$\mathcal{C}$ – any abelian category.

Definition 1 (n-suspension functor). *$X \in \mathrm{Kom}(\mathcal{C})$, then $X[n] \in \mathrm{Kom}(\mathcal{C})$, $(X[n])_i = X_{n+i}$, $d_{X[n]} = (-1)^n d_X$.*

$f : X \to Y$, $f[n] : X[n] \to Y[n]$ defined in an obvious way.

$T : \mathrm{Kom}(\mathcal{C}) \to \mathrm{Kom}(\mathcal{C})$, $T(X) = X[1]$, is called a translation / shift/ suspension *functor.*

Definition 2. $\mathrm{Kom}(\mathcal{C})$ *is known already.*

$\mathrm{Kom}^+(\mathcal{C}) = \{X \in \mathrm{Kom}(\mathcal{C}) : \exists_{i_0} \forall_{i \leqslant i_0} X_i = 0\}$

$\mathrm{Kom}^-(\mathcal{C})$ *obvious.*

$\mathrm{Kom}^b(\mathcal{C}) = \mathrm{Kom}^+(\mathcal{C}) \cap \mathrm{Kom}^-(\mathcal{C})$

Remark 3. T is well defined in each of these.

Definition 4 (cone). *$f : X \to Y$, $\mathrm{Cone}(f) = \mathrm{C}(F) \in \mathrm{Kom}(\mathcal{C})$ is the* cone *of f.*

$\mathrm{C}(f)_i = X[1]_i \oplus Y_i$, $d_{\mathrm{C}(f)} = (-d_X \pi_1, f[1]\pi_1 + d_y \pi_2)$.

Definition 5 (cylinder). *$f : X \to Y$, $\mathrm{Cyl}(f) \in \mathrm{Kom}(\mathcal{C})$ is the* cylinder *of f.*

$\mathrm{Cyl}(f)_i = X_i \oplus X[1]_i \oplus Y_i$, $d_{\mathrm{Cyl}(f)} = (d_x \pi_1 - \pi_2, -d_X \pi_2, f[1]\pi_2 + d_Y \pi_3)$.

Remark 6. Once in life it is worth to check that $d^2 = 0$ for the cone and the cylinder.

Fact 7. *For any $f : X \to Y$ the following diagram has exact rows and is functorial in f:*

$$\begin{tikzcd}
 & 0 \arrow[r] & Y \arrow[r, "\pi"] \arrow[d, "\alpha"] & \mathrm{C}(f) \arrow[r] \arrow[d, "="] & X[1] \arrow[r] & 0 \\
0 \arrow[r] & X \arrow[r, "\bar{f}"] \arrow[d, "="] & \mathrm{Cyl}(f) \arrow[r] \arrow[d, "\beta"] & \mathrm{C}(f) \arrow[r] & 0 & \\
 & X \arrow[r, "f"] & Y & & &
\end{tikzcd}$$

where $\beta = f\pi_1 + \pi_3$ and other maps are obvious.

Also, α, β are quasi-isomorphism, with $\beta\alpha = \mathrm{id}_Y$ (therefore $Y \sim \mathrm{Cyl}(f)$ in $\mathrm{D}(\mathcal{C})$.

Definition 8 (triangle). *In the category $\mathrm{Kom}(\mathcal{C})$ a* triangle *is any sequence of the form $X \to Y \to Z \to X[1]$.*

A map of triangles is given by a commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "h"] & X[1] \arrow[d, "f[1]"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

A triangle $X \to Y \to Z \to X[1]$ is distinguished if it is isomorphic to a triangle $X' \to \mathrm{Cyl}(f) \to \mathrm{C}(f) \to X'[1]$ for some $f : X' \to Y'$.

Fact 9. *Every exact sequence in $\mathrm{Kom}(\mathcal{C})$ is quasi-isomorphic to a sequence $0 \to X \to \mathrm{Cyl}(f) \to C(f) \to 0$.*

Fact 10. *If $X \to Y \to Z \to X[1]$ is distinguished, then it induces a long exact sequence of cohomology groups:* $\ldots \to H^i(X) \to H^i(Y) \to H^i(Z) \to H^{i+1}(X) \to \ldots$

Definition 11 (homotopy category)**.** $\mathrm{K}(\mathcal{C})$ *is the* homotopy category *of* $\mathrm{Kom}(\mathcal{C})$*, defined via* $\mathrm{Ob}(\mathrm{K}(\mathcal{C})) = \mathrm{Ob}(\mathrm{Kom}(\mathcal{C}))$ *and* $\mathrm{Mor}_{\mathrm{K}(\mathcal{C})}(X,Y) = \mathrm{Mor}_{\mathrm{Kom}(\mathcal{C})}(X,Y)/\sim$*, where* $\sim$ *is a chain homotopy relation.*

Theorem 12. *Let S be a class of quasi-isomorphisms in* $\mathrm{K}(\mathcal{C})$*. Then* $\mathrm{K}(\mathcal{C})[S^{-1}]$ *is isomorphic to* $\mathrm{D}(\mathcal{C})$ *in a canonical way.*

This applies to any of $\mathrm{Kom}^*(\mathcal{C})$*.*

Lemma 13. *Assume $f, g : X \to Y$ are chain homotopic in* $\mathrm{Kom}(\mathcal{C})$*. Then* $\mathcal{Q}(f) = \mathcal{Q}(g)$*.*

Theorem 14. *In any of* $\mathrm{K}^*(\mathcal{C})$ *the class of quasi-isomorphisms is localising.*

Theorem 15. $\mathrm{D}(\mathcal{C})$ *is an additive category.*