Beijing-Dublin International College (BDIC)

FIRST SEMESTER, Academic Year 2015–2016

Campus: Beijing University of Technology (BJUT)

Linear Algebra — Midterm Exam

Honesty Pledge:

I have read and clearly understand the Examination Rules of Beijing University of Technology and University College Dublin and am aware of the Punishment for Violating the Rules of Beijing University of Technology and University College Dublin. I hereby promise to abide by the relevant rules and regulations by not giving or receiving any help during the exam. If caught violating the rules, I would accept the punishment thereof.

Pledger: ______________________ Class NO: ______________________

BJUT Student ID: ______________ UCD Student ID: ______________

NOTE: Answer **ALL** questions.

Time allowed is **90** minutes.

The exam paper has **2** sections on **15** pages, with a full score of 100 marks.

You are required to use the provided **Examination Book** only for answers.

SECTION A — MULTIPLE CHOICE QUESTIONS

In each question, choose **at most one** option.
Circle the preferred choice on the **Examination Book** provided.
This section is worth a total of **60** marks, with each question worth **4** marks.

1. Let A and B be two sets,

$$A = \left\{ a \middle| \, a = n^2 + 1, \quad n \in \mathbb{N} \right\}, \qquad B = \left\{ b \middle| \, b = k^2 - 4k + 5, \quad k \in \mathbb{N} \right\},$$

where $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N} = \{1, 2, 3, \cdots\}$. Then which statement below about the relationship between A and B is correct?

(a) $A \cap B = \varnothing$; (b) $A \cup B = \varnothing$; (c) $A \subseteq B$; (d) $A \supseteq B$.

Solution: Since $k^2 - 4k + 5 = (k-2)^2 + 1$, where $k - 2 = 0$ is permitted, we have $A \subseteq B$.

Hence (C).

2. Evaluate the sum $\sum_{k=0}^{n} ar^{2k}$, where $a = 3$ and $r = 2$.

(a) $3\left(2^{n+1} - 1\right)$; (b) $4^{n+1} - 1$; (c) $3\frac{1-2^{2n+2}}{1-2^n}$; (d) -1.

Solution:

$$\sum_{k=0}^{n} ar^{2k} = a \sum_{k=0}^{n} r^{2k} = a \frac{1 - \left(r^2\right)^{n+1}}{1 - r^2} = 3 \frac{1 - 4^{n+1}}{1 - 4} = 4^{n+1} - 1.$$

Hence (B).

3. Determine the relative positions of the following two planes:

$$\Sigma_1 : \ 3x + 9y + 6z - 3 = 0, \qquad \Sigma_2 : \ 2x + 6y + 4z - 9 = 0.$$

They are

(a) parallel; (b) intersecting; (c) coincident （重合）; (d) perpendicular.

Solution:

$$\Sigma_1 : \ x + 3y + 2z - 1 = 0, \qquad \Sigma_2 : \ x + 3y + 2z - \frac{9}{2} = 0, \qquad \text{parallel.}$$

Hence (A).

4. Determine the relative positions of the following line and plane:

$$\mathcal{L} : \ \frac{x-3}{2} = \frac{y+4}{7} = z, \qquad \Sigma : \ 4x - y - z = 1.$$

Which statement is correct?

(a) $\mathcal{L}$ is perpendicular to Σ;

(b) $\mathcal{L}$ lies within Σ;

(c) $\mathcal{L}$ obliquely（倾斜）intersects with Σ;

(d) $\mathcal{L}$ is parallel to Σ.

Solution: The direction of $\mathcal{L}$ is $(2, 7, 1)$, and the normal direction of Σ is $(4, -1, -1)$ which is perpendicular to $(2, 7, 1)$. So $\mathcal{L}$ is parallel to Σ.

Hence (D).

5. Given

$$\mathbf{a} = \mathbf{i} - \mathbf{j}, \qquad \mathbf{b} = -\mathbf{j} + \mathbf{k}, \qquad \mathbf{a} \cdot \mathbf{c} = 2, \qquad \mathbf{b} \cdot \mathbf{c} = -1,$$

evaluate $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$.

