Growth Models

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1 Quick Review from General Equilibrium

- Homogeneous Functions/CRS (Inputs x, y)
  - If a function $F(x, y)$ has:

$$F(\gamma x, \gamma y) = \gamma \cdot F(x, y) \tag{1}$$

    it is called constant returns to scale or homogeneous of degree 1.

- We can also get from CRS:

$$F(x, y) = \overbrace{\underbrace{x}_{\text{quantity } x} \cdot \underbrace{\boldsymbol{\partial}_x F(x, y)}_{\substack{\text{marginal}\\\text{product}\\\text{of } x}}}^{\text{Also be the "competitive factor payments"}} + \underbrace{y}_{\text{quantity } y} \cdot \underbrace{\boldsymbol{\partial}_y F(x, y)}_{\substack{\text{marginal}\\\text{product}\\\text{of } y}} \tag{2}$$

$$CRS \Rightarrow \begin{cases} \boldsymbol{\partial}_x F(\gamma x, \gamma y) = \boldsymbol{\partial}_x F(x, y) \\ \boldsymbol{\partial}_y F(\gamma x, \gamma y) = \boldsymbol{\partial}_y F(x, y) \end{cases} \text{(homogeneous of degree 0)} \tag{3}$$

- Define $f\left(\frac{x}{y}\right) \equiv F\left(\frac{x}{y}, 1\right) \Rightarrow F(x, y) = yf\left(\frac{x}{y}\right)$, i.e. let $\gamma = \frac{1}{y}$
  - Then from (3)

$$\boldsymbol{\partial}_x F = f'\left(\frac{x}{y}\right) \tag{4}$$

  - And we can also show that

$$\boldsymbol{\partial}_y F = f\left(\frac{x}{y}\right) - f'\left(\frac{x}{y}\right)\left(\frac{x}{y}\right) \tag{5}$$

We will use these to express everything in the capital-labor ratio with CRS.

2 Malthusian Growth Model

- Setup:

$$Y_t = zF(\underbrace{L_t}_{\text{Land}}, \underbrace{N_t}_{\text{Labor}}) \text{ where } F \text{ is CRS (e.g. food products)} \tag{6}$$

  - Let $L_t = L$ be fixed, i.e. can't create land.
  - Let $C_t = Y_t$, consume all productions.
  - Also assume all population, N_t, works.

- Population grows:

  - Based on output (i.e. food) supply.

$$N_{t+1} = N_t\, G\left(\frac{C_t/N_t}{c^*}\right) \tag{7}$$

    where $G(\cdot)$ is growth function, c^* is a constant. And $c_t \equiv C_t/N_t$ is consumption per capita.

  - And growth factor: $G(1) = 1, G' > 0, G'' < 0$
  - Example Growth Factor: $G(\frac{c_t}{c^*}) = \left(\frac{c_t}{c^*}\right)^\gamma$ for $\gamma \in (0, 1)$

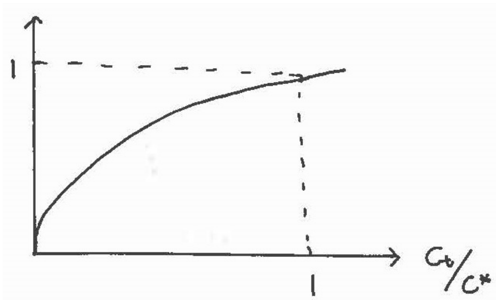

Figure 1: Malthusian Growth Model

- Interpretation

$$\frac{N_{t+1}}{N_t} = \left(\frac{c_t}{c^*}\right)^\gamma \tag{8}$$

So we have:

$$\begin{cases} c_t > c^* & \Rightarrow \frac{N_{t+1}}{N_t} > 1 \quad \text{, population grows} \\ c_t < c^* & \Rightarrow \frac{N_{t+1}}{N_t} < 1 \quad \text{, population shrinks} \end{cases} \tag{9}$$

so c^* is a <u>subsistence level of consumption</u>

- <u>Production</u>:

$$C_t = zF(L, N_t) \tag{10}$$

$$\Rightarrow C_t = N_t zF\left(\frac{L}{N_t}, 1\right) \text{ by CRS} \tag{11}$$

$$\Rightarrow c_t \equiv zF(\ell_t, 1) \text{ where } \ell_t \equiv \frac{L}{N_t} \text{ (land-per-capita)} \tag{12}$$

