

Undergraduate Topology
Robert H. Kasriel (Dover Publication)
Solutions to exercises
Part I
Chapters I to IV

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Figure 1

Remarks and warnings

You're welcome to use these notes, but they may contain errors, so proceed with caution : I graduated in 1979, went straight in the industry (where I didn't have to use fancy maths), and picked mathematics and physics again after I retired, so my mathematics got rusty for sure. If you do find an error, typo's , I'd be happy to receive bug reports, suggestions, and the like, through Github.

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Sets, Functions, and Relations

1.1 Sets and Membership

1.1.1

List explicitly the elements of the set

$$\{x : x < 0 \text{ and } (x-1)(x+2)(x+3) = 0\}$$

$$\{-3, -2\}$$



1.1.2

List the elements of the set

$$\{x : 3x - 1 \text{ is a multiple of } 3\}$$

$$\{x : x = k + \frac{1}{3}, k \in \mathbb{Z}\}$$



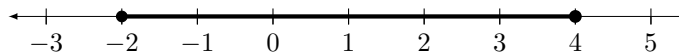
1.1.3

Sketch on a number line each of the following sets.

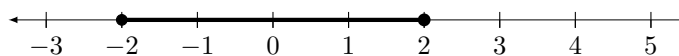
(a) $\{x : |x - 1| \leq 3\}$

(b) $\{x : |x - 1| \leq 3 \text{ and } |x| \leq 2\}$

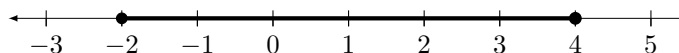
(c) $\{x : |x - 1| \leq 3 \text{ or } |x| \leq 2\}$



(a)



(b)



(c)



1.2 Some remarks on the use of the connectives *and*, *or*, *implies*

1.2.1

Demonstrate by means of a table showing truth values that the following is a true statement for any choice of p and q . Thus show that it is a tautology.

$$(\neg q \Rightarrow \neg p) \Rightarrow (p \Rightarrow q)$$

p	q	$\neg q$	$\neg p$	$\neg q \Rightarrow \neg p$	$p \Rightarrow q$	$(\neg q \Rightarrow \neg p) \Rightarrow (p \Rightarrow q)$
T	T	F	F	T	T	T
T	F	T	F	F	F	T
F	T	F	T	T	T	T
F	F	T	T	T	T	T



1.2.2

Show by means of a truth table that the statement

$$((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$$

is a tautology.

p	q	r	$p \Rightarrow q$	$q \Rightarrow r$	$(p \Rightarrow q) \wedge (q \Rightarrow r)$	$p \Rightarrow r$	$((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T



1.2.3

Show by means of a truth table that

$$(p \wedge q) \Rightarrow (p \vee q)$$

is a tautology.

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \Rightarrow (p \vee q)$
T	T	T	T	T
T	F	F	F	T
F	T	F	T	T
F	F	F	F	T



1.2.4

Suppose that p and q are statements such that $(p \wedge q)$ is a false statement. Does it follow that the statement

$$(p \text{ is false}) \vee (q \text{ is false})$$

is a true statement?

p	q	$p \wedge q$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	F	F	F	T	T
F	T	F	T	F	T
F	F	F	T	T	T

The answer is Yes.



1.2.5

Negate the following statement: *If two angles of a triangle have equal measure, then the length of two sides of that triangle are equal.*

First we note that $\neg(p \Rightarrow q) \Leftrightarrow (p \wedge \neg q)$. Indeed,

p	q	$p \Rightarrow q$	$\neg(p \Rightarrow q)$	$\neg q$	$p \wedge \neg q$	$\neg(p \Rightarrow q) \Leftrightarrow (p \wedge \neg q)$
T	T	T	F	F	F	T
T	F	F	T	T	T	T
F	T	T	F	F	F	T
F	F	T	F	T	F	T

Putting p as *two angles of a triangle have equal measure* and $\neg q$ as *no two sides of that triangle have equal length* we get the true 'false' statement:

Two angles of a triangle have equal measure \wedge no two sides of that triangle have equal length.



1.2.6

Write the contrapositive of the statement in Exercise 5.

The contrapositive of $p \Rightarrow q$ is $\neg q \Rightarrow \neg p$. Putting $\neg p$ as *no two angles of a triangle have equal measure* and $\neg q$ as *no two sides of that triangle have equal length* we get

If no two sides of that triangle have equal length then no two angles of a triangle have equal measure.



1.2.7

Write the converse of the statement in Exercise 5.

The converse of $p \Rightarrow q$ is $q \Rightarrow p$, giving

If two sides of a triangle have equal length then two angles of a that triangle have equal measure.



1.2.8

Write the contrapositive of the following statement

If a person belongs to Committee A, then he must be a member of Committee B and he must be a member of Committee C.

Lets put

$p \equiv$ a person belongs to Committee A

$q \equiv$ a person belongs to Committee B

$r \equiv$ a person belongs to Committee C

then the given statement translates as

$$p \Rightarrow (q \wedge r)$$

and the contrapositive

$$\neg(q \wedge r) \Rightarrow \neg p$$

This last statement is equivalent with

$$(\neg q \vee \neg r) \Rightarrow \neg p$$

or in plain text:

If a person does not belong to Committee B or C , then he is not a member of Committee A.



1.2.9

Write the contrapositive of the following statement

If $x \in A$ and $x \in B$, then $x \in C$

Lets put

$$p \equiv x \in A$$

$$q \equiv x \in B$$

$$r \equiv x \in C$$

then the given statement translates as

$$p \wedge q \Rightarrow r$$

and the contrapositive

$$\neg(r) \Rightarrow \neg(p \wedge q)$$

This last statement is equivalent with

$$\neg(r) \Rightarrow (\neg p \vee \neg q)$$

i.e:

$$x \notin C \Rightarrow (x \notin A \vee x \notin B)$$



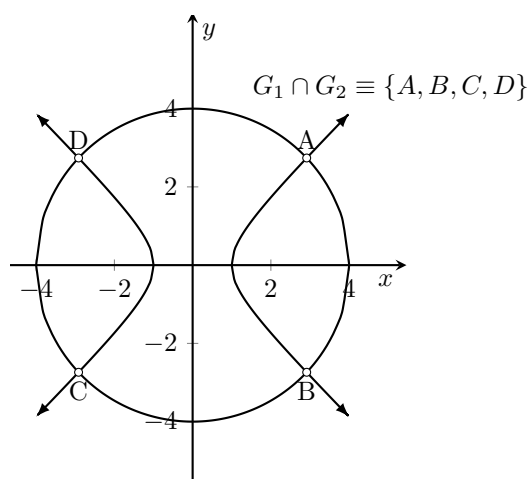
1.3 Subsets

No exercises!

1.4 Union and Intersection of sets

1.4.1

Let G_1 be the graph of the equation $x^2 + y^2 = 16$, and let G_2 be the graph of the equation $x^2 - y^2 = 1$. Sketch the sets $G_1 \cup G_2$ and $G_1 \cap G_2$.



$G_1 \cup G_2$ contains all the points defined by the graphs G_1 and G_2 . $G_1 \cap G_2 \equiv \{A, B, C, D\}$ contains the 4 points at the intersection of the two graphs.



1.4.2

We define the sets A , B , C as follows: $A = \{(x, y) : x^2 + y^2 \leq 9\}$, $B = \{(x, y) : x + y \geq 3\}$, $C = \{(x, y) : x \geq 0\}$.

