UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Homework 1
Spring 2021

1. **Choosing from Any Jar Makes No Difference**

   Each of k jars contains w white and b black balls. A ball is randomly chosen from jar 1 and transferred to jar 2, then a ball is randomly chosen from jar 2 and transferred to jar 3, etc. Finally, a ball is randomly chosen from jar k. Show that the probability that the last ball is white is the same as the probability that the first ball is white, i.e., it is $w/(w+b)$.

2. **Coin Flipping & Symmetry**

   Alice and Bob have $2n+1$ fair coins (where $n \geq 1$), each coin with probability of heads equal to $1/2$. Bob tosses $n+1$ coins, while Alice tosses the remaining n coins. Assuming independent coin tosses, show that the probability that, after all coins have been tossed, Bob will have gotten more heads than Alice is $1/2$.

   Hint: Consider the event $A = \{\text{more heads in the first } n+1 \text{ tosses than the last } n \text{ tosses}\}$.

3. **Passengers on a Plane**

   There are N passengers in a plane with N assigned seats (N is a positive integer), but after boarding, the passengers take the seats randomly. Assuming all seating arrangements are equally likely, what is the probability that no passenger is in their assigned seat? Compute the probability when $N \to \infty$.

   Hint: Use the inclusion-exclusion principle and the power series $e^x = \sum_{j=0}^{\infty} x^j/j!$.

4. **Expanding the NBA**

   The NBA is looking to expand to another city. In order to decide which city will receive a new team, the commissioner interviews potential owners from each of the N potential cities (N is a positive integer), one at a time. Unfortunately, the owners would like to know immediately after the interview whether their city will receive the team or not. The commissioner decides to use the following strategy: she will interview the first m owners and reject all of them ($m \in \{1, \ldots, N\}$). After the mth owner is interviewed, she will pick the first city that is better than all previous cities. The cities are interviewed in a uniformly random order. What is the probability that the best city is selected? Assume that the commissioner has an objective method of scoring each city and that each city receives a unique score.

   You should arrive at an exact answer for the probability in terms of a summation. Approximate your answer using $\sum_{i=1}^{n} i^{-1} \approx \ln n$ and find the optimal value of m that maximizes the probability that the best city is selected. You can also say $\ln(n-1) \approx \ln n$.

   Hint: Consider the events $B_i = \{\text{i-th city is the best}\}$ and $A = \{\text{best city is chosen}\}$.

5. **Coupling: Choosing a Sample Space**

   Consider a *random* graph with 100 vertices; since there are $\binom{100}{2}$ possible edges, there are $2^{\binom{100}{2}}$ possible graphs. So, $\Omega = \{\text{all } 2^{\binom{100}{2}} \text{ possible graphs}\}$ and we define the probability of a graph $G \in \Omega$ to be $p^{E(G)}(1-p)^{\binom{100}{2}-E(G)}$, where $E(G)$ is the number of edges present in the graph. We call this the **random graph with parameter** p (abbreviated RG(p)).

A **triangle** is a set of three vertices in the graph with all possible edges between them. Prove that the probability that there are at least ten triangles in the random graph is greater in RG(p) than under RG(q), if $p > q$. [*Hint*: Think about using a sequence of two coin flips to generate $RG(q)$ followed by $RG(p)$ on the same vertex set.]