3.4 Denotational Semantic

We will now expore the semantics of the formal language. To do so, we first define a new context.

Definition 3.1: Variable Context

Let Γ be a type context.

—

$\Delta : \mathcal{V} \nrightarrow \bigcup_{T\in\mathcal{T}} \text{value}_\Gamma(T)$ is called a *variable context*.

The semantics of the type signature was already defined in the last section, as the semantic of a type signature is its type. We therefore define the same concept but now in a denotational style.

Definition 3.2: Type Signature Semantic

Let $T, T' \in \mathcal{T}$, $c, a_0, a \in \mathcal{V}$. Let $t_0, t_1, t_2 \in$ `<type>`, $ltf \in$ `<list-type-fields>` and $lt \in$ `<list-type>`. Let Γ be a type context.

—

Let $s \in (\mathcal{V} \times \mathcal{T})^*$ for the following function.

$$
\begin{aligned}
[\![.]\!]_\Gamma &: \texttt{<list-type-fields>} \to (\mathcal{V} \times \mathcal{T})^* \\
[\![\texttt{""}]\!]_\Gamma = s &:\Leftrightarrow s = (\,) \\
[\![a_0 \quad \texttt{":"} \quad t_0 \quad \texttt{","} \quad ltf]\!]_\Gamma = s &:\Leftrightarrow T_0 = [\![t_0]\!]_\Gamma \\
&\quad\wedge s = ((a_0, T_0), \ldots, (a_n, T_n)) \\
&\quad\wedge [\![ltf]\!]_\Gamma = ((a_1, T_1), \ldots, (a_n, T_n)) \\
&\quad\text{where } n \in \mathbb{N} \text{ and } T_i \in \mathcal{T}, a_i \in \mathcal{V} \text{ for all } i \in \mathbb{N}_0^n
\end{aligned}
$$

Let $s \in \mathcal{T}^*$ for the following function.

$$
\begin{aligned}
[\![.]\!]_\Gamma &: \texttt{<list-type>} \to \mathcal{T}^* \\
[\![\texttt{""}]\!]_\Gamma = s &:\Leftrightarrow s = (\,) \\
[\![t_0 \quad lt]\!]_\Gamma = s &:\Leftrightarrow T_0 = [\![t_0]\!]_\Gamma \\
&\quad\wedge [\![lt]\!]_\Gamma = (T_1, \ldots, T_n) \\
&\quad\wedge s = (T_0, \ldots, T_n) \\
&\quad\text{where } n \in \mathbb{N} \text{ and } T_i \in \mathcal{T} \text{ for all } i \in \mathbb{N}_0^n
\end{aligned}
$$

Let $s \in \mathcal{T}$ for the following function.

$$
\begin{aligned}
\llbracket . \rrbracket_\Gamma &: \texttt{<type>} \to \mathcal{T} \\
\llbracket \texttt{"Bool"} \rrbracket_\Gamma = s &:\Leftrightarrow s = \mathit{Bool} \\
\llbracket \texttt{"Int"} \rrbracket_\Gamma = s &:\Leftrightarrow s = \mathit{Int} \\
\llbracket \texttt{"List"} \quad t \rrbracket_\Gamma = s &:\Leftrightarrow T = \llbracket \texttt{t} \rrbracket_\Gamma \wedge s = \mathit{List}\ T \\
&\quad \text{where } T \in \mathcal{T} \\
\llbracket \texttt{"("} \quad t_1 \quad \texttt{","} \quad t_2 \quad \texttt{")"} \rrbracket_\Gamma = s &:\Leftrightarrow T_1 = \llbracket t_1 \rrbracket_\Gamma \wedge T_2 = \llbracket t_2 \rrbracket_\Gamma \wedge s = (T_1, T_2) \\
&\quad \text{where } T_1, T_2 \in \mathcal{T} \\
\llbracket \texttt{"\{"} \quad \mathit{ltf} \quad \texttt{"\}"} \rrbracket_\Gamma = s &:\Leftrightarrow \llbracket \mathit{ltf} \rrbracket_\Gamma = ((a_1, T_1), \ldots, (a_n, T_n)) \\
&\quad \wedge s = \{a_1 : T_1, \ldots, a_n : T_n\} \\
&\quad \text{where } n \in \mathbb{N} \text{ and } T_i \in \mathcal{T}, a_i \in \mathcal{V} \text{ for all } i \in \mathbb{N}_0^n \\
\llbracket t_1 \quad \texttt{"->"} \quad t_2 \rrbracket_\Gamma = s &:\Leftrightarrow \llbracket t_1 \rrbracket_\Gamma = T_1 \wedge \llbracket t_2 \rrbracket_\Gamma = T_2 \wedge s = T_1 \to T_2 \\
\llbracket c \ \ \mathit{lt} \rrbracket_\Gamma = s &:\Leftrightarrow (c, T) \in \Gamma \\
&\quad \wedge (T_1, \ldots, T_n) = \llbracket \mathit{lt} \rrbracket_\Gamma \\
&\quad \wedge T' = \overline{T}\ T_1 \ldots T_n \\
&\quad \wedge s = T' \\
&\quad \text{where } n \in \mathbb{N}, T, T' \in \mathcal{T} \text{ and } T_i \in \mathcal{T} \text{ for all } i \in \mathbb{N}_1^n \\
\llbracket a \rrbracket_\Gamma = s &:\Leftrightarrow s = \forall b.b
\end{aligned}
$$

