FACTORIZACIÓN LU

Métodos Numéricos y Estadísticos

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Sistemas de ecuaciones lineales

$$A \cdot x = b$$

$$\left.\begin{array}{l} a_{11}x_{11} + a_{12}x_{12} + \ldots + a_{1n}x_{1n} = b_1 \\ a_{21}x_{21} + a_{22}x_{22} + \ldots + a_{2n}x_{2n} = b_2 \\ \vdots \\ a_{n1}x_{n1} + a_{n2}x_{n2} + \ldots + a_{nn}x_{nn} = b_n \end{array}\right\} \rightarrow \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

Operaciones elementales

- Intercambio de dos ecuaciones
  $(E_i) \rightarrow (E_j)$
- Multiplicar una ecuación por una constante distinta de cero
  $(\lambda E_i) \rightarrow (E_j)$
- Multiplicar a una ecuación por una constante distinta de cero y sumarla a otra ecuación
  $(E_i + \lambda E_j) \rightarrow (E_j)$

Operaciones elementales y sus matrices

Matriz identidad $\xrightarrow{\text{Operaciones elementales}}$ Matriz elemental

- $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{E_2' \to E_2 + 6E_1} \begin{pmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
- $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{E_3' \to E_3 - 5E_1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$E_2' \to E_2 - 3E_1 \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 6 & -1 & 1 \\ 7 & 2 & 3 \\ 0 & -3 & -2 \end{pmatrix} = \begin{pmatrix} 6 & -1 & 2 \\ -11 & 5 & -3 \\ 0 & -3 & -2 \end{pmatrix}$$

$$E \cdot E^{-1} = Id \Rightarrow \qquad E = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad E^{-1} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Métodos directos de resolucion de ecuaciones lineales

- Eliminación gaussiana $Ax = b \rightarrow Ux = c$
- Descomposición LU $Ax = b \rightarrow LUx = b$
- Eliminación Gauss-Jordan $Ax = b \rightarrow Ix = c$

Métodos iterativos

- Método de Jacobi
- Método de Gauss-Seidel
- Método de relajación (SOR)

Fase de eliminación gaussiana

$$E_i \rightarrow E_i - \lambda \cdot E_j \qquad (multiplicador : \lambda = \frac{a_{ij}}{a_{ii}})$$

$$\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

Sustitución regresiva

$$x_n = \frac{1}{u_{nn}} c_n$$

$$x_i = \frac{1}{u_{ii}} \left(c_i - \sum_{j=i+1}^{n} u_{ij} x_j \right) \rightarrow i = n-1, , 1$$

Fase de eliminación:

$$\begin{pmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 11 \\ -16 \\ 17 \end{pmatrix} \xrightarrow{E_2' \to E_2 - \frac{-2}{4} E_1} \begin{pmatrix} 4 & -2 & 1 \\ 0 & 3 & -1.5 \\ 1 & -2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 11 \\ -10.5 \\ 17 \end{pmatrix}$$

$$\begin{pmatrix} 4 & -2 & 1 \\ 0 & 3 & -1.5 \\ 1 & -2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 11 \\ -10.5 \\ 17 \end{pmatrix} \xrightarrow{E_3' \to E_3 - \frac{1}{4} E_1} \begin{pmatrix} 4 & -2 & 1 \\ 0 & 3 & -1.5 \\ 0 & -1.5 & 3.75 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 11 \\ -10.5 \\ 14.25 \end{pmatrix}$$

$$\begin{pmatrix} 4 & -2 & 1 \\ 0 & 3 & -1.5 \\ 0 & -1.5 & 3.75 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 11 \\ -10.5 \\ 14.25 \end{pmatrix} \xrightarrow{E_3' \to E_3 - \frac{-1.5}{3} E_2} \begin{pmatrix} 4 & -2 & 1 \\ 0 & 3 & -1.5 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 11 \\ -10.5 \\ 9 \end{pmatrix}$$

$$\left\{ \begin{array}{cccccc} 4x_1 & -2x_2 & +x_3 & = & 11 \\ & 3x_2 & -1.5x_3 & = & -10.5 \\ & & 3x_3 & = & 9 \end{array} \right\}$$

Sustitución regresiva

$$\left.\begin{array}{rrrcr} 4x_1 & -2x_2 & +x_3 & = & 11 \\ & 3x_2 & -1.5x_3 & = & -10.5 \\ & & 3x_3 & = & 9 \end{array}\right\} \rightarrow x_3 = \frac{1}{3} \cdot 9 = 3$$

$$\left.\begin{array}{rrrcr} 4x_1 & -2x_2 & +x_3 & = & 11 \\ & 3x_2 & -1.5x_3 & = & -10.5 \\ & & x_3 & = & 3 \end{array}\right\} \rightarrow x_2 = \frac{1}{3} \cdot (-10.5 + 1.5 \cdot 3) = -2$$

$$\left.\begin{array}{rrrcr} 4x_1 & -2x_2 & +x_3 & = & 11 \\ & & x_2 & = & -2 \\ & & x_3 & = & 3 \end{array}\right\} \rightarrow x_1 = \frac{1}{4} \cdot (11 - 1 \cdot 3 + 2 \cdot (-2)) = 1$$

$$\left\{\begin{array}{rcr} x_1 & = & 1 \\ x_2 & = & -2 \\ x_3 & = & 3 \end{array}\right\}$$

Eliminación de Gauss-Jordan

$$\left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} l_{11} & l_{12} & l_{13} & b_1' \\ 0 & l_{22} & l_{23} & b_2' \\ 0 & 0 & l_{33} & b_3' \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} d_{11} & 0 & 0 & b_1'' \\ 0 & d_{22} & 0 & b_2'' \\ 0 & 0 & d_{33} & b_3'' \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & c_1 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_3 \end{array}\right)$$

Uso en el cálculo de inversas

$$\left(\begin{array}{c|ccc} & 1 & 0 & 0 \\ A & 0 & 1 & 0 \\ & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & A^{-1} \\ 0 & 0 & 1 & \end{array}\right)$$

Ejemplo de Gauss-Jordan:

$$\left\{ \begin{array}{r} x+y+z=5 \\ 2x+3y+5z=8 \\ 4x+5z=2 \end{array} \right\} \xrightarrow{\textit{forma} \quad \textit{matricial}} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array} \right) \xrightarrow[E_3'=E_3-4E_1]{E_2'=E_2-2E_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & -4 & 1 & -18 \end{array} \right) \xrightarrow{E_3'=E_3+4E_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 13 & -26 \end{array} \right) \xrightarrow{E_3'=\frac{E_3}{13}}$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{array} \right) \xrightarrow[E_1'=E_1-3E_3]{E_2'=E_2-3E_3} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array} \right) \xrightarrow{E_1'=E_1-E_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array} \right)$$

SOLUCIÓN: $\left\{ \begin{array}{r} x=3 \\ y=4 \\ z=-2 \end{array} \right\}$

Algoritmo de la sustitución progresiva

$$x_1 = \frac{1}{u_{11}} b_1$$

$$x_i = \frac{1}{u_{ii}} \left(b_i - \sum_{j=1}^{n} u_{ij} x_j \right) \rightarrow i = 2, 3, \ldots, n$$

