Cutting Plane Selection with Analytic Centers and Multiregression

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Outline

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MIP notation

Mixed-Integer Linear Program (MILP):

$$\underset{\mathbf{x}}{\operatorname{argmin}}\{\mathbf{c}^\intercal \mathbf{x} \;\mid\; \mathbf{A}\mathbf{x} \leq \mathbf{b}, \;\; \mathbf{l} \leq \mathbf{x} \leq \mathbf{u}, \;\; \mathbf{x} \in \mathbb{Z}^{|\mathcal{J}|} \times \mathbb{R}^{n-|\mathcal{J}|}\}$$

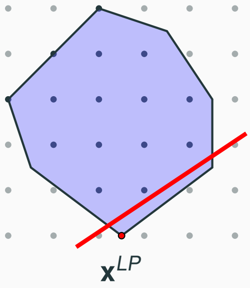
- $\mathbf{c} \in \mathbb{R}^n$ - Objective coefficient vector
- $\mathbf{A} \in \mathbb{R}^{m \times n}$ - Constraint matrix
- $\mathbf{b} \in \mathbb{R}^m$ - RHS constraint vector
- $\mathbf{l}, \mathbf{u} \in \{\mathbb{R}, -\infty, \infty\}^n$ - Lower and upper variable bound vectors
- $\mathcal{J} \subseteq \{1, \ldots, n\}$ - Set of indices of integer variables

Cut definition

A *cut* is a linear constraint that does not increase the optimal value when added.

Cut $(\alpha, \beta) \in \mathbb{R}^n \times \mathbb{R}$ noted:

$$\alpha^\top \mathbf{x} \leq \beta,$$



Cut selection

Purpose of cuts: tighten the convex relaxation (intentionally vague).

Cuts generated in *separation rounds*: separate and re-solve.
Most separators cheaper than relaxation solves:
$\rightarrow$ MIP solvers generate more cuts than used.

Key trade-off:
Adding all cuts $\Rightarrow$ Expensive LPs, numerically unstable
$\Leftrightarrow$
Adding no cut $\Rightarrow$ More nodes needed to solve

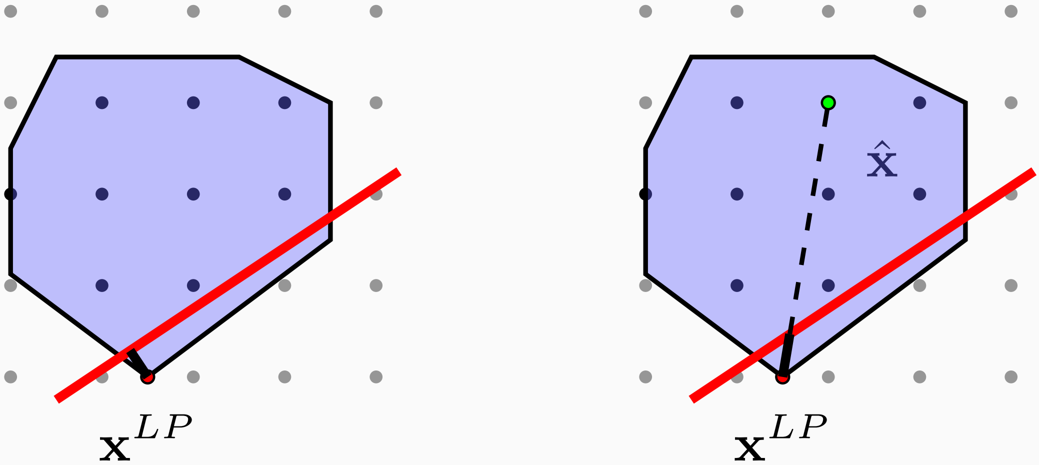
Cut scoring

Different aspects for selecting a subset of cuts - here those of SCIP:

- *Strength*: how much are we removing from the relaxation?
- *Parallelism* / orthogonality (to previous cuts),
- *Sparsity*,
- *Objective parallelism*.

This talk $\Rightarrow$ What is a good measure or proxy for strength?

Cut scoring: efficacy & directed cutoff distance



Other measures

Expected improvement:

$$\texttt{expimprov}(\alpha,\beta;\mathbf{c},\mathbf{x}^{LP}) := ||\mathbf{c}|| \cdot \frac{\alpha^\top \mathbf{c}}{||\alpha||||\mathbf{c}||} \cdot \texttt{eff}(\alpha,\beta;\mathbf{x}^{LP})$$

Actual improvement:
Re-solve the relaxation with the cut added a.k.a. *look-ahead*, analogue to strong branching.

Existing literature for cut selection

Older comprehensive computational experiments on cut selection:

- [Achterberg, 2007, Andreello et al., 2007, Wesselmann and Stuhl, 2012],
- Focus on cheap ranking measures, mostly **efficacy**,
- Filtering **parallel** cuts is most important.

