Road map!

- Module 1- Introduction to Deep Forecasting
- Module 2- Setting up Deep Forecasting Environment
- Module 3- Exponential Smoothing
- Module 4- ARIMA models
- Module 5- Machine Learning for Time series Forecasting
- Module 6- Deep Neural Networks
- Module 7- Deep Sequence Modeling (RNN, LSTM)
- Module 8- Prophet and Neural Prophet

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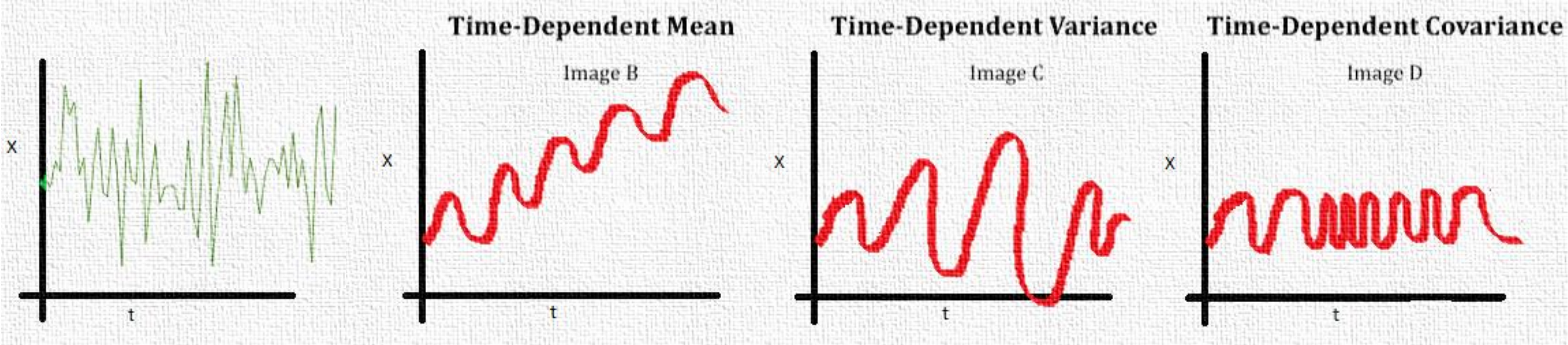
Module 4 – Part I
ARIMA models' Prerequisites
ACF, PACF, Stationarity, Differencing



Stationary | Non-Stationary | Non-Stationary | Non-Stationary

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ARIMA models prerequisites

- ARIMA stands for AutoRegressive Integrated Moving Average. It is a class of statistical models for analyzing and forecasting time series data.
- ETS and ARIMA models are two popular models for forecasting time series data. They offer complementary approaches to addressing the challenges of time series forecasting.
- ARIMA models describe autocorrelations in the data, whereas ETS models describe trends and seasonality.
- Let's review some prerequisites before moving forward with the models:

ACF → PACF → Stationarity → Differencing

Autocorrelation

- Autocorrelation, also known as serial correlation, is a measure of the correlation between a time series and a lagged version of itself.
- It is used to assess the degree to which the past values of a time series are predictive of its future values.

$$r_k = \frac{\sum_{t=k+1}^{T} (y_t - \bar{y})(y_{t-k} - \bar{y})}{\sum_{t=1}^{T} (y_t - \bar{y})^2}$$

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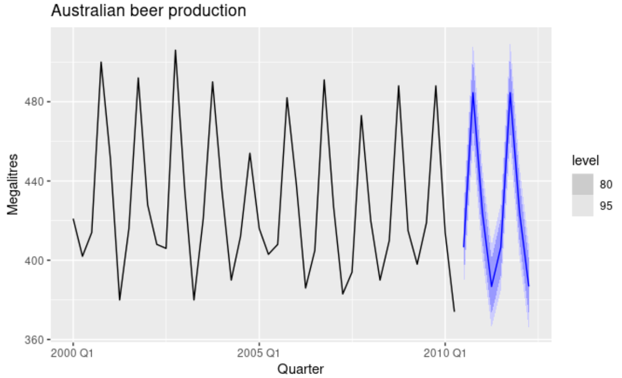
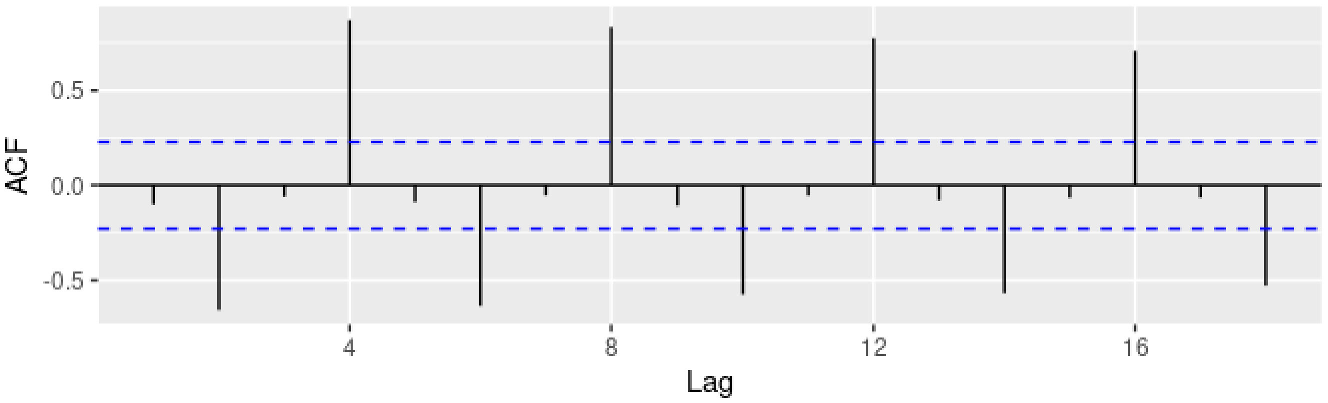
ACF: Autocorrelation Function

- The autocorrelation function (ACF) is a statistical tool that can be used to measure the autocorrelation of a time series.
- It calculates the correlation between the time series and lagged versions of itself at different lag periods.

r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	r_9
-0.102	-0.657	-0.060	0.869	-0.089	-0.635	-0.054	0.832	-0.108

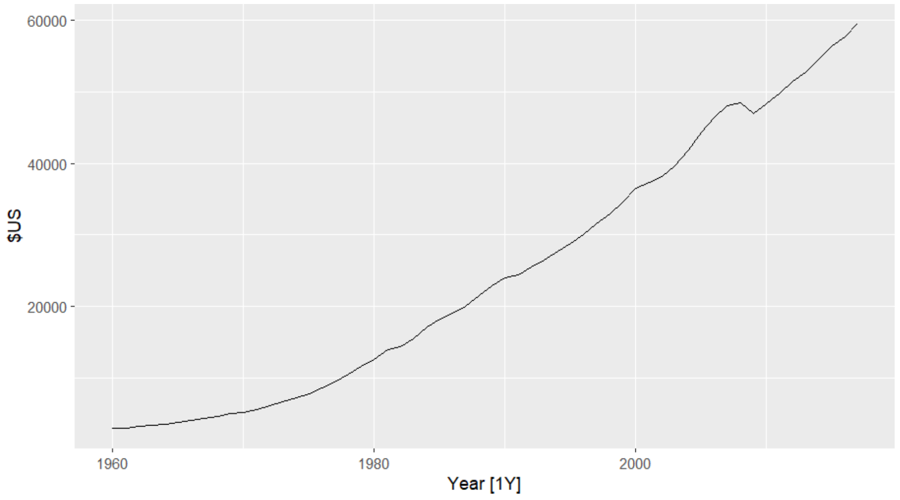




Partial Autocorrelation

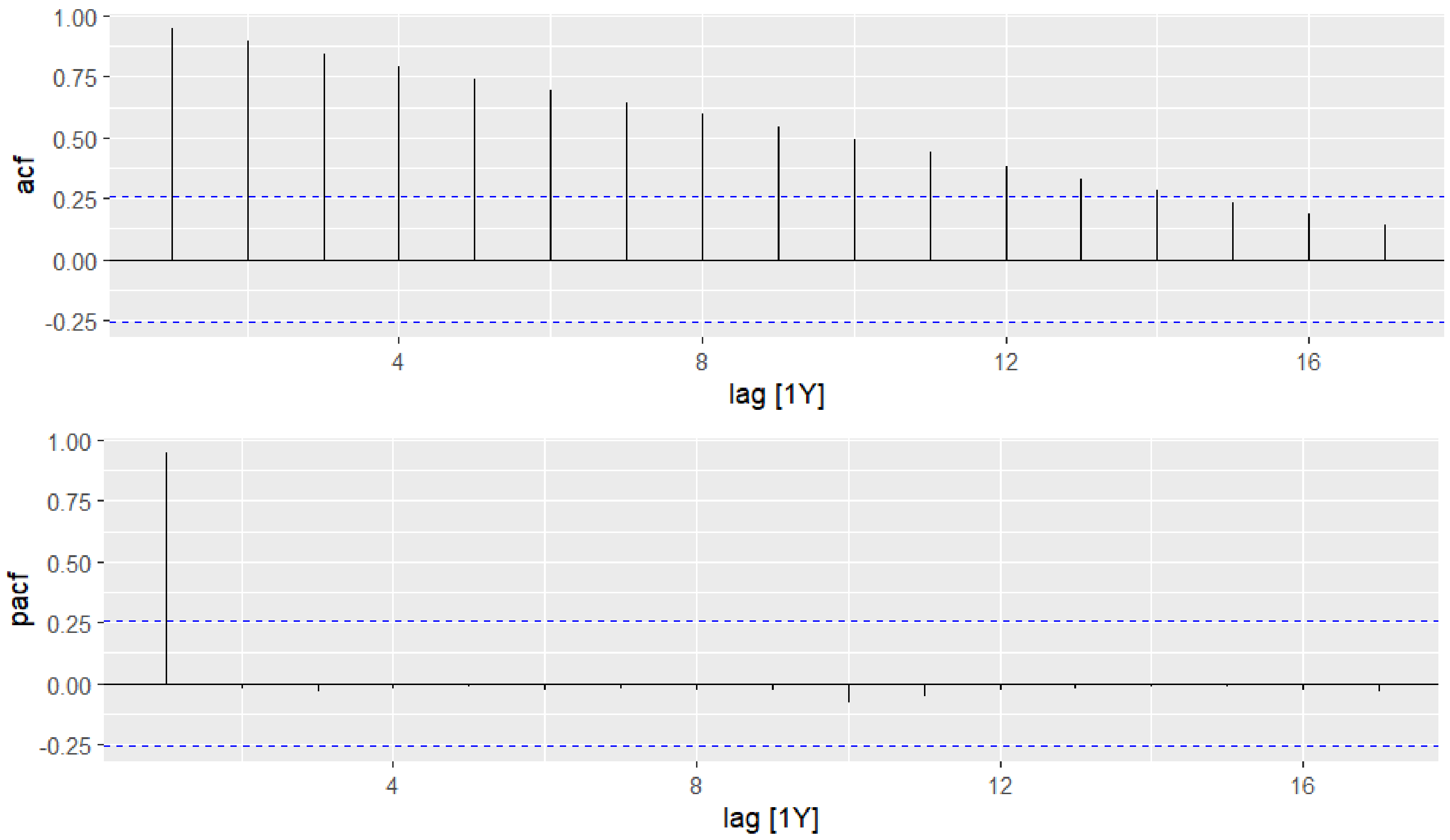
- Partial autocorrelation, also known as partial serial correlation, is a measure of the correlation between a time series and a lagged version of itself, controlling for the effects of intermediate lag periods.
- y_t and y_{t-2} might be correlated, simply because they are both connected to y_{t-1}, rather than because of any new information contained in y_{t-2}. Partial autocorrelation overcomes this problem.

