

Simplicial objects in categories

Definition 1 (simplicial object). A simplicial object X in \mathcal{C} consists of:

- $\forall_{n \geq 0} X_n \in \text{Ob } \mathcal{C}$ – n -simplices of X ,
- $\forall_{n \geq 0} \forall_{0 \leq i \leq n} d_i : X_n \rightarrow X_{n-1}$ – boundaries (faces),
- $\forall_{n \geq 0} \forall_{0 \leq i \leq n} s_i : X_n \rightarrow X_{n+1}$ – degeneracies,

such that

- $\forall_{i < j} d_i d_j = d_{j-1} d_i$,
- $\forall_{i > j} s_i s_j = s_j s_{i-1}$,
- $d_i s_j = \begin{cases} s_{j-1} d_i & \forall_{i < j} \\ \text{id} & \forall_{i=j \vee i=j+1} \\ s_i d_{i-1} & \forall_{i > j+1} \end{cases}$.

Definition 2 (simplicial map). A simplicial map between simplicial objects $X \rightarrow Y$ consists of the sequence of $f_n : X_n \rightarrow Y_n$ which commute with boundaries and degeneracies.

Definition 3 (simplicial category). Denote by $s\mathcal{C}$ the category of simplicial objects in \mathcal{C} .

Remark 4. Any functor $F : \mathcal{C} \rightarrow \mathcal{C}'$ extends to $F : s\mathcal{C} \rightarrow s\mathcal{C}'$

Definition 5. Let Δ denote the subcategory of sets $\text{Ob}(\Delta) = \{[n]\} = \{\{0, 1, \dots, n\} : n \geq 0\}$, $\text{Mor}_\Delta([n], [m]) = \text{nondecreasing maps } [n] \rightarrow [m]$.

Definition 6 (simplicial object again). Any functor $X : \Delta^{op} \rightarrow \mathcal{C}$ is called a simplicial object in \mathcal{C} .

Definition 7 (simplicial maps). For X, Y simplicial objects, $\text{Mor}_{s\mathcal{C}}(X, Y) = \text{Mor}_{F(\Delta^{op}, \mathcal{C})}(X, Y)$.

Definition 8. Let $\varepsilon^i : [n-1] \rightarrow [n]$ be defined as $\varepsilon^i(j) = \begin{cases} j & \forall_{j < i} \\ j+1 & \forall_{j \geq i} \end{cases}$ and $\eta^i : [n+1] \rightarrow [n]$

be defined as $\eta^i(j) = \begin{cases} j & \forall_{j \leq i} \\ j-1 & \forall_{j > i} \end{cases}$.

Remark 9. These correspond to d_i, s_i respectively.

Proposition 10. Any morphism $\alpha \in \Delta$ can be uniquely expressed as $\varepsilon \circ \eta$, where ε is a composition of ε^i 's, and η is a composition of η^i 's.

Remark 11. A bunch of examples appear:

- \tilde{K} – simplicial set of a geometric simplicial complex K ,
- Δ_n – topological simplices, and $S : \text{Top} \rightarrow \text{sSet}$ singular simplicial set functor,
- $\Delta[n] = \text{Hom}_\Delta(\cdot, [n])$,
- nerve of a small category $N(\mathcal{C})$,
- functor $\text{sSet} \rightarrow \text{sR-mod}$ induced by a functor $\text{Set} \rightarrow \text{R-mod}$ mapping $X \mapsto R[X]$.

Remark 12. If $X, Y \in \text{sSet}$, then there is a simplicial product $(X \times Y)_n = X_n \times Y_n$, $d_i = d_i^X \times d_i^Y$ and $s_i = s_i^X \times s_i^Y$.

Definition 13 (geometric realization). *Define*

$\sigma_i : \Delta_n \rightarrow \Delta_{n-1}$, $\sigma_i(t_0, \dots, t_n) = (t_0, \dots, t_i + t_{i+1}, \dots, t_n)$ and

$\delta_i : \Delta_n \rightarrow \Delta_{n+1}$, $\delta_i(t_0, \dots, t_n) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_n)$.

Assume $X \in \text{sSet}$. We can define a geometric realization of X

$$|X_\bullet| = \sqcup X_n \times \Delta_n / \sim,$$

where $(d_i(x), s) \sim (x, \delta_i(s))$ for $(x, s) \in X_n \times \Delta_{n-1}$

and $(s_i(x), s) \sim (x, \sigma_i(s))$ for $(x, s) \in X_n \times \Delta_{n+1}$.

Remark 14. If a category \mathcal{C} has a faithful functor to Set , then for $X_\bullet \in \text{s}\mathcal{C}$ we define its $|X_\bullet|$.

Theorem 15 (properties of $|\bullet| : \text{sSet} \rightarrow \text{Top}$). 1. $|X \times Y| \simeq |X| \times |Y|$ homeomorphism (in CW topology),

2. K – geometric simplicial complex, then $|\tilde{K}_\bullet| \simeq K$ homeomorphic,

3. \mathcal{C} is group G , i.e. $\text{Ob}(\mathcal{C}) = *$, $\text{Mor}_{\mathcal{C}}(*, *) = G$, then $|N(\mathcal{C})| = K(G, 1)$,

4. Functors $S : \text{Top} \rightarrow \text{sSet}$ and $|\bullet| : \text{sSet} \rightarrow \text{Top}$ are adjoint.