On The Infinity of Twin Primes and other K-tuples

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December 13, 2019

Abstract

The paper uses the structure and math of Prime Generators to show there are an infinity of twin primes, proving the Twin Prime Conjecture, as well as establishing the infinity of other k-tuples of primes.

1 Introduction

In number theory ***Polignac's Conjecture*** (1849) [6] states there are infinitely many consecutive primes (prime pairs) that differ by any even number n. The ***Twin Prime Conjecture*** derives from it for prime pairs that differ by 2, the so called twin primes, e.g. (11, 13) and (101, 103).

K-tuples are groupings of primes adhering to specific patterns, usually designated as (k, d) groupings, where k is the number of primes in the group and d the total spacing between its first and last prime [4]. Thus, Polignac's pairs are type (2, n), where n is any even number, and twin primes are the specific named k-tuples of type (2, 2).

Various types of k-tuples form a constellation of groupings for $k \geq 2$. Triplets are type (3, 6) which have two forms, (0, 2, 6) and (0, 4, 6). The smallest occurrence for each form are (5, 7, 11) and (7, 11, 13).

Some k-tuples have been given names. Three named (2, d) tuples are Twin Primes (2, 2), Cousin Primes (2, 4), and Sexy Primes (2, 6). The paper shows there are many more Sexy Primes than Twins or Cousins, though an infinity of each.

The nature of the proof presented herein takes a totally different approach than usually taken. It is a straightforward ***proof by logic*** derived from the natural structure and properties of mathematical expressions I've named ***Prime Generators (PG)***, which as their name implies, generate all the primes. Each larger PG is more efficient by reducing the ***number space*** primes can possibly exist within. They thus structurally squeeze the primes into a smaller set of integers that contain fewer composites.

Each PG has a characteristic ***Prime Generator Sequence (PGS)***, a repeating pattern of gaps between the ***residue*** elements of its PG. These gap patterns reveal, and adhere to, a ***deterministic structured set of properties***. I systematically use these properties to show that once a prime gap of any even size comes into existence it will be repeated forever for all other PGS. This will be used to establish the infinity of twin pairs, and other k-tuples. I provide data and graphs to empirically show this.

At the time of writing, the largest known twin prime is $2996863034895 \cdot 2^{1290000} \pm 1$ [5] (2016), which resides on ***restracks*** P5[29:31] and P7[29:31] for those PG. There are an infinity of larger twin primes, which will reside on some twin pair restracks for every PG. The same will be true for other k-tuples.

I have previously used Prime Generators to construct, and implement in software, efficient and very fast ***prime sieves***, to find all the primes up to a finite N, or within a finite range, including the fastest and most efficient prime sieve method to find twin primes. See [1], [2], [3]

2 Prime Generators

A prime generator **Pn** is composed of a modulus **modpn** and a set of residues $\boldsymbol{r_i}$ with residue count **rescntpn** (determined by the Euler's Totient Function, $\phi(n)= n\prod(1-1/p_i)$) which have the form:

$$P_n = modpn \cdot k + \{r_i\} \tag{1}$$

$$modpn = p_n\# = \prod p_i = 2 \cdot 3 \cdot 5 \cdot ... \cdot p_n \tag{2}$$

$$rescntpn = (p_n - 1)\# = \prod (p_i - 1) = (2-1) \cdot (3-1) \cdot (5-1) \cdot ... \cdot (p_n - 1) \tag{3}$$

where $\boldsymbol{p_n}$ is the last PG prime. A PG's **residues** are the set of integers $\boldsymbol{r_i}$ **ε {1...modpn-1}** coprime (no common factors) to its **modpn**, i.e. their greatest common divisor is 1: gcd(r_i, modpn) = 1. They exist as ***modular complement pairs***, such that **modpn** $= \boldsymbol{r_i} + \boldsymbol{r_j}$ and therefore $(\boldsymbol{r_i} + \boldsymbol{r_j})$ **mod modpn** $\equiv \mathbf{0}$. Thus, we only need to generate the residues $\boldsymbol{r_i} <$ **modpn/2**, and the other half are $\boldsymbol{r_j}$ = **modpn -** $\boldsymbol{r_i}$.

For P5 then, modp5 = 2 · 3 · 5 = 30, with rescntp5 = (2 - 1) · (3 - 1) · (5 - 1) = 8. P5's 8 residues are {1, 7, 11, 13, 17, 19, 23, 29}, which are used as {7, 11, 13, 17, 19, 23, 29, 31}, to always have the first residue in its sequence be prime, with the last set to **1** ≡ **(modpn + 1) mod modpn**. Thus we have:

$$P5 = 30 \cdot k + \{7, 11, 13, 17, 19, 23, 29, 31\} \tag{4}$$

We can now construct P5's **prime candidates (pc) table**, here up to N = 541, the 100th prime, where each **k ≥ 0** index **residue group (resgroup)** contains pc values along each **residue track (restrack|rt)**.

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
rt0	7	37	67	97	127	157	187	217	247	277	307	337	367	397	427	457	487	517
rt1	11	41	71	101	131	161	191	221	251	281	311	341	371	401	431	461	491	521
rt2	13	43	73	103	133	163	193	223	253	283	313	343	373	403	433	463	493	523
rt3	17	47	77	107	137	167	197	227	257	287	317	347	377	407	437	467	497	527
rt4	19	49	79	109	139	169	199	229	259	289	319	349	379	409	439	469	499	529
rt5	23	53	83	113	143	173	203	233	263	293	323	353	383	413	443	473	503	533
rt6	29	59	89	119	149	179	209	239	269	299	329	359	389	419	449	479	509	539
rt7	31	61	91	121	151	181	211	241	271	301	331	361	391	421	451	481	511	541

Fig 1.

A pc table of prime candidates can be created for every PG. All the primes $> p_n$ occur along the residue restracks in a statistically uniform manner. The marked cells are prime multiples (composites) of only the residue primes, which can be sieved out to reveal the primes within any range. See [1], [2]. P5 is the largest Pn for which all its residues are prime. All larger will also have residues consisting of multiples of its prime residues < modpn.

