Fiscal Policy in Growth Models

Jesse Perla
University of British Columbia

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1 Tax and Government Policy

1.1 Basic Setup

- Price System: $\{q_t, w_t, r_t\}$
- Tax Policy: $\{\tau_{ct}, \tau_{nt}, \tau_{kt}, \tau_{ht}, \tau_{it}\}$ and government expenditure $\{g_t\}$ (exogenous sequence).
  - Where τ_{it} is the investment tax credit, τ_{ct} is the consumption tax, τ_{kt} is the capital tax, and τ_{ht} is the head tax.
  - All are proportional except τ_{ht} (which is denominated in consumption goods)

Assume inelastic labor supply. For the version with elastic labor supply, see Appendix A

1.2 Households

Given prices and government policies, households maximize the PDV of utility of consumption given the lifetime budget constraint:

$$\max_{\{c_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{1}$$

$$\text{s.t.} \sum_{t=0}^{\infty} q_t \left[(1+\tau_{ct})c_t + (1-\tau_{it})\left(k_{t+1} - (1-\delta)k_t\right) + \tau_{ht}\right] \tag{2}$$

$$= \sum_{t=0}^{\infty} w_t(1-\tau_{nt}) \times 1 + \sum_{t=0}^{\infty} r_t(1-\tau_{kt})k_t \tag{3}$$

Lagrangian of maximization problem:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \tag{4}$$

$$\lambda \sum_{t=0}^{\infty} \left[\underbrace{(1-\tau_{nt})w_t}_{\text{labor income}} + \underbrace{(1-\tau_{kt})r_t k_t}_{\text{capital income}} - \underbrace{(1+\tau_{ct})q_t c_t}_{\text{consumption goods}} - \underbrace{(1-\tau_{it})q_t\left(k_{t+1}-(1-\delta)k_t\right)}_{\text{investment } \tau_{it} \text{ \underline{lowers} cost}} - \underbrace{q_t\tau_{ht}}_{\text{head tax}} \right] \tag{5}$$

Note:

FONC:

$$[c_t] : q_t = \lambda \frac{\beta^t u'(c_t)}{1+\tau_{ct}} \text{ (tax distorts the marginal utility of consumption)} \tag{6}$$

$$[k_{t+1}] : -(1-\tau_{it})q_t + (1-\tau_{k,t+1})r_{t+1} + (1-\tau_{i,t+1})q_{t+1}(1-\delta) = 0 \tag{7}$$

Rearrange (7) and divide by q_{t+1}:

$$1 = \frac{q_{t+1}}{q_t}(1-\tau_{it})^{-1}\left[(1-\tau_{i,t+1})(1-\delta) + (1-\tau_{k,t+1})\frac{r_{t+1}}{q_{t+1}}\right] \tag{8}$$

$$\Rightarrow 1 = \beta \underbrace{(1-\tau_{it})^{-1}}_{\substack{\text{subsidy}\\ \text{to investment}}} \underbrace{\frac{(1+\tau_{ct})}{(1+\tau_{c,t+1})}}_{\text{tax adjusted}} \underbrace{\frac{u'(c_{t+1})}{u'(c_t)}}_{\text{MUC Ratio}} \left[\underbrace{\overbrace{(1-\tau_{i,t+1})}^{\substack{\text{would need to}\\ \text{replenish,}\\ \text{subsidized}}}(1-\delta) + (1-\tau_{k,t+1}) \overbrace{\frac{r_{t+1}}{q_{t+1}}}^{\substack{\text{real pre-tax}\\ \text{rental rate}}}}_{\text{real return on investment}} \right] \tag{9}$$

1.3 Firms

As before, no direct distortions, households owns firm, which pays no direct taxes. Consider a price taking firm choosing N_t and K_t,

- From solves:

$$\max_{N_t, K_t} \{q_t F(K_t, N_t) - w_t N_t - r_t K_t\}, \text{ Or in real terms:} \tag{10}$$

$$\max_{N_t, K_t} \left\{ F(K_t, N_t) - \frac{w_t}{q_t} N_t - \frac{r_t}{q_t} K_t \right\} \tag{11}$$

- FOCs. Assume CRS production function, and using the $k_t \equiv K_t/N_t$ ratio.

$$\frac{r_t}{q_t} = \boldsymbol{\partial}_k F(K_t, N_t) = f'(k_t) \tag{12}$$
$$\frac{w_t}{q_t} = \boldsymbol{\partial}_n F(K_t, N_t) = f(k_t) - f'(k_t)k_t \tag{13}$$

As all firms have identical k_t, we will use a representative price taking firm using (inelastic) labor input $N_t = 1$ (and hence, $k_t = K_t$).

1.4 Government

- Given price system and expenditures, government has budget constraint:

$$\sum_{t=0}^{\infty}\left[-q_t g_t - q_t \tau_{it} \underbrace{(k_{t+1} - (1-\delta)k_t)}_{\text{investment}} + \tau_{ct} q_t c_t + \tau_{nt} w_t \times 1 + \tau_{kt} r_t k_t + \tau_{ht} q_t\right] = 0 \tag{14}$$

- Feasibility:

$$g_t + c_t + \underbrace{k_{t+1} - (1-\delta)k_t}_{=x_t} \leq F(K_t, 1) = f(k_t) \tag{15}$$

where allocation is $\{c_t, k_{t+1}\}$

1.5 Competitive equilibrium

- Definition:

  A C.E. is a set of government policy, price system, feasible allocation such that:

  - Government budget constraint holds
  - Given prices/policy, allocation solves Households' problem
  - Given prices/policy, allocation solves Firm's problem.

1.6 Summary

- Allocations: $\{k_t, c_t\}$ solve the system of two equations:
- Feasibility:

$$\boxed{g_t + c_t + \underbrace{k_{t+1} - (1-\delta)k_t}_{\text{Investment:}x_t} = f(k_t)} \tag{16}$$

- Euler Equation:

$$\boxed{1 = \beta \left(\frac{1+\tau_{ct}}{1+\tau_{c,t+1}} \right) \frac{u'(c_{t+1})}{u'(c_t)} \left[\frac{1-\tau_{i,t+1}}{1-\tau_{it}}(1-\delta) + \frac{1-\tau_{k,t+1}}{1-\tau_{it}} f'(k_{t+1}) \right]} \tag{17}$$

where k_0 is given, and a k_∞ boundary value coming from the steady state.

