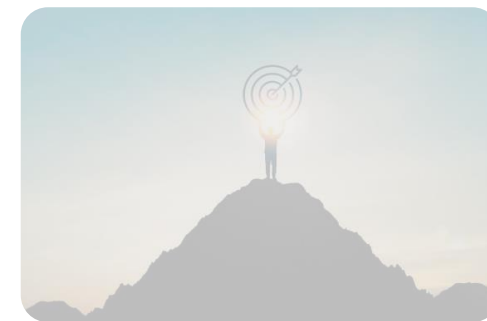


# ➔ Road map!

- Module 1- Introduction to Deep Forecasting
- Module 2- Setting up Deep Forecasting Environment
- Module 3- Exponential Smoothing
- **Module 4- ARIMA models**
- Module 5- Machine Learning for Time series Forecasting
- Module 6- Deep Neural Networks
- Module 7- Deep Sequence Modeling (RNN, LSTM)
- Module 8- Transformers (Attention is all you need!)

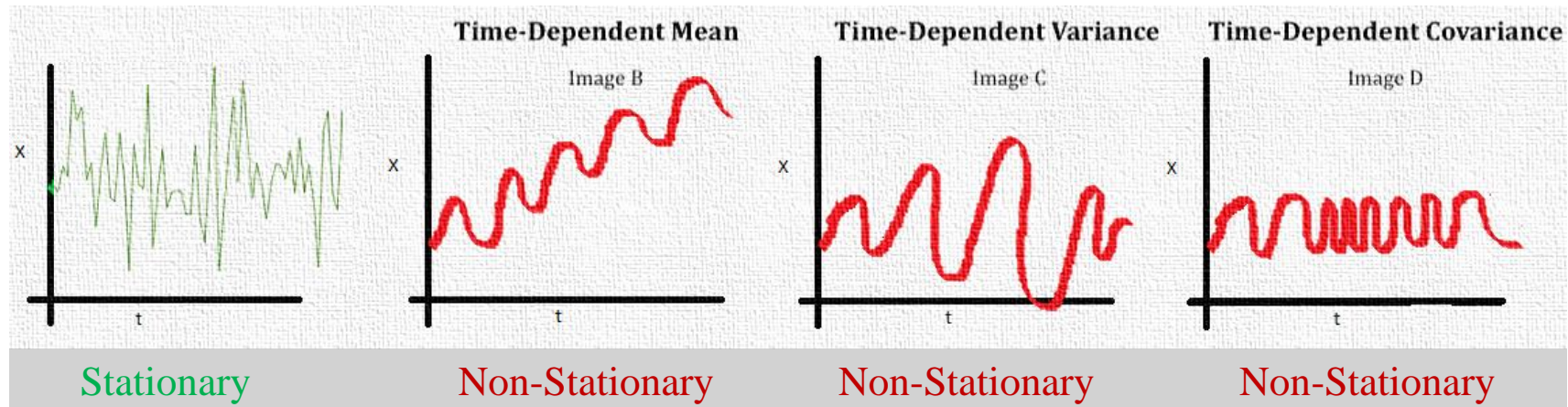


# Module 4 – Part I

## ARIMA models' Prerequisites

### ACF, PACF, Stationarity, Differencing

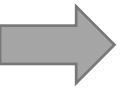
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# → ARIMA models prerequisites

- ARIMA stands for AutoRegressive Integrated Moving Average. It is a class of **statistical models** for analyzing and forecasting time series data.
- ETS and ARIMA models are two popular models for forecasting time series data. They offer **complementary approaches** to addressing the challenges of time series forecasting.
- **ARIMA** models describe **autocorrelations** in the data, whereas **ETS** models describe **trends** and **seasonality**.
- Let's review some prerequisites before moving forward with the models:

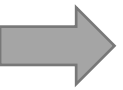




# Autocorrelation

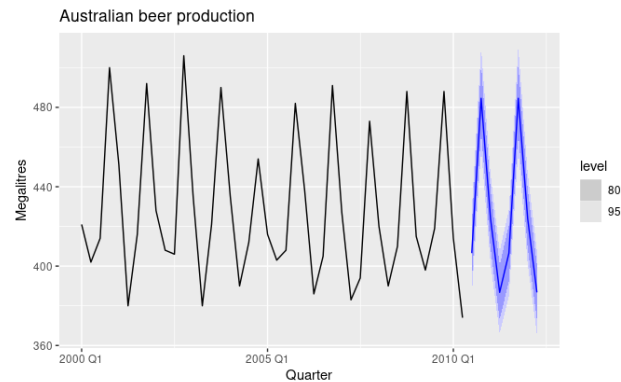
- **Autocorrelation**, also known as **serial correlation**, is a measure of the correlation between a time series and a lagged version of itself.
- It is used to assess the **degree to which** the past values of a time series **are predictive** of its future values.

$$r_k = \frac{\sum_{t=k+1}^T (y_t - \bar{y})(y_{t-k} - \bar{y})}{\sum_{t=1}^T (y_t - \bar{y})^2}$$

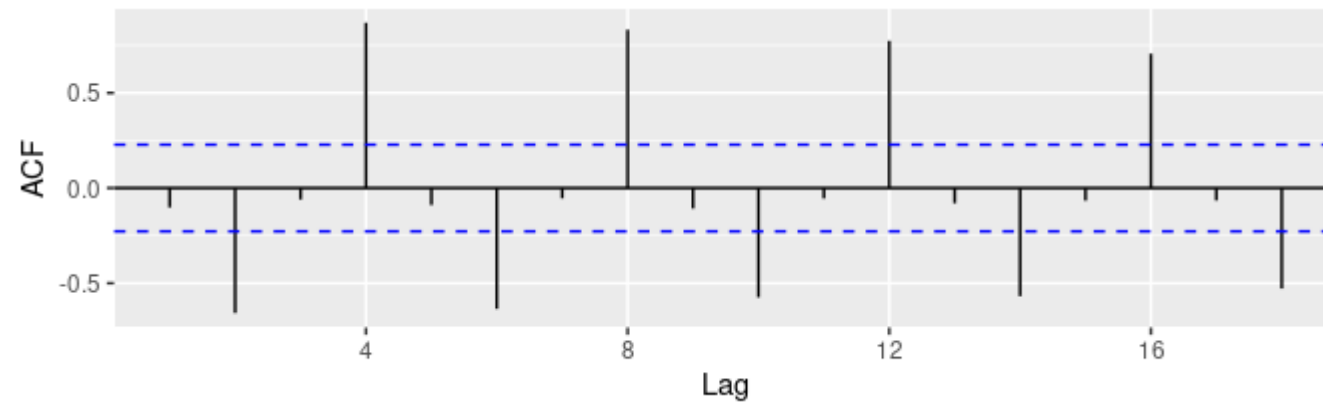


# ACF: Autocorrelation Function

- The autocorrelation function (**ACF**) is a statistical tool that can be used to measure the autocorrelation of a time series.
- It calculates the correlation between the time series and lagged versions of itself at different lag periods.



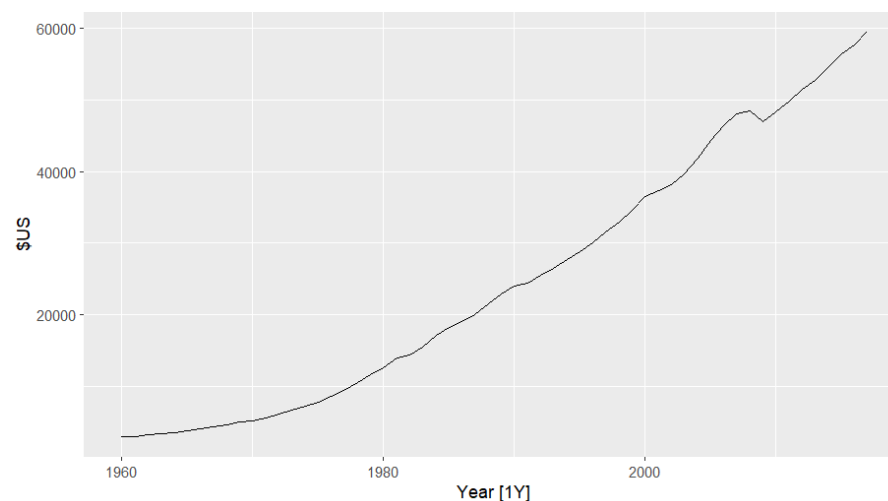
$r_1$	$r_2$	$r_3$	$r_4$	$r_5$	$r_6$	$r_7$	$r_8$	$r_9$
-0.102	-0.657	-0.060	0.869	-0.089	-0.635	-0.054	0.832	-0.108

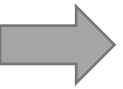


# ➔ Partial Autocorrelation

- **Partial autocorrelation**, also known as **partial serial correlation**, is a measure of the correlation between a time series and a lagged version of itself, **controlling for the effects of intermediate lag periods**.
- $y_t$  and  $y_{t-2}$  might be correlated, simply because they are both connected to  $y_{t-1}$ , rather than because of any new information contained in  $y_{t-2}$ . **Partial autocorrelation** overcomes this problem.

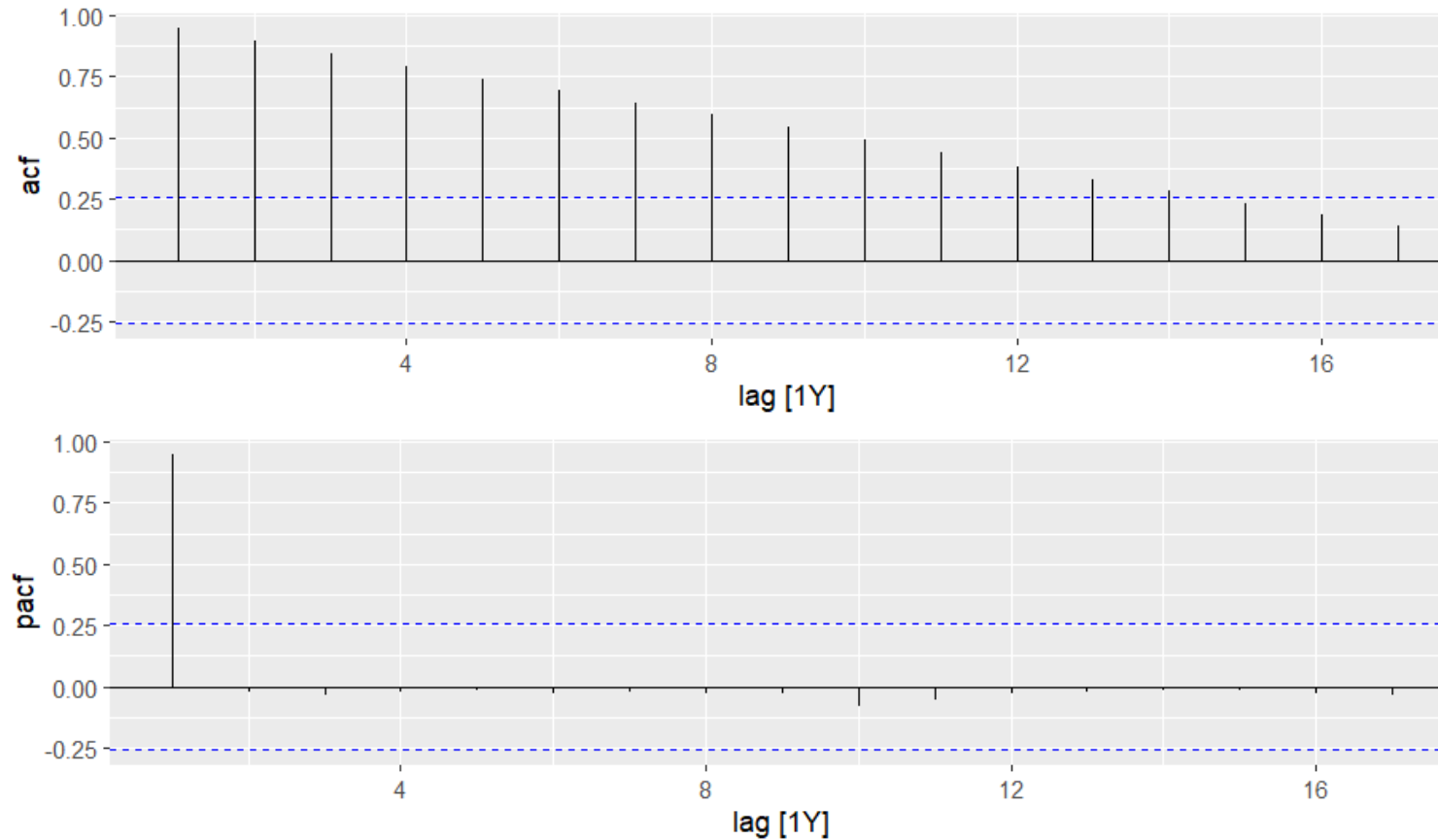
US Annual GDP per capita (1960-2017)



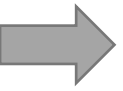


# PACF: Partial Autocorrelation Function

- PACF is a statistical tool that can be used to measure the partial autocorrelation of a time series.

