

Let \mathcal{C} be abelian, remind $s\mathcal{C}$ – simplicial objects in \mathcal{C} , $C_*(\mathcal{C})$ – chain complexes over \mathcal{C} .

Definition 1 (*s-morphism*). $X_\bullet, Y_\bullet \in s\mathcal{C}$. For a simplicial set $K \in s\text{Set}$ a map which associates $F(\sigma) : X_n \rightarrow Y_n$ to any $\sigma \in K_n$ is called *s-morphism* (denote $F : K \times X_\bullet \rightarrow Y_\bullet$) if for any $\alpha : [m] \rightarrow [n]$ in Δ we have $F(K(\alpha)(\sigma)) \circ X(\alpha) = Y(\alpha)F(\sigma)$.

Observe that when X_\bullet, Y_\bullet are in $s\text{Set}$ then *s-morphisms* are simplicial maps $K_\bullet \times X_\bullet \rightarrow Y_\bullet$.

Example 2. If $K = \Delta[0]$, then *s-morphism* is just a simplicial morphism $X_\bullet \rightarrow Y_\bullet$.

Example 3. If $K = \Delta[1]$ then *s-morphism* is called *a homotopy* between $F(0)$ and $F(1)$.

Remark 4. $T : \mathcal{C} \rightarrow \mathcal{C}'$ functor induces $T : s\mathcal{C} \rightarrow s\mathcal{C}'$, if $X_\bullet \in s\mathcal{C}$, then $T(X)_n = T(X_n), T(X)(\alpha) = T(X(\alpha))$.

Definition 5. If F is an *s-morphism* $F : K_\bullet \times X_\bullet \rightarrow Y_\bullet$, then it defines $TF : K_\bullet \times T(X_\bullet) \rightarrow T(Y_\bullet)$ defined by $TF(\sigma) = T(F(\sigma))$.

Remark 6. Any functor $T : \mathcal{C} \rightarrow \mathcal{C}'$ sends homotopic maps to homotopic ones.

Definition 7. Let \mathcal{C} be abelian, then there are functors

$$s\mathcal{C} \xrightleftharpoons[K]{N} C_*(\mathcal{C})$$

defined as follows.

Normalization N is defined, for $X_\bullet \in s\mathcal{C}$, as

$$N(X)_n = \bigcap_{i=1}^n \ker(d_i : X_n \rightarrow X_{n-1})$$

(e.g. $\ker \left(X_n \xrightarrow{\prod d_i} \prod X_{n-1} \right)$), with

$$d : N(X)_n \rightarrow N(X)_{n-1}$$

induced by d_0 .

K is defined in such a way. If $\alpha : [n] \rightarrow [q]$, then $d(\alpha) = n$ and $r(\alpha) = q$. Notice for any α there is unique $\alpha = \varepsilon \circ \eta$, where ε is an injection and η is a surjection. For $C \in C_*(\mathcal{C})$, take

$$K(C)_n = \bigoplus_{\eta: d(\eta)=n} C_{r(\eta)},$$

Now for $\alpha : [m] \rightarrow [n]$ define

$$KC(\alpha) : K(C)_n \rightarrow K(C)_m$$

on every $C_{r(\eta)}$ in such a way: $\eta\alpha = \varepsilon'\eta'$, let $KC(\alpha)$ map $C_{r(\eta)}$ into $C_{r(\eta')}$ via the formula

$$K(\eta, \alpha) = \begin{cases} \text{id}_{C_{r(\eta)}} & \text{for } \varepsilon' = \text{id}_{[r(\eta)]} \\ d : C_{r(\eta)} \rightarrow C_{r(\eta)-1} = C_{r(\eta')} & \text{for } \varepsilon' = \varepsilon^0 \\ 0 & \text{otherwise} \end{cases}.$$

Remark 8. Observe that if $f : C \rightarrow D$ in $C_*(\mathcal{C})$, then the induced map $KC \rightarrow KD$ is simplicial.

Theorem 9 (Dold-Kan). The functors N and K give an equivalence of $s\mathcal{C}$ and $C_*(\mathcal{C})$.

Remark 10. It was somehow convenient to define, for $X \in s\mathcal{C}$, $\bar{X} \in s\mathcal{C}$ via $\bar{X}_n = \ker(d_{n+1} : X_{n+1} \rightarrow X_n)$.

Lemma 11. Let $f : X_\bullet \rightarrow Y_\bullet$ be a simplicial morphism which satisfies $(Nf)_i$ is mono(epi) for $i \leq n$. Then $f_n : X_n \rightarrow Y_n$ is mono(epi) for $i \leq n$.