Markov Chains and Unemployment

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1 Markov Chains

A model of a stochastic process with discrete number of states.

1.1 Random Variable and Mathematical Expectation

Notation for discrete states:

- $n = 1, \ldots, N$ represents for possible "states of the world" (e.g. individual unemployed, employed,...)
- $\pi_n = \mathbb{P}\left(\text{state of the world is n}\right)$

  $\pi_n \geq 0$, $\sum\limits_{n=1}^{N} \pi_n = 1$, i.e. world must be in one of the states

  Stack as a vector: $\pi \equiv \begin{bmatrix} \pi_1 & \ldots & \pi_N \end{bmatrix}$
- Random variable $Y \in \{y_1, \ldots y_N\}$
- Values mapping states of the world for r.v. Y: $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$

  e.g. If event n is unemployed, then income if unemployed is y_n
- $\mathbb{E}\left[Y\right] = \sum\limits_{n=1}^{N} \mathbb{P}\left(Y = y_n\right) y_n = \sum\limits_{n=1}^{N} \pi_n y_n = \pi \cdot y$ (i.e., inner product)
  - i.e. weight the realizations with the probabilities

- e.g., if the probability of unemployment is $\pi_1 = 0.1$, income from unemployment insurance is $y_1 = 15,000$;
  probability of employment is $\pi_2 = 0.9$, income from employment is $y_2 = 40,000$.
  Then expected income (or average across states of world):

  $$\mathbb{E}\left[Y\right] = (0.1 \times 15,000) + (0.9 \times 40,000)$$

- We could use this to find an individuals expected income at some point in the future. Alternatively, we can use this to find averages for a continuum of population. A step towards aggregation.

- e.g., if 10 % of population is unemployed at \$15,000 and 90 % of population is employed at \$40,000. Then the average income is $\mathbb{E}\left[Y\right]$

1.2 Transitions

For example, Let ϕ = probability to become employed. Let State 1 $\leftrightarrow E$, State 2 $\leftrightarrow U$

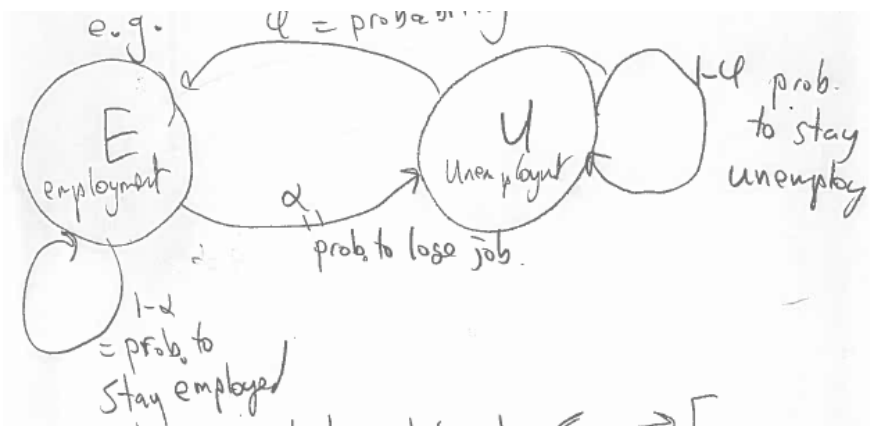

Figure 1: Markov Chain

Transition Matrix:

$$P = \begin{array}{cc} & \begin{array}{cc} state_{1,t+1} & state_{2,t+1} \end{array} \\ \begin{array}{c} state_{1,t} \\ state_{2,t} \end{array} & \begin{pmatrix} 1-\alpha & \alpha \\ \phi & 1-\phi \end{pmatrix} \end{array} \tag{1}$$

Let π_t be the probability mass function (pmf) of a random variable of an agent's employment status at time t. This is a probability mass function (pmf) since the possible events is discrete.

- If employed at time 0, $\pi_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}$

- If 50% chance of employment at time 3, $\pi_3 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

1.3 Evolution of Probability Distribution

Find the evolution of the probability mass function for the random variable with the transition matrix P. A property of markov chains:

$$\pi_{t+1} = \pi_t \cdot P \tag{2}$$

Careful with the order of the matrix product!