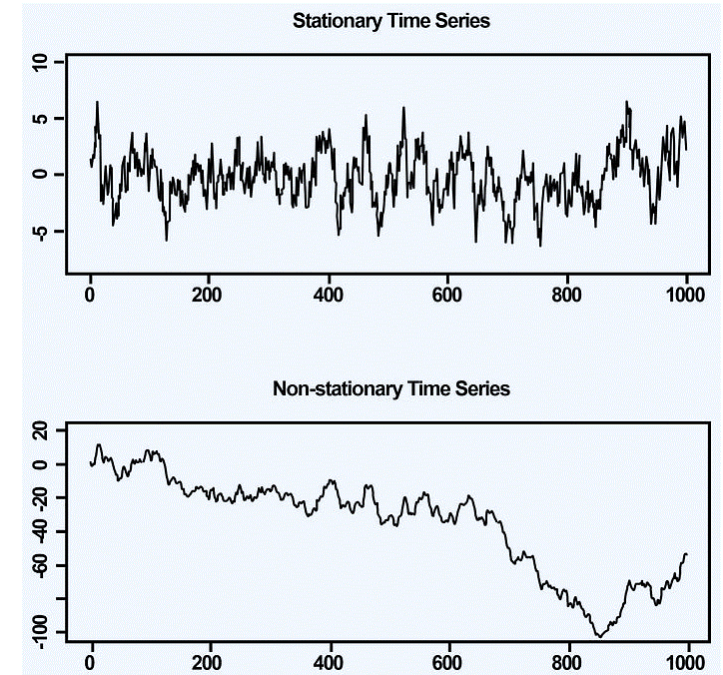






# Stationarity

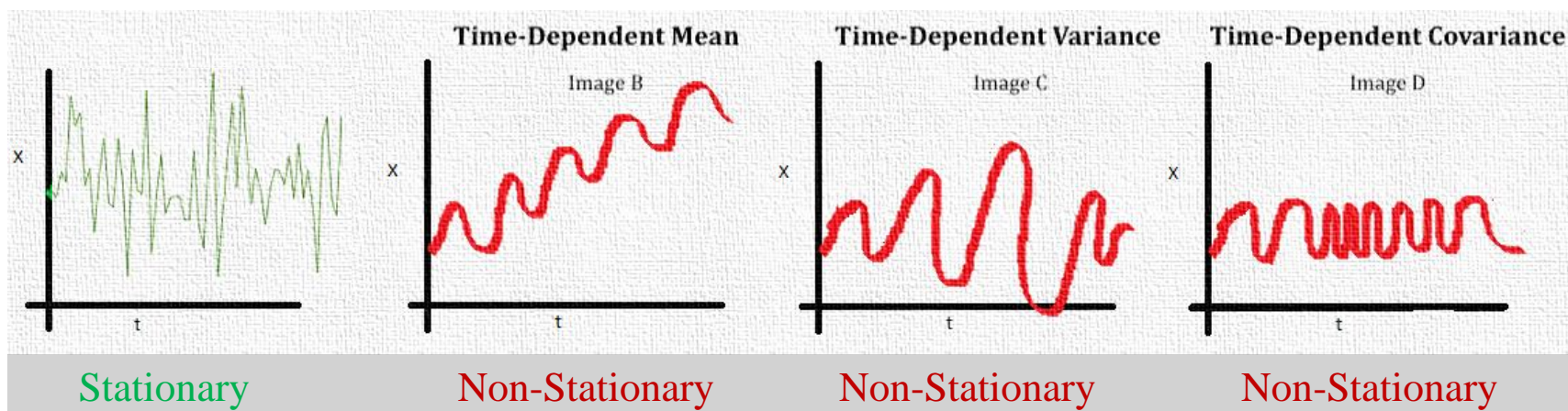
- Stationary vs Non-Stationary Data. What makes a data set **Stationary**?
- In a stationary timeseries, the statistical properties **do not depend on the time**
- **Predictability**: Stationary time series are easier to predict because you can assume that future statistical properties will not change.
- This doesn't mean we cannot predict non-stationary data!
- Data with **trend** and **seasonality** are **NOT** stationary!
- **Data granularity matters**: The level of detail in your data can impact its stationarity





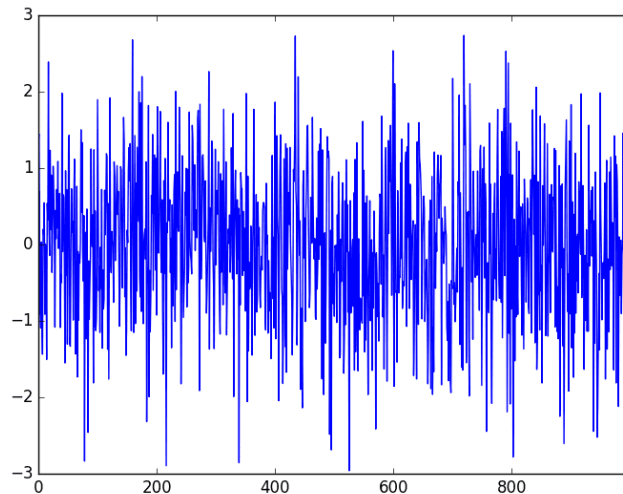
# → Weak vs Strong Stationarity

- **Weak Stationarity (Covariance Stationary):** A time series is considered weakly stationary if the following conditions hold:
  1. Constant **Mean**: The mean of the process is constant over time.
  2. Constant **Variance**: The variance of the process is constant over time.
  3. **Covariance Depends Only on Lag**: The covariance between two points in the time series depends only on the time difference (lag) between the points, not their absolute position in time.

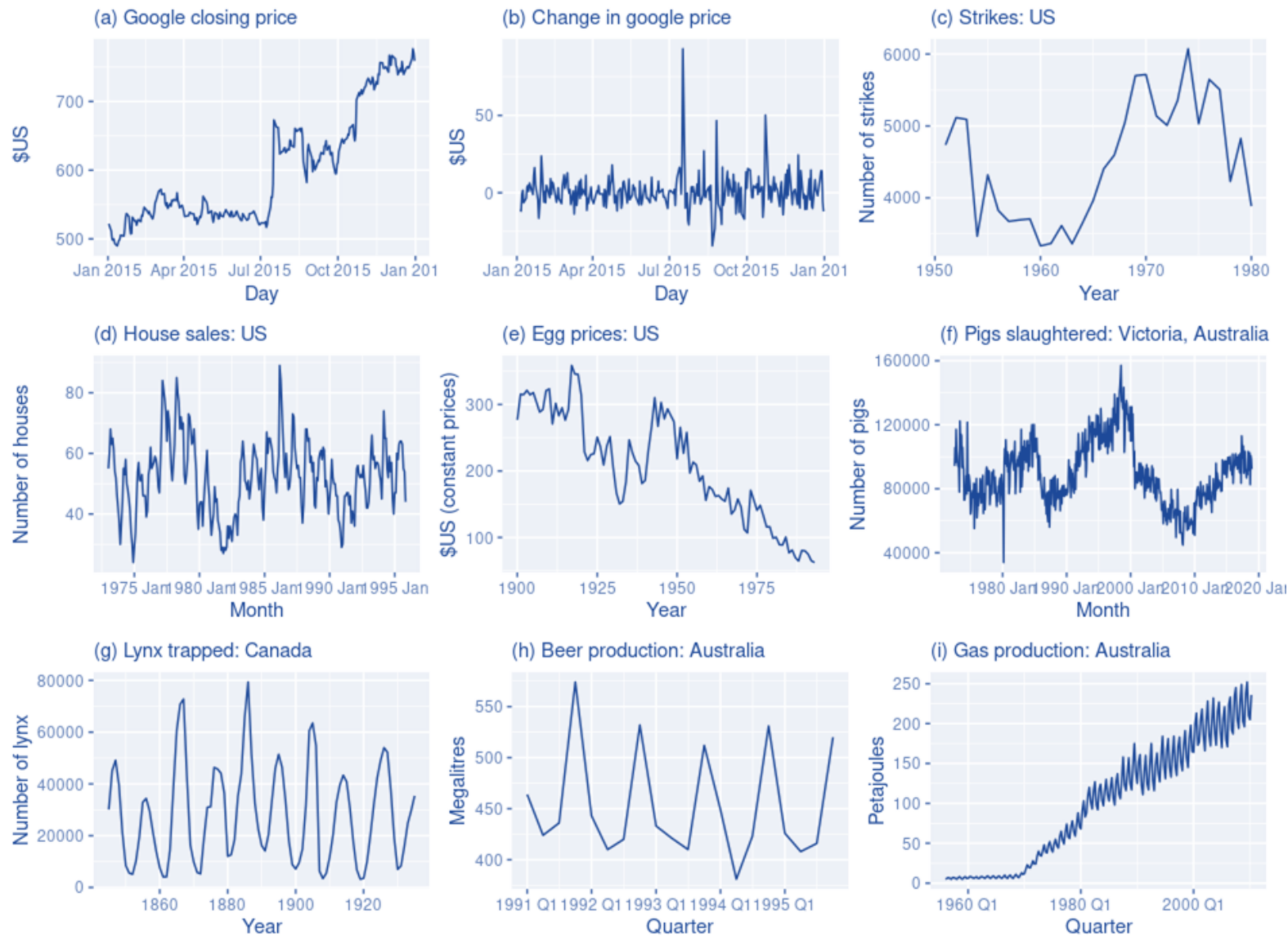


# → Weak vs Strong Stationarity

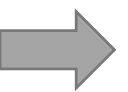
- **Strong stationarity:** A time series is considered strongly stationary if its **joint probability distribution** does not change when shifted in time
  - **All moments** of the series (mean, variance, skewness, kurtosis, etc) and joint distributions remain constant, irrespective of the time period



# ➔ Which ones are stationary?

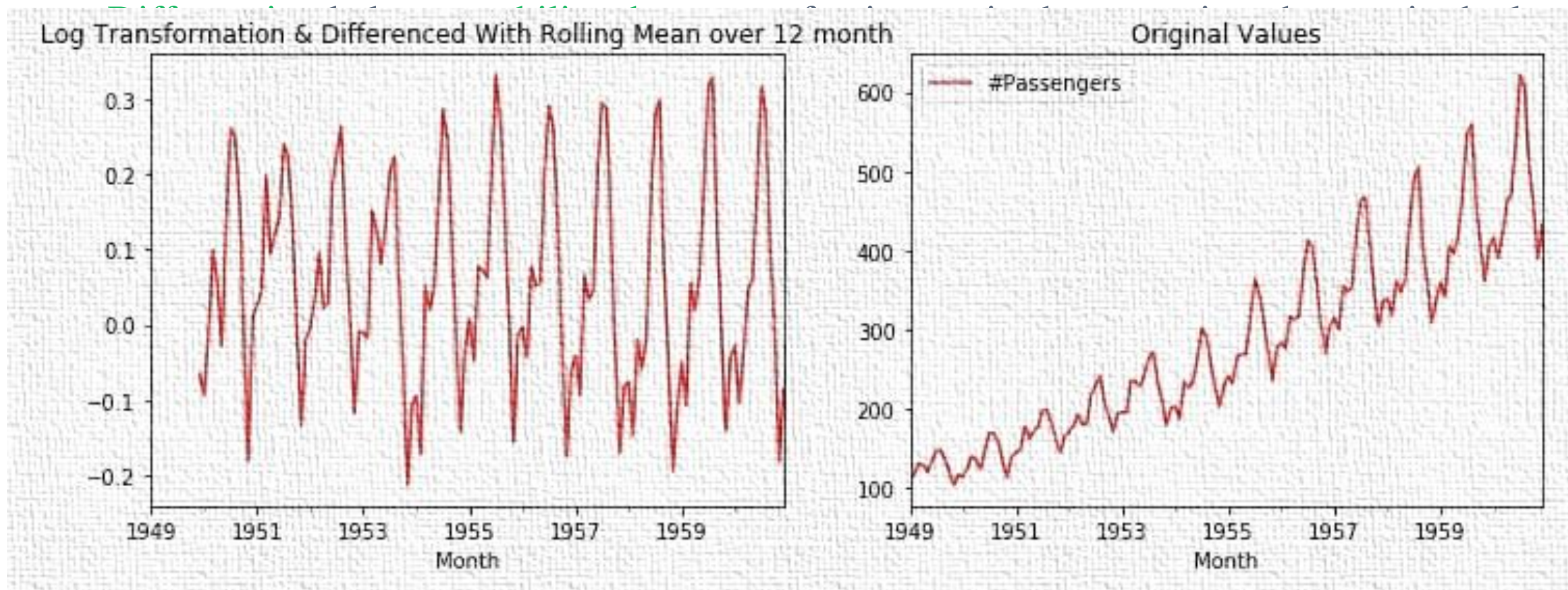


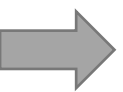




# Differencing

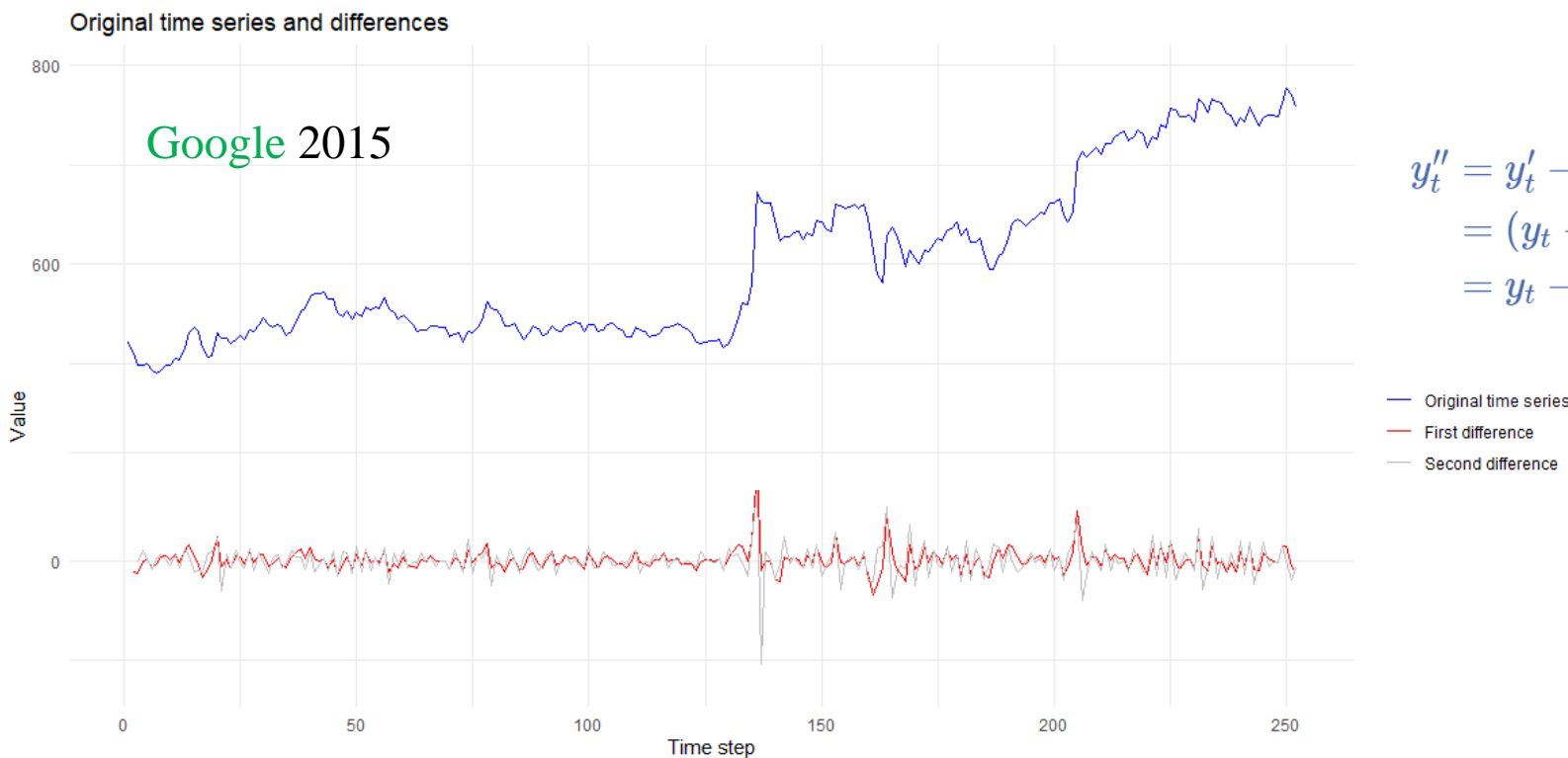
- Differencing: Computing the difference between consecutive observations.





