4 Soundness

We now prove the soundness of liquid types.

Theorem 4.1

Let $iet \in$ `<int-exp-type>` and $exp \in IntExp$. Assume $iet : exp$ can be derived.

—

Then $[\![iet]\!] = exp$.

Proof. Let $iet \in$ `<int-exp-type>` and $exp \in IntExp$. Assume $iet : exp$ can be derived.

- **Case** $iet = i$ for $i \in Int$: Then $[\![iet]\!] = i$ and therefore the conclusion holds.
- **Case** $iet = iet_1$ `+` iet_2 for $iet_1, iet_2 \in$ `<int-exp-type>`: From the premise of the inference rule, we assume that $iet_1 : exp_1$ and $iet_2 : exp_2$ hold. By induction hypothesis $[\![iet_1]\!] = exp_1$ and $[\![iet_2]\!] = exp_2$. Thus $[\![iet]\!] = exp_1 + exp_2$ and therefore the conclusion holds.
- **Case** $iet = iet_1$ `*` i for $iet_1 \in$ `<int-exp-type>` and $i \in Int$: From the premise of the inference rule, we assume that $iet_1 : exp_1$ holds. By induction hypothesis $[\![iet_1]\!] = exp_1$. Thus $[\![iet]\!] = exp_1 \cdot i$ and therefore the conclusion holds.
- **Case** $iet = a$ for $a \in \mathcal{A}$: Then $[\![a]\!] = a$ and therefore the conclusion holds.

□

Theorem 4.2

Let $qt \in$ `<qualifier-type>` and $q \in \mathcal{Q}$. Assume $qt : q$ can be derived.

—

Then $[\![qt]\!] = q$.

Proof. Let $qt \in$ `<qualifier-type>` and $q \in \mathcal{Q}$. Assume $qt : q$ can be derived.

- **Case** $qt =$ `True`: Then $[\![qt]\!] = True$ and therefore the conclusion holds.
- **Case** $qt =$ `False`: Then $[\![qt]\!] = False$ and therefore the conclusion holds.
- **Case** $qt =$ `(<)` iet `v`: From the premise of the inference rule, we assume that $iet : exp$. By Theorem 4.1 $[\![iet]\!] = exp$ for $exp \in IntExp$. Then $[\![qt]\!] = exp < \nu$ and therefore the conclusion holds.
- **Case** $qt =$ `(<)` `v` iet: From the premise of the inference rule, we assume that $iet : exp$. By Theorem 4.1 $[\![iet]\!] = exp$ for $exp \in IntExp$. Then $[\![qt]\!] = \nu < exp$ and therefore the conclusion holds.
- **Case** $qt =$ `(=)` `v` iet: From the premise of the inference rule, we assume that $iet : exp$. By Theorem 4.1 $[\![iet]\!] = exp$ for $exp \in IntExp$. Then $[\![qt]\!] = (\nu = exp)$ and therefore the conclusion holds.

- **Case** $qt =$ `(&&)` qt_1 qt_2 for $qt_1, qt_2 \in$ `<qualifier-type>`: From the premise of the inference rule, we assume that $qt_1 : q_1$ and $qt_2 : q_2$ hold for $q_1, q_2 \in \mathcal{Q}$. By induction hypothesis $[\![qt_1]\!] = q_1$ and $[\![qt_2]\!] = q_2$. Thus $[\![qt]\!] = q_1 \wedge q_2$ and therefore the conclusion holds.

- **Case** $qt =$ `(||)` qt_1 qt_2 for $qt_1, qt_2 \in$ `<qualifier-type>`: From the premise of the inference rule, we assume that $qt_1 : q_1$ and $qt_2 : q_2$ hold for $q_1, q_2 \in \mathcal{Q}$. By induction hypothesis $[\![qt_1]\!] = q_1$ and $[\![qt_2]\!] = q_2$. Thus $[\![qt]\!] = q_1 \vee q_2$ and therefore the conclusion holds.

- **Case** $qt =$ `not` qt_1 for $qt_1 \in$ `<qualifier-type>`: From the premise of the inference rule, we assume that $qt_1 : q_1$ holds for $q_1 \in \mathcal{Q}$. By induction hypothesis $[\![qt_1]\!] = q_1$. Thus $[\![qt]\!] = \neg\, q_1$ and therefore the conclusion holds.

□

Theorem 4.3

Let $\Theta : \mathcal{V} \nrightarrow \mathcal{T}$. Let $lt \in$ `<liquid-type>` and $\hat{T} \in \mathcal{T}$. Assume $lt :_\Theta \hat{T}$ can be derived.

—

Then $[\![lt]\!] = \hat{T}$.

Proof. Let $\Theta : \mathcal{V} \nrightarrow \mathcal{T}$. Let $lt \in$ `<liquid-type>` and $\hat{T} \in \mathcal{T}$. Assume $lt :_\Theta \hat{T}$ can be derived.

- **Case** $lt =$ `"{v:Int|"` qt `"}"` for $qt \in$ `<qualifier-type>`: From the premise of the inference rule, we assume that $qt : q$ for $q \in \mathcal{Q}$ holds. By Theorem 4.2 $[\![qt]\!] = q$. Then $[\![lt]\!] = \{\nu : Int |\ q\}$ and therefore the conclusion holds.

- **Case** $lt = a$ `":"` `"{v:Int|"` qt `"}"` `"->"` lt_2 for $a \in \mathcal{V}$, $qt \in$ `<qualifier-type>` and $lt \in$ `<liquid-type>`: From the premise of the inference rule, we assume that `"{v:Int|"` qt `"}"` $:_\Theta \hat{T}_1$ and $lt_2 :_{\Theta \cup \{(a, \hat{T}_1)\}} \hat{T}_2$ for liquid types $\hat{T}_1, \hat{T}_2$. By induction hypothesis $[\![lt_2]\!] = \hat{T}_2$. Then $[\![lt]\!] = a : \hat{T}_1 \to \hat{T}_2$ and therefore the conclusion holds.

□

Theorem 4.4

Let Γ be a type context, $t \in$ `<type>` and $T \in \mathcal{T}$. Assume $\Gamma \vdash t : T$ can be derived.

—

Then $[\![t]\!]_\Gamma = T$.

Proof. Let Γ be a type context, $t \in$ `<type>` and $T \in \mathcal{T}$. Assume $\Gamma \vdash t : T$ can be derived.

- **Case** $t = lt$ for $lt \in$ `<liquid-type>`: From the premise of the inference rule, we assume that $lt :_\Theta \hat{T}$ for liquid type $\hat{T}$ holds. By Theorem 4.3 $[\![lt]\!] = \hat{T}$. Then $[\![t]\!] = \hat{T}$ and therefore the conclusion holds.

All other cases have been proven in Theorem **??**. $\square$