<u>Iterate forward:</u>

$$\pi_{t+j} = \pi_t \cdot P^j \tag{3}$$

Example:

- Started employed at $t = 0$, i.e. $\pi_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}$
- Probability of unemployment/employment at $t = 1$:

$$\pi_1 = \pi_0 \cdot P = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1-\alpha & \alpha \\ \phi & 1-\phi \end{bmatrix} = \begin{bmatrix} 1-\alpha & \alpha \end{bmatrix} \tag{4}$$

At time 2:

$$\begin{aligned} \pi_2 &= \pi_1 \cdot P && (5) \\ &= \begin{bmatrix} 1-\alpha & \alpha \end{bmatrix} \cdot \begin{bmatrix} 1-\alpha & \alpha \\ \phi & 1-\phi \end{bmatrix} && (6) \\ &= \begin{bmatrix} (1-\alpha)^2 + \alpha\phi \\ (1-\alpha)\alpha + \alpha(1-\phi) \end{bmatrix}' && (7) \end{aligned}$$

Interpret:

$$= \begin{bmatrix} \mathbb{P}\left(E \to E, E \to E\right) + \mathbb{P}\left(E \to U, U \to E\right) \\ \mathbb{P}\left(E \to E, E \to U\right) + \mathbb{P}\left(E \to U, U \to U\right) \end{bmatrix} \tag{8}$$

Iterating Forward:

$$\pi_{t+j} = \pi_t \cdot \underbrace{P \cdot P \ldots P}_{\text{j times}} = \pi_t \cdot P^j \tag{9}$$

Stationarity and asymptotics. One possibility:

$$\pi_\infty = \lim_{j\to\infty} \pi_{t+j} = \lim_{j\to\infty} \pi_t \cdot P^j \tag{10}$$

Another is to find a π_∞ which doesn't change, i.e.

$$\pi_\infty = \pi_\infty P \tag{11}$$

Questions :
- Does a unique limit exist? Is it independent of π_t?
- Is there an absorbing state? (e.g., all end up unemployed forever)
- These answers depend on P.
- In some cases, we will refer to π_∞ as the stationary distribution.

2 Example

2.1 Non-Degenerate Stationary Distribution

What if:

$$P = \begin{bmatrix} 1-\alpha & \alpha \\ \phi & 1-\phi \end{bmatrix},\ 0<\alpha<1,\ 0<\phi<1 \tag{12}$$

Definition of stationary random variable, π_∞:

$$\pi_\infty = \pi_\infty \cdot P \tag{13}$$

- i.e. It's the R.V. associated with P such that it doesn't change between periods.
Remark: In linear algebra, the left eigenvector associated with the unit eigenvalue.

To find π_∞:
- Use software to find the left eigenvector, or
- Solve system for simple examples:
Let $\bar{\pi}$ = prob of being employed; $\pi_\infty = \begin{bmatrix} \bar{\pi} & 1-\bar{\pi} \end{bmatrix}$.

Then:

$$\begin{bmatrix}\bar{\pi} & 1-\bar{\pi}\end{bmatrix} = \begin{bmatrix}\bar{\pi} & 1-\bar{\pi}\end{bmatrix} \cdot \begin{bmatrix}1-\alpha & \alpha \\ \phi & 1-\phi\end{bmatrix} \tag{14}$$

$$\Rightarrow \begin{bmatrix}\bar{\pi} \\ 1-\bar{\pi}\end{bmatrix}' = \begin{bmatrix}\bar{\pi}(1-\alpha)+(1-\bar{\pi})\phi \\ \bar{\pi}\cdot\alpha+(1-\bar{\pi})(1-\phi)\end{bmatrix}' \begin{matrix}\leftrightarrow \text{ equation 1} \\ \leftrightarrow \text{ equation 2}\end{matrix} \tag{15}$$

