Road map!

- Module 1- Introduction to Deep Forecasting
- Module 2- Setting up Deep Forecasting Environment
- **Module 3- Exponential Smoothing**
- Module 4- ARIMA models
- Module 5- Machine Learning for Time series Forecasting
- Module 6- Deep Neural Networks
- Module 7- Deep Sequence Modeling (RNN, LSTM)
- Module 8- Transformers (Attention is all you need!)
- Module 9- Prophet and Neural Prophet

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Exponential Smoothing

- Exponential smoothing was proposed in the late 1950s (Brown, 1959; Holt, 1957; Winters, 1960), and has motivated some of the most successful forecasting methods
- A forecast generated by exponential smoothing uses weighted averages of past observations, with the weights decaying exponentially over time.
- In other words, the more recent the observation the higher the associated weight.
- In this module:
  - First, we present the mechanics of the most important exponential smoothing methods
  - Then, we present the statistical models that underlie exponential smoothing methods. These models generate identical point forecasts to the methods discussed in the first part of the chapter, but also generate prediction intervals.

Module 3- Part I

Exponential Smoothing (methods)

Method	Data Pattern		Forecast Equation
SES	No trend	, No seasonality	$\hat{y}_{t+h\|t} = \boldsymbol{l_t}$
Holt's linear trend	Trend	, No seasonality	$\hat{y}_{t+h\|t} = \boldsymbol{l_t} + h b_t$
Damped trend	Damped Trend	, No seasonality	$\hat{y}_{t+h\|t} = \boldsymbol{l_t} + (\phi + \phi^2 + \cdots + \phi^h) b_t$
Holt Winter	Trend	, Seasonality	$\hat{y}_{t+h\|t} = \boldsymbol{l_t} + h b_t + s_{t+h-m(k+1)}$ $\hat{y}_{t+h\|t} = (\boldsymbol{l_t} + h b_t) * s_{t+h-m(k+1)}$
Holt-Winter's Damped	Damped Trend	, Seasonality	$\hat{y}_{t+h\|t} = [\boldsymbol{l_t} + (\phi + \phi^2 + \cdots + \phi^h)]\, b_t * s_{t+h-m(k+1)}$

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Forecasting notation

$$\hat{y}_{t+h|t} = \boldsymbol{f}(y_t)$$

- y_t itself can be decomposed into different components (level, trend, seasonality)
- Fitted values at time $\mathrm{t} = 1 \ldots \mathrm{T}$, are $\hat{y}_{t|t-1}$ $(h = 0)$
- One-step ahead forecast at time $\mathrm{T} + 1$ (T last observation in train data) and $h = 1$.
- Multi-step ahead forecast: $h = 2, 3, 4, \ldots$
  - One-output at a time
  - Multi-output at once

1	2	3	4	5	6	7	8	9	T=10	11	12	13	14	15

Train

Forecast

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Simple Exponential Smoothing (SES)

- SES is the simples exponential smoothing method.
- SES is suitable for forecasting data with no clear trend or seasonal pattern
- Naïve forecast can be thought of as a weighted average where all the weight is given to the last observation.

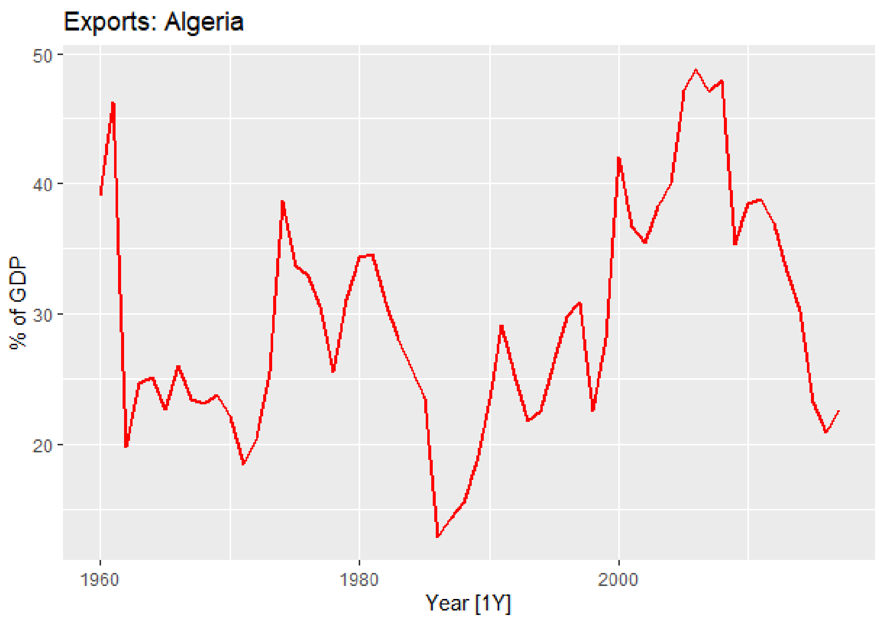
$$\hat{y}_{T+h|T} = y_T$$

- Mean forecast assumes all observations are of equal importance and give them same weights.

$$\hat{y}_{T+h|T} = \frac{1}{T}\sum_{t=1}^{T} y_t$$

- We want something in between these two extremes. ex, attach larger weights to more recent observations! This is SES.

$$\hat{y}_{T+1|T} = \alpha y_T + \alpha(1-\alpha) y_{T-1} + \alpha(1-\alpha)^2 y_{T-2} + \cdots$$



SES weights

$$\hat{y}_{T+1|T} = \alpha y_T + \alpha(1-\alpha)y_{T-1} + \alpha(1-\alpha)^2 y_{T-2} + \cdots$$

- $0 < \alpha < 1$ is the smoothing parameter.
- For any α between 0 and 1, the weights attached to the observations decrease exponentially as we go back in time, hence the name “exponential smoothing”.
- Sum of the weights is approximately one.

