3.5 Soundness

In this section we want to prove the soundness of the inference rules with respect to the semantics. This means we want to ensure that a if we can infer the type of a program, the program has also got a semantic.

3.5.1 Soundness of the type signiture

The inference rules and the semantics for the type signitures are build structually similar. Thus we will now show that the inference rules have the same result as the semantics.

Theorem 3.1

Let Γ be a type context, $\mathit{ltf} \in$ <list-type-fields>, $a_i \in \mathcal{V}$,$T_i \in \mathcal{T}$ for $i \in \mathbb{N}_1^n$ and $n \in \mathbb{N}_0$. Let $\Gamma \vdash \mathit{ltf} : \{a_1 : T_1, \ldots, a_n : T_n\}$ hold.
—

Then $[\![\mathit{lft}]\!]_\Gamma = \{a_1 : T_1, \ldots, a_n : T_n\}$.

Proof. Let Γ be a type context, $\mathit{ltf} \in$ <list-type-fields>, $a_i \in \mathcal{V}$,$T_i \in \mathcal{T}$ for $i \in \mathbb{N}_1^n$ and $n \in \mathbb{N}_0$. Let $\mathit{ltf} : \{a_1 : T_1, \ldots, a_n : T_n\}$ hold.

- **Case** $ltf =$ "" for $n = 0$: Then by the definition of the semantic the hypothesis holds.
- **Case** $lft = a_1$ ":" T_1 "," lft_1 for $\mathit{ltf}_1 \in$ <list-type-field>: Then by the premise of the inference rule $\Gamma \vdash \mathit{ltf}_1 : \{a_2 : T_2, \ldots, a_n : T_n\}$ holds and by induction hypothesis $[\![\mathit{ltf}_1]\!]_\Gamma = \{a_2 : T_2, \ldots, a_n : T_n\}$. Therefore by the definition of the semantic the hypothesis follows.

□

Theorem 3.2

Let Γ be a type context,$\mathit{lt} \in$ <list-type>,$T_i \in \mathcal{T}$ for $i \in \mathbb{N}_1^n$ and $n \in \mathbb{N}_0$. Let $\Gamma \vdash \mathit{lt} : (T_1, \ldots, T_n)$ hold. Let for all occurences $t \in$ <type> in lt exists $T \in \mathcal{T}$ such that if $\Gamma \vdash t : T$ holds then $[\![t]\!]_\Gamma = T$.
—

Then $[\![\mathit{lt}]\!]_\Gamma = (T_1, \ldots, T_n)$.

Proof. Let Γ be a type context, $T_i \in \mathcal{T}$ for $i \in \mathbb{N}_1^n$ and $n \in \mathbb{N}_0$. Let $\Gamma \vdash \mathit{lt} : (T_1, \ldots, T_n)$ hold.

- **Case** $l =$ "" for $n = 0$: Then by the definition of the semantic the hypothesis holds.

- **Case** $l = t_1\ l_1$ for $l_1 \in$ `<list-type>`: Then from the premise of the inference rule $\Gamma \vdash l_1 : (T_2, \ldots, T_n)$ and $\Gamma \vdash t_1 : T_1$ hold and by our premise $[\![t_1]\!]_\Gamma = T_1$ for $T_1 \in \mathcal{T}$ and therefore by the definition of the semantic the hypothesis holds.

$\square$

Theorem 3.3

Let Γ be a type context, $t \in$ `<type>` and $T \in \mathcal{T}$. Let $\Gamma \vdash t : T$ hold.

—

Then $[\![t]\!]_\Gamma = T$.

Proof. Let Γ be a type context, $t \in$ `<type>` and $T \in \mathcal{T}$. Let $\Gamma \vdash t : T$ hold.

