Undergraduate Topology

Robert H. Kasriel (Dover Publication)

Solutions to exercises

Part I

Chapters I to IV

Bernard Carrette

May 12, 2023

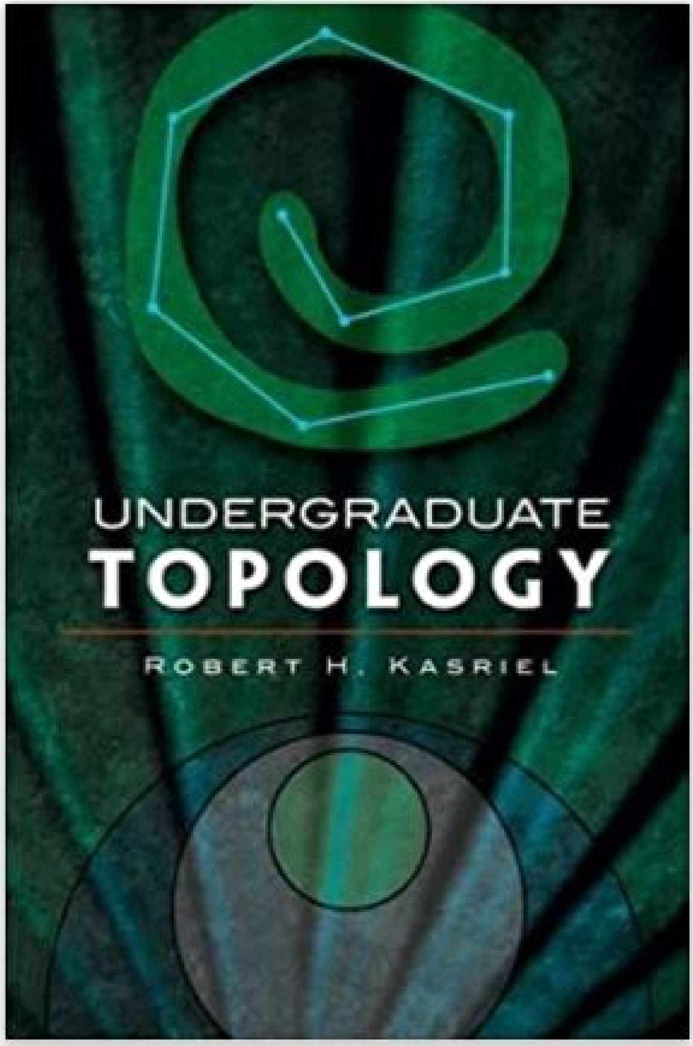


Figure 1

Remarks and warnings

You're welcome to use these notes, but they may contain errors, so proceed with caution : I graduated in 1979, went straight in the industry (where I didn't have to use fancy maths), and picked mathematics and physics again after I retired, so my mathematics got rusty for sure. If you do find an error, typo's , I'd be happy to receive bug reports, suggestions, and the like, through Github.

Contents

List of Figures

Sets, Functions, and Relations

1.1 Sets and Membership

1.1.1

List explicitly the elements of the set

$$\{x : x < 0 \text{ and } (x-1)(x+2)(x+3) = 0\}$$

$$\{-3, -2\}$$

♦

1.1.2

List the elements of the set

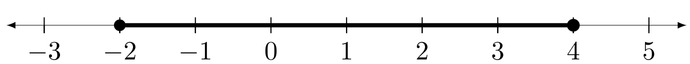
$$\{x : 3x - 1 \text{ is a multiple of } 3\}$$

$$\{x : x = k + \frac{1}{3},\ k \in \mathbb{Z}\}$$

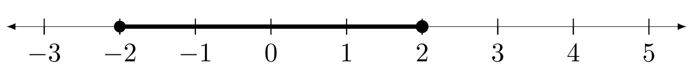
♦

1.1.3

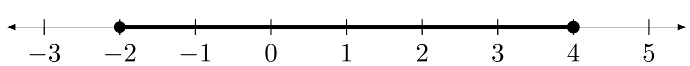
Sketch on a number line each of the following sets.
(a) $\{x : |x-1| \leq 3\}$
(b) $\{x : |x-1| \leq 3 \text{ and } |x| \leq 2\}$
(c) $\{x : |x-1| \leq 3 \text{ or } |x| \leq 2\}$



(a)



(b)



(c)

♦

1.2 Some remarks on the use of the connectives *and*, *or*, *implies*

1.2.1

Demonstrate by means of a table showing truth values that the following is a true statement for any choice of p and q. Thus show that it is a tautology.

$$(\neg q \Rightarrow \neg p) \Rightarrow (p \Rightarrow q)$$

p	q	$\neg q$	$\neg p$	$\neg q \Rightarrow \neg p$	$p \Rightarrow q$	$(\neg q \Rightarrow \neg p) \Rightarrow (p \Rightarrow q)$
T	T	F	F	T	T	T
T	F	T	F	F	F	T
F	T	F	T	T	T	T
F	F	T	T	T	T	T

♦

1.2.2

Show by means of a truth table that the statement

$$((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$$

is a tautology.

p	q	r	$p \Rightarrow q$	$q \Rightarrow r$	$(p \Rightarrow q) \wedge (q \Rightarrow r))$	$p \Rightarrow r$	$((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

♦

1.2.3

Show by means of a truth table that

$$(p \wedge q) \Rightarrow (p \vee q)$$

is a tautology.

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \Rightarrow (p \vee q)$
T	T	T	T	T
T	F	F	F	T
F	T	F	T	T
F	F	F	F	T

♦

1.2.4

Suppose that p and q are statements such that $(p \wedge q)$ is a false statement. Does it follow that the statement

$$(p \text{ is false}) \vee (q \text{ is false})$$

is a true statement?

p	q	$p \wedge q$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	F	F	F	T	T
F	T	F	T	F	T
F	F	F	T	T	T

The answer is Yes.

♦

1.2.5

Negate the following statement: *If two angles of a triangle have equal measure, then the length of two sides of that triangle are equal.*

First we note that $\neg(p \Rightarrow q) \Leftrightarrow (p \wedge \neg q)$. Indeed,

p	q	$p \Rightarrow q$	$\neg(p \Rightarrow q)$	$\neg q$	$p \wedge \neg q$	$\neg(p \Rightarrow q) \Leftrightarrow (p \wedge \neg q)$
T	T	T	F	F	F	T
T	F	F	T	T	T	T
F	T	T	F	F	F	T
F	F	T	F	T	F	T

Putting p as *two angles of a triangle have equal measure* and $\neg q$ as *no two sides of that triangle have equal length* we get the true 'false' statement:
Two angles of a triangle have equal measure $\wedge$ no two sides of that triangle have equal length.

♦

1.2.6

Write the contrapositive of the statement in Exercise 5.

The contrapositive of $p \Rightarrow q$ is $\neg q \Rightarrow \neg p$. Putting $\neg p$ as *no two angles of a triangle have equal measure* and $\neg q$ as *no two sides of that triangle have equal length* we get
If no two sides of that triangle have equal length then no two angles of a triangle have equal measure.

♦

1.2.7

Write the converse of the statement in Exercise 5.

The converse of $p \Rightarrow q$ is $q \Rightarrow p$, giving
If two sides of a triangle have equal length then two angles of a that triangle have equal measure.

♦

1.2.8

Write the contrapositive of the following statement
If a person belongs to Committee A, then he must be a member of Committee B and he must be a member of Committee C.

Lets put

$$p \equiv \text{a person belongs to Committee A}$$
$$q \equiv \text{a person belongs to Committee B}$$
$$r \equiv \text{a person belongs to Committee C}$$

then the given statement translates as

$$p \Rightarrow (q \wedge r)$$

and the contrapositive

$$\neg(q \wedge r) \Rightarrow \neg p$$

This last statement is equivalent with

$$(\neg q \vee \neg r) \Rightarrow \neg p$$

or in plain text:

If a person does not belong to Committee B or C , then he is not a member of Committee A.

♦

1.2.9

Write the contrapositive of the following statement

If $x \in A$ and $x \in B$, then $x \in C$

Lets put

$$p \equiv x \in A$$
$$q \equiv x \in B$$
$$r \equiv x \in C$$

then the given statement translates as

$$p \wedge r \Rightarrow r$$

and the contrapositive

$$\neg(r) \Rightarrow \neg(p \wedge q)$$

This last statement is equivalent with

$$\neg(r) \Rightarrow (\neg p \vee \neg q)$$

i.e:

$$x \notin C \Rightarrow (x \notin A \vee x \notin B)$$

♦

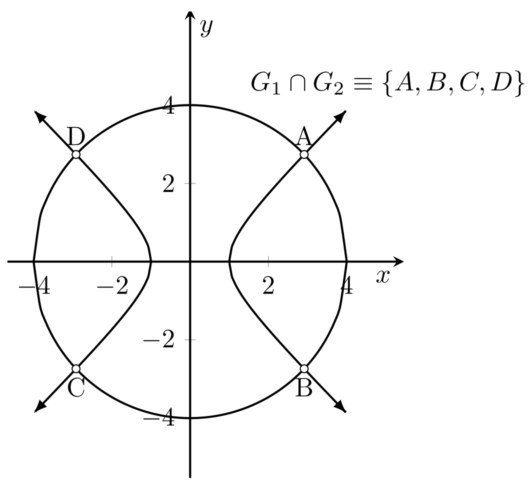
1.3 Subsets

No exercises!

1.4 Union and Intersection of sets

1.4.1

Let G_1 be the graph of the equation $x^2 + y^2 = 16$, and let G_2 be the graph of the equation $x^2 - y^2 = 1$. Sketch the sets $G_1 \cup G_2$ and $G_1 \cap G_2$.



$G_1 \cup G_2$ contains all the points defined by the graphs G_1 and G_2. $G_1 \cap G_2 \equiv \{A, B, C, D\}$ contains the 4 points at the intersection of the two graphs.

♦

1.4.2

We define the sets A, B, C as follows: $A = \{(x,y) : x^2 + y^2 \leq 9\}$, $B = \{(x,y) : x + y \geq 3\}$, $C = \{(x,y) : x \geq 0\}$.
Draw sketches of each of the following sets:

$$
\begin{array}{ll}
(a) & A \cup (B \cup C) \\
(b) & A \cap (B \cup C) \\
(c) & (A \cap B) \cup (A \cap C) \\
(d) & (A \cup B) \cup C \\
(e) & A \cup (B \cap C) \\
(f) & (A \cup B) \cap (A \cup C)
\end{array}
$$

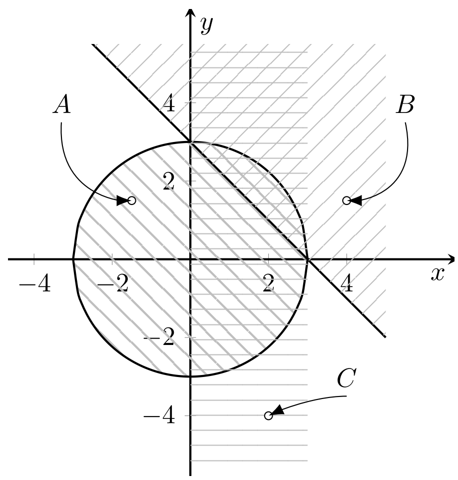
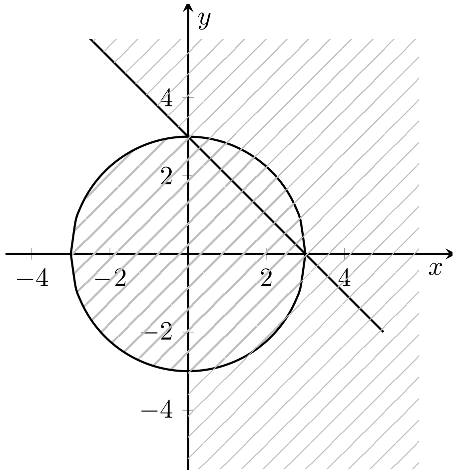
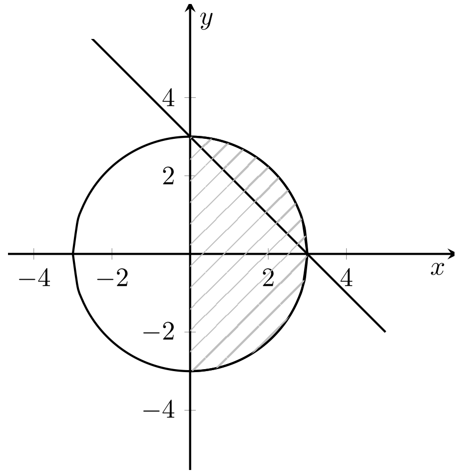