	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$
y_T	0.2000	0.4000	0.6000	0.8000
y_{T-1}	0.1600	0.2400	0.2400	0.1600
y_{T-2}	0.1280	0.1440	0.0960	0.0320
y_{T-3}	0.1024	0.0864	0.0384	0.0064
y_{T-4}	0.0819	0.0518	0.0154	0.0013
y_{T-5}	0.0655	0.0311	0.0061	0.0003

SES weights

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Equivalent forms of SES

- There are two equivalent forms of SES:

1. Weighted Average Form: the forecast is equal to weighted average between the most recent observation and the previous forecast.

$$\hat{y}_{T+1|T} = \alpha y_T + (1-\alpha)\hat{y}_{T|T-1}$$

2. Component Form: For simple exponential smoothing, the only component included is the level, l_t

Forecast equation	$\hat{y}_{t+h\|t} = \ell_t$
Smoothing equation	$\ell_t = \alpha y_t + (1-\alpha)\ell_{t-1}$

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SES forecasts

- Simple exponential smoothing has a "flat" forecast function:

$$\hat{y}_{T+h|T} = \hat{y}_{T+1|T} = \ell_T, \qquad h = 2, 3, \dots.$$

- All forecasts take the same value, equal to the last level component.
- Remember that these forecasts will only be suitable if the time series has no trend or seasonal component.
- The parameters of SES model (alpha and level zero) can be optimized by minimizing SSE.

Year	Time	Observation	Level	Forecast
	t	y_t	ℓ_t	$\hat{y}_{t\|t-1}$
1959	0		39.54	
1960	1	39.04	39.12	39.54
1961	2	46.24	45.10	39.12
1962	3	19.79	23.84	45.10
2016	57	20.86	21.43	24.39
2017	58	22.64	22.44	21.43
	h			$\hat{y}_{T+h\|T}$
2018	1			22.44
2019	2			22.44
2020	3			22.44
2021	4			22.44
2022	5			22.44

Exponential Smoothing

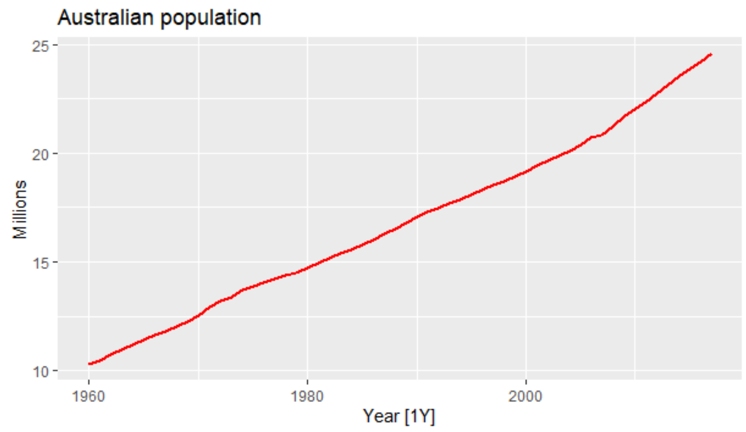
Methods with trend

Holt's linear trend method

- Holt (1957) extended simple exponential smoothing to allow the forecasting of data with a trend.
- This method is suitable for forecasting data with clear trend but no seasonal pattern
- This method involves a forecast equation and two smoothing equations:

$$
\begin{aligned}
\text{Forecast equation} \qquad \hat{y}_{t+h|t} &= \ell_t + hb_t \\
\text{Level equation} \qquad \ell_t &= \alpha y_t + (1-\alpha)(\ell_{t-1} + b_{t-1}) \\
\text{Trend equation} \qquad b_t &= \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1},
\end{aligned}
$$

- l_t denotes an estimate of the level of the series at time t
- b_t denotes an estimate of the trend (slope) of the series at time t
- α and β^* are the smoothing parameters for level and trend.



