Undergraduate Topology

Robert H. Kasriel (Dover Publication)

Solutions to exercises

Part I

Chapters I to IV

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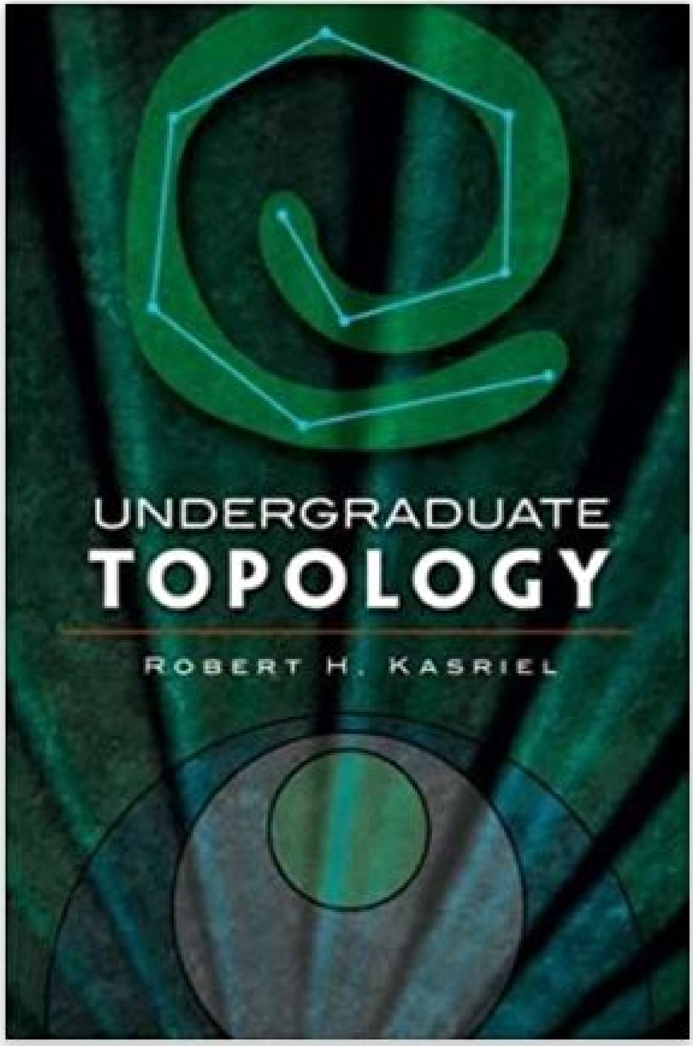

Figure 1

Remarks and warnings

You're welcome to use these notes, but they may contain errors, so proceed with caution : I graduated in 1979, went straight in the industry (where I didn't have to use fancy maths), and picked mathematics and physics again after I retired, so my mathematics got rusty for sure. If you do find an error, typo's , I'd be happy to receive bug reports, suggestions, and the like, through Github.

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Sets, Functions, and Relations

1.1 Sets and Membership

1.1.1

List explicitly the elements of the set

$$\{x:\ x<0 \text{ and } (x-1)(x+2)(x+3)=0\}$$

$$\{\,-3,\ -2\}$$

♦

1.1.2

List the elements of the set

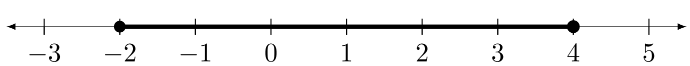
$$\{x:\ 3x-1 \text{ is a multiple of } 3\}$$

$$\{x:\ x=k+\frac{1}{3},\ k\in\mathbb{Z}\}$$

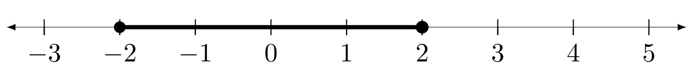
♦

1.1.3

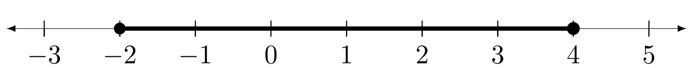
Sketch on a number line each of the following sets.
(a) $\{x:\ |x-1|\le 3\}$
(b) $\{x:\ |x-1|\le 3 \text{ and } |x|\le 2\}$
(c) $\{x:\ |x-1|\le 3 \text{ or } |x|\le 2\}$



