

4.4 Formulating SMT Statements

So far we have described the inference rules and the subtyping rule. We have yet to describe an algorithm that can derive a valid type for a set of given subtyping rules.

Definition 4.1: Liquid Type Variable

We say $\mathcal{K} := \{\kappa_i \mid i \in \mathbb{N}\}$ is the set of all *liquid type variables*.

Note that κ is a special character.

Definition 4.2: Template

We say \hat{T} is a *template* $:\Leftrightarrow$

$$\begin{aligned} &\hat{T} \text{ is of form } \{\nu : Int \mid [k]_S\} \\ &\text{where } k \in \mathcal{K} \text{ and } S : \mathcal{V} \rightarrow \mathcal{Q} \\ \vee \hat{T} \text{ is of form } a : \{\nu : Int \mid [k]_S\} \rightarrow \hat{T} \\ &\text{where } k \in \mathcal{K}, \hat{T} \text{ is a template and } S : \mathcal{V} \rightarrow \mathcal{Q}. \end{aligned}$$

We define $\mathcal{T}^? := \{\hat{T} \mid \hat{T} \text{ is a template}\}$

A template will be used for a liquid type with unknown refinement. Note that the inference rule for function applications introduces a refinement substitution S . For templates this substitution is not defined and needs to be delayed until the corresponding liquid type has been derived. We will point out whenever the substitution $[k]_S$ will be applied.

Definition 4.3: Type Variable Context

Let $K := \{[k]_S \mid k \in \mathcal{K} \wedge S : \mathcal{V} \rightarrow \mathcal{Q}\}$.

We say $\Theta : \mathcal{V} \rightarrow (\mathcal{Q} \cup K)$ is a type variable context.

Our algorithm will resolve a set of subtyping conditions:

Definition 4.4: Subtyping Condition

We say c is a *Subtyping Condition* $:\Leftrightarrow$

c is of form $\hat{T}_1 <_{\Theta, \Lambda} \hat{T}_2$
 where \hat{T}_1, \hat{T}_2 are a liquid types or templates, Θ is a type variable context and $\Lambda \subset \mathcal{Q}$.

We define $\mathcal{C} := \{c \mid c \text{ is a subtyping condition}\}$

We will also need a function to obtain the set of all liquid type variables of a template or subtyping condition.

Definition 4.5: Vars

$$\begin{aligned}
& \text{Vars} : (\mathcal{T} \cup \mathcal{T}^?) \rightarrow \mathcal{P}(\mathcal{K}) \\
& \text{Vars}(\{\nu \in \text{Int} \mid r\}) = \{\} \\
& \text{Vars}(\{\nu \in \text{Int} \mid \kappa_i\}) = \{\kappa_i\} \\
& \text{Vars}(a : \{\nu \in \text{Int} \mid \kappa_i\} \rightarrow \hat{T}) = \{\kappa_i\} \cup \text{Vars}(\hat{T}) \\
\\
& \text{Vars} : \mathcal{C} \rightarrow \mathcal{P}(\mathcal{K}) \\
& \text{Vars}(\hat{T}_1 <_{\Theta, \Lambda} \hat{T}_2) = \text{Vars}(\hat{T}_1) \cup \text{Vars}(\hat{T}_2) \\
& \cup \{k \mid (_, q) \in \Theta \wedge q = [k]_S \text{ for } k \in \mathcal{K} \text{ and } S : \mathcal{V} \nrightarrow \mathcal{Q}\}
\end{aligned}$$

The main idea of the algorithm is to first generate a set of predicates and then exclude elements until all subtyping conditions are valid for the remaining predicates. By conjunction over all remaining predicates we result in a valid refinement.

We therefore need a function, depending on a set of variable \mathcal{Q} , that will generate a set of predicates. Note that the resulting set should be finite and a subset of \mathcal{Q} . If the generated set is too small, then our resulting subtyping conditions might be too weak.

$$\begin{aligned}
& \text{Init} : \mathcal{P}(\mathcal{V}) \rightarrow \mathcal{P}(\mathcal{Q}) \\
& \text{Init}(V) ::= \{0 < \nu\} \\
& \quad \cup \{a < \nu \mid a \in V\} \\
& \quad \cup \{\nu < 0\} \\
& \quad \cup \{\nu < a \mid a \in V\} \\
& \quad \cup \{\nu = a \mid a \in V\} \\
& \quad \cup \{\nu = 0\} \\
& \quad \cup \{a < \nu \vee \nu = a \mid a \in V\} \\
& \quad \cup \{\nu < a \vee \nu = a \mid a \in V\} \\
& \quad \cup \{0 < \nu \vee \nu = 0\} \\
& \quad \cup \{\nu < 0 \vee \nu = 0\} \\
& \quad \cup \{\neg(\nu = a) \mid a \in V\} \\
& \quad \cup \{\neg(\nu = 0)\}
\end{aligned}$$

We can always extend the realm of predicates if the resulting refinements are too weak.

4.4.1 The Inference Algorithm

$\text{Infer} : \mathcal{P}(\mathcal{C}) \rightarrow (\mathcal{K} \multimap \mathcal{Q})$

$\text{Infer}(C) =$

Let $V := \bigcup_{\hat{T}_1 <_{\Theta, \Lambda} \hat{T}_2 \in C} \{a \mid (a, _) \in \Theta\}$

$Q_0 := \text{Init}(V),$

$A_0 := \{(\kappa, Q_0) \mid \kappa \in \bigcup_{c \in C} \text{Var}(c)\},$

$A := \text{Solve}(\bigcup_{c \in C} \text{Split}(c), A_0)$

in $\{(\kappa, \bigwedge Q) \mid (\kappa, Q) \in A\}$

where $V \subseteq \mathcal{V}, Q_0, Q \subseteq \mathcal{Q}, A_0, A \in \mathcal{K} \multimap \mathcal{Q}, \Theta$ be a type variable context and $\Lambda \subseteq \mathcal{Q}$.

We first split the subtyping conditions for functions into subtyping conditions for simpler templates:

$$\begin{aligned} \mathcal{C}^- := & \{ \{ \nu : \text{Int}|q_1 \} <_{\Theta, \Lambda} \{ \nu : \text{Int}|q_2 \} \\ & \mid (q_1 \in \mathcal{Q} \vee q_1 = [k_1]_{S_1} \text{ for } k_1 \in \mathcal{K}, S_1 \in \mathcal{V} \multimap \mathcal{Q}) \\ & \wedge (q_2 \in \mathcal{Q} \vee q_2 = [k_2]_{S_2} \text{ for } k_2 \in \mathcal{K}, S_2 \in \mathcal{V} \multimap \mathcal{Q}) \}. \end{aligned}$$

With this we can now define the Split function.