We have already seen the pattern matching predicate in the last section. It is now time to actually define its meaning.

Definition 3.3: Pattern Semantic

Let Γ be a type context and let $\Theta, \Theta_1, \Theta_2, \Theta_3$ be variable contexts. Let $p, p_1, p_2 \in$ `<pattern>`, $lpl \in$ `<list-pattern-list>`, $lps \in$ `<list-pattern-sort>` and $lpv \in$ `<list-pattern-vars>`. Let $b \in$ `<bool>` and $i \in$ `<int>`. Let $c, a \in \mathcal{V}$.

—

Let $s \in \bigcup_{T \in \mathcal{T}} \text{value}_\Gamma(\mathit{List}\ T))$ for the following predicate.

$$
\begin{aligned}
\text{match}_\Theta &: (\bigcup_{T \in \mathcal{T}} \text{value}_\Gamma(\mathit{List}\ T)) \times \texttt{<list-pattern-list>} \\
\text{match}_\Theta(s, \texttt{""}) &:\Leftrightarrow [\,] = s \\
\text{match}_{\Theta_3}(s, p \ \texttt{","} \ \mathit{lpl}) &:\Leftrightarrow [a_0, \ldots, a_n] = s \\
&\quad \wedge \text{match}_{\Theta_1}(a_0, p) \wedge \text{match}_{\Theta_2}(a_1, \ldots, a_n, \mathit{lpl}) \\
&\quad \wedge \varnothing = \Theta_1 \cap \Theta_2 \wedge \Theta_3 = \Theta_1 \cup \Theta_2 \\
&\quad \text{where } n \in \mathbb{N} \text{ and } a_i \in \mathcal{V} \text{ for all } i \in \mathbb{N}_0^n.
\end{aligned}
$$

Let $s \in \bigcup_{T \in \mathcal{T}} \text{value}_\Gamma(T)^*$ for the following predicate.

$$
\begin{aligned}
\mathrm{match}_\Theta : \bigcup_{T\in\mathcal{T}} \mathrm{value}_\Gamma(T)^* &\times \texttt{<list-pattern-sort>} \\
\mathrm{match}_\Theta(s, \texttt{""}) &:\Leftrightarrow (\,) = s \\
\mathrm{match}_{\Theta_3}(s, p\ \mathit{lps}) &:\Leftrightarrow (s_1, \dots, s_n) = s \\
&\quad \wedge \mathrm{match}_{\Theta_1}(s_1, p) \wedge \mathrm{match}_{\Theta_2}(s_2, \dots, s_n, \mathit{lps}) \\
&\quad \wedge \varnothing = \Theta_1 \cap \Theta_2 \wedge \Theta_3 = \Theta_1 \cup \Theta_2 \\
&\quad \text{where } n \in \mathbb{N} \text{ and } s_i \in \mathrm{value}(T) \text{ with } T \in \mathcal{T} \text{ for all } i \in \mathbb{N}_0^n.
\end{aligned}
$$

Let $s \in \mathcal{V}^*$ for the following function.

$$
\begin{aligned}
[\![\,.\,]\!] &: \texttt{<list-pattern-vars>} \to \mathcal{V}^* \\
[\![\texttt{""}]\!] = s &:\Leftrightarrow s = (\,) \\
[\![a_0\ \ \mathit{lpv}]\!] = s &:\Leftrightarrow (a_1, \dots, a_n) = [\![\mathit{lpv}]\!] \wedge (a_0, \dots, a_n) = s \\
&\quad \text{where } n \in \mathbb{N} \text{ and } a_i \in \mathcal{V} \text{ for all } i \in \mathbb{N}_0^n.
\end{aligned}
$$

Let $s \in \bigcup_{T\in\mathcal{T}} \mathrm{value}_\Gamma(T)$ for the following predicate.