1.7 Steady State

- $k_t = \bar{k}, c_t = \bar{c}$. All taxes constant, $g_t = \bar{g}$
- Take (17), and use $\beta \equiv \frac{1}{1+\rho}$

$$\boxed{\rho + \delta = \frac{1-\tau_k}{1-\tau_i} f'(\bar{k}) \text{ (solve for } \bar{k})} \tag{18}$$

- With this, you can solve for $\bar{k}$ directly. From (16)

$$\boxed{\bar{c} = f(\bar{k}) - \underbrace{\delta \bar{k}}_{=\bar{x}} - \bar{g}} \tag{19}$$

- Government budget divide by q_t and use w_t/q_t and r_t/q_t from (12) and (13). Must hold period by period if constant:

$$\boxed{\tau_c \bar{c} + \tau_n \left(f(\bar{k}) - \bar{k} f'(\bar{k}) \right) + \tau_k \bar{k} f'(\bar{k}) + \tau_h = \bar{g} + \tau_i \delta \bar{k}} \tag{20}$$

2 Steady state and taxes

2.1 Setup

- Consider a steady state with taxes

$$\frac{1-\tau_i}{1-\tau_k}(\rho + \delta) = f'(\bar{k}^{\text{tax}}) \tag{21}$$

$$\rho + \delta = f'(\bar{k}) \tag{22}$$

- Assume $g = 0$, $c = f(k) - \delta k$ is output.

$$\max_k \{ f(k) - \delta k \} \Rightarrow \boxed{\delta = f'(\bar{k}^m)} \tag{23}$$

- Compare if $\frac{1-\tau_i}{1-\tau_k} > 1$, i.e. positive net taxes:

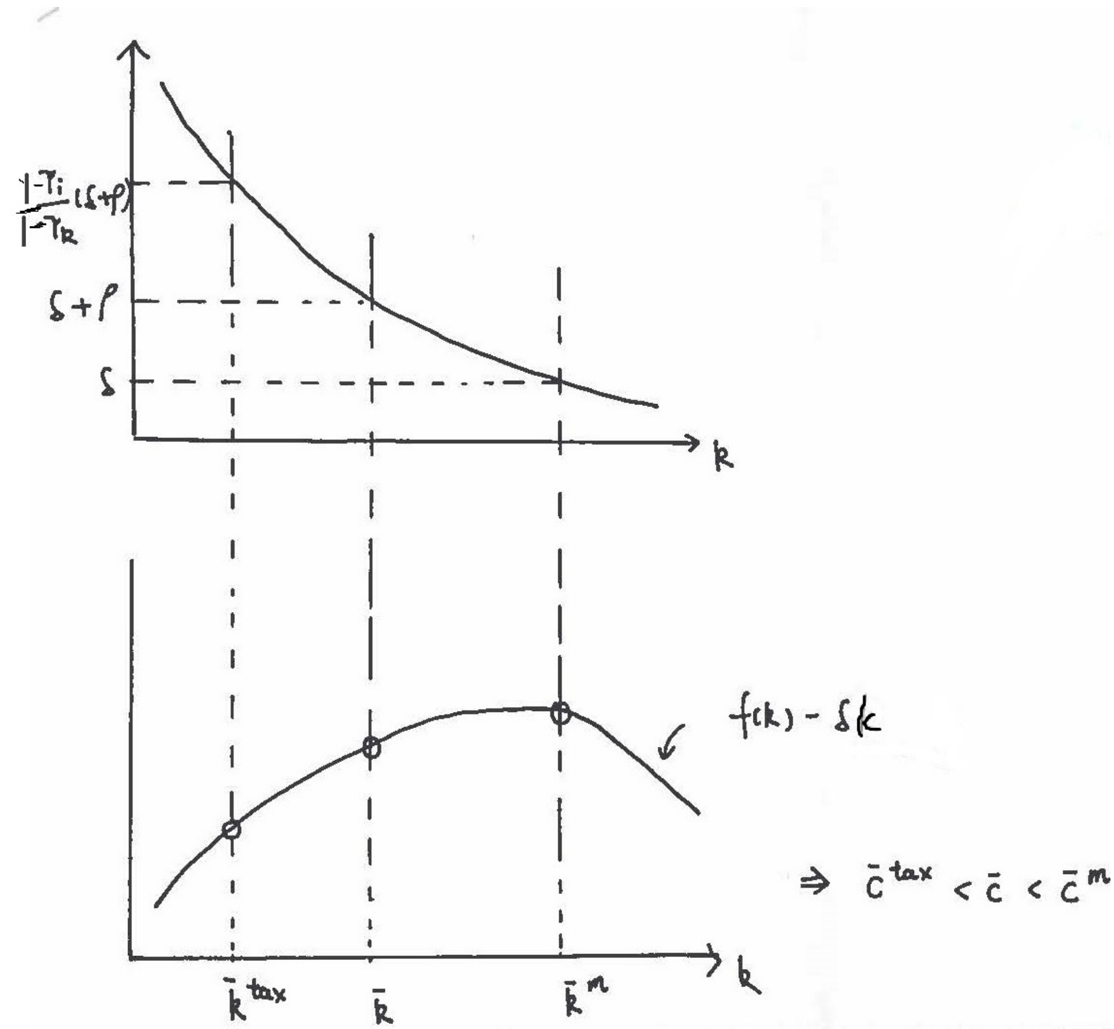


Figure 1: Distorted Equilibrium k

2.2 Interest rates

- From (17),

$$1 = \beta \left(\frac{1+\tau_{ct}}{1+\tau_{c,t+1}} \right) \frac{u'(c_{t+1})}{u'(c_t)} \left[\frac{1-\tau_{it+1}}{1-\tau_{it}}(1-\delta) + \frac{1-\tau_{k,t+1}}{1-\tau_{it}} f'(k_{t+1}) \right] \tag{24}$$

- Then:
  - Recall PIH, interest rates,etc.

– Recall pricing of risk-free bond:

$$1 = \beta R_t \frac{u'(c_{t+1})}{u'(c_t)} \tag{25}$$

$$\Rightarrow \boxed{\bar{R} = \frac{1}{\beta}} \tag{26}$$

For example, with power utility, if $\gamma > 0$, i.e. $\frac{c_{t+1}}{c_t} > 1 \Rightarrow R_t > \frac{1}{\beta}$

- Futhermore, comparing (24) and (25), we see a condition for the absence of arbitrage between investment in financial assets and real assets is,

$$R_t = \left(\frac{1+\tau_{ct}}{1+\tau_{c,t+1}}\right)\left[\frac{1-\tau_{it+1}}{1-\tau_{it}}(1-\delta) + \frac{1-\tau_{k,t+1}}{1-\tau_{it}} f'(k_{t+1})\right] \tag{27}$$

This equation connects the interest rate formulas we derived in the interest rates section (which used an endowment economy) with this model (where the endowments are endogenized through investment in physical capital)

2.3 Unforeseen changes in policy

- Assume inelastic supply, $\bar{g}_{t<0} = 0$ before time 0, economy is in steady state.

$$\rho + \delta = \frac{1-\tau_k}{1-\tau_i} f'(\bar{k}) \tag{28}$$

$$\bar{c} = f(\bar{k}) - \bar{g} - \delta \bar{k} \tag{29}$$

- Now assume at time 0, the government suddenly changes $\bar{g} > 0$ forever. Effective immediately. What is the transition? If financed through lump-sum tax.

- New steady state:
  - From (28), same $\bar{k}$. No changes in output or capital.
  - From (29), $\bar{c} = \bar{c}_{t<0} - \bar{g}$

2.4 Foreseen changes in policy

- Instead, what if government announces that at $t = 10$, $g = \bar{g}$ forever. Announce at $t = 0$

- New steady state? Same as $t = 0$ implementation; Announcement timing wears off.

