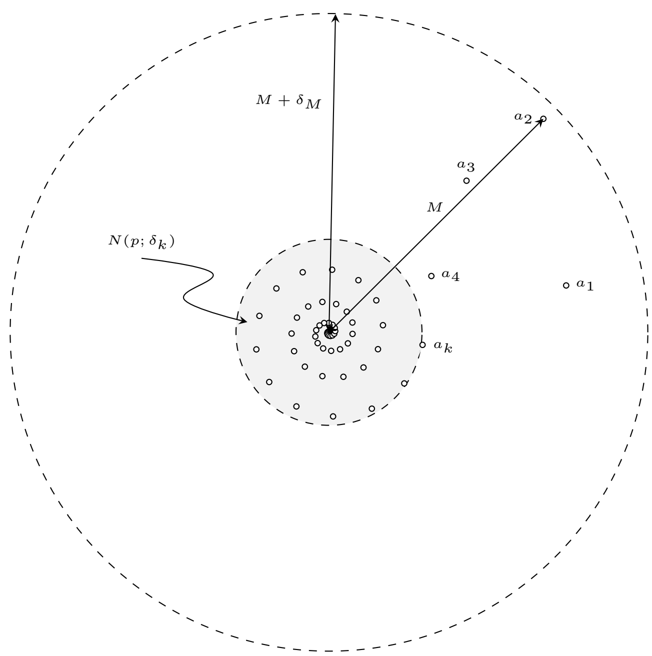


Figure 2.6: Example in $\mathbb{R}^2$ of a convergent and hence, bounded sequence

In the book, an $\epsilon = 1$ is taken to prove that a convergent sequence is bounded. But any positive number can be chosen to prove the boundness as seen in the figure 2.6.
The sequence begins with a_1 and we choose an arbitrary δ_k. As the sequence is convergent this means that for this $\epsilon = \delta_k$ there must be a positive integer k such that $d(L; a_i) < \delta_k$ for all $i > k+1$.
And, if we take as $M = max\{d(a_i, L) : i \in \mathbb{P}\}$ then $\forall i \in \mathbb{P}, \forall \delta_M > 0 : d(a_i, L) < M + \delta_m$

Remark on Theorem 41.4 on the convergence of a subsequence of a bounded sequence

Proof. *Suppose the range of (a_i) is a finite set. Then there is a strictly increasing sequence of positive integers (n_i) and a point $z \in \mathbb{R}^n$ such that $a_{n_i} = z$ for $i \in \mathbb{P}$. Hence,* $\lim(a_{n_i}) = z$ *and we are through.*
As an example take the sequence in $\mathbb{R}^2$ defined by

$$a_i = \left(\sin\left(\frac{2\pi}{T}i\right), \cos\left(\frac{2\pi}{T}i\right)\right) \text{ with } T \in \mathbb{P}, i \in \mathbb{P}$$

Then the range of the (bounded) sequence is finite with T possible values. Define the subsequence by defining the map

$$n_i = T(i-1) + T^{'} \text{ with } T^{'} \in \mathbb{P} \text{ and } T^{'} \leq T$$

will map a_{n_i} on one of the T values and we have a convergent sequence with $a_{n_i} = C^t \quad \forall n_i$.

$\Diamond$

Next suppose the range S of (a_i) is an infinite set. By the Bolzano-Weierstrass theorem there is a point z that is a limit point of S. (We shall find a subsequence (a_{n_i}) of (a_i) by choosing for each $i \in \mathbb{P}$, an (a_{n_i})such that $d(a_{n_i}, z) < \frac{1}{i}$. However, we must be careful to choose these in such a way that (n_i) is a strictly increasing sequence of positive integers. We shall do this inductively.)

Since z is a limit point of S, there is a positive integer such that

$$0 < d(a_{n_1}, z) < 1$$

Assume that for $1, 2, \ldots, h$, increasing integers $n_1, n_2, \ldots, n_h$ have been chosen such that

$$0 < d(a_{n_i}, z) < \frac{1}{i}$$

Since** $S \cap N\left(z; \frac{1}{i+1}\right)$ **is an infinite set (see Exercise 7, page 71), there is an integer** n_{h+1} **such that** $n_h < n_{h+1}$ **and for which** $d(a_{n_{h+1}}, z) < \frac{1}{h+1}$**.

Is it so obvious that there is an integer n_{h+1} such that $n_h < n_{h+1}$ and for which $d(a_{n_{h+1}}, z) < \frac{1}{h+1}$?
By assumption we have a set $K = \{a_{n_1}, a_{n_2}, \ldots, a_{n_h}\}$ such that $0 < d(a_{n_i}, z) < \frac{1}{i}$. As $U = S \cap N\left(z; \frac{1}{i+1}\right)$ is an infinite set $U - K$ still will be an infinite set because K is a finite set.
So, is it possible that for all elements $a_{n_p} \in U - K$ we have $n_p < n_h$? The answer is definitely negative, as the number of 'available slots' in the set $\{1, 2, \ldots, n_h\}$ is finite. Hence after excluding all n_j such that $n_j < n_h$ we will still be able to find an infinite number of $n_k > n_h : k \geq h+1$ for which $d(a_{n_k}, z) < \frac{1}{k} : \quad n_k > n_h$

Hence, by induction (see 18.4), a strictly increasing sequence of integers n_i has been chosen such that $d(a_{n_i}, z) < \frac{1}{i}$. It is clear from the way in which the integers (n_i) were chosen that (a_{n_i}) is a subsequence of (a_i) and that $\lim(a_{n_i}) = z$.

2.41.1

Prove that if S is a subset of $\mathbb{R}^n$ and $z \in \mathbb{R}^n$, then z is a limit point of S if and only if there exists a sequence (a_i) in S that converges to z and which is such that $a_i \neq a_j$ for $i \neq j$.

We have to prove

$$\forall \epsilon > 0 : (N(z;\epsilon) - \{z\}) \cap S \neq \varnothing \quad \Longleftrightarrow \quad \exists \quad (a_i) \text{ in } S : a_i \neq a_j \text{ for } i \neq j \text{ and } \lim a_i = z$$

$$\Diamond$$

$$\Rightarrow$$

We have that z is a limit point. So,

$$\forall \epsilon > 0 : (N(z;\epsilon) - \{z\}) \cap S \neq \varnothing \tag{1}$$

Let's put $\epsilon = 1$ and denote

$$D_1 = (N(z;1) - \{z\}) \cap S$$

Then we can write D_1 as

$$D_1 = \{x \in S : d(x,z) < 1\} \tag{2}$$

We know that D_1 is an infinite set (see Exercise 7, page 71).
Let's define

$$\mathscr{D}_1 = \{d(x,z) : x \in D_1\} \tag{3}$$

Obviously $\mathscr{D}_1 \subset \mathbb{R}$ and is (upper) bounded and is a linearly ordered subset of $\mathbb{R}$. Using Zorn's Lemmma (page 47) we know that $\mathscr{D}_1$ has a maximal element in $\mathscr{D}_1$. Let's choose the element a_1 of D_1 for which $d(a_1,z) = \max \mathscr{D}_1$ (a_1 is unique as S, being a set, will contain only one element of the same object).
Consider now

$$D_2 = \{x \in S : 0 < d(x,z) < \frac{1}{2}\} - \{a_1\} \tag{4}$$

D_2 is still infinite. Let's repeat the procedure initiated in (3) by defining $\mathscr{D}_2 = \{d(x,z) : x \in D_2\}$ etc.. and choose an element a_2 such that $d(a_2,z) = \max \mathscr{D}_2$.

Repeating this procedure gives us a sequence

$$\{(1,a_1),(2,a_2),\ldots,(i,a_i),\ldots\} \text{ such that } 0 < d(a_i,z) < \frac{1}{i} \text{ for } i \in \mathbb{P} \tag{5}$$

The definition of convergence of a sequence (a_i) to z is:

$$\forall \epsilon > 0, \exists\, N \in \mathbb{P}:\ d(a_k,z) < \epsilon \text{ whenever } k \geq N$$

If we choose an arbitrary $\epsilon > 0$ it suffice in (5) to put $\frac{1}{i} \leq \epsilon$ or $i \geq \frac{1}{\epsilon}$ to see that the sequence in (5) converges to z.

Finally, by the way we constructed the sequence by defining the different subsets D_k as

$$D_k = \{x \in S : 0 < d(x,z) < \frac{1}{k}\} - \{a_1,\ldots,a_{k-1}\}$$

we are assured that $a_i \neq a_j$ for $i \neq j$.

$\Diamond$

$\Leftarrow$

We know that for the sequence (a_i) we have

$$\lim a_i = z \quad \text{ and } a_i \neq a_j \text{ for } i \neq j \tag{6}$$

this implies

$$\forall \epsilon > 0, \exists\, N \in \mathbb{P}:\ d(a_k,z) < \epsilon \text{ whenever } k \geq N \tag{7}$$

The requirement $a_i \neq a_j$ for $i \neq j$ in (6) makes that if z occurs in the sequence, it can only occur once. Be M the position in the sequence were z occurs, then the subsequence $(a_k) : k > M$ will still converges to z (construct the subsequence $(i,a_i) : i > M$ and replace N in (7) by $N - M$).
So, (7) can be replaced by

$$\forall \epsilon > 0, \exists\, N \in \mathbb{P}:\ 0 < d(a_k,z) < \epsilon \text{ whenever } k \geq N \tag{8}$$

which is equivalent to state as $\forall \epsilon > 0 : (N(z;\epsilon) - \{z\}) \cap S \neq \varnothing$

$\blacklozenge$

2.41.2

Suppose (a_i) is a sequence in $\mathbb{R}^n$ that has the following property: For each $\epsilon > 0$, there is an integer N such that $d(a_m, a_n) < \epsilon$ for $m \geq N$ and $n \geq N$. Show that (a_i) is a bounded sequence.

The given property of the sequence can be stated as

$$\forall \epsilon > 0, \exists\, N \in \mathbb{P} : d(a_m, a_n) < \epsilon,\ m \geq N \text{ and } n \geq N \tag{1}$$

If the sequence is bounded then we have

$$\exists\, M > 0 : d(a_m, a_n) \leq M,\ \forall m,\ n \in \mathbb{P} \tag{2}$$

Note that by (1), choosing an ϵ (giving a N), and putting $\epsilon = M$ the statement(2) is true on a subsequence of (a_n) i.e. when $m \geq N$ and $n \geq N$. What about the finite subsequence $(a_k) : k < N$? Can we be sure that

$$\exists\, M^{'} > 0 : d(a_m, a_n) \leq M^{'},\ m < N \text{ and } n < N \tag{3}$$

Yes, as $K = \{(d(a_m, a_n),\ m < N \text{ and } n < N\}$ is a finite subset of $\mathbb{R}$ and hence is a totally ordered finite set. Therefore (being finite) has a first and last element L and $M^{'}$ with the property that $L \leq d(a_m, a_n) \leq M^{'}$ for all $d(a_m, a_n)$ in K. So K is bounded by L and $M^{'}$.
So, choosing $\hat{M} = \max(M, M^{'})$ the statement in (2) is true.

