Beijing-Dublin International College

MIDTERM - SPRING TRIMESTER EXAMINATION - 2023/2024

MATH1002J Introduction to Analysis

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Time Allowed: <u>**80 minutes**</u>

Instructions for Candidates

Full marks will be awarded for complete answers and correct steps to all 4 questions. The number of marks per question is indicated.

Show and explain carefully all the steps of your workings in the Answer Sheet.

At the end of the exam you need to submit ONLY the Answer Sheet (keep the questions sheet).

If you finish the exam early, you still are required to stay until the end.

IMPORTANT: at the end of the exam stop writing. The invigilators will allow to orderly retrieve your phones in an organized manner. You shall take a picture of your own answer sheet, and afterwards you will submit the original answer sheet to the invigilators. You will need your own solutions for the Self-Assessment Report.

Instructions for Invigilators

This examination is closed-book and closed-notes. Non-programmable calculators are permitted. No rough-work paper is to be provided to the candidate.

Students who finish the exam cannot leave before the end of the allowed time.

At the end of the exams, the students should stop writing; afterwards, they will orderly allowed to retrieve their phones and they will be given 2 minutes to quickly take pictures of their own solutions, and then submit the answer sheet.

About the Grading Scheme

The grading scheme below is given step by step. It is fairly complete. However it does not account for full details and minituae, or alternative solutions that may exist. For these reasons:

1. Follow the solution in dark blue to check the necessary step to solve each problem.
2. Follow the "[+5, explaination...]" to decide how many marks attribute to yourself for each step/portion of the exercise.
3. If your solution was different, but still fundamentally correct, you can assign to that step/portion marks, in increasing way according to the fact that your solution is incomplete, partially complete, or fully complete.
4. If some details are missing in your solution are missing, decide the penalty according to the importance of the missing step (if not explicitly stated already in the grading scheme).
5. Check that the total mark you give to each exercise does not exceed the maximum that you can obtain for that component.
6. (Suggestion) Quickly review your self-grading after you complete the Self-Assessment Report.

1. In this question you have to find where is the **mistake** in the given "proof" of the clearly paradoxical statement: "*All points are on the same line*".

— BEGINNING OF THE PROOF —

- Proposition $\mathcal{P}(n)$: "if we have a set of n points, then they are collinear (on the same line)".
- Induction Base: for $n = 1$ the statement $\mathcal{P}(n)$ becomes "if there is only one point, it is on a line", which is obviously true.
- Induction Hypothesis: assume that $\mathcal{P}(k)$ is true, i.e. that any set of k points, is on a single line.
- Induction Step: We want to use $\mathcal{P}(k)$ to show $\mathcal{P}(k+1)$ is true. Take any set of $k+1$ points; call this set $\{P_1, P_2, P_3, \ldots, P_k, P_{k+1}\}$. Consider its two subsets $A = \{P_1, P_2, P_3, \ldots, P_k\}$ and $B = \{P_2, P_3, \ldots, P_k, P_{k+1}\}$. Each is a set of k points, therefore, using the induction hypothesis, we conclude that each of this two sets must be on the same line, i.e. all points in A must be on the line ℓ_A, and all points of B must be on the line ℓ_B. But the two

sets overlap because their intersection is $\{P_2, \ldots, P_k\}$ which then belongs to both lines ℓ_A and ℓ_B, so they must be the same, therefore there must be only one line containing all $k+1$ points.

- Since we did all the necessary steps, by mathematical induction the statement $\mathcal{P}(n)$ is true for all $n \geq 1$, so all sets of n points must be collinear.

