UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 2
Spring 2021

1. **Suspicious Game**

You are playing a card game with your friend: you take turns picking a card from a deck (you may assume that you never run out of cards). If you draw one of the special "bullet" cards, then you lose the game. Unfortunately, you do not know the contents of the deck. Your friend claims that 1/3 of the deck is filled with "bullet" cards. You don't trust your friend fully, however: you believe that he is lying with probability 1/4. You assume that if your friend is lying, then the opposite is true: 2/3 of the deck is filled with "bullet" cards!

What is the probability that you win the game if you go first? **Solution:**

Let's denote the probability of randomly selecting a "bullet" card by p (which is always the same since you never run out of cards). Since the game ends when the first "bullet" card is drawn, the number of turns before the game ends, X, is a geometric random variable with probability p. The probability that you win is the probability that X is even, so we have

$$\begin{aligned}\Pr(\text{Win} \mid p) = \Pr(X \text{ is even} \mid p) = \sum_{\substack{k=1 \\ k \text{ is even}}}^{\infty} p(1-p)^{k-1} &= p(1-p)\sum_{j=0}^{\infty}(1-p)^{2j} = \frac{p(1-p)}{1-(1-p)^2} \\ &= \frac{p(1-p)}{2p-p^2} = \frac{1-p}{2-p}.\end{aligned}$$

Now, we don't know whether $p = 1/3$ or $2/3$, so we can use the Law of Total Probability,

$$\begin{aligned}\Pr(\text{Win}) &= \Pr(\text{Win} \mid p = 1/3)\Pr(p = 1/3) + \Pr(\text{Win} \mid p = 2/3)\Pr(p = 2/3) \\ &= \left(\frac{1-1/3}{2-1/3}\right)(3/4) + \left(\frac{1-2/3}{2-2/3}\right)(1/4) \\ &= 0.3625\end{aligned}$$

Note: In the above step, it may be tempting to first calculate p as

$$\Pr(B) = \Pr(B \mid L)\Pr(L) + \Pr(B \mid \bar{L})\Pr(\bar{L}) = \frac{2}{3}\cdot\frac{1}{4} + \frac{1}{3}\cdot\frac{3}{4} = \frac{5}{12}.$$

where B is the event of drawing a "bullet" card and L is the event that your friend is lying. Then plug in $p = 5/12$ to get the probability of of $7/19 \approx 0.3684$. However, this does not work since the order of conditioning is reversed in this case.

Remark: The reason why the game is suspicious is because

$$\frac{1-p}{2-p} \le \frac{1}{2},$$

since $0 \le p \le 1$, so your chances of winning are always unfavorable!

2. **Packet Routing**

Consider a system with n inputs and n outputs. At each input, a packet appears independently with probability p. If a packet appears, it is destined for one of the n outputs uniformly randomly, independently of the other packets.

(a) Let X_1 denote the number of packets destined for the first output. What is the distribution of X_1?

(b) A collision happens when two or more packets are destined for the same output. What is the expected number of total collisions C?

Solution:

(a) The probability that there exists a packet at an input and the packet is destined for the first output is p/n. By the independence over inputs, X_1 has the binomial distribution $(n, p/n)$.

(b) Let C_i be an indicator that the ith output has a collision, then the number of total collisions $C = \sum_{i=1}^{n} C_i$. By the linearity of expectation,

$$\mathbb{E}[C] = \sum_{i=1}^{n} \mathbb{E}[C_i] = \sum_{i=1}^{n} \Pr(X_i \geq 2),$$

where X_i is the number of packets destined for the ith output. Unconditioned on the others, each output $X_i \sim \text{Binomial}(n, p/n)$, for $i = 1, \ldots, n$, as shown in part (a) for $i = 1$ (the rest follows the same arguments). Each marginal probability is

$$\begin{aligned}
\Pr(X_i \geq 2) &= 1 - \Pr(X_i = 0) - \Pr(X_i = 1) \\
&= 1 - (1 - p/n)^n - n(p/n)(1 - p/n)^{n-1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}[C] &= \sum_{i=1}^{n} 1 - (1 - p/n)^n - n(p/n)(1 - p/n)^{n-1} \\
&= n - n(1 - p/n)^n - np(1 - p/n)^{n-1}.
\end{aligned}$$

3. **Sampling without Replacement**

Suppose you have N items, G of which are good and B of which are bad (B, G, and N are positive integers, $B + G = N$). You start to draw items without replacement, and suppose that the first good item appears on draw X. Compute $\mathbb{E}[X]$ and $\mathbb{E}[(X-1)^2]$ **Solution:**

The expectation is computed with a clever trick: let X_i be the indicator that the ith bad item appears before the first good item, for $i = 1, \ldots, B$. Then, $X = 1 + \sum_{i=1}^{B} X_i$, and by linearity of expectation,

$$\mathbb{E}[X] = 1 + B\,\mathbb{E}[X_1] = 1 + \frac{B}{G+1} = \frac{N+1}{G+1}.$$

Using the same indicators, we compute $\mathbb{E}[(X-1)^2]$.

$$\mathbb{E}[(X-1)^2] = B\,\mathbb{E}[X_1^2] + B(B-1)\,\mathbb{E}[X_1 X_2]$$

$$= \frac{B}{G+1} + \frac{2B(B-1)}{(G+1)(G+2)}$$

Note: The reason we calculated $\mathbb{E}[(X-1)^2]$ instead of $\mathbb{E}[X^2]$ is to more easily calculate $\operatorname{var}(X)$ since $\mathbb{E}[(X-1)^2]$ is easier to calculate as the -1 cancels. Observe that $\operatorname{var}(X) = \operatorname{var}(X-1)$, so $\operatorname{var}(X) = \mathbb{E}[(X-1)^2] - \mathbb{E}[X-1]^2$. Therefore

$$\operatorname{var} X = \frac{B}{G+1} + \frac{2B(B-1)}{(G+1)(G+2)} - \left(\frac{B}{G+1}\right)^2.$$

With a little algebra, we can actually simplify the result.

$$\begin{aligned}
\operatorname{var} X &= \frac{B(G+1)(G+2) + 2B(B-1)(G+1) - B^2(G+2)}{(G+1)^2(G+2)} \\
&= \frac{BG(N+1)}{(G+1)^2(G+2)}
\end{aligned}$$