$\text{Split} : \mathcal{C} \multimap \mathcal{P}(\mathcal{C}^-)$

$$\begin{aligned} \text{Split}(a : \{ \nu : \text{Int}|q_1 \} \rightarrow \hat{T}_2 <_{\Theta, \Lambda} a : \{ \nu : \text{Int}|q_3 \} \rightarrow \hat{T}_4) = \\ & \{ \{ \nu : \text{Int}|q_3 \} <_{\Theta, \Lambda} \{ \nu : \text{Int}|q_1 \} \} \cup \text{Split}(\hat{T}_2 <_{\Theta \cup \{(a, q_3)\}, \Lambda} \hat{T}_4) \\ \text{Split}(\{ \nu : \text{Int}|q_1 \} <_{\Theta, \Lambda} \{ \nu : \text{Int}|q_2 \}) = \\ & \{ \{ \nu : \text{Int}|q_1 \} <_{\Theta, \Lambda} \{ \nu : \text{Int}|q_2 \} \} \end{aligned}$$

Note that Split will result in an error, if the subtyping condition is not one of the two cases above.

We resolve the obtained subtyping conditions by repeatably checking if a subtyping condition is not valid and removing all predicates that contradict it. By removing the predicate we weaken the resulting refinement.

$\text{Solve} : \mathcal{P}(\mathcal{C}^-) \times (\mathcal{K} \multimap \mathcal{Q}) \rightarrow (\mathcal{K} \multimap \mathcal{Q})$

$\text{Solve}(C, A) =$

Let $S := \{(k, \bigwedge Q) \mid (k, Q) \in A\}$.

If there exists $(\{v : \text{Int} \mid q_1\} <_{\Theta, \Lambda} \{v : \text{Int} \mid [k_2]_{S_2}\}) \in C$ such that

$$\neg(\forall i_1 \in \text{value}_\Gamma(\{\nu : \text{Int} \mid r'_1\}) \dots \forall i_n \in \text{value}_\Gamma(\{\nu : \text{Int} \mid r'_n\})).$$

$$\llbracket r_1 \wedge p \rrbracket_{\{(b_1, i_1), \dots, (b_n, i_n)\}} \Rightarrow \llbracket r_2 \rrbracket_{\{(b_1, i_1), \dots, (b_n, i_n)\}}$$

for $r_2 := \bigwedge [S(\kappa_2)]_{S_2}$, $p := \bigwedge \Lambda$,

$$r_1 := \begin{cases} \bigwedge [S(k_1)]_{S_1} & \text{if } q_1 \text{ has the form } [k_1]_{S_1} \text{ for } k \in \mathcal{K} \text{ and } S_1 \in \mathcal{V} \multimap \mathcal{Q}, \\ q_1 & \text{if } q_1 \in \mathcal{Q} \end{cases},$$

$$\Theta' := \{ (a, r)$$

$$\mid r \text{ has the form } q \wedge (a, q) \in \Theta \wedge q \in \mathcal{Q}$$

$$\vee r \text{ has the form } [[k]_S]_{S_0} \wedge (a, q) \in \Theta$$

$$\wedge q \text{ has the form } [k]_{S_0} \wedge k \in \mathcal{K} \wedge S_0 \in \mathcal{V} \multimap \mathcal{Q} \}$$

$$\{(b_1, r'_1), \dots, (b_n, r'_n)\} = \Theta'$$

then $\text{Solve}(C, \text{Weaken}(c, A))$ else A

where $k, k_2 \in \mathcal{K}, S : \mathcal{K} \multimap \mathcal{Q}, Q, \Lambda \subseteq \mathcal{Q}, S_1, S_2 : \mathcal{V} \multimap \mathcal{Q}, q_1 \in \mathcal{K} \cup \mathcal{Q}$,

Θ be a type variable context, $r_1, p, r_2 \in \mathcal{Q}, a \in \mathcal{V}, \Theta' : \mathcal{V} \multimap \mathcal{Q}, r \in \mathcal{Q}, n \in \mathbb{N}, b_i \in \mathcal{V}$,

$r_i \in \mathcal{Q}$ for $i \in \mathbb{N}_0^n$ and $[t]_A$ denotes the substitution for the term t with a

substitution A .

Note that we can use a SMT solver to validate

$$\neg(\forall i_1 \in \text{value}_\Gamma(\{\nu : \text{Int} \mid r'_1\}) \dots \forall i_n \in \text{value}_\Gamma(\{\nu : \text{Int} \mid r'_n\})).$$

$$\llbracket r_1 \wedge p \rrbracket_{\{(b_1, i_1), \dots, (b_n, i_n)\}} \Rightarrow \llbracket r_2 \rrbracket_{\{(b_1, i_1), \dots, (b_n, i_n)\}}$$

by deciding the satisfiability of

$$(\bigwedge_{j=0}^n r'_j) \wedge r_1 \wedge p \wedge \neg r_2$$

with free variables $b_i \in \mathbb{N}$ for $i \in \mathbb{N}_1^n$.

Weaken : $\mathcal{C}^- \times (\mathcal{K} \multimap \mathcal{Q}) \multimap (\mathcal{K} \multimap \mathcal{Q})$

Weaken($\{\nu : Int|x\} <_{\Theta, \Lambda} \{\nu : Int|[k_2]_{S_2}\}, A) =$

Let $S := \{(k, \bigwedge Q) \mid (k, Q) \in A\},$

$$r_1 := \begin{cases} \bigwedge [S(k_1)]_{S_1} & \text{if } q_1 \text{ has the form } [k_1]_{S_1} \text{ for } k \in \mathcal{K} \text{ and } S_1 \in \mathcal{V} \multimap \mathcal{Q} \\ q_1 & \text{if } q_1 \in \mathcal{Q} \end{cases},$$

$$p := \bigwedge \{[q]_S \mid q \in \Lambda\},$$

$$\Theta' := \{ (a, r)$$

$$\mid r \text{ has the form } q \wedge (a, q) \in \Theta \wedge q \in \mathcal{Q}$$

$$\vee r \text{ has the form } [[k]_S]_{S_0} \wedge (a, q) \in \Theta$$

$$\wedge q \text{ has the form } [k]_{S_0} \wedge k \in \mathcal{K} \wedge S_0 \in \mathcal{V} \multimap \mathcal{Q}\}$$

$$\{(b_1, r'_1), \dots, (b_n, r'_n)\} = \Theta'$$

$$Q_2 := \{ q$$

$$\mid q \in A(k_2) \wedge \text{wellFormed}(q, \{(b_1, \{\nu : Int|r'_1\}), \dots, (b_n, \{\nu : Int|r'_n\})\})$$

$$\wedge (\forall i_1 \in \text{value}_\Gamma(\{\nu : Int|r'_1\}) \dots \forall i_n \in \text{value}_\Gamma(\{\nu : Int|r'_n\})).$$

$$\llbracket r_1 \wedge p \rrbracket_{\{(b_1, i_1), \dots, (b_n, i_n)\}} \Rightarrow \llbracket [q]_{S_2} \rrbracket_{\{(b_1, i_1), \dots, (b_n, i_n)\}} \}$$

in $\{(k, Q) \mid (k, Q) \in A \wedge k \neq k_2\} \cup \{(k_2, Q_2)\}$

where $k, k_1 \in \mathcal{K}, Q, Q_2 \subseteq \mathcal{Q}, S : \mathcal{K} \multimap \mathcal{Q}, r_1 \in \mathcal{Q}, p \in \mathcal{Q}, S_1, S_2 : \mathcal{V} \multimap \mathcal{Q}, \Theta' : \mathcal{V} \multimap \mathcal{T},$

$a \in \mathcal{V}, T' \in \mathcal{T} \cup \mathcal{T}^?, n \in \mathbb{N}, b_i \in \mathcal{V}, T_i \in \mathcal{T}$ for $i \in \mathbb{N}_0^n$ and $[t]_A$ denotes the

substitution for the term t with a substitution A .