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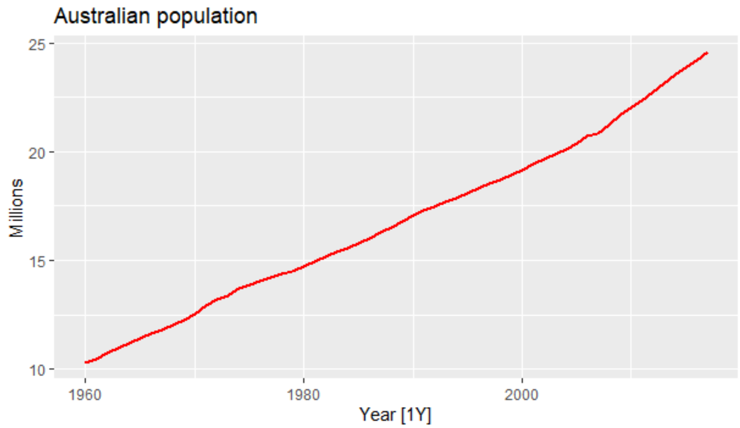
Holt's linear trend forecasts

- The forecast function is no longer flat but trending. $\hat{y}_{t+h|t} = \ell_t + hb_t$
- The h-step-ahead forecast is equal to the last estimated level + h times the last estimated trend value. Hence the forecasts are a linear function of h
- The smoothing parameters, and the initial values are estimated by minimizing the SSE for the one-step training errors



Year	Time	Observation	Level	Slope	Forecast
	t	y_t	ℓ_t		$\hat{y}_{t+1\|t}$
1959	0		10.05	0.22	
1960	1	10.28	10.28	0.22	10.28
1961	2	10.48	10.48	0.22	10.50
1962	3	10.74	10.74	0.23	10.70
2016	57	24.21	24.21	0.36	24.21
2017	58	24.60	24.60	0.37	24.57
	h				$\hat{y}_{T+h\|T}$
2018	1				24.97
2019	2				25.34
2020	3				25.71
2021	4				26.07
2022	5				26.44

$$\hat{y}_{t+h|t} = \ell_t + hb_t$$
$$\ell_t = \alpha y_t + (1-\alpha)(\ell_{t-1} + b_{t-1})$$
$$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1}$$

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Damped trend methods

- The forecasts generated by Holt's linear method display a constant trend (increasing or decreasing) indefinitely into the future
- Empirical evidence indicates that these methods tend to over-forecast, especially for longer forecast horizons.
- Gardner & McKenzie (1985) introduced a parameter that "dampens" the trend to a flat line some time in the future (adding a damping parameter $0 < \phi < 1$)
- ϕ dampens the trend so that it approaches a constant some time in the future. This means that short-run forecasts are trended a while long-run forecasts are constant.

$$
\begin{aligned}
\text{Forecast equation} \quad & \hat{y}_{t+h|t} = \ell_t + h b_t \\
\text{Level equation} \quad & \ell_t = \alpha y_t + (1-\alpha)(\ell_{t-1} + b_{t-1}) \\
\text{Trend equation} \quad & b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*) b_{t-1},
\end{aligned}
$$

$$
\begin{aligned}
\hat{y}_{t+h|t} &= \ell_t + (\phi + \phi^2 + \cdots + \phi^h) b_t \\
\ell_t &= \alpha y_t + (1-\alpha)(\ell_{t-1} + \phi b_{t-1}) \\
b_t &= \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)\phi b_{t-1}
\end{aligned}
$$

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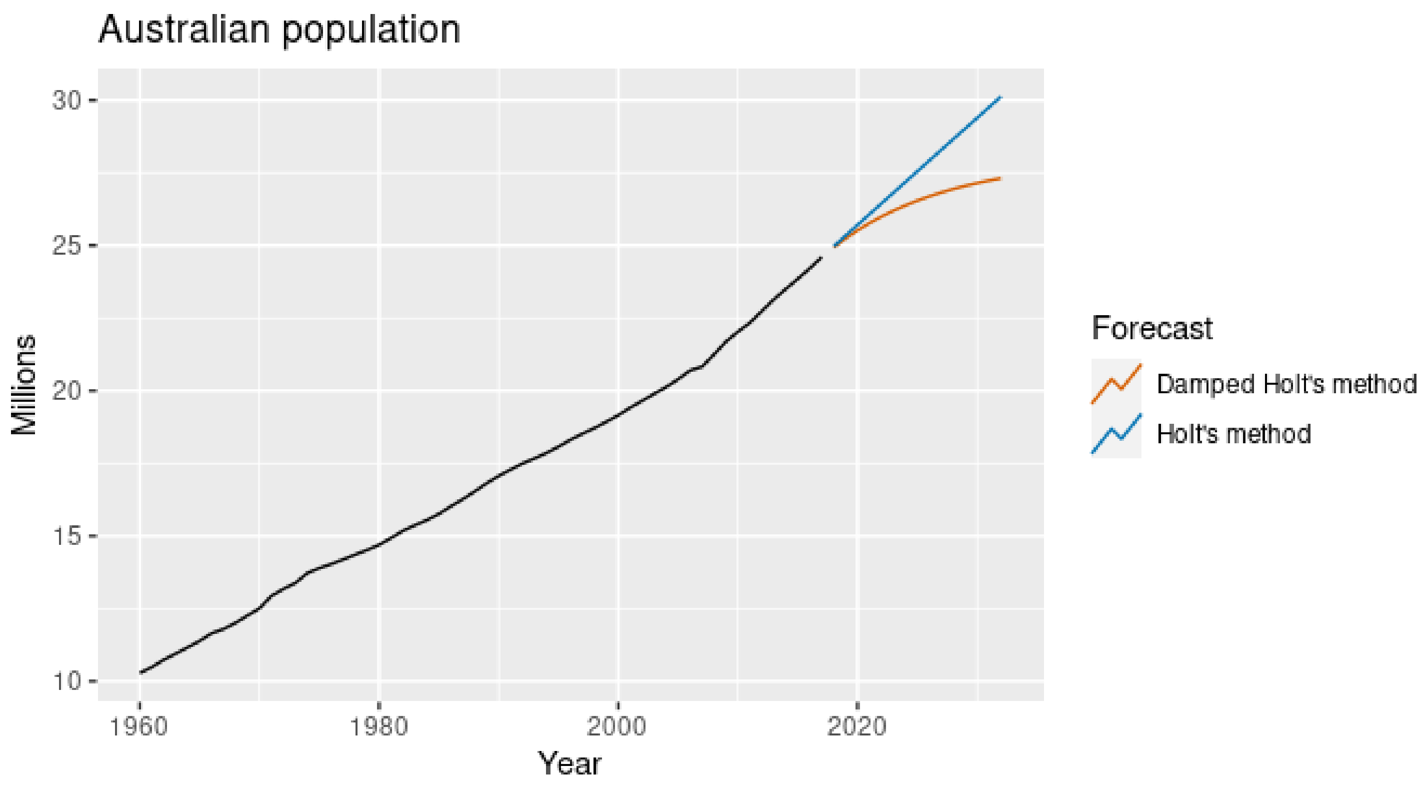
Methods with trend, Example

