Beijing-Dublin International College

SPRING TRIMESTER EXAMINATION - 2020/2021

MATH1002J Introduction to Analysis

MODULE COORDINATOR: Kwok Chi Chim

Time Allowed: **120 minutes**

Instructions for Candidates

Full marks will be awarded for complete answers to all five questions. Each question is worth 20 marks in total. The number of marks per question part is indicated. Unless otherwise stated, please show all your steps.

BJUT Student ID: ____________________ **UCD Student ID:** ____________________

I have read and clearly understand the Examination Rules of both Beijing University of Technology and University College Dublin. I am aware of the Punishment for Violating the Rules of Beijing University of Technology and/or University College Dublin. I hereby promise to abide by the relevant rules and regulations by not giving or receiving any help during the exam. If caught violating the rules, I accept the punishment thereof.

Honesty Pledge: ____ **(Signature)**

Instructions for Invigilators

This examination is closed-book and closed-notes.

Non-programmable calculators are permitted.

1. (a) Determine whether each of the following statements is True or False.
No explanation is needed when answering 1(a)(i) to 1(a)(v).

(i) For any $n \in \mathbb{N}$, we have

$$\binom{2n+1}{n} = \binom{2n+1}{n+1}.$$ [1]

(ii) The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = (2+x)^3$$

is not surjective. [1]

(iii) The function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = |x+8|$$

is strictly increasing. [1]

(iv) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two real-valued functions.
Then $\mathrm{Im}(f \circ g) \subset \mathrm{Im}(f)$. [1]

(v) Let (a_n) be a sequence.
If (a_n) is convergent and $\lim_{n\to\infty} a_{2n} = 5$, then $\lim_{n\to\infty} a_{2n+1} = 5$. [1]

(b) Determine whether each of the following statements is True or False.
Briefly justify your answers to questions 1(b)(i) to 1(b)(v).

(i) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two real-valued functions.
If f and g are both bijective, then $f - g$ is also bijective. [3]

(ii) The coefficient of x^3 in $(1+2x)^5$ is 80. [3]

(iii) Let $f : [2, 10] \to \mathbb{R}$ and $h : [2, 10] \to \mathbb{R}$ be two continuous functions with $f(2) = 3$, $f(10) = 5$ and h is defined by

$$h(x) = f(x) \cdot x^2.$$

Then $50 \in \mathrm{Im}(h)$. [3]

(iv) Let (a_n) be a sequence. If the subsequences (a_{3n}) and (a_{3n+1}) are both convergent, then (a_n) is also convergent. [3]

(v) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ |x| & \text{if } x > 0. \end{cases}$$

Then f is differentiable at 0. [3]

2. (a) Let x be an integer. Prove that if $x^2 + 6x + 15$ is a multiple of 3, then x is a multiple of 3. **[7]**

(b) Examine if the function $g : \mathbb{R} \to \mathbb{R}$, defined by

$$g(x) = \binom{18}{0} + \binom{18}{1}x + \cdots + \binom{18}{k}x^k + \cdots + \binom{18}{18}x^{18}$$

is injective, surjective or bijective. If g is bijective, find its inverse. **[6]**

(c) Give an example of a function $f : [0, 10] \to [-60, 0]$ which is bijective. Justify that f is bijective, and find its inverse. **[7]**

3. (a) Use the definition of the limit of a sequence to find the limit of the sequence (a_n) defined by

$$a_n = 3 + \frac{2}{5n^3}.$$ **[6]**

(b) Determine whether the limit of the sequence (a_n) defined by

$$a_n = \frac{2}{n^2}\sin\left(n^2\right)$$

exists. **[4]**

(c) Give an example of a sequence (a_n) which is strictly decreasing and bounded below. Determine the limit of this sequence (if it exists). **[5]**

(d) Consider the sequence (a_n) defined by

$$a_1 = a_2 = 3 \quad \text{and} \quad a_n = \sqrt{2a_{n-2} + 3a_{n-1}} \quad \text{for } n > 2.$$

By using mathematical induction, show that

$$a_n \leq 6$$

for all $n \in \mathbb{N}$. **[5]**

4. (a) In each of the following, determine whether the series is convergent.

(i) $\sum_{n=1}^{\infty} \frac{2^n}{9^n} \cos\left(\frac{3}{2^n}\right)$ **[5]**

(ii) $\sum_{n=1}^{\infty} \frac{2n^2 + 1}{n^2 + n + 1}$ **[5]**

(iii) $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n + 9}$ **[5]**

(b) Evaluate the sum of the telescoping series

$$\sum_{n=1}^{\infty} \frac{20}{(2n+1)(2n+3)}.$$ **[5]**

5. (a) Justify whether the following statement is true or false:
The series $\sum_{n=1}^{\infty} a_n$ converges if

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \geq 1.$$

Give an example to support your justification. **[6]**

(b) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \dfrac{(x-6)(x+2)^2}{4(x-6)} & \text{if } x < 6, \\ b & \text{if } x = 6, \\ ce^{x-6} & \text{if } x > 6, \end{cases}$$

where b and c are real numbers. Determine the value(s) of b and c such that f is continuous at 6. **[7]**

(c) Compute the radius of convergence of the following power series:

(i) $\sum_{n=0}^{\infty} 3^n x^n$ **[3]**

(ii) $\sum_{n=0}^{\infty} \dfrac{(3n)!}{(n!)^3} x^n$ **[4]**

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