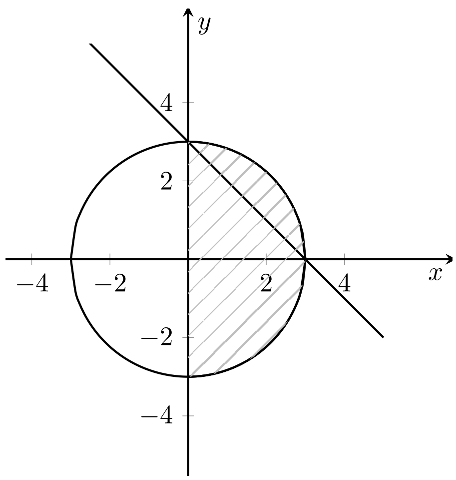
Figure 1.1: The 3 sets A, B, C



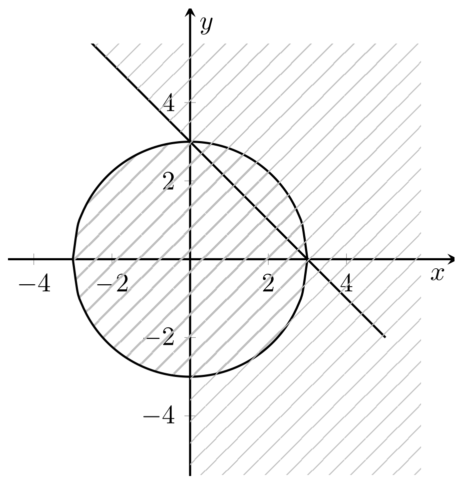
(a) $A \cup (B \cup C)$



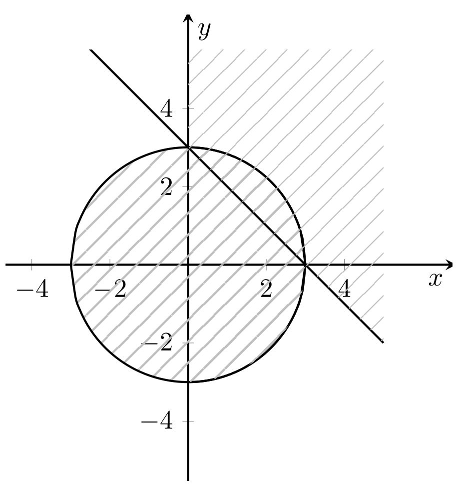
(b) $A \cap (B \cup C)$



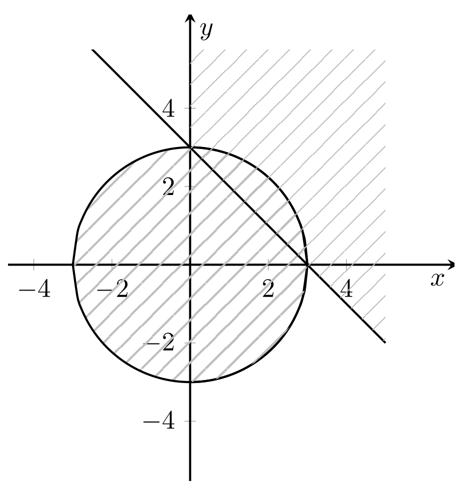
(c) $(A \cap B) \cup (A \cap C)$



(d) $(A \cup B) \cup C$



(e) $A \cup (B \cap C)$



(f) $(A \cup B) \cap (A \cup C)$

♦

1.4.3

Let A, B, C as follows: $A = \{(x, y) : x+y \leq 5\}$, $B = \{(x, y) : x+y \geq 3\}$, $C = \{(x, y) : x \geq 3\}$, and $D = \{(x, y) : y \geq 3\}$.
Draw a sketch for each of the following sets:

$$(a) \quad (A \cap B) \cap C$$
$$(b) \quad [(A \cap B) \cap C] \cap D$$

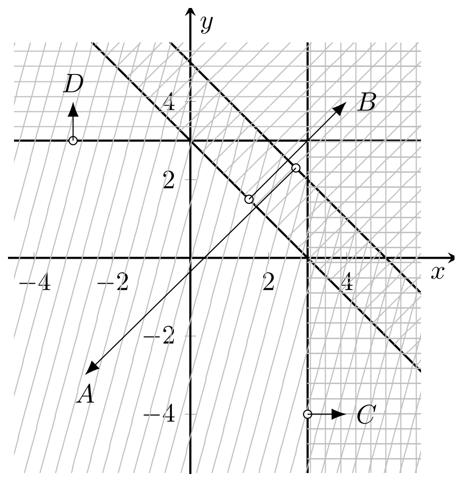
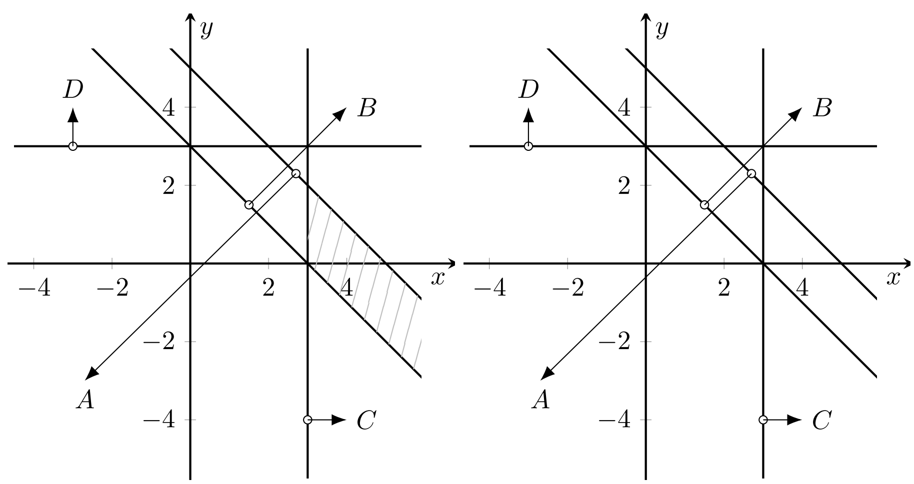


Figure 1.2: The 4 sets A, B, C, D



(a) $(A \cap B) \cap C$ (b) $((A \cap B) \cap C) \cap D = \emptyset$

♦

1.5 Complementation

1.5.1

Sketch each of the following sets: (the sets A, B, C are defined as in exercise 3page 8)

$$
\begin{array}{ll}
(a) & \sim (A \cap B) \\
(b) & (\sim A) \cup (B) \\
(c) & \sim (A \cup B) \\
(d) & (\sim A) \cap (B) \\
(e) & C - A \\
(f) & \sim (A \cap C) \\
(g) & (\sim A) \cup (\sim B) \\
(h) & (\sim A) \cap (A) \\
(i) & C - (A \cup B) \\
(j) & (C - A) \cap (C - B) \\
(k) & \sim (\sim A)
\end{array}
$$

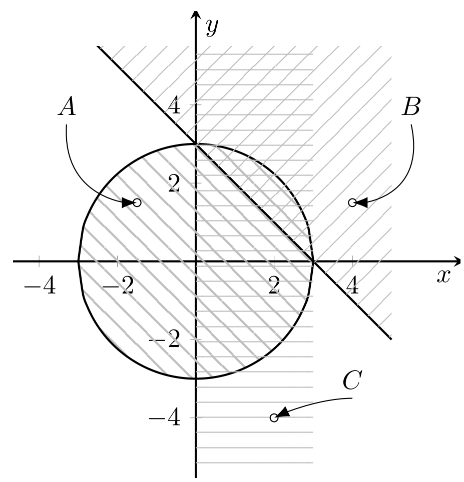
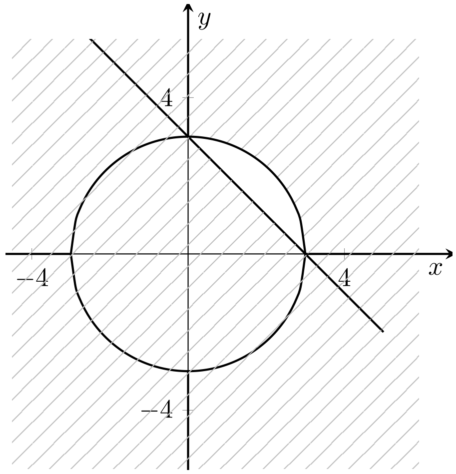
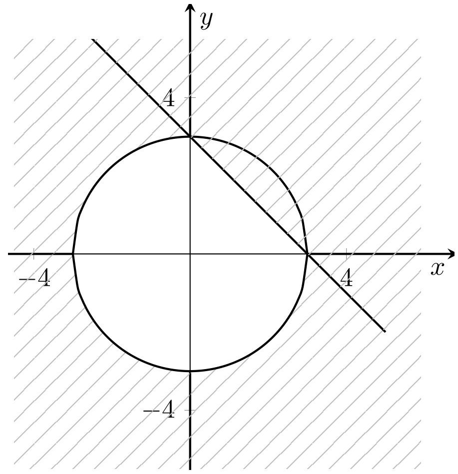