Ejemplo de sustitución progresiva:

$$\left.\begin{array}{lllcr} 3x_1 & & & = & 9 \\ -1.5x_1 & 3x_2 & & = & -10.5 \\ x_1 & -2x_2 & +4x_3 & = & 11 \end{array}\right\} \quad x_1 = \frac{1}{3} \cdot 9 = 3$$

$$\left.\begin{array}{lllcr} x_1 & & & = & 3 \\ -1.5x_1 & 3x_2 & & = & -10.5 \\ x_1 & -2x_2 & +4x_3 & = & 11 \end{array}\right\} \quad x_2 = \frac{1}{3} \cdot (-10.5 + 1.5 \cdot 3) = -2$$

$$\left.\begin{array}{lllcr} x_1 & & & = & 3 \\ & x_2 & & = & -2 \\ x_1 & -2x_2 & +4x_3 & = & 11 \end{array}\right\} \quad x_3 = \frac{1}{4} \cdot (11 - 1 \cdot 3 + 2 \cdot (-2)) = 1$$

$$SOLUCIN : \left\{ \begin{array}{ccc} x_1 & = & 3 \\ x_2 & = & -2 \\ x_3 & = & 1 \end{array} \right\}$$

Ejemplo de la importancia de tener un buen pivote:

Resolución del sistema S por Gauss (con mal pivote):

$$S = \left\{ \begin{array}{l} 0.003x_1 + 59.14x_2 = 59.17 \\ 5.291x_1 - 6.130x_2 = 46.78 \end{array} \right\}$$

- Redondeando a 4 cifras:

$$\left\{ \begin{array}{l} 0.003x_1 + 59.14x_2 = 59.17 \\ 5.291x_1 - 6.130x_2 = 46.78 \end{array} \right\} \xrightarrow[m_{21}=\frac{a_{21}}{a_{11}}=\frac{5.291}{0.003}=1764]{E_2'=E_2-m_{21}\cdot E_1} \left\{ \begin{array}{l} 0.003x_1 + 59.14x_2 \approx 59.17 \\ -104300x_2 \approx -104400 \end{array} \right\} \rightarrow$$

$$\rightarrow \left\{ \begin{array}{l} x_1 \approx -10 \\ x_2 \approx 1.001 \end{array} \right\}$$

- Sin redondear:

$$\left\{ \begin{array}{l} 0.003x_1 + 59.14x_2 = 59.17 \\ 5.291x_1 - 6.130x_2 = 46.78 \end{array} \right\} \xrightarrow[m_{21}=1763.\hat{6}]{E_2'=E_2-m_{21}\cdot E_1} \left\{ \begin{array}{l} 0.003x_1 + 59.14x_2 = 59.17 \\ -104309.37\hat{6}x_2 = -104309.37\hat{6} \end{array} \right\} \rightarrow$$

$$\rightarrow \left\{ \begin{array}{l} x_1 = 10 \\ x_2 = 1 \end{array} \right\}$$

Pivoteo parcial

$a_{pj} = \max_{i=j,\ldots,n} \{|a_{ij}|\} \rightarrow intercambio filas E_p \leftrightharpoons E_i$

Ejemplo de pivoteo parcial:

$$S = \left\{ \begin{array}{l} 0.003x_1 + 59.14x_2 = 59.17 \\ 5.291x_1 - 6.130x_2 = 46.78 \end{array} \right\}$$

$$a_{pj} = \max\{|a_{11}|, |a_{21}|\} = \max\{|0.003|, |5.291|\} = |5.291| = |a_{21}|$$

$$\left\{ \begin{array}{l} 5.291x_1 - 6.130x_2 = 46.78 \\ 0.003x_1 + 59.14x_2 = 59.17 \end{array} \right\} \xrightarrow[m_{21}=\frac{a_{21}}{a_{11}}=\frac{0.003}{5.291}=0.000567]{E_2'=E_2-m_{21}E_1} \left\{ \begin{array}{l} 5.291x_1 - 6.130x_2 = 46.78 \\ 59.14x_2 = 59.14 \end{array} \right\} \rightarrow$$

$$\rightarrow \left\{ \begin{array}{l} x_1 = 10 \\ x_2 = 1 \end{array} \right\}$$

Matriz de permutación

Para hacer un cambio de filas, se emplea una matriz P_n (matriz identidad modificada) para que al multiplicarla por la izquierda los cambios afecten a las filas :

$$\left.\begin{array}{l} Id_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ S = \begin{pmatrix} 0.003 & 59.14 \\ 5.291 & -6.130 \end{pmatrix} \end{array}\right\} \xrightarrow{E_1^{(Id)} \leftrightharpoons E_2^{(Id)}} P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow P \cdot S = \begin{pmatrix} 5.291 & -6.130 \\ 0.003 & 59.14 \end{pmatrix}$$

Una buena matriz P cumple $P^{-1} = P^t$

Fallo del pivoteo parcial

Vamos a multiplicar ahora E_1 por 10^4 y a resolver el sistema obtenido.

$$\left\{ \begin{array}{l} 30.00x_1 + 591400x_2 = 591700 \\ 5.291x_1 - 6.130x_2 = 46.78 \end{array} \right\} \xrightarrow[m_{21}=\frac{5.291}{30.00}=0.1764]{E_2'=E_2-m_{21}\cdot E_1} \left\{ \begin{array}{l} 30.00x_1 + 591400x_2 = 591700 \\ -104300x_2 = -104400 \end{array} \right\} \rightarrow$$

$$\left\{ \begin{array}{l} x_1 = -10 \\ x_2 = 1.001 \end{array} \right\}$$

Volvemos a obtener resultados incorrectos por tener un mal pivote, ya que a_{11} es muy grande respecto a_{21}.

Elección de pivote

- $S_i = \max_{j=1,\ldots,n} \{|a_{ij}|\}$
- $\dfrac{|a_{p1}|}{S_p} = \max_{k=1,\ldots,n} \dfrac{|a_{k1}|}{S_k}$ $\quad (p = \text{nueva fila pivote })$
- *intercambio* $E_p \leftrightharpoons E_k$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{2n} & a_{3n} & \ldots & a_{nn} \end{pmatrix}$$

Ejemplo:

$$\frac{|a_{11}|}{S_1} = \frac{30.00}{\max\{|30.00|,|591400|\}} = 0.5073 \times 10^{-4}$$

$$\frac{|a_{21}|}{S_2} = \frac{5.291}{\max\{|5.291|,|6.130|\}} = 0.8631$$

$$S = \left\{ \begin{array}{l} 30.00x_1 + 591400x_2 = 591700 \\ 5.291x_1 - 6.130x_2 = 46.78 \end{array} \right\}$$

Intercambio $E_1 \leftrightharpoons E_2$

$$\left\{ \begin{array}{l} 5.291x_1 - 6.130x_2 = 46.78 \\ 30.00x_1 + 591400x_2 = 591700 \end{array} \right\} \xrightarrow[m_{21}=\frac{30.00}{5.291}=5.670]{E_2'=E_2-m_{21}\cdot E_1} \left\{ \begin{array}{l} 5.291x_1 - 6.130x_2 = 46.78 \\ 591434.76x_2 = 591434.76 \end{array} \right\} \rightarrow$$

$$\left\{ \begin{array}{l} x_1 = 10 \\ x_2 = 1 \end{array} \right\}$$

Descomposición de A

$$A = L \cdot U$$

Algoritmo de resolución

$$A \cdot X = b \xrightarrow{A=L\cdot U} L \cdot U \cdot X = b \rightarrow \left\{ \begin{array}{c} L \cdot Y = b \\ U \cdot X = Y \end{array} \right\}$$