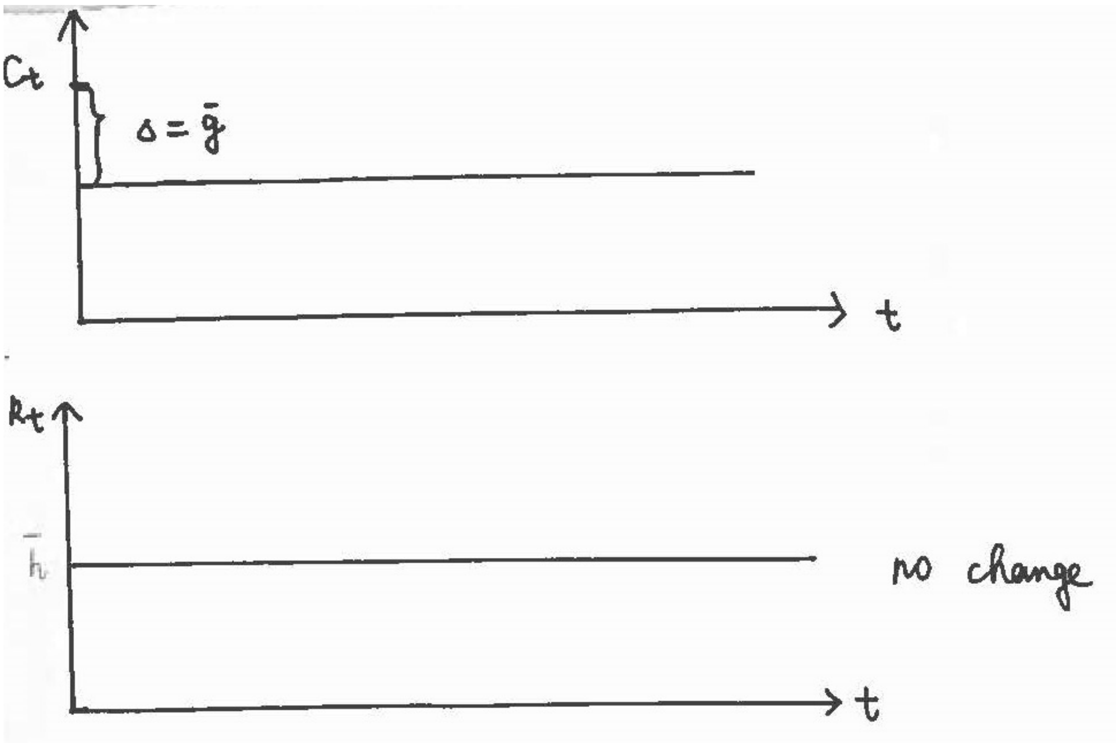


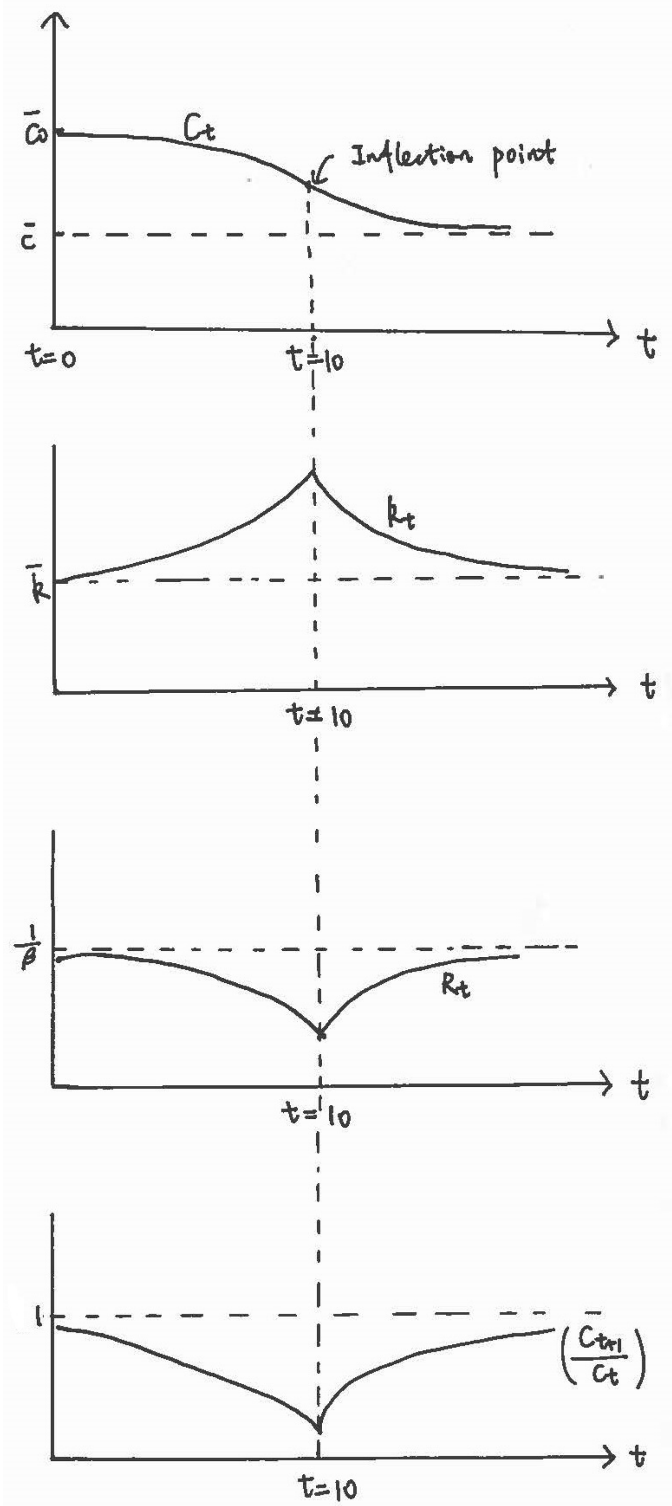
Figure 2: Unanticipated Change (and no Transition Dynamics)

- So same capital stock in steady state, but lower consumption.
- So want to smooth with concave utility. How? Capital!
- See Figure 3

2.5 Feedback and Feedforward

- General principle: Effects of a change in policy depend on if it is anticipated vs. unanticipated.
- Feedback vs Feedforward
  - Feedback: Anticipation of change at $t = 10$. Then gone.
  - Feedforward: Transitory efforts of being off steady state at $t = 10$. Only $t > 10$, wear off. See Figure 4

$$\underbrace{R_t^{-1} \equiv \beta \frac{u'(c_{t+1})}{u'(c_t)}}_{\text{Definition}} = \left[\underbrace{(1-\delta) + f'(k_{t+1})}_{\text{real return, from (17)}} \right]^{-1} \tag{30}$$



Note : $R_t = \frac{1}{\beta}\left(\frac{c_{t+1}}{c_t}\right)^{\gamma}$

$= 1 - \delta + f'(k_{t+1})$

Figure 3: Anticipated (with Transition Dynamics)