Draw sketches of each of the following sets:

- (a) $A \cup (B \cup C)$
- (b) $A \cap (B \cup C)$
- (c) $(A \cap B) \cup (A \cap C)$
- (d) $(A \cup B) \cup C$
- (e) $A \cup (B \cap C)$
- (f) $(A \cup B) \cap (A \cup C)$

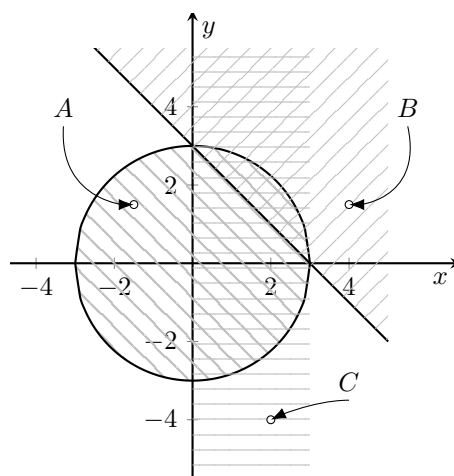
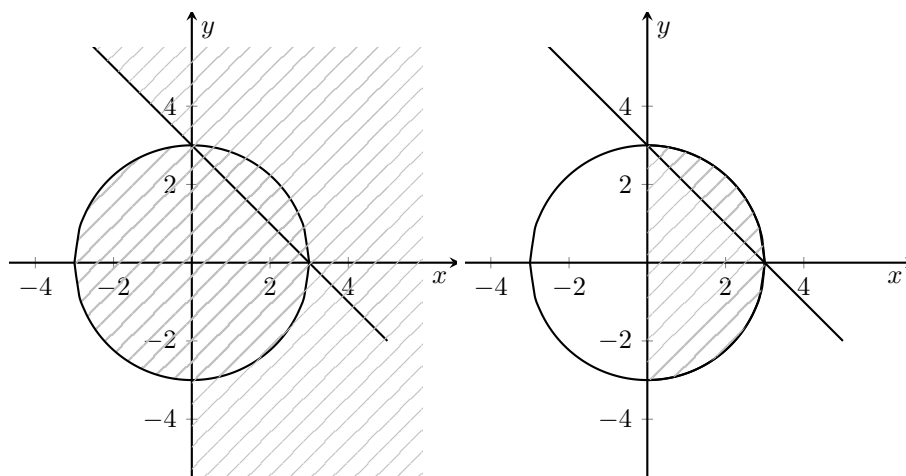
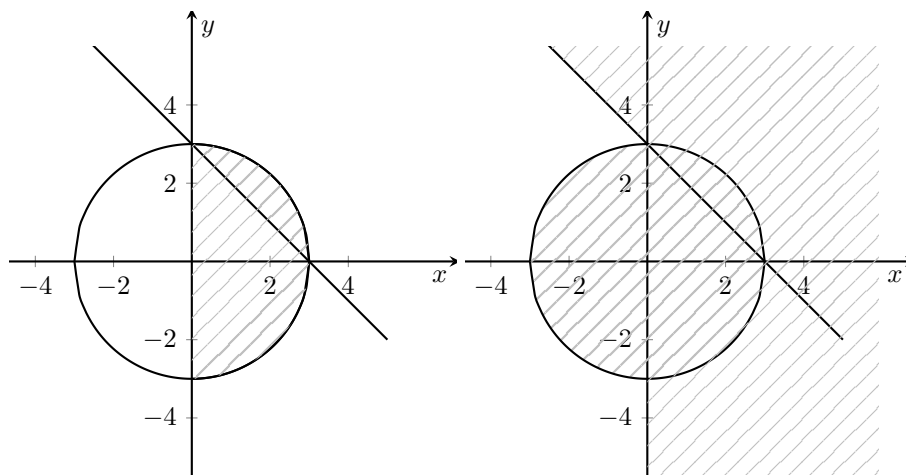
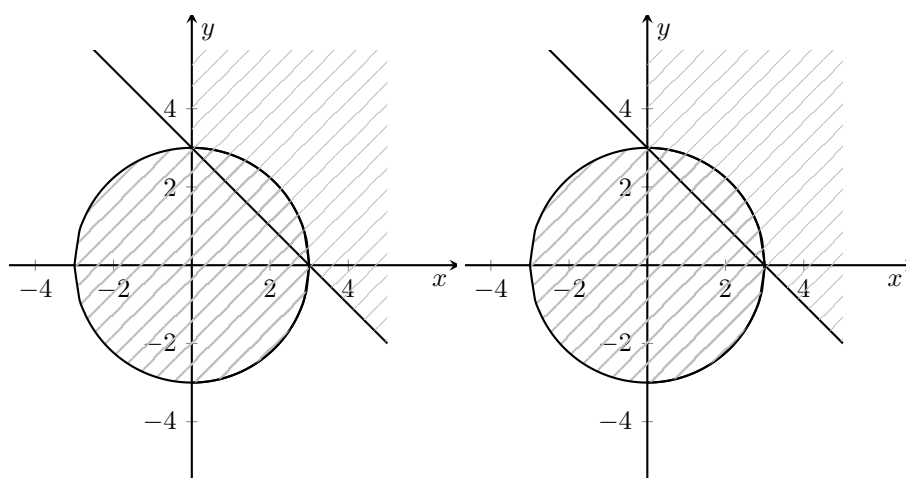


Figure 1.1: The 3 sets A , B , C

(a) $A \cup (B \cup C)$ (b) $A \cap (B \cup C)$ (c) $(A \cap B) \cup (A \cap C)$ (d) $(A \cup B) \cup C$ (e) $A \cup (B \cap C)$ (f) $(A \cup B) \cap (A \cup C)$ 

1.4.3

Let A, B, C as follows: $A = \{(x, y) : x + y \leq 5\}$, $B = \{(x, y) : x + y \geq 3\}$, $C = \{(x, y) : x \geq 3\}$, and $D = \{(x, y) : y \geq 3\}$.

Draw a sketch for each of the following sets:

- (a) $(A \cap B) \cap C$
 (b) $[(A \cap B) \cap C] \cap D$

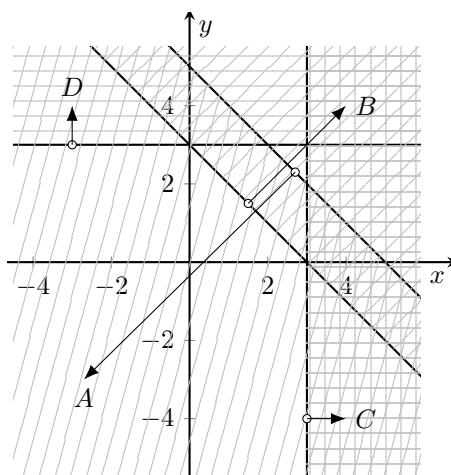
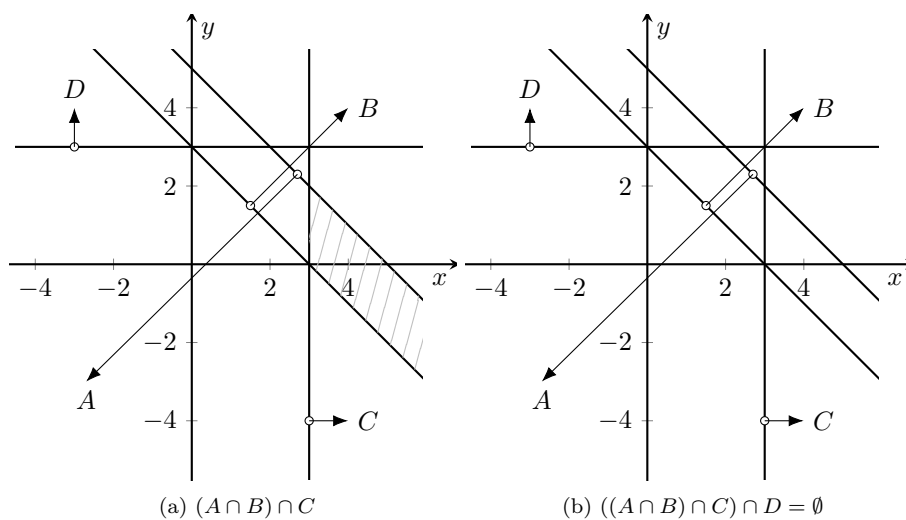