3 Prime Generator Sequences

Each prime generator has a characteristic ***Prime Generator Sequence (PGS)***. This is the sequence of the differences (gaps) between consecutive residues defined over the range $\boldsymbol{r_0}$ to $\boldsymbol{r_0}$ + **modpn** where $\boldsymbol{r_0}$ is the first residue of Pn, and the next prime $> p_n$.

Let's construct the first prime generator P2, and its PGS.

For P2: modp2 = 2, with rescntp2 = (2 - 1) = 1, with residue {1}, but use its congruent value {3}.

Thus, P2 $= 2 \cdot k + 3$, produces the pc sequence: 3 5 7 9 11 13 15 17... ∞, i.e the odd numbers. So for P2, its PGS is a single element of gap size ($\boldsymbol{r_0}$ - 1) = (3 - 1) = 2: PGS P2: [$\boldsymbol{r_0}$ = 3] 2 |

Now let's construct P3: modp3 = 2 · 3 = 6; rescntp3 = (2 - 1) · (3 - 1) = 2, with residues {1, 5}. P3, thus, has the functional form: P3 = 6·k + {5, 7}. Its pcs table is shown below up to k = 16.

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
rt0	5	11	17	23	29	35	41	47	53	59	65	71	77	83	89	95	101
rt1	7	13	19	25	31	37	43	49	55	61	67	73	79	85	91	97	103

Fig 2.

For P3, each resgroup (column) contains prime candidates forming a possible twin pair, extending into infinity. Execpt for (3, 5), every twin prime can be written as 6n ± 1 for some n ≥ 1 values.

The last two residues for all prime generators > P2 are **modpn ± 1**, thus they have at least one twin pair set of residues. For larger prime generators there are more twin pair residues, and others. To illustrate this, we examine the PGS for increasing prime generators Pn.

For P3 we see its PGS contains the gaps 2 and 4, which occur one each, with the last ($\boldsymbol{r_0}$ - 1) = 4.

```
PGS P3: 5  7  11  13  17  19  23  25  29  31  35 ...  ∞
          2   4 |  2   4 |  2   4 |  2   4 |  2   4 |
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For P5 we see from Fig 1. its sequence of prime candidates, with its PGS spacing.

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PGS P5: 7  11  13  17  19  23   29  31  37  41  43  47  49  53  59  61  67 ...  ∞
          4   2   4   2   4   6   2   6 |  4   2   4   2   4   6   2   6 |
```

Again we see the gaps 2 and 4 occurring with the same (odd) frequency, with the last three gaps now having the form **($\boldsymbol{r_0}$ - 1) 2 ($\boldsymbol{r_0}$ - 1)**, where $\boldsymbol{r_0}$ = 7 is the first residue for P5.

We are beginning to see some of the inherent properties of prime generators emerge. Each larger Pn (P7, P11, P13, P17, etc) will conform to these properties, producing an increasing number of gaps, with a defined number of specific gap sizes, distributed within the sequence in a structured manner.

4 Characterizing PGS

Each prime generator sequence is defined over the range $\boldsymbol{r_0}$ to $\boldsymbol{r_0}$ + **modpn**, therefore the number of gaps equals the number of residues, and the sum of the gap sizes equals the modulus. Let $\boldsymbol{a_i}$ be the frequency coefficients for each gap of size $2i$, $i \geq 1$, thus:

$$rescntpn = \sum a_i \tag{5}$$

$$modpn = \sum gap_i = \sum a_i \cdot 2i \tag{6}$$

Therefore for PGS P3: [$\boldsymbol{r_0}$ = 5] 2 4 | – modp3 = 6 = (1) · 2 + (1) · 4
and PGS P5: [$\boldsymbol{r_0}$ = 7] 4 2 4 2 4 6 2 6 | – modp5 = 30 = (3) · 2 + (3) · 4 + (2) · 6

For P7, modp7 = modp5 · 7 = 210, and rescntp7 = rescntp5 · (7 - 1) = 48, with the residues:

{11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 121, 127, 131, 137, 139, 143, 149, 151, 157, 163, 167, 169, 173, 179, 181, 187, 191, 193, 197, 199, 209, 211}

PGS P7: [$\boldsymbol{r_0}$ = 11] 2 4 2 4 6 2 6 4 2 4 6 6 2 6 4 2 6 4 6 8 4 2 4 2
4 8 6 4 6 2 4 6 2 6 6 4 2 4 6 2 6 4 2 4 2 10 2 10 |

With: modp7 = 210 = (15) · 2 + (15) · 4 + (14) · 6 + (2) · 8 + (2) · 10

Again we see for P7, there are an equal odd number of occurrences for gaps 2 and 4. This is observed to be true for every odd numbered prime generator, where coefficients $a_1 = a_2$ have form:

$$a_1 = a_2 = (p_n - 2)\# = \prod (p_{2+i} - 2) = (3 - 2) \cdot (5 - 2) \cdot (7 - 2) \cdot \ldots \cdot (p_n - 2) \tag{7}$$

We also see the consistent pattern that the last gap term is **($\boldsymbol{r_0}$ - 1)**, and starting with P5, the last three gaps have the pattern **($\boldsymbol{r_0}$ - 1) 2 ($\boldsymbol{r_0}$ - 1)**. This occurs because the last two residues are always twin pairs of form **modpn ± 1**, and the second from last is the modular complement of $\boldsymbol{r_0}$, i.e. **(modpn - $\boldsymbol{r_0}$)**.

We now also notice that the number of unique gap sizes for each generator Pn are of order p_{n-1}. Through observation of increasing Pn this is seen as a consistent property (for nonzero coefficients). Thus the PGS for P3 has two (2) gaps, for P5 three (3) gaps, for P7 five (5) gaps sizes, and so on.

5 PGS Symmetry and Distribution

Because the residues exist as ***modular complement pairs***, they produce a mirror image gap distribution around a midpoint pivot term. The PGS pattern up to the pivot will exist as its mirror image after.

Starting with P5, we know the last 3 gaps for all Pn have the form **($\boldsymbol{r_0}$ - 1) 2 ($\boldsymbol{r_0}$ - 1)**, thus their sum is **2$\boldsymbol{r_0}$**, and the remaining odd number **(rescntpn - 3)** gaps must equal **(modpn - 2$\boldsymbol{r_0}$)**.

