Cooperative Mollow: equations for baby-Mollow with truncation

Nicola

(Dated: October 17, 2018)

MASTER EQUATION APPROACH

The dynamics of a system of atoms driven by an external laser beam and undergoing cooperative re-emission into vacuum modes can be described by a master equation approach [1] in the Born-Markov approximation:

$$\begin{aligned}\dot{\rho} &= -i\frac{\Omega_0}{2}\sum_{j=1}\left[e^{i\Delta_0 t - i\mathbf{k}_0\cdot\mathbf{r}_j}\sigma_j^- + e^{-i\Delta_0 t + i\mathbf{k}_0\cdot\mathbf{r}_j}\sigma_j^+, \rho\right] \\ &+ i\frac{\Gamma}{2}\sum_{j=1}\sum_{m\neq j}[\Omega_{jm}[\sigma_j^+\sigma_m^-, \rho] + \frac{\Gamma}{2}\sum_{j=1}\sum_{m\neq j}\Gamma_{jm}\left\{2\sigma_m^-\rho\sigma_j^+ - \sigma_j^+\sigma_m^-\rho - \rho\sigma_j^+\sigma_m^-\right\}\end{aligned} \tag{1}$$

where $[\sigma_j^+, \sigma_m^-] = \delta_{jm}\sigma_{zj}$, $[\sigma_j^\pm, \sigma_{zm}] = \mp 2\delta_{jm}\sigma_j^\pm$, $\Omega_{jm} = \cos(k_0 r_{jm})/(k_0 r_{jm})$, $\Gamma_{jm} = \sin(k_0 r_{jm})/(k_0 r_{jm})$, and $G_{jm} = \Gamma_{jm} - i\Omega_{jm}$, where $r_{jm} = |\mathbf{r}_j - \mathbf{r}_m|$.

We approximate the density operator for N atoms as

$$\rho = \bigotimes_i \rho^{(i)} + \sum_{j<k}\left(\rho^{(j,k)} \otimes \bigotimes_{i\neq j,k}\rho^{(i)}\right) \tag{2}$$

where the first term is a product state and describes the mean field approximation. The second term accounts for pair correlations, accounted by the two-particle density operators $\rho^{(j,k)}$, chosen to generate vanishing single-atom expectation values, i.e. $\mathrm{Tr}\left\{\sigma_i\rho^{(j,k)}\right\} = 0$. With this assumption, the third-order expectation values are [2]

$$\langle\sigma_i^\alpha\sigma_j^\beta\sigma_k^\gamma\rangle = -2\langle\sigma_i^\alpha\rangle\langle\sigma_j^\beta\rangle\langle\sigma_k^\gamma\rangle + \langle\sigma_i^\alpha\rangle\langle\sigma_j^\beta\sigma_k^\gamma\rangle + \langle\sigma_i^\alpha\sigma_j^\beta\rangle\langle\sigma_k^\gamma\rangle + \langle\sigma_j^\beta\rangle\langle\sigma_i^\alpha\sigma_k^\gamma\rangle \tag{3}$$

Defining the fluctuation operators

$$A_{ij}^{\alpha,\beta} = \langle\sigma_i^\alpha\sigma_j^\beta\rangle - \langle\sigma_i^\alpha\rangle\langle\sigma_j^\beta\rangle \tag{4}$$

we can write

$$\langle\sigma_i^\alpha\sigma_j^\beta\sigma_k^\gamma\rangle = \langle\sigma_i^\alpha\rangle\langle\sigma_j^\beta\rangle\langle\sigma_k^\gamma\rangle + \langle\sigma_i^\alpha\rangle A_{jk}^{\beta\gamma} + \langle\sigma_j^\beta\rangle A_{ik}^{\alpha\gamma} + \langle\sigma_k^\gamma\rangle A_{ij}^{\alpha\beta}. \tag{5}$$

Equations for the first-order expectation values

The equations for the first-order expectation values are:

$$\langle\dot{\sigma}_j\rangle = \left(i\Delta_0 - \frac{\Gamma}{2}\right)\langle\sigma_j\rangle + \frac{i\Omega_0}{2}e^{i\mathbf{k}_0\cdot\mathbf{r}_j}\langle\sigma_{zj}\rangle + \frac{\Gamma}{2}\sum_{m\neq j}^{N}G_{jm}\langle\sigma_{zj}\sigma_m\rangle \tag{6}$$

$$\langle\dot{\sigma}_{zj}\rangle = i\Omega_0\left\{e^{-i\mathbf{k}_0\cdot\mathbf{r}_j}\langle\sigma_j\rangle - \mathrm{h.c.}\right\} - \Gamma(1 + \langle\sigma_{zj}\rangle) - \Gamma\sum_{m\neq j}^{N}\{G_{jm}\langle\sigma_j^\dagger\sigma_m\rangle + \mathrm{h.c.}\} \tag{7}$$