- **Case** $t =$ `"Bool"`: Then by the definition of the semantic the hypothesis holds.
- **Case** $t =$ `"Int"`: Then by the premise of the inference rule $\Gamma \vdash t : \mathit{Int}$ holds and therefore by the definition of the semantic the hypothesis holds.
- **Case** $t =$ `"List"` t_2, for $t_2 \in$ `<type>`: By the premise of the inference rule $\Gamma \vdash t_2 : T_2$ holds and by induction hypothesis $[\![t_2]\!]_\Gamma = T_2$ for given $T_2 \in \mathcal{T}$. Then by the definition of the semantic the hypothesis holds.
- **Case** $t =$ `"("` t_1 `","` t_2 `")"`, for $t_1, t_2 \in$ `<type>`: By the premise of the inference rule $\Gamma \vdash t_1 : T_1$ and $\Gamma \vdash t_2 : T_2$ hold for given $T_1, T_2 \in \mathcal{T}$. Then by induction hypothesis $[\![t_1]\!]_\Gamma = T_1$ and $[\![t_2]\!]_\Gamma = T_2$. Thus by the definition of the semantic the hypothesis holds.
- **Case** $t =$ `"{"` ltf `"}"`, for $\mathit{ltf} \in$ `<list-type-field>`: Then by the premise of the inference rule $\Gamma \vdash \mathit{ltf} : \{a_1 : T_1, \ldots, a_n : T_n\}$ for $a_i \in \mathcal{V}$,$T_i \in \mathcal{T}$,$i \in \mathbb{N}_1^n$ and $n \in \mathbb{N}_0$. Thus by Theorem 3.1 $[\![\mathit{ltf}]\!]_\Gamma = T$ and therefore the hypothesis holds.
- **Case** $t = t_1$ `"->"` t_2, for $t_1, t_2 \in$ `<type>`: By the premise of the inference rule $\Gamma \vdash t_1 : T_1$ and $\Gamma \vdash t_2 : T_2$ hold for given $T_1, T_2 \in \mathcal{T}$. By induction hypothesis $[\![t_i]\!]_\Gamma = T_i$ for $i \in \{1, 2\}$. Thus by the definition of the semantic the hypothesis holds.
- **Case** $t = c\ \mathit{lt}$ for $\mathit{lt} \in$ `<list-type>` and $c \in$ `<upper-var>`: By the premise of the inference rule $(c, T') \in \Gamma$ with $T' \in \mathcal{T}$ and $\Gamma \vdash \mathit{lt} : (T_0, \ldots, T_n)$. By applying the induction hypothesis and Theorem 3.2 we know $[\![\mathit{lt}]\!]_\Gamma = (T_1, \ldots, T_n)$ for $T_i \in \mathcal{T}$,$i \in \mathbb{N}^n$ and $n \in \mathbb{N}_0$. Thus by the definition of the semantic the hypothesis holds.
- **Case** $t = a$ for $a \in \mathcal{V}$: Then by the definition of the semantic the hypothesis holds.

$\square$

3.5.2 Soundness of the variable context

In our previous sections we had two different meanings for Δ. We now want to show, that these two definitions correlate.

Definition 3.1: Well formed variable context

Let Γ, Δ be type contexts and Δ' a variable context.

—

We say Δ' is *well defined with respect to* Δ $:\Leftrightarrow$

$$\forall T \in \mathcal{T}. \forall a \in \mathcal{V}. (a, T) \in \Delta \Rightarrow \exists e \in \text{value}_\Gamma(T). (a, e) \in \Delta'.$$

We will now show that our semantic ensures that all used variable contexts are well formed.

//Todo

3.5.3 Soundness of the expression semantic

We can now use the definition of well formed variable contexts, to prove the soundness of the expression semantics.

Theorem 3.4

Let Γ,Δ be type contexts, Δ' be a well formed variable context with respect to Δ and $\mathit{lef} \in$ `<list-exp-field>`. Let the judgment $\Gamma, \Delta \vdash \mathit{lef} : T$ hold for $T = \{a_1 : T_1, \dots, a_n : T_n\} \in \mathcal{T}$, $a_i \in \mathcal{V}, T_i \in \mathcal{T}$, given $i \in \mathbb{N}_1^n$ and $n \in \mathbb{N}_0$. Let for all occurences $e \in$ `<exp>` in lef exist a $T \in \mathcal{T}$ such that $[\![e]\!]_{\Gamma,\Delta'} \in \text{value}_\Gamma(T)$.

—

Then $[\![\mathit{lef}]\!]_{\Gamma,\Delta'} \in \text{value}_\Gamma(T)$.

Proof. Let Γ,Δ be type contexts, Δ' be a well formed variable context with respect to Δ and $\mathit{lef} \in$ `<list-exp-field>`. Let the judgment $\Gamma, \Delta \vdash \mathit{lef} : T$ hold for $T = \{a_1 : T_1, \dots, a_n : T_n\} \in \mathcal{T}$, $a_i \in \mathcal{V}, T_i \in \mathcal{T}$, given $i \in \mathbb{N}_1^n$ and $n \in \mathbb{N}_0$.