Note that we can use a SMT solver to validate

$$\forall i_1 \in \text{value}_\Gamma(\{\nu : Int|r'_1\}) \dots \forall i_n \in \text{value}_\Gamma(\{\nu : Int|r'_n\}).$$

$$\llbracket r_1 \wedge p \rrbracket_{\{(b_1, i_1), \dots, (b_n, i_n)\}} \Rightarrow \llbracket [q]_{S_2} \rrbracket_{\{(b_1, i_1), \dots, (b_n, i_n)\}}$$

To do so, we first need to compute $r_2 := [q]_{S_2}$, with that we can now use a SMT solver to decide the satisfiability of

$$\neg((\bigwedge_{j=0}^n r'_j) \wedge r_1 \wedge p) \vee r_2$$

with free variables $b_i \in \mathbb{N}$ for $i \in \mathbb{N}_1^n$.

Example 4.1

Assume that we have given the following suptyping conditions:

$$\Theta := \{(a, \{Int|\kappa_1\}), (b, \{Int|\kappa_2\})\}$$

$$C_0 := \{\{\nu : Int|\nu = b\} <_{\Theta, \{a < b\}} \{\nu : Int|\kappa_3\},$$

$$\{\nu : Int|\nu = a\} <_{\Theta, \{\neg(a < b)\}} \{\nu : Int|\kappa_3\},$$

$$a : \{\nu : Int|\kappa_1\} \rightarrow b : \{\nu : Int|\kappa_2\} \rightarrow \{\nu : Int|\kappa_3\}$$

$$<_{\{\cdot, \cdot\}} a : \{\nu : Int|True\} \rightarrow b : \{\nu : Int|True\} \rightarrow \{\nu : Int|\kappa_4\}$$

Then $V := \{a, b\}$ and $A_0 := \{(\kappa_1, \text{Init}(V)), (\kappa_2, \text{Init}(V)), (\kappa_3, \text{Init}(V)), (\kappa_4, \text{Init}(V))\}$.

Splitting the conditions

We will only consider the last subtyping condition of C_0 , all other conditions do not need to be split.

$$\begin{aligned}
& \text{Split}(a : \{\nu : \text{Int}|\kappa_1\} \rightarrow b : \{\nu : \text{Int}|\kappa_2\} \rightarrow \{\nu : \text{Int}|\kappa_3\} \\
& \quad <_{:\{\},\{\}} a : \{\nu : \text{Int}|\text{True}\} \rightarrow b : \{\nu : \text{Int}|\text{True}\} \rightarrow \{\nu : \text{Int}|\kappa_4\}) \\
& = \text{Split}(a : \{\nu : \text{Int}|\kappa_1\} <_{:\{\},\{\}} a : \{\nu : \text{Int}|\text{True}\}) \\
& \quad \cup \text{Split}(b : \{\nu : \text{Int}|\kappa_2\} \rightarrow \{\nu : \text{Int}|\kappa_3\} \\
& \quad \quad <_{:(a, \{\nu : \text{Int}|\text{True}\}), \{\}} b : \{\nu : \text{Int}|\text{True}\} \rightarrow \{\nu : \text{Int}|\kappa_4\}) \\
& = \{a : \{\nu : \text{Int}|\text{True}\} <_{:\{\},\{\}} a : \{\nu : \text{Int}|\kappa_1\}\} \\
& \quad \cup \text{Split}(b : \{\nu : \text{Int}|\text{True}\} <_{:(a, \{\nu : \text{Int}|\text{True}\}), \{\}} b : \{\nu : \text{Int}|\kappa_2\}) \\
& \quad \cup \text{Split}(\{\nu : \text{Int}|\kappa_3\} <_{:\Theta, \{\}} \{\nu : \text{Int}|\kappa_4\}) \\
& = \{\{\nu : \text{Int}|\text{True}\} <_{:\{\},\{\}} \{\nu : \text{Int}|\kappa_1\}, \\
& \quad \{\nu : \text{Int}|\text{True}\} <_{:(a, \{\nu : \text{Int}|\text{True}\}), \{\}} \{\nu : \text{Int}|\kappa_2\}, \\
& \quad \{\nu : \text{Int}|\kappa_3\} <_{:\Theta, \{\}} \{\nu : \text{Int}|\kappa_4\}\}
\end{aligned}$$

So in conclusion we have the following set of subtypings conditions:

$$\begin{aligned}
C := & \{\{\nu : \text{Int}|\nu = b\} <_{:\Theta, \{a < b\}} \{\nu : \text{Int}|\kappa_3\}, \\
& \{\nu : \text{Int}|\nu = a\} <_{:\Theta, \{\neg(a < b)\}} \{\nu : \text{Int}|\kappa_3\}, \\
& \{\nu : \text{Int}|\text{True}\} <_{:\{\},\{\}} \{\nu : \text{Int}|\kappa_1\}, \\
& \{\nu : \text{Int}|\text{True}\} <_{:(a, \{\nu : \text{Int}|\text{True}\}), \{\}} \{\nu : \text{Int}|\kappa_2\}, \\
& \{\nu : \text{Int}|\kappa_3\} <_{:\Theta, \{\}} \{\nu : \text{Int}|\kappa_4\}\}
\end{aligned}$$

We therefore now will go through each condition $c \in C$ and check its validity.

Iteration 1, Case $c = \{\nu : \text{Int}|\nu = b\} <_{:\Theta, \{a < b\}} \{\nu : \text{Int}|\kappa_3\}$:

We define $S := \{(\kappa_1, \wedge \text{Init}(V)), (\kappa_2, \wedge \text{Init}(V)), (\kappa_3, \wedge \text{Init}(V)), (\kappa_4, \wedge \text{Init}(V))\}$.

$\text{Init}(V)$ contains $\nu = 0$ and $\neg \nu = 0$, so we know that $\wedge \text{Init}(V)$ can be simplified to *False*.