Question recently revisited with machine learning angle:

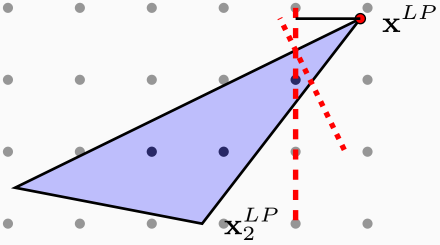
- [Baltean-Lugojan et al., 2019, Paulus et al., 2022]: learning objective change (strong cutting/look-ahead),
- [Tang et al., 2020]: selection of Gomory cuts with neural networks,
- [Turner et al., 2022]: learn to weigh different scores.

Summary: Improvement is possible, but non-trivial and difficult to generalize.

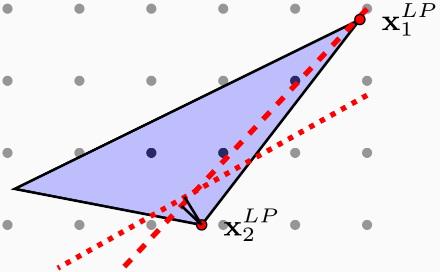
What is wrong with efficacy?

Efficacy $\approx$ how far does a cut remove current LP solution?

Infeasible projection on cut:
what is removed from relaxation?



Dual degeneracy:
Arbitrary LP solution on optimal face



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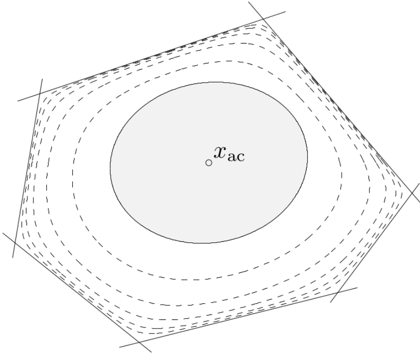
Analytic Centers

Solution to log-barrier minimization:

$$\arg\min_{\mathbf{x}} - \sum_i \log(\mathbf{b}_i - \mathbf{A}_i\mathbf{x}) - \sum_j (\log(\mathbf{x}_j - \mathbf{l}_j) + \log(\mathbf{u}_j - \mathbf{x}_j))$$



- "Central-most" point of the polytope.
- Easily extendable to conic optimization.
- Robust to bad conditioning (unlike Chebyshev)
- Not independent of the formulation for a given set (e.g. redundant inequalities) $\rightarrow$ importance of presolving

Figure: Convex Optimization, Boyd & Vandenberghe

Analytic centers in MIPs

- Presolving: deducing tight bounds, fixing variables [Berthold et al., 2018]
- Branching: which variables are close to integer values [Berthold et al., 2018]
- Heuristics: exploiting the AC as central direction [Baena and Castro, 2011, Naoum-Sawaya, 2014]
- Cut generation: [Fischetti and Salvagnin, 2009]

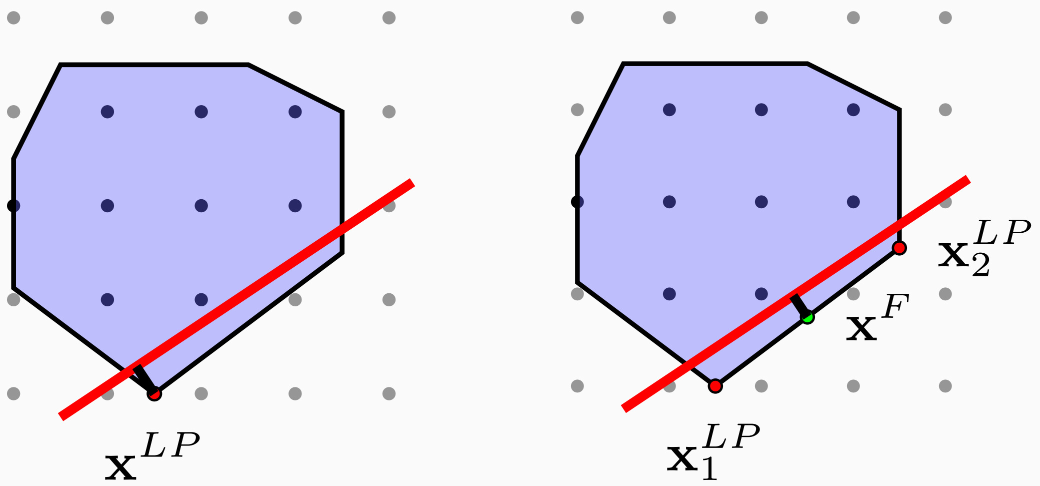
Analytic centers: the computational case

Analytic centers: (almost) a “co-product” of LP solves with Interior Points.

