Undergraduate Topology

Robert H. Kasriel (Dover Publication)

Solutions to exercises

Part I

Chapters I to IV

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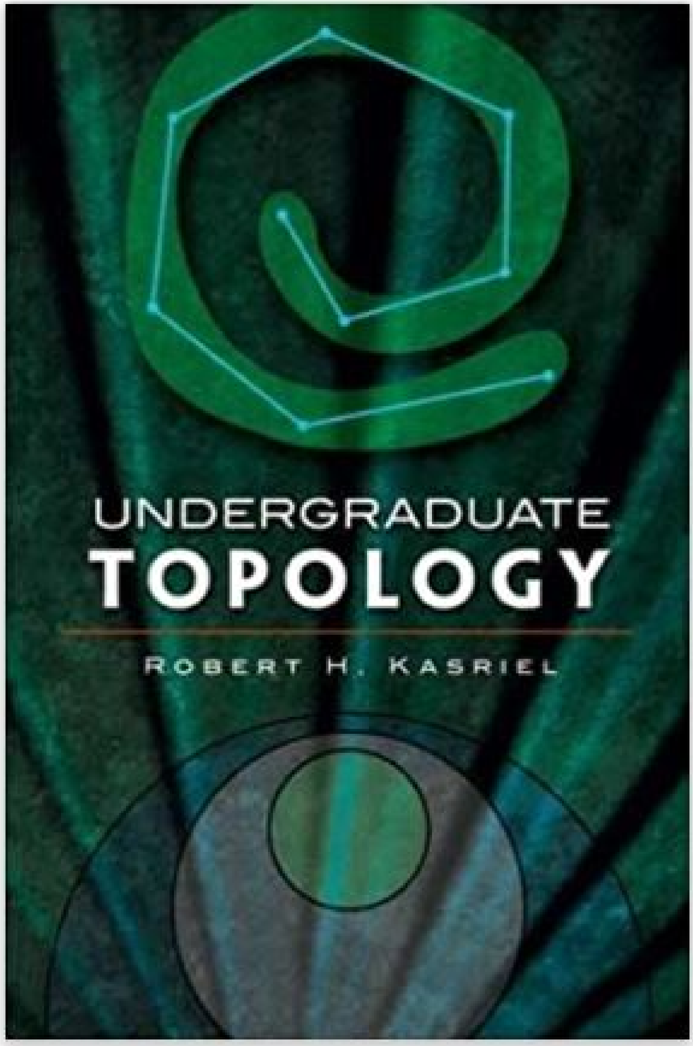


Figure 1

Remarks and warnings

You're welcome to use these notes, but they may contain errors, so proceed with caution : I graduated in 1979, went straight in the industry (where I didn't have to use fancy maths), and picked mathematics and physics again after I retired, so my mathematics got rusty for sure. If you do find an error, typo's , I'd be happy to receive bug reports, suggestions, and the like, through Github.

Contents

List of Figures

Sets, Functions, and Relations

1.1 Sets and Membership

1.1.1

List explicitly the elements of the set

$$\{x : x < 0 \text{ and } (x-1)(x+2)(x+3) = 0\}$$

$$\{-3, -2\}$$

♦

1.1.2

List the elements of the set

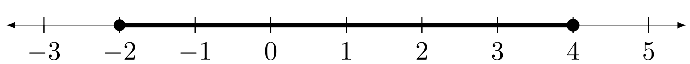
$$\{x : 3x - 1 \text{ is a multiple of } 3\}$$

$$\{x : x = k + \frac{1}{3}, k \in \mathbb{Z}\}$$

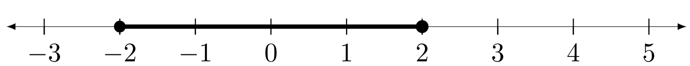
♦

1.1.3

Sketch on a number line each of the following sets.
(a) $\{x : |x-1| \leq 3\}$
(b) $\{x : |x-1| \leq 3 \text{ and } |x| \leq 2\}$
(c) $\{x : |x-1| \leq 3 \text{ or } |x| \leq 2\}$



(a)



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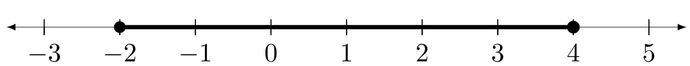


(c)

♦

1.2 Some remarks on the use of the connectives *and, or, implies*

1.2.1

Demonstrate by means of a table showing truth values that the following is a true statement for any choice of p and q. Thus show that it is a tautology.

$$(\neg q \Rightarrow \neg p) \Rightarrow (p \Rightarrow q)$$

p	q	$\neg q$	$\neg p$	$\neg q \Rightarrow \neg p$	$p \Rightarrow q$	$(\neg q \Rightarrow \neg p) \Rightarrow (p \Rightarrow q)$
T	T	F	F	T	T	T
T	F	T	F	F	F	T
F	T	F	T	T	T	T
F	F	T	T	T	T	T

♦

1.2.2

Show by means of a truth table that the statement

$$((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$$

is a tautology.

p	q	r	$p \Rightarrow q$	$q \Rightarrow r$	$(p \Rightarrow q) \wedge (q \Rightarrow r))$	$p \Rightarrow r$	$((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

♦

1.2.3

Show by means of a truth table that

$$(p \wedge q) \Rightarrow (p \vee q)$$

is a tautology.

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \Rightarrow (p \vee q)$
T	T	T	T	T
T	F	F	F	T
F	T	F	T	T
F	F	F	F	T

♦

1.2.4

Suppose that p and q are statements such that $(p \wedge q)$ is a false statement. Does it follow that the statement

$$(p \text{ is false}) \vee (q \text{ is false})$$

is a true statement?

p	q	$p \wedge q$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	F	F	F	T	T
F	T	F	T	F	T
F	F	F	T	T	T

The answer is Yes.

♦

1.2.5

Negate the following statement: *If two angles of a triangle have equal measure, then the length of two sides of that triangle are equal.*

First we note that $\neg(p \Rightarrow q) \Leftrightarrow (p \wedge \neg q)$. Indeed,

p	q	$p \Rightarrow q$	$\neg(p \Rightarrow q)$	$\neg q$	$p \wedge \neg q$	$\neg(p \Rightarrow q) \Leftrightarrow (p \wedge \neg q)$
T	T	T	F	F	F	T
T	F	F	T	T	T	T
F	T	T	F	F	F	T
F	F	T	F	T	F	T

Putting p as *two angles of a triangle have equal measure* and $\neg q$ as *no two sides of that triangle have equal length* we get the true 'false' statement:
Two angles of a triangle have equal measure $\wedge$ no two sides of that triangle have equal length.

♦

1.2.6

Write the contrapositive of the statement in Exercise 5.

The contrapositive of $p \Rightarrow q$ is $\neg q \Rightarrow \neg p$. Putting $\neg p$ as *no two angles of a triangle have equal measure* and $\neg q$ as *no two sides of that triangle have equal length* we get
If no two sides of that triangle have equal length then no two angles of a triangle have equal measure.

♦

1.2.7

Write the converse of the statement in Exercise 5.

The converse of $p \Rightarrow q$ is $q \Rightarrow p$, giving
If two sides of a triangle have equal length then two angles of a that triangle have equal measure.

♦

1.2.8

Write the contrapositive of the following statement
If a person belongs to Committee A, then he must be a member of Committee B and he must be a member of Committee C.

Lets put

$$\begin{aligned} p &\equiv \text{a person belongs to Committee A} \\ q &\equiv \text{a person belongs to Committee B} \\ r &\equiv \text{a person belongs to Committee C} \end{aligned}$$

then the given statement translates as

$$p \Rightarrow (q \wedge r)$$

and the contrapositive

$$\neg(q \wedge r) \Rightarrow \neg p$$

This last statement is equivalent with

$$(\neg q \vee \neg r) \Rightarrow \neg p$$

or in plain text:

If a person does not belong to Committee B or C , then he is not a member of Committee A.

♦

1.2.9

Write the contrapositive of the following statement

If $x \in A$ and $x \in B$, then $x \in C$

Lets put

$$p \equiv x \in A$$
$$q \equiv x \in B$$
$$r \equiv x \in C$$

then the given statement translates as

$$p \wedge r \Rightarrow r$$

and the contrapositive

$$\neg(r) \Rightarrow \neg(p \wedge q)$$

This last statement is equivalent with

$$\neg(r) \Rightarrow (\neg p \vee \neg q)$$

i.e:

$$x \notin C \Rightarrow (x \notin A \vee x \notin B)$$

♦

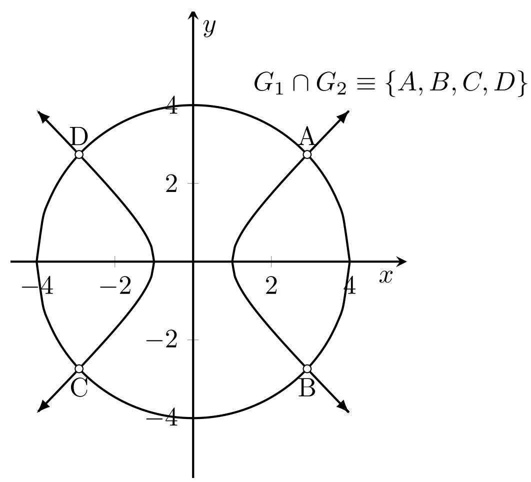
1.3 Subsets

No exercises!

1.4 Union and Intersection of sets

1.4.1

Let G_1 be the graph of the equation $x^2 + y^2 = 16$, and let G_2 be the graph of the equation $x^2 - y^2 = 1$. Sketch the sets $G_1 \cup G_2$ and $G_1 \cap G_2$.



$G_1 \cup G_2$ contains all the points defined by the graphs G_1 and G_2. $G_1 \cap G_2 \equiv \{A, B, C, D\}$ contains the 4 points at the intersection of the two graphs.

♦

1.4.2

We define the sets A, B, C as follows: $A = \{(x, y) : x^2 + y^2 \leq 9\}$, $B = \{(x, y) : x + y \geq 3\}$, $C = \{(x, y) : x \geq 0\}$.
Draw sketches of each of the following sets:

(a) $A \cup (B \cup C)$
(b) $A \cap (B \cup C)$
(c) $(A \cap B) \cup (A \cap C)$
(d) $(A \cup B) \cup C$
(e) $A \cup (B \cap C)$
(f) $(A \cup B) \cap (A \cup C)$

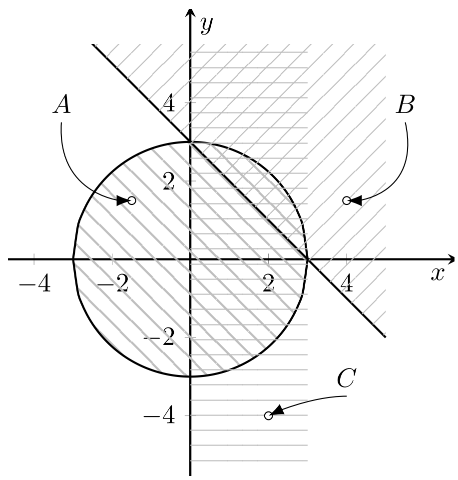
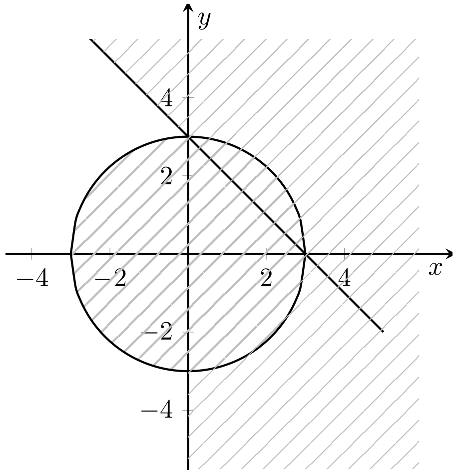
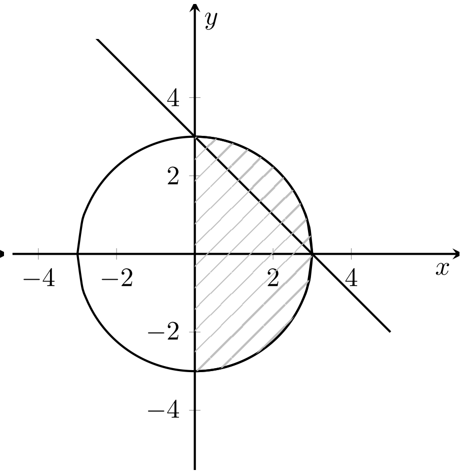


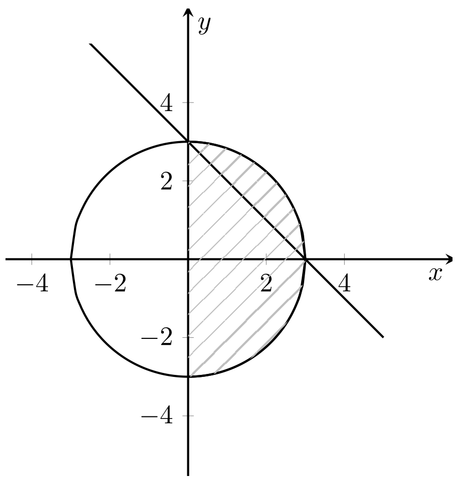
Figure 1.1: The 3 sets A, B, C



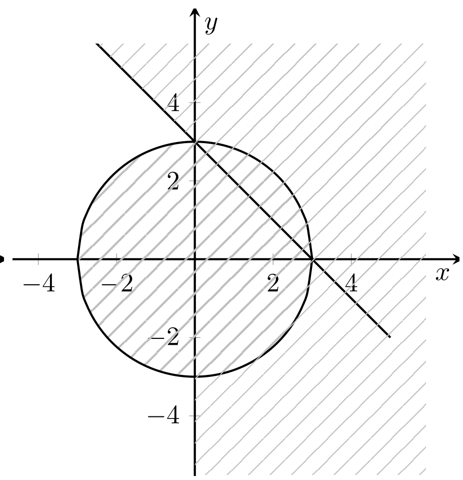
(a) $A \cup (B \cup C)$



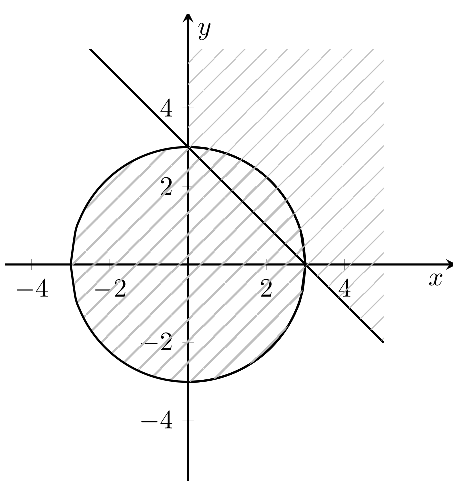
(b) $A \cap (B \cup C)$



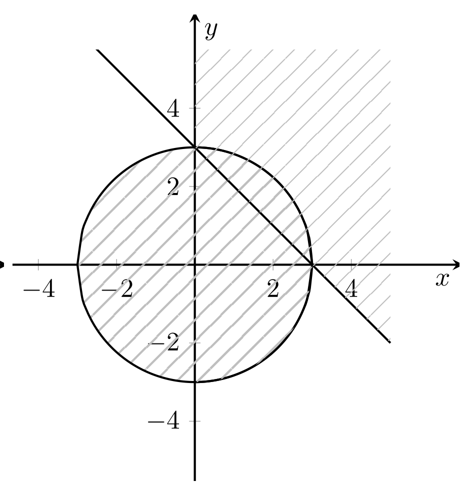
(c) $(A \cap B) \cup (A \cap C)$



(d) $(A \cup B) \cup C$



(e) $A \cup (B \cap C)$



(f) $(A \cup B) \cap (A \cup C)$

♦

1.4.3

Let A, B, C as follows: $A = \{(x, y) : x+y \leq 5\}$, $B = \{(x, y) : x+y \geq 3\}$, $C = \{(x, y) : x \geq 3\}$, and $D = \{(x, y) : y \geq 3\}$.
Draw a sketch for each of the following sets:

$$\begin{array}{ll} (a) & (A \cap B) \cap C \\ (b) & [(A \cap B) \cap C] \cap D \end{array}$$

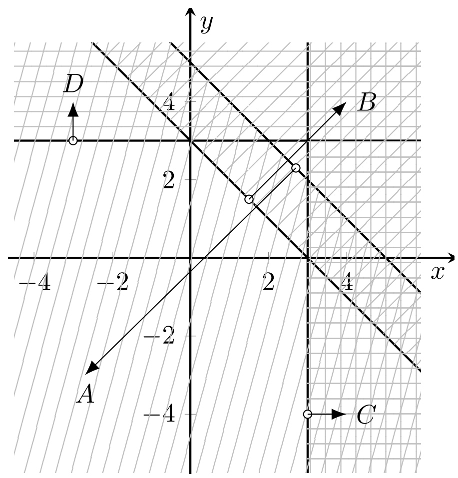
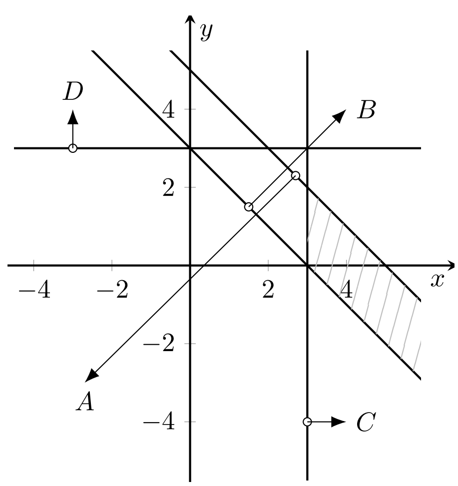
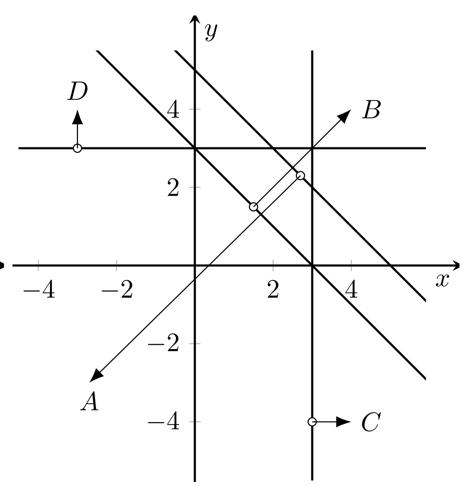


Figure 1.2: The 4 sets A, B, C, D



(a) $(A \cap B) \cap C$



(b) $((A \cap B) \cap C) \cap D = \emptyset$

♦

1.5 Complementation

1.5.1

Sketch each of the following sets: (the sets A, B, C are defined as in exercise 3page 8)

$$\begin{array}{ll}
(a) & \sim(A \cap B) \\
(b) & (\sim A) \cup (B) \\
(c) & \sim(A \cup B) \\
(d) & (\sim A) \cap (B) \\
(e) & C - A \\
(f) & \sim(A \cap C) \\
(g) & (\sim A) \cup (\sim B) \\
(h) & (\sim A) \cap (A) \\
(i) & C - (A \cup B) \\
(j) & (C - A) \cap (C - B) \\
(k) & \sim(\sim A)
\end{array}$$

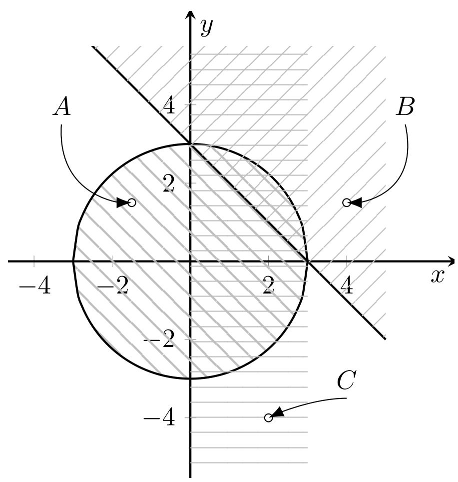
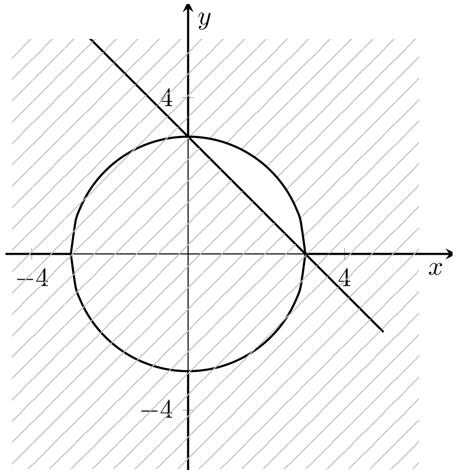
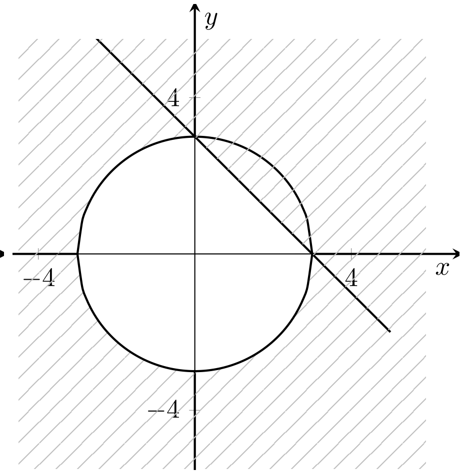


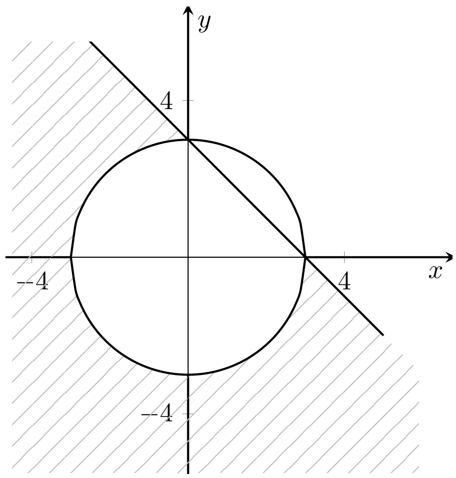
Figure 1.3: The 3 sets A, B, C



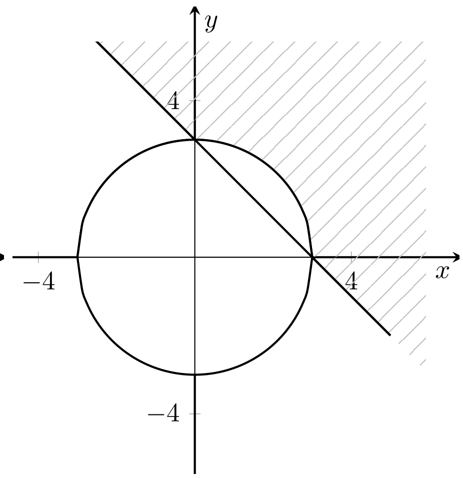
(a) $\sim (A \cap B)$



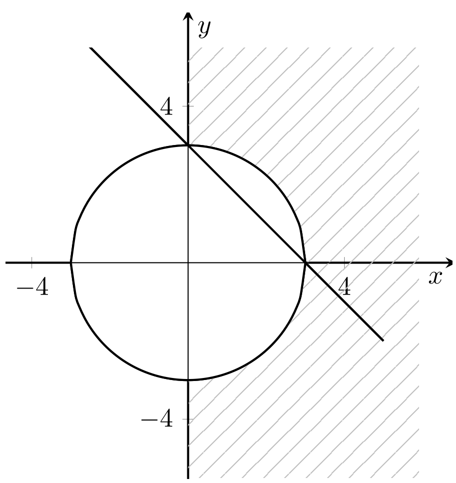
(b) $(\sim A) \cup (B)$



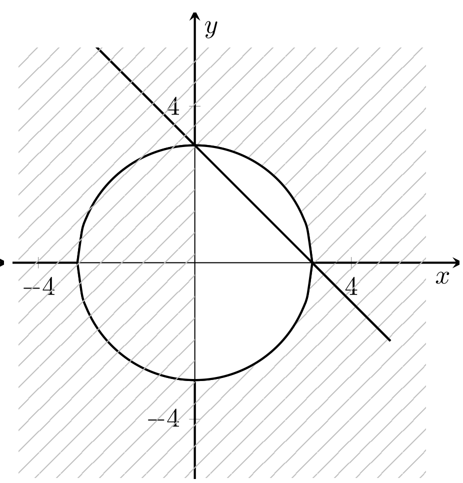
(c) $\sim (A \cup B)$



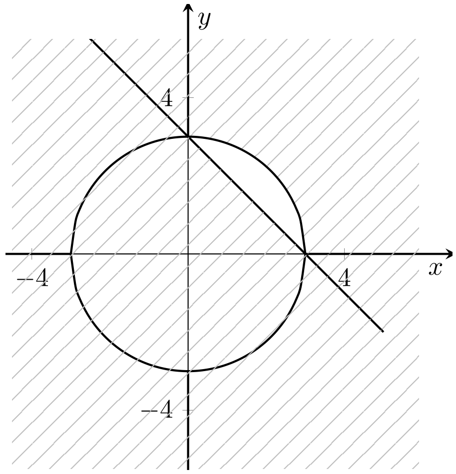
(d) $(\sim A) \cap (B)$



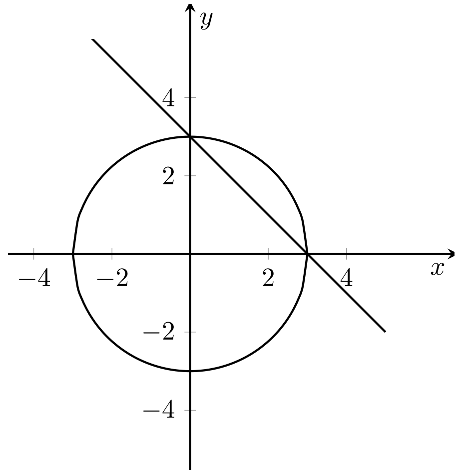
(e) $C - A$



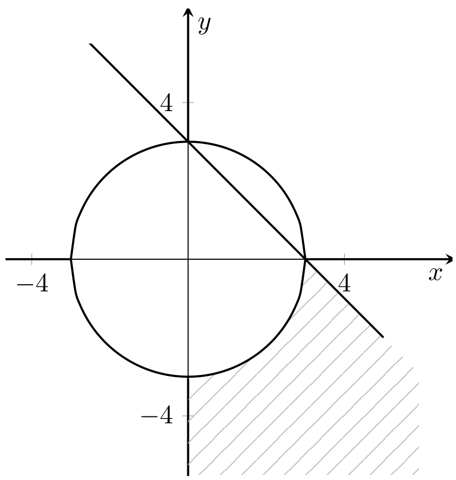
(f) $\sim (A \cap C)$



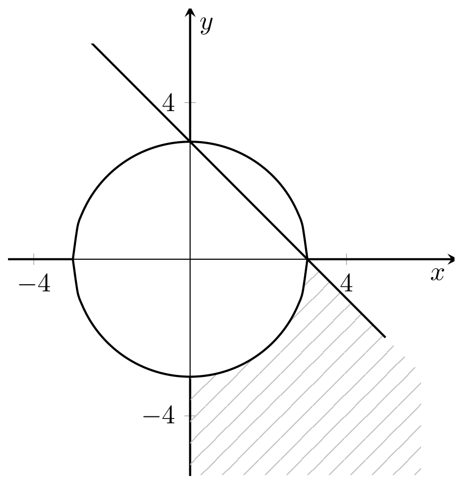
(g) $(\sim A) \cup (\sim B)$



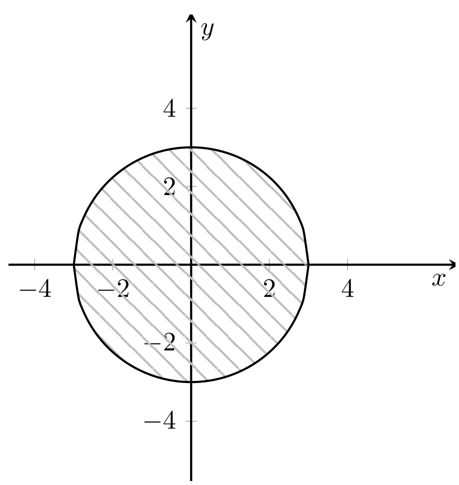
(h) $(\sim A) \cap (A) = \emptyset$



(i) $C - (A \cup B)$



(j) $(C - A) \cap (C - B)$



(k) $\sim (\sim A)$

1.5.2

On the basis of the sketches made in the previous exeercise, formulate a proposition about relation that exist concerning complementation, union, and intersection. Try out your conjecture on other examples. In subsequent exercises you will be asked to try to prove such conjectures.