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$\begin{pmatrix} L_{11} & 0 \\ L_{12} & L_{22} \end{pmatrix} \cdot \begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} L_{11} & 0 \\ L_{12} & L_{22} \end{pmatrix} \cdot \overbrace{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}}^{U\cdot X=Y} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}}_{L\cdot Y=b} \qquad \underbrace{\begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}}_{U\cdot X=Y}$$

Algoritmo de resolución (y II)

$$L \cdot Y = b \rightarrow \left\{ \begin{array}{l} L_{11} y_1 = b_1 \\ L_{12} y_1 + L_{22} y_2 = b_2 \end{array} \right\} \rightarrow \left[\begin{array}{l} y_1 = \dfrac{1}{L_{11}} b_1 \\ \\ y_2 = \dfrac{1}{L_{22}} \left(b_2 - L_{12} y_1 \right) \end{array} \right]$$

- $y_i = \dfrac{1}{L_{ii}} \left(b_i - \displaystyle\sum_{j=1}^{i-1} L_{ij} y_j \right) \rightarrow \textit{siendo } i = 2, \ldots, n$

$$U \cdot X = Y \rightarrow \left\{ \begin{array}{l} U_{11} x_1 + U_{12} x_2 = y_1 \\ U_{22} x_2 = y_2 \end{array} \right\} \rightarrow \left[\begin{array}{l} x_2 = \dfrac{1}{U_{22}} y_2 \\ \\ x_1 = \dfrac{1}{U_{11}} \left(y_1 - U_{12} x_2 \right) \end{array} \right] \rightarrow x_n = \frac{1}{U_{ii}} y_n$$

- $x_n = \dfrac{1}{U_{ii}} \left(y_i - \displaystyle\sum_{j=i+1}^{n} U_{ij} x_j \right) \rightarrow \textit{siendo } i = n-1, \ldots, 1$

Ejemplo de obtención de matrices L y U:

$$A = \begin{pmatrix} -2 & 5 & 2 \\ 1 & 3 & -6 \\ 4 & 6 & 9 \end{pmatrix} \qquad B = \begin{pmatrix} b1 \\ b2 \\ b3 \end{pmatrix} \qquad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$U_{inicial} = \begin{pmatrix} -2 & 5 & 2 \\ 1 & -3 & 5 \\ 4 & 1 & 2 \end{pmatrix} \xrightarrow{F_2' \to F_1 + 2F_2} \begin{pmatrix} -2 & 5 & 2 \\ 0 & -1 & 12 \\ 4 & 1 & 3 \end{pmatrix} \xrightarrow{F_3' \to F_3 + 2F_1} \begin{pmatrix} -2 & 5 & 2 \\ 0 & -1 & 12 \\ 0 & 11 & 6 \end{pmatrix} \to$$

$$\xrightarrow{F_3' \to 11F_2 + F_3} \begin{pmatrix} -2 & 4 & 9 \\ 0 & -1 & 12 \\ 0 & 0 & 22 \end{pmatrix}$$

$$L_{inicial} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{Factor\ 1} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{Factor\ 2} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \to$$

$$\xrightarrow{Factor\ 3} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & -11 & 1 \end{pmatrix}$$

Tipos de descomposiciones

- Descomposición de Doolitle $L_{ii} = 1 \rightarrow i = 1, 2, \ldots, n$
- Descomposición de Court $U_{ii} = 1 \rightarrow i = 1, 2, \ldots, n$
- Descomposición de Choleski $L = U^T$

$$\left\{ \begin{array}{lcl} l_{11} \cdot u_{11} & = & a_{11} \\ l_{11} \cdot u_{12} & = & a_{12} \\ l_{11} \cdot u_{13} & = & a_{13} \\ l_{21} \cdot u_{11} & = & a_{21} \\ l_{21} \cdot u_{12} + l_{22} \cdot u_{22} & = & a_{22} \\ l_{21} \cdot u_{13} + l_{22} \cdot u_{23} & = & a_{23} \\ l_{31} \cdot u_{11} & = & a_{31} \\ l_{31} \cdot u_{12} + l_{32} \cdot u_{22} & = & a_{32} \\ l_{31} \cdot u_{13} + l_{32} \cdot u_{23} + l_{33} \cdot u_{33} & = & a_{33} \end{array} \right\}$$

$$\begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \cdot \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Ejemplo:

$$A = \begin{pmatrix} -4 & -3 & 1 \\ 8 & 11 & -1 \\ 4 & 18 & 5 \end{pmatrix} \Rightarrow L \cdot U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -4 & -3 & 1 \\ 8 & 11 & -1 \\ 4 & 18 & 5 \end{pmatrix}$$

$$E_2' = E_2 + 2E_1 \Rightarrow E = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow A = LE^{-1}EU$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -4 & -3 & 1 \\ 8 & 11 & -1 \\ 4 & 18 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -4 & -3 & 1 \\ 0 & 5 & 1 \\ 4 & 18 & 5 \end{pmatrix}$$

$$E_3' = E_3 + E_1 \Rightarrow E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \Rightarrow A = LE^{-1}EU$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -4 & -3 & 1 \\ 0 & 5 & 1 \\ 4 & 18 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -4 & -3 & 1 \\ 0 & 5 & 1 \\ 0 & 15 & 6 \end{pmatrix}$$

Ejemplo (y II):

$$E_3' = E_3 - 3E_2 \Rightarrow E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix} \Rightarrow A = LE^{-1}EU$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix} \begin{pmatrix} -4 & -3 & 1 \\ 0 & 5 & 1 \\ 0 & 15 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 3 & 1 \end{pmatrix} \begin{pmatrix} -4 & -3 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

$$A = L \cdot U \Rightarrow \begin{pmatrix} -4 & -3 & 1 \\ 8 & 11 & -1 \\ 4 & 18 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 3 & 1 \end{pmatrix} \begin{pmatrix} -4 & -3 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

Matriz de permutación

$$PA = LU$$

Permutación de matriz A de AX=b

$$A \cdot X = b \rightarrow P \cdot A \cdot X = P \cdot b \xrightarrow{A=L\cdot U} L \cdot U \cdot X = P \cdot b \rightarrow \left\{ \begin{array}{c} L \cdot Y = P \cdot b \\ U \cdot X = Y \end{array} \right\}$$

Elección de una buena matriz P

$$P^{-1} = P^t$$

Número de operaciones

$$\sum_{i=1}^{n} 1 + 2\sum_{i=1}^{n}(i-1) = 2\sum_{i=1}^{n}(i-n) = n^2 \; operaciones$$

Ejemplo de factorización LU con permutación

$$A = \begin{pmatrix} 0 & 5 & 2 \\ 1 & 3 & -6 \\ -2 & 4 & 9 \end{pmatrix}$$

- $U = \begin{pmatrix} 0 & 5 & 2 \\ 1 & 3 & -6 \\ -2 & 4 & 9 \end{pmatrix} \rightarrow \begin{pmatrix} -2 & 4 & 9 \\ 1 & 3 & -6 \\ 0 & 5 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} -2 & 4 & 9 \\ 0 & 10 & -3 \\ 0 & 5 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} -2 & 4 & 9 \\ 0 & 10 & -3 \\ 0 & 0 & -7 \end{pmatrix}$
- $L = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} + Id$
- $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$$P \cdot A = L \cdot U \rightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 5 & 2 \\ 1 & 3 & -6 \\ -2 & 4 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} -2 & 4 & 9 \\ 0 & 10 & -3 \\ 0 & 0 & -7 \end{pmatrix}$$

