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REALISTIC MATHEMATICAL MODELING OF THE RATTLEBACK

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Abstract—The rattleback (also called a Celt or wobblestone) is an object which, when placed on a horizontal surface and caused to rotate about a vertical axis, sometimes begins to oscillate, stops turning, and then starts rotating in the direction opposite to that associated with the original motion. Earlier analyses dealing with this phenomenon have been based on a variety of assumptions. In the present work, it is shown by means of numerical solutions of full, non-linear equations of motion that one can construct a realistic mathematical model by assuming rolling without slipping and employing a torque proportional to angular velocity to provide for energy dissipation.

1. INTRODUCTION

"If you spin a certain type of stone in the 'wrong' direction, it will stop, rattle up and down for a few seconds and then spin in the opposite direction." So begins an article [1] in a recent issue of *Scientific American* in which Jearl Walker gives a detailed, qualitative account of observations made of a phenomenon that has been the subject of analytical studies [2, 3, 4] dating from 1896 to the present. Each of these works is concerned with questions of stability, but is based on a different mathematical model. G. T. Walker [2] assumed rolling without slipping, Magnus [3] invoked the Contensou friction law, and Caughey [4], working with a mathematical model the physical significance of which he himself questions, performed a non-linear stability analysis. Hence, it is natural to wonder whether or not it is, in fact, necessary to go beyond G. T. Walker's model and if so what modifications are appropriate, particularly since neither Magnus nor Caughey stated any explicit objections to assumptions made by their predecessors. The present paper has the two-fold purpose of answering this question and of providing a comprehensive theory for the exploration of the phenomenon under consideration. To these ends, a numerical simulation algorithm is devised, and comparisons are made between motions predicted by simulations, on the one hand, and motions observed by Jearl Walker and by Crabtree [5], on the other hand.

The paper is organized as follows. We begin with a description of the input and output quantities used in a computer program based on the assumption of rolling without slipping and containing no provisions for energy dissipation. Next, we perform the aforementioned comparisons of predicted and observed motions and, after noting good agreement between these, we modify the analysis so as to provide for energy dissipation, whereupon we find even better agreement. Finally, we give a detailed account of the kinematic and dynamic analyses underlying our computer program, and we set forth a computational algorithm that can be implemented by readers interested in performing their own numerical experiments.

2. INPUT AND OUTPUT

Figure 1 shows a dextral set of mutually perpendicular axes X_1, X_2, X_3 intersecting at a point O and presumed to be fixed in a rigid body R. These lines are the axes of an ellipsoid that forms a portion S of the bounding surface of R, S being the locus of

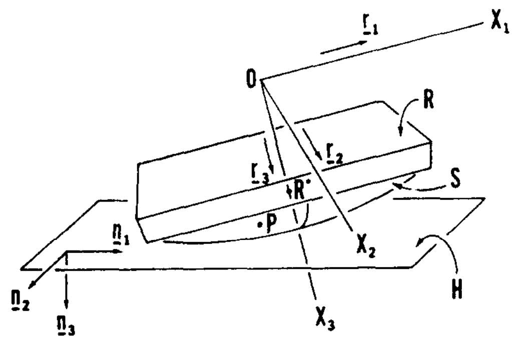

Fig. 1. Rattleback.

points whose equation is

$$\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}+\frac{x_3^2}{c^2}=1 \tag{1}$$

where x_1, x_2, x_3 are, respectively, the X_1, X_2, X_3 coordinates of a generic point P of S, and a, b, c $(a > b > c)$ are semi-diameters of the ellipsoid.

R has a mass M, and R^*, the mass center of R, lies on X_3, at a distance h from O. To specify the inertia properties of R, we let $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ be unit vectors parallel to X_1, X_2, X_3, respectively, and express $\mathbf{I}$, the inertia dyadic of R for R^*, as

$$\mathbf{I} = A\mathbf{r}_1\mathbf{r}_1 + B\mathbf{r}_2\mathbf{r}_2 + C\mathbf{r}_3\mathbf{r}_3 + D(\mathbf{r}_1\mathbf{r}_2 + \mathbf{r}_2\mathbf{r}_1). \tag{2}$$

In other words, we assume that X_3 is one of the central principal axes of inertia of R, but that X_1 and X_2 are not central principal axes of inertia of R.