(a) $\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$; (b) $-\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$; (c) $\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$; (d) $-\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$.

Solution:

$$\begin{aligned} \mathbf{a} \times \mathbf{b} \times \mathbf{c} &= (\mathbf{a} \cdot \mathbf{c})\, \mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\, \mathbf{a} = 2\,(-\mathbf{j} + \mathbf{k}) - (-1)\,(\mathbf{i} - \mathbf{j}) \\ &= -2\mathbf{j} + 2\mathbf{k} + \mathbf{i} - \mathbf{j} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}. \end{aligned}$$

Hence (A).

6. For the equation

$$ax + by + cz + d = 0 \qquad \text{with } b = 0 \quad \text{and} \quad a, c, d \neq 0,$$

which of the following statements about this equation is **true**?

(a) It describes a plane perpendicular to the xy-plane, and passing through the origin（原点）.

(b) It describes a plane parallel to the y axis.

(c) It describes a line parallel to the x-axis.

(d) It describes a plane parallel to the x and z axes, and not passing through the origin.

Solution: Since $b = 0$ and $a, c, d \neq 0$, this describes a plane perpendicular to the xz-plane (or equivalently, parallel to the y-axis), but not passing through the origin.

Hence (B).

7. Let $\mathbf{a} = 2\mathbf{i} - 2\mathbf{j}$ and $\mathbf{b} = 3\mathbf{j} + \mathbf{k}$ be two vectors. Find a unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

(a) $\frac{1}{\sqrt{11}}(\mathbf{i} + \mathbf{j} - 3\mathbf{k})$; (b) $-2(\mathbf{i} + \mathbf{j} - 3\mathbf{k})$; (c) $\frac{1}{\sqrt{11}}(\mathbf{i} - \mathbf{j} - 3\mathbf{k})$; (d) $-2(\mathbf{i} - \mathbf{j} - 3\mathbf{k})$.

Solution:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 0 \\ 0 & 3 & 1 \end{vmatrix} = -2(\mathbf{i} + \mathbf{j} - 3\mathbf{k}).$$

Thus the required unit vector is $\pm\frac{1}{\sqrt{11}}(\mathbf{i} + \mathbf{j} - 3\mathbf{k})$.

Hence (A).

8. Let $\mathbf{a} = -5\mathbf{i} + \alpha\mathbf{j}$ and $\mathbf{b} = \mathbf{i} - 5\mathbf{j}$ be two vectors, with α being a nonzero real number. If $\mathbf{a}$ and $\mathbf{b}$ are parallel, determine the value of α.

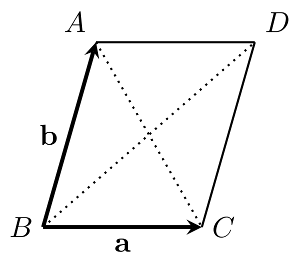
(a) 25; (b) 1; (c) 0; (d) 10.

Solution:

$$\mathbf{a} = t\mathbf{b} \Longrightarrow -5\mathbf{i} + \alpha\mathbf{j} = t(\mathbf{i} - 5\mathbf{j}) \Longrightarrow \begin{cases} -5 = t \\ \alpha = -5t \end{cases}, \quad \text{i.e.,} \quad \alpha = 25.$$

Hence (A).

9. Let $\square ABCD$ be a parallelogram with two adjacent edges represented by two vectors $\mathbf{a}$ and $\mathbf{b}$, i.e., $\overrightarrow{BC} = \mathbf{a}$, $\overrightarrow{BA} = \mathbf{b}$. Suppose $|\mathbf{a}| = 2$ and $|\mathbf{b}| = 5$. Evaluate the inner product $\overrightarrow{BD} \cdot \overrightarrow{AC}$.



(a) 29; (b) 0; (c) -21; (d) 21.