Método de descomposición de Court

$A = L \cdot U \quad (\ cuando \ U_{ii} = 1)$

Forma de las matrices

$$L = \begin{bmatrix} L_{11} & 0 & 0 & \dots & 0 \\ L_{21} & L_{22} & 0 & \dots & 0 \\ L_{31} & L_{32} & L_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ L_{n1} & L_{n2} & L_{n3} & \dots & L_{nn} \end{bmatrix} \qquad U = \begin{bmatrix} 1 & U_{12} & U_{13} & \dots & U_{1n} \\ 0 & 1 & U_{23} & \dots & U_{2n} \\ 0 & 0 & 1 & \dots & U_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & U_{nn} \end{bmatrix}$$

Aplicación del método de Court

$$\begin{pmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} \end{pmatrix} \cdot \begin{pmatrix} 1 & U_{12} & U_{13} & U_{14} \\ 0 & 1 & U_{23} & U_{24} \\ 0 & 0 & 1 & U_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

- Primera columna de L $\rightarrow \left\{ \begin{array}{rl} L_{11} = & a_{11} \\ L_{21} = & a_{21} \\ L_{31} = & a_{31} \\ L_{41} = & a_{41} \end{array} \right\}$
- Primera fila de U $\rightarrow \left\{ \begin{array}{l} L_{11} \cdot U_{12} = a_{12} \rightarrow U_{12} = \dfrac{a_{12}}{L_{11}} \\ \\ L_{11} \cdot U_{13} = a_{13} \rightarrow U_{13} = \dfrac{a_{12}}{L_{11}} \\ \\ L_{11} \cdot U_{14} = a_{14} \rightarrow U_{14} = \dfrac{a_{12}}{L_{11}} \end{array} \right\}$
- ...

Ejemplo:

$$A = \begin{pmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{pmatrix} = L \cdot U = \begin{pmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{pmatrix} \cdot \begin{pmatrix} 1 & U_{12} & U_{13} \\ 0 & 1 & U_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left\{ \begin{array}{l} L_{11} = 1 \\ L_{21} = -1 \\ L_{31} = 3 \end{array} \right\}$$

$$\left. \begin{array}{l} U_{12} = \dfrac{5}{L_{11}} \\ U_{13} = \dfrac{2}{L_{11}} \end{array} \right\} \to \left\{ \begin{array}{l} U_{12} = 5 \\ U_{13} = 2 \end{array} \right\}$$

$$L_{21} \cdot U_{12} + L_{22} = 0 \quad \to \left\{ \; L_{22} = 5 \; \right\}$$

$$L_{21} \cdot U_{13} + L_{22} \cdot U_{23} = 1 \quad \to \left\{ \; U_{23} = \frac{3}{5} \; \right\}$$

$$\left. \begin{array}{l} L_{31} \cdot U_{12} + L_{32} = 2 \\ L_{31} \cdot U_{13} + L_{32} \cdot U_{23} + L_{33} = 4 \end{array} \right\} \to \left\{ \begin{array}{l} L_{32} = -13 \\ L_{33} = 29/5 \end{array} \right\}$$

$$A = L \cdot U \to \begin{pmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 5 & 0 \\ 3 & -13 & 29/5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 5 & 2 \\ 0 & 1 & 3/5 \\ 0 & 0 & 1 \end{pmatrix}$$

Método de descomposición de Doolittle

$A = L \cdot U \quad (\ cuando\ L_{ii} = 1)$

Forma de las matrices

$$L = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ L_{21} & 1 & 0 & \dots & 0 \\ L_{31} & L_{32} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ L_{n1} & L_{n2} & L_{n3} & \dots & 1_{nn} \end{bmatrix} \qquad U = \begin{bmatrix} U_{11} & U_{12} & U_{13} & \dots & U_{1n} \\ 0 & U_{22} & U_{23} & \dots & U_{2n} \\ 0 & 0 & U_{33} & \dots & U_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & U_{nn} \end{bmatrix}$$

Descomposición de Doolittle en matrices 3x3

$$\left.\begin{array}{l} L = \begin{pmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{pmatrix} \\ \\ U = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{pmatrix} \end{array}\right\} A = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{11}L_{21} & U_{12}L_{21} + U_{22} & U_{13}L_{21} + U_{23} \\ U_{11}L_{31} & U_{12}L_{31} + U_{22}L_{32} & U_{13}L_{31} + U_{23}L_{32} + U_{33} \end{pmatrix}$$

Aplicando eliminación gaussiana:

$$A = L \cdot U = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{11}L_{21} & U_{12}L_{21} + U_{22} & U_{13}L_{21} + U_{23} \\ U_{11}L_{31} & U_{12}L_{31} + U_{22}L_{32} & U_{13}L_{31} + U_{23}L_{32} + U_{33} \end{pmatrix} \xrightarrow[F_3' = F_3 - L_{31}F_1]{F_2' = F_2 - L_{21}F_1}$$

$$\to \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & U_{22}L_{32} & U_{23}L_{32} + U_{33} \end{pmatrix} \xrightarrow{F_3' = F_3 - L_{32}F_2} \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{pmatrix}$$

Ejemplo:

$$A=\begin{pmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 5 & 4 \end{pmatrix} \qquad b=\begin{pmatrix} 5 \\ 8 \\ -7 \end{pmatrix} \qquad X=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$A \cdot X = b \xrightarrow{A=L \cdot U} L \cdot U \cdot X = b \rightarrow \begin{cases} L \cdot Y = b \\ U \cdot X = Y \end{cases}$$

$$U_{inicial} = \begin{pmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 5 & 4 \end{pmatrix} \xrightarrow{F_2'=F_1+F_2} \begin{pmatrix} 1 & 5 & 2 \\ 0 & 5 & 3 \\ 3 & 5 & 4 \end{pmatrix} \xrightarrow{F_3'=F_3-3F_1} \begin{pmatrix} 1 & 5 & 2 \\ 0 & 5 & 3 \\ 0 & -10 & -2 \end{pmatrix} \rightarrow$$

$$\xrightarrow{F_3'=F_3+2F_2} \begin{pmatrix} 1 & 5 & 2 \\ 0 & 5 & 3 \\ 0 & 0 & 4 \end{pmatrix}$$

$$L_{inicial} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{Factor\ 1} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{Factor\ 2} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \xrightarrow{Factor\ 3}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & -2 & 1 \end{pmatrix}$$

Ejemplo (y II):

$$U = \begin{pmatrix} 1 & 5 & 2 \\ 0 & 5 & 3 \\ 0 & 0 & 4 \end{pmatrix} \quad L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & -2 & 1 \end{pmatrix} \quad Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \; X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad b = \begin{pmatrix} 5 \\ 8 \\ 15 \end{pmatrix}$$

$$L \cdot Y = b \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ -y_1 + y2 \\ 3y_1 - 2y_2 + y_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ -7 \end{pmatrix} \rightarrow \ldots \rightarrow$$

$$\rightarrow \left\{ \begin{array}{l} y_1 = 5 \\ y_2 = 13 \\ y_3 = 4 \end{array} \right\} \rightarrow Y = \begin{pmatrix} 5 \\ 13 \\ 4 \end{pmatrix}$$

$$U \cdot X = Y \rightarrow \begin{pmatrix} 1 & 5 & 2 \\ 0 & 5 & 3 \\ 0 & 0 & 4 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + 5x_2 + 2x_3 \\ 5x_2 + 3x_3 \\ 4x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 13 \\ 4 \end{pmatrix} \rightarrow \ldots$$