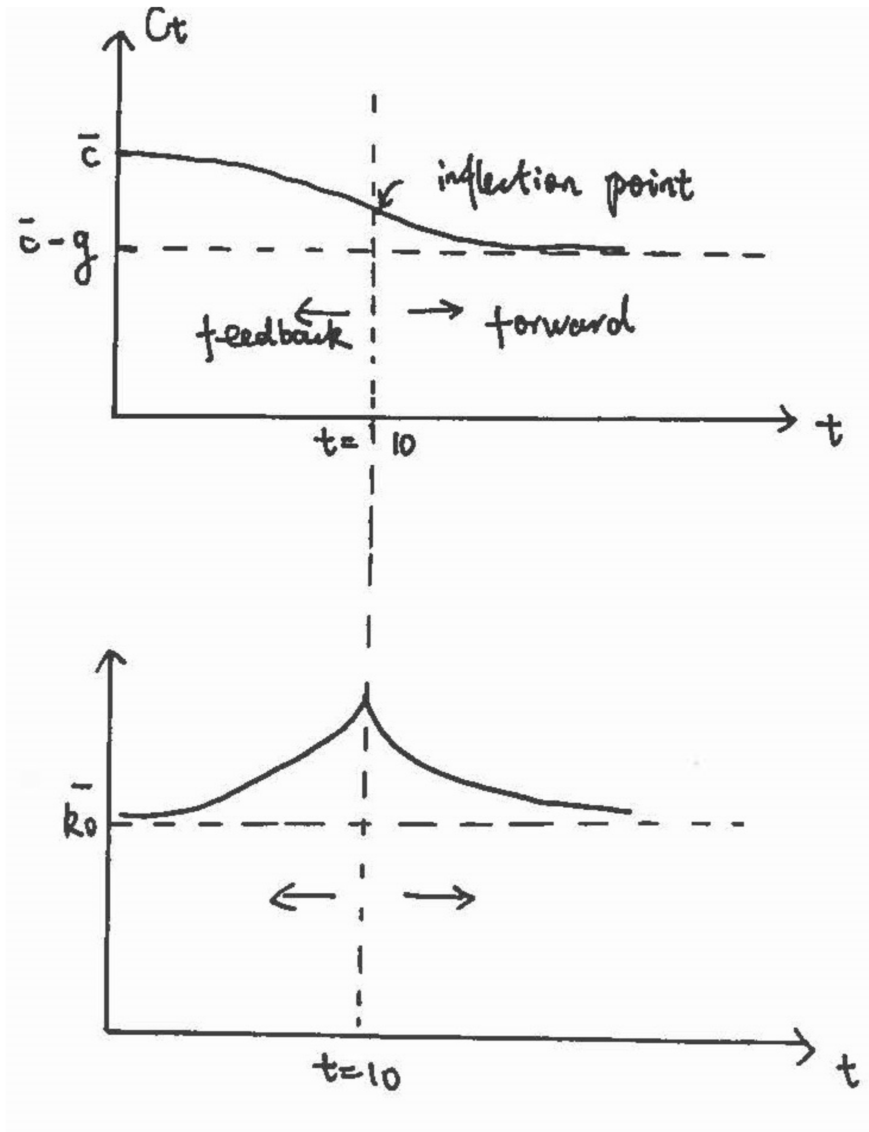


Figure 4: Feedback and Feedforward with Anticipated Changes

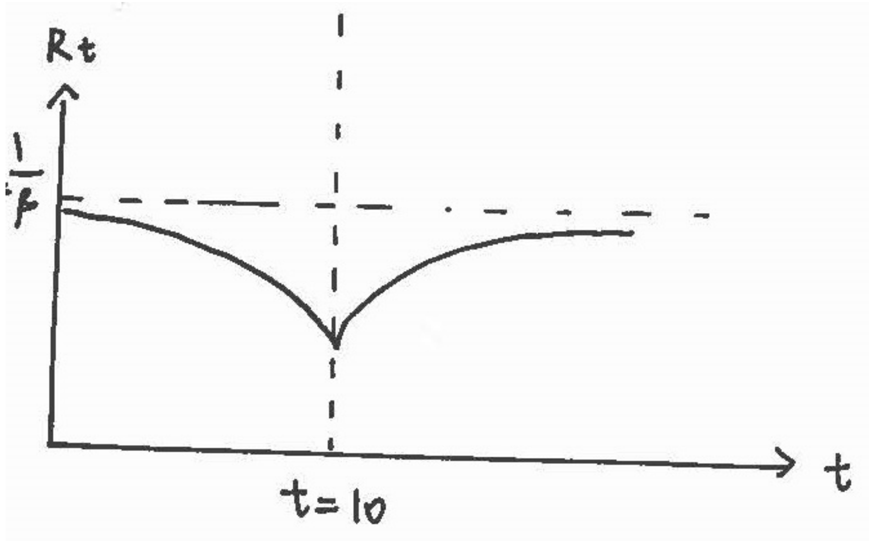


Figure 5: Interest Rate Transitions with Foresight

- Think of R_t in the PIH model. In interest rate that "reconciles the household to substitute c_t vs. c_{t+1} away from perfect smoothing. R_t is rate at which market and tax system allows substitution.

3 Examples

3.1 Foreseen changes in policy

- As before, $g_t = \bar{g}$ for $t \geq 10$, announced at $t = 0$ For this, could use direct formula

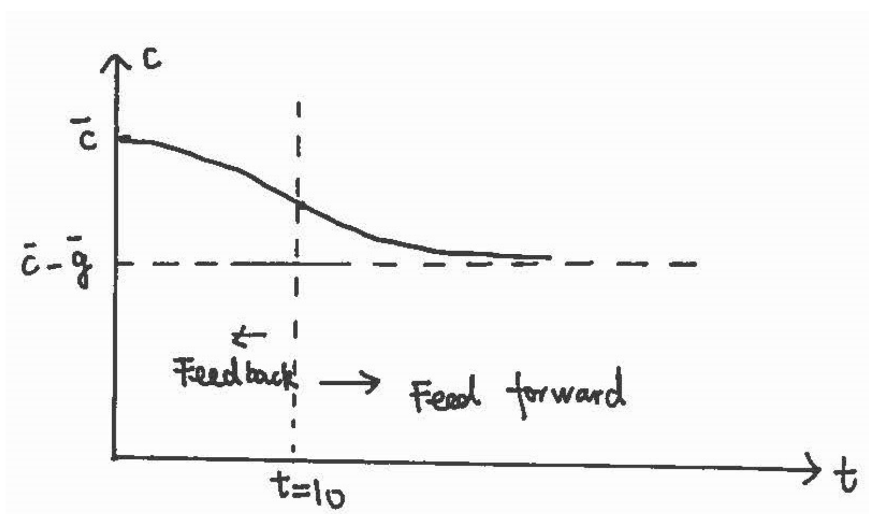


Figure 6: Forseen Government Spending

or log approximation. See Figure 6 for consumption guess and Figure 7 for interest rates/yield curve.