where $\sigma_j = \sigma_j^- e^{i\Delta_0 t}$

Equations for the second-order expectation values

Using the master equation (1) we obtain, for $j \neq m$,

$$\begin{aligned}
\frac{d}{dt}\langle\sigma_{zj}\sigma_m\rangle &= (i\Delta_0 - 3\Gamma/2)\langle\sigma_{zj}\sigma_m\rangle - \Gamma\langle\sigma_m\rangle + i\Omega_0\left\{e^{-i\mathbf{k}_0\cdot\mathbf{r}_j}\langle\sigma_j\sigma_m\rangle - e^{i\mathbf{k}_0\cdot\mathbf{r}_j}\langle\sigma_j^\dagger\sigma_m\rangle + \frac{1}{2}e^{i\mathbf{k}_0\cdot\mathbf{r}_m}\langle\sigma_{zj}\sigma_{zm}\rangle\right\} \\
&- \Gamma\sum_{k\neq j,m}\left\{G_{jk}\langle\sigma_j^\dagger\sigma_m\sigma_k\rangle + G_{jk}^*\langle\sigma_k^\dagger\sigma_m\sigma_j\rangle\right\} + \frac{\Gamma}{2}\sum_{k\neq j,m}G_{mk}\langle\sigma_{zm}\sigma_{zj}\sigma_k\rangle \\
&- \Gamma\Gamma_{jm}\langle\sigma_j\sigma_{zm}\rangle - \frac{\Gamma}{2}G_{jm}^*\langle\sigma_j\rangle \qquad (8)
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}\langle\sigma_j^\dagger\sigma_m\rangle &= -\Gamma\langle\sigma_j^\dagger\sigma_m\rangle - i\frac{\Omega_0}{2}\left(e^{-i\mathbf{k}_0\cdot\mathbf{r}_j}\langle\sigma_{zj}\sigma_m\rangle - e^{i\mathbf{k}_0\cdot\mathbf{r}_m}\langle\sigma_j^\dagger\sigma_{zm}\rangle\right) \\
&+ \frac{\Gamma}{2}\sum_{k\neq j,m}\left[G_{jk}^*\langle\sigma_k^\dagger\sigma_m\sigma_{zj}\rangle + G_{mk}\langle\sigma_j^\dagger\sigma_k\sigma_{zm}\rangle\right] \\
&+ \frac{\Gamma}{4}(G_{jm}\langle\sigma_{zm}\rangle + G_{jm}^*\langle\sigma_{zj}\rangle) + \frac{\Gamma}{2}\Gamma_{jm}\langle\sigma_{zj}\sigma_{zm}\rangle \qquad (9)
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}\langle\sigma_j\sigma_m\rangle &= (2i\Delta_0 - \Gamma)\langle\sigma_j\sigma_m\rangle + i\frac{\Omega_0}{2}\left(e^{i\mathbf{k}_0\cdot\mathbf{r}_j}\langle\sigma_{zj}\sigma_m\rangle + e^{i\mathbf{k}_0\cdot\mathbf{r}_m}\langle\sigma_{zm}\sigma_j\rangle\right) \\
&+ \frac{\Gamma}{2}\sum_{k\neq j,m}\left[G_{jk}\langle\sigma_{zj}\sigma_m\sigma_k\rangle + G_{mk}\langle\sigma_{zm}\sigma_j\sigma_k\rangle\right] \qquad (10)
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}\langle\sigma_{zj}\sigma_{zm}\rangle &= -\Gamma(\langle\sigma_{zj}\rangle + \langle\sigma_{zm}\rangle + 2\langle\sigma_{zj}\sigma_{zm}\rangle) + i\Omega_0\left\{e^{-i\mathbf{k}_0\cdot\mathbf{r}_j}\langle\sigma_{zm}\sigma_j\rangle + e^{-i\mathbf{k}_0\cdot\mathbf{r}_m}\langle\sigma_{zj}\sigma_m\rangle - c.c.\right\} \\
&- \Gamma\sum_{k\neq j,m}\left\{G_{jk}\langle\sigma_j^\dagger\sigma_{zm}\sigma_k\rangle + G_{mk}\langle\sigma_{zj}\sigma_m^\dagger\sigma_k\rangle + c.c.\right\} \\
&+ 2\Gamma\Gamma_{jm}\left(\langle\sigma_j^\dagger\sigma_m\rangle + c.c.\right). \qquad (11)
\end{aligned}$$

QUANTUM REGRESSION THEOREM

Using the quantum regression theorem, we can write:

$$\begin{aligned}
\frac{d}{d\tau}\langle\sigma_m^\dagger(t)\sigma_j(t+\tau)\rangle &= \left(i\Delta_0 - \frac{\Gamma}{2}\right)\langle\sigma_m^\dagger(t)\sigma_j(t+\tau)\rangle + \frac{i\Omega_0}{2}e^{i\mathbf{k}_0\cdot\mathbf{r}_j}\langle\sigma_m^\dagger(t)\sigma_{zj}(t+\tau)\rangle \\
&+ \frac{\Gamma}{2}\sum_{k\neq j}^{N}G_{jk}\langle\sigma_m^\dagger(t)\sigma_{zj}(t+\tau)\sigma_k(t+\tau)\rangle \qquad (12)
\end{aligned}$$

$$\begin{aligned}
\frac{d}{d\tau}\langle\sigma_m^\dagger(t)\sigma_j^\dagger(t+\tau)\rangle &= \left(-i\Delta_0 - \frac{\Gamma}{2}\right)\langle\sigma_m^\dagger(t)\sigma_j^\dagger(t+\tau)\rangle - \frac{i\Omega_0}{2}e^{-i\mathbf{k}_0\cdot\mathbf{r}_j}\langle\sigma_m^\dagger(t)\sigma_{zj}(t+\tau)\rangle \\
&+ \frac{\Gamma}{2}\sum_{k\neq j}^{N}G_{jk}^*\langle\sigma_m^\dagger(t)\sigma_{zj}(t+\tau)\sigma_k^\dagger(t+\tau)\rangle \qquad (13)
\end{aligned}$$

$$\begin{aligned}
\frac{d}{d\tau}\langle\sigma_m^\dagger(t)\sigma_{zj}(t+\tau)\rangle &= i\Omega_0\left\{e^{-i\mathbf{k}_0\cdot\mathbf{r}_j}\langle\sigma_m^\dagger(t)\sigma_j(t+\tau)\rangle - e^{i\mathbf{k}_0\cdot\mathbf{r}_j}\langle\sigma_m^\dagger(t)\sigma_j^\dagger(t+\tau)\rangle\right\} \\
&- \Gamma(\langle\sigma_m^\dagger(t)\rangle + \langle\sigma_m^\dagger(t)\sigma_{zj}(t+\tau)\rangle) \\
&- \Gamma\sum_{k\neq j}^{N}\{G_{jk}\langle\sigma_m^\dagger(t)\sigma_j^\dagger(t+\tau)\sigma_k(t+\tau)\rangle + G_{jk}^*\langle\sigma_m^\dagger(t)\sigma_j(t+\tau)\sigma_k^\dagger(t+\tau)\rangle\} \qquad (14)
\end{aligned}$$

