Refinement Types for Elm

Master Thesis Report

Lucas Payr

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Topics of this Talk

- Revisiting the Max Function
- The Inference Algorithm
- Demonstration

Revisiting the Max Function

```
max : a:{ v:Int|True } -> b:{ v:Int|True } -> { v:Int|k4 }
max =
  \a -> \b ->
    if
      (<) a b
    then
      b
    else
      a
```

We want to derive the refinement label as $k4$.

Revisiting the Max Function

We remained with the following problem:

Find refinements $\kappa_1, \kappa_2, \kappa_3$ and κ_4 such that:

$$\{\nu : Int | \nu = b\} <:_{\{(a,\{Int|True\}),(b,\{Int|True\})\},\{a<b\}} \{\nu : Int|\kappa_3\},$$
$$\{\nu : Int | \nu = a\} <:_{\{(a,\{Int|True\}),(b,\{Int|True\})\},\{\neg(a<b)\}} \{\nu : Int|\kappa_3\},$$

$$a : \{\nu : Int|\kappa_1\} \to b : \{\nu : Int|\kappa_2\} \to \{\nu : Int|\kappa_3\}$$
$$<:_{\{\},\{\}} a : \{\nu : Int|True\} \to b : \{\nu : Int|True\} \to \{\nu : Int|\kappa_4\}$$

The Inference Algorithm: Definitions

Subtyping Condition

We say c is a *Subtyping Condition* $:\Leftrightarrow$ c is of form $\hat{T}_1 <:_{\Theta,\Lambda} \hat{T}_2$
where $\hat{T}_1, \hat{T}_2$ are a liquid types or templates, Θ is a type variable context and $\Lambda \subset \mathcal{Q}$.

Template

We say $\hat{T}$ is a *template* $:\Leftrightarrow$ $\hat{T}$ is of form $\{\nu : \mathit{Int} \mid [k]_S\}$
where $k \in \mathcal{K}$ and $S : \mathcal{V} \nrightarrow \mathcal{Q}$
$\vee$ $\hat{T}$ is of form $a : \{\nu : \mathit{Int} \mid [k]_S\} \rightarrow \hat{T}$
where $k \in \mathcal{K}$, $\hat{T}$ is a template and $S : \mathcal{V} \nrightarrow \mathit{IntExp}$.

We define $\mathcal{K} := \{\kappa_i \mid i \in \mathbb{N}\}$.

The Inference Algorithm

$$\mathsf{Infer} : \mathcal{P}(\mathcal{C}) \to \ (\mathcal{K} \nrightarrow \mathcal{Q})$$

$$\mathsf{Infer}(C) =$$

$$\text{Let } V := \bigcup_{\hat{T}_1 <:_{\Theta,\Lambda} \hat{T}_2 \in C} \{a \mid (a, \_) \in \Theta\}$$

$$Q_0 := \mathit{Init}(V),$$

$$A_0 := \{(\kappa, Q_0) \mid \kappa \in \bigcup_{c \in C} \mathsf{Var}(c)\},$$

$$A := \mathsf{Solve}(\bigcup_{c \in C} \mathsf{Split}(c), A_0)$$

$$\text{in } \{(\kappa, \bigwedge Q) \mid (\kappa, Q) \in A\}$$

The Inference Algorithm: Step 1 (Split)

$$\mathrm{Split} : \mathcal{C} \nrightarrow \mathcal{P}(\mathcal{C}^-)$$

$$\mathrm{Split}(a : \{\nu : \mathit{Int}|q_1\} \rightarrow \hat{T}_2 <:_{\Theta,\Lambda} a : \{\nu : \mathit{Int}|q_3\} \rightarrow \hat{T}_4) =$$

$$\{\{\nu : \mathit{Int}|q_3\} <:_{\Theta,\Lambda} \{\nu : \mathit{Int}|q_1\}\} \cup \mathrm{Split}(\hat{T}_2 <:_{\Theta\cup\{(a,q_3)\},\Lambda} \hat{T}_4\})$$

$$\mathrm{Split}(\{\nu : \mathit{Int}|q_1\} <:_{\Theta,\Lambda} \{\nu : \mathit{Int}|q_2\}) =$$

$$\{\{\nu : \mathit{Int}|q_1\} <:_{\Theta,\Lambda} \{\nu : \mathit{Int}|q_2\}\}$$

$$\mathcal{C} := \{c \mid c \text{ is a subtyping condition}\}$$

$$\begin{aligned}\mathcal{C}^- := \{\ &\{\nu : \mathit{Int}|q_1\} <:_{\Theta,\Lambda} \{\nu : \mathit{Int}|q_2\}\\ &\mid (q_1 \in \mathcal{Q} \vee q_1 = [k_1]_{S_1} \text{ for } k_1 \in \mathcal{K}, S_1 \in \mathcal{V} \nrightarrow \mathit{IntExp})\\ &\wedge (q_2 \in \mathcal{Q} \vee q_2 = [k_2]_{S_2} \text{ for } k_2 \in \mathcal{K}, S_2 \in \mathcal{V} \nrightarrow \mathit{IntExp})\}.\end{aligned}$$

