Beijing-Dublin International College

SEMESTER II FINAL EXAMINATION – 2017/2018

School of Mathematics and Statistics

BDIC1027J Maths 3 (Advanced Mathematics; Finance)

HEAD OF SCHOOL: Wenying Wu
MODULE LECTURER: Xin Liu

Time Allowed: 90 minutes

Instructions for Candidates

Answer ALL questions. The marks that each question carry is written as shown.

BJUT Student ID: ________________ **UCD Student ID: ________________**

I have read and clearly understand the Examination Rules of both Beijing University of Technology and University College Dublin. I am aware of the Punishment for Violating the Rules of Beijing University of Technology and/or University College Dublin. I hereby promise to abide by the relevant rules and regulations by not giving or receiving any help during the exam. If caught violating the rules, I accept the punishment thereof.

Honesty Pledge：____ (Signature)

Instructions for Invigilators

Non-programmable calculators are permitted. NO dictionaries are permitted.
No rough-work paper is to be provided for candidates.

SECTION A — MULTIPLE CHOICE QUESTIONS

In each question, choose **at most one** option.
Write your answers on the **Examination Book** provided.
This section is worth a total of **45** marks, with each question worth **3** marks.

1. Express the complex number $\frac{i}{1-i}$ in the Cartesian form:

(a) $\frac{1}{2} - \frac{i}{2}$; (b) $1 + \frac{i}{2}$; (c) $-1 + \frac{i}{2}$; (d) $-\frac{1}{2} + \frac{i}{2}$.

2. Consider two complex numbers: $z_1 = 6\left(\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right)$ and $z_2 = 3e^{i\frac{\pi}{3}}$. Evaluate the quotient $\frac{z_1}{z_2}$.

(a) -2, (d) 2, (c) $2\left(\cos\frac{\pi}{3} - \sin\frac{\pi}{3}\right)$, (b) $2e^{i\frac{\pi}{3}}$.

3. Let $f(x)$ be a periodic function with a typical period $[-\pi, \pi]$, which can be expanded into a Fourier series

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos nx + \sum_{n=1}^{\infty} B_n \sin nx, \qquad f(x) \not\equiv 0.$$

Which of the following statement is correct? ($n \in \mathbb{Z}$, $n \geq 1$ below)

(a) If $f(x)$ is an odd function, $A_0 \neq 0$ and $B_n = 0$.
(b) If $f(x)$ is an odd function, $A_0 = 0$ and $A_n = 0$.
(c) If $f(x)$ is an even function, $A_n \neq 0$ and $B_n \neq 0$.
(d) If $f(x)$ is an even function, $A_0 = 0$, $A_n = 0$ and $B_n = 0$.

4. Consider a periodic function $f(x) = \sin\left(x + \frac{\pi}{4}\right)$ with period $T = 2\pi$. If its Fourier series reads

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos nx + \sum_{n=1}^{\infty} B_n \sin nx,$$

we have the coefficients:

(a) $A_0 = 0$; $A_n = 0$ for $n = 1, 2, 3, \cdots$; $B_1 = \frac{\sqrt{2}}{2}$, and $B_m = 0$ for $m = 2, 3, 4, \cdots$.

(b) $A_0 = \frac{\sqrt{2}}{2}$; $A_n = 0$ for $n = 1, 2, 3, \cdots$; $B_1 = 1$, and $B_m = 0$ for $m = 2, 3, 4, \cdots$.

(c) $A_0 = 0$; $A_1 = \frac{\sqrt{2}}{2}$, $B_1 = \frac{\sqrt{2}}{2}$; and $A_m = B_m = 0$ for $m = 2, 3, 4, \cdots$.

(d) $A_0 = \frac{\sqrt{2}}{2}$; $A_n = B_n = 0$ for $n = 1, 2, 3, \cdots$.

5. Regarding the following Fourier transform formulae

$$\begin{cases} f(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(\omega) e^{i\omega t} d\omega, \\ G(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt, \end{cases}$$

which statement/understanding is NOT correct?

(a) Fourier transform can be done for not only an aperiodic function, but also for a periodic function actually.

(b) For an aperiodic function, "aperiodic" means its period $T \to 0$.

(c) If regarding $f(t)$ as an original function, then $G(\omega)$ is the corresponding kernel of the Fourier transform. On the contrary, if $G(\omega)$ is regarded as the original function, then $f(t)$ becomes the kernel of the transform.

(d) Fourier transform implies a duality relationship between t and ω, and that between $f(t)$ and $G(\omega)$.

6. Find the plane passing through three points $A\,(2, -2, 1)$, $B\,(1, 3, -2)$ and $C\,(1, 2, -1)$ in space.

(a) $2x + y + z - 3 = 0$;

(b) $x + 2y + z + 3 = 0$;

(c) $\frac{x-2}{2} = y + 2 = \frac{z-1}{2}$;

(d) $\mathbf{r} = (\mathbf{i} + 2\mathbf{j} - \mathbf{k}) + t\,(2\mathbf{i} + \mathbf{j} + \mathbf{k})$.

