Implementing Symbolic Mathematics in a Statically-Typed Language

SymboMath

Symbolic Mathematics in C++

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August 26, 2022

Abstract

While computers excel at solving numerical calculations, it's incredibly difficult to efficiently implement symbolic calculations – i.e., equations including variables, derivatives and integrals. For numeric calculations and the vast majority of software, symbolic mathematics provides very limited benefits, however in certain applications, such as visual calculators and scientific programs, the ability to manipulate equations algebraically can be incredibly useful.

This paper investigates how symbolic mathematics can be implemented in C++ , an object-oriented, statically-typed language which operates closer to the hardware than languages like Python and Java.

Summary

1 What is Symbolic Mathematics?

Symbolic mathematics is "the use of computers to manipulate mathematical equations and expressions in symbolic form, as opposed to manipulating the numerical quantities represented by those symbols." [1] In practice, this means the computer can deal with an expression like $5x + 7$, instead of just $5 \cdot 2 + 7$. Additionally, the computer would be able to tell you that

$$\frac{\mathrm{d}}{\mathrm{d}x} 2x^2 = 4x \tag{1}$$

instead of evaluating/approximating the derivative at a given point by using the definition for the derivative (or similar):

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{2}$$

In addition, it is possible to integrate certain equations symbolically to obtain exact results, as opposed to being constrained to definite integration using Riemann Sums or other approximation methods.

1.1 Why is it Useful?

The most common use-case for a symbolic maths is for solving incredibly large, tedious problems, which would be prone to trivial errors if solved by a human [2]. By providing a computer with instructions and operations to manipulate and solve equations step-by-step, it's possible to answer questions that had been ignored for millennia due to their scale and complexity.

Another common use for symbolic maths engine is to present data in a more human-readable form. For example, WolframAlpha is capable of taking complex, multi-variable equations as inputs and manipulating them to solve equations, plot interactive graphs and provide general solutions to queries.

As mentioned earlier, symbolic maths engines can also directly integrate or differentiate an expression, providing exact answers as opposed to approximations. The approximations generated from naive methods are often slow to compute (especially for integration), and require precise calculations and, in some cases, vast numbers of iterations to converge on an accurate value. A symbolic expression, however, can be evaluated very efficiently and, as there are no approximations present in the integration/differentiation, give exact results.

1.2 Why C++ ?

C++ is a statically-typed language based on C, providing classes, structs, template-meta-programming and more. It is also possible to write highly efficient code in C++ , making it the programming language of choice for many high-performance applications.

Being statically typed, using C++ adds another layer of complexity to implementing a symbolic maths library, as it makes it vastly more complicated to deal with different types for numbers, variables, functions, operations, etc.

If this project were to be implemented in Python, which is dynamically typed and hence does not care about data types, it would be much easier to design a data-structure which could store a symbolic equation and operate on it accordingly.

Ultimately, I chose C++ for this project as it would pose more of a challenge, as well as allowing more room for optimisation of performance and efficiency. C++ is also the language I am most comfortable in using, and, fortunately, appears to have a limited number of fully-featured symbolic math libraries available for it [3].

1.3 What About Automatic Differentiation?

When researching symbolic mathematics, particularly symbolic *differentiation*, one is bound to come across the term "automatic differentiation". This varies from symbolic mathematics in a few significant ways.

Firstly, automatic differentiation does not operate on symbolic equations. It operates entirely numerically, though may operate on vectors, matrices or arrays – not just scalar values.

Secondly, automatic differentiation aims to provide a means of computing a derivative of a function at a given point, not in the general case. This may seem exactly like the approximation methods spoken about earlier, but it does still compute a function for the derivative (just in a very different manor).

Automatic differentiation is most commonly used in machine-learning applications, where a complex activation function needs to be differentiated in order to make use of the gradient descent algorithm. The reason it's used here is because, given a static, compile-time definition for a function, it's possible to generate another function at compile-time which represents the derivative of the initial function, meaning no time needs to be spent at runtime parsing the input and differentiating it before evaluating the result.

