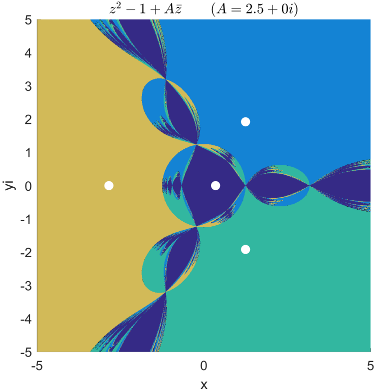
Newton's Method for 2 Dimensional Functions



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Advisor: Bruce Peckham

Newton's Method

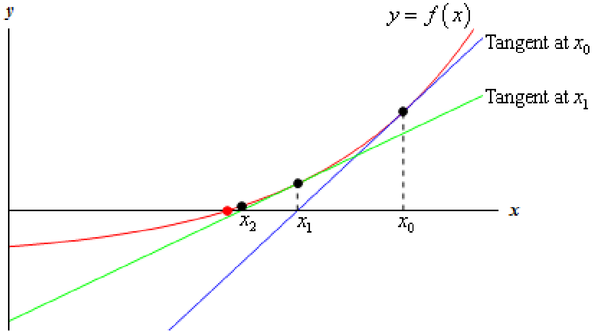

Figure: Example of Newton's Method

Newton's Method Formula

Newton's method for the function $H : \mathbb{R} \to \mathbb{R}$

$$N_H(x) = x - \frac{H(x)}{H'(x)}$$

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Orbits are defined as $x_0, x_1 = N_H(x_0), x_2 = N_H(x_1), \ldots x_{n+1} = N_H(x_n), \ldots$

The *nth* term in the sequence is the iterative function applied n times.

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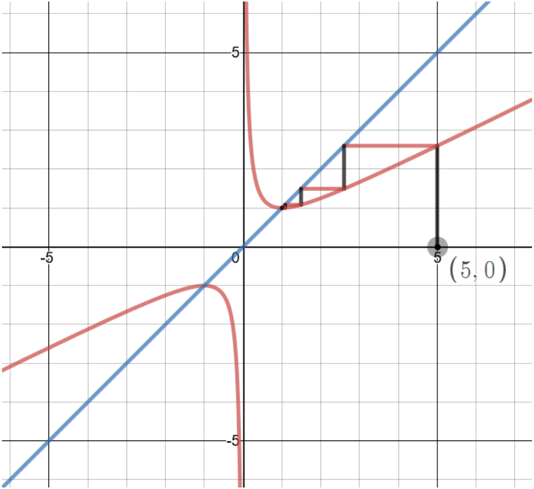
If $x_0 = \text{-}\frac{1}{3}$ we get the sequence of $\text{-}\frac{1}{3}, \text{-}\frac{5}{3}, \text{-}\frac{17}{15}, \text{-}\frac{257}{255}, \cdots \to \text{-}1$

Cobweb Diagrams



Graph of $N_H(x) = \frac{x^2+1}{2x}$ and reference line

Attracting Fixed Points

Iterating a **fixed point** returns the fixed point so x is fixed if

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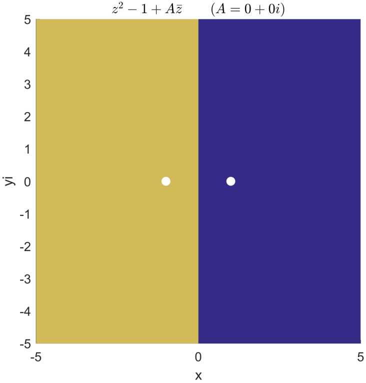
A fixed point p for N_H is **attracting** if $|N'_H(p)| < 1$.

$$N'_H(x) = \frac{H(x)H''(x)}{(H'(x))^2}$$

If $H(p) = 0$ and $H'(p) \neq 0$ then $N'_H(p) = 0$.

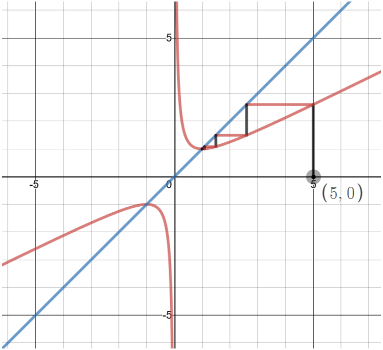
Newton's Fractal for $H(z) = z^2 - 1$

The different color represent the **basin of attraction** for each fixed point. The basin of attraction is the set of all points which converge to the given fixed point.