We now check if

$$\begin{aligned}
& \forall a \in \text{values}_{\{\}}(\{\nu : \text{Int} | \text{False}\}). \\
& \forall b \in \text{values}_{\{\}}(\{\nu : \text{Int} | \text{False}\}). \\
& \forall \nu \in \text{values}_{\{\}}(\{\nu : \text{Int} | \text{True}\}). \\
& \nu = b \wedge a < b \\
& \models \forall a \in \text{values}_{\{\}}(\{\nu : \text{Int} | \text{False}\}). \\
& \quad \forall b \in \text{values}_{\{\}}(\{\nu : \text{Int} | \text{False}\}). \\
& \quad \forall \nu \in \text{values}_{\{\}}(\{\nu : \text{Int} | \text{True}\}). \\
& \quad \text{False}
\end{aligned}$$

is not valid.

We know that $\text{values}_{\{\}}(\{\nu : \text{False}\}) = \{\}$, and therefore this can be simplified to $\text{True} \models \text{True}$, which is valid.

Iteration 1, Case $c = \{\nu : \text{Int} | \nu = a\} <_{:\Theta, \{\neg(a < b)\}} \{\nu : \text{Int} | \kappa_3\}$:

The argument is analogously to the previous case.

Iteration 1, Case $c = \{\nu : \text{Int} | \text{True}\} <_{:\{\}, \{\}} \{\nu : \text{Int} | \kappa_1\}$:

We now check if

$$\forall \nu \in \text{values}_{\{\}}(\{\nu : \text{Int} | \text{True}\}). \text{True} \models \forall \nu \in \text{values}_{\{\}}(\{\nu : \text{Int} | \text{True}\}). \text{False}$$

is valid. This time we can ignore the quantifiers and thus it simplifies to $\text{True} \models \text{False}$, which is not valid.

We therefore will now weaken A_0 . To do so we compute all $q \in A_0(\kappa_1)$ such that $\text{wellFormed}(q)$ and

$$\forall \nu \in \text{values}_{\{\}}(\{\nu : \text{Int} | \text{True}\}). \llbracket \text{True} \rrbracket_{\{\}} \models \forall \nu \in \text{values}_{\{\}}(\{\nu : \text{Int} | \text{True}\}). \llbracket q \rrbracket_{\{\}}.$$

There are only two values for q that are well formed: True and False .

The resulting set is $Q_2 := \{\text{True}\}$ and thus we replace A_0 with

$$A := \{(\kappa_1, \{\text{True}\}), (\kappa_2, \text{Init}(V)), (\kappa_3, \text{Init}(V)), (\kappa_4, \text{Init}(V))\}$$

Iteration 1, Case $c = \{\nu : \text{Int} | \text{True}\} <_{:\{(a, \{\nu : \text{Int} | \text{True}\})\}, \{\}} \{\nu : \text{Int} | \kappa_2\}$:

The argument is analogously to the previous case, resulting in the updated value for A :

$$A = \{(\kappa_1, \{\text{True}\}), (\kappa_2, \{\text{True}\}), (\kappa_3, \text{Init}(V)), (\kappa_4, \text{Init}(V))\}$$

Iteration 1, Case $c = \{\nu : \text{Int} | \kappa_3\} <_{:\Theta, \{\}} \{\nu : \text{Int} | \kappa_4\}$:

The suptyping condition is valid, analogously to the first case of this iteration.

Iteration 2, Case $c = \{\nu : Int | \nu = b\} <_{\Theta, \{a < b\}} \{\nu : Int | \kappa_3\}$:

We check the validity of

$$\begin{aligned}
& \forall a \in \text{values}_{\{\}}(\{\nu : Int | True\}). \\
& \forall b \in \text{values}_{\{\}}(\{\nu : Int | True\}). \\
& \forall \nu \in \text{values}_{\{\}}(\{\nu : Int | True\}). \\
& \nu = b \wedge a < b \\
& \models \forall a \in \text{values}_{\{\}}(\{\nu : Int | True\}). \\
& \quad \forall b \in \text{values}_{\{\}}(\{\nu : Int | True\}). \\
& \quad \forall \nu \in \text{values}_{\{\}}(\{\nu : Int | True\}). \\
& \quad False.
\end{aligned}$$

It is easy to see, that it is not valid.

Thus we now compute all $q \in A(\kappa_3)$ such that $\text{wellFormed}(q)$ and

$$\begin{aligned}
& \forall a \in \text{values}_{\{\}}(\{\nu : Int | True\}). \\
& \forall b \in \text{values}_{\{\}}(\{\nu : Int | True\}). \\
& \forall \nu \in \text{values}_{\{\}}(\{\nu : Int | True\}). \\
& \nu = b \wedge a < b \\
& \models \forall a \in \text{values}_{\{\}}(\{\nu : Int | True\}). \\
& \quad \forall b \in \text{values}_{\{\}}(\{\nu : Int | True\}). \\
& \quad \forall \nu \in \text{values}_{\{\}}(\{\nu : Int | True\}). \\
& \quad q.
\end{aligned}$$

is valid. The resulting set Q_2 is the following.

$$\begin{aligned}
Q_2 := \{ & a < \nu, \nu = b, \neg(\nu = a), \nu < b \vee \nu = b, b < \nu \vee \nu = b, \nu < a \vee \nu = a, \\
& a < \nu \vee \nu = a \}
\end{aligned}$$

Therefore we update A :

$$\begin{aligned}
A = \{ & (\kappa_1, \{True\}), (\kappa_2, \{True\}), \\
& (\kappa_3, \{a < \nu, \nu = b, \neg(\nu = a), \nu < b \vee \nu = b, b < \nu \vee \nu = b, \nu < a \vee \nu = a, \\
& \quad a < \nu \vee \nu = a\}), \\
& (\kappa_4, \text{Init}(V)) \}
\end{aligned}$$

Iteration 2, Case $c = \{\nu : Int | \nu = a\} <_{\Theta, \{\neg(a < b)\}} \{\nu : Int | \kappa_3\}$:

We check the validity of

$$\begin{aligned}
& \forall a \in \text{values}_{\{\}}(\{\nu : \text{Int} \mid \text{True}\}). \\
& \forall b \in \text{values}_{\{\}}(\{\nu : \text{Int} \mid \text{True}\}). \\
& \forall \nu \in \text{values}_{\{\}}(\{\nu : \text{Int} \mid \text{True}\}). \\
& \nu = a \wedge \neg(a < b) \\
& \models \forall a \in \text{values}_{\{\}}(\{\nu : \text{Int} \mid \text{True}\}). \\
& \quad \forall b \in \text{values}_{\{\}}(\{\nu : \text{Int} \mid \text{True}\}). \\
& \quad \forall \nu \in \text{values}_{\{\}}(\{\nu : \text{Int} \mid \text{True}\}). \\
& \quad a < \nu \wedge \nu = b \wedge \neg(\nu = a) \wedge \nu < b \vee \nu = b \\
& \quad \wedge b < \nu \vee \nu = b \wedge \nu < a \vee \nu = a \wedge a < \nu \vee \nu = a.
\end{aligned}$$

It is not valid, because $\nu = a \wedge \neg(a < b) \models \nu = b$ is not valid.