Figure 1.2: The 4 sets A, B, C, D



1.5 Complementation

1.5.1

Sketch each of the following sets: (the sets A , B , C are defined as in exercise 3page 8)

- (a) $\sim (A \cap B)$
- (b) $(\sim A) \cup (B)$
- (c) $\sim (A \cup B)$
- (d) $(\sim A) \cap (B)$
- (e) $C - A$
- (f) $\sim (A \cap C)$
- (g) $(\sim A) \cup (\sim B)$
- (h) $(\sim A) \cap (A)$
- (i) $C - (A \cup B)$
- (j) $(C - A) \cap (C - B)$
- (k) $\sim (\sim A)$

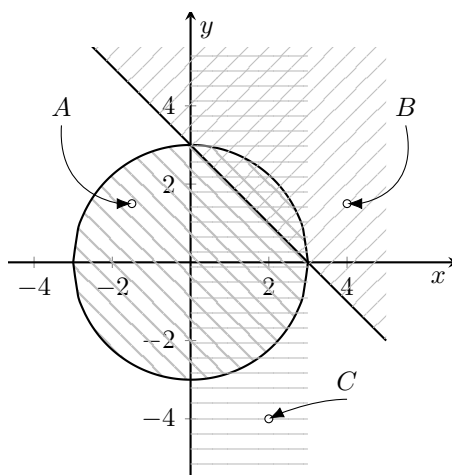
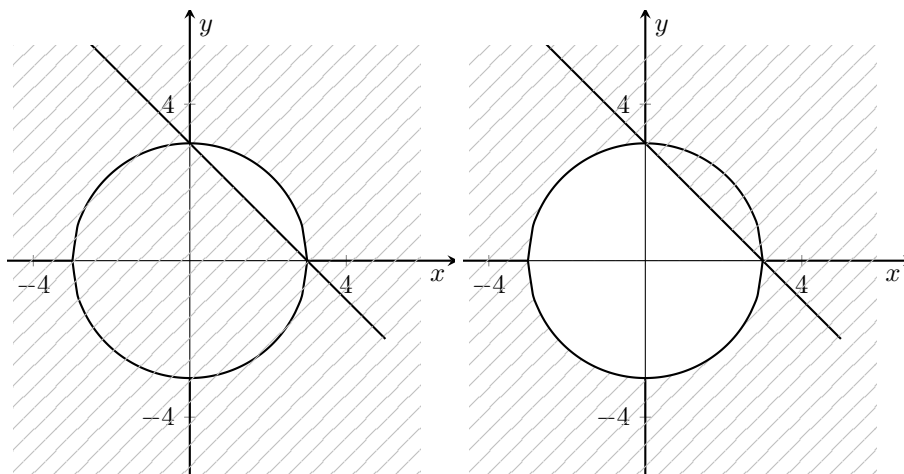
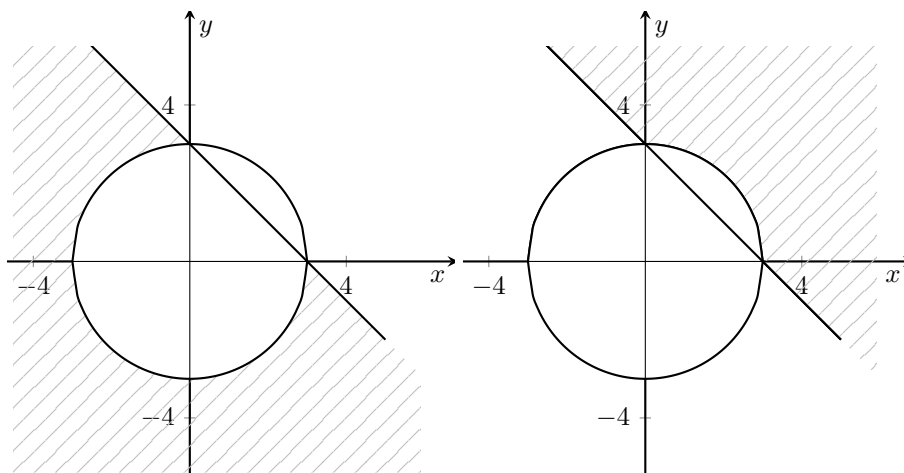