♦

2.41.3

Prove that if (a_i) is a convergent sequence in $\mathbb{R}^n$, then for each subsequence (a_{N_i}) of (a_i), $\lim(a_i) = \lim(a_{N_i})$.

$\lim(a_i) = a$ means

$$\forall \epsilon > 0, \exists\, N \in \mathbb{P} : d(a_k, a) < \epsilon,\ k \geq N$$

For a chosen ϵ we will have a $N \in \mathbb{P}$ from which on the elements of the sequence will fulfil the condition $d(a_i, a) < \epsilon$. Let's choose an arbitrary strictly increasing sequence $L : \mathbb{P} \to \mathbb{P} : L(i) > L(j) \Rightarrow i > j$. Then the set of elements of $\{a_k : k < N\}$ is finite and thus countable. This means that there will exist a $P \in \mathbb{P}$ such that $L(P) = N$ (see figure).

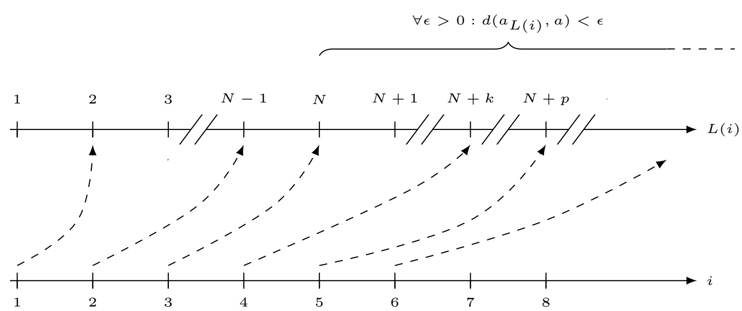


Figure 2.7: Converging subsequence of a convergent sequence, constructed with the strictly increasing sequence $L : \mathbb{P} \to \mathbb{P} : L(k) > L(j) \Rightarrow k > j$

The sequence L being strictly increasing and putting $(b_i) = (a_{L(i)})$ we will have

$$\forall\epsilon > 0, \exists\, P \in \mathbb{P} : d(b_i, a) < \epsilon,\, i \geq P$$

proving the theorem.

♦

2.41.4

Give the details of the proof of **41.2**.

41.2. *Theorem. Suppose that* $x_i = (x_{i,1}, x_{i,2}, \ldots, x_{i,n})$ *for each* $i \in \mathbb{P}$ *and* $x_0 = (x_{0,1}, x_{0,2}, \ldots, x_{0,n})$. *Then* $\lim(x_i) = x_0$ *if and only* $\lim_{i\to\infty} x_{i,k} = x_{0,k}$ *for each* $k \in \mathbb{P}_n$.

By **36.3**. page 69, we know that

$$\forall k \in \mathbb{P}_n : |x_k - y_k| \leq d(x, y),\, \forall x, y \in \mathbb{R}^n$$

$\lim(x_i) = x_0$ means

$$\forall \epsilon > 0,\, \exists\, N \in \mathbb{P} : d(x_i, x_0) < \epsilon,\, i \geq N$$

As $|x_{i,k} - x_{0,k}| \leq d(x_i, x_0)$, $k \in \mathbb{P}_n$ and thus

$$\forall \epsilon > 0,\, \exists\, N \in \mathbb{P} : |x_{i,k} - x_{0,k}| < \epsilon,\, i \geq N,\, k \in \mathbb{P}_n$$

which means

$$\lim_{i\to\infty} x_{i,k} = x_{0,k}, \quad \forall k \in \mathbb{P}_n$$

♦

2.41.5

Prove that a subset S of $\mathbb{R}^n$ is closed and bounded if and only if every sequence in S has a convergent subsequence whose limit is a point in S.

$$\Rightarrow$$

We know that S is bounded. Hence, any sequence (a_i) with $a_i \in S$ will also be bounded and by theorem **41.4**. there is a subsequence (a_{n_i}) of (a_i) that converges. So, the existence of a convergent subsequence has been proven. This means, if this subsequence converges to a:

$$\forall \epsilon > 0, \exists N \in \mathbb{P} : d(a_{n_i}, a) < \epsilon, \, n_i \geq N \tag{1}$$

What is also given, is that S is closed. This means that a limit point $p \in S$ (as S is closed) exists for which yields

$$\forall \epsilon > 0 : (N(p; \epsilon) - \{p\}) \cap S \neq \varnothing$$

Can we prove that a must be a limit point of S?

From (1) we see that the set $A = \{a_{n_i} : n_i \geq N\}$ is a subset of $N(a; \epsilon)$ and as $A \subset S$ we will have

$$\forall \epsilon > 0 : N(a; \epsilon) \cap S \neq \varnothing$$

We can not conclude without further arguments that $N(a; \epsilon) - \{a\} \neq \varnothing$. Indeed consider the case where $a_{n_i} = a$, $n_i \geq N$. But this case is trivial as $a_{n_i} = a$, $n_i \geq N$ is an element of S and thus $a \in S$ (in this case, the requirement S is closed is not necessary).

Consider now the case $\exists n_k \in \mathbb{P} : a_{n_i} \neq a$, $n_k \geq N$. This implies that the ϵ-neighbourhood $N(a; \epsilon)$ contains at least one element different from a and thus we have indeed

$$\forall \epsilon > 0 : (N(a; \epsilon) - \{a\}) \cap S \neq \varnothing$$

meaning that in that case a is a limit point of S and S being closed, a is a point of S, proving the proposed statement.

$$\Leftarrow$$

We first prove that S must be bounded.
Suppose S is unbounded. This means

$$\neg\left(\exists\, M \in \mathbb{R} : \forall x, y \in S,\ d(x, y) \leq M\right) \tag{2}$$
$$\Rightarrow \quad \forall M \in \mathbb{R}, \exists\, x, y \in S :\ d(x, y) > M \tag{3}$$

We know that $a \in S$ and so we can put $y = a$ and write (3) as

$$\forall M \in \mathbb{R}, \exists\, x, \in S :\ d(a, x) > M$$

In the supposition that S in not bounded, we are able to consider the set $D = \{x \in S | d(a, x) > M\}$ and construct a sequence (a_i) with $a_i \in D$. This sequence has a convergent subsequence (a_{n_i}) for which we have

$$\forall \epsilon > 0, \exists\, N \in \mathbb{P} : d(a, a_{n_i}) < \epsilon, \quad n_i \geq N$$

remember that the elements (a_{n_i}) are elements of D for which we have

$$d(a, a_{n_i}) > M$$

with M an arbitrary value in $\mathbb{R}$. This means that it also true for $\epsilon = M$. We get a contradiction as we would have at the same time $d(a, a_{n_i}) > M$ and $d(a, a_{n_i}) < M$.

Conclusion: S is bounded.

$$\Diamond$$

We now want to prove that:

If z is a limit point of S than we can find a sequence in S such that a subsequence of it converges to z or stated in another way: all limit points of S are limits of sequences in S.
(*But the converse of the latter is not necessary true. Indeed, consider $S = \{p\}$ i.e. a singleton. Obviously, S has no limit point - and thus is closed - yet, we can construct a convergent sequence (a_i) with $a_i = a, \forall i \in \mathbb{P}$. This sequence converges to a but a is not a limit point of S!*)

$$\bullet$$

Suppose z is a limit point of S. We will try to prove that then we can find a sequence in S who has a subsequence that converges to z. As this limit of the sequence is in S (by assumption), then S will be closed .

An element $z \in \mathbb{R}^n$ is a limit point of $S \subset \mathbb{R}^n$ provided that

$$\forall\epsilon > 0 : (N(z;\epsilon) - \{z\}) \cap S \neq \varnothing$$

We know (see Exercise 7 page 71) that $B = N(z;\epsilon) \cap S$ is an infinite set. Hence choose an arbitrary set $B^{'} \subset B$ and define a sequence (b_i) out of the elements of B. We know that we then have a convergent subsequence (b_{n_i}) for which we have $\lim(b_{n_i}) = a$, which means

$$\forall\epsilon > 0, \exists\, N \in \mathbb{P} : d(a, b_{n_i}) < \epsilon,\, n_i \geq N$$

But we also know that by construction, the elements $b_{n_i} \in N(z;\epsilon)$ and thus

$$\forall\epsilon > 0 : d(z, b_{n_i}) < \epsilon,\, \forall\, b_{n_i} \tag{4}$$

As (4) is true for all b_{n_i}, $n_i \in \mathbb{P}$, it will also be true for the b_{n_i}, $n_i \geq N$ and get $d(z, b_{n_i}) < \epsilon$ and $d(a, b_{n_i}) < \epsilon$ and thus $a = z$.
Indeed, suppose $a \neq z \;\Rightarrow\; d(a,z) = \delta > 0$ and choose an arbitrary $\epsilon < \frac{\delta}{3}$ e.g. $\epsilon = \frac{\delta}{3}$, then

$$\begin{aligned} & d(a,z) \leq d(a, b_{n_i}) + d(z, b_{n_i}) \\ \Rightarrow \quad & \delta < \frac{\delta}{3} + \frac{\delta}{3} \\ \Rightarrow \quad & 1 < \frac{2}{3} \end{aligned}$$

which is of course a false statement, so δ mus be zero and we conclude $a = z$. As a is a limit of a sequence and by assumption an element of S we may conclude that all limit points of S are also limits of a convergent sequence with a limit in S.

Conclusion: S is closed.

♦

2.41.6

Suppose (S_i) is a sequence of nonempty bounded and closed subsets of $\mathbb{R}^n$ such that for each $i \in \mathbb{P}$, $S_{i+1} \subset S_i$. Prove that there is at least one point $p \in \bigcap\{S_i : i \in \mathbb{P}\}$. (Note that this is a generalization of the nested interval theorem. After proving the proposition, see if your proof depended on some consequence of the nested interval theorem.)