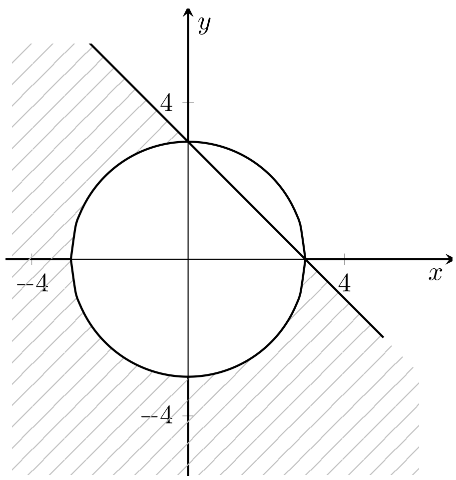
Figure 1.3: The 3 sets A, B, C



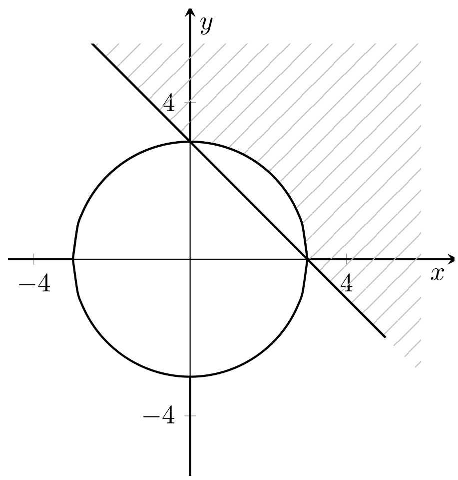
(a) $\sim (A \cap B)$



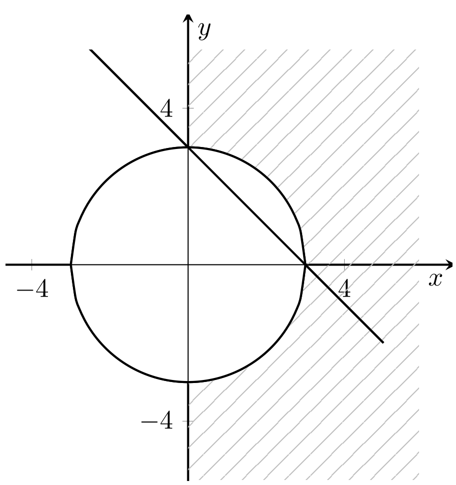
(b) $(\sim A) \cup (B)$



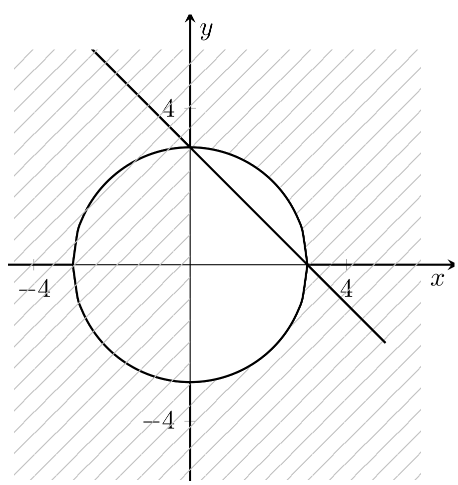
(c) $\sim (A \cup B)$



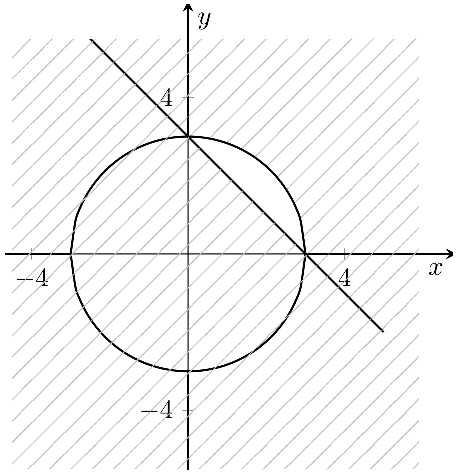
(d) $(\sim A) \cap (B)$



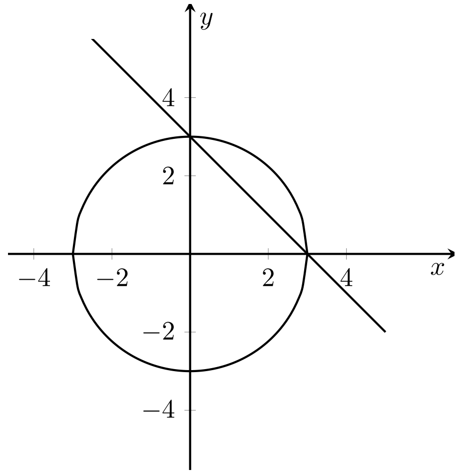
(e) $C - A$



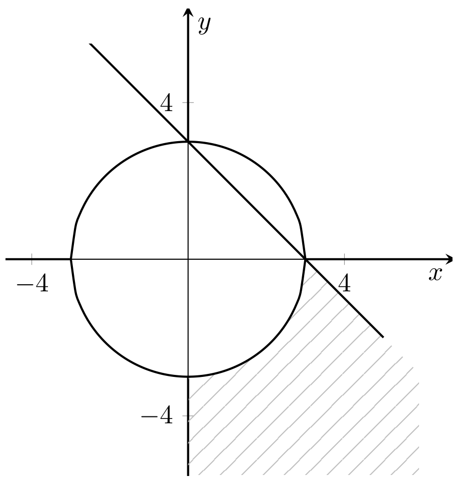
(f) $\sim (A \cap C)$



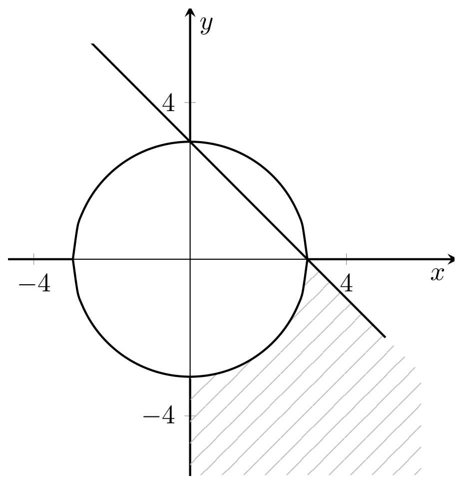
(g) $(\sim A) \cup (\sim B)$



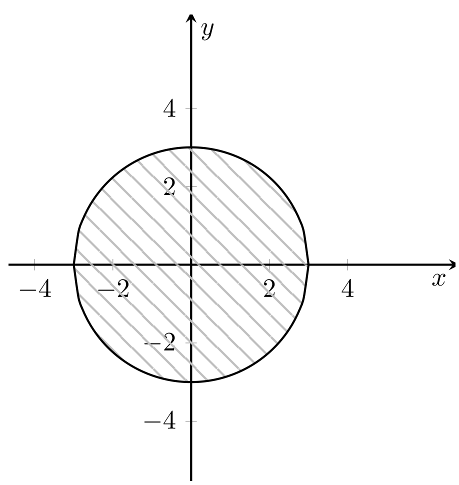
(h) $(\sim A) \cap (A) = \emptyset$



(i) $C - (A \cup B)$



(j) $(C - A) \cap (C - B)$



(k) $\sim (\sim A)$

1.5.2

On the basis of the sketches made in the previous exeercise, formulate a proposition about relation that exist concerning complementation, union, and intersection. Try out your conjecture on other examples. In subsequent exercises you will be asked to try to prove such conjectures.

1.4.2 (a) and (d)	$A \cup (B \cup C) = (A \cup B) \cup C)$
1.4.2 (b) and (c)	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
1.4.2(e) and (f)	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
1.5.1(a) and (g)	$\sim (A \cap B) = (\sim A) \cup (\sim B)$
1.5.1(h)	$(\sim A) \cap A = \emptyset$
1.5.1(i) and (j)	$C - (A \cup B) = (C - A) \cap (C - B)$
1.5.1(k)	$\sim (\sim A) = A$

♦

1.6 Set identities and other set relations

1.6.1

Prove that if $A \subset B$, then:

$$\begin{array}{ll} (a) & A \cap C \subset B \cap C \\ (b) & \sim B \subset \sim A \\ (c) & A \cap B = A \\ (d) & A \cup C \subset B \cup C \end{array}$$

a) $A \cap C \subset B \cap C$

Given is $x \in B$ if $x \in A$. Suppose $x \in A \cap C$, then $x \in A$ (given) and $x \in C$ but $x \in B$ (given) and as $x \in C$ follows that $x \in B \cap C$. And we conclude that $A \cap C \subset B \cap C$.

$\lozenge$

b) $\sim B \subset \sim A$

Given is $x \in B$ if $x \in A$. If $x \notin B$ then $x \in \sim B$. As $A \subset B$, x will not be in A but $x \in \sim A$. So $x \in \sim B \Rightarrow \quad x \in \sim A$ and thus $\sim B \subset \sim A$.

$\lozenge$

c) $A \cap B = A$

Given is $x \in B$ if $x \in A$. Suppose $x \in A \cap B$, then $x \in A$ and thus $A \cap B \subset A$. Suppose $x \in A$, then $x \in B$ as $A \subset B$ and thus $x \in A \cap B$ from which we conclude $A \subset A \cap B$.

$\lozenge$

d) $A \cup C \subset B \cup C$

Given is $x \in B$ if $x \in A$. Suppose $x \in A \cup C$, then $x \in A$ or $x \in C$. But $x \in B$ (given), so $x \in B$ or $x \in C$ and thus $x \in B \cup C$, from which we conclude $A \cup C \subset B \cup C$.

$\blacklozenge$

1.6.2

Verify that each of the following is an an identity:

$$\begin{array}{ll}
(a) & A \cup \emptyset = A \\
(b) & A \cap \emptyset = \emptyset \\
(c) & A \cap A = A \\
(d) & A \cup A = A \\
(e) & (A \cup B) \cup C = A \cup (B \cup C) \\
(f) & (A \cap B) \cap C = A \cap (B \cap C) \\
(g) & A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \\
(h) & X - (A \cup B) = (X - A) \cap (X - B) \\
(i) & A \cap \sim A = \emptyset \\
(j) & A \cup \sim A = U
\end{array}$$

a) $A \cup \emptyset = A$

This is a consequence of remark 3.3 page 7: the empty set $\emptyset$ is a subset of every set. So, $\emptyset \subset A$ giving the asked identity.

$\Diamond$

b) $A \cap \emptyset = \emptyset$

If $x \in A \cap \emptyset$ then $x \in A$ and x must also be in $\emptyset$ which is impossible by definition. So there is no element $x \in \emptyset$ which can satisfy $x \in A \cap \emptyset$ giving the proposed identity.

$\Diamond$

c) $A \cap A = A$

Suppose $x \in A \cap A$, then $x \in A$ and $x \in A$ and thus $x \in A$, giving $A \cap A \subset A$. Suppose $x \in A$, then obviously $x \in A$ and $x \in A$, giving $A \subset A \cap A$. Hence $A \cap A = A$

$\Diamond$

d) $A \cup A = A$

Suppose $x \in A \cup A$, then $x \in A$ or $x \in A$ and thus $x \in A$, giving $A \cup A \subset A$. Suppose $x \in A$, then obviously $x \in A$ or $x \in A$, giving $A \subset A \cup A$. Hence $A \cup A = A$

$\Diamond$

e) $(A \cup B) \cup C = A \cup (B \cup C)$

Suppose $x \in (A \cup B) \cup C$, then $x \in (A \cup B)$ or $x \in C$ and thus $x \in A$ or $x \in B$ or $x \in C$. So $x \in B$ or $x \in C$ can be written as $x \in (B \cup C)$. So $x \in A$ or $x \in (B \cup C)$, giving $(A \cup B) \cup C \subset A \cup (B \cup C)$. The same reasoning yields for $x \in A \cup (B \cup C)$ giving the identity.

$\Diamond$

f) $(A \cap B) \cap C = A \cap (B \cap C)$

Suppose $x \in (A \cap B) \cap C$, then $x \in (A \cup B)$ and $x \in C$ and thus $x \in A$ and $x \in B$ and $x \in C$. So $x \in B$ and $x \in C$ can be written as $x \in (B \cap C)$. So $x \in A$ and $x \in (B \cup C)$, giving $(A \cap B) \cap C \subset A \cap (B \cap C)$. The same reasoning yields for $x \in A \cap (B \cap C)$ giving the identity.

$\Diamond$

g) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Suppose $x \in A \cup (B \cap C)$, then $x \in A$ or $x \in (B \cap C)$. Take the case $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$ which implies $x \in (A \cup B) \cap (A \cup C)$, giving $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$. The other case: if $x \in B \cap C$ then $x \in B$ and $x \in C$. So, $x \in A \cup B$ and $x \in A \cup C$ giving also $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$.
On the other hand, be $x \in (A \cup B) \cap (A \cup C)$ then $x \in (A \cup B)$ and $x \in (A \cup C)$. Let's first take the case $x \in A$ then obviously $x \in A \cup (B \cap C)$ even if $x \notin B \cap C$. Alternatively, be $x \notin A$ then we must have $x \in B$ and $x \in C$ which implies $x \in B \cap C$, giving again $x \in A \cup (B \cap C)$.

$\Diamond$

h) $X - (A \cup B) = (X - A) \cap (X - B)$

Suppose $x \in X - (A \cup B)$, then $x \notin A$ and $x \notin B$ which implies $x \in X - A$ and $x \in X - B$ and thus $x \in X - A \cap X - B$ giving $X - (A \cup B) \subset (X - A) \cap (X - B)$.
The other way around. Suppose $x \in (X - A) \cap (X - B)$. Then $x \in (X - A)$ and $x \in (X - B)$ which implies $x \notin A$ and $x \notin B$ giving $x \notin A \cup B$ which in turn implies $x \in X - (A \cup B)$ giving $(X - A) \cap (X - B) \subset X - (A \cup B)$.
Conclusion: $X - (A \cup B) = (X - A) \cap (X - B)$

$\Diamond$

i) $A \cap \sim A = \emptyset$

Suppose $x \in A \cap \sim A$, then $x \in A$ and $x \notin A$ which is a contradiction, so the only element which is always an element of any set is the empty set, so $A \cap \sim A \subset \emptyset$. Suppose on the contrary that $x \in \emptyset$. This implies that x correspond to the empty set and as the empty set is an element of any set, we have $\emptyset \subset A \cap \sim A$

$\lozenge$

j) $A \cup \sim A = U$

Suppose $x \in A \cup \sim A$, then $x \in A$ or $x \notin A$. So, in any case $x \in U$ and thus $A \cup \sim A \subset U$.
On the opposite way suppose that $x \in U$. Then obviously $x \in A$ or $x \in \sim A$ and thus $U \subset A \cup \sim A$.

$\blacklozenge$

1.6.3

Prove that if $A \subset C$ and $B \subset C$, then $A \cup B \subset C$.

Given is $A \subset C$ and $B \subset C$. Take $x \in A$, then $x \in C$, so even if $x \notin B$, then $x \in A \cup B$ reduces to $x \in A$ and thus $x \in C$. The same reasoning yields for $x \in B$, giving $A \cup B \subset C$.

$\blacklozenge$

1.6.4

Prove that if $A \subset B$ and $A \subset C$, then $A \subset B \cap C$.

Given is $A \subset B$ and $A \subset C$. Take $x \in A$, then $x \in C$ and $x \in B$, which implies $x \in C \cap B$. giving indeed $A \subset B \cap C$.

$\blacklozenge$

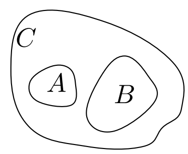
1.7 Counterexamples

In each of the following exercises state whether the statement is necessarily true. Assume that A, B and C are subsets of a universal set U. Justify with a proof or a counterexample.