- Forecasting annual Australian population (millions) over 2018-2032. For the damped trend method, ϕ =0.90



Exponential Smoothing

Methods with trend and seasonality

Holt-Winters method

- Holt (1957) and Winters (1960) extended Holt's method to capture seasonality.
- This method comprises the forecast equation and three smoothing equations:
  1. l_t for the level component with corresponding smoothing parameter α
  2. b_t for the trend component with corresponding smoothing parameter β^*
  3. l_t for the seasonal component with corresponding smoothing parameter γ

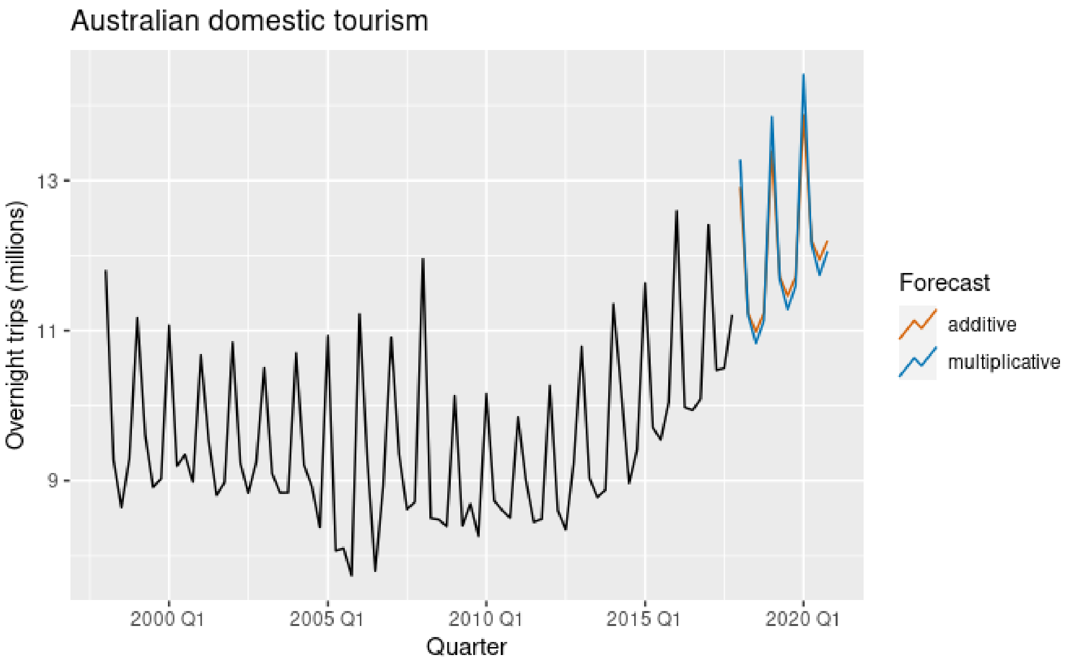
- There are two variations to this method that differ in the nature of the seasonal component.
  1. The additive method: when the seasonal variations are roughly constant through the series
  2. the multiplicative method: when the seasonal variations are changing proportional to the level of the series.