- From Euler Equation,

$$r_{t,t+1} = \frac{1}{\beta}\left(\frac{c_{t+1}}{c_t}\right)^{\gamma} - 1 = f'(k_{t+1}) - \delta \tag{31}$$

Announcement:

  - Major shift due to feedback at $t = 0$. Returns decrease.
  - No more feedback. Returns increasing as $k_{t+1} \downarrow$
  - Flat since average of constants

3.2 Foreseen/Unforeseen $\uparrow$ in Productivity

- Let $f(k_t) = z_t k_t^{\alpha}$ where z_t is Total Factor of Productivity. So we have:

$$f'(k_t) = z_t \alpha k_t^{\alpha-1} \tag{32}$$

which is linearly increasing marginal product.

- Let $z_0 = 1$, economy in steady state and then $\uparrow \bar{z} > 1$. In steady state:

$$\rho + \delta = z\alpha \bar{k}^{(\alpha-1)} \Rightarrow \bar{k} > \bar{k_0} \text{ if } \bar{z} \uparrow \tag{33}$$

$$\bar{c} = z\bar{k}^{\alpha} - \delta\bar{k} \Rightarrow \bar{c} > \bar{c_0} \text{ since } \bar{k} \uparrow \text{ and } z > 1 \tag{34}$$

- See the following pages for simulation and discussion of anticipated vs. unanticipated.
- Keys:
  - If anticipated, see path of c_t that would smooth the transition between the steady states.
  - Figure out capital that would enable the transition. Usually an inflection at anticipated $t = 10$.
  - Remember that k_t needs to change slowly since accumulated.
  - From k_t or $\frac{c_{t+1}}{c_t}$, figure out R_t and get $r_{t,t+1}$.
  - Average for yield curve.

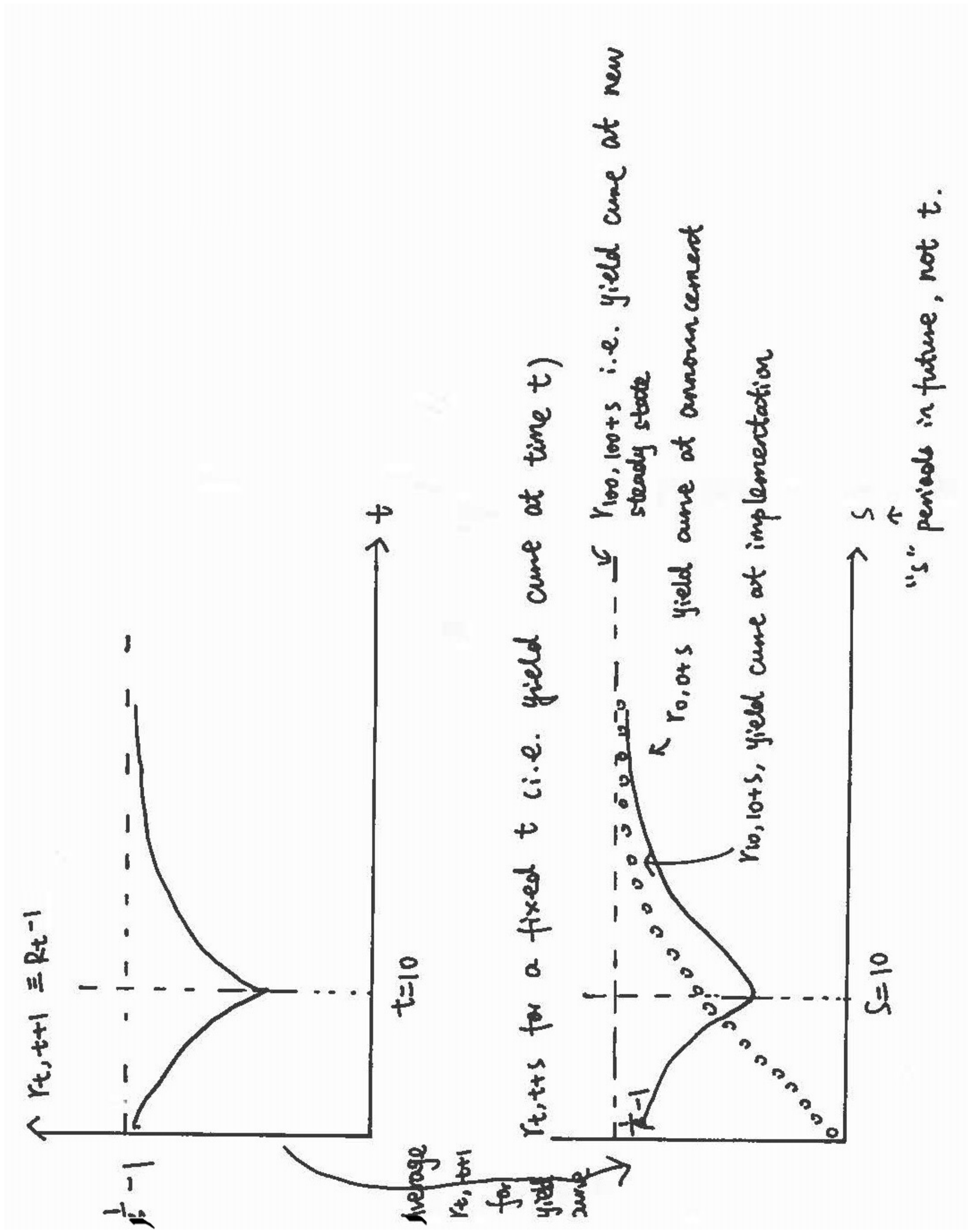
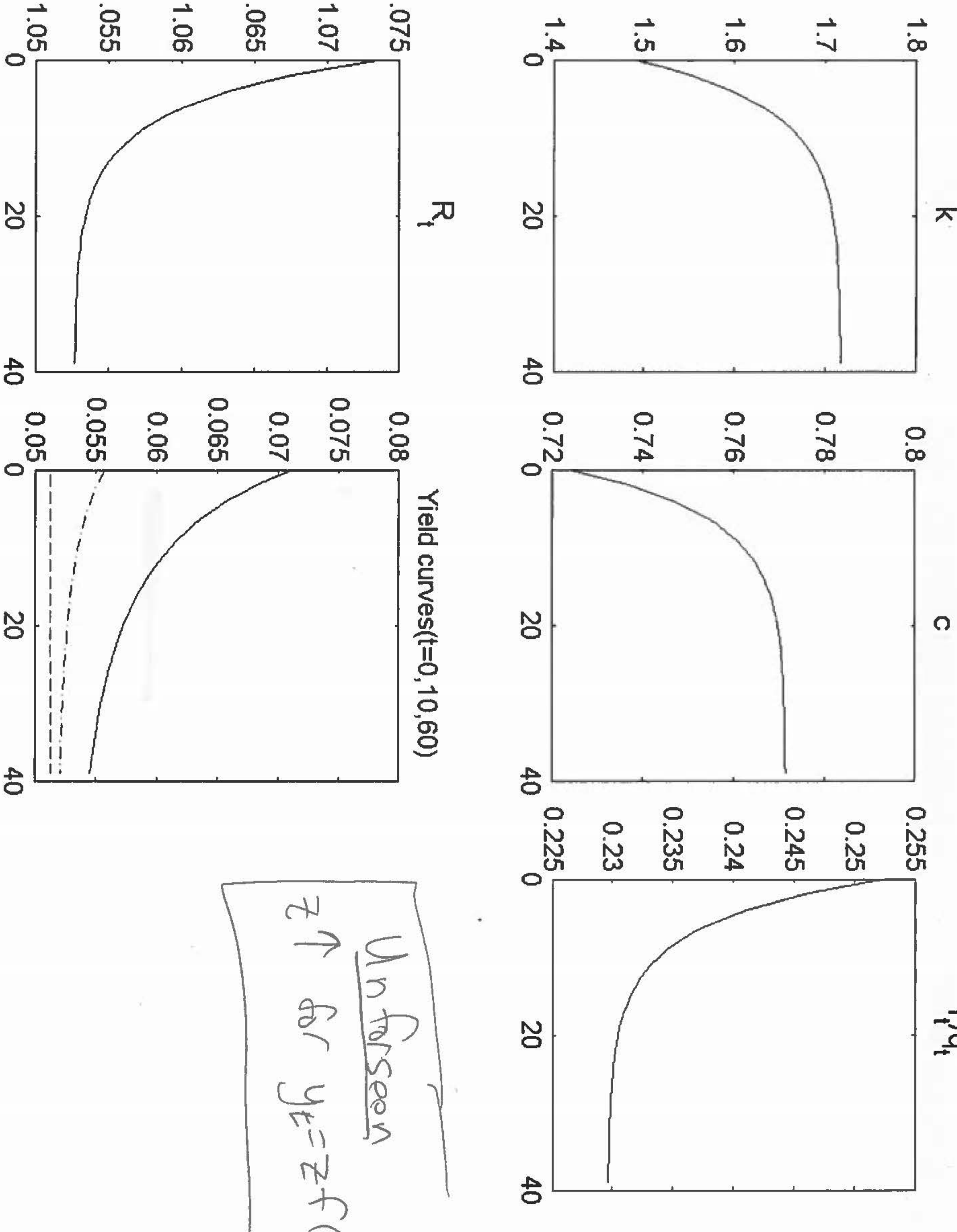


