3.3 Type Inference

In the first section of this chapter we defined a type system, in the second section we introduced a syntax for our language. Now we want to define rules how to obtain the type of a given program written in our language.

3.3.1 Typing Judgments

A type system is a set of inference rules to derive various kinds of typing judgments. These *inference rules* have the following form

$$\frac{P_1 \dots P_n}{C}$$

where the judgments P_1 up to P_n are the premises of the rule and the judgment C is its conclusion.

We can read it in two ways:

- “If all premises hold then the conclusion holds as well” or
- “To prove the conclusion we need to prove all premises”.

We will now provide a judgment for every production rule defined in the last section. Ultimately, we will have a judgment $p : T$ which indicates that a program p is of a type T and therefore well-formed.

If the type T is known then we talk about *type checking* else we call the process of finding the judgment *type inference*.

TYPE SIGNATURE JUDGMENTS

For type signature judgments, let Γ be a type context, $T \in \mathcal{T}$ and $a_i \in \mathcal{V}, T_i \in \mathcal{T}$ for all $i \in \mathbb{N}_1^n$ and $n \in \mathbb{N}$.

For $\mathit{ltf} \in$ `<list-type-fields>` the judgment has the form

$$\Gamma \vdash \mathit{ltf} : \{a_1 : T_1, \dots, a_n : T_n\}$$

which can be read as “given Γ, ltf has the type $\{a_1 : T_1, \dots, a_n : T_n\}$”.

For $\mathit{lt} \in$ `<list-type>` the judgment has the form

$$\Gamma \vdash \mathit{lt} : (T_1, \dots, T_n)$$

which can be read as “given Γ, lt defines the list $(T_1, \dots, T_n)$”.

For $t \in$ `<type>` the judgment has the form

$$\Gamma \vdash t : T$$

which can be read as “given Γ, t has the type T”.

EXPRESSION JUDGMENTS

For expression judgments, let Γ, Δ be type contexts, $T \in \mathcal{T}$, $a \in \mathcal{V}$ and $T_i \in \mathcal{T}, a_i \in \mathcal{V}$ for all $i \in \mathbb{N}_0^n, n \in \mathbb{N}$.

For *lef* $\in$ `<list-exp-field>` the judgment has the form

$$\Gamma, \Delta \vdash \mathit{lef} : \{a_1 : T_1, \ldots, a_n : T_n\}$$

which can be read as "given Γ and Δ, *lef* has the type $\{a_1 : T_1, \ldots, a_n : T_n\}$".

For *mes* $\in$ `<maybe-exp-sign>` the judgment has the form

$$\Gamma, \mathit{mes} \vdash a : T$$

which can be read as "given Γ, a has the type T under the assumption *mes*".

For $b \in$ `<bool>` the judgment has the form

$$b : T$$

which can be read as "b has the type T".

For $i \in$ `<int>` the judgment has the form

$$e : T$$

which can be read as "i has the type T".

For *le* $\in$ `<list-exp>` the judgment has the form

$$\Gamma, \Delta \vdash \mathit{le} : \mathit{List}\ T$$

which can be read as "given Γ and Δ, *le* has the type *List* T".

For $e \in$ `<exp>` the judgment has the form

$$\Gamma, \Delta \vdash e : T$$

which can be read as "given Γ and Δ, e is of type T".

STATEMENT JUDGMENTS

For statement judgments, let $\Gamma, \Gamma_1, \Gamma_2, \Delta, \Delta_1, \Delta_2$ be a type contexts, $T, T_1, T_2 \in \mathcal{T}$, $a \in \mathcal{V}$ and $T_i, A_i \in \mathcal{T}, a_i \in \mathcal{V}$ for $i \in \mathbb{N}_0^n$ and $T_{i,j} \in \mathcal{T}$ for $i \in \mathbb{N}_0^n, n \in \mathbb{N}, j \in \mathbb{N}_0^{k_i}$ and $k_i \in \mathbb{N}$.