We know that each set S_i is bounded and closed. By exercise 2.41.5 we know that there will be a convergent (sub)sequence (a_j^i), $(a_j^i) \in S_i$, $\forall j \in \mathbb{P}$ whose limit is a point in S_i

$$\lim(a_j^i) = \hat{a}_i, \quad \hat{a}_i \in S_i$$

Consider now the same for the set S_{i+1}. We will have

$$\lim(a_j^{i+1}) = \hat{a}_{i+1}, \quad \hat{a}_{i+1} \in S_{i+1}$$

But we know that $i \in \mathbb{P}$, $S_{i+1} \subset S_i$. Thus we have $\hat{a}_{i+1} \in S_{i+1}$ and $\hat{a}_{i+1} \in S_i$ or

$$\hat{a}_{i+1} \in S_i \cap S_{i+1}$$

and by recursion

$$\hat{a}_i \in S_1 \cap S_2 \cap \cdots \cap S_i \cap S_i \quad \forall i \in \mathbb{P}$$

which we can write

$$\bigcap\{S_i : i \in \mathbb{P}\} \neq \varnothing$$

which proves the proposition.

$\Diamond$

The dependence on the nested interval theorem is :

33.1	Nested interval theorem for $\mathbb{R}$
$\downarrow$	
40.3	Nested interval theorem for $\mathbb{R}^n$
$\downarrow$	
40.4	Bolzano-Weierstrass theorem
$\downarrow$	
41.4	Existance of a convergent subsequence of a bounded sequence
$\downarrow$	
$Ex.2.41.5$	Existance of a convergent sequence with its limit in the set
$\downarrow$	
$Ex.2.41.6$	

$\blacklozenge$

2.41.7

Suppose that (a_i) and (b_i) are sequences in $\mathbb{R}^n$. Suppose also that $c_i = a_i + b_i$ for $i \in \mathbb{P}$. Show that if two of the sequences $(a_i), (b_i), (c_i)$ converge, then so does the remaining one and further $\lim(c_i) = \lim(a_i) + \lim(b_i)$.

Put $\lim(a_i) = A$ and $\lim(b_i) = B$.

First we notice that for a certain $i \in \mathbb{P}$, we have

$$d(A + B, a_i + b_i) = d(A - b_i, a_i - B)$$

indeed

$$\begin{aligned} d^2(A + B, a_i + b_i) &= \sum_{j=1}^{n} (A_{,j} + B_{,j} - a_{i,j} - b_{i,j})^2 \\ &= \sum_{j=1}^{n} (A_{,j} - b_{i,j} - (a_{i,j} - B_{,j}))^2 \\ &= d^2(A - b_i, a_i - B) \end{aligned}$$

In the triangle inequality $d(x, y) \leq d(x, p) + d(p, y)$ put $x = A - b_i$, $y = a_i - B$ and $p = A - B$. We get

$$\begin{aligned} d(A + B, a_i + b_i) &= d(A - b_i, a_i - B) \\ &\leq d(A - b_i, A - B) + d(A - B, a_i - B) \\ &= \sqrt{\sum_{j=1}^{n} (A_{,j} - b_{i,j} - (A_{,j} - B_{,j}))^2} + \sqrt{\sum_{j=1}^{n} (A_{,j} - B_{,j} - (a_{i,j} - B_{,j}))^2} \\ &= \sqrt{\sum_{j=1}^{n} (B_{,j} - b_{i,j})^2} + \sqrt{\sum_{j=1}^{n} (A_{,j} - (a_{i,j})^2} \\ &= d(A, a_i) + d(B, b_i) \end{aligned}$$

We know that $d(A, a_i) < \frac{\epsilon}{2}$ and $d(B, b_i) < \frac{\epsilon}{2}$ for all $\epsilon > 0$ and for $i \geq max(N_a, N_b)$ where N_a, $N_b \in \mathbb{P}$ are the indices in the sequences from which on the inequalities are true. Hence we get

$$\begin{aligned} d(A + B, a_i + b_i) &\leq d(A, a_i) + d(B, b_i) \\ &< \epsilon \quad \forall \epsilon, \quad i \geq max(N_a, N_b) \end{aligned}$$

So $\lim(a_i) + \lim(b_i) = \lim(c_i = a_i + b_i)$ if (a_i) and (b_i) converge. (In the case that we know that e.g. (a_i) and (c_i) converge it suffice to put $\lim(a_i) + \lim(-c_i) = \lim(-b_i)$).

♦

2.41.8

Prove that $\lim(a_i) = a$ if and only if $\lim d(a_i, a) = 0$.

As $\lim(a_i) = a$ we have

$$\begin{aligned}
&\forall \epsilon > 0, \exists\, N \in \mathbb{P} : d(a, a_i) < \epsilon,\ i \geq N \\
\iff\quad &\forall \epsilon > 0, \exists\, N \in \mathbb{P} : |d(a, a_i) - 0| < \epsilon,\ i \geq N
\end{aligned}$$

This is the definition a convergent sequence with elements in $\mathbb{R}$ and thus we conclude (as we have the $\iff$ predicate):

$$\lim(a_i) = a \iff \lim d(a_i, a) = 0$$

♦

2.42 Cauchy criterion for convergence

2.42.1

Show directly from the definition that the sequence $\left(\frac{1}{i}\right)$ satisfies the Cauchy criterion.

We have $d(a_{i+k}, a_i) = |\frac{1}{i+k} - \frac{1}{i}| \leq \frac{1}{i+k} + \frac{1}{i}$.
For a given $\epsilon > 0$ chose an N such that $\frac{1}{N} < \frac{\epsilon}{2}$. We then will also have $\frac{1}{N+1} < \frac{\epsilon}{2}$ and get $d(a_{i+k}, a_i) < \frac{1}{i+1} + \frac{1}{i} < \epsilon, \quad i \geq N$. Hence $\left(\frac{1}{i}\right)$ converges.

♦

2.42.2

Define the sequence (x_n) in $\mathbb{R}$ inductively as follows: Let $x_1 = a$, $x_2 = b$, $x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$, $n \geq 3$. Show that (x_n) converges.

Be $k \in \mathbb{P}$, we have :

$$\begin{aligned}
d(x_{n+k}, x_n) &\leq d(x_{n+k}, x_{n+k-1}) + d(x_{n+k-1}, x_n) \\
&\leq d(x_{n+k}, x_{n+k-1}) + d(x_{n+k-1}, x_{n+k-2}) + d(x_{n+k-2}, x_n) \\
&\leq d(x_{n+k}, x_{n+k-1}) + d(x_{n+k-1}, x_{n+k-2}) + d(x_{n+k-2}, x_{n+k-3}) + d(x_{n+k-3}, x_n) \\
&\vdots \\
&\leq \underbrace{d(x_{n+k}, x_{n+k-1}) + d(x_{n+k-1}, x_{n+k-2}) + \cdots + d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)}_{k \text{ terms}}
\end{aligned}$$

giving

$$d(x_{n+k}, x_n) \leq \sum_{i=0}^{k-1} d(x_{n+k-i}, x_{n+k-i-1}) \tag{1}$$

Lemma :

$$d(x_i, x_{i-1}) = \frac{1}{2^{i-2}}|b-a| \tag{2}$$

We prove this by induction:

For $i = 2$ we have $\frac{1}{2^{i-2}}|b-a| = |b-a|$.

Suppose that indeed : $d(x_{i-1}, x_{i-2}) = \frac{1}{2^{i-3}}|b-a|$.
Then

$$\begin{aligned} d(x_i, x_{i-1}) &= |x_i - x_{i-1}| \\ &= \left|\frac{1}{2}(x_{i-1} + x_{i-2}) - x_{i-1}\right| \\ &= \frac{1}{2} \underbrace{|x_{i-1} - x_{i-2})|}_{=d(x_{i-1},x_{i-2})=\frac{1}{2^{i-3}}|b-a|} \\ &= \frac{1}{2^{i-2}}|b-a| \end{aligned}$$

$\Diamond$

Hence (1) becomes

$$\begin{aligned} d(x_{n+k}, x_n) &\le \sum_{i=0}^{k-1} d(x_{n+k-i}, x_{n+k-i-1}) \\ &= \sum_{i=0}^{k-1} \frac{1}{2^{n+k-i-2}}|b-a| \\ &= \frac{|b-a|}{2^{n-2}} \sum_{i=0}^{k-1} \frac{1}{2^{k-i}} \\ &= \frac{|b-a|}{2^{n-2}} \frac{1}{2} \sum_{j=1}^{k} \frac{1}{2^j} \\ &= \frac{|b-a|}{2^{n-1}} \left(2 - \frac{1}{2^{k-1}}\right) \\ &< \frac{|b-a|}{2^{n-1}} (2) \\ &= \frac{|b-a|}{2^n} \end{aligned}$$

So for a given $\epsilon > 0$ it suffice to chose n such that $\frac{|b-a|}{2^n} < \epsilon$ and the Cauchy criterion is fulfilled.

$\blacklozenge$

2.42.3

Suppose that (x_i) is a sequence in $\mathbb{R}^n$. For each i, let $x_i = (x_{i,1}, x_{i,2}, \ldots, x_{i,n})$. Show that (x_i) satisfies the Cauchy condition for $\mathbb{R}^n$ stated in **42.1(a)** if and only if for each $j \in \mathbb{P}_n$, the sequence $(x_{i,j})_{i=1}^{\infty}$ satisfies the condition for $\mathbb{R}$.

$$\Rightarrow$$

Suppose (x_i) satisfies the Cauchy condition for $\mathbb{R}^n$. This means

$$\forall \epsilon > 0, \exists M \in \mathbb{P} : d(x_m, x_p) < \epsilon, \quad m, p \geq M \tag{1}$$

By **36.3**. page 69, we know that

$$\forall k \in \mathbb{P}_n : |x_k - y_k| \leq d(x, y), \forall x, y \in \mathbb{R}^n$$

Then from (1) we get

$$\begin{aligned} &\forall k \in \mathbb{P}_n : |x_{m,k} - x_{p,k}| \leq d(x_m, x_p) < \epsilon, \quad m, p \geq M \\ \Rightarrow \quad &\forall k \in \mathbb{P}_n : d(x_{m,k}, x_{p,k}) < \epsilon, \quad m, p \geq M \end{aligned}$$

which means that the sequence $(x_{i,k})$ converges for all $k \in \mathbb{P}_n$.

$$\Diamond$$

$$\Leftarrow$$

Suppose $(x_{i,k})$ satisfies the Cauchy condition for $\mathbb{R}$ and for all $k \in \mathbb{P}_n$. This means

$$\forall \frac{\epsilon}{\sqrt{n}} > 0, \exists M \in \mathbb{P} : d(x_{m,k}, x_{p,k}) < \frac{\epsilon}{\sqrt{n}}, \quad m, p \geq M$$

For $\mathbb{R}$ we have $d(x_{m,k}, x_{p,k}) = |x_{m,k} - x_{p,k}|$ and all quantities being positive in both sides of the inequality we can write

$$\forall \frac{\epsilon}{\sqrt{n}}, \exists M \in \mathbb{P} : d^2(x_{m,k}, x_{p,k}) = |x_{m,k} - x_{p,k}|^2 < \frac{\epsilon^2}{n}, \quad m, p \geq M$$

Summing all these inequalities over n gives

$$\begin{aligned} &\forall \frac{\epsilon}{\sqrt{n}}, \exists M \in \mathbb{P} : \sum_{k=1}^{n} |x_{m,k} - x_{p,k}|^2 < n\frac{\epsilon^2}{n}, \quad m, p \geq M \\ \Rightarrow \quad &\forall \frac{\epsilon}{\sqrt{n}}, \exists M \in \mathbb{P} : \underbrace{\sqrt{\sum_{k=1}^{n} |x_{m,k} - x_{p,k}|^2}}_{=d(x_m,x_p)} < \epsilon, \quad m, p \geq M \end{aligned}$$

which means that the sequence (x_i) in $\mathbb{R}^n$ converges.