US Annual GDP per capita (1960-2017)



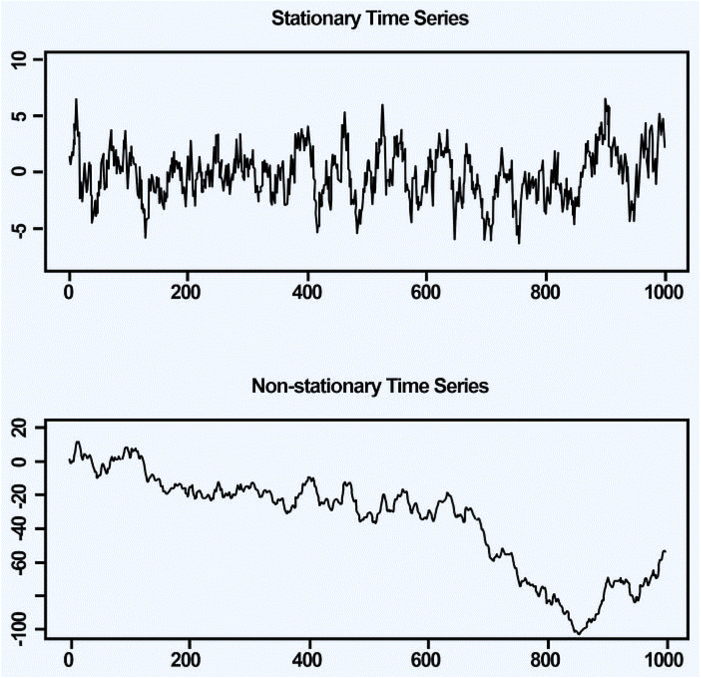
PACF: Partial Autocorrelation Function

- PACF is a statistical tool that can be used to measure the partial autocorrelation of a time series.



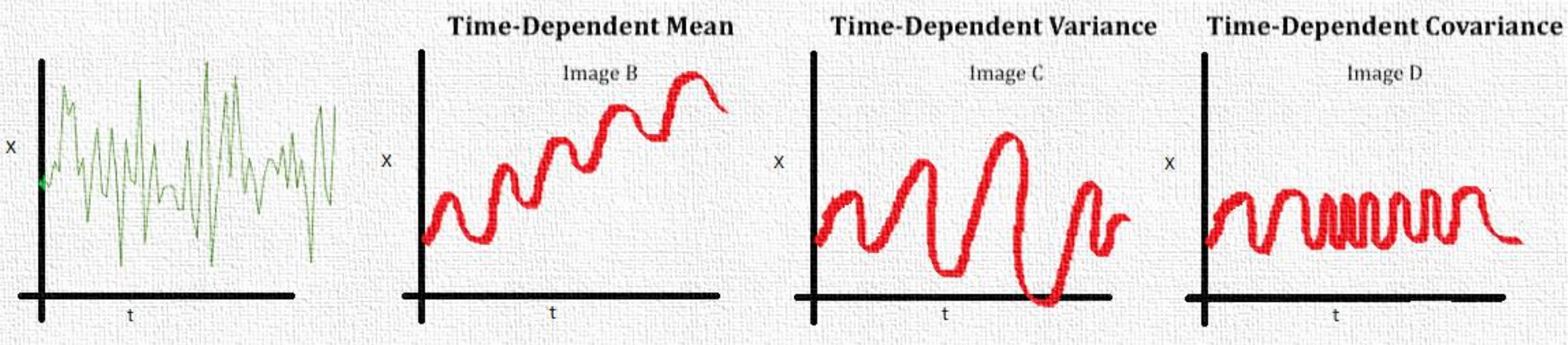
Stationarity

- Stationary vs Non-Stationary Data. What makes a data set Stationary?
- In a stationary timeseries, the statistical properties do not depend on the time
- Predictability: Stationary time series are easier to predict because you can assume that future statistical properties will not change.
- This doesn't mean we cannot predict non-stationary data!
- Data with trend and seasonality are NOT stationary!
- Data granularity matters: The level of detail in your data can impact its stationarity



Weak vs Strong Stationarity

- Weak Stationarity (Covariance Stationary): A time series is considered weakly stationary if the following conditions hold:
  1. Constant Mean: The mean of the process is constant over time.
  2. Constant Variance: The variance of the process is constant over time.
  3. Covariance Depends Only on Lag: The covariance between two points in the time series depends only on the time difference (lag) between the points, not their absolute position in time.



Stationary | Non-Stationary | Non-Stationary | Non-Stationary

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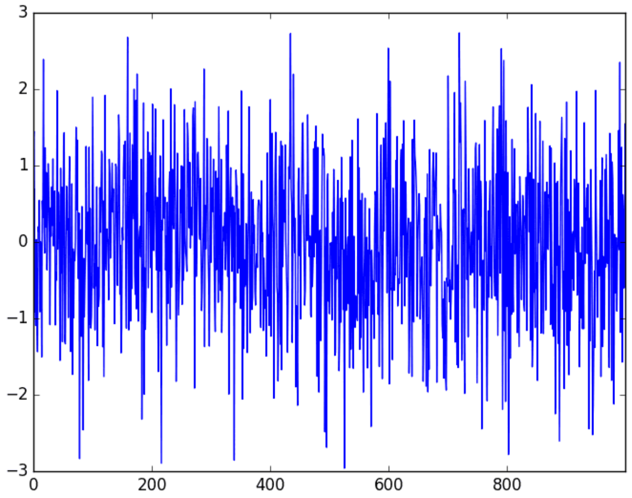
Weak vs Strong Stationarity

- Strong stationarity: A time series is considered strongly stationary if its joint probability distribution does not change when shifted in time
  - **All moments** of the series (mean, variance, skewness, kurtosis, etc) and joint distributions remain constant, irrespective of the time period



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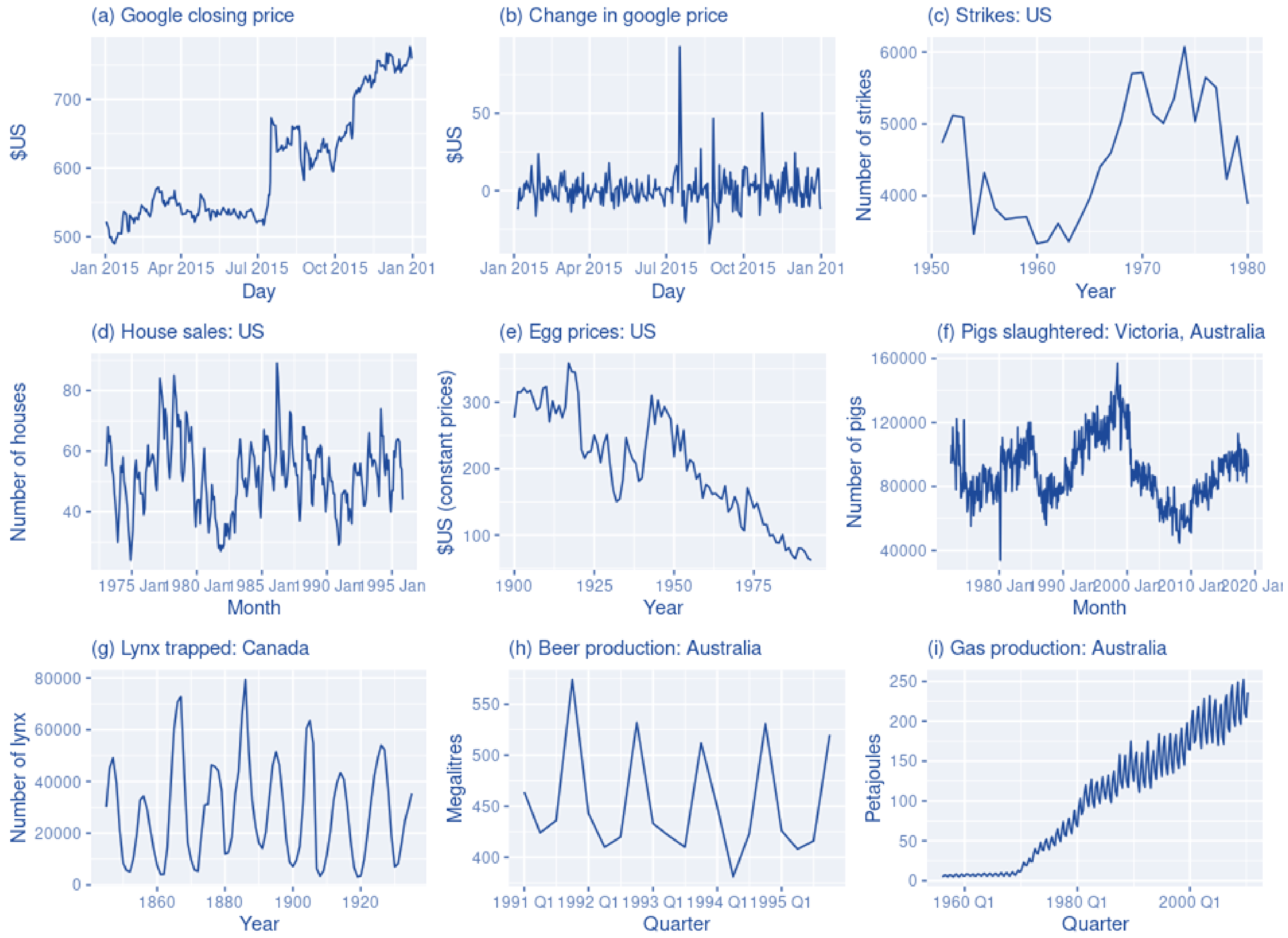
Which ones are stationary?



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Differencing

- Differencing: Computing the difference between consecutive observations.
- Differencing helps to stabilize the mean of a time series by removing changes in the level and therefore reducing the trend and seasonality.
- Recall: Transformations help to stabilize the variance of a time series.

Original data

Time1	10
Time2	12
Time3	8
Time4	14
Time5	7

1st differenced

Time1	NA
Time2	2
Time3	-4
Time4	6
Time5	-7

2nd differenced

Time1	NA
Time2	NA
Time3	-6
Time4	10
Time5	-13

2^{nd} Differencing

- Occasionally the differenced data will not appear to be stationary, and it may be necessary to difference the data a second time to obtain a stationary series.
- Second differencing is change in change.
- In practice, it is almost never necessary to go beyond second-order differences.

$$
\begin{aligned}
y_t'' &= y_t' - y_{t-1}' \\
&= (y_t - y_{t-1}) - (y_{t-1} - y_{t-2}) \\
&= y_t - 2y_{t-1} + y_{t-2}.
\end{aligned}
$$

Seasonal Differencing

- A seasonal difference is the difference between an observation and the previous observation from the same season.

$$y'_t = y_t - y_{t-m}$$

- m is the number of seasons. This is also called lag-m difference.
- If seasonal differenced is white noise, then

$$y_t = y_{t-m} + \varepsilon_t$$

- Recall:
  - Seasonal Naïve forecast: each forecast set to be equal to the last observed value from the same season

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Put it together!