1.4.2 (a) and (d)	$A \cup (B \cup C) = (A \cup B) \cup C)$
1.4.2 (b) and (c)	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
1.4.2(e) and (f)	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
1.5.1(a) and (g)	$\sim (A \cap B) = (\sim A) \cup (\sim B)$
1.5.1(h)	$(\sim A) \cap A = \emptyset$
1.5.1(i) and (j)	$C - (A \cup B) = (C - A) \cap (C - B)$
1.5.1(k)	$\sim (\sim A) = A$

♦

1.6 Set identities and other set relations

1.6.1

Prove that if $A \subset B$, then:

$$\begin{array}{ll} (a) & A \cap C \subset B \cap C \\ (b) & \sim B \subset \sim A \\ (c) & A \cap B = A \\ (d) & A \cup C \subset B \cup C \end{array}$$

a) $A \cap C \subset B \cap C$

Given is $x \in B$ if $x \in A$. Suppose $x \in A \cap C$, then $x \in A$ (given) and $x \in C$ but $x \in B$ (given) and as $x \in C$ follows that $x \in B \cap C$. And we conclude that $A \cap C \subset B \cap C$.

◊

b) $\sim B \subset \sim A$

Given is $x \in B$ if $x \in A$. If $x \notin B$ then $x \in \sim B$. As $A \subset B$, x will not be in A but $x \in \sim A$. So $x \in \sim B \Rightarrow \quad x \in \sim A$ and thus $\sim B \subset \sim A$.

◊

c) $A \cap B = A$

Given is $x \in B$ if $x \in A$. Suppose $x \in A \cap B$, then $x \in A$ and thus $A \cap B \subset A$. Suppose $x \in A$, then $x \in B$ as $A \subset B$ and thus $x \in A \cap B$ from which we conclude $A \subset A \cap B$.

◊

d) $A \cup C \subset B \cup C$

Given is $x \in B$ if $x \in A$. Suppose $x \in A \cup C$, then $x \in A$ or $x \in C$. But $x \in B$ (given), so $x \in B$ or $x \in C$ and thus $x \in B \cup C$, from which we conclude $A \cup C \subset B \cup C$.

♦

1.6.2

Verify that each of the following is an an identity:

$$\begin{array}{ll}
(a) & A \cup \emptyset = A \\
(b) & A \cap \emptyset = \emptyset \\
(c) & A \cap A = A \\
(d) & A \cup A = A \\
(e) & (A \cup B) \cup C = A \cup (B \cup C) \\
(f) & (A \cap B) \cap C = A \cap (B \cap C) \\
(g) & A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \\
(h) & X - (A \cup B) = (X - A) \cap (X - B) \\
(i) & A \cap \sim A = \emptyset \\
(j) & A \cup \sim A = U
\end{array}$$

a) $A \cup \emptyset = A$

This is a consequence of remark 3.3 page 7: the empty set $\emptyset$ is a subset of every set. So, $\emptyset \subset A$ giving the asked identity.

$\Diamond$

b) $A \cap \emptyset = \emptyset$

If $x \in A \cap \emptyset$ then $x \in A$ and x must also be in $\emptyset$ which is impossible by definition. So there is no element $x \in \emptyset$ which can satisfy $x \in A \cap \emptyset$ giving the proposed identity.

$\Diamond$

c) $A \cap A = A$

Suppose $x \in A \cap A$, then $x \in A$ and $x \in A$ and thus $x \in A$, giving $A \cap A \subset A$. Suppose $x \in A$, then obviously $x \in A$ and $x \in A$, giving $A \subset A \cap A$. Hence $A \cap A = A$

$\Diamond$

d) $A \cup A = A$

Suppose $x \in A \cup A$, then $x \in A$ or $x \in A$ and thus $x \in A$, giving $A \cup A \subset A$. Suppose $x \in A$, then obviously $x \in A$ or $x \in A$, giving $A \subset A \cup A$. Hence $A \cup A = A$

$\Diamond$

e) $(A \cup B) \cup C = A \cup (B \cup C)$

Suppose $x \in (A \cup B) \cup C$, then $x \in (A \cup B)$ or $x \in C$ and thus $x \in A$ or $x \in B$ or $x \in C$. So $x \in B$ or $x \in C$ can be written as $x \in (B \cup C)$. So $x \in A$ or $x \in (B \cup C)$, giving $(A \cup B) \cup C \subset A \cup (B \cup C)$. The same reasoning yields for $x \in A \cup (B \cup C)$ giving the identity.

$\Diamond$

f) $(A \cap B) \cap C = A \cap (B \cap C)$

Suppose $x \in (A \cap B) \cap C$, then $x \in (A \cup B)$ and $x \in C$ and thus $x \in A$ and $x \in B$ and $x \in C$. So $x \in B$ and $x \in C$ can be written as $x \in (B \cap C)$. So $x \in A$ and $x \in (B \cup C)$, giving $(A \cap B) \cap C \subset A \cap (B \cap C)$. The same reasoning yields for $x \in A \cap (B \cap C)$ giving the identity.

$\Diamond$

g) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Suppose $x \in A \cup (B \cap C)$, then $x \in A$ or $x \in (B \cap C)$. Take the case $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$ which implies $x \in (A \cup B) \cap (A \cup C)$, giving $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$. The other case: if $x \in B \cap C$ then $x \in B$ and $x \in C$. So, $x \in A \cup B$ and $x \in A \cup C$ giving also $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$.
On the other hand, be $x \in (A \cup B) \cap (A \cup C)$ then $x \in (A \cup B)$ and $x \in (A \cup C)$. Let's first take the case $x \in A$ then obviously $x \in A \cup (B \cap C)$ even if $x \notin B \cap C$. Alternatively, be $x \notin A$ then we must have $x \in B$ and $x \in C$ which implies $x \in B \cap C$, giving again $x \in A \cup (B \cap C)$.

$\Diamond$

h) $X - (A \cup B) = (X - A) \cap (X - B)$

Suppose $x \in X - (A \cup B)$, then $x \notin A$ and $x \notin B$ which implies $x \in X - A$ and $x \in X - B$ and thus $x \in X - A \cap X - B$ giving $X - (A \cup B) \subset (X - A) \cap (X - B)$.
The other way around. Suppose $x \in (X - A) \cap (X - B)$. Then $x \in (X - A)$ and $x \in (X - B)$ which implies $x \notin A$ and $x \notin B$ giving $x \notin A \cup B$ which in turn implies $x \in X - (A \cup B)$ giving $(X - A) \cap (X - B) \subset X - (A \cup B)$.
Conclusion: $X - (A \cup B) = (X - A) \cap (X - B)$

$\Diamond$

i) $A \cap \sim A = \emptyset$

Suppose $x \in A \cap \sim A$, then $x \in A$ and $x \notin A$ which is a contradiction, so the only element which is always an element of any set is the empty set, so $A \cap \sim A \subset \emptyset$. Suppose on the contrary that $x \in \emptyset$. This implies that x correspond to the empty set and as the empty set is an element of any set, we have $\emptyset \subset A \cap \sim A$

$\lozenge$

j) $A \cup \sim A = U$

Suppose $x \in A \cup \sim A$, then $x \in A$ or $x \notin A$. So, in any case $x \in U$ and thus $A \cup \sim A \subset U$.
On the opposite way suppose that $x \in U$. Then obviously $x \in A$ or $x \in \sim A$ and thus $U \subset A \cup \sim A$.

$\blacklozenge$

1.6.3

Prove that if $A \subset C$ and $B \subset C$, then $A \cup B \subset C$.

Given is $A \subset C$ and $B \subset C$. Take $x \in A$, then $x \in C$, so even if $x \notin B$, then $x \in A \cup B$ reduces to $x \in A$ and thus $x \in C$. The same reasoning yields for $x \in B$, giving $A \cup B \subset C$.

$\blacklozenge$

1.6.4

Prove that if $A \subset B$ and $A \subset C$, then $A \subset B \cap C$.

Given is $A \subset B$ and $A \subset C$. Take $x \in A$, then $x \in C$ and $x \in B$, which implies $x \in C \cap B$. giving indeed $A \subset B \cap C$.

$\blacklozenge$

1.7 Counterexamples

In each of the following exercises state whether the statement is necessarily true. Assume that A, B and C are subsets of a universal set U. Justify with a proof or a counterexample.

1.7.1

If $A \cup C = B \cup C$, then $A = B$

Not TRUE.

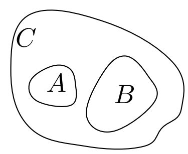


Figure 1.4: $A \cup C = B \cup C \not\Rightarrow A = B$

Be $A \subset C$ and $B \subset C$, then we have $A \cup C = B \cup C \equiv C = C$ even if $A \cap B = \emptyset$.

♦

1.7.2

$(A \cup B) - B = A$

Not TRUE.

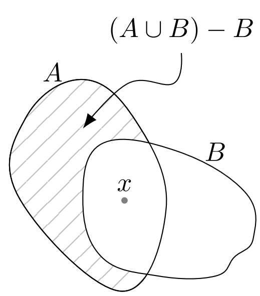


Figure 1.5: $(A \cup B) - B \neq A$

Be $A \cap B \neq \emptyset$, take $x \in A$ and $x \in B$, then x can't be $x \in (A \cup B) - B$ although it is an element of A.

♦

1.7.3

$(A - B) \cup B = A$

Not TRUE.

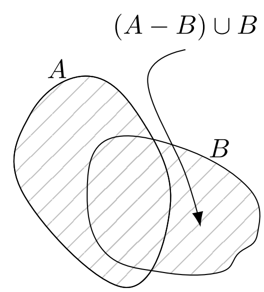


Figure 1.6: $(A - B) \cup B \neq A$

This is only true if $B \subset A$

♦

1.7.4

$\sim (A - B) = \sim (A \cap \sim B)$

TRUE.

Suppose first that A and B are disjoint, i.e. $A \cap B = \emptyset$, then $A - B = A$ and $\sim (A - B) = \sim A$. On the other hand $A \subset \sim B$, so $A \cap \sim B = A$, giving $\sim (A \cap \sim B) = \sim A$, giving indeed $\sim (A - B) = \sim (A \cap \sim B)$.

Suppose now that A and B are not disjoint, i.e. $A \cap B \neq \emptyset$. Be $x \in A - B \subset A$. This is equivalent with the statement $x \in A \wedge x \notin B$. Negating this statement: $\neg(x \in A \wedge x \notin B) \Leftrightarrow x \notin A \vee x \in B$. This give $\sim (A - B) \equiv x \notin A \vee x \in B$.
Be now $x \in A \cap \sim B$. This is equivalent with the statement $x \in A \wedge x \notin B$. Negating this statement: $\neg(x \in A \cap \sim B) \Leftrightarrow x \notin A \vee x \in B$. This give $\sim (A \cap \sim B) \equiv x \notin A \vee x \in B$, resulting in $\sim (A - B) = \sim (A \cap \sim B)$.

♦

1.7.5

$\sim (\sim (\sim A)) = \sim A$

TRUE.
Be $x \in \sim (\sim (\sim A))$. This is equivalent to $x \notin \sim (\sim A)$. Which on it's turn is equivalent with $x \in \sim A$. So, $\sim (\sim (\sim A)) \subset \sim A$.

Be $x \in \sim A$. This is equivalent to $x \notin \sim (\sim A)$. Which on it's turn is equivalent with $x \in \sim (\sim (\sim A))$. So, $\sim A \subset \sim (\sim (\sim A))$.
Both cases reduce to $\sim (\sim (\sim A)) = \sim A$.

♦

1.7.6

$A \cup (B - C) = (A \cup B) - C$

Not TRUE.
Be $x \in A \cup (B - C)$. This is equivalent to $x \in A \vee x \in (B - C)$. Suppose $x \in A$, then $x \in A \cup B$. Let's consider the set C so that $(A \cup B) \subset C$, then $(A \cup B) - C = \emptyset$. We get a contradiction and the proposed statement is not true.

♦

1.7.7

$\sim (A - B) = (\sim A) \cup B$

TRUE.
Be $x \in (A - B)$. This is equivalent to $x \in A \wedge x \notin B$. Negating this statement: $\neg(x \in A \wedge x \notin B) \Leftrightarrow x \notin A \vee x \in B$. This is equivalent to the statement $x \in (\sim A) \cup B$. So $\sim (A - B) \subset (\sim A) \cup B$.
Consider now $x \in (\sim A) \cup B$. So $x \notin A \vee x \in B$. If we have the case $x \notin A$ then also $x \notin (A - B)$ as x can not be one of the remaining elements of A after the complement of B relative to A. Also, if $x \in B$ then also $x \notin (A - B)$ as x is an element of B and thus can not be an element of $(A - B)$. Thus, in both cases we have, $x \notin (A - B)$ which implies $x \in \sim (A - B)$. So $(\sim A) \cup B \subset \sim (A - B)$.

♦

1.7.8

If $A - B = C - B$, then $A = C$.

Not TRUE.

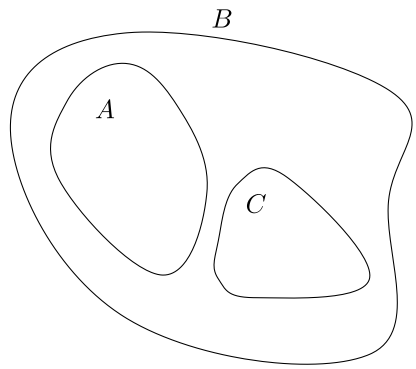


Figure 1.7: If $A - B = C - B \not\Rightarrow A = C$

Suppose $A \subset B$, then $A - B = \emptyset$. Choose a C such that $C \subset B$ and also $A \cap C = \emptyset$, then also $C - B = \emptyset$ and get $A - B = C - B$ although $A \neq C$.

♦

1.7.9

If $A - (B \cap C) = (A - B) \cap (A - C)$.

TRUE.

Suppose $x \in A - (B \cap C)$, then $x \in A \wedge x \notin B \cap C$. As x can not be simultaneously in B and C, then also x must be simultaneously in $A - B$ and $A - C$ as the "complementation of A with B and C will not "subtract" x out of A, and considering that $x \in A$ we have $A - (B \cap C) \subset (A - B) \cap (A - C)$ Suppose $x \in (A - B) \cap (A - C)$, then x must be an element of A but not an element of B and C. This means that $x \notin B \cap C$ and thus the complementation of A by $B \cap C$ has no effect on x. Thus, $\underbrace{(A - B) \cap (A - C)}_{=A} \subset A - (B \cap C)$.

♦

1.8 Collections of Sets

1.8.1

Suppose that A, B and C are the following subsets of the plane:
$A = \{(x, y) : x^2 + y^2 \leq 16\}$, $B = \{(x, y) : x \geq 0 \text{ and } y \leq 0\}$, $C = \{(x, y) : y \leq x\}$. If $\mathscr{K}$ is the collection of sets $\{A, B, C\}$, sketch each of the following sets:

(a) $\bigcap \mathscr{K}$
(b) $\bigcup \mathscr{K}$
(c) $\bigcup \mathscr{K} - \bigcap \mathscr{K}$

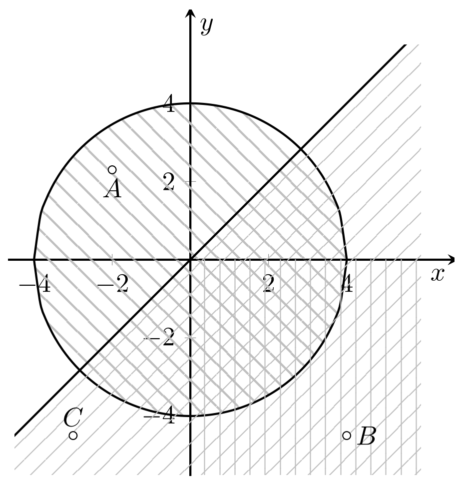


Figure 1.8: The sets A, B, C

a) $\bigcap \mathscr{K}$

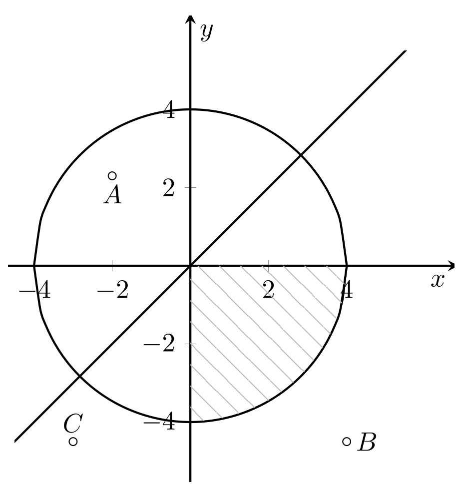


Figure 1.9: $\bigcap \mathscr{K}$

$\Diamond$

b) $\bigcup \mathscr{K}$

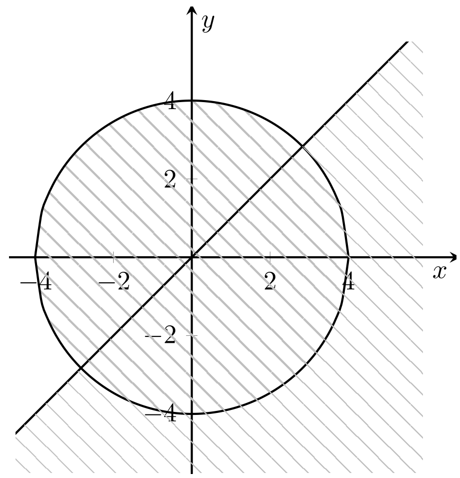


Figure 1.10: $\bigcup \mathscr{K}$

$\Diamond$

c) $\bigcup \mathscr{K} - \bigcap \mathscr{K}$

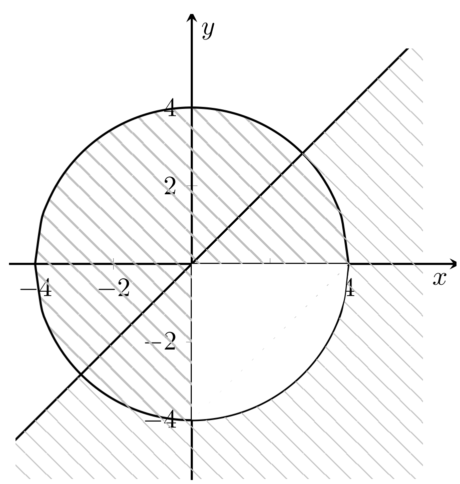


Figure 1.11: $\bigcup \mathscr{K} - \bigcap \mathscr{K}$

$\blacklozenge$

1.8.2

Recall that $\mathbb{P}$ is the symbol for the set of positive integers. Suppose that for each $n \in \mathbb{P}$, we let $A_n = \{x \in \mathbb{R} : x \geq n\}$. Describe the sets $\bigcup\{A_n : n \in \mathbb{P}\}$ and $\bigcap\{A_n : n \in \mathbb{P}\}$.

$S = \bigcup\{A_n : n \in \mathbb{P}\}$

$$S = [1, +\infty)$$

$$\Diamond$$

$S = \bigcap\{A_n : n \in \mathbb{P}\}$

$$S = \emptyset$$

This can be understood by the fact that for every $x \in \mathbb{R}$, you can find a $n \in \mathbb{P}$ so that $x \notin A_n$. So, no x can be an element of S.

$$\blacklozenge$$

1.8.3

Suppose that for each $n \in \mathbb{P}$, K_n is a non-empty set such that $K_{n+1} \subset K_n$. Let $\mathscr{K} = \{K_n : n \in \mathbb{P}\}$.
In each of the following, if the statement is necessarily true, say so and justify your answer. If the statement is not necessarily true, give a counterexample to justify your answer.

$(a)\quad \bigcup \mathscr{K} = K_1$
$(b)\quad \bigcap\{K_i : i = 1, 2, \ldots, n\} = K_n$
$(c)\quad \bigcap \mathscr{K} \neq \emptyset$

(a) $\bigcup \mathscr{K} = K_1$.
TRUE.
Be $x \in K_n$ for any arbitrary n. So, $x \in K_n \cup K_{n-1}$. But $K_n \cup K_{n-1} = K_{n-1}$, giving $x \in K_{n-1}$. Repeating that process with $K_{n-1} \subset K_{n-2} \subset \ldots K_2 \subset K_1$ we get $x \in K_n$.

$$\Diamond$$

(b) $\bigcap\{K_i : i = 1, 2, \ldots, n\} = K_n$.
TRUE.
Suppose first that for all n we have K_n is a *proper* subset of K_{n-1}. Then $K_n \cap K_{n-1} = K_n$. Be $x \in K_n$ but not in K_{n-1} for any arbitrary n. Then, $x \in K_n \cap K_{n-1}$ is equivalent to $x \in K_n$. Repeating that process with we have $K_n \cap K_{n-1} \cap K_{n-2} \cap \ldots K_2 \cap K_1 = K_n$ and get $x \in K_n$. Hence, $\bigcup \mathscr{K} = K_1$.
In the case that for some or all n we have $K_n = K_{n-1}$ we could also state that $\bigcap\{K_i : i = 1, 2, \ldots, n\} = K_{n-1}$ but as $K_n = K_{n-1}$ we can write $\bigcap\{K_i : i = 1, 2, \ldots, n\} = K_{n-1} = K_n$.

The same is true in the case that a sequence of the subsets are proper subset of each other i.e. $K_{n+p} = K_{n+p-1} = \dots K_{n+1} = K_n = K_{n-1} = \cdots = K_{n-t}$. then one could write $\bigcap\{K_i : i = 1, 2, \dots, n\} = K_{n+p}$ but as $K_{n+p} = K_n$, the original statement holds.

$\lozenge$

(c) $\bigcap \mathscr{K} \neq \emptyset$.
TRUE.
As no K_n is an empty set, K_n will always contain at least one element and due to (b) we get indeed $\bigcap \mathscr{K} \neq \emptyset$: suppose that for a given n, K_n contains only one element x, then all subsequent K_{n+p} must also have only one element i.e. x and we will get $\bigcap \mathscr{K} = \{x\}$

$\blacklozenge$

1.8.4

For each real number $r > 0$, let $L_r = \{x : x \geq r\}$. Sketch the set $\bigcup\{L_r : r > 0\}$ and $\bigcap\{L_r : r > 0\}$ on a number line. If a set happens to be empty, say so.