This requires for P5, the (8 - 3) = 5 gaps at the front of its PGS must sum to (30 - 2·7) = 16. If all the gaps were 2 you would need 8, which is too many, if all were 4 you need just 4, which is too few. So the structure adapts to generate the necessary combination of gap sizes to satisfy both requirements.

In addition, these (rescntpn - 3) odd gaps exist with a symmetric mirror image distribution around a mid pivot gap that is always of size 4.

To show this, excluding the last 3 term of PGS P5 we have the gap sequence: 4 2 **4** 2 4
Here the terms 4 2 are the mirror image of 2 4 and are symmetric around midterm **4**.

For PGS P7 we get: 2 4 2 4 6 2 6 4 2 4 6 6 2 6 4 2 6 4 6 8 4 2 **4**
2 4 8 6 4 6 2 4 6 2 6 6 4 2 4 6 2 6 4 2 4 2

and again see a similar mirror image symmetry of each half around the midterm 4.

For P7, in order for the (48 - 3) = 45 gaps in its PGS front to sum to (210 - 2·11) = 188 we see new gaps of 8 are introduced (mirrored in both halves) close to the middle pivot point.

As the PG moduli increase, new larger gaps will emerge and be included toward the pivot element. This amounts to pushing the preexisting gaps toward the front and back. This ***expansion process*** ensures all preexisting residue gaps will eventually exist for the primes $< \boldsymbol{r_0}^2$ for some Pn.

The reason $a_1 = a_2$ are odd is because gap size 4 is the pivot term and a gap 2 is part of the last three sequence terms. Every other gap term is part of each mirror image and occur in even numbers. Thus as similar to the residues, we only need to (computationally) determine the first (rescntpn - 4)/2 gap terms.

6 The Infinity of Primes

Starting with just the first two primes 2 and 3, we can show the ***infinite progression of primes***.

Using the first two primes we create: P3 = 6·k + {5, 7}, k ≥ 0.

From Fig 2. the pc $< \boldsymbol{r_0}^2 = 5^2 = 25$ are prime, which are the primes {5, 7, 11, 13, 17, 19, 23}.

We now use the new found primes 5. . . 23 to construct P23, with modp23 = 223092870, whose $\boldsymbol{r_0} = 29$. All the residues between 29 and $29^2 = 841$ will be primes. The primes counting function $\pi(x)$ tells us there are exactly 137 primes from 29...841, the last being 839. ***We can now repeat this process as many times as we want to find new primes, into infinity***.

Thus, any prime p can be treated as $\boldsymbol{r_0}$ of a Pn composed of all the primes $< p$. All the residues of such Pn from p to p^2 are new primes, and we can repeat this ***progression of primes*** process forever, always sure we will generate new primes. Thus we've established this process will generate all the primes, into infinity.

In fact, an estimate of the number of new primes generated in any range p to p^2 will be of order:

$$\pi_{est}(p, p^2) = \frac{p^2}{log(p^2)} - \frac{p}{log(p)} = \frac{p \cdot (p-2)}{2 \cdot log(p)} \tag{8}$$

For $p = 29$, this produces an estimate of 116 primes from 29 to 841, compared to the actual of 137.

7 Prime Generator Properties

Given what we've observed, and now know about prime generators and their sequences, we can codify their inherent and immutable properties, and use them in a logically consistent manner to empirically establish and project the nature, numbers, and distribution of all prime gap k-tuples.

Though simple mathematical expressions, prime generators reveal an astounding breadth of knowledge about the nature of prime numbers, embedded in their inherent immutable properties. When I refer to their properties as being 'inherent' these are natural aspects and characteristics of their structure that are discernible primarily through visual observation. Once observed though, I could then mathematically describe them, and formulate a consistent framework for application.

As an example, it is an inherent property of base ten numbers that the *least significant digit* (*lsd*) of an even integer must (only) be the digits, 0, 2, 4, 6, 8, and conversely 1, 3, 5, 7, 9 for odd. However when we change the base system, say to a binary (base two) system, even|odd has a different expression, i.e. the *least significant bit* (*lsb*) of an even number is a '0' and a '1' for odd. We performed no calculation to determine this, these are observable characteristics that are inherently associated with the concepts of even and odd for each base system.

Using these inherent properties of even|odd for base ten numbers, we can apply them through observation to 'prime' numbers. It is an inherent property of prime numbers that, other than for the prime 2, all others are odd, which means their *lsd* aren't 0, 2, 4, 6, or 8. So by mere observation you know 341786 isn't prime. You didn't *need* to perform a calculation to confirm this, if you understood this natural inherent property of prime numbers it's observably obvious.

Also, other than for the prime 5, all other primes *lsd* can only be 1, 3, 7, or 9. This means at minimum 60% of all integers (those with *lsd* of 0, 2, 4, 5, 6, and 8) can't be primes. This is an inherent property of numbers. If you know a little bit more number theory, you'd know that while 11 and 101 could be primes (they are) 111, 1011, and 1101 observably could not. Why? Because for base ten numbers, if the sum of their digits is a mutiple of 3 then it's divisible by 3, and thus not prime.

Thus it is an inherent property of Twin Primes their *lsd* can only be $\{1, 3\}$, $\{7, 9\}$, or $\{9, 1\}$ e.g. for (11, 13), (17, 19), and (29, 31). It's also inherent for all prime numbers > 2, the gaps between them are even because each is odd. You don't have to 'prove' this (though the proof is simple), it's an inherent property of their oddness.

Thus, when I refer to the inherent properties of prime generators, these are observable characteristics and patterns that fall out naturally from their stucture which I have mathematically codified. They are also immutable because they are the same for all generators constructed as shown, and can't change.

Probably the most amazing consequence of creating a modulus of the product of consecutive primes is the unleashing of the otherwise hidden properties that can explain the fabric of prime numbers. However to do this, ***you have to draw pictures***, e.g. Fig 1. and generator sequences, and produce enough examples to reveal their patterns. ***You cannot just think these properties into existence, you have to observe them***.

Now that I have described and given examples of prime generators and their sequences, I will list their observable inherent properties, which I have codified into a mathematically consistent framework for application.