To characterize the orientation of R in a Newtonian reference frame N, we first introduce $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$, a dextral set of mutually perpendicular unit vectors fixed in N, letting $\mathbf{n}_1$ and $\mathbf{n}_2$ be parallel to the horizontal plane H with which S remains in contact throughout the motion of R in N, and requiring $\mathbf{n}_3$ to be directed vertically downward. Next, we bring R into a general orientation in N by aligning $\mathbf{r}_k$ with $\mathbf{n}_k$ $(k = 1, 2, 3)$ and then subjecting R successively to rotations described in magnitude and direction by the vectors $\gamma\mathbf{r}_3$, $\alpha\mathbf{r}_1$, $\beta\mathbf{r}_2$, where α, β, γ are the radian measures of angles designated by the same letters in Fig. 2, which shows $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ in their final orientations. Also shown in Fig. 2 is the angle δ between $\mathbf{n}_3$ and $\mathbf{r}_3$. Lastly, we define angular velocity measure numbers ω_k $(k = 1, 2, 3)$ as

$$\omega_k \stackrel{\Delta}{=} \boldsymbol{\omega} \cdot \mathbf{r}_k \qquad (k = 1, 2, 3) \tag{3}$$

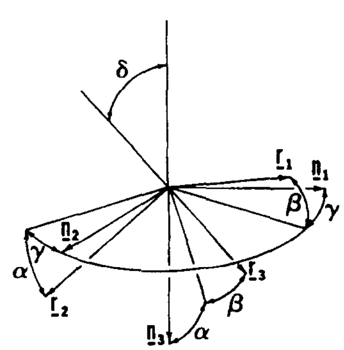

Fig. 2. Angles α, β, γ, and δ.

where ω is the angular velocity of R in N. Now we are in a position to state that we shall use as *input* parameters the quantities a, b, c, M, h, A, B, C, D and the initial values of α, β, γ and ω_1, ω_2, ω_3. (Without loss of generality, we shall take $\gamma = 0$ at time $t = 0$ in all cases.) For *output* variables, we shall employ γ and δ. These two angles furnish precise measures of what one may call 'rotation' and 'wobble', respectively.

3. COMPARISONS

The first test to which we subject our simulation program is to see whether or not it is capable of predicting motions of the kind described in the opening sentence of Jearl Walker's (as well as our) paper. To this end, we take $a = 0.20$ m, $b = 0.03$ m, $c = 0.02$ m, $M = 1.00$ kg, $h = 0.01$ m, $A = 2 \times 10^{-4}$ kg m^2, $B = 1.6 \times 10^{-3}$ kg m^2, $C = 1.7 \times 10^{-3}$ kg m^2, $D = -2 \times 10^{-5}$ kg m^2, and let $\alpha(0) = 0.5°$, $\beta(0) = 0.5°$, $\omega_1(0) = \omega_2(0) = 0$ and $\omega_3(0) = -5$ rad/s. Initially, we thus have R rotating counterclockwise as seen from above, and the initial angular velocity vector is nearly vertical. Figure 3 shows the resulting plots of γ and δ versus t, the first of which contains a sign change in the slope, which corresponds to a reversal of the rotation sense, while the second is in accord with Jearl Walker's "rattle up and down for a few seconds". The theory based on rolling without slipping has thus passed its first test.

Jearl Walker asserts that "some of the rattlebacks display a second reversal, from clockwise to counterclockwise, near the end of spinning just as the friction is about to remove the last of the stone's energy. The second reversal is usually not as strong as the first and does not display the same type of vertical oscillation". We shall take up the question of energy dissipation presently, but first we extend the computer run that

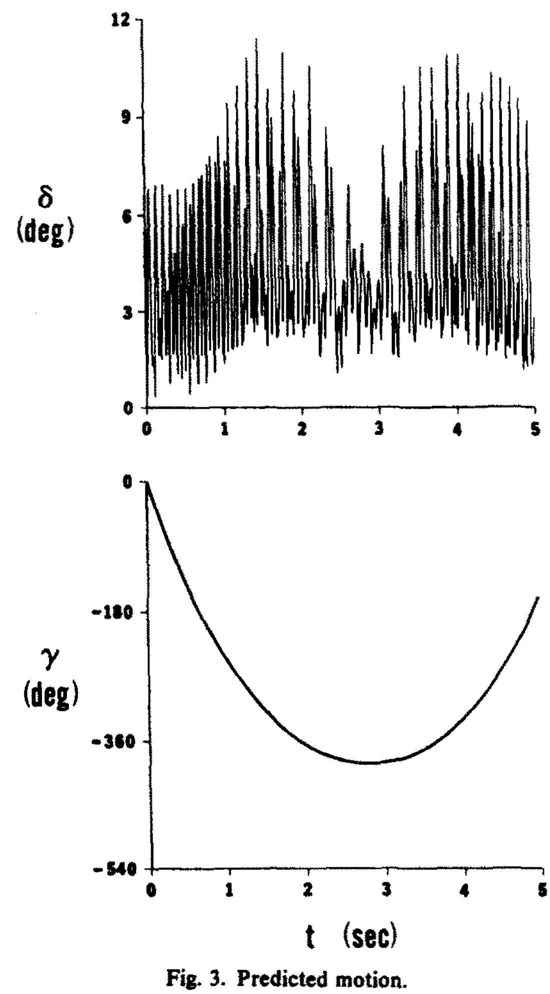

Fig. 3. Predicted motion.