Figure 1.3: The 3 sets A , B , C



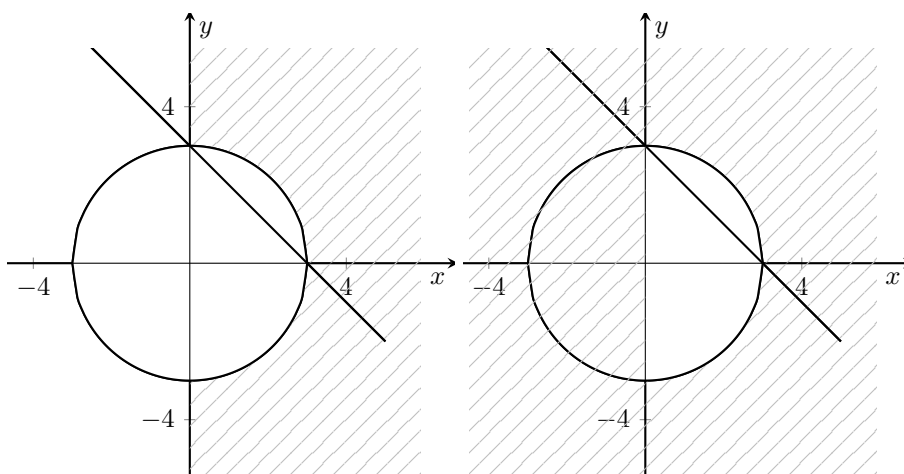
(a) $\sim (A \cap B)$

(b) $(\sim A) \cup (B)$



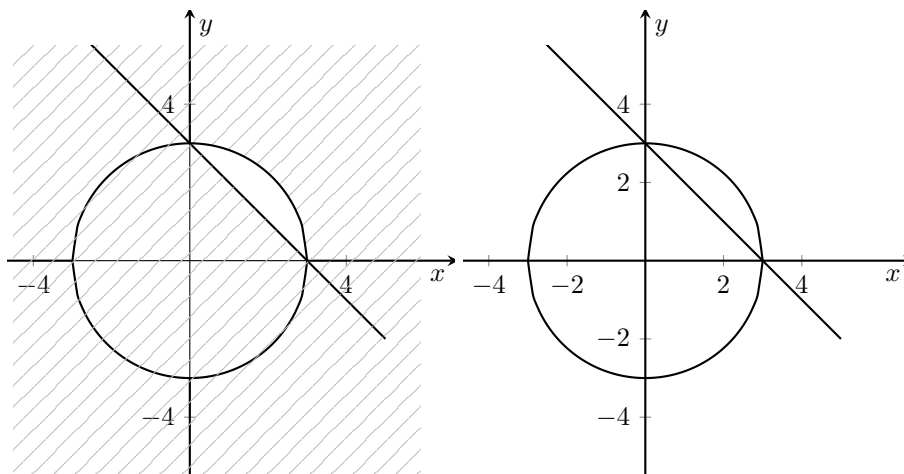
(c) $\sim (A \cup B)$

(d) $(\sim A) \cap (B)$



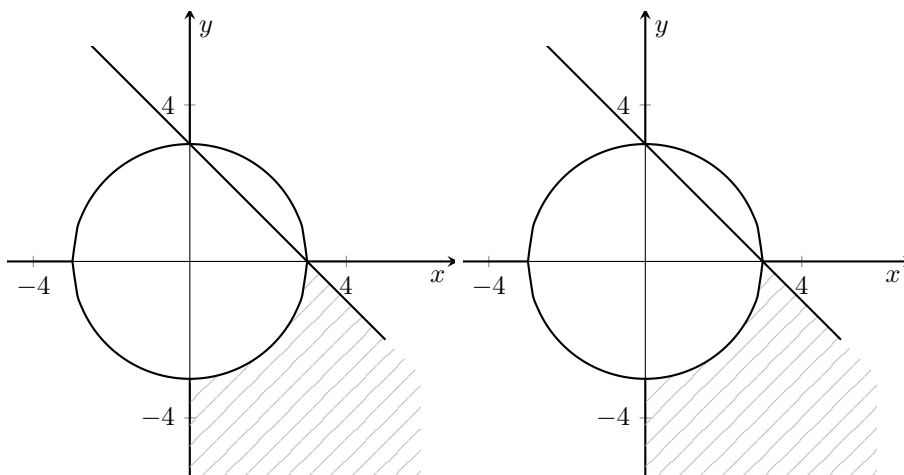
(e) $C - A$

(f) $\sim (A \cap C)$



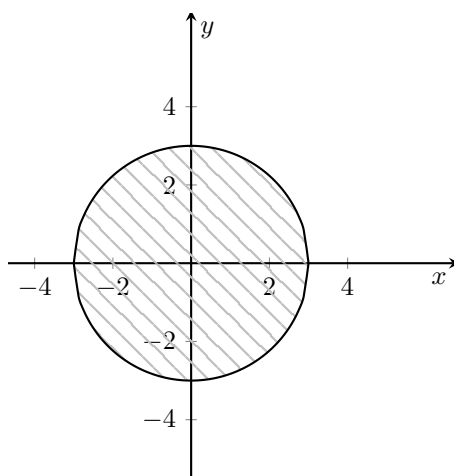
(g) $(\sim A) \cup (\sim B)$

(h) $(\sim A) \cap (A) = \emptyset$



(i) $C - (A \cup B)$

(j) $(C - A) \cap (C - B)$



(k) $\sim(\sim A)$



1.5.2

On the basis of the sketches made in the previous exercise, formulate a proposition about relation that exist concerning complementation, union, and intersection. Try out your conjecture on other examples. In subsequent exercises you will be asked to try to prove such conjectures.

$$1.4.2(a) \text{ and } (d) \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$1.4.2(b) \text{ and } (c) \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$1.4.2(e) \text{ and } (f) \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$1.5.1(a) \text{ and } (g) \quad \sim (A \cap B) = (\sim A) \cup (\sim B)$$

$$1.5.1(h) \quad (\sim A) \cap A = \emptyset$$

$$1.5.1(i) \text{ and } (j) \quad C - (A \cup B) = (C - A) \cap (C - B)$$

$$1.5.1(k) \quad \sim (\sim A) = A$$



1.6 Set identities and other set relations

1.6.1

Prove that if $A \subset B$, then:

$$(a) \quad A \cap C \subset B \cap C$$

$$(b) \quad \sim B \subset \sim A$$

$$(c) \quad A \cap B = A$$

$$(d) \quad A \cup C \subset B \cup C$$

a) $A \cap C \subset B \cap C$

Given is $x \in B$ if $x \in A$. Suppose $x \in A \cap C$, then $x \in A$ (given) and $x \in C$ but $x \in B$ (given) and as $x \in C$ follows that $x \in B \cap C$. And we conclude that $A \cap C \subset B \cap C$.

◇

b) $\sim B \subset \sim A$

Given is $x \in B$ if $x \in A$. If $x \notin B$ then $x \in \sim B$. As $A \subset B$, x will not be in A but $x \in \sim A$. So $x \in \sim B \Rightarrow x \in \sim A$ and thus $\sim B \subset \sim A$.

◇

c) $A \cap B = A$

Given is $x \in B$ if $x \in A$. Suppose $x \in A \cap B$, then $x \in A$ and thus $A \cap B \subset A$. Suppose $x \in A$, then $x \in B$ as $A \subset B$ and thus $x \in A \cap B$ from which we conclude $A \subset A \cap B$.

◇

d) $A \cup C \subset B \cup C$

Given is $x \in B$ if $x \in A$. Suppose $x \in A \cup C$, then $x \in A$ or $x \in C$. But $x \in B$ (given), so $x \in B$ or $x \in C$ and thus $x \in B \cup C$, from which we conclude $A \cup C \subset B \cup C$.

◆

1.6.2

Verify that each of the following is an identity:

- (a) $A \cup \emptyset = A$
- (b) $A \cap \emptyset = \emptyset$
- (c) $A \cap A = A$
- (d) $A \cup A = A$
- (e) $(A \cup B) \cup C = A \cup (B \cup C)$
- (f) $(A \cap B) \cap C = A \cap (B \cap C)$
- (g) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- (h) $X - (A \cup B) = (X - A) \cap (X - B)$
- (i) $A \cap \sim A = \emptyset$
- (j) $A \cup \sim A = U$

a) $A \cup \emptyset = A$

This is a consequence of remark 3.3 page 7: the empty set \emptyset is a subset of every set. So, $\emptyset \subset A$ giving the asked identity.

◇

b) $A \cap \emptyset = \emptyset$

If $x \in A \cap \emptyset$ then $x \in A$ and x must also be in \emptyset which is impossible by definition. So there is no element $x \in \emptyset$ which can satisfy $x \in A \cap \emptyset$ giving the proposed identity.

◇

c) $A \cap A = A$

Suppose $x \in A \cap A$, then $x \in A$ and $x \in A$ and thus $x \in A$, giving $A \cap A \subset A$. Suppose $x \in A$, then obviously $x \in A$ and $x \in A$, giving $A \subset A \cap A$. Hence $A \cap A = A$

◇

d) $A \cup A = A$

Suppose $x \in A \cup A$, then $x \in A$ or $x \in A$ and thus $x \in A$, giving $A \cup A \subset A$. Suppose $x \in A$, then obviously $x \in A$ or $x \in A$, giving $A \subset A \cup A$. Hence $A \cup A = A$

◇

e) $(A \cup B) \cup C = A \cup (B \cup C)$

Suppose $x \in (A \cup B) \cup C$, then $x \in (A \cup B)$ or $x \in C$ and thus $x \in A$ or $x \in B$ or $x \in C$. So $x \in B$ or $x \in C$ can be written as $x \in (B \cup C)$. So $x \in A$ or $x \in (B \cup C)$, giving $(A \cup B) \cup C \subset A \cup (B \cup C)$. The same reasoning yields for $x \in A \cup (B \cup C)$ giving the identity.