For *lsv* $\in$ `<list-statement-var` the judgment has the form

$$\mathit{lsv} : (a_1, \ldots, a_n)$$

which can be read as "*lsv* describes the list $(a_1, \ldots, a_n)$".

For *ls* $\in$ `<list-statement>` the judgment has the form

$$\Gamma_1, \Delta_2, \mathit{ls} \vdash \Gamma_2, \Delta_2$$

which can be read as “the list of statements ls maps Γ_1 to Γ_2 and Δ_1 to Δ_2”.

For $mss \in$ `<maybe-statement-sign>` the judgment has the form

$$\Gamma, mss \vdash a : T$$

which can be read as “given Γ, a has the type T_2 under the assumption mss”.

For $s \in$ `<statement>` the judgment has the form

$$\Gamma_1, \Delta_1, s \vdash \Gamma_2, \Delta_2$$

which can be read as “the statement s maps Γ_1 to Γ_2 and Δ_1 to Δ_2”.

For $mms \in$ `<maybe-main-sign>` the judgment has the form

$$\Gamma, mms \vdash \text{main} : T$$

which can be read as “the main function has type T under the assumtion mms”.

For $prog \in$ `<program>` the judgment has the form

$$prog : T$$

which can be read as “the program $prog$ is wellformed and has the type T”.

3.3.2 Auxiliary Definitions

We will assume that T is a mono type, T is a type variable and $T_1 = T_2$ denotes the equiality of two given types T_1 and T_2.

We will write $a_1, \ldots, a_n = \text{free}(T)$ to denote all free variables $a_1, \ldots, a_n$ of T.

INSTANTIATION, GENERALIZATION

The type system that we are using is polymorphic, meaning that whenever a judgment holds for a type, it will also hold for a more specific type. To counter this polymorphism we will force the types in a judgment to be unique by explicitly stating whenever we want to use a more specific or general type.

Definition 3.1: Instantiation

Let $\Delta : \mathcal{V} \nrightarrow \mathcal{T}$ be a type context, $T \in \mathcal{T}$ and e be an expression.

—

Then we define

$$e \sqsubseteq_\Delta T :\Leftrightarrow \exists T_0 \in \mathcal{T}.(e, T_0) \in \Delta \wedge T_0 \sqsubseteq T$$

Note that Δ is a partial function and therefore $\Delta(e)$ would only be defined if T_0 exists. If T_0 does not exist, then this predicate will be false.

The act of replacing T_0 with the more specific type T is called *Instantiation* and is typically in the text books introduced as an additional inference rule.

Definition 3.2: Uniquely Quantified Poly Type

Let Δ be a type context. $T_1, T_2 \in \mathcal{T}. a_i \in \mathcal{V}$ for $i \in \mathbb{N}_0^n$. Let T' be the mono type of T_1.

—

We say $\forall a_1 \dots \forall a_n.T'$ is a *uniquely quantified* poly type of T_1 in Δ, iff the following holds:

$$(a, \forall a_1 \dots \forall a_n.T') \in \Delta_2 \wedge \{a_1, \dots, a_n\} = \{a \mid a \in \text{free}(T') \wedge (a, \_) \notin \Delta_2\}$$

A uniquely quantified poly type ensures that all type variable are renamed in order to not clash with free variables in Delta and also ensure that all currently free variables are being bound.

Definition 3.3: Generalization

Let Δ_1, Δ_2 be type contexts, $a \in \mathcal{V}$.

—

We define

$$\begin{aligned}&\text{insert}_{\Delta_1}(\Delta_2) := \\ &\qquad \Delta_1 \cup \left\{ \ (a, T') \ \middle| \ \begin{array}{l} T \in \mathcal{T} \wedge (a, T) \in \Delta_2 \\ \wedge T' \text{ is a uniquely quantified poly type of } T \text{ in } \Delta \end{array} \right\}\end{aligned}$$

This definition essentially states that all quantified variables of T, that occur in Δ_2, will be dropped and any free variables will be quantified. The act of removing a quantified variable that is already in the type context is called *Generalization* and is also typically found as an inference rule in text books.