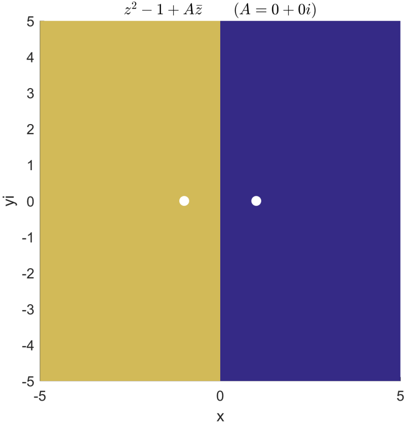


Newton's Fractal for $H(z) = z^2 - 1$

One-Dimensional vs. Two-Dimensional



Graph of $N_H(x) = \frac{x^2+1}{2x}$ and reference line



Newton's Fractal for $H(z) = z^2 - 1$

Complex Newton's Method

Complex version has a similar function

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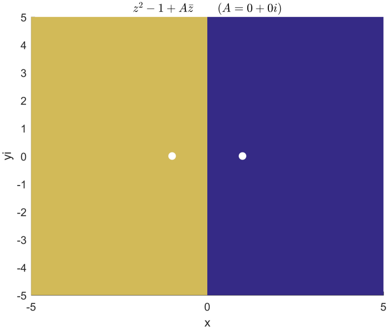

The line $y = 0$ is **invariant**, any point that starts on this line its orbit will remain on the line $y = 0$.

Complex Perturbation

The function we are exploring

$$F(z) = z^2 - 1 + A\bar{z}$$

where $\bar{z}$ is the complex conjugate and A is a complex **parameter**, $A = a + bi$.

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Complex Newton's Method for $F(z)$

$$N_F(z) = z - \frac{F(z)}{F'(z)}$$

The real two-dimensional version of $F(z)$ is

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 - y^2 - 1 + ax + by \\ 2xy - ay + bx \end{pmatrix}$$

2D Newton's Method

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Newton's Method for the two dimensional function F is

$$N_F \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} - JF^{-1} \begin{pmatrix} x \\ y \end{pmatrix} F \begin{pmatrix} x \\ y \end{pmatrix}$$

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JF^{-1} represents the inverse of the Jacobian Matrix of F. The Jacobian is given by

$$JF \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix}$$

$F(z) = z^2 - 1 + A\bar{z}$

The corresponding two-dimensional function for $F(z)$ is

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 - y^2 - 1 + ax + by \\ 2xy - ay + bx \end{pmatrix}$$

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Using Newton's Method for 2-dimensional functions

$$N\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} - JF^{-1}\begin{pmatrix} x \\ y \end{pmatrix} F\begin{pmatrix} x \\ y \end{pmatrix}$$

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This has the Jacobian Matrix of

$$JF\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a + 2x & b - 2y \\ b + 2y & -a + 2x \end{pmatrix}$$

Newton's Method for $F(z) = z^2 - 1 + A\bar{z}$

Newton's 2 Dimensional Iteration Function

$$N_F \binom{x}{y} = \begin{pmatrix} \frac{a(1+x^2-y^2)-2x(1-by+x^2+y^2)}{a^2+b^2-4(x^2+y^2)} \\ \frac{b(1+x^2-y^2)-2y(-1+ax+x^2+y^2)}{a^2+b^2-4(x^2+y^2)} \end{pmatrix}$$

where x and y represents the real and imaginary components of the original $F(z)$ respectively.

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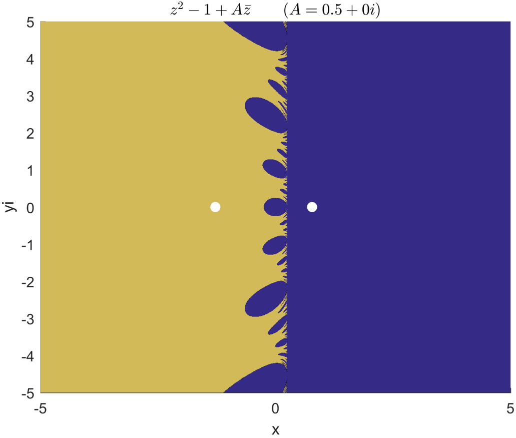
We restrict our complex parameter to be real, $b = 0$, resulting in a simpler system we can analyze.