1.7.1

If $A \cup C = B \cup C$, then $A = B$

Not TRUE.



(l) $A \cup C = B \cup C \not\Rightarrow A = B$

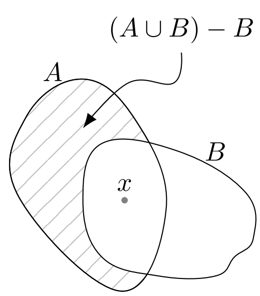
Be $A \subset C$ and $B \subset C$, then we have $A \cup C = B \cup C \equiv C = C$ even if $A \cap B = \emptyset$.

♦

1.7.2

$(A \cup B) - B = A$

Not TRUE.



(m) $(A \cup B) - B \neq A$

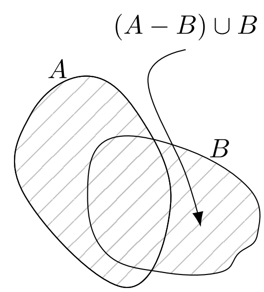
Be $A \cap B \neq \emptyset$, take $x \in A$ and $x \in B$, then x can't be $x \in (A \cup B) - B$ although it is an element of A.

♦

1.7.3

$(A - B) \cup B = A$

Not TRUE.



(n) $(A - B) \cup B \neq A$

This is only true if $B \subset A$

♦

1.7.4

$\sim (A - B) = \sim (A \cap \sim B)$

TRUE.

Suppose first that A and B are disjoint, i.e. $A \cap B = \emptyset$, then $A - B = A$ and $\sim (A - B) = \sim A$. On the other hand $A \subset \sim B$, so $A \cap \sim B = A$, giving $\sim (A \cap \sim B) = \sim A$, giving indeed $\sim (A - B) = \sim (A \cap \sim B)$.

Suppose now that A and B are not disjoint, i.e. $A \cap B \neq \emptyset$. Be $x \in A - B \subset A$. This is equivalent with the statement $x \in A \wedge x \notin B$. Negating this statement: $\neg(x \in A \wedge x \notin B) \Leftrightarrow x \notin A \vee x \in B$. This give $\sim (A - B) \equiv x \notin A \vee x \in B$.
Be now $x \in A \cap \sim B$. This is equivalent with the statement $x \in A \wedge x \notin B$. Negating this statement: $\neg(x \in A \cap \sim B) \Leftrightarrow x \notin A \vee x \in B$. This give $\sim (A \cap \sim B) \equiv x \notin A \vee x \in B$, resulting in $\sim (A - B) = \sim (A \cap \sim B)$.

♦

1.7.5

$\sim (\sim (\sim A)) = \sim A$

TRUE.
Be $x \in \sim (\sim (\sim A))$. This is equivalent to $x \notin \sim (\sim A)$. Which on it's turn is equivalent with $x \in \sim A$. So, $\sim (\sim (\sim A)) \subset \sim A$.

Be $x \in \sim A$. This is equivalent to $x \notin \sim (\sim A)$. Which on it's turn is equivalent with $x \in \sim (\sim (\sim A))$. So, $\sim A \subset \sim (\sim (\sim A))$.
Both cases reduce to $\sim (\sim (\sim A)) = \sim A$.

♦

1.7.6

$A \cup (B - C) = (A \cup B) - C$

Not TRUE.
Be $x \in A \cup (B - C)$. This is equivalent to $x \in A \vee x \in (B - C)$. Suppose $x \in A$, then $x \in A \cup B$. Let's consider the set C so that $(A \cup B) \subset C$, then $(A \cup B) - C = \emptyset$. We get a contradiction and the proposed statement is not true.

♦

1.7.7

$\sim (A - B) = (\sim A) \cup B$

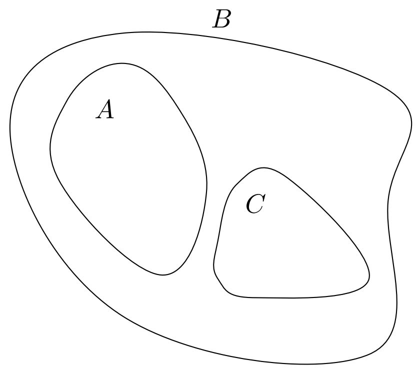
TRUE.
Be $x \in (A - B)$. This is equivalent to $x \in A \wedge x \notin B$. Negating this statement: $\neg(x \in A \wedge x \notin B) \Leftrightarrow x \notin A \vee x \in B$. This is equivalent to the statement $x \in (\sim A) \cup B$. So $\sim (A - B) \subset (\sim A) \cup B$.
Consider now $x \in (\sim A) \cup B$. So $x \notin A \vee x \in B$. If we have the case $x \notin A$ then also $x \notin (A - B)$ as x can not be one of the remaining elements of A after the complement of B relative to A. Also, if $x \in B$ then also $x \notin (A - B)$ as x is an element of B and thus can not be an element of $(A - B)$. Thus, in both cases we have, $x \notin (A - B)$ which implies $x \in \sim (A - B)$. So $(\sim A) \cup B \subset \sim (A - B)$.

♦

1.7.8

If $A - B = C - B$, then $A = C$.

Not TRUE.



(o) If $A - B = C - B \not\Rightarrow A = C$

Suppose $A \subset B$, then $A - B = \emptyset$. Choose a C such that $C \subset B$ and also $A \cap C = \emptyset$, then also $C - B = \emptyset$ and get $A - B = C - B$ although $A \neq C$.

♦

1.7.9

If $A - (B \cap C) = (A - B) \cap (A - C)$.

TRUE.

Suppose $x \in A - (B \cap C)$, then $x \in A \wedge x \notin B \cap C$. As x can not be simultaneously in B and C, then also x must be simultaneously in $A - B$ and $A - C$ as the "complementation of A with B and C will not "subtract" x out of A, and considering that $x \in A$ we have $A - (B \cap C) \subset (A - B) \cap (A - C)$ Suppose $x \in (A - B) \cap (A - C)$, then x must be an element of A but not an element of B and C. This means that $x \notin B \cap C$ and thus the complementation of A by $B \cap C$ has no effect on x. Thus, $\underbrace{(A - B) \cap (A - C)}_{=A} \subset A - (B \cap C)$.

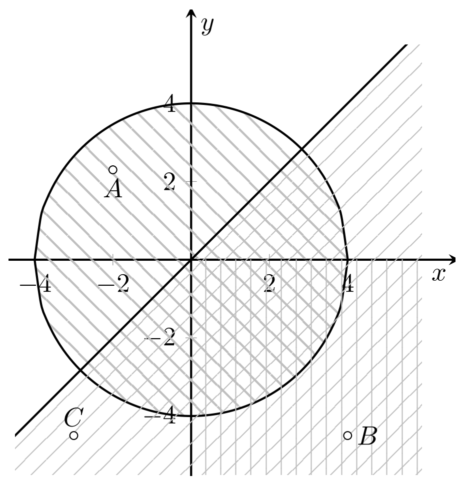
♦

1.8 Collections of Sets

1.8.1

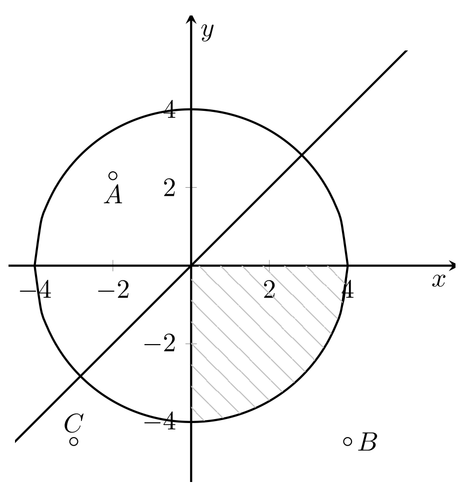
Suppose that A, B and C are the following subsets of the plane:
$A = \{(x, y) : x^2 + y^2 \leq 16\}$, $B = \{(x, y) : x \geq 0 \text{ and } y \leq 0\}$, $C = \{(x, y) : y \leq x\}$. If $\mathscr{K}$ is the collection of sets $\{A, B, C\}$, sketch each of the following sets:

$$\begin{aligned} &(a) \quad \bigcap \mathscr{K} \\ &(b) \quad \bigcup \mathscr{K} \\ &(c) \quad \bigcup \mathscr{K} - \bigcap \mathscr{K} \end{aligned}$$



(p) The sets A, B, C

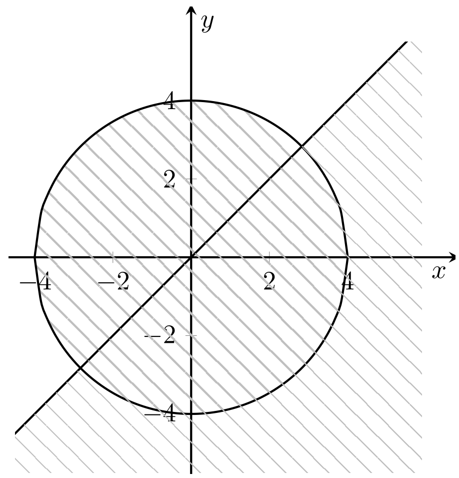
a) $\bigcap \mathscr{K}$



(q) $\bigcap \mathscr{K}$

$\Diamond$

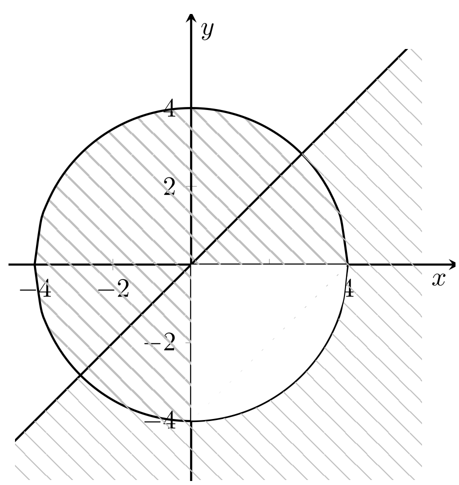
b) $\bigcup \mathscr{K}$



(r) $\bigcup \mathscr{K}$

$\Diamond$

c) $\bigcup \mathscr{K} - \bigcap \mathscr{K}$



(s) $\bigcup \mathscr{K} - \bigcap \mathscr{K}$

$\blacklozenge$

1.8.2

Recall that $\mathbb{P}$ is the symbol for the set of positive integers. Suppose that for each $n \in \mathbb{P}$, we let $A_n = \{x \in \mathbb{R} : x \geq n\}$. Describe the sets $\bigcup\{A_n : n \in \mathbb{P}\}$ and $\bigcap\{A_n : n \in \mathbb{P}\}$.

$S = \bigcup\{A_n : n \in \mathbb{P}\}$

$$S = [1, +\infty)$$

$$\Diamond$$

$S = \bigcap\{A_n : n \in \mathbb{P}\}$

$$S = \emptyset$$

This can be understood by the fact that for every $x \in \mathbb{R}$, you can find a $n \in \mathbb{P}$ so that $x \notin A_n$. So, no x can be an element of S.

$$\blacklozenge$$

1.8.3

Suppose that for each $n \in \mathbb{P}$, K_n is a non-empty set such that $K_{n+1} \subset K_n$. Let $\mathscr{K} = \{K_n : n \in \mathbb{P}\}$.
In each of the following, if the statement is necessarily true, say so and justify your answer. If the statement is not necessarily true, give a counterexample to justify your answer.

$$\begin{array}{ll} (a) & \bigcup \mathscr{K} = K_1 \\ (b) & \bigcap\{K_i : i = 1, 2, \ldots, n\} = K_n \\ (c) & \bigcap \mathscr{K} \neq \emptyset \end{array}$$

(a) $\bigcup \mathscr{K} = K_1$.

TRUE.

Be $x \in K_n$ for any arbitrary n. So, $x \in K_n \cup K_{n-1}$. But $K_n \cup K_{n-1} = K_{n-1}$, giving $x \in K_{n-1}$. Repeating that process with $K_{n-1} \subset K_{n-2} \subset \ldots K_2 \subset K_1$ we get $x \in K_n$.

$$\Diamond$$

(b) $\bigcap\{K_i : i = 1, 2, \ldots, n\} = K_n$.

TRUE.

Suppose first that for all n we have K_n is a *proper* subset of K_{n-1}. Then $K_n \cap K_{n-1} = K_n$. Be $x \in K_n$ but not in K_{n-1} for any arbitrary n. Then, $x \in K_n \cap K_{n-1}$ is equivalent to $x \in K_n$. Repeating that process with we have $K_n \cap K_{n-1} \cap K_{n-2} \cap \ldots K_2 \cap K_1 = K_n$ and get $x \in K_n$. Hence, $\bigcup \mathscr{K} = K_1$.