— END OF THE PROOF —

Carefully read the argument, decide if the result is true or false. Explain and defend your decision using clear logical arguments. **[10]**

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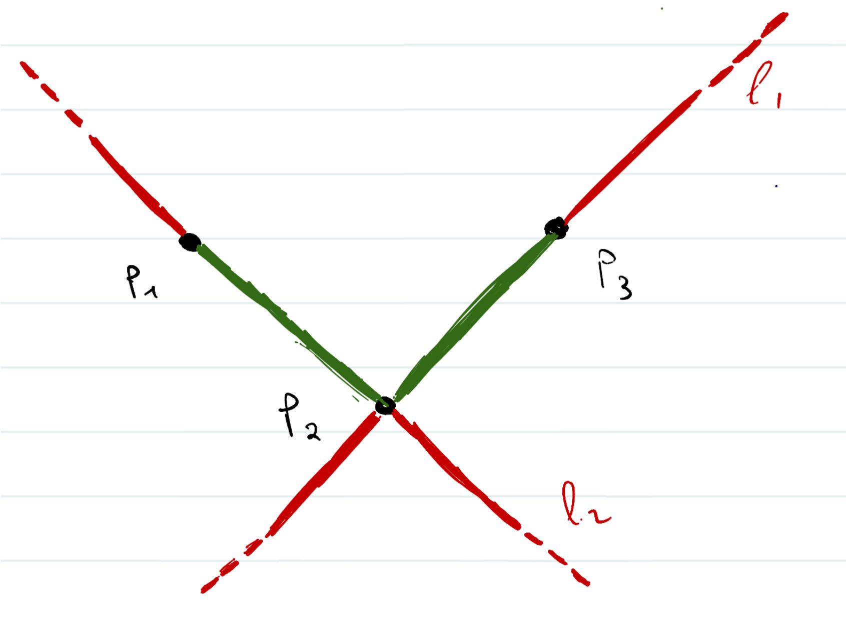
SOLUTION:

[If you correctly identify the step where we have the issue (Induction Step 2 → 3) grants +6, if you fully correctly explain WHY that is the issue, +4, for a total of +10 marks.]

The problem arises when we apply the inductive step to pass from $k = 2$ to $k + 1 = 3$ [+6]. In fact, in that case:

$$\{P_1, P_2\} \cap \{P_2, P_3\} = \{P_2\}$$

In the proof this is used to say that the lines P_1P_2 and P_2P_3 must be the same line, but since in this case we only have one point of intersection (and from a single point there are infinitely many lines passing) we cannot conclude that P_1P_2 and P_2P_3 are the same [+4]. In fact, they could be different lines (see example in the picture):



Therefore we cannot conclude that the lines P_1P_2 and P_2P_3 are the same, which implies that the Inductive Step is not correct when passing from step 2 to step 3. Since $2 \to 3$ fails to be correct, the argument falls from there onwards and therefore it is false. $\square$

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2. Prove that $\sqrt{10}$ is an irrational number. **[30]**

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SOLUTION:

Assume for sake of contradiction that $\sqrt{10} = \frac{m}{n}$ with $m, n \in \mathbb{N}$, $n \neq 0$, and $\gcd(m, n) = 1$.

[+5 for the correctly stated assumption in order to start the contradiction.]

Then $m^2 = 10n^2$, which means that m^2 is even, so that $m = 2a$ is even as well.

[+5. +3 to identify that m^2 must be even (divisible by 2). +2 to identify that m must be even.

If you use divisibility by 5 instead, i.e. 5 divides m^2 therefore 5 divides m, it is ok too (we have seen this for prime numbers so it is ok).

If you use divisibility by 10 instead, i.e. 10 divides m^2 therefore 10 divides m, you have to justify an extra step, which is $\gcd(2, 5) = 1$. If you do not, there is a -1 penalty, so the maximum you can get in this full step is $+4$.]

Then $2a^2 = 5n^2$, and since 5 is odd, for $5n^2$ to be even we need n^2 even, so that $n = 2b$ is even as well.

[+5. +3 to identify that n^2 must be even (divisible by 2). +2 to identify that n must be even.

If you use divisibility by 5 instead, i.e. 5 divides n^2 therefore 5 divides n since 5 does not divide 2, it is ok too (we have seen this for prime numbers so it is ok).