Major Properties of Prime Generators

- the modulus of every prime generator with last prime $\boldsymbol{p_n}$ has primorial form: **modpn** $= \boldsymbol{p_n}\#$
- the number of residues are even with form: **rescntpn** $=$ **($\boldsymbol{p_n}$ - 1)**$\#$
- the residues occur as ***modular complement pairs*** to its modulus: **modpn** $= \boldsymbol{r_i} + \boldsymbol{r_j}$
- the last two residues of a generator are constructed as: **(modpn - 1) (modpn + 1)**
- the residues include all the ***coprime primes*** up to **modpn**
- the first residue $\boldsymbol{r_0}$ is the next prime $> \boldsymbol{p_n}$
- the residues from $\boldsymbol{r_0}$ to $\boldsymbol{r_0}^2$ are primes
- each prime generator has a ***characteristic sequence*** of even sized residue gaps
- the last 3 sequence gaps have form: **($\boldsymbol{r_0}$ - 1) 2 ($\boldsymbol{r_0}$ - 1)**
- the gaps are distributed in a ***symmetric mirror image*** around a pivot gap of size $\underline{\mathbf{4}}$
- the residue gaps sum from $\boldsymbol{r_0}$ to $\boldsymbol{r_0}$ + **modpn** equals the modulus: **modpn** $= \boldsymbol{\Sigma a_i \cdot 2i}$
- the coefficients $\boldsymbol{a_i}$ are the ***frequency*** of each gap of size $\boldsymbol{2i}$
- the sum of the coefficients $\boldsymbol{a_i}$ equal the number of residues: **rescntpn** $= \boldsymbol{\Sigma a_i}$
- coefficients $\boldsymbol{a_1} = \boldsymbol{a_2}$ are odd and equal with form: $\boldsymbol{a_1} = \boldsymbol{a_2} =$ **($\boldsymbol{p_n}$ - 2)**$\#$
- the coefficients $\boldsymbol{a_i}$ are even for $\boldsymbol{i} > \mathbf{2}$
- the number of nonzero coefficients $\boldsymbol{a_i}$ in the sequence for **Pn** is of order $\boldsymbol{p_{n-1}}$

These inherent and immutable properties form a bounded set of constraints which characterize the formation and distribution of primes, and thus also the distribution of all their prime k-tuples.

These discrete mathematical properties and operations form a striking correlation to calculus, where for distance x(t) its first derivative is velocity = dx(t)/dt and its second derivative is acceleration = dv(t)/dt. For prime generators, distance is the number span covered by modpn, and its derivative are the number of residues|gaps. Taking the derivative of the number of gaps gives us the actual gap size coefficients.

Calculus	Prime Generators
x(t) = ∫ v(t)/dt	modpn $= \Sigma a_i \cdot 2i = \prod p_i$
v(t) = ∫ a(t)/dt	rescntpn $= \Sigma a_i = \prod (p_i$ - 1)
a(t) = ∫ A(t)/dt	$a_1 = a_2 = \prod (p_i$ - 2)

While calculus integration is analogous to discrete summation, it is not intuitive that discrete summation correlates to ***primorial operators*** for prime generators. Or is it? Actually we see a similar relationship with the Riemann Zeta series and its equivalent Euler primes product form.

$$\sum \frac{1}{n^s} = \prod \left(1 - p^{-s}\right)^{-1} = \frac{\prod p^s}{\prod (p^s - 1)} \Rightarrow \frac{modp^s}{rescntp^s} \tag{9}$$

8 The Infinity of Twin Primes and other k-tuples

The ***simplest*** way to establish the infinity of k-tuples is to merely show their progression within the range $\boldsymbol{r_0}$ to $\boldsymbol{r_0^2}$ for increasing Pn. Since all these residues are primes their gaps constitute k-tuples. As Pn increases, more gaps of coefficients a_i will eventually come into the range for any even gap size, which we can then directly examine and count.

We start by noting again, the residue gaps sum equals modpn for all Pn, and have the form:

$$modpn = \sum a_i \cdot 2i = a_1 \cdot 2 + a_2 \cdot 4 + a_3 \cdot 6 + ... + a_n \cdot 2n \tag{10}$$

Polignac's conjecture is thus equivalent to stating: for all residue gap coefficients a_i once they come into existence (become > 0) for some Pn, they will remain > 0 (in fact increase) for all larger Pn, and eventually appear and remain within the interval $\boldsymbol{r_0}$ to $\boldsymbol{r_0^2}$ (containing all primes) for all larger Pn.

We've previously seen the PGS for P2 to P7, and here list their coefficients sum form, with P11.

modp2 $= 2 = (1) \cdot 2$

modp3 $= 6 = (1) \cdot 2 + (1) \cdot 4$

modp5 $= 30 = (3) \cdot 2 + (3) \cdot 4 + (2) \cdot 6$

modp7 $= 210 = (15) \cdot 2 + (15) \cdot 4 + (14) \cdot 6 + (2) \cdot 8 + (2) \cdot 10$

modp11 $= 2310 = (135) \cdot 2 + (135) \cdot 4 + (142) \cdot 6 + (28) \cdot 8 + (30) \cdot 10 + (8) \cdot 12 + (2) \cdot 14$

For P7 with $\boldsymbol{r_0} = 11$, we see its prime residues from 11 to 121 are {11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113}, which show prime gaps of 2, 4, 6, and 8 in the range. Even though P7's residues includes other primes, e.g. the cluster {191, 193, 197, 199}, they're outside this range, but eventually will fall within $\boldsymbol{r_0}$ to $\boldsymbol{r_0^2}$ for a larger Pn and be accounted for.

Continuing, P11 extends the range up to 169, finding new primes {127, 131, 137, 139, 149, 151, 157, 163, 167}, which also contain gaps 2, 4, 6, 8, with new gaps 14 (113, 127) and 10 (139, 149). ***This process can be logically continued indefinitely***, to always find new primes, and new larger prime gaps.

Fig 3. shows the frequency coefficients a_i for the first few PG. It shows once a gap coefficient comes into existence for a Pn it increases in frequency for every larger Pn. (There are other interesting properties shown too, though they're not necessary to elucidate upon for the purposes here.) We note every Pn preserves and increases the frequency of prior existing gaps, while creating new larger gaps.

Fig 4. shows the declining percentage of residues in $\boldsymbol{r_0}$ to $\boldsymbol{r_0^2}$ as Pn become larger. This affects the rate of gaps a_i coming into range, but not their eventuality.