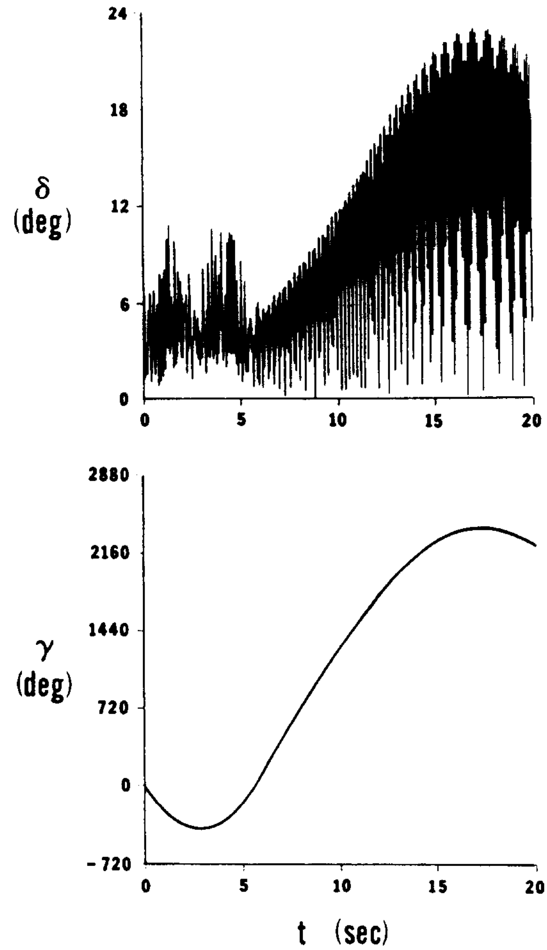

Fig. 4. Two reversals.

led to Fig. 3, which permits us to draw Fig. 4. Here, the hoped for second reversal of the rotation sense is seen to be predicted; the second reversal is weaker than the first insofar as the time interval between the two reversals is considerably greater than that between the initial instant and the first reversal; and the second reversal is accompanied by a δ versus t behavior differing noticeably from that associated with the first reversal.

When the computations leading to Fig. 4 are continued beyond twenty seconds, it is found that additional reversals occur, apparently periodically. However, in most experiments, motion ceases either before any reversal has occurred or soon after the first one. Conjecturing that we can sharpen our results in this regard by making provisions for energy dissipation, we now modify our mathematical model by assuming that R is at all times subjected to the action of a couple whose torque is given by

$$\mathbf{T} = -\sigma\boldsymbol{\omega} \tag{4}$$

where σ is a positive constant, the underlying idea being that air resistance may be the principal energy dissipating mechanism that must be taken into account; equation (4) may serve quite adequately since we are necessarily dealing with relatively small angular velocities. Taking $\sigma = 10^{-4}$ N s/m, but leaving all other values unaltered, we obtain the plots shown in Fig. 5. As can be seen, we now no longer have a second reversal, and the wobble angle decays with time, which is precisely what is frequently observed. Moreover, if the sense of the initial spin is reversed, that is, if $\omega_3(0) = 5$ rad/s (rather than -5 rad/s), then computations predict that no reversals will occur, which is exactly what one generally finds experimentally.

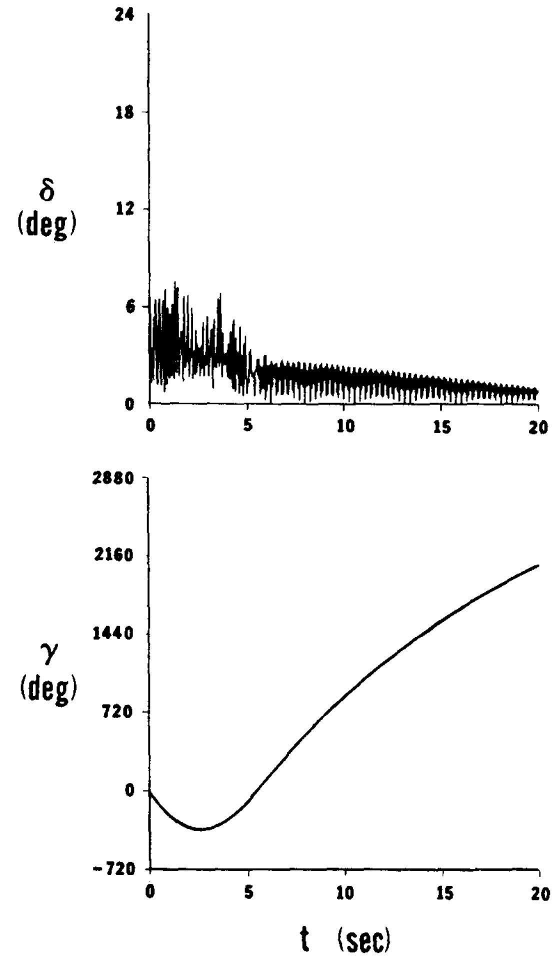

Fig. 5. Effects of energy dissipation.