Equations of the 3-point correlators in τ

The equations for the expectation values of three operators can be obtained from the second-order expectation value equations:

$$\begin{aligned}
\frac{d}{dt}\langle\sigma_m^\dagger(t)\sigma_{zj}(t+\tau)\sigma_k(t+\tau)\rangle &= (i\Delta_0-3\Gamma/2)\langle\sigma_m^\dagger(t)\sigma_{zj}(t+\tau)\sigma_k(t+\tau)\rangle-\Gamma\langle\sigma_m^\dagger(t)\sigma_k(t+\tau)\rangle\\
&+i\Omega_0\left\{e^{-i\mathbf{k}_0\cdot\mathbf{r}_j}\langle\sigma_m^\dagger(t)\sigma_j(t+\tau)\sigma_k(t+\tau)\rangle-e^{i\mathbf{k}_0\cdot\mathbf{r}_j}\langle\sigma_m^\dagger(t)\sigma_j^\dagger(t+\tau)\sigma_k(t+\tau)\rangle\right.\\
&\left.+\frac{1}{2}e^{i\mathbf{k}_0\cdot\mathbf{r}_k}\langle\sigma_m^\dagger(t)\sigma_{zj}(t+\tau)\sigma_{zk}(t+\tau)\rangle\right\}\\
&-\Gamma\sum_{l\neq j,k}\left\{G_{jl}\langle\sigma_m^\dagger(t)\sigma_j^\dagger(t+\tau)\sigma_k(t+\tau)\sigma_l(t+\tau)\rangle+G_{jl}^*\langle\sigma_m^\dagger(t)\sigma_l^\dagger(t+\tau)\sigma_k(t+\tau)\sigma_j(t+\tau)\rangle\right\}\\
&+\frac{\Gamma}{2}\sum_{l\neq j,k}G_{lk}\langle\sigma_m^\dagger(t)\sigma_{zk}(t+\tau)\sigma_{zj}(t+\tau)\sigma_l(t+\tau)\rangle\\
&-2\Gamma\Gamma_{jk}\langle\sigma_m^\dagger(t)\sigma_j(t+\tau)\sigma_{zk}(t+\tau)\rangle-\Gamma G_{jk}^*\langle\sigma_m^\dagger(t)\sigma_j(t+\tau)\rangle
\end{aligned}\tag{15}$$

$$\begin{aligned}
\frac{d}{dt}\langle\sigma_m^\dagger(t)\sigma_{zj}(t+\tau)\sigma_k^\dagger(t+\tau)\rangle &= (-i\Delta_0-3\Gamma/2)\langle\sigma_m^\dagger(t)\sigma_{zj}(t+\tau)\sigma_k^\dagger(t+\tau)\rangle-\Gamma\langle\sigma_m^\dagger(t)\sigma_k^\dagger(t+\tau)\rangle\\
&-i\Omega_0\left\{e^{i\mathbf{k}_0\cdot\mathbf{r}_j}\langle\sigma_m^\dagger(t)\sigma_j^\dagger(t+\tau)\sigma_k^\dagger(t+\tau)\rangle-e^{-i\mathbf{k}_0\cdot\mathbf{r}_j}\langle\sigma_m^\dagger(t)\sigma_j(t+\tau)\sigma_k^\dagger(t+\tau)\rangle\right.\\
&\left.+\frac{1}{2}e^{-i\mathbf{k}_0\cdot\mathbf{r}_k}\langle\sigma_m^\dagger(t)\sigma_{zj}(t+\tau)\sigma_{zk}(t+\tau)\rangle\right\}\\
&-\Gamma\sum_{l\neq j,k}\left\{G_{jl}^*\langle\sigma_m^\dagger(t)\sigma_j(t+\tau)\sigma_k^\dagger(t+\tau)\sigma_l^\dagger(t+\tau)\rangle+G_{jl}\langle\sigma_m^\dagger(t)\sigma_l(t+\tau)\sigma_k^\dagger(t+\tau)\sigma_j^\dagger(t+\tau)\rangle\right\}\\
&+\frac{\Gamma}{2}\sum_{l\neq j,k}G_{lk}^*\langle\sigma_m^\dagger(t)\sigma_{zk}(t+\tau)\sigma_{zj}(t+\tau)\sigma_l^\dagger(t+\tau)\rangle\\
&-2\Gamma\Gamma_{jk}\langle\sigma_m^\dagger(t)\sigma_j^\dagger(t+\tau)\sigma_{zk}(t+\tau)\rangle-\Gamma G_{jk}\langle\sigma_m^\dagger(t)\sigma_j^\dagger(t+\tau)\rangle
\end{aligned}\tag{16}$$

$$\begin{aligned}
\frac{d}{dt}\langle\sigma_m^\dagger(t)\sigma_j^\dagger(t+\tau)\sigma_k(t+\tau)\rangle &= -\Gamma\langle\sigma_m^\dagger(t)\sigma_j^\dagger(t+\tau)\sigma_k(t+\tau)\rangle\\
&-i\frac{\Omega_0}{2}\left(e^{-i\mathbf{k}_0\cdot\mathbf{r}_j}\langle\sigma_m^\dagger(t)\sigma_{zj}(t+\tau)\sigma_k(t+\tau)\rangle-e^{i\mathbf{k}_0\cdot\mathbf{r}_k}\langle\sigma_m^\dagger(t)\sigma_j^\dagger(t+\tau)\sigma_{zk}(t+\tau)\rangle\right)\\
&+\frac{\Gamma}{2}\sum_{l\neq j,k}\left[G_{jl}^*\langle\sigma_m^\dagger(t)\sigma_l^\dagger(t+\tau)\sigma_k(t+\tau)\sigma_{zj}(t+\tau)\rangle+G_{lk}\langle\sigma_m^\dagger(t)\sigma_j^\dagger(t+\tau)\sigma_l(t+\tau)\sigma_{zk}(t+\tau)\rangle\right]\\
&+\frac{\Gamma}{4}(G_{jk}\langle\sigma_m^\dagger(t)\sigma_{zk}(t+\tau)\rangle+G_{jk}^*\langle\sigma_m^\dagger(t)\sigma_{zj}(t+\tau)\rangle)+\frac{\Gamma}{2}\Gamma_{jk}\langle\sigma_m^\dagger(t)\sigma_{zj}(t+\tau)\sigma_{zk}(t+\tau)\rangle
\end{aligned}\tag{17}$$