(a)



(b)



(c)

♦

1.2 Some remarks on the use of the connectives *and, or, implies*

1.2.1

Demonstrate by means of a table showing truth values that the following is a true statement for any choice of p and q. Thus show that it is a tautology.

$$(\neg q \Rightarrow \neg p) \Rightarrow (p \Rightarrow q)$$

p	q	$\neg q$	$\neg p$	$\neg q \Rightarrow \neg p$	$p \Rightarrow q$	$(\neg q \Rightarrow \neg p) \Rightarrow (p \Rightarrow q)$
T	T	F	F	T	T	T
T	F	T	F	F	F	T
F	T	F	T	T	T	T
F	F	T	T	T	T	T

♦

1.2.2

Show by means of a truth table that the statement

$$((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$$

is a tautology.

p	q	r	$p \Rightarrow q$	$q \Rightarrow r$	$(p \Rightarrow q) \wedge (q \Rightarrow r))$	$p \Rightarrow r$	$((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

♦

1.2.3

Show by means of a truth table that

$$(p \wedge q) \Rightarrow (p \vee q)$$

is a tautology.

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \Rightarrow (p \vee q)$
T	T	T	T	T
T	F	F	F	T
F	T	F	T	T
F	F	F	F	T

♦

1.2.4

Suppose that p and q are statements such that $(p \wedge q)$ is a false statement. Does it follow that the statement

$$(p \text{ is false}) \vee (q \text{ is false})$$

is a true statement?

p	q	$p \wedge q$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	F	F	F	T	T
F	T	F	T	F	T
F	F	F	T	T	T

The answer is Yes.

♦

1.2.5

Negate the following statement: *If two angles of a triangle have equal measure, then the length of two sides of that triangle are equal.*

First we note that $\neg(p \Rightarrow q) \Leftrightarrow (p \wedge \neg q)$. Indeed,

p	q	$p \Rightarrow q$	$\neg(p \Rightarrow q)$	$\neg q$	$p \wedge \neg q$	$\neg(p \Rightarrow q) \Leftrightarrow (p \wedge \neg q)$
T	T	T	F	F	F	T
T	F	F	T	T	T	T
F	T	T	F	F	F	T
F	F	T	F	T	F	T

Putting p as *two angles of a triangle have equal measure* and $\neg q$ as *no two sides of that triangle have equal length* we get the true 'false' statement:
Two angles of a triangle have equal measure $\wedge$ no two sides of that triangle have equal length.

♦

1.2.6

> Write the contrapositive of the statement in Exercise 5.

The contrapositive of $p \Rightarrow q$ is $\neg q \Rightarrow \neg p$. Putting $\neg p$ as *no two angles of a triangle have equal measure* and $\neg q$ as *no two sides of that triangle have equal length* we get
If no two sides of that triangle have equal length then no two angles of a triangle have equal measure.

♦

1.2.7

> Write the converse of the statement in Exercise 5.

The converse of $p \Rightarrow q$ is $q \Rightarrow p$, giving
If two sides of a triangle have equal length then two angles of a that triangle have equal measure.

♦

1.2.8

> Write the contrapositive of the following statement
> *If a person belongs to Committee A, then he must be a member of Committee B and he must be a member of Committee C.*

Lets put

$$\begin{aligned} p &\equiv \text{a person belongs to Committee A} \\ q &\equiv \text{a person belongs to Committee B} \\ r &\equiv \text{a person belongs to Committee C} \end{aligned}$$

then the given statement translates as

$$p \Rightarrow (q \wedge r)$$

and the contrapositive

$$\neg(q \wedge r) \Rightarrow \neg p$$

This last statement is equivalent with

$$(\neg q \vee \neg r) \Rightarrow \neg p$$

or in plain text:

If a person does not belong to Committee B or C , then he is not a member of Committee A.