Crabtree describes two experiments that we can simulate by taking, first, $\alpha(0) = 5°$, $\beta(0) = \omega_1(0) = \omega_2(0) = \omega_3(0) = 0$, and second, $\beta(0) = 5°$, $\alpha(0) = \omega_1(0) = \omega_2(0) = \omega_3(0) = 0$. He asserts that one may expect to observe initial oscillations followed by rotations of opposite sense in the two cases. The two curves in Fig. 6 predict precisely these motions. (Note that the vertical scales in these two figures are not the same.)

The authors of [1–3] all point out that the way in which the central principal axes of inertia of R are oriented relative to the axes X_1, X_2, X_3 of the ellipsoid of which S is a part has a decisive effect on motions of R. Jearl Walker describes how one can change the relative orientation by attaching objects to R and how one can thus alter the behavior of R. Making corresponding changes in A, B, C and D, we find that we can simulate such experiments readily. Thus, on the basis of all the evidence now in hand, we conclude that one can construct a very satisfactory theory for dealing with the rattleback phenomenon by simply assuming that R rolls on H without slipping and that air resistance can be modeled as in equation (4).

The analysis set forth in the next section goes beyond that provided by G. T. Walker [2], not only by accounting for dissipation effects, but also by supplementing the small number of equations† reported in Section 1 of his paper with the additional equations required for a simulation, but not needed for the linear stability analyses performed by him.

†Some of these equations are incorrect because the relationship $r = 1$ in [2] is invalid.

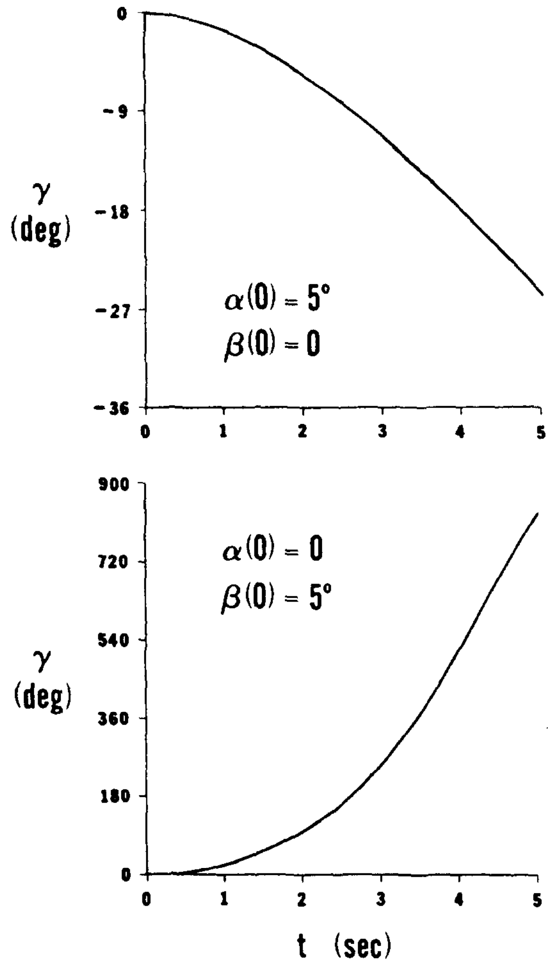

Fig. 6. Crabtree's experiments.

4. THEORY

The development that follows is divided into two major portions, namely, a kinematical analysis and the formulation of dynamical equations of motion. The first of these involves, primarily, finding the X_1, X_2, X_3 coordinates of the point of contact between S and H (see Fig. 1) and expressing the velocity and the acceleration of the mass center R^* of R in suitable forms. In addition, consideration is given to the angles α, β, γ, and δ.

The point of contact between S and H is that point of S at which the normal to S is parallel to the unit vector $\mathbf{n}_3$. Defining $f(x_1, x_2, x_3)$ as

$$f \triangleq \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} - 1 \tag{5}$$

and taking advantage of the fact that the normal to S is parallel to ∇f, the gradient of f, one can write

$$\nabla f = 2\lambda \mathbf{n}_3 \tag{6}$$

or

$$\frac{x_1}{a^2}\mathbf{r}_1 + \frac{x_2}{b^2}\mathbf{r}_2 + \frac{x_3}{c^2}\mathbf{r}_3 = \lambda \mathbf{n}_3 \tag{7}$$

where λ is a scalar quantity that depends on the orientation of R in N and x_1, x_2, x_3

are now the coordinates of the contact point. After introducing μ_k as

$$\mu_k \overset{\Delta}{=} \mathbf{r}_k \cdot \mathbf{n}_3 \qquad (k = 1, 2, 3) \tag{8}$$

one can dot-multiply equation (7) first with $\mathbf{n}_3$, which gives, with the aid of equation (8),