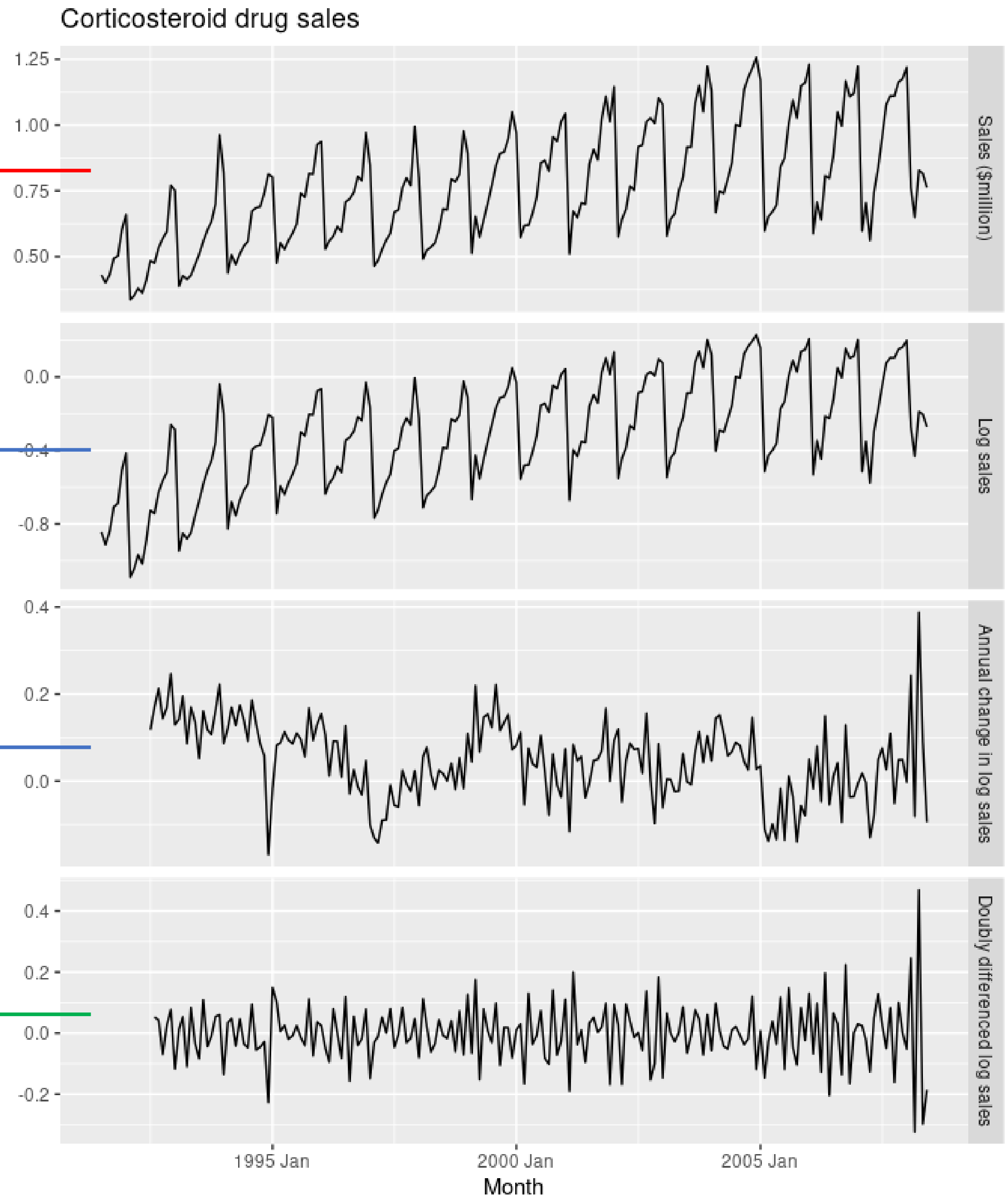
Original data

Log transformed

Seasonal difference

1st differenced seasonal difference

Interpretable?



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Random Walk

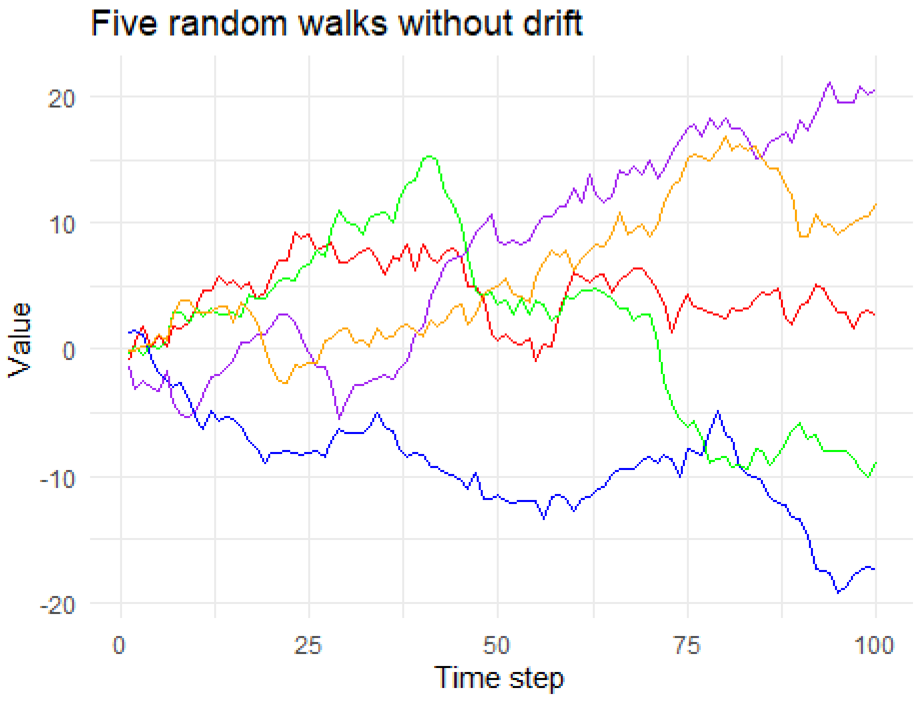
- Random Walk: When the 1st differenced series is white noise

$$y_t - y_{t-1} = \varepsilon_t \longrightarrow \boxed{y_t = y_{t-1} + \varepsilon_t}$$

- Random walk models are widely used for non-stationary data, particularly financial and economic data.

- Random walks typically have long periods of up or down trend + sudden change in direction.

- Random walk with no drift = Naïve forecasting model



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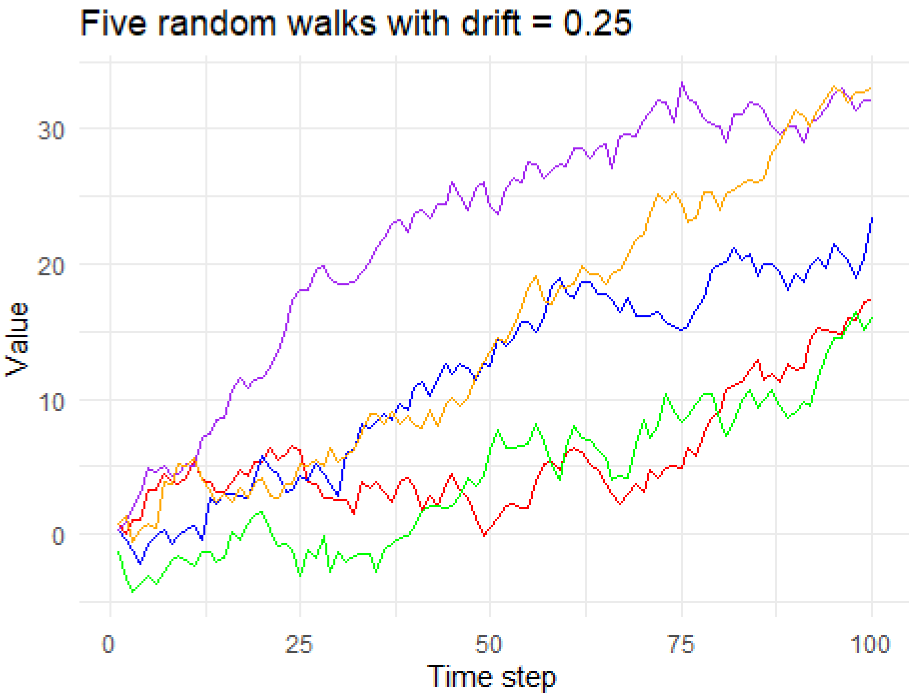
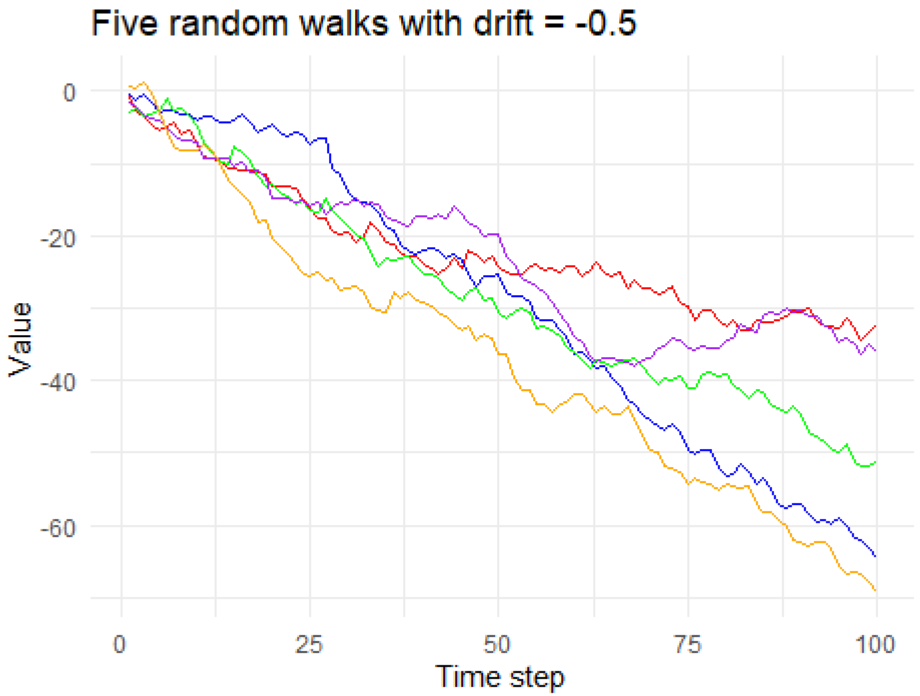
Random Walk with Drift

- Random walk with drift c (the 1st difference does not have zero average):

$$y_t - y_{t-1} = c + \varepsilon_t \quad \text{or} \quad y_t = c + y_{t-1} + \varepsilon_t$$

- C is the average change between consecutive observations.





Testing for Stationarity

- Unite root test is a statistical test used to determine whether a time series has a unit root, which is a characteristic of a non-stationary time series
- There are several different unit root tests including:
  1. Augmented Dickey-Fuller (ADF) test.
  2. Kwiatkowski-Phillips-Schmidt-Shin (KPSS) test.

Hypothesis Test	Null	Alternative	P-value to get stationarity
ADF	Non-Stationary	Stationary	Small
KPSS	Stationary	Non-Stationary	Large

	ADF	KPSS
ADF statistic	-12.533939	0.012944
p-value	0.0	0.1
should we difference?	?	
conclusion		

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Testing for Stationarity

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Hypothesis Test	Null	Alternative	P-value to get stationarity
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	ADF	KPSS
ADF statistic	-12.533939	0.012944
p-value	0.0	0.1
should we difference?	False	False
conclusion	stationary	stationary

OPTIONAL

Dickey-Fuller Test Variants

There are three main variants of the DF test, depending on the inclusion of a constant (α) and a deterministic trend (βt):

1. **Model without a drift and without a trend:**
   $\Delta y_t = \gamma y_{t-1} + \varepsilon_t$
2. **Model with a drift but without a trend:**
   $\Delta y_t = \alpha + \gamma y_{t-1} + \varepsilon_t$
3. **Model with both a drift and a trend:**
   $\Delta y_t = \alpha + \beta t + \gamma y_{t-1} + \varepsilon_t$

In each case, the focus is on testing the coefficient γ. The null hypothesis is that $\gamma = 0$, indicating a unit root (non-stationarity). Rejecting the null hypothesis suggests that the series does not have a unit root and is stationary.

The Dickey-Fuller test's limitation, which the Augmented Dickey-Fuller test addresses, is its inability to handle higher-order serial correlation. The ADF extends the DF test by including lagged differences of the dependent variable to account for this serial correlation, thus providing a more general approach to testing for unit roots.

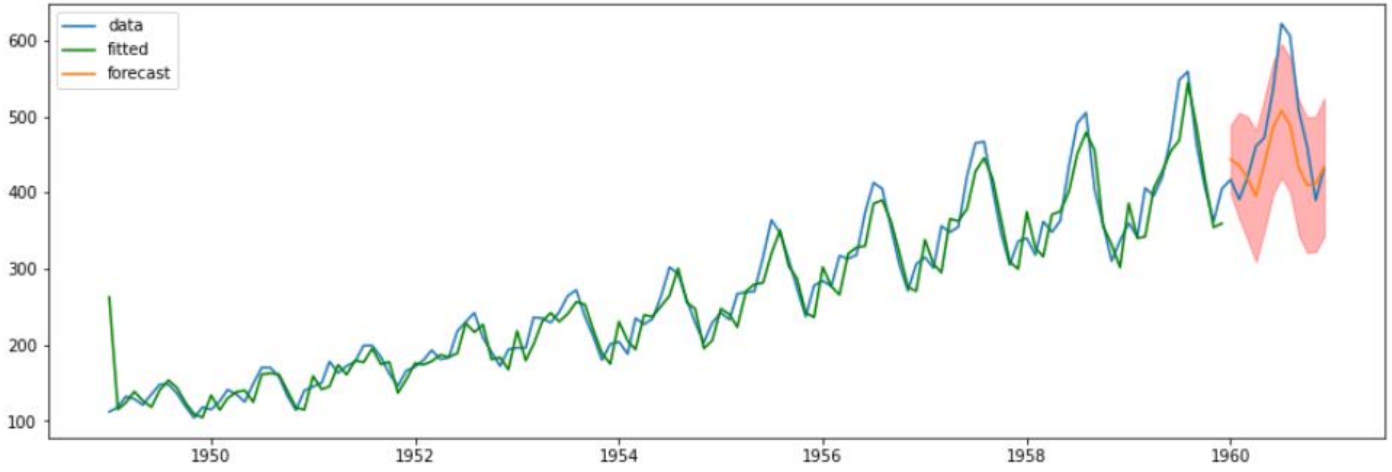
$$\Delta y_t = \alpha + \beta t + \gamma(y_{t-1} - y_t) + \delta_1 \Delta y_{t-1} + \delta_2 \Delta y_{t-2} + \cdots + \delta_{p-1} \Delta y_{t-p+1} + \varepsilon_t$$

Components of ARIMA model

1. Autoregressive (AR) term - captures the autocorrelation in the data
2. Integrated (I) term - removes the non-stationarity in the data
3. Moving Average (MA) term - captures the error term or noise in the data

How it works?