Solution:

$$\overrightarrow{BD} \cdot \overrightarrow{AC} = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = |\mathbf{a}|^2 - |\mathbf{b}|^2 = 2^2 - 5^2 = -21.$$

Hence (C).

10. Given $\mathbf{a} \cdot \mathbf{b} = 10$, $|\mathbf{a}| = 5$ and $|\mathbf{b}| = 3$. Evaluate $|\mathbf{a} \times \mathbf{b}|$.

(a) $3\sqrt{3}$; (b) $5\sqrt{5}$; (c) 15; (d) 125.

Solution:

$$\begin{aligned} |\mathbf{a} \times \mathbf{b}| &= |\mathbf{a}|\,|\mathbf{b}|\,|\sin\theta| = |\mathbf{a}|\,|\mathbf{b}|\sqrt{1 - \cos^2\theta} = \sqrt{|\mathbf{a}|^2\,|\mathbf{b}|^2 - |\mathbf{a}|^2\,|\mathbf{b}|^2 \cos^2\theta} \\ &= \sqrt{|\mathbf{a}|^2\,|\mathbf{b}|^2 - |\mathbf{a} \cdot \mathbf{b}|^2} = \sqrt{15^2 - 10^2} = 5\sqrt{5}. \end{aligned}$$

Hence (B).

11. Let $\triangle ABC$ be a triangle in 3 dimensions, where the vertices are $A\,(3, 2, 1)$, $B\,(7, 5, 2)$ and $C\,(6, 3, 5)$. Find the area of $\triangle ABC$.

(a) $3\sqrt{35}$; (b) 19; (c) $\frac{19}{2}$; (d) $\frac{3}{2}\sqrt{35}$.

Solution: Noticing $\overrightarrow{AB} = 4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\overrightarrow{AC} = 3\mathbf{i} + \mathbf{j} + 4\mathbf{k}$, we have

$$\text{Area} = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 3 & 1 \\ 3 & 1 & 4 \end{vmatrix} = \frac{1}{2}\left|11\mathbf{i} - 13\mathbf{j} - 5\mathbf{k}\right| = \frac{1}{2}\sqrt{11^2 + 13^2 + 5^2} = \frac{3}{2}\sqrt{35}.$$

Hence (D).

12. Let $\mathcal{L}$ be a line in 3 dimensions, $\mathcal{L}:\ \frac{x-2}{2} = \frac{y-7}{-2} = \frac{z+3}{3}$. Which of the following equations is an equivalent expression for $\mathcal{L}$?

(a) $\begin{cases} x = 2 + 2t, \\ y = -2 + 7t, \\ z = 3 + 3t; \end{cases}$ (b) $2x - 2y + 3z = -19$; (c) $\mathbf{r} = 9\mathbf{j} - 6\mathbf{k} + t\,(2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$;

(d) $(2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \cdot [(x-2)\,\mathbf{i} - (y-7)\,\mathbf{j} + (z+3)\,\mathbf{k}] = 0$.

Solution: (B) and (D) are equations of planes. (A) has wrong data for the coefficients of t: $(2, 7, 3) \neq (2, -2, 3)$. For (C),

$$\begin{cases} x = & 2 + 2t = 0 + 2\,(t+1) = 0 + 2t' \\ y = & 7 - 2t = 9 - 2\,(t+1) = 9 - 2t' \\ z = & -3 + 3t = -6 + 3\,(t+1) = -6 + 3t' \end{cases}$$

Hence (C).

13. Let $\mathbf{w} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = \mathbf{i} + 3\mathbf{k}$. Find the component of $\mathbf{w}$ orthogonal to $\mathbf{v}$.

(a) $-\frac{1}{2}\,(\mathbf{i} + 3\mathbf{k})$; (b) $6\mathbf{i} + 2\mathbf{j} + 13\mathbf{k}$; (c) $\frac{3}{2}\mathbf{i} + 2\mathbf{j} - \frac{1}{2}\mathbf{k}$; (d) $-5\,(\mathbf{i} + 3\mathbf{k})$.