♦

2.42.4

> Suppose that (a_n) is a sequence in $\mathbb{R}^n$ that satisfies the following property.
>
> There exists a θ, where $0 \leq \theta < 1$, such that for each $n \in \mathbb{P}$, $d(a_{n+1}, a_{n+2}) < \theta d(a_n, a_{n+1})$. Show that (a_n) converges.

From the property of (a_n) we have for $n = 1$

$$d(a_2, a_3) < \theta d(a_1, a_2)$$

Let's put $d(a_1, a_2) = \Delta$, then by iteration we see that

$$d(a_{n+1}, a_{n+2}) < \theta^n \Delta \tag{1}$$

In the Cauchy criterion, let's put for a given a_n, $a_m = a_{n+k}$ (as the distance is symmetric in both elements, we can switch n and m such that we always have $k \geq 0$).
We have

$$\begin{aligned} d(a_{n+1}, a_{n+k}) &\leq d(a_{n+1}, a_{n+2}) + d(a_{n+2}, a_{n+k}) \\ &\leq d(a_{n+1}, a_{n+2}) + d(a_{n+2}, a_{n+3}) + d(a_{n+3}, a_{n+k}) \\ &\vdots \\ &\leq d(a_{n+1}, a_{n+2}) + d(a_{n+2}, a_{n+3}) + d(a_{n+3}, a_{n+k}) + \cdots + d(a_{n+k-2}, a_{n+k-1}) + d(a_{n+k-1}, a_{n+k}) \end{aligned}$$

Using (1) we get

$$\begin{aligned} d(a_{n+1}, a_{n+k}) &< \theta^n \Delta + \theta^{n+1}\Delta + \cdots + \theta^{n+k-3}\Delta + \theta^{n+k-2}\Delta \\ &= \theta^n \Delta \left(\theta^0 + \theta^1 + \cdots + \theta^{k-3} + \theta^{k-2}\right) \end{aligned}$$

Using

$$\sum_{i=0}^{m-1} x^i = \frac{1-x^m}{1-x}$$

we get

$$d(a_{n+1}, a_{n+k}) < \theta^n \Delta \frac{1-\theta^{k-1}}{1-\theta}$$

For a given $\epsilon > 0$ can we find a n (for any $k \in \mathbb{P}$) such that $\theta^n \Delta \frac{1-\theta^{k-1}}{1-\theta} \leq \epsilon$? Note that $\frac{1-\theta^{k-1}}{1-\theta}$ is bounded as $\theta < 1$. So is Δ. Hence, it suffice to take

$$n \geq \left[\frac{\log \epsilon - \log \Delta - \log\left(1-\theta^{k-1}\right) - \log\left(1-\theta\right)}{\log \theta} \right]$$

and the sequence is convergent.

♦

2.43 Some additional properties for $\mathbb{R}^n$

2.43.1

Prove the theorems in this section.

43.1. **Theorem**. Let D be the collection of all points $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ such that for $i \in \mathbb{P}_n$, x_i is rational. Then D is dense in $\mathbb{R}^n$. Hence, $\mathbb{R}^n$ contains a countable dense subset.

Consider an arbitrary nonempty open subset P in $\mathbb{R}^n$. Be p a point in P. We have

$$\exists\, \epsilon > 0 : N(p;\epsilon) \subset P$$

Consider another point q in $N(p;\epsilon)$ such that for each $i \in \mathbb{P}_n$ we have $q_i = p_i - \frac{\epsilon}{n}$ (obvioulsy $d(p,q) = \frac{\epsilon}{\sqrt{n}} < \epsilon$).
Suppose for a given $i \in \mathbb{P}_n$ we have $q_i < p_i$. Then by Theorem **30.5**. (Density of rationals in the reals) there is a rational number r_i such that $q_i < r_i < p_i$.
The point $r = (r_1, r_2, \ldots, r_n)$ will be in $N(p;\epsilon)$ and as $N(p;\epsilon) \subset P$ we have $r \in P$.
The theorem is proven as for an arbitrary nonempty open set in $\mathbb{R}^n$ we are able to find a point $(r_1, r_2, \ldots, r_n)$ in $\mathbb{R}^n$ such that for $i \in \mathbb{P}_n$, r_i is rational and thus r is an element of D.

$\Diamond$

43.2. Theorem. There exists a countable collection $\mathscr{B} = \{B_i : i \in \mathbb{P}\}$ of open subsets of $\mathbb{R}^n$ that have the following property.
If U is open in $\mathbb{R}^n$ and $p \in U$, then there is a $B_i \in \mathscr{B}$ such that

$$p \in B_i \subset U$$

Hence, each open set $U \subset \mathbb{R}^n$ is the union of a countable subcollection of $\mathscr{B}$.

From theorem **43.1** page 79 we know that D, the collection of all points $(x_1, x_2, \dots, x_n)$ in $\mathbb{R}^n$ such that for $i \in \mathbb{P}_n$, x_i is rational, is dense in $\mathbb{R}^n$.

Consider the map $f : \mathbb{P} \to D \to \mathbb{R}^n$ such that $f(i)$ is an open subset A_i of $\mathbb{R}^n$ with $q_i \in D$ is also in A_i. Let's denote $\mathscr{A} = \{A_i : i \in \mathbb{P}\}$. By construction, $\mathscr{A}$ is countable.
Be U an open set in $\mathbb{R}^n$ with an element $p \in U$. U is an nonempty open set in $\mathbb{R}_n$ hence, as D is dense, there will be at least one element q of D which is also in U. We can associate (with the map f) an open subset $A_i \in \mathscr{A}$ such that $q \in A_i$ and have $q \in U \cap A_i$. Yet, p is not necessary an element of A_i. We consider two cases:
$U \subset A_i$. In that case we can replace A_i by the set $B_i = U$. This set contains both p and q and $B_i \subset U$.

If $A_i \not\subset U$, consider the set $B_i = U \cup (U \cap A_i)$. This set will contain both p and q (see figure).

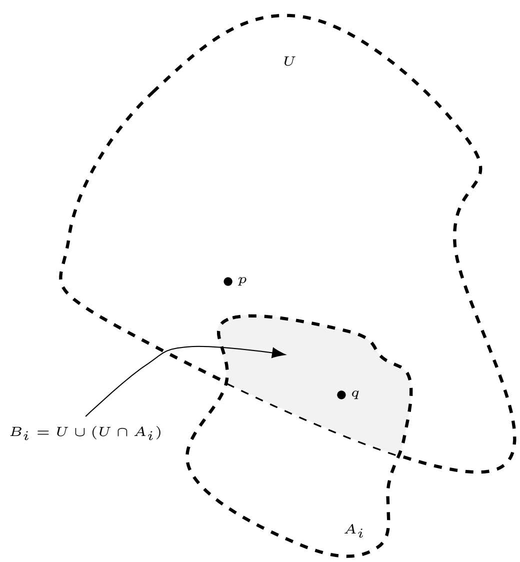


Figure 2.8: $B_i = U \cup (U \cap A_i)$

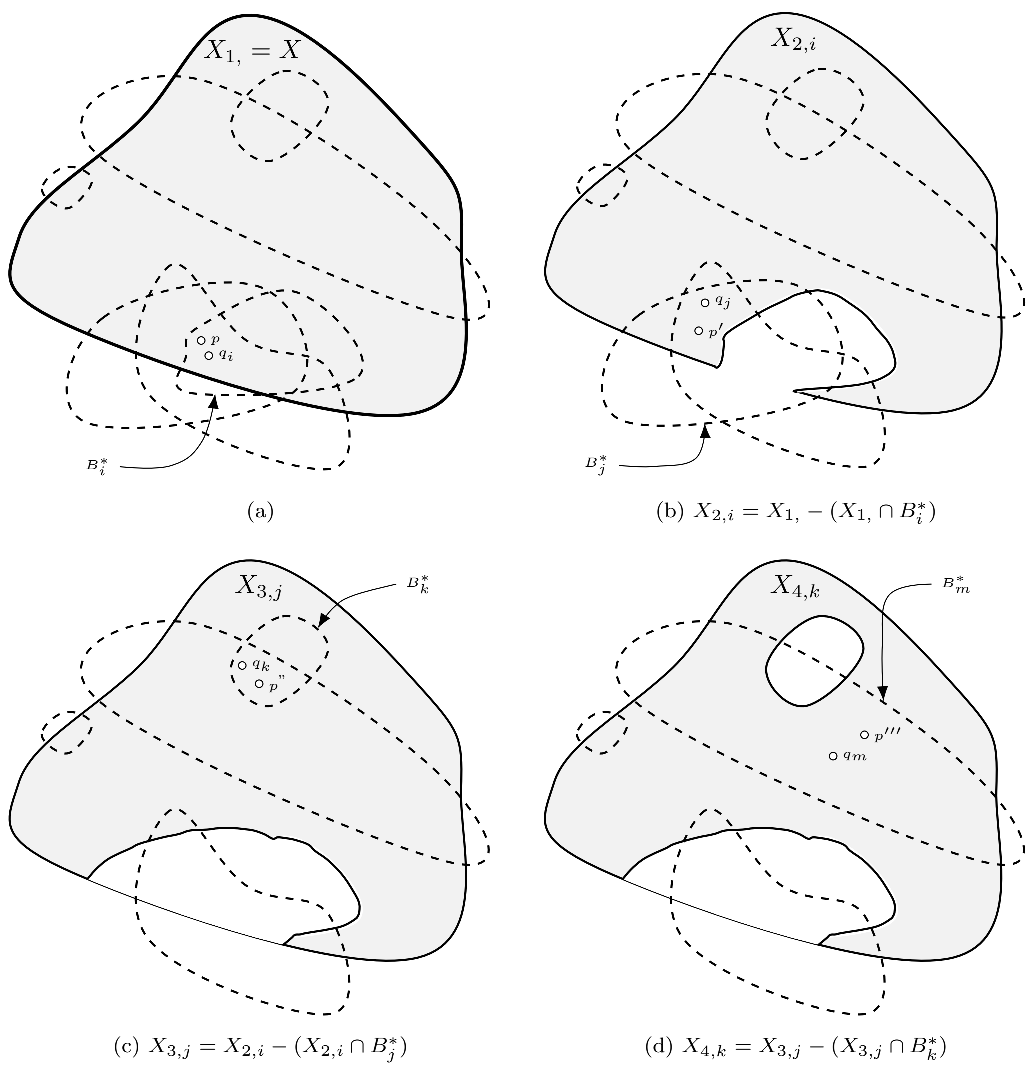
Hence, in both cases, we are able to construct a countable collection of subsets B_i i.e. $\mathscr{B} = \{B_i : i \in \mathbb{P}\}$ such that

$$p \in B_i \subset U$$

$\Diamond$

43.3. **Lindelöf Theorem for $\mathbb{R}^{\mathbf{n}}$**. Let X be a subset of $\mathbb{R}^n$. Suppose that $\mathscr{B}$ is a collection of open subsets of $\mathbb{R}^n$ such that $\bigcup \mathscr{B} \supset X$. Then for some countable subcollections $\mathscr{B}^* \subset \mathscr{B}$, $\bigcup \mathscr{B}^* \supset X$.

