Simpliecial objects in categories

Definition 1 (simplicial object)**.** *A* simplicial object X *in* $\mathcal{C}$ *consists of:*

- $\forall_{n\geqslant 0}\, X_n \in \operatorname{Ob}\mathcal{C}$ – n-simplices *of* X*,*
- $\forall_{n\geqslant 0}\forall_{0\leqslant i\leqslant n}\, d_i : X_n \to X_{n-1}$ – boundaries (faces)*,*
- $\forall_{n\geqslant 0}\forall_{0\leqslant i\leqslant n}\, s_i : X_n \to X_{n+1}$ – degeneracies*,*

such that

- $\forall_{i<j}\, d_i d_j = d_{j-1} d_i$,
- $\forall_{i>j}\, s_i s_j = s_j s_{i-1}$,
- $d_i s_j = \begin{cases} s_{j-1} d_i & \forall_{i<j} \\ \mathrm{id} & \forall_{i=j \vee i=j+1} \\ s_i d_{i-1} & \forall_{i>j+1} \end{cases}.$

Definition 2 (simplicial map)**.** *A* simplicial map *between simplicial objects* $X \to Y$ *consists of the sequence of* $f_n : X_n \to Y_n$ *which commute with boundaries and degeneracies.*

Definition 3 (simplicial category)**.** *Denote by* $s\mathcal{C}$ *the category of simplicial objects in* $\mathcal{C}$*.*

Remark 4. Any functor $F : \mathcal{C} \to \mathcal{C}'$ extends to $F : s\mathcal{C} \to s\mathcal{C}'$

Definition 5. *Let* Δ *denote the subcategory of sets* $\operatorname{Ob}(\Delta) = \{[n]\} = \{\{0, 1, \ldots, n\} : n \geqslant 0\}$*,* $\operatorname{Mor}_\Delta([n], [m]) =$ *nondecreasing maps* $[n] \to [m]$*.*

Definition 6 (simplicial object again)**.** *Any functor* $X : \Delta^{op} \to \mathcal{C}$ *is called a* simplicial object *in* $\mathcal{C}$*.*

Definition 7 (simplicial maps)**.** *For* X, Y *simplicial objects,* $\operatorname{Mor}_{s\mathcal{C}}(X, Y) = \operatorname{Mor}_{F(\Delta^{op},\mathcal{C})}(X, Y)$*.*

Definition 8. *Let* $\varepsilon^i : [n-1] \to [n]$ *be defined as* $\varepsilon^i(j) = \begin{cases} j & \forall_{j<i} \\ j+1 & \forall_{j\geqslant i} \end{cases}$ *and* $\eta^i : [n+1] \to [n]$ *be defined as* $\eta^i(j) = \begin{cases} j & \forall_{j\leqslant i} \\ j-1 & \forall_{j>i} \end{cases}.$

Remark 9. These correspond to d_i, s_i respectively.

Proposition 10. *Any morphism* $\alpha \in \Delta$ *can be uniquely expressed as* $\varepsilon \circ \eta$*, where* ε *is a composition of* ε^i*'s, and* η *is a composition of* η^i*'s.*

Remark 11. A bunch of examples appear:

- $\tilde{K}$ – simplicial set of a geometric simplicial complex K,
- Δ_n – topological simplices, and $S : \mathrm{Top} \to s\mathrm{Set}$ singular simplicial set functor,
- $\Delta[n] = \mathrm{Hom}_\Delta(\cdot, [n])$,
- nerve of a small category $N(\mathcal{C})$,
- functor $s\mathrm{Set} \to s\mathrm{R}-\mathrm{mod}$ induced by a functor $\mathrm{Set} \to \mathrm{R}-\mathrm{mod}$ mapping $X \mapsto R[X]$.

Remark 12. If $X, Y \in s\mathrm{Set}$, then there is a simplicial product $(X \times Y)_n = X_n \times Y_n$, $d_i = d_i^X \times d_i^Y$ and $s_i = s_i^X \times s_i^Y$.

Definition 13 (geometric realization)**.** *Define* $\sigma_i : \Delta_n \to \Delta_{n-1}$, $\sigma_i(t_0, \ldots, t_n) = (t_0, \ldots, t_i + t_{i+1}, \ldots, t_n)$ *and* $\delta_i : \Delta_n \to \Delta_{n+1}$, $\delta_i(t_0, \ldots, t_n) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_n)$.

Assume $X \in s\mathrm{Set}$. *We can define a geometric realization of* X

$$|X_\bullet| = \bigsqcup X_n \times \Delta_n \big/ \sim,$$

where $(d_i(x), s) \sim (x, \delta_i(s))$ *for* $(x, s) \in X_n \times \Delta_{n-1}$ *and* $(s_i(x), s) \sim (x, \sigma_i(s))$ *for* $(x, s) \in X_n \times \Delta_{n+1}$.

Remark 14. If a category $\mathcal{C}$ has a faithful functor to Set, then for $X_\bullet \in s\mathcal{C}$ we define its $|X_\bullet|$.

Theorem 15 (properties of $|\bullet| : s\mathrm{Set} \to \mathrm{Top}$)**.**

1. $|X \times Y| \simeq |X| \times |Y|$ *homeomorphism (in CW topology),*
2. K *– geometric simplicial complex, then* $|\tilde{K}_\bullet| \simeq K$ *homeomorphic,*
3. $\mathcal{C}$ *is group* G, *i.e.* $\mathrm{Ob}(\mathcal{C}) = *, \mathrm{Mor}_\mathcal{C}(*, *) = G$, *then* $|N(\mathcal{C})| = K(G, 1)$,
4. *Functors* $S : \mathrm{Top} \to s\mathrm{Set}$ *and* $|\bullet| : s\mathrm{Set} \to \mathrm{Top}$ *are adjoint.*