2 SymboMath's Goals

`SymboMath`[1] should be a library usable in any `CMake` project, and should be wrapped in an easy-to-use, C++ -styled class for easy reuse of variables, equations and functions.

`SymboMath` *could* also support some form of basic equation solving, though this might be outside the scope of simple algebraic manipulation of equations.

2.1 Numeric Evaluation

The program should be able to evaluate any numeric expression involving the four basic arithmetic operations, as well as exponentiation, trigonometry and roots.

2.2 Functions

It should be easy to register custom functions with procedures to evaluate, differentiate and potentially integrate them. There should also be a wide set of functions already implemented, including trigonometry, logarithms and exponents, square roots, etc.

2.3 Variables

`SymboMath` must be able to support user-defined variables, including value substitution, manipulation of equations including variables and potentially simplification of expressions containing variables (see 2.5). Variables should act like any other object within the data structure of an equation, and operations applied to them should perform as expected, with no unwanted side-effects.

2.4 Calculus

The primary data structure used in `SymboMath` should be easily differentiated given that every function being used has a well-defined derivative. Integration of functions is more complex, but should be possible in most simple cases, and some more advanced scenarios. Calculus should also be performed with respect to a specified variable, meaning multi-variable equations can still be

[1]`SymboMath` GitHub repository: https://github.com/pencilcaseman/symbomath

operated on correctly. In such cases, all other variables will be treated as constants.

2.5 Simplification

Ideally, `SymboMath` should be able to simplify a given expression and reduce it to it's most fundamental form. This means $2 \cdot 5x$ will automatically be transformed into $10x$, and $\frac{10}{2}x$ will be transformed into $5x$. This will be most important in the result of differentiation/integration, as, depending on how these are implemented, the resulting expressions may include many redundant operations and values.

Simplification is likely to be the most difficult part of the `SymboMath` project and hence will be implemented as the last step. The equations returned by the program will/must still be valid regardless of whether they've been simplified or not, so the program can still be verified without this step.

3 Existing Symbolic Mathematics Libraries

Pre-existing symbolic mathematics libraries are a good place to start to identify implementation techniques and overall design architectures, as they are shown to work effectively.

`mathiu.cpp` is a C++ symbolic maths library based heavily on pattern matching. By making use of the `match(it)` library, `mathiu.cpp` can combine and compare operations and expressions to form an equation. Differentiation is performed recursively, applying a small set of rules to individual functions and combining them to produce the final result.

Wolfram Mathematica is another program capable of symbolic manipulation of equations. Although Mathematica's code is not available online, there are detailed explanations of how it's internal systems handle the data on the Wolfram Reference website [5]. According to their explanations, they have four fundamental data types:

```
Numbers
Strings
Symbols
General Expressions
```

Everything in the Wolfram Language keeps a reference count, tracking how many other Wolfram Language objects are pointing to it. This allows the language to free anything immediately after the last thing pointing to it is destroyed, saving memory and improving efficiency.

To process the inputs provided to the language, the Wolfram Language first creates an expression representing the input. It then applies all known rules for the contained objects before outputting the result. The important thing to note here is that *all* known rules are applied. This implies that no single rule can achieve the desired result, and that intelligently analysing the expression to identify the most effective rule is not always possible.

4 Overall Program Design

4.1 Data Types

Based on previous research and extensive testing, a polymorphic approach using C++ pointers and virtual functions appears to be the most efficient way of implementing the required features. Additionally, instead of using raw pointers, the program will make use of the C++ standard library's `std::shared_ptr<T>` type, as it can automatically handle reference counting and memory allocation/deallocation, reducing the scope for memory leaks and bugs in the final program.

Another benefit of using `std::shared_ptr<T>` is that, if all classes inherit from a single base type, it's possible to cast between the base type and the inherited types at runtime. This will allow for the creation of trees, but with no branching or data type requirements, which will be elaborated on later.