SOLUCIÓN: $\left\{ \begin{array}{l} x_1 = -7 \\ x_2 = 2 \\ x_3 = 1 \end{array} \right\}$

Método de descomposición de Choleski

$A = L \cdot U \quad \left(\textit{siendo } U = L^T \right)$

Forma de las matrices

$$L = \begin{bmatrix} L_{11} & 0 & 0 & \dots & 0 \\ L_{21} & L_{22} & 0 & \dots & 0 \\ L_{31} & L_{32} & L_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ L_{n1} & L_{n2} & L_{n3} & \dots & L_{nn} \end{bmatrix} \qquad U = L^T = \begin{bmatrix} L_{11} & L_{21} & L_{31} & \dots & L_{n1} \\ 0 & L_{22} & L_{32} & \dots & L_{n2} \\ 0 & 0 & L_{33} & \dots & L_{n3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & L_{nn} \end{bmatrix}$$

Limitaciones de la descomposición de Choleski

Choleski no es muy utilizado porque posee ciertas limitaciones de uso, ya que se aplica a matrices simétricas y emplea raíces cuadradas

Descomposición de Choleski en matrices 3x3

$$L = \begin{pmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{pmatrix} \qquad U = L^T = \begin{pmatrix} L_{11} & L_{21} & L_{31} \\ 0 & L_{22} & L_{32} \\ 0 & 0 & L_{33} \end{pmatrix}$$

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = L \cdot U = L \cdot L^T = \begin{pmatrix} L_{11}^2 & L_{11}L_{21} & L_{11}L_{31} \\ L_{11}L_{21} & L_{21}^2 + L_{22}^2 & L_{21}L_{31} + L_{22}L_{32} \\ L_{11}L_{31} & L_{21}L_{31} + L_{22}L_{32} & L_{31}^2 + L_{32}^2 + L_{33}^2 \end{pmatrix}$$

$$\left\{ \begin{array}{lcl} A_{11} = L_{11}^2 & \rightarrow & L_{11} = \sqrt{A_{11}} \\ A_{21} = L_{11}L_{21} & \rightarrow & L_{21} = A_{21}/L_{11} \\ A_{31} = L_{11}L_{31} & \rightarrow & L_{31} = A_{31}/L_{11} \\ A_{22} = L_{21}^2 + L_{22}^2 & \rightarrow & L_{22} = \sqrt{A_{22} - L_{21}^2} \\ A_{32} = L_{21}L_{31} + L_{22}L_{32} & \rightarrow & L_{32} = (A_{32} - L_{21}L_{31})/L_{22} \\ A_{33} = L_{31}^2 + L_{32}^2 + L_{33}^2 & \rightarrow & L_{33} = \sqrt{A_{33} - L_{31}^2 - L_{32}^2} \end{array} \right\} \rightarrow \text{Generalización} \rightarrow \ldots$$

Descomposición de Choleski en matrices 3x3 (y II)

- Cualquier elemento de la matriz triangular inferior

$$(LL^T)_{ij} = A_{ij} = L_{i1}L_{j1} + L_{i2}L_{j2} + \ldots + L_{ij}L_{jj} = \sum_{k=1}^{j} L_{ik}L_{jk} \rightarrow i \geqslant j \rightarrow \left\{ \begin{array}{c} j = 1, 2, \ldots, n \\ i = j, j+1, \ldots, n \end{array} \right.$$

- Primera columna

$$\left\{ \begin{array}{l} L_{11} = \sqrt{A_{11}} \\ L_{i1} = A_{i1}/L_{11} \rightarrow i = 2, 3, \ldots, n \end{array} \right.$$

- Otra columna

$$A_{ij} = \sum_{k=1}^{j-1} L_{ik}L_{jk} + L_{ij}L_{jj}$$

- Término de la diagonal

$$L_{ij} = \sqrt{A_{jj} - \sum_{k=1}^{j-1} L_{jk}^2} \rightarrow j = 2, 3, \ldots, n$$

- Término no perteneciente a la diagonal principal

$$L_{ij} = \left(A_{ij} - \sum_{k=1}^{j-1} L_{ik}L_{jk} \right) / L_{jj}; \rightarrow \left\{ \begin{array}{c} i = j+1, j+2, \ldots, n \\ j = 2, 3, \ldots, n-1 \end{array} \right.$$

Ejemplo:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

$$A = L \cdot L^T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} L_{11}^2 & L_{11}L_{21} & L_{11}L_{31} \\ L_{11}L_{21} & L_{21}^2 + L_{22}^2 & L_{21}L_{31} + L_{22}L_{32} \\ L_{11}L_{31} & L_{21}L_{31} + L_{22}L_{32} & L_{31}^2 + L_{32}^2 + L_{33}^2 \end{pmatrix} \rightarrow$$

$$\rightarrow \left\{ \begin{array}{rcl} 1 = L_{11}^2 & \rightarrow & L_{11} = \sqrt{1} = 1 \\ 1 = L_{11}L_{21} & \rightarrow & L_{21} = 1/L_{11} = 1/1 = 1 \\ 1 = L_{11}L_{31} & \rightarrow & L_{31} = 1/L_{11} = 1 \\ 2 = L_{21}^2 + L_{22}^2 & \rightarrow & L_{22} = \sqrt{2 - L_{21}^2} = \sqrt{2-1} = 1 \\ 2 = L_{21}L_{31} + L_{22}L_{32} & \rightarrow & L_{32} = (2 - L_{21}L_{31})/L_{22} = (2-1)/1 = 1 \\ 3 = L_{31}^2 + L_{32}^2 + L_{33}^2 & \rightarrow & L_{33} = \sqrt{3 - L_{31}^2 - L_{32}^2} = \sqrt{3-1-1} = 1 \end{array} \right\}$$

SOLUCIÓN: $L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ $\quad U = L^T = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Tipos de matrices dispersas

- Matrices en banda $\rightarrow \begin{pmatrix} X & X & X & \circ & \circ & \circ & \circ \\ X & X & \circ & X & X & \circ & \circ \\ X & \circ & X & \circ & X & \circ & \circ \\ \circ & X & \circ & X & \circ & X & \circ \\ \circ & X & X & \circ & X & X & X \\ \circ & \circ & \circ & X & X & X & \circ \\ \circ & \circ & \circ & \circ & X & \circ & X \end{pmatrix}$
- Matrices diagonales $\rightarrow \begin{pmatrix} X & X & \circ & \circ & \circ & \circ & \circ \\ X & X & X & \circ & \circ & \circ & \circ \\ \circ & X & X & X & \circ & \circ & \circ \\ \circ & \circ & X & X & X & \circ & \circ \\ \circ & \circ & \circ & X & X & X & \circ \\ \circ & \circ & \circ & \circ & X & X & X \\ \circ & \circ & \circ & \circ & \circ & X & X \end{pmatrix}$
- Matrices simétricas

2.7 Matrices dispersas

Almacenamiento de datos

$$A = \begin{pmatrix} d_1 & e_1 & \cdots & 0 \\ c_1 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & c_{n-1} & d_n \end{pmatrix} \rightarrow c = \begin{pmatrix} 0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix}, d = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}, e = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ 0 \end{pmatrix}$$

Factorización LU en matrices tridiagonales

$$A = L \cdot U \rightarrow \begin{pmatrix} X & X & \circ & \circ \\ X & X & X & \circ \\ \circ & X & X & X \\ \circ & \circ & X & X \end{pmatrix} = \begin{pmatrix} 1 & \circ & \circ & \circ \\ X & 1 & \circ & \circ \\ \circ & X & 1 & \circ \\ \circ & \circ & X & 1 \end{pmatrix} \cdot \begin{pmatrix} X & X & \circ & \circ \\ \circ & X & X & \circ \\ \circ & \circ & X & X \\ \circ & \circ & \circ & X \end{pmatrix}$$

