Beijing-Dublin International College

SEMESTER I FINAL EXAMINATION – 2018/2019

School of Mathematics and Statistics
BDIC1014J & BDIC1044J Linear Algebra

HEAD OF SCHOOL: Wenying WU
MODULE COORDINATOR: Xin LIU

Time Allowed: 90 minutes

Instructions for Candidates

Answer ALL questions. The marks that each question carry is written as shown.

BJUT Student ID: ________________ **UCD Student ID:** ________________

I have read and clearly understand the Examination Rules of both Beijing University of Technology and University College Dublin. I am aware of the Punishment for Violating the Rules of Beijing University of Technology and/or University College Dublin. I hereby promise to abide by the relevant rules and regulations by not giving or receiving any help during the exam. If caught violating the rules, I accept the punishment thereof.

Honesty Pledge: ____ **(Signature)**

Instructions for Invigilators

Non-programmable calculators are permitted. NO dictionaries are permitted.
No rough-work paper is to be provided for candidates.

SECTION A — MULTIPLE CHOICE QUESTIONS

In each question, choose **at most one** option.

Circle the preferred choice on the **Examination Book** provided.

This section is worth a total of **30** marks, with each question worth **3** marks.

1. For a linear system $\begin{cases} x \; -y \; +z = -2, \\ \quad\; y \; -2z = 1, \\ -x \qquad\quad z = -1, \end{cases}$ determine the number of its solution(s):

(a) inconsistent; (b) unique solution; (c) two solutions; (d) infinitely many solutions.

2. A matrix $M = \begin{pmatrix} 1 & k-2 \\ k-2 & 1 \end{pmatrix}$ is nonsingular, if and only if

(a) $k = 1$; (b) $k = 3$; (c) $k \neq 1$ or $k \neq 3$; (d) $k \neq 1$ and $k \neq 3$.

3. For a matrix $A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 0 & 0 & -1 \\ 0 & 0 & 5 \end{pmatrix}$, we have rank $A =$ _______________.

(a) 1; (b) 2; (c) 3; (d) 4.

4. Supposing A and B are two 3×3 matrices, with

$$\det A = 2, \qquad \det B = \frac{1}{3},$$

we have $\det\left(-2AB^T\right) =$ _______________.

(a) $\frac{4}{3}$; (b) -48; (c) $-\frac{4}{3}$; (d) $-\frac{16}{3}$.

5. Let I be an identity matrix, and A an $n \times n$ matrix satisfying $A^2 = A + I$. Then A^{-1} is

(a) $-A - I$; (b) $A + I$; (c) $A - I$; (d) $-A + I$.

6. Let A and B be two $n \times n$ matrices. Which of the following statements holds true for always?

(a) $(AB)^2 = A^2B^2$; (b) $(A+B)^2 = A^2 + 2AB + B^2$; (c) $\frac{A}{B}B = A$; (d) $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

7. Given that

$$\begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} + \begin{pmatrix} 5 \\ -2 \\ 8 \end{pmatrix} - 3\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 9 \\ -4 \\ 8 \end{pmatrix}$$

try to solve the unknowns (x, y, z) in the equation

$$\begin{pmatrix} 3 & 1 & 9 \\ -3 & 1 & -4 \\ 3 & -3 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ 2 \\ -8 \end{pmatrix}$$

(a) $(1, 1, -1)$; (b) $(1, 1, -3)$; (c) $(1, 3, 1)$; (d) $(1, -1, -1)$.

8. Let A be an $n \times n$ matrix. Let λ be an eigenvalue of A, with $\vec{v}$ the eigenvector corresponding to λ. Then which of the following is an eigenvalue of A^m? $(m \in \mathbb{Z},\ \ m \geq 1)$

(a) λ; (b) λ^{-m}; (c) λ^m; (d) λ^{m-1}.

9. Given $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 1$, compute $\begin{vmatrix} 3a_{11} & 2a_{11} & -a_{12} + a_{13} \\ 3a_{21} & 2a_{21} & -a_{22} + a_{23} \\ 3a_{31} & 2a_{31} & -a_{32} + a_{33} \end{vmatrix} =$ __________.

(a) 0; (b) 2; (c) 3; (d) 6.

10. Let A be a 3×3 matrix. If

$$A\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} - 3a_{31} & a_{22} - 3a_{32} & a_{23} - 3a_{33} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

try to determine $A =$__________.

(a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}$; (b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$; (c) $\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; (d) $\begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

SECTION B — GAP-FILLING QUESTIONS

Write your answers on the **Examination Book** provided. Only **brief** answers are needed. This section is worth a total of **20** marks, with each question worth **4** marks.

11. Consider an invertible matrix $M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. Let A_{ij} be the cofactor of a_{ij}, $i, j = 1, 2$. Then:

(a) $a_{11}A_{11} + a_{12}A_{12} =$ ________________,

(b) $a_{11}A_{21} + a_{12}A_{22} =$ ________________.