`SymboMath` will make use of the following data types:

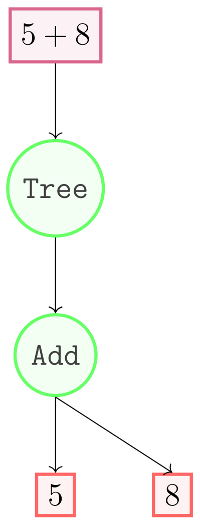
```
Component:  The most fundamental type in the system
Number:  Stores a single numeric value
Variable:  Stores a string value representing a variable name
Function:  An operation taking n arguments, returning a scalar
Tree:  Stores an equation and provides some utility functions
```

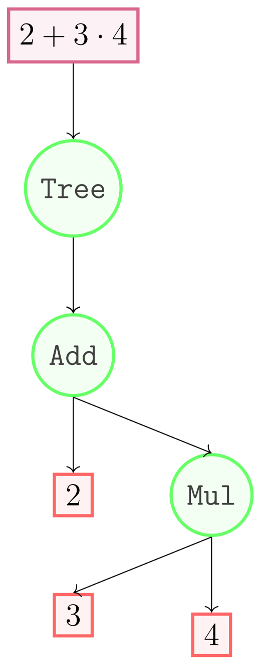
All objects will store only the data and instructions necessary for their operation, and will not access global variables unless it is necessary (searching for a function, for example).

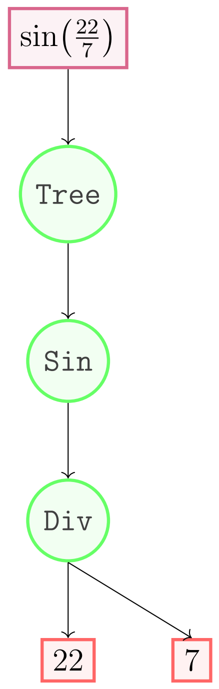
An equation will be a `Tree` object storing a single `Function` object, which will be the highest-level function/operation in the equation. The function objects will have n branches, where n represents the number of arguments they take. For an arithmetic operation such as addition or subtraction, $n = 2$. For a single-argument function like $\sin(x)$, $n = 1$.

Each branch of the function will contain yet another object – this may be a `Number`, `Variable` or a `Function`. Thankfully, the data structure identified earlier (elaborated on later) allows for any of these types to be stored, effectively circumventing the C++ static-typing system which makes this problem hard in the first place.

4.1.1 Examples







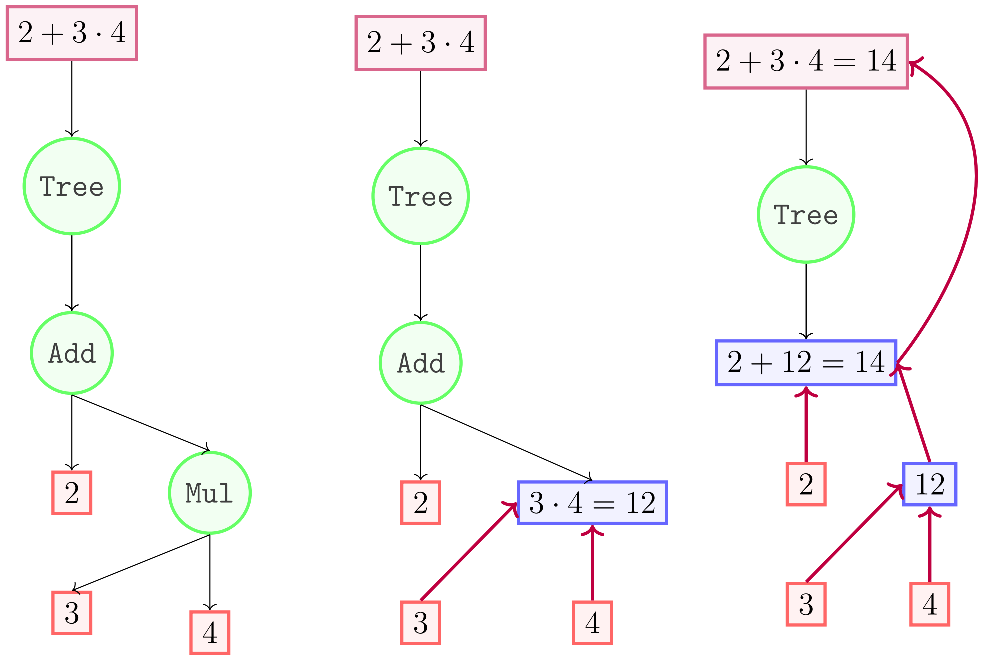
4.2 Algorithms

All high-level algorithms in `SymboMath` will operate recursively on the data they contain, and should not require external information to operate correctly.