$$
\begin{aligned}
\mathrm{match}_\Theta : & \bigcup_{T \in \mathcal{T}} \mathrm{value}_\Gamma(T) \times \texttt{<pattern>} \\
\mathrm{match}_\Theta(s, b) :\Leftrightarrow & \wedge\, b \in \texttt{<bool>} \wedge s = [\![b]\!]_{\Gamma,\varnothing} \\
\mathrm{match}_\Theta(s, i) :\Leftrightarrow & \wedge\, i \in \texttt{<int>} \wedge s = [\![b]\!]_{\Gamma,\varnothing} \\
\mathrm{match}_\Theta(s, \texttt{"["}\ lpl\ \texttt{"]"}) :\Leftrightarrow & \mathrm{match}_\Theta(s, lpl) \\
\mathrm{match}_{\Theta_3}(s, \texttt{"("}\ p_1\ \texttt{","}\ p_2\ \texttt{")"}) :\Leftrightarrow & (s_1, s_2) = s \\
& \wedge \mathrm{match}_{\Theta_1}(s_1, p_1) \wedge \mathrm{match}_{\Theta_2}(s_2, p_2) \\
& \wedge \varnothing = \Theta_1 \cap \Theta_2 \wedge \Theta_3 = \Theta_1 \cup \Theta_2 \\
& \text{where } s_1 \in \mathrm{value}(T_1), s_2 \in \mathrm{value}(T_2) \text{ for} \\
& T_1, T_2 \in \mathcal{T} \\
\mathrm{match}_\Theta(s, c\ lps) :\Leftrightarrow & c\ s_1\ \ldots\ s_n = s \wedge \mathrm{match}_\Theta((s_1, \ldots, s_n), lps) \\
& \text{where } n \in \mathbb{N} \text{ and } s_i \in \mathrm{value}(T) \text{ with } T \in \mathcal{T} \text{ for} \\
& \text{all } i \in \mathbb{N}_0^n. \\
\mathrm{match}_\Theta(s, a) :\Leftrightarrow & s \in \mathcal{V} \wedge \Theta = \{(a, s)\} \\
\mathrm{match}_{\Theta_2}(s, p\ \texttt{"as"}\ a) :\Leftrightarrow & \mathrm{match}_{\Theta_1}(s, p) \\
& \wedge \varnothing = \Theta_1 \cap \{(a, s)\} \wedge \Theta_2 = \Theta_1 \cup \{(a, s)\} \\
\mathrm{match}_\Theta(s, \texttt{"\{"}\ lpv\ \texttt{"\}"}) :\Leftrightarrow & (a_1, \ldots, a_n) = [\![lpv]\!] \\
& \wedge \{a_1 = e_1, \ldots, a_n = e_n\} = s \\
& \wedge \Theta = \{(a_1, e_1), \ldots, (a_n, e_n)\} \\
& \text{where } n \in \mathbb{N} \text{ and } a_i \in \mathcal{V} \text{ for all } i \in \mathbb{N}_0^n. \\
\mathrm{match}_{\Theta_3}(s, p_1\ \texttt{"::"}\ p_2) :\Leftrightarrow & (s_1, \ldots, s_n) = s \wedge \mathrm{match}_{\Theta_1}(s_1, p_1) \\
& \wedge \mathrm{match}_{\Theta_2}((s_2, \ldots, s_n), p_2) \\
& \wedge \varnothing = \Theta_1 \cap \Theta_2 \wedge \Theta_3 = \Theta_1 \cup \Theta_2 \\
& \text{where } n \in \mathbb{N} \text{ and } s_i \in \mathrm{value}(T) \text{ with } T \in \mathcal{T} \text{ for} \\
& \text{all } i \in \mathbb{N}_0^n. \\
\mathrm{match}_\Theta(\texttt{"\_"}) :\Leftrightarrow & \varnothing = \Theta
\end{aligned}
$$

An Elm program is nothing more than an expression. Semantics of an expression is therefore the heart piece of this section.

Definition 3.4: Expression Semantic

Let Γ be a type context and let Δ, Θ be variable contexts. Let $a, a_0, a_1 \in \mathcal{V}$, $e, e_1, e_2, e_3 \in$ <exp>. Let $lef \in$ <list-exp-field>, $t \in$ <type>, $p \in$ <pattern>, $lc \in$ <list-case>, $b \in$ <bool>, $nr \in \mathbb{N}$, $le \in$ <list-exp> and $mes \in$ <maybe-expression-sign>.