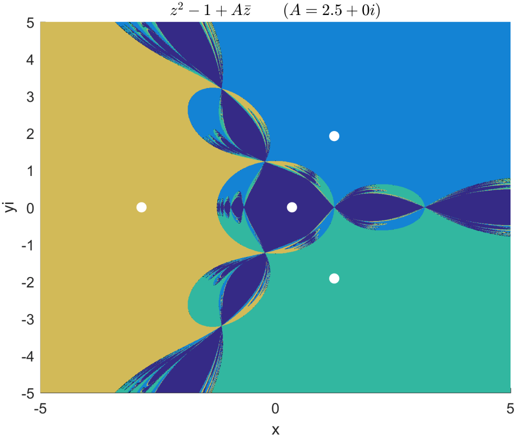
Newton's Fractal



Newton's Fractal for $F(z) = z^2 - 1 + 0.5\bar{z}$

Four Root Case

When $|a| > \frac{2\sqrt{3}}{3}$ there are four roots.



Animation Real Axis

Critical Circle

The critical points of $F(z)$

$$Det(JF\begin{pmatrix} x \\ y \end{pmatrix}) = a^2 + b^2 - 4(x^2 + y^2) = 0$$

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$$\frac{a^2 + b^2}{4} = x^2 + y^2$$

$$(\frac{|A|}{2})^2 = x^2 + y^2$$

This is the **critical circle** centered at the origin with radius half of the magnitude of the parameter A.

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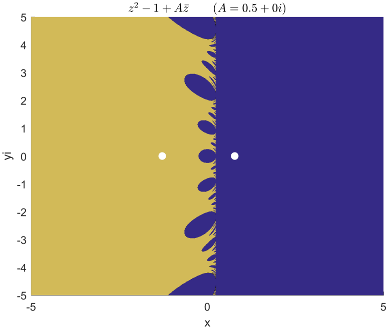
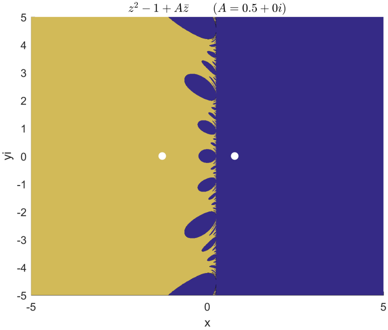
$$N_A\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\frac{a(1+x^2-y^2)-2x(1-by+x^2+y^2)}{a^2+b^2-4(x^2+y^2)}\\ \frac{b(1+x^2-y^2)-2y(-1+ax+x^2+y^2)}{a^2+b^2-4(x^2+y^2)}\end{pmatrix}$$

Invariant Line $x = \frac{a}{2}$



There is an invariant line $l = \{(x, y) | x = \frac{a}{2}\}$. Points on l follow the equation.

Invariant Line $x = \frac{a}{2}$



There is an invariant line $I = \{(x, y) | x = \frac{a}{2}\}$. Points on I follow the equation.

$$N_F \begin{pmatrix} a/2 \\ y \end{pmatrix} = \begin{pmatrix} a/2 \\ \frac{y^2 - 1 + \frac{3a^2}{4}}{2y} \end{pmatrix}$$

Invariant Line One Dimensional

For $y \in I$

$$N_I(y) = \frac{y^2 - 1 + \frac{3a^2}{4}}{2y}$$

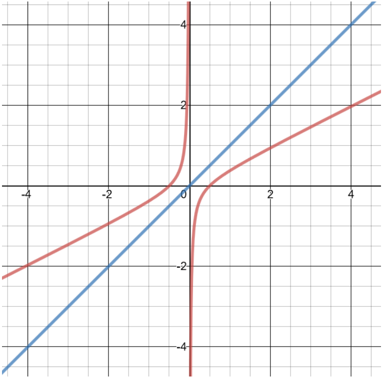
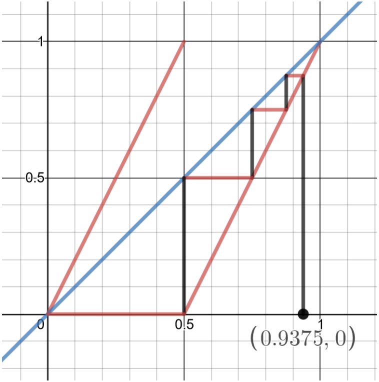
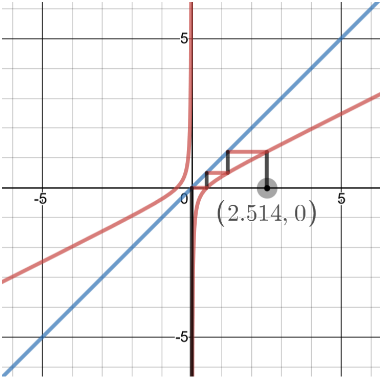