PREDEFINED TYPES

Additionally, we define

$$\begin{aligned} Bool &:= \mu\_.\,True|False \\ Nat &:= \mu C.1|Succ\ C \\ Int &:= \mu\_.0 \mid Pos\ Nat \mid Neg\ Nat \\ List &:= \forall a.\mu C.[\,]\ |Cons\ a\ C \end{aligned}$$

3.3.3 Inference Rules for Type Signatures

LIST-TYPE-FIELDS

Judgment: $\Gamma \vdash ltf : \{a_1 : T_1, \dots, a_n : T_n\}$

$$\Gamma \vdash \texttt{""} : \{\}$$

$$\frac{\Gamma \vdash t : T_0 \quad \Gamma \vdash \mathit{ltf} : \{a_1 : T_1, \ldots, a_n : T_n\} \quad \{a_0 : T_0, a_1 : T_1, \ldots, a_n : T_n\} = T}{\Gamma \vdash a_0 \ \texttt{":"}\ t\ \texttt{","}\ \mathit{ltf} : T}$$

The type context Γ is used in the judgment $\Gamma \vdash t : T_0$ that turns the type signature t into a type T_0.

LIST-TYPE

Judgment: $\Gamma \vdash lt : (T_1, \ldots, T_n)$

$$\Gamma \vdash \texttt{""} : ()$$

$$\frac{\Gamma \vdash t : T_0 \quad \Gamma \vdash lt : (T_1, \ldots, T_n) \quad (T_0, T_1, \ldots, T_n) = T}{\Gamma \vdash t\ lt : T}$$

TYPE

Judgment: $\Gamma \vdash t : T$

$$\frac{\mathit{Bool} = T}{\Gamma \vdash \texttt{"Bool"} : T}$$

$$\frac{\mathit{Int} = T}{\Gamma \vdash \texttt{"Int"} : T}$$

$$\frac{\mathit{List}\ T_2 = T_1 \quad \Gamma \vdash t : T_2}{\Gamma \vdash \texttt{"List"}\ \texttt{t} : T_1}$$

$$\frac{(T_1, T_2) = T_0 \quad \Gamma \vdash t_1 : T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash \texttt{"("}\ t_1\ \texttt{","}\ t_2\ \texttt{")"} : T_0}$$

$$\frac{\Gamma \vdash ltf : T}{\Gamma \vdash \texttt{"\{"}\ ltf\ \texttt{"\}"} : T}$$

$$\frac{T_1 \to T_2 = T_0 \quad \Gamma \vdash t_1 : T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash t_1 \to t_2 : T_0}$$

$$\frac{(c, T') \in \Gamma \quad \Gamma \vdash l : (T_1, \ldots, T_n) \quad \overline{T'}\ T_1 \ldots T_n = T}{\Gamma \vdash c\ l : T}$$

For a given type T we write the application constructor as $\overline{T}$.

$$\frac{\forall a.a = T}{\Gamma \vdash a : T}$$

Example 3.1

In example **??** we have looked at the syntax for a list reversing function.

The type signiture for the `reverse` function was `List a -> List a`. We will now show how we can obtain the curresponding type T_0. For that, let $\Gamma = \varnothing$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\forall a.a = T_3}}{\varnothing \vdash a : T_3} \quad \dfrac{}{\mathit{List}\ T_3 = T_1}}{\varnothing \vdash \mathit{Lista} : T_1} \quad \dfrac{\dfrac{\dfrac{}{\forall a.a = T_4}}{\varnothing \vdash a : T_4} \quad \dfrac{}{\mathit{List}\ T_4 = T_2}}{\varnothing \vdash \mathit{Lista} : T_2}}{\dfrac{T_1 \to T_2 = T_0}{\varnothing \vdash \texttt{List a -> List a} : T_0}}$$

We can therefore conclude that $T_0 = \mathit{List}\ (\forall a.a) \to \mathit{List}\ (\forall a.a) = \forall a.\mathit{List}\ a \to \mathit{List}\ a$.