For example, to evaluate the numeric result of an expression, the `eval` function would be called on the `Tree` object containing the equation. The `Tree` will then call `eval` on the primary function/operation, which, in turn, calls `eval` on each of it's branches. The numeric result of these function calls can then be propagated back up the tree, applying the relevant operations where necessary. This results in a final numeric result being returned from the `Tree` object.

4.2.1 Example

For example, let's evaluate the middle example from 4.1.1.



Hopefully, the example above shows that it's possible to evaluate a tree recursively, from the bottom up, to calculate a numeric result. These steps can be extended to differentiation, simplification and potentially integration.

4.3 External Libraries

In order to reduce the chance of bugs appearing and to improve functionality, `SymboMath` makes use of a few external libraries.

```
SymboMath
├── LibRapid [6] -- Provides advanced maths functionality
│   ├── MPFR [7] -- Support for multi-precision floating point types
│   └── MPIR [8] -- Required by MPFR
├── {fmt} [9] -- For faster, easier string manipulation and printing
└── scnlib [10] -- Provides fast string scanning and casting routines
```

5 Implementing the Data Types

5.1 The Component Type

For the data structure identified in 4.1, it's necessary to have a single base object which all other types will extend. This data type should have very minimal functionality, and all the functions it provides should ultimately be overwritten.

Internally, the `Component` type contains virtual functions which throw an error when called, ensuring they cannot be called accidentally and cause unwanted results.

To start with, none of the classes will support any advanced features like simplification or calculus, but will support a simple, numeric evaluation function so we can check that the whole system is working correctly.

```cpp
virtual Scalar eval() const;
virtual std::string type() const;
virtual std::string name() const;
virtual std::string str(uint64_t indent) const;
virtual std::string repr(uint64_t indent, uint64_t typeWidth,
                         uint64_t valWidth) const;
```

5.1.1 Evaluation

The `eval` function returns a scalar value, which is the result of evaluating the object. The evaluation function for each type will be covered in more detail later on.

5.2 Type Detection

The `type` function returns the type of the object (i.e. `"NUMBER"`, `"VARIABLE"`, `"FUNCTION"`, etc.). This is used in the cases where more information needs to be known about a specific object, as the program cannot apply a general algorithm. It's also useful for debugging purposes, and is closely linked to the helper functions.

5.3 Naming

Objects like functions and variables have names associated with them, and hence a procedure to extract the name of a given object is required. For objects which do not have a name associated with them, calling this function will raise an error, preventing unwanted results.

5.4 Helpers

The `str` and `repr` functions are purely to debug the algorithms and ensure that everything is working correctly. They both return string representations of the object, but in slightly different formats, showing different information.

5.5 The Number Type

The `Number` class is incredibly simple, and stores a single `Scalar` object (this is a type definition to allow different types to be used). The `eval` function simply returns the value stored in the class.

```
class Number : public Component {
public:
  /* [ code omitted ] */

  Scalar eval() const override { return m_value; }

  /* [ code omitted ] */
private:
  Scalar m_value = 0;
};
```

5.6 Variables

Variables such as x and y cannot be evaluated numerically without some form of substitution, so for now, the class will act as a stub which can be expanded on later as more features are added.

It is still important to implement the `name` function, however, as this will be needed for variable look up further down the line.

```
class Variable : public Component {
public:
  /* [ code omitted ] */

  std::string name() const override { return m_name; }

  /* [ code omitted ] */
private:
  std::string m_name = "NONAME";
};
```

5.7 Functions

The `Function` class is responsible for handing arithmetic operations such as addition and subtraction and single-argument functions like $\sin(x)$ and $\cos(y)$. The type can also be used to handle multi-argument functions like $\max(x, y)$ if required.