In the case that for some or all n we have $K_n = K_{n-1}$ we could also state that $\bigcap\{K_i : i = 1, 2, \ldots, n\} = K_{n-1}$ but as $K_n = K_{n-1}$ we can write $\bigcap\{K_i : i = 1, 2, \ldots, n\} = K_{n-1} = K_n$.

The same is true in the case that a sequence of the subsets are proper subset of each other i.e. $K_{n+p} = K_{n+p-1} = \ldots K_{n+1} = K_n = K_{n-1} = \cdots = K_{n-t}$. then one could write $\bigcap\{K_i : i = 1, 2, \ldots, n\} = K_{n+p}$ but as $K_{n+p} = K_n$, the original statement holds.

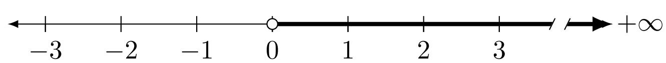
$\lozenge$

(c) $\bigcap \mathscr{K} \neq \emptyset$.
TRUE.
As no K_n is an empty set, K_n will always contain at least one element and due to (b) we get indeed $\bigcap \mathscr{K} \neq \emptyset$: suppose that for a given n, K_n contains only one element x, then all subsequent K_{n+p} must also have only one element i.e. x and we will get $\bigcap \mathscr{K} = \{x\}$

$\blacklozenge$

1.8.4

For each real number $r > 0$, let $L_r = \{x : x \geq r\}$. Sketch the set $\bigcup\{L_r : r > 0\}$ and $\bigcap\{L_r : r > 0\}$ on a number line. If a set happens to be empty, say so.



(t) $\bigcup\{L_r : r > 0\}$

$\lozenge$

$\bigcap\{L_r : r > 0\} = \emptyset$.

Indeed, take an arbitrary r and be $\epsilon > 0$ then $\exists x \in L_r : x \notin L_{r+\epsilon}$. Then, $L_r \cap L_{r+\epsilon} = \emptyset$. So, whatever L_r we choose in the collection $\mathscr{L} = \{L_r : r \in \mathbb{R}^+\}$ there always be a $L_{r'}$ for which $L_r \cap L_{r'} = \emptyset$ and hence $\bigcap\{L_r : r > 0\} = \emptyset$.

$\blacklozenge$

1.8.5

Let U be a set and let $\mathscr{K}$ be a non-empty collection of subsets of U. $\sim$ will signify the complement with respect to U. Prove the following set identities. The identities are quite important and are known as De Morgan's Laws.

$$\begin{array}{ll}(a) & \sim (\bigcup\{K : K \in \mathscr{K}\}) = \bigcap\{\sim K : K \in \mathscr{K}\} \\ (b) & \sim (\bigcap\{K : K \in \mathscr{K}\}) = \bigcup\{\sim K : K \in \mathscr{K}\}\end{array}$$

(a) $\sim (\bigcup\{K : K \in \mathscr{K}\}) = \bigcap\{\sim K : K \in \mathscr{K}\}$

Suppose $x \in \sim (\bigcup\{K : K \in \mathscr{K}\})$, then $x \notin \bigcup\{K : K \in \mathscr{K}\}$. This means that x is not an element of any $K \in \mathscr{K}$ i.e. $\forall K \in \mathscr{K} : x \notin K$. This can also be expressed as $\forall K \in \mathscr{K} : x \in \sim K)$. This means that x is an element of all $\sim K$ giving $x \in \bigcap\{\sim K : K \in \mathscr{K}\}$ and thus $\sim (\bigcup\{K : K \in \mathscr{K}\}) \subset \bigcap\{\sim K : K \in \mathscr{K}\}$.

Suppose now that $x \in \bigcap\{\sim K : K \in \mathscr{K}\}$. This means that x is an element of $\{\sim K : K \in \mathscr{K}\}$ for all K i.e. $x \notin \{K : K \in \mathscr{K}\}$ for all K, (indeed if x would be an element of a $K \in \mathscr{K}$ then x would not be an element of its complement and so x could not be an element of $\bigcap\{\sim K : K \in \mathscr{K}\}$). The conclusion is that $x \notin \bigcup\{K : K \in \mathscr{K}\}$ and thus $x \in \sim \bigcup\{K : K \in \mathscr{K}\}$. Hence, $\bigcap\{\sim K : K \in \mathscr{K}\} \subset \sim (\bigcup\{K : K \in \mathscr{K}\})$.

Conclusion $\sim (\bigcup\{K : K \in \mathscr{K}\}) = \bigcap\{\sim K : K \in \mathscr{K}\}$.

$\lozenge$

(b) $\sim (\bigcap\{K : K \in \mathscr{K}\}) = \bigcup\{\sim K : K \in \mathscr{K}\}$

Suppose $x \in \sim (\bigcap\{K : K \in \mathscr{K}\})$, then $x \notin \bigcap\{K : K \in \mathscr{K}\}$. This means that there exists at least one $K \in \mathscr{K}$ so that x is not an element of this K i.e. $\exists K \in \mathscr{K} : x \notin K$. This can also be expressed as $\exists K \in \mathscr{K} : x \in \sim K$. This means that x is an element of $\bigcup\{\sim K : K \in \mathscr{K}\}$ and thus $\sim (\bigcap\{K : K \in \mathscr{K}\}) \subset \bigcup\{\sim K : K \in \mathscr{K}\}$.

Suppose now that $x \in \bigcup\{\sim K : K \in \mathscr{K}\}$. This means that x is an element of at least one $\sim K : K \in \mathscr{K}$. Stated differently, there exist at least one $K : K \in \mathscr{K}$ for which $x \notin K$. This means that x can not be an element of $\bigcap\{K : K \in \mathscr{K}\}$ and thus $x \in \sim \bigcap\{K : K \in \mathscr{K}\}$ which means $\bigcup\{\sim K : K \in \mathscr{K}\} \subset \sim (\bigcup\{K : K \in \mathscr{K}\})$

Conclusion $\sim (\bigcap\{K : K \in \mathscr{K}\}) = \bigcup\{\sim K : K \in \mathscr{K}\}$.

$\blacklozenge$

1.8.6

Let $S = \{1, 2, 3, 4, 5\}$ and let $\mathscr{P}(S)$ be the power set of S. List the elements in $\mathscr{P}(S)$.

We order them according to the number of elements in the subsets. We check the number of subsets by using the $\binom{5}{m}$ formula (i.e. combination without repetition).

$$\text{5 elements} \quad \binom{5}{5} = 1$$

$$\{1, 2, 3, 4, 5\}$$

$$\text{4 elements} \quad \binom{5}{4} = 5$$

$$\{1, 2, 3, 4\}$$

$$\{1, 2, 3, 5\}$$

$$\{1, 2, 4, 5\}$$

$$\{1, 3, 4, 5\}$$

$$\{2, 3, 4, 5\}$$

$$\text{3 elements} \quad \binom{5}{3} = 10$$

$$\{1, 2, 3\}$$

$$\{1, 2, 4\}$$

$$\{1, 2, 5\}$$

$$\{1, 3, 4\}$$

$$\{1, 3, 5\}$$

$$\{1, 4, 5\}$$

$$\{2, 3, 4\}$$

$$\{2, 3, 5\}$$

$$\{2, 4, 5\}$$

$$\{3, 4, 5\}$$

$$\text{2 elements} \quad \binom{5}{2} = 10$$

$$\{1,2\}$$
$$\{1,3\}$$
$$\{1,4\}$$
$$\{1,5\}$$
$$\{2,3\}$$
$$\{2,4\}$$
$$\{2,5\}$$
$$\{3,4\}$$
$$\{3,5\}$$
$$\{4,5\}$$

$$\text{1 element} \quad \binom{5}{1} = 5$$

$$\{1\}$$
$$\{2\}$$
$$\{3\}$$
$$\{4\}$$
$$\{5\}$$

$$\text{0 elements} \quad \binom{5}{0} = 1$$

$$\emptyset$$

Note that the total number of subsets in $\mathscr{P}(S)$ is $1 + 5 + 10 + 10 + 5 + 1 = 32$ which corresponds to 2^5.

♦

1.9 Cartesian Product

1.9.1

Suppose that $A \subset B$ and C is a set. Prove that $A \times C \subset B \times C$.

Be $x \in A$ and $y \in C$. As $A \subset B$, then x is also in B. Thus $\underbrace{(x, y)}_{x \in A,\, y \in C} \in A \times C$ means also that $\underbrace{(x, y)}_{x \in B,\, y \in C} \in B \times C$

♦

1.9.2

Let $A = \{1, 2, 3\}$, $B = \{a, b\}$, and $C = \{\alpha, \beta\}$. List the elements of each of the following sets:

(a) $A \times (B \cup C)$
(b) $(A \times B) \cup (A \times C)$
(c) $(A \cup B) \times C$
(d) $(A \times C) \cup (B \times C)$

(a) $A \times (B \cup C)$

$(1, a)$, $(1, b)$, $(2, a)$, $(2, b)$, $(3, a)$, $(3, b)$
$(1, \alpha)$, $(1, \beta)$, $(2, \alpha)$, $(2, \beta)$, $(3, \alpha)$, $(3, \beta)$

◊

(b) $(A \times B) \cup (A \times C)$

$(1, a)$, $(1, b)$, $(2, a)$, $(2, b)$, $(3, a)$, $(3, b)$
$(1, \alpha)$, $(1, \beta)$, $(2, \alpha)$, $(2, \beta)$, $(3, \alpha)$, $(3, \beta)$

◊

(c) $(A \cup B) \times C$

$(1, \alpha)$, $(1, \beta)$, $(2, \alpha)$, $(2, \beta)$, $(3, \alpha)$, $(3, \beta)$ $(1, \alpha)$, $(1, \beta)$, $(2, \alpha)$, $(2, \beta)$, $(3, \alpha)$, $(3, \beta)$
(a, α), (a, β), (b, α), (b, β)

◊

(d) $(A \times C) \cup (B \times C)$

$(1, \alpha)$, $(1, \beta)$, $(2, \alpha)$, $(2, \beta)$, $(3, \alpha)$, $(3, \beta)$ $(1, \alpha)$, $(1, \beta)$, $(2, \alpha)$, $(2, \beta)$, $(3, \alpha)$, $(3, \beta)$
(a, α), (a, β), (b, α), (b, β)

♦

1.9.3

Are any of the sets in Exercise 2 the same? If so write the set identities that are suggested by your observations. Try to prove your conjecture.

In exercise 2 we can see that that the set (a) and (b) are the same. Also (c) and (d) are the same. This suggests the following identities $A \times (B \cup C) = (A \times B) \cup (A \times C)$ and $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

$\mathbf{A} \times (\mathbf{B} \cup \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cup (\mathbf{A} \times \mathbf{C})$
Proof:
Be $x \in A$ and $y \in B \cup C$, so y is an element of B or C. Consider $(x, y) \in A \times (B \cup C)$. As the y can be an element of B or C follows immediately that $(x, y) \in (A \times B)$ or $(x, y) \in A \times C)$ and thus $(x, y) \in (A \times B) \cup (A \times C)$. And get $A \times (B \cup C) \subset (A \times B) \cup (A \times C)$
Suppose now that $(x, y) \in (A \times B) \cup (A \times C)$. The (x, y) is an element of $A \times B$ or $A \times C$. For the same $x \in A$ this implies that $y \in B$ or $y \in C$ and thus $(x, y) \in A \times (B \cup C)$, giving $(A \times B) \cup (A \times C) \subset A \times (B \cup C)$ leading with the previous $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

◊

$(\mathbf{A} \cup \mathbf{B}) \times \mathbf{C} = (\mathbf{A} \times \mathbf{C}) \cup (\mathbf{B} \times \mathbf{C})$
Proof:
Be $x \in A \cup B$ and $y \in C$, so x is an element of A or B. Consider $(x, y) \in (A \cup B) \times C$. As the x can be an element of A or B follows immediately that $(x, y) \in (A \times C)$ or $(x, y) \in (B \times C)$ and thus $(x, y) \in (A \times B) \cup (\mathbf{A} \times C)$. And get $(A \cup B) \times C \subset (A \times C) \cup (B \times C)$
Suppose now that $(x, y) \in (A \times B) \cup (\mathbf{A} \times C)$. The (x, y) is an element of $A \times C$ or $B \times C$. For the same $y \in C$ this implies that $x \in A$ or $x \in B$ and thus $(x, y) \in (A \times C) \cup (B \times C)$, giving $(A \times C) \cup (B \times C) \subset (A \cup B) \times C$ leading with the previous $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

♦

1.9.4

Suppose that A is a set consisting of five elements and B is a set consisting of three elements. How many elements does the set $A \times B$ have? The set $B \times A$?