3.3.4 Inference Rules for Expressions

LIST-EXP-FIELD

Judgment: $\Gamma, \Delta \vdash \mathit{lef} : \{a_1 : T_1, \ldots, a_n : T_n\}$

$$\frac{\Gamma, \Delta \vdash e : T}{\Gamma, \Delta \vdash a \texttt{ "=" } e : \{a : T\}}$$

$$\frac{\Gamma, \Delta \vdash \mathit{lef} : T \quad \Gamma, \Delta \vdash e : T_0 \quad \{a_0 : T_0, \ldots, a_n : T_n\} = T}{\Gamma, \Delta \vdash a_0 \texttt{ "=" } e \texttt{ "," } \mathit{lef} : T}$$

MAYBE-EXP-SIGN

Judgment: $\Gamma, \mathit{mes} \vdash a : T$

$$\Gamma, \texttt{""} \vdash a : T$$

If no argument is given, then we do nothing.

$$\frac{\Gamma \vdash t : T a_1 = a_2}{\Gamma, a_1 \texttt{":"} t \texttt{";"} \vdash a_2 : T}$$

If we have a variable a_1 and a type T, then the variables a_2 need to match. The type signature t defines the type of a_2.

BOOL

Judgment: $b : T$

$$b : Bool$$

INT

Judgment: $i : T$

$$i : Int$$

We have proven in theorem **??** that *Nat* is isomorph to $\mathbb{N}$. Is should be trivial to therefore conclude that *Int* is isomorph to $\mathbb{Z}$. And therefore this rule is justified.

LIST-EXP

Judgment: $\Gamma, \Delta \vdash le : List\ T$

$$\Gamma, \Delta \vdash \texttt{""} : \forall a.List\ a$$

$$\frac{\Gamma, \Delta \vdash e : T \quad \Gamma, \Delta \vdash le : List\ T}{\Gamma, \Delta \vdash e\ \texttt{","}\ le : List\ T}$$

EXP

Judgment: $\Gamma, \Delta \vdash e : T$

$$\Gamma, \Delta \vdash \texttt{"foldl"} : \forall a.\forall b.(a \to b \to b) \to b \to List\ a \to b$$

$$\Gamma, \Delta \vdash \texttt{"(::)"} : \forall a.a \to List\ a \to List\ a$$

$$\Gamma, \Delta \vdash \texttt{"(+)"} : Int \to Int \to Int$$

$$\Gamma, \Delta \vdash \texttt{"(-)"} : Int \to Int \to Int$$

$$\Gamma, \Delta \vdash \texttt{"(*)"} : Int \to Int \to Int$$

$$\Gamma, \Delta \vdash \texttt{"(//)"} : Int \to Int \to Int$$

$$\Gamma, \Delta \vdash \texttt{"(<)"} : Int \to Int \to Bool$$

$$\Gamma, \Delta \vdash \texttt{"(==)"} : \mathit{Int} \to \mathit{Int} \to \mathit{Bool}$$

$$\Gamma, \Delta \vdash \texttt{"not"} : \mathit{Bool} \to \mathit{Bool}$$

$$\Gamma, \Delta \vdash \texttt{"(\&\&)"} : \mathit{Bool} \to \mathit{Bool} \to \mathit{Bool}$$

$$\Gamma, \Delta \vdash \texttt{"(||)"} : \mathit{Bool} \to \mathit{Bool} \to \mathit{Bool}$$

