

**Definition 1** (“exactness”). A functor  $D^+(\mathcal{C}) \rightarrow D^+(\mathcal{C})$  is “exact” if it maps distinguished triangles to distinguished triangles.

**Proposition 2.** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be exact. Then it takes quasi-isomorphisms to quasi-isomorphisms.

Therefore it defines a functor  $DF : D^+(\mathcal{C}) \rightarrow D^+(\mathcal{D})$  and the functor is “exact”.

**Definition 3** (adapted class). We say that a class  $\mathcal{R}$  of objects in  $\mathcal{C}$  is adapted to a functor  $F$  if it satisfies:

- $\mathcal{R}$  is closed under finite coproducts,
- $F$  takes acyclic complexes from  $K^+(\mathcal{R})$  to acyclic ones,
- every object of  $\mathcal{C}$  embeds into an object of  $\mathcal{R}$ .

**Proposition 4.** Let  $\mathcal{R}$  be an adapted class of objects of  $\mathcal{C}$  for a left exact additive functor  $F : \mathcal{C} \rightarrow \mathcal{D}$ . Let  $S_{\mathcal{R}}$  be the class of quasi-isomorphisms in  $K^+(\mathcal{R})$ . Then  $S_{\mathcal{R}}$  is localizing and the canonical functor  $K^+(\mathcal{R})[S_{\mathcal{R}}^{-1}] \rightarrow D^+(\mathcal{C})$  is an equivalence of categories.

**Definition 5.** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be an additive, left exact functor between abelian categories. Its derived functor  $D^+F$  consists of a functor

$$RF : D^+(\mathcal{C}) \rightarrow D^+(\mathcal{D})$$

which is exact and a morphism of functors

$$\varepsilon_F : \mathcal{Q}_{\mathcal{D}} \circ K^+(F) \rightarrow D^+(F) \circ \mathcal{Q}_{\mathcal{C}}$$

such that for any exact  $F : D^+(\mathcal{C}) \rightarrow D^+(\mathcal{D})$  and any natural  $\varepsilon : \mathcal{Q}_{\mathcal{D}} \circ K^+(F) \rightarrow G \circ \mathcal{Q}_{\mathcal{C}}$  there exists a unique morphism  $\eta : D^+F \rightarrow G$  such that the diagram is commutative:

$$\begin{array}{ccc} \mathcal{Q}_{\mathcal{D}} \circ K^+(F) & \xrightarrow{\varepsilon} & G \circ \mathcal{Q}_{\mathcal{C}} \\ & \searrow \varepsilon_F & \uparrow \eta \circ \text{id}_{\mathcal{Q}_{\mathcal{C}}} \\ & & D^+F \circ \mathcal{Q}_{\mathcal{C}} \end{array}$$

**Theorem 6.** If for  $F$  there exists an adapted class  $\mathcal{R}$  of objects in  $\mathcal{C}$ , then  $D^+F$  exists and is unique up to an isomorphism.

**Theorem 7.**  $RF$  may be defined via the composition  $D^+(\mathcal{C}) \rightarrow K^+(\mathcal{R})[S_{\mathcal{R}}^{-1}] \xrightarrow{F} D^+(\mathcal{D})$ .

**Theorem 8.** If  $\mathcal{C}$  contains enough injective objects, then the class  $\mathcal{I}$  of them is adapted to any left exact functor  $F : \mathcal{C} \rightarrow \mathcal{D}$ .

**Proposition 9.** When  $\mathcal{R} = \mathcal{I}$ , then  $\varepsilon_F$  may be defined in the following way.

$T : K^+(\mathcal{I}) \rightarrow D^+(\mathcal{C})$  is an equivalence of categories,  $U : D^+(\mathcal{C}) \rightarrow K^+(\mathcal{I})$  inverse equivalence, we know that for  $Y$  injective  $\text{Hom}_{D^+(\mathcal{C})}(X, Y) \simeq \text{Hom}_{K^+(\mathcal{C})}(X, Y)$ , so take  $f : X \rightarrow I$  a quasi-isomorphism in  $\text{Hom}_{D^+(\mathcal{C})}(\mathcal{Q}X, TU\mathcal{Q}X)$  and denote by  $f_X$  its image under the bijection in  $\text{Hom}_{K^+(\mathcal{C})}(X, U\mathcal{Q}X)$ , and define  $(\varepsilon_F)_X = \mathcal{Q}F(f_X)$ .