$$\begin{aligned}
\frac{d}{dt}\langle\sigma_m^\dagger(t)\sigma_j(t+\tau)\sigma_k(t+\tau)\rangle &= (2i\Delta_0-\Gamma)\langle\sigma_m^\dagger(t)\sigma_j(t+\tau)\sigma_k(t+\tau)\rangle\\
&+i\frac{\Omega_0}{2}\left(e^{i\mathbf{k}_0\cdot\mathbf{r}_j}\langle\sigma_m^\dagger(t)\sigma_{zj}(t+\tau)\sigma_k(t+\tau)\rangle+e^{i\mathbf{k}_0\cdot\mathbf{r}_k}\langle\sigma_m^\dagger(t)\sigma_{zk}(t+\tau)\sigma_j(t+\tau)\rangle\right)\\
&+\frac{\Gamma}{2}\sum_{l\neq j,k}\left[G_{jl}\langle\sigma_m^\dagger(t)\sigma_{zj}(t+\tau)\sigma_k(t+\tau)\sigma_l(t+\tau)\rangle+G_{kl}\langle\sigma_m^\dagger(t)\sigma_{zk}(t+\tau)\sigma_j(t+\tau)\sigma_l(t+\tau)\rangle\right]
\end{aligned}\tag{18}$$

$$\frac{d}{dt}\langle\sigma_m^\dagger(t)\sigma_j^\dagger(t+\tau)\sigma_k^\dagger(t+\tau)\rangle = (-2i\Delta_0-\Gamma)\langle\sigma_m^\dagger(t)\sigma_j^\dagger(t+\tau)\sigma_k^\dagger(t+\tau)\rangle$$
$$- i\frac{\Omega_0}{2}\left(e^{-i\mathbf{k}_0\cdot\mathbf{r}_j}\langle\sigma_m^\dagger(t)\sigma_{zj}(t+\tau)\sigma_k^\dagger(t+\tau)\rangle + e^{-i\mathbf{k}_0\cdot\mathbf{r}_k}\langle\sigma_m^\dagger(t)\sigma_{zk}(t+\tau)\sigma_j^\dagger(t+\tau)\rangle\right)$$
$$+ \frac{\Gamma}{2}\sum_{l\neq j,k}\left[G_{jl}^*\langle\sigma_m^\dagger(t)\sigma_{zj}(t+\tau)\sigma_k^\dagger(t+\tau)\sigma_l^\dagger(t+\tau)\rangle + G_{kl}^*\langle\sigma_m^\dagger(t)\sigma_{zk}(t+\tau)\sigma_j^\dagger(t+\tau)\sigma_l^\dagger(t+\tau)\rangle\right] \tag{19}$$

$$\frac{d}{dt}\langle\sigma_m^\dagger(t)\sigma_{zj}(t+\tau)\sigma_{zk}(t+\tau)\rangle = -\Gamma(\langle\sigma_m^\dagger(t)\sigma_{zj}(t+\tau)\rangle + \langle\sigma_m^\dagger(t)\sigma_{zk}(t+\tau)\rangle + 2\langle\sigma_m^\dagger(t)\sigma_{zj}(t+\tau)\sigma_{zk}(t+\tau)\rangle)$$
$$+ i\Omega_0\left\{e^{-i\mathbf{k}_0\cdot\mathbf{r}_j}\langle\sigma_m^\dagger(t)\sigma_{zk}(t+\tau)\sigma_j(t+\tau)\rangle + e^{-i\mathbf{k}_0\cdot\mathbf{r}_k}\langle\sigma_m^\dagger(t)\sigma_{zj}(t+\tau)\sigma_k(t+\tau)\rangle\right.$$
$$\left.- e^{i\mathbf{k}_0\cdot\mathbf{r}_j}\langle\sigma_m^\dagger(t)\sigma_{zk}(t+\tau)\sigma_j^\dagger(t+\tau)\rangle - e^{i\mathbf{k}_0\cdot\mathbf{r}_k}\langle\sigma_m^\dagger(t)\sigma_{zj}(t+\tau)\sigma_k^\dagger(t+\tau)\rangle\right\}$$
$$- \Gamma\sum_{l\neq j,k}\left\{G_{jl}\langle\sigma_m^\dagger(t)\sigma_j^\dagger(t+\tau)\sigma_{zk}(t+\tau)\sigma_l(t+\tau)\rangle\right.$$
$$+ G_{lk}\langle\sigma_m^\dagger(t)\sigma_{zj}(t+\tau)\sigma_k^\dagger(t+\tau)\sigma_l(t+\tau)\rangle$$
$$+ G_{jl}^*\langle\sigma_m^\dagger(t)\sigma_j(t+\tau)\sigma_{zk}(t+\tau)\sigma_l^\dagger(t+\tau)\rangle$$
$$\left.+ G_{lk}^*\langle\sigma_m^\dagger(t)\sigma_{zj}(t+\tau)\sigma_k(t+\tau)\sigma_l^\dagger(t+\tau)\rangle\right\}$$
$$+ 2\Gamma\Gamma_{jk}\left(\langle\sigma_m^\dagger(t)\sigma_j^\dagger(t+\tau)\sigma_k(t+\tau)\rangle + \langle\sigma_m^\dagger(t)\sigma_j(t+\tau)\sigma_k^\dagger(t+\tau)\rangle\right). \tag{20}$$