$$\frac{\Gamma, \Delta \vdash e_1 : T_1 \quad \Gamma, \Delta \vdash e_2 : T_1 \to T_2}{\Gamma, \Delta \vdash e_1 \texttt{ "|>" } e_2 : T_2}$$

$$\frac{\Gamma, \Delta \vdash e_1 : T_1 \to T_2 \quad \Gamma, \Delta \vdash e_2 : T_2 \to T_3}{\Gamma, \Delta \vdash e_1 \texttt{ ">>" } e_2 : T_1 \to T_3}$$

$$\frac{\Gamma, \Delta \vdash e_1 : \mathit{Bool} \quad \Gamma, \Delta \vdash e_2 : T \quad \Gamma, \Delta \vdash e_3 : T}{\Gamma, \Delta \vdash \texttt{"if"}\ e_1 \texttt{ "then" } e_2 \texttt{ "else" } e_3 : T}$$

$$\frac{\Gamma, \Delta \vdash \mathit{lef} : \{a_1 : T_1, \ldots, a_n : T_n\}}{\Gamma, \Delta \vdash \texttt{"\{"}\mathit{lef}\texttt{"\}"} : \{a_1 : T_1, \ldots, a_n : T_n\}}$$

$$\Gamma, \Delta \vdash \texttt{"\{\}"} : \{\}$$

$$\frac{\begin{array}{c}\Gamma, \Delta \vdash \mathit{lef} : \{a_1 : T_1, \ldots, a_n : T_n\} \\ \Gamma, \Delta \vdash a \sqsubseteq_\Delta T_0 \quad T_0 = \{a_1 : T_1, \ldots, a_n : T_n, \ldots\}\end{array}}{\Gamma, \Delta \vdash \texttt{"\{"}\ a \texttt{ "|" } \mathit{lef} \texttt{ "\}"} : T_0}$$

Since Elm version 0.19, released in 2018, setters are not allowed to change the type of a field in a record.

$$\frac{(a_1, \{a_2 : T, \ldots\}) \in \Delta}{\Gamma, \Delta \vdash a_1\texttt{"."}a_2 : T}$$

$$\frac{\begin{array}{c}(a, \_) \notin \Delta \quad \Gamma, \Delta \vdash e_1 : T_1 \quad \mathit{mes} : T_1 \vdash a : T_1 \\ \Gamma, \mathrm{insert}_\Delta(\{(a, T_1)\}) \vdash e_2 : T_2\end{array}}{\Gamma, \Delta \vdash \texttt{"let"}\ \mathit{mes}\ a\texttt{"="}\ e_1 \texttt{ "in" } e_2 : T_2}$$

$$\frac{\Gamma, \Delta \vdash e_1 : T_1 \quad \Gamma, \Delta, T_1 \vdash \mathit{lc} : T_2}{\Gamma, \Delta \vdash \texttt{"case"}\ e_1 \texttt{ "of" } \texttt{ "[" } \mathit{lc} \texttt{ "]"} : T_2}$$

$$\frac{\Gamma, \Delta \vdash e_1 : T_1 \to T_2 \quad \Gamma, \Delta \vdash e_2 : T_1}{\Gamma, \Delta \vdash e_1\ e_2 : T_2}$$

$$\frac{b : T}{\Gamma, \Delta \vdash b : T}$$

$$\frac{i : T}{\Gamma, \Delta \vdash i : T}$$

$$\frac{\Gamma, \Delta \vdash le : T}{\Gamma, \Delta \vdash \texttt{"["}le\texttt{"]"} : T}$$

$$\frac{\Gamma, \Delta \vdash e_1 : T_1 \quad \Gamma, \Delta \vdash e_2 : T_2}{\Gamma, \Delta \vdash \texttt{"("}\ e_1\ \texttt{","}\ e_2\texttt{")"} : (T_1, T_2)}$$

$$\frac{\Gamma, \mathrm{insert}_\Delta(\{(a, T_1)\}) \vdash e : T_2}{\Gamma, \Delta \vdash \texttt{"\textbackslash"}a\texttt{"->"}e : T_1 \to T_2}$$