Thus we compute all $q \in A(\kappa_3)$ such that $\text{wellFormed}(q)$ and

$$\begin{aligned}
& \forall a \in \text{values}_{\{\}}(\{\nu : \text{Int} \mid \text{True}\}). \\
& \forall b \in \text{values}_{\{\}}(\{\nu : \text{Int} \mid \text{True}\}). \\
& \forall \nu \in \text{values}_{\{\}}(\{\nu : \text{Int} \mid \text{True}\}). \\
& \nu = a \wedge \neg(a < b) \\
& \models \forall a \in \text{values}_{\{\}}(\{\nu : \text{Int} \mid \text{True}\}). \\
& \quad \forall b \in \text{values}_{\{\}}(\{\nu : \text{Int} \mid \text{True}\}). \\
& \quad \forall \nu \in \text{values}_{\{\}}(\{\nu : \text{Int} \mid \text{True}\}). \\
& \quad q.
\end{aligned}$$

is valid. The resulting set Q_2 is the following.

$$Q_2 := \{\nu < b \vee \nu = b, \nu < a \vee \nu = a\}$$

Thus we update A :

$$A = \{(\kappa_1, \{\text{True}\}), (\kappa_2, \{\text{True}\}), (\kappa_3, \{\nu < b \vee \nu = b, \nu < a \vee \nu = a\}), (\kappa_4, \text{Init}(V))\}$$

Iteration 2, Case $\{\nu : \text{Int} \mid \text{True}\} <_{\{\}, \{\}} \{\nu : \text{Int} \mid \kappa_1\}$:

Nothing has changed since the last iteration, therefore this case can be skipped.

Iteration 2, Case $\{\nu : \text{Int} \mid \text{True}\} <_{\{(a, \{\nu : \text{Int} \mid \text{True}\})\}, \{\}} \{\nu : \text{Int} \mid \kappa_2\}$:

The argument is analogously to the previous case, therefore this case can be skipped.

Iteration 2, Case : $\{\nu : \text{Int} \mid \kappa_3\} <_{\Theta, \{\}} \{\nu : \text{Int} \mid \kappa_4\}$:

We check the validity of

$$\begin{aligned}
& \forall a \in \text{values}_{\{\}}(\{\nu : \text{Int} \mid \text{True}\}). \\
& \forall b \in \text{values}_{\{\}}(\{\nu : \text{Int} \mid \text{True}\}). \\
& \forall \nu \in \text{values}_{\{\}}(\{\nu : \text{Int} \mid \text{True}\}). \\
& \{\nu < b \vee \nu = b \wedge \nu < a \vee \nu = a\} \\
& \models \forall a \in \text{values}_{\{\}}(\{\nu : \text{Int} \mid \text{True}\}). \\
& \quad \forall b \in \text{values}_{\{\}}(\{\nu : \text{Int} \mid \text{True}\}). \\
& \quad \forall \nu \in \text{values}_{\{\}}(\{\nu : \text{Int} \mid \text{True}\}). \\
& \text{False.}
\end{aligned}$$

We see that this is not valid, therefore we derive the new set Q_2 . Note that $A(\kappa_3) \subseteq \text{Init}(V)$ and therefore $Q_2 = A(\kappa_3)$.

We update the corresponding entry in A :

$$\begin{aligned}
A = & \{(\kappa_1, \{\text{True}\}), (\kappa_2, \{\text{True}\}), \\
& (\kappa_3, \{\nu < b \vee \nu = b, \nu < a \vee \nu = a\}), \\
& (\kappa_4, \{\nu < b \vee \nu = b, \nu < a \vee \nu = a\})\}
\end{aligned}$$

Iteration 3:

In this iteration all subtyping conditions are valid, thus the algorithm stops. The resulting set of substitutions is therefore the following

$$\begin{aligned}
& \{(\kappa_1, \text{True}), (\kappa_2, \text{True}), \\
& (\kappa_3, (\nu < b \vee \nu = b) \wedge (\nu < a \vee \nu = a)), \\
& (\kappa_4, (\nu < b \vee \nu = b) \wedge (\nu < a \vee \nu = a))\}
\end{aligned}$$

4.4.2 Correctness

The algorithm that we described can fail if the subtyping conditions are not well-formed.

Definition 4.6: Wellformed Subtyping Condition

We say a subtyping condition c is well-formed if the following holds

$$\begin{aligned}
& c \text{ has the form } \{\nu : \text{Int} \mid [k_1]_{S_1}\} <_{:\Theta, \Lambda} \{\nu : \text{Int} \mid [k_2]_{S_2}\} \\
& \vee c \text{ has the form } \{\nu : \text{Int} \mid r\} <_{:\Theta, \Lambda} \{\nu : \text{Int} \mid [k_2]_{S_2}\} \\
& \vee c \text{ has the form } \{\nu : \text{Int} \mid [k_1]_{S_1}\} \rightarrow \hat{T}_1 <_{:\Theta, \Lambda} \{\nu : \text{Int} \mid r\} \rightarrow \hat{T}_2 \\
& \text{such that } \hat{T}_1 <_{:\Theta, \Lambda} \hat{T}_2 \text{ is well-formed.}
\end{aligned}$$

where $r \in \mathcal{Q}, k_1, k_2 \in \mathcal{K}, S_1, S_2 : \mathcal{V} \rightarrow \mathcal{Q}, \Theta$ is a type variable context, $\Lambda \subset \mathcal{Q}$ and $T_1, T_2 \in (\mathcal{T} \cup \mathcal{T}^?)$

Theorem 4.1

Let C be a set of well-formed conditions, $S := \text{Infer}(C)$ and $V := \bigcup_{\hat{T}_1 <_{\Theta, \Lambda} \hat{T}_2 \in C} \{a \mid (a, _) \in \Theta\}$.