12. Consider a 5×5 matrix A, with five eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$, $\lambda_4 = 4$ and $\lambda_5 = 5$. Try to evaluate the trace: $\operatorname{Tr} A =$ ________________.

13. Given that

$$\begin{pmatrix} 7 & 11 & 5 & 2 \\ 0 & -1 & 4 & 3 \\ -3 & 7 & 13 & 1 \\ 2 & -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & -1 & 1 & 0 \\ 3 & 2 & 11 & 7 \\ 9 & -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 40 & -3 & 59 & 51 \\ 39 & 6 & 43 & 31 \\ 45 & 18 & 153 & 86 \\ 11 & 1 & -4 & 5 \end{pmatrix}$$

try to compute

$$\begin{pmatrix} 1 & 0 & 3 & 9 \\ 0 & -1 & 2 & -1 \\ -1 & 1 & 11 & 0 \\ 2 & 0 & 7 & 1 \end{pmatrix} \begin{pmatrix} 7 & -3 & 2 \\ 11 & 7 & -2 \\ 5 & 13 & 0 \\ 2 & 1 & 1 \end{pmatrix} = \underline{\hspace{4cm}}.$$

14. Let A be an orthogonal matrix,

$$A = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{pmatrix}.$$

By definition, we immediately have its inverse $A^{-1} =$ _______________.

15. The *Fibonacci sequence* are the numbers in the following integer sequence:

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ \cdots.$$

Denoting the i^{th} Fibonacci number as F_i, this sequence can be generated by the following recursive relation:

$$F_{n+2} = F_{n+1} + F_n, \qquad \text{with initial numbers } F_1 = F_2 = 1, \quad \text{and } n \in \mathbb{Z}, \quad n \geq 0.$$

This recursive relation can be expressed in terms of a matrix M,

$$\begin{pmatrix} F_{n+2} \\ F_{n+1} \end{pmatrix} = M \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}, \qquad \text{thus} \qquad \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = M^n \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}, \qquad \text{where } n \geq 0.$$

This matrix M is given by $M =$ _______________.

SECTION C — EXTENDED ANSWER QUESTIONS

Write your answers on the **Examination Book** provided.
This section is worth a total of **50** marks. The marks of each question are as shown.

16. (9 marks) Use three methods to solve the following linear system:

(a) row operations; *(3 marks)*

(b) inverse of the coefficient matrix, i.e., $\vec{x} = M^{-1}\vec{b}$; *(3 marks)*

(c) the Cramer's rule. *(3 marks)*

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

17. (7 marks) (*Elementary matrices*) A transformation, for instance, $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix}$, can be realized by left-multiplying an elementary matrix as

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix}.$$

(a) Find appropriate elementary matrices to realize the following transformation: *(3 marks)*

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix} \longrightarrow \begin{pmatrix} 7 & 8 \\ 5 & 6 \\ 3 & 4 \\ 1 & 2 \end{pmatrix}$$

(b) Try to write out the *lower-upper (LU) decomposition* of the following matrix A, i.e., try to *LU-decompose* the matrix A: *(4 marks)*

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}.$$

18. (9 marks)

(a) Evaluate the following power of matrix: *(3 marks)*

$$A^{2019} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ & \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}^{2019}.$$

(b) Evaluate the following power of matrix: *(6 marks)*

$$B^{2019} = \begin{pmatrix} 5 & -10 & 5 \\ 3 & -6 & 3 \\ 2 & -4 & 2 \end{pmatrix}^{2019}.$$

*Hint: Try to write this matrix B as a column matrix times a row matrix.

19. (12 marks) Let M be a 4×4 matrix,

$$M = \begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 2 & -1 & 0 \\ 1 & 0 & -2 & 1 \\ -1 & 2 & 0 & -1 \end{pmatrix}.$$

Compute the following <u>adjoint of adjoint matrix</u>:

$$\operatorname{adj}(\operatorname{adj} M).$$

20. (13 marks) Use the method of <u>matrix diagonalization</u> to find the following power of matrix A:

$$A^{10} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{10}.$$

Glossary

Adjoint matrix	伴随矩阵
Coefficient	系数
Column matrix	列矩阵
Cramer's rule	克莱姆法则
Determinant	行列式
Diagonalization	对角化
Eigenvalue	本征值
Eigenvector	本征矢量
Elementary matrix	初等矩阵
Fibonacci sequence	斐波那契数列
Identity (matrix)	恒等（矩阵）
Inconsistent	无解，不相容
Inverse	逆（矩阵）
Invertible	可逆
Linearly dependent	线性相关，线性依赖
Linearly independent	线性独立
Lower-upper decomposition	上下分解
Nonsingular	非奇异
Orthogonal	正交
Rank	秩
Recursive relation	递推关系
Row operations	行操作
Unique	唯一