◇

f) $(A \cap B) \cap C = A \cap (B \cap C)$

Suppose $x \in (A \cap B) \cap C$, then $x \in (A \cap B)$ and $x \in C$ and thus $x \in A$ and $x \in B$ and $x \in C$. So $x \in B$ and $x \in C$ can be written as $x \in (B \cap C)$. So $x \in A$ and $x \in (B \cap C)$, giving $(A \cap B) \cap C \subset A \cap (B \cap C)$. The same reasoning yields for $x \in A \cap (B \cap C)$ giving the identity.

◇

g) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Suppose $x \in A \cup (B \cap C)$, then $x \in A$ or $x \in (B \cap C)$. Take the case $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$ which implies $x \in (A \cup B) \cap (A \cup C)$, giving $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$. The other case: if $x \in B \cap C$ then $x \in B$ and $x \in C$. So, $x \in A \cup B$ and $x \in A \cup C$ giving also $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$.

On the other hand, be $x \in (A \cup B) \cap (A \cup C)$ then $x \in (A \cup B)$ and $x \in (A \cup C)$. Let's first take the case $x \in A$ then obviously $x \in A \cup (B \cap C)$ even if $x \notin B \cap C$. Alternatively, be $x \notin A$ then we must have $x \in B$ and $x \in C$ which implies $x \in B \cap C$, giving again $x \in A \cup (B \cap C)$.

◇

h) $X - (A \cup B) = (X - A) \cap (X - B)$

Suppose $x \in X - (A \cup B)$, then $x \notin A$ and $x \notin B$ which implies $x \in X - A$ and $x \in X - B$ and thus $x \in X - A \cap X - B$ giving $X - (A \cup B) \subset (X - A) \cap (X - B)$.

The other way around. Suppose $x \in (X - A) \cap (X - B)$. Then $x \in (X - A)$ and $x \in (X - B)$ which implies $x \notin A$ and $x \notin B$ giving $x \notin A \cup B$ which in turn implies $x \in X - (A \cup B)$ giving $(X - A) \cap (X - B) \subset X - (A \cup B)$.

Conclusion: $X - (A \cup B) = (X - A) \cap (X - B)$

◇

i) $A \cap \sim A = \emptyset$

Suppose $x \in A \cap \sim A$, then $x \in A$ and $x \notin A$ which is a contradiction, so the only element which is always an element of any set is the empty set, so $A \cap \sim A \subset \emptyset$. Suppose on the contrary that $x \in \emptyset$. This implies that x correspond to the empty set and as the empty set is an element of

any set, we have $\emptyset \subset A \cap \sim A$

◇

j) $A \cup \sim A = U$

Suppose $x \in A \cup \sim A$, then $x \in A$ or $x \notin A$. So, in any case $x \in U$ and thus $A \cup \sim A \subset U$.

On the opposite way suppose that $x \in U$. Then obviously $x \in A$ or $x \in \sim A$ and thus $U \subset A \cup \sim A$.

◆

1.6.3

Prove that if $A \subset C$ and $B \subset C$, then $A \cup B \subset C$.

Given is $A \subset C$ and $B \subset C$. Take $x \in A$, then $x \in C$, so even if $x \notin B$, then $x \in A \cup B$ reduces to $x \in A$ and thus $x \in C$. The same reasoning yields for $x \in B$, giving $A \cup B \subset C$.

◆

1.6.4

Prove that if $A \subset B$ and $A \subset C$, then $A \subset B \cap C$.

Given is $A \subset B$ and $A \subset C$. Take $x \in A$, then $x \in C$ and $x \in B$, which implies $x \in C \cap B$. giving indeed $A \subset B \cap C$.

◆

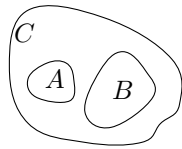
1.7 Counterexamples

In each of the following exercises state whether the statement is necessarily true. Assume that A , B and C are subsets of a universal set U . Justify with a proof or a counterexample.

1.7.1

If $A \cup C = B \cup C$, then $A = B$

Not TRUE.



(I) $A \cup C = B \cup C \not\Rightarrow A = B$

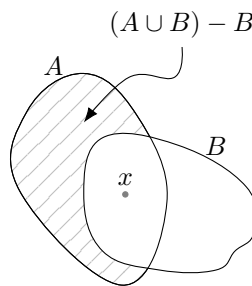
Be $A \subset C$ and $B \subset C$, then we have $A \cup C = B \cup C \equiv C = C$ even if $A \cap B = \emptyset$.



1.7.2

$(A \cup B) - B = A$

Not TRUE.



(m) $(A \cup B) - B \neq A$

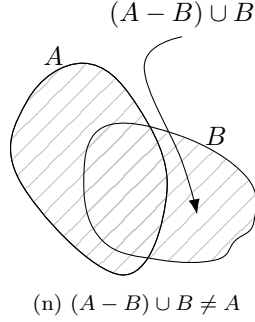
Be $A \cap B \neq \emptyset$, take $x \in A$ and $x \in B$, then x can't be $x \in (A \cup B) - B$ although it is an element of A .



1.7.3

$$(A - B) \cup B = A$$

Not TRUE.



This is only true if $B \subset A$



1.7.4

$$\sim (A - B) = \sim (A \cap \sim B)$$

TRUE.

Suppose first that A and B are disjoint, i.e. $A \cap B = \emptyset$, then $A - B = A$ and $\sim (A - B) = \sim A$. On the other hand $A \subset \sim B$, so $A \cap \sim B = A$, giving $\sim (A \cap \sim B) = \sim A$, giving indeed $\sim (A - B) = \sim (A \cap \sim B)$.

Suppose now that A and B are not disjoint, i.e. $A \cap B \neq \emptyset$. Be $x \in A - B \subset A$. This is equivalent with the statement $x \in A \wedge x \notin B$. Negating this statement: $\neg(x \in A \wedge x \notin B) \Leftrightarrow x \notin A \vee x \in B$. This give $\sim (A - B) \equiv x \notin A \vee x \in B$.

Be now $x \in A \cap \sim B$. This is equivalent with the statement $x \in A \wedge x \notin B$. Negating this statement: $\neg(x \in A \cap \sim B) \Leftrightarrow x \notin A \vee x \in B$. This give $\sim (A \cap \sim B) \equiv x \notin A \vee x \in B$, resulting in $\sim (A - B) = \sim (A \cap \sim B)$.



1.7.5

$$\sim (\sim (\sim A)) = \sim A$$

TRUE.

Be $x \in \sim (\sim (\sim A))$. This is equivalent to $x \notin \sim (\sim A)$. Which on it's turn is equivalent with $x \in \sim A$. So, $\sim (\sim (\sim A)) \subset \sim A$.

Be $x \in \sim A$. This is equivalent to $x \notin \sim (\sim A)$. Which on it's turn is equivalent with $x \in \sim (\sim (\sim A))$. So, $\sim A \subset \sim (\sim (\sim A))$.

Both cases reduce to $\sim (\sim (\sim A)) = \sim A$.



1.7.6

$$A \cup (B - C) = (A \cup B) - C$$

Not TRUE.

Be $x \in A \cup (B - C)$. This is equivalent to $x \in A \vee x \in (B - C)$. Suppose $x \in A$, then $x \in A \cup B$. Let's consider the set C so that $(A \cup B) \subset C$, then $(A \cup B) - C = \emptyset$. We get a contradiction and the proposed statement is not true.