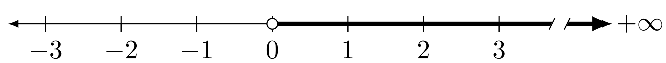


Figure 1.12: $\bigcup\{L_r : r > 0\}$

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$\bigcap\{L_r : r > 0\} = \emptyset$.

Indeed, take an arbitrary r and be $\epsilon > 0$ then $\exists x \in L_r : x \notin L_{r+\epsilon}$. Then, $L_r \cap L_{r+\epsilon} = \emptyset$. So, whatever L_r we choose in the collection $\mathscr{L} = \{L_r : r \in \mathbb{R}^+\}$ there always be a $L_{r'}$ for which $L_r \cap L_{r'} = \emptyset$ and hence $\bigcap\{L_r : r > 0\} = \emptyset$.

$\blacklozenge$

1.8.5

Let U be a set and let $\mathscr{K}$ be a non-empty collection of subsets of U. $\sim$ will signify the complement with respect to U. Prove the following set identities. The identities are quite important and are known as De Morgan's Laws.

$$\begin{array}{ll} (a) & \sim(\bigcup\{K : K \in \mathscr{K}\}) = \bigcap\{\sim K : K \in \mathscr{K}\} \\ (b) & \sim(\bigcap\{K : K \in \mathscr{K}\}) = \bigcup\{\sim K : K \in \mathscr{K}\} \end{array}$$

(a) $\sim(\bigcup\{K : K \in \mathscr{K}\}) = \bigcap\{\sim K : K \in \mathscr{K}\}$
Suppose $x \in \sim(\bigcup\{K : K \in \mathscr{K}\})$, then $x \notin \bigcup\{K : K \in \mathscr{K}\}$. This means that x is not an element of any $K \in \mathscr{K}$ i.e. $\forall K \in \mathscr{K} : x \notin K$. This can also be expressed as $\forall K \in \mathscr{K} : x \in \sim K)$. This means that x is an element of all $\sim K$ giving $x \in \bigcap\{\sim K : K \in \mathscr{K}\}$ and thus $\sim(\bigcup\{K : K \in \mathscr{K}\}) \subset \bigcap\{\sim K : K \in \mathscr{K}\}$.
Suppose now that $x \in \bigcap\{\sim K : K \in \mathscr{K}\}$. This means that x is an element of $\{\sim K : K \in \mathscr{K}\}$ for all K i.e. $x \notin \{K : K \in \mathscr{K}\}$ for all K, (indeed if x would be an element of a $K \in \mathscr{K}$ then x would not be an element of its complement and so x could not be an element of $\bigcap\{\sim K : K \in \mathscr{K}\}$). The conclusion is that $x \notin \bigcup\{K : K \in \mathscr{K}\}$ and thus $x \in \sim \bigcup\{K : K \in \mathscr{K}\}$. Hence, $\bigcap\{\sim K : K \in \mathscr{K}\} \subset \sim(\bigcup\{K : K \in \mathscr{K}\})$.
Conclusion $\sim(\bigcup\{K : K \in \mathscr{K}\}) = \bigcap\{\sim K : K \in \mathscr{K}\}$.

$\Diamond$

(b) $\sim(\bigcap\{K : K \in \mathscr{K}\}) = \bigcup\{\sim K : K \in \mathscr{K}\}$
Suppose $x \in \sim(\bigcap\{K : K \in \mathscr{K}\})$, then $x \notin \bigcap\{K : K \in \mathscr{K}\}$. This means that there exists at least one $K \in \mathscr{K}$ so that x is not an element of this K i.e. $\exists K \in \mathscr{K} : x \notin K$. This can also be expressed as $\exists K \in \mathscr{K} : x \in \sim K$. This means that x is an element of $\bigcup\{\sim K : K \in \mathscr{K}\}$ and thus $\sim(\bigcap\{K : K \in \mathscr{K}\}) \subset \bigcup\{\sim K : K \in \mathscr{K}\}$.
Suppose now that $x \in \bigcup\{\sim K : K \in \mathscr{K}\}$. This means that x is an element of at least one $\sim K : K \in \mathscr{K}$. Stated differently, there exist at least one $K : K \in \mathscr{K}$ for which $x \notin K$. This means that x can not be an element of $\bigcap\{K : K \in \mathscr{K}\}$ and thus $x \in \sim \bigcap\{K : K \in \mathscr{K}\}$ which means $\bigcup\{\sim K : K \in \mathscr{K}\} \subset \sim(\bigcup\{K : K \in \mathscr{K}\})$
Conclusion $\sim(\bigcap\{K : K \in \mathscr{K}\}) = \bigcup\{\sim K : K \in \mathscr{K}\}$.

$\blacklozenge$

1.8.6

Let $S = \{1, 2, 3, 4, 5\}$ and let $\mathscr{P}(S)$ be the power set of S. List the elements in $\mathscr{P}(S)$.

We order them according to the number of elements in the subsets. We check the number of subsets by using the $\binom{5}{m}$ formula (i.e. combination without repetition).

$$\begin{aligned}
\text{5 elements} \quad \binom{5}{5} &= 1 \\
\{1, 2, 3, 4, 5\} \\
\text{4 elements} \quad \binom{5}{4} &= 5 \\
\{1, 2, 3, 4\} \\
\{1, 2, 3, 5\} \\
\{1, 2, 4, 5\} \\
\{1, 3, 4, 5\} \\
\{2, 3, 4, 5\} \\
\text{3 elements} \quad \binom{5}{3} &= 10 \\
\{1, 2, 3\} \\
\{1, 2, 4\} \\
\{1, 2, 5\} \\
\{1, 3, 4\} \\
\{1, 3, 5\} \\
\{1, 4, 5\} \\
\{2, 3, 4\} \\
\{2, 3, 5\} \\
\{2, 4, 5\} \\
\{3, 4, 5\}
\end{aligned}$$

$$2 \text{ elements} \quad \binom{5}{2} = 10$$

$$\{1, 2\}$$
$$\{1, 3\}$$
$$\{1, 4\}$$
$$\{1, 5\}$$
$$\{2, 3\}$$
$$\{2, 4\}$$
$$\{2, 5\}$$
$$\{3, 4\}$$
$$\{3, 5\}$$
$$\{4, 5\}$$

$$1 \text{ element} \quad \binom{5}{1} = 5$$

$$\{1\}$$
$$\{2\}$$
$$\{3\}$$
$$\{4\}$$
$$\{5\}$$

$$0 \text{ elements} \quad \binom{5}{0} = 1$$

$$\emptyset$$

Note that the total number of subsets in $\mathscr{P}(S)$ is $1 + 5 + 10 + 10 + 5 + 1 = 32$ which corresponds to 2^5.

♦

1.9 Cartesian Product

1.9.1

Suppose that $A \subset B$ and C is a set. Prove that $A \times C \subset B \times C$.

Be $x \in A$ and $y \in C$. As $A \subset B$, then x is also in B. Thus $\underbrace{(x,y)}_{x\in A,\, y\in C} \in A \times C$ means also that $\underbrace{(x,y)}_{x\in B,\, y\in C} \in B \times C$

♦

1.9.2

Let $A = \{1,2,3\}$, $B = \{a,b\}$, and $C = \{\alpha,\beta\}$. List the elements of each of the following sets:

$$\begin{array}{ll} (a) & A \times (B \cup C) \\ (b) & (A \times B) \cup (A \times C) \\ (c) & (A \cup B) \times C \\ (d) & (A \times C) \cup (B \times C) \end{array}$$

(a) $A \times (B \cup C)$

$(1,a)$, $(1,b)$, $(2,a)$, $(2,b)$, $(3,a)$, $(3,b)$
$(1,\alpha)$, $(1,\beta)$, $(2,\alpha)$, $(2,\beta)$, $(3,\alpha)$, $(3,\beta)$

◊

(b) $(A \times B) \cup (A \times C)$

$(1,a)$, $(1,b)$, $(2,a)$, $(2,b)$, $(3,a)$, $(3,b)$
$(1,\alpha)$, $(1,\beta)$, $(2,\alpha)$, $(2,\beta)$, $(3,\alpha)$, $(3,\beta)$

◊

(c) $(A \cup B) \times C$

$(1,\alpha)$, $(1,\beta)$, $(2,\alpha)$, $(2,\beta)$, $(3,\alpha)$, $(3,\beta)$ $(1,\alpha)$, $(1,\beta)$, $(2,\alpha)$, $(2,\beta)$, $(3,\alpha)$, $(3,\beta)$
(a,α), (a,β), (b,α), (b,β)

◊

(d) $(A \times C) \cup (B \times C)$

$(1,\alpha)$, $(1,\beta)$, $(2,\alpha)$, $(2,\beta)$, $(3,\alpha)$, $(3,\beta)$ $(1,\alpha)$, $(1,\beta)$, $(2,\alpha)$, $(2,\beta)$, $(3,\alpha)$, $(3,\beta)$
(a,α), (a,β), (b,α), (b,β)

♦

1.9.3

> Are any of the sets in Exercise 2 the same? If so write the set identities that are suggested by your observations. Try to prove your conjecture.

In exercise 2 we can see that that the set (a) and (b) are the same. Also (c) and (d) are the same. This suggests the following identities $A \times (B \cup C) = (A \times B) \cup (A \times C)$ and $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

$\mathbf{A} \times (\mathbf{B} \cup \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cup (\mathbf{A} \times \mathbf{C})$
Proof:
Be $x \in A$ and $y \in B \cup C$, so y is an element of B or C. Consider $(x, y) \in A \times (B \cup C)$. As the y can be an element of B or C follows immediately that $(x, y) \in (A \times B)$ or $(x, y) \in A \times C)$ and thus $(x, y) \in (A \times B) \cup (A \times C)$. And get $A \times (B \cup C) \subset (A \times B) \cup (A \times C)$
Suppose now that $(x, y) \in (A \times B) \cup (A \times C)$. The (x, y) is an element of $A \times B$ or $A \times C$. For the same $x \in A$ this implies that $y \in B$ or $y \in C$ and thus $(x, y) \in A \times (B \cup C)$, giving $(A \times B) \cup (A \times C) \subset A \times (B \cup C)$ leading with the previous $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

$\lozenge$

$(\mathbf{A} \cup \mathbf{B}) \times \mathbf{C} = (\mathbf{A} \times \mathbf{C}) \cup (\mathbf{B} \times \mathbf{C})$
Proof:
Be $x \in A \cup B$ and $y \in C$, so x is an element of A or B. Consider $(x, y) \in (A \cup B) \times C$. As the x can be an element of A or B follows immediately that $(x, y) \in (A \times C)$ or $(x, y) \in (B \times C)$ and thus $(x, y) \in (A \times B) \cup (\mathbf{A} \times C)$. And get $(A \cup B) \times C \subset (A \times C) \cup (B \times C)$
Suppose now that $(x, y) \in (A \times B) \cup (\mathbf{A} \times C)$. The (x, y) is an element of $A \times C$ or $B \times C$. For the same $y \in C$ this implies that $x \in A$ or $x \in B$ and thus $(x, y) \in (A \times C) \cup (B \times C)$, giving $(A \times C) \cup (B \times C) \subset (A \cup B) \times C$ leading with the previous $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

$\blacklozenge$

1.9.4

> Suppose that A is a set consisting of five elements and B is a set consisting of three elements. How many elements does the set $A \times B$ have? The set $B \times A$?

$A \times B$ has $5 \times 3 = 15$ elements. Indeed in the element $(x, y) \in A \times B$ we can choose for x out of the five elements of A and for each choice of x we are free to choose one element out of the 3 elements of B.
For $B \times A$, the reasoning is the same and get $3 \times 5 = 15$ elements.

$\blacklozenge$

1.9.5

> Suppose that A is a set consisting of m elements and B is a set consisting of n elements, where m and n are positive integers. How many elements are there in $A \times B$?

$A \times B$ has $m \times m$ elements. Indeed in the element $(x, y) \in A \times B$ we can choose for x out of the m elements of A and for each choice of x we are free to choose one element out of the n elements of B.

♦

1.9.6

> Suppose that A is a set consisting of three elements, B consists of four elements and C consists of two element. How many elements are there in the set $(A \times B)\times$?

$(A \times B) \times C$ has $(3 \times 4) \times 2 = 24$ elements. Indeed in the element $((x, y), z) \in (A \times B) \times C$ we have for (x, y), $3 \times 4 = 12$ elements (see Exercise 1.9.5) and for each choice of this (x, y) we are free to choose one element out of the 2 elements of C.

♦

1.10 Functions

1.10.1

In each of the following, a set of ordered pairs Γ is given. In each case, determine whether Γ is a function and, if it is, determine if it is a one-to-one function.

(a) Let $\Gamma = \{(x, y) : -1 \leq x \leq 1 \text{ and } x^2 + y^2 = 1\}$.
(b) Let $\Gamma = \{(x, y) : -1 \leq x \leq 1,\ y \geq 0, \text{ and } x^2 + y^2 = 1\}$.
(c) Let $\Gamma = \{(x, y) : 0 \leq x \leq 1 \text{ and } x^2 + y^2 = 1\}$.
(d) Let $\mathscr{F}$ be the collection of all real-valued differentiable functions defined on the open interval (a, b).
Let $\Gamma = \{(f, f^{'}) : f \in \mathscr{F} \text{ and } f^{'} \text{ is the derivative of } f\}$.
(e) Let X be the collection of all continuous real-valued functions defined on the closed interval $[a, b]$.
Let $\Gamma = \left\{\left(f, \int_a^b f(x)dx\right) : f \in X\right\}$.

(a) Let $\Gamma = \{(x, y) : -1 \leq x \leq 1 \text{ and } x^2 + y^2 = 21\}$

Γ is not a function due to the ambiguity of the $\sqrt{\ }$ function. E.g. take $x = 0$ then $y = \pm 1$.

$\Diamond$

(b) Let $\Gamma = \{(x, y) : -1 \leq x \leq 1,\ y \geq 0, \text{ and } x^2 + y^2 = 1\}$.

This time, as the ambiguity on the range has been removed by the condition $y \geq 0$ Γ is a function. Yet, it is not one-to-one e.g. for $x = -1$ and $x = 1$ we get the same value for y.

$\Diamond$

(c) Let $\Gamma = \{(x, y) : 0 \leq x \leq 1 \text{ and } x^2 + y^2 = 2\}$.

This time, as the ambiguity on the range has been removed by the condition $y \geq 0$ Γ is a function. And, it is a one-to-one function as with the restriction on the domain $x \in [0, 1]$, y is well and uniquely defined.

$\Diamond$

(d) Let $\mathscr{F}$ be the collection of all real-valued differentiable functions defined on the open interval (a, b).Let $\Gamma = \{(f, f^{'}) : f \in \mathscr{F} \text{ and } f^{'} \text{ is the derivative of } f\}$.

Γ is a function as f is a real-valued differentiable function, meaning that $\forall f \in \mathscr{F}, \exists f^{'}$.
Yet, it is not one-to-one. E.g. take $f_1 = x + 1$ and $f_2 = x + 2$, both function give $f^{'} = 1$ meaning that Γ is not one-to-one.

$\Diamond$

(e) Let X be the collection of all continuous real-valued functions defined on the closed interval $[a, b]$.
Let $\Gamma = \left\{ \left(f, \int_a^b f(x)dx \right) : f \in X \right\}$.

Γ is a function as f is a continuous real-valued function, and from calculus we know that every continuous is Riemann-integrable, meaning that for every f there exist a real number $\int_a^b f(x)dx$. Yet, Γ is not one-to-one as two different functions f_1 and f_2 could have the same value of their integral on the given domain e.g. take $f_1 = \frac{x-a}{b-a}$ and $f_2 = \frac{b-x}{b-a}$, both have the same value for the integral over $[a, b]$ namely $\frac{1}{2}(b-a)$.

♦

1.10.2

Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ be the function defined as follows:
For each $(x, y) \in \mathbb{R}$, let $f(x, y) = (a, b)$ where

$$a = x + 2y$$

and

$$b = 2x + 4y$$

Which of the following terms applies to $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$?
(a) surjective, (b) bijective, (c) injective.

f in not injective. Indeed the given function definition can be considered as a system of linear equations with x and y as unknowns and a, b as parameters. So for a given $(a, b) \in \mathbb{R} \times \mathbb{R}$ (the domain) the range will only span $\mathbb{R} \times \mathbb{R}$ only if the system of equations is not degenerated i.e. if the determinant of the system is not 0, but we have

$$det \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = 0$$

Hence, f is not surjective. It is however one-to-one (injective) as for a given $(x,\, y)$, due to linear form of the function, there will be only one $(a,\, b)$ on which $(x,\, y)$ is mapped. As f is not surjective, f can not be bijective.

♦

1.10.3

Repeat the question in Ecercise 2 for the system

$$a = 3x + 2y$$

$$b = 6x - 2y$$

f is injective as we see that this time the determinant of the system is

$$det \begin{pmatrix} 3 & 2 \\ 6 & -2 \end{pmatrix} = -18$$

Hence, f is surjective. It is also one-to-one (injective) for the same reason mentioned in Exercise 2. . As f is surjective and injective , f is also bijective.

♦

1.10.4

Let f be a map from the set of all reals $\mathbb{R}$ into $\mathbb{R}$. Suppose furthermore that if x_1 and x_2 are in $\mathbb{R}$ and $x_1 < x_2$, then $f(x_1) < f(x_2)$. Is it necessarily true that f is one-to-one? Is it necessarily true that $f[\mathbb{R}] = \mathbb{R}$? Justify your answer.

It is necessarily true that f is one-to-one. (At each point x_1, the function for a given x_2 could be re-written as $f(x_2) = f(x_1) + \phi(x_1)(x_2 - x_1)$ with $\phi(x_1) > 0$. So $f(x_2)$ can not be equal to $f(x_1)$ unless $x_2 = x_1$.)
On the other hand $f[\mathbb{R}]$ is not necessarily equal to $\mathbb{R}$. As a counterexample, consider the function $f(x) = e^{-x}$, which is a monotone increasing function but the range is $(-\infty, 0) \neq \mathbb{R}$

♦

1.10.5

Consider the function $f : X \to Y$. Suppose that A and B are subsets of X. Decide which of the following statements are necessarily true. Justify your answers.

$$\begin{array}{ll}
(a) & \text{If } A \cap B = \emptyset, \text{ then } f[A] \cap f[B] = \emptyset. \\
(b) & \text{If } f[A] \cap f[B] = \emptyset, \text{ then } A \cap B = \emptyset. \\
(c) & \text{If } A \subset B, \text{ then } f[A] \subset f[B]. \\
(d) & f[A - B] = f[A] - f[B]. \\
(e) & f[A \cup B] = f[A] \cup f[B]. \\
(f) & f[A \cap B] \subset f[A] \cap f[B]. \\
(g) & f[A \cap B] = f[A] \cap f[B].
\end{array}$$

(a) If $A \cap B = \emptyset,$ then $f[A] \cap f[B] = \emptyset.$

This is not necessarily true. Take for example a non injective function like $f(x) = \sin(x)$ then $f[[0, \frac{\pi}{4}]] \cap f[\frac{3\pi}{4}, \pi] = [0, \frac{\sqrt{2}}{2}].$

$\Diamond$

(b) If $f[A] \cap f[B] = \emptyset,$ then $A \cap B = \emptyset$

This is necessarily true as for f being a function we have $(x_2, f(x_2)) \in f$ and $(x_1, f(x_1)) \in f \Rightarrow f(x_1) = f(x_2)$ and $A \cap B \neq \emptyset$ would mean that $\exists x \in A \cap B$ for which x has two different images.

$\Diamond$

(c) If $A \subset B,$ then $f[A] \subset f[B].$

This is necessarily true as for the same reason as in (b).

$\Diamond$

(d) $f[A - B] = f[A] - f[B]$

This is not necessarily true. Let's take the same counterexample as in (a) i.e. $f(x) = \sin(x)$ and let's define $A = [0, 2\pi],\ B = [0, \frac{\pi}{4}],$ then $f[A] = [-1, 1]$ and $f[B] = [0, \frac{\sqrt{2}}{2}]$ and $f[A] - f[B] = [-1, 0) \cup (\frac{\sqrt{2}}{2}, 1]$ while $f[A - B] = [-1, 1].$

$\Diamond$

(e) $f[A \cup B] = f[A] \cup f[B]$

This is true.

Suppose first that $A \cap B = \emptyset$ and take $x \in A$, then $f(x) \in f[A]$ and $x \notin f[B]$ giving $f(x) \in f[A] \cup f[B]$. On the other hand it is obvious that if $A \cap B \neq \emptyset$ then $f(x) \in f[A]$ and-or $f(x) inf[B]$ giving $f(x) \in f[A] \cup f[B]$. Hence, $f[A \cup B] \subset f[A] \cup f[B]$.

Suppose now that $f(x) \in f[A]$ this means that $x \in A$ regardless of $x \in B$ or not. So, $f[A] \cup f[B] \subset f[A \cup B]$ and with the previous we get $f[A \cup B] = f[A] \cup f[B]$.

$\Diamond$

(f) $f[A \cap B] \subset f[A] \cap f[B]$

True as if $f(x) \in f[A \cap B]$ means that $x \in A \cap B$ so x will be mapped in the image $f[A]$ and in the image $f[B]$ and thus $f[A \cap B] \subset f[A] \cap f[B]$.

$\Diamond$

(g) $f[A \cap B] = f[A] \cap f[B]$

Not true. Suppose $f(x) \in f[A] \cap f[B]$. But if f is not injective the possibility exists that for a given $x_a \in A$ and another $x_b \in B$ we have $f[x_a] = f[x_b]$ even if A and B are disjoint sets which would give $f[A \cap B] = f[\emptyset] = \emptyset$.

$\blacklozenge$

1.11 Relations

In Exercises 1 to 5, all relations are subsets of the plane. In each case, draw a sketch of R, and give Dom$R$, Range$R$, $R[0]$ and $R^{-1}[0]$.

1.11.1

Let $(x, y) \in R$ provided that (x, y) satisfies each of the following inequalities: $x + y \leq 3$, $y - x \geq 0$, $x \geq -3$.

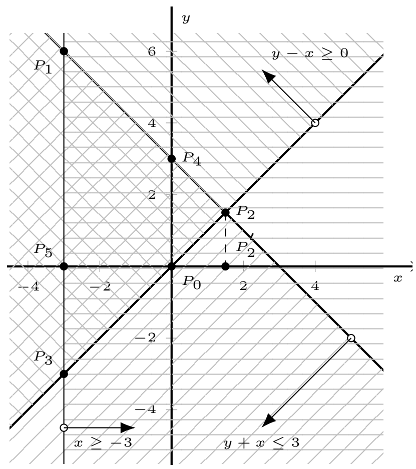


Figure 1.13: $x + y \leq 3$, $y - x \geq 0$, $x \geq -3$

Dom$R$ = segment $[P_5, P_2']$
Range$R$ = segment $[P_3, P_1]$
$R[0]$: = segment $[P_0, P_4]$
$R^{-1}[0]$: = segment $[P_5, P_0]$

♦

1.11.2

Let R be the set of all (x, y) that satisfy $x^2 - y^2 \leq 1$ and $y^2 - x^2 \leq 1$.

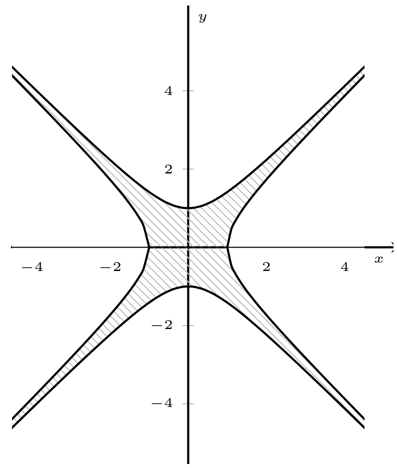


Figure 1.14: $x^2 - y^2 \leq 1$ and $y^2 - x^2 \leq 1$

$\mathbf{Dom}R = (-\infty, +\infty)$
$\mathbf{Range}R = (-\infty, +\infty)$
$R[0] := [-1, 1]$
$R^{-1}[0]\colon = [-1, 1]$

⧫

1.11.3

Let R be the set of all (x, y) such that $x - y$ is a multiple of 3.