Solution: With $\hat{\mathbf{v}} = \frac{1}{\sqrt{10}}(\mathbf{i}+3\mathbf{k})$, we have

$$\begin{aligned}\mathbf{w} - (\mathbf{w}\cdot\hat{\mathbf{v}})\,\hat{\mathbf{v}} &= (\mathbf{i}+2\mathbf{j}-2\mathbf{k}) - \frac{1}{10}\left[(\mathbf{i}+2\mathbf{j}-2\mathbf{k})\cdot(\mathbf{i}+3\mathbf{k})\right](\mathbf{i}+3\mathbf{k}) \\ &= (\mathbf{i}+2\mathbf{j}-2\mathbf{k}) + \frac{1}{2}(\mathbf{i}+3\mathbf{k}) = \frac{3}{2}\mathbf{i}+2\mathbf{j}-\frac{1}{2}\mathbf{k}\end{aligned}$$

Hence (C).

14. Let Σ be a plane, containing three non-collinear points A, B and C:

$$A\,(1,-1,2)\,, \qquad B\,(0,-1,-1)\,, \qquad C\,(-1,1,0)\,.$$

Find the equation of the plane Σ.

(a) $x+y-z+3=0$;

(b) $-x+y-z-3=0$;

(c) $2x+3y-2z+1=0$;

(d) $3x+2y-z+1=0$.

Solution: Let A be $\mathbf{r}_0$, then

$$\begin{vmatrix} x-1 & y+1 & z-2 \\ 0-1 & -1+1 & -1-2 \\ -1-1 & 1+1 & 0-2 \end{vmatrix} = \begin{vmatrix} x-1 & y+1 & z-2 \\ -1 & 0 & -3 \\ -2 & 2 & -2 \end{vmatrix} = 6x+4y-2z+2=0,$$

i.e.,

$$3x+2y-z+1=0.$$

Hence (D).

15. Consider three vectors in 3 dimensions:

$$\mathbf{a} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}, \qquad \mathbf{a} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}, \qquad \mathbf{a} = \mathbf{i} + \beta\mathbf{j} - \mathbf{k}.$$

If they are linearly dependent, determine the value of the constant β.

(a) 0; (b) $\frac{1}{2}$; (c) 2; (d) There is no appropriate evaluation for β.

Solution: It is seen that if $\beta = \frac{1}{2}$, there will be $\mathbf{b} = 2\mathbf{c}$. Hence the answer is (B). Another method is

$$\begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & -2 \\ 1 & \beta & -1 \end{vmatrix} = 6\beta - 3 = 6\left(\beta - \frac{1}{2}\right).$$

If we ask the determinant is 0, there is $\beta = \frac{1}{2}$.

Hence (B).

SECTION B — EXTENDED ANSWER QUESTIONS

Write your answers on the **Examination Book** provided.
This section is worth a total of **40** marks. The marks of each question is as shown.

16. **(6 marks)** Prove the following formula by means of mathematical induction（数学归纳法）:

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6} n\,(n+1)\,(2n+1)\,, \qquad n \in \mathbb{N},\ n \geq 1.$$

Solution:

Step 0: The question is re-stated as

$$\sum_{i=1}^{n} i^2 = \frac{1}{6} n\,(n+1)\,(2n+1)\,.$$

Step 1: For $n = 1$, the statement is true:

$$1 = \frac{1}{6} \times 1 \times (1+1)\,(2 \times 1 + 1)\,.$$

Step 2: Suppose the statement is true for $n = k$, i.e.,

$$\sum_{i=1}^{k} i^2 = \frac{1}{6} k\,(k+1)\,(2k+1)\,.$$

Step 3: We need to prove the statement is true for $n = k+1$, i.e.,

$$\sum_{i=1}^{k+1} i^2 = \frac{1}{6}\,(k+1)\,(k+2)\,(2k+3)\,.$$

Proof:

$$\begin{aligned}\sum_{i=1}^{k+1} i^2 &= \sum_{i=1}^{k} i^2 + (k+1)^2 \overset{\text{ind hypo}}{=} \frac{1}{6} k\,(k+1)\,(2k+1) + (k+1)^2 \\ &= \frac{1}{6}\,(k+1)\,[k\,(2k+1) + 6\,(k+1)] = \frac{1}{6}\,(k+1)\,\left[2k^2 + 7k + 6\right] = \frac{1}{6}\,(k+1)\,(k+2)\,(2k+3)\,.\end{aligned}$$

Done.