- The AR term models the current **value** of the time series as a linear combination of its past **values**.
- The I term models the differences between the current **value** and the past **value**.
- The MA term models the current **error** term as a linear combination of the past **error** terms.



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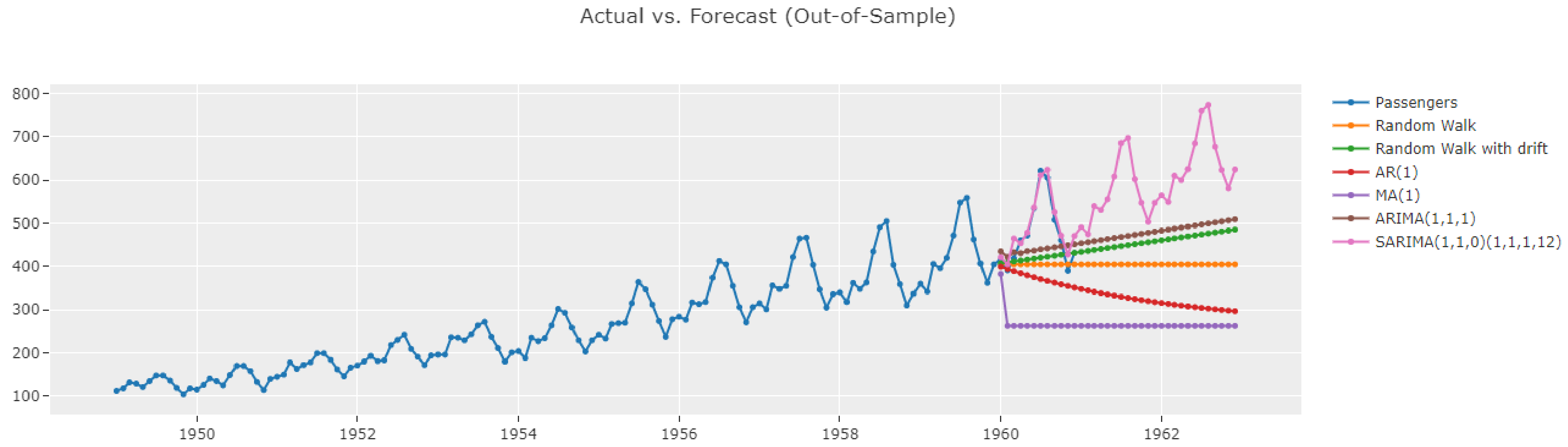
Module 4 – Part II

ARIMA models



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Components of ARIMA model

ARIMA

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Autoregressive models

- An autoregressive (AR) model is a statistical model (multiple linear regression model) that uses lagged variable as predictors
- Autoregression = regression of the variable against itself
- AR(p) model, autoregressive model of order p.

$$\boxed{y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t}$$

- In AR(1) model:
  - when $\phi_1 = 0$ and $c = 0$, y_t is equivalent to ?
  - when $\phi_1 = 1$ and $c = 0$, y_t is equivalent to ?
  - when $\phi_1 = 1$ and $c \neq 0$, y_t is equivalent to ?
  - when $\phi_1 < 0$, y_t tends to oscillate around the mean.

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Autoregressive models

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- Autoregression = regression of the variable against itself
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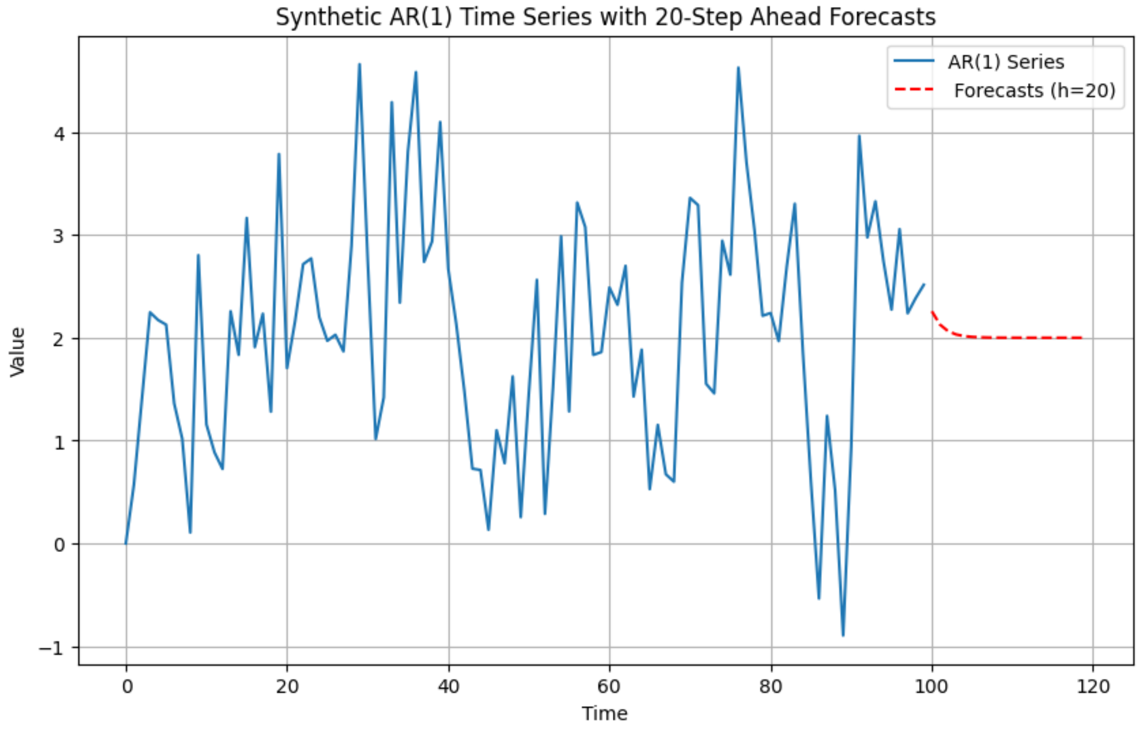
$$\boxed{y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t}$$

- In AR(1) model:
  - when $\phi_1 = 0$ and $c = 0$, y_t is equivalent to white noise;
  - when $\phi_1 = 1$ and $c = 0$, y_t is equivalent to a random walk;
  - when $\phi_1 = 1$ and $c \neq 0$, y_t is equivalent to a random walk with drift;
  - when $\phi_1 < 0$, y_t tends to oscillate around the mean.

Autoregressive Models (Forecasting)

$$y_t = c + \phi y_{t-1} + \varepsilon_t$$



$c = 1, \quad \phi = 0.5, \quad \epsilon_t \sim N(0, \sigma = 1)$

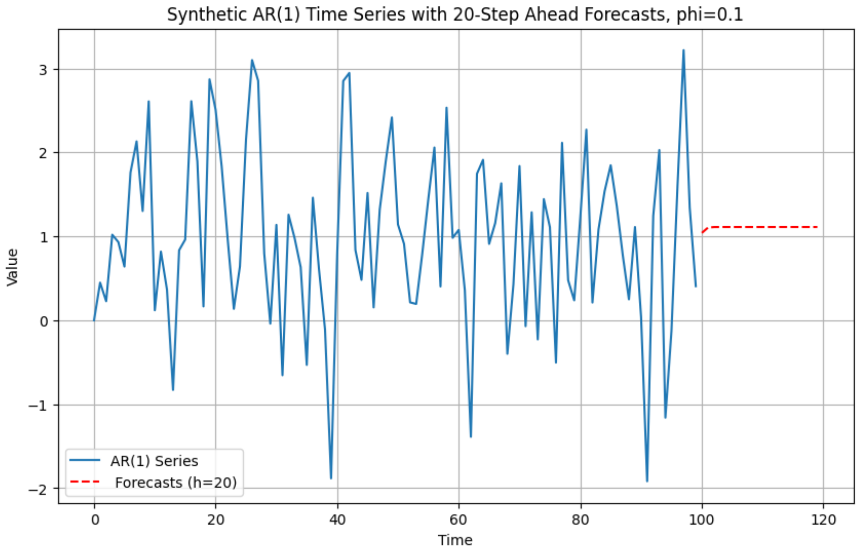
Forecasting Equation:

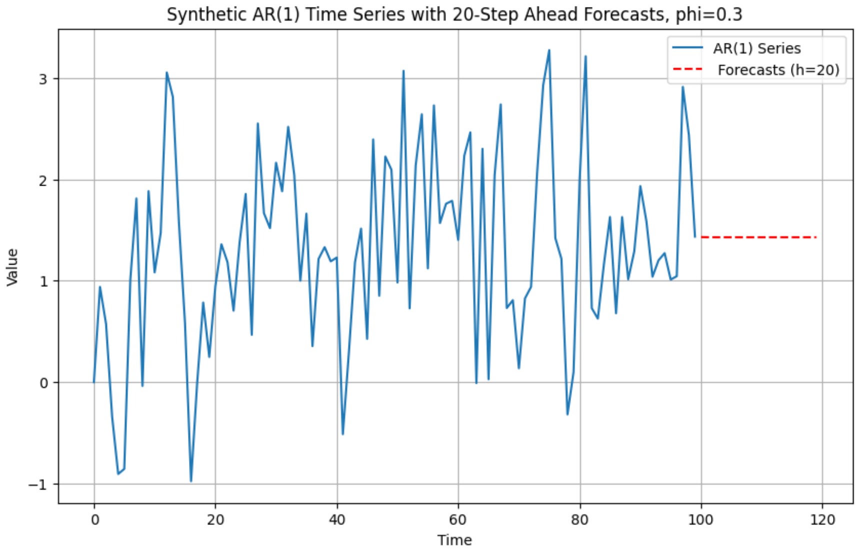
$$\hat{y}_{t+1} = c + \phi y_t$$

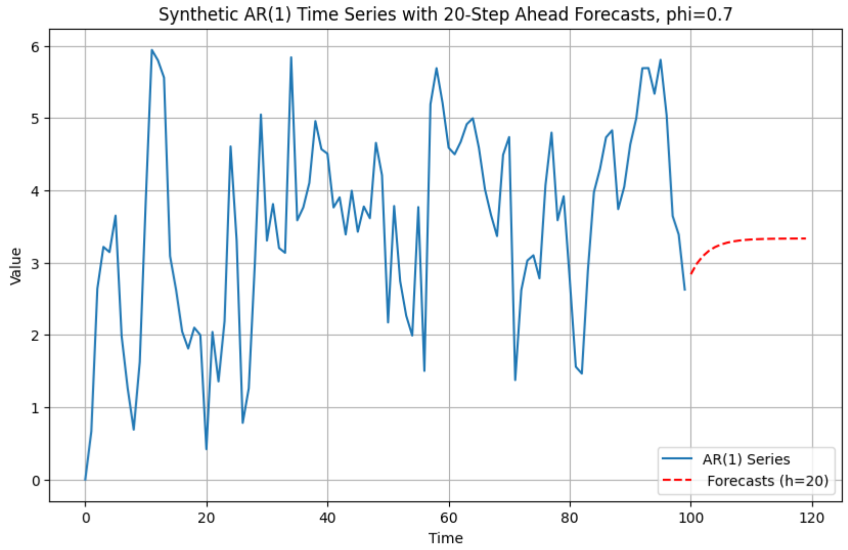
$$\hat{y}_{t+h|t} = c(1 + \phi + \phi^2 + \cdots + \phi^{h-1}) + \phi^h y_t$$