Resolución sistema Ax=b siendo A una matriz tridiagonal

$$[L|b] = \left(\begin{array}{ccccc|c} 1 & 0 & 0 & \cdots & 0 & b_1 \\ c_1 & 1 & 0 & \cdots & 0 & b_2 \\ 0 & c_2 & 1 & \cdots & 0 & b_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & c_{n-1} & 1 & b_n \end{array}\right) \qquad [U|y] = \left(\begin{array}{ccccc|c} d_1 & e_1 & 0 & \cdots & 0 & y_1 \\ 0 & d_2 & e_2 & \cdots & 0 & y_2 \\ 0 & 0 & d_3 & \cdots & 0 & y_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & d_n & y_n \end{array}\right)$$

Ejemplo de almacenamiento de matrices tridiagonales:

$$A = \begin{pmatrix} 1 & 4 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

Almacenamiento de datos en vectores:

$$A = \begin{pmatrix} 1 & 4 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \end{pmatrix} \rightarrow c = \begin{pmatrix} 0 \\ 3 \\ 2 \\ 1 \end{pmatrix}, d = \begin{pmatrix} 1 \\ 4 \\ 3 \\ 3 \end{pmatrix}, e = \begin{pmatrix} 4 \\ 1 \\ 4 \\ 0 \end{pmatrix}$$

Ejemplo de factorización LU en matrices tridiagonales:

$$A = \begin{pmatrix} 1 & 4 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \end{pmatrix} \xrightarrow{F_2' \to F_2 - \left(\frac{c_1}{d_1}\right)\cdot F_1} \begin{pmatrix} 1 & 4 & 0 & 0 \\ 0 & -8 & 1 & 0 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \end{pmatrix} \xrightarrow{F_3' \to F_3 - \left(\frac{c_2}{d_2}\right)\cdot F_2}$$

$$\to \begin{pmatrix} 1 & 4 & 0 & 0 \\ 0 & -8 & 1 & 0 \\ 0 & 0 & 3.25 & 4 \\ 0 & 0 & 1 & 3 \end{pmatrix} \xrightarrow{F_4' \to F_4 - \left(\frac{c_3}{d_3}\right)\cdot F_3} U = \begin{pmatrix} 1 & 4 & 0 & 0 \\ 0 & -8 & 1 & 0 \\ 0 & 0 & 3.25 & 4 \\ 0 & 0 & 0 & \frac{23}{13} \end{pmatrix}$$

$$\lambda = \frac{c_{k-1}}{d_{k-1}} \to L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{3}{1} & 1 & 0 & 0 \\ 0 & \frac{2}{-8} & 1 & 0 \\ 0 & 0 & \frac{1}{3.25} & 1 \end{pmatrix}$$

$$A = L \cdot U \to \begin{pmatrix} 1 & 4 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & -\frac{1}{4} & 1 & 0 \\ 0 & 0 & \frac{1}{3.25} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 4 & 0 & 0 \\ 0 & -8 & 1 & 0 \\ 0 & 0 & 3.25 & 4 \\ 0 & 0 & 0 & \frac{23}{13} \end{pmatrix}$$

Ejemplo de resolución Ax=b siendo A una matriz tridiagonal:

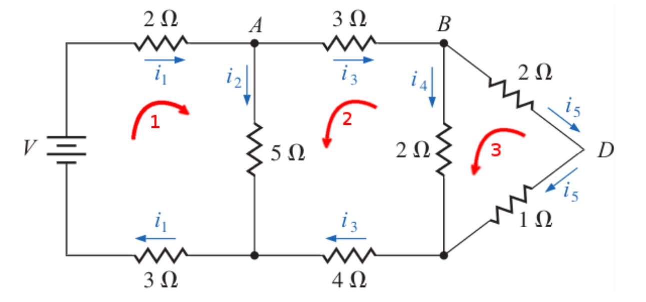
$$[L|b] = \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 3 & 1 & 0 & 0 & 2 \\ 0 & -\frac{1}{4} & 1 & 0 & 3 \\ 0 & 0 & \frac{1}{3.25} & 1 & 4 \end{array}\right) \rightarrow \left\{\begin{array}{l} y_1 = b_1 \\ y_2 = b_2 - y_1 c_1 \\ \vdots \\ y_n = b_n - y_{n-1} c_{n-1} \end{array}\right\} \rightarrow \left\{\begin{array}{l} y_1 = 1 \\ y_2 = -1 \\ y_3 = 11/4 = 2.75 \\ y_4 = 41/13 \end{array}\right.$$

$$[U|y] = \left(\begin{array}{cccc|c} 1 & 4 & 0 & 0 & 1 \\ 0 & -8 & 1 & 0 & -1 \\ 0 & 0 & 3.25 & 4 & 2.75 \\ 0 & 0 & 0 & \frac{23}{13} & \frac{41}{13} \end{array}\right) \rightarrow \left\{\begin{array}{l} x_n = y_n \\ x_{n-1} = \dfrac{y_{n-1} - e_{n-1} y_n}{d_{n-1}} \\ \vdots \\ x_1 = \dfrac{y_1 - e_1 y_2}{d_2} \end{array}\right\} \rightarrow \mathit{SOLUCIONES}$$

SOLUCIONES:

$$x_1 = \frac{27}{13} \qquad x_2 = -\frac{1}{23} \qquad x_3 = -\frac{31}{23} \qquad x_4 = \frac{41}{23}$$

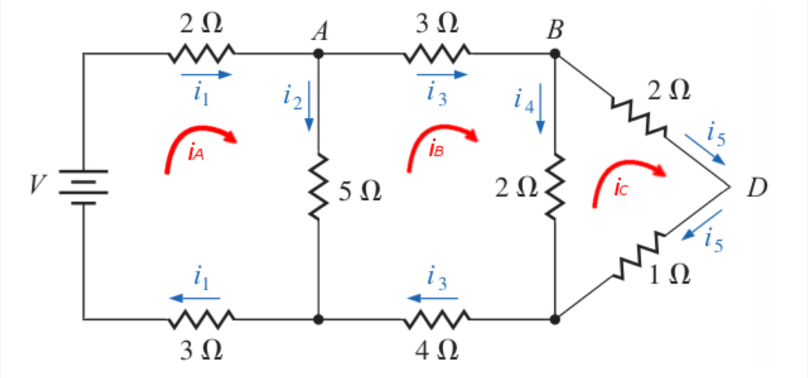
Kirchhoff



$$\left\{ \begin{aligned} 5i_1 + 5i_2 &= 5.5 \\ 5i_2 - 7i_3 - 2i_4 &= 0 \\ 2i_4 - 3i_5 &= 0 \\ i_1 &= i_2 + i_3 \\ i_3 &= i_4 + i_5 \end{aligned} \right\}$$