♦

1.2.9

Write the contrapositive of the following statement

If $x \in A$ and $x \in B$, then $x \in C$

Lets put

$$p \equiv x \in A$$
$$q \equiv x \in B$$
$$r \equiv x \in C$$

then the given statement translates as

$$p \wedge r \Rightarrow r$$

and the contrapositive

$$\neg(r) \Rightarrow \neg(p \wedge q)$$

This last statement is equivalent with

$$\neg(r) \Rightarrow (\neg p \vee \neg q)$$

i.e:

$$x \notin C \Rightarrow (x \notin A \vee x \notin B)$$

♦

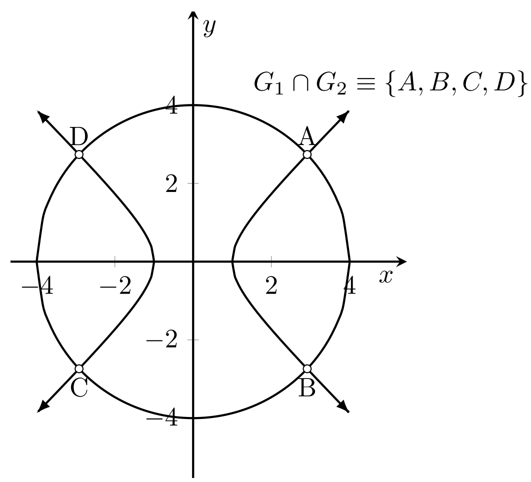
1.3 Subsets

No exercises!

1.4 Union and Intersection of sets

1.4.1

Let G_1 be the graph of the equation $x^2 + y^2 = 16$, and let G_2 be the graph of the equation $x^2 - y^2 = 1$. Sketch the sets $G_1 \cup G_2$ and $G_1 \cap G_2$.



$G_1 \cup G_2$ contains all the points defined by the graphs G_1 and G_2. $G_1 \cap G_2 \equiv \{A, B, C, D\}$ contains the 4 points at the intersection of the two graphs.

♦

1.4.2

We define the sets A, B, C as follows: $A = \{(x, y) : x^2 + y^2 \leq 9\}$, $B = \{(x, y) : x + y \geq 3\}$, $C = \{(x, y) : x \geq 0\}$.
Draw sketches of each of the following sets:

$$\begin{array}{ll}
(a) & A \cup (B \cup C) \\
(b) & A \cap (B \cup C) \\
(c) & (A \cap B) \cup (A \cap C) \\
(d) & (A \cup B) \cup C \\
(e) & A \cup (B \cap C) \\
(f) & (A \cup B) \cap (A \cup C)
\end{array}$$

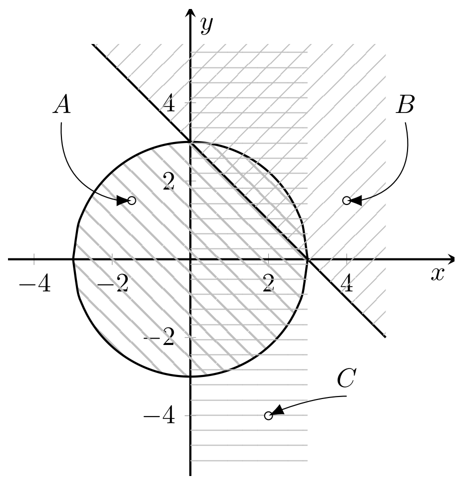
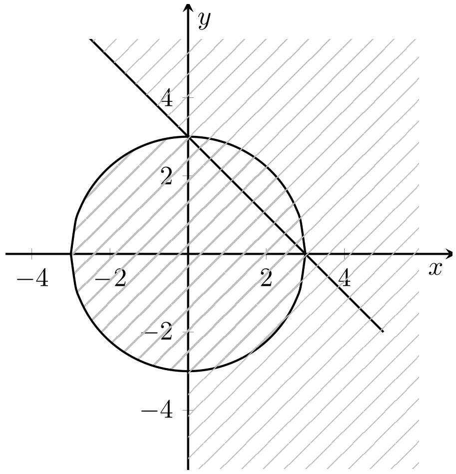
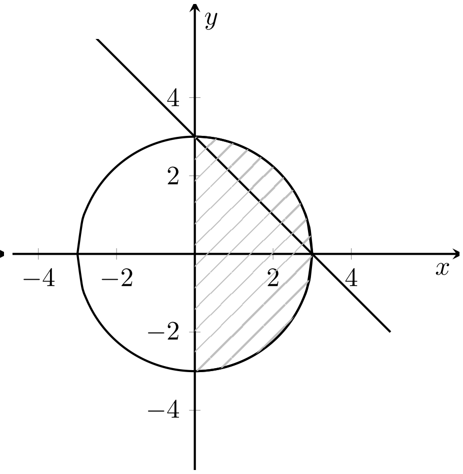