1st Equation:

$$\bar{\pi} = (1-\alpha)\bar{\pi} - \phi\bar{\pi} + \phi \tag{16}$$

$$\Rightarrow (1-(1-\alpha)+\phi)\bar{\pi} = \phi \tag{17}$$

$$\boxed{\Rightarrow \bar{\pi} = \frac{\phi}{\alpha+\phi}} \tag{18}$$

$$\boxed{\Rightarrow \pi_\infty = \begin{bmatrix}\frac{\phi}{\alpha+\phi} \\ \frac{\alpha}{\alpha+\phi}\end{bmatrix}'} \tag{19}$$

2nd Equation: would find identical solution. (luckily, since there is only 1 variable and 2 equations)

2.2 Example: Unemployment

Assume: $P = \begin{bmatrix}1-\alpha & \alpha \\ \phi & 1-\phi\end{bmatrix} \begin{matrix}\leftrightarrow E \\ \leftrightarrow U\end{matrix}$

Invariant Distribution (i.e. "long run")

$$\bar{\pi} = \mathbb{P}(E),\ 1-\bar{\pi} = \mathbb{P}(U) \tag{20}$$

Solves:

$$\begin{bmatrix}\bar{\pi} & 1-\bar{\pi}\end{bmatrix} \begin{bmatrix}1-\alpha & \alpha \\ \phi & 1-\phi\end{bmatrix} = \begin{bmatrix}\bar{\pi} & 1-\bar{\pi}\end{bmatrix} \tag{21}$$

Equation:

$$\bar{\pi}(1-\alpha)+\phi(1-\bar{\pi})=\bar{\pi} \tag{22}$$
$$\Rightarrow \bar{\pi}=\frac{\phi}{\alpha+\phi} \tag{23}$$
$$1-\bar{\pi}=\frac{\alpha}{\alpha+\phi} \tag{24}$$

Divding top and bottom by $\phi\alpha$:

$$1-\bar{\pi}=\frac{1/\phi}{1/\phi+1/\alpha} \tag{25}$$

What is the average unemployment spell?

- In each period an unemployed person gets a job with probability $1-\phi$. Otherwise, stays unemployed. - Let N be the random variable "length of time it takes to find a job". $N=1$

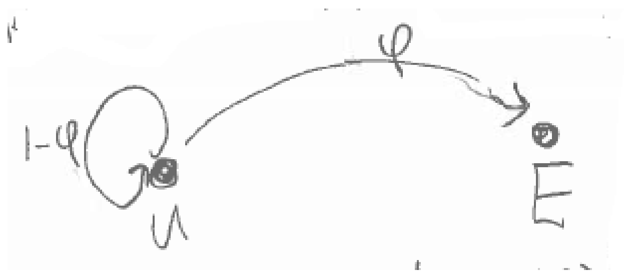

Figure 2: Employment Chain

means 1 person.

- Let $p_j=\mathbb{P}\left(N=j\right)$.

Then:

$$p_1=\phi, \qquad \text{(success)} \tag{26}$$
$$p_2=\phi(1-\phi), \quad \text{(fail, success)} \tag{27}$$
$$p_3=\phi(1-\phi)^2, \quad \text{(fail, fail, success)} \tag{28}$$
$$\boxed{p_j=\phi(1-\phi)^{j-1}} \tag{29}$$
$$\Rightarrow \sum_{j=1}^{\infty} p_j=\phi\sum_{j=1}^{\infty}(1-\phi)^{j-1}=\phi\sum_{j=0}^{\infty}(1-\phi)^j=\frac{\phi}{1-(1-\phi)}$$

i.e., a proper probability distribution.

Another Geometric Series Result:

$$\boxed{\sum_{j=1}^{\infty} j a^{j-1} = \frac{1}{(1-a)^2} \text{ for } |a| < 1, \text{ (can derive from } Z\text{-transforms)}} \tag{30}$$

Back to the question:

$$p_j = \mathbb{P}\left(N = j\right) = \phi(1-\phi)^{j-1} \tag{31}$$

$$\mathbb{E}\left[N\right] = \text{expected / mean time to find a job} \tag{32}$$

$$= \sum_{j=1}^{\infty} j \cdot p_j = \phi \sum_{j=1}^{\infty} j(1-\phi)^{j-1} = \phi \cdot \frac{1}{(1-(1-\phi))^2} = \frac{1}{\phi} \tag{33}$$

- So the average # of periods in unemployment $= \frac{1}{\phi}$.
- More generally, this is the mean waiting time for a geometric distribution, i.e., if arrivals happen with probability a, then the expected wait time $= \frac{1}{a}$.

