UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 1
Spring 2021

1. Miscellaneous Review

(a) Show that the probability that exactly one of the events A and B occurs is $\Pr(A) + \Pr(B) - 2\Pr(A \cap B)$.

(b) If A is independent of itself, show that $\Pr(A) = 0$ or 1.

Solution:

(a) The probability of the event that exactly one of A and B occur is

$$\begin{aligned}\Pr(A \cap B^c) + \Pr(A^c \cap B) &= \Pr(A) - \Pr(A \cap B) + \Pr(B) - \Pr(A \cap B) \\ &= \Pr(A) + \Pr(B) - 2\Pr(A \cap B).\end{aligned}$$

(b) $\Pr(A \cap A) = \Pr(A)\Pr(A)$, so $\Pr(A) = \Pr(A)^2$; this implies that $\Pr(A) \in \{0, 1\}$. Alternatively, suppose for the sake of contradiction that $0 < \Pr(A) < 1$. Then, $\Pr(A \mid A) = 1 \neq \Pr(A)$, which contradicts the supposed independence of A with itself. Hence, $\Pr(A) \in \{0, 1\}$.

2. Balls & Bins

Let $n \in \mathbb{Z}_{>1}$ (i.e. n is an integer greater than 1). You throw n balls, one after the other, into n bins, so that each ball lands in one of the bins uniformly at random.

(a) What is an appropriate sample space to model this scenario?

(b) What is the probability that ball i falls in bin i, for each $i = 1, \dots, n$.

Solution:

(a) An appropriate sample space is to take $\Omega = \{1, \dots, n\}^n$, the set of n-tuples where each coordinate is a number in $\{1, \dots, n\}$. An outcome $\omega \in \Omega$ represents a scenario as follows: the first coordinate gives the label of the bin into which the first ball fell; the second coordinate gives the label of the bin into which the second ball fell; and so on.

Note: Notice that this choice of sample space treats all of the balls as distinguishable and all of the bins as distinguishable. The reason for making this choice is so that the probability distribution over sample space is *uniform*, that is, all outcomes have the same probability.

In contrast, if we chose a sample space corresponding to *indistinguishable balls* (and distinguishable bins), then the probability distribution would *not* be uniform, which makes the problem harder to analyze. The reason why the distribution over the sample space is no longer uniform is that some outcomes can happen in more ways, so they have higher probabilities.

Concretely, the outcome that all balls land in the first bin will have a smaller probability than the outcome that half the balls land in the first bin and the other half land in the second bin, because in the latter outcome you have the freedom to change *which* balls land in first bin (because the balls are indistinguishable).

(b) With our sample space from part (a), notice that the event "ball i falls in bin i, for each $i = 1, \dots, n$" is simply a single sample from our sample space. Our sample space has size n^n, and due to our choice from part (a), the sample is drawn uniformly at random, so the probability of the event is n^{-n}.

3. Colored Sphere

Consider a sphere that has $\frac{1}{10}$ of its surface colored blue, and the rest is colored red. Show that, no matter how the colors are distributed, it is possible to inscribe a cube in the sphere with all of its vertices red.

Hint: Carefully define some relevant events. **Solution:**

Pick an inscribed cube uniformly at random, enumerate its vertices, and let B_i be the event that vertex i is blue. Note that:

$$\Pr(B_1 \cup \cdots \cup B_8) \leq \sum_{i=1}^{8} \Pr(B_i) = \sum_{i=1}^{8} \frac{1}{10} = \frac{8}{10} < 1$$

In other words, the probability of at least one vertex being blue is *less* than 1, so there must exist an inscribed cube where each vertex is red.

Note: This is an example of a powerful tool known as the probabilistic method.

4. [Extra] The Countable Union Bound

Let $A_1, A_2, \dots$ be a countable sequence of events. Prove that the union bound holds for countably many events:

$$\Pr\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \Pr(A_i).$$

Solution:

Define $A_1' = A_1$ and $A_i' = A_i \setminus \bigcup_{j=1}^{i-1} A_i$ for $i \in \mathbb{N}$, $i \geq 2$. Now, the A_i' for $i \in \mathbb{Z}_{>0}$ are disjoint, and $\bigcup_{i=1}^{\infty} A_i' = \bigcup_{i=1}^{\infty} A_i$, so $\Pr(\bigcup_{i=1}^{\infty} A_i) = \Pr(\bigcup_{i=1}^{\infty} A_i') = \sum_{i=1}^{\infty} \Pr(A_i')$. Also for all $i \in \mathbb{Z}_{>0}$ we have $\Pr(A_i') \leq \Pr(A_i)$ since $A_i' \subseteq A_i$, so $\Pr(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \Pr(A_i)$.

Note: The fact we used above is that if $B \subseteq A$, then $\Pr(B) \leq \Pr(A)$; this follows because $A = B \cup (A \setminus B)$ is a disjoint union, so $\Pr(A) = \Pr(B) + \Pr(A \setminus B) \geq \Pr(B)$.