Fig 5. shows empirically how the gap sizes, and the max gap, grow over increasing ranges from $\boldsymbol{r_0}$ to $\boldsymbol{r_0^2}$ for larger Pn. Here again, the key features shown are the steady increasing frequency of every gap size once it comes into range for some Pn, and the increase in the max gap within each range.

Thus while the process is not rapid, it is conceptually and logically unequivocal for establishing the infinity of all possible k-tuples. (It's computationally more efficient to use a fast prime sieve, such as in [1], to actually identify|count twin pairs, et al.)

Because coefficients $a_1 = a_2$ have a clear deterministic expression for all Pn, we can formulate a good estimate for prime gaps 2 and 4 (Twins|Cousins) for all Pn. We can simply say it's the percentage of their gaps to its residue count times the number of primes from $\boldsymbol{r_0}$ to $\boldsymbol{r_0}^{\mathbf{2}}$, i.e. $\pi(p, p^2)$. For computational simplicity we can use $\pi_{est}(p, p^2) = p \cdot (p-2)/2 \cdot log(p)$, for a weaker estimate.

$$Twins|Cousins\ count \simeq (a_1/rescntpn) \cdot \pi(p, p^2) \tag{11}$$

If we substitute the expressions for $\boldsymbol{a_1}$, **rescntpn**, and $\boldsymbol{\pi_{est}(p, p^2)}$ we get:

$$Twins|Cousins\ count \simeq \frac{\prod(p_i - 2)}{\prod(p_i - 1)} \cdot \frac{p \cdot (p-2)}{2 \cdot log(p)} \tag{12}$$

To verify it works, let's first use the parameters for P7, with $\boldsymbol{r_0} = p = 11$, rescntp7 = 48, and a_1= 15. The actual primes count π(11,121) = 26, thus: Twins|Cousins count $\simeq$ (15·26)/48 = 8.15. Using the weaker primes estimate of (11)(11 - 2) / 2·log(11), we get (15)(11)(9) / 96·log(11) = 6.45 Twins|Cousins primes. We see previously for P7 (and Fig 5.) the actual Twins|Cousins counts are 8|9 in the range 11 to 121, thus we get good (realistic) estimates from both calculations.

To test for a larger range, let's use P97, whose $\boldsymbol{r_0} = p = 101$.

rescntp97 = $\prod(p_i - 1) = (2-1) \cdot (3-1) \cdot (5-1) \cdot \ldots \cdot (97-1)$ = 2773996900427737839953078806118400000

$a_{1|2} = \prod(p_{2+i} - 2) = (3-2) \cdot (5-2) \cdot (7-2) \cdot \ldots \cdot (97-2)$ = 441482155429401516282749679126093 75

$\pi(101, 101^2) = 1227$ $\qquad\qquad$ $\pi_{est}(101, 101^2) = (101) \cdot (99)/2 \cdot log(101) = 1083.3$

Stong estimate: Twins|Cousins $\simeq$ (a_1/rescntp97)·1227 = 195.3
Weaker estimate: Twins|Cousins $\simeq$ (a_1/rescntp97)·1083 = 172.4

From Fig 5. we see the computed Twins|Cousins counts are 220|197 in the range 101 to 101^2.

Thus, any prime p can be considered $\boldsymbol{r_0}$ of a Pn whose range p to p^2 will always contain Twins|Cousins. Thus it's empirically logical there must be an infinity of Twins|Cousins primes, because within any finite range p to p^2 we know we can empirically compute a good (minimum) estimate of their numbers, for the infinity of primes p.

The properties of PGS also logically mandate the same outcome for every residue gap coefficient a_i as they too are the consequences of the structure of the residues count, their modular complement pairing, and mirror image distribution within all PGS. Once an a_i comes into existence it can not then disappear (go to 0) because that would mean the residue structure of some PGS would have to change (mutate). Thus, as there are an infinity of primes there are also an infinity of all even gaps a_i among them.

9 Proof By Contradiction

To say there are ***not an infinity*** of all k-tuples (i.e. there is a finite number) means empirically for some a_i it becomes, and remains, zero (0) for some Pn, and all larger. This empirically requires some Pn residues structure to change, i.e. mutate. How can a PG structure possibly mutate?

For residues gaps of 2 and 4, i.e. for gap coefficients $a_1 = a_2$, this is clearly a contradiction, as they conform to a deterministic relationship solely based on the modulus primes, and clearly only (must) increase for increasing Pn. Thus $a_1 = a_2$ can never go to zero (0), and thus there will always be an increasing number of their residue gaps, which will always exist within the range $\boldsymbol{r_0}$ to $\boldsymbol{r_0^2}$ for every Pn.

To say the residue structure of a PG will (can) mutate requires either the number of residues no longer are determined by the Euler totient function (which means there are either more, or fewer, residues per modulus), and/or the residues no longer adhere to the modular complement property (which means the residues gaps distribution symmetry has changed). Can this logically|theoretically happen?

Until otherwise shown not to be true, let's accept the Euler totient function works as formulated, thus the number of residues can't mutate. This leaves only the possibility that the residues stop adhering to the modular complement property, and thus mutate the distribution symmetry of the residue gaps.

However, we know the residues of every Pn consists of all the coprime primes up to modpn, and any of their composite multiples < modpn necessary to sum to the Euler totient count. Thus, the number and values of the residues are all (solely) determined by modpn, no matter how large.

This means clusters of consecutive prime residues outside the range $\boldsymbol{r_0}$ to $\boldsymbol{r_0^2}$ will always exist within a residue set too (as Pn increases), and thus their a_i k-tuples values must be nonzero. These consecutive prime clusters can't mutate their gap sizes, and will eventually be seen in a larger range for some Pn.

Thus, the inherent properties of prime generators are, in fact, empirically immutable as presented. Consequently, their properties establish there logically must exist an infinity of gap sizes.

10 Predictive Results

Ultimately, any proof must be able to explain known empirical results, and predict future ones. It's shown we can compute a good minimum estimate for Twins|Cousins for any Pn. We can also establish when any residue gap first appears in some Pn, and then determine when it migrates into the $\boldsymbol{r_0}$ to $\boldsymbol{r_0^2}$ range primes for some larger Pn.