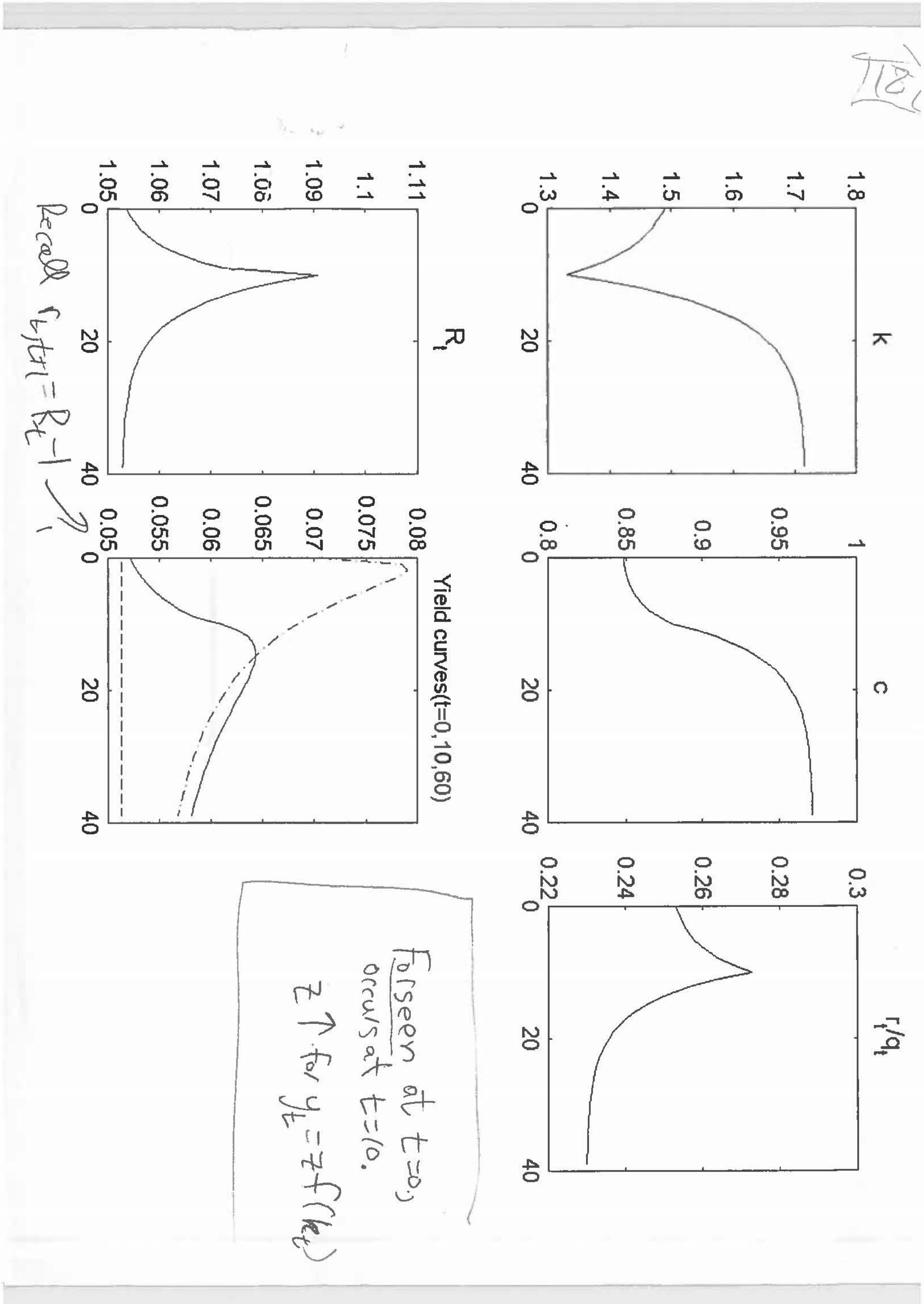
Figure 7: Interest Rates with Forseen Government Spending

(17)



Unforseen

$z\uparrow$ for $y_t = z f(k_t)$


k
c
r_t/q_t
R_t
Yield curves(t=0,10,60)
Recall $r_{t,t+1} = R_t - 1$?
Forseen at t=0, occurs at t=10.
z ↑ for $y_t = z f(k_t)$

4 Term Structure and Forecasts

Recall from the endowment economy in the interest rates section that the price of a claim to consumption at time t delivered s periods in the future was

$$\frac{q_{t+s}^t}{q_t^t} = \beta^s \frac{u'(c_{t+s})}{u'(c_t)} \tag{35}$$

This is assuming consumption taxes are 0. As before, we can define the interest rate to maturity,

$$\frac{q_{t+s}^t}{q_t^t} \equiv \frac{1}{(1+r_{t,t+s})^s} \tag{36}$$

Use $\beta \equiv \frac{1}{1+\rho}$ and solve for $r_{t,t+s}$

$$1 + r_{t,t+s} = (1+\rho)\left(\frac{u'(c_{t+s})}{u'(c_t)}\right)^{-1/s} \tag{37}$$

Hence, given interest rates from financial markets for different bond lengths (i.e., s), we can determine consumer's expectations of future consumption. See Appendix B for more on connection of the term structure to the real return on capital.