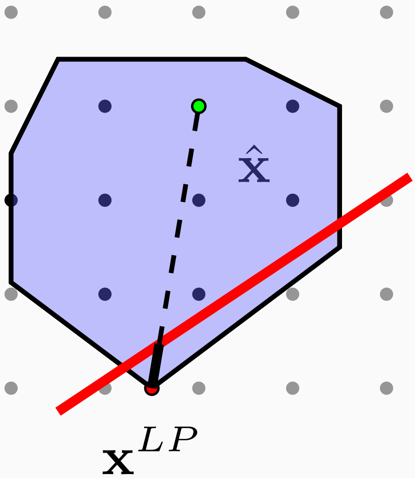
- Original LP with feasibility objective: AC of polyope
- Direct LP solve without cross-over: AC of optimal face.

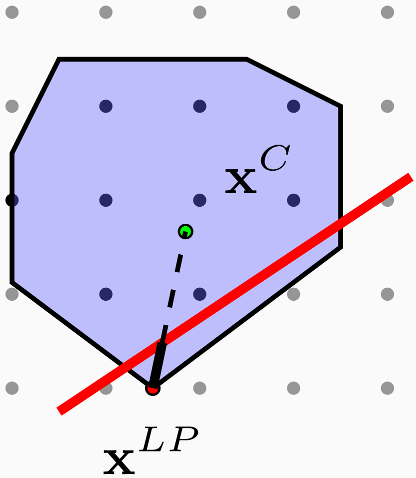
Latter crucial to exploit dual degeneracy of current relaxation.

New efficacy measures



New directed cutoff measure





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Cut dominance consistency

Cut $(A) = (\alpha_A, \beta_A)$ *dominates* cut $(B) = (\alpha_B, \beta_B)$
$\Rightarrow$ points of the polytope cut by (A) are cut by (B), $\exists$ point cut by (A) not cut by (B).

Dominance consistency:
Given a relaxation point to cut off $\mathbf{x}$, a distance measure $d(\mathbf{x}, \alpha_X, \beta_X)$ is *dominance-consistent* w.r.t. cuts (A) and (B) if
$d(\mathbf{x}, \alpha_A, \beta_A) > d(\mathbf{x}, \alpha_B, \beta_B) \ \Rightarrow \ (A)$ is not dominated by (B).

Results

Projection-based methods are dominance-consistent if cut with smallest measure has **feasible projection** and positive measure.

Directed cutoff measures are dominance-consistent **if all cuts separate** the relaxation point.

Proof: white board depending on time
TL;DR: we draw circles.

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Setup and questions

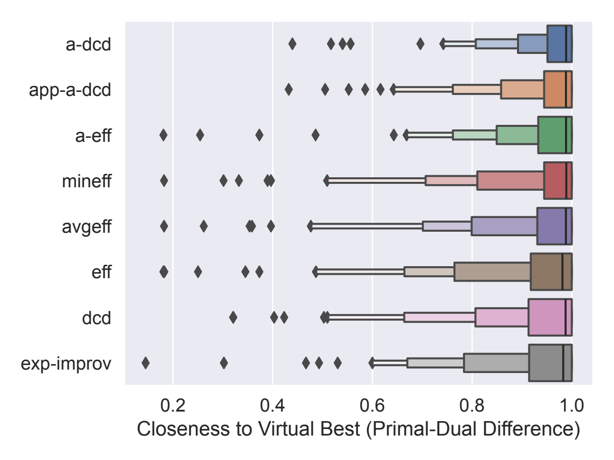
Experiments are performed on 162 MIPLIB instances with SCIP 8.
Analytic centers computed in the transformed (presolved) space.
Separation at the root node only.

Questions:

- Does the cut selection method influence root relaxation strength?
- Does it influence solver performance?
- Are the two tied?

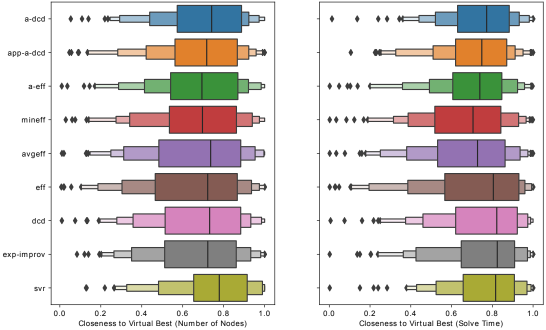
Root node experiments setup

Most measures are usually similar.
Effect on worst cases only.
Influence of performance variability:
→ “trickle down” effect of early cuts.



