Beijing-Dublin International College

SEMESTER I FINAL EXAMINATION – 2019/2020

BDIC1014J & BDIC1044J Linear Algebra

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Time Allowed: 90 minutes

Instructions for Candidates

Answer ALL questions. The marks that each question carry are written as shown.

BJUT Student ID: ________________ **UCD Student ID:** ________________

I have read and clearly understand the Examination Rules of both Beijing University of Technology and University College Dublin. I am aware of the Punishment for Violating the Rules of Beijing University of Technology and/or University College Dublin. I hereby promise to abide by the relevant rules and regulations by not giving or receiving any help during the exam. If caught violating the rules, I accept the punishment thereof.

Honesty Pledge：____ **(Signature)**

Instructions for Invigilators

Non-programmable calculators are permitted. NO dictionaries are permitted.
No rough-work paper is to be provided for candidates.

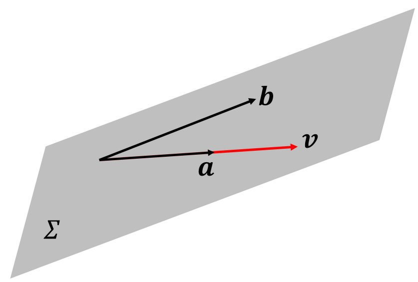
SECTION A — MULTIPLE CHOICE QUESTIONS

In each question, choose **at most one** option.

Circle the preferred choice on the **Examination Book** provided.

This section is worth a total of **20** marks, with each question worth **4** marks.

1. Let Σ be a plane in 3 dimensions, as shown below. Let $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{v}$ be three non-zero vectors in Σ: $\boldsymbol{a} = (a_1 \;\; a_2 \;\; a_3)^T$, $\boldsymbol{b} = (b_1 \;\; b_2 \;\; b_3)^T$ and $\boldsymbol{v} = (v_1 \;\; v_2 \;\; v_3)^T$, where $\boldsymbol{a}$ and $\boldsymbol{v}$ are collinear, $\boldsymbol{a} \parallel \boldsymbol{v}$.



Which of the following statements is CORRECT?

(a) $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{v}$ are linearly independent.

(b) $\boldsymbol{a} \in \mathrm{span}\{\boldsymbol{v}\}$, but $\boldsymbol{a} \notin \mathrm{span}\{\boldsymbol{b}, \boldsymbol{v}\}$.

(c) Σ, as a 2-dimensional sub-space, can be generated by $\{\boldsymbol{a}, \boldsymbol{b}\}$.

(d) Σ can be generated by $\{\boldsymbol{a}, \boldsymbol{v}\}$.

2. Consider the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{v}$ in **Question 1**. Which of the following statements is CORRECT?

(a) For the linear system $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, the unknowns $x \neq 0$ and $y = 0$.

(b) For the linear system $\begin{pmatrix} a_1 & v_1 \\ a_2 & v_2 \\ a_3 & v_3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, the solution $x = 0$ and $y \neq 0$.

(c) For the linear system $x\boldsymbol{a} + y\boldsymbol{v} = \boldsymbol{b}$, the unknowns $x = 0$ and $y \neq 0$.

(d) None of the above is correct.

3. Let I be an identity matrix, and A an $n \times n$ matrix satisfying $A^2 = A - I$. Then A^{-1} is

(a) $-A - I$; (b) $A + I$; (c) $A - I$; (d) $-A + I$.

4. Let A and B be two invertible $n \times n$ matrices. Which of the following statements are always true?

(a) $(AB)^2 = A^2B^2$; (c) $(A+B)^2 = A^2 + 2AB + B^2$;

(b) $\det(-AB) = -\det(AB)$; (d) $\mathrm{Tr}(AB^{-1}) = \mathrm{Tr}(B^{-1}A)$.

5. Let A be a 3×3 matrix. If

$$A \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} - 3a_{31} & a_{22} - 3a_{32} & a_{23} - 3a_{33} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

try to determine $A =$__________.