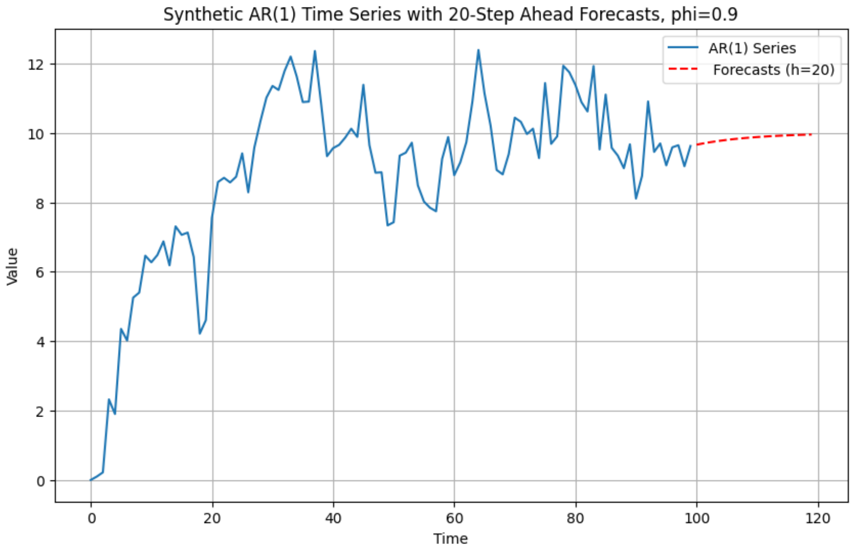
$$\boxed{\hat{y}_{t+h|t} = c\frac{1-\phi^h}{1-\phi} + \phi^h y_t}$$

Autoregressive Models (Examples)









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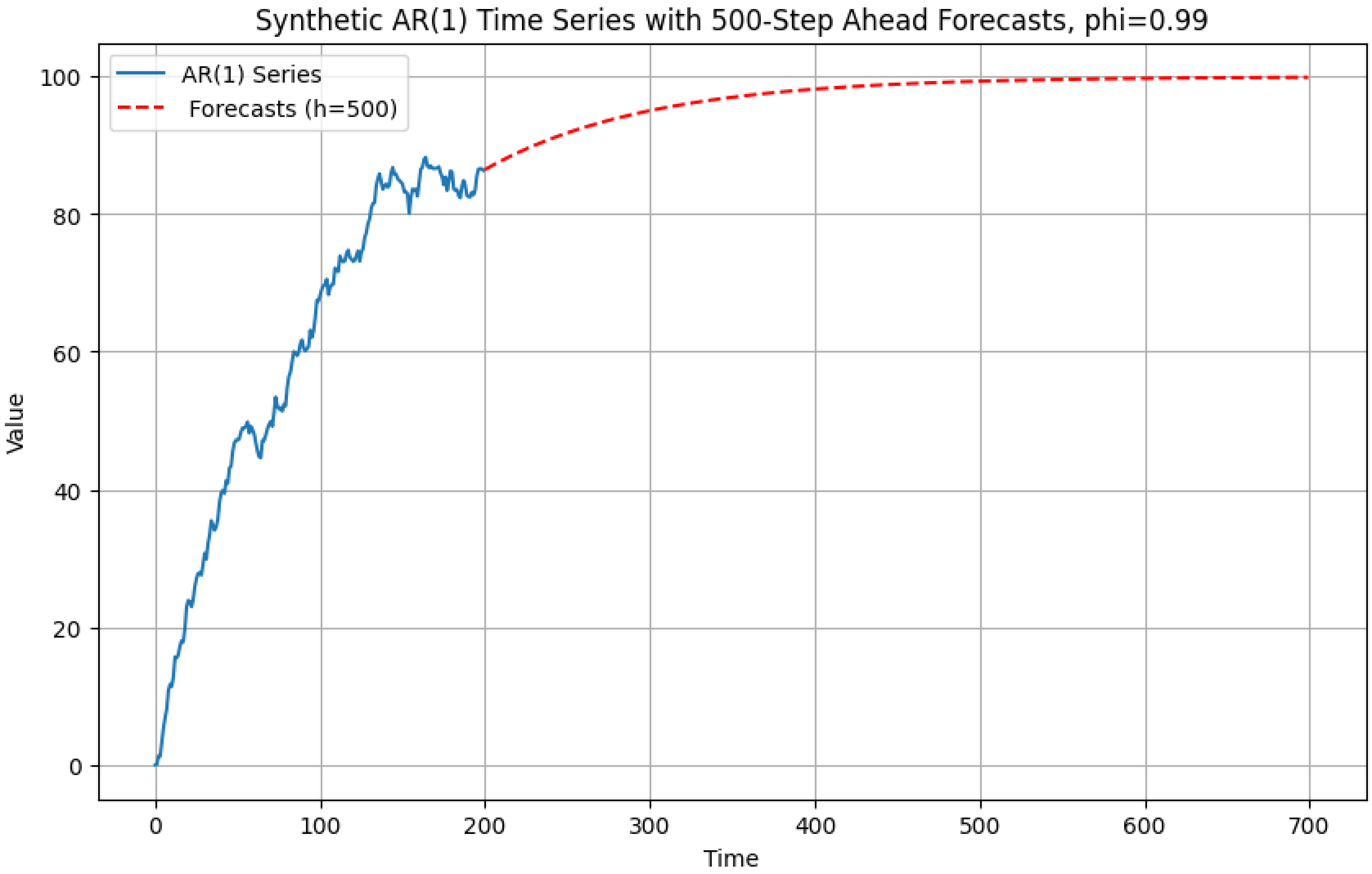
Synthetic AR(1) Time Series with 500-Step Ahead Forecasts, phi=0.99
AR(1) Series
Forecasts (h=500)
Value
Time
0
20
40
60
80
100
0
100
200
300
400
500
600
700

Moving Average Models

- A moving average model uses past forecast errors in a regression-like model
- MA(q) model, a moving average model of order q.

$$\boxed{y_t = c + \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2} + \cdots + \theta_q\varepsilon_{t-q}}$$

- y_t can be thought of as a weighted moving average of the past few forecast errors
- We require $|\theta| < 1$, the most recent observations carry a greater weight than those from the distant past.

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Moving Average Models

$$y_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots + \theta_q \varepsilon_{t-q}$$

- Do NOT confuse this model with simple moving average method or exponentially weighted moving average method.
- Moving average models is used for forecasting future values!
- Moving average smoothing (SMA, EWMA, …) is used for estimating the trend-cycle of past values.

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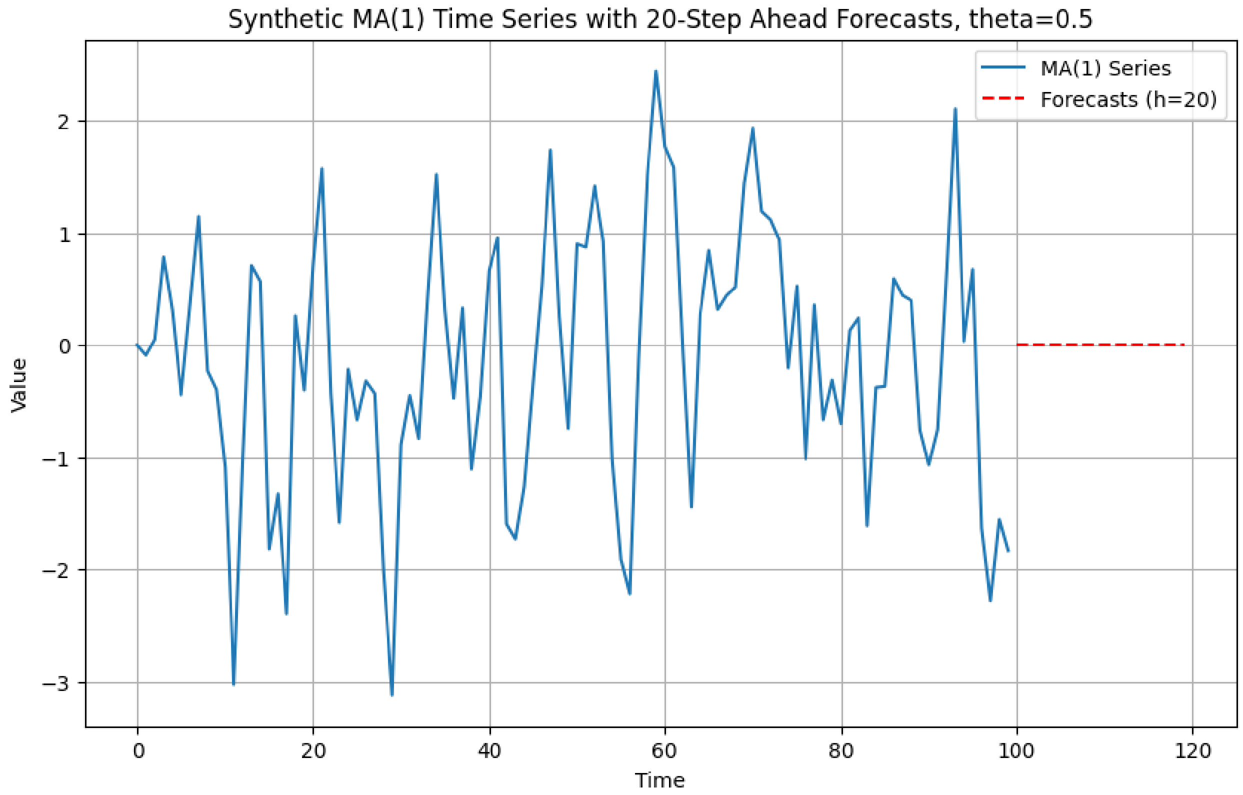
Moving Average Models (Forecasting)

$$y_t = \mu + \theta \varepsilon_{t-1} + \varepsilon_t$$



$\mu = 0, \qquad \theta = 0.5, \qquad \epsilon_t \sim N(0, \sigma = 1)$

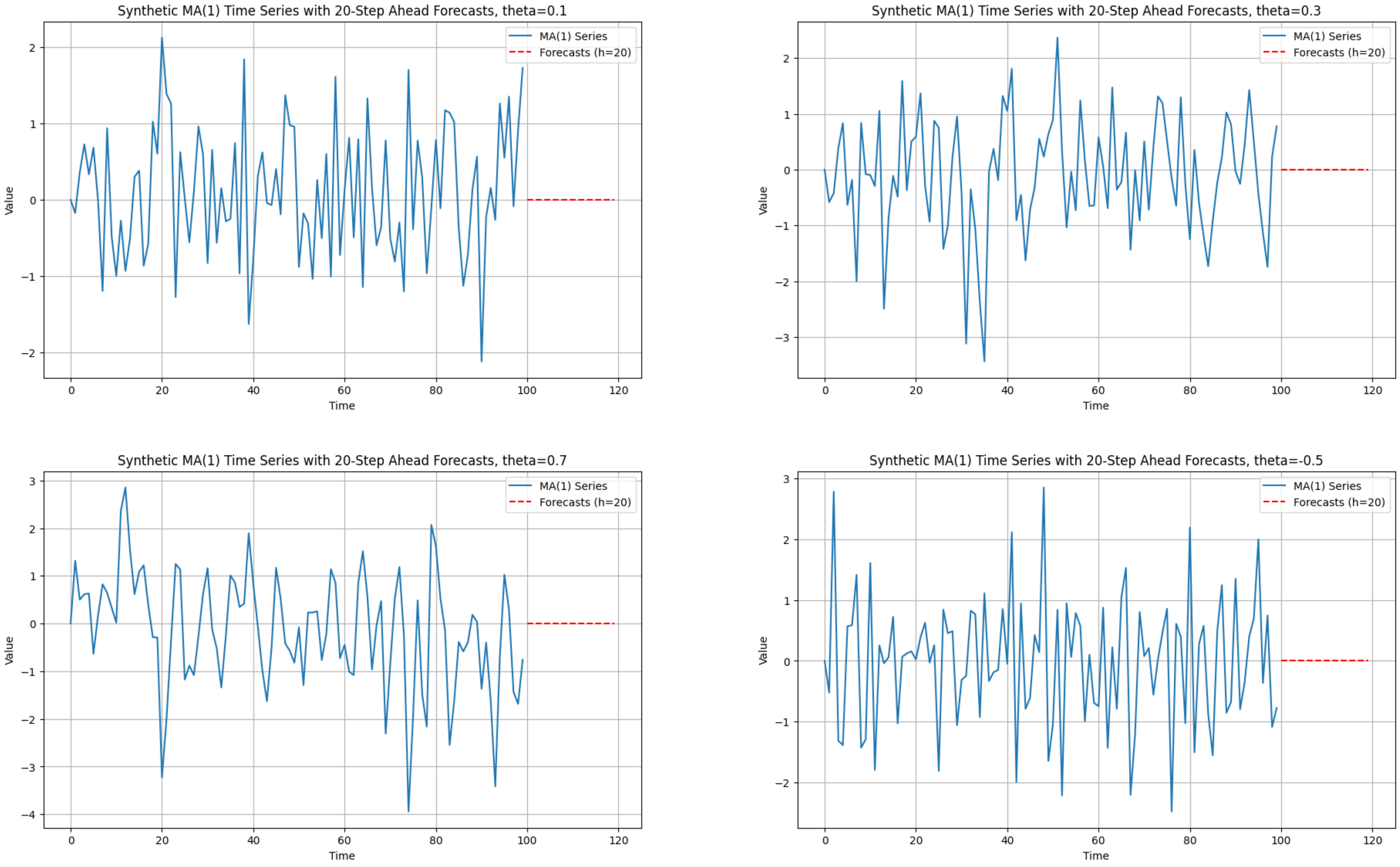
Forecasting Equation:

$$\hat{y}_{t+1|t} = \mu + \theta \varepsilon_t$$

Since future error terms are unknown

$$\hat{y}_{t+h|t} = \mu$$

Moving Average Models (Examples)