For example, a_{50}, which denotes residues gaps of 100, can first occur for P59 (because its PGS has on order 53 coefficients). From Fig 5. a max gap size of 100 occurs for $503 < p < 1009$. The exact value is $p = 631$, i.e. between 631 and 631^2 the first prime pair of gap size 100 occurs among the 33,599 primes within this range. Thus, while a_{50} can possibly come into existence for P59 it takes until P619 to show up in its range, a span of 98 consecutive prime generators. While the process is not rapid, it is certain.

The following list are the first prime pairs with the first multiple of 100 gaps sizes shown.

- first instance of prime gap of 100 is (396,733; 396,833)
- first instance of prime gap of 200 is (378,043,979; 378,044,179)
- first instance of prime gap of 300 is (4,758,958,741; 4,758,959,041)
- first instance of prime gap of 400 is (47,203,303,159; 47,203,303,559)
- first instance of prime gap of 500 is (303,371,455,241; 303,371,455,741)

(It should be noted, the gaps don't necessarily occur in linear order, as the first prime gap for 210, for the pair (20,831,323; 20,831,533), occurs well before the first prime pair gap 200.)

Again, we see it takes time for larger and larger gap sizes to migrate to the range $\boldsymbol{r_0}$ to $\boldsymbol{r_0}^{\mathbf{2}}$ of larger Pn, but that's ok. My intent is not to find the actual values of the prime pairs, but to establish with certainty with this simple process that they exist, and there are an infinity of them of any even size.

11 Conclusion

The properties of Prime Generators allow for direct examination of the structure of the gaps between the primes. They empirically show prime numbers, and their gaps, conform to a deterministic structure that determines their nature, numbers, and distribution. We see that once any residue gap size comes into existence for some Pn it exists in larger numbers for all larger ones. By showing that every residue gap eventually will exist for some Pn within its residues range $\boldsymbol{r_0}$ to $\boldsymbol{r_0}^{\mathbf{2}}$, which are all primes, I establish every residue gap will eventually become a prime gap, which will increase in numbers for all larger Pn. Thus through this simple process, it has been established since all residue gaps only increase, and they eventually with certainty become prime gaps, and as there are an infinity of primes, by the logical extension of this process into infinity, we know there are an infinity of prime gaps of any even size.

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[6] Polignac's Conjecture - `https://en.wikipedia.org/wiki/Polignac%27s_conjecture`

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Data

The following data was derived using Ruby|Crystal scripts to generate and count the prime gaps.

Listed here are all the residue gap coefficients a_i for the first few prime generators. We observe: the sum of the columns for each Pn equals its residues count; the sum of the products of each a_i by its gap size $2i$ equals modpn; and for each Pn there are on order p_{n-1} unique coefficients. Also for the Pn shown, the first instance for a_{prime} (a_3, a_5, a_7, etc) equal 2.

We also see that the gaps oscillate up and down in their frequency as they linearly increase, and through the expansion process, the smaller gaps are numerically dominant in their frequency, and larger gaps initially occur with relatively much much lower frequency.

	Residue gap coefficients a_i for all gaps $2i$ for given Pn									
p_n	3	5	7	11	13	17	19	23	29	31
$a_1 \cdot 2$	1	3	15	135	1,485	22,275	378,675	7,952,175	214,708,725	6,226,553,025
$a_2 \cdot 4$	1	3	15	135	1,485	22,275	378,675	7,952,175	214,708,725	6,226,553,025
$a_3 \cdot 6$		2	14	142	1,690	26,630	470,630	10,169,950	280,232,050	8,278,462,850
$a_4 \cdot 8$			2	28	394	6,812	128,810	2,918,020	83,120,450	2,524,575,200
$a_5 \cdot 10$			2	30	438	7,734	148,530	3,401,790	97,648,950	2,985,436,650
$a_6 \cdot 12$				8	188	4,096	90,124	2,255,792	68,713,708	2,206,209,208
$a_7 \cdot 14$				2	58	1,406	33,206	871,318	27,403,082	903,350,042
$a_8 \cdot 16$					12	432	12,372	362,376	12,199,404	423,955,224
$a_9 \cdot 18$					8	376	12,424	396,872	14,123,368	512,670,088
$a_{10} \cdot 20$					0	24	1,440	61,560	2,594,160	106,604,280
$a_{11} \cdot 22$					2	78	2,622	88,614	3,324,402	126,682,650
$a_{12} \cdot 24$						20	1,136	48,868	2,100,872	88,337,252
$a_{13} \cdot 26$						2	142	7,682	386,554	18,298,102
$a_{14} \cdot 28$							72	5,664	324,792	16,461,600
$a_{15} \cdot 30$							20	2,164	154,220	9,169,532
$a_{16} \cdot 32$							0	72	10,128	833,688
$a_{17} \cdot 34$							2	198	15,942	1,075,458
$a_{18} \cdot 36$								56	7,228	620,632
$a_{19} \cdot 38$								2	570	77,042
$a_{20} \cdot 40$								12	1,464	128,988
$a_{21} \cdot 42$									272	40,636
$a_{22} \cdot 44$									12	3,516
$a_{23} \cdot 46$									2	1,795
$a_{24} \cdot 48$										1,296
$a_{25} \cdot 50$										504
$a_{26} \cdot 52$										20
$a_{27} \cdot 54$										84
$a_{28} \cdot 56$										12
$a_{29} \cdot 58$										2

Fig 3.

As new larger gaps appear within a PGS, it takes some time for them to migrate within the range from p to p^2 of larger Pn. The number of these residues constitute a dwindling percentage of the residue count for larger Pn, as shown below. This affects the rate they will become (with absolute certainty though) primes gaps.

Pn	7	11	13	17	19	23	29
residues count	48	480	5,760	92,160	1,658,880	36,495,360	1,021,870,080
r_0 to ${r_0}^2$ count	26	34	55	65	91	137	152
% of total residues	54.2	7.08	0.955	0.071	0.055	0.000375	0.0000149

Fig 4.

Below shows the progression of gaps frequency within p to p^2 for gap sizes shown, and the max gap.