17. (6 marks) Let Σ_1 and Σ_2 be two planes in 3 dimensions,

$$\Sigma_1 : \ 2x + y + z = 3, \qquad \Sigma_2 : \ x + y - z = 2.$$

Find a plane which contains the point $P\,(1, 0, 2)$ and is perpendicular to both Σ_1 and Σ_2.

Solution:

Let the plane we want be Σ. The normal directions of Σ_1 and Σ_2 are, respectively, $\mathbf{a} = (2, 1, 1)$ and $\mathbf{b} = (1, 1, -1)$. Their cross product will give the normal direction of Σ:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = -2\mathbf{i} + 3\mathbf{j} + \mathbf{k}.$$

Hence, together with the given point $P\,(1, 0, 2)$, we have the equation of the plane Σ:

$$-2\,(x-1) + 3y + (z-2) = 0, \qquad \text{i.e.,} \qquad -2x + 3y + z = 0.$$

18. **(8 marks)** Consider three vectors

$$\begin{aligned} \mathbf{a} &= \mathbf{i} - 2\mathbf{j} + \mathbf{k}, \\ \mathbf{b} &= 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}, \\ \mathbf{c} &= 5\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}. \end{aligned}$$

(a) Are they linearly dependent or independent?

(b) If they are linearly independent, try to compute the scalar triple product of **a**, **b** and **c**; if linearly dependent, try to express **c** in terms of **a** and **b**.

Solution:

(a) Method 1: $\begin{vmatrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 5 & 4 & -5 \end{vmatrix} = 0$, hence **a**, **b** and **c** are linearly dependent.

Method 2: Let $\lambda_1 \mathbf{a} + \lambda_2 \mathbf{b} + \lambda_3 \mathbf{c} = \mathbf{0}$, i.e.,

$$\lambda_1 (\mathbf{i} - 2\mathbf{j} + \mathbf{k}) + \lambda_2 (3\mathbf{i} + \mathbf{j} - 2\mathbf{k}) + \lambda_3 (5\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}) = 0, \tag{1}$$

i.e.,

$$(\lambda_1 + 3\lambda_2 + 5\lambda_3)\,\mathbf{i} + (-2\lambda_1 + \lambda_2 + 4\lambda_3)\,\mathbf{j} + (\lambda_1 - 2\lambda_2 - 5\lambda_3)\,\mathbf{k} = 0.$$

Thus

$$\begin{cases} \lambda_1 + 3\lambda_2 + 5\lambda_3 &= 0, \\ -2\lambda_1 + \lambda_2 + 4\lambda_3 &= 0, \\ \lambda_1 - 2\lambda_2 - 5\lambda_3 &= 0. \end{cases}$$

This array of equations has a solution

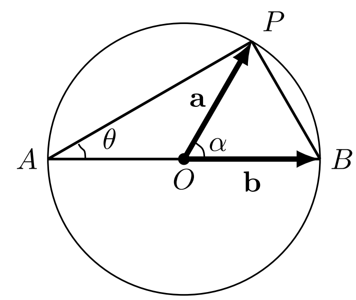
$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = t \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \qquad t \in \mathbb{R}. \tag{2}$$

Hence the vectors **a**, **b** and **c** are linearly dependent.