Figure: Graphical Iteration for $N_I(y)$ with $a = 1$

Conjugacy

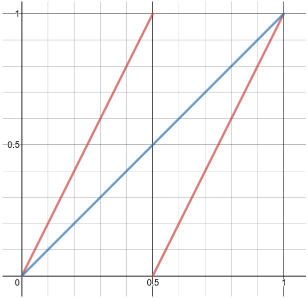


Doubling Map $D(x) = 2x \pmod 1$



Newton's Method on I

Doubling Map



Doubling Map $D(x) = 2x \pmod 1$ for $x \in [0, 1)$

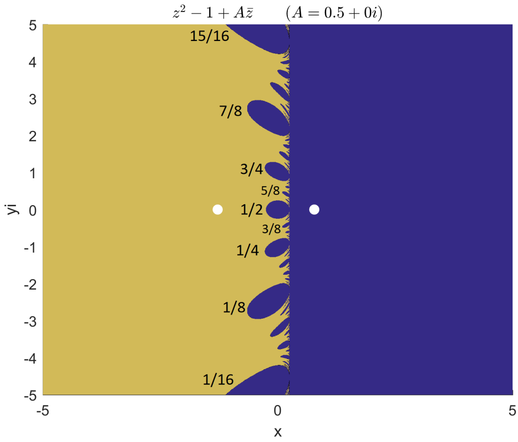
Eventually fixed points q are points that after a certain number of iterations become fixed.

$$q = \frac{k}{2^n} \qquad k = 0, 1, \ldots, 2^n - 1$$

$$D^n(q) = 2^n\left(\frac{k}{2^n}\right) = k \pmod 1 = 0$$

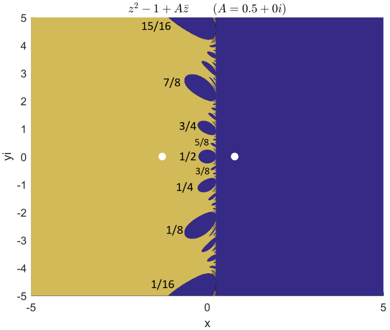
.

Doubling Map Conjecture



Newton's Method with Bulbs labeled

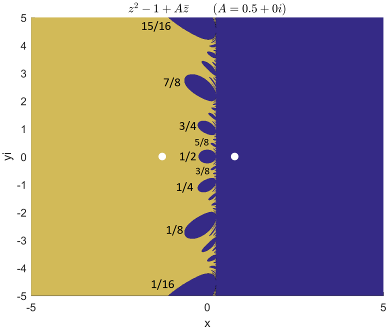
Doubling Map Conjecture



Newton's Method with Bulbs labeled

The critical circle maps to the region to the right of the invariant line l.

Doubling Map Conjecture



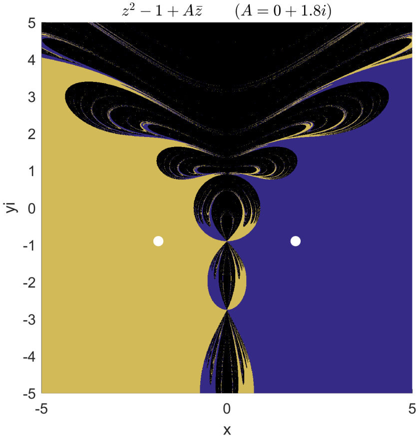
Newton's Method with Bulbs labeled

The critical circle maps to the region to the right of the invariant line I. Bulbs are preimages of the critical circle.

Pure imaginary animation

Future Work

Questions: Why are there large portions of non-convergence for certain values of b.



Future Work

Explain the dynamics for all values of A.

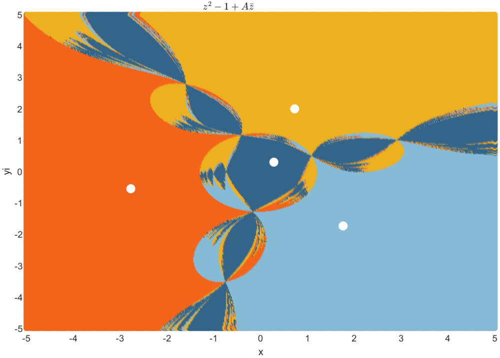


Figure: Newton's Fractal for $f(z) = z^2 - 1 + (2 + 1.5i)\bar{z}$

The End

Thank you!