1.7.7

$$\sim (A - B) = (\sim A) \cup B$$

TRUE.

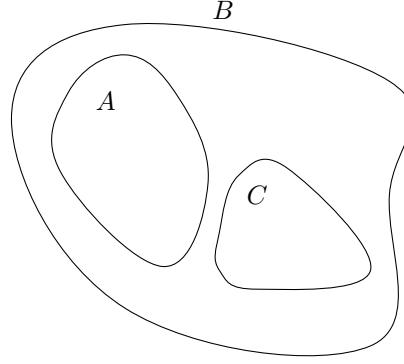
Be $x \in (A - B)$. This is equivalent to $x \in A \wedge x \notin B$. Negating this statement: $\neg(x \in A \wedge x \notin B) \Leftrightarrow x \notin A \vee x \in B$. This is equivalent to the statement $x \in (\sim A) \cup B$. So $\sim (A - B) \subset (\sim A) \cup B$. Consider now $x \in (\sim A) \cup B$. So $x \notin A \vee x \in B$. If we have the case $x \notin A$ then also $x \notin (A - B)$ as x can not be one of the remaining elements of A after the complement of B relative to A . Also, if $x \in B$ then also $x \notin (A - B)$ as x is an element of B and thus can not be an element of $(A - B)$. Thus, in both cases we have, $x \notin (A - B)$ which implies $x \in \sim (A - B)$. So $(\sim A) \cup B \subset \sim (A - B)$.



1.7.8

$$\text{If } A - B = C - B, \text{ then } A = C.$$

Not TRUE.



(o) If $A - B = C - B \not\Rightarrow A = C$

Suppose $A \subset B$, then $A - B = \emptyset$. Choose a C such that $C \subset B$ and also $A \cap C = \emptyset$, then also $C - B = \emptyset$ and get $A - B = C - B$ although $A \neq C$.



1.7.9

If $A - (B \cap C) = (A - B) \cap (A - C)$.

TRUE.

Suppose $x \in A - (B \cap C)$, then $x \in A \wedge x \notin B \cap C$. As x can not be simultaneously in B and C , then also x must be simultaneously in $A - B$ and $A - C$ as the "complementation of A with B and C will not "subtract" x out of A , and considering that $x \in A$ we have $A - (B \cap C) \subset (A - B) \cap (A - C)$. Suppose $x \in (A - B) \cap (A - C)$, then x must be an element of A but not an element of B and C . This means that $x \notin B \cap C$ and thus the complementation of A by $B \cap C$ has no effect on x . Thus, $\underbrace{(A - B) \cap (A - C)}_{=A} \subset A - (B \cap C)$.



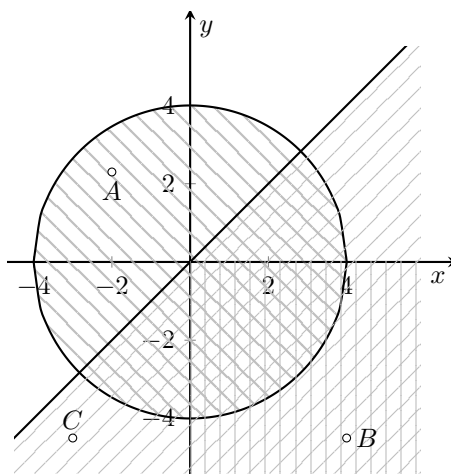
1.8 Collections of Sets

1.8.1

Suppose that A , B and C are the following subsets of the plane:

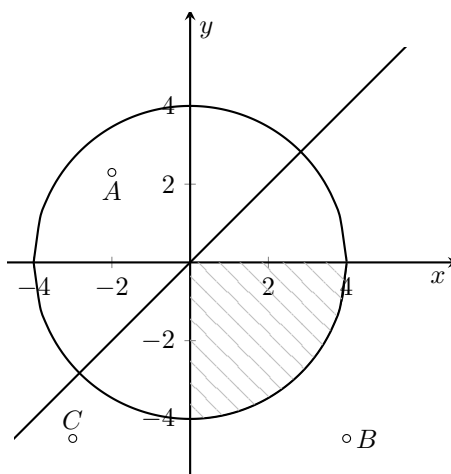
$A = \{(x, y) : x^2 + y^2 \leq 16\}$, $B = \{(x, y) : x \geq 0 \text{ and } y \leq 0\}$, $C = \{(x, y) : y \leq x\}$. If \mathcal{K} is the collection of sets $\{A, B, C\}$, sketch each of the following sets:

- (a) $\bigcap \mathcal{K}$
- (b) $\bigcup \mathcal{K}$
- (c) $\bigcup \mathcal{K} - \bigcap \mathcal{K}$



(p) The sets A , B , C

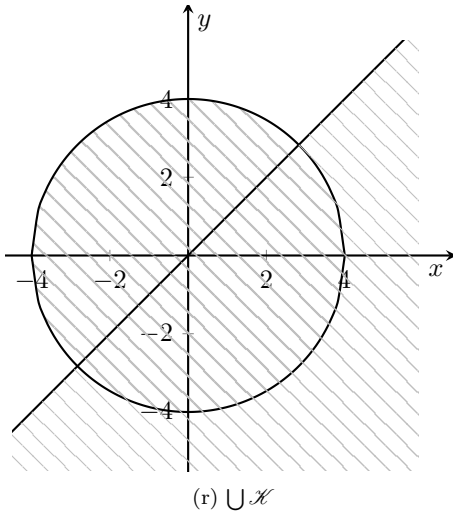
a) $\bigcap \mathcal{K}$



(q) $\bigcap \mathcal{K}$

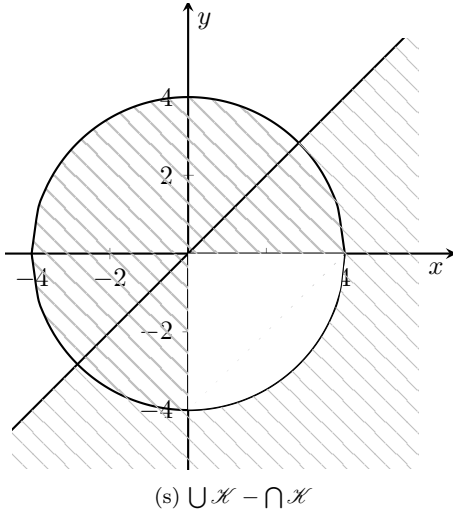
◇

b) $\cup \mathcal{K}$



◇

c) $\cup \mathcal{K} - \cap \mathcal{K}$



◆

1.8.2

Recall that \mathbb{P} is the symbol for the set of positive integers. Suppose that for each $n \in \mathbb{P}$, we let $A_n = \{x \in \mathbb{R} : x \geq n\}$. Describe the sets $\bigcup\{A_n : n \in \mathbb{P}\}$ and $\bigcap\{A_n : n \in \mathbb{P}\}$.

$$S = \bigcup\{A_n : n \in \mathbb{P}\}$$

$$S = [1, +\infty)$$

◇

$$S = \bigcap\{A_n : n \in \mathbb{P}\}$$

$$S = \emptyset$$

This can be understood by the fact that for every $x \in \mathbb{R}$, you can find a $n \in \mathbb{P}$ so that $x \notin A_n$. So, no x can be an element of S .

◆

1.8.3

Suppose that for each $n \in \mathbb{P}$, K_n is a non-empty set such that $K_{n+1} \subset K_n$. Let $\mathcal{K} = \{K_n : n \in \mathbb{P}\}$.