- **Case** $\mathit{lef} = a_1$ `"="` e for $e \in$ `<exp>` and $n = 1$: Then by the premise of the inference rule $\Gamma, \Delta \vdash e : T_1$ and therfore by our premise $[\![e]\!]_{\Gamma,\Delta'} \in \text{value}_\Gamma(T)$. Thus by the definition of the semantic the hypothesis holds.
- **Case** $\mathit{lef} = a_1$ `"="` e `","` lef_0 for $e \in$ `<exp>` and $\mathit{lef}_0 \in$ `<list-exp-field>`: Then by the premise of the inference rule $\Gamma, \Delta \vdash \mathit{lef}_0 : T$ and $\Gamma, \Delta \vdash e : T_1$ and therefore by our premise $[\![e]\!]_{\Gamma,\Delta'} \in \text{value}_\Gamma(T_1)$ and by induction hypothesis $[\![\mathit{lef}_0]\!]_{\Gamma,\Delta'} \in \text{value}_\Gamma(\{a_2 : T_2, \dots, a_n : T_n\})$. Thus by the definition of the semantic the hypothesis holds.

□

Theorem 3.5

Let $b \in$ `<bool>`. Let $b : Bool$.

—

Then $[\![b]\!] \in \text{value}_{\varnothing}(Bool)$.

Proof. //Todo □

Theorem 3.6

Let $i \in$ `<int>`. Let $i : Int$.

—

Then $[\![i]\!] \in \text{value}_{\varnothing}(Int)$.

Proof. //Todo □

Theorem 3.7

Let Γ, Δ be type contexts, Δ' be a well formed variable context with respect to Δ and $le \in$ `<list-exp>`. Let $\Gamma, \Delta \vdash le : List\ T$ for $T \in \mathcal{T}$. Let $[\![e]\!] = T$ for all occurences $e \in$ `<exp>` in le.

—

Then $[\![le]\!]_{\Gamma,\Delta'} \in \text{value}_{\Gamma}(List\ T)$.

Proof. //Todo □

Theorem 3.8

Let Γ, Δ be type contexts, Δ' be a well formed variale context with respect to Δ. Let $e \in$ `<exp>` and $T \in \mathcal{T}$. Let $\Delta, \Gamma \vdash e : T$ be valid.

—

Then $[\![e]\!]_{\Gamma,\Delta'} \in \text{value}_{\Gamma}(T)$.

Proof. //Todo □

3.5.4 Soundness of the Statement Semantic

Statements are modeled as operations on either the type context or the variable context. We will now show that their definitions described in the semantics are the same as the definition in the inference rules.

Theorem 3.9

Let $lsv \in$ <list-statement-var>, $a_i \in \mathbb{N}_1^n$ for $n \in \mathbb{N}_0$. Let $lsv : (a_1, \ldots, a_n)$ hold.

—

Then $[\![\texttt{lsv}]\!] \in \mathcal{V}*$.

Proof. //Todo □

Theorem 3.10

Let $\Gamma_1, \Delta_1, \Gamma_2, \Delta_2$ be type contexts and Δ_1', Δ_2' be well formed variable contexts with respect to Δ_1 and Δ_2 respectively. Let $ls \in$ <list-statement> such that $\Gamma_1, \Delta_1, ls \vdash \Gamma_2, \Delta_2$ holds. Let for all occurences $s \in$ <statement> in ls exist a type context Γ_3 and variable context Δ_3 such that $[\![ls]\!](\Gamma_1, \Delta_1) = (\Gamma_3, \Delta_3)$.

—

Then $[\![ls]\!](\Gamma_1, \Delta_1') = (\Gamma_2, \Delta_2')$.

Proof. //Todo □

Theorem 3.11

Let $\Gamma_1, \Gamma_2, \Delta_1, \Delta_2$ be type contexts and Δ_1', Δ_2' be well defined variable contexts with respect to Δ_1, Δ_2. Let $s \in$ <statement> and $\Gamma_1, \Delta_1, s \vdash \Gamma_2, \Delta_2$ hold.

—

Then $[\![s]\!](\Gamma_1, \Delta_1') = (\Gamma_2, \Delta_2')$.

Proof. //Todo □

3.5.5 Soundness of the Program Semantic

A program is a sequence of statements. Starting with an empty type context, and an empty variable context, one statement at the time will be applied, resulting in a value e, a type T and a type context Γ such that $e \in \text{value}_\Gamma(T)$.

Theorem 3.12

Let $p \in$ <program> and $T \in \mathcal{T}$ such that $p : T$.

—

Then there exists a type context Γ such that $[\![p]\!] \in \text{value}_\Gamma(T)$.

Proof. //Todo □