(a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}$; (b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$; (c) $\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; (d) $\begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

SECTION B — GAP-FILLING QUESTIONS

Write your answers on the **Examination Book** provided. Only **brief** answers are needed. This section is worth a total of **40** marks, with each question worth **4** marks.

6. Let $M = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$ be a 4×4 matrix, with $\det M = 7$. Let A_{ij} be the cofactor of the entry a_{ij}, $i, j = 1, 2, 3, 4$. Compute the following matrix multiplication:

$$\begin{pmatrix} A_{41} & A_{42} & A_{43} & A_{44} \\ A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \end{pmatrix} \begin{pmatrix} a_{31} & a_{11} & a_{21} & a_{41} \\ a_{32} & a_{12} & a_{22} & a_{42} \\ a_{33} & a_{13} & a_{23} & a_{43} \\ a_{34} & a_{14} & a_{24} & a_{44} \end{pmatrix} = \underline{\qquad\qquad}$$

7. A square matrix A is **idempotent** if $A^2 = A$. If A is a non-zero idempotent matrix find all values of k such that $(I - kA)$ is also idempotent.

$k = \underline{\qquad\qquad}$

8. Consider the matrix

$$A = \begin{pmatrix} 2 & 4 & 1 \\ -1 & 1 & -2 \\ 1 & 1 & 1 \\ -1 & -3 & 0 \end{pmatrix}, \quad \operatorname{rank} A = \underline{\qquad\qquad}$$

9. Consider the vectors $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j}$, $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{c} = -\boldsymbol{i} - 3\boldsymbol{k}$.

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \underline{\qquad\qquad}$$

10. Consider an $n \times n$ diagonalisable matrix with characteristic polynomial

$$p(\lambda) = (\lambda - 1)^4(\lambda - 2)^3(\lambda - 3)^2(\lambda - 4)\,.$$

What is the value of n? _____________

11. Find the eigenvalues λ_1 and λ_2 of the matrix

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad 0 \le \theta < 2\pi\,.$$

$\lambda_1, \lambda_2 =$ _____________

12. A matrix A has an eigenvector $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ with eigenvalue 3, and an eigenvector $v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ with eigenvalue -2.

Find $A^2 \begin{pmatrix} 4 \\ 3 \end{pmatrix} =$ _____________

13. Consider a 2×2 matrix A which satisfies $\operatorname{Tr} A = 3$ and $\det A = 2$.

Find the eigenvalues of A: _____________

14. Find an elementary matrix E which realises the following transformation:

$$\begin{pmatrix} 3 & 2 \\ 4 & 1 \\ 4 & 7 \\ 7 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 2 \\ 4 & 1 \\ -5 & 1 \\ 7 & 3 \end{pmatrix}$$

$E =$ _____________

15. Consider the matrix

$$A = \begin{pmatrix} 0 & 0 & c \\ 0 & b & 0 \\ a & 0 & 0 \end{pmatrix}.$$

If A is an orthogonal matrix, find the values of a, b and c: _____________.

SECTION C — EXTENDED ANSWER QUESTIONS

Write your answers on the **Examination Book** provided.
This section is worth a total of **40** marks. The marks of each question are as shown.