# 2<sup>nd</sup> Differencing

- Occasionally the differenced data will not appear to be stationary, and it may be necessary to difference the data a **second time** to obtain a stationary series.
- Second differencing is **change in change**.
- In practice, it is almost never necessary to go beyond second-order differences.



$$\begin{aligned}y_t'' &= y_t' - y_{t-1}' \\&= (y_t - y_{t-1}) - (y_{t-1} - y_{t-2}) \\&= y_t - 2y_{t-1} + y_{t-2}.\end{aligned}$$

# ➔ Seasonal Differencing

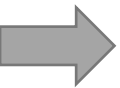
- A seasonal difference is the difference between an observation and the previous observation **from the same season**.

$$y'_t = y_t - y_{t-m}$$

- **m** is the number of seasons. This is also called lag-**m** difference.
- If seasonal differenced is white noise, then

$$y_t = y_{t-m} + \varepsilon_t$$

- Recall:
  - **Seasonal Naïve forecast**: each forecast set to be equal to the last observed value **from the same season**



# Put it together!

Original data



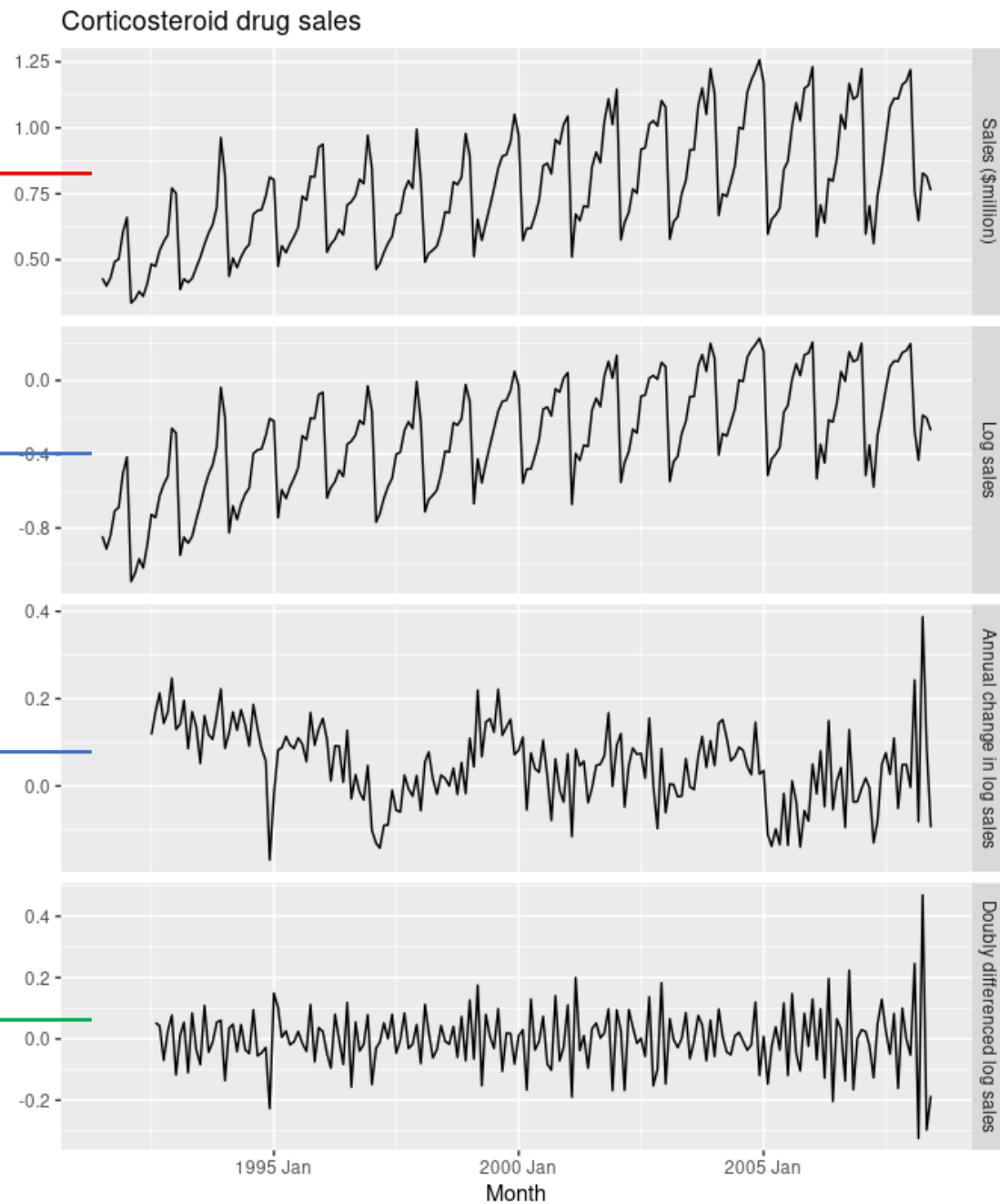
Log transformed



Seasonal difference

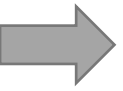


1<sup>st</sup> differenced seasonal difference



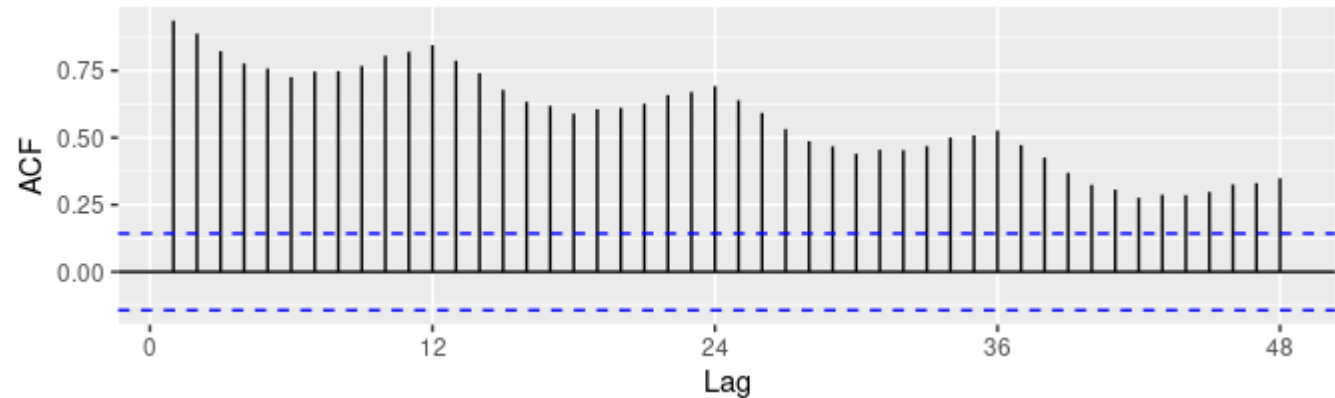
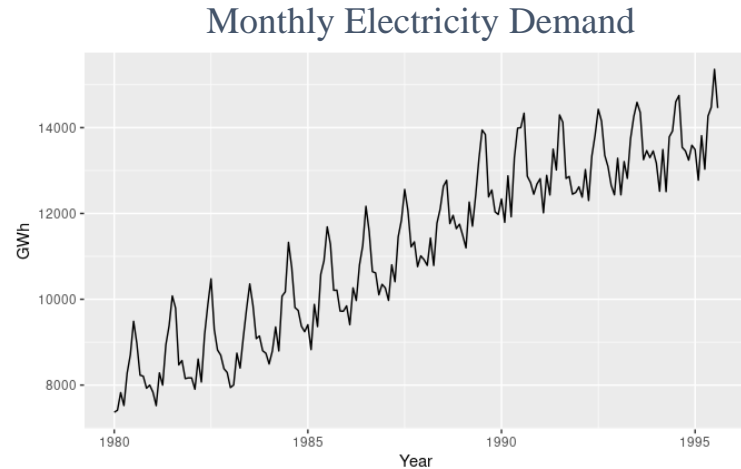
Interpretable?





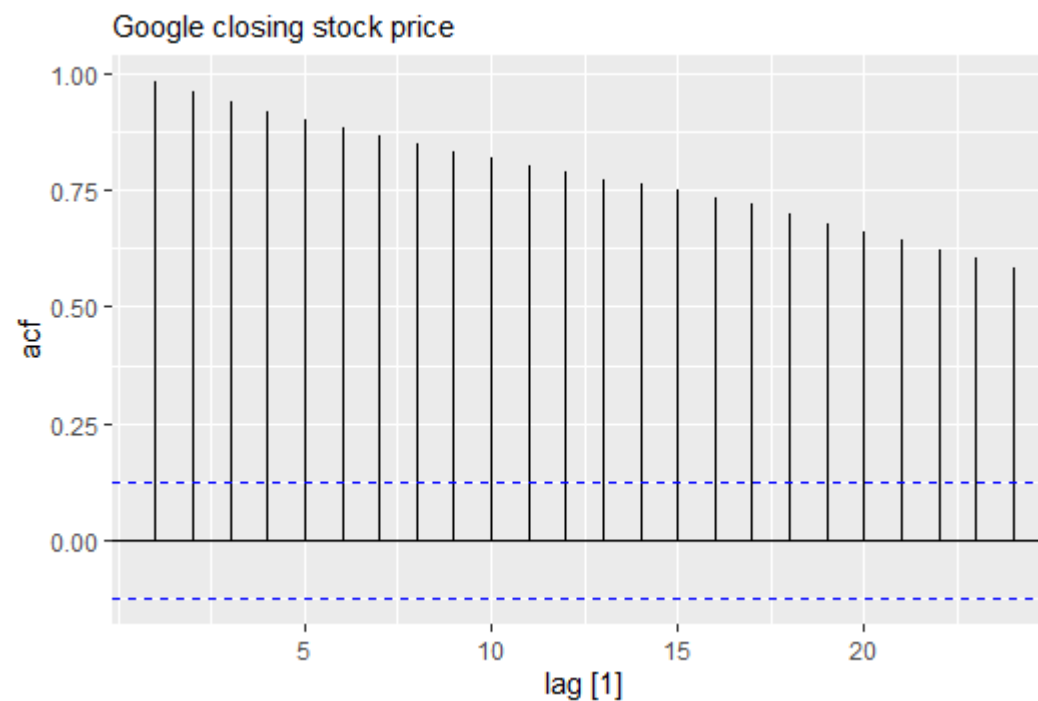
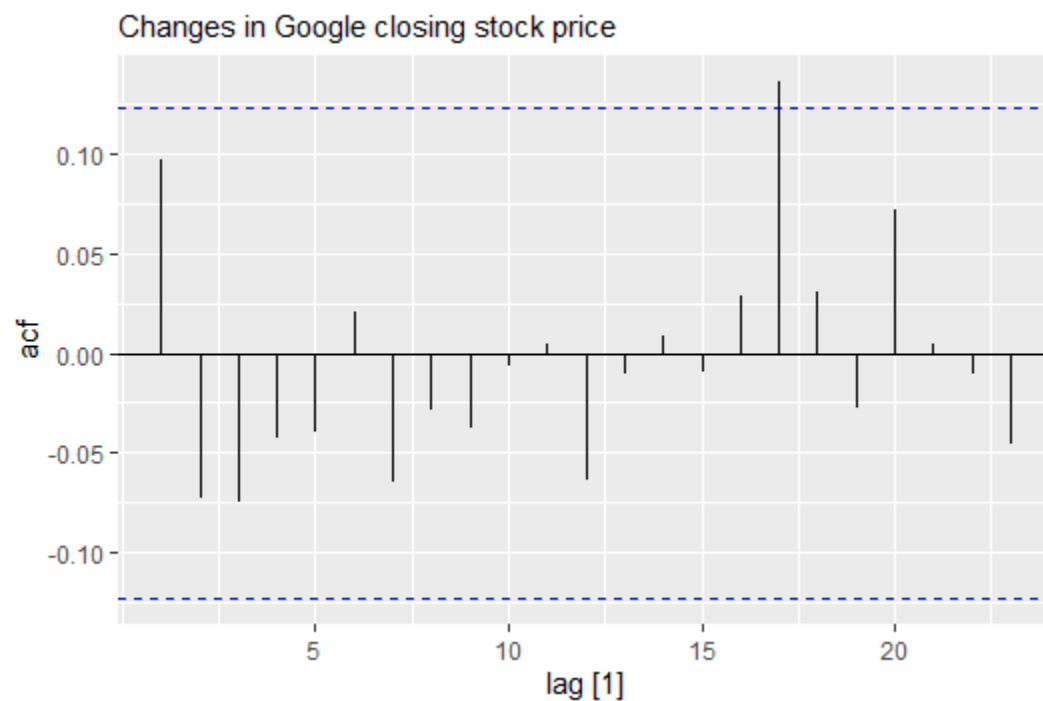
# Recall: Trend and seasonality in ACF plots

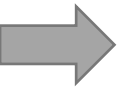
- Autocorrelation can be useful for **identifying patterns and trends** in time series data.
- The ACF of **trended** time series tend to have **positive values that slowly decrease** as the lags increase.
- For seasonal data, the autocorrelations **are larger** for the **seasonal lags** than for other lags.



# ➔ ACF plots and Stationarity

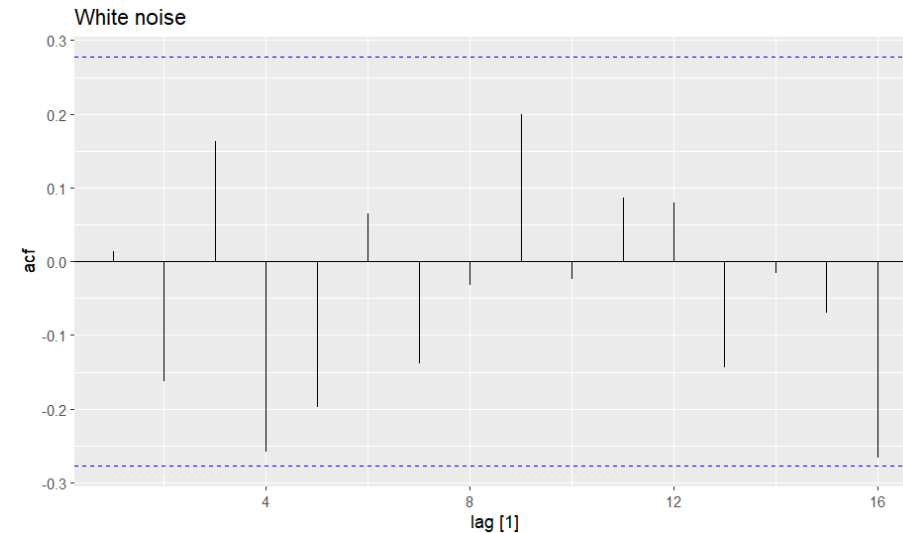
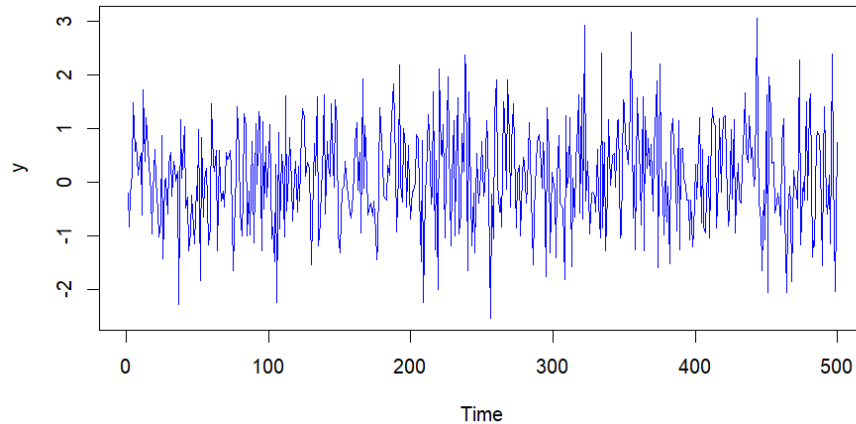
- For stationary data,
  - The ACF plot drops to **zero quickly**.
  - $r_1$  is mostly large and positive.