For example, assume

$$F(\ell_t, 1) = {\ell_t}^{\alpha},\ \alpha \in (0, 1) \tag{13}$$

- <u>Equations</u>

$$c_t = z{\ell_t}^{\alpha} \tag{14}$$

$$\frac{N_{t+1}}{N_t} = \left(\frac{c_t}{c^*}\right)^{\gamma} \tag{15}$$

- <u>Substitute:</u>

$$\frac{N_{t+1}}{N_t} = \left(\frac{z(\frac{L}{N_t})^{\alpha}}{c^*}\right)^{\gamma} = \frac{z^{\gamma}(\frac{L}{N_t})^{\alpha\gamma}}{(c^*)^{\gamma}} \tag{16}$$

$$\Rightarrow \boxed{N_{t+1} = \left(\frac{z}{c^*}\right)^{\gamma} L^{\alpha\gamma} N_t^{1-\alpha\gamma}} \tag{17}$$

- <u>Steady State</u>:

$$\bar{N} = \left(\frac{z}{c^*}\right)^{\gamma} L^{\alpha\gamma} \bar{N}^{1-\alpha\gamma} \tag{18}$$

$$\Rightarrow 1 = \left(\frac{z}{c^*}\right)^{\gamma} \left(\frac{L}{\bar{N}}\right)^{\alpha\gamma} \tag{19}$$

$$\Rightarrow \bar{N} = \frac{L}{\left(\frac{c^*}{z}\right)^{\frac{1}{\alpha}}} \tag{20}$$

Substitute (20) into (14) to show that $c_t = c^*$. So only equilibrium is one of subsistence per-capita consumption <u>for all</u>. $\uparrow z \Rightarrow \bar{N} \uparrow$ but $c = c^*$. Pessimistic comment on technological change?

- Dynamics
  - Take logs of (1):

$$\log N_{t+1} = \underbrace{\gamma\left(\log z - \log c^* + \alpha \log L\right)}_{\phi_0} + \underbrace{(1-\alpha\gamma)}_{\phi_1} \log N_t \tag{21}$$

  - Or of form:

$$n_{t+1} = \phi_0 + \phi_1 n_t \text{ if } n_t \equiv \log N_t \tag{22}$$

  - Can use our old toolset of linear difference equations:

$$x_t = \begin{pmatrix} 1 \\ n_t \end{pmatrix}, A = \begin{pmatrix} 1 & 0 \\ \phi_0 & \phi_1 \end{pmatrix}, \boxed{x_t = A^t x_0} \tag{23}$$

3 Solow Growth Model

- Setup:
  - Let N_t be exogenous: $N_{t+1} = (1+g)N_t$, N_0 given and g a constant population growth rate.
  - Other factor of production is capital, not land.

$$\underbrace{K_{t+1}}_{\substack{\text{Next}\\\text{periods}\\\text{capital}}} = \underbrace{(1-\delta)}_{\substack{\text{depreciation}\\\text{of capital}}} K_t + \underbrace{X_t}_{\substack{\text{investment}\\\text{in new}\\\text{capital}}}, \delta \in (0,1) \tag{24}$$

- Production:
  - We have:

$$Y_t = zF(K_t, N_t) \text{ with C.R.S} \tag{25}$$

  which can be used for consumption or for capital.

  - And:

$$C_t + X_t = zF(K_t, N_t) \tag{26}$$

  which is the resource constraint for economy.

- Define the capital-labor ratio:

$$k_t \equiv \frac{K_t}{N_t} \tag{27}$$

- And investment per capital:

$$x_t \equiv \frac{X_t}{N_t} \tag{28}$$

- Consumption/capital:

$$c_t \equiv \frac{C_t}{N_t} \tag{29}$$

- Converting equations, divide (24) by N_t:

$$\frac{K_{t+1}}{N_t} = (1-\delta)\frac{K_t}{N_t} + \frac{X_t}{N_t}, \text{ or:} \tag{30}$$

$$\frac{N_{t+1}}{N_{t+1}}\frac{K_{t+1}}{N_t} = \left(\frac{N_{t+1}}{N_t}\right)\left(\frac{K_{t+1}}{N_{t+1}}\right) = (1-\delta)\frac{K_t}{N_t} + \frac{X_t}{N_t}, \text{ where } N_{t+1} \text{ is the growth of population} \tag{31}$$

$$\Rightarrow \boxed{k_{t+1}(1+g) = (1-\delta)k_t + x_t} \tag{32}$$

- Next, <u>assume</u> (not based on optimization!):
  - Agents save fraction $s \in (0,1)$ of output. i.e. by economy resource constraint.