(b) Substituting (2) into (1) we have

$$\mathbf{a} - 2\mathbf{b} + \mathbf{c} = \mathbf{0}, \qquad \text{i.e.,} \qquad \mathbf{c} = -\mathbf{a} + 2\mathbf{b}.$$

19. **(10 marks)** Consider a circle centered at the point O, with radius 1. Let AB be a diameter. Consider an arbitrary point（任意点）P on the circle. Denote the vector $\overrightarrow{OP}$ to be $\mathbf{a}$, and $\overrightarrow{OB}$ to be $\mathbf{b}$, and denote $\angle BOP$ as $\angle\alpha$, and $\angle BAP$ as $\angle\theta$.



(a) Use $\mathbf{a}$ and $\mathbf{b}$ to express $\cos\alpha$.

(b) Use $\mathbf{a}$ and $\mathbf{b}$ to express $\cos\theta$.

(c) Try to prove the following relation in terms of the results of (a) and (b):

$$\angle\alpha = 2\angle\theta.$$

Solution:

(a) Noticing $|\mathbf{a}| = |\mathbf{b}| = 1$, we have

$$\cos\alpha = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \mathbf{a}\cdot\mathbf{b},$$

(b)

$$\cos\theta = \frac{\overrightarrow{AP}\cdot\overrightarrow{AB}}{\left|\overrightarrow{AP}\right|\left|\overrightarrow{AB}\right|} = \frac{(\mathbf{a}+\mathbf{b})\cdot(2\mathbf{b})}{|\mathbf{a}+\mathbf{b}|\,|2\mathbf{b}|} = \frac{\mathbf{a}\cdot\mathbf{b}+|\mathbf{b}|^2}{|\mathbf{a}+\mathbf{b}|\,|\mathbf{b}|} = \frac{\mathbf{a}\cdot\mathbf{b}+1}{|\mathbf{a}+\mathbf{b}|}.$$

(c)

$$\begin{aligned} 2\cos^2\theta - 1 &= 2\left(\frac{\mathbf{a}\cdot\mathbf{b}+1}{|\mathbf{a}+\mathbf{b}|}\right)^2 - 1 = 2\frac{(\mathbf{a}\cdot\mathbf{b})^2+1+2(\mathbf{a}\cdot\mathbf{b})}{|\mathbf{a}+\mathbf{b}|^2} - 1 \\ &= \frac{1}{|\mathbf{a}+\mathbf{b}|^2}\left[2(\mathbf{a}\cdot\mathbf{b})^2+2+4(\mathbf{a}\cdot\mathbf{b})-(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})\right] \end{aligned}$$

where $|\mathbf{a}+\mathbf{b}|^2 = 2+2(\mathbf{a}\cdot\mathbf{b})$, and

$$\begin{aligned} & 2(\mathbf{a}\cdot\mathbf{b})^2+2+4(\mathbf{a}\cdot\mathbf{b})-(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) \\ = \; & 2(\mathbf{a}\cdot\mathbf{b})^2+2+4(\mathbf{a}\cdot\mathbf{b})-(2+2\mathbf{a}\cdot\mathbf{b}) = 2(\mathbf{a}\cdot\mathbf{b})^2+2(\mathbf{a}\cdot\mathbf{b}). \end{aligned}$$

Hence

$$2\cos^2\theta - 1 = \frac{2(\mathbf{a}\cdot\mathbf{b})^2+2(\mathbf{a}\cdot\mathbf{b})}{2+2(\mathbf{a}\cdot\mathbf{b})} = \mathbf{a}\cdot\mathbf{b} = \cos\alpha,$$

therefore,

$$\angle\alpha = 2\angle\theta.$$

20. (10 marks) Let Σ be a plane in 3 dimensions, and P a point outside Σ,

$$\Sigma : 2x - y - z + 2 = 0, \qquad\qquad P\,(1, 0, 1)\,.$$

Find the symmetric point of P about the plane Σ.

Solution 1:

Suppose the mirror point of P is $P'\,(x, y, z)$. Then:

(i) $\overrightarrow{PP'}$ should be parallel to the normal vector of Σ, i.e., $\mathbf{v} = (2, -1, -1)$;

(ii) The mid-point between P and P' should be on the plane Σ.

