Beijing-Dublin International College

SEMESTER II FINAL EXAMINATION – 2016/2017

School of Mathematics and Statistics

BDIC1027J Maths 3 (Advanced Mathematics; Finance)

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MODULE COORDINATOR: Xin Liu

Time Allowed: 90 minutes

Instructions for Candidates

Answer ALL questions. The marks that each question carry is written as shown.

BJUT Student ID: ________________ **UCD Student ID:** ________________

I have read and clearly understand the Examination Rules of both Beijing University of Technology and University College Dublin. I am aware of the Punishment for Violating the Rules of Beijing University of Technology and/or University College Dublin. I hereby promise to abide by the relevant rules and regulations by not giving or receiving any help during the exam. If caught violating the rules, I accept the punishment thereof.

Honesty Pledge： ____ **(Signature)**

Instructions for Invigilators

Non-programmable calculators are permitted. NO dictionaries are permitted.
No rough-work paper is to be provided for candidates.

SECTION A — MULTIPLE CHOICE QUESTIONS

In each question, choose **at most one** option.
Circle the preferred choice on the **Examination Book** provided.
This section is worth a total of **60** marks, with each question worth **3** marks.

1. Consider a parallelepiped Ω generated by three vectors in space $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = -\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{c} = \mathbf{i} + \mathbf{j} - \mathbf{k}$; namely, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three adjacent edges of Ω. Compute the volume of Ω.

(a) -4; (b) 4; (c) $2\mathbf{i} - 2\mathbf{k}$; (d) $-2\mathbf{i} + 2\mathbf{k}$.

2. Let l be a line and Σ a plane parallel to l:

$$l: \quad x - 2 = \frac{y+2}{-4} = \frac{z-1}{2}, \qquad \Sigma: \quad 2x + y + z - 1 = 0.$$

If there is another line l' residing in Σ and parallel to l, which of the following could be the equation of this l'?

(a) $x - 1 = \frac{y}{-4} = \frac{z+1}{2}$; (b) $x - 1 = \frac{y+2}{-4} = \frac{z}{2}$;

(c) $\frac{x-1}{2} = \frac{y}{-2} = z + 1$; (d) $\frac{x-1}{2} = \frac{y+1}{-2} = z$.

3. Find the plane passing through three points $A\,(2, -2, 1)$, $B\,(1, 3, -2)$ and $C\,(1, 2, -1)$ in space.

(a) $2x + y + z - 3 = 0$; (b) $x + 2y + z + 3 = 0$;

(c) $\frac{x-2}{2} = y + 2 = \frac{z-1}{2}$; (d) $\mathbf{r} = (\mathbf{i} + 2\mathbf{j} - \mathbf{k}) + t\,(2\mathbf{i} + \mathbf{j} + \mathbf{k})$.

4. Determine the shape of the surface given by the following equation

$$z^2 - 5x^2 - 5y^2 - 2z + 1 = 0.$$

(a) An ellipsoid; (b) A paraboloid; (c) A hyperboloid; (d) A cone.

5. Determine the shape of the surface given by the implicit function

$$F(x, y, z) = x^2 + y^2 + z^2 - 2x + 4y + 6z + 13 = 0.$$

(a) A sphere, centered at $(1, -3, 2)$, with radius 2.

(b) A sphere, centered at $(1, -3, 2)$, with radius 1.

(c) A sphere, centered at $(1, -2, -3)$, with radius 1.

(d) An ellipsoid, with the three half-axes being $a = 1$, $b = 2$ and $c = 3$.

6. The surface of a revolution body Σ is generated by rotating a curve C about a symmetric axis L in space. If the equation of Σ is given by

$$y = e^{x^2+z^2} - \tan\sqrt{x^2+z^2},$$

determine the curve C and the symmetric axis L.

(a) C is $z = e^{x^2} - \tan|x|$, and L the z-axis.

(b) C is $y = e^{x^2} - \tan|z|$, and L the y-axis.

(c) C is $y = e^{z^2} - \tan|z|$, and L the y-axis.

(d) C is $x = e^{z^2} - \tan|y|$, and L the x-axis.

7. Which of the following statements is correct?

(a) On the one-dimensional x-axis, there is only 1 path to approach a point x_0;

(b) On the one-dimensional x-axis, there are infinitely many paths to approach a point x_0;

(c) In the two-dimensional xy-plane, there are only 2 paths to approach a point (x_0, y_0);

(d) In the two-dimensional xy-plane, there are infinitely many paths to approach a point (x_0, y_0).

8. Consider two surfaces

$$C_1: \quad 3x^2 - 2y^2 + z - 7 = 0, \qquad C_1: \quad 3x^2 - 2y^2 + z + 1 = 0.$$

Their intersection curve is

(a) a line; (b) a parabola; (c) a hyperbola;

(d) C_1 and C_2 are parallel and have no intersection points.

9. Let C be a curve in space

$$C: \begin{cases} x^2 + y^2 + z^2 - 4 &= 0, \\ -x^2 + y^2 - z + 1 &= 0. \end{cases}$$

Find the equation of the projection of C onto the xz-plane.

