(Static) General Equilibrium

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1 Basic setup and Consumer Preferences

- Recall notation: $\boldsymbol{\partial}_c u \equiv \frac{\partial u(c,\ell)}{\partial c}$
- Commodities:
  - c: consumption good
  - ℓ: labor
  - k: capital, exogenously given for now
- Households:
  - Preferences over $\{c, \ell\}$: $u(c, \ell)$, where $0 \leq \ell \leq 1$ with $\boldsymbol{\partial}_\ell u(c, \ell) \leq 0$ and $\boldsymbol{\partial}_c u(c, \ell) > 0$
  - c is consumption, ℓ is hours of working (labor)
  - Total Derivative gives indifference curves:
    $\boldsymbol{\partial} u = \boldsymbol{\partial}_c u \boldsymbol{\partial} c + \boldsymbol{\partial}_\ell u \boldsymbol{\partial} \ell = 0 \Rightarrow \frac{\boldsymbol{\partial} c}{\boldsymbol{\partial} \ell} = \frac{-\boldsymbol{\partial}_\ell u}{\boldsymbol{\partial}_c u} = MRS$, where MRS is "Marginal Rate of Substitution" between leisure and consumption
  - Example: $u(c, \ell) = \log c - B\ell$, where B is the disutility of labor
    $\Rightarrow \frac{\boldsymbol{\partial} c}{\boldsymbol{\partial} l} = cB = \frac{-\boldsymbol{\partial}_\ell u}{\boldsymbol{\partial}_c u}$,

2 Production and Feasibility

- Setup production function:

$$Y = F(k, \ell) \tag{1}$$

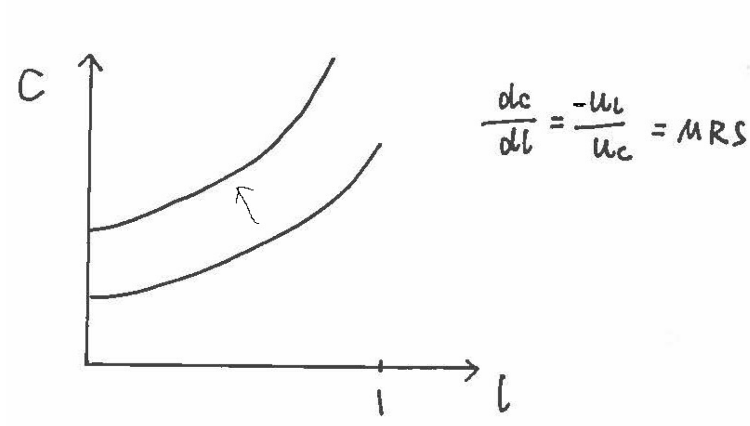

Figure 1: Indifference Curve

where Y is the total output, F is the production function, k is the capital inputs. ℓ is the labor inputs

- Assume $F(k,\ell)$ has constant returns to scale (e.g. double inputs, double outputs) i.e. $\lambda F(k,\ell) = F(\lambda k, \lambda \ell)$

- Also note, differentiating with respect to ℓ:

$$\lambda \boldsymbol{\partial}_\ell F(k,\ell) = \lambda \boldsymbol{\partial}_\ell F(\lambda k, \lambda \ell) \tag{2}$$

$$\Rightarrow \boldsymbol{\partial}_\ell F(k,\ell) = \boldsymbol{\partial}_\ell F(\lambda k, \lambda \ell), \forall \lambda \text{ which called ``homogeneous of degree zero''} \tag{3}$$

Since works for any λ, let $\lambda = \frac{1}{k}$ in (3).[1]

$$\Rightarrow \boldsymbol{\partial}_\ell F(k,\ell) = \boldsymbol{\partial}_\ell F\left(1, \frac{\ell}{k}\right) \tag{4}$$

So homogenous of degree 0 $\Rightarrow$ partials only depends on ratios of inputs

- Example:

$$F(k,\ell) = Ak^\alpha \ell^{1-\alpha} \text{ for } \alpha \in (0,1) \tag{5}$$

Note: $\underbrace{\boldsymbol{\partial}_\ell F > 0, \boldsymbol{\partial}_k F > 0}_{\text{positive marginal products}}, \underbrace{\boldsymbol{\partial}_{kk} F < 0, \boldsymbol{\partial}_{\ell\ell} F < 0}_{\text{diminishing returns}}$

Also: $\boldsymbol{\partial}_\ell F(k,\ell) = (1-\alpha)Ak^\alpha \ell^{-\alpha} = (1-\alpha)A\left(\frac{\ell}{k}\right)^{-\alpha}$, $\boldsymbol{\partial}_k F(k,\ell) = \alpha A \left(\frac{\ell}{k}\right)^{1-\alpha}$

- Feasibility: $c + G \leq F(k,\ell)$, where c is consumption, G is exogenously given government expenditure in real goods, $F(k,\ell)$ is total output given choice ℓ.

