

UC Berkeley
Department of Electrical Engineering and Computer Sciences
EECS 126: PROBABILITY AND RANDOM PROCESSES

Problem Set 2
Spring 2021

1. Among Us

In the game of Among Us, there are 9 players. 4 of them are imposters and 5 of them are crewmates. There is also a deck of 17 cards containing 11 “sabotage” cards and 6 “task” cards. Imposters want to play sabotage cards, and crewmates want to play task cards. Here’s how the play proceeds.

- A captain and a first mate are chosen uniformly at random from the 9 players.
- The captain draws 3 cards from the deck and gives 2 to the first mate, discarding the third.
- The first mate chooses one to play.

Now suppose you are the first mate, but the captain gave you 2 sabotage cards. Being a crewmate, you wonder, did the captain just happen to have 3 sabotage cards, or was the captain an imposter who secretly discarded a task card. In this scenario, what’s the probability that the captain is an imposter? Let’s assume that imposter captains always try to discard task cards, and crewmate captains always try to discard sabotage cards.

2. Lightbulbs

Consider an $n \times n$ array of switches. Each row i of switches corresponds to a single lightbulb L_i , so that L_i lights up if at least i switches in row i are flipped on. All of the switches start in the “off” position, and each are flipped “on” with probability p , independently of all others. What is the expected number of lightbulbs that will be lit up? Express your answer in closed form without any summations.

3. Random Bipartite Graph

Consider a random bipartite graph with, K left nodes and M right nodes. Each of the $K \cdot M$ possible edges of this graph is present with probability p independently.

- (a) Find the distribution of the degree of a particular right node.
- (b) Now, pick a left node u and right node v . Conditioned on the event that the edge (u, v) is present, what is the distribution of the degree of the right node v ? Is it the same as in part (a)?
- (c) We call a right node with degree one a *singleton*. What is the average number of singletons in a random bipartite graph?
- (d) Find the average number of left nodes that are connected to at least one singleton.

4. Compact Arrays

Consider an array of n entries, where n is a positive integer. Each entry is chosen uniformly randomly from $\{0, \dots, 9\}$. We want to make the array more compact, by putting all of the non-zero entries together at the front of the array. As an example, suppose we have the array

$$[6, 4, 0, 0, 5, 3, 0, 5, 1, 3].$$

After making the array compact, it now looks like

$$[6, 4, 5, 3, 5, 1, 3, 0, 0, 0].$$

Let i be a fixed positive integer in $\{1, \dots, n\}$. Suppose that the i th entry of the array is non-zero (assume that the array is indexed starting from 1). Let X be a random variable which is equal to the index that the i th entry has been moved after making the array compact. Calculate $\mathbb{E}[X]$ and $\text{var}(X)$.

5. Clustering Coefficient

This problem will explore an important probabilistic concept of clustering that is widely used in machine learning applications today. Consider n students, where n is a positive integer. For each pair of students $i, j \in \{1, \dots, n\}$, $i \neq j$, they are friends with probability p , independently of other pairs. We assume that friendship is mutual. We can see that the friendship among the n students can be represented by an undirected graph G . Let $N(i)$ be the number of friends of student i and $T(i)$ be the number of triangles attached to student i . We define the **clustering coefficient** $C(i)$ for student i as follows:

$$C(i) = \frac{T(i)}{\binom{N(i)}{2}}.$$

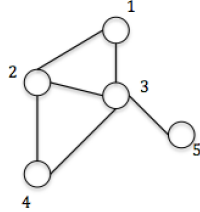


Figure 1: Friendship and clustering coefficient.

The clustering coefficient is not defined for the students who have no friends. An example is shown in Figure ???. Student 3 has 4 friends (1, 2, 4, 5) and there are two triangles attached to student 3, i.e., triangle 1-2-3 and triangle 2-3-4. Therefore $C(3) = 2/\binom{4}{2} = 1/3$.

Find $\mathbb{E}[C(i) \mid N(i) \geq 2]$.