In each of the following, if the statement is necessarily true, say so and justify your answer. If the statement is not necessarily true, give a counterexample to justify your answer.

- (a) $\bigcup \mathcal{K} = K_1$
- (b) $\bigcap\{K_i : i = 1, 2, \dots, n\} = K_n$
- (c) $\bigcap \mathcal{K} \neq \emptyset$

(a) $\bigcup \mathcal{K} = K_1$.

TRUE.

Be $x \in K_n$ for any arbitrary n . So, $x \in K_n \cup K_{n-1}$. But $K_n \cup K_{n-1} = K_{n-1}$, giving $x \in K_{n-1}$. Repeating that process with $K_{n-1} \subset K_{n-2} \subset \dots \subset K_2 \subset K_1$ we get $x \in K_1$.

◇

(b) $\bigcap\{K_i : i = 1, 2, \dots, n\} = K_n$.

TRUE.

Suppose first that for all n we have K_n is a *proper* subset of K_{n-1} . Then $K_n \cap K_{n-1} = K_n$. Be $x \in K_n$ but not in K_{n-1} for any arbitrary n . Then, $x \in K_n \cap K_{n-1}$ is equivalent to $x \in K_n$. Repeating that process with we have $K_n \cap K_{n-1} \cap K_{n-2} \cap \dots \cap K_2 \cap K_1 = K_n$ and get $x \in K_n$. Hence, $\bigcap \mathcal{K} = K_1$.

In the case that for some or all n we have $K_n = K_{n-1}$ we could also state that $\bigcap\{K_i : i = 1, 2, \dots, n\} = K_{n-1}$ but as $K_n = K_{n-1}$ we can write $\bigcap\{K_i : i = 1, 2, \dots, n\} = K_{n-1} = K_n$.

The same is true in the case that a sequence of the subsets are proper subset of each other i.e. $K_{n+p} = K_{n+p-1} = \dots K_{n+1} = K_n = K_{n-1} = \dots = K_{n-t}$. then one could write $\bigcap \{K_i : i = 1, 2, \dots, n\} = K_{n+p}$ but as $K_{n+p} = K_n$, the original statement holds.

◇

(c) $\bigcap \mathcal{K} \neq \emptyset$.

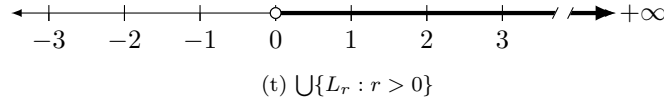
TRUE.

As no K_n is an empty set, K_n will always contain at least one element and due to (b) we get indeed $\bigcap \mathcal{K} \neq \emptyset$: suppose that for a given n , K_n contains only one element x , then all subsequent K_{n+p} must also have only one element i.e. x and we will get $\bigcap \mathcal{K} = \{x\}$

◆

1.8.4

For each real number $r > 0$, let $L_r = \{x : x \geq r\}$. Sketch the set $\bigcup \{L_r : r > 0\}$ and $\bigcap \{L_r : r > 0\}$ on a number line. If a set happens to be empty, say so.



◇

$\bigcap \{L_r : r > 0\} = \emptyset$.

Indeed, take an arbitrary r and be $\epsilon > 0$ then $\exists x \in L_r : x \notin L_{r+\epsilon}$. Then, $L_r \cap L_{r+\epsilon} = \emptyset$. So, whatever L_r we choose in the collection $\mathcal{L} = \{L_r : r \in \mathbb{R}^+\}$ there always be a $L_{r'}$ for which $L_r \cap L_{r'} = \emptyset$ and hence $\bigcap \{L_r : r > 0\} = \emptyset$.

◆

1.8.5

Let U be a set and let \mathcal{K} be a non-empty collection of subsets of U . \sim will signify the complement with respect to U . Prove the following set identities. The identities are quite important and are known as De Morgan's Laws.

- (a) $\sim (\bigcup \{K : K \in \mathcal{K}\}) = \bigcap \{\sim K : K \in \mathcal{K}\}$
 (b) $\sim (\bigcap \{K : K \in \mathcal{K}\}) = \bigcup \{\sim K : K \in \mathcal{K}\}$

(a) $\sim (\bigcup \{K : K \in \mathcal{K}\}) = \bigcap \{\sim K : K \in \mathcal{K}\}$

Suppose $x \in \sim (\bigcup \{K : K \in \mathcal{K}\})$, then $x \notin \bigcup \{K : K \in \mathcal{K}\}$. This means that x is not an element of any $K \in \mathcal{K}$ i.e. $\forall K \in \mathcal{K} : x \notin K$. This can also be expressed as $\forall K \in \mathcal{K} : x \in \sim K$. This means that x is an element of all $\sim K$ giving $x \in \bigcap \{\sim K : K \in \mathcal{K}\}$ and thus $\sim (\bigcup \{K : K \in \mathcal{K}\}) \subset \bigcap \{\sim K : K \in \mathcal{K}\}$.

Suppose now that $x \in \bigcap \{\sim K : K \in \mathcal{K}\}$. This means that x is an element of $\{\sim K : K \in \mathcal{K}\}$ for all K i.e. $x \notin \{K : K \in \mathcal{K}\}$ for all K , (indeed if x would be an element of a $K \in \mathcal{K}$ then x would not be an element of its complement and so x could not be an element of $\bigcap \{\sim K : K \in \mathcal{K}\}$). The conclusion is that $x \notin \bigcup \{K : K \in \mathcal{K}\}$ and thus $x \in \sim \bigcup \{K : K \in \mathcal{K}\}$. Hence, $\bigcap \{\sim K : K \in \mathcal{K}\} \subset \sim (\bigcup \{K : K \in \mathcal{K}\})$.

Conclusion $\sim (\bigcup \{K : K \in \mathcal{K}\}) = \bigcap \{\sim K : K \in \mathcal{K}\}$.

◇

(b) $\sim (\bigcap \{K : K \in \mathcal{K}\}) = \bigcup \{\sim K : K \in \mathcal{K}\}$

Suppose $x \in \sim (\bigcap \{K : K \in \mathcal{K}\})$, then $x \notin \bigcap \{K : K \in \mathcal{K}\}$. This means that there exists at least one $K \in \mathcal{K}$ so that x is not an element of this K i.e. $\exists K \in \mathcal{K} : x \notin K$. This can also be expressed as $\exists K \in \mathcal{K} : x \in \sim K$. This means that x is an element of $\bigcup \{\sim K : K \in \mathcal{K}\}$ and thus $\sim (\bigcap \{K : K \in \mathcal{K}\}) \subset \bigcup \{\sim K : K \in \mathcal{K}\}$.

Suppose now that $x \in \bigcup \{\sim K : K \in \mathcal{K}\}$. This means that x is an element of at least one $\sim K : K \in \mathcal{K}$. Stated differently, there exist at least one $K : K \in \mathcal{K}$ for which $x \notin K$. This means that x can not be an element of $\bigcap \{K : K \in \mathcal{K}\}$ and thus $x \in \sim \bigcap \{K : K \in \mathcal{K}\}$ which means $\bigcup \{\sim K : K \in \mathcal{K}\} \subset \sim (\bigcap \{K : K \in \mathcal{K}\})$.

Conclusion $\sim (\bigcap \{K : K \in \mathcal{K}\}) = \bigcup \{\sim K : K \in \mathcal{K}\}$.

