Jacobian

d = difference vector of length $m \equiv$ # states s $\times$ # characteristics k

β = parameters vector of length $n \equiv m$

$$J = \begin{bmatrix} \frac{\partial d_1}{\partial \beta_1} & \cdots & \frac{\partial d_1}{\partial \beta_n} \\ \vdots & & \vdots \\ \frac{\partial d_m}{\partial \beta_1} & \cdots & \frac{\partial d_m}{\partial \beta_n} \end{bmatrix}$$

$$d_i = \underset{\text{(constant)}}{\overset{\text{actual target}}{t_i}} - \overset{\text{calculated target } i}{ct_i}$$

In computer programs, d, t, and ct have been $s \times k$ matrices. For example, with 3 states and 2 characteristics, we might have

$$\underbrace{t = \begin{bmatrix} 10 & 20 \\ 30 & 40 \\ 50 & 60 \end{bmatrix}}_{\text{constant}} \qquad \underbrace{ct = \begin{bmatrix} 8 & 24 \\ 27 & 41 \\ 50 & 59 \end{bmatrix}}_{\text{calculated with a given set of estimated weights}}$$

$$d = \begin{bmatrix} 2 & -4 \\ 3 & -1 \\ 0 & 1 \end{bmatrix}$$

In this exercise we need to convert these to vectors with all of 1 characteristics of one state, then the next, then the next, so that we express this as

$$t = (10, 30, 50, 20, 40, 60)$$
$$ct = (8, 27, 50, 24, 41, 59)$$
$$d = (2, 3, 0, -4, -1, 1)$$

Similarly, β (I am making this up) might be <u>estimated</u> as:

$$\beta \approx \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

and so for purposes of calculating the Jacobian we would express β as

$$\beta \cong (1, 3, 5, 2, 4, 6)$$

The Jacobian is $\left[\frac{\partial d}{\partial \beta}\right]$ which is a 6×6 symmetric matrix. It always will have dimension $(s \cdot k) \times (s \cdot k)$.

It's useful to think about how to map the s, k indexes to the $i \,\&\, j$ indexes in this example

i, j	s	k
1	1	1
2	2	1
3	3	1
4	1	2
5	2	2
6	3	2

We want an analytic expression, ultimately in matrix notation, for J.

We can start by deriving the partial derivative of any single difference, d_i with respect to any β_j .. for example $\frac{\partial d_1}{\partial \beta_2}$, and then generalizing it.

Let's say, as above, $K = 2$ and $S = 3$ and $h = 4$.

Then for $\frac{\partial d_1}{\partial \beta_2}$

Map of indexes

	index	s	k
d_1	1	1	1
β_2	2	2	1

$$d_1 = \underset{\left\{\substack{s=1 \\ K=1}\right\}}{t_1} - \sum_{h=1}^{4} w_{h,s=1} X_h^{s=1}$$

$$\frac{\partial d_1}{\partial \beta_2} = - \sum_{h=1}^{4} w_{h,s=1} X_h^{s=1}$$

Let's look just at $h=1$ -- one household, Then we have

$$w_{h=1,s=1} \cdot X_{h=1}^{s=1}$$

as one of the 4 elements in the second term of the RHS of d_1. We need

$$\frac{\partial}{\partial \beta_2 \left\{ \begin{smallmatrix} s=2 \\ k=1 \end{smallmatrix} \right\}} w_{h=1,s=1} \cdot X_{h=1}^{s=1}$$

which is

$$X_{h=1}^{s=1} \frac{\partial}{\partial \beta_2}\left(w_{h=1,s=1}\right)$$

because X is constant)

(after we get this we need to add the results for the 4 households to get $\frac{\partial d_1}{\partial \beta_2}$

So we need

$$\frac{\partial}{\partial \beta_{\substack{s=2\\k=1}}} \left(W_{h=1,s=1} \right).$$

where

$$W_{h=1,s=1} = e^{\overset{\beta_{j=1}}{\beta_{\substack{s=1\\k=1}}} \cdot X^{k=1}_{h=1} + \overset{\beta_{j=4}}{\beta_{\substack{s=1\\k=2}}} \cdot X^{k=2}_{h=1} + \delta_{h=1}}$$

$$\boxed{\underline{\underline{\text{where}}}\ \delta_{h=1} = f(\beta)}$$

This is the part TPC left out of their paper, as I read it, although their software might deal with it properly.

Re-express

$$W_{h=1,s=1} = \underbrace{e^{\left(\beta_{s=1,k=1} X_{h=1}^{k=1} + \beta_{s=1,k=2} X_{h=1}^{k=2}\right)}}_{f} \cdot \underbrace{e^{\delta}}_{g} \qquad \delta = f(\beta)$$

By product rule:

$$\frac{\partial W_{h=1,s=1}}{\partial \beta_{2}} \quad = \quad f'g + fg'$$

$\begin{bmatrix} s=2 \\ k=1 \end{bmatrix}$

$$= \underbrace{0}_{\text{This will not be zero on the diagonal}} + f \cdot \frac{\partial g}{\partial \beta_2}$$

on the diagonal this will simplify $X_{h=1}^{k=1} \cdot W_{h=1, s=1}$ I think