$$\frac{x_1}{a^2}\mu_1 + \frac{x_2}{b^2}\mu_2 + \frac{x_3}{c^2}\mu_3 = \lambda \tag{9}$$

and next with $\mathbf{r}_1$ and $\mathbf{r}_2$, which produces [again with the aid of equation (8)]

$$\frac{x_1}{a^2} = \lambda\mu_1 \qquad \frac{x_2}{b^2} = \lambda\mu_2 \tag{10}$$

or, after elimination of λ by reference to equation (9),

$$\frac{x_1}{a^2}(1 - \mu_1^2) - \frac{x_2}{b^2}\mu_1\mu_2 = \frac{x_3}{c^2}\mu_3\mu_1 \tag{11}$$

$$-\frac{x_1}{a^2}\mu_1\mu_2 + \frac{x_2}{b^2}(1 - \mu_2^2) = \frac{x_3}{c^2}\mu_2\mu_3. \tag{12}$$

Solving these equations for x_1/a^2 and x_2/b^2, and using the fact that

$$\mu_1^2 + \mu_2^2 + \mu_3^2 = 1 \tag{13}$$

one arrives at

$$\frac{x_1}{a^2} = \frac{x_3}{c^2}\frac{\mu_1}{\mu_3} \qquad \frac{x_2}{b^2} = \frac{x_3}{c^2}\frac{\mu_2}{\mu_3} \tag{14}$$

and substitution into equation (1) then yields

$$\left(\frac{x_3}{c}\right)^2 = \frac{(c\mu_3)^2}{(a\mu_1)^2 + (b\mu_2)^2 + (c\mu_3)^2}. \tag{15}$$

Now, for motions of interest to us, x_3 is necessarily positive. Hence, after defining ϵ as

$$\epsilon \overset{\Delta}{=} [(a\mu_1)^2 + (b\mu_2)^2 + (c\mu_3)^2]^{1/2} \tag{16}$$

we have‡

$$x_3 \underset{(15,16)}{=} c^2\mu_3/\epsilon \tag{17}$$

and substitution into equation (14) leads to

$$x_1 = a^2\mu_1/\epsilon \qquad x_2 = b^2\mu_2/\epsilon. \tag{18}$$

‡Numbers beneath signs of equality refer to equations numbered correspondingly.

Equations (17) and (18) are the desired relationships governing the coordinates of the point of contact between S and H.

Before long, we shall need expressions for the time-derivatives of ϵ and x_1, x_2, x_3. These are

$$\dot{\epsilon} \underset{(16)}{=} (a^2\mu_1\dot{\mu}_1 + b^2\mu_2\dot{\mu}_2 + c^2\mu_3\dot{\mu}_3)/\epsilon \tag{19}$$

and

$$\dot{x}_1 \underset{(18)}{=} \frac{a^2(\epsilon\dot{\mu}_1 - \dot{\epsilon}\mu_1)}{\epsilon^2}, \quad \dot{x}_2 \underset{(18)}{=} \frac{b^2(\epsilon\dot{\mu}_2 - \dot{\epsilon}\mu_2)}{\epsilon^2}, \quad \dot{x}_3 \underset{(17)}{=} \frac{c^2(\epsilon\dot{\mu}_3 - \dot{\epsilon}\mu_3)}{\epsilon^2}. \tag{20}$$

The time-derivatives of μ_1, μ_2, μ_3 that appear in these equations are themselves functions of μ_k and ω_k $(k = 1, 2, 3)$, found as follows.

The time derivatives of $\mathbf{n}_3$ in N and in R are related to each other by the equation

$$\frac{{}^N d\mathbf{n}_3}{dt} = \frac{{}^R d\mathbf{n}_3}{dt} + \boldsymbol{\omega} \times \mathbf{n}_3. \tag{21}$$

Now, the left-hand side vanishes because $\mathbf{n}_3$ is fixed in N. With

$$\boldsymbol{\omega} \underset{(3)}{=} \omega_1\mathbf{r}_1 + \omega_2\mathbf{r}_2 + \omega_3\mathbf{r}_3, \quad \mathbf{n}_3 \underset{(8)}{=} \mu_1\mathbf{r}_1 + \mu_2\mathbf{r}_2 + \mu_3\mathbf{r}_3 \tag{22}$$

one thus obtains from equation (21),

$$\dot{\mu}_1 = \omega_3\mu_2 - \omega_2\mu_3, \quad \dot{\mu}_2 = \omega_1\mu_3 - \omega_3\mu_1, \quad \dot{\mu}_3 = \omega_2\mu_1 - \omega_1\mu_2. \tag{23}$$