ARIMA (AutoRegressive Integrated Moving Average)

- ARIMA model combines three models, autoregressive (AR) model, an integrated (I) model, and a moving average (MA) model.
- ARIMA(p, d, q) model.

$$\boxed{y'_t = c + \phi_1 y'_{t-1} + \cdots + \phi_p y'_{t-p} + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} + \varepsilon_t}$$

- y'_t is the differenced time series.
- d degree of first difference involved.
- Note: p, d, and q are estimated using MLE.

?	ARIMA(0,0,0) with no constant
?	ARIMA(0,1,0) with no constant
?	ARIMA(0,1,0) with a constant
?	ARIMA(p,0,0)
?	ARIMA(0,0,q)

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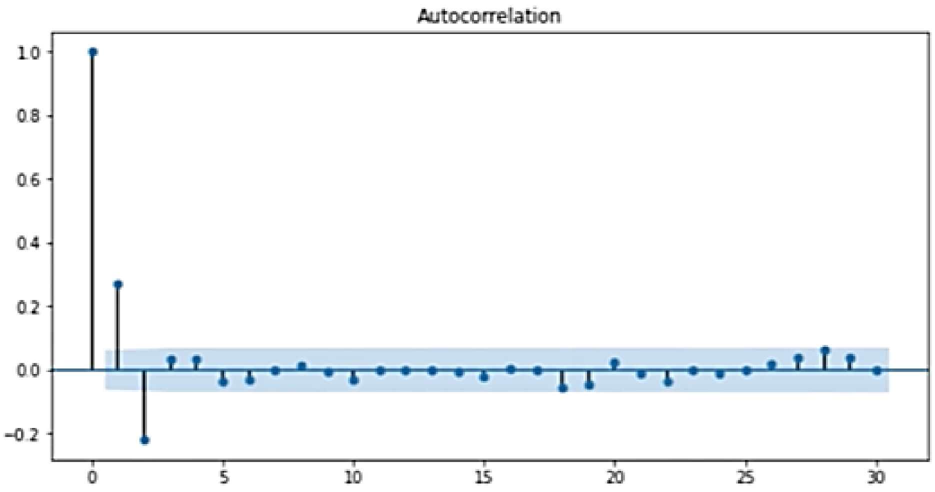
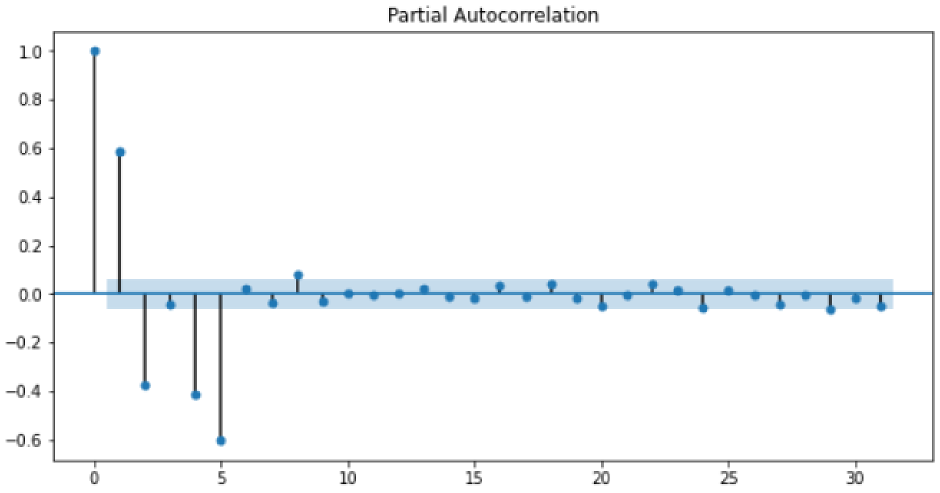
- y'_t is the differenced time series.
- d degree of first difference involved.
- Note: p, d, and q are estimated using MLE.

White noise	ARIMA(0,0,0) with no constant
Random walk	ARIMA(0,1,0) with no constant
Random walk with drift	ARIMA(0,1,0) with a constant
Autoregression	ARIMA(p,0,0)
Moving average	ARIMA(0,0,q)

Selecting (p, q) orders using ACF and PAC

- Some rough guidelines:
- Identification of an AR model is often best done with the PACF
  - p set to be the maximum significant non-zero lag in PACF typically followed by a sharp decline.
- Identification of an MA model is often best done with the ACF
  - q set to be the maximum significant non-zero lag in ACF typically followed by a sharp decline.





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Model selection

- Fore model selection we can either use information criteria or any cross validated performance metrics like R^2, MSE, RMSE, MAPE, sMAPE.

Information Criteria	Formula
Akaike's Information Criterion (AIC)	$\text{AIC} = -2\log(L) + 2k$
AIC corrected for small sample bias (AICc)	$\text{AIC}_c = \text{AIC} + \frac{2k(k+1)}{T-k-1}$
Bayesian Information Criterion (BIC)	$\text{BIC} = \text{AIC} + k[\log(T) - 2]$

- L is the likelihood of the model and K is the total number of parameters (including the variance of residuals)
- The model with the minimum information criteria is often the best model for forecasting

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ARIMA model estimation (optional)

SARIMA (Seasonal ARIMA) models

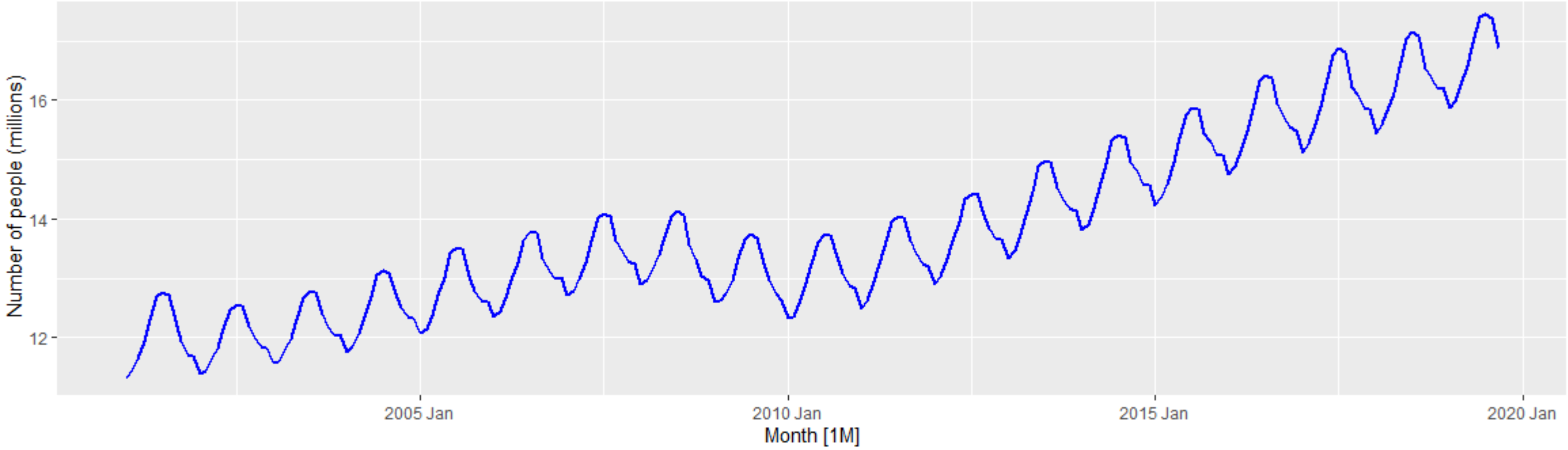
- SARIMA is an extension of an ARIMA model that includes additional seasonal terms.
- It is used to model time series data that exhibits seasonal patterns

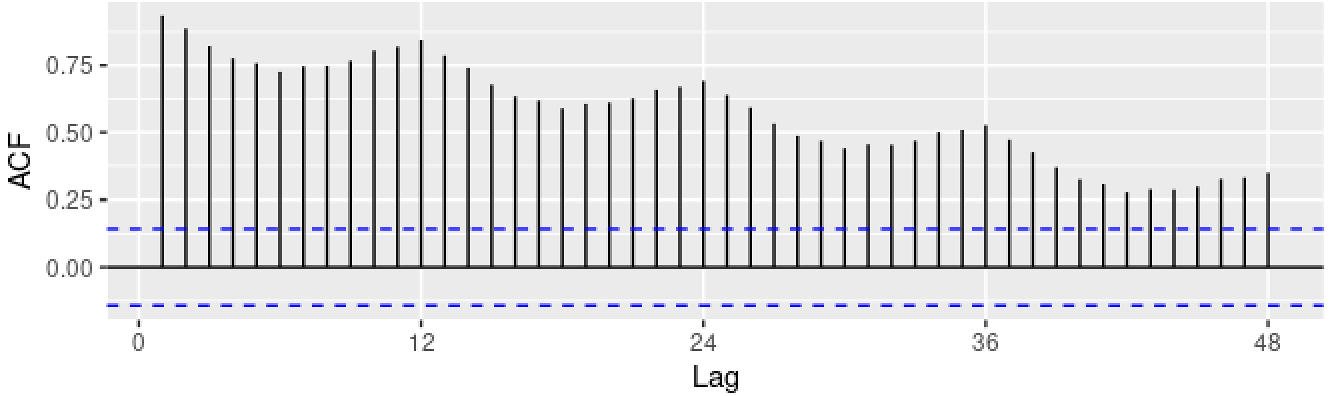
$$\text{ARIMA} \quad \underbrace{(p, d, q)}_{\substack{\uparrow \\ \text{Non-seasonal part} \\ \text{of the model}}} \quad \underbrace{(P, D, Q)_m}_{\substack{\uparrow \\ \text{Seasonal part} \\ \text{of the model}}}$$

- p, d, q are defined as before.
- P is the order of the seasonal autoregressive component
- D is the degree of seasonal differencing
- Q is the order of the seasonal moving average component
- m is the period of the seasonality. m = 4, 12 is for quarterly and monthly seasonality, respectively.

SARIMA example

Monthly US leisure and hospitality employment, 2001-2019.