If you use divisibility by 10 instead, i.e. 10 divides n^2 therefore 10 divides n, you have to justify an extra step, which is $\gcd(2, 5) = 1$. If you do not, like in the previous step, there is a -1 penalty, so the maximum you can get in this full step is $+4$.

Therefore 2 divides both m and n, so that $2 \mid \gcd(m, n) = 1$, which is absurd (2 does not divide 1 in $\mathbb{Z}$!). Since this contradicts the initial assumption in the proof that $\sqrt{10}$ is rational, then it must be irrational. $\square$

[+5, for the correct identification of the contradiction/absurd and conclusion of the argument.]

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3. For any number $c \in \mathbb{R} \setminus \{0\}$, the function $f_c : \mathbb{R} \setminus \{-c\} \to \mathbb{R}$ given by

$$f_c(x) = \frac{c^{2024}}{x + c} \left(\binom{2024}{0} + \binom{2024}{1} \frac{x}{c} + \cdots + \binom{2024}{r} \left(\frac{x}{c}\right)^r + \cdots + \binom{2024}{2024} \left(\frac{x}{c}\right)^{2024} \right)$$

Find the image $\mathrm{Im}(f_c)$ (notice: you do not have to choose a number c). **[30]**

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SOLUTION:

We can see that this function is equal to:

$$f_c(x) = \frac{c^{2024}}{x+c}\sum_{r=0}^{2024}\binom{2024}{r}\left(\frac{x}{c}\right)^r.$$

Using the binomial theorem [+5, to quote the theorem correctly] we can write:

$$f(x) = \frac{c^{2024}}{x+c}\left(1+\frac{x}{c}\right)^{2024} = \frac{(x+c)^{2024}}{x+c},$$

[+10, for the correct application of the binomial theorem]
which is equal to $(x+c)^{2023}$ when $x \neq -c$. In particular:

$$y \in \operatorname{Im}(f) \iff y = (x+c)^{2023}, \text{ and } x \neq -c, \iff x = \sqrt[2023]{y} - c, \text{and } x \neq -c.$$

[+10, for the correct steps in the above chain. The above chain shows that $y \in \operatorname{Im}(f) = \mathbb{R} \setminus \{0\}$, since all steps are equivalences.
If you go with $\Longleftarrow$ only, you show that $\operatorname{Im}(f) \supset \mathbb{R}\setminus\{0\}$ is also ok since we already know that $\operatorname{Im}(f) \subset \mathbb{R} \setminus \{0\}$ is true.
If you go with $\Longrightarrow$ only, you show that $\operatorname{Im}(f) \subset \mathbb{R}\setminus\{0\}$ is not sufficient, we already know it, the image is always a subset of the codomain.]
which has solutions for every $y \in \mathbb{R}$ with $y \neq 0$ (because $x \neq -c$), so that IFF $y \in \mathbb{R} \setminus \{0\}$. In particular we find that $\operatorname{Im}(f) = R \setminus \{0\} = (-\infty, 0) \cup (0, +\infty)$. □
[+5, for the correct conclusion after the computations. If you do not state the final conclusion but it is still VERY VERY CLEAR in your solution from the previous lines, you may receive a penalty of 1 mark (so only +4 for this last step).]

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4. Decide if the following statements are True or False, and justify your answer.