$$\frac{\Delta(c) \sqsubseteq T}{\Gamma, \Delta \vdash c : T}$$

$$\frac{\Delta(a) \sqsubseteq T}{\Gamma, \Delta \vdash a : T}$$

Example 3.2

In example **??** we have looked at the syntax for a list reversing function. We can now check the type $T_0 = \forall a.List\ a \to List\ a$ of the `reverse` function for $\Gamma = \Delta = \varnothing$, $\Delta = \varnothing$. The body of the *reverse* function is as follows:

```
foldl (::) []
```

$$\dfrac{\dfrac{\dfrac{}{\varnothing, \varnothing \vdash \texttt{"foldl"} : T_2} \quad \dfrac{}{\varnothing, \varnothing \vdash \texttt{"(::)"} : \forall a.List\ a \to List\ a}}{\varnothing, \varnothing \vdash \texttt{"foldl (::)"} : T_1} \quad \dfrac{\dfrac{}{\varnothing, \varnothing \vdash \texttt{""} : \forall a.a}}{\varnothing, \varnothing \vdash \texttt{"[]"} : \forall a.List\ a}}{\varnothing, \varnothing \vdash \texttt{"foldl (::) []"} : T_0}$$

where $T_1 = \forall a.List\ a \to List\ a \to List\ a$ and $T_2 = \forall a.(List\ a \to List\ a) \to List\ a \to List\ a \to List\ a$.

3.3.5 Inference Rules for Statements

LIST-STATEMENT-VAR

Judgment: $lsv : (a_1, \ldots, a_n)$

$$\texttt{""} : ()$$

$$\frac{\mathit{lsv} : (a_1, \ldots, a_n)}{a_0\ \mathit{lsv} : (a_0, a_1, \ldots, a_n)}$$

LIST-STATEMENT-SORT

Judgment: $\mathit{lss} : (c_1 : (T_{1,1}, \ldots, T_{1,k_1}), \ldots, c_n : (T_{n,1}, \ldots, T_{n,k_n}))$

$$\frac{\Gamma \vdash \mathit{lt} : (T_0, \ldots, T_n)}{c\ \mathit{lt} : (c : (T_0, \ldots, T_n))}$$

$$\frac{\Gamma \vdash \mathit{lt} : (T_{0,1}, \ldots, T_{0,k_n}) \quad \mathit{lss} : \begin{pmatrix} a_1 : (T_{1,1}, \ldots, T_{1,k_1}), \\ \vdots \\ a_n : (T_{n,1}, \ldots, T_{n,k_n}) \end{pmatrix}}{c\ \mathit{lt}\ \texttt{"|"}\ \mathit{lss} : \begin{pmatrix} a_0 : (T_{0,1}, \ldots, T_{0,k_0}), \\ a_1 : (T_{1,1}, \ldots, T_{1,k_1}), \\ \vdots \\ a_n : (T_{n,1}, \ldots, T_{n,k_n}) \end{pmatrix}}$$

LIST-STATEMENT

Judgment: $\Gamma_1, \Delta_1, \mathit{ls} \vdash \Gamma_2, \Delta_2$

$$\frac{\Gamma_1 = \Gamma_2 \quad \Delta_1 = \Delta_2}{\Gamma_1, \Delta_1 \texttt{""} \vdash \Gamma_2, \Delta_2}$$

$$\frac{\Gamma_1, \Delta_1, s \vdash \Gamma_2, \Delta_2 \quad \Gamma_2, \Delta_2, \mathit{ls} \vdash \Gamma_3, \Delta_3}{\Gamma_1, \Delta_1, s\ \texttt{";"}\ \mathit{ls} \vdash \Gamma_3, \Delta_3}$$