Holt-Winters' additive vs multiplicative methods

- m to denote the period of the seasonality. For quarterly data $m = 4$ and monthly, $m = 12$.

$$
\begin{aligned}
\hat{y}_{t+h|t} &= \ell_t + hb_t + s_{t+h-m(k+1)} \\
\ell_t &= \alpha(y_t - s_{t-m}) + (1-\alpha)(\ell_{t-1} + b_{t-1}) \\
b_t &= \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1} \\
s_t &= \gamma(y_t - \ell_{t-1} - b_{t-1}) + (1-\gamma)s_{t-m},
\end{aligned}
$$

$$
\begin{aligned}
\hat{y}_{t+h|t} &= (\ell_t + hb_t)s_{t+h-m(k+1)} \\
\ell_t &= \alpha\frac{y_t}{s_{t-m}} + (1-\alpha)(\ell_{t-1} + b_{t-1}) \\
b_t &= \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1} \\
s_t &= \gamma\frac{y_t}{(\ell_{t-1} + b_{t-1})} + (1-\gamma)s_{t-m}.
\end{aligned}
$$



$$\alpha = 0.2237, \beta^* = 0.1360, \gamma = 0.0001$$

Holt-Winters' damped methods

- Damping is possible with both additive and multiplicative Holt-Winters' methods.
- A method that often provides accurate and robust forecasts for seasonal data is the Holt-Winters method with a damped trend and multiplicative seasonality:

$$\begin{aligned}
\hat{y}_{t+h|t} &= \left[\ell_t + (\phi + \phi^2 + \cdots + \phi^h) b_t\right] s_{t+h-m(k+1)} \\
\ell_t &= \alpha(y_t / s_{t-m}) + (1-\alpha)(\ell_{t-1} + \phi b_{t-1}) \\
b_t &= \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)\phi b_{t-1} \\
s_t &= \gamma \frac{y_t}{(\ell_{t-1} + \phi b_{t-1})} + (1-\gamma) s_{t-m}.
\end{aligned}$$

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Summary

Method	Data Pattern		Forecast Equation
SES	No trend	, No seasonality	$\hat{y}_{t+h\|t} = \boldsymbol{l_t}$
Holt's linear trend	Trend	, No seasonality	$\hat{y}_{t+h\|t} = \boldsymbol{l_t} + hb_t$
Damped trend	Damped Trend	, No seasonality	$\hat{y}_{t+h\|t} = \boldsymbol{l_t} + (\phi + \phi^2 + \cdots + \phi^h) b_t$
Holt Winter	Trend	, Seasonality	$\hat{y}_{t+h\|t} = \boldsymbol{l_t} + hb_t + s_{t+h-m(k+1)}$ $\hat{y}_{t+h\|t} = (\boldsymbol{l_t} + hb_t) * s_{t+h-m(k+1)}$
Holt-Winter's Damped	Damped Trend	, Seasonality	$\hat{y}_{t+h\|t} = [\boldsymbol{l_t} + (\phi + \phi^2 + \cdots + \phi^h)]\, b_t * s_{t+h-m(k+1)}$

- we study the statistical models that underlie the exponential smoothing methods we have considered so far

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A taxonomy of exponential smoothing methods

Table 8.5: A two-way classification of exponential smoothing methods.

Trend Component	Seasonal Component		
	N	**A**	**M**
	(None)	**(Additive)**	**(Multiplicative)**
N (None)	(N,N)	(N,A)	(N,M)
A (Additive)	(A,N)	(A,A)	(A,M)
A_d (Additive damped)	(A_d,N)	(A_d,A)	(A_d,M)

Some of these methods we have already seen using other names:

Short hand	Method
(N,N)	Simple exponential smoothing
(A,N)	Holt's linear method
(A_d,N)	Additive damped trend method
(A,A)	Additive Holt-Winters' method
(A,M)	Multiplicative Holt-Winters' method
(A_d,M)	Holt-Winters' damped method

Module 3- Part II
Exponential Smoothing-based models
ETS

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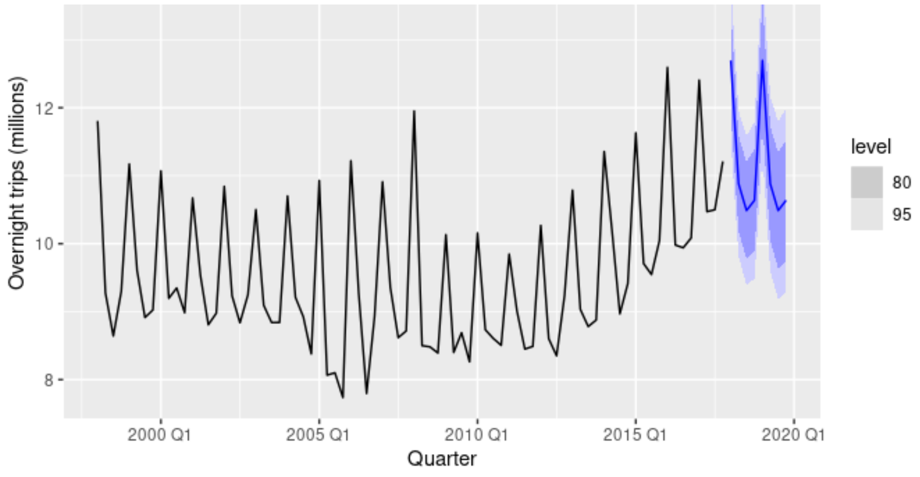
State Space Models for Exponential Smoothing

- The exponential smoothing methods are algorithms which generate point forecasts.
- The statistical models generate the same point forecasts but can also generate prediction (or forecast) intervals. i.e., producing the entire forecast distribution.
- These models are referred to as state space models because they describe how the unobserved components or states (level, trend, seasonal) change over time.
- We label each state space model as ETS (. , . , .)!
- ETS stands for Error, Trend, Seasonality! Also thought of as ExponenTial Smoothing.