5.7.1 The Functor

In order to maintain runtime-compatible evaluation and to reduce the complexity of the program, the functor member of the function objects will be an instance of the C++ standard library `function` type, which provides a simple way to wrap a lambda function.

The functor will take a list of `Scalar` objects and will return another `Scalar` object, which is the result of the calculation.

5.7.2 Functor Arguments

The `Function` class also needs to know what arguments it's operating on, and these must be set at runtime. For this reason, each instance of the class will store a list of `Component` pointers (remember, these could be numbers, variables or functions) – one for each argument to the function.

One downside to this approach is that the number of arguments cannot be determined from the functor argument itself, so another member variable is required to store this information, with a function to access it.

5.7.3 Function Evaluation

In order to evaluate the result of the function, the arguments, which are still `Component` pointers, must first be evaluated and stored in a list. That list can then be fed directly into the functor object and the result can be returned.

```
class Function : public Component {
public:
  /* [ code omitted ] */

  Scalar eval() const override {
    std::vector<Scalar> operands;
    for (const auto &val : m_values)
      operands.push_back(val->eval());
    return m_functor(operands);
  }

  uint64_t numOperands() const { return m_numOperands; }

  std::string name() const override { return m_name; }

  /* [ code omitted ] */

private:
  std::string m_name   = "NULLOP";
  std::function<Scalar(const std::vector<Scalar> &)>
   m_functor;
  uint64_t m_numOperands = 0;

  std::vector<std::shared_ptr<Component>> m_values = {};
};
```

5.8 The Tree Type

The Tree class is incredibly simple, mainly intended to store a single function object which can then be evaluated. It also provides a few helper functions to format equations more nicely, though these are purely visual and have no impact on the program itself.

In the code listing, you may notice that the m_tree member is a list of pointers; this is done for future-proofing to allow variable values to be stored on a per-equation basis, for example. Currently it serves no purpose and can easily be changed to store a single object.

```
class Tree : public Component {
public:
  /* [ code omitted ] */

  Scalar eval() const override { return m_tree[0]->eval(); }

  /* [ code omitted ] */
private:
  std::vector<std::shared_ptr<Component>> m_tree;
};
```

6 Processing the Equation Input

Before it's possible to generate the tree from the input equation (in this case, a `std::string`), the text must first be processed into a form that can be understood more easily by the computer.

6.1 Tokenizing

The first step in transforming the user's input into something the computer can process is called "tokenizing" [2]. It's the process of splitting the input string into a list of "tokens", which, in this case, are individual characters [3] the program is allowed to use.

The process is very simple, and involves iterating over each character in the input text, checking whether it's a valid character to be in an equation (e.g. emojis are invalid) and, if it is, storing it in a list. This process also removes spaces from the input, as it's not important to the equation itself.

Another data type is required to store the result of this process, which is very simple and contains information about the type of the value stored, along with the actual character.

```
struct Token {
  uint64_t type;
  char val;
};
```

Listing 1: The token object definition

6.1.1 Example

For example, the input $41 + 1$ will produce the list of tokens shown below. Note, however, that `TYPE_DIGIT`, `TYPE_ADD` and `TYPE_OPERATION` are all numeric values.

```
// ===== [ 0 ] =====
{
  uint64_t type = TYPE_DIGIT;
  char val      = '4';
}
```

[2]Tokenizing and lexing are technically the same process, however, in **SymboMath** the process is split into two parts, and "tokenizing" fits the first part more closely than "lexing"

[3]Character as in an ASCII character – not just letters of the alphabet

```
// ===== [ 1 ] =====
{
  uint64_t type = TYPE_DIGIT;
  char val      = '1';
}
// ===== [ 2 ] =====
{
  uint64_t type = TYPE_ADD | TYPE_OPERATION;
  char val      = '+';
}
// ===== [ 3 ] =====
{
  uint64_t type = TYPE_DIGIT;
  char val      = '1';
}
```

6.2 Lexing

The individual characters returned from the tokenizing step are entirely useless unless we can make sense of what they mean in a larger context. This is the process of the "lexer", which takes the characters identified in the previous step and produces a list of objects which the computer can easily make sense of.