Appendix A Elastic Labor Supply

In this section, we solve a version of the model where labor, n_t, is elastically supplied.

A.1 Households

- Given prices and tax policy, utility:

$$u(c,n) = \underbrace{\frac{c^{1-\gamma}}{1-\gamma}}_{\equiv \log c \ \text{ if } \gamma=1} + B \underbrace{\frac{(\overbrace{1-n}^{\text{leisure}})^{1-\phi}}{1-\phi}}_{\equiv \log(1-n) \ \text{ if } \phi=1} \tag{A.1}$$

  where $B > 0$, $\gamma > 0$, and ϕ is related to Frisch elasticity of labor supply. But in general macro and micro labor elasticities don't match.

$$\max_{\{c_t, k_{t+1}, n_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t, n_t) \tag{A.2}$$

$$\text{s.t. } \sum_{t=0}^{\infty} q_t \left[(1+\tau_{ct})c_t + (1-\tau_{it})\left(k_{t+1} - (1-\delta)k_t\right) + \tau_{ht}\right] \tag{A.3}$$

$$= \sum_{t=0}^{\infty} w_t(1-\tau_{nt})n_t + \sum_{t=0}^{\infty} r_t(1-\tau_{kt})k_t \tag{A.4}$$

$$0 < n_t \leq 1 \tag{A.5}$$

- Lagrangian of maximization problem:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t, n_t) + \tag{A.6}$$

$$\lambda \sum_{t=0}^{\infty} \left[\underbrace{(1-\tau_{nt})w_t n_t}_{\text{labor income}} + \underbrace{(1-\tau_{kt})r_t k_t}_{\text{capital income}} - \underbrace{(1+\tau_{ct})q_t c_t}_{\text{consumption goods}} - \underbrace{(1-\tau_{it})q_t\left(k_{t+1} - (1-\delta)k_t\right)}_{\text{investment } \tau_{it} \text{ \underline{lowers} cost}} - \underbrace{q_t \tau_{ht}}_{\text{head tax}} \right] \tag{A.7}$$

Note:

$$\boldsymbol{\partial}_c u = c^{-\gamma} \tag{A.8}$$

$$-\boldsymbol{\partial}_n u = B(1-n)^{-\phi} \tag{A.9}$$

- FONC:

$$[c_t] : q_t = \beta^t \frac{c_t^{-\gamma}}{1+\tau_{ct}} = \frac{\beta^t \boldsymbol{\partial}_c u}{1+\tau_{ct}} \text{ (normalize } \lambda = 1 \text{ price level)} \tag{A.10}$$

$$[n_t] : -\beta^t \boldsymbol{\partial}_n u = \beta^t B \cdot (1-n_t)^{-\phi} = (1-\tau_{nt})w_t \tag{A.11}$$

$$[k_{t+1}] : -(1-\tau_{it})q_t + (1-\tau_{k,t+1})r_{t+1} + (1-\tau_{i,t+1})q_{t+1}(1-\delta) = 0 \tag{A.12}$$

Dividing (A.11) by (6). Static Decision:

$$\boxed{\frac{B(1-n_t)^{-\phi}}{c_t^{-\gamma}} = \frac{-\boldsymbol{\partial}_n u}{\boldsymbol{\partial}_c u} = \underbrace{\frac{1-\tau_{nt}}{1+\tau_{ct}}}_{\substack{\text{relative tax} \\ \text{distortion}}} \underbrace{\frac{w_t}{q_t}}_{\substack{\text{real} \\ \text{wage,} \\ \text{pre-tax}}}} \tag{A.13}$$

Rearrange (7) and divide by q_{t+1}:

$$1 = \frac{q_{t+1}}{q_t}(1-\tau_{it})^{-1}\left[(1-\tau_{i,t+1})(1-\delta) + (1-\tau_{k,t+1})\frac{r_{t+1}}{q_{t+1}}\right] \tag{A.14}$$

$$\Rightarrow 1 = \beta \underbrace{(1-\tau_{it})^{-1}}_{\substack{\text{subsidy}\\ \text{to investment}}} \underbrace{\frac{(1+\tau_{ct})}{(1+\tau_{c,t+1})}}_{\text{tax adjusted}} \underbrace{\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}}_{\text{MUC}} \left[\underbrace{\overbrace{(1-\tau_{i,t+1})}^{\substack{\text{would need to}\\ \text{replenish,}\\ \text{subsidized}}}(1-\delta) + (1-\tau_{k,t+1}) \overbrace{\frac{r_{t+1}}{q_{t+1}}}^{\substack{\text{real pre-tax}\\ \text{rental rate}}}}_{\text{real return on investment}}\right] \tag{A.15}$$

A.2 Firms

As before, no direct distortions, households owns firm, which pays no direct taxes. Hence, kthe firms will all choose the same ratio $\tilde{k}_t \equiv k_t/n_t$, where n_t is the endogenously supplied hours per capita, and k_t is the capital per capita.

- FOCs, 0-profit if CRS:

$$\frac{w_t}{q_t} = \boldsymbol{\partial}_N F(K_t, N_t) \tag{A.16}$$

$$\frac{r_t}{q_t} = \boldsymbol{\partial}_K F(K_t, N_t) \tag{A.17}$$

- Using properties of CRS, with the capital-labor ratio $\tilde{k}_t$ and labor supplied n_t,

$$\frac{w_t}{q_t} = f(\tilde{k}_t) - \tilde{k}_t f'(\tilde{k}_t) \tag{A.18}$$

$$\frac{r_t}{q_t} = f'(\tilde{k}_t) \tag{A.19}$$

A.3 Government

- Given price system and expenditures, government has budget constraint:

$$\sum_{t=0}^{\infty}\left[-q_t g_t - q_t \tau_{it} \underbrace{(k_{t+1} - (1-\delta)k_t)}_{\text{investment}} + \tau_{ct} q_t c_t + \tau_{nt} w_t n_t + \tau_{kt} r_t k_t + \tau_{ht} q_t\right] = 0 \tag{A.20}$$

- Feasibility:

$$g_t + c_t + \underbrace{k_{t+1} - (1-\delta)k_t}_{=x_t} \leq n_t f(k_t/n_t) \tag{A.21}$$

where allocation is $\{c_t, k_{t+1}, n_t\}$

A.4 Competitive equilibrium

A C.E. is a set of government policy, price system, feasible allocation such that:

- Government budget constraint holds
- Given prices/policy, allocation solves Households' problem
- Given prices/policy, allocation solves Firm's problem.