State Space Models for Exponential Smoothing

- For each method there exist two models: one with additive errors and one with multiplicative errors.
- Models with multiplicative errors are useful when the data are strictly positive but are not numerically stable when the data contain zeros or negative values.
- State possibilities notation:

  Error= {A,M}   Trend={N, A, A_d}   Seasonal={N, A, M}

- We can write 2*3*3=18 different state space model for each of the exponential smoothing methods.

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ETS (A, N, N): Simple Exponential Smoothing with additive errors

- Recall SES components :

Forecast equation $\hat{y}_{t+h|t} = \ell_t$

Smoothing equation $\ell_t = \alpha y_t + (1-\alpha)\ell_{t-1}$

- Re-arrange the smoothing equation for the level and get the error correction from.

$$\begin{aligned}\ell_t &= \ell_{t-1} + \alpha(y_t - \ell_{t-1}) \\ &= \ell_{t-1} + \alpha e_t,\end{aligned}$$

- Where $e_t = y_t - \ell_{t-1} = y_t - \hat{y}_{t|t-1}$ is the residual at time t. Remember, level is what the ETS(A,N,N) model predicts.
- If the model is over/under shooting, the level will adjust in the next period. The magnitude of adjustment depends on α. Smaller α means smoother adjustment.
- We can also write $y_t = \ell_{t-1} + e_t$, so each observation = previous level + error

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ETS (A, N, N): Simple Exponential Smoothing with additive errors

- So far, we showed that $y_t = \ell_{t-1} + e_t$
- The only thing left out is to specify the probability distribution for errors. With that, we have our first innovations state space model.
- For a model with additive errors, we assume $e_t = \varepsilon_t \sim \text{NID}(0, \sigma^2)$, errors are normally and independently distributed. (white noise)
- The final model can be written as:

Measurement (observation) equation

$$y_t = \ell_{t-1} + \varepsilon_t$$
$$\ell_t = \ell_{t-1} + \alpha\varepsilon_t$$

State (transition) equation

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ETS(A,A,N): Holt's linear method with additive errors

- In this model, the training errors are given by $\varepsilon_t = y_t - \ell_{t-1} - b_{t-1} \sim \text{NID}(0, \sigma^2)$

- Substituting this into the level equation and trend equation, we get:

$$
\begin{aligned}
y_t &= \ell_{t-1} + b_{t-1} + \varepsilon_t \\
\ell_t &= \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t \\
b_t &= b_{t-1} + \beta\varepsilon_t,
\end{aligned}
$$

- Note that $\beta = \alpha\beta^*$ where α and β^* are the smoothing parameters for the level and trend components, respectively.

ETS(M,A,N): Holt's linear method with multiplicative errors

- In this model, the training errors are given by $\varepsilon_t = \dfrac{y_t - (\ell_{t-1} + b_{t-1})}{(\ell_{t-1} + b_{t-1})} \sim \text{NID}(0, \sigma^2)$

- Substituting this into the level equation and trend equation, we get:

$$
\begin{aligned}
y_t &= (\ell_{t-1} + b_{t-1})(1 + \varepsilon_t) \\
\ell_t &= (\ell_{t-1} + b_{t-1})(1 + \alpha\varepsilon_t) \\
b_t &= b_{t-1} + \beta(\ell_{t-1} + b_{t-1})\varepsilon_t,
\end{aligned}
$$

- Note that $\beta = \alpha\beta^*$ where α and β^* are the smoothing parameters for the level and trend components, respectively.

Other ETS models

- Recall: Error= {A,M}, Trend={N, A, *A_d*}, Seasonal={N, A, M}
- We can write 2*3*3=18 different state space model for each of the exponential smoothing methods.

ADDITIVE ERROR MODELS

Trend	Seasonal N	Seasonal A	Seasonal M
N	$y_t = \ell_{t-1} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \alpha\varepsilon_t$	$y_t = \ell_{t-1} + s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \alpha\varepsilon_t$ $s_t = s_{t-m} + \gamma\varepsilon_t$	$y_t = \ell_{t-1}s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \alpha\varepsilon_t/s_{t-m}$ $s_t = s_{t-m} + \gamma\varepsilon_t/\ell_{t-1}$
A	$y_t = \ell_{t-1} + b_{t-1} + \varepsilon_t$ $\ell_t = \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t$ $b_t = b_{t-1} + \beta\varepsilon_t$	$y_t = \ell_{t-1} + b_{t-1} + s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t$ $b_t = b_{t-1} + \beta\varepsilon_t$ $s_t = s_{t-m} + \gamma\varepsilon_t$	$y_t = (\ell_{t-1} + b_{t-1})s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t/s_{t-m}$ $b_t = b_{t-1} + \beta\varepsilon_t/s_{t-m}$ $s_t = s_{t-m} + \gamma\varepsilon_t/(\ell_{t-1} + b_{t-1})$
A$_\mathbf{d}$	$y_t = \ell_{t-1} + \phi b_{t-1} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha\varepsilon_t$ $b_t = \phi b_{t-1} + \beta\varepsilon_t$	$y_t = \ell_{t-1} + \phi b_{t-1} + s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha\varepsilon_t$ $b_t = \phi b_{t-1} + \beta\varepsilon_t$ $s_t = s_{t-m} + \gamma\varepsilon_t$	$y_t = (\ell_{t-1} + \phi b_{t-1})s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha\varepsilon_t/s_{t-m}$ $b_t = \phi b_{t-1} + \beta\varepsilon_t/s_{t-m}$ $s_t = s_{t-m} + \gamma\varepsilon_t/(\ell_{t-1} + \phi b_{t-1})$