◆

1.8.6

Let $S = \{1, 2, 3, 4, 5\}$ and let $\mathcal{P}(S)$ be the power set of S . List the elements in $\mathcal{P}(S)$.

We order them according to the number of elements in the subsets. We check the number of subsets by using the $\binom{5}{m}$ formula (i.e. combination without repetition).

$$5 \text{ elements} \quad \binom{5}{5} = 1$$

$$\{1, 2, 3, 4, 5\}$$

$$4 \text{ elements} \quad \binom{5}{4} = 5$$

$$\{1, 2, 3, 4\}$$

$$\{1, 2, 3, 5\}$$

$$\{1, 2, 4, 5\}$$

$$\{1, 3, 4, 5\}$$

$$\{2, 3, 4, 5\}$$

$$3 \text{ elements} \quad \binom{5}{3} = 10$$

$$\{1, 2, 3\}$$

$$\{1, 2, 4\}$$

$$\{1, 2, 5\}$$

$$\{1, 3, 4\}$$

$$\{1, 3, 5\}$$

$$\{1, 4, 5\}$$

$$\{2, 3, 4\}$$

$$\{2, 3, 5\}$$

$$\{2, 4, 5\}$$

$$\{3, 4, 5\}$$

$$2 \text{ elements} \quad \binom{5}{2} = 10$$

$$\{1, 2\}$$

$$\{1, 3\}$$

$$\{1, 4\}$$

$$\{1, 5\}$$

$$\{2, 3\}$$

$$\{2, 4\}$$

$$\{2, 5\}$$

$$\{3, 4\}$$

$$\{3, 5\}$$

$$\{4, 5\}$$

$$1 \text{ element} \quad \binom{5}{1} = 5$$

$$\{1\}$$

$$\{2\}$$

$$\{3\}$$

$$\{4\}$$

$$\{5\}$$

$$0 \text{ elements} \quad \binom{5}{0} = 1$$

$$\emptyset$$

Note that the total number of subsets in $\mathcal{P}(S)$ is $1 + 5 + 10 + 10 + 5 + 1 = 32$ which corresponds to 2^5 .



1.9 Cartesian Product

1.9.1

Suppose that $A \subset B$ and C is a set. Prove that $A \times C \subset B \times C$.

Be $x \in A$ and $y \in C$. As $A \subset B$, then x is also in B . Thus $\underbrace{(x, y)}_{x \in A, y \in C} \in A \times C$ means also that $\underbrace{(x, y)}_{x \in B, y \in C} \in B \times C$

◆

1.9.2

Let $A = \{1, 2, 3\}$, $B = \{a, b\}$, and $C = \{\alpha, \beta\}$. List the elements of each of the following sets:

- (a) $A \times (B \cup C)$
- (b) $(A \times B) \cup (A \times C)$
- (c) $(A \cup B) \times C$
- (c) $(A \times C) \cup (B \times C)$

(a) $A \times (B \cup C)$

$(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)$
 $(1, \alpha), (1, \beta), (2, \alpha), (2, \beta), (3, \alpha), (3, \beta)$

◇

(b) $(A \times B) \cup (A \times C)$

$(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)$
 $(1, \alpha), (1, \beta), (2, \alpha), (2, \beta), (3, \alpha), (3, \beta)$

◇

(c) $(A \cup B) \times C$

$(1, \alpha), (1, \beta), (2, \alpha), (2, \beta), (3, \alpha), (3, \beta)$ $(1, \alpha), (1, \beta), (2, \alpha), (2, \beta), (3, \alpha), (3, \beta)$
 $(a, \alpha), (a, \beta), (b, \alpha), (b, \beta)$

◇

(c) $(A \times C) \cup (B \times C)$

$(1, \alpha), (1, \beta), (2, \alpha), (2, \beta), (3, \alpha), (3, \beta)$ $(1, \alpha), (1, \beta), (2, \alpha), (2, \beta), (3, \alpha), (3, \beta)$
 $(a, \alpha), (a, \beta), (b, \alpha), (b, \beta)$

◆

1.9.3

Are any of the sets in Exercise 2 the same? If so write the set identities that are suggested by your observations. Try to prove your conjecture.

In exercise 2 we can see that that the set (a) and (b) are the same. Also (c) and (d) are the same. This suggests the following identities $A \times (B \cup C) = (A \times B) \cup (A \times C)$ and $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

$$\mathbf{A} \times (\mathbf{B} \cup \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cup (\mathbf{A} \times \mathbf{C})$$

Proof:

Be $x \in A$ and $y \in B \cup C$, so y is an element of B or C . Consider $(x, y) \in A \times (B \cup C)$. As the y can be an element of B or C follows immediately that $(x, y) \in (A \times B)$ or $(x, y) \in (A \times C)$ and thus $(x, y) \in (A \times B) \cup (A \times C)$. And get $A \times (B \cup C) \subset (A \times B) \cup (A \times C)$

Suppose now that $(x, y) \in (A \times B) \cup (A \times C)$. The (x, y) is an element of $A \times B$ or $A \times C$. For the same $x \in A$ this implies that $y \in B$ or $y \in C$ and thus $(x, y) \in A \times (B \cup C)$, giving $(A \times B) \cup (A \times C) \subset A \times (B \cup C)$ leading with the previous $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

◇

$$(\mathbf{A} \cup \mathbf{B}) \times \mathbf{C} = (\mathbf{A} \times \mathbf{C}) \cup (\mathbf{B} \times \mathbf{C})$$

Proof:

Be $x \in A \cup B$ and $y \in C$, so x is an element of A or B . Consider $(x, y) \in (A \cup B) \times C$. As the x can be an element of A or B follows immediately that $(x, y) \in (A \times C)$ or $(x, y) \in (B \times C)$ and thus $(x, y) \in (A \times C) \cup (B \times C)$. And get $(A \cup B) \times C \subset (A \times C) \cup (B \times C)$

Suppose now that $(x, y) \in (A \times C) \cup (B \times C)$. The (x, y) is an element of $A \times C$ or $B \times C$. For the same $y \in C$ this implies that $x \in A$ or $x \in B$ and thus $(x, y) \in (A \cup B) \times C$, giving $(A \times C) \cup (B \times C) \subset (A \cup B) \times C$ leading with the previous $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

◆

1.9.4

Suppose that A is a set consisting of five elements and B is a set consisting of three elements. How many elements does the set $A \times B$ have? The set $B \times A$?

$A \times B$ has $5 \times 3 = 15$ elements. Indeed in the element $(x, y) \in A \times B$ we can choose for x out of the five elements of A and for each choice of x we are free to choose one element out of the 3 elements of B .

For $B \times A$, the reasoning is the same and get $3 \times 5 = 15$ elements.

◆

1.9.5

Suppose that A is a set consisting of m elements and B is a set consisting of n elements, where m and n are positive integers. How many elements are there in $A \times B$?

$A \times B$ has $m \times n$ elements. Indeed in the element $(x, y) \in A \times B$ we can choose for x out of the m elements of A and for each choice of x we are free to choose one element out of the n elements of B .

**1.9.6**

Suppose that A is a set consisting of three elements, B consists of four elements and C consists of two elements. How many elements are there in the set $(A \times B) \times C$?

$(A \times B) \times C$ has $(3 \times 4) \times 2 = 24$ elements. Indeed in the element $((x, y), z) \in (A \times B) \times C$ we have for (x, y) , $3 \times 4 = 12$ elements (see Exercise 1.9.5) and for each choice of this (x, y) we are free to choose one element out of the 2 elements of C .



1.10 Functions

1.10.1