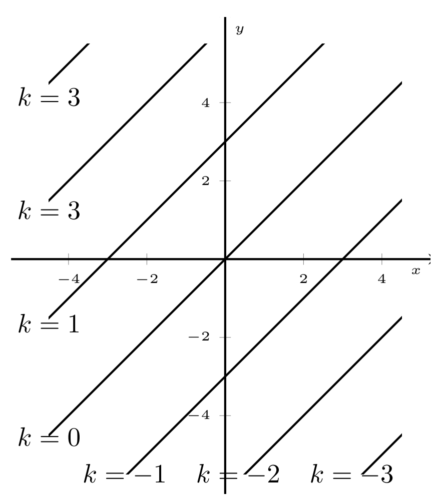


Figure 1.15: The relation $\{(x, y) : y = x - 3k, k \in \mathbb{Z}$

Dom$R = (-\infty, +\infty)$
Range$R = (-\infty, +\infty)$
$R[0] \, : = \{y : y = 3k, k \in \mathbb{Z}\}$
$R^{-1}[0]\!: = \{x : x = 3k, k \in \mathbb{Z}\}$

♦

1.11.4

Let R be a subset of the plane such that $(x, y) \in R$ provided that $x - y \leq \frac{1}{2}$

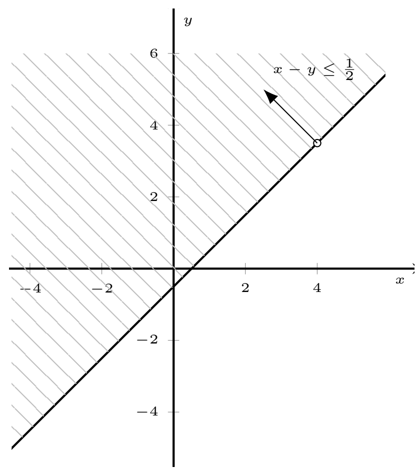


Figure 1.16: The relation $x - y \leq \frac{1}{2}$

Dom$R = (-\infty, +\infty)$
Range$R = (-\infty, +\infty)$
$R[0] \, : = [-\frac{1}{2}, \infty)$
$R^{-1}[0]\!: = (-\infty, \frac{1}{2}]$

♦

1.11.5

Let R be a subset of the plane such that $(x, y) \in R$ provided that $y = x^4$

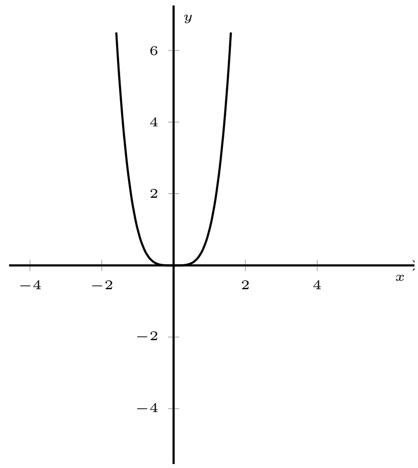


Figure 1.17: The relation $y = x^4$

Dom$R = (-\infty, +\infty)$
Range$R = [0, +\infty)$
$R[0] := \{0\}$
$R^{-1}[0] := \{0\}$

♦

1.11.6

Let $R = \{(x, y) : x \geq 0,\, x^2 + y^2 = 26\}$. Find $R[0]$, $R[5]$, and $R[I]$, where $I = \{r : 0 \leq r \leq 1\}$; $R^{-1}[J]$ where $J = \{r : -1 \leq r \leq 1\}$.

$R[0] := \{-\sqrt{26},\, \sqrt{26}\}$
$R[5] := \{-1,\, 1\}$
$R[I] := [-\sqrt{26}, -5] \cup [5, \sqrt{26}]$
$R^{-1}[J] := [5, \sqrt{26}]$

♦

1.11.7

Let $R = \{(x, y) : x \text{ is real and } y = x(x-1)(x-2)\}$. Find $R[0]$, $R[1]$, $R[2]$, $R^{-1}[0]$ and $R[I]$, where $I = \{x : 0 \leq x \leq 2\}$.

$R[0] := \{0\}$
$R[1] := \{0\}$
$R[2] := \{0\}$
$R^{-1}[0] := \{0, 1, 2\}$
$R[I] := [-1, 2]$

♦

1.11.8

Let R be a relation between sets X and Y, and suppose that A and B are subsets of X. In each of the following, tell wether the statement is necessarily true and give a justification of your answer.

$$\begin{array}{ll} (a) & R[A \cap B] = R[A] \cap R[B]. \\ (b) & R[A \cap B] \subset R[A] \cap R[B]. \\ (c) & R[A \cap B] \supset R[A] \cap R[B]. \end{array}$$

(a) $R[A \cap B] = R[A] \cap R[B]$
This is not necessarily true. Take for example a non injective function as the relation R with $A \cap B = \emptyset$. This means that $R[A \cap B] = \emptyset$ but the relation being a non injective function it also possible that $R[A] \cap R[B] \neq \emptyset$. So, $R[A \cap B] \not\supset R[A] \cap R[B]$ and we can't have $R[A \cap B] = R[A] \cap R[B]$.

♢

(b) $R[A \cap B] \subset R[A] \cap R[B]$
This is necessarily true. Take $(x, y) : x \in A \cap B, y = R(x)$. Then we have obviously $y \in R[A]$ and also $y \in R[A \cap B]$ but as $x \in B$ (because $x \in A \cap B$) we have also $y \in R[B]$. So $y \in R[A]$ and $y \in R[B]$ and thus $y \in R[A] \cap R[B]$ giving $R[A \cap B] \subset R[A] \cap R[B]$.

♢

(c) $R[A \cap B] \supset R[A] \cap R[B]$
See (a).

♦

1.11.9

Let $\mathbb{Z}$ be the set of all integers. For each m and $n \in \mathbb{Z}$, let us write mRn if and only $m - n$ is an even integer. Thus this relation R is the set $\{(m, n) : m - n = 2k,\ k \in \mathbb{Z}\}$. Find $R[1]$ and $R[2]$. How many distinct sets of the form $R[i]$ are there?

$R[0] := \{n : n = 1 - 2k,\ k \in \mathbb{Z}\}$ i.e. the set of all odd integers.
$R[1] := \{n : n = 2 - 2k,\ k \in \mathbb{Z}\} \Leftrightarrow \{n : n = 2k,\ k \in \mathbb{Z}\}$ i.e. the set of all even integers.
There are 2 distinct sets in total.

♦

1.11.10

Let R be the relation defined as follows: For each ordered pair of integers (m, n), let mRn if and only $m-n$ is an integral multiple of 5 (including negative multiples of 5). Find $R[1]$, $R[2]$, and $R[6]$. How many distinct sets of the form $R[i]$ are there? Find $R^{-1}[1]$ and $R^{-1}[2]$. Is $R^{-1}[i] = R[i]$ for each i? For this relation R, if iRj and jRk, does it follow that iRk?

$R[1] := \{\dots, -9, -4, 1, 6, 11, \dots\}$
$R[2] := \{\dots, -8, -3, 2, 7, 12, \dots\}$
$R[6] := \{\dots, -9, -4, 1, 6, 11, \dots\}$
There are 5 distinct sets in total.
$R^{-1}[1] := \{\dots, -9, -4, 1, 6, 11, \dots\} = R[1]$
$R^{-1}[2] := \{\dots, -8, -3, 2, 7, 12, \dots\} = R[2]$
$R^{-1}[i] = R[i]$ for each i as the relation $n = m - 5k, \forall k \in \mathbb{Z}$ can be written as $m = n - 5p, \forall p \in \mathbb{Z}$. So the sets $R^{-1}[i]$ and $R[i]$ are not distinguishable.
If iRj and jRk, does it follow that iRk? Yes, as the composed relation $(jRk) \circ (iRj)$ has the relation $k = i - 5(p+q), p, q \in \mathbb{Z}$ and as $p+q \in \mathbb{Z}$ we can rewrite the relation $(jRk) \circ (iRj)$ as $j = i - 5p, p \in \mathbb{Z}$.

♦

1.12 Set inclusions for image and inverse image sets

1.12.1

Prove that
12.5 Suppose that R is a relation between X and Y. Then, if $\{A_\alpha : \alpha \in \Lambda\}$ is a non-empty collection of subsets of X, the following hold:

$$\mathbf{12.5(a)} \quad R\left[\bigcup\{A_\alpha : \alpha \in \Lambda\}\right] = \bigcup\{R[A_\alpha] : \alpha \in \Lambda\}.$$
$$\mathbf{12.5(b)} \quad R\left[\bigcap\{A_\alpha : \alpha \in \Lambda\}\right] \subset \bigcap\{R[A_\alpha] : \alpha \in \Lambda\}.$$

12.5(a) $\quad R\left[\bigcup\{A_\alpha : \alpha \in \Lambda\}\right] = \bigcup\{R[A_\alpha] : \alpha \in \Lambda\}$.
Be $y \in R\left[\bigcup\{A_\alpha : \alpha \in \Lambda\}\right]$, then there must be an x that is an element of at least one of the A_α and hence y must be in $R[A_\alpha]$, so y will also be in $\bigcup\{R[A_\alpha] : \alpha \in \Lambda\}$ and thus $R\left[\bigcup\{A_\alpha : \alpha \in \Lambda\}\right] \subset \bigcup\{R[A_\alpha] : \alpha \in \Lambda\}$.
Suppose now that $yin \bigcup\{R[A_\alpha] : \alpha \in \Lambda\}$. Then y must be an element of at least one of the $R[A_\alpha]$ and hence there mus be an x that is in a set A_α, so x will also be in $\bigcup\{A_\alpha : \alpha \in \Lambda\}$ and thus $\bigcup\{R[A_\alpha] : \alpha \in \Lambda\} \subset R\left[\bigcup\{A_\alpha : \alpha \in \Lambda\}\right]$.
From this and the previous conclusion follows $R\left[\bigcup\{A_\alpha : \alpha \in \Lambda\}\right] = \bigcup\{R[A_\alpha] : \alpha \in \Lambda\}$.

$\Diamond$

12.5(b) $\quad R\left[\bigcap\{A_\alpha : \alpha \in \Lambda\}\right] \subset \bigcap\{R[A_\alpha] : \alpha \in \Lambda\}$.

We prove first that $R\left[\bigcap\{A_\alpha : \alpha \in \Lambda\}\right] \subset \bigcap\{R[A_\alpha] : \alpha \in \Lambda\}$
Take $y \in R\left[\bigcap\{A_\alpha : \alpha \in \Lambda\}\right]$. Then there must be an $x \in A_\alpha, \forall \alpha \in \Lambda$ and thus $y \in R[A_\alpha], \forall \alpha \in \Lambda$ giving $y \in \bigcap\{R[A_\alpha] : \alpha \in \Lambda\}$ and thus $R\left[\bigcap\{A_\alpha : \alpha \in \Lambda\}\right] \subset \bigcap\{R[A_\alpha] : \alpha \in \Lambda\}$

We prove now that $R\left[\bigcap\{A_\alpha : \alpha \in \Lambda\}\right] \not\supset \bigcap\{R[A_\alpha] : \alpha \in \Lambda\}$
Be a non injective function as the relation R with $\bigcap\{R[A_\alpha] : \alpha \in \Lambda\} \neq \emptyset$. As R is a non injective function, we can have a $x \in \bigcap\{R[A_\alpha] : \alpha \in \Lambda\}$ but with $\bigcap\{A_\alpha : \alpha \in \Lambda\} = \emptyset$, which means that x can't be an element of $\bigcap\{A_\alpha : \alpha \in \Lambda\}$ and thus $R\left[\bigcap\{A_\alpha : \alpha \in \Lambda\}\right] \not\supset \bigcap\{R[A_\alpha] : \alpha \in \Lambda\}$.
(*Take for example the relation defined by* $R = \{(n, 1) : n \in \mathbb{N}\}$ *and the subsets of* $\mathbb{N}$, $A_\alpha = \{\alpha\} : \alpha \in \mathbb{N}\}$. *We have* $\bigcap\{R[A_\alpha] : \alpha \in \Lambda\} = \{1\}$ *with* $\bigcap\{A_\alpha : \alpha \in \Lambda\} = \emptyset$.)

$\blacklozenge$

1.12.2

Prove that

12.6 Let $f : X \to Y$ be a function. Let $\{A_\delta : \delta \in \Delta\}$ and $\{B_\lambda : \lambda \in \Lambda\}$ be non empty collections of subsets of X and Y respectively. Then,

$$\begin{array}{ll} \mathbf{12.6(a)} & f\left[\bigcup\{A_\delta : \delta \in \Delta\}\right] = \bigcup\{f[A_\delta] : \delta \in \Delta\}. \\ \mathbf{12.6(b)} & f\left[\bigcap\{A_\delta : \delta \in \Delta\}\right] \subset \bigcap\{f[A_\delta] : \delta \in \Delta\}. \\ \mathbf{12.6(c)} & f^{-1}\left[\bigcup\{B_\lambda : \lambda \in \Lambda\}\right] = \bigcup\{f^{-1}[B_\lambda] : \lambda \in \Lambda\}. \\ & f^{-1}\left[\bigcap\{B_\lambda : \lambda \in \Lambda\}\right] = \bigcap\{f^{-1}[B_\lambda] : \lambda \in \Lambda\}. \end{array}$$

12.6(a) $f\left[\bigcup\{A_\delta : \delta \in \Delta\}\right] = \bigcup\{f[A_\delta] : \delta \in \Delta\}$.

This a direct consequence of **12.6** with $R = f$.

◊

12.6(b) $f\left[\bigcap\{A_\delta : \delta \in \Delta\}\right] \subset \bigcap\{f[A_\delta] : \delta \in \Delta\}$.

This a direct consequence of **12.6** with $R = f$.

◊

12.6(c) $f^{-1}\left[\bigcup\{B_\lambda : \lambda \in \Lambda\}\right] = \bigcup\{f^{-1}[B_\lambda] : \lambda \in \Lambda\}$.

As f^{-1} is a relation and by **12.6** with $R = f^{-1}$ we get the asked identity.

◊

12.6(c′) $f^{-1}\left[\bigcap\{B_\lambda : \lambda \in \Lambda\}\right] = \bigcap\{f^{-1}[B_\lambda] : \lambda \in \Lambda\}$.

As f^{-1} is a relation and by **12.6** with $R = f^{-1}$ we get $f^{-1}\left[\bigcap\{B_\lambda : \lambda \in \Lambda\}\right] \subset \bigcap\{f^{-1}[B_\lambda] : \lambda \in \Lambda\}$.
We prove now that $f^{-1}\left[\bigcap\{B_\lambda : \lambda \in \Lambda\}\right] \supset \bigcap\{f^{-1}[B_\lambda] : \lambda \in \Lambda\}$.
Suppose $x \in \bigcap\{f^{-1}[B_\lambda] : \lambda \in \Lambda\}$ then $x \in f^{-1}[B_\lambda] : \forall \lambda \in \Lambda$. This means that there must be an unique $y = f(x)$ (f being a function) for which yields $y \in B_\lambda : \forall \lambda \in \Lambda$. Hence y must be in $\bigcap\{B_\lambda : \lambda \in \Lambda\}$ and thus $x \in f^{-1}\left[\bigcap\{B_\lambda : \lambda \in \Lambda\}\right]$ giving $f^{-1}\left[\bigcap\{B_\lambda : \lambda \in \Lambda\}\right] \supset \bigcap\{f^{-1}[B_\lambda] : \lambda \in \Lambda\}$.

♦

1.12.3

Prove that
12.7 Let $f : X \to Y$ be a function. Then, each of the following holds

$$\begin{array}{ll} \mathbf{12.7(a)} & \forall x \in X,\ x \in f^{-1}[f[x]]. \\ \mathbf{12.7(b)} & \forall A \subset X,\ A \subset f^{-1}[f[A]]. \\ \mathbf{12.7(c)} & \forall y \in \text{Range} f,\ f[f^{-1}[y]] = \{y\}. \end{array}$$

12.7(a) $\forall x \in X,\ x \in f^{-1}[f[x]]$.
Be $y = f(x)$, then obviously there is at least one x (there could be more if f is not injective), so that $x = f^{-1}[y]$. Hence, $\forall x \in X,\ x \in f^{-1}[f[x]]$.

◊

12.7(b) $A \subset X,\ A \subset f^{-1}[f[A]]$.
This is a consequence of the previous statement but with the remark that we could have (for a non injective function) a $x \in B$ with $A \cap B = \emptyset$ for which we have $f(x) \in f[A]$. So A is not always equal to $f^{-1}[f[A]]$ and get $A \subset X,\ A \subset f^{-1}[f[A]]$.

◊

12.7(c) $\forall y \in \text{Range} f,\ f[f^{-1}[y]] = \{y\}$.
This a direct consequence of f being a function. Indeed suppose for a given y we have the set $A = f^{-1}[\{y\}]$, so this set will contain all x as element which f maps (uniquely, f being a function) to y. So $f[A] = f[f^{-1}(y)] = \{y\}$.

♦

1.12.4

Prove that
Suppose that $f : X \to Y$ is a function and A and B are subsets of X. Suppose also that C and D are subsets of Y. For each of the following, determine whether the statement is necessarily true. In any case for which the statement is not necessarily true, determine whether it is under any of the following conditions: $f : X \to Y$ is a surjection,$f : X \to Y$ is a injection,$f : X \to Y$ is a bijection.

$$\begin{array}{ll} \mathbf{(a)} & f[A - B] = f[A] - f[B]. \\ \mathbf{(b)} & f^{-1}[D - C] = f^{-1}[D] - f^{-1}[C]. \\ \mathbf{(c)} & f^{-1}[f[A]] = A. \\ \mathbf{(b)} & f[f^{-1}[C]] = C. \end{array}$$

(a) $f[A - B] = f[A] - f[B]$.
This is not necessarily True.

Suppose, $x \in A$ but not in B and $y = f(x)$, so $y \in f[A - B]$ but if f is not a surjection then it is possible that a $x^{'} \in B$ exists which is mapped to y, meaning that y will not be an element of $f[A] - f[B]$ i.e. $y \notin f[A] - f[B]$ and thus $f[A - B] \not\subset f[A] - f[B]$ meaning that not always $f[A - B] = f[A] - f[B]$. So this identity can only be true if f is a surjection or a bijection as a bijection has to be an injection. Of course $f : X \to Y$ is a surjection, is not a sufficient condition for the identity to be true as a surjection is not necessarily an injection.

$\Diamond$

(**b**) $f^{-1}[D - C] = f^{-1}[D] - f^{-1}[C]$.
This is True if f is a surjection (or by extension a bijection).
Suppose, $x \in f^{-1}[D - C]$, so there is a $y \in D - C$ for which $x = f^{-1}(y)$. Also, $y \in D$ but not in C. Can there be $y^{'} \in C, \notin D$ for which $y^{'} = f(x)$? Obviously not, as f is a function meaning that $y^{'} = f(x)$ and $y = f(x) \Rightarrow y^{'} = y$. This means that x can't be an element of $f^{-1}[C]$ and thus that $x \in f^{-1}[D] - f^{-1}[C]$. Hence, $f^{-1}[D - C] \subset f^{-1}[D] - f^{-1}[C]$.
Suppose now that $x \in f^{-1}[D] - f^{-1}[C]$, so there is no $x \in f^{-1}[C]$ for which $y = f(x)$, $y \in C$. This means that $y \notin C$ but y must be in D. i.e. $y \in D - C$ and thus $x \in f^{-1}[D - C]$ or $f^{-1}[D] - f^{-1}[C] \subset f^{-1}[D - C]$, leading to the identity.
Remark that the reasoning deployed implies that f is a surjection as if $y \in D - C$ has no inverse image in X, this would mean that $f^{-1}[D - C] = \emptyset$ and thus no x would exist for the given identity.

$\Diamond$

c) $f^{-1}[f[A]] = A$.
This is not necessarily True.
Be $y \in f[A]$, if f is not an injection then it is possible that there exist a $x^{'} \in B \not\subset A$ so that $f(x^{'}) = y$, so $x^{'}$ will be an element of $f^{-1}[f[A]]$ and as $x^{'} \notin A$ the identity can't be true. So, f needs to be an injection (and by extension a bijection) for the identity to be true.

$\Diamond$

(**d**) $f[f^{-1}[C]] = C$.
This is True if f is a surjection (or by extension a bijection).
Be $y \in C$ and $x \in f^{-1}[C]$, as f is a function then $f(x)$ will be in C and the set $f^{-1}[C]$ will contain all x for which $f(x) \in C$. But note that C may contain elements which are not mapped by f. In that case $f[f^{-1}[C]] \not\subset C$. So f needs to be a surjection (or by extension a bijection).
On the other hand, suppose $y \in C$, then $f^{-1}[C]$ will contain all $x \in X$ which are mapped to C and f being a function we will have $f[f^{-1}[C]] \subset C$.
Conclusion, the identity is true if f is a surjection (or by extension a bijection).
To illustrate this, take A as the subset $\mathbb{N} \subset \mathbb{R}$ and C as the subset of $\mathbb{R}$ with the even natural numbers as elements. Define now the function $f = \{(n, 4n) : n \in A\}$. Obviously $f[A] \neq C$ as $f[A] = \{4, 8, 12, \dots\} \neq \{2, 4, 6, 8, \dots\}$. Then as $f^{-1}[C] = A$, we have $f[f^{-1}[C]] \neq C$

$\blacklozenge$

1.12.5

Let $M : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be the map from $\mathbb{R}\times\mathbb{R}$ into $\mathbb{R}$ defined as follows: For each $(a, b) \in \mathbb{R}\times\mathbb{R}$, let $M((a, b)) = ab$. Is M a map from $\mathbb{R} \times \mathbb{R}$ onto $\mathbb{R}$? Representing $\mathbb{R} \times \mathbb{R}$ s a plane, draw a sketch for each of the following sets: $M^{-1}[0]$, $M^{-1}[1]$, $M^{-1}[I]$, where I is the closed interval $[0, 1]$.

Yes, M is a map from $\mathbb{R} \times \mathbb{R}$ onto $\mathbb{R}$ as every $x \in \mathbb{R}$ can be expressed as the product of two real numbers.

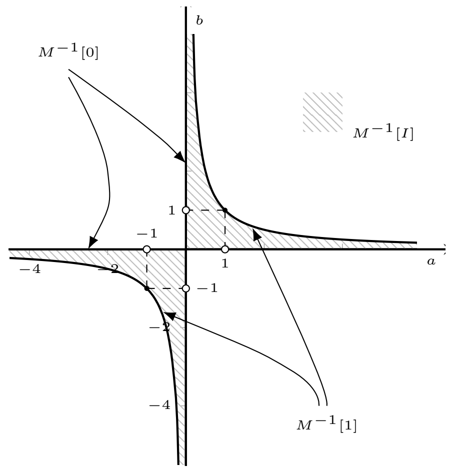


Figure 1.18: Inverse image of $M^{-1}(0)$, $M^{-1}(1)$, $M^{-1}(I)$

♦

1.12.6

Examine carefully the content of Theorem **12.6** and your answer to Exercise 4(a) and (b). Which seems to have a nicer behaviour on collections of sets, f or f^{-1}?

Putting aside the notion of 'nicer', we still could put forward that f^{-1} requires less restrictions in order to have certain identities.
Take first **12.6**(b) $\quad f\left[\bigcap\{A_\delta : \delta \in \Delta\}\right] \subset \bigcap\{f[A_\delta] : \delta \in \Delta\}$ compared to **12.6**($c^{'}$) $\quad f^{-1}\left[\bigcap\{B_\lambda : \lambda \in \Lambda\}\right] = \bigcap\{f^{-1}[B_\lambda] : \lambda \in \Lambda\}$. f^{-1} requires no special condition (except for f being a function) in order to have an equality for the intersection of the sets in the collection.
Moreover in Exercise 4, for having the identity 4(a) $\quad f[A - B] = f[A] - f[B]$, we need f to be at least an injection while for 4(b) $\quad f^{-1}[D - C] = f^{-1}[D] - f^{-1}[C]$ we "only need af to be a surjection which can be achieved by restricting the target Y to $D \cup C$.

♦

1.13 The restriction of a function

1.13.1

Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ be mappings such that $f(x)=\sin x$ for each $x\in\mathbb{R}$ and $g(x)=\sqrt{1-\cos^2 x}$ for each $x\in\mathbb{R}$. Find the largest interval of real numbers, I, whose left endpoint is 0 and which satisfies $f|I=g|I$.

As $g(x)$ can be expressed as $g(x)=|\sin x|$ the condition $f(x)=g(x)$ will only be met in the intervals $\{[2\pi k,\pi(2k+1)] : k\in\mathbb{P}\cup\{0\}\}$. Putting $k=0$ we get $I=[0,\pi]$.

♦

1.13.2

Let $f:\mathbb{R}\to\mathbb{R}$ be defined as follows. For each $x\in\mathbb{R}$, let $f(x)=|x-1|$. Let $g:\mathbb{R}\to\mathbb{R}$ be defined by and $g(x)=x-1$ for each $x\in\mathbb{R}$. Find the largest set $S\subset\mathbb{R}$ for which $f|I=g|I$.

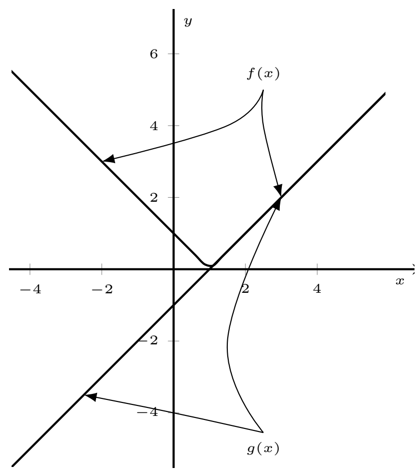


Figure 1.19: $f(x)=|x-1|$ and $g(x)=x-1$

From the figure we get $S=[1,+\infty)$.

♦

1.13.3

Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ be given by $g(x)=\cos x$ and $f(x)=\sqrt{1-\sin^2 x}$ for $x\in\mathbb{R}$. Find the largest set $S\subset\mathbb{R}$ for which $f|I=g|I$.

As $f(x)\equiv|\cos x|$, $f(x)=g(x)$ implies $S=\{[-\frac{\pi}{2}k,-\frac{\pi}{2}k+\pi] : k\in\mathbb{Z}_{/\{0\}}\}$.

1.14 Composition of functions

1.14.1

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \frac{x}{x+1}$ and $g(x) = x^2$. Give explicit formulas for $g \circ f(x)$ and $f \circ g(x)$. Determine the ranges of $g \circ f$ and $f \circ g$.