$A \times B$ has $5 \times 3 = 15$ elements. Indeed in the element $(x, y) \in A \times B$ we can choose for x out of the five elements of A and for each choice of x we are free to choose one element out of the 3 elements of B.
For $B \times A$, the reasoning is the same and get $3 \times 5 = 15$ elements.

♦

1.9.5

Suppose that A is a set consisting of m elements and B is a set consisting of n elements, where m and n are positive integers. How many elements are there in $A \times B$?

$A \times B$ has $m \times m$ elements. Indeed in the element $(x, y) \in A \times B$ we can choose for x out of the m elements of A and for each choice of x we are free to choose one element out of the n elements of B.

♦

1.9.6

Suppose that A is a set consisting of three elements, B consists of four elements and C consists of two element. How many elements are there in the set $(A \times B)\times$?

$(A \times B) \times C$ has $(3 \times 4) \times 2 = 24$ elements. Indeed in the element $((x, y), z) \in (A \times B) \times C$ we have for (x, y), $3 \times 4 = 12$ elements (see Exercise 1.9.5) and for each choice of this (x, y) we are free to choose one element out of the 2 elements of C.

♦

1.10 Functions

1.10.1

In each of the following, a set of ordered pairs Γ is given. In each case, determine whether Γ is a function and, if it is, determine if it is a one-to-one function.

(a) Let $\Gamma = \{(x, y) : -1 \leq x \leq 1 \text{ and } x^2 + y^2 = 1\}$.
(b) Let $\Gamma = \{(x, y) : -1 \leq x \leq 1,\ y \geq 0, \text{ and } x^2 + y^2 = 1\}$.
(c) Let $\Gamma = \{(x, y) : 0 \leq x \leq 1 \text{ and } x^2 + y^2 = 1\}$.
(d) Let $\mathscr{F}$ be the collection of all real-valued differentiable functions defined on the open interval (a, b).
Let $\Gamma = \{(f, f^{'}) : f \in \mathscr{F} \text{ and } f^{'} \text{ is the derivative of } f\}$.
(e) Let X be the collection of all continuous real-valued functions defined on the closed interval $[a, b]$.
Let $\Gamma = \left\{\left(f, \int_a^b f(x)dx\right) : f \in X\right\}$.

(a) Let $\Gamma = \{(x, y) : -1 \leq x \leq 1 \text{ and } x^2 + y^2 = 21\}$

Γ is not a function due to the ambiguity of the $\sqrt{}$ function. E.g. take $x = 0$ then $y = \pm 1$.

$\Diamond$

(b) Let $\Gamma = \{(x, y) : -1 \leq x \leq 1,\ y \geq 0, \text{ and } x^2 + y^2 = 1\}$.

This time, as the ambiguity on the range has been removed by the condition $y \geq 0$ Γ is a function. Yet, it is not one-to-one e.g. for $x = -1$ and $x = 1$ we get the same value for y.

$\Diamond$

(c) Let $\Gamma = \{(x, y) : 0 \leq x \leq 1 \text{ and } x^2 + y^2 = 2\}$.

This time, as the ambiguity on the range has been removed by the condition $y \geq 0$ Γ is a function. And, it is a one-to-one function as with the restriction on the domain $x \in [0, 1]$, y is well and uniquely defined.

$\Diamond$

(d) Let $\mathscr{F}$ be the collection of all real-valued differentiable functions defined on the open interval (a, b).Let $\Gamma = \{(f, f^{'}) : f \in \mathscr{F} \text{ and } f^{'} \text{ is the derivative of } f\}$.

Γ is a function as f is a real-valued differentiable function, meaning that $\forall f \in \mathscr{F}, \exists f^{'}$.
Yet, it is not one-to-one. E.g. take $f_1 = x + 1$ and $f_2 = x + 2$, both function give $f^{'} = 1$ meaning that Γ is not one-to-one.

$\Diamond$

(e) Let X be the collection of all continuous real-valued functions defined on the closed interval $[a, b]$.
Let $\Gamma = \left\{ \left(f, \int_a^b f(x)dx \right) : f \in X \right\}$.

Γ is a function as f is a continuous real-valued function, and from calculus we know that every continuous is Riemann-integrable, meaning that for every f there exist a real number $\int_a^b f(x)dx$. Yet, Γ is not one-to-one as two different functions f_1 and f_2 could have the same value of their integral on the given domain e.g. take $f_1 = \frac{x-a}{b-a}$ and $f_2 = \frac{b-x}{b-a}$, both have the same value for the integral over $[a, b]$ namely $\frac{1}{2}(b-a)$.

♦

1.10.2

Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ be the function defined as follows:
For each $(x, y) \in \mathbb{R}$, let $f(x, y) = (a, b)$ where

$$a = x + 2y$$

and

$$b = 2x + 4y$$

Which of the following terms applies to $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$?
(a) surjective, (b) bijective, (c) injective.

f in not injective. Indeed the given function definition can be considered as a system of linear equations with x and y as unknowns and a, b as parameters. So for a given $(a, b) \in \mathbb{R} \times \mathbb{R}$ (the domain) the range will only span $\mathbb{R} \times \mathbb{R}$ only if the system of equations is not degenerated i.e. if the determinant of the system is not 0, but we have

$$det \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = 0$$

Hence, f is not surjective. It is however one-to-one (injective) as for a given (x, y), due to linear form of the function, there will be only one (a, b) on which (x, y) is mapped. As f is not surjective, f can not be bijective.

♦

1.10.3

Repeat the question in Ecercise 2 for the system

$$a = 3x + 2y$$

$$b = 6x - 2y$$

f is injective as we see that this time the determinant of the system is

$$det \begin{pmatrix} 3 & 2 \\ 6 & -2 \end{pmatrix} = -18$$

Hence, f is surjective. It is also one-to-one (injective) for the same reason mentioned in Exercise 2. . As f is surjective and injective , f is also bijective.

♦

1.10.4

Let f be a map from the set of all reals $\mathbb{R}$ into $\mathbb{R}$. Suppose furthermore that if x_1 and x_2 are in $\mathbb{R}$ and $x_1 < x_2$, then $f(x_1) < f(x_2)$. Is it necessarily true that f is one-to-one? Is it necessarily true that $f[\mathbb{R}] = \mathbb{R}$? Justify your answer.

It is necessarily true that f is one-to-one. (At each point x_1, the function for a given x_2 could be re-written as $f(x_2) = f(x_1) + \phi(x_1)(x_2 - x_1)$ with $\phi(x_1) > 0$. So $f(x_2)$ can not be equal to $f(x_1)$ unless $x_2 = x_1$.)
On the other hand $f[\mathbb{R}]$ is not necessarily equal to $\mathbb{R}$. As a counterexample, consider the function $f(x) = e^{-x}$, which is a monotone increasing function but the range is $(-\infty, 0) \neq \mathbb{R}$

♦

1.10.5

Consider the function $f : X \to Y$. Suppose that A and B are subsets of X. Decide which of the following statements are necessarily true. Justify your answers.

$$\begin{array}{ll}
(a) & \text{If } A \cap B = \emptyset, \text{ then } f[A] \cap f[B] = \emptyset. \\
(b) & \text{If } f[A] \cap f[B] = \emptyset, \text{ then } A \cap B = \emptyset. \\
(c) & \text{If } A \subset B, \text{ then } f[A] \subset f[B]. \\
(d) & f[A - B] = f[A] - f[B]. \\
(e) & f[A \cup B] = f[A] \cup f[B]. \\
(f) & f[A \cap B] \subset f[A] \cap f[B]. \\
(g) & f[A \cap B] = f[A] \cap f[B].
\end{array}$$

(a) If $A \cap B = \emptyset$, then $f[A] \cap f[B] = \emptyset$.
This is not necessarily true. Take for example a non injective function like $f(x) = \sin(x)$ then $f[[0, \frac{\pi}{4}]] \cap f[\frac{3\pi}{4}, \pi] = [0, \frac{\sqrt{2}}{2}]$.

$\Diamond$

(b) If $f[A] \cap f[B] = \emptyset$, then $A \cap B = \emptyset$
This is necessarily true as for f being a function we have $(x_2, f(x_2)) \in f$ and $(x_1, f(x_1)) \in f \Rightarrow f(x_1) = f(x_2)$ and $A \cap B \neq \emptyset$ would mean that $\exists x \in A \cap B$ for which x has two different images.

$\Diamond$

(c) If $A \subset B$, then $f[A] \subset f[B]$.
This is necessarily true as for the same reason as in (b).

$\Diamond$

(d) $f[A - B] = f[A] - f[B]$
This is not necessarily true. Let's take the same counterexample as in (a) i.e. $f(x) = \sin(x)$ and let's define $A = [0, 2\pi]$, $B = [0, \frac{\pi}{4}]$, then $f[A] = [-1, 1]$ and $f[B] = [0, \frac{\sqrt{2}}{2}]$ and $f[A] - f[B] = [-1, 0) \cup (\frac{\sqrt{2}}{2}, 1]$ while $f[A - B] = [-1, 1]$.

$\Diamond$

(e) $f[A \cup B] = f[A] \cup f[B]$
This is true.
Suppose first that $A \cap B = \emptyset$ and take $x \in A$, then $f(x) \in f[A]$ and $x \notin f[B]$ giving $f(x) \in f[A] \cup f[B]$. On the other hand it is obvious that if $A \cap B \neq \emptyset$ then $f(x) \in f[A]$ and-or $f(x) inf[B]$ giving $f(x) \in f[A] \cup f[B]$. Hence, $f[A \cup B] \subset f[A] \cup f[B]$.
Suppose now that $f(x) \in f[A]$ this means that $x \in A$ regardless of $x \in B$ or not. So, $f[A] \cup f[B] \subset f[A \cup B]$ and with the previous we get $f[A \cup B] = f[A] \cup f[B]$.

$\Diamond$

(f) $f[A \cap B] \subset f[A] \cap f[B]$
True as if $f(x) \in f[A \cap B]$ means that $x \in A \cap B$ so x will be mapped in the image $f[A]$ and in the image $f[B]$ and thus $f[A \cap B] \subset f[A] \cap f[B]$.

$\Diamond$

(g) $f[A \cap B] = f[A] \cap f[B]$
Not true. Suppose $f(x) \in f[A] \cap f[B]$. But if f is not injective the possibility exists that for a given $x_a \in A$ and another $x_b \in B$ we have $f[x_a] = f[x_b]$ even if A and B are disjoint sets which would give $f[A \cap B] = f[\emptyset] = \emptyset$.

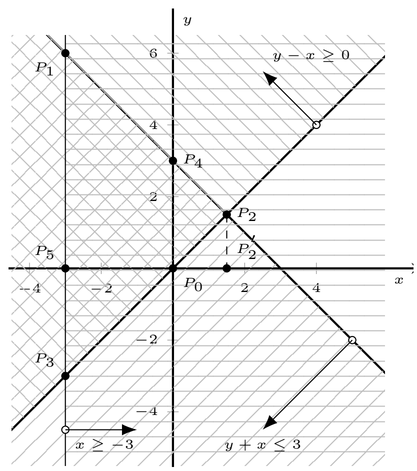
$\blacklozenge$

1.11 Relations

In Exercises 1 to 5, all relations are subsets of the plane. In each case, draw a sketch of R, and give Dom$R$, Range$R$, $R[0]$ and $R^{-1}[0]$.

1.11.1

Let $(x, y) \in R$ provided that (x, y) satisfies each of the following inequalities: $x + y \leq 3$, $y - x \geq 0$, $x \geq -3$.



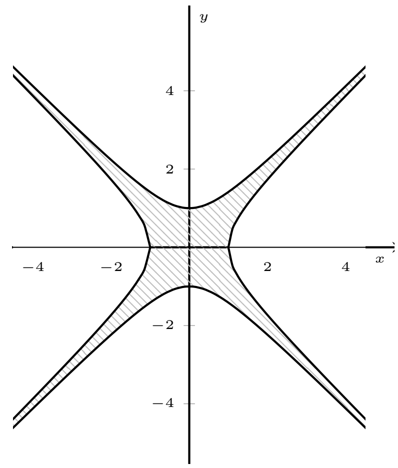
(u) $x + y \leq 3$, $y - x \geq 0$, $x \geq -3$

Dom$R$ = segment $[P_5, P_2']$
Range$R$ = segment $[P_3, P_1]$
$R[0]$: = segment $[P_0, P_4]$
$R^{-1}[0]$: = segment $[P_5, P_0]$

♦

1.11.2

Let R be the set of all (x, y) that satisfy $x^2 - y^2 \leq 1$ and $y^2 - x^2 \leq 1$.