—

Let $s \in (\mathcal{V} \times \bigcup_{T \in \mathcal{T}} \mathrm{value}_\Gamma(T))^*$ for the following function.

$$\llbracket . \rrbracket_{\Gamma,\Delta} : \texttt{<list-exp-field>} \to (\mathcal{V} \times \bigcup_{T\in\mathcal{T}} \text{value}_\Gamma(T))^*$$

$$\begin{aligned}
\llbracket a \quad \texttt{"="} \quad e \rrbracket_{\Gamma,\Delta} = s_1 &:\Leftrightarrow s_2 = \llbracket e \rrbracket_{\Gamma,\Delta} \wedge ((a, s_2)) = s_1 \\
&\quad \text{where } s_2 \in \text{value}_\Gamma(T) \text{ for } T \in \mathcal{T} \\
\llbracket a_1 \quad \texttt{"="} \quad e \quad \texttt{","} \quad lef \rrbracket_{\Gamma,\Delta} = s_3 &:\Leftrightarrow ((a_1, s_1)) = \llbracket a \quad \texttt{"="} \quad e \rrbracket_{\Gamma,\Delta} \\
&\quad \wedge ((a_2, s_2), \dots, (a_n, s_n)) = \llbracket lef \rrbracket_{\Gamma,\Delta} \\
&\quad \wedge ((a_1, s_1), \dots, (a_n, s_n)) = s_3 \\
&\quad \text{where } \ n \in \mathbb{N} \text{ and } a_i \in \mathcal{V}, s_i \in \text{value}_\Gamma(T_i) \\
&\quad \text{for } T_i \in \mathcal{T} \text{ for } i \in \mathbb{N}_0^n
\end{aligned}$$

$$\begin{aligned}
\llbracket . \rrbracket_{\Gamma,\Delta} &: \texttt{<maybe-exp-sign>} \to (\,) \\
\llbracket \texttt{""} \rrbracket_{\Gamma,\Delta} = s &:\Leftrightarrow (\,) = s \\
\llbracket a \quad \texttt{":"} \quad t \quad \texttt{";"} \rrbracket_{\Gamma,\Delta} = s &:\Leftrightarrow (\,) = s
\end{aligned}$$

Let $s \in \bigcup_{T\in\mathcal{T}} \text{value}_\Gamma(T))$ for the following function.

$$\llbracket . \rrbracket_{\Gamma,\Delta} : \texttt{<exp>} \to \texttt{<list-case>} \to \bigcup_{T\in\mathcal{T}} \text{value}_\Gamma(T))$$

$$\begin{aligned}
\llbracket e_1, p \quad \texttt{"->"} \quad e_2 \rrbracket_{\Gamma,\Delta} = s &:\Leftrightarrow \text{match}_\Theta(e_1, p) \wedge \llbracket e_2 \rrbracket_{\Gamma,\Delta\cup\Theta} = s \\
\llbracket e_1, p \quad \texttt{"->"} \quad e_2 \quad \texttt{";"} \quad lc \rrbracket_{\Gamma,\Delta} = s &:\Leftrightarrow s = \begin{cases} \llbracket e_2 \rrbracket_{\Gamma,\Delta\cup\Theta} & \text{if match}_\Theta(e_1, p) \\ \llbracket e_1, lc \rrbracket_{\Gamma,\Delta} & \text{else} \end{cases}
\end{aligned}$$

Let $s \in \text{value}_\varnothing(Bool)$ for the following function.

$$\begin{aligned}
\llbracket . \rrbracket &: \texttt{<Bool>} \to \text{value}_\varnothing(Bool) \\
\llbracket b \rrbracket = s &:\Leftrightarrow \begin{cases} True & \text{if } b = \texttt{"True"} \\ False & \text{if } b = \texttt{"False"} \end{cases}
\end{aligned}$$

Let $s \in \text{value}_\varnothing(Int)$ for the following function.

$$\begin{aligned}
\llbracket . \rrbracket &: \texttt{<int>} \to \text{value}_\varnothing(Int) \\
\llbracket \texttt{"0"} \rrbracket = s &:\Leftrightarrow 0 = s \\
\llbracket \texttt{"-"} \quad nr \rrbracket = s &:\Leftrightarrow Neg\ Succ^{nr}\ 1 \\
\llbracket nr \rrbracket = s &:\Leftrightarrow Pos\ Succ^{nr}\ 1
\end{aligned}$$

Let $s \in \bigcup_{T\in\mathcal{T}} \text{value}_\Gamma(T))^*$ for the following function.

$$[\![.]\!]_{\Gamma,\Delta} : \texttt{<list-exp>} \to \bigcup_{T\in\mathcal{T}} \text{value}_\Gamma(T))^*$$

$$[\![\texttt{""}]\!]_{\Gamma,\Delta} = s :\Leftrightarrow (\,) = s$$

$$[\![e \quad \texttt{","} \quad le]\!]_{\Gamma,\Delta} = s :\Leftrightarrow s_1 = [\![e]\!]_{\Gamma,\Delta} \wedge (s_2,\ldots,s_n) = [\![le]\!]_{\Gamma,\Delta} \wedge (s_1,\ldots,s_n) = s$$
$$\text{where } \; n \in \mathbb{N} \; \text{ and } \; s_i \in \text{value}_\Gamma(T_i), T_i \in \mathcal{T} \text{ for each } i \in \mathbb{N}_0^n$$