7. Determine the shape of the surface described by the following equation

$$2x^2 + y^2 + 2z^2 + 8x - 2y + 4z + 3 = 0.$$

(a) A cone; (b) A paraboloid; (c) A hyperboloid; (d) An ellipsoid.

8. The surface of a revolution body Σ is generated by rotating a curve C about a symmetric axis L in space. If the equation of Σ is given by

$$z = e^{x^2+y^2} - \sin\sqrt{x^2+y^2},$$

determine the curve C and the symmetric axis L.

(a) C could be $z = e^{x^2} - \sin|x|$, and L the z-axis.

(b) C could be $y = e^{x^2} - \sin|z|$, and L the z-axis.

(c) C could be $y = e^{z^2} - \sin|z|$, and L the y-axis.

(d) C could be $z = e^{y^2} - \sin|y|$, and L the x-axis.

9. Which of the following statements is correct?

(a) On the one-dimensional x-axis, there is only 1 path to approach a point x_0;

(b) On the one-dimensional x-axis, there are only 2 paths to approach a point x_0;

(c) In the two-dimensional xy-plane, there are only 2 paths to approach a point (x_0, y_0);

(d) In the three-dimensional space, there are only 3 paths to approach a point (x_0, y_0, z_0).

10. Let C be a curve in space

$$C : \left\{ \begin{array}{rcl} z & = & \sin\left(x^2+y^2\right), \\ z & = & x+2y. \end{array} \right.$$

Find the equation of the projection of C onto the xy-plane.

(a) $\left\{ \begin{array}{rcl} \sin\left(x^2+y^2\right) & = & 0, \\ z & = & x+2y. \end{array} \right.$ (b) $\left\{ \begin{array}{rcl} \sin\left(x^2+y^2\right) & = & x+2y, \\ x & = & 0. \end{array} \right.$

(c) $\left\{ \begin{array}{rcl} \sin\left(x^2+y^2\right) & = & 0, \\ x+2y & = & 0. \end{array} \right.$ (d) $\left\{ \begin{array}{rcl} \sin\left(x^2+y^2\right) & = & x+2y, \\ z & = & 0. \end{array} \right.$

11. Let C be a horizontal curve given by $\begin{cases} e^{3(x+y)}\cos(x-y) &= 1, \\ \qquad\qquad z &= 2. \end{cases}$ Suppose Σ is a cone formed by moving a line L along C — that is, regarding C as a trajectory — while keeping L passing through the origin point $(0,0,0)$. Determine the cone Σ.

(a) $(0,0,0)\cup\left\{(x,y,z)\middle| e^{3(x+y)}\cos(x-y)=1,\quad z\neq 0\right\}$;

(b) $(0,0,0)\cup\left\{(x,y,z)\middle| e^{\frac{6}{z}(x+y)}\cos\frac{2(x-y)}{z}=1,\quad z\neq 0\right\}$;

(c) $(0,0,0)\cup\left\{(x,y,z)\middle| e^{\frac{3z}{2}(x+y)}\cos\frac{z(x-y)}{2}=1,\quad z\neq 0\right\}$;

(d) $(0,0,0)\cup\left\{(x,y,z)\middle| 3e^{3(x+y)}\cos(x-y)-e^{3(x+y)}\sin(x-y)=0,\quad z\neq 0\right\}$.

12. Find the natural domain of the following function in $\mathbb{R}^2$:

$$z=f(x,y)=\sqrt{\frac{1}{x^2+y^2-2x-8}}.$$

(a) Let C be the circle centered at $(0,0)$ with radius $2\sqrt{2}$. The domain is the region outside and including C;

(b) Let C be the circle centered at $(0,0)$ with radius $2\sqrt{2}$. The domain is the region outside but not including C;

(c) Let C be the circle centered at $(1,0)$ with radius 3. The domain is the region outside but not including C;

(d) The domain is the upper z-axis including the origin point.

13. Evaluate the limit:

$$\lim_{(x,y)\to(0,0)}\frac{\sqrt{3}xy}{x+\sqrt{3}y}.$$

(a) 0; (b) $\frac{\sqrt{3}}{2}$; (c) $\sqrt{3}$; (d) The limit does not exist.

14. Evaluate the limit:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{3\sqrt{x^2+y^2}}.$$

(a) 0; (b) $\frac{1}{3}$; (c) ∞; (d) The limit does not exist.

15. If $f(x,y) = 4x^2y - y^3x + y^2$, then $f_y(1,-1)$ equals

(a) -9; (b) 1; (c) -1; (d) 3.