NOTATIONS

Let define

$$A_{mj}(\tau) = \langle\sigma_m^\dagger(t)\sigma_j(t+\tau)\rangle \tag{21}$$
$$B_{mj}(\tau) = \langle\sigma_m^\dagger(t)\sigma_j^\dagger(t+\tau)\rangle \tag{22}$$
$$C_{mj}(\tau) = \langle\sigma_m^\dagger(t)\sigma_{zj}(t+\tau)\rangle \tag{23}$$
$$H_{mjk}(\tau) = \langle\sigma_m^\dagger(t)\sigma_{zj}(t+\tau)\sigma_k(t+\tau)\rangle \tag{24}$$
$$I_{mjk}(\tau) = \langle\sigma_m^\dagger(t)\sigma_{zj}(t+\tau)\sigma_k^\dagger(t+\tau)\rangle \tag{25}$$
$$J_{mjk}(\tau) = \langle\sigma_m^\dagger(t)\sigma_j(t+\tau)\sigma_k(t+\tau)\rangle \tag{26}$$
$$K_{mjk}(\tau) = \langle\sigma_m^\dagger(t)\sigma_j^\dagger(t+\tau)\sigma_k(t+\tau)\rangle \tag{27}$$
$$L_{mjk}(\tau) = \langle\sigma_m^\dagger(t)\sigma_j^\dagger(t+\tau)\sigma_k^\dagger(t+\tau)\rangle \tag{28}$$
$$M_{mjk}(\tau) = \langle\sigma_m^\dagger(t)\sigma_{zj}(t+\tau)\sigma_{zk}(t+\tau)\rangle \tag{29}$$

Eqs.(12)-14)) become:

$$\frac{d}{d\tau}A_{mj}(\tau) = \left(i\Delta_0 - \frac{\Gamma}{2}\right)A_{mj}(\tau) + \frac{i\Omega_0}{2}e^{i\mathbf{k}_0\cdot\mathbf{r}_j}C_{mj}(\tau) + \frac{\Gamma}{2}\sum_{k\neq j}^{N}G_{jk}H_{mjk}(\tau) \tag{30}$$

$$\frac{d}{d\tau}B_{mj}(\tau) = \left(-i\Delta_0 - \frac{\Gamma}{2}\right)B_{mj}(\tau) - \frac{i\Omega_0}{2}e^{-i\mathbf{k}_0\cdot\mathbf{r}_j}C_{mj}(\tau) + \frac{\Gamma}{2}\sum_{k\neq j}^{N}G_{jk}^*I_{mjk}(\tau) \tag{31}$$

$$\frac{d}{d\tau}C_{mj}(\tau) = i\Omega_0\left\{e^{-i\mathbf{k}_0\cdot\mathbf{r}_j}A_{mj}(\tau) - e^{i\mathbf{k}_0\cdot\mathbf{r}_j}B_{mj}(\tau)\right\} - \Gamma(\langle\sigma_m^\dagger\rangle_s + C_{mj}(\tau))$$
$$- \Gamma\sum_{k\neq j}^{N}\{G_{jk}K_{mjk}(\tau) + G_{jk}^*K_{mkj}(\tau)\}. \tag{32}$$

Eqs.(15)-(20) become

$$
\begin{aligned}
\frac{d}{dt}H_{mjk}(\tau) &= (i\Delta_0 - 3\Gamma/2)H_{mjk}(\tau) - \Gamma A_{mk}(\tau) \\
&\quad + i\Omega_0\left\{e^{-i\mathbf{k}_0\cdot\mathbf{r}_j}J_{mjk}(\tau) - e^{i\mathbf{k}_0\cdot\mathbf{r}_j}K_{mjk}(\tau) + \frac{1}{2}e^{i\mathbf{k}_0\cdot\mathbf{r}_k}M_{mjk}(\tau)\right\} - 2\Gamma\Gamma_{jk}H_{mkj}(\tau) - \Gamma G^*_{jk}A_{mj}(\tau) \\
&\quad - \Gamma\sum_{l\neq j,k}\left\{G_{jl}\langle\sigma^\dagger_m(t)\sigma^\dagger_j(t+\tau)\sigma_k(t+\tau)\sigma_l(t+\tau)\rangle + G^*_{jl}\langle\sigma^\dagger_m(t)\sigma^\dagger_l(t+\tau)\sigma_k(t+\tau)\sigma_j(t+\tau)\rangle\right\} \\
&\quad + \frac{\Gamma}{2}\sum_{l\neq j,k}G_{lk}\langle\sigma^\dagger_m(t)\sigma_{zk}(t+\tau)\sigma_{zj}(t+\tau)\sigma_l(t+\tau)\rangle
\end{aligned} \tag{33}
$$

$$
\begin{aligned}
\frac{d}{dt}I_{mjk}(\tau) &= (-i\Delta_0 - 3\Gamma/2)I_{mjk}(\tau) - \Gamma B_{mk}(\tau) \\
&\quad - i\Omega_0\left\{e^{i\mathbf{k}_0\cdot\mathbf{r}_j}L_{mjk}(\tau) - e^{-i\mathbf{k}_0\cdot\mathbf{r}_j}K_{mkj}(\tau) + \frac{1}{2}e^{-i\mathbf{k}_0\cdot\mathbf{r}_k}M_{mjk}(\tau)\right\} - 2\Gamma\Gamma_{jk}I_{mkj}(\tau) - \Gamma G_{jk}B_{mj}(\tau) \\
&\quad - \Gamma\sum_{l\neq j,k}\left\{G^*_{jl}\langle\sigma^\dagger_m(t)\sigma_j(t+\tau)\sigma^\dagger_k(t+\tau)\sigma^\dagger_l(t+\tau)\rangle + G_{jl}\langle\sigma^\dagger_m(t)\sigma_l(t+\tau)\sigma^\dagger_k(t+\tau)\sigma^\dagger_j(t+\tau)\rangle\right\} \\
&\quad + \frac{\Gamma}{2}\sum_{l\neq j,k}G^*_{lk}\langle\sigma^\dagger_m(t)\sigma_{zk}(t+\tau)\sigma_{zj}(t+\tau)\sigma^\dagger_l(t+\tau)\rangle
\end{aligned} \tag{34}
$$