When R rolls on H without slipping, the velocity of the point of contact between S and H is equal to zero and the velocity $\mathbf{v}$ of R^* can be expressed as the cross-product of $\boldsymbol{\omega}$ and the position vector from the contact point to R^*; that is,

$$\mathbf{v} = \boldsymbol{\omega} \times [-x_1\mathbf{r}_1 - x_2\mathbf{r}_2 + (h - x_3)\mathbf{r}_3] \tag{24}$$

or, in view of the first of equations (22),

$$\mathbf{v} = [\omega_2(h - x_3) + \omega_3x_2]\mathbf{r}_1 + [-\omega_3x_1 - \omega_1(h - x_3)]\mathbf{r}_2 + (-\omega_1x_2 + \omega_2x_1)\mathbf{r}_3. \tag{25}$$

To simplify the determination of the acceleration of R^* in N, we define v_k $(k = 1, 2, 3)$ as

$$v_1 \overset{\Delta}{=} \omega_2(h - x_3) + \omega_3x_2, \quad v_2 \overset{\Delta}{=} -\omega_3x_1 - \omega_1(h - x_3), \quad v_3 \overset{\Delta}{=} -\omega_1x_2 + \omega_2x_1 \tag{26}$$

which permits us to write

$$\mathbf{v} = v_1\mathbf{r}_1 + v_2\mathbf{r}_2 + v_3\mathbf{r}_3 \tag{27}$$

and we note that differentiation of equations (26) yields

$$\dot{v}_1 = \dot{\omega}_2(h - x_3) + \dot{\omega}_3x_2 - \omega_2\dot{x}_3 + \omega_3\dot{x}_2 \tag{28}$$

$$\dot{v}_2 = -\dot{\omega}_3x_1 - \dot{\omega}_1(h - x_3) - \omega_3\dot{x}_1 + \omega_1\dot{x}_3 \tag{29}$$

$$\dot{v}_3 = -\dot{\omega}_1x_2 + \dot{\omega}_2x_1 - \omega_1\dot{x}_2 + \omega_2\dot{x}_1. \tag{30}$$

Differentiating equation (27) with respect to t in N, we can now express the acceleration $\mathbf{a}$ of R^* in N as

$$\mathbf{a} = \dot{v}_1\mathbf{r}_1 + \dot{v}_2\mathbf{r}_2 + \dot{v}_3\mathbf{r}_3 + \boldsymbol{\omega} \times (v_1\mathbf{r}_1 + v_2\mathbf{r}_2 + v_3\mathbf{r}_3) \tag{31}$$

and, after using equations (28)–(30) and the first of equations (22), we find with the aid of the definitions

$$\zeta_1 \stackrel{\Delta}{=} \omega_2(v_2 - \dot{x}_3) - \omega_3(v_2 - \dot{x}_2) \tag{32}$$

$$\zeta_2 \stackrel{\Delta}{=} \omega_3(v_1 - \dot{x}_1) - \omega_1(v_3 - \dot{x}_3) \tag{33}$$

$$\zeta_3 \stackrel{\Delta}{=} \omega_1(v_2 - \dot{x}_2) - \omega_2(v_1 - \dot{x}_1) \tag{34}$$

that

$$\mathbf{a} = [\dot{\omega}_2(h - x_3) + \dot{\omega}_3 x_2 + \zeta_1]\mathbf{r}_1 + [-\dot{\omega}_3 x_1 - \dot{\omega}_1(h - x_3) + \zeta_2]\mathbf{r}_2 + (-\dot{\omega}_1 x_2 + \dot{\omega}_2 x_1 + \zeta_3)\mathbf{r}_3. \tag{35}$$

Finally, we turn to the consideration of the angles α, β, γ and δ. Specifically, we seek to express μ_1, μ_2 and μ_3 in terms of α, β, and γ, we need the kinematic differential equations relating α, β and γ to ω_1, ω_2 and ω_3, and we wish to construct a convenient expression for the evaluation of δ.

It follows directly from the way α and β were introduced (see Section 2) that

$$\mathbf{n}_3 = -\cos\alpha \sin\beta \mathbf{r}_1 + \sin\alpha \mathbf{r}_2 + \cos\alpha \cos\beta \mathbf{r}_3. \tag{36}$$

Dot-multiplying this equation successively with $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ and referring to equation (8), one finds that

$$\mu_1 = -\cos\alpha \sin\beta,\ \ \mu_2 = \sin\alpha,\ \ \mu_3 = \cos\alpha\cos\beta. \tag{37}$$

As for δ, the angle between $\mathbf{n}_3$ and $\mathbf{r}_3$, one can write

$$\delta = \cos^{-1}(\mathbf{n}_3 \cdot \mathbf{r}_3) \tag{38}$$

or, after using equation (36) and the third of equations (37),

$$\delta = \cos^{-1}\mu_3. \tag{39}$$

Finally, the kinematical differential equations relating α, β, γ to ω_1, ω_2, ω_3 are [6]

$$\dot{\alpha} = \omega_3 \sin\beta + \omega_1 \cos\beta \tag{40}$$

$$\dot{\beta} = (-\omega_3\cos\beta + \omega_1\sin\beta)\tan\alpha + \omega_2 \tag{41}$$

$$\dot{\gamma} = (\omega_3\cos\beta - \omega_1\sin\beta)\sec\alpha. \tag{42}$$

This concludes the kinematic portion of our analysis.