16. (7 marks) Try to find M^{-1}, where the matrix M is given by

$$M = \frac{1}{75}\begin{pmatrix} +\begin{vmatrix} 3 & 2 & 1 \\ -1 & 2 & -2 \\ -3 & -1 & -1 \end{vmatrix} & -\begin{vmatrix} -3 & -1 & 4 \\ -1 & 2 & -2 \\ -3 & -1 & -1 \end{vmatrix} & +\begin{vmatrix} -3 & -1 & 4 \\ 3 & 2 & 1 \\ -3 & -1 & -1 \end{vmatrix} & -\begin{vmatrix} -3 & -1 & 4 \\ 3 & 2 & 1 \\ -1 & 2 & -2 \end{vmatrix} \\ \\ -\begin{vmatrix} -3 & 2 & 1 \\ 3 & 2 & -2 \\ 2 & -1 & -1 \end{vmatrix} & +\begin{vmatrix} 3 & -1 & 4 \\ 3 & 2 & -2 \\ 2 & -1 & -1 \end{vmatrix} & -\begin{vmatrix} 3 & -1 & 4 \\ -3 & 2 & 1 \\ 2 & -1 & -1 \end{vmatrix} & +\begin{vmatrix} 3 & -1 & 4 \\ -3 & 2 & 1 \\ 3 & 2 & -2 \end{vmatrix} \\ \\ +\begin{vmatrix} -3 & 3 & 1 \\ 3 & -1 & -2 \\ 2 & -3 & -1 \end{vmatrix} & -\begin{vmatrix} 3 & -3 & 4 \\ 3 & -1 & -2 \\ 2 & -3 & -1 \end{vmatrix} & +\begin{vmatrix} 3 & -3 & 4 \\ -3 & 3 & 1 \\ 2 & -3 & -1 \end{vmatrix} & -\begin{vmatrix} 3 & -3 & 4 \\ -3 & 3 & 1 \\ 3 & -1 & -2 \end{vmatrix} \\ \\ -\begin{vmatrix} -3 & 3 & 2 \\ 3 & -1 & 2 \\ 2 & -3 & -1 \end{vmatrix} & +\begin{vmatrix} 3 & -3 & -1 \\ 3 & -1 & 2 \\ 2 & -3 & -1 \end{vmatrix} & -\begin{vmatrix} 3 & -3 & -1 \\ -3 & 3 & 2 \\ 2 & -3 & -1 \end{vmatrix} & +\begin{vmatrix} 3 & -3 & -1 \\ -3 & 3 & 2 \\ 3 & -1 & 2 \end{vmatrix} \end{pmatrix}$$

17. (9 marks) Use three methods to solve the following linear system:

$$\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}.$$

(a) row operations; *(3 marks)*

(b) inverse of coefficient matrix, i.e., $\boldsymbol{x} = M^{-1}\boldsymbol{b}$, where $\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$; *(3 marks)*

(c) Cramer's rule. *(3 marks)*

18. (10 marks) The Taylor series of an exponential function e^α is given by

$$e^\alpha = \sum_{n=0} \frac{1}{n!}\alpha^n = 1 + \alpha + \frac{1}{2!}\alpha^2 + \frac{1}{3!}\alpha^3 + \cdots,$$

where $\alpha \in \mathbb{R}$. Replacing α with an $n \times n$ matrix A, we have a similar formula

$$e^A = \sum_{n=0} \frac{1}{n!}A^n = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots.$$

Try to prove: If A is diagonalisable, then

$$\ln\left[\det\left(e^A\right)\right] = \operatorname{Tr} A, \qquad \ln \text{ — natural logarithm.}$$

*Hint: Take benefit of the following formulae

$$\ln(AB) = \ln A + \ln B, \quad \ln\left(e^A\right) = A.$$

19. (14 marks) Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Use the method of matrix diagonalization to show that

$$A^{n+1} - A^n = 2^{n-1}(A^2 - A).$$

Glossary

Adjoint matrix	伴随矩阵
Characteristic polynomial	示性多项式
Coefficient	系数
Cofactor	代数余子式
Collinear	共线
Cramer’s rule	克莱姆法则
Diagonalizable	可对角化的
Diagonalization	对角化
Eigenvalue	本征值
Eigenvector	本征矢量
Elementary matrix	初等矩阵
Exponential function	指数函数
Generate	生成
Idempotent	等幂的
Invertible	可逆
Linearly dependent	线性相关，线性依赖
Linearly independent	线性独立
Logarithm	对数
Orthogonal	正交
Rank	秩
Row operations	行操作
Span	张开
Taylor series	泰勒级数
Unknown	未知数