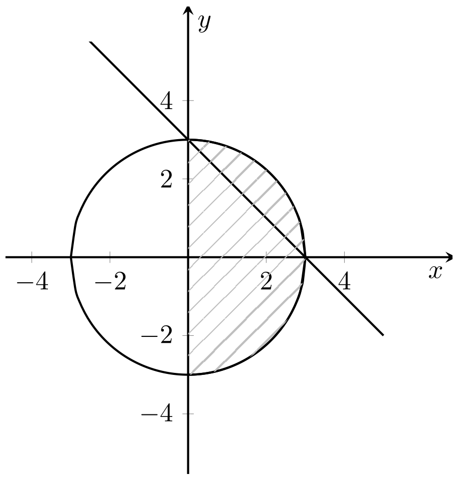
Figure 1.1: The 3 sets A, B, C



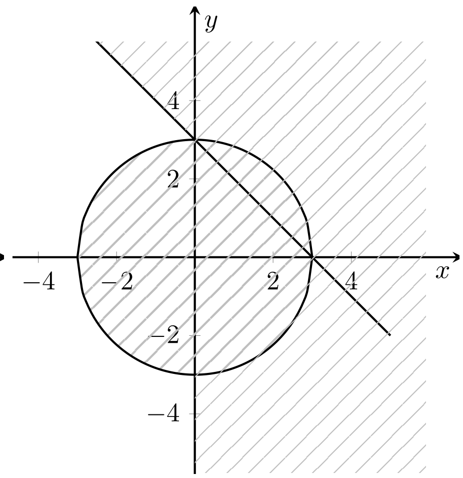
(a) $A \cup (B \cup C)$



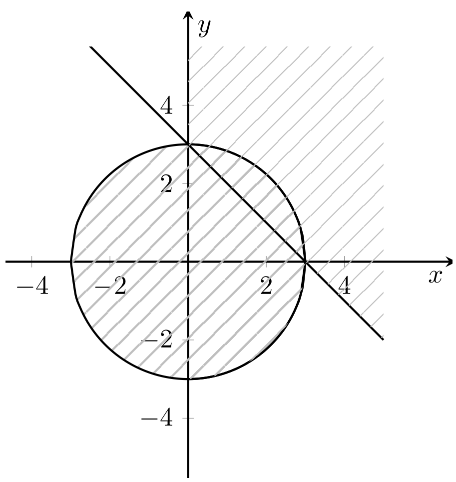
(b) $A \cap (B \cup C)$



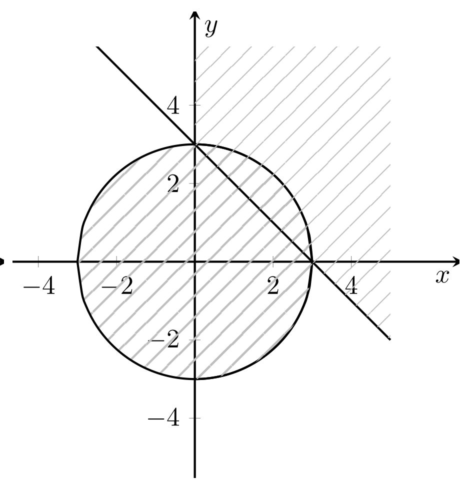
(c) $(A \cap B) \cup (A \cap C)$



(d) $(A \cup B) \cup C$



(e) $A \cup (B \cap C)$



(f) $(A \cup B) \cap (A \cup C)$

♦