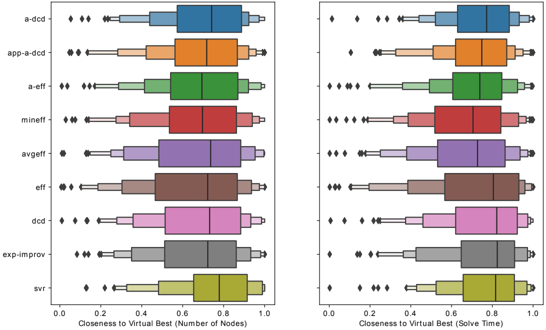
Tree experiments (1)

Analytic directed cutoff distance:→
best median # nodes.
Directed cutoff distance:
→ best median time, but more tail.
Choosing a scoring function matters.



Tree experiments (2)

Using approximate analytic center:
→ clear degradation.
26.5% of analytic centers infeasible with new cuts.
Efficacy: high variance of results.
Expected improvement & multi-solution efficacy: poor results overall.
90.4% of LP-infeasible projections.
87.5% dual degeneracy.



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Are instance features sufficient?

Given instance features, which score would produce the minimum tree size/runtime?

Features of the transformed problem:

- Dual degeneracy: % non-basic variables with zero-reduced cost;
- Primal degeneracy: % basic variables at bounds;
- Solution fractionality at root node;
- Thinness: % equality constraints;
- Density of the whole constraint matrix.

Learning to select cuts

Similar to all Learning to XYZ in MIP $\Rightarrow$ dual goal:

1. **Discovering** rules working well on representative instances;
2. **Building** practical adaptive algorithms within MIP solving.

1. can use sophisticated techniques, high-dimensional kernels, neural networks.
2. requires interpretable, light-weight models, simpler parameter space.

Multiregression model

Selecting best measure to score cuts:
Isn’t that a classification task?

Selecting one as the best is tricky for many instances:
$\rightarrow$ no cut selected, “obvious” cuts always selected, etc...

Ties allowed? What subset of measures included in “best”?

Multiregression model

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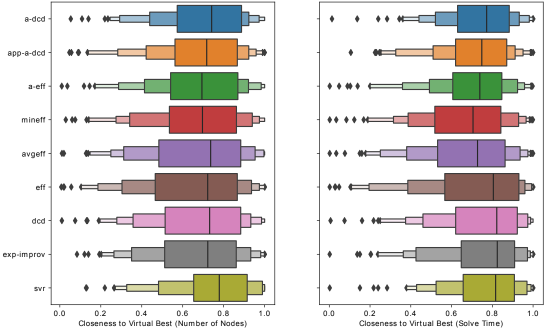
Ties allowed? What subset of measures included in "best"?

Multi-output regression: # nodes method / # nodes best
Several attempted models: regression trees, support vectors, random forests.
Final choice: support vector regression with cubic kernel.

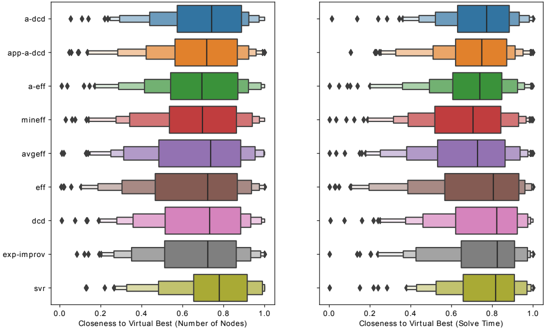
Results

12% fewer nodes on average compared to best method.
But 8% increase in time.



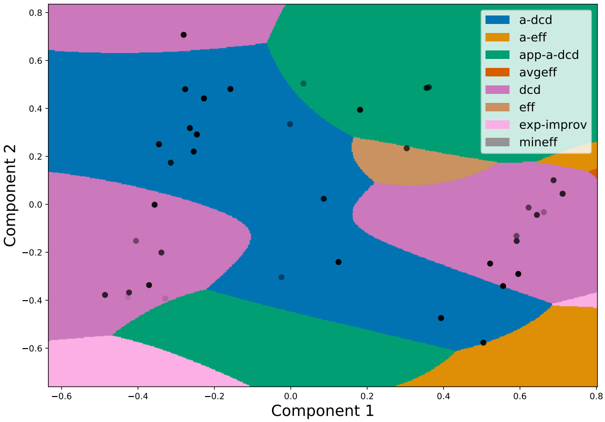
Results

12% fewer nodes on average compared to best method.
But 8% increase in time.
Learning with runtime much harder and not conclusive.
→ time-specific features?



Decision boundaries on projected space (with PCA)

Two PCA components: 71% of the explained variance.
Data points for validation instances.
Opacity
→ relative performance of the predicted measure.



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Conclusion

Highlights:

- Scoring cuts for their selection matters
- Robust analytic measures avoid pitfalls of **dual degeneracy**, **infeasible projections**
- Ability to predict best score for tree size.

Future direction:

- Work on scoring individual cuts: how to pick subsets?
- Not all separation rounds may matter: finer adaptive method?
- Faster and exact analytic centers?

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