The algorithm itself is quite simple, and joins together strings of digits to form numbers, strings of characters to form text and identifies arithmetic operations and brackets.

Another data type is also required for the output of the lexing process. It's almost exactly the same as the token object, except it stores a string instead of a single character.

Taking the example given in 6.1.1, the algorithm will output the following.

```
// ===== [ 0 ] =====
{
  uint64_t type   = TYPE_NUMBER | TYPE_VARIABLE;
  std::string val = "41";
}
// ===== [ 1 ] =====
{
  uint64_t type   = TYPE_ADD | TYPE_OPERATION;
  std::string val = "+";
}
// ===== [ 2 ] =====
{
  uint64_t type    = TYPE_NUMBER | TYPE_VARIABLE;
  std::string val = "1";
}
```

This example is incredibly simple, and could be achieved simply by splitting the equation by spaces, for example. In more complex situations, however, such a technique will not work an may lead to unwanted edge-case scenarios – this method avoids those completely by interpreting the input character by character.

6.3 Processing the Lexed Results

This section of the program isn't necessarily required, but it will serve a very important purpose later down the line, as well as allowing for a few more mathematical inputs.

When writing an equation by hand, it's not uncommon to write something like $5(3x-2)$ or $2\sin(x)$. In these cases, we understand that the term preceded by a coefficient should be multiplied by that coefficient, however the computer doesn't understand this.

In order to simplify things, we need to insert a `Lexed{TYPE_MUL, "*"}` between the coefficient and the rest of the term.

We also need to check for equations like $x(5+y)$, however this is a little more complicated because it's not possible to simply check for text followed by an open bracket. To show why, take the expression $\sin(x)$ – using the naive method from above will lead to $\sin \cdot x$, which is obviously not a valid equation.

To mitigate this issue, `SymboMath` first checks whether the string maps to a function or a variable. If the value is a variable, we can safely insert a `Lexed{TYPE_MUL, "*"}`, otherwise we'll leave it in it's current form [4].

6.4 Conversion to Postfix Notation

Currently, the equations are in "infix" form, which is a fancy term for how we normally write equations, where the operator sits between the two operands.

[4]Due to the way function calls will work, we need to move the function so it sits after the closing parenthesis. This will be covered later.

One downside to this method of representing equations is that the order of precedence of operators plays a large role in the final result. While it is possible to process equations in this form directly, converting them to "postfix" notation dramatically simplifies the tree generation algorithm.

6.4.1 Postfix Notation

Postfix notation (sometimes called "Reverse Polish Notation") places the operator after the operands, and relies on a stack-based approach to calculate the final result. Postfix notation also removes the requirement for brackets, as the order of precedence is implied by the order of the operations in the equation.

For example, the equation $5+5$ would become $5,5,+$, $(1+2)\cdot 3$ becomes $1,2,+,3,\cdot$, and $2+3\cdot 4^x$ would become $2,3,4,x,^{exp},\cdot,+$. To evaluate the result, start on the first number and push it to the stack – repeat this until an operator is found. The operator acts on two numbers, so pop off the first and second items from the stack, apply the operation to those two values and push that number back onto the stack. Repeat this until the end of the equation is reached.

6.4.2 Infix to Postfix

[11]

To convert from infix to postfix notation, we can use Dijkstra's "Shunting Yard Algorithm", which acts as follows:

Algorithm 1 Dijkstra's Shunting Yard Algorithm

```
1: function TO_POSTFIX($input$)                          ▷ Convert from infix to postfix
2:     $postfix \leftarrow \{\}$
3:     $stack \leftarrow \{\}$
4:     for $lex$ in $input$ do
5:         if $lex \rightarrow type$ is $variable$ then                       ▷ Number or string
6:             $postfix$ push back $lex$
7:         else if $lex \rightarrow type$ is $(operator, function)$ then
8:             while $stack$ not empty and $stack \rightarrow back \rightarrow precedence \geq lex \rightarrow precedence$ do
9:                 $postfix$ push back $stack \rightarrow back$
10:                pop $stack$
11:            end while
12:            $stack$ push back $lex$
13:        else if $lex \rightarrow type$ is $leftparen$ then
14:            $stack$ push back $lex$                             ▷ Store the bracket for later
15:        else if $lex \rightarrow type$ is $rightparen$ then
16:            while $stack \rightarrow back$ is not $leftparen$ do
17:                $postfix$ push back $stack \rightarrow back$
18:            end while
19:            pop $stack$                                         ▷ Remove the left bracket
20:        end if
21:    end for
22:    while $stack$ not empty do                                  ▷ Pop remaining operators
23:        $postfix$ push back $stack \rightarrow back$
24:        pop $stack$
25:    end while
26: end function
```

6.5 Parsing

The final stage is to parse the resulting objects. This process involves identifying whether the object is a number, a variable or a function, and mapping it to the correct type for the system to understand it.

6.5.1 Numbers

If the lexed type is a number, `SymboMath` simply outputs a `std::make_shared<Number>(lex.val)`.

6.5.2 Strings

A string could represent either a variable or a function. `SymboMath` operates on the premise that functions must be declared before the equation is constructed, but variables can be specified afterwards, meaning the program must check if the string exists as a function and otherwise assumes it's a variable.

```
auto func = findFunction(lex.val);
if (func != functions.end()) {
  res.emplace_back(*func);
} else {
  res.emplace_back(std::make_shared<Variable>(lex.val));
}
```

Listing 2: Convert from text to either a function or variable

6.5.3 Operators

Arithmetic operators are also functions, so a simple lookup is all that's needed. To reduce bugs, the code still checks whether the functions were successfully found, as it's possible the functions haven't been registered at this point in the program. `SymboMath` will throw an error if this is the case.

```
auto func = findFunction("_"); // Default to a nullary
    function
if (lex.type & TYPE_ADD) func = findFunction("ADD");
if (lex.type & TYPE_SUB) func = findFunction("SUB");
if (lex.type & TYPE_MUL) func = findFunction("MUL");
if (lex.type & TYPE_DIV) func = findFunction("DIV");
if (lex.type & TYPE_CARET) func = findFunction("POW");
LR_ASSERT(func != functions.end(), "Operator not found");
```

```
res.emplace_back(*func);
```

Listing 3: Identify arithmetic operators and convert them to function objects

7 Equation Tree Generation

With the equations in postfix notation, it'd be quite trivial to evaluate a numeric result with no further manipulation, however, to implement calculus and more advanced manipulations, it'll be necessary to transform the equation into a tree-like structure.

To do this, we can evaluate the postfix equation and apply the typing system implemented earlier to store the final equation, instead of evaluating it numerically.

Algorithm 2 Postfix to Tree

```
function GEN_TREE(input)                       ▷ Generate a tree from postfix notation
    tree ← empty tree
    stack ← empty stack
    for lex in input do
        if lex → type is (number, variable) then
            stack push back lex
        else if lex → type is function then
            args ← {}
            for i in 0..(lex → numOperands) do
                args push back stack → back
                pop stack
            end for
            node ← copy lex                             ▷ lex is a function
            for arg in reverse args do
                node add value arg
            end for
            stack push back node                  ▷ Push node back onto stack
        end if
    end for
    tree set node stack → back
end function
```

As an example, we can test the algorithm on the trees from 4.1.1. Each of the three examples below shows the input to the program, the generated tree and the numeric result after calling `eval` on the main tree object.

7.1 Example 1

```
Input: "5 + 8"
[ TREE ]
    [ FUNCTION ] [ ADD ]
        [ NUMBER ] [ 5.0000000000 ]
        [ NUMBER ] [ 8.0000000000 ]

Result: 13
```

7.2 Example 2

```
Input: "2 + 3 * 4"
[ TREE ]
    [ FUNCTION ] [ ADD ]
        [  NUMBER  ] [ 2.0000000000 ]
        [ FUNCTION ] [     MUL      ]
            [ NUMBER ] [ 3.0000000000 ]
            [ NUMBER ] [ 4.0000000000 ]

Result: 14
```

7.3 Example 2

```
Input: "sin(22 / 7)"
[ TREE ]
    [ FUNCTION ] [ sin ]
        [ FUNCTION ] [ DIV ]
            [ NUMBER ] [ 22.0000000000 ]
            [ NUMBER ] [ 7.0000000000  ]

Result: -0.00126448893037729
```

8 Variables

Currently, `SymboMath` can store variables in the tree, but cannot process them in any way. The first step to having full support for variables to allow an equation to be evaluated given known values for variables.