[1] The choice of $\lambda = \frac{1}{k}$ is arbitrary. We could use $\lambda = \frac{1}{\ell}$ and do similar algebra in the capital-to-labor ratio. In fact, it will be more convenient when we do the neo-classical growth model with evolving capital.

3 Planning Problem (Command Economy)

- Dictators do not need prices!
- Setup:

$$\max_{c,\ell} \{u(c,\ell)\} \tag{6}$$

$$\text{s.t. } c + G \leq F(k,\ell) \tag{7}$$

- Lagrangian:

$$L = u(c,\ell) + \lambda \left[F(k,\ell) - c - G\right] \tag{8}$$

- FONC: (Assume Interior)

$$[c] : \boldsymbol{\partial}_c u - \lambda = 0 \tag{9}$$

$$[l] : \boldsymbol{\partial}_\ell u + \lambda \boldsymbol{\partial}_\ell F(k,\ell) = 0 \Rightarrow \frac{-\boldsymbol{\partial}_\ell u}{\boldsymbol{\partial}_c u} = \boldsymbol{\partial}_\ell F \tag{10}$$

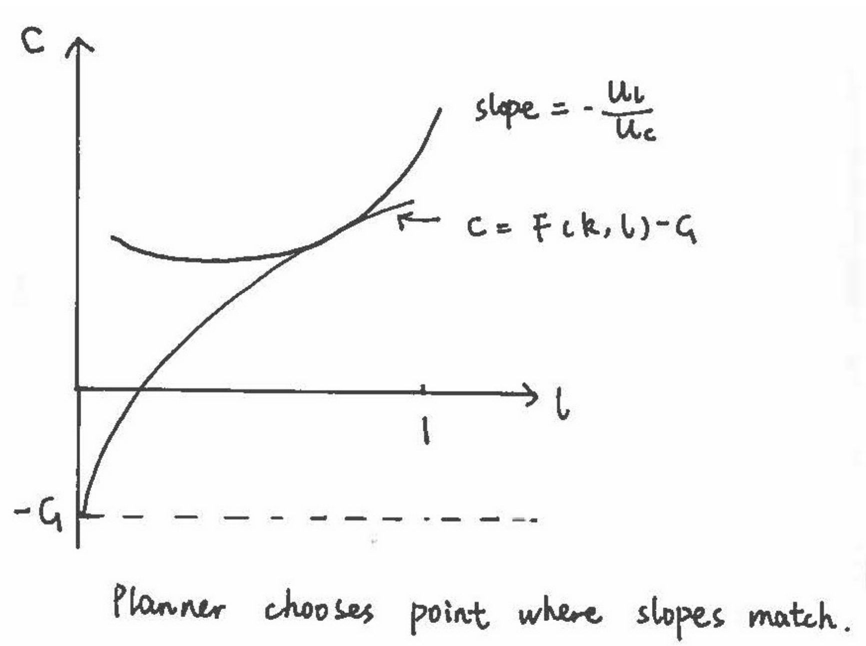

Figure 2: Planner

4 Competitive Equilibrium

4.1 Households in Market Economy

Assume that consumers are price takers

- Assume that consumers own the capital (it doesn't matter if they do or the firms) and rent labor/capital at market prices.[2]
- Nominal prices mean denoted in \$, whereas real prices are relative to the consumption goods price.
- The nominal price of the consumption good is $\tilde{p}$, the nominal wage is $\tilde{w}$, and the nominal rental rate of capital is $\tilde{r}$.
- Let the real wage and rental rate be $w \equiv \tilde{w}/\tilde{p}$ and $r \equiv \tilde{r}/\tilde{p}$
- Leave in a marginal tax rate on labor, τ_ℓ
- Given prices, the consumer's problem is

$$\max_{c,\ell} \{u(c,\ell)\} \tag{11}$$

$$\text{s.t. } \tilde{p}c \leq (1-\tau_\ell)\tilde{w}\ell + \tilde{r}k \tag{12}$$