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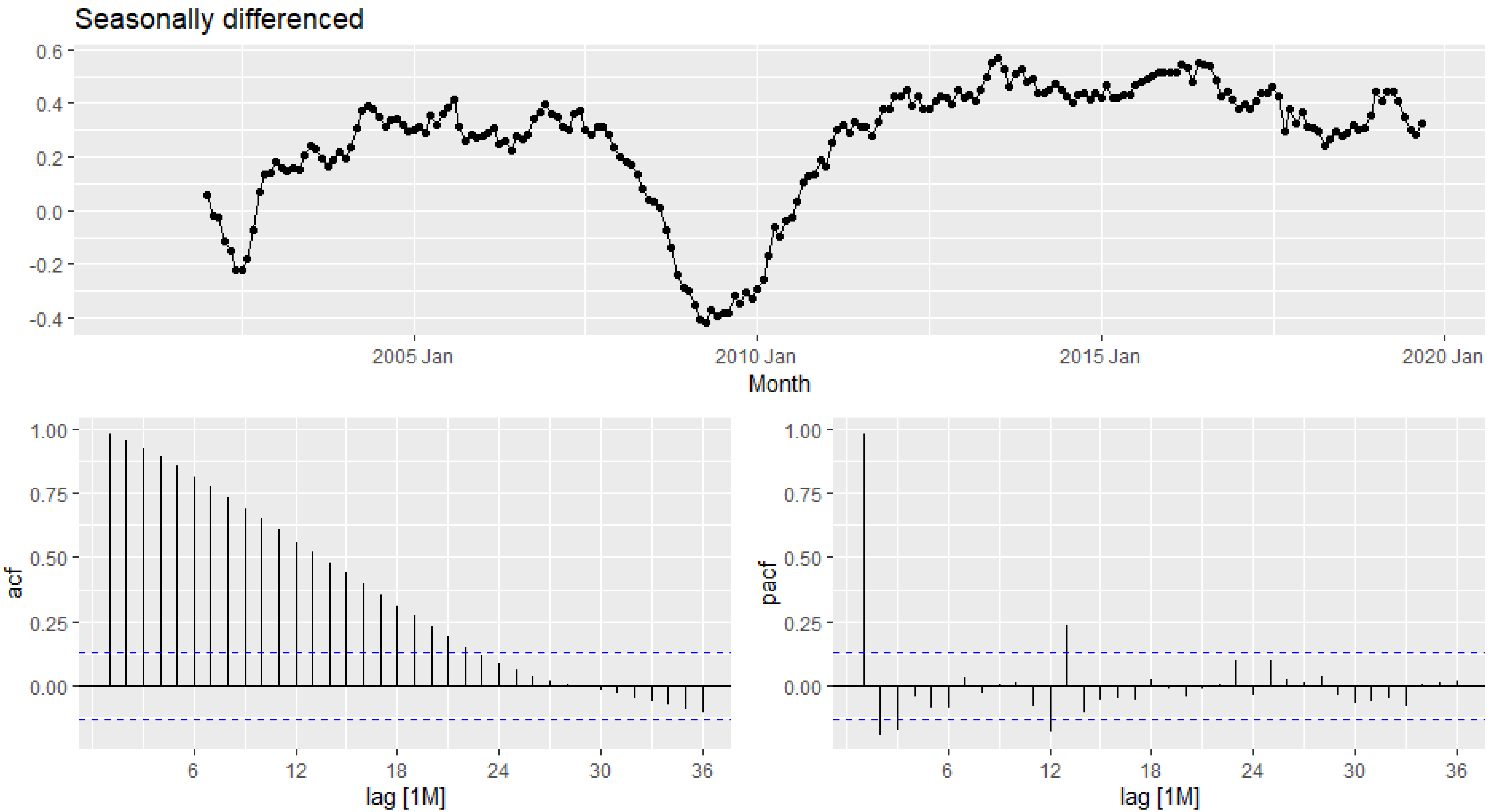
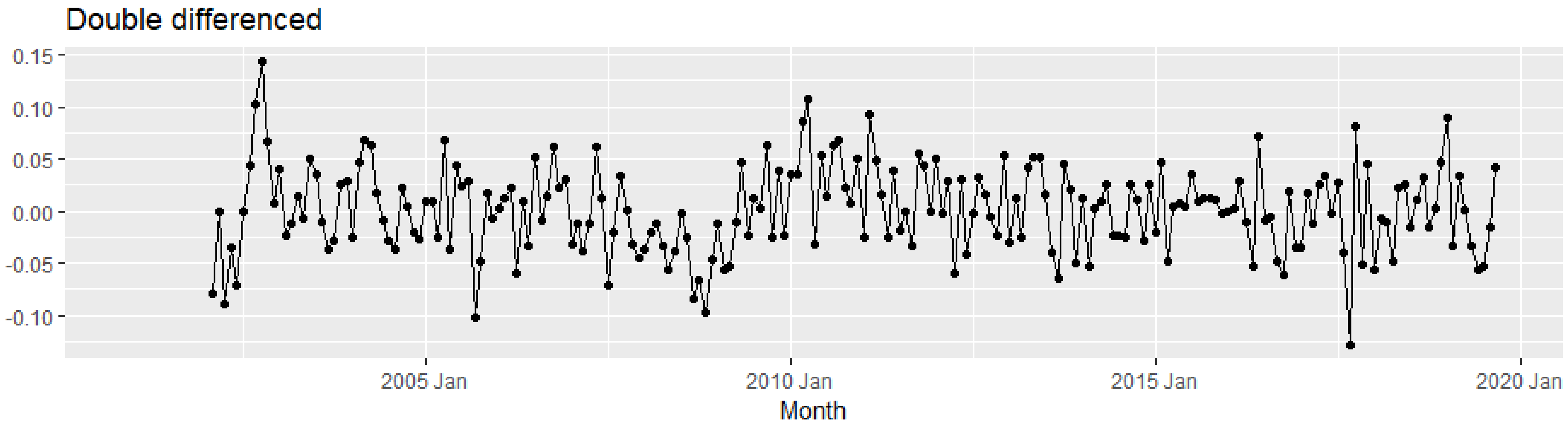
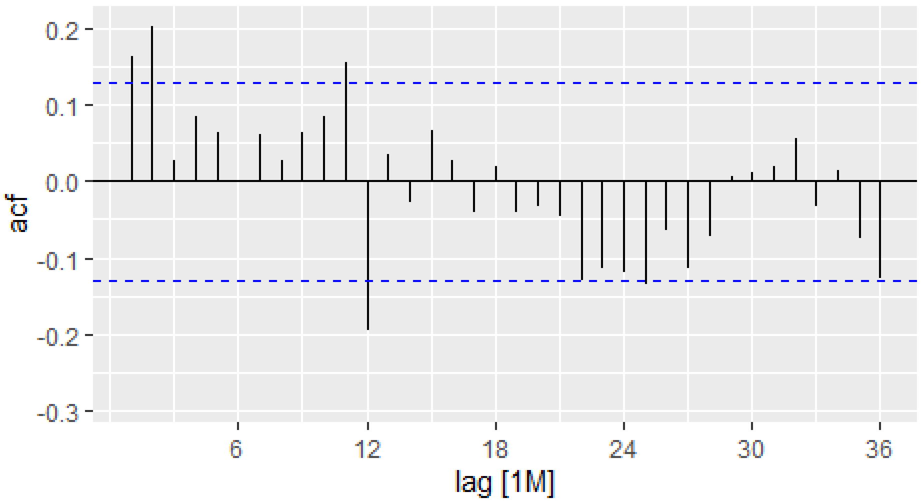
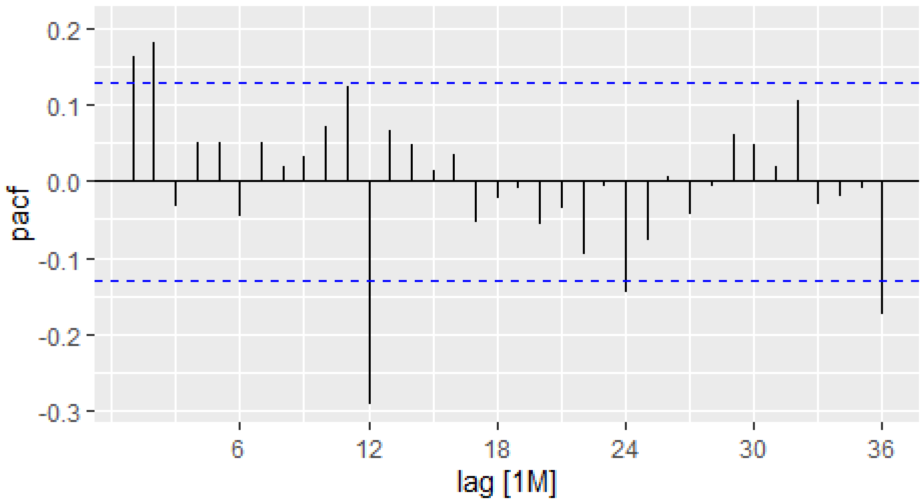
SARIMA example



SARIMA example







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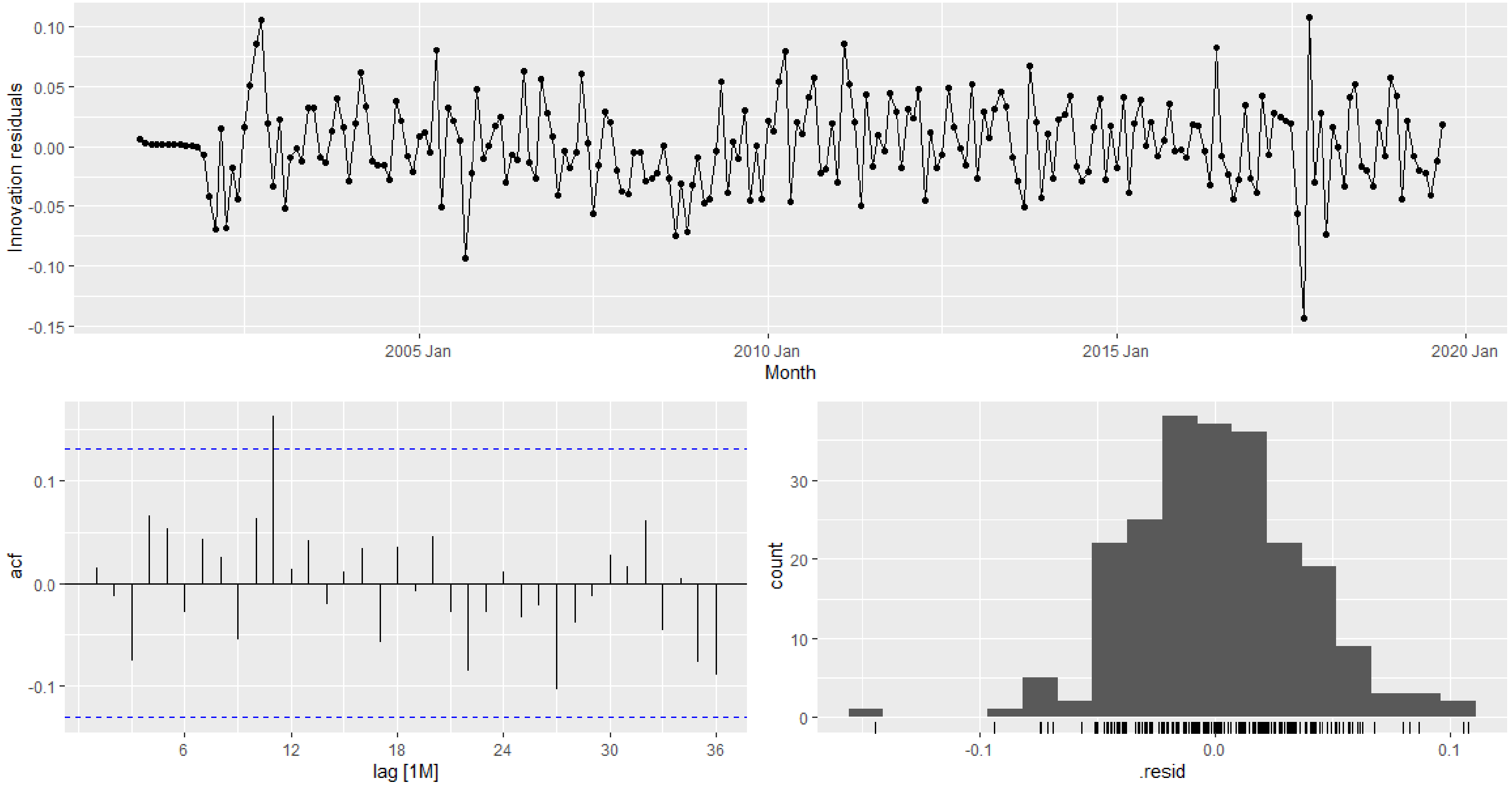
SARIMA example

- Using Auto-SARIMA, the winning model is $\text{SARIMA}(2,1,0)(1,1,1)_{12}$
- Plotting residuals to confirm they are like Gaussian white noise.