Other ETS models

- Recall: Error= {A,M}, Trend={N, A, *A_d*}, Seasonal={N, A, M}
- We can write 2*3*3=18 different state space model for each of the exponential smoothing methods.

MULTIPLICATIVE ERROR MODELS

Trend	Seasonal N	Seasonal A	Seasonal M
N	$y_t = \ell_{t-1}(1+\varepsilon_t)$ $\ell_t = \ell_{t-1}(1+\alpha\varepsilon_t)$	$y_t = (\ell_{t-1}+s_{t-m})(1+\varepsilon_t)$ $\ell_t = \ell_{t-1}+\alpha(\ell_{t-1}+s_{t-m})\varepsilon_t$ $s_t = s_{t-m}+\gamma(\ell_{t-1}+s_{t-m})\varepsilon_t$	$y_t = \ell_{t-1}s_{t-m}(1+\varepsilon_t)$ $\ell_t = \ell_{t-1}(1+\alpha\varepsilon_t)$ $s_t = s_{t-m}(1+\gamma\varepsilon_t)$
A	$y_t = (\ell_{t-1}+b_{t-1})(1+\varepsilon_t)$ $\ell_t = (\ell_{t-1}+b_{t-1})(1+\alpha\varepsilon_t)$ $b_t = b_{t-1}+\beta(\ell_{t-1}+b_{t-1})\varepsilon_t$	$y_t = (\ell_{t-1}+b_{t-1}+s_{t-m})(1+\varepsilon_t)$ $\ell_t = \ell_{t-1}+b_{t-1}+\alpha(\ell_{t-1}+b_{t-1}+s_{t-m})\varepsilon_t$ $b_t = b_{t-1}+\beta(\ell_{t-1}+b_{t-1}+s_{t-m})\varepsilon_t$ $s_t = s_{t-m}+\gamma(\ell_{t-1}+b_{t-1}+s_{t-m})\varepsilon_t$	$y_t = (\ell_{t-1}+b_{t-1})s_{t-m}(1+\varepsilon_t)$ $\ell_t = (\ell_{t-1}+b_{t-1})(1+\alpha\varepsilon_t)$ $b_t = b_{t-1}+\beta(\ell_{t-1}+b_{t-1})\varepsilon_t$ $s_t = s_{t-m}(1+\gamma\varepsilon_t)$
A$_\mathbf{d}$	$y_t = (\ell_{t-1}+\phi b_{t-1})(1+\varepsilon_t)$ $\ell_t = (\ell_{t-1}+\phi b_{t-1})(1+\alpha\varepsilon_t)$ $b_t = \phi b_{t-1}+\beta(\ell_{t-1}+\phi b_{t-1})\varepsilon_t$	$y_t = (\ell_{t-1}+\phi b_{t-1}+s_{t-m})(1+\varepsilon_t)$ $\ell_t = \ell_{t-1}+\phi b_{t-1}+\alpha(\ell_{t-1}+\phi b_{t-1}+s_{t-m})\varepsilon_t$ $b_t = \phi b_{t-1}+\beta(\ell_{t-1}+\phi b_{t-1}+s_{t-m})\varepsilon_t$ $s_t = s_{t-m}+\gamma(\ell_{t-1}+\phi b_{t-1}+s_{t-m})\varepsilon_t$	$y_t = (\ell_{t-1}+\phi b_{t-1})s_{t-m}(1+\varepsilon_t)$ $\ell_t = (\ell_{t-1}+\phi b_{t-1})(1+\alpha\varepsilon_t)$ $b_t = \phi b_{t-1}+\beta(\ell_{t-1}+\phi b_{t-1})\varepsilon_t$ $s_t = s_{t-m}(1+\gamma\varepsilon_t)$

ETS model estimation

- Maximum Likelihood Estimation is used to optimize the smoothing parameters and the initial values for level, trend and seasonal components.
- The smoothing parameters are restricted to be between 0 and 1. $0 < \alpha, \beta^*, \gamma^*, \phi < 1$ so that the equations can be interpreted as weighted averages.
- The parameters are constrained in order to prevent observations in the distant past having a continuing effect on current forecasts.

- Reminder: Maximum likelihood estimation (MLE) is a method used to estimate the parameters of a statistical model, given observations. It involves finding the parameter values that maximize the likelihood function, which is a function that describes the probability of a set of observations given the parameter values.

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ETS model selection

- Fore model selection we can either use information criteria or any cross validated performance metrics like R^2, MSE, RMSE, MAPE, sMAPE.

Information Criteria	Formula
Akaike's Information Criterion (AIC)	$\text{AIC} = -2\log(L) + 2k$
AIC corrected for small sample bias (AICc)	$\text{AIC}_{\text{c}} = \text{AIC} + \frac{2k(k+1)}{T-k-1}$
Bayesian Information Criterion (BIC)	$\text{BIC} = \text{AIC} + k[\log(T) - 2]$

- L is the likelihood of the model and K is the total number of parameters and initials states that have been estimated (including the residual variance)
- The model with the minimum information criteria is often the best model for forecasting

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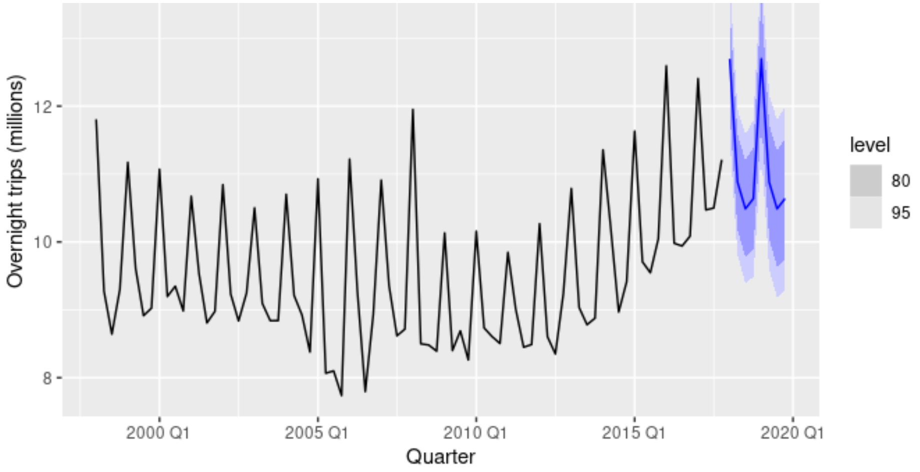
Forecasting with ETS models

- Point forecasts can be obtained from the models by iterating the equations for the forecasting horizon. $t = T+1, \dots T+h$
- Setting all $\epsilon_t = 0$ for $t > T$
- These point forecasts are identical to the forecasts from the exponential smoothing methods.
- Prediction intervals: for most ETS models, a prediction interval can be written as:

Critical values

$$\hat{y}_{T+h|T} \pm c\sigma_h$$

Forecast standard error



Forecast variance: σ_h^2

Table 8.8: Forecast variance expressions for each additive state space model, where σ^2 is the residual variance, m is the seasonal period, and k is the integer part of $(h-1)/m$ (i.e., the number of complete years in the forecast period prior to time $T+h$).

Model	Forecast variance: σ_h^2
(A,N,N)	$\sigma_h^2 = \sigma^2\left[1+\alpha^2(h-1)\right]$
(A,A,N)	$\sigma_h^2 = \sigma^2\left[1+(h-1)\left\{\alpha^2+\alpha\beta h+\frac{1}{6}\beta^2 h(2h-1)\right\}\right]$
(A,A$_d$,N)	$\sigma_h^2 = \sigma^2\Big[1+\alpha^2(h-1)+\frac{\beta\phi h}{(1-\phi)^2}\{2\alpha(1-\phi)+\beta\phi\} -\frac{\beta\phi(1-\phi^h)}{(1-\phi)^2(1-\phi^2)}\left\{2\alpha(1-\phi^2)+\beta\phi(1+2\phi-\phi^h)\right\}\Big]$
(A,N,A)	$\sigma_h^2 = \sigma^2\left[1+\alpha^2(h-1)+\gamma k(2\alpha+\gamma)\right]$
(A,A,A)	$\sigma_h^2 = \sigma^2\Big[1+(h-1)\left\{\alpha^2+\alpha\beta h+\frac{1}{6}\beta^2 h(2h-1)\right\} +\gamma k\left\{2\alpha+\gamma+\beta m(k+1)\right\}\Big]$
(A,A$_d$,A)	$\sigma_h^2 = \sigma^2\Big[1+\alpha^2(h-1)+\gamma k(2\alpha+\gamma) +\frac{\beta\phi h}{(1-\phi)^2}\{2\alpha(1-\phi)+\beta\phi\} -\frac{\beta\phi(1-\phi^h)}{(1-\phi)^2(1-\phi^2)}\left\{2\alpha(1-\phi^2)+\beta\phi(1+2\phi-\phi^h)\right\} +\frac{2\beta\gamma\phi}{(1-\phi)(1-\phi^m)}\left\{k(1-\phi^m)-\phi^m(1-\phi^{mk})\right\}\Big]$

Title

- We need some example plots for the second half.

Road map!

- ✓ Module 1- Introduction to Deep Forecasting
- ✓ Module 2- Setting up Deep Forecasting Environment
- ✓ Module 3- Exponential Smoothing
- Module 4- ARIMA models
- Module 5- Machine Learning for Time series Forecasting
- Module 6- Deep Neural Networks
- Module 7- Deep Sequence Modeling (RNN, LSTM)
- Module 8- Transformers (Attention is all you need!)
- Module 9- Prophet and Neural Prophet

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