$$
\begin{aligned}
\frac{d}{dt}K_{mjk}(\tau) &= -\Gamma K_{mjk}(\tau) - i\frac{\Omega_0}{2}\left(e^{-i\mathbf{k}_0\cdot\mathbf{r}_j}H_{mjk}(\tau) - e^{i\mathbf{k}_0\cdot\mathbf{r}_k}I_{mkj}(\tau)\right) \\
&\quad + \frac{\Gamma}{4}(G_{jk}C_{mk}(\tau) + G^*_{jk}C_{mj}(\tau)) + \frac{\Gamma}{2}\Gamma_{jk}M_{mjk}(\tau) \\
&\quad + \frac{\Gamma}{2}\sum_{l\neq j,k}\left[G^*_{jl}\langle\sigma^\dagger_m(t)\sigma^\dagger_l(t+\tau)\sigma_k(t+\tau)\sigma_{zj}(t+\tau)\rangle + G_{lk}\langle\sigma^\dagger_m(t)\sigma^\dagger_j(t+\tau)\sigma_l(t+\tau)\sigma_{zk}(t+\tau)\rangle\right]
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
\frac{d}{dt}J_{mjk}(\tau) &= (2i\Delta_0 - \Gamma)J_{mjk}(\tau) + i\frac{\Omega_0}{2}\left(e^{i\mathbf{k}_0\cdot\mathbf{r}_j}H_{mjk}(\tau) + e^{i\mathbf{k}_0\cdot\mathbf{r}_k}H_{mkj}(\tau)\right) \\
&\quad + \frac{\Gamma}{2}\sum_{l\neq j,k}\left[G_{jl}\langle\sigma^\dagger_m(t)\sigma_{zj}(t+\tau)\sigma_k(t+\tau)\sigma_l(t+\tau)\rangle + G_{kl}\langle\sigma^\dagger_m(t)\sigma_{zk}(t+\tau)\sigma_j(t+\tau)\sigma_l(t+\tau)\rangle\right]
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
\frac{d}{dt}L_{mjk}(\tau) &= (-2i\Delta_0 - \Gamma)L_{mjk}(\tau) - i\frac{\Omega_0}{2}\left(e^{-i\mathbf{k}_0\cdot\mathbf{r}_j}I_{mjk}(\tau) + e^{-i\mathbf{k}_0\cdot\mathbf{r}_k}I_{mkj}(\tau)\right) \\
&\quad + \frac{\Gamma}{2}\sum_{l\neq j,k}\left[G^*_{jl}\langle\sigma^\dagger_m(t)\sigma_{zj}(t+\tau)\sigma^\dagger_k(t+\tau)\sigma^\dagger_l(t+\tau)\rangle + G^*_{kl}\langle\sigma^\dagger_m(t)\sigma_{zk}(t+\tau)\sigma^\dagger_j(t+\tau)\sigma^\dagger_l(t+\tau)\rangle\right]
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
\frac{d}{dt}M_{mjk}(\tau) &= -\Gamma(C_{mj}(\tau) + C_{mk}(\tau) + 2M_{mjk}(\tau)) \\
&\quad + i\Omega_0\left\{e^{-i\mathbf{k}_0\cdot\mathbf{r}_j}H_{mkj}(\tau) + e^{-i\mathbf{k}_0\cdot\mathbf{r}_k}H_{mjk}(\tau) - e^{i\mathbf{k}_0\cdot\mathbf{r}_j}I_{mkj}(\tau) - e^{i\mathbf{k}_0\cdot\mathbf{r}_k}I_{mjk}(\tau)\right\} \\
&\quad + 2\Gamma\Gamma_{jk}\left(K_{mjk}(\tau) + K_{mkj}(\tau)\right) \\
&\quad - \Gamma\sum_{l\neq j,k}\left\{G_{jl}\langle\sigma^\dagger_m(t)\sigma^\dagger_j(t+\tau)\sigma_{zk}(t+\tau)\sigma_l(t+\tau)\rangle + G_{lk}\langle\sigma^\dagger_m(t)\sigma_{zj}(t+\tau)\sigma^\dagger_k(t+\tau)\sigma_l(t+\tau)\rangle\right. \\
&\quad \left. + G^*_{jl}\langle\sigma^\dagger_m(t)\sigma_j(t+\tau)\sigma_{zk}(t+\tau)\sigma^\dagger_l(t+\tau)\rangle + G^*_{lk}\langle\sigma^\dagger_m(t)\sigma_{zj}(t+\tau)\sigma_k(t+\tau)\sigma^\dagger_l(t+\tau)\rangle\right\}.
\end{aligned} \tag{38}
$$

APPROXIMATIONS

In order to reduce the number of equations to be integrated, we neglect the 3-operator terms with $m \neq j, k$. Furthermore, we neglect the 4-operator terms, since they involve three atoms (with $j \neq k \neq l$).

The approximated equations are

$$\frac{d}{d\tau}A_{jj}(\tau) = \left(i\Delta_0 - \frac{\Gamma}{2}\right)A_{jj}(\tau) + \frac{i\Omega_0}{2}e^{i\mathbf{k}_0\cdot\mathbf{r}_j}C_{jj}(\tau) + \frac{\Gamma}{2}\sum_{k\neq j}^{N}G_{jk}H_{jjk}(\tau) \tag{39}$$

$$\frac{d}{d\tau}B_{jj}(\tau) = \left(-i\Delta_0 - \frac{\Gamma}{2}\right)B_{jj}(\tau) - \frac{i\Omega_0}{2}e^{-i\mathbf{k}_0\cdot\mathbf{r}_j}C_{jj}(\tau) + \frac{\Gamma}{2}\sum_{k\neq j}^{N}G^*_{jk}I_{jjk}(\tau) \tag{40}$$