Let $s \in \bigcup_{T\in\mathcal{T}} \text{value}_\Gamma(T)$ for the following function.

$$[\![.]\!]_{\Gamma,\Delta} : \texttt{<exp>} \to \bigcup_{T\in\mathcal{T}} \text{value}_\Gamma(T)$$

$$[\![\texttt{"foldl"}]\!]_{\Gamma,\Delta} = s :\Leftrightarrow s = \lambda f.\lambda e_1.\lambda l_1.$$
$$\begin{cases} e_1 & \text{if } [\,] = l_1 \\ f(e_2, s(f, e_1, l_2)) & \text{if } \mathit{Cons}\; e_2\; l_2 = l_1 \end{cases}$$
$$\text{where } \; e_1 \in \text{value}_\Gamma(T_1), e_2 \in \text{value}_\Gamma(T_2)$$
$$\text{and } l_1, l_2 \in \text{value}_\Gamma(\mathit{List}\; T_2) \text{ and}$$
$$f \in \text{value}_\Gamma(T_2 \to T_1 \to T_1) \text{ for } T_1, T_2 \in \mathcal{T}$$

$$[\![\texttt{"(::)"}]\!]_{\Gamma,\Delta} = s :\Leftrightarrow s = \lambda e.\lambda l.\mathit{Cons}\; e\; l$$
$$\text{where } \; e \in \text{value}_\Gamma(T) \text{ and } \; l \in \text{value}_\Gamma(\mathit{List}\; T)$$
$$\text{for } T \in \mathcal{T}$$

$$[\![\texttt{"(+)"}]\!]_{\Gamma,\Delta} = s :\Leftrightarrow s = \lambda n.\lambda m.n + m$$
$$\text{where } n, m \in \mathbb{N}$$

$$[\![\texttt{"(-)"}]\!]_{\Gamma,\Delta} = s :\Leftrightarrow s = \lambda n.\lambda m.n - m$$
$$\text{where } n, m \in \mathbb{N}$$

$$[\![\texttt{"(*)"}]\!]_{\Gamma,\Delta} = s :\Leftrightarrow s = \lambda n.\lambda m.n * m$$
$$\text{where } n, m \in \mathbb{N}$$

$$[\![\texttt{"(//)"}]\!]_{\Gamma,\Delta} = s :\Leftrightarrow s = \lambda n.\lambda m.\left\lfloor \frac{n}{m} \right\rfloor$$
$$\text{where } n, m \in \mathbb{N}$$

$$[\![\texttt{"(<)"}]\!]_{\Gamma,\Delta} = s :\Leftrightarrow s = \lambda n.\lambda m.n < m$$
$$\text{where } n, m \in \mathbb{N}$$

$$[\![\texttt{"(==)"}]\!]_{\Gamma,\Delta} = s :\Leftrightarrow s = \lambda n.\lambda m.(n = m)$$
$$\text{where } n, m \in \mathbb{N}$$

$$[\![\texttt{"not"}]\!]_{\Gamma,\Delta} = s :\Leftrightarrow s = \lambda b.\neg b$$
$$\text{where } b \in \text{value}_\Gamma(\mathit{Bool})$$

$$[\![\texttt{"(\&\&)"}]\!]_{\Gamma,\Delta} = s :\Leftrightarrow s = \lambda b_1.\lambda b_2.b_1 \wedge b_2$$
$$\text{where } b_1, n_2 \in \text{value}_\Gamma(\mathit{Bool})$$

$$[\![\texttt{"(||)"}]\!]_{\Gamma,\Delta} = s :\Leftrightarrow s = \lambda b_1.\lambda b_2.b_1 \vee b_2$$
$$\text{where } b_1, n_2 \in \text{value}_\Gamma(\mathit{Bool})$$

$$[\![e_1 \quad \texttt{"|>"} \quad e_2]\!]_{\Gamma,\Delta} = s :\Leftrightarrow s' = [\![e_1]\!]_{\Gamma,\Delta} \wedge f = [\![e_2]\!]_{\Gamma,\Delta} \wedge f(s_1) = s$$
$$\text{where } f \in \text{value}(T_1 \to T_2), s' \in \text{value}(T_1) \text{ for } T_1, T_2 \in \mathcal{T}$$

$$[\![e_1 \quad \texttt{">>"} \quad e_2]\!]_{\Gamma,\Delta} = s :\Leftrightarrow g = [\![e_1]\!]_{\Gamma,\Delta} \wedge f = [\![e_2]\!]_{\Gamma,\Delta} \wedge f \circ g = s$$
$$\text{where } g \in \text{value}(T_1 \to T_2), f \in \text{value}(T_2 \to T_3) \text{ for } T_1, T_2, T_2 \in \mathcal{T}$$