$$C_t = (1-s)zF(K_t, N_t) \tag{33}$$

$$\Rightarrow \text{dividing by } N_t\ , c_t = (1-s)zF(k_t) \tag{34}$$

  - Plug into equation (32):

$$\boxed{k_{t+1} = \frac{(1-\delta)k_t}{1+g} + \frac{sz}{1+g}f(k_t)} \tag{35}$$

  - This is a non-linear difference equation in k_t

– Find a steady state solution:

$$\bar{k} = \frac{1-\delta}{1+g}\bar{k} + \frac{sz}{1+g}f(\bar{k}) \tag{36}$$

$$\Rightarrow \left(\frac{1+g}{1+g} - \frac{1-\delta}{1+g}\right)\bar{k} = \frac{sz}{1+g}f(\bar{k}) \tag{37}$$

$$\Rightarrow \boxed{(g+\delta)\bar{k} = szf(\bar{k})} \tag{38}$$

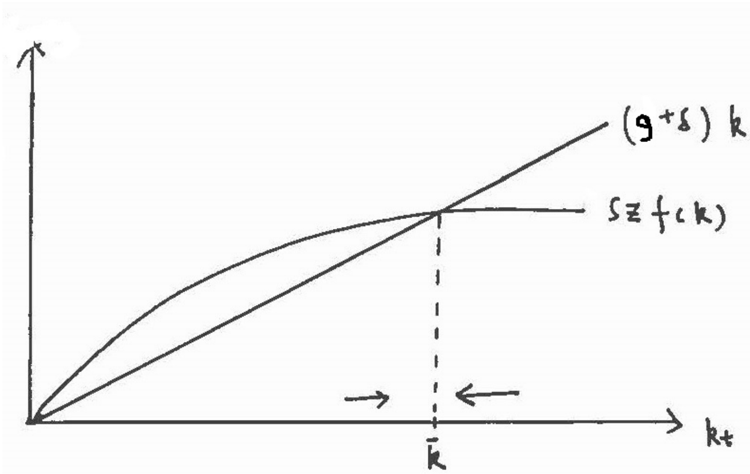

Figure 2: Convergence in a Solow Growth Model

- Use to understand effect on $\bar{k}$ of:
  - increase in z
  - increase in s
  - increase in g

4 Neoclassical growth

- Setup:
  - Solow took savings rate, s, as constant
  - Instead, make it choice of a representative consumer
  - First solve planner's problem, then competition equilibrium.
- Technology:
  - $Y = F(K, N)$ with CRS.

– Let $k \equiv \frac{K}{N}$ then per capita like in Solow

– Let $f(k) \equiv F(k, 1)$, then $Nf(k) = F(K, N)$ from CRS:

$$c_t + \underbrace{x_t}_{\equiv k_{t+1}-(1-\delta)k_t} = f(k_t) \tag{39}$$

or,

$$\underbrace{c_t + k_{t+1}}_{\substack{\text{consume of} \\ \text{capital} \\ \text{next period}}} = \underbrace{f(k_t)}_{\text{production}} + \underbrace{(1-\delta)k_t}_{\substack{\text{last period's capital,} \\ \text{depreciated}}} \quad \text{i.e., the gross output per capita} \tag{40}$$

$$\Leftrightarrow C_t + K_{t+1} = F(K_t, N_t) + (1-\delta)K_t \tag{41}$$

where $c_t = \frac{C_t}{N}$, replace K_{t+1} as k_{t+1}, $f(k) = \frac{1}{N}F(K, N)$

– Note: we are leaving out a possible irreversibility constraint on capital.

  * If you don't want the consumer to be able to "eat" their machines, then don't let capital contract more than the depreciation rate.

$$x_t \geq 0 \tag{42}$$

  or,

$$K_{t+1} \geq (1-\delta)K_t \tag{43}$$

- Preferences:

$$u(c) \text{ (no disutility of labor)} \tag{44}$$

where $u'(c) > 0$, $u"(c) < 0$, $u'(0) = \infty$
Welfare:

$$\sum_{t=0}^{T} \beta^t u(c_t), \text{ where } T \leq \infty \tag{45}$$

- Planning Problem:

$$\max_{c_t, k_{t+1}} \sum_{t=0}^{T} \beta^t u(c_t) \tag{46}$$

$$\text{s.t. } c_t + k_{t+1} = f(k_t) + (1-\delta)k_t, \text{ where } k_0 \text{ was given} \tag{47}$$

$$\text{Transversality or } k_{T+1} \geq 0 \tag{48}$$

  - Lagrangian (without transversality):

$$L = \sum_{t=0}^{\infty} \beta^t \left(u(c_t) + \lambda_t \left(f(k_t) + (1-\delta)k_t - c_t - k_{t+1} \right) \right) \tag{49}$$