First note that if $X = \varnothing$ then the theorem is true as $\varnothing$ is a subset of any set.
Also if $\mathscr{B}$ is a finite collection or countable, then it is countable and the theorem is anyway true with $\mathscr{B}^* = \mathscr{B}$.
So, be $\mathscr{B}$ an infinite, non-countable, collection of open subsets $B \in \mathscr{B}$. Be B^* an arbitrary element of $\mathscr{B}$ such that $X \cap B^* \neq \varnothing$. $X \cap B^*$ has at least one point in X and by theorem **43.1** we know that we can find a point q (with all rational numbers as coordinates), which is also a point of B^*. As $q \in \mathbb{Q}^n$ with $\mathbb{Q}^n$ countable, there is a bijection $f : \mathbb{P} \to \mathbb{Q}^n$ and can find a unique $i \in \mathbb{P}$ such that $f(i) = q$ and can label the set B^* and put $B^*_i = B^*$. Let's put $X_{1,} = X$. Consider now $X_{2,i} = X_{1,} - (X_{1,} \cap B^*_i)$.



(a)

(b) $X_{2,i} = X_{1,} - (X_{1,} \cap B^*_i)$

(c) $X_{3,j} = X_{2,i} - (X_{2,i} \cap B^*_j)$

(d) $X_{4,k} = X_{3,j} - (X_{3,j} \cap B^*_k)$

Figure 2.9: Lindelöf Theorem for $\mathbb{R}^2$

Choose an arbitrary point $p^{'} \in X_{2,i}$ and a B^{**} such that $p^{'} \in (X_{2,i} \cap B^{**})$. Consider the subset $X_{2,i} \cap B^{**}$. As previously seen (notice that $p^{'} \notin B_i^*$), this subset will contain an element $q_j \in \mathbb{Q}^n$ and we can label the set B^{**} and put $B_j^* = B^{**}$. The next step is than to consider $X_{3,j} = X_{2,i} - (X_{2,i} \cap B_j^*)$ and repeat the procedure i.e. choose an arbitrary point $p^{"} \in X_{3,j}$ and a $B^{***} \in \mathscr{B}$ such that $p^{"} \in (X_{3,j} \cap B^{***})$. Consider the set $X_{3,j} \cap B^{***}$. There will be a $q_k \in \mathbb{Q}^n$ and $\in X_{3,j} \cap B^{***}$ and we label $B_k^* = B^{***}$. Doing that recursively (see figure as illustration), as at each step X is reduced in elements and knowing that every point in X is covered by a open subset $B \in \mathscr{B}$, as long as $X_{m,k}$ is not an empty set, there will be a $B \in \mathscr{B}$ so that a point in $X_{m,k}$ is also in B. Hence, we will be able to "cover" X with open subsets $B_m^* \in \mathscr{B}$ but for each of these subsets we can map a $m \in \mathbb{P}$ such that $\bigcup \mathscr{B}^*$ covers X and hence

$$\mathscr{B}^* \text{ is countable and } X \subset \bigcup \mathscr{B}^*$$

$\Diamond$

43.4. Heine-Borel Theorem. Suppose that X is a bounded and closed subset of $\mathbb{R}^n$. Let $\mathscr{K}$ be a collection of open subsets of $\mathbb{R}^n$ that covers X. Then some finite subcollection of $\mathscr{K}$ also covers X.

By the Lindelöf Theorem (**43.3.**), we know that there is a countable subset $\mathscr{K}^*$ of $\mathscr{K}$ that covers X.
Remember that in the proof of this theorem (see previous exercises) we had to choose for each $X_{m,k}$ a point $p \in X_{m,k}$ and a $K^* \in \mathscr{K}$ such that $p \in X_{m,k} \cap K^*$. By construction, the sets $\{X_{m,k}\}$, $\{p\}$ and $\{K^*\}$ were countable and thus we can label the elements in the sets and consider the three following (finite or infinite) sequences: (X_m), (p_m) and (K^*_m) related with the following properties

$$X_{m+1} = X_m - (X_m \cap K^*_m), \quad p_m \in X_m \cap K^*_m, \tag{1}$$

Assumption: the sequences (X_m), (p_m) and (K^*_m) are infinite.
(This means that $\forall m \in \mathbb{P} : X_m \neq \varnothing$ as otherwise the sequence by construction, would be finite.)

We know that X is bounded, hence the sequence (p_m) will also be bounded and from theorem **41.4** page 76 we know that there is a subsequence (p_{n_m}) of (p_m) that converges. Moreover, as X is bounded and closed, by exercise **2.41.5** page 77, we also know that the limit of this sequence will be in X. So we have, considering the sub-sequences (X_{n_m}), (p_{n_m}) and $(K^*_{n_m})$ (with n_m a strictly increasing sequence of positive integers):

i) $\lim(p_{n_m}) = p \in X$ means that we have

$$\forall \epsilon > 0, \exists\, N \in \mathbb{P} : d(p_{n_m}, p) < \epsilon,\, n_m \geq N \tag{2}$$

ii) by (1) each element $p_{n_m} \in X_{n_m} \cap K^*_{n_m}$ must also be in the open subset $K^*_{n_m} \in \mathscr{K}^*$, so:

$$\exists\ \epsilon > 0 : N(p_{n_m}; \epsilon) \subset K^*_{n_m} \tag{3}$$

iii) from (2) and (3) we can conclude that for a given $\epsilon > 0$ and a $N \in \mathbb{P}$ we will have

$$p \in N(p_{n_m}; \epsilon) \text{ for } n_m \geq N \Rightarrow \begin{cases} p \in K^*_{n_m} \\ p_{n_m} \in N(p; \epsilon) \end{cases} \tag{4}$$

Note that we also have $p \notin X_{n_m}$, $n_m \geq N$.
Indeed, suppose that we had $p \in X_{n_{m-1}}$, then as $p \in K^*_{n_{m-1}}$ we would have $p \in X_{n_{m-1}} \cap K^*_{n_{m-1}}$. So p can't be an element of X_{n_m} as it is the result of the complement with respect to $X_{n_{m-1}}$ of the intersection of some X_k and K^*_k, $k = n_{m-1}, n_{m-1}+1, \ldots, n_m - 1$:

$$X_{n_m} = X_{n_{m-1}} - (X_{n_{m-1}} \cap K^*_{n_{m-1}}) - (X_{n_{m-1}+1} \cap K^*_{n_{m-1}+1}) - (X_{n_{m-1}+2} \cap K^*_{n_{m-1}+2}) - \cdots - (X_{n_m-1} \cap K^*_{n_m-1})$$

of which one contained p, so p can not be an element of $X_{n_{m+1}}$.

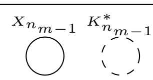
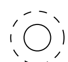
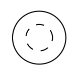
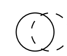
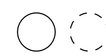
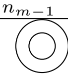
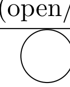
We conclude

$$\begin{cases} p \in K^*_{n_m} \\ \\ p \notin X_{n_m} \end{cases}, n_m \geq N$$

We also notice that p is a limit point of X_{n_m}, $n_m \geq N$. Indeed, we know that for $n_m \geq N$ we have by (1) and (4):

$$\forall \epsilon > 0 : (N(p;\epsilon) - \{p\}) \cap X_{n_m} = \{p_{n_m}, \dots\} \neq \varnothing,\ n_m \geq N$$

confirming that p is not only the limit of the sequence (p_{n_m}) but also a limit point of X_{n_m} but not an element of it. Hence, X_{n_m} must be an open subset.
We know that $p \in K^*_{n_m}$, $n_m \geq N$ and for the element p_{n_m} we have $p \in N(p_{n_m};\epsilon)$ but we concluded that X_{n_m} is open subset with p as a limit point but not an element of it, meaning that for the point $p_{n_m} \in X_{n_m} \cap K^*_{n_m}$ we have $\exists\, \epsilon > 0 : N(p_{n_m};\epsilon) \subset X_{n_m}$. But $p \in N(p_{n_m};\epsilon)$. We get a contradiction as p becomes a point of X_{n_m} which we proved could not be.
As a matter of fact X_{n_m} becomes an empty set as illustrated hereunder (starting with $X_{n_{m-1}}$, a closed susbet).

				
$X_{n_{m-1}} \cap K^*_{n_{m-1}}$	$= X_{n_{m-1}}$ (closed)	$= K^*_{n_{m-1}}$ (open)	$\neq \varnothing$ (open)	$= \varnothing$ (open/closed)
$X_{n_{m-1}} - (X_{n_{m-1}} \cap K^*_{n_{m-1}})$	$\varnothing$			
X_{n_m}	open/closed	closed	closed	closed

Conclusion: our assumption that the sequences (X_m), (p_m) and (K^*_m) are infinite is incorrect and there must be some finite subcollection of $\mathscr{K}$ that also covers X.

♦

2.43.2

Suppose $\mathscr{B}$ is a collection of pairwise disjoint open subsets of $\mathbb{R}^n$. Show that $\mathscr{B}$ is a countable collection.

We know from Theorem **43.1** that $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$. This means:

$$\forall B \in \mathscr{B}, \exists\, q \in \mathbb{Q}^n : q \in B$$

As $\forall B,\ B^{'} \in \mathscr{B} : B \cap B^{'} = \varnothing$ it follows that if $q \in B$ then $q \notin B^{'}$. We conclude that we can associate with each $B \in \mathscr{B}$ a unique element $q \in \mathbb{Q}^n$ and as $\mathbb{Q}^n$ is countable, $\mathscr{B}$ is countable.