8.1 Numeric Variable Evaluation

Providing values for variables involves passing in a set of name-value pairs, such as `("x", 123)`, which can tell `SymboMath` what to substitute in place of variables. To make this process easier, `SymboMath` needs a function to convert an input value into one of a number, a variable or a tree, depending on what is passed to it.

Algorithm 3 Intelligent auto parse system

```
1: function AUTO_PARSE(input)
2:     if textitinput is (number, variable) then
3:         return input                          ▷ Leave the input as a number of variable
4:     else
5:         return gentree input                  ▷ Generate a new tree object
6:     end if
7: end function
```

By developing a function similar to `eval` which, instead of generating numeric results, produces a new `Tree` object where all instances of a given variable are replaced with their corresponding expansions.

Additionally, this implementation allows for another equation to be passed as a variable. This could be used to substitute a rearranged equation into another, for example.

8.1.1 Example

```
Input: "a * b" with {"a": "5", "b": "2 + 4"}

Original Tree:
[ TREE ]
    [ FUNCTION ] [ MUL ]
        [ VARIABLE ] [ a ]
        [ VARIABLE ] [ b ]
```

```

Substituted Tree:
[ TREE ]
    [ FUNCTION ] [ MUL ]
        [ NUMBER ] [ 5 ]
        [ FUNCTION ] [ ADD ]
            [ NUMBER ] [ 2 ]
            [ NUMBER ] [ 4 ]

Result: 30
```

8.2 Pretty Printing an Equation

While the tree representation of an equation is useful in some ways, it's not how we are used to viewing an equation. For this reason, a `prettyPrint` function is needed, which will format a `Tree` object more nicely.

The internal workings of the function are not important here, but it uses some logic to ensure brackets are placed where they are needed (and sometimes where they are *not* needed, though are still valid).

8.2.1 Example

```
Input: "(2 * 3 + 4) / sin(x)"
Pretty Printed: ((2 * 3) + 4) / (sin(123))
(note the extra brackets)
```

9 Differentiation

Differentiation is a very useful tool in mathematics, and, fortunately, can be implemented into `SymboMath` relatively easily.

Luckily, almost any equation can be differentiated using a small subset of simple rules, as these rules can be applied to smaller and smaller parts of the equation until the final result is generated.

`SymboMath` will differentiate the tree object recursively, similar to how the tree is evaluated.

9.1 Differentiation Rules

Below is a minimal list of rules which can be used to differentiate a wide

$$\frac{\mathrm{d}}{\mathrm{d}x}a = 0,\ \text{for } a \in \mathbb{R} \tag{3}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x) \pm h(x)) = \frac{\mathrm{d}f}{\mathrm{d}x} \pm \frac{\mathrm{d}h}{\mathrm{d}x} \tag{4}$$

$$\frac{\mathrm{d}x}{\mathrm{d}y}f(x)h(x) = v\frac{\mathrm{d}h}{\mathrm{d}x} - u\frac{\mathrm{d}f}{\mathrm{d}x} \tag{5}$$

$$\frac{\mathrm{d}x}{\mathrm{d}y}\frac{f(x)}{h(x)} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2} \tag{6}$$

$$\frac{\mathrm{d}x}{\mathrm{d}y}f(x)^a = \frac{\mathrm{d}a}{\mathrm{d}x} \tag{7}$$

$$\frac{\mathrm{d}x}{\mathrm{d}y}sin(x) = cos(x) \tag{8}$$

$$\frac{\mathrm{d}x}{\mathrm{d}y}cos(x) = -sin(x) \tag{9}$$

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