The Inference Algorithm: Step 1 (Split)

$$\Theta := \{(a, \{Int|\kappa_1\}), (b, \{Int|\kappa_2\})\}$$

$$\begin{aligned}
C_0 := \{&\{\nu : Int|\nu = b\} <:_{\Theta,\{a<b\}} \{\nu : Int|\kappa_3\},\\
&\{\nu : Int|\nu = a\} <:_{\Theta,\{\neg(a<b)\}} \{\nu : Int|\kappa_3\},\\
&a : \{\nu : Int|\kappa_1\} \to b : \{\nu : Int|\kappa_2\} \to \{\nu : Int|\kappa_3\}\\
&\quad <:_{\{\},\{\}} a : \{\nu : Int|True\} \to b : \{\nu : Int|True\} \to \{\nu : Int|\kappa_4\}
\end{aligned}$$

After Step 1:

$$\begin{aligned}
C := \{&\{\nu : Int|\nu = b\} <:_{\Theta,\{a<b\}} \{\nu : Int|\kappa_3\},\\
&\{\nu : Int|\nu = a\} <:_{\Theta,\{\neg(a<b)\}} \{\nu : Int|\kappa_3\},\\
&\{\nu : Int|True\} <:_{\{\},\{\}} \{\nu : Int|\kappa_1\},\\
&\{\nu : Int|True\} <:_{\{(a,\{\nu:Int|True\})\},\{\}} \{\nu : Int|\kappa_2\},\\
&\{\nu : Int|\kappa_3\} <:_{\Theta,\{\}} \{\nu : Int|\kappa_4\}\}
\end{aligned}$$

The Inference Algorithm: Step 2 (Solve)

$$
\begin{aligned}
\mathit{Init} : \mathcal{P}(\mathcal{V}) &\rightarrow \mathcal{P}(\mathcal{Q}) \\
\mathit{Init}(V) ::= &\{0 < \nu\} \\
&\cup \{a < \nu \mid a \in V\} \\
&\cup \{\nu < 0\} \\
&\cup \{\nu < a \mid a \in V\} \\
&\cup \{\nu = a \mid a \in V\} \\
&\cup \{\nu = 0\} \\
&\cup \{a < \nu \vee \nu = a \mid a \in V\} \\
&\cup \{\nu < a \vee \nu = a \mid a \in V\} \\
&\cup \{0 < \nu \vee \nu = 0\} \\
&\cup \{\nu < 0 \vee \nu = 0\} \\
&\cup \{\neg(\nu = a) \mid a \in V\} \\
&\cup \{\neg(\nu = 0)\}
\end{aligned}
$$

In our example $V := \{a, b\}$

The Inference Algorithm: Step 2 (Solve)

$\mathsf{Solve} : \mathcal{P}(\mathcal{C}^-) \times (\mathcal{K} \nrightarrow \mathcal{P}(\mathcal{Q})) \to (\mathcal{K} \nrightarrow \mathcal{P}(\mathcal{Q}))$

$\mathsf{Solve}(C, A) =$

Let $S := \{(k, \bigwedge Q) \mid (k, Q) \in A\}$.

If there exists $(\{\nu : Int \mid q_1\} <:_{\Theta,\Lambda} \{\nu : Int \mid [k_2]_{S_2}\}) \in C$ such that

$\neg(\forall z \in \mathbb{Z}.\forall i_1 \in \mathsf{value}_\Gamma(\{\nu : Int|r'_1\}).\dots.\forall i_n \in \mathsf{value}_\Gamma(\{\nu : Int|r'_n\}).$

$[[r_1 \wedge p]]_{\{(\nu,z),(b_1,i_1),\dots,(b_n,i_n)\}} \Rightarrow [[r_2]]_{\{(\nu,z),(b_1,i_1),\dots,(b_n,i_n)\}})$

then $\mathsf{Solve}(C, \mathsf{Weaken}(c, A))$ else A

SMT statement:

$$((\bigwedge_{j=0}^{n} [r'_j]_{\{(\nu,b_j)\}}) \wedge r_1 \wedge p) \wedge \neg r_2$$

with free variables $\nu \in \mathbb{Z}$ and $b_i \in \mathbb{Z}$ for $i \in \mathbb{N}_1^n$.