SECTION B — GAP-FILLING QUESTIONS

Write your answers on the **Examination Book** provided.
This section is worth a total of **15** marks, with each question worth **3** marks.

16. Consider a complex number $z = 3 + 4i$. Let $|z|$ and $\bar{z}$ be the modulus and complex conjugate of z, respectively. Then compute

$$|z|\bar{z} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_.$$

17. Evaluate the limit:

$$\lim_{(x,y)\to(0,-1)} \frac{e^{2xy}\sin(2xy)}{x} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_.$$

18. The total differential of a two-variable function $z = e^{x^2 - \sin y}$ is: ______________

19. Let $f(x,y) = ye^{\sin x} + xe^{\sin y}$. The second order partial derivative $f_{yy} = $ ______________.

20. For $f(x,y) = \ln\sin\left(x^3y^2\right)$, find its partial derivative f_y over the natural domain.

$$f_y = \_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

SECTION C — EXTENDED ANSWER QUESTIONS

Write your answers on the **Examination Book** provided.
This section is worth a total of **40** marks. The marks of each question are as shown.

21. **(8 marks)** Consider a real coefficient polynomial equation in the unknown z:

$$z^5 - 2z^4 + 6z^3 - 2z^2 + 5z = 0, \qquad \text{where} \quad z \in \mathbb{C}.$$

If $z_1 = i$ is a root of this equation, try to find all the other roots.

22. **(7 marks)** Let l be a line and $\mathbf{v}$ a direction in space,

$$l: \quad \frac{x-2}{2} = y + 1 = -z, \qquad \mathbf{v} = \mathbf{j} - \mathbf{k}.$$

Find the plane swapped by l along the direction $\pm\mathbf{v}$; i.e., the plane formed by moving l along $\pm\mathbf{v}$.

23. **(5 marks)** Let C_1 and C_2 be two surfaces in space:

$$C_1: \quad z = x^2 + (y-1)^2 + 2, \qquad C_2: \quad x^2 + (y-1)^2 + (z+3)^2 = 1.$$

Find the shortest distance between C_1 and C_2.

24. (12 marks) (*Thermal conductivity*)

Consider a one-dimensional heat equation

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = \Phi(x,t),$$

where:

- k is a constant, called the thermal conductivity coefficient, $k > 0$.
- $u = u(x,t)$ is a function of the spatial coordinate x and the time t, describing the temperature field distribution.
- $\Phi(x,t)$ describes the distribution of the heat source.

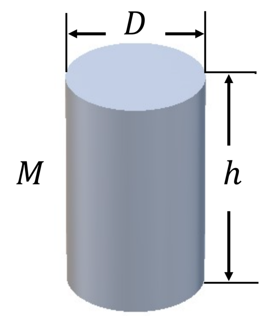
If there is a solution to this heat equation,

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}}e^{-\frac{x^2}{4kt}}, \qquad t > 0, \qquad x > 0,$$

try to determine the heat source function $\Phi(x,t)$.

25. (8 marks) (*Error in physical experiments*)

Consider an experiment to measure the density of a type of metal. A sample is as follows, with the shape being a regular cylinder. The transection is a standard disk, with diameter measured to be D. The height of the cylinder is measured to be h. The mass of the sample is scaled to be M.



With these data, the density of this metal is obtained as

$$\rho = \rho(M, D, h) = \frac{M}{\frac{\pi}{4} D^2 h}.$$

(a) Regarding M, D and h as the variables, find the total differential of ρ.

(b) Suppose the errors of M, D and h are: (below "*unit*" means an appropriate unit)

$$\Delta M = \pm 0.05 \text{ unit}, \qquad \Delta D = \pm 0.002 \text{ unit}, \qquad \Delta h = \pm 0.002 \text{ unit},$$

and your computation gives the error of measurement for the density ρ:

$$\Delta\rho = 5.00\Delta M - 10.000\Delta D + 80.000\Delta h.$$

Try to **evaluate/compute explicitly the error $\Delta\rho$.**

Glossary

Aperiodic	非周期的
Approach	逼近
Cartesian	笛卡尔
Coefficient	系数
Complex conjugate	复共轭
Cone	锥面
Cylinder	柱体
Density	密度
Duality	对偶
Ellipsoid	椭球面
Error	误差
Even function	偶函数
Fourier series	傅立叶级数
Fourier transform	傅立叶变换
Heat equation	热（传导）方程
Heat source	热源
Horizontal	水平
Hyperboloid	双曲面
Kernel	核（函数）
Mass	质量
Modulus	模长
Odd function	奇函数
Paraboloid	抛物面
Partial derivative	偏导数
Periodic	周期的
Polynomial	多项式
Projection	投影
Swap	扫过
Symmetric axis	对称轴
Temperature field distribution	温度场分布
Thermal conductivity	热传导
Trajectory	轨迹