$g \circ f(x) = \frac{x^2}{x^2+2x+1}$
$f \circ g(x) = \frac{x^2}{x^2+1}$
$Range\, g \circ f = [0, +\infty)$
$Range\, f \circ g = [0, 1)$

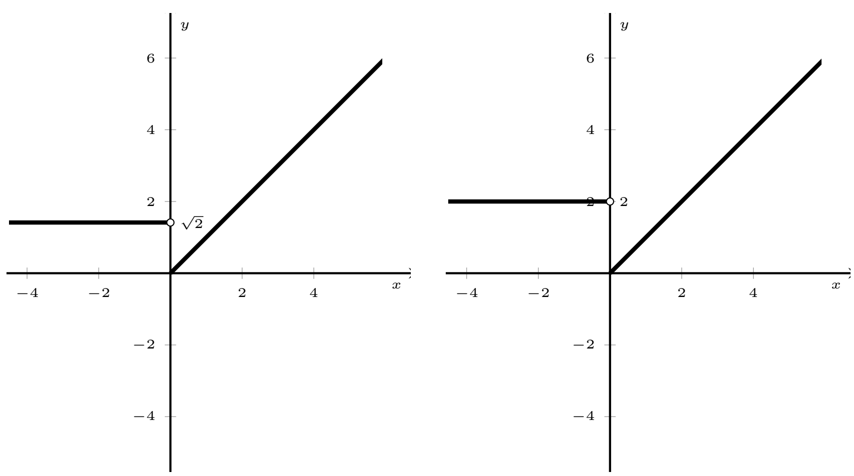
♦

1.14.2

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$\begin{aligned} f(x) &= x^2 && \text{for } x \geq 0 \\ &= 2 && \text{for } x < 0 \\ g(x) &= \sqrt{x} && \text{for } x \geq 0 \\ &= x && \text{for } x < 0 \end{aligned}$$

(a) Sketch the graph of $g \circ f$
(b) Sketch the graph of $f \circ g$
(c) Find $(f \circ g)^{-1}[x]$ for each $x \in \mathbb{R}$



(a) $g \circ f$ (b) $f \circ g$

(c) Find $(f \circ g)^{-1}[x]$ for each $x \in \mathbb{R}$

$(f \circ g)^{-1}[x] = \emptyset, \forall x < 0$

$(f \circ g)^{-1}[x] = \{x\}, \forall x \geq 0$

$(f \circ g)^{-1}[x] = (-\infty, 0) \cup \{2\}$ for $x = 2$

♦

1.14.3

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \sin x$ and $g(x) = |x|$. Write explicit expressions for $g \circ f(x)$ and $f \circ g(x)$ and find the range of each.

$g \circ f(x) = |\sin x|$
$f \circ g(x) = \sin |x|$
$Range\, g \circ f = [0, 1]$
$Range\, f \circ g = [-1, 1]$

♦

1.14.4

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2 + 2$ and $g(x) = x - 1$. Find expressions for $(g \circ f)(x)$ and $(f \circ g)(x)$ and note that $g \circ f \neq f \circ g$.

$(g \circ f)(x) = x^2 + 1$
$(f \circ g)(x) = x^2 - 2x + 3$

♦

1.14.5

Let $f:\mathbb{R}\to\mathbb{R}$,$g:\mathbb{R}\to\mathbb{R}$ and $h:\mathbb{R}\to\mathbb{R}$ be given by $f(x)=x^2+x$, $g(x)=(x-1)^2$ and $h(x)=x+1$ for each $x\in\mathbb{R}$. Find an expression for $h\circ g\circ f(x)$ for $x\in\mathbb{R}$.

$$h\circ g\circ f(x)=\underbrace{\underbrace{\left((\underbrace{x^2+x}_{f(x)})-1\right)^2}_{g(x)}+1}_{h(x)}=x^4+2x^3-x^2+2x+2$$

♦

1.14.6

Suppose $f:X\to Y$ is a bijection. Show that $f^{-1}\circ f=i$ where $i:X\to X$ is the identity map on X and $f\circ f^{-1}=j$ where j is the identity map on Y.

$\mathbf{f^{-1}\circ f=i}$
Let $x\in X$. Then, $(x,f(x))\in f$. Hence, $(f(x),x)\in f^{-1}$. thus, $f^{-1}(f(x))=x$ and $\left(f^{-1}\circ f\right)(x)=x$ for each $x\in X$. hence, $f-1\circ f=i$, the identity map on X.
$\mathbf{f=i\circ f^{-1}=j}$
Let $y\in Y$. Then, $\left(y,f^{-1}(y)\right)\in f^{-1}$. Hence, $\left(f^{-1}(y),y\right)\in f$. thus, $f\left(f^{-1}(y)\right)=y$ and $\left(f\circ f^{-1}\right)(y)=y$ for each $y\in Y$. hence, $f\circ f^{-1}=j$, the identity map on Y.
Note that in the step $\left(y,f^{-1}(y)\right)\in f^{-1}$ we implicitly use the fact that f is a bijection.

♦

1.14.7

Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ be surjections. Suppose $g\circ f=i$ where $i:X\to X$ is the identity map on X. Show:

(a) f is one-to-one
(b) g is one-to-one
(c) $f\circ g=j$ where $j:X\to X$ is the identity map fromY onto Y
(d) $f=g^{-1}$
(d) $g=f^{-1}$

(a) **f** is one-to-one

Suppose that f is not one-to-one. This means that there exist 2 or more $x_1, x_2, \ldots$ which are mapped by f to an element $y^* \in Y$. But as g is a function, the image $g[\{y^*\}]$ can only have one element. Hence, x_1 or x_2 will not be included in this image. Suppose that x_2 is not an element of $g[\{y^*\}]$. As g is a surjection, there must be an $y^{'} \in Y$ so that $g(y^{'}) = x_2$. But also f is a surjection, so there must be an $x_3 \in X$ so that $f(x_3) = y^{'}$. So $g(y^{'}) = x_2$ implies $g(f(x_3)) = x_2$. But $g(f(x_3)) = g \circ f(x^3)$ and given that $g \circ f = i$ (the identity map) we get $g \circ f(x^3) = x_3$ and can conclude that $x_2 = x_3$ resulting in $f(x_2) = y^{'}$. But we started with the assumption that $f(x_2) = y^*$ and as f is a function, this means $y^{'} = y^*$, so there is no other $(y, x_2) \in g$ that maps y to x_2, meaning that only the element x_1 is mapped to the chosen y^*. Hence f is one-to-one.

$\Diamond$

(b) **g** is one-to-one

Suppose that g is not one-to-one. This means that for a given x^* we could have 2 (or more) $y_1, y_2, \ldots,$ so that $g(y_1) = x^*$ and $g(y_2) = x^*$. But f is on-to-one (see $1.14.7(a)$) so we must have two distinct x_1, x_2 for which we have $f(x_1) = y_1$ and $f(x_2) = y_2$, but $g(y_1) = x^*$ and $g(y_2) = x^*$, so $g \circ f(x_1) = x^*$ and $g \circ f(x_2) = x^*$ and given that $g \circ f = i$ (the identity map) we get $x_1 = x_2 = x^*$ and also $y_1 = y_2$ and conclude that g must be one-to-one.

$\Diamond$

(c) $\mathbf{f} \circ \mathbf{g} = \mathbf{j}$ where $\mathbf{j} : \mathbf{X} \to \mathbf{X}$ is the identity map from **Y** onto **Y**

We have $g \circ f = i$, so $g \circ f(x) = x$. As $x \in X$ we can apply f to x and this gives $(f \circ g \circ f)(x) = f(x)$ or $(f \circ g) \circ f(x) = f(x)$. Put $f(x) = y$, we can rewrite this as $(f \circ g)(y) = y$. Hence $f \circ g$ is the identity map $j : Y \to Y$.

$\Diamond$

(d) $\mathbf{f} = \mathbf{g}^{-1}$

Be $(y, x) \in g$. Then, $(x, y) \in g^{-1}$. As f is a bijection, we have one $x \in X$ and one $y \in Y$ so that $y = f(x)$. Then, $(x, y) \in g^{-1}$ is equivalent to $(x, f(x)) \in g^{-1}$. But, by definition , we have $(x, f(x)) \in f$, and conclude $f \subset g^{-1}$.

Be now, $(x, y) \in f$. As g is one-to-one (see above), there is just one (x, y) so that $x = g(y)$. Thus, $(g(y), y) \in f$. We notice that $(y, g(y)) \in g$, or $(g(y), y) \in g^{-1}$. This give with $(g(y), y) \in f$, $g^{-1} \subset f$. Combining with the first subset we get $f = g^{-1}$.

$\Diamond$

(e) $\mathbf{g} = \mathbf{f}^{-1}$

Be $(x, y) \in f$. Then, $(y, x) \in f^{-1}$. As g is a bijection, we have one $x \in X$ and one $y \in Y$ so that $x = g(y)$. Then, $(y, x) \in f^{-1}$is equivalent to $(y, g(y)) \in f^{-1}$. But, by definition , we have $(y, g(y)) \in g$, and conclude $g \subset f^{-1}$.

Be now, $(y, x) \in g$. As g is one-to-one (see above), there is just one (y, x) so that $x = g(y)$. Thus,

$(y, g(y)) \in g$. But $y = f(x)$. This gives $(f(x), \underbrace{f \circ g}_{=j}(y)) \in g$. But we notice that $(x, f(x)) \in f$ or $(f(x), x) \in f^{-1}$ and conclude with $(f(x), x) \in g$ that $f^{-1} \subset g$.
Combining with the first subset we get $g = f^{-1}$.

♦

1.14.8

Recall that $\mathbb{R}_+ = \{x : x \in \mathbb{R} \text{ and } x > 0\}$. Recall also that the natural logarithm function , ln, is defined on $\mathbb{R}_+$, with range $\mathbb{R}$; the exponential function, $\{(x, e^x) : x \in \mathbb{R}\}$, is the inverse of the of the ln function.; $\cosh x = \frac{e^x + e^{-x}}{2}$ for each $x \in \mathbb{R}$.
Sketch the cosh function and note that although cosh is not one-to-one, the restriction $\cosh|\mathbb{R}_+$ is one-to-one and hence, its inverse is a function. By using the results of Exercise 1.14.7, prove that $\cosh^{-1}(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$ for $x \geq 1$.

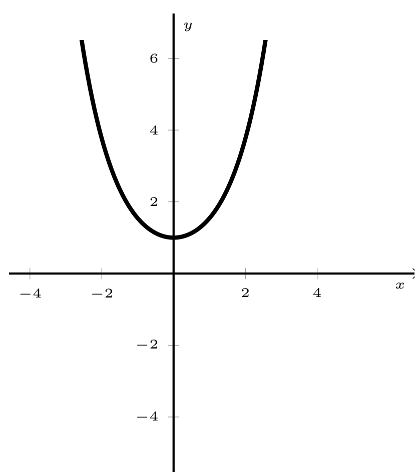


Figure 1.20: The function $\cosh x$

For two functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ for which yields $g \circ f = i$, we know from Exercise 1.14.7(c) that $f \circ g = j$ (the unit map). We put $f(x) = \cosh x = \frac{e^x + e^{-x}}{2}$ and try to find $g = f^{-1}$. From the sketch, we see that if we restrict x to $[0, +\infty)$ that f is bijective with $\mathit{Range}\, f = [1, +\infty)$. What we have to do, is to solve the functional equation $g\left(\frac{e^x + e^{-x}}{2}\right) = x$ (with $g = f^{-1}$) (this expression represent $g \circ f = i$).
We use the identity $a = e^{\ln a}$ and use the equivalence of $g \circ f = i$ and $f \circ g = j$, i.e.

$$\frac{e^{g(y(} + e^{-g(y)}}{2} = y$$

We put tentatively $g(y) = \ln p(y)$ with $p : [1, +\infty) \to \mathbb{R}$ and get from $a = e^{\ln a}$:

$$
\begin{aligned}
& \frac{e^{g(y)} + e^{-g(y)}}{2} = y \\
\Leftrightarrow \quad & \frac{e^{\ln p(y)} + e^{-g(y)}}{2} = y \\
\Leftrightarrow \quad & \frac{p(y) + \frac{1}{p(y)}}{2} = y \\
\Leftrightarrow \quad & p^2(y) - 2p(y)y = -1 \\
\Leftrightarrow \quad & p^2(y) - 2p(y)y + y^2 = y^2 - 1 \\
\Leftrightarrow \quad & (p(y) - y)^2 = y^2 - 1 \\
\Rightarrow \quad & p(y) = y \pm \sqrt{y^2 - 1}
\end{aligned}
$$

From our definition $g(y) = \ln p(y)$, we get $g(y) = \ln\left(y \pm \sqrt{y^2 - 1}\right)$. To get a real valued g we need obviously $y \geq 1$ but also need that $g(y) \geq 0$ as we restricted the function f to $\mathbb{R}_+$ which requires $y \pm \sqrt{y^2 - 1} \geq 1$ and thus only the solution $g(y) = \ln\left(y + \sqrt{y^2 - 1}\right), \quad y \geq 1$ can be retained.

♦

1.14.9

Consider the map $P : \mathbb{R}^2 \to \mathbb{R}$ such that for each $(x, y) \in \mathbb{R}^2$, $\alpha((x, y))$ is the element $(u, v) \in \mathbb{R}^2$ given by $u = 2x - y$, $v = 5x + y$. Recall that $\mathbb{R}^2$ denotes $\mathbb{R} \times \mathbb{R}$.

(a) Does $\alpha[\mathbb{R}^2] = \mathbb{R}^2$?.
(b) Is α one-to-one?.
(c) If α is one-to-one, find a rule for α^{-1} analogous to the rule given for α.

(a) Does $\alpha[\mathbb{R}^2] = \mathbb{R}^2$?
(b)Is α one-to-one?.
The answer is yes to both questions as we can consider the map as a system of linear equation with (x, y) as unknowns and (u, v) as parameters i.e.

$$
\begin{cases}
2x - y = u \\
5x + y = v
\end{cases}
$$

The determinant of this system is not zero (it is 7) and so for every (u, v) we have a unique (x, y). From the definition it is also clear that $u, v \in \mathbb{R}$ (it is in fact a map from one plane to another plane).

So, $\alpha[\mathbb{R}^2] = \mathbb{R}^2$ and α is a bijection.

♢

(c) If α is one-to-one, find a rule for α^{-1} analogous to the rule given for α.

$$\begin{cases} x = & \frac{u+v}{7} \\ y = & \frac{2v-5u}{7} \end{cases}$$

♦

1.14.10

> Consider the map $P : \mathbb{R}^2 \to \mathbb{R}$ such that for each $(x, y) \in \mathbb{R}^2$, $P((x, y)) = x$. Note that P is not one-to-one. Find a subset S of $\mathbb{R}^2$ such that $P|S : S \to \mathbb{R}$ is a one-to-one map from S to $\mathbb{R}$.

We can find two types of restrictions:
A type with y a constant i.e. $P_r|S_r : S_r \to \mathbb{R}$ with $P_r = \{(x, r) : x \in \mathbb{R},\, r \in \mathbb{R} = \text{ constant}\}$.

And a type with $y = f(x)$ where $f : \mathbb{R} \to \mathbb{R}$ is a one-to-one function.

♦

1.14.11

> Suppose $f : A \to B$ is a one-to-one map from A to B and $g : B \to C$ is a one-to-one map from B into C. Prove that $g \circ f : A \to C$ is one-to-one map from A into C.

Suppose $g \circ f$ is not one-to-one. This means that for at least one $z \in C$ there exist two (or more) x_1, x_2 such that (x_1, z^*) and $(x_2, z^*) \in g \circ f$. But $f : A \to B$ and $g : B \to C$ are one-to-one maps which means that $x_1 \in A$ is mapped to a unique $y_1 \in B$ (the same for x_2 mapped to a unique y_2) and also $y_1 \in B$ is mapped to a unique $z_1 \in C$ (the same for y_2 mapped to a unique z_2). So the map $g \circ f$ will map x_1 into z_1 and x_2 into z_2. So supposition that there exist a z^* is in contradiction with our result that $z_1 \neq z_2$.
Is $g \circ f$ a surjection ?
As f and g are one-to-one, every $y \in B$ and every $z \in C$ are covered by f, g respectively, so every $z \in C$ is also covered by $g \circ f$. So, $g \circ f$ is a surjection and by the first argument also a bijection.

♦

1.14.12

Let $f : \mathbb{R} \to \mathbb{R}$ be a map such that for each pair of numbers x and y, $f(x+y) = f(x) + f(y)$.

(a) Show that $f(0) = 0$.
(b) Show that $f(-x) = -f(x), \quad \forall x \in \mathbb{R}$.
(c) Show that $f(mx) = mf(x), \quad \forall m \in \mathbb{Z}, x \in \mathbb{R}$.
(d) Show that $f(rx) = rf(x), \quad \forall r \in \mathbb{Q}, x \in \mathbb{R}$.

(a) Show that $f(0) = 0$.
Put $x = 0$ and $y = 0$. We have, $\underbrace{f(0+0)}_{=f(0)} = \underbrace{f(0) + f(0)}_{=2f(0)}$. This implies that $f(0) = 0$.

(b) Show that $f(-x) = -f(x), \quad \forall x \in \mathbb{R}$.
Put $y = -x$. We have, $\underbrace{f(x + (-x))}_{=f(0)=0} = f(x) + f(-x)$. This implies that $f(-x) = -f(x)$.

(c) Show that $f(mx) = mf(x), \quad \forall m \in \mathbb{Z}, x \in \mathbb{R}$.

$$\begin{aligned} f(mx) &= f(x + (m-1)x) \\ &= f(x) + f((m-1)x) \\ &= f(x) + f(x + (m-2)x) \\ &= f(x) + f(x) + f((m-2)x) \\ &= \quad \vdots \\ &= \underbrace{f(x) + f(x) + \ldots f(x)}_{=mf(x)} \end{aligned}$$

and get the required result.
(d) Show that $f(rx) = rf(x), \quad \forall r \in \mathbb{Q}, x \in \mathbb{R}$.

$$\begin{array}{ccc} & f(rx) & = f(\frac{m}{n}x) \\ \Leftrightarrow & f(rx) & = mf(\frac{1}{n}x) \\ \times n & nf(rx) & = mnf(\frac{1}{n}x) \\ \Leftrightarrow & nf(rx) & = mf(\frac{n}{n}x) \\ \Rightarrow & f(rx) & = \frac{m}{n}f(x) \\ \Rightarrow & f(rx) & = rf(x) \end{array}$$

♦

1.15 Sequences

1.15.1

In each of the following find a formula for the nth term a_n of an infinite sequence whose first five terms are given.

$$\begin{array}{cc} (a) & a_l = 1, a_2 = \frac{1}{2}, a_3 = \frac{1}{4}, a_4 = \frac{1}{8}, a_5 = \frac{1}{16} \\ (b) & a_1 = 1, a_2 = 0, a_3 = 1, a_4 = 0, a_5 = 1. \\ (c) & a_1 = 1, a_2 = 0, a_3 = -1, a_4 = 0, a_5 = 1. \\ (d) & a_1 = 1, a_2 = 3, a_3 = 6, a_4 = 10, a_5 = 15. \end{array}$$

(a) $a_l = 1, a_2 = \frac{1}{2}, a_3 = \frac{1}{4}, a_4 = \frac{1}{8}, a_5 = \frac{1}{16}$

$$a_n = \frac{1}{2^{n-1}}$$

$\Diamond$

(b) $a_1 = 1, a_2 = 0, a_3 = 1, a_4 = 0, a_5 = 1$

$$a_n = n \pmod 2$$

$\Diamond$

(c) $a_1 = 1, a_2 = 0, a_3 = -1, a_4 = 0, a_5 = 1$

$$a_n = \left(\sin \frac{\pi}{2} n\right) (n \pmod 2)$$

$\Diamond$

(d) $a_1 = 1, a_2 = 3, a_3 = 6, a_4 = 10, a_5 = 15$

$$a_{n+1} = a_n + (n+1), \quad a_1 = 1$$

$\blacklozenge$

1.15.2

Let $f = (f_i)_{i=1}^{+\infty}$ g, be the sequence defined as follows: Let $g(x) = \sin x$. For each positive integer i, let $f_i = g^{(i)}(0)$, where $g^{(i)}$ is the i^{th} derivative of g. Write the terms off sufficiently far to see the pattern followed.

We have $g^{(1)}(x) = \cos x$, $g^{(2)}(x) = -\sin x$, $g^{(3)}(x) = -\cos x$, $g^{(4)}(x) = \sin x$, $g^{(5)}(x) = \cos x$, $\dots$, so we get $f = (1, 0, -1, 0, 1, 0, -1, \dots$, so from Exercise 1.15.1(c), we get

$$f_i = \left(\sin \frac{\pi}{2} i\right) (i \pmod 2)$$

♦

1.15.3

For each $n \in P$, let $a_n = \sum_{j=1}^{n} j^2$. Try to discover a formula for a_n.

We can suppose that when $n \to +\infty$ that $a_n = \sum_{j=1}^{n} j^2$ and $\int_0^n x^2 dx$ will tend to a common value. As $\int_0^n x^2 dx = \frac{1}{3}n^3$ we tentatively write a_n as a cubic expression $a_n = an^3 + bn^2 + cn + d$. As $a_{n+1} = a_n + (n+1)^2$ we try to find the parameters a, b, c, d by solving the expression

$$a(n+1)^3 + b(n+1)^2 + c(n+1) + d = an^3 + bn^2 + cn + d + (n+1)^2$$

for some values of n. Expanding gives

$$(3a-1)n^2 + (3a+2b-2)n + a + b + c - 1 = 0$$

For $n = 1, 2, 3$ we get the following system of linear equations

$$\begin{cases} 7a + 3b + c = 4 \\ 19a + 5b + c = 9 \\ 37a + 7b + c = 16 \end{cases}$$

and get $a_n = \frac{1}{3}$, $b = \frac{1}{2}$, $c = \frac{1}{6}$ giving $a_n = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n + d$ and for $n = 1$, $a_n = 1$ we get $1 = \underbrace{\frac{1}{3} + \frac{1}{2} + \frac{1}{6}}_{=1} + d$, giving $d = 0$. So, we get the expression $a_n = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$ which can be rewritten as

$$a_n = \frac{n(2n+1)(n+1)}{6}$$

Yet, we didn't prove formally that this equation works for every n:
We notice that this equation is exact for $n = 1$. Suppose that it is also correct for a n. Then:

$$
\begin{aligned}
a_{n+1} &= a_n + (n+1)^2 \\
&= \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n + (n+1)^2 \\
&= \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n + n^2 + 2n + 1 \\
&= \frac{1}{3}n^3 + \frac{3}{2}n^2 + \frac{13}{6}n + 1
\end{aligned}
$$

Using the proposed closed expression for $n+1$ gives

$$
\begin{aligned}
a_{n+1} &= \frac{1}{3}(n+1)^3 + \frac{1}{2}(n+1)^2 + \frac{1}{6}(n+1) \\
&= \frac{1}{3}n^3 + n^2 + n + \frac{1}{3} + \frac{1}{2}n^2 + n + \frac{1}{2} + \frac{1}{6}n + \frac{1}{6} \\
&= \frac{1}{3}n^3 + \frac{3}{2}n^2 + \frac{13}{6}n + 1
\end{aligned}
$$

So, both expression are the same and by the axiom of induction , we conclude that the expression is correct for every n.

♦

1.15.4

Suppose that a is a sequence such that $a_l = 1$, $a_2 = 3$, $a_3 = a_1 + a_2$, and for $j \geq 3$, $a_j = a_{j-1} + a_{j-2}$. Find a_4, a_5, a_6 and a_7.

$a_4 = 6$, $a_5 = 10$, $a_6 = 16$, $a_7 = 26$.

♦

1.15.5

Suppose a sequence a is given by $a_n = 2^n$ for each positive integer n. For which values of n is it true that $a_n \geq 10,000$?

We have to find a n such that $\log_2 2^n \geq \log_2 10,000$ (the logarithm function is a strict increasing function).

So, $n \geq \log_{10} 10,000 \log_2 10$ or $n \geq \underbrace{4 \times 3.3}_{=13.3}$ and hence $n \geq 14$.

♦

1.15.6

Let a be the sequence given by $a_n = \frac{n}{(n+1)}$. Find the smallest integer N such that for $n \geq N$, $a_n > \frac{9}{10}$.

We need $\frac{n}{(n+1)} > \frac{9}{10}$ or $n > 9$, hence $N = 10$.

♦

1.15.7

Let a be the sequence given by $a_n = \sqrt{n+1} - \sqrt{n}$. Find an integer N such that for $n \geq N$, $a_{n+1} < a_n$. Find an integer M such that for $n \geq M$, $a_n \leq \frac{1}{10}$.

$a_{n+1} < a_n$ gives $\sqrt{n+1} - \sqrt{n} < \sqrt{n} - \sqrt{n-1}$ or $\sqrt{n+1} + \sqrt{n-1} < 2\sqrt{n}$. As both sides are positive, we can take the power of thus inequality and get $2n + 2\sqrt{n^2-1} < 4n$ or $\sqrt{n^2-1} < n$ giving the trivial inequality $0 > -1$ meaning that the inequality yields for all $N > 0$.

$a_n \leq \frac{1}{10}$
We have

$$
\begin{aligned}
& \sqrt{n+1} - \sqrt{n} \leq \frac{1}{10} \\
\Rightarrow \quad & n + 1 \leq \left(\sqrt{n} + \frac{1}{10}\right)^2 \\
\Rightarrow \quad & (4.95)^2 \leq n
\end{aligned}
$$

from which we conclude that n must be greater or equal to 25.