To formulate dynamical equations of motion, we use as generalized speeds [7] the angular velocity measure numbers ω_1, ω_2, ω_3 defined in equation (3), and we form

Kane's equations,

$$F_k + F_k^* = 0 \qquad (k = 1, 2, 3) \tag{43}$$

after noting that the partial angular velocities $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$, $\boldsymbol{\omega}_3$ of R corresponding to ω_1, ω_2, ω_3, respectively, are [see the first of equations (22)]

$$\boldsymbol{\omega}_k = \mathbf{r}_k \qquad (k = 1, 2, 3) \tag{44}$$

while the partial velocities $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ of R^* corresponding to ω_1, ω_2, ω_3, respectively, are [see equation (25)]

$$\mathbf{v}_1 = (x_3 - h)\mathbf{r}_2 - x_2\mathbf{r}_3, \;\; \mathbf{v}_2 = (h - x_3)\mathbf{r}_1 + x_1\mathbf{r}_3, \;\; \mathbf{v}_3 = x_2\mathbf{r}_1 - x_1\mathbf{r}_2. \tag{45}$$

Hence, expressing the generalized active force F_k as

$$F_k = \boldsymbol{\omega}_k \cdot \mathbf{T} + \mathbf{v}_k \cdot (Mg\mathbf{n}_3) \qquad (k = 1, 2, 3) \tag{46}$$

where g is the acceleration of gravity, we find with the aid of equations (3), (4) and (8) that

$$F_1 = -\sigma\omega_1 + Mg[(x_3 - h)\mu_2 - x_2\mu_3] \tag{47}$$

$$F_2 = -\sigma\omega_2 + Mg[(h - x_3)\mu_1 + x_1\mu_3] \tag{48}$$

$$F_3 = -\sigma\omega_3 + Mg(x_2\mu_1 - x_1\mu_2). \tag{49}$$

As for the generalized inertia force F_k^*, we write

$$F_k^* = \boldsymbol{\omega}_k \cdot \mathbf{T}^* - M\mathbf{v}_k \cdot \mathbf{a} \qquad (k = 1, 2, 3) \tag{50}$$

where $\mathbf{T}^*$, the inertia torque for R in N, is given by

$$\mathbf{T}^* = -\mathbf{I} \cdot \boldsymbol{\alpha} - \boldsymbol{\omega} \times (\mathbf{I} \cdot \boldsymbol{\omega}) \tag{51}$$

and $\boldsymbol{\alpha}$, the angular acceleration of R in N, may be expressed as

$$\boldsymbol{\alpha} = \dot{\omega}_1\mathbf{r}_1 + \dot{\omega}_2\mathbf{r}_2 + \dot{\omega}_3\mathbf{r}_3. \tag{52}$$

Substituting from equations (2) and (52) into equation (51) and introducing the abbreviations

$$R_1 \triangleq [D\omega_1 + (B - C)\omega_2]\omega_3 \tag{53}$$

$$R_2 \triangleq [(C - A)\omega_1 - D\omega_2]\omega_3 \tag{54}$$

$$R_3 \triangleq D(\omega_2^2 - \omega_1^2) + (A - B)\omega_1\omega_2 \tag{55}$$

we thus have

$$\mathbf{T}^* = [-(A\dot{\omega}_1 + D\dot{\omega}_2) + R_1]\mathbf{r}_1 + [-(B\dot{\omega}_2 + D\dot{\omega}_1) + R_2]\mathbf{r}_2 + (-C\dot{\omega}_3 + R_3)\mathbf{r}_3 \tag{56}$$

and, referring to equations (44), (45) and (35) for $\boldsymbol{\omega}_k$, $\mathbf{v}_k$ $(k = 1, 2, 3)$, and $\mathbf{a}$, respectively, we substitute into equation (50) to arrive at

$$F_k^* = -(I_{k1}\dot{\omega}_1 + I_{k2}\dot{\omega}_2 + I_{k3}\dot{\omega}_3) + S_k + R_k \qquad (k = 1, 2, 3) \tag{57}$$

where I_{jk} $(j, k = 1, 2, 3)$, defined as

$$I_{11} \triangleq A + M[x_2^2 + (h - x_3)^2] \tag{58}$$

$$I_{22} \triangleq B + M[x_1^2 + (h - x_3)^2] \tag{59}$$

$$I_{33} \triangleq C + M(x_1^2 + x_2^2) \tag{60}$$

$$I_{12} \triangleq I_{21} = D - Mx_1x_2 \tag{61}$$

$$I_{23} \triangleq I_{32} = M(h - x_3)x_2 \tag{62}$$

$$I_{31} \triangleq I_{13} = M(h - x_3)x_1 \tag{63}$$

can be recognized as moments and products of inertia of R with respect to lines parallel to X_1, X_2, X_3 and passing through the point of contact between S and H. S_k $(k = 1, 2, 3)$ are defined as