The Inference Algorithm: Step 3 (Weaken)

$$\text{Weaken} : \mathcal{C}^- \times (\mathcal{K} \nrightarrow \mathcal{P}(\mathcal{Q})) \nrightarrow (\mathcal{K} \nrightarrow \mathcal{P}(\mathcal{Q}))$$

$$\begin{aligned}
&\text{Weaken}(\{\nu : \mathit{Int} | x\} <:_{\Theta,\Lambda} \{\nu : \mathit{Int} | [k_2]_{S_2}\}, A) = \\
&\quad \text{Let } S := \{(k, \bigwedge Q) \mid (k, Q) \in A\}, \\
&\qquad Q_2 := \{\ q \\
&\qquad\qquad \mid q \in A(k_2) \\
&\qquad\qquad \wedge(\forall z \in \mathbb{Z}.\forall i_1 \in \text{value}_\Gamma(\{\nu : \mathit{Int} | r_1'\}). \ldots .\forall i_n \in \text{value}_\Gamma(\{\nu : \mathit{Int} | r_n'\}). \\
&\qquad\qquad\quad [[r_1 \wedge p]]_{\{(\nu,z),(b_1,i_1),\ldots,(b_n,i_n)\}} \Rightarrow [[[q]_{S_2}]]_{\{(\nu,z),(b_1,i_1),\ldots,(b_n,i_n)\}})\} \\
&\quad \text{in } \{(k, Q) \mid (k, Q) \in A \wedge k \neq k_2\} \cup \{(k_2, Q_2)\}
\end{aligned}$$

SMT statement:

$$\neg((\bigwedge_{j=0}^{n} [r_j']_{\{(\nu,b_j)\}}) \wedge r_1 \wedge p) \vee r_2$$

with free variables $\nu \in \mathbb{Z}$ and $b_i \in \mathbb{Z}$ for $i \in \mathbb{N}_1^n$ and $r_2 := [q]_{S_2}$.

Demonstration

$$\begin{aligned}
\Theta &:= \{(a, \{\nu : Int | True\}), (b, \{\nu : Int | True\})\} \\
C_0 &:= \{\{\nu : Int | \nu = b\} <:_{\Theta, \{a<b\}} \{\nu : Int | \kappa_3\}, \\
&\qquad \{\nu : Int | \nu = a\} <:_{\Theta, \{\neg(a<b)\}} \{\nu : Int | \kappa_3\}, \\
&\qquad a : \{\nu : Int | \kappa_1\} \to b : \{\nu : Int | \kappa_2\} \to \{\nu : Int | \kappa_3\} \\
&\qquad\quad <:_{\{\},\{\}} a : \{\nu : Int | True\} \to b : \{\nu : Int | True\} \to \{\nu : Int | \kappa_4\}
\end{aligned}$$

Demonstration: Example 1

$$
\begin{aligned}
\Theta &:= \{(a, \{Int|Int\}), (b, \{Int|Int\})\} \\
C_0 &:= \{\{\nu : Int|\nu = b\} <:_{\Theta,\{a<b\}} \{\nu : Int|\kappa_3\}, \\
&\qquad \{\nu : Int|\nu = a\} <:_{\Theta,\{\neg(a<b)\}} \{\nu : Int|\kappa_3\}, \\
&\qquad a : \{\nu : Int|\kappa_1\} \to b : \{\nu : Int|\kappa_2\} \to \{\nu : Int|\kappa_3\} \\
&\qquad\quad <:_{\{\},\{\}} a : \{\nu : Int|True\} \to b : \{\nu : Int|True\} \to \{\nu : Int|\kappa_4\}
\end{aligned}
$$

Example 2: abs

```
abs : {v:Int|True} -> {v:Int|k4}
abs =
  let
    max =
      \a -> \b ->
        if a < b then
          b
        else
          a
  in
  \z ->
    max ((*) z -1) z
```

Example 2: abs

$$\frac{\Gamma, \Delta, \Theta, \Lambda \vdash e_1 : (a : \hat{T}_1 \rightarrow \hat{T}_2) \\ \Gamma, \Delta, \Theta, \Lambda \vdash e_2 : \hat{T}_1 \quad e_2 : e_2' \quad [\hat{T}_2]_{\{(a, e_2')\}} = \hat{T}_3}{\Gamma, \Delta, \Theta, \Lambda \vdash e_1 \ e_2 : \hat{T}_3}$$

Example 3: abs

$$
\begin{aligned}
\Theta &:= \{(a, \{\mathit{Int}|\kappa_1\}), (b, \{\mathit{Int}|(\kappa_2)\})\} \\
C_0 &:= \{\{\nu : \mathit{Int}|\nu = b\} <:_{\Theta,\{a<b\}} \{\nu : \mathit{Int}|\kappa_3\}, \\
&\qquad \{\nu : \mathit{Int}|\nu = a\} <:_{\Theta,\{\neg(a<b)\}} \{\nu : \mathit{Int}|\kappa_3\}, \\
&\qquad z : \{\nu : \mathit{Int}|\kappa_1\} \rightarrow \{\nu : \mathit{Int}|[\kappa_3]_{\{(a,z\ \cdot\ -1),(b,z)\}}\} \\
&\qquad\quad <:_{\{\},\{\}} z : \{\nu : \mathit{Int}|\mathit{True}\} \rightarrow \{\nu : \mathit{Int}|\kappa_4\}
\end{aligned}
$$

Current State

1. Formal language similar to Elm **(DONE)**
2. Extension of the formal language with Liquid Types
    2.1 A formal syntax **(DONE)**
    2.2 A formal type system **(DONE)**
    2.3 Proof that the extension infers the correct types. **(DONE)**
    2.4 Implementation of the inference algorithm. **(DONE)**

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