  - FONC:

$$[c_t] : u'(c_t) - \lambda_t = 0 \tag{50}$$

$$[k_{t+1}] : -\lambda_t + \beta \lambda_{t+1} \left[f'(k_{t+1}) + (1-\delta) \right] = 0 \text{ for } t = 0, \cdots, T-1 \tag{51}$$

$$[k_{T+1}] : -\beta^T \lambda_T \leq 0, = 0 \text{ if } k_{T+1} > 0 \tag{52}$$

$$\text{Or if } T = \infty, \lim_{T \to \infty} \beta^T \lambda_T k_{T+1} = 0 \tag{53}$$

- Summary of Equations:
  - Equations in c_t, λ_t, k_t given k_0:

$$\boxed{\begin{aligned} & u'(c_t) = \lambda_t && (54) \\ & \lambda_t = \beta \lambda_{t+1} \left[f'(k_{t+1}) + (1-\delta) \right] && (55) \\ & k_{t+1} + c_t = f(k_t) + (1-\delta)k_t && (56) \end{aligned}}$$

  - (54) shows the interpretation of the present-value Lagrange multiplier λ_t as the marginal utility of consumption on relaxing the resource constraint
  - Assume $T = \infty$, look for steady state:

$$\lambda_t = \bar{\lambda} \tag{57}$$

$$c_t = \bar{c} \tag{58}$$

$$k_t = \bar{k} \tag{59}$$

- Also:

$$1 + \underbrace{\rho}_{\substack{\text{discount} \\ \text{rate}}} \equiv \frac{1}{\beta} = f'(\bar{k}) + (1-\delta) \tag{60}$$

$$\text{Or: } \boxed{f'(\bar{k}) = \rho + \delta} \text{ (Modified golden rule!)} \tag{61}$$

- Note: $\bar{k}$ is independent of preference except β. Then:

$$\bar{k} + \bar{c} = f(\bar{k}) + (1-\delta)\bar{k} \Rightarrow \boxed{\bar{c} = f(\bar{k}) - \delta\bar{k}} \tag{62}$$

- Dynamics: Shooting Method:
  - Finite horizon: $k_{T+1} = 0$
  - Equations:

$$u'(c_t) = \lambda_t \tag{63}$$

$$\lambda_t = \beta\lambda_{t+1}\left[f'(k_{t+1}) + (1-\delta)\right] \tag{64}$$

$$k_{t+1} + c_t = f(k_t) + (1-\delta)k_t \tag{65}$$

  - Boundaries: k_0 given, $k_{T+1} = 0$ if $T < \infty$.
  - Shooting method:
    * Guess λ_0
    * Use (63) to get c_0
    * Use (65) to get k_1 since k_0 given
    * Use (64) to get λ_1
    * Use (63) to get c_1
    * keep repeating until k_{T+1} calculated
    * Check that $k_{T+1} \approx 0$
    * If not, change λ_0 and repeat
      If $k_{T+1} > 0$, "eat more" by lowering λ_0
  - If infinite horizon, know that $k_t \to k_\infty$, where $f'(k_\infty) = \rho + \delta$
  - Can try to "shoot" at $k_\infty = k_{T+1}$ for very large T
- “Turnpike" Theorem For large T, will spend more of the time near k_∞