MAYBE-STATEMENT-SIGN

Judgment: $\Gamma, \mathit{mss} \vdash a : T$

$$\Gamma, \texttt{""} \vdash a : T$$

$$\frac{\Gamma \vdash t : T a_1 = a_2}{\Gamma, a_1\ \texttt{":"}\ t\ \texttt{";"} \vdash a_2 : T}$$

STATEMENT

Judgment: $\Gamma_1, \Delta_1, s \vdash \Gamma_2, \Delta_2$

$$\frac{\begin{array}{c} \Gamma_1 = \Gamma_2 \quad (a, \_) \notin \Delta_1 \\ \Gamma_1, \mathit{mss} \vdash e : T \quad \Gamma_1, \Delta_1 \vdash e : T \quad \Delta_2 = \mathrm{insert}_{\Delta_1}(\{(a, T)\}) \end{array}}{\Gamma_1, \Delta_1, \mathit{mss}\ a\ \texttt{"="} e \vdash \Gamma_2, \Delta_2}$$

$$\frac{\Delta_1 = \Delta_2 \quad (c, \_) \notin \Gamma_1 \quad \Gamma \vdash t : T_1 \\ T_2 \text{ is a mono type} \quad lsv : (a_1, \dots, a_n) \quad \{a_1 \dots a_n\} = \text{free}(T_2) \\ \forall a_1 \dots . \forall a_n . T_2 = T_1 \quad \Gamma_2 = \Gamma_2 \cup \{(c, T_1)\}}{\Gamma_1, \Delta_1, \texttt{"type alias"}\ c\ lsv\ \texttt{"="}\ t \vdash \Gamma_2, \Delta_2}$$

$$\frac{\begin{array}{c}(c, \_) \notin \Gamma_1 \quad lsv : (a_1, \dots, a_n) \\ lss : (c_1 : (T_{1,1}, \dots, T_{1,k_1}), \dots, c_n : (T_{n,1}, \dots, T_{n,k_n})) \\ \Delta_1 \cap \{(c_1, \_), \dots, (c_n, \_)\} = \varnothing \quad \{a_1 \dots a_n\} = \text{free}(T_2) \\ \mu C . c_1\ T_{1,1}\ \dots\ T_{1,k_1} \mid \dots \mid c_n\ T_{n,1}\ \dots\ T_{n,k_n} = T_2 \quad \forall a_1 \dots . \forall a_n . T_2 = T_1 \\ \Gamma_1 \cup \{(c, T_1)\} = \Gamma_2 \quad \text{insert}_{\Delta_1}(\left\{\begin{array}{l}(c_1, T_{1,1} \to \dots \to T_{1,k_1} \to T_1), \\ \vdots \\ (c_n, T_{n,1} \to \dots \to T_{n,k_n} \to T_1)\end{array}\right\}) = \Delta_2\end{array}}{\Gamma_1, \Delta_1, \texttt{"type"}\ c\ lsv\texttt{"="}lss \vdash \Gamma_2, \Delta_2}$$

The list *lss* provides us with the structure of the type. From there we construct the type T_2 and bind all variables, thus creating the poly type T_1. Additionally, every sort c_i for $i \in \mathbb{N}_1^n$ has its own constructor that gets added to Δ_1 under the name c_i. In Elm these constructors are the only constants beginning with an upper-case letter.

MAYBE-MAIN-SIGN

Judgment: $\Gamma, mms \vdash \text{main} : T$

$$\Gamma, \texttt{""} \vdash \text{main} : T$$

$$\frac{\Gamma \vdash t : T}{\Gamma, \texttt{"main :"}t\texttt{";"} \vdash \text{main} : T}$$

PROGRAM

Judgment: $prog : T$

$$\frac{\varnothing, \varnothing, ls \vdash \Gamma, \Delta \quad \Gamma, mms \vdash \text{main} : T \quad \Gamma, \Delta \vdash e : T}{ls\ mms\ \texttt{"main = "}\ e : T}$$