$$\left[\!\!\left[\begin{array}{l}\texttt{"if"}\; e_1 \;\texttt{"then"} \\ e_2 \;\texttt{"else"}\; e_3\end{array}\right]\!\!\right]_{\Gamma,\Delta} = s :\Leftrightarrow b = [\![e_1]\!]_{\Gamma,\Delta} \wedge s = \begin{cases} [\![e_2]\!]_{\Gamma,\Delta} & \text{if } b \\ [\![e_3]\!]_{\Gamma,\Delta} & \text{if } \neg b \end{cases}$$
$$\text{where } b \in \text{value}(\mathit{Bool})$$

$$[\![\texttt{"\{"} \quad \mathit{lef} \quad \texttt{"\}"}]\!]_{\Gamma,\Delta} = s :\Leftrightarrow ((a_1, s_1), \ldots, (a_n, s_n)) = [\![\mathit{lef}]\!]_{\Gamma,\Delta}$$
$$\wedge \{a_1 = s_1, \ldots, a_n = s_n\} = s$$
$$\text{where } n \in \mathbb{N} \text{ and } a_i \in \mathcal{V}, s_i \in \text{value}(T_i), T_i \in \mathcal{T} \text{ for } i \in \mathbb{N}_0^n$$

$$[\![\texttt{"\{ \}"}]\!]_{\Gamma,\Delta} = s :\Leftrightarrow \{\} = s$$

$$[\![\texttt{"\{"} \quad a \quad \texttt{"|"} \quad \mathit{lef} \quad \texttt{"\}"}]\!]_{\Gamma,\Delta} = s :\Leftrightarrow ((a_1, s_1), \ldots, (a_n, s_n)) = [\![\mathit{lef}]\!]_{\Gamma,\Delta}$$
$$\wedge \left(a, \begin{Bmatrix} a_1 = \_, \ldots, a_n = \_, \\ a_{n+1} = s_{n+1}, \ldots, a_m = s_m \end{Bmatrix}\right) \in \Delta$$
$$\wedge \{a_1 = s_1, \ldots, a_m = s_m\} = s$$
$$\text{where } n, m \in \mathbb{N} \text{ such that } n \leq m \text{ and } a_i \in \mathcal{V}, s_i \in \text{value}(T_i), T_i \in \mathcal{T} \text{ for } i \in \mathbb{N}_0^m$$

$$[\![a_0 \quad \texttt{"."} \quad a_1]\!]_{\Gamma,\Delta} = s :\Leftrightarrow (a_0, \{a_1 : s_1, \ldots\})\Delta \wedge s' = s$$
$$\text{where } s' \in \text{value}(T) \text{for } T \in \mathcal{T}$$

$$\left[\!\!\left[\begin{array}{l}\texttt{"let"}\; \mathit{mes}\; a \;\texttt{"="}\; e_1 \\ \texttt{"in"}\; e_2\end{array}\right]\!\!\right]_{\Gamma,\Delta} = s :\Leftrightarrow s' = [\![e_1]\!]_{\Gamma,\Delta} \;\wedge [\![e_2]\!]_{\Gamma,\Delta \cup \{(a,s')\}} = s$$
$$\text{where } s' \in \text{value}(T) for T \in \mathcal{T}$$

$$\left[\!\!\left[\begin{array}{l}\texttt{"case"}\; e \;\texttt{"of"} \\ \texttt{"["}\; \mathit{lc} \;\texttt{"]"}\end{array}\right]\!\!\right]_{\Gamma,\Delta} = s :\Leftrightarrow [\![e, \mathit{lc}]\!]_{\Gamma,\Delta} = s$$

$$[\![e_1 \;\; e_2]\!]_{\Gamma,\Delta} = s :\Leftrightarrow s_1 = [\![e_1]\!]_{\Gamma,\Delta} \wedge s_2 = [\![e_2]\!]_{\Gamma,\Delta} \wedge s_1(s_2) = s$$
$$\text{where } s_1 \in \text{value}_\Gamma(T_1 \to T_2) \text{ and } s_2 \in \text{value}_\Gamma(T_1) \text{for } T_1, T_2 \in \mathcal{T}$$

$$[\![b]\!]_{\Gamma,\Delta} = s :\Leftrightarrow [\![b]\!] = s$$

$$[\![i]\!]_{\Gamma,\Delta} = s :\Leftrightarrow [\![i]\!] = s$$

$$\llbracket \texttt{"["} \quad le \quad \texttt{"]"} \rrbracket_{\Gamma,\Delta} = s :\Leftrightarrow (s_1, \ldots, s_n) = \llbracket le \rrbracket_{\Gamma,\Delta} \wedge [s_1, \ldots, s_n] = s$$
$$\text{where } n \in \mathbb{N} \text{ and } s_i \in \text{value}_\Gamma(T) \text{ for } T \in \mathcal{T}$$