	Frequency of prime gaps (not complete) between p and p^2								
p	11	53	101	503	1,009	5,003	10,007	50,021	100,003
max gap	8	34	36	86	114	210	220	320	354
gaps of 2	8	74	202	2,585	8,278	130,543	440,666	7,816,170	27,412,929
gaps of 4	9	78	197	2,575	8,239	130,201	440,606	7,816,884	27,410,258
gaps of 6	7	99	296	4,165	13,715	224,001	769,338	13,979,458	49,393,480
gaps of 8	1	37	103	1,692	5,643	96,432	334,491	6,221,667	22,161,302
gaps of 10		39	121	2,120	7,169	123,641	430,458	8,059,613	28,765,142
gaps of 12		27	107	2,267	8,134	151,420	530,008	10,420,167	37,589,303
gaps of 14		15	54	1,199	4,302	81,767	293,529	5,774,452	20,944,700
gaps of 16		6	33	795	2,929	59,224	216,032	4,347,314	15,888,865
gaps of 18		8	40	1,283	4,995	104,769	385,207	7,933,971	29,190,859
gaps of 20		2	15	601	2,433	53,704	203,194	4,366,505	16,296,757
gaps of 22		4	18	555	2,211	46,822	176,170	3,748,342	13,954,841
gaps of 24		2	15	604	2,278	66,815	257,882	5,701,980	21,488,356
gaps of 26		1	3	274	1,195	30,588	119,624	2,720,294	10,348,264
gaps of 28		0	6	271	1,261	32,971	129,739	2,963,462	11,288,578
gaps of 30		0	11	414	1,959	55,436	223,137	5,345,019	20,707,409
gaps of 32		0	1	97	558	16,563	68,384	1,695,929	6,641,679
gaps of 34		1	3	113	563	17,262	71,351	1,785,000	6,997,115
gaps of 36			1	149	779	27,127	114,180	2,927,973	11,593,976
gaps of 38				75	337	12,068	51,843	1,38,1811	5,518,125
gaps of 40				90	436	14,320	60,853	1,640,477	6,576,788
gaps of 42				83	486	19,568	86,754	2,438,771	9,920,126
gaps of 44				23	205	7,745	34,939	1,001,765	4,107,209
gaps of 46				24	158	6,514	29,372	866,337	3,580,246
gaps of 48				29	203	10,790	49,904	1,501,630	6,251,179
gaps of 50				16	110	5,803	27,544	857,165	3,607,941

Fig 5.

Here I use the data for $p = 101$ to graphically show the polynomial distributive nature of the gap sizes. We see from the curve, local maxima are (close to) multiples of gap size 6, while local minima are (close to) multiples of 4. We see from the data in Fig 5. this characteristic becomes more pronounced for larger p gap ranges. Larger ranges will have more local maxima/minima as they will generate more larger gaps. Each generator, thus, will have its own signature curve.

	Prime gaps from p to p^2 for $p = 101$																	
gaps	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36
freq	202	197	296	103	121	107	54	33	40	15	18	15	3	6	11	1	3	1

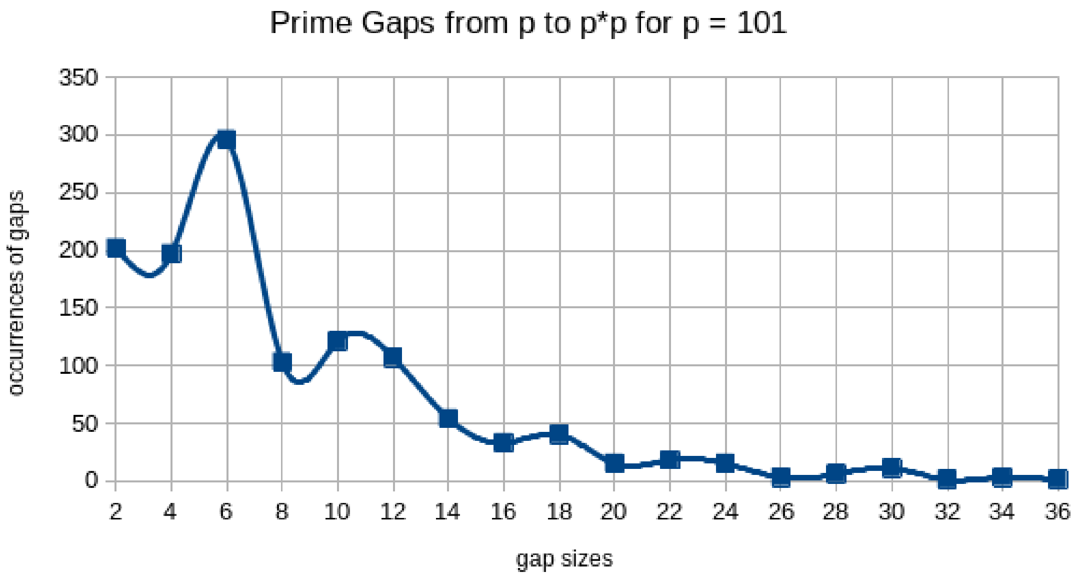

Fig 6.

We also clearly see the affect of the expansion property. All the preexisting gaps are pushed toward the front for the first half mirrored gaps (as new ones are included toward the middle) and they will appear first, and in greater frequency than larger gaps, for each larger generator.

The data also clearly shows there will always be more gaps of 6 than 2 and 4 (Twins|Cousins), or any other individual k-tuple. According to [4] gaps of 6 are called Sexy primes, which includes any gap of 6 between non-consecutive primes, e.g. (5, 11) and (7, 13). For the purpose of nomenclature then, I'll define Super (or Strictly) Sexy Primes as consecutive primes with gaps of 6, e.g. (23, 29) and (53, 59), which I think look better anyway. (Sex is so complicated.)

Here I show in more detail the slow growth rate of max gap sizes for increasing ranges p to p^2.