♦

1.15.8

For each $(x, y) \in \mathbb{R}^2$, let $f(x, y) = (x^2 - y^2, 2xy)$. Let the point p_1 in the plane be given by $f(\frac{1}{2}, \frac{1}{2})$, $p_2 = f(p_1)$, $p_3 = f(p_2)$, $p_4 = f(p_3)$, $p_5 = f(p_4)$. Calculate and plot the points $p_1, p_2, \ldots, p_5$.

$p(1) = (0, \frac{1}{2})$, $p(2) = (-\frac{1}{2^2}, 0)$, $p(3) = (\frac{1}{2^4}, 0)$, $p(4) = (\frac{1}{2^8}, 0)$, $p(5) = (\frac{1}{2^{16}}, 0)$

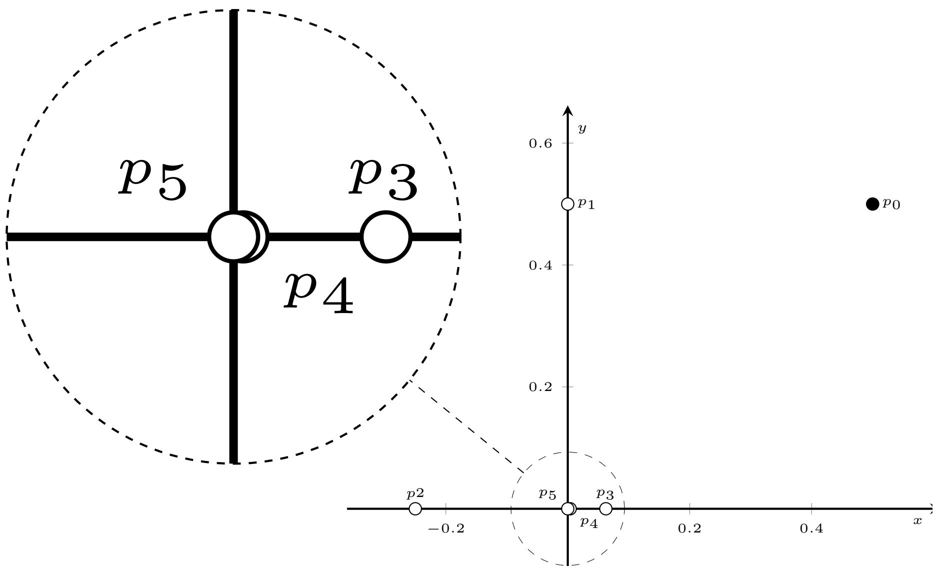


Figure 1.21: The dynamical system $f(x, y) = (x^2 - y^2, 2xy)$ with starting point $(\frac{1}{2}, \frac{1}{2})$

♦

1.15.9

Let $A_l = \{1, 2, 3\}, A_2 = \{1, 2\}, A_3 = \{a, b\}$. Write out the elements of $A_1 \times A_2 \times A_3$. Write out the elements of $(A_1 \times A_2) \times A_3$. Show that there is a ”natural” one-to-one correspondence between the elements of these two sets.

$A_1 \times A_2 \times A_3$

$$\begin{Bmatrix} (1,1,a) & (1,1,b) & (1,2,a) & (1,2,b) \\ (2,1,a) & (2,1,b) & (2,2,a) & (2,2,b) \\ (3,1,a) & (3,1,b) & (3,2,a) & (3,2,b) \end{Bmatrix}$$

$(A_1 \times A_2) \times A_3$

$$\begin{Bmatrix} ((1,1),a) & ((1,1),b) & ((1,2),a) & ((1,2),b) \\ ((2,1),a) & ((2,1),b) & ((2,2),a) & ((2,2),b) \\ ((3,1),a) & ((3,1),b) & ((3,2),a) & ((3,2),b) \end{Bmatrix}$$

As the two sets have the same number of elements, we can define a bijection between the two sets. The most natural bijection is the one which maps an element $(x, y, z) \in A_1 \times A_2 \times A_3$ to $((x, y), z) \in (A_1 \times A_2) \times A_3$ according to the following scheme:

$$(x, y, z) \in A_1 \times A_2 \times A_3$$

$$((x, y), z) \in (A_1 \times A_2) \times A_3$$

Figure 1.22: "natural" one-to-one correspondence between the elements of $A_1 \times A_2 \times A_3$ and $(A_1 \times A_2) \times A_3$

1.15.10

Suppose that for each $i \in \mathbb{P}$, $A_i = \{0,\ 1\}$. Describe in words the set $\bigtimes\{A_i : i \in \mathbb{P}\}$.

Each element of $\bigtimes\{A_i : i \in \mathbb{P}\}$ is of the form $(1, 0, 0, \dots, 1, 1 \dots 0, 1 \dots)$ and can be interpreted as the binary representation of any $n \in \mathbb{P} \cup \{0\}$. (In the given example $n = 1 \times 2^0 + 0 \times 2^1 + 0 \times 2^2 + \dots + 1 \times 2^k + 1 \times 2^{k+1} + \dots + 0 \times 2^p + 1 \times 2^{p+1} + \dots$).

1.15.11

Let A_1 be the set of all real numbers $\mathbb{R}$. For each $i \in \mathbb{P}$ such that $i \geq 2$, let $A_i = \{0\}$. Describe in words the set $\bigtimes\{A_i : i \in \mathbb{P}\}$. Show that there exists a bijection from $\bigtimes\{A_i : i \in \mathbb{P}\}$ onto $\mathbb{R}$.

Each element of $\bigtimes\{A_i : i \in \mathbb{P}\}$ is of the form $(r, 0, 0, 0, \dots)$, $r \in \mathbb{R}$ and can be interpreted in different ways. One way is to consider an element of $\bigtimes\{A_i : i \in \mathbb{P}\}$ as the coordinates of a family of points in an infinite dimensional vector space where the points are strictly collinear with the first base vector. The bijection is trivial as for each point of this family, only one real number is assigned along the first base vector

♦

1.16 Sequences and Subsequences

1.16.1

Consider the sequence $S = (1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots)$. Find a map $N : \mathbb{P} \to \mathbb{P}$ such that $S \circ N$ is the sequence $(\frac{1}{3}, \frac{1}{6}, \ldots, \frac{1}{3n}, \ldots)$.

Define the map $N : \mathbb{P} \to \mathbb{P}$ as $N = \{n, 3n) : n \in \mathbb{P}\}$.

♦

1.16.2

Consider the sequence S such that $S(n) = (-1)^n \frac{1}{2^n}$. Find the nth term of the subsequence of S whose terms consist of all the positive terms of S and none of the negative terms of S.

Define the map $N : \mathbb{P} \to \mathbb{P}$ as $N = \{n, 2n) : n \in \mathbb{P}\}$ (each $n\mathbb{P}$ is mapped to an even element of $\mathbb{P}$). Hence $(S \circ N)_n = \frac{1}{4^n}$.

♦

1.16.3

For each positive integer n, let h_n be the function given by $h_n = \{(x, x^{n+1}) : 0 \leq x \leq 1\}$. Suppose that k is the sequence such that for each $n \in \mathbb{P}$, $k_n = \int_0^1 h_n dt$. Find the nth term of k.

$$\begin{aligned} k_n &= \int_0^1 t^{n+1} dt \\ &= \frac{1}{n+2} (x^{n+2})_0^1 \\ &= \frac{1}{n+2} \end{aligned}$$

♦

1.16.4

In each of the following determine whether or not the sequence is a strictly increasing sequence.

$$
\begin{array}{ll}
(a) & \left(\frac{n}{n+1}\right)_{n=1}^{+\infty} \\
(b) & ((n-\frac{1}{2})^2)_{n=1}^{+\infty} \\
(c) & f=(50n-n^2)_{n=1}^{+\infty} \\
(d) & g\circ f, \text{where } f \text{ is a strictly increasing sequence of positive integers} \\
 & \text{and } g \text{ is a strictly increasing sequence of real numbers.}
\end{array}
$$

(a) $\left(\frac{n}{n+1}\right)_{n=1}^{+\infty}$
We note that for $n > 1$, $\Delta_n = \frac{n}{n+1} - \frac{n-1}{n} = \frac{n^2-(n+1)(n-1)}{(n+1)n} = \frac{1}{(n+1)n}$, so $\Delta_n > 0, \forall n \in \mathbb{P}$ proving that the sequence is strictly increasing.

$\Diamond$

(b) $((n-\frac{1}{2})^2)_{n=1}^{+\infty}$
We note that $\Delta_{n+1} = (n+1-\frac{1}{2})^2 - (n-\frac{1}{2})^2 = (n+\frac{1}{2})^2 - (n-\frac{1}{2})^2 = 2n$, so $\Delta_n > 0, \forall n > 1 \in \mathbb{P}$ proving that the sequence is strictly increasing.

$\Diamond$

(c) $f = (50n-n^2)_{n=1}^{+\infty}$
We note that $\Delta_{n+1} = (50(n+1)-(n+1)^2)-(50n-n^2) = (50n+50-n^2-2n-1-50n+n^2) = 49-2n$, so $\Delta_n < 0, \forall n \geq 25 \in \mathbb{P}$ proving that the sequence is not strictly increasing.

$\Diamond$

(d) $g \circ f,$ where f is a strictly increasing sequence of positive integers
and g is a strictly increasing sequence of real numbers.
For f we have $f : \mathbb{P} \to \mathbb{P}$, such that $f = \{(n,i) : n,i \in \mathbb{P} \text{ and } f(n+p) > f(n) \quad \forall n, p \in \mathbb{P}\}$. Analogously, for g we we have $g : \mathbb{P} \to \mathbb{R}$, such that $g = \{(k,x) : k \in \mathbb{P}, x \in \mathbb{R} \text{ and } g(k+q) > g(k) \quad \forall k, q \in \mathbb{P}\}$.
Suppose that for a given n we have $f(n) = k$, then for $(g \circ f)(n)$ we have $(g \circ f)(n) = g(k)$. But we know that for a $n^{'} < n$ we will have $f(n^{'}) = k^{'} < k$, so $(g \circ f)(n^{'}) = g(k^{'}) < g(k)$ and notice that $g \circ f$ is a strictly increasing sequence.

$\blacklozenge$

1.16.5

Suppose that h is a subsequence of a sequence k and f is a subsequence of h. Is f a subsequence of k?

For h we have $\exists N : \mathbb{P} \to \mathbb{P}$, such that $h = k \circ N$. Analogously, for f we have $\exists M : \mathbb{P} \to \mathbb{P}$, such that $f = h \circ M$. Using the composition of functions, we have $f = k \circ N \circ M$. Be $Q = N \circ M$. Is Q a strictly increasing sequence? The answer is yes (see Exercise 1.16.4(d)). As Q is strictly increasing, f will be a subsequence of k.

♦

1.17 Finite induction and well-ordering for positive integers.

1.17.1

Prove that the following statement is equivalent to 17.1.
Suppose that h is an integer. Suppose further that $S(n)$ is a statement for each integer $n \geq h$, $S(h)$ is true, and $S(n)$ implies $S(n+1)$ for each integer $n \geq h$. Then $S(n)$ is true for each integer $n \geq h$.

Without loss of generality we take $h = 1$ (we could also define for a given h the map $N_h : \mathbb{P} \to \mathbb{Z}$ defined as $N = \{(p, h+p-1) : p \in \mathbb{P},\, h \in \mathbb{Z}\}$ and consider the statements $S \circ N_h(p)$.).
Let's define

$$M = \{m \in \mathbb{P} : S(m) \text{ is true}\}$$

Then the assumptions
i) $S(1)$ is true, is equivalent to $1 \in M$
ii) $S(n)$ implies $S(n+1)$ for each integer $n \geq 1$, is equivalent to $n \in M$ implies $n+1 \in M$ for each integer $n \geq 1$
iii) Then $S(n)$ is true for each integer $n \geq 1$, is equivalent to $m \in M$ every $m \geq 1$. This last statement implies that m contains all positive integers and thus $M = \mathbb{P}$.

♦

1.17.2

Prove that the sum of the first n positive integers is $\frac{1}{2}n(n+1)$.

Take $n = 1$, then $\frac{1}{2}n(n+1) = 1$ which corresponds to $\sum_{n=1}^{1} n = 1$.
Suppose that $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$ for a give n. Then $\sum_{k=1}^{n+1} k = \underbrace{\sum_{k=1}^{n} k}_{=\frac{1}{2}n(n+1)} + n + 1$, giving

$$
\begin{aligned}
\sum_{k=1}^{n+1} k &= \frac{1}{2}n(n+1) + n + 1 \\
&= \frac{1}{2}\left[n(n+1) + 2n + 2\right] \\
&= \frac{1}{2}\left[n^2 + n + 2n + 2\right] \\
&= \frac{1}{2}\left[(n^2 + 2n + 1) + (n+1)\right] \\
&= \frac{1}{2}\left[(n+1)^2 + (n+1)\right] \\
&= \frac{1}{2}(n+1)(n+2)
\end{aligned}
$$

confirming the expression.

♦

1.17.3

Prove that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$ for each $n \in \mathbb{P}$.

See Exercise **1.15.3**5

♦

1.17.4

Prove or disprove the following statement: For each $n \in \mathbb{P}$,

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$$

Take $n = 1$, then $\frac{1}{4}n^2(n+1)^2 = 1$ which corresponds to 1^3.

Suppose that $\sum_{k=1}^{n-1} k^3 = \frac{1}{4}(n-1)^2(n)^2$ for a give $n-1$. Then $\sum_{k=1}^{n} k^3 = \underbrace{\sum_{k=1}^{n-1} k}_{=\frac{1}{4}(n-1)^2(n)^2} + n^3$, giving

$$
\begin{aligned}
\sum_{k=1}^{n} k^3 &= \frac{1}{4}(n-1)^2 n^2 + n^3 \\
&= \frac{1}{4} n^2 \left[(n-1)^2 + 4n\right] \\
&= \frac{1}{4} n^2 \left[(n^2 + 2n + 1\right] \\
&= \frac{1}{4} n^2 (n+1)^2
\end{aligned}
$$

confirming the expression.

♦

1.17.5

> Is the following statement true? Justify your answer.
> For each positive integer n, $2^n - 1 \geq n$.

Take $n = 1$, then $2^n - 1 \geq 1$ which is a true statement.
Suppose now that for given $n-1$ we have $2^{n-1} - 1 \geq n - 1$, then for n we have

$$
\begin{aligned}
2^n - 1 &= \left(2^{n-1}\right) 2 - 1 \\
&= \left(2^{n-1} - \frac{1}{2}\right) 2 \\
&= \left(\underbrace{2^{n-1} - 1}_{\geq n-1} + \frac{1}{2}\right) 2 \\
&\geq 2\left(n - \frac{1}{2}\right) \\
&\geq 2n - 1
\end{aligned}
$$

We prove that $2n - 1 \geq n$ for all positive integers. For $n = 1$ we have $2n - 1 \geq 1$.
Suppose now that for given n we have $2n - 1 \geq n$, then for $n + 1$ we have $2(n+1) - 1 = 2n + 1$ which is $\geq 2n$ which is $\geq n$ as n is a positive integer, confirming the inequality stated.

♦

1.17.6

Either prove or disprove the following statement: For each positive integer n,

$$1.2 + 2.3 + 3.4 + \cdots + n(n+1) = \frac{1}{3}\left[n(n+1)(n+2)+3\right]$$

.

Take $n = 1$, then $\frac{1}{3}\left[n(n+1)(n+2)+3\right] = 3$. But this should give $1.2 = 2$, so even for $n = 1$, the expression is wrong, hence can't be correct for all $n \in \mathbb{P}$.

♦

1.17.7

Either prove or disprove the following statement: For each $n \in \mathbb{P}$, $7^n - 3^n$ is divisible by 4.

For each $n \in \mathbb{P}$, $7^n - 3^n$ is divisible by 4 means that there exist a $k \in \mathbb{P}$ so that $7^n - 3^n = 4k$.
take $n = 1$, then $7^n - 3^n = 4$ and with $k = 1$ the statement is true .
Suppose now that for given $n - 1$ we have $7^{n-1} - 3^{n-1} = 4k$, this means that $7^{n-1} = 3^{n-1} + 4k$.
Then for n we get

$$\begin{aligned}
7^n - 3^n &= 7^{n-1}7 - 3^{n-1}3 \\
&= \left(3^{n-1} + 4k\right)7 - 3^{n-1}3 \\
&= 3^{n-1}7 + 28k - 3^{n-1}3 \\
&= 3^{n-1}(7-3) + 28k \\
&= 3^{n-1}4 + 28k \\
&= 4\left(3^{n-1} + 7k\right)
\end{aligned}$$

and get $7^n - 3^n = 4p^{'}$ with $p^{'} = 3^{n-1} + 7k$ where $p^{'} \in \mathbb{P}$. Hence, $7^n - 3^n$ is divisible by 4.

♦

1.17.8

Suppose that K is a nonempty collection of negative integers. Prove that there is a largest element in K.

Suppose K has no biggest element. Be $N = \mathbb{Z}_- - K$.
Note that $-1 \in N$ as otherwise -1 would be an element of K making this the largest element of K which we assumed had none. Consider now the set $N_n = \{n, n+1, \ldots, -2, -1\}$. Obviously $N_n \subset N$ as otherwise one of the element of N_n would be in K and thus K would have a largest element. Be $N_n \subset N$, then $N_{n-1} \subset N$ for otherwise $n-1$ would be the largest element of K. Continuing that process with $n-k$, $k \in \mathbb{P}$ we see that $N_{n-k} - > \mathbb{Z}_-$ meaning that $\mathbb{Z}_- \subset N$ and hence $N = \mathbb{Z}_-$. Thus K must be the empty-set which is a contradiction. Hence K must have a largest element.

♦

1.17.9

Is $3n^2 + n$ an even integer for each positive integer n? Justify your answer.

Take $n = 1$, then $3n^2 + n = 4$, an even integer.
Suppose that for a give n we have $3n^2 + n = 2p$ with $p \in \mathbb{P}$. Then for $n+1$ we have

$$\begin{aligned} 3(n+1)^2 + n + 1 &= 3n^2 + 6n + 3 + n + 1 \\ &= \underbrace{3n^2 + n}_{=2p} + 6n + 4 \\ &= 2p + 2(3n+2) \\ &= 2(p + 3n + 2) \end{aligned}$$

Hence, $3n^2 + n$ an even integer for each positive integer n.

♦

1.17.10

Try to discover a formula for the number of subsets (including the empty set) of a set of n objects. Then prove by induction that your conjecture is correct.

Be S the considered set with n objects, then the number of elements in the powerset $\mathscr{P}$ is, $\#\mathscr{P} = 2^n$.
Be $n = 0$ ($S = \emptyset$), then $\#\mathscr{P} = 1$ which is correct as $\mathscr{P} = \{\emptyset\}$, containing only one subset.
Suppose that for a n we have $\#\mathscr{P}_n = 2^n$. Adding one object $x \notin S$ to the set S we get a new set $S^{'} = S \cup \{x\}$. Be $\mathscr{P}_n = \{P_1, P_2, \ldots P_{2^n}\}$, so the new powerset $\mathscr{P}_{n+1}$ can be expressed as $\mathscr{P}_{n+1} = \{P_1, P_2, \ldots P_{2^n}, P_1 \cup \{x\}, P_2 \cup \{x\}, \ldots P_{2_n} \cup \{x\}\} = \{P_1, P_2, \ldots P_{2_n}\} \cup \{P_1 \cup \{x\}, P_2 \cup \{x\}, \ldots P_{2^n} \cup \{x\}\}$ meaning that $\mathscr{P}_{n+1}$ will contain $2(2^n) = 2^{n+1}$ subsets.

♦

1.17.11

Is $n(n+1)(n+2)$ divisible by 3 for each positive integer n? Justify your answer.

We could perform the usual induction, but we can use another reasoning.
Suppose n is divisible by 3, then we are done. If n is not divisible by 3 then n will be of the form $n = 3k+1$ or $n = 3k+2$ for a given $k \in \mathbb{P}$. We don't need to consider the forms beyond $n = 3k+2$ because $n = 3k+3$, obviously is divisible by 3.
So, in the case n is not divisible by 3 , $n(n+1)(n+2)$ can be expressed as $(3k+1)(3k+2)(3k+3)$ or $(3k+2)(3k+3)(3k+4)$, so in both cases, one of the numbers in the product is divisble by 3, hence the total product also.

♦

1.17.12

Is $\frac{[n(n+1)(n+2)(n+3)]}{24}$ an integer for each positive integer n? Justify your answer.

We prove by induction that $\forall n \in \mathbb{P} : n(n+1)(n+2)(n+3) = 24m,\ m \in \mathbb{P}$
For $n = 1$, we have $n(n+1)(n+2)(n+3) = 24$.
Suppose $\exists n \in \mathbb{P} :\ n(n+1)(n+2)(n+3) = 24k,\ k \in \mathbb{P}$. The for $n+1$ have

$$(n+1)(n+2)(n+3)(n+4) = \underbrace{n(n+1)(n+2)(n+3)}_{=24k} + 4(n+1)(n+2)(n+3) \tag{1}$$

From **1.17.11** we know that $(n+1)(n+2)(n+3)$ is divisible by 3 for each positive integer n. Hence

(1) can be written as

$$(n+1)(n+2)(n+3)(n+4) = 24k + 4 \times 3 \times m \tag{2}$$

We claim that m is even. Indeed, for $n = 1$, $n(n+1)(n+2) = 2 \times 3$
Suppose $\exists n \in \mathbb{P} : \, n(n+1)(n+2) = 3 \times 2 \times q$, $q \in \mathbb{P}$. Then, for $n+1$ have

$$(n+1)(n+2)(n+3) = \underbrace{n(n+1)(n+2)}_{=3\times 2\times q} + 3(n+1)(n+2) \tag{3}$$

In the second term of (3), or $(n+1)$ or $(n+2)$ will be even, so (3) can be written as

$$\begin{aligned} (n+1)(n+2)(n+3) &= 3 \times 2 \times q + 3 \times 2 \times q^{'} \times n^* \quad (n^* = n+1 \text{ or } n+2) \\ &= 3 \times \underbrace{2 \times \left(q + q^{'} \times n^*\right)}_{=m} \end{aligned}$$

By induction, we conclude that m indeed, must be even. Hence (1) becomes

$$\begin{aligned} (n+1)(n+2)(n+3)(n+4) &= 24k + 4 \times 3 \times 2 \times \left(q + q^{'} \times n^*\right) \\ &= 24 \left(k + q + q^{'} \times n^*\right) \end{aligned}$$

proving, by the induction principle (remember the initialisation with $n = 1$) that $\frac{n(n+1)(n+2)(n+3)}{24}$ is an integer for each positive integer n

♦

1.18 Sequences defined inductively

Clarification to example **18.3**

18.3. Example. Let $f(1) = 2$, $f(2) = 7$. Furthermore, let it be given that for each positive integer $n \geq 3$, $f(n) = \frac{1}{2}[(f(n-I) + f(n-2)]$. At this point it would be instructive for the reader to calculate a few terms off f. It is intuitively clear that there should exist a unique function satisfying the above properties. It furthermore seems reasonable that we should be able to prove by induction that such a function exists. This we can do, but not in as straightforward a manner as we might guess. To establish the existence of a function f satisfying the required properties, we proceed as follows: Let $f_2 = \{(1,2),\ (2,7)\}$. and for $n \geq 3$, let $S(n)$ be the following statement.
18.3(a). There exists a map $f_n : \mathbb{P}_n \to \mathbb{R}$ such that $f_n(1) = 2$ and $f_n(2) = 7$, and for $i \in \{3, 4, \ldots, n\}$,

$$f_n(i) = \frac{1}{2}[(f_n(i-1) + f_n(i-2)]$$

We see that $S(3)$ is true by considering the function $f_3 = \{(1,2), (2,7), (3, \frac{9}{2})\}$. Let $h \geq 3$ and assume that $S(h)$ is true. We show that $S(h+1)$ is true. To see this let

$$f_{h+1} = f_h \cup \{(h+1, \frac{1}{2}[(f_h(h) + f_h(h-1)]\}$$

It is easy to show that f_{h+1} satisfies the properties required of it so that $S(h+1)$ is true. Hence, by induction, $S(n)$ is true for each integer $n \geq 3$. Thus, there exists a collection of functions $\{f_n : n \geq 3\}$, each of which satisfies the conditions stated in **18.1(a)**. We next prove that for each f_n in the collection, $f_{n+1}|\mathbf{P}_n = f_n$. To see this, suppose that for some fixed integer $n \geq 3$, $f_{n+1}|\mathbf{P}_n \neq f_n$. Let j be the first positive integer for which $f_{n+1}(j) \neq f_n(j)$. We observe that $3 \leq j \leq n$. Then

$$f_{n+1}(j-1) = f_n(j-1) \text{ and } f_{n+1}(j-2) = f_n(j-2)$$

From this, we obtain

$$f_{n+1}(j) = \frac{1}{2}[(f_{n+1}(j-1) + f_{n+1}(j-2)] = \frac{1}{2}[(f_n(j-1) + f_n(j-2)] = f_n(j)$$

and we have arrived at a contradiction. For each $n \in \mathbb{P}$, let f_n be a function satisfying **18.3(a)**. We may apply **18.1** to $\{f_n : n \in \mathbb{P}\}$ and conclude that $f = \bigcup\{f_i = i \geq 3\}$ is a function. Next recall the remark in **18.2**, about the union of functions theorem. which states that $f(x) = f_n(x)$ for all x for which $x \in Dom f_n$. Thus, $f(1) = f_3(1) = 2$, $f(2) = f_3(2) = 7$, and $f(3) = f_3(3) = \frac{1}{2}[(f_3(2) + f_3(1)]$. Also, for $i \geq 3$,

$$f(i) = f_i(i) = \frac{1}{2}[(f_i(i-1) + f_i(i-2)] \frac{1}{2}[(f(i-1) + f(i-2)]$$

That f is a unique function that satisfies the given conditions is seen as follows: Suppose that there is another function f^* that satisfies the required conditions. Then the set $K = \{j : f^*(j) \neq f(j)\}$ would be a nonempty set of positive integers. Let m be the first element in K. Obviously $m \geq 4$. But then $f(i) = f^*(i)$ for $i < m$. However,

$$f(m) = \frac{1}{2}[(f(m-1) + f(m-2)] = \frac{1}{2}[(f^*(m-1) + f^*(m-2)] = f^*(m)$$

, and we have a contradiction.