$$\llbracket \texttt{"("} \quad e_1 \quad \texttt{","} \quad e_2 \quad \texttt{")"} \rrbracket_{\Gamma,\Delta} = s :\Leftrightarrow s_1 = \llbracket e_1 \rrbracket \wedge s_2 = \llbracket e_2 \rrbracket \wedge (s_1, s_2) = s$$
$$\text{where } s_1 \in \text{value}_\Gamma(T_1) \text{ and } s_2 \in \text{value}_\Gamma(T_1)$$

$$\llbracket \texttt{"\textbackslash"} \quad p \quad \texttt{"->"} \quad e \rrbracket_{\Gamma,\Delta} = s :\Leftrightarrow \text{match}_\Theta(s_1, p) \wedge s_2 = \llbracket e \rrbracket \wedge \lambda s_1 . s_2 = s$$
$$\text{where } s_1 \in \text{value}_\Gamma(T_1) \text{ and } s_2 \in \text{value}_\Gamma(T_1)$$

$$\llbracket c \rrbracket_{\Gamma,\Delta} = s :\Leftrightarrow (c, s) \in \Delta$$
$$\llbracket a \rrbracket_{\Gamma,\Delta} = s :\Leftrightarrow (a, s) \in \Delta$$

Statements are, semantically speaking, just functions that either map the type- or variable-context.

Definition 3.5: Statement Semantic

Let Γ be a type context. Let $a, a_0 \in \mathcal{V}$, $t \in$ `<type>`, $lsv \in$ `<list-statement-var>`, $lt \in$ `<list-type>`, $lss \in$ `<list-statement-sort>`, $st \in$ `<statement>`, $ls \in$ `<list-statement>`, $mss \in$ `<maybe-statement-sign>` and $mms \in$ `<maybe-main-sign>`. Let $\mathcal{S}$ be the class of all finite sets.

—

Let $s \in \mathcal{V}^*$ for the following function.

$$\llbracket . \rrbracket : \texttt{<list-statement-var>} \to \mathcal{V}^*$$
$$\llbracket \texttt{""} \rrbracket = s :\Leftrightarrow (\,) = s$$
$$\llbracket a_0 \quad lsv \rrbracket = s :\Leftrightarrow (a_1, \ldots, a_n) = \llbracket lsv \rrbracket \wedge (a_0, \ldots, a_n) = s$$
$$\text{where } \; n \in \mathbb{N} \text{ and } a_i \in \mathcal{V} \text{ for } i \in \mathbb{N}_0^n$$

Let $s \in (\mathcal{V} \times \mathcal{T}^*)^*$ for the following function.

$$\llbracket . \rrbracket_\Gamma : \texttt{<list-statement-sort>} \to (\mathcal{V} \times \mathcal{T}^*)^*$$
$$\llbracket a \quad lt \rrbracket_\Gamma = s :\Leftrightarrow l = \llbracket lt \rrbracket_\Gamma \wedge ((a, l)) = s$$
$$\text{where } l \in \mathcal{T}^*$$
$$\llbracket a_0 \quad lt \quad lss \rrbracket_\Gamma = s :\Leftrightarrow ((a_1, l_1), \ldots, (a_n, l_n)) = \llbracket lss \rrbracket$$
$$\wedge \; l_0 = \llbracket lt \rrbracket_\Gamma$$
$$\wedge \; ((a_0, l_0), \ldots, (a_n, l_n)) = s$$
$$\text{where } n \in \mathbb{N} \text{ and } a_i \in \mathcal{V}, l_i \in \mathcal{T}^* \text{ for } i \in \mathbb{N}$$

Let $s \in ((\mathcal{V} \nrightarrow \mathcal{T}) \times (\mathcal{V} \nrightarrow \mathcal{S})) \to ((\mathcal{V} \nrightarrow \mathcal{T}) \times (\mathcal{V} \nrightarrow \mathcal{S}))$ for the following function.

$$
\begin{aligned}
[\![.]\!] : \texttt{<list-statement>} &\to ((\mathcal{V} \nrightarrow \mathcal{T}) \times (\mathcal{V} \nrightarrow \mathcal{S})) \to ((\mathcal{V} \nrightarrow \mathcal{T}) \times (\mathcal{V} \nrightarrow \mathcal{S}))\\
[\![\texttt{""}]\!] = s &:\Leftrightarrow id = s\\
[\![st \quad \texttt{","} \quad ls]\!] = s &:\Leftrightarrow f = [\![st]\!] \wedge g = [\![ls]\!] \wedge g \circ f = s\\
&\quad \text{where } f, g \in ((\mathcal{V} \nrightarrow \mathcal{T}) \times (\mathcal{V} \nrightarrow \mathcal{S})) \to ((\mathcal{V} \nrightarrow \mathcal{T}) \times (\mathcal{V} \nrightarrow \mathcal{S}))
\end{aligned}
$$

$$
\begin{aligned}
[\![.]\!] : &\texttt{<maybe-statement-sign>} \to (\,)\\
[\![\texttt{""}]\!] = s &:\Leftrightarrow (\,) = s\\
[\![a \quad \texttt{":"} \quad t \ \texttt{";"}]\!] = s &:\Leftrightarrow (\,) = s
\end{aligned}
$$