(a) If $f : [0, 10] \to [3, 5]$ is injective and $g : [1, 6] \to [0, 1]$ is injective, then the composition $g \circ f$ and it is injective too. [6]

(b) The function $f : \mathbb{R} \to \mathbb{R}$ defined by:

$$f(x) = \begin{cases} x^2, & \text{if } x \leq 1, \\ x, & \text{if } x \geq 1, \end{cases}$$

is invertible. [6]

(c) The rule $f(a/b) = b$ gives a function $f : \mathbb{Q} \to \mathbb{Z}$? [6]

(d) If I have 7 players, I can form a 2 players Badminton team in 25 different ways. [6]

(e) The equation $y = x + \frac{8}{x^2}$ has only 3 solutions with $x, y \in \mathbb{Z}$. [6]

(Bonus) There is no injective function from $(0,1)$ to $\mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \dots$ (product of infinitely many copies of $\mathbb{N}$. [+6]

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SOLUTION:

[The exercise has to be evaluated as follows:

- +3 for the correct choice "True/False"
- +3 for the fully correct justification/proof (or counterexample). If no justification is given, this part is 0, if important details are missing, only +1 is given. If there is only some minor mistakes/imprecisions or details missing but the core of the argument is correct, +2.

  If the argument you present is different from the one given in this solution, but FULLY correct, you can get full marks.

There are 5 questions, plus the bonus, for a total of $5 \times 6 + 6 = 30 + 6$ marks.]

(a) True. The composition surely makes sense since $\mathrm{Cod}(f) = [3,5] \subset [1,6] = \mathrm{Dom}(g)$. Also, in PS4 Q1a we showed that composition of injective function is injective too, as follows:

$$g(f(x_1)) = g(f(x_2)) \implies f(x_1) = f(x_2) \implies x_1 = x_2,$$

where the first $\implies$ is true because g is injective, and the second $\implies$ is true because f is injective.

(b) False. We see that $f(1) = f(-1) = 1$, so the function is not injective, therefore it cannot be bijective. Since we know that invertible is equivalent to bijective, f cannot be invertible.

(c) False. In fact, in rational numbers $q = 1/2 = 2/4$, but then the outcome of the function should be the same. However, $f(q) = f(1/2) = 2 \neq 4 = f(2/4) = f(q)$, which is impossible. This is not a function because it can associate multiple outcomes to the same element in the domain.

(d) False. In fact we know that this number must be equal to $\binom{7}{2} = 7 \cdot 6/2 = 21$. □

(e) False. In fact, the expression $x + \frac{8}{x^2}$ is an integer any time that $\frac{8}{x^2}$ is an integer too. This only happens for $x^2 = 1$, $x^2 = 4$ ($x^2 = 2$ or 8 are impossible), and each gives $x \in \{\pm 1, \pm 4\}$, so that we have 4 solutions $(1,9), (-1,7), (2,4), (-2,0)$.

[Alternative] If there is just a list of solutions $(1,9), (-1,7), (2,4), (-2,0)$, explaining that they are solutions!, and say that they are AT LEAST 4, then it is acceptable. If the proof says that there are EXACYLY 4 solutions but does not explain why, not sufficient for full marks.

(f) **(Bonus):** False. In fact, any number $x \in (0,1)$ can be written as a decimal expansion as follows:

$$x = 0, d_1 d_2 d_3 d_4 d_5 d_6 \ldots, \qquad d_i \in \{0,1,2,3,4,5,6,7,8,9\}.$$

So we define $f : (0,1) \to \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \ldots$ as follows:

$$f(x) = (d_1, d_2, d_3, d_4, \ldots).$$

(One should be careful defining this function since, for example, $0.5 = 0.4\overline{9}...$ It is an important detail for the correctedness of the argument; however, no penalty will be given if this is not noticed)

If we consider two numbers $x = 0.d_1d_2d_3...$ and $y = 0.e_1e_2e_3...$, then

$$\begin{aligned} f(x) = f(y) \quad &\Longleftrightarrow \quad (d_1, d_2, d_3, \ldots) = (e_1, e_2, e_3, \ldots) \quad \Longleftrightarrow \quad d_k = e_k, \ \forall k \geq 1, \\ &\Longrightarrow \quad 0.d_1d_2d_3... = 0.e_1e_2e_3... \quad \Longrightarrow \quad x = y \end{aligned}$$

which means that the function f is injective. $\square$

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