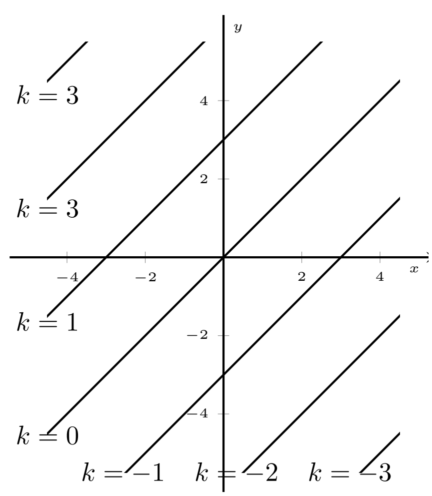
(v) $x^2 - y^2 \leq 1$ and $y^2 - x^2 \leq 1$

$\mathbf{Dom}R = (-\infty, +\infty)$
$\mathbf{Range}R = (-\infty, +\infty)$
$R[0] := [-1, 1]$
$R^{-1}[0] := [-1, 1]$

♦

1.11.3

Let R be the set of all (x, y) such that $x - y$ is a multiple of 3.



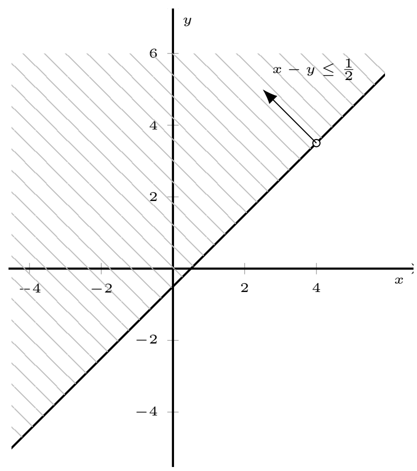
(w) The relation $\{(x, y) : y = x - 3k, k \in \mathbb{Z}$

$\mathbf{Dom}R = (-\infty, +\infty)$
$\mathbf{Range}R = (-\infty, +\infty)$
$R[0] := \{y : y = 3k, k \in \mathbb{Z}\}$
$R^{-1}[0] := \{x : x = 3k, k \in \mathbb{Z}\}$

♦

1.11.4

Let R be a subset of the plane such that $(x, y) \in R$ provided that $x - y \leq \frac{1}{2}$



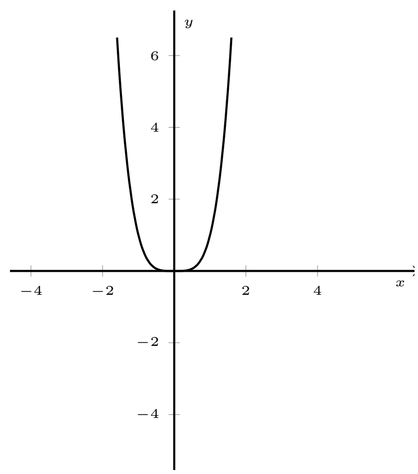
(x) The relation $x - y \leq \frac{1}{2}$

$\mathbf{Dom}R = (-\infty, +\infty)$
$\mathbf{Range}R = (-\infty, +\infty)$
$R[0] := [-\frac{1}{2}, \infty)$
$R^{-1}[0] := (-\infty, \frac{1}{2}]$

♦

1.11.5

Let R be a subset of the plane such that $(x, y) \in R$ provided that $y = x^4$



(y) The relation $y = x^4$

$\mathbf{Dom}R = (-\infty, +\infty)$
$\mathbf{Range}R = [0, +\infty)$
$R[0] := \{0\}$
$R^{-1}[0] := \{0\}$

♦

1.11.6

Let $R = \{(x, y) : x \geq 0,\ x^2 + y^2 = 26\}$. Find $R[0]$, $R[5]$, and $R[I]$, where $I = \{r : 0 \leq r \leq 1\}$; $R^{-1}[J]$ where $J = \{r : -1 \leq r \leq 1\}$.

$R[0] := \{-\sqrt{26},\ \sqrt{26}\}$
$R[5] := \{-1,\ 1\}$
$R[I] := [-\sqrt{26}, -5] \cup [5, \sqrt{26}]$
$R^{-1}[J] := [5, \sqrt{26}]$

♦

1.11.7

Let $R = \{(x, y) : x \text{ is real and } y = x(x-1)(x-2)\}$. Find $R[0]$, $R[1]$, $R[2]$, $R^{-1}[0]$ and $R[I]$, where $I = \{x : 0 \leq x \leq 2\}$.

$R[0] := \{0\}$
$R[1] := \{0\}$
$R[2] := \{0\}$
$R^{-1}[0] := \{0, 1, 2\}$
$R[I] := [-1, 2]$

♦

1.11.8

Let R be a relation between sets X and Y, and suppose that A and B are subsets of X. In each of the following, tell wether the statement is necessarily true and give a justification of your answer.

$$\begin{array}{ll} (a) & R[A \cap B] = R[A] \cap R[B]. \\ (b) & R[A \cap B] \subset R[A] \cap R[B]. \\ (c) & R[A \cap B] \supset R[A] \cap R[B]. \end{array}$$

(a) $R[A \cap B] = R[A] \cap R[B]$
This is not necessarily true. Take for example a non injective function as the relation R with $A \cap B = \emptyset$. This means that $R[A \cap B] = \emptyset$ but the relation being a non injective function it also possible that $R[A] \cap R[B] \neq \emptyset$. So, $R[A \cap B] \not\supset R[A] \cap R[B]$ and we can't have $R[A \cap B] = R[A] \cap R[B]$.

◊

(b) $R[A \cap B] \subset R[A] \cap R[B]$
This is necessarily true. Take $(x, y) : x \in A \cap B, y = R(x)$. Then we have obviously $y \in R[A]$ and also $y \in R[A \cap B]$ but as $x \in B$ (because $x \in A \cap B$) we have also $y \in R[B]$. So $y \in R[A]$ and $y \in R[B]$ and thus $y \in R[A] \cap R[B]$ giving $R[A \cap B] \subset R[A] \cap R[B]$.

◊

(c) $R[A \cap B] \supset R[A] \cap R[B]$
See (a).

♦

1.11.9

Let $\mathbb{Z}$ be the set of all integers. For each m and $n \in \mathbb{Z}$, let us write mRn if and only $m-n$ is an even integer. Thus this relation R is the set $\{(m,n) : m-n = 2k,\ k \in \mathbb{Z}\}$. Find $R[1]$ and $R[2]$. How many distinct sets of the form $R[i]$ are there?

$R[0] := \{n : n = 1-2k,\ k \in \mathbb{Z}\}$ i.e. the set of all odd integers.
$R[1] := \{n : n = 2-2k,\ k \in \mathbb{Z}\} \Leftrightarrow \{n : n = 2k,\ k \in \mathbb{Z}\}$ i.e. the set of all even integers.
There are 2 distinct sets in total.

♦

1.11.10

Let R be the relation defined as follows: For each ordered pair of integers (m,n), let mRn if and only $m-n$ is an integral multiple of 5 (including negative multiples of 5). Find $R[1]$, $R[2]$, and $R[6]$. How many distinct sets of the form $R[i]$ are there? Find $R^{-1}[1]$ and $R^{-1}[2]$. Is $R^{-1}[i] = R[i]$ for each i? For this relation R, if iRj and jRk, does it follow that iRk?

$R[1] := \{\dots, -9, -4, 1, 6, 11, \dots\}$
$R[2] := \{\dots, -8, -3, 2, 7, 12, \dots\}$
$R[6] := \{\dots, -9, -4, 1, 6, 11, \dots\}$
There are 5 distinct sets in total.
$R^{-1}[1] := \{\dots, -9, -4, 1, 6, 11, \dots\} = R[1]$
$R^{-1}[2] := \{\dots, -8, -3, 2, 7, 12, \dots\} = R[2]$
$R^{-1}[i] = R[i]$ for each i as the relation $n = m - 5k, \forall k \in \mathbb{Z}$ can be written as $m = n - 5p, \forall p \in \mathbb{Z}$. So the sets $R^{-1}[i]$ and $R[i]$ are not distinguishable.
If iRj and jRk, does it follow that iRk? Yes, as the composed relation $(jRk)\circ(iRj)$ has the relation $k = i - 5(p+q), p, q \in \mathbb{Z}$ and as $p+q \in \mathbb{Z}$ we can rewrite the relation $(jRk)\circ(iRj)$ as $j = i - 5p, p \in \mathbb{Z}$.

♦

1.12 Set inclusions for image and inverse image sets

1.12.1

Prove that
12.5 Suppose that R is a relation between X and Y . Then, if $\{A_\alpha : \alpha \in \Lambda\}$ is a non-empty collection of subsets of X, the following hold:

$$\textbf{12.5(a)} \quad R\left[\bigcup\{A_\alpha : \alpha \in \Lambda\}\right] = \bigcup\{R[A_\alpha] : \alpha \in \Lambda\}.$$
$$\textbf{12.5(b)} \quad R\left[\bigcap\{A_\alpha : \alpha \in \Lambda\}\right] \subset \bigcap\{R[A_\alpha] : \alpha \in \Lambda\}.$$

12.5(a) $R\left[\bigcup\{A_\alpha : \alpha \in \Lambda\}\right] = \bigcup\{R[A_\alpha] : \alpha \in \Lambda\}$.
Be $y \in R\left[\bigcup\{A_\alpha : \alpha \in \Lambda\}\right]$, then there must be an x that is an element of at least one of the A_α and hence y must be in $R[A_\alpha]$, so y will also be in $\bigcup\{R[A_\alpha] : \alpha \in \Lambda\}$ and thus $R\left[\bigcup\{A_\alpha : \alpha \in \Lambda\}\right] \subset \bigcup\{R[A_\alpha] : \alpha \in \Lambda\}$.
Suppose now that $yin \bigcup\{R[A_\alpha] : \alpha \in \Lambda\}$. Then y must be an element of at least one of the $R[A_\alpha]$ and hence there mus be an x that is in a set A_α, so x will also be in $\bigcup\{A_\alpha : \alpha \in \Lambda\}$ and thus $\bigcup\{R[A_\alpha] : \alpha \in \Lambda\} \subset R\left[\bigcup\{A_\alpha : \alpha \in \Lambda\}\right]$.
From this and the previous conclusion follows $R\left[\bigcup\{A_\alpha : \alpha \in \Lambda\}\right] = \bigcup\{R[A_\alpha] : \alpha \in \Lambda\}$.

$\Diamond$

12.5(b) $R\left[\bigcap\{A_\alpha : \alpha \in \Lambda\}\right] \subset \bigcap\{R[A_\alpha] : \alpha \in \Lambda\}$.

We prove first that $R\left[\bigcap\{A_\alpha : \alpha \in \Lambda\}\right] \subset \bigcap\{R[A_\alpha] : \alpha \in \Lambda\}$
Take $y \in R\left[\bigcap\{A_\alpha : \alpha \in \Lambda\}\right]$. Then there must be an $x \in A_\alpha, \forall\alpha \in \Lambda$ and thus $y \in R[A_\alpha], \forall\alpha \in \Lambda$ giving $y \in \bigcap\{R[A_\alpha] : \alpha \in \Lambda\}$ and thus $R\left[\bigcap\{A_\alpha : \alpha \in \Lambda\}\right] \subset \bigcap\{R[A_\alpha] : \alpha \in \Lambda\}$

We prove now that $R\left[\bigcap\{A_\alpha : \alpha \in \Lambda\}\right] \not\supset \bigcap\{R[A_\alpha] : \alpha \in \Lambda\}$
Be a non injective function as the relation R with $\bigcap\{R[A_\alpha] : \alpha \in \Lambda\} \neq \emptyset$. As R is a non injective function, we can have a $x \in \bigcap\{R[A_\alpha] : \alpha \in \Lambda\}$ but with $\bigcap\{A_\alpha : \alpha \in \Lambda\} = \emptyset$, which means that x can't be an element of $\bigcap\{A_\alpha : \alpha \in \Lambda\}$ and thus $R\left[\bigcap\{A_\alpha : \alpha \in \Lambda\}\right] \not\supset \bigcap\{R[A_\alpha] : \alpha \in \Lambda\}$.
(*Take for example the relation defined by* $R = \{(n, 1) : n \in \mathbb{N}\}$ *and the subsets of* $\mathbb{N}$, $A_\alpha = \{\alpha\} : \alpha \in \mathbb{N}\}$. *We have* $\bigcap\{R[A_\alpha] : \alpha \in \Lambda\} = \{1\}$ *with* $\bigcap\{A_\alpha : \alpha \in \Lambda\} = \emptyset$.)