- Alternatively, in real terms, just divide budget by $\tilde{p}$

$$\max_{c,\ell} \{u(c,\ell)\} \tag{13}$$

$$\text{s.t. } c \leq (1-\tau_\ell)w\ell + rk \tag{14}$$

- Lagrangian:

$$\mathcal{L} = u(c,\ell) + \lambda\left[w(1-\tau_\ell)\ell + rk - c\right] \tag{15}$$

- FONC: (The $c \geq 0$ and $l \geq 0$ multipliers implied)

$$[c] : \boldsymbol{\partial}_c u - \lambda \leq 0, = 0 \text{ if } c > 0 \tag{16}$$

$$[l] : \boldsymbol{\partial}_\ell u + \lambda w(1-\tau_\ell) \leq 0, = 0 \text{ if } \ell > 0 \tag{17}$$

[2] At this point, we are assuming that consumer's are identical. In the "Interest Rates" notes, we will prove conditions under which this doesn't matter.

- At equality (i.e., if both are interior):

$$\frac{-\boldsymbol{\partial}_\ell u}{\boldsymbol{\partial}_c u} = \underbrace{w(1-\tau_\ell)}_{\substack{\text{marginal rate}\\ \text{of substitution}}} \tag{18}$$

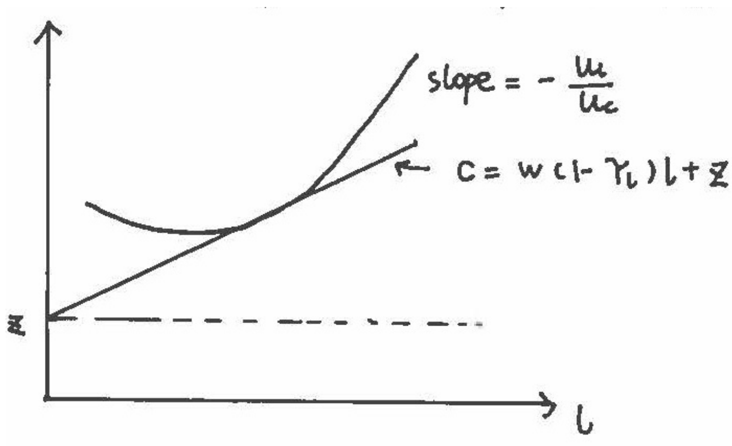

Figure 3: At equality, budget matches the indifference curve

4.2 Firm's Problem

- Instead of a planner, assume firm labeled i rents capital and labor from the consumer. Firm is competitive, i.e. price taker

$$\max_{\{Y(i),k(i),\ell(i)\}} \{\tilde{p}Y(i) - \tilde{r}k(i) - \tilde{w}\ell(i)\} \text{ (maximizing profits)} \tag{19}$$
$$Y(i) = F(k(i), \ell(i)), \text{ where } Y(i) \text{ is outputs if rents } k(i),\ \ell(i) \tag{20}$$

- So the problem is transformed to:

$$\max_{\{k(i),\ell(i)\}} \{\tilde{p}F(k(i),\ell(i)) - \tilde{r}k(i) - \tilde{w}\ell(i)\} \tag{21}$$

- FONC:

$$[k] : \tilde{p}\boldsymbol{\partial}_k F(k(i),\ell(i)) - \tilde{r} = 0 \Rightarrow \frac{\tilde{r}}{\tilde{p}} = \boldsymbol{\partial}_k F(k(i),\ell(i)) \tag{22}$$
$$[l] : \tilde{p}\boldsymbol{\partial}_\ell F(k(i),\ell(i)) - \tilde{w} = 0 \Rightarrow \frac{\tilde{w}}{\tilde{p}} = \boldsymbol{\partial}_\ell F(k(i),\ell(i)) \tag{23}$$

Using the real prices of inputs, $\boldsymbol{\partial}_k F(k(i),\ell(i)) = r$, $\boldsymbol{\partial}_\ell F(k(i),\ell(i)) = w$. But since "homogeneous of degree 0", we have:

$$\boldsymbol{\partial}_k F\left(1, \frac{\ell(i)}{k(i)}\right) = r, \boldsymbol{\partial}_\ell F\left(1, \frac{\ell(i)}{k(i)}\right) = w \tag{24}$$

i.e. the size of the particular firm, i, and the levels of $k(i)$ and $\ell(i)$, cannot be determined by these equations. Just the ratio $\frac{k(i)}{\ell(i)}$, which must be identical for all firms. Because of this, drop the i index.