Hence we have

$$\begin{cases} x - 1 &= 2t \\ y - 0 &= -t \\ z - 1 &= -t \end{cases}, \qquad \text{and} \qquad 2\left(\frac{x+1}{2}\right) - \left(\frac{y+0}{2}\right) - \left(\frac{z+1}{2}\right) + 2 = 0,$$

i.e.,

$$\begin{cases} x = 1 + 2t, \\ y = -t, \\ z = 1 - t, \\ 2x - y - z + 5 = 0. \end{cases}$$

The solution to this coupled set of equations is $t = -1$, hence

$$x = -1, \qquad y = 1, \qquad z = 2, \qquad \text{i.e.,} \qquad (x, y, z) = (-1, 1, 2)\,.$$

Solution 2:

The normal direction of Σ is $\mathbf{v} = \pm(2,-1,-1)$. Let $Q(x_0,y_0,z_0)$ be an arbitrary point on Σ. Then the distance between P and Σ is computed as

$$\begin{aligned} d &= \left|\overrightarrow{PQ}\cdot\hat{\mathbf{v}}\right| = \left|\left[(x_0-1)\,\mathbf{i} + y_0\mathbf{j} + (z_0-1)\,\mathbf{k}\right]\cdot\frac{(2\mathbf{i}-\mathbf{j}-\mathbf{k})}{\sqrt{6}}\right| \\ &= \frac{1}{\sqrt{6}}\left|2(x_0-1) - y_0 - (z_0-1)\right| = \frac{1}{\sqrt{6}}\left|2x_0 - y_0 - z_0 - 1\right| = \frac{1}{\sqrt{6}}\left|-2-1\right| = \frac{\sqrt{6}}{2}. \end{aligned}$$

Now, letting the required mirror point of P be $P'(x,y,z)$, we should have

(a) $\overrightarrow{PP'} = t\mathbf{v}$, that is

$$\frac{x-1}{2} = \frac{y}{-1} = \frac{z-1}{-1}. \tag{3}$$

(b) $d = \left|\overrightarrow{P'Q}\cdot\hat{\mathbf{v}}\right|$, i.e.,

$$\begin{aligned} \frac{\sqrt{6}}{2} &= \left|\frac{(2\mathbf{i}-\mathbf{j}-\mathbf{k})}{\sqrt{6}}\cdot\left[(x-x_0)\,\mathbf{i} + (y-y_0)\,\mathbf{j} + (z-z_0)\,\mathbf{k}\right]\right| \\ &= \frac{1}{\sqrt{6}}\left|2(x-x_0) - (y-y_0) - (z-z_0)\right| \\ &= \frac{1}{\sqrt{6}}\left|(2x-y-z) - (2x_0-y_0-z_0)\right| = \frac{1}{\sqrt{6}}\left|2x-y-z+2\right|, \end{aligned}$$

i.e., $|2x-y-z+2| = 3$.

Case I: $2x-y-z+2 = 3$. This case is just the original point $(1,0,1)$. Hence abandoned.

Case II: $2x-y-z+2 = -3$. This equation can be coupled to the above (3):

$$2x-y-z+2 = -3, \qquad x-1 = -2y, \qquad y = z-1,$$

whose solution is $(x,y,z) = (-1,1,2)$. Therefore, **the mirror point is**

$$\underline{P' = (-1,1,2)\,.}$$

Linear Algebra

EXAMINATION BOOK

Pledger: ____________________ Class NO: ____________________

BJUT Student ID: ______________ UCD Student ID: ______________

SECTION A

Circle the preferred answer.

If you make a mistake, mark a cross through your wrong choice and circle your next alternative.

1.	a	b	c	d
2.	a	b	c	d
3.	a	b	c	d
4.	a	b	c	d
5.	a	b	c	d
6.	a	b	c	d
7.	a	b	c	d
8.	a	b	c	d
9.	a	b	c	d
10.	a	b	c	d
11.	a	b	c	d
12.	a	b	c	d
13.	a	b	c	d
14.	a	b	c	d
15.	a	b	c	d