	Max prime gap sizes from p to p^2											
p	11	19	31	59	101	179	317	563	1,009	1,783	3,163	5,623
$log10(p)$	1.0	1.25	1.5	1.75	2.0	2.25	2.5	2.75	3.0	3.25	3.5	3.75
max gap	8	14	20	34	36	72	72	86	114	148	154	210

p	10007	17783	31627	56237	100003	177823	316233	526337	1000003
$log10(p)$	4.0	4.25	4.5	4.75	5.0	5.25	5.5	5.75	6.0
max gap	220	248	282	320	354	456	464	486	540

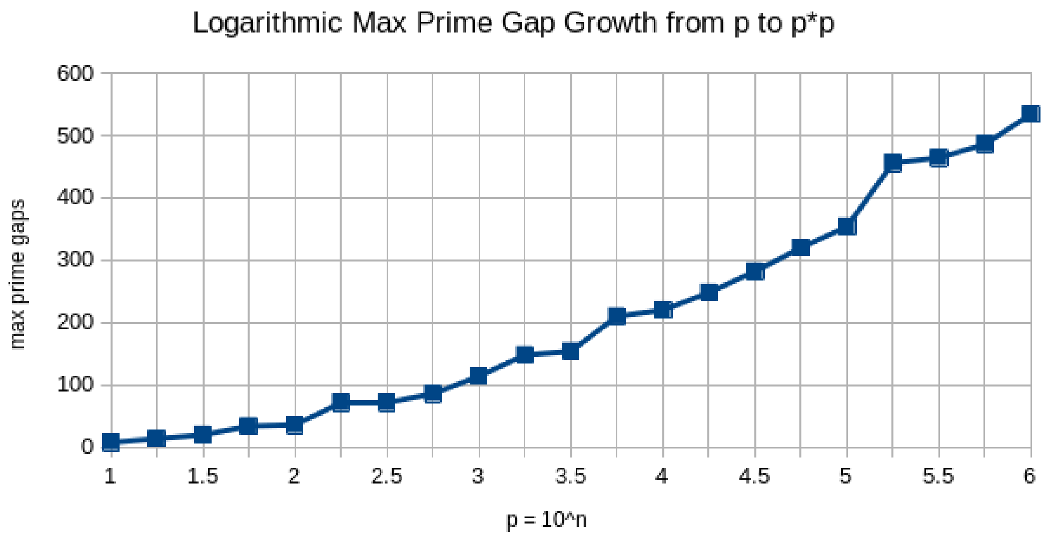

Fig 7.

This graph quantifies the slow expansion. As p increases orders of magnitude its PGS max gap grows much slower. For p of order 10^3 the max gap reaches 10^2, but only increases to $5 \cdot 10^2$ for p of order 10^6.

We can similarly create growth curves for all the other gap sizes to visually see their growth rate.

The data shows the distribution of primes is not random, but in fact is deterministic, and conform to the described properties manifested within the structure of prime generators. The primes exist in clusters. Prime gaps with (relatively) small gaps will cluster, then larger jump occurs, then other clusters, etc.

It, again, should be noted, though while this graph is technically accurate, it doesn't tell the whole story, as the gaps don't neccessarily occur in linear order. For example, the first occurrences of prime gaps 210, 220, 248, etc, occur for prime values much smaller than for the first prime pair with gap 200.

Appendix

Infinite Progression of Primes

From the Prime Number Theorem (PNT) (https://en.wikipedia.org/wiki/Prime_number_theorem) it has been proved the number of primes up to any value x is on order $x/log(x)$, or better $Li(x)$ (log integral x). Equation 8 (for computational simplicity) uses $x/log(x)$ to estimate the number of primes between any random prime p (or really any value x) and p^2, per the PNT.

The Pn residues are the integers $p_n < r_i <$ modpn coprime to modpn. The Euler Totient Funcition (ETF) tells us their exact number. Thus it's clear, the $\{r_i\}$ must include all the coprime primes (and any of their prime multiples) $<$ modpn, necessary to satisfy the ETF residues count.

Each Pn eliminates all its modulus primes multiples from consideration. Since the first residue $\boldsymbol{r_0}$ of every Pn is the next prime $> p_n$, its first multiple in its residue set (pc table) is the multiple with itself, i.e. $\boldsymbol{r_0}^{\mathbf{2}}$. Therefore, the residues between $\boldsymbol{r_0}$ to $\boldsymbol{r_0}^{\mathbf{2}}$ can only be the coprime primes in that interval, as they are not multiples (the only non-multiples) of the modulus primes $< \boldsymbol{r_0}^{\mathbf{2}}$. And the PNT tells us their numbers are of order $p^2/log(p^2)$ - $p/log(p)$, or better $Li(p^2)$ - $Li(p)$.

However, for each specific generator Pn we can compute easier a better estimate. We know the number of modulus primes for any Pn, I'll note as $\pi(modpn)$. Thus the primes $< \boldsymbol{r_0}^{\mathbf{2}}$, for $\boldsymbol{r_0} = p$ are: $p^2/log(p^2)$ - $\pi(modpn)$. For the previous example for P23, with $\boldsymbol{r_0} = 29$, a better estimate is then: $(841)/log(841)$ - $9 = 117.87$ (118), better than pror 116. In fact, we can just use $p^2/log(p^2)$, here $841/log(841) = 124.88$ (125), as $\pi(modpn)$ is relatively so much smaller as p^2 becomes larger.

Thus, since we know each generator Pn always generates the consecutive primes $\boldsymbol{r_0}$ to $\boldsymbol{r_0}^{\mathbf{2}}$, we can use these primes to construct a larger Pn, and keep bootstrapping this process as many times as we want to generate as many of the consecutive primes we want, and thus can also then observe, record, and count, the exact gap structure of all the primes we generate, into infinity.

Modular Complement Property

Using ***clock math***, we see residues exist as ***modular complement pairs***, and ***prime generator sequences*** have ***mirror image symmetry***, as a direct property of their ***modular forms***.

Any even n can be the modulus for a ***cyclic integer generator*** (I use only moduli of form $\boldsymbol{p_n}\#$) we can visualize as a clock of n hours. A 12 hour analogue clock has a modulus of 12 with residues $1 - 12$, placed equidistance around the clock. It's easy to see, if we draw horizontal lines between residues (hours), left-to-right, their sums equals 12 (the top|bottom residues are really (0:12) and (6:6)), and also see this if we fold the clock on its vertical axis.

When we form the prime generator P12, for mod12 we only use the residues coprime to 12, i.e. $\{1, 5, 7, 11\}$, where (1, 11) and (5, 7) are modular complement pairs. Eliminating the non-coprime residues shows the P12 generator, with its 4 residues, with its mirror image gap distribution. Any even $n > 2$ will have a ***modular form*** with these modular complement properties, for every Pn.