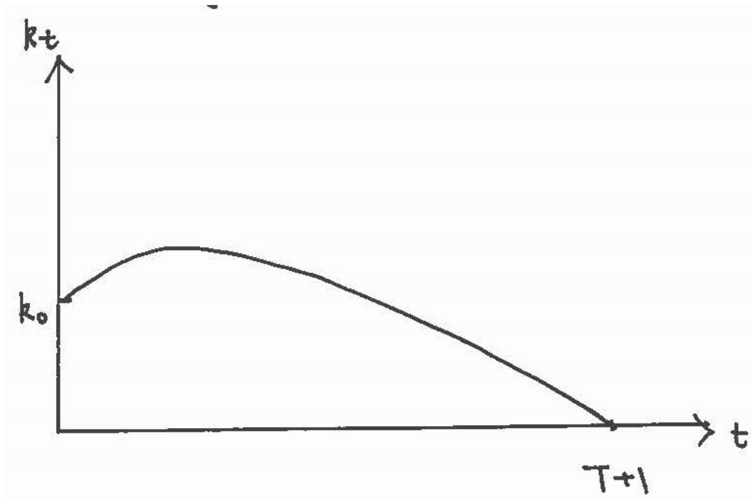

Figure 3: Shooting Method

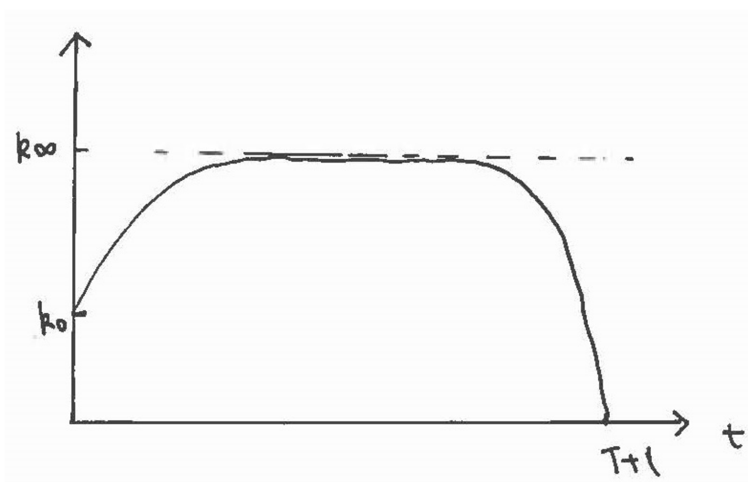

Figure 4: Turnpike Theorem, with large T

5 Decentralization with a Competitive Equilibrium

5.1 Basic setup

- Take the same technology $F(K_t, N_t)$
- Add in potential for endogenous labor supply N_t
- Formulate with large number of households, firms.
- Price system: $\{q_t, w_t, r_t\}_{t=0}^{\infty}$ at date 0 (i.e. time 0 budget), where q_t is the price of good, w_t is wage and r_t is the capital rental.

5.2 Households

- Owns labor, capital. Rent both to firms.
- Purchases goods for consumption or capital investment.
- Include disutility of labor in utility: $u(c, n)$
- Solve problem:

$$\max_{\{c_t, n_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t, n_t) \tag{66}$$

$$\text{s.t.} \sum_{t=0}^{\infty} \underbrace{q_t}_{\substack{\text{Price of}\\ \text{homogeneous}\\ \text{good at } t\\ \text{in time 0}}} \left(c_t + \underbrace{k_{t+1} - (1-\delta)k_t}_{\equiv x_t,\text{ net saving}} \right) \leq \sum_{t=0}^{\infty} (\underbrace{w_t}_{\substack{\text{wages}\\ \text{on labor}\\ \text{supplied}}} n_t + \underbrace{r_t}_{\substack{\text{rents}\\ \text{from}\\ \text{capital}\\ \text{supplied}}} k_t) \tag{67}$$

$$0 \leq n_t \tag{68}$$

$$n_t \leq 1 \quad \text{1 unit of time endowment} \tag{69}$$

- Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[u(c_t, n_t) + \lambda_t (1 - n_t) + \phi_t n_t \right] \tag{70}$$

$$+ \mu \left[\sum_{t=0}^{\infty} (w_t n_t + r_t k_t) - \sum_{t=0}^{\infty} q_t \left(c_t + k_{t+1} - (1-\delta) k_t \right) \right] \tag{71}$$

where λ_t is LM on $n_t \leq 1$, ϕ_t is LM on $n_t \geq 0$ and μ is LM on time 0 budget.

- FONC:

$$[c_t] : \beta^t \boldsymbol{\partial}_c u(c_t, n_t) - \mu q_t = 0 \tag{72}$$

$$[n_t] : \beta^t \left(\boldsymbol{\partial}_n u(c_t, n_t) + \phi_t - \lambda_t\right) + \mu w_t = 0 \tag{73}$$

$$[k_{t+1}] : r_{t+1} + q_{t+1}(1-\delta) - q_t = 0 \tag{74}$$

$$\text{Complimentary slackness conditions:} \begin{cases} \lambda_t(1-n_t) = 0 \\ \phi_t n_t = 0 \end{cases} \tag{75}$$