$\blacklozenge$

1.12.2

Prove that

12.6 Let $f : X \to Y$ be a function. Let $\{A_\delta : \delta \in \Delta\}$ and $\{B_\lambda : \lambda \in \Lambda\}$ be non empty collections of subsets of X and Y respectively. Then,

12.6(a) $f\left[\bigcup\{A_\delta : \delta \in \Delta\}\right] = \bigcup\{f[A_\delta] : \delta \in \Delta\}$.
12.6(b) $f\left[\bigcap\{A_\delta : \delta \in \Delta\}\right] \subset \bigcap\{f[A_\delta] : \delta \in \Delta\}$.
12.6(c) $f^{-1}\left[\bigcup\{B_\lambda : \lambda \in \Lambda\}\right] = \bigcup\{f^{-1}[B_\lambda] : \lambda \in \Lambda\}$.
$f^{-1}\left[\bigcap\{B_\lambda : \lambda \in \Lambda\}\right] = \bigcap\{f^{-1}[B_\lambda] : \lambda \in \Lambda\}$.

12.6(a) $f\left[\bigcup\{A_\delta : \delta \in \Delta\}\right] = \bigcup\{f[A_\delta] : \delta \in \Delta\}$.

This a direct consequence of **12.6** with $R = f$.

♢

12.6(b) $f\left[\bigcap\{A_\delta : \delta \in \Delta\}\right] \subset \bigcap\{f[A_\delta] : \delta \in \Delta\}$.

This a direct consequence of **12.6** with $R = f$.

♢

12.6(c) $f^{-1}\left[\bigcup\{B_\lambda : \lambda \in \Lambda\}\right] = \bigcup\{f^{-1}[B_\lambda] : \lambda \in \Lambda\}$.

As f^{-1} is a relation and by **12.6** with $R = f^{-1}$ we get the asked identity.

♢

12.6(c$'$) $f^{-1}\left[\bigcap\{B_\lambda : \lambda \in \Lambda\}\right] = \bigcap\{f^{-1}[B_\lambda] : \lambda \in \Lambda\}$.

As f^{-1} is a relation and by **12.6** with $R = f^{-1}$ we get $f^{-1}\left[\bigcap\{B_\lambda : \lambda \in \Lambda\}\right] \subset \bigcap\{f^{-1}[B_\lambda] : \lambda \in \Lambda\}$.
We prove now that $f^{-1}\left[\bigcap\{B_\lambda : \lambda \in \Lambda\}\right] \supset \bigcap\{f^{-1}[B_\lambda] : \lambda \in \Lambda\}$.
Suppose $x \in \bigcap\{f^{-1}[B_\lambda] : \lambda \in \Lambda\}$ then $x \in f^{-1}[B_\lambda] : \forall\lambda \in \Lambda$. This means that there must be an unique $y = f(x)$ (f being a function) for which yields $y \in B_\lambda : \forall\lambda \in \Lambda$. Hence y must be in $\bigcap\{B_\lambda : \lambda \in \Lambda\}$ and thus $x \in f^{-1}\left[\bigcap\{B_\lambda : \lambda \in \Lambda\}\right]$ giving $f^{-1}\left[\bigcap\{B_\lambda : \lambda \in \Lambda\}\right] \supset \bigcap\{f^{-1}[B_\lambda] : \lambda \in \Lambda\}$.

♦

1.12.3

Prove that
12.7 Let $f : X \to Y$ be a function. Then, each of the following holds

$$\begin{array}{ll} \mathbf{12.7(a)} & \forall x \in X,\, x \in f^{-1}[f[x]]. \\ \mathbf{12.7(b)} & \forall A \subset X,\, A \subset f^{-1}[f[A]]. \\ \mathbf{12.7(c)} & \forall y \in \text{Range} f,\, f[f^{-1}[y]] = \{y\}. \end{array}$$

12.7(a) $\forall x \in X$, $x \in f^{-1}[f[x]]$.
Be $y = f(x)$, then obviously there is at least one x (there could be more if f is not injective), so that $x = f^{-1}[y]$. Hence, $\forall x \in X$, $x \in f^{-1}[f[x]]$.

♢

12.7(b) $A \subset X$, $A \subset f^{-1}[f[A]]$.
This is a consequence of the previous statement but with the remark that we could have (for a non injective function) a $x \in B$ with $A \cap B = \emptyset$ for which we have $f(x) \in f[A]$. So A is not always equal to $f^{-1}[f[A]]$ and get $A \subset X$, $A \subset f^{-1}[f[A]]$.

♢

12.7(c) $\forall y \in \text{Range} f$, $f[f^{-1}[y]] = \{y\}$.
This a direct consequence of f being a function. Indeed suppose for a given y we have the set $A = f^{-1}[\{y\}]$, so this set will contain all x as element which f maps (uniquely, f being a function) to y. So $f[A] = f[f^{-1}(y)] = \{y\}$.

♦

1.12.4

Prove that
Suppose that $f : X \to Y$ is a function and A and B are subsets of X. Suppose also that C and D are subsets of Y. For each of the following, determine whether the statement is necessarily true. In any case for which the statement is not necessarily true, determine whether it is under any of the following conditions: $f : X \to Y$ is a surjection,$f : X \to Y$ is a injection,$f : X \to Y$ is a bijection.

$$\begin{array}{ll} \mathbf{(a)} & f[A - B] = f[A] - f[B]. \\ \mathbf{(b)} & f^{-1}[D - C] = f^{-1}[D] - f^{-1}[C]. \\ \mathbf{(c)} & f^{-1}[f[A]] = A. \\ \mathbf{(b)} & f[f^{-1}[C]] = C. \end{array}$$

(a) $f[A - B] = f[A] - f[B]$.
This is not necessarily True.

Suppose, $x \in A$ but not in B and $y = f(x)$, so $y \in f[A - B]$ but if f is not a surjection then it is possible that a $x^{'} \in B$ exists which is mapped to y, meaning that y will not be an element of $f[A] - f[B]$ i.e. $y \notin f[A] - f[B]$ and thus $f[A - B] \not\subset f[A] - f[B]$ meaning that not always $f[A - B] = f[A] - f[B]$. So this identity can only be true if f is a surjection or a bijection as a bijection has to be an injection. Of course $f : X \to Y$ is a surjection, is not a sufficient condition for the identity to be true as a surjection is not necessarily an injection.

$\Diamond$

(**b**) $f^{-1}[D - C] = f^{-1}[D] - f^{-1}[C]$.

This is True if f is a surjection (or by extension a bijection).

Suppose, $x \in f^{-1}[D - C]$, so there is a $y \in D - C$ for which $x = f^{-1}(y)$. Also, $y \in D$ but not in C. Can there be $y^{'} \in C, \notin D$ for which $y^{'} = f(x)$? Obviously not, as f is a function meaning that $y^{'} = f(x)$ and $y = f(x) \Rightarrow y^{'} = y$. This means that x can't be an element of $f^{-1}[C]$ and thus that $x \in f^{-1}[D] - f^{-1}[C]$. Hence, $f^{-1}[D - C] \subset f^{-1}[D] - f^{-1}[C]$.

Suppose now that $x \in f^{-1}[D] - f^{-1}[C]$, so there is no $x \in f^{-1}[C]$ for which $y = f(x), y \in C$. This means that $y \notin C$ but y must be in D. i.e. $y \in D - C$ and thus $x \in f^{-1}[D - C]$ or $f^{-1}[D] - f^{-1}[C] \subset f^{-1}[D - C]$, leading to the identity.

Remark that the reasoning deployed implies that f is a surjection as if $y \in D - C$ has no inverse image in X, this would mean that $f^{-1}[D - C] = \emptyset$ and thus no x would exist for the given identity.

$\Diamond$

c) $f^{-1}[f[A]] = A$.

This is not necessarily True.

Be $y \in f[A]$, if f is not an injection then it is possible that there exist a $x^{'} \in B \not\subset A$ so that $f(x^{'}) = y$, so $x^{'}$ will be an element of $f^{-1}[f[A]]$ and as $x^{'} \notin A$ the identity can't be true. So, f needs to be an injection (and by extension a bijection) for the identity to be true.

$\Diamond$

(**d**) $f[f^{-1}[C]] = C$.

This is True if f is a surjection (or by extension a bijection).

Be $y \in C$ and $x \in f^{-1}[C]$, as f is a function then $f(x)$ will be in C and the set $f^{-1}[C]$ will contain all x for which $f(x) \in C$. But note that C may contain elements which are not mapped by f. In that case $f[f^{-1}[C]] \not\subset C$. So f needs to be a surjection (or by extension a bijection).

On the other hand, suppose $y \in C$, then $f^{-1}[C]$ will contain all $x \in X$ which are mapped to C and f being a function we will have $f[f^{-1}[C]] \subset C$.

Conclusion, the identity is true if f is a surjection (or by extension a bijection).

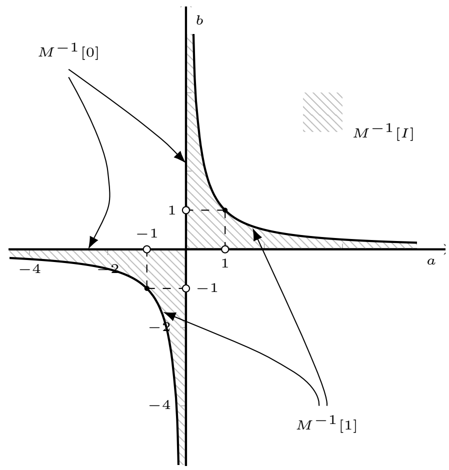
To illustrate this, take A as the subset $\mathbb{N} \subset \mathbb{R}$ and C as the subset of $\mathbb{R}$ with the even natural numbers as elements. Define now the function $f = \{(n, 4n) : n \in A\}$. Obviously $f[A] \neq C$ as $f[A] = \{4, 8, 12, \dots\} \neq \{2, 4, 6, 8, \dots\}$. Then as $f^{-1}[C] = A$, we have $f[f^{-1}[C]] \neq C$

$\blacklozenge$

1.12.5

Let $M : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the map from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$ defined as follows: For each $(a, b) \in \mathbb{R} \times \mathbb{R}$, let $M((a, b)) = ab$. Is M a map from $\mathbb{R} \times \mathbb{R}$ onto $\mathbb{R}$? Representing $\mathbb{R} \times \mathbb{R}$ s a plane, draw a sketch for each of the following sets: $M^{-1}[0]$, $M^{-1}[1]$, $M^{-1}[I]$, where I is the closed interval $[0, 1]$.

Yes, M is a map from $\mathbb{R} \times \mathbb{R}$ onto $\mathbb{R}$ as every $x \in \mathbb{R}$ can be expressed as the product of two real numbers.



(z) Inverse image of $M^{-1}(0)$, $M^{-1}(1)$, $M^{-1}(I)$

♦

1.12.6

Examin carefully the content of Theorem **12.6** and your answer to Exercise $4(a)$ and (b). Which seems to have a nicer behaviour on collections of sets, f or f^{-1}?

Putting aside the notion of 'nicer', we still could put forward that f^{-1} requires less restrictions in order to have certain identities.
Take first $\mathbf{12.6}(b)$ $\quad f\left[\bigcap\{A_\delta : \delta \in \Delta\}\right] \subset \bigcap\{f[A_\delta] : \delta \in \Delta\}$ compared to $\mathbf{12.6}(c^{'})$ $\quad f^{-1}\left[\bigcap\{B_\lambda : \lambda \in \Lambda\}\right] = \bigcap\{f^{-1}[B_\lambda] : \lambda \in \Lambda\}$. f^{-1} requires no special condition (except for f being a function) in order to have an equality for the intersection of the sets in the collection.
Moreover in Exercise 4, for having the identity $4(a)$ $\quad f[A - B] = f[A] - f[B]$, we need f to be at least an injection while for $4(b)$ $\quad ff^{-1}[D - C] = f^{-1}[D] - f^{-1}[C]$ we "only need af to be a surjection which can be achieved by restricting the target Y to $D \cup C$.

♦