♦

2.44 Some further remarks about $\mathbb{R}^n$ and Review exercises

2.44.I

Suppose that K is an uncountable collection of real numbers. Prove that there exists at least one $z \in K$ such that z is a limit point of K. Prove that there is an uncountable collection of limit points of K in K.

Let's consider the statement

$$\exists\, p \in \mathbb{R} : \forall \epsilon > 0 : (N(p;\epsilon) - \{p\}) \cap K \neq \varnothing$$

i.e. K has at least one limit point.
Suppose K has no limit point, then

$$\begin{aligned} &\neg\, (\exists\, p \in \mathbb{R} : \forall \epsilon > 0 : (N(p;\epsilon) - \{p\}) \cap K \neq \varnothing) \\ \Rightarrow \quad &\forall p \in \mathbb{R} : \exists\, \epsilon > 0 : (N(p;\epsilon) - \{p\}) \cap K = \varnothing \end{aligned}$$

As this is for all $p \in \mathbb{R}$, choose an arbitrary point $p_1 \in K \subset \mathbb{R}$. This means that there exists $\epsilon_1 > 0$ such that $N(p_1;\epsilon_1)$ does not contain other points of K except p_1.
Let's put $K_1^* = N(p_1;\epsilon_1)$. All other points of K will be in the set $K - K_1^*$.
Be

$$p_2 \in K - K_1^* \text{ such that } p_2 = \max(K_1^* \cap (-\infty, p_1))$$

and

$$p_3 \in K - K_1^* \text{ such that } p_3 = \min(K_1^* \cap (p_1, +\infty,))$$

(i.e. choose the closest point left and right of p_1 that are elements of K).
As p_2, p_3 can't be limit points of of K (by assumption), they also will have ϵ-neighbourhoods that only contain themselves.
Put

$$K_2^* = N(p_2;\epsilon_2) \text{ and } K_3^* = N(p_3;\epsilon_3)$$

Continuing that way with

$$p_4 = \max\left(K - (K_1^* \cup K_2^* \cup K_3^*) \cap (-\infty, p_2)\right) \text{ and } p_5 = \min\left(K - (K_1^* \cup K_2^* \cup K_3^*) \cap (p_3, -\infty)\right)$$

we are able to construct a countable collection $\mathscr{K} = \{K_i^* : i \in \mathbb{P}\}$ of set, each containing only 1 point of K.

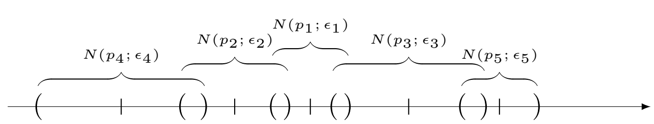


Figure 2.10: $p_{2i} = \max\left(K - \left(K_1^* \cup K_2^* \cup \cdots \cup K_{2i-1}^*\right) \cap (-\infty, p_{2i-2})\right)$

We prove that $\bigcup \mathscr{K} = K$.
Suppose

$$\exists\, q_1 \in K : q_1 \notin \bigcup \mathscr{K}$$

As q_1 can't be a limit point of K, q_1 has also an ϵ-neighbourhood that contains no other point of K, except itself.
q_1 can not be between two points p_i, p_k as constructed. Indeed, without loss of generality, suppose $p_i < q_1 < p_k$. Then there will be a $p_{k-1} \in \bigcup \mathscr{K}$ such that $p_i < q_1 < p_{k-1} < p_k$ as otherwise q_1 would have been the closest point to p_k which was not an element of the ϵ-neighbourhood of p_k and hence, by construction would be an element of $\bigcup \mathscr{K}$. Continuing that way we will arrive at a point where $p_i < q_1 < p_{i+1}$ which is not possible as otherwise q_1 would be the closest element near p_{i+1} and again by construction would have been an element of $\bigcup \mathscr{K}$ and we conclude that q_1 can not be between two element of $\bigcup \mathscr{K}$. So q_1 must be either left or right of all elements of $\bigcup \mathscr{K}$.
Suppose q_1 is on the left of all point of $\bigcup \mathscr{K}$. As $q_1 \in K$, q_1 is not a limit point (by assumption, K has none) and so there exists an ϵ-neighbourhood of q_1 that has only one element of K i.e. q_1 itself. Furthermore, there must be another q_2 at the right of q_1 that is not an element of $\bigcup \mathscr{K}$ as otherwise q_1 would have been the closest element to one element of $\bigcup \mathscr{K}$. Continuing that way, as K is infinite (uncountable), we can generate an infinite subset $Q = \{q_1, q_2, \dots\}$ that is bounded by all elements of $\bigcup \mathscr{K}$. By the Bolzano-Weierstrass theorem, this subset must have at least one limit point. But $Q \subset K$ which means that Q can't have a limit point and we get a first contradiction and conclude $\nexists q_1 \in K : q_1 \notin \bigcup \mathscr{K}$ and thus

$$\bigcup \mathscr{K} = K$$

and get the final contradiction as K is uncountable while by construction $\bigcup \mathscr{K} = K$ is, so our initial assumption that K had no limit points was wrong.
Conclusion: K has at least one limit point.

$\lozenge$

We have now to prove that some of the limit point are in K and that there is an uncountable set of limit points of K in K.

Let $\tilde{K}$ be the set of elements of K that are not limit points of K (which means that $K - \tilde{K}$ contains only limit points of K or is the empty set). Then for each $x \in \tilde{K}$ there exists $\epsilon_x > 0$ such that the neighbourhood $N(x; \epsilon_x)$ contains no other points of K but x. The elements of the set of neighbourhoods $\{N(x; \epsilon_x) | x \in \tilde{K}\}$ are disjoint, and each contains a rational number (see **30.5** page 60 - Density of rationals in the reals), so there can only be a countable number of them.
Hence $\tilde{K}$ is countable, and the set of limit points of K, which contains $K - \tilde{K}$, is uncountable.
Note that this also leads to the conclusion that $K - \tilde{K} \neq \varnothing$ as otherwise $K = \tilde{K}$ and K would then be countable, which leads to a contradiction.

Conclusion: There is an uncountable collection of limit points of K in K.

$\blacklozenge$

2.44.II

Suppose that $S \subset \mathbb{R}$, $S \neq \varnothing$ and $S \neq \mathbb{R}$. Prove that S is not both open and closed.

Suppose $S \subset \mathbb{R}$ is nonempty and $S \neq \mathbb{R}$. Suppose further that S is open and closed.
Let $p \in S$. As $S \neq \mathbb{R}$, then there is some $q \in \sim S$. Whithout loss of generality, we can assume that $q > p$. Then the set $\tilde{S} = \{x \in \sim S : x > p\}$ is bounded below (by p) and nonempty ($q \in \tilde{S}$). Therefore $\tilde{S}$ has a g.l.b. m (see Exercise 2.30.1 page 62).

Can m be in S ? Since S is open, it contains an open neighbourhood $N(m;\epsilon)$. This contradicts the definition of $m = g.l.b. \tilde{S}$, because $[m, m+\epsilon) \subset S$ and thus m would not be the g.l.b. of $\tilde{S}$

Suppose $m \notin S$. Then since S is closed, its complement $\sim S$ is open with $m \in \sim S$, therefore there is an open neighbourhood $N(m;\epsilon)$ contained in $\sim S$. Then $(m - \frac{\epsilon}{2}, m] \subset N(m;\epsilon)$ contradicts the $g.l.b$ definition of m as $m - \frac{\epsilon}{2} \notin S$ and $> p$ (as otherwise p would be in $N(m;\epsilon)$ which is not possible as $p \notin \sim S$) and thus m would not be the g.l.b. of $\tilde{S}$

It follows that $S = \mathbb{R}$.

♦

2.44.III

Suppose that $\{K_i : i \in \mathbb{P}\}$ is a collection of nonempty bounded and closed subsets of $\mathbb{R}^n$ such that $K_{i+1} \subset K_i$, for $i \in\in \mathbb{P}$. Is $\bigcap\{K_i : i \in \mathbb{P}\}$ necessarily nonempty?

Yes, it is necessarily nonempty (see exercise 6 page 77).

♦

2.44.IV

Suppose that K is a nonempty closed and bounded subset of $\mathbb{R}$. Do there exist points x and y in K such that $d(x,y)) = diam(K)$?

As K is closed, by exercise 1 page 77, there exist sequences in K that converge to a limit point (element of K).
Be $\tilde{x}_k,,\ \tilde{x}_m$ two limits points then

$$\begin{cases}\lim x_k = \tilde{x}_k\\ \lim x_m = \tilde{x}_m\end{cases}$$

We first prove that

$$\lim d(x_k, x_m) = d(\tilde{x}_k, \tilde{x}_m)$$

By the triangle inequality , we have

$$\begin{aligned}&\begin{cases}d(x_k,x_m) \le d(x_k,\tilde{x}_k) + d(\tilde{x}_k,\tilde{x}_m) + d(\tilde{x}_m, x_m)\\ d(\tilde{x}_k,\tilde{x}_m) \le d(\tilde{x}_k, x_k) + d(x_k,x_m) + d(x_m,\tilde{x}_m)\end{cases}\\ \Rightarrow\quad &\begin{cases}d(x_k,x_m) - d(\tilde{x}_k,\tilde{x}_m) \le d(x_k,\tilde{x}_k) + d(\tilde{x}_m,x_m)\\ d(x_k,x_m) - d(\tilde{x}_k,\tilde{x}_m) \ge -\left[d(\tilde{x}_k,x_k) + d(x_m,\tilde{x}_m)\right]\end{cases}\\ \Rightarrow\quad & |d(x_k,x_m) - d(\tilde{x}_k,\tilde{x}_m)| \le d(x_k,\tilde{x}_k) + d(\tilde{x}_m,x_m)\end{aligned}$$

By the definition of the limit:

$$\forall \epsilon,\ \exists\, N \in \mathbb{P} : d(x_k,\tilde{x}_k) < \frac{\epsilon}{2} \text{ and } d(x_m,\tilde{x}_m) < \frac{\epsilon}{2},\ k,\ m \ge N$$

and thus

$$|d(x_k,x_m) - d(\tilde{x}_k,\tilde{x}_m)| < \epsilon \quad k,\ m \ge N$$

and thus $d(x_k,x_m)$ converges to $d(\tilde{x}_k,\tilde{x}_m)$.
Consider the set $P = \{d(\tilde{x}_k,\tilde{x}_m)\}$. This set is a totally ordered set (with elements in $\mathbb{R}$) and bounded (by $diam\, K$) and thus has a maximal element $d(\tilde{x}_p,\tilde{x}_q)$.
Suppose $d(\tilde{x}_p,\tilde{x}_q) \neq\ diam\, K$, then as $diam\, K = l.u.b.\, d(x,y)$, $x, y \in K$ there must be two element in K such that

$$d(\tilde{x}_p,\tilde{x}_q) < d(x,y) \le\ diam\, K$$

If $d(x,y) =\ diam\, K$, we are done.
If $d(x,y) \neq\ diam\, K$ then there must be other elements $x^{'}, y^{'}$ such that

$$d(\tilde{x}_p,\tilde{x}_q) < d(x,y) < d(x^{'},y^{'}) \le\ diam\, K$$

as long as $d(x^{'},y^{'}) \neq\ diam\, K$ we will have

$$d(\tilde{x}_p,\tilde{x}_q) < d(x,y) < d(x^{'},y^{'}) < d(x^{''},y^{''}) < \cdots \le\ diam\, K$$