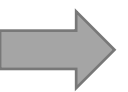




# White Noise

- White noise can be thought of as a random **sequence of iid values** (independent and identically distributed) characterized by a distribution.
- White noise has **zero mean** and **finite variance**.  $\epsilon_t \sim D(0, \sigma^2)$
- White noise data show **no autocorrelation**.





# Random Walk

- Random Walk: When the 1<sup>st</sup> differenced series is white noise

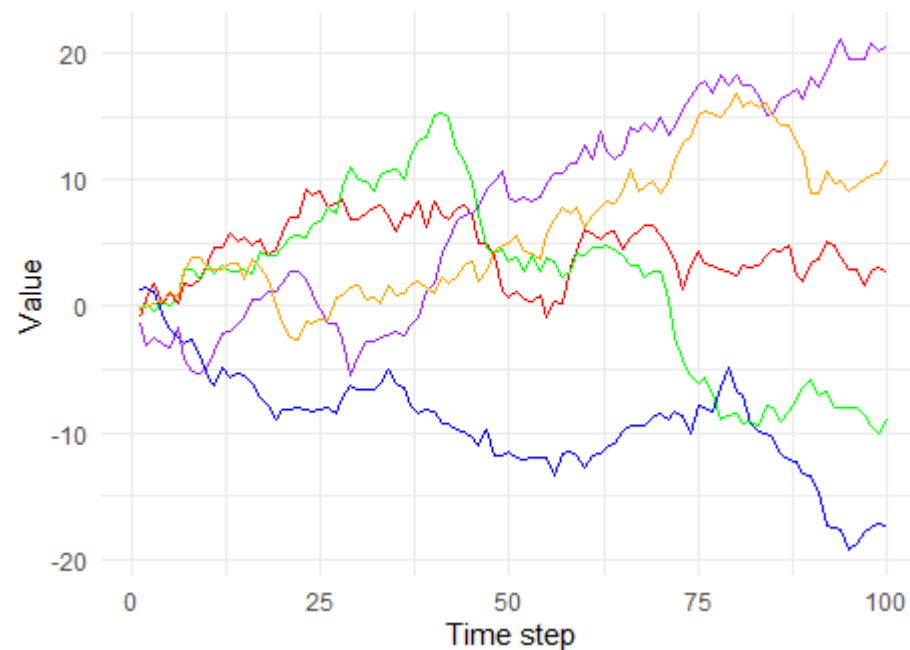
$$y_t - y_{t-1} = \varepsilon_t$$



$$y_t = y_{t-1} + \varepsilon_t$$

- Random walk models are widely used for non-stationary data, particularly **financial** and economic data.
- Random walks typically have **long periods of up or down trend** + **sudden change in direction**.
- Random walk with **no drift** = **Naïve** forecasting model

Five random walks without drift

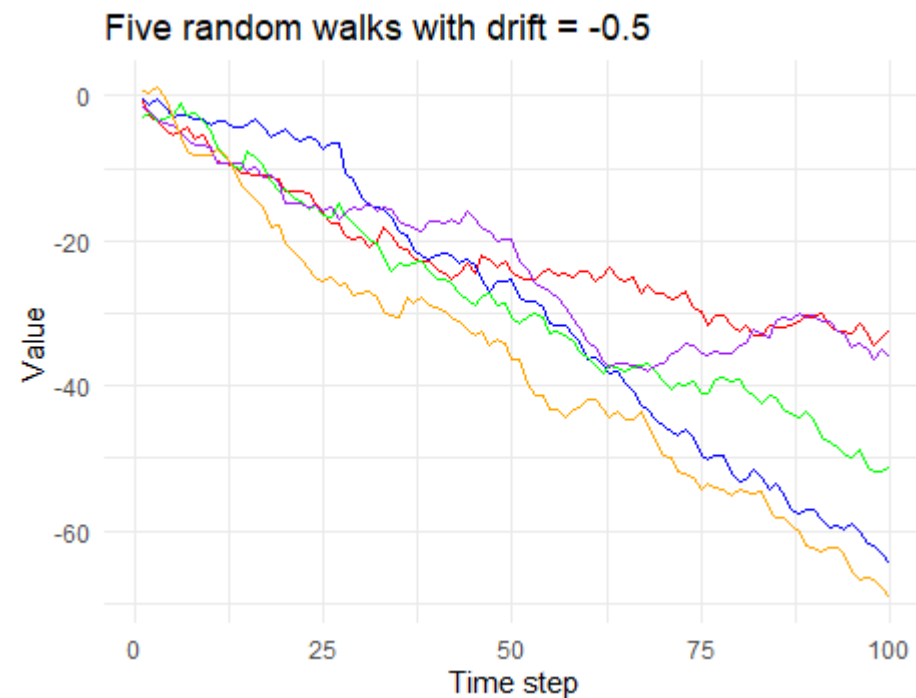
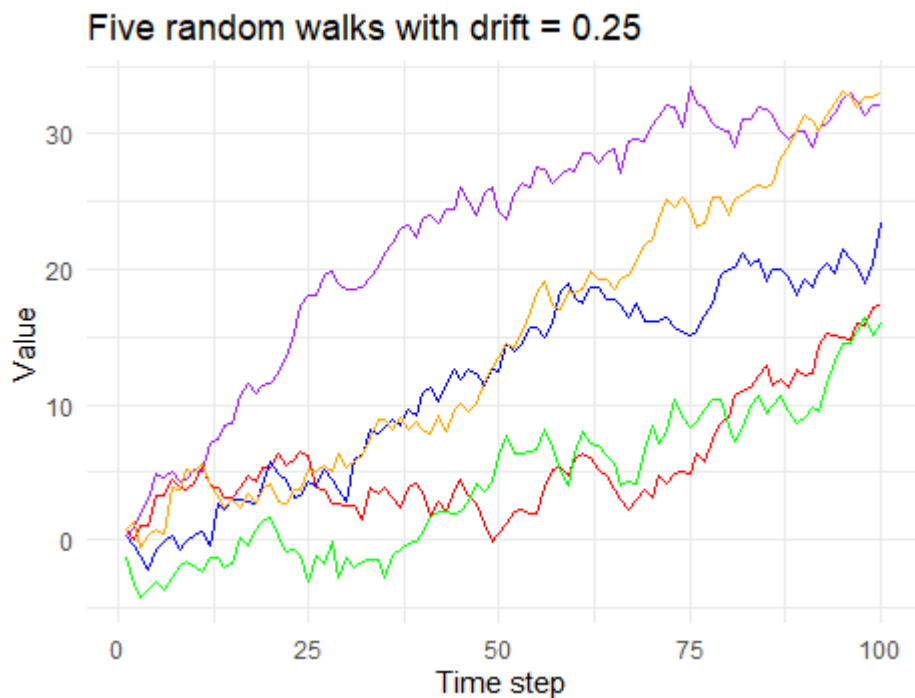


# ➔ Random Walk with Drift

- Random walk with drift  $c$  (the 1<sup>st</sup> difference does not have zero average):

$$y_t - y_{t-1} = c + \varepsilon_t \quad \text{or} \quad y_t = c + y_{t-1} + \varepsilon_t$$

- $C$  is the average change between consecutive observations.



# → Testing for Stationarity

- Unit root test is a statistical test used to determine whether a time series **has a unit root**, which is a characteristic of a **non-stationary time series**
- There are several different unit root tests including:
  1. Augmented Dickey-Fuller (**ADF**) test.
  2. Kwiatkowski-Phillips-Schmidt-Shin (**KPSS**) test.

Hypothesis Test	Null	Alternative	P-value to get stationarity
ADF	Non-Stationary	Stationary	Small
KPSS	Stationary	Non-Stationary	Large

	ADF	KPSS
ADF statistic	-12.533939	0.012944
p-value	0.0	0.1
should we difference?	?	
conclusion		

# → Testing for Stationarity

- Unit root test is a statistical test used to determine whether a time series **has a unit root**, which is a characteristic of a **non-stationary time series**
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Hypothesis Test	Null	Alternative	P-value to get stationarity
ADF	Non-Stationary	Stationary	Small
KPSS	Stationary	Non-Stationary	Large

	ADF	KPSS
ADF statistic	-12.533939	0.012944
p-value	0.0	0.1
should we difference?	False	False
conclusion	stationary	stationary

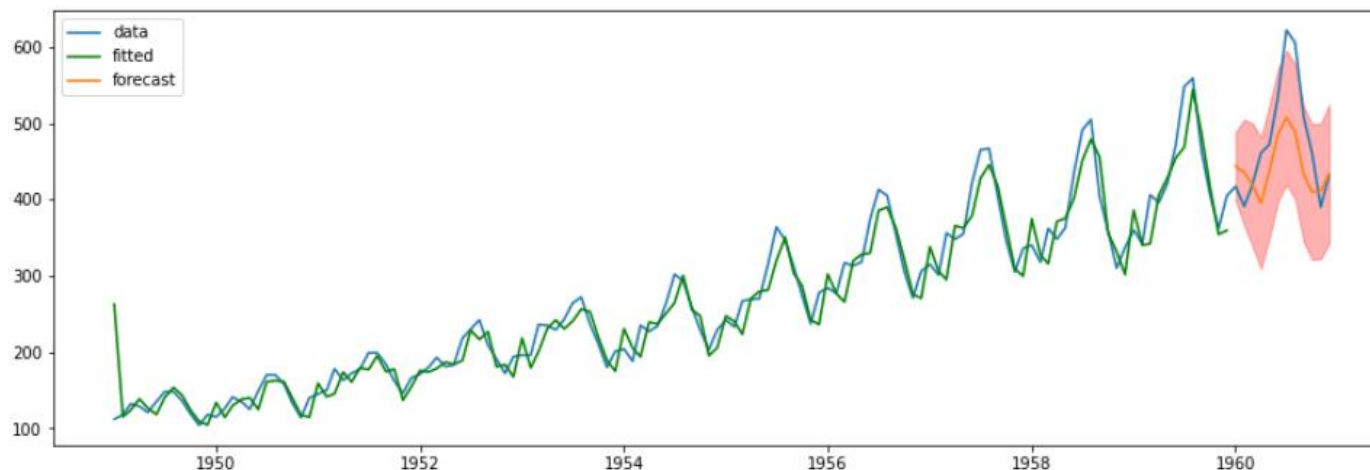


# → Components of **ARIMA** model

1. Autoregressive (**AR**) term - captures the autocorrelation in the data
2. Integrated (**I**) term - removes the non-stationarity in the data
3. Moving Average (**MA**) term - captures the error term or noise in the data

## How it works?

- The **AR** term models the current **value** of the time series as a **linear combination** of its past **values**.
- The **I** term models the **differences** between the current **value** and the past **value**.
- The **MA** term models the current **error** term as a **linear combination** of the past **error** terms.

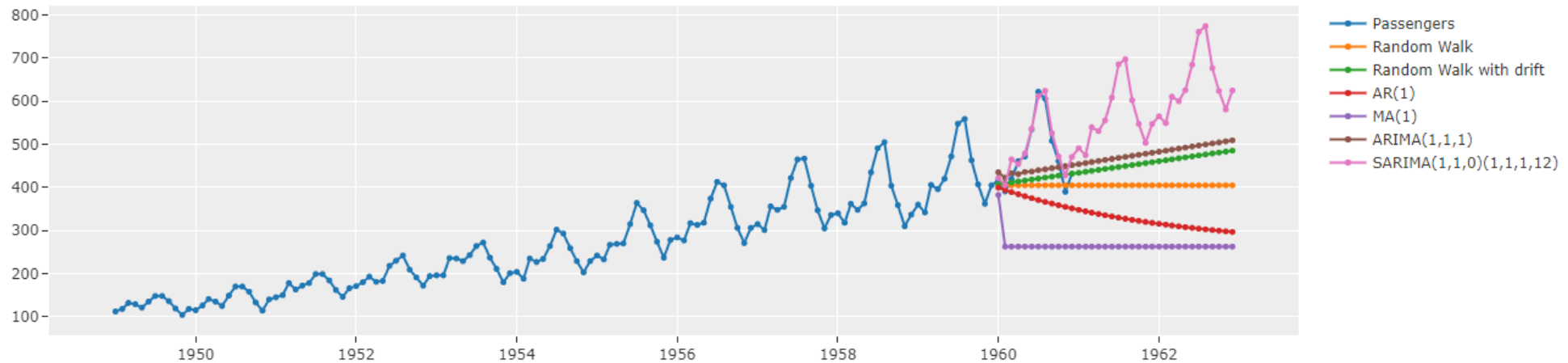


# Module 4 – Part II

## ARIMA models



Actual vs. Forecast (Out-of-Sample)



# → Components of ARIMA model

---

ARIMA

1. Autoregressive (**AR**) term - captures the autocorrelation in the data
2. Integrated (**I**) term - removes the non-stationarity in the data
3. Moving Average (**MA**) term - captures the error term or noise in the data

# → Autoregressive models

- An autoregressive (AR) model is a statistical model (multiple linear regression model) that uses **lagged** variable as **predictors**
- Autoregression = regression of the variable against **itself**
- AR(**p**) model, autoregressive model of order **p**.

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t$$

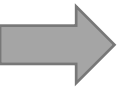
- In AR(1) model:
  - when  $\phi_1 = 0$  and  $c = 0$ ,  $y_t$  is equivalent to ?
  - when  $\phi_1 = 1$  and  $c = 0$ ,  $y_t$  is equivalent to ?
  - when  $\phi_1 = 1$  and  $c \neq 0$ ,  $y_t$  is equivalent to ?
  - when  $\phi_1 < 0$ ,  $y_t$  tends to oscillate around the mean.