$$\begin{aligned}\frac{d}{d\tau}C_{jj}(\tau) &= i\Omega_0\left\{e^{-i\mathbf{k}_0\cdot\mathbf{r}_j}A_{jj}(\tau) - e^{i\mathbf{k}_0\cdot\mathbf{r}_j}B_{jj}(\tau)\right\} - \Gamma(\langle\sigma_j^\dagger\rangle_s + C_{jj}(\tau))\\ &- \Gamma\sum_{k\neq j}^{N}\{G_{jk}K_{jjk}(\tau) + G^*_{jk}K_{jkj}(\tau)\}.\end{aligned} \tag{41}$$

$$\frac{d}{d\tau}A_{jk}(\tau) = \left(i\Delta_0 - \frac{\Gamma}{2}\right)A_{jk}(\tau) + \frac{i\Omega_0}{2}e^{i\mathbf{k}_0\cdot\mathbf{r}_k}C_{jk}(\tau) + \frac{\Gamma}{2}G_{jk}H_{jkj}(\tau) \tag{42}$$

$$\frac{d}{d\tau}B_{jk}(\tau) = \left(-i\Delta_0 - \frac{\Gamma}{2}\right)B_{jk}(\tau) - \frac{i\Omega_0}{2}e^{-i\mathbf{k}_0\cdot\mathbf{r}_k}C_{jk}(\tau) + \frac{\Gamma}{2}G^*_{jk}I_{jkj}(\tau) \tag{43}$$

$$\begin{aligned}\frac{d}{d\tau}C_{jk}(\tau) &= i\Omega_0\left\{e^{-i\mathbf{k}_0\cdot\mathbf{r}_k}A_{jk}(\tau) - e^{i\mathbf{k}_0\cdot\mathbf{r}_k}B_{jk}(\tau)\right\} - \Gamma(\langle\sigma_j^\dagger\rangle_s + C_{jk}(\tau))\\ &- \Gamma\{G_{jk}K_{jkj}(\tau) + G^*_{jk}K_{jjk}(\tau)\}.\end{aligned} \tag{44}$$

where $m \neq j$, and

$$\frac{d}{dt}H_{jjk}(\tau) = (i\Delta_0 - 3\Gamma/2)H_{jjk}(\tau) - \Gamma A_{jk}(\tau) + i\Omega_0\left\{e^{-i\mathbf{k}_0\cdot\mathbf{r}_j}J_{jjk}(\tau) - e^{i\mathbf{k}_0\cdot\mathbf{r}_j}K_{jjk}(\tau) + \frac{1}{2}e^{i\mathbf{k}_0\cdot\mathbf{r}_k}M_{jjk}(\tau)\right\} - 2\Gamma\Gamma_{jk}H_{jkj}(\tau) - \Gamma G^*_{jk}A_{jj}(\tau) \tag{45}$$

$$\frac{d}{dt}I_{jjk}(\tau) = (-i\Delta_0 - 3\Gamma/2)I_{jjk}(\tau) - \Gamma B_{jk}(\tau) - i\Omega_0\left\{e^{i\mathbf{k}_0\cdot\mathbf{r}_j}L_{jjk}(\tau) - e^{-i\mathbf{k}_0\cdot\mathbf{r}_j}K_{jkj}(\tau) + \frac{1}{2}e^{-i\mathbf{k}_0\cdot\mathbf{r}_k}M_{jjk}(\tau)\right\} - 2\Gamma\Gamma_{jk}I_{jkj}(\tau) - \Gamma G_{jk}B_{jj}(\tau) \tag{46}$$

$$\frac{d}{dt}K_{jjk}(\tau) = -\Gamma K_{jjk}(\tau) - i\frac{\Omega_0}{2}\left(e^{-i\mathbf{k}_0\cdot\mathbf{r}_j}H_{jjk}(\tau) - e^{i\mathbf{k}_0\cdot\mathbf{r}_k}I_{jkj}(\tau)\right) + \frac{\Gamma}{4}(G_{jk}C_{jk}(\tau) + G^*_{jk}C_{jj}(\tau)) + \frac{\Gamma}{2}\Gamma_{jk}M_{jjk}(\tau) \tag{47}$$

$$\frac{d}{dt}J_{jjk}(\tau) = (2i\Delta_0 - \Gamma)J_{jjk}(\tau) + i\frac{\Omega_0}{2}\left(e^{i\mathbf{k}_0\cdot\mathbf{r}_j}H_{jjk}(\tau) + e^{i\mathbf{k}_0\cdot\mathbf{r}_k}H_{jkj}(\tau)\right) \tag{48}$$

$$\frac{d}{dt}L_{jjk}(\tau) = (-2i\Delta_0 - \Gamma)L_{jjk}(\tau) - i\frac{\Omega_0}{2}\left(e^{-i\mathbf{k}_0\cdot\mathbf{r}_j}I_{jjk}(\tau) + e^{-i\mathbf{k}_0\cdot\mathbf{r}_k}I_{jkj}(\tau)\right) \tag{49}$$

$$\frac{d}{dt}M_{jjk}(\tau) = -\Gamma(C_{jj}(\tau) + C_{jk}(\tau) + 2M_{jjk}(\tau)) + i\Omega_0\left\{e^{-i\mathbf{k}_0\cdot\mathbf{r}_j}H_{jkj}(\tau) + e^{-i\mathbf{k}_0\cdot\mathbf{r}_k}H_{jjk}(\tau) - e^{i\mathbf{k}_0\cdot\mathbf{r}_j}I_{jkj}(\tau) - e^{i\mathbf{k}_0\cdot\mathbf{r}_k}I_{jjk}(\tau)\right\} + 2\Gamma\Gamma_{jk}\left(K_{jjk}(\tau) + K_{jkj}(\tau)\right) \tag{50}$$