So we get a sequence in $\mathbb{R}$ such that $a_n < a_{n+1}$ and that is bounded above. By Exercise 1 page 64, this sequence has a limit $d(x_0, y_0) = \,diam\, K$ meaning that x_0 and y_0 are limits of sequences of elements in K so x_0 and y_0 are limits points of K (by Exercise 1 page 77). As K is closed, x_0 and y_0 $\in K$ en we can conclude that

$$\exists x_0,\, y_0 \,\in K : d(x_0, y_0) = \,diam\, K$$

♦

2.44.V

Suppose that M is a countable subset of $\mathbb{R}^2$ and $p \in \mathbb{R}^2 - M$. Does there necessarily exist a line $L \subset \mathbb{R}^2$ such that $p \in L \subset \mathbb{R}^2 - M$

Yes, it exists.

By exercise 4 page 51 we know that if two sets M_1 and M_2 are countable sets than $M = M_1 \times M_2$ is countable. So we can construct the given countable set $M \subset \mathbb{R}^2$ with two countable subset $M1$, $M2$ which are subsets of $\mathbb{R}$.

If a line $L \subset \sim M$ exists, it will be a subset of $(\sim M_1) \times\, (\sim M_2)$, $(\sim M_1) \times \mathbb{R}$ or $\mathbb{R} \times\, (\sim M_2)$.

Choose $p = (p_1,\, p_2) : p_1 \in\, \mathbb{R}$ and $p_2 \in \sim M_2$. Obviously, the line will contain no point of M because $L \subset \mathbb{R} \times\, (\sim M_2)$ and thus $L \subset\, (\sim M)$.

♦

2.44.VI

Suppose that A and B are subsets of $\mathbb{R}^n$.
Is $cl(A \cup B) = cl(A) \cup cl(B)$ (see 39.4)?
Is $cl(A) \cap cl(B) = cl(A \cap B)$?

a) $cl(A \cup B) = cl(A) \cup cl(B)$

We first prove that $cl(A \cup B) \subset cl(A) \cup cl(B)$.

i) Suppose $z \in cl(A \cup B)$ and $z \notin cl(A \cup B) - (A \cup B)$ then $z \in A \cup B$, which implies

$$\begin{aligned} & z \in A \text{ or } z \in B \\ \Rightarrow\quad & z \in cl(A) \text{ or } z \in cl(B) \\ \Rightarrow\quad & z \in cl(A) \cup z \in cl(B) \end{aligned}$$

ii) Suppose $z \in cl(A \cup B)$ and $z \notin A \cup B$, which implies that z is a limit point of $A \cup B$, and thus

$$\begin{aligned}
& && \forall \epsilon > 0 : (N(z;\epsilon) - \{z\}) \cap (A \cup B) \neq \varnothing \\
& \underset{X\cap(A\cup B)=(X\cap A)\cup(X\cap B)}{} && \Rightarrow \quad \forall \epsilon > 0 : ((N(z;\epsilon) - \{z\}) \cap A) \cup ((N(z;\epsilon) - \{z\}) \cap B) \neq \varnothing \\
& && \Rightarrow \quad \forall \epsilon > 0 : ((N(z;\epsilon) - \{z\}) \cap A) \cup ((N(z;\epsilon) - \{z\}) \cap B) \neq \varnothing \\
& && \Rightarrow \quad \forall \epsilon > 0 : ((N(z;\epsilon) - \{z\}) \cap A) \text{ or } ((N(z;\epsilon) - \{z\}) \cap B) \neq \varnothing
\end{aligned}$$

which means that z must be a limit point of A or B and thus an element of $cl(A)$ or $cl(B)$ $\Rightarrow$ $z \in cl(A) \cup cl(B)$.
By **i)** and **ii)** we come to the conclusion

$$cl(A \cup B) \subset cl(A) \cup cl(B)$$

We now prove that $cl(A \cup B) \supset cl(A) \cup cl(B)$.

Suppose $z \in cl(A)$ (without any assertion about z being an element of $cl(B)$), then $z \in cl(A) \cup cl(B)$.
i) If $z \notin cl(A) - A$ then we must have $z \in A$ and thus $z \in A \cup B \Rightarrow \quad z \in cl(A \cup B)$.

ii) If $z \in cl(A) - A$ then z must be a limit point of A which implies

$$\begin{aligned}
& && \forall \epsilon > 0 : (N(z;\epsilon) - \{z\}) \cap A \neq \varnothing \\
& && \Rightarrow \quad \forall \epsilon > 0 : ((N(z;\epsilon) - \{z\}) \cap A) \cup ((N(z;\epsilon) - \{z\}) \cap B) \neq \varnothing \\
& \underset{X\cap(A\cup B)=(X\cap A)\cup(X\cap B)}{} && \Rightarrow \quad \forall \epsilon > 0 : (N(z;\epsilon) - \{z\}) \cap (A \cup B) \neq \varnothing
\end{aligned}$$

so, z is a limit point of $A \cup B$ and thus an element of its closure $cl(A \cup B)$ and conclude

$$cl(A \cup B) \supset cl(A) \cup cl(B)$$

Combining the two previous conclusion we get

$$cl(A \cup B) = cl(A) \cup cl(B)$$

$\Diamond$

b) Is $cl(A) \cap cl(B) = cl(A \cap B)$? The answer is no as can be seen by this counterexample in $\mathbb{R}$:

Be $A = (-1, 0)$ and $B = (0, 1)$. Obviously $A \cap B = \varnothing$ so $cl\,(A \cap B) = \varnothing$.
On the other hand $cl(A) = [-1, 0]$ and $cl(B) = [0, 1]$ and thus $cl(A) \cap cl(B) = \{0\} \neq \varnothing$ and conclude that in general

$$cl(A) \cap cl(B) \neq cl(A \cap B)$$

$\blacklozenge$

2.44.VII

Determine whether the following proposition is correct.
Suppose that X is a bounded subset of $\mathbb{R}^n$, and $\epsilon > 0$.
Then there exists a finite set $F \subset X$ such that

$$X \subset \bigcup \{N(x;\epsilon) : x \in F\}$$

We proceed in the same vein as in the proof of the Bolzano-Weierstrass Theorem (see **40.4**. page 75 and see figure 2.11. as illustration).
As X is bounded, there is a closed interval K_0 in $\mathbb{R}^n$ that contains X and that can be taken such that all sides have equal length (let's say $|\beta - \alpha|$). We further partition this closed interval in $\mathbb{R}^n$ by subdividing each side of the interval in two equal subintervals in $\mathbb{R}$. This gives us 2^n subintervals in $\mathbb{R}^n$. We continue this process. After k steps the length of the sides of the subintervals will have a length of $\frac{|\beta-\alpha|}{2^k}$ and get 2^{nk} subintervals. These intervals K_k are bounded with a diameter of $diam\, K_k = \sqrt{n}\frac{|\beta-\alpha|}{2^k}$. We proceed further until we get $diam\, K_k < \epsilon$ (the latter being given).

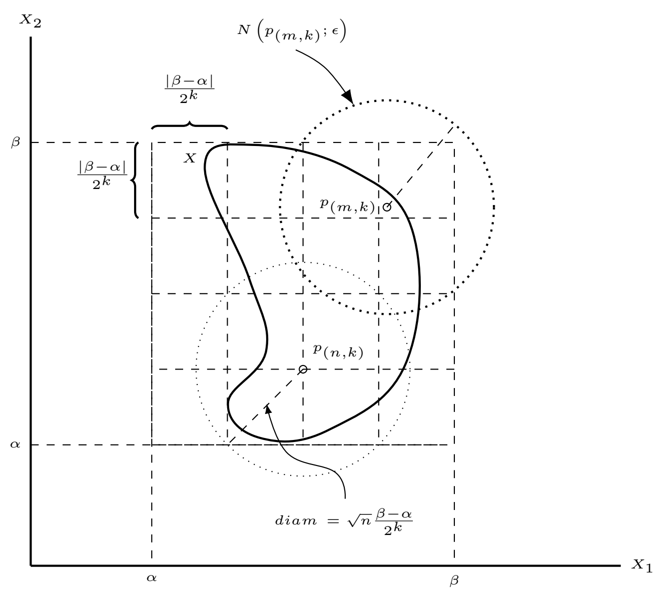


Figure 2.11: Finite covering of a bounded subset in $\mathbb{R}^2$

Be $\mathscr{K} = \{K_{i,k} : i = 1, \cdots, 2^{nk}\}$ the collection of subintervals created that way. We have $\bigcup \mathscr{K} = K_0$ and thus $X \subset \bigcup \mathscr{K}$.
For each i we choose a point $p_{(i,k)} \in X$ such that $p_{(i,k)} \in X \cap K_{i,k}$ (if $X \cap K_{i,k} = \varnothing$ we skip that interval) and get a set of points $F = \{p_{(i,k)} : p_{(i,k)} \in X \cap K_{i,k},\, i \leq 2^{nk}\}$. For each of these points in P, consider the ϵ-neighbourhoods $N(p_{(i,k)};\epsilon)$. These neighbourhoods will each contain the set $X \cap K_{i,k}$ as $diam\, K_{i,k} < \epsilon$. So we have $X \cap K_{i,k} \subset N(p_{(i,k)};\epsilon)$.
Hence, the collection $\mathscr{N} = \{N(p_{(i,k)};\epsilon) : i \leq 2^{nk}\}$ will cover the subset X as $\bigcup \mathscr{K}$ covers X and $N(p_{(i,k)};\epsilon) \supset K_{i,k}$, $\forall i \leq 2^{nk}$.
The set F is finite with less than $\log_2\left(\frac{\sqrt{n}}{\epsilon}|\beta - \alpha|\right)$ elements and conclude that there is a finite set

$F \subset X$ such that

$$X \subset \bigcup \{N(x;\epsilon) : x \in F\}$$

♦

Metric Spaces: Introduction

3.45 Distance function and metric spaces

3.45.1

Verify that the functions given in 45.2 through 45.6 are metrics.