# → Autoregressive models

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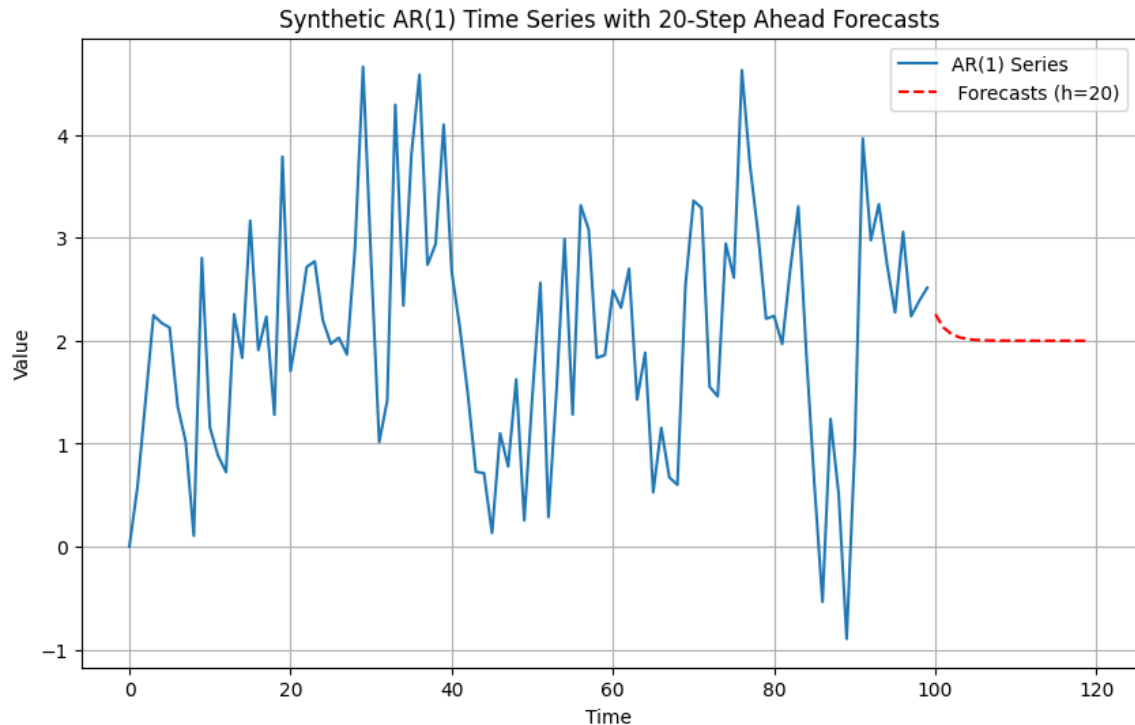
$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t$$

- In AR(1) model:
  - when  $\phi_1 = 0$  and  $c = 0$ ,  $y_t$  is equivalent to white noise;
  - when  $\phi_1 = 1$  and  $c = 0$ ,  $y_t$  is equivalent to a random walk;
  - when  $\phi_1 = 1$  and  $c \neq 0$ ,  $y_t$  is equivalent to a random walk with drift;
  - when  $\phi_1 < 0$ ,  $y_t$  tends to oscillate around the mean.



# Autoregressive Models (Forecasting)

$$y_t = c + \phi y_{t-1} + \varepsilon_t$$



$$c = 1, \quad \phi = 0.5, \quad \varepsilon_t \sim N(0, \sigma = 1)$$

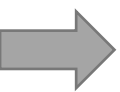
Forecasting Equation:

$$\hat{y}_{t+1} = c + \phi y_t$$

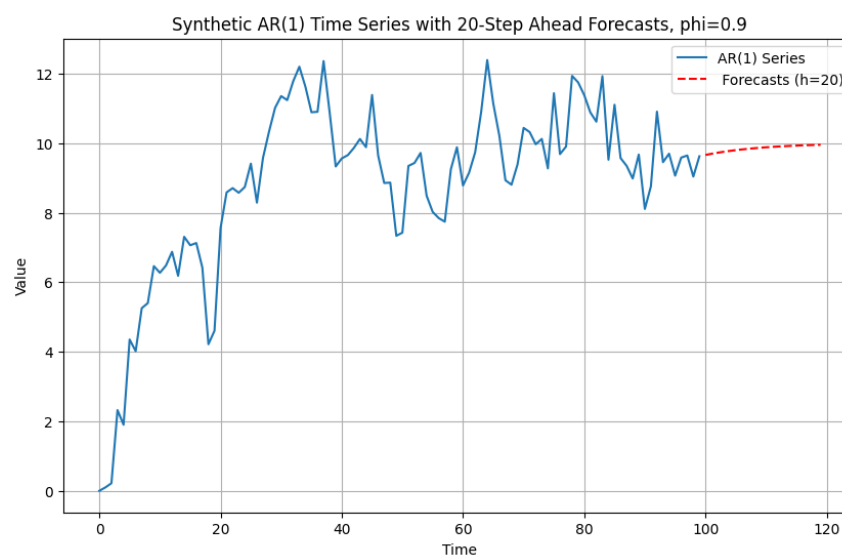
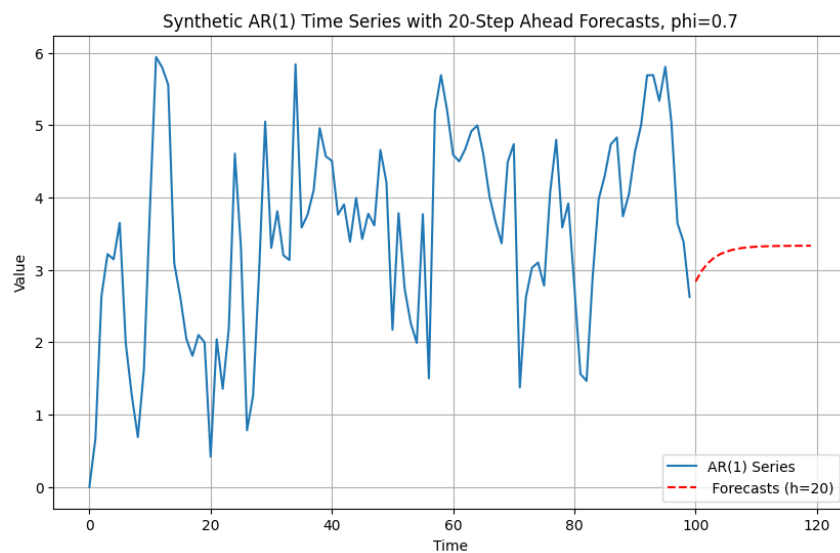
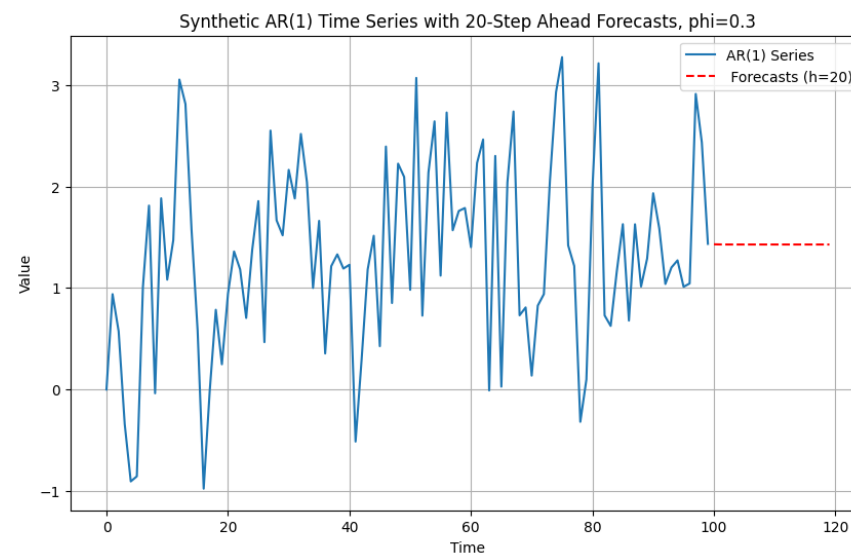
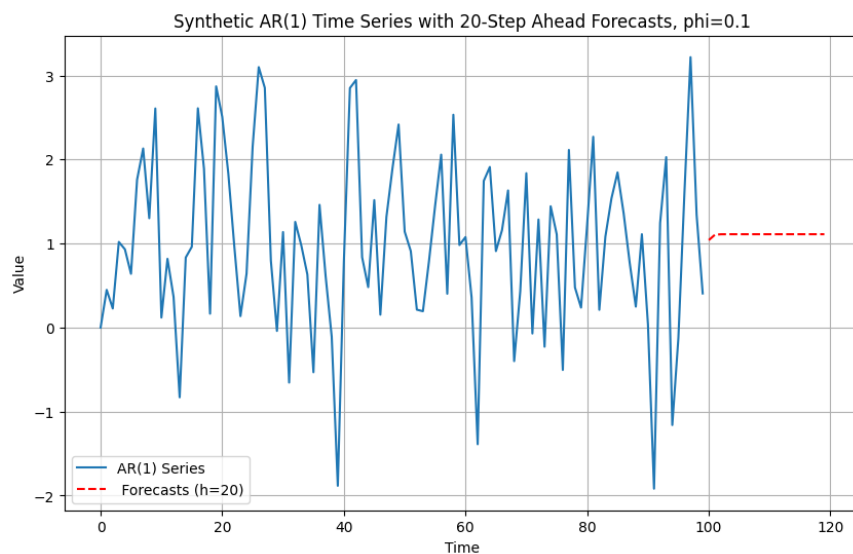


$$\hat{y}_{t+h|t} = c(1 + \phi + \phi^2 + \dots + \phi^{h-1}) + \phi^h y_t$$

$$\hat{y}_{t+h|t} = c \frac{1 - \phi^h}{1 - \phi} + \phi^h y_t$$

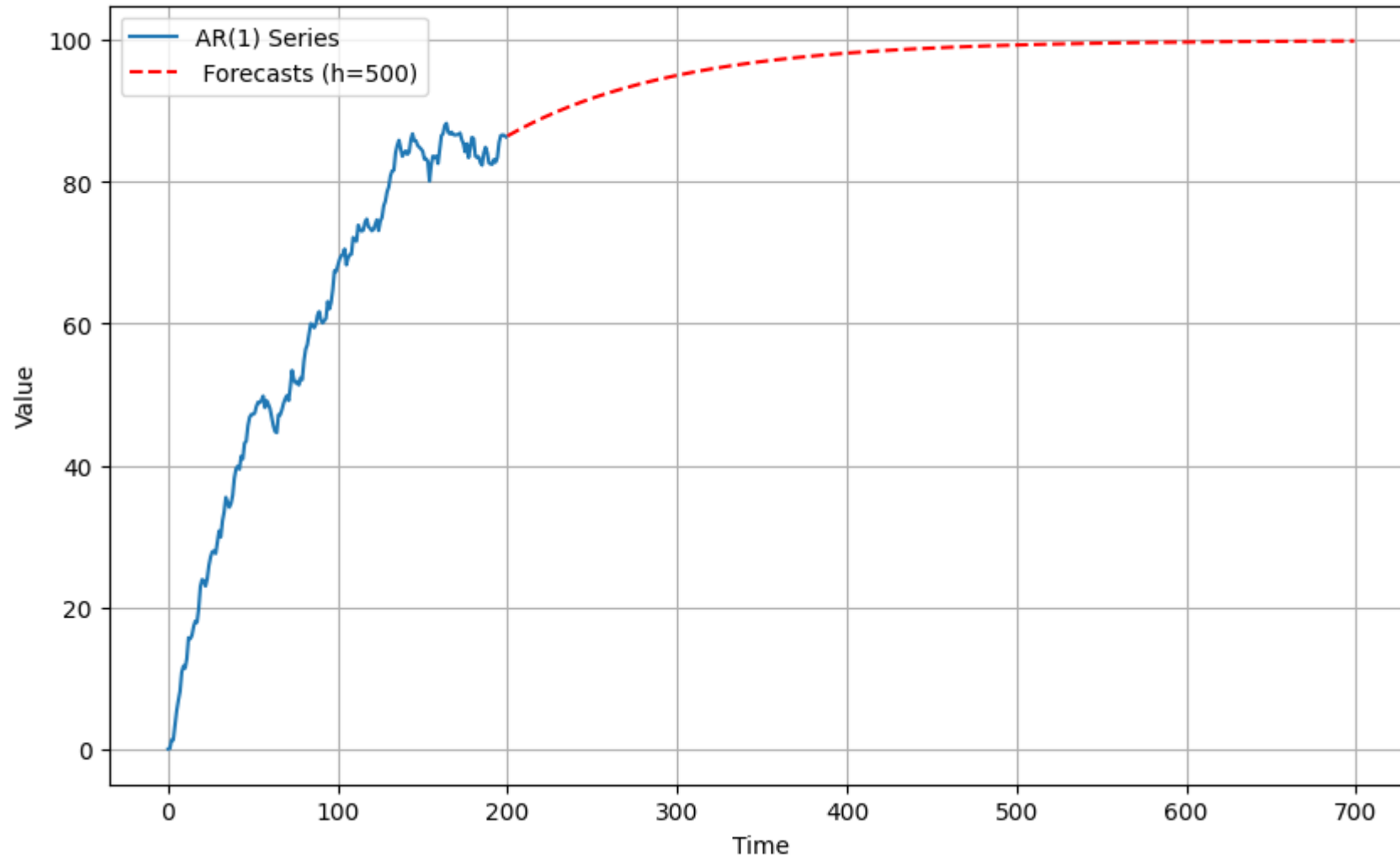


# Autoregressive Models (Examples)





Synthetic AR(1) Time Series with 500-Step Ahead Forecasts,  $\phi=0.99$



# → Moving Average Models

- A moving average model uses past forecast **errors** in a regression-like model
- MA(**q**) model, a moving average model of order **q**.

$$y_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots + \theta_q \varepsilon_{t-q}$$

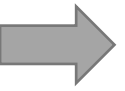
- $y_t$  can be thought of as a weighted moving average of the past few forecast errors
- We require  $|\phi| < 1$ , the most recent observations carry a greater weight than those from the distant past.

# → Moving Average Models

$$y_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots + \theta_q \varepsilon_{t-q}$$

- Do **NOT** confuse this model with simple moving average method or exponentially weighted moving average method.
- Moving average **models** is used for forecasting future values!
- Moving average **smoothing** (SMA, EWMA, ...) is used for estimating the trend-cycle of past values.