(a) $2x^2 + 2z^2 - 5 = 0.$

(b) $\begin{cases} 2x^2 + z^2 + z - 5 &= 0, \\ y &= 0. \end{cases}$

(c) $\begin{cases} 2y^2 + z^2 - z - 3 &= 0, \\ x &= 0. \end{cases}$

(d) $\begin{cases} x^2 + y^2 + z^2 - 4 &= 0, \\ -x^2 + y^2 - z + 1 &= 0, \\ y &= 0. \end{cases}$

10. Let C be the intersection curve between a surface Σ_1 and a plane Σ_2,

$$\Sigma_1: \quad \frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1; \qquad \Sigma_2: \quad x + y = 0.$$

What is the shape of the projection of C onto the xz plane?

(a) An ellipse; (b) A circle; (c) A parabola; (d) A line.

11. Suppose the function

$$f(x, y) = \begin{cases} \frac{1}{x^2+y^2} \tan\left(\frac{x^2+y^2}{2}\right), & \text{if} \quad (x, y) \neq (0, 0), \\ C, & \text{if} \quad (x, y) = (0, 0). \end{cases}$$

is continuous over the domain $\mathbb{R}^2$. Determine the value of the real number C.

(a) 0; (b) $\frac{1}{2}$; (c) 2; (d) ∞.

12. Let C be a horizontal curve given by $\begin{cases} \ln(x+y)+\sin(x-y)+2 &= 0, \\ z &= 2. \end{cases}$ Suppose Σ is a cone formed by moving a line L along C — that is, regarding C as a trajectory — while keeping L passing through the origin point $(0,0,0)$. Determine the cone Σ.

(a) $(0,0,0) \cup \{(x,y,z)\,|\ln(x+y)+\sin(x-y)+2=0\}$;

(b) $(0,0,0) \cup \left\{(x,y,z)\,\middle|\ln\left[\frac{z}{2}(x+y)\right]+\sin\left[\frac{z}{2}(x-y)\right]+2=0\right\}$;

(c) $(0,0,0) \cup \left\{(x,y,z)\,\middle|\ln\left[\frac{2}{z}(x+y)\right]+\sin\left[\frac{2}{z}(x-y)\right]+2=0\right\}$;

(d) $(0,0,0) \cup \left\{(x,y,z)\,\middle|\,\frac{1}{x+y}+\cos(x-y)+2=0\right\}$.

13. Evaluate the limit:

$$\lim_{(x,y)\to(0,0)} \frac{x-y}{3\sqrt{x^2+y^2}}.$$

(a) $\frac{2}{3}$; (b) 0; (c) ∞; (d) The limit does not exist.

14. Evaluate the limit:

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{3\sqrt{x^2+y^2}}.$$

(a) 0; (b) $\frac{2}{3}$; (c) ∞; (d) The limit does not exist.

15. Evaluate the limit:

$$\lim_{(x,y)\to(0,3)} \frac{e^{xy}\sin(xy)}{x}.$$

(a) -3; (b) 3; (c) $-\frac{1}{3}$; (d) $\frac{1}{3}$.

16. Let $f(x,y) = ye^{\sin x} + xe^{\sin y}$. Find f_{yy}.

(a) $e^{\sin x} + x\cos y e^{\sin y}$; (b) $y\left(-\sin x+\cos^2 x\right)e^{\sin x}$;

(c) $\cos x e^{\sin x} + \cos y e^{\sin y}$; (d) $x\left(-\sin y+\cos^2 y\right)e^{\sin y}$.

17. The total differential of a two-variable function $z = e^{x^2-y^2}$ is given by

(a) $dz = dxdye^{x^2-y^2}$;

(b) $dz = 2xy\,(dx + dy)\,e^{x^2-y^2}$;

(c) $dz = (2xdx - 2ydy)\,e^{x^2-y^2}$;

(d) $dz = (2xdx + 2ydy)\,e^{x^2-y^2}$.

18. Find the natural domain of the following function in $\mathbb{R}^2$:

$$z = f(x, y) = \sqrt{\ln\left(x^2 + y^2 - 4\right)}.$$

(a) Let C be the circle centered at $(0, 0)$ with radius $\sqrt{5}$. The domain is the region outside and including C;

(b) Let C be the circle centered at $(0, 0)$ with radius $\sqrt{5}$. The domain is the region outside but not including C;

(c) Let C be the circle centered at $(0, 0)$ with radius 2. The domain is the region outside but not including C;

(d) The domain is the upper z-axis including the origin point.

19. If $f\,(x, y) = 4x^2y - y^3x + y^2$, then $f_y\,(1, -1)$ equals

(a) -9; (b) -7; (c) -1; (d) 3.

20. For $f\,(x, y) = \ln \sin\left(x^3y^2\right)$, find its partial derivative f_y over the natural domain.

(a) $\frac{\cos x^3y^2}{\sin x^3y^2}$; (b) $\left(2yx^3 + 3x^2y^2\right)\frac{\cos x^3y^2}{\sin x^3y^2}$; (c) $3x^2y^2\frac{\cos x^3y^2}{\sin x^3y^2}$; (d) $2x^3y\frac{\cos x^3y^2}{\sin x^3y^2}$.