$$\frac{d}{dt}H_{kjk}(\tau) = (i\Delta_0 - 3\Gamma/2)H_{kjk}(\tau) - \Gamma A_{kk}(\tau) + i\Omega_0\left\{e^{-i\mathbf{k}_0\cdot\mathbf{r}_j}J_{kjk}(\tau) - e^{i\mathbf{k}_0\cdot\mathbf{r}_j}K_{kjk}(\tau) + \frac{1}{2}e^{i\mathbf{k}_0\cdot\mathbf{r}_k}M_{kjk}(\tau)\right\} - 2\Gamma\Gamma_{jk}H_{kkj}(\tau) - \Gamma G^*_{jk}A_{kj}(\tau) \tag{51}$$

$$\frac{d}{dt}I_{kjk}(\tau) = (-i\Delta_0 - 3\Gamma/2)I_{kjk}(\tau) - \Gamma B_{kk}(\tau) - i\Omega_0\left\{e^{i\mathbf{k}_0\cdot\mathbf{r}_j}L_{kjk}(\tau) - e^{-i\mathbf{k}_0\cdot\mathbf{r}_j}K_{kkj}(\tau) + \frac{1}{2}e^{-i\mathbf{k}_0\cdot\mathbf{r}_k}M_{kjk}(\tau)\right\} - 2\Gamma\Gamma_{jk}I_{kkj}(\tau) - \Gamma G_{jk}B_{kj}(\tau) \tag{52}$$

$$\frac{d}{dt}K_{kjk}(\tau) = -\Gamma K_{kjk}(\tau) - i\frac{\Omega_0}{2}\left(e^{-i\mathbf{k}_0\cdot\mathbf{r}_j}H_{kjk}(\tau) - e^{i\mathbf{k}_0\cdot\mathbf{r}_k}I_{kkj}(\tau)\right) + \frac{\Gamma}{4}(G_{jk}C_{kk}(\tau) + G^*_{jk}C_{kj}(\tau)) + \frac{\Gamma}{2}\Gamma_{jk}M_{kjk}(\tau) \tag{53}$$

$$\frac{d}{dt}J_{kjk}(\tau) = (2i\Delta_0 - \Gamma)J_{kjk}(\tau) + i\frac{\Omega_0}{2}\left(e^{i\mathbf{k}_0\cdot\mathbf{r}_j}H_{kjk}(\tau) + e^{i\mathbf{k}_0\cdot\mathbf{r}_k}H_{kkj}(\tau)\right) \tag{54}$$

$$\frac{d}{dt}L_{kjk}(\tau) = (-2i\Delta_0 - \Gamma)L_{kjk}(\tau) - i\frac{\Omega_0}{2}\left(e^{-i\mathbf{k}_0\cdot\mathbf{r}_j}I_{kjk}(\tau) + e^{-i\mathbf{k}_0\cdot\mathbf{r}_k}I_{kkj}(\tau)\right) \tag{55}$$

$$\frac{d}{dt}M_{kjk}(\tau) = -\Gamma(C_{kj}(\tau) + C_{kk}(\tau) + 2M_{kjk}(\tau)) + i\Omega_0\left\{e^{-i\mathbf{k}_0\cdot\mathbf{r}_j}H_{kkj}(\tau) + e^{-i\mathbf{k}_0\cdot\mathbf{r}_k}H_{kjk}(\tau) - e^{i\mathbf{k}_0\cdot\mathbf{r}_j}I_{kkj}(\tau) - e^{i\mathbf{k}_0\cdot\mathbf{r}_k}I_{kjk}(\tau)\right\} + 2\Gamma\Gamma_{jk}\left(K_{kjk}(\tau) + K_{kkj}(\tau)\right) \tag{56}$$

INITIAL CONDITIONS

At $\tau = 0$:

$$A_{jj}(0) = \frac{1}{2}(1 + \langle\sigma_{zj}\rangle_s) \qquad A_{jk}(0) = \langle\sigma_j^\dagger\sigma_k\rangle_s \tag{57}$$

$$B_{jj}(0) = 0 \qquad B_{jk}(0) = \langle\sigma_j^\dagger\sigma_k^\dagger\rangle_s \tag{58}$$

$$C_{jj}(0) = -\langle\sigma_j^\dagger\rangle_s \qquad C_{jk}(0) = \langle\sigma_j^\dagger\sigma_{zk}\rangle_s \tag{59}$$

$$H_{jjk}(0) = -\langle\sigma_j^\dagger\sigma_k\rangle \qquad H_{kjk}(0) = \frac{1}{2}(\langle\sigma_{zj}\rangle_s + \langle\sigma_{zj}\sigma_k\rangle_s) \tag{60}$$

$$I_{jjk}(0) = -\langle\sigma_j^\dagger\sigma_k^\dagger\rangle_s \qquad I_{kjk}(0) = 0 \tag{61}$$

$$K_{jjk}(0) = 0 \qquad K_{kjk}(0) = \frac{1}{2}(\langle\sigma_j^\dagger\rangle_s + \langle\sigma_j^\dagger\sigma_{zk}\rangle_s) \tag{62}$$

$$J_{jjk}(0) = \frac{1}{2}(\langle\sigma_k\rangle_s + \langle\sigma_{zj}\sigma_k\rangle)_s \qquad J_{kjk}(0) = \frac{1}{2}(\langle\sigma_j\rangle_s + \langle\sigma_j\sigma_{zk}\rangle_s) \tag{63}$$

$$L_{jjk}(0) = 0 \qquad L_{kjk}(0) = 0 \tag{64}$$

$$M_{jjk}(0) = -\langle\sigma_j^\dagger\sigma_{zk}\rangle_s \qquad M_{kjk}(0) = -\langle\sigma_{zj}\sigma_k^\dagger\rangle_s \tag{65}$$

[1] T. Bienaimé, R. Bachelard, N. Piovella, and R. Kaiser, Fortschr. Phys. **61** (2013) 377.
[2] S. Kramer, and H. Ritsch, Eur. Phys. J.D. **69** (2015) 282.
[3] R. Paškauskas, and M. Kastner, J. Stat. Mech. (2012) P20005.