Summarizing Formula:
- $\bar{\pi}$ =proportion employed, $1-\bar{\pi}$ = proportion unemployed.
- $\phi = \dfrac{\phi}{\alpha+\phi}$, $1-\bar{\pi} = \dfrac{\alpha}{\alpha+\phi} = \dfrac{1/\phi}{1/\phi + 1/\alpha}$.
- $\mathbb{E}\left[\# \text{ of periods to become employed} \mid \text{start unemployed}\right] = \frac{1}{\phi}$
- $\mathbb{E}\left[\# \text{ of periods to become unemployed} \mid \text{start employed}\right] = \frac{1}{\alpha}$

2.2.1 Example with Data (approx. 2007 US Data)

- Average unemployment duration = 16.8 weeks = 3.87 months.
- Civilian unemployment: 4.7%
- Employment / population: 63%
- Labour force / population: 66%
- Civilian population: 231 million
- Civilian labour force: 153 million = 231 × 66% (not institutional military, etc.)
- Unemployment: 7 million = 153 million × 4.7%

Stationary Distribution:

$$1-\bar{\pi} = 0.47 \text{ (proportion unemployed)} \tag{34}$$

$$\frac{1}{\phi} = 3.87 \text{ (average unemployment length, in months)} \tag{35}$$

Equation for stationary distribution:

$$1 - \bar{\pi} = \frac{1/\phi}{1 + \phi + 1/\alpha} \tag{36}$$
$$\Rightarrow 0.047 = \frac{3.87}{3.87 + 1/\alpha} \tag{37}$$

Solve for $\frac{1}{\alpha}$:

$$\frac{1}{\alpha} = 78.8 \tag{38}$$

i.e., average job length is 78 months.

So, the transition matrix is:

$$P = \begin{bmatrix} 1 - \frac{1}{78.8} & \frac{1}{78.8} \\ \frac{1}{3.87} & 1 - \frac{1}{3.87} \end{bmatrix} \approx \begin{bmatrix} 0.987 & 0.013 \\ 0.258 & 0.742 \end{bmatrix} \tag{39}$$

Stationary:

$$\pi_\infty = \begin{bmatrix} 0.953 & 0.047 \end{bmatrix} \tag{40}$$

Question

(a) Total Jobs Destroyed/Month: 0.013×146 million ≈ 1.85 million

(b) If employed worker today, what is the probability to be employed in j months?

$$\mathbb{P}\left(E \text{ at } j\right) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\substack{\text{need} \\ \text{employment} \\ \text{start}}} \cdot \left(\underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\pi_0 = E} P^j \right)' \tag{41}$$

What about as $j \to \infty$? $\mathbb{P}\left(E \text{ at } j \to \infty\right) = \bar{\pi}$

(c) The economy is away from its stationary equilibrium: $\pi_0 \neq \pi_\infty$.

What is the predicted sequence of unemployment rates?

$\pi_j = \begin{bmatrix} 0 & 1 \end{bmatrix} \cdot [\pi_0 P^j]'$

Appendix A Degenerate Markov Chains

A.1 Example: Absorbing State of Unemployment

Let $P = \begin{bmatrix} 1-\alpha & \alpha \\ 0 & 1 \end{bmatrix}$,
i.e. α chance to stay employed, and in unemployment never get a job ("absorbing").
Let $\pi_0 = \begin{bmatrix} a & 1-a \end{bmatrix}$