1.18.1

Prove Theorem **18.4**, using the discussion in **18.3** as a hint.

Note that S is the set of all finite sequences with X as domain. Hence, the following sequence will be in S:

$$\{(1,a_1),\,(2,a_2),\,(3,a_3),\,\ldots,\,(h,a_h),\,(h+1,x_{h+1}),\,\ldots,\,(n,x_n)\}\,,\quad (a_i,\,x_k\in X)$$

The given map G is a map from S into X. So, for the element (a sequence of length h)

$$\{(1,a_1),\,(2,a_2),\,(3,a_3),\,\ldots,\,(h,a_h)\}$$

there exist an element $f_{h+1} = G\left(\{(1,a_1),\,(2,a_2),\,(3,a_3),\,\ldots,\,(h,a_h)\}\right) \in X$. As $f_{h+1} \in X$ the sequence

$$\{(1,a_1),\,(2,a_2),\,(3,a_3),\,\ldots,\,(h,a_h),\,(h+1,f_{h+1})\}$$

will be also in S. Again this element (sequence of length $h+1$) will be mapped by G to an element $f_{h+2} \in X$. hence the sequence

$$\{(1,a_1),\,(2,a_2),\,(3,a_3),\,\ldots,\,(h,a_h),\,(h+1,f_{h+1}),\,(h+2,f_{h+2})\}$$

will be in S.
By induction, we conclude that there exists a sequence f such that

$$f_i = \begin{cases} a_i & \text{for } i \in \{1,2,\ldots,h\} \\ G\left(\{f_1,\,f_2,\,\ldots,\,f_h,\,f_{h+1},\,\ldots,\,f_{i-1}\}\right) & \text{for } i \geq h+1 \end{cases}$$

Is this sequence, uniquely defined by $G : S \to X$? Yes, as G is a mapping (a function), so for a given element $\{f_1,\,f_2,\,\ldots,\,f_n\}$, there is a unique $G\left(\{f_1,\,f_2,\,\ldots,\,f_n\}\right) \in X$.

♦

1.18.2

Using Theorem **18.4**, prove that there exists a unique function f on the set of all nonnegative integers that satisfies the following conditions.
$f(0) = 1$ and $f(n) = nf(n-1)$ for each positive integer n. (Recall that common notation for $f(j)$ as defined inductively in this exercise is $j!$, read "j factorial.")

Be $X = \mathbb{N}$, $h = 1$, $a_1 = 1 \in \mathbb{N}$ and $G : S \to X$ with S a set of all finite-sequences with ranges in $\mathbb{N}$. From theorem **18.4** we know that theer is a unique sequence f such that

$$f_i = \begin{cases} a_i & \text{for } i \in \{1, 2, \ldots, h\} \\ G\left(\{f_1,\, f_2,\, \ldots,\, f_h,\, f_{h+1},\, \ldots,\, f_{i-1}\}\right) & \text{for } i \geq h+1 \end{cases}$$

As $h_1 = 1$ and $a_1 = 1$ and defining G as $G\left(\{f_1,\, f_2,\, \ldots f_{i-1}\}\right) = i\, f_{i-1}$, this simplifies to

$$f_i = \begin{cases} 1 & \text{for } i = 1 \\ i\, f(i-1) & \text{for } i \geq 2 \end{cases}$$

We can extend the function by adding one element $f_0 = 1$ to the sequence, giving as definition

$$f_i = \begin{cases} 1 & \text{for } i = 0 \\ 1 & \text{for } i = 1 \\ i\, f(i-1) & \text{for } i \geq 2 \end{cases}$$

♦

Following are some exercises concerning the factorial function that are useful in various branches of mathematics.

1.18.3

For each positive integer n and each nonnegative integer r such that $r \leq n$, define $\begin{pmatrix} n \\ r \end{pmatrix} = \frac{n!}{r!(n-r)!}$. Verify each of the following:

(a) $\begin{pmatrix} n \\ 0 \end{pmatrix} = 1$ for each positive integer n.

(b) $\begin{pmatrix} n \\ n \end{pmatrix} = 1$ for each positive integer n.

(c) For each positive integer h and for each positive integer $j \leq h$

$$\begin{pmatrix} h \\ j \end{pmatrix} + \begin{pmatrix} h \\ j-1 \end{pmatrix} = \begin{pmatrix} h+1 \\ j \end{pmatrix}$$

We can make use of Exercise 3 to prove the binomial expansion theorem in the next exercise.

(a) $\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{(n)!} = 1$

(b) $\binom{n}{n} = \frac{n!}{n!(n-n)!} = \frac{n!}{(n)!} = 1$

(c) $\binom{h}{j} + \binom{h}{j-1} = \binom{h+1}{j}$

$$\binom{h}{j} + \binom{h}{j-1} = \frac{h!}{j!(h-j)!} + \frac{h!}{(j-1)!(h-j+1)!}$$

$$= \frac{h!}{(j-1)!}\left(\frac{1}{j(h-j)!} + \frac{1}{(h-j+1)!}\right)$$

$$= \frac{h!}{(j-1)!}\left(\frac{(h-j+1)+j}{j(h-j+1)!}\right)$$

$$= \frac{h!}{(j-1)!}\left(\frac{(h+1)}{j(h-j+1)!}\right)$$

$$= \frac{(h+1)!}{j!(h+1-j)!}$$

$$= \binom{h+1}{j}$$

♦

1.18.4

Prove that for each positive integer n,

$$(a+b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n-1} a^1 b^{n-1} + \binom{n}{n} a^0 b^n$$

Note that a short form for writing this, using summation notation, is

$$\sum_{i=0}^{n} = \binom{n}{i} a^{n-i} b^i$$

Initialize with $n = 1$, we have $(a+b)^1 = \underbrace{\binom{1}{0}}_{=1} a^1 b^0 + \underbrace{\binom{1}{1}}_{=1} a^{1-1} b^1 = a+b$. So, the proposed expression is true for $n = 1$.

Suppose now that it is alos correct for a certain n:

$$(a+b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n-1} a^1 b^{n-1} + \binom{n}{n} a^0 b^n$$

Then $(a+b)^{n+1} = (a+b)^n (a+b)$ giving

$$(a+b)^{n+1} = \begin{cases} \binom{n}{0} a^{n+1} b^0 + \binom{n}{1} a^n b^1 + \binom{n}{2} a^{n-1} b^2 + \cdots + \binom{n}{n-1} a^2 b^{n-1} + \binom{n}{n} a^1 b^n + \\ \\ \binom{n}{0} a^n b^1 + \binom{n}{1} a^{n-1} b^2 + \binom{n}{2} a^{n-2} b^3 + \cdots + \binom{n}{n-1} a^1 b^n + \binom{n}{n} a^0 b^{n+1} \end{cases}$$

regrouping as much as possible terms of the same power and using the results (a), (b), (c) of the previous exercise we get

$$(a+b)^{n+1} = \begin{cases} \binom{n}{0} a^{n+1} b^0 + \left[\binom{n}{1} + \binom{n}{0}\right] a^n b^1 + \left[\binom{n}{2} + \binom{n}{1}\right] a^{n-1} b^2 + \dots \\ \\ + \left[\binom{n}{n-2} + \binom{n}{n-1}\right] a^2 b^{n-1} + \left[\binom{n}{n-1} + \binom{n}{n}\right] a^1 b^n + \binom{n}{n} a^0 b^{n+1} \end{cases}$$

$$= \binom{n+1}{0} a^{n+1} b^0 + \binom{n+1}{1} a^n b^1 + \binom{n+1}{2} a^{n-1} b^2 + \cdots + \binom{n+1}{n} a^1 b^n + \binom{n+1}{n+1} a^0 b^{n+1}$$

Hence, by the induction principle, the proposed expression is true.

♦

1.19 Some important properties of relations

1.19.1

In each of the following, classify the relation as to which of the properties discussed in Section 19 it possesses.

(a) Let S be the set of all triangles in the plane.
Let R be the relation in S defined as follows: for all a and b in S,
$a\,R\,b$ if and only if a is congruent tob.

(b) Let R be the set of all real numbers.
Let $S = \{(x, y) : (x, y) \in \mathbb{R} \times \mathbb{R} \text{ and } y \neq 0\}$.
For all (a, b) and $(c, d) \in S$, let $(a, b)R(c, d)$ provided that $ad = be$.

(c) Suppose j is a fixed positive integer.
For each a and $b \in Z$, let aRb if and only if $a - b = jk$ for some integer k.
(See Exercises 9 and 10, page 23 .)

(a)
- R is reflexive (a triangle is congruent to itself).
- R is symmetric (if a triangle A is congruent to a triangle B then the triangle B will also be congruent to A).
- R is transitive (if a triangle A is congruent to a triangle B and this triangle B is congruent to a triangle C then the triangle A will also be congruent to C).

(b) $(a, b)R(c, d) \Rightarrow ad = bc$
- R is reflexive $((a, b)R(a, b) \Rightarrow ad = ab)$.
- R is symmetric $((a, b)R(c, d) \Rightarrow ad = bc$ and $(c, d)R(a, b) \Rightarrow cb = ad,\ )$.
- R is transitive $([(a, b)R(c, d) \Rightarrow ad = bc\ ,\ (c, d)R(e, f) \Rightarrow cf = de] \Rightarrow af = be$ and $(a, b)R(e, f) \Rightarrow af = be\ )$.

(c) $aRb \Rightarrow a - b = jk, k \in \mathbb{Z}$
- R is reflexive $(aRb \Rightarrow a - b = j \times 0, 0 \in \mathbb{Z})$.
- R is symmetric $(aRb \Rightarrow a - b = j \times k, k \in \mathbb{Z}$ and $bRa \Rightarrow b - a = j \times (-k), -k \in \mathbb{Z})$.
- R is transitive $(aRb \Rightarrow a - b = j \times k, k \in \mathbb{Z}$, $bRac \Rightarrow b - c = j \times (k^{'}), k^{'} \in \mathbb{Z} \Rightarrow a - b + b - c = a - c = j(k + k^{'}), \in \mathbb{Z}$ and $aRc \Rightarrow a - c = j \times k^{"}, k^{"} \in \mathbb{Z}$ with $k^{"} = k + k^{'}\ )$.

♦

1.19.2

Suppose that R is a relation that is transitive in a set S. Let us define a new relation in S as follows: For each a and b in S, let aR^*b if and only if $a = b$ or aRb. Is R^* transitive in S? Is R^* reflexive in S? Illustrate with an R that is not reflexive.

(a) Is R^* transitive?
Be aR^*b and bR^*c, we can have 4 possibilities:

- $a = b$, $b = c$, then $a = c$ and by definition of the relation aR^*c.
- $a = b$, $b \neq c$, then $aR^*b \Leftrightarrow aR^*c$
- $a \neq b$, $b = c$, then $aR^*b \Leftrightarrow aR^*c$
- $a \neq b$, $b \neq c$, then aRb and bRc and as R is transitive, thus aRc and by definition of the relation we have aR^*c.

(b) Is R^* reflexive?
By the definition of R^* (case a=b) , we have indeed aR^*a.

$\Diamond$

Consider the set $S = \mathbb{R}$ and the relation $aRb : a < b$. R is obviously not reflexive, yet is transitive ($a < b$, $b < c \Rightarrow a < c$), but $aR^*a \equiv (a, a) \in R^*$.

$\blacklozenge$

1.19.3

Suppose that R is a relation in a set S. Let us define a new relation R^* as follows: For each a and b in S, let aR^*b if and only if aRb is true and bRa is false. Suppose R is transitive. Is R^* also transitive ? Is R^* necessarily antisymmetric?

a) Is R^* transitive?
We have

$$\begin{cases} aR^*b & \Leftrightarrow & aRb & \wedge & b\not{R}a \\ \\ bR^*c & \Leftrightarrow & bRc & \wedge & c\not{R}b \end{cases}$$

R is transitive, so aR^*c is transitive provided that $b\not{R}a \quad \wedge \quad c\not{R}b$. So, provided that R is antisymmetric and transitive , R^* will be transitive.

b) Is R^* necessarily antisymmetric?
Suppose

$$\begin{cases} aR^*b & \Leftrightarrow & aRb & \wedge & b\not{R}a \\ \\ bR^*a & \Leftrightarrow & bRa & \wedge & a\not{R}b \end{cases}$$

This is a contradiction, hence the answer is, yes, R^* is necessarily antisymmetric.

♦

1.19.4

Suppose that a relation R in a set S is transitive and antireflexive. Is it necessarily antisymmetric?

R transitive means aRb, $bRc \Rightarrow aRc$, R antireflexive means $(a, a) \notin R$ and antisymmetric means aRb, $bRa \Rightarrow a = b$.
Be $c = a$, then transitivity implies aRb, $bRa \Leftrightarrow aRa$, but as R is antireflexive we can't have aRa, meaning that $aRb \wedge bRa$ can not be true. So R needs definitely to be antysymmetric.

♦

1.19.5

Are the following propositions true?

(a) Suppose that R is a relation in a set S. Then R is symmetric if and only if $R \subset R^{-1}$.
(b) Suppose that R is a relation in a set S. Then R is symmetric if and only if $R = R^{-1}$.

(a)
Suppose R is symmetric. Then $aRb \Leftrightarrow bRa$ or stated differently $(a, b) \in R$ and $(b, a) \in R$ for all $a, b \in S$. For R^{-1} we have $(b, a) \in R^{-1}$ if $(a, b) \in R$. So we have $(b, a) \in R^{-1}$ and provided that R is symmetric $(b, a) \in R$, hence, $R \subset R^{-1}$ (starting from R is symmetric and noting that $(a, b) \in R \Rightarrow (b, a) \in R^{-1}$ we can also conclude that $R^{-1} \subset R$).
Suppose now that $R \subset R^{-1}$, this means that if $(b, a) \in R$ we also have also $(b, a) \in R^{-1}$, the latter implying $(a, b) \in R$, so we have $(a, b) \in R$ and $(b, a) \in R$, thus R is symmetric.

(b)

Suppose R is symmetric. From (a) we have $R = R^{-1}$.
We must prove that also $R^{-1} \subset R$.
Suppose now that $R = R^{-1}$, this means that if $(b, a) \in R$ we also have also $(b, a) \in R^{-1}$, the latter implying $(a, b) \in R$, so we have $(a, b) \in R$ and $(b, a) \in R$, thus R is symmetric.

♦

1.19.6

Suppose that R is a relation defined in S. Is $R \cap R^{-1}$ a symmetric relation in S?

Suppose, $R \cap R^{-1}$ is symmetric, then $\forall a, b \in S : aR \cap R^{-1}b \Rightarrow bR \cap R^{-1}a$ which means $(a, b) \in R \cap R^{-1} \wedge (b, a) \in R \cap R^{-1}$. This means that (b, a) also must be an element of R or bRa. If R is not symmetric then the statement is not true and $R \cap R^{-1}$ can not be symmetric in a general case.

♦

1.19.7

Suppose that R is a transitive relation in S. Is R^{-1} transitive in S?

We have $(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R$. But, $(a, c) \in R$ implies that $(c, a) \in R^{-1}$. For R^{-1} to be transitive, we need $(b, a) \in R^{-1}, (c, b) \in R^{-1} \Rightarrow (c, a) \in R^{-1}$.
So, yes indeed, R^{-1} is transitive in S

♦

1.19.8

Suppose that R is symmetric in S. Is R^{-1} symmetric in S?

We have $(a, b) \in R \Rightarrow (b, a) \in R$. But, $(b, a) \in R$ implies that $(a, b) \in R^{-1}$. For R^{-1} to be symmetric, we need $(b, a) \in R^{-1} \Rightarrow (a, b) \in R^{-1}$.
So, yes indeed, R^{-1} is symmetric in S

♦

1.19.9

Suppose that R is reflexive in S. Is R^{-1} reflexive in S?

$(a, a) \in R \Rightarrow (a, a) \in R^{-1}$. Hence, R^{-1} is reflexive in S

♦

1.19.10

Does there exist a nonempty set S and a relation R in S such that R is both symmetric and antisymmetric in S?

Be $S = \{a\}$ an arbitrary non-empty set with 1 element. Let's define R as the identity map i.e. $R = \{(a, a) : a \in S\}$.
Symmetry means

$$\forall x, y \in S : (x, y) \in R \Rightarrow (y, x) \in R$$

which is a true statement as $(x = a, y = x)$, $(y = x, x = a) \in R$ because R is the identity map.

Anti-symmetry means

$$\forall x, y \in S : (x, y) \in R \wedge (y, x) \in R \Rightarrow x = y$$

which is a true statement as $x = a,\ y = a \quad \forall x, y \in S$ and R is the identity map.

♦

1.20 Decomposition of a set

1.20.1

For each real number r, let $F_r = \{(r, y) : y \in \mathbb{R}\}$. Is $\{F_r : r \in \mathbb{R}\}$ a partition of $\mathbb{R} \times \mathbb{R}$?

First note that $F_r \cap F_{r'} = \emptyset$ if $r \neq r^{'}$.
Also $\bigcup\{(r, y) : y \in \mathbb{R}\}$ will cover, for a given r, the whole $\mathbb{R}$-axis. Also, as F_r is defined for each real number, for each of this number will be associated the whole real number axis, and hence $\{F_r : r \in \mathbb{R}\}$ a partition of $\mathbb{R} \times \mathbb{R}$.

♦

1.20.2

Let $A_0 = \{x : -1 \leq x \leq 1\}$. For each $x \in \mathbb{R} - A_0$ let $A_x = \{x\}$. Is the following collection $\mathscr{K}$ a decomposition of the real line $\mathbb{R}$?

$$\mathscr{K} = \{A_x : x = 0 \text{ or } |x| > 1\}$$

- $\bigcup\{A_x : x = 0 \text{ or } |x| > 1\} = \mathbb{R}$. This can be seen as A_0 is the closed interval $[-1, 1]$ and for all other $x \in \mathbb{R} - A_0$ i.e. $x \in (-\infty, -1) \cup (1, +\infty)$, each A_x, $x \geq 1$ is $\{x\}$ itself. So, $\bigcup\{A_x : x = 0 \text{ or } |x| > 1\} = (-\infty, 1) \cup [-1, 1] \cup (1, +\infty) = \mathbb{R}$
- $A_0 \cap A_x = \emptyset$, $\forall\, |x| > 1$.
- $A_x \cap A_{x'} = \{x\} \cap \{x^{'}\} = \emptyset$, $\forall\, |x|, |x^{'}| > 1$.

Conclusion: $\mathscr{K}$ a decomposition of the real line $\mathbb{R}$

♦

1.20.3

Suppose X is a nonempty set and $f : X \to Y$ is a surjection. Is $\{f^{-1}[y] : y \in Y\}$ a decomposition of X?

f is a surjection, so every $y \in Y$ has at least one element of $x \in X$ such that $f(x) = y$. But for two different y_1, $y_2 \in Y$, as f is a function, we must have $f^{-1}[y_1] \cap f^{-1}[y_2] = \emptyset$ as otherwise we

would have a x^*, such that $f(x^*) = y_1$ and $f(x^*) = y_2$, which is excluded as f is a function. So, one condition for $\{f^{-1}[y] : y \in Y\}$ a being a decomposition of X is fulfilled
What about $\bigcup\{f^{-1}[y] : y \in Y\}$? We notice that $\{f^{-1}[y] : y \in Y\}$ corresponds to $dom\, f$ which is not necessarily equal to X ($dom\, f \subset X$, see definition **10.3** page 16). So, $\{f^{-1}[y] : y \in Y\}$ a not always a decomposition of X.

♦

1.21 Equivalence classes

1.21.1

For each ordered pair of real numbers (a, b) such that $a \neq 0$ and $b \neq 0$, let $E(a, b)$ be the equation $ax + by = 0$. Let $\xi = \{E(a, b) : a \neq 0 \text{ and } b \neq 0\}$. For $E(a, b)$ and $E(c, d)$ in ξ, let us define a relation as follows. $E(a, b) \simeq E(c, d)$ if and only if every solution (x, y) of $ax + by = 0$ is a solution of $cx + dy = 0$, and every solution of $cx + dy = 0$ is a solution of $ax + by = 0$. Note that $\simeq$ is an equivalence relation. Is it true that $E(2, 3) \simeq E(4, 6)$? Try to discover an equation relating $a, b,\ c,$ and d so that $E(a, b) \simeq E(c, d)$ provided that $a, b,\ c,$ and d satisfy your equation. Justify your conjecture. Write an equation that is in the same equivalence class as is $E(3, 2)$ but which is not the same as $E(3, 2)$.

We first check that $\simeq$ is an equivalence relation.
i) reflexivity: $E(a, b) \simeq E(a, b)$ is obvious as (x, y) is of course a solution of the same equation $ax + by = 0$ on both sides of the relation $\simeq$.
ii) symmetric: $E(a, b) \simeq E(c, d) \Leftrightarrow E(c, d) \simeq E(a, b)$ is obvious as a solution (x, y) of $E(a, b)$ will be a solution of $E(c, d)$ and vice-versa, a solution of $E(c, d)$ will also be a solution of $E(a, b)$.
iii) transitivity $E(a, b) \simeq E(c, d)$, $E(c, d) \simeq E(e, f) \Rightarrow E(a, b) \simeq E(e, f)$: also obvious as a solution of $ax + by = 0$ will be a solution of $cx + dy = 0$ but then also a solution of $ex + fy = 0$.

a) $E(2, 3) \simeq E(4, 6)$

$$\begin{cases} 2x + 3y = 0 \\ 4x + 6y = 0 \Leftrightarrow 2x + 3y = 0 \end{cases}$$

so, yes, $E(2, 3) \simeq E(4, 6)$

b) Try to discover an equation relating $a, b,\ c,$ and d so that $E(a, b) \simeq E(c, d)$

Suppose (x, y) is a solution of $ax + by = 0$ and a solution of $cx + dy = 0$.

$$\begin{cases} ax + by = 0 \\ cx + dy = 0 \end{cases}$$
$$\Rightarrow \begin{cases} cax + cby = 0 \\ acx + ady = 0 \end{cases}$$
$$\Rightarrow \begin{cases} cax + cby = 0 \\ (ad - bc)y = 0 \end{cases}$$

If we exclude the trivial solution $(x, y) = (0, 0)$ we see that $ad - bc$ must be zero, hence the condition needed is

$$ad - bc = 0$$

c)Write an equation that is in the same equivalence class as is $E(3, 2)$ but which is not the same as $E(3, 2)$.

Using the previous result we need $3d - 2c = 0$ or $c = \frac{3}{2}d$. Hence another equation is with $d = 1$ $\frac{3}{2}x + y = 0$

♦

1.21.2

Let $\mathbb{R}_+$, be the collection of all positive real numbers. For $a \in \mathbb{R}_+$, and $b \in \mathbb{R}_+$,, let aRb if and only if $\frac{a}{b}$ is a rational number. Is R an equivalence relation on $\mathbb{R}_+$? Justify your answer. What is the form of all the numbers b such that $b \in R[\sqrt{2}]$? If a is an irrational positive number and bRa, is b necessarily an irrational number?

a) Is R an equivalence relation on $\mathbb{R}_+$?

i) reflexivity: aRa implies $\frac{a}{a} = 1 \in \mathbb{Q}$.

ii) symmetric : aRb implies $\frac{a}{b} \in \mathbb{Q}$ We can then express $\frac{a}{b}$ as $\frac{a}{b} = \frac{p}{q}$ with p, $q \in \mathbb{Z}_+$. Obviously, $\frac{b}{a} = \frac{q}{p} \in \mathbb{Q}$ and thus bRa.

iii) transitivity : We need $aRb, bRc \Rightarrow aRc$.

aRb implies $\frac{a}{b} =\in \mathbb{Q}$ and cRd implies $\frac{c}{d} =\in \mathbb{Q}$. We can then express $\frac{a}{b}$ as $\frac{a}{b} = \frac{p}{q}$ with p, $q \in \mathbb{Z}_+$.
Also $\frac{c}{d} = \frac{r}{s}$ with r, $s \in \mathbb{Z}_+$.
Hence, we have $\frac{a}{b}\frac{b}{c} = \frac{pr}{qs}$ or $\frac{a}{c} = \frac{u}{v}$ with $(u = pr, v = qs) \in \mathbb{Q}$ and thus aRc.

b) What is the form of all the numbers b such that $b \in R[\sqrt{2}]$?

First remember (see section 11) that $R[a] = \{b : aRb\}$, so $R[\sqrt{2}] = \{b : \sqrt{2}Rb\}$ and from the definition of R we must have $\frac{\sqrt{2}}{b} \in \mathbb{Q}$. As $\sqrt{2}$ is irrational we need b to be of the form $q\sqrt{2}$ with $q \in \mathbb{Q}$.

c) If a is an irrational positive number and bRa, is b necessarily an irrational number?