Let $s \in ((\mathcal{V} \nrightarrow \mathcal{T}) \times (\mathcal{V} \nrightarrow \mathcal{S})) \to ((\mathcal{V} \nrightarrow \mathcal{T}) \times (\mathcal{V} \nrightarrow \mathcal{S}))$ for the following function.

$$
\begin{aligned}
[\![.]\!] : \texttt{<statement>} \to &((\mathcal{V} \nrightarrow \mathcal{T}) \times (\mathcal{V} \nrightarrow \mathcal{S})) \to ((\mathcal{V} \nrightarrow \mathcal{T}) \times (\mathcal{V} \nrightarrow \mathcal{S}))\\
[\![mss \quad a \quad \texttt{"="} \quad e]\!](\Gamma, \Delta) = s :\Leftrightarrow &s' = [\![e]\!]_{\Gamma,\Delta} \wedge (\Gamma, \Delta \cup \{(a, s')\}) = s\\
&\text{where } s' \in \text{value}(T) \text{ for } T \in \mathcal{T}\\
\left[\!\!\left[\begin{matrix}\texttt{"type alias"}\\ c \; lsv \; \texttt{"="} \; t\end{matrix}\right]\!\!\right](\Gamma, \Delta) = s :\Leftrightarrow &T = [\![\texttt{t}]\!]_{\Gamma} \wedge (\Gamma \cup \{(c, T)\}, \Delta) = s\\
\left[\!\!\left[\begin{matrix}\texttt{"type"} \; c\\ lsv \; \texttt{"="} \; lss\end{matrix}\right]\!\!\right](\Gamma_1, \Delta_2) = s :\Leftrightarrow &((c_1, (T_{1,1}, \dots, T_{1,k_1})), \dots, (c_n, (T_{n,1}, \dots, T_{n,k_n})))\\
&\wedge T_1 = \mu C.c_1 T_{1,1} \dots T_{1,k_1} | \dots | c_n T_{n,1} \dots T_{n,k_n}\\
&\wedge (a_1, \dots, a_m) = [\![lsv]\!]\\
&\wedge T_2 = \forall a_1. \dots \forall a_m T_1\\
&\wedge \Gamma_2 = \Gamma_1 \cup \{(c, T_2)\}\\
&\wedge \Delta_2 = \left\{\begin{matrix}(c_1, \lambda t_{1,1}. \dots, \lambda t_{1,k_1}.c_1 t_{1,1} \dots t_{1,k_1})\\ \vdots\\ (c_n, \lambda t_{n,1}. \dots, \lambda t_{n,k_n}.c_1 t_{n,1} \dots t_{n,k_n})\end{matrix}\right\}\\
&\wedge \Delta_3 = \Delta_1 \cup \Delta_2\\
&\wedge (\Gamma_2, \Delta_3) = s\\
&\text{where } n, m \in \mathbb{N} \text{ and } k_i \in \mathbb{N}, c_i, a_j \in \mathcal{V}, T_{i,l_i} \in \mathcal{T}\\
&\text{for } i \in \mathbb{N}_1^n, j \in \mathbb{N}_1^m, l_i \in \mathbb{N}_0^{k_i}
\end{aligned}
$$

$$
\begin{aligned}
[\![.]\!] : &\texttt{<maybe-main-sign>} \to (\,)\\
[\![\texttt{""}]\!] = s &:\Leftrightarrow (\,) = s\\
[\![\texttt{"main : "} \quad t \quad \texttt{";"}]\!] = s &:\Leftrightarrow (\,) = s
\end{aligned}
$$

Let $s \in \bigcup_{T \in \mathcal{T}} \text{value}_{\Gamma}(T)$ for the following function.

$$[\![.]\!] : \texttt{<program>} \to \bigcup_{T \in \mathcal{T}} \text{value}_\Gamma(T)$$

$$[\![ls \quad mms \quad \texttt{"main = "} \quad e]\!] = s :\Leftrightarrow (\Gamma, \Delta) = [\![ls]\!](\varnothing, \varnothing) \wedge [\![e]\!]_{\Gamma,\Delta} = s$$

where Γ is a type context and Δ is a variable context.