- $$\pi_1 = \pi_0 P \tag{A.1}$$
$$= \begin{bmatrix} a & 1-a \end{bmatrix} \begin{bmatrix} 1-\alpha & \alpha \\ 0 & 1 \end{bmatrix} \tag{A.2}$$
$$= \begin{bmatrix} \alpha(1-a) \\ \alpha a + 1 - a \end{bmatrix}' \tag{A.3}$$
$$= \begin{bmatrix} \text{kept job} \\ \text{lost job or never had one} \end{bmatrix}' \tag{A.4}$$

- $$\pi_2 = \pi_1 P \tag{A.5}$$
$$= \begin{bmatrix} \alpha a & (1-\alpha)a + (1-a) \end{bmatrix} \begin{bmatrix} \alpha & 1-\alpha \\ 0 & 1 \end{bmatrix} \tag{A.6}$$
$$= \begin{bmatrix} \alpha^2 a \\ (1-\alpha)\cdot \alpha a + (1-\alpha)a + (1-a) \end{bmatrix}' \tag{A.7}$$

Note:

- $\alpha^2 a$ represents for kept job twice

- Nominator and denominator must sum to 1

Example continued

$$\pi_j = \pi_0 \cdot P^j \tag{A.8}$$

$$= \begin{bmatrix} \alpha^j \cdot a \\ 1 - \alpha^j \cdot a \end{bmatrix} \tag{A.9}$$

$$\lim_{j\to\infty} \pi_j = \begin{bmatrix} 0 & 1 \end{bmatrix}, \text{ (i.e. all end up unemployed, independent of } \pi_0) \tag{A.10}$$

$$\text{or: } P \cdot P = \begin{bmatrix} \alpha & 1-\alpha \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha & 1-\alpha \\ 0 & 1 \end{bmatrix} \tag{A.11}$$

$$= \begin{bmatrix} \alpha^2 & \alpha(1-\alpha) + (1-\alpha) \\ 0 & 1 \end{bmatrix} \tag{A.12}$$

$$= \begin{bmatrix} \alpha^2 & 1-\alpha^2 \\ 0 & 1 \end{bmatrix} \tag{A.13}$$

Generalize:

$$P^j = \begin{bmatrix} \alpha^j & 1-\alpha^j \\ 0 & 1 \end{bmatrix} \tag{A.14}$$

$$\lim_{j\to\infty} P^j = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \tag{A.15}$$

$$\pi_\infty = \lim_{j\to\infty} \pi_0 P^j = \begin{bmatrix} \alpha & 1-\alpha \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \tag{A.16}$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix}, \text{ (i.e. all unemployed independent of } \pi_0) \tag{A.17}$$

Alternatively :

$$\pi_\infty = \begin{bmatrix} \bar{\pi} & 1-\bar{\pi} \end{bmatrix} \tag{A.18}$$

$$\Rightarrow \pi_\infty = \pi_\infty \cdot P \tag{A.19}$$

$$= \begin{bmatrix} \bar{\pi} & 1-\bar{\pi} \end{bmatrix} \begin{bmatrix} \alpha & 1-\alpha \\ 0 & 1 \end{bmatrix} \tag{A.20}$$

Equation:

$$\bar{\pi} = \alpha \cdot \bar{\pi} + 0 \tag{A.21}$$

If $\alpha < 1$, then this

$$\Rightarrow \bar{\pi} = 0, \ \Rightarrow \pi_\infty = \begin{bmatrix} 0 & 1 \end{bmatrix} \tag{A.22}$$

A.2 Example: No Ergodic Distribution

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \text{(i.e., switch from whatever you had)} \tag{A.23}$$

$$P^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{A.24}$$

$$P^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{A.25}$$

$$\ldots P^j = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{if j even} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \text{if j odd} \end{cases} \tag{A.26}$$

$$\tag{A.27}$$

$\lim_{j\to\infty} P^j$ doesn't exist in general.

Alternatively :

$$\begin{bmatrix} a & 1-a \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & 1-a \end{bmatrix} \tag{A.28}$$

Equations:

$$\begin{bmatrix} 1-a \\ a \end{bmatrix}' = \begin{bmatrix} a \\ 1-a \end{bmatrix}' \Rightarrow \pi_\infty = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}' \tag{A.29}$$

i.e. must start out with 50/50% probability.