- With constant returns to scale, we can use the output of a single "representative" firm with these competitive prices. This is an example of a proof of "aggregation" to a representative agent. We will do similar derivations for using a representative consumer.

4.3 Competitive Equilibrium (With $G = 0$ and $\tau_\ell = 0$)

- A feasible allocation is a bundle of $\{k, \ell, c\}$ that satisfies $c \leq F(k, \ell)$ with k given
- A price system is a pair $\{w, r\}$
- A competitive equilibrium is a feasible allocation and price system such that:
  - (1) Given $\{w, r\}$, $\{c, \ell\}$ solves the household's problem.
  - (2) Given $\{w, r\}$, $\{\ell, k\}$ solves the firm's problem.

4.4 Example

4.4.1 Setup:

- Compute competitive equilibrium, where

$$u(c, \ell) = \ln c - B\ell \tag{25}$$
$$F(k, \ell) = Ak^\alpha \ell^{1-\alpha}, G = 0 \tag{26}$$

- Method:
  - (a) Solve planner's problem
  - (b) Reverse engineer required prices to support that equilibrium
  - (c) Verify competitive equilibrium conditions hold.

4.4.2 Steps:

- (a) Planning Problem:
  Recall from FONC for planner:

$$\boldsymbol{\partial}_\ell F = \frac{-\boldsymbol{\partial}_\ell u}{\boldsymbol{\partial}_c u} \Rightarrow (1-\alpha)A\left(\frac{\ell}{k}\right)^{-\alpha} = cB \tag{27}$$

  And from feasibility:

$$c = Ak^\alpha \ell^{1-\alpha} \tag{28}$$

  So we have:

$$(1-\alpha)A\ell^{-\alpha}k^\alpha = BAk^\alpha \ell^{1-\alpha} \Rightarrow \tag{29}$$

$$\boxed{l = \frac{1-\alpha}{B}} \tag{30}$$

  Substitute into feasibility,

$$\boxed{c = Ak^\alpha \left(\frac{1-\alpha}{B}\right)^{1-\alpha}} \tag{31}$$

  This is the feasible allocation(c, l, k)

- (b) Reverse engineer prices:

$$F(k,\ell) = Ak^\alpha \ell^{1-\alpha} \Rightarrow \tag{32}$$
$$\boldsymbol{\partial}_k F(k,\ell) = \alpha Ak^{\alpha-1}\ell^{1-\alpha} \tag{33}$$
$$F_l(k,\ell) = (1-\alpha)Ak^\alpha \ell^{-\alpha} \tag{34}$$

  From Firm's FOC:

$$\boldsymbol{\partial}_k F(k,\ell) = \alpha Ak^{\alpha-1}\ell^{1-\alpha} = \boxed{\alpha Ak^{\alpha-1}\left(\frac{1-\alpha}{B}\right)^{1-\alpha} = r} \tag{35}$$

$$\boldsymbol{\partial}_\ell F(k,\ell) = (1-\alpha)Ak^\alpha \ell^{-\alpha} = \boxed{(1-\alpha)Ak^\alpha \left(\frac{1-\alpha}{B}\right)^{-\alpha} = w} \tag{36}$$

- (c) Verify: Since $\alpha \in (0,1)$, $A > 0$, $B > 0$, $k > 0$, then $r, w > 0$ are prices that are strictly positive

4.4.3 Verification of CE conditions:

1. Feasibility: Yes, since used same feasibility to solve planners problem.

2. FONC of Firms: Yes, used to reverse engineer prices directly.

3. FONC of household:
   Note, FONC used for planner:

$$\frac{-\boldsymbol{\partial}_\ell u}{\boldsymbol{\partial}_c u} = \boldsymbol{\partial}_\ell F \tag{37}$$

   FONC for household:

$$\frac{-\boldsymbol{\partial}_\ell u}{\boldsymbol{\partial}_c u} = w \tag{38}$$

   Plug in FONC of firm for $w = \boldsymbol{\partial}_\ell F$

$$\frac{-\boldsymbol{\partial}_\ell u}{\boldsymbol{\partial}_c u} = \boldsymbol{\partial}_\ell F \tag{39}$$

   Same FOC used as planner.

In nominal terms, this holds for any $\tilde{p} > 0$, but is unique in real terms.