$$S_1 \triangleq M[(h - x_3)\zeta_2 + x_2\zeta_3] \tag{64}$$

$$S_2 \triangleq M[(x_3 - h)\zeta_1 - x_1\zeta_3] \tag{65}$$

$$S_3 \triangleq M(x_1\zeta_2 - x_2\zeta_1). \tag{66}$$

Now we are ready to form the dynamic equations of motion by substituting from equation (57) into equation (43), which gives

$$I_{k1}\dot{\omega}_1 + I_{k2}\dot{\omega}_2 + I_{k3}\dot{\omega}_3 = Q_k \qquad (k = 1, 2, 3) \tag{67}$$

where Q_k is defined as

$$Q_k \triangleq F_k + R_k + S_k \qquad (k = 1, 2, 3). \tag{68}$$

For the purpose of performing numerical integrations of equations (67), it is convenient first to solve these equations for $\dot{\omega}_1$, $\dot{\omega}_2$, $\dot{\omega}_3$. Hence, we let G and E_k $(k = 1, 2, 3)$ be the determinants

$$G \triangleq \begin{vmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{vmatrix} \tag{69}$$

$$E_1 \triangleq \begin{vmatrix} Q_1 & I_{12} & I_{13} \\ Q_2 & I_{22} & I_{23} \\ Q_3 & I_{32} & I_{33} \end{vmatrix}, \quad E_2 \triangleq \begin{vmatrix} I_{11} & Q_1 & I_{13} \\ I_{21} & Q_2 & I_{23} \\ I_{31} & Q_3 & I_{33} \end{vmatrix}, \quad E_3 \triangleq \begin{vmatrix} I_{11} & I_{12} & Q_1 \\ I_{21} & I_{22} & Q_2 \\ I_{31} & I_{32} & Q_3 \end{vmatrix} \tag{70}$$

Table 1

Quantity	Equation	Quantity	Equation	Quantity	Equation
μ_k $(k = 1, 2, 3)$	(37)	$\dot{x}_k$ $(k = 1, 2, 3)$	(20)	I_{jk} $(j, k = 1, 2, 3)$	(58)–(63)
ϵ	(16)	v_k $(k = 1, 2, 3)$	(26)	S_k $(k = 1, 2, 3)$	(64)–(66)
$x_1, x_2; x_3$	(18); (17)	ζ_k $(k = 1, 2, 3)$	(32)–(34)	Q_k $(k = 1, 2, 3)$	(68)
$\dot{\mu}_k$ $(k = 1, 2, 3)$	(23)	F_k $(k = 1, 2, 3)$	(47)–(49)	E_k $(k = 1, 2, 3)$	(70)
$\dot{\epsilon}$	(19)	R_k $(k = 1, 2, 3)$	(53)–(55)	G	(69)

after we can replace equation (67) with

$$\dot{\omega}_k = E_k/G \quad (k = 1, 2, 3). \tag{71}$$

This concludes the dynamic analysis.

One may now proceed as follows to perform a numerical simulation of a motion of R that takes place in N during the time interval $0 \leq t \leq t^*$:

(1) Specify the parameters a, b, c, M, h, A, B, C, D (see Section 2); σ (see Section 3); the initial values of α, β, γ and ω_1, ω_2, ω_3 (see Section 2); and t^*, the final value of the time t.

(2) Form the quantities listed in Table 1 in accordance with the equations there indicated.

(3) Integrate equations (40)–(42) simultaneously with equations (71).

(4) Use as output any one or more of α, β, γ, and δ [see equation (39)], ω_1, ω_2, ω_3, or any quantity, or combination of the quantities, appearing in Table 1.

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Zusammenfassung:

Ein "Rattleback" (auch "Celt" oder "Wobblestone" genannt) ist ein Gegenstand der das folgende Verhalten aufweisst: Auf einer horizontalen Ebene in Drehungen um eine vertikale Achse versetzt beginnt er manchmal zu schwingen, hoert auf sich zu drehen und beginnt dann eine Drehung in die der urspruenglichen Bewegung entgegengesetzte Richtung. Fruehere Untersuchungen dieser Erscheinung basierten auf einer Reihe von Annahmen. In dieser Arbeit wird mit Hilfe numerischer Loesungen vollstaendiger, nichtlinearer Bewegungsgleichungen gezeigt, dass man ein realistisches mathematisches Modell konstruieren kann, wenn Rollbewegung ohne Gleiten angenommen wird und ein der Kreisgeschwindigkeit proportionales Drehmoment fuer Energiedissipation sorgt.