Note: (75) can also be written as:

$$-\beta^t \boldsymbol{\partial}_n u(c_t, n_t) \begin{cases} = \mu w_t & \text{if } 0 < n_t < 1 \\ > \mu w_t & \text{if } n_t = 1 \\ < \mu w_t & \text{if } n_t = 0 \end{cases} \tag{76}$$

- Summarizing Equations for $\{c_t, k_t, n_t\}$ in terms of $\{q_t, r_t, w_t\}$:

$$q_t = q_{t+1}(1-\delta) + r_{t+1} \tag{77}$$

$$\mu q_t = \beta^t \boldsymbol{\partial}_c u(c_t, n_t) \tag{78}$$

$$-\beta^t \boldsymbol{\partial}_n u(c_t, n_t) \begin{cases} = \mu w_t & \text{if } 0 < n_t < 1 \\ > \mu w_t & \text{if } n_t = 1 \\ < \mu w_t & \text{if } n_t = 0 \end{cases} \tag{79}$$

Combining (78) and (79) with interior solutions gives us the same as our labor supply decision in the static general equilibrium,

$$\frac{-\boldsymbol{\partial}_n u(c_t, n_t)}{\boldsymbol{\partial}_c u(c_t, n_t)} = \frac{w_t}{q_t} \tag{80}$$

For now, assume that the agent gains no utility from leisure, so:

$$u(c, n) = u(c) \Rightarrow \boldsymbol{\partial}_n u = 0 \Rightarrow n_t = 1 \tag{81}$$

i.e. Inelastic supply of all labor

5.3 Firms (Price taker)

- Face prices : $\{q_t, w_t, r_t\}$, choose inputs for production.

- Solve Problem:

$$\max_{K_t,N_t} \sum_{t=0}^{\infty} \left[\underbrace{q_t F(K_t,N_t)}_{\text{revenue}} \underbrace{-r_t K_t - w_t N_t}_{\text{input costs}} \right] \tag{82}$$

Use $F(K,N) = \boldsymbol{\partial}_K F(K,N)K + \boldsymbol{\partial}_N F(K,N)N$ with CRS:

$$= \max_{K_t,N_t} \sum_{t=0}^{\infty} \left[\left(q_t \boldsymbol{\partial}_K F(K_t,N_t) - r_t \right) K_t + \left(q_t \boldsymbol{\partial}_N F(K_t,N_t) - w_t \right) N_t \right] \tag{83}$$

- Since CRS, we have a 0-profit condition as before. Necessary conditions:

$$q_t \boldsymbol{\partial}_K F(K_t,N_t) = r_t \tag{84}$$
$$q_t \boldsymbol{\partial}_N F(K_t,N_t) = w_t \tag{85}$$

Since CRS, use $F\left(\frac{K}{N},1\right) = f(k)$ for $k \equiv \frac{K}{N}$, and other expressions from this sections:

$$\boldsymbol{\partial}_k F(K,N) = \boldsymbol{\partial}_k F\left(\frac{K}{N},1\right) = f'(k) \tag{86}$$
$$\boldsymbol{\partial}_N F(K,N) = f(k) - kf'(k) \tag{87}$$

So:

$$\frac{r_t}{q_t} = f'(k_t) \tag{88}$$
$$\frac{w_t}{q_t} = f(k_t) - k_t f'(k_t) \tag{89}$$

So workers paid marginal products of inputs and firms of indeterminate size, but 0 profits.

5.4 Solve for Competitive Equilibrium

- Definition

  A C.E. is a Price system $\{q_t.w_t, r_t\}_{t=0}^{\infty}$ and a Feasible allocation $\{k_{t+1}, c_t\}_{t=0}^{\infty}$, such that:

  - Given price system, allocations solve firm problem.
  - Given price system, allocations solve Households problem.

- Solving for the equilibrium:

  - Solve planning problems $\{c_t, k_{t+1}\}$

- Guess $\mu = 1$ (indeterminate), plug in with c_t:

$$q_t = \beta^t u'(c_t) \text{ (from consumer's problem)} \tag{90}$$

$$r_t = q_t f'(k_t) \text{ (from firm's problem)} \tag{91}$$

$$w_t = q_t \left(f(k_t) - k_t f'(k_t) \right) \text{ (from firm's problem)} \tag{92}$$

- Verify that this price system and allocations fulfills all of the conditions of the Firm and HH problems (Budget constraints, etc).

• Conclusion:

  - For this economy, the planning problem and the competitive equilibrium have the same allocations.
  - The prices decentralize the planning problem.