For every subtyping condition $(\hat{T}_1 <_{\Theta, \Lambda} \hat{T}_2) \in C$, let

$$\begin{aligned} \Theta' := \{ & (a, r) \\ & \mid r \text{ has the form } q \wedge (a, q) \in \Theta \wedge q \in \mathcal{Q} \\ & \vee r \text{ has the form } [[k]_S]_{S_0} \wedge (a, q) \in \Theta \\ & \wedge q \text{ has the form } [k]_{S_0} \wedge k \in \mathcal{K} \wedge S_0 \in \mathcal{V} \rightarrow \mathcal{Q} \} \end{aligned}$$

and $\{(b_1, r'_1), \dots, (b_n, r'_n)\} := \Theta'$. Then we have the following correctness property:

$$\begin{aligned} & [\hat{T}_1]_S \in \mathcal{T} \wedge [\hat{T}_2]_S \in \mathcal{T} \\ & \wedge [\hat{T}_1]_S <_{\Theta', \Lambda} [\hat{T}_2]_S \\ & \wedge \forall S' \in (\mathcal{V} \rightarrow \mathcal{Q}). (\forall a \in \mathcal{V}. S'(a) \text{ is well defined} \Rightarrow \exists Q \in \text{Init}(V). S'(a) = \bigwedge Q) \\ & \wedge [\hat{T}_1]_{S'} \in \mathcal{T} \wedge [\hat{T}_2]_{S'} \in \mathcal{T} \\ & \wedge ([\hat{T}_1]_{S'} <_{\Theta', \Lambda} [\hat{T}_2]_{S'}) \\ & \Rightarrow (\forall a \in \mathcal{V}. S(a) \text{ and } S'(a) \text{ are well defined} \\ & \Rightarrow (\forall i_1 \in \text{value}_\Gamma(\{\nu : \text{Int} \mid r'_1\}) \dots \forall i_n \in \text{value}_\Gamma(\{\nu : \text{Int} \mid r'_n\}). \\ & \quad \llbracket S(a) \rrbracket_{(b_1, i_1), \dots, (b_n, i_n)} \Rightarrow \llbracket S'(a) \rrbracket_{(b_1, i_1), \dots, (b_n, i_n)})) \end{aligned}$$

The first post-condition states that $[\hat{T}_1]_S$ and $[\hat{T}_2]_S$ are not templates. The second condition states that S is a solution, meaning that $[\hat{T}_1]_S$ is a subtype of $[\hat{T}_2]_S$. The third condition states that S is the sharpest solution.

Proof. Let C be a set of well-formed conditions, $S := \text{Infer}(C)$ and $V := \bigcup_{\hat{T}_1 <_{\Theta, \Lambda} \hat{T}_2 \in C} \{a \mid (a, _) \in \Theta\}$. Let $Q_0 := \text{Init}(V)$, $A_0 := \{(\kappa, Q_0) \mid \kappa \in \bigcup_{c \in C} \text{Var}(c)\}$ and $C_1 := \bigcup_{c \in C} \text{Split}(c)$. Then by Theorem 4.2 $C_1 \subset \mathcal{C}^-$ is a set of well-formed conditions. Let $A := \text{Solve}(C_1, A_0)$ and therefore $S = \{(\kappa, \bigwedge Q) \mid (\kappa, Q) \in A\}$. Then by Theorem 4.3 the conclusion holds. \square

Theorem 4.2

Let c is a well-formed condition and $C := \text{Split}(c)$.

Then $C \subseteq \mathcal{C}^-$ and, for every $c \in C$, c is a well-formed condition.

Proof. Let c is a well-formed condition. Then $C := \text{Split}(c)$ is well-defined. We prove the theorem by induction on well-formed conditions.

Case $c = \{\nu : \text{Int}|q\} <_{\Theta, \Lambda} \{\nu : \text{Int}|[k_2]_{S_2}\}$: c is well-formed and $c \in C^-$. Then $C = c$ and therefore the conclusion holds.

Case $c = \{\nu : \text{Int}|[k_1]_{S_1}\} \rightarrow \hat{T}_1 <_{\Theta, \Lambda} \{\nu : \text{Int}|r\} \rightarrow \hat{T}_2$ such that $\hat{T}_1 <_{\Theta, \Lambda} \hat{T}_2$ is well-formed: Let $C_0 := \text{Split}(\hat{T}_1 <_{\Theta, \Lambda} \hat{T}_2)$. By the induction hypothesis $C_0 \subseteq \mathcal{C}^-$ and for every $c \in C_0$, c is a well-formed condition. Therefore, by applying Theorem ?? to each element in $C := \{\{\nu : \text{Int}|r\} <_{\Theta, \Lambda} \{\nu : \text{Int}|[k_1]_{S_1}\}\} \cup C_0$, the conclusion holds. \square

Theorem 4.3

Let $C \subseteq \mathcal{C}^-$ be a set of well-formed conditions, $A_1, A_2 : \mathcal{K} \rightarrow \mathcal{Q}$ and $V := \bigcup_{\hat{T}_1 <_{\Theta, \Lambda} \hat{T}_2 \in C} \{a \mid (a, _) \in \Theta\}$. Let for all $a \in V$, $A_1(a)$ be well defined. Let $A_2 = \text{Solve}(C, A_1)$ and $S = \{(\kappa, \wedge Q) \mid (\kappa, Q) \in A_2\}$.

Then for every $a \in V$, $A_2(a) \subseteq A_1(a)$

For every subtyping condition $(\hat{T}_1 <_{\Theta, \Lambda} \hat{T}_2) \in C$, let

$$\begin{aligned} \Theta' := \{ & (a, r) \\ & \mid r \text{ has the form } q \wedge (a, q) \in \Theta \wedge q \in \mathcal{Q} \\ & \vee r \text{ has the form } [[k]_S]_{S_0} \wedge (a, q) \in \Theta \\ & \wedge q \text{ has the form } [k]_{S_0} \wedge k \in \mathcal{K} \wedge S_0 \in \mathcal{V} \rightarrow \mathcal{Q} \} \end{aligned}$$

and $\{(b_1, r'_1), \dots, (b_n, r'_n)\} = \Theta'$. We then have the following correctness property.

$$\begin{aligned} & [\hat{T}_1]_S \in \mathcal{T} \wedge [\hat{T}_2]_S \in \mathcal{T} \\ & \wedge [\hat{T}_1]_S <_{\Theta', \Lambda} [\hat{T}_2]_S \\ & \wedge \forall S' \in (\mathcal{V} \rightarrow \mathcal{Q}). (\forall a \in V. \exists Q \in \mathcal{P}(A_1(a)). S'(a) = \bigwedge Q) \\ & \wedge [\hat{T}_1]_{S'} \in \mathcal{T} \wedge [\hat{T}_2]_{S'} \in \mathcal{T} \\ & \wedge ([\hat{T}_1]_{S'} <_{\Theta', \Lambda} [\hat{T}_2]_{S'}) \\ & \Rightarrow \forall a \in V. \forall i_1 \in \text{value}_\Gamma(\{\nu : \text{Int}|r'_1\}) \dots \forall i_n \in \text{value}_\Gamma(\{\nu : \text{Int}|r'_n\}). \\ & \quad \llbracket S(a) \rrbracket_{(b_1, i_1), \dots, (b_n, i_n)} \Rightarrow \llbracket S'(a) \rrbracket_{(b_1, i_1), \dots, (b_n, i_n)} \end{aligned}$$

The post-condition is the same as for Infer.