45.2. Let $k > 0$ and $\rho : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be given by

$$\rho(x, y) = k|x - y| \text{ for all } x \text{ and } y \text{ in } \mathbb{R}^n$$

$\triangledown$

$$\begin{array}{llcll}
45.1(a). & d(x,y) = 0 \Leftrightarrow x = y & : & \rho(x,x) & = k|x - x| = 0 \\
45.1(b). & d(x,y) = d(y,x) & : & \rho(x,y) & = k|x - y| = k|y - x| = \rho(y,x) \\
45.1(c). & d(x,y) + d(y,z) \geq d(x,z) & : & \rho(x,z) & = k|x - y| \\
 & & & & = k|x - z + z - y| \\
 & & & & \leq k|x - z| + k|z - y| \\
 & & & & = \rho(x,y) + \rho(y,z)
\end{array}$$

$\lozenge$

45.3. The function $g : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by

$$g(x, y) = |x_1 - y_1| + |x_2 - y_2|$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$

$\triangledown$

$$\begin{array}{llcll}
45.1(a). & d(x,y) = 0 \Leftrightarrow x = y & : & g(x,x) & = |x_1 - x_1| + |x_2 - x_2| = 0 \\
45.1(b). & d(x,y) = d(y,x) & : & g(x,y) & = |x_1 - y_1| + |x_2 - y_2| \\
 & & & & = |y_1 - x_1| + |y_2 - x_2| \\
 & & & & = g(y,x) \\
45.1(c). & d(x,y) + d(y,z) \geq d(x,z) & : & g(x,z) & = |x_1 - z_1| + |x_2 - z_2| \\
 & & & & = |x_1 - y_1 + y_1 - z_1| + |x_2 - y_2 + y_2 - z_2| \\
 & & & & \leq |x_1 - y_1| + |y_1 - z_1| + |x_2 - y_2| + |y_2 - z_2| \\
 & & & & = |x_1 - y_1| + |x_2 - y_2| + |y_1 - z_1| + |y_2 - z_2| \\
 & & & & = g(x,y) + g(y,z)
\end{array}$$

$\lozenge$

45.4. Let X be an arbitrary set. Define $m : X \times X \to \mathbb{R}$ as follows:

$$m(x, y) = 0 \text{ if } x = y$$

$$m(x, y) = 1 \text{ if } x \neq y$$

$\triangledown$

$$\begin{array}{llcll}
45.1(a). & d(x,y)=0 \Leftrightarrow x=y & : & m(x,x) & = 0 \text{ (by definition)} \\
45.1(b). & d(x,y)=d(y,x) & : & m(x,y) & = 1 \\
 & & & & = m(x,y) \text{ (by definition)} \\
45.1(c). & d(x,y)+d(y,z) \geq d(x,z) & : & m(x,z) & = 1 \text{ if } x \neq z \\
 & & & & \leq 1+1 \\
 & & & & = m(x,y)+m(y,z) \text{ if } y \neq x \text{ and } y \neq z \\
 & & & m(x,z) & = 1 \text{ if } x \neq z \\
 & & & & = 1+0 \\
 & & & & = m(x,y)+m(y,z) \text{ if } y \neq x \text{ and } y = z \\
 & & & m(x,z) & = 0 \text{ if } x = z \\
 & & & & \leq 1+1 \\
 & & & & = m(x,y)+m(y,z) \text{ if } y \neq x \text{ and } y \neq z \\
 & & & m(x,z) & = 0 \text{ if } x = z \\
 & & & & \leq 1+0 \\
 & & & & = m(x,y)+m(y,z) \text{ if } y \neq x \text{ and } y = z
\end{array}$$

$\Diamond$

45.5. Let p be the real-valued function defined on $\mathbb{R}^2 \times \mathbb{R}^2$ by

$$p(x,y) = \max\{|x_i - y_i| : i = 1,2\}$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. The function p is a metric for $\mathbb{R}^2$.

$\triangledown$

$$\begin{array}{llcll}
45.1(a). & d(x,y)=0 \Leftrightarrow x=y & : & p(x,x) & = \max\{|x_i - x_i| : i=1,2\} \\
 & & & & = \max\{0,\ 0\} \\
 & & & & = 0 \\
45.1(b). & d(x,y)=d(y,x) & : & p(x,y) & = \max\{|x_i - y_i| : i=1,2\} \\
 & & & & = \max\{|y_i - x_i| : i=1,2\} \\
 & & & & = p(y,x) \\
45.1(c). & d(x,y)+d(y,z) \geq d(x,z) & : & p(x,z) & = \max\{|x_i - z_i| : i=1,2\} \\
 & & & & = \max\{|x_i - y_i + y_i - z_i| : i=1,2\} \\
 & & & & \leq \max\{|x_i - y_i| + |y_i - z_i| : i=1,2\} \\
 & & & & \leq \max\{\max\{|x_i - y_i|\} + \max\{|y_i - z_i|\} : i=1,2\} \\
 & & & & = \max\{|x_i - y_i| : i=1,2\} + \max\{|y_i - z_i| : i=1,2\} \\
 & & & & = p(x,y)+p(y,z)
\end{array}$$

$\Diamond$

45.6. Let the function h be defined as follows: For all x and y in $\mathbb{R}^2$, let

$$h(x,y) = \frac{|x-y|}{1+|x-y|}$$

The function is a metric for $\mathbb{R}^2$

$\triangledown$

$$\begin{array}{llcll}
45.1(a). & d(x,y)=0 \Leftrightarrow x=y & : & h(x,x) & = \frac{|x-x|}{1+|x-x|} \\
 & & & & = 0 \\
45.1(b). & d(x,y)=d(y,x) & : & h(x,y) & = \frac{|x-y|}{1+|x-y|} \\
 & & & & = \frac{|y-x|}{1+|y-x|} \\
 & & & & = h(y,x) \\
45.1(c). & d(x,y)+d(y,z) \geq d(x,z) & : & h(x,z) & = \frac{|x-z|}{1+|x-z|}
\end{array}$$

For this last requirement, consider the function $g(t) = \frac{t}{1+t}$, $t \geq 0$. Its derivative is $g'(t) = \frac{t}{(1+t)^2}$, $t \geq 0$. As $g'(t) > 0$, $t > 0$, $g(t)$ is a monotone increasing function and thus, using this fact in our function $h(x,y)$ we find that

$$\begin{aligned}
h(x,z) &= \frac{|x-z|}{1+|x-z|} \\
&= \frac{|x-y+y-z|}{1+|x-y+y-z|} \\
&\leq \frac{|x-y|+|y-z|}{1+|x-y|+|y-z|} \quad (\text{ because } |x-y|+|y-z| \geq |x-z|) \\
&= \frac{|x-y|}{1+|x-y|+|y-z|} + \frac{|y-z|}{1+|x-y|+|y-z|} \\
&\leq \frac{|x-y|}{1+|x-y|} + \frac{|y-z|}{1+|y-z|} \\
&= h(x,y)+h(y,z)
\end{aligned}$$

♦

3.45.2

Let (X,d) be a metric space. Define $d^* : X \times X \to \mathbb{R}$ as follows:

$$d^*(x,y) = \min\{1, d(x,y)\}$$

for all x and y in X. Show that d^* is a metric for X.

$$\begin{array}{llcll}
45.1(a). & d(x,y)=0 \Leftrightarrow x=y & : & d^*(x,x) & = \min\{1, d(x,x)\} \\
 & & & & = \min\{1,0\} \\
 & & & & = 0 \\
45.1(b). & d(x,y)=d(y,x) & : & d^*(y,x) & = \min\{1, d(x,y)\} \\
 & & & & = \min\{1, d(y,x)\} \\
 & & & & = d^*(x,y) \\
45.1(c). & d(x,y)+d(y,z) \geq d(x,z) & : & d^*(x,z) & = \min\{1, d(x,z)\}
\end{array}$$

For this last requirement, let's consider the following cases
i) $d(x,z) \leq 1$. Then we have $d^*(x,z) = d(x,z)$ and thus $d^*(x,z) \leq 1$.
As $d(x,z) \leq d(x,y) + d(y,z)$ we have $d^*(x,z) \leq d(x,y) + d(y,z)$.
Let's examine the following possibilities:

$$\begin{array}{lcl}
d(x,y) \text{ and } d(y,z) \geq 1 & : & d^*(x,y)+d^*(y,z) = 2 > d^*(x,z) \\
d(x,y) \leq 1 \text{ and } d(y,z) \geq 1 & : & d^*(x,y)+d^*(y,z) = 1 + d(x,y) \geq 1 \geq d^*(x,z) \\
d(x,y) \geq 1 \text{ and } d(y,z) \leq 1 & : & d^*(x,y)+d^*(y,z) = 1 + d(y,z) \geq 1 \geq d^*(x,z) \\
d(x,y) \text{ and } d(y,z) \leq 1 & : & d^*(x,y)+d^*(y,z) = d(x,y)+d(y,z) \geq d^*(x,z)
\end{array}$$

and can conclude

$$d^*(x,z) \leq d^*(x,z) + d^*(x,z)$$

ii) $d(x,z) \geq 1$. Then we have $d^*(x,z) = 1$. By the triangle inequality we have $d(x,z) \leq d(x,y) + d(y,z)$ and as $d(x,z) \geq 1$, we are sure that $d(x,y) + d(y,z) \geq 1$.
Let's examine the following possibilities:

$$\begin{array}{lcl}
d(x,y) \text{ and } d(y,z) \geq 1 & : & d^*(x,y)+d^*(y,z) = 2 > 1 \\
d(x,y) \leq 1 \text{ and } d(y,z) \geq 1 & : & d^*(x,y)+d^*(y,z) = 1 + d(x,y) \geq 1 \\
d(x,y) \geq 1 \text{ and } d(y,z) \leq 1 & : & d^*(x,y)+d^*(y,z) = 1 + d(y,z) \geq 1 \\
d(x,y) \text{ and } d(y,z) \leq 1 & : & d^*(x,y)+d^*(y,z) = d(x,y)+d(y,z) \geq 1
\end{array}$$

and get, as $d^*(x,z) = 1$,

$$d^*(x,z) \leq d^*(x,z) + d^*(x,z)$$

♦

3.45.3

Let $d^* : X \times X \to \mathbb{R}$ be a function that satisfies the following:

$$d(x, y) = 0 \text{ if and only if } x = y$$
$$d(z, x) + d(z, y) \geq d(x, y)$$

Show that for ll x and y in X, $d(z, y) \geq 0$ and $d(x, y) = d(y, x)$. (Hence, a function satisfying the two given properties is a metric for X.)

♦

3.46 Open sets and closed sets

3.46.1