

We began with some motivation-discussion. Ain't got time for writing that nicely (in a motivating way).

Theorem 1 (definition of derived category). *Let \mathcal{C} be an abelian category and $\text{Kom}(\mathcal{C})$ denote the category of cochain complexes over \mathcal{C} . Then there is a category $\text{D}(\mathcal{C})$ (derived category of \mathcal{C}) and a functor $\mathcal{Q} : \text{Kom}(\mathcal{C}) \rightarrow \text{D}(\mathcal{C})$ such that*

1. *For every quasi-isomorphism $f \in \text{Mor}(\text{Kom}(\mathcal{C}))$, $\mathcal{Q}(f)$ is an isomorphism.*
2. *\mathcal{Q} is universal with respect to 1, i.e. for every \mathcal{A} and $F : \text{Kom}(\mathcal{C}) \rightarrow \mathcal{A}$, such that for every quasi-isomorphism f the map $F(f)$ is invertible, there exists $\mathcal{Q}F$ making the diagram commutative:*

$$\begin{array}{ccc} \text{Kom}(\mathcal{C}) & \xrightarrow{\mathcal{Q}} & \text{D}(\mathcal{C}) \\ F \searrow & & \swarrow \mathcal{Q}F \\ & \mathcal{A} & \end{array}$$

$\text{D}(\mathcal{C})$ is called the derived category of \mathcal{C} .

Definition 2 (localisation of a category). *\mathcal{B} is a category, S a class of morphisms in \mathcal{B} . We can find a new category $\mathcal{B}[S^{-1}]$ and a functor $L : \mathcal{B} \rightarrow \mathcal{B}[S^{-1}]$ such that for any functor $F : \mathcal{B} \rightarrow \mathcal{B}'$ which takes any $s \in S$ to an isomorphism there exists a functor $LF : \mathcal{B}[S^{-1}] \rightarrow \mathcal{B}'$ such that*

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{L} & \mathcal{B}[S^{-1}] \\ F \searrow & & \swarrow LF \\ & \mathcal{B}' & \end{array}$$

Fact 3. $\text{D}(\mathcal{C}) = \text{Kom}(\mathcal{C})[(q - \text{iso})^{-1}]$

Definition 4. *The class $S \subset \text{Mor}(\mathcal{B})$ is localising if it satisfies*

- $\forall_{X \in \text{Ob}(\mathcal{B})} \text{id}_X \in S$,
- $s, t \in S \implies s \circ t \in S$,
- $\forall_{s \in S, f \text{ any}} \exists_{t \in S, g \text{ any}}$

$$\begin{array}{ccc} W & \xrightarrow{g} & Z \\ \downarrow t & & \downarrow s \\ X & \xrightarrow{f} & Y \end{array}$$

- $\forall_{t \in S, g \text{ any}} \exists_{s \in S, f \text{ any}}$ as above,
- $f, g : X \rightarrow Y$, then $\exists_{s \in S} sf = sg \iff \exists_{t \in S} ft = gt$.

Lemma 5. *If S is localizing in \mathcal{B} , then we can present any morphism in $\mathcal{B}[S^{-1}]$ as a triangle $X \xleftarrow{s} Z \xrightarrow{f} Y$ with equivalence $(s, f) \sim (t, g) \iff \exists r \in S, h$*

$$\begin{array}{ccc}
 & Z'' & \\
 \swarrow r & & \searrow h \\
 Z & & Z' \\
 \downarrow s & \xrightarrow{f} & \downarrow g \\
 X & & Y
 \end{array}$$

Also, an equivalent statement with left fractions is true.

Lemma 6 (composition). *Like that.*

$$\begin{array}{ccccc}
 & & X'' & & \\
 & \swarrow t' & & \searrow f' & \\
 X' & & & & Y' \\
 \swarrow s & \searrow f & & \swarrow t & \searrow g \\
 X & & Y & & Z
 \end{array}$$

Remark 7. Class of quasi-isomorphisms is not localising in $\text{Kom}(\mathcal{C})$.

\mathcal{C} – any abelian category.

Definition 1 (n -suspension functor). $X \in \text{Kom}(\mathcal{C})$, then $X[n] \in \text{Kom}(\mathcal{C})$, $(X[n])_i = X_{n+i}$, $d_{X[n]} = (-1)^n d_X$.

$f : X \rightarrow Y$, $f[n] : X[n] \rightarrow Y[n]$ defined in an obvious way.

$T : \text{Kom}(\mathcal{C}) \rightarrow \text{Kom}(\mathcal{C})$, $T(X) = X[1]$, is called a translation / shift / suspension functor.

Definition 2. $\text{Kom}(\mathcal{C})$ is already known.

$\text{Kom}^+(\mathcal{C}) = \{X \in \text{Kom}(\mathcal{C}) : \exists_{i_0} \forall_{i \leq i_0} X_i = 0\