Proof. Let $C \subseteq \mathcal{C}^-$ be a set of well-formed conditions, $A_1, A_2 : \mathcal{K} \rightarrow \mathcal{Q}$ and $V := \bigcup_{\hat{T}_1 <_{\Theta, \Lambda} \hat{T}_2 \in C} \{a \mid (a, _) \in \Theta\}$. Let for all $a \in V$, $A_1(a)$ be well defined. Let $A_2 = \text{Solve}(C, A_1)$ and $S = \{(\kappa, \wedge Q) \mid (\kappa, Q) \in A_2\}$.

Show that the precondition of $\text{Solve}(C, \text{Weaken}(c, A_1))$ holds.

By Theorem 4.4 $A_2(a)$ is defined for all a for which A_1 is defined, namely for all $a \in V$, thus the precondition holds.

Show that the then-branch ensures the postcondition

We know that $A_2 = \text{Solve}(C, \text{Weaken}(c, A_1))$. By the postcondition of $\text{Solve}(C, \text{Weaken}(c, A_1))$, we know that for every $a \in V$, $A_2(a) \subseteq \text{Weaken}(c, A_1)$. By Theorem 4.4 we also know that for every $a \in V$, $\text{Weaken}(c, A_1)(a) \subseteq A_1(a)$. Thus for every $a \in V$, $A_2(a) \subseteq A_1(a)$.

We have left to show that for every $a \in V$, if the sharpest solution is generated by subsets of $A_2(a)$, then it is also the sharpest solution over all subsets of $A_1(a)$.

By theorem 4.4 we know that for every $a \in V$, $A_2(a)$ contains the smallest subset of $A_1(a)$ such that the properties hold. Thus $A_2(a)$ is its definition the sharpest solution over all subsets of $A_1(a)$.

Show that the else-branch ensures the postcondition

Once the recursion is done, we can assume that for all $c \in C$, c has the form $(\{\nu : \text{Int}|q_1\} <_{\Theta, \Lambda} \{\nu : \text{Int}|[k_2]_{S_2}\})$ and the following holds:

$$(\forall i_1 \in \text{value}_\Gamma(\{\nu : \text{Int}|r'_1\}) \dots \forall i_n \in \text{value}_\Gamma(\{\nu : \text{Int}|r'_n\})) \\ \llbracket r_1 \wedge p \rrbracket_{\{(b_1, i_1), \dots, (b_n, i_n)\}} \Rightarrow \llbracket r_2 \rrbracket_{\{(b_1, i_1), \dots, (b_n, i_n)\}}$$

for

$$\begin{aligned} r_2 &:= \bigwedge [S(k_2)]_{S_2}, \quad p := \bigwedge \Lambda, \\ r_1 &:= \begin{cases} \bigwedge [S(k_1)]_{S_1} & \text{if } q_1 \text{ has the form } [k_1]_{S_1} \text{ for } k \in \mathcal{K} \text{ and } S_1 \in \mathcal{V} \multimap \mathcal{Q} \\ q_1 & \text{if } q_1 \in \mathcal{Q} \end{cases}, \\ \Theta' &:= \{ (a, r) \\ &\quad | r \text{ has the form } q \wedge (a, q) \in \Theta \wedge q \in \mathcal{Q} \\ &\quad \vee r \text{ has the form } [[k]_S]_{S_0} \wedge (a, q) \in \Theta \\ &\quad \wedge q \text{ has the form } [k]_{S_0} \wedge k \in \mathcal{K} \wedge S_0 \in \mathcal{V} \multimap \mathcal{Q} \} \\ \{(b_1, r'_1), \dots, (b_n, r'_n)\} &= \Theta'. \end{aligned}$$

where $k, k_2 \in \mathcal{K}$, $S : \mathcal{K} \multimap \mathcal{Q}$, $Q, \Lambda \subseteq \mathcal{Q}$, $S_1, S_2 : \mathcal{V} \multimap \mathcal{Q}$, $q_1 \in \mathcal{K} \cup \mathcal{Q}$, Θ be a type variable context, $r_1, p, r_2 \in \mathcal{Q}$, $a \in \mathcal{V}$, $\Theta' : \mathcal{V} \multimap \mathcal{Q}$, $r \in \mathcal{Q}$, $n \in \mathbb{N}$, $b_i \in \mathcal{V}$, $r_i \in \mathcal{Q}$ for $i \in \mathbb{N}_0^n$ and $[t]_A$ denotes the substitution for the term t with a substitution A . This is the same as saying $[\hat{T}_1]_S <_{\Theta', \Lambda} [\hat{T}_2]_S$ if $[\hat{T}_1]_S \in \mathcal{T}$ and $[\hat{T}_2]_S \in \mathcal{T}$. We know by Theorem 4.4 that $A_2(a)$ is defined for all $a \in V$ and thus $[\hat{T}_1]_S \in \mathcal{T} \wedge [\hat{T}_2]_S \in \mathcal{T}$.

We still need to show that it is the sharpest solution. To do so, we will sketch a proof by contradiction. Assume there exists S' and $a \in V$ such that $S'(a)$ is sharper than $S(a)$. If $S(a) = \text{False}$ then there can not exist a sharper refinement $S'(a)$ and therefore this is a contradiction. If $S(a) \neq \text{False}$ then it must have been produced by calling Weaken . This is a contradiction, as by Theorem 4.4 we know that Weaken produces the sharpest solution therefore $S'(a) = S(a)$. \square

Theorem 4.4

Let $c \in \mathcal{C}^-$ be a well-formed condition and therefore c has the form

$$\{\nu : Int|x\} <_{\Theta, \Lambda} \{\nu : Int|[k_2]_{S_2}\}$$

where x has the form $[k_1]_{S_1}$ or r . Let $A_1, A_2 : \mathcal{K} \rightarrow \mathcal{Q}$. Let for all $a \in V$, $A_1(a)$ be well defined and $A_2 = \text{Weaken}(c, A_1)$. Let

$$\begin{aligned} S &:= \{(k, \bigwedge Q) | (k, Q) \in A_2\} \\ r_1 &:= \begin{cases} \bigwedge [S(k_1)]_{S_1} & \text{if } q_1 \text{ has the form } [k_1]_{S_1} \text{ for } k \in \mathcal{K} \text{ and } S_1 \in \mathcal{V} \rightarrow \mathcal{Q} \\ q_1 & \text{if } q_1 \in \mathcal{Q} \end{cases} \end{aligned}$$