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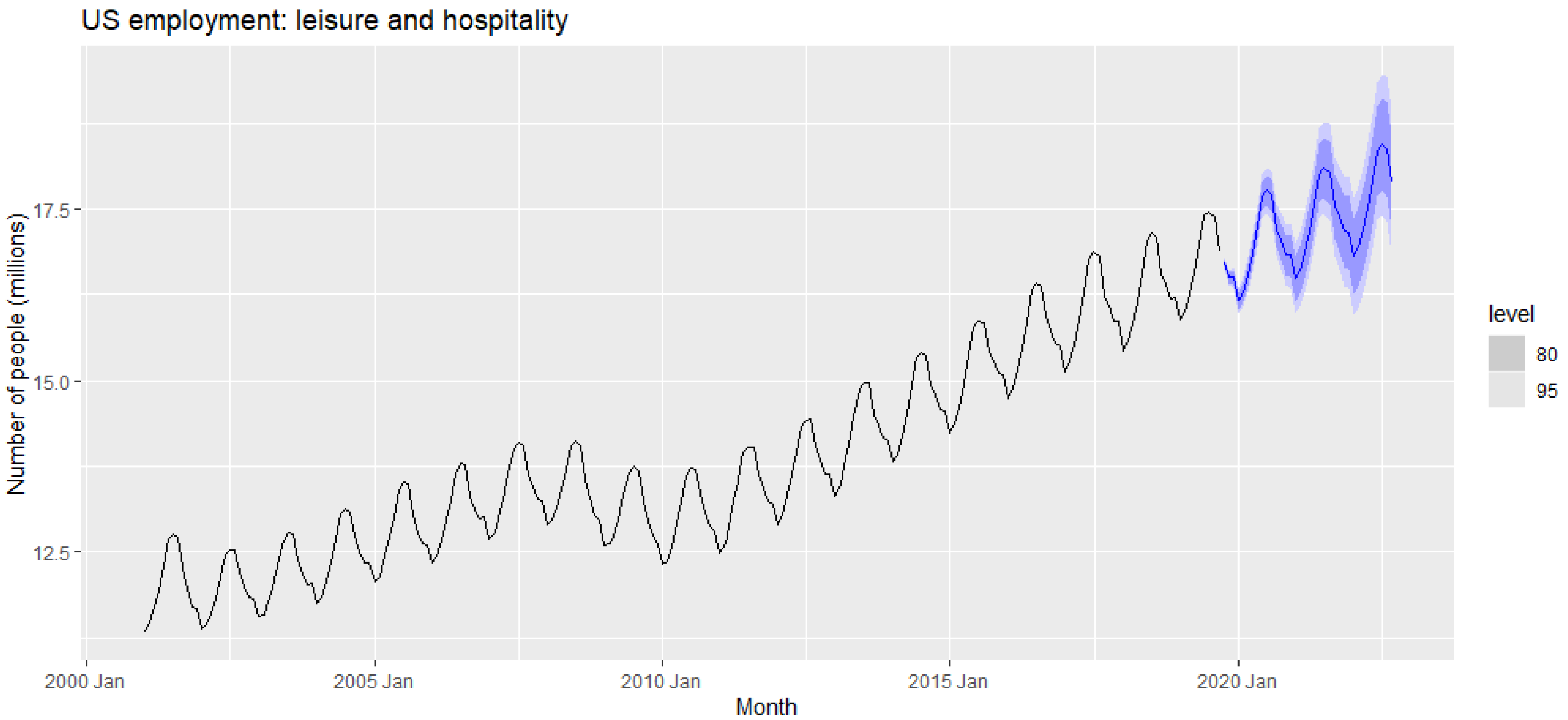
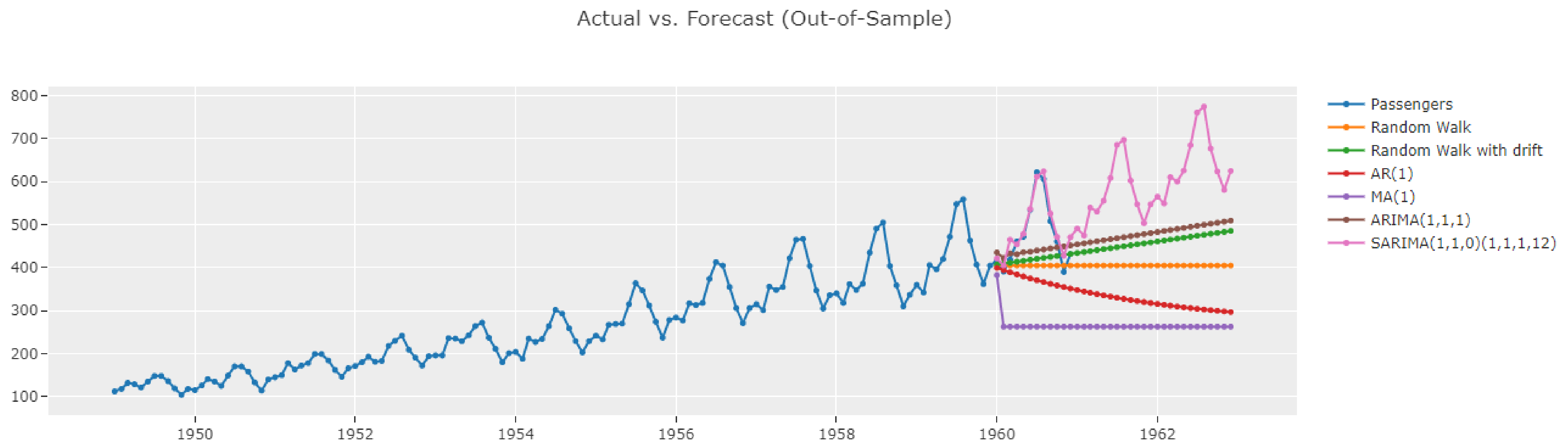
SARIMA example, Forecasting

- We now have a seasonal ARIMA model that passes the required checks and is ready for forecasting.
- The forecasts have captured the seasonal pattern very well, and the increasing trend extends the recent pattern. The trend in the forecasts is induced by the double differencing.



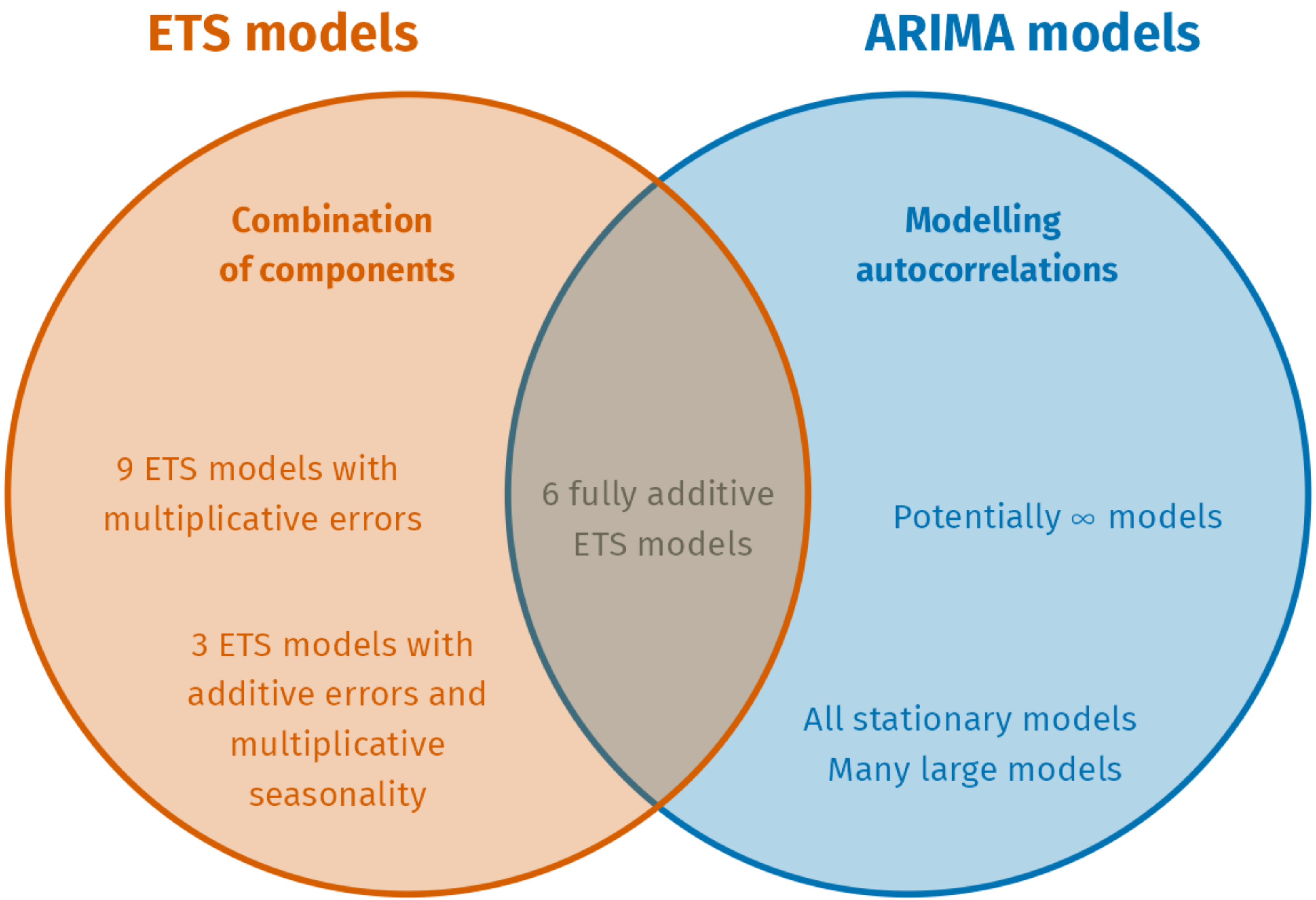
Comparing all the models



ARIMA vs ETS

- ARIMA and ETS models can be used together to enhance forecasting accuracy
- **Modeling Approach:**
    - ARIMA: Focuses on describing the **autocorrelations** in the data. It models the time series as a **linear** function of its past values (AR part), the past error terms (MA part), and differences of the series (I part) to ensure stationarity.
    - ETS: Models the time series by **explicitly decomposing it into error, trend, and seasonal** components, which can be combined additively or multiplicatively. ETS directly models the series' level, trend, and seasonality.

ARIMA vs ETS


ETS models
ARIMA models
Combination of components
Modelling autocorrelations
9 ETS models with multiplicative errors
6 fully additive ETS models
Potentially ∞ models
3 ETS models with additive errors and multiplicative seasonality
All stationary models
Many large models

ARIMA vs ETS

- ARIMA and ETS models can be used together to enhance forecasting accuracy
- **Data Characteristics:**
  - ARIMA is well-suited for time series that can be made stationary through differencing and that have significant autocorrelation patterns but may not have a clear trend or seasonal component.
  - ETS excels with time series that have a pronounced trend and/or seasonality.

- In summary, ARIMA and ETS models are complementary because they offer different approaches to modeling and forecasting time series data, each with its own set of advantages.
- The choice between them, or the decision to use them together, depends on the specific characteristics of the time series data and the forecasting goals.

Handling Non-Linearities

- Data Transformation: Logarithms, Box-Cox, or similar transformations can sometimes linearize **mild non-linear relationships**, making the data suitable for ARIMA or ETS

- Approximating with ARIMA: Combining **multiple AR and MA terms** with differencing allows ARIMA models to **partially** capture some forms of non-linear trends or patterns.

- Approximating with ETS: selecting between **additive** and **multiplicative** components allows for **some** flexibility in handling non-linear trends and seasonal patterns

- Limitations of ARIMA and ETS: For data with strong, complex non-linear relationships, standard ARIMA and ETS models often prove insufficient

- **Transition to Machine Learning**: ML models like tree-based methods (e.g., Random Forests, Gradient Boosting) or neural networks (given enough data) are inherently designed to capture complex non-linear patterns in time series.

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Road map!

- ✓ Module 1- Introduction to Deep Forecasting
- ✓ Module 2- Setting up Deep Forecasting Environment
- ✓ Module 3- Exponential Smoothing
- ✓ Module 4- ARIMA models
- Module 5- Machine Learning for Time series Forecasting
- Module 6- Deep Neural Networks
- Module 7- Deep Sequence Modeling (RNN, LSTM)
- Module 8- Prophet and Neural Prophet

Module 4 – Part III

ARIMA models in Python (Pycaret)