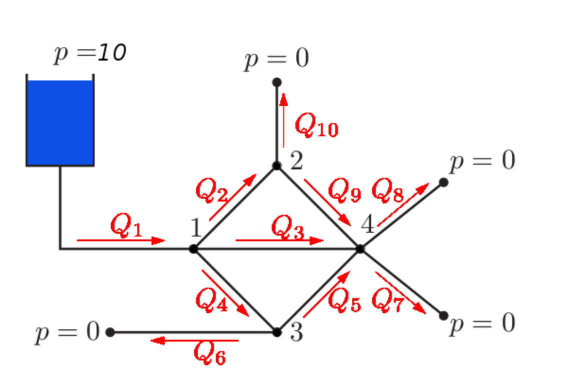
$$\begin{pmatrix} 5 & 5 & 0 & 0 & 0 \\ 0 & 5 & -7 & -2 & 0 \\ 0 & 0 & 0 & 2 & -3 \\ 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \end{pmatrix} \cdot \begin{pmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{pmatrix} = \begin{pmatrix} 5.5 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \left\{ \begin{aligned} i_1 &= 0.6785 \\ i_2 &= 0.4215 \\ i_3 &= 0.2570 \\ i_4 &= 0.1542 \\ i_5 &= 0.1028 \end{aligned} \right\}$$

Mallas



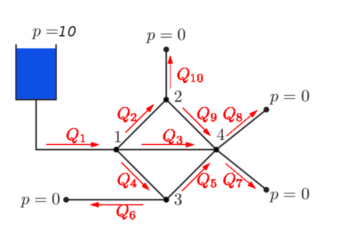
$$\begin{pmatrix} 10 & -5 & 0 \\ -5 & 14 & -2 \\ 0 & -2 & 5 \end{pmatrix} \cdot \begin{pmatrix} i_A \\ i_B \\ i_C \end{pmatrix} = \begin{pmatrix} 5.5 \\ 0 \\ 0 \end{pmatrix} \rightarrow \left\{ \begin{array}{l} i_A = 0.6785 \\ i_B = 0.2570 \\ i_C = 0.1028 \end{array} \right\} \rightarrow \left\{ \begin{array}{l} i_1 = i_A = 0.6785 \\ i_2 = i_A - i_B = 0.4215 \\ i_3 = i_B = 0.2570 \\ i_4 = i_B - i_C = 0.1542 \\ i_5 = i_C = 0.1028 \end{array} \right\}$$

Cálculo de presiones en nudos de circuito hidráulico cerrado



$$
\left.\begin{array}{ll}
\textit{Nudo } 1: & Q_1 = Q_2 + Q_3 + Q_4 \\
\textit{Nudo } 2: & Q_2 = Q_{10} + Q_9 \\
\textit{Nudo } 3: & Q_4 = Q_5 + Q_6 \\
\textit{Nudo } 4: & Q_9 + Q_3 + Q_5 = Q_8 + Q_7
\end{array}\right\} \qquad Q_j = kL\Delta p_j
$$

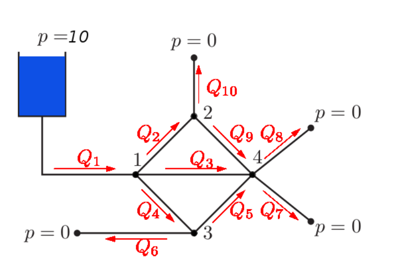
Cálculo de presiones en nudos de circuito hidráulico cerrado (y II)



tuberia	k	L
1	0.01	20
2	0.005	10
3	0.005	14
4	0.005	10
5	0.005	10
6	0.002	8
7	0.002	8
8	0.002	8
9	0.005	10
10	0.002	8

$$\rightarrow Q_j = kL\Delta p_j \rightarrow \begin{cases} Q_1 = 0.2(10 - P_1) \\ Q_2 = 0.05(P_1 - P_2) \\ Q_3 = 0.07(P_1 - P_4) \\ Q_4 = 0.05(P_1 - P_3) \\ Q_5 = 0.05(P_3 - P_4) \\ Q_6 = 0.016(P_3) \\ Q_7 = 0.016(P_4) \\ Q_8 = 0.016(P_4) \\ Q_9 = 0.05(P_2 - P_4) \\ Q_{10} = 0.016(P_2) \end{cases}$$

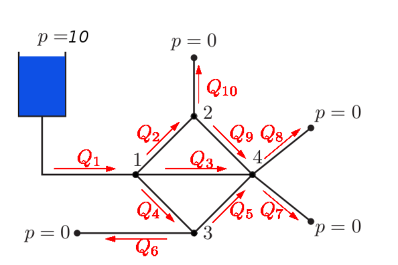
Cálculo de presiones en nudos de circuito hidráulico cerrado (y III)



$$\left\{\begin{array}{l} 0.2(10-P_1)=0.05(P_1-P_2)+0.07(P_1-P_4)+0.05(P_1-P_3) \\ 0.05(P_1-P_2)=0.05(P_2-P_4)+0.016(P_2) \\ 0.05(P_1-P_3)=0.05(P_3-P_4)+0.016(P_3) \\ 0.05(P_2-P_4)+0.07(P_1-P_4)+0.05(P_3-P_4)=0.016(P_4)+0.016(P_4) \end{array}\right\} \rightarrow$$

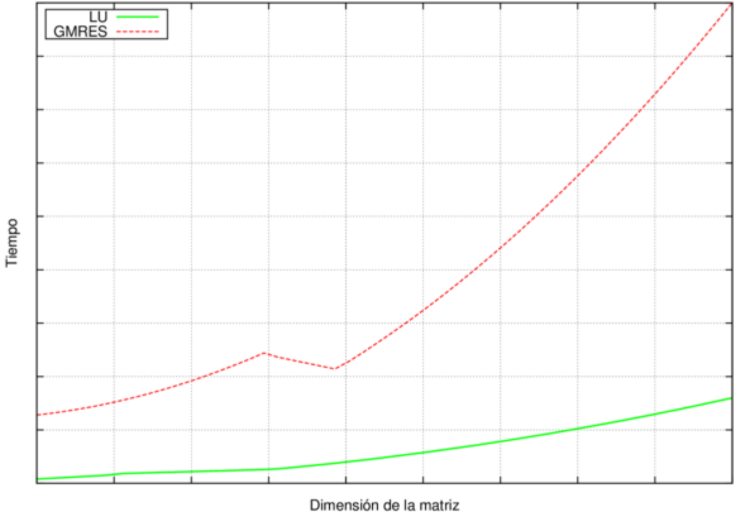
$$\rightarrow \left\{\begin{array}{l} 2-0.2P_1=0.05P_1-0.05P_2+0.07P_1-0.07P_4+0.05P_1-0.05P_3 \\ 0.05P_1-0.05P_2=0.05P_2-0.05P_4+0.016P_2 \\ 0.05P_1-0.05P_3=0.05P_3-0.05P_4+0.016P_3 \\ 0.05P_2-0.05P_4+0.07P_1-0.07P_4+0.05P_3-0.05P_4=2\cdot 0.016P_4 \end{array}\right\} \rightarrow$$

Cálculo de presiones en nudos de circuito hidráulico cerrado (y IV)