Then the following holds

$$\begin{aligned} &(\forall k \neq k_2. A_1(k) = A_2(k)) \\ &\wedge A_2(k_2) \subseteq A_1(k_2) \\ &\wedge [k_2]_S \in \mathcal{Q} \wedge \{\nu : Int|r_1\} <_{\Theta, \Lambda} \{\nu : Int|[[k_2]_S]_{S_2}\} \\ &\wedge \forall A'_2 : \mathcal{V} \rightarrow \mathcal{P}(\mathcal{Q}). (\forall k \neq k_2. A_1(k) = A'_2(k)) \\ &\quad \wedge A'_2(k_2) \subseteq A_1(k_2) \\ &\quad \wedge \text{Let } S' := \{(k, \bigwedge Q) | (k, Q) \in A'_2\}, \\ &\quad \quad r'_1 := \begin{cases} \bigwedge [S(k_1)]_{S_1} & \text{if } q_1 \text{ has the form } [k_1]_{S_1} \text{ for } k \in \mathcal{K} \text{ and } S_1 \in \mathcal{V} \rightarrow \mathcal{Q} \\ q_1 & \text{if } q_1 \in \mathcal{Q} \end{cases} \\ &\quad \text{in } [k_2]_{S'} \in \mathcal{Q} \\ &\quad \wedge \{\nu : Int|r'_1\} <_{\Theta, \Lambda} \{\nu : Int|[[k_2]_{S'}]_{S_2}\} \Rightarrow A'_2 \subseteq A_2 \end{aligned}$$

The first post-condition states that by updating A_1 to A_2 only the value for k_2 changes. The second condition states that the updated value $A_2(k_2)$ needs to be a subset of the old value $A_1(k_2)$. The third condition states that the resulting substitution S generates a refinement $[k_1]_S$ that makes the subtyping condition valid. The fourth condition states that the value $A_2(k_2)$ is the smallest subset of $A_1(k_2)$ such that the previous conditions hold.

Proof. Let $c \in \mathcal{C}^-$ be a well-formed condition and therefore c has the form $\{\nu : Int|x\} <_{\Theta, \Lambda} \{\nu : Int|[k_2]_{S_2}\}$ where x has the form $[k_1]_{S_1}$ or r . Let $A_1, A_2 : \mathcal{K} \rightarrow \mathcal{Q}$.

Let for all $a \in V$, $A_1(a)$ be well defined and $A_2 = \text{Weaken}(c, A_1)$. Let

$$\begin{aligned}
S &:= \{(k, \bigwedge Q) \mid (k, Q) \in A_2\} \\
r_1 &:= \begin{cases} \bigwedge [S(k_1)]_{S_1} & \text{if } q_1 \text{ has the form } [k_1]_{S_1} \text{ for } k \in \mathcal{K} \text{ and } S_1 \in \mathcal{V} \multimap \mathcal{Q} \\ q_1 & \text{if } q_1 \in \mathcal{Q} \end{cases} \\
p &:= \bigwedge \{[q]_S \mid q \in \Lambda\}, \\
\Theta' &:= \{ (a, r) \\
&\quad \mid r \text{ has the form } q \wedge (a, q) \in \Theta \wedge q \in \mathcal{Q} \\
&\quad \vee r \text{ has the form } [[k]_S]_{S_0} \wedge (a, q) \in \Theta \\
&\quad \wedge q \text{ has the form } [k]_{S_0} \wedge k \in \mathcal{K} \wedge S_0 \in \mathcal{V} \multimap \mathcal{Q} \} \\
\{(b_1, r'_1), \dots, (b_n, r'_n)\} &= \Theta' \\
Q_2 &:= \{ q \\
&\quad \mid q \in A(k_2) \wedge \text{wellFormed}(q, \{(b_1, \{\nu : \text{Int}|r'_1\}), \dots, (b_n, \{\nu : \text{Int}|r'_n\})\}) \\
&\quad \wedge (\forall i_1 \in \text{value}_\Gamma(\{\nu : \text{Int}|r'_1\}) \dots \forall i_n \in \text{value}_\Gamma(\{\nu : \text{Int}|r'_n\})) \\
&\quad \llbracket r_1 \wedge p \rrbracket_{\{(b_1, i_1), \dots, (b_n, i_n)\}} \Rightarrow \llbracket [q]_{S_2} \rrbracket_{\{(b_1, i_1), \dots, (b_n, i_n)\}} \}
\end{aligned}$$

where $k, k_1 \in \mathcal{K}, Q, Q_2 \subseteq \mathcal{Q}, S : \mathcal{K} \multimap \mathcal{Q}, r_1 \in \mathcal{Q}, p \in \mathcal{Q}, S_1, S_2 : \mathcal{V} \multimap \mathcal{Q}, \Theta' : \mathcal{V} \multimap \mathcal{T}, a \in \mathcal{V}, T' \in \mathcal{T} \cup \mathcal{T}^2 n \in \mathbb{N}, b_i \in \mathcal{V}, T_i \in \mathcal{T}$ for $i \in \mathbb{N}_0^n$ and $[t]_A$ denotes the substitution for the term t with a substitution A .

Then

$$A_2 = \{(k, Q) \mid (k, Q) \in A \wedge k \neq k_2\} \cup \{(k_2, Q_2)\}$$

and therefore $\forall k \neq k_2. A_1(k) = A_2(k)$.

We know $A_2(k_2) = Q_2$ and $Q_2 \subseteq A_1(k_2)$ and therefore $A_2(k_2) \subseteq A_1(k_2)$.

We know by the definition of S that $S(k_2) = \bigwedge Q_2$. Therefore it is easy to see that $\{\nu : \text{Int}|r_1\} <_{:\Theta, \Lambda} \{\nu : \text{Int}|[[k_2]_S]_{S_2}\}$ is true because by definition of Q_2 it is true for all $q \in Q_2$, thus it is true for $\bigwedge Q_2$ as well.

We finish off by proving that the result is the sharpest, meaning that A_2 is the smallest subset such that the premise holds. So let there exists a A'_2 such that $\forall k \neq k_2. A_1(k) = A'_2(k), A'_2(k_2) \subseteq A_1(k_2), [k_2]_{S'} \in \mathcal{Q}, \{\nu : \text{Int}|r'_1\} <_{:\Theta, \Lambda} \{\nu : \text{Int}|[[k_2]_{S'}]_{S_2}\}$ and $A'_2 \subseteq A_2$ for given S' and r_1 .

This means for every $q \in A'_2(k_2)$ that $q \in A_1(k_2)$ and by the definition of subtyping, $\text{wellFormed}(q, \{(b_1, \{\nu : \text{Int}|r'_1\}), \dots, (b_n, \{\nu : \text{Int}|r'_n\})\})$ and

$$\begin{aligned}
&\forall i_1 \in \text{value}_\Gamma(\{\nu : \text{Int}|r'_1\}) \dots \forall i_n \in \text{value}_\Gamma(\{\nu : \text{Int}|r'_n\}). \\
&\llbracket r_1 \wedge p \rrbracket_{\{(b_1, i_1), \dots, (b_n, i_n)\}} \Rightarrow \llbracket [q]_{S_2} \rrbracket_{\{(b_1, i_1), \dots, (b_n, i_n)\}}.
\end{aligned}$$

By the definition of A_2 , this means that $q \in A_2(k_2)$. Thus A_2 is the smallest subset. \square