SECTION B — EXTENDED ANSWER QUESTIONS

Write your answers on the **Examination Book** provided.
This section is worth a total of **40** marks. The marks of each question are as shown.

21. (6 marks) Let l be a line and $\mathbf{v}$ a direction in space,

$$l: \quad \frac{x-2}{2} = y+1 = -z, \qquad\qquad \mathbf{v} = \mathbf{j} - \mathbf{k}.$$

Find the plane swapped by l along the direction $\pm\mathbf{v}$; i.e., the plane formed by moving l along $\pm\mathbf{v}$.

22. (4 marks) Let C_1 and C_2 be two surfaces in space:

$$C_1: \quad x^2 + y^2 + (z-2)^2 = 1, \qquad\qquad C_2: \quad z = -\frac{1}{4}(x^2+y^2) - 2.$$

Find the shortest distance between C_1 and C_2.

23. (10 marks) Let $\mathcal{L}_1$ be a helix in space:

$$\mathcal{L}_1: \quad \begin{cases} x = & \sin t, \\ y = & \cos t, \\ z = & t^2, \end{cases} \qquad\qquad t > 0.$$

(a) Find the projection of $\mathcal{L}$ onto the xy plane. *(3 marks)*

(b) Find the projections of $\mathcal{L}$ onto the yz and xz planes. *(2 marks)*

(c) Considering a line $\mathcal{L}_2: \begin{cases} x = & 0, \\ y = & 0, \\ z = & t, \end{cases}$ find the shortest distance between $\mathcal{L}_1$ and $\mathcal{L}_2$.

Specify: Under what condition/when this shortest distance can be taken. *(5 marks)*

24. **(8 marks)** It is known that $u_0(x, y) = e^x \sin y$ is a solution of the *Laplace equation* in 2 dimensions:

$$\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} = 0.$$

Consider another partial differential equation, the so-called *Poisson equation*,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \rho(x, y),$$

where $\rho(x, y)$ is a density distribution function.

If there is a solution to this Poisson equation,

$$u(x, y) = u_0(x, y) + \tilde{u}(x, y) = e^x \sin y + \cos(x - y),$$

try to find the expression of the density distribution function $\rho(x, y)$.

25. (12 marks) *Black–Scholes–Merton model (Nobel Prize in Economic Sciences, 1997):*
This is a mathematical model of a financial market containing derivative investment instruments. From the model, one can deduce the Black–Scholes formula which gives a theoretical estimate of the price of *European-style options.* The formula leads to a boom in options trading and provided mathematical legitimacy to the activities of the *Chicago Board Options Exchange* and other options markets around the world. lt is widely used, although often with adjustments and corrections, by options market participants. Many empirical tests have shown that the Black–Scholes price is "fairly close" to the observed prices (*although there are well-known discrepancies such as the "option smile").

The Black–Scholes formula reads

$$\begin{aligned} C(S,t) &= N(d_1)S - N(d_2)Ke^{-rt}, \\ P(S,t) &= -N(-d_1)S + N(-d_2)Ke^{-rt}, \end{aligned}$$

where,

$C(S,t)$	— price of an European call option;
$P(S,t)$	— price of an European put option;
K	— strike price of the option;
r	— annualized risk-free interest rate, continuously compounded (the force of interest);
t	— time left before maturity (expressed in years);
$N(d_1),\ N(d_2)$	— probabilities of the option expiring in-the-money (standard normal distributions).

Suppose we have a setting:

When $d_1 = 0.8$ and $d_2 = 0.6$, there are

$$N(d_1) = 0.8, \quad N(-d_1) = 0.2; \qquad N(d_2) = 0.7, \quad N(-d_2) = 0.3.$$

Taking $S = 42$, $K = 40$, $r = 0.1$ and $t = 0.5$, we have $Ke^{-rt} = 38$.

Then, if S has an increment $\Delta S = 1$ and t an increment $\Delta t = -0.01$, try to compute:

(a) the increment of the price $C(S,t)$, namely, ΔC; *(8 marks)*

(b) the increment of the price $P(S,t)$, namely, ΔP. *(4 marks)*

Glossary

Adjacent edges	相邻的边
Approach	逼近
Call option	看涨期权
Cone	锥面
Density distribution	密度分布
Domain	定义域
Ellipsoid	椭球体
Estimate	估测值
Half axis	半轴
Helix	螺旋线
Hyperbola	双曲线
Hyperboloid	双曲面
Increment	增量
Interest rate	利率
Maturity	到期，成熟
Normal distributions	正则分布
Options	期权（期货与选择权的合称）
Parabola	抛物线
Paraboloid	抛物面
Parallelepiped	平行六面体
Probabilities	概率
Projection	投影
Put option	看跌期权
Revolution	旋转
Specify	说明，阐明
Strike price of the option	期权执行价格，又称协议价格
Swap	扫过
Trajectory	轨迹