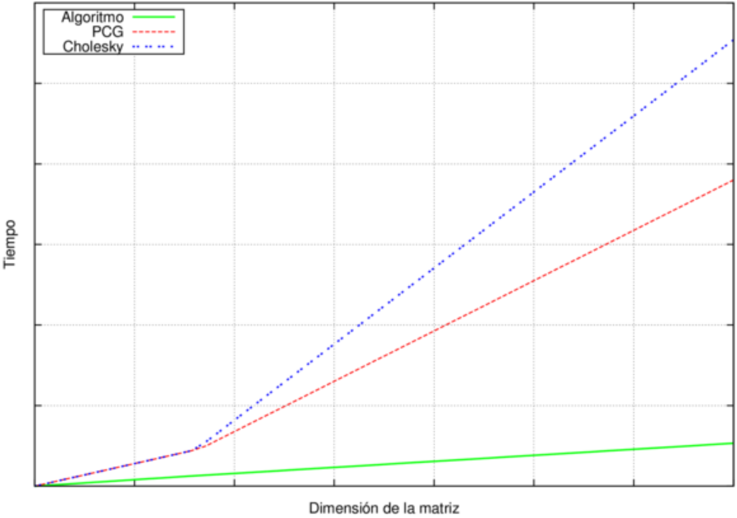


$$\left\{ \begin{array}{r} -0.37P_1 + 0.05P_2 + 0.05P_3 + 0.07P_4 = -2 \\ 0.05P_1 - 0.016P_2 + 0.05P_4 = 0 \\ 0.05P_1 - 0.116P_3 + 0.05P_4 = 0 \\ 0.07P_1 + 0.05P_2 + 0.05P_3 - 0.202P_4 = 0 \end{array} \right\}$$

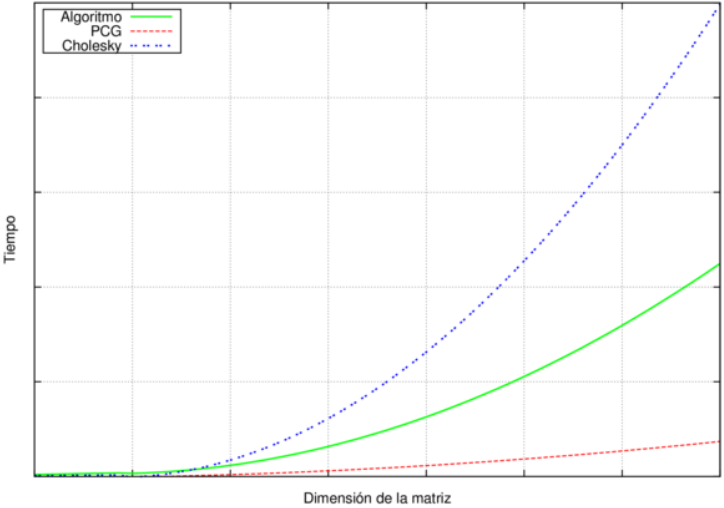
$$\begin{pmatrix} -0.37 & 0.05 & 0.05 & 0.07 \\ 0.05 & -0.016 & 0 & 0.05 \\ 0.05 & 0 & -0.116 & 0.05 \\ 0.07 & 0.05 & 0.05 & -0.202 \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \left\{ \begin{array}{l} P_1 = 8.1172 bar \\ P_2 = 5.9893 bar \\ P_3 = 5.9893 bar \\ P_4 = 5.7779 bar \end{array} \right.$$


LU
GMRES
Tiempo
Dimensión de la matriz

4.0 ¿Directo o iterado? - Matriz de banda estrecha

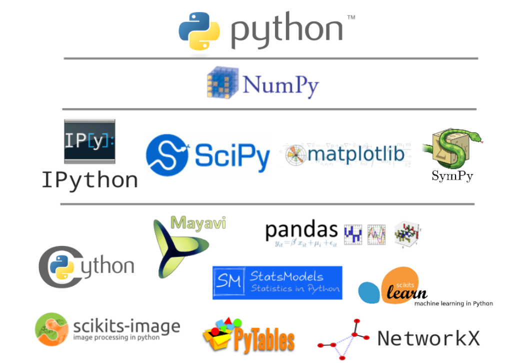


4.0 ¿Directo o iterado? - Matriz de banda ancha



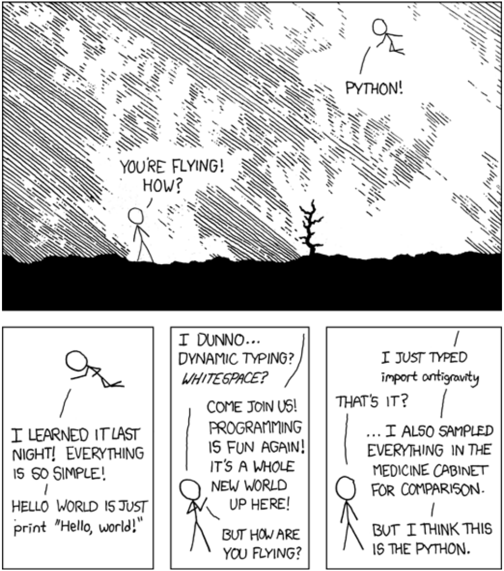
Python

- Python tiene una sintaxis clara
- Python es de propósito general
- Python es dinámico
- Python es amigo de C/C++ y Fortran
- Python es libre



Cómo comenzar en Python

- Tutorial oficial:
  http://docs.python.org.ar/tutorial/2/contenido.html
- Pybonacci: blog de Python científico
- Aeropython:
  https://github.com/AeroPython/Curso_AeroPython
- Python para ingenieros:
  http://cacheme.org/curso-online-python-cientifico-ingenieros/
- Lorena Barba: profesora el GWU
- Numerical MOOC:
  http://openedx.seas.gwu.edu/courses/GW/MAE6286/2014_fall/about
- Stack Overflow:
  http://stackoverflow.com/


YOU'RE FLYING!
HOW?
PYTHON!
I LEARNED IT LAST NIGHT! EVERYTHING IS SO SIMPLE!
HELLO WORLD IS JUST print "Hello, world!"
I DUNNO... DYNAMIC TYPING? WHITESPACE?
COME JOIN US! PROGRAMMING IS FUN AGAIN! IT'S A WHOLE NEW WORLD UP HERE!
BUT HOW ARE YOU FLYING?
I JUST TYPED import antigravity
THAT'S IT?
... I ALSO SAMPLED EVERYTHING IN THE MEDICINE CABINET FOR COMPARISON.
BUT I THINK THIS IS THE PYTHON.

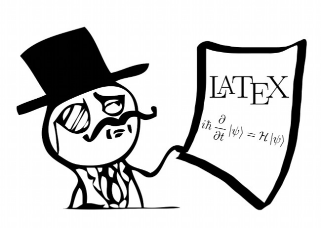
LaTeX

Ventajas:

- Es estable y multiplataforma
- Alta calidad en la edición de ecuaciones
- Facilita la creación de documentos estructurados
- Es gratis

Inconvenientes:

- Dificultad avanzada y curva de aprendizaje lenta
- No se ven los resultados hasta que se compila el archivo



Cómo comenzar en LaTeX

- TeXmacs → LyX
- TeXStudio / TeXLive
- ShareLatex
  `https://es.sharelatex.com`
- Pagina TeX español...
  `http://www.cervantex.es`
- ...y lo que mas útil es, su sección de FAQ
  `http://www.aq.upm.es/Departamentos/Fisica/agmartin/webpublico/latex/FAQ-CervanTeX/FAQ-CervanTeX.html`
- Videotutoriales en castellano:
  `https://es.sharelatex.com/blog/latex-guides/beginners-tutorial.html`
- Tutorial desde cero:
  `http://mate.dm.uba.ar/~pdenapo/tutorial-latex/tutorial-latex.html`
- Stack Overflow:
  `http://stackoverflow.com/`

Bibliografía

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- MOIN, P; 2010. Fundamentals of engineering - Numerical Analysis; Cambridge
- QUARTERONI, A; SALERI, F; 2006. Cálculo matemático con MATLAB y Octave; Springer
- FANGOHR, H; 2014. Python for computational science and engineering; Southampton