Suppose $\frac{b}{a} \in \mathbb{Q}$. Then $\frac{b}{a} = \frac{p}{q}$ with $p, q \in \mathbb{Z}_+$. Thus $b = a\frac{p}{q}$. Suppose b is rational then $b = \frac{u}{v}$,

$u, v \in \mathbb{Z}_+$, giving $\frac{u}{v} = a\frac{p}{q}$. Hence $\underbrace{uq}_{=s\in\mathbb{Z}_+} = a \underbrace{pv}_{=t\in\mathbb{Z}_+}$ from which follows that $a = \frac{s}{t}$ and thus rational.

We get a contradiction and hence b is necessarily irrational.

♦

1.21.3

Let $\mathbb{R}$ be the set of all real numbers and let $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. Let m be a fixed real number. For each (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$ let $(x_1, y_1)R(x_2, y_2)$ provided that $y_1 - mx_1 = y_2 - mx_2$. Is R an equivalence relation ? Let $m = 3$. Sketch $R[(1, 2)]$.

a) Is R an equivalence relation?

i) reflexive: obviously $y_1 - mx_1 = y_1 - mx_1$ hence R is a reflexive relation.

ii) symmetric:

$$\underbrace{(x_1, y_1)R(x_2, y_2)}_{y_1 - mx_1 = y_2 - mx_2} \Rightarrow \underbrace{(x_2, y_2)R(x_1, y_1)}_{y_2 - mx_2 = y_1 - mx_1}$$

hence R is a symmetric relation.

iii) transitive Obviously from $y_1 - mx_1 = y_2 - mx_2$ and $y_2 - mx_2 = y_3 - mx_3$ follows $y_1 - mx_1 = y_3 - mx_3$ and thus $(x_1, y_1)R(x_3, y_3)$ hence R is a transitive relation.

Conclusion: R is an equivalence relation.

b) The set generated by $R^[(1, 2)]$ with $m = 3$ is defined by $y_1 - 3x_1 = y_2 - 3x_2$ and $R^[(1, 2)] = \{(x, y) : (1, 2)R(x, y)\} = \{(x, y) : y - 3x = -1\}$ giving the following sketch:

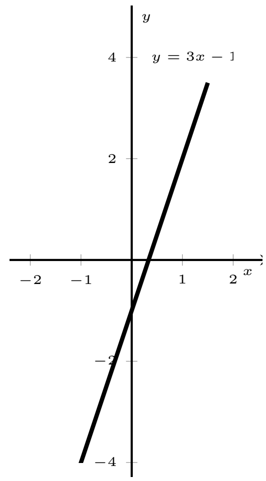


Figure 1.23: Sketch of the set generated by $R[(1,2)]$ with $m=3$

♦

1.21.4

Let $\mathscr{D}$ be the set of all real-valued functions which are defined and have derivatives on the open interval (a,b). For $f \in \mathscr{D}$ and $g \in \mathscr{D}$, let fRg provided that $f^{'} = g^{'}$. Is R an equivalence relation in $\mathscr{D}$? Let $f(x) = x^2$ for $x \in (a,b)$. Find $R[f]$.

a) Is R an equivalence relation?

i) reflexive: obviously $fRf \Rightarrow f^{'} = f^{'}$ hence R is a reflexive relation.

ii) symmetric:

$$\underbrace{fRg}_{f'=g'} \Rightarrow \underbrace{gRf}_{g'=f'}$$

hence R is a symmetric relation.

iii) transitive Obviously from fRg and gRh follows $f^{'} = g^{'}$ and $g^{'} = h^{'}$ and thus $f^{'} = h^{'}$ or fRh, hence R is a transitive relation.

Conclusion: R is an equivalence relation.

b) Let $f(x) = x^2$ for $x \in (a,b)$. Find $R[f]$.

$f(x) = x^2 \Rightarrow f^{'}(x) = 2x$.
From $R[f] = \{h : fRh\}$ follows $2x = h^{'}(x) \Rightarrow h = x^2 + C$ with C a constant in $\mathbb{R}$.

♦

1.23 Partially ordered and totally ordered sets

1.23.1

Consider the system $(\mathbb{R}, \leq)$ of the real line $\mathbb{R}$ together with the usual ordering $\leq$.
(a) Give an example of subset A of $\mathbb{R}$ that is bounded below but not above. Similarly, give an example of a subset B of $\mathbb{R}$ that is bounded above but not below.
(b) Give an example of a subset S of $\mathbb{R}$ that has a least upper bound but whose least upper bound does not belong to S.

(a) Consider $A \subset \mathbb{R}$ defined by $A = \{x^2 : x \in \mathbb{R}\}$. Then A is bounded below by the element 0, but A is not bounded above.

(a') Consider $A \subset \mathbb{R}$ defined by $A^{'} = \{e^{-x} : x \in \mathbb{R}\}$ Then $A^{'}$ in bounded above by the element 0, but $A^{'}$ is not bounded below.

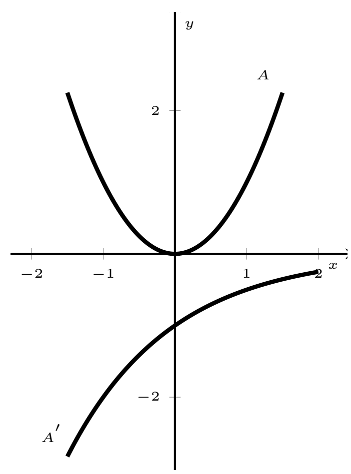


Figure 1.24: The sets $A = \{x^2 : x \in \mathbb{R}\}$ and $A^{'} = \{-e^{-x} : x \in \mathbb{R}\}$

(b) Consider the set $A^{'}$ above. $l.u.b.(A^{'}) = 0$ but $0 \notin A^{'}$.

♦

1.23.2

Give an example of a collection of sets $\mathscr{K}$ such that the partially ordered set $(\mathscr{K}, \subset)$ satisfies the following two conditions:
(a) $\mathscr{K}$ is not linearly ordered.
(b) Every linearly ordered subset of $\mathscr{K}$ has an upper bound in In your example, does $\mathscr{K}$ have a maximal element ?

(a)Consider the collection of sets $\mathscr{K}_+ = \{[0, e^{-p^2}] : p \in \mathbb{Z}_+\}$ and $\mathscr{K}_- = \{[0, -e^{-p^2}] : p \in \mathbb{Z}_-\}$ and the union of them $\mathscr{K} = \mathscr{K}_+ \cup \mathscr{K}_-$

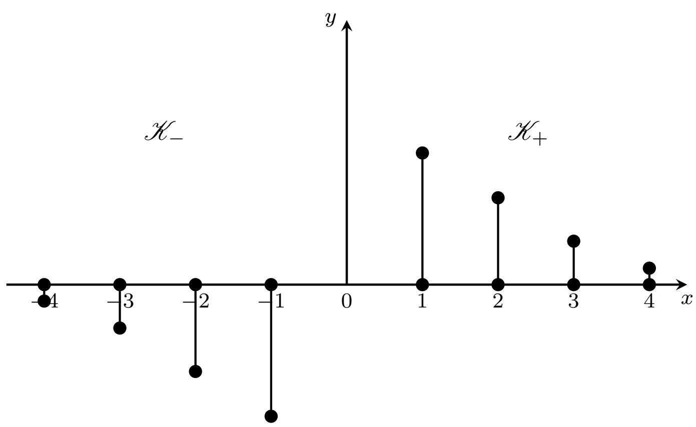


Figure 1.25: The collection $\mathscr{K} = \{[0, e^{-p^2}] : p \in \mathbb{Z}_+\} \cup \{[0, -e^{-p^2}] : p \in \mathbb{Z}_-\}$

We see that $\mathscr{K}_+ \cap \mathscr{K}_- = \emptyset$. Hence for any arbitrary element K_n of $\mathscr{K}_+$ and any arbitrary element K_m of $\mathscr{K}_-$we can't have $K_n \subset K_m$ or $K_n \supset K_m$. Hence, $\mathscr{K}$ is not linearly ordered.

◊

(b)For $\mathscr{K}_-$ we have $[-e, 0]$ as upper bound and for $\mathscr{K}_+$ we have $[0, e]$ as upper bound.
Does $\mathscr{K}$ has a maximal element? For $\mathscr{K}$ having a maximal element we need

$$\exists M \in \mathscr{K}, \forall K \in \mathscr{K} : M \subset K \Rightarrow M = K$$

It is clear that $\mathscr{K}$ does not have a maximal element as any candidate M will be either an element of $\mathscr{K}_+$ or an element of $\mathscr{K}_-$. Suppose $M \in \mathscr{K}_+$ than for any element $K \in \mathscr{K}_-$ which is also an element of $\mathscr{K}$we statement $M \subset K$ is always false, giving that the necessary condition for a maximal element is never true, hence no maximal element can be defined.

⧫

1.23.3

Suppose that S is a set and R is a relation in S that is transitive and antireflexive (19.2). Define $\underset{=}{R}$ in S as follows: For all x and y in S, $x\underset{=}{R}y$ if and only if xRy or $x = y$.
Is $\underset{=}{R}$ a partial ordering in S?

We check whether $\underset{=}{R}$ is transitive, anti-symmetric and reflexive.

i) transitive: $x\underset{=}{R}y,\ y\underset{=}{R}z \Rightarrow x\underset{=}{R}z$
We look at three possible cases:
• $x = z$: then, although $x\not{R}z$, we have by definition $x\underset{=}{R}z$.
• $x \neq y$ and $y \neq z$: we have in that case xRy and yRz, so by definition of $\underset{=}{R}$ we have $x\underset{=}{R}y$ and $y\underset{=}{R}z$. as R is transitive we have $xRyz$, so by definition of $\underset{=}{R}$ we have also $x\underset{=}{R}z$.
• ($x \neq y$ and $y = z$) or ($x = y$ and $y \neq z$): Let's take the case $x = y$ and $y \neq z$.
xRy is not defined as R is anti-reflexive. But, by definition, $x\underset{=}{R}y$ is defined. Also $y\underset{=}{R}z$ is defined because yRz is defined. Then obviously $y\underset{=}{R}z \Leftrightarrow x\underset{=}{R}z$, because $x = y$.

So, $\underset{=}{R}$ is transitive.

ii) reflexive: $\forall x \in S : x\underset{=}{R}x$
This is true by the definition of $\underset{=}{R}$.

So, $\underset{=}{R}$ is reflexive.

iii) anti-symmetric : $\forall x, y \in S : x\underset{=}{R}y$ and $y\underset{=}{R}x \Rightarrow x = y$

Recall: ($x\underset{=}{R}y$ then or xRy or $x = y$) and also ($y\underset{=}{R}x$ then or yRx or $y = x$).

Suppose we have $x\underset{=}{R}y$ and $y\underset{=}{R}x$. Then, by definition of $\underset{=}{R}$ we would have xRy and yRx if both exist. Yet, nothing is said about the symmetrical properties of R. If R is symmetric then xRy and yRx is possible, even if $x \neq y$. Hence, $\underset{=}{R}$ is not necessarily anti-symmetric.

Conclusion: No, $\underset{=}{R}$ is not necessarily a partial ordering of S.

♦

1.23.4

4. Let (A, R) and (B, Γ) be partially ordered sets. Define a relation $\leq$ in $A \times B$,

$$(a_l, b_1) \leq (a_2, b_2) \text{ if and only if } a_l R a_2 \text{ and } b_1 \Gamma b_2.$$

(a) Show that $\leq$ is a partial ordering for $A \times B$.
(b) Suppose that R and Γ are total orderings for A and B, respectively. Is $\leq$ necessarily a total ordering for $A \times B$?

(a) $\leq$ is a partial ordering of $A \times B$.

i) transitive: To prove $(a_1, b_1) \leq (a_2, b_2)$, $(a_2, b_2) \leq (a_3, b_3) \Rightarrow (a_1, b_1) \leq (a_3, b_3)$.

We have $(a_1, b_1) \leq (a_2, b_2) \Leftrightarrow a_1 R a_2$ and $b_1 \Gamma b_2$, and also $(a_2, b_2) \leq (a_3, b_3) \Leftrightarrow a_2 R a_3$ and $b_2 \Gamma b_3$.
We know that R and Γ are transitive relations so $a_1 R a_2$ and $a_2 R a_3 \Rightarrow a_1 R a_3$, and also $b_1 \Gamma b_2$ and $b_2 \Gamma b_3 \Rightarrow b_1 \Gamma b_3$. So we have indeed $(a_1, b_1) \leq (a_2, b_2)$, $(a_2, b_2) \leq (a_3, b_3) \Rightarrow a_1 R a_3$ and $b_1 \Gamma b_3$ and thus $(a_1, b_1) \leq (a_2, b_2)$, $(a_2, b_2) \leq (a_3, b_3) \Rightarrow (a_1, b_1) \leq (a_3, b_3)$ Conclusion : $\leq$ is a transitive relation on $A \times B$.

ii) reflexive: To prove $\forall (a, b) \in A \times B : (a, b) \leq (a, b)$.

Both R and Γ are reflexive, so $\forall a \in A : aRa$ and $\forall b \in B : b \Gamma b$. By definition $\leq$ we need $(a, b) \leq (a, b) \Rightarrow aRa$ and $b \Gamma b$. Hence, the right part of the statement is true, so $\leq$ is a reflexive relation.

iii) anti-symmetric: To prove $\forall (a_1,\, b_1), (a_2,\, b_2) \in A \times B : (a_1,\, b_1) \leq (a_2,\, b_2)$ and $(a_2,\, b_2) \leq (a_1,\, b_1) \Leftrightarrow a_1 = a_2$ and $b_1 = b_2$.

• $\Rightarrow$:
We know that R and Γ are anti-symmetric in A respectively B. By definition we have:
∘ $(a_1,\, b_1) \leq (a_2,\, b_2) \Leftrightarrow a_1 R a_2$ and $b_1 \Gamma b_2$.
∘ $(a_2,\, b_2) \leq (a_1,\, b_1) \Leftrightarrow a_2 R a_1$ and $b_2 \Gamma b_1$.
So, if $(a_1,\, b_1) \leq (a_2,\, b_2)$ and $(a_2,\, b_2) \leq (a_1 2,\, b_1)$ we have

$$(a_1 R a_2 \text{ and } b_1 \Gamma b_2) \text{ and } (a_2 R a_1 \text{ and } b_2 \Gamma b_1)$$

This implies (R and Γ being anti-symmetric)

$$a_1 = a_2 \text{ and } b_1 = b_2$$

an thus we conclude

$$\forall (a_1,\, b_1), (a_2,\, b_2) \in A \times B : (a_1,\, b_1) \leq (a_2,\, b_2) \text{ and } (a_2,\, b_2) \leq (a_1,\, b_1) \Rightarrow a_1 = a_2 \text{ and } b_1 = b_2$$

- $\Leftarrow$: This requires

$$\forall a_1 \text{ and } b_1 \Rightarrow (a_1,\, b_1), (a_1,\, b_1) \in A \times B : (a_1,\, b_1) \leq (a_1,\, b_1) \text{ and } (a_1,\, b_1) \leq (a_1,\, b_1)$$

As A and Γ are reflexive $(a_1,\, b_1) \leq (a_1,\, b_1)$ is a true statement and conclude

$$\forall (a_1,\, b_1), (a_2,\, b_2) \in A \times B : (a_1,\, b_1) \leq (a_2,\, b_2) \text{ and } (a_2,\, b_2) \leq (a_1,\, b_1) \Leftarrow a_1 = a_2 \text{ and } b_1 = b_2$$

Conclusion: From i), ii) and ii) $\leq$ is an partial ordering in $A \times B$.

$\Diamond$

(b) Suppose that R and Γ are total orderings for A and B, respectively. Is $\leq$ necessarily a total ordering for $A \times B$?

The answer is yes as $aRa^{'}$ and $b\Gamma b^{'}$ are defined for all elements in A and B, hence $\leq$ is defined for all elements in $A \times B$.

♦

1.23.5

Let $\mathscr{L}$ be the relation defined in $\mathbb{R} \times \mathbb{R}$ as follows. For all (a_1, a_2) and (b_1, b_2) in $\mathbb{R} \times \mathbb{R}$, let $(a_1,\, a_2)\mathscr{L}(b_1, b_2)$ if and only if

$$a_1 \leq b_1, \text{ and if } a_l = b_1, \text{ then } a_2 \leq b_2$$

For obvious reasons this relation $\mathscr{L}$, is called a dictionary or lexicographical order for $\mathbb{R} \times \mathbb{R}$.
(a) Is $\mathscr{L}$ a partial ordering for $\mathbb{R} \times \mathbb{R}$?
(b) If the answer to (a) is yes, is $\mathscr{L}$ a total ordering for $\mathbb{R} \times \mathbb{R}$?

♦

1.23.6

Prove that if $(S, \leq)$ is a well-ordered set, then it is a linearly ordered set.

• Let's remind what a well ordered set means:

If $(S, \leq)$ is a partially ordered set such that every nonempty subset of S has a first element, then $\neq$ is said to be a well ordering for S and $(S, \neq)$ is said to be a well-ordered set.

• And a maximal element m of a partially ordered set $(S, \leq)$ implies that $\forall s \in S : m \leq s \Rightarrow m = s$.

• A linearly ordered set means that $\forall x, y \in S : x \leq y$ or $y \leq x$.

♦

1.24 Axiom of choice and Zorn's lemma

1.24.1

Show that the two forms of Zorn's lemma given in **24.3** are equivalent.

♦

1.24.2

Prove the following variation of **24.4**. Let $(S, \leq)$ be a partially ordered set. If A is a linearly ordered subset of S, then there exists a maximal linearly ordered subset M of S such that $A \subset M$.

♦

1.24.3

Let $\mathscr{K}$ be a collection of sets. Prove that there exists a maximal nested subcollection $\mathscr{K}$.

♦

1.25 Cardinality of sets (Introduction)

1.25.1

Prove Theorem **25.3**.

♦

1.25.2

Show that the set $\mathbb{P}$ of all positive integers is equivalent to the set of all positive even integers.

♦

1.25.3

Show that the set $\mathbb{P}$ of all positive integers is equivalent to the set $\mathbb{P} - \{1\}$.

♦

1.25.4

Recall that $\mathbb{P}_n = \{1, 2, ..., n\}$. Show that $\mathbb{P} \sim (\mathbb{P} - \mathbb{P}_n)$.

♦

1.26 Countable sets

1.26.1

Prove that the set of all rational real numbers is a countably infinite set.

♦

1.26.2

Prove that if X is a set that is equivalent to a countable set, then X is also a countable set.

♦

1.26.3

Suppose that $f : X \to Y$ is a map from a set X onto a countable set Y. Suppose that for each $y \in Y$, $f^{-1}[v]$ is a countable set. Is X necessarily a countable set?

♦

1.26.4

Prove that if A and B are countable sets, then so is $A \times B$.

♦

1.26.5

Let $f : X \to Y$ be a surjection. Show that there is a subset of X that is equivalent to Y.

♦

1.27 Uncountable sets

1.27.1

It is known that every real number between 0 and 1 inclusive has a (binary) representation in the form $.a_l a_2 a_3 \ldots a_n \ldots$ where each a, is either 0 or 1. However, as in the decimal system, the representation is not unique. For example, .01100 (remaining terms 0) represents the same number as .0101111 (remaining terms 1). But each real number between 0 and 1 has at least one and no more than two representations. Use this information to prove that the reals are uncountable. Point out why this implies that the set of all irrational numbers is uncountable.

♦

1.27.2

Suppose that A is an uncountable set and C is a countable subset of A. Show that $A - C$ is an uncountable set.

♦

1.27.3

Suppose that A, B, C, and D are sets such that $A \cap C = B \cap D$. Suppose further that $A \sim B$ and $C \sim D$. Is $(A \cup C) \sim (B \cup D)$?

♦

1.27.4

Prove that every infinite set contains a countably infinite subset.

♦

1.27.5

Prove that if S is an infinite set and $x \in S$, then $S - \{x\} \sim S$.

♦

1.27.6

Suppose S is an uncountable set and C is a countable set. Show that $S - C \sim S$.

♦

1.29 Review exercises

1.29.I

Let p and q, r, and s be statements. Consider the following compound statement: If (p and q), then (r or s). Choose the statement or statements below which would be the correct way to state the contrapositive of the given compound statement.

1.29.I.1

1. If (r is false and s is false), then (p is false or q is false).
2. If r is false or s is false, then p is false and q is false.
3. If (p and q) is a false statement, then (r or s) is a false statement.
4. If (r or s), then (p and q).
5. None of the previous choices is correct, but a correct one is ...

♦

1.29.II

1.29.II.1

Let A and B and C be nonempty sets. Either prove the following statement or give a counterexample.

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$$

.

♦

1.29.III

1.29.III.1

Give the names of the three properties that a relation must possess in order to be called an equivalence relation.

Associative, reflexive and symmetric.

♦

1.29.III.2

Suppose that R is a relation defined on the set of real numbers as follows: xRy if and only there exists an integer k such that $x - y = k$. Is R an equivalence relation ?
Justify your answers.

♦

1.29.III.3

Suppose that R is an equivalence relation defined in a set A. Which of the following is necessarily true? Justify your answers.

(a) $R = A \times A$
(b) $R \subset A \times A$
(c) $\{(x, x) : x \in A\} \subset R$
(d) If a and b are distinct elements in A, then $R[a] \cap R[b] = \emptyset$
(e) If $R[a] \cap R[b] \neq \emptyset$, , then $R[a] = R[b]$
(f) $R = R^{-1}$

♦

1.29.IV

Let R be the relation defined as follows:

$$R = \{(x, y) : x \text{ is real}, y \text{ is real}, and|x - y| = 5\}$$

1.29.IV.1

Is R a symmetric relation?

♦

1.29.IV.2

2. Is R a transitive relation?

♦

1.29.IV.3

Determine $R[2]$.

♦

1.29.IV.4

Is R a function?

♦

1.29.IV.5

Find the domain of R^{-1}

♦

1.29.V

Let $f : X \to Y$ be a bijection. Let A and B be subsets of X.

1.29.V.1

Prove that $f[A \cap B] = f[A] \cap f[B]$.

♦

1.29.V.2

Give a counterexample to show that the preceding is not true if f is not a one-to-one function but simply a function.

♦

1.29.V.3

Explain what step in your proof of the first part breaks down if the hypothesis does not say ”one-to-one.”

1.29.VI

Consider the map $f : \mathbb{R} \to \mathbb{R}$ given by the following: $f(x) = x(x-2)$ for each $x \in \mathbb{R}$.

1.29.VI.1

Find the range of f.

♦

1.29.VI.2

Determine the set $f^{-1}[\{y : -1 \leq y \leq 0\}]$.

♦

1.29.VI.3

Find the largest number z such that f restricted to the set $\{x : 0 \leq x \leq z\}$ is a one-to-one function.

♦

1.29.VII

1.29.VII.1

Give a precise statement of the principle of finite induction.

♦

1.29.VII.2

Give a precise statement of the well-ordering principle for integers.

♦

1.29.VII.3

Prove that the well-ordering principle implies the principle of finite induction.

♦

1.29.VII.4

Prove that $9^n - 8n - 1$ is divisible by 64, if n is any positive integer.

♦

1.29.VIII

1.29.VIII.1

Define what is meant by an infinite sequence.

♦

1.29.VIII.2

Define what is meant by a subsequence of an infinite sequence.

♦

1.29.VIII.3

Define what is meant by a decomposition of a set.

♦

1.29.VIII.4

Define what is meant by a function that is one-to-one.

♦

1.29.VIII.5

Suppose that $f : X \to Y$ is a one-to-one function, and $g : Y \to Z$ is also a one-to-one function. Prove that the composition $g \circ f$ is a one-to-one function.

♦

1.29.IX

Is the following statement necessarily true? Justify your answer.
If R is a relation and $R^{-1} \subset R$, then R is a symmetric relation.

♦

1.29.X

1.29.X.1

Define what is meant by a partially ordered set.

♦

1.29.X.2

Define what is meant by a totally ordered set.

♦

1.29.X.3

Give an example of a partially ordered set that is not totally ordered.

♦

1.29.X.4

Give an example of a partially ordered set that has a maximal element but no greatest element.

♦

1.29.X.5

Prove that a set has at most one greatest lower bound.

♦

1.29.XI

1.29.XI.1

Use the axiom of choice to give an alternate proof of Theorem **26.6**.

♦

1.29.XI.2

Let $\mathscr{S}$ be the collection of all finite-sequences of integers. Is $\mathscr{S}$ a countable set?

♦

1.29.XII

Let $\mathscr{F}$ be the set of all functions that map the closed interval $[0,1]$ into $[0,1]$. Prove that $[0,1] < \mathscr{F}$ (Hint: Imitate somewhat the proof of **27.1**.)

♦

1.29.XIII

1.29.XIII.1

Give an example of a set A and a relation R in A that is symmetric and transitive in A but is not reflexive.

♦

1.29.XIII.2

Tell what is wrong with the following argument, which claims that to show that if a relation R in a set A is transitive, then it is reflexive.
Let $a \in A$. Choose an element $b \in A$ such that aRb. Since R is symmetric, it then follows that bRa. Since R is transitive, aRb and bRa imply that aRa. hence, we have shown that R is reflexive.

♦

1.29.XIV

Let $(S, \leq)$ be a nonempty partially ordered set such that every linearly ordered subset has an upper bound. Show that if $a \in A$, then there is a maximal element $m \in S$ such that $a \leq m$.

♦