Recall that $k_t = \tilde{k}_t n_t$. The system of equations is easier to write in terms of of $\tilde{k}_t$ than k_t

- Allocations: $\left\{n_t, \tilde{k}_t, c_t\right\}$
- Altenative Feasibility:

$$\boxed{g_t + c_t + \underbrace{n_{t+1}\tilde{k}_{t+1} - (1-\delta)n_t\tilde{k}_t}_{=x_t} = n_t f(\tilde{k}_t)} \tag{A.22}$$

- Labor supply/demand:

$$\boxed{B\frac{(1-n_t)^{-\phi}}{c_t^{-\gamma}} = \frac{1-\tau_{nt}}{1+\tau_{ct}}\underbrace{\left(f(\tilde{k}_t) - \tilde{k}_t f'(\tilde{k}_t)\right)}_{=w_t/q_t}} \tag{A.23}$$

- Euler Equation:

$$\boxed{1 = \beta\left(\frac{1+\tau_{ct}}{1+\tau_{c,t+1}}\right)\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\left[\frac{1-\tau_{i,t+1}}{1-\tau_{it}}(1-\delta) + \frac{1-\tau_{k,t+1}}{1-\tau_{it}}\underbrace{f'(\tilde{k}_{t+1})}_{=r_t/q_t}\right]} \tag{A.24}$$

where k_0 is given, $f(\tilde{k}_t)$ is given. Nests inelastic labor supply.

A.5 Steady State

- $\tilde{k}_t = \bar{k}, c_t = \bar{c}, n_t = \bar{n}$. All taxes constant, $g_t = \bar{g}$
- Take (A.24), divide by $1-\tau$ with $\beta = \frac{1}{1+\rho}$

$$\boxed{\rho + \delta = \frac{1-\tau_k}{1-\tau_i}f'(\bar{k}) \text{ (solve for } \bar{k}\text{)}} \tag{A.25}$$

- Then, from (A.22)

$$\boxed{\bar{g} + \bar{c} + \delta \bar{n} \bar{k} = \bar{n} f(\bar{k})} \tag{A.26}$$

- And from (A.23),

$$\boxed{B \frac{(1-\bar{n})^{-\phi}}{\bar{c}^{-\gamma}} = \frac{1-\tau_n}{1+\tau_c} \left(f(\bar{k}) - \bar{k} f'(\bar{k}) \right)} \tag{A.27}$$

  Solve for $\bar{c}, \bar{n}$ with 2 equations.

- Government budget divide by q_t. Must hold period by period if constant:

$$\boxed{\tau_c \bar{c} + \tau_n \bar{n} \left(f(\bar{k}) - \bar{k} f'(\bar{k}) \right) + \tau_k \bar{n} \bar{k} f'(\bar{k}) + \tau_h = \bar{g} + \tau_i \delta \bar{n} \bar{k}} \tag{A.28}$$

- Questions:
  - Balance τ_k and τ_i?
  - Why τ_c now distorts?
  - Balance τ_n and τ_c?

Appendix B More on the Term Structure in a Growth Model

This section elaborates on the role of the short and long rates in a growth model

B.1 Setup

- Assume complete markets for t contingent assets exist.
- Recall Lucas model, can price based on aggregate consumption.
- Recall:

$$q_t = \beta^t \frac{u'(c_t)}{1+\tau_{ct}} \tag{B.1}$$

  Decompose:

$$q_t = q_0 \cdot \frac{q_1}{q_0} \cdot \frac{q_2}{q_1} \cdots \frac{q_t}{q_{t-1}} \tag{B.2}$$

Define:

$$m_{t,t+1} \equiv \frac{q_{t+1}}{q_t} \text{ (one-period discount factor)} \tag{B.3}$$

Then:

$$q_t = q_0 m_{0,1} m_{1,2} m_{2,3} \cdots m_{t-1,t} \tag{B.4}$$

$$R_{t,t+1}^{-1} \equiv \underbrace{m_{t,t+1}}_{\substack{\text{gross} \\ \text{one period} \\ \text{rate of} \\ \text{interest} \\ \text{from } t \to t+1}} \equiv \underbrace{\frac{1}{1 + r_{t,t+1}}}_{\substack{\text{net} \\ \text{one-period} \\ \text{rate of} \\ \text{interest}}} \tag{B.5}$$

As before, $R_{t,t+1}^{-1} \approx e^{(-r_{t,t+1})}$

B.2 Solve for the short/long rate

- Using q_t decomposition:

$$q_t = q_0 e^{-r_{0,1}} e^{-r_{1,2}} \cdots e^{-r_{t-1,t}} \tag{B.6}$$

Define the net t-period rate of interest between 0 and t.

$$r_{0,t} \equiv \frac{1}{t}\left(r_{0,1} + \cdots r_{t,t-1}\right) \tag{B.7}$$

$$\Rightarrow \frac{q_t}{q_0} = e^{-tr_{0,t}} \tag{B.8}$$

Note: $r_{0,t}$ is the yield to maturity on a 0-coupon bond that matures at time t. $r_{t,t+1}$ is the implied short-rate between $t \to t+1$. So interest rates on $t-$period loans are averages of rates of one-period loans expected to prevail over the horizon of the long run.

- Results from the model:

$$r_{t.t+1} = -\log\left(\frac{q_{t+1}}{q_t}\right) = -\log\left(\beta \frac{1 + \tau_{ct}}{1 + \tau_{c,t+1}} \frac{u'(c_{t+1})}{u'(c_t)}\right) \tag{B.9}$$

But from Euler Equation:

$$1 = \beta\left(\frac{1 + \tau_{ct}}{1 + \tau_{c,t+1}}\right) \frac{u'(c_{t+1})}{u'(c_t)} \left[1 - \delta + (1 - \tau_{k,t+1}) f'(k_{t+1})\right] \tag{B.10}$$

$$\Rightarrow \text{“Short-rate"}: \boxed{r_{t,t+1} \equiv \log(1 - \delta + (1 - \tau_{k,t+1}) f'(k_{t+1}))} \tag{B.11}$$

where more taxes could be added here. And the yield curve at time t plots $r_{t,t+s}$, $\forall s$, and $r_{t,t+s}$ is the s-period long rate. From previous notes and above:

$$r_{t,t+s} \equiv \frac{1}{s}\left(r_{t,t+1} + r_{t+1,t+2} + \cdots r_{t+s-1,t+s}\right) \tag{B.12}$$

where $\frac{1}{s}$ is the average of short rate.

B.3 Fiscal Policy

- Can affect the contemporaneous short rate directly (e.g. τ_{ct} changes k_{t+1}, etc)
- Can affect the short rate indirectly through c_t, k_{t+1} allocations.
- Affects the long rate through averages of short rates.
- Why is this useful?
  - $r_{t,t+s}$ includes expectations of future allocations up to time s periods in the future. Priced today.
  - Many different assets exist with long rates in the real world (e.g. 10-year government bond; 30-year government bond, etc).
  - So with an empirical yield curve from bloomberg, can reverse engineer the expectations of k_{t+s}, c_{t+s} for all s. e.g. Does private sector think a recession is coming?