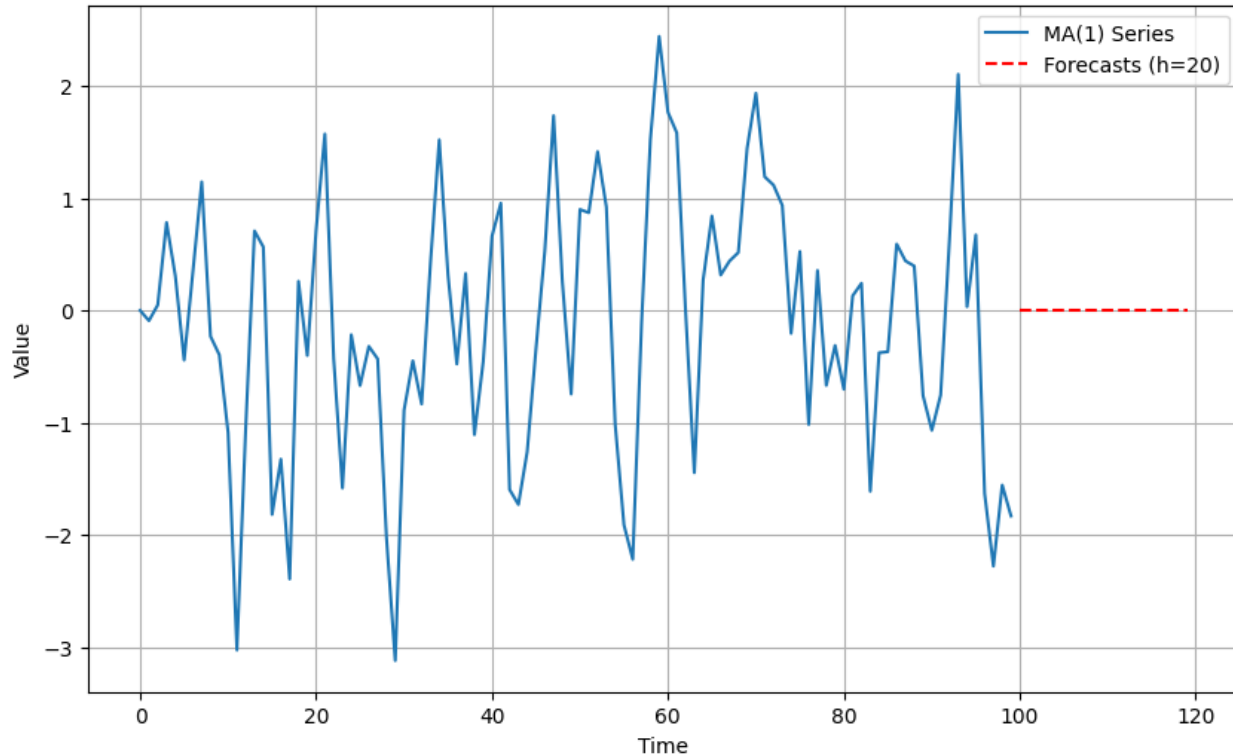




# Moving Average Models (Forecasting)

$$y_t = \mu + \theta \varepsilon_{t-1} + \varepsilon_t$$

Synthetic MA(1) Time Series with 20-Step Ahead Forecasts, theta=0.5



$$\mu = 0, \quad \theta = 0.5, \quad \varepsilon_t \sim N(0, \sigma = 1)$$

Forecasting Equation:

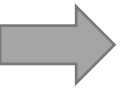
$$\hat{y}_{t+1|t} = \mu + \theta \varepsilon_t$$



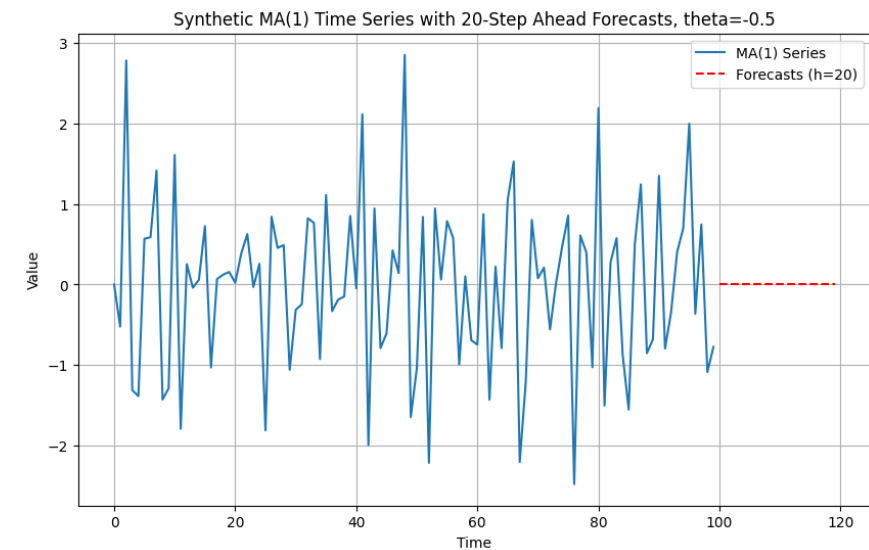
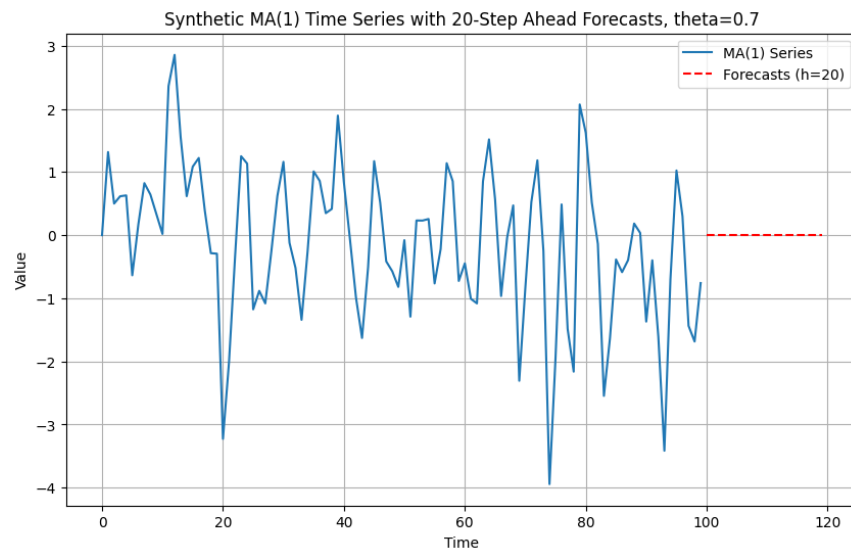
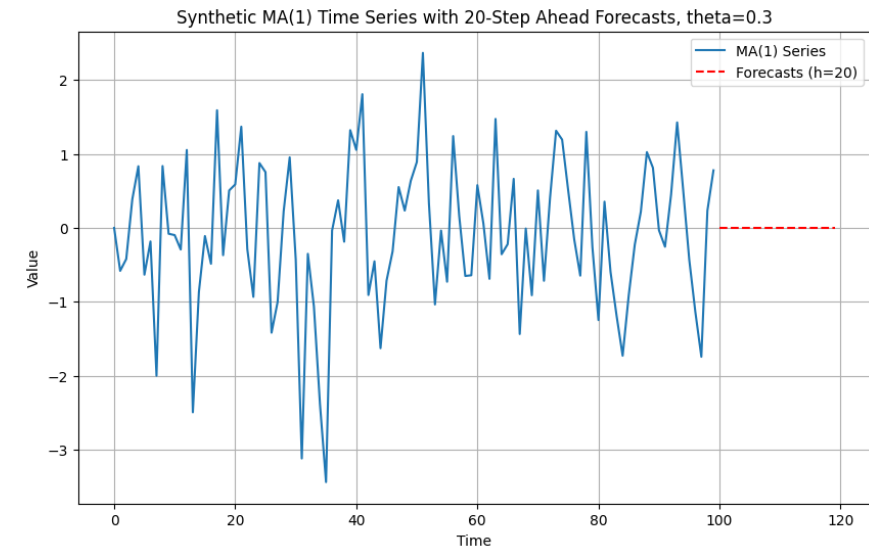
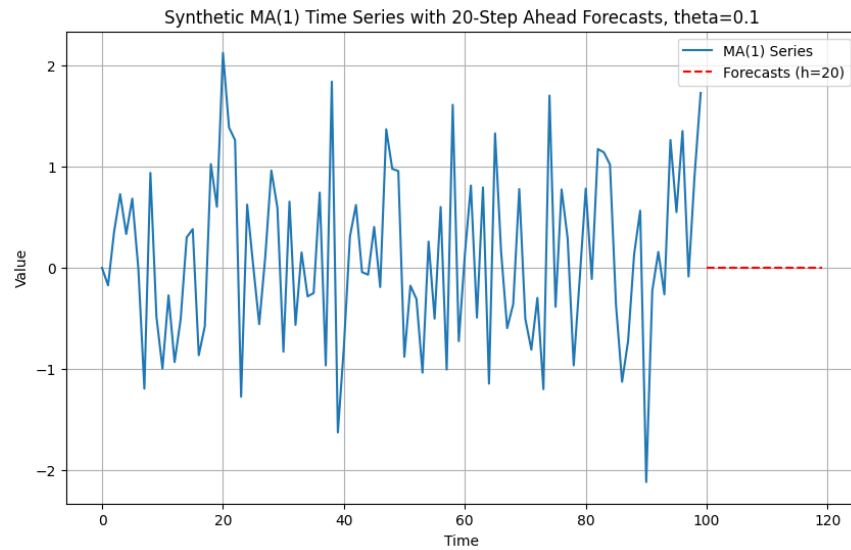
Since future error terms are unknown

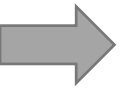


$$\hat{y}_{t+h|t} = \mu$$



# Moving Average Models (Examples)





# ARIMA (AutoRegressive Integrated Moving Average)

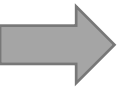
- ARIMA model combines three models, autoregressive (**AR**) model, an integrated (**I**) model, and a moving average (**MA**) model.
- ARIMA(**p**, **d**, **q**) model.

$$y'_t = c + \phi_1 y'_{t-1} + \cdots + \phi_p y'_{t-p} + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} + \varepsilon_t$$

- $y'_t$  is the differenced time series.
- d degree of first difference involved.
- Note: p, d, and q are estimated using **MLE**.

?	ARIMA(0,0,0) with no constant
?	ARIMA(0,1,0) with no constant
?	ARIMA(0,1,0) with a constant
?	ARIMA(p,0,0)
?	ARIMA(0,0,q)





# ARIMA (AutoRegressive Integrated Moving Average)

- ARIMA model combines three models, autoregressive (**AR**) model, an integrated (**I**) model, and a moving average (**MA**) model.
- ARIMA(**p**, **d**, **q**) model.

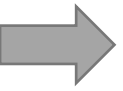
$$y'_t = c + \phi_1 y'_{t-1} + \cdots + \phi_p y'_{t-p} + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} + \varepsilon_t$$

- $y'_t$  is the differenced time series.
- $d$  degree of first difference involved.
- Note:  $p$ ,  $d$ , and  $q$  are estimated using **MLE**.

White noise	ARIMA(0,0,0) with no constant
Random walk	ARIMA(0,1,0) with no constant
Random walk with drift	ARIMA(0,1,0) with a constant
Autoregression	ARIMA( $p$ ,0,0)
Moving average	ARIMA(0,0, $q$ )

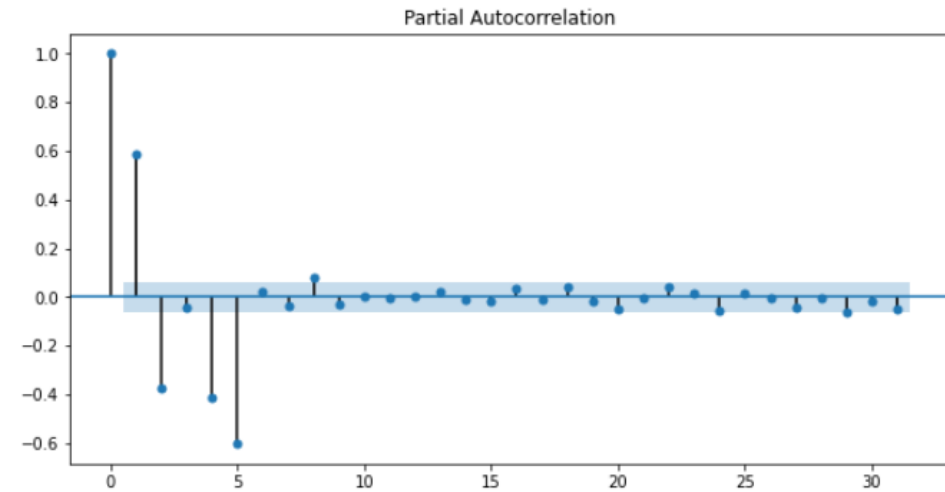
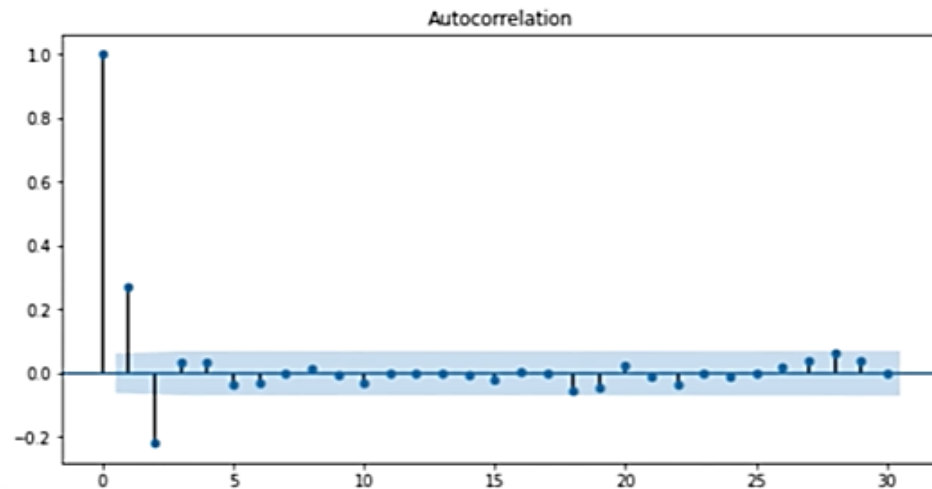






# Selecting (p, q) orders using ACF and PAC

- Some rough guidelines:
- Identification of an **AR** model is often best done with the **PACF**
  - **p** set to be the maximum significant non-zero lag in PACF typically followed by a **sharp decline**.
- Identification of an **MA** model is often best done with the **ACF**
  - **q** set to be the maximum significant non-zero lag in ACF typically followed by a **sharp decline**.





# Model selection

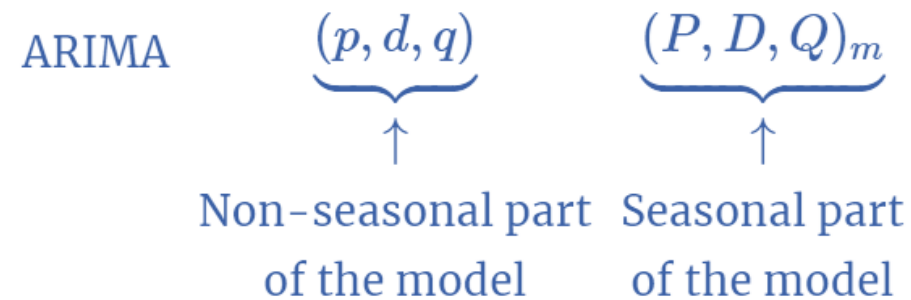
- For model selection we can either use **information criteria** or any **cross validated** performance metrics like  $R^2$ , MSE, RMSE, MAPE, sMAPE.

Information Criteria	Formula
Akaike's Information Criterion (AIC)	$AIC = -2 \log(L) + 2k$
AIC corrected for small sample bias (AICc)	$AIC_c = AIC + \frac{2k(k+1)}{T-k-1}$
Bayesian Information Criterion (BIC)	$BIC = AIC + k[\log(T) - 2]$

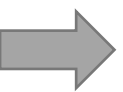
- L** is the likelihood of the model and **K** is the total number of parameters (including the variance of residuals)
- The model with the **minimum information criteria** is often the best model for forecasting

➔ SARIMA (Seasonal ARIMA) models

- SARIMA is an extension of an ARIMA model that includes **additional seasonal terms**.
- It is used to model time series data that exhibits seasonal patterns

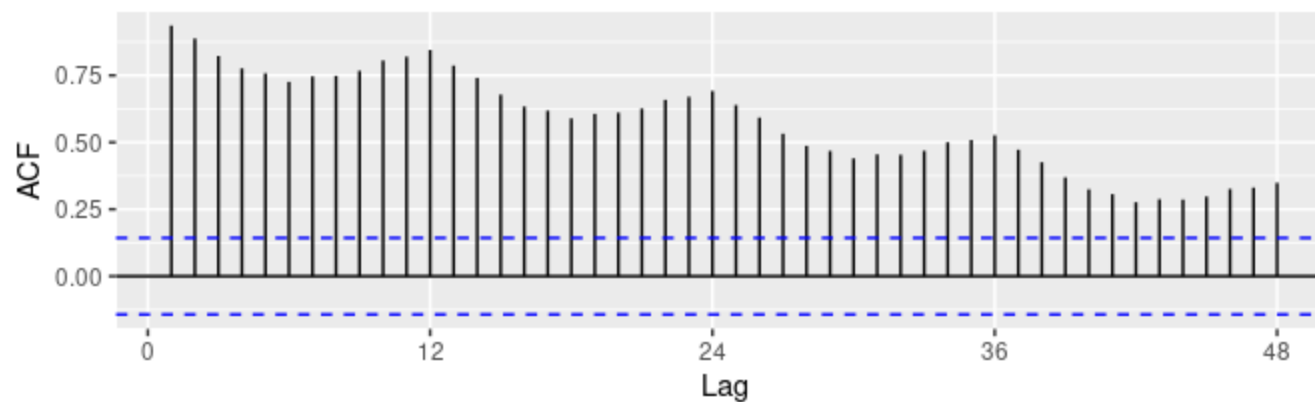
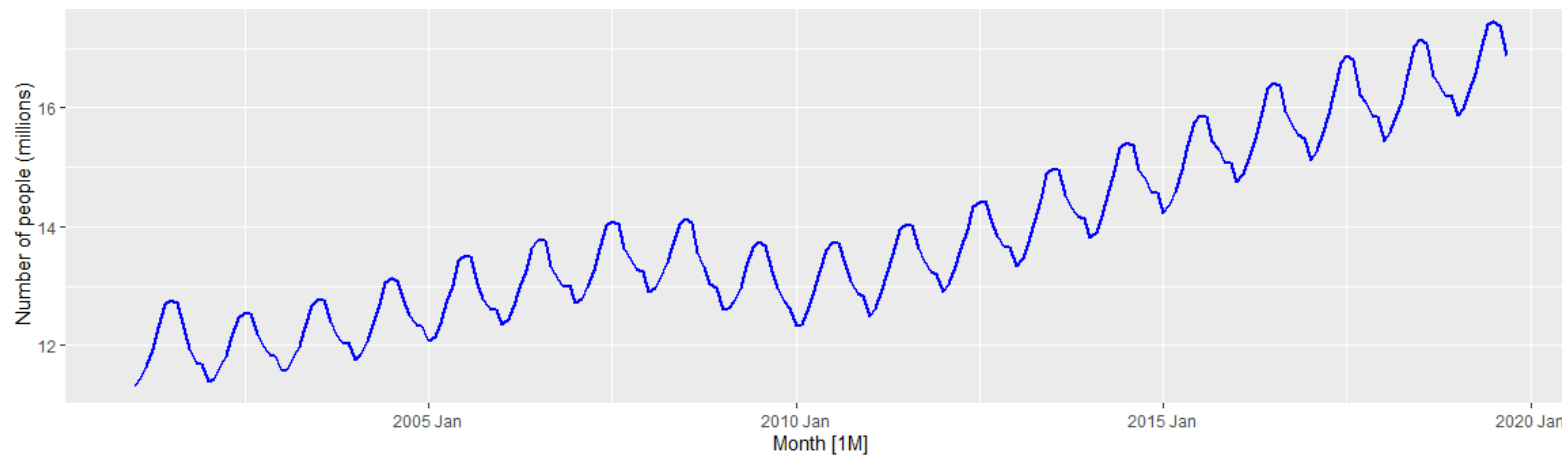


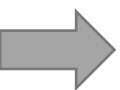
- $p, d, q$  are defined as before.
- $P$  is the order of the **seasonal** autoregressive component
- $D$  is the degree of **seasonal** differencing
- $Q$  is the order of the **seasonal** moving average component
- $m$  is the **period of the seasonality**.  $m = 4, 12$  is for quarterly and monthly seasonality, respectively.



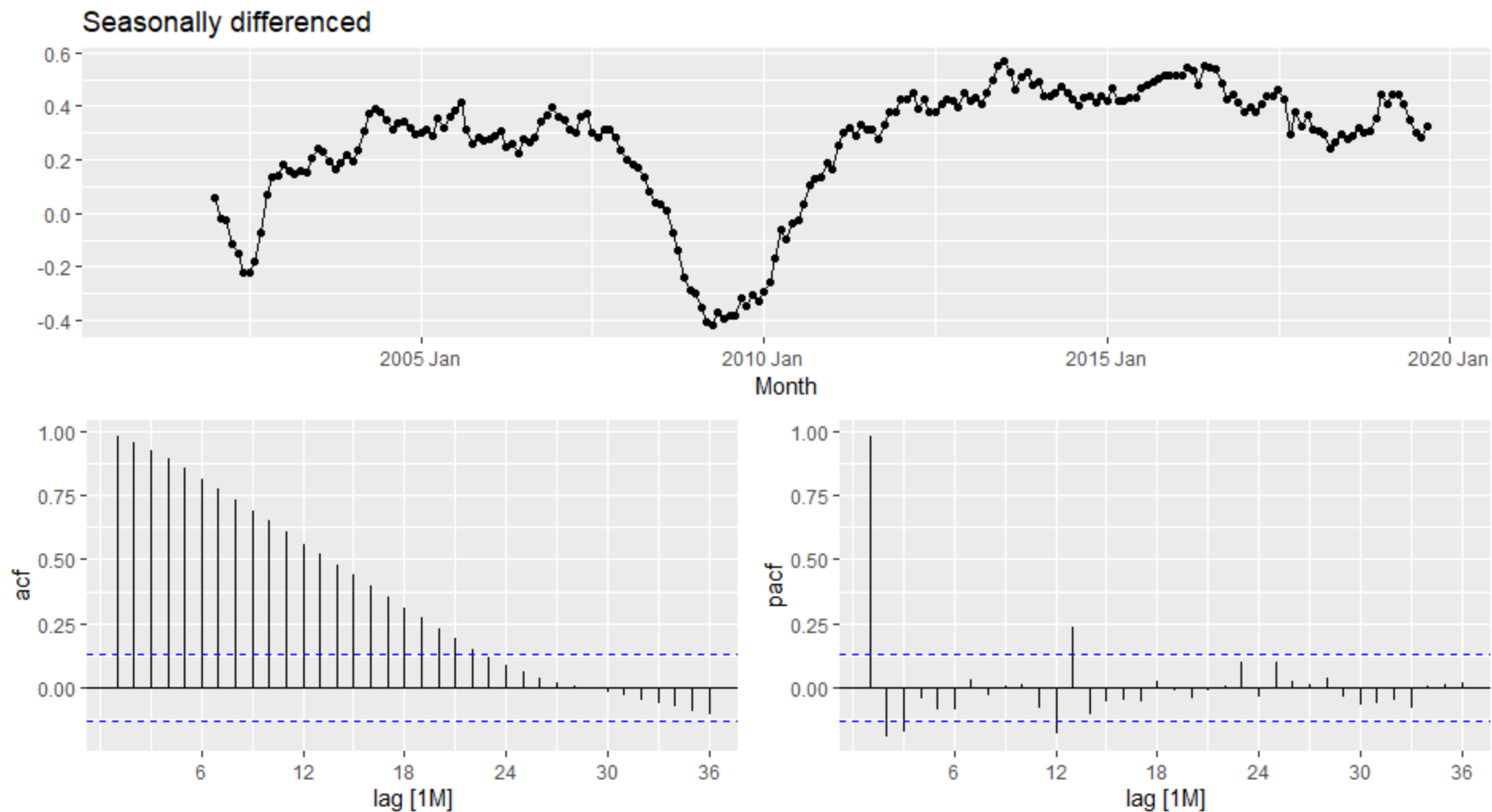
# SARIMA example

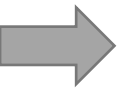
Monthly US leisure and hospitality employment, 2001-2019.



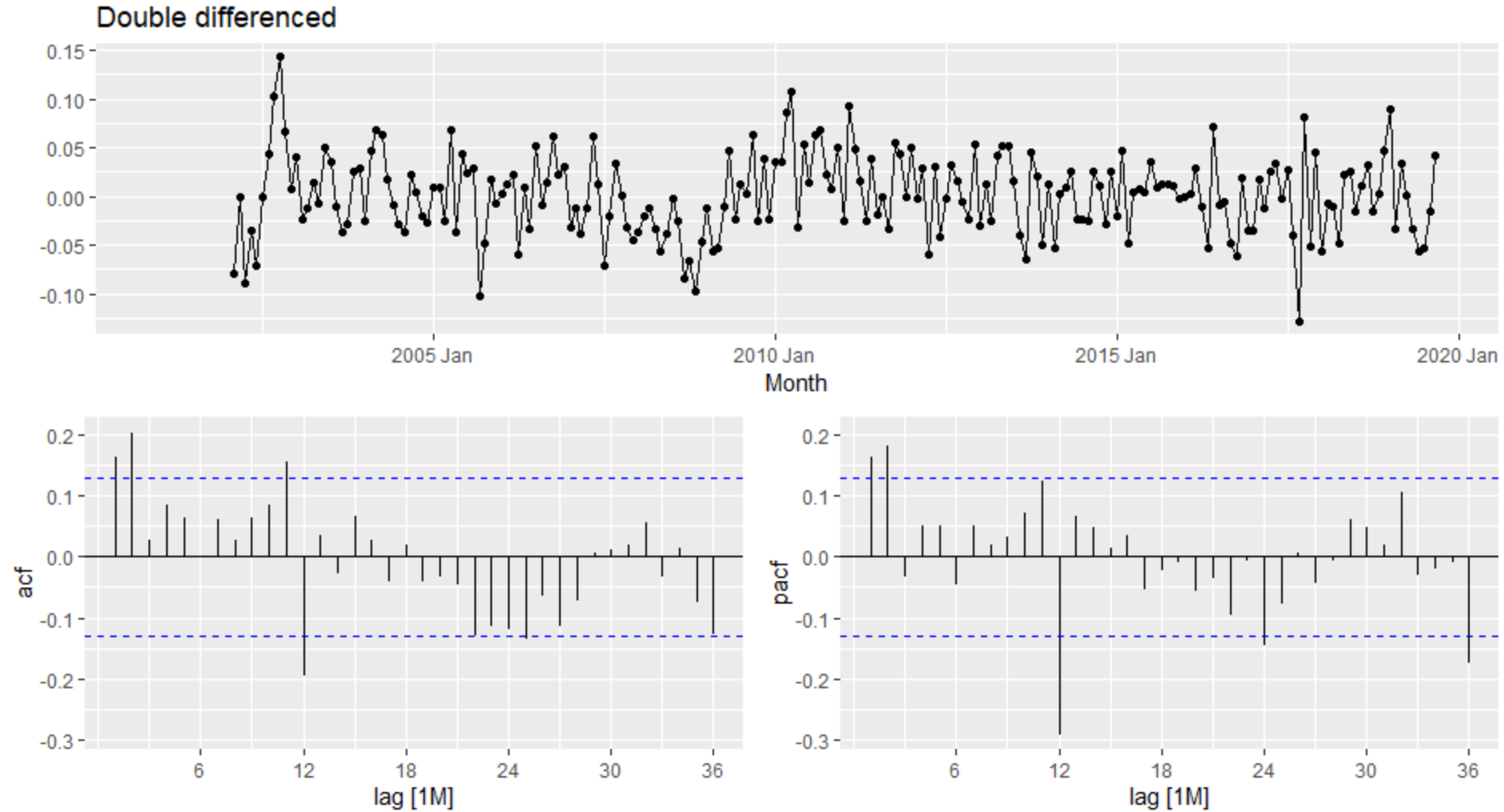


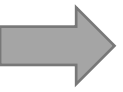
# SARIMA example





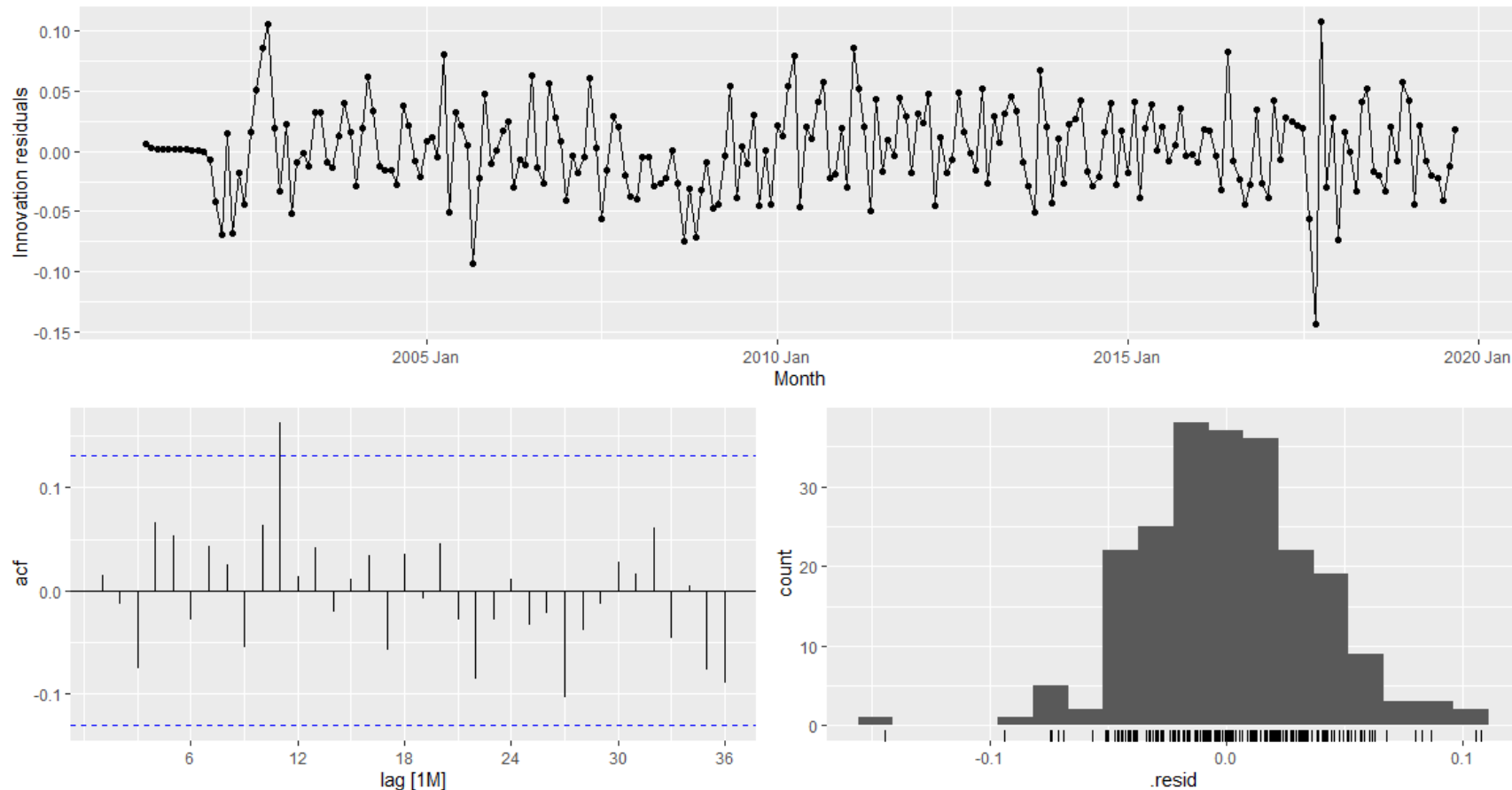
# SARIMA example





# SARIMA example

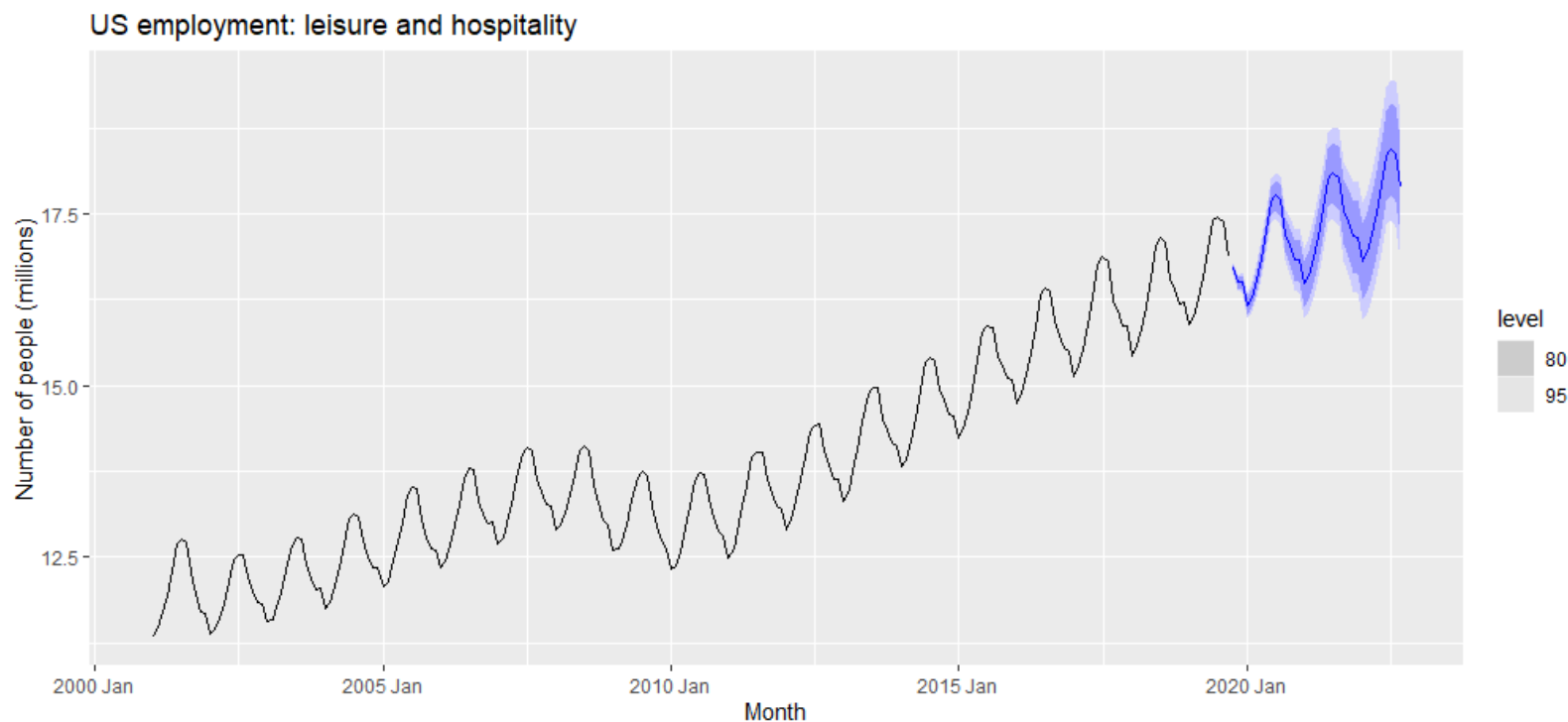
- Using Auto-SARIMA, the winning model is **SARIMA(2,1,0)(1,1,1)<sub>12</sub>**
- Plotting residuals to confirm they are like Gaussian white noise.





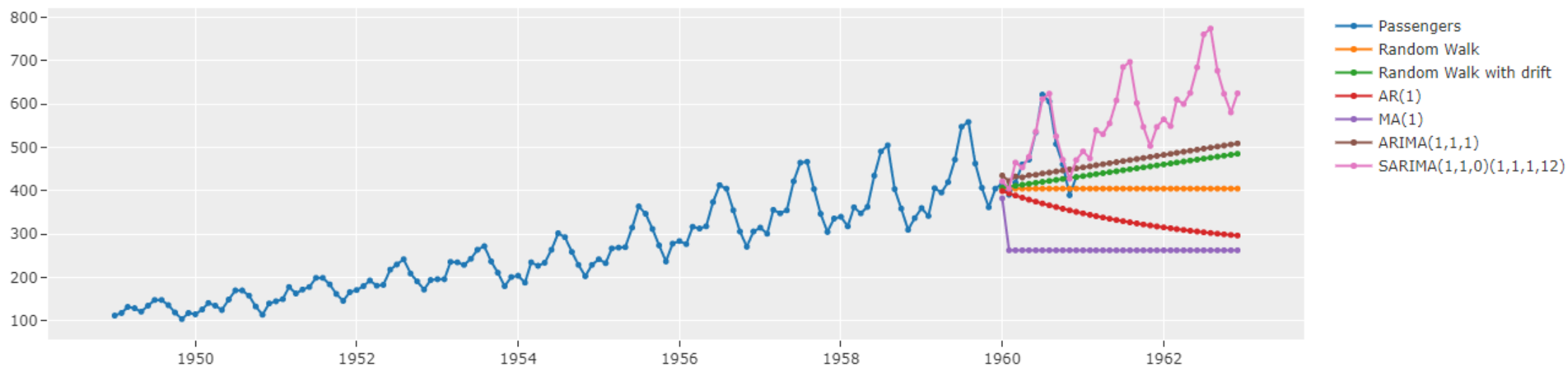
# ➔ SARIMA example, Forecasting

- We now have a seasonal ARIMA model that passes the required checks and is ready for **forecasting**.
- The forecasts have captured the **seasonal** pattern very well, and the increasing **trend** extends the recent pattern. The trend in the forecasts is induced by the double differencing.



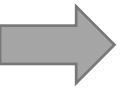
# ➔ Comparing all the models

Actual vs. Forecast (Out-of-Sample)



# → ARIMA vs ETS (parameter estimation)

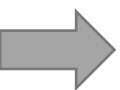
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# ARIMA vs ETS

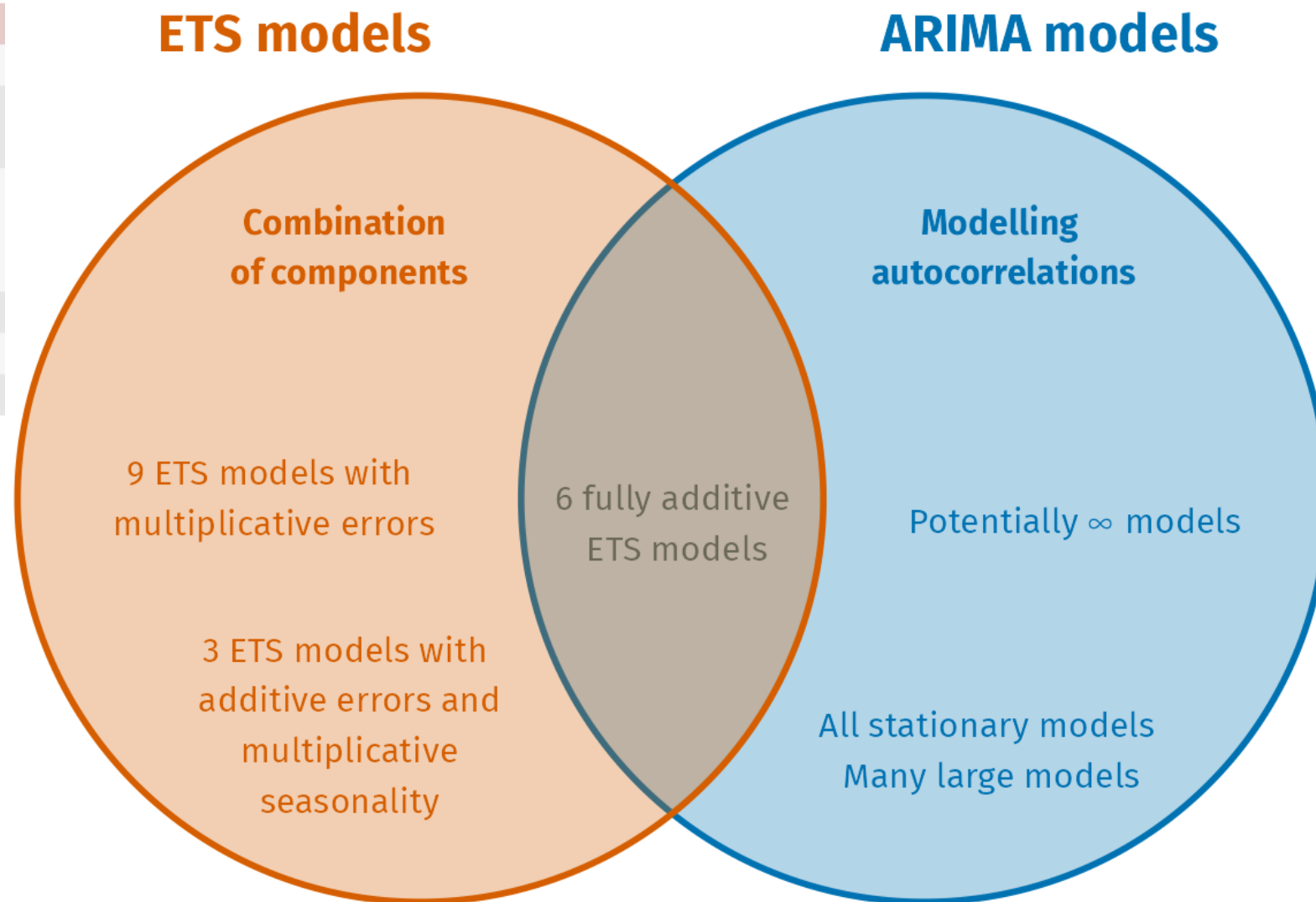
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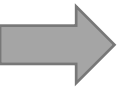
- ARIMA and ETS models can be used **together** to enhance forecasting accuracy
- **Modeling Approach:**
  - **ARIMA**: Focuses on describing the **autocorrelations** in the data. It models the time series as a **linear** function of its past values (AR part), the past error terms (MA part), and differences of the series (I part) to ensure stationarity.
  - **ETS**: Models the time series by **explicitly decomposing it into error, trend, and seasonal** components, which can be combined additively or multiplicatively. ETS directly models the series' level, trend, and seasonality.



# ARIMA vs ETS

ETS model	ARIMA model
ETS(A,N,N)	ARIMA(0,1,1)
ETS(A,A,N)	ARIMA(0,2,2)
ETS(A,A <sub>d</sub> ,N)	ARIMA(1,1,2)
ETS(A,N,A)	ARIMA(0,1, <i>m</i> )(0,1,0) <sub><i>m</i></sub>
ETS(A,A,A)	ARIMA(0,1, <i>m</i> + 1)(0,1,0) <sub><i>m</i></sub>
ETS(A,A <sub>d</sub> ,A)	ARIMA(1,0, <i>m</i> + 1)(0,1,0) <sub><i>m</i></sub>





# ARIMA vs ETS

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- ARIMA and ETS models can be used **together** to enhance forecasting accuracy
- **Data Characteristics:**
  - **ARIMA** is well-suited for time series that can be made stationary through differencing and that have significant autocorrelation patterns but may not have a clear trend or seasonal component.
  - **ETS** excels with time series that have a pronounced trend and/or seasonality.
- In summary, ARIMA and ETS models are **complementary** because they offer different approaches to modeling and forecasting time series data, each with its own set of advantages.
- The choice between them, or the decision to use them together, depends on the specific characteristics of the time series data and the forecasting goals.

# ➔ Handling Non-Linearities

- **Data Transformation:** Logarithms, Box-Cox, or similar transformations can sometimes linearize **mild non-linear relationships**, making the data suitable for ARIMA or ETS
- Approximating with **ARIMA**: Combining **multiple AR and MA terms** with differencing allows ARIMA models to **partially** capture some forms of non-linear trends or patterns.
- Approximating with **ETS**: selecting between **additive** and **multiplicative** components allows for **some** flexibility in handling non-linear trends and seasonal patterns
- **Limitations of ARIMA and ETS:** For data with strong, complex non-linear relationships, standard ARIMA and ETS models often prove insufficient
- **Transition to Machine Learning:** ML models like tree-based methods (e.g., Random Forests, Gradient Boosting) or neural networks (given enough data) are inherently designed to capture complex non-linear patterns in time series.



# ➔ Road map!

- ✓ Module 1- Introduction to Deep Forecasting
- ✓ Module 2- Setting up Deep Forecasting Environment
- ✓ Module 3- Exponential Smoothing
- ✓ Module 4- ARIMA models
- Module 5- Machine Learning for Time series Forecasting
- Module 6- Deep Neural Networks
- Module 7- Deep Sequence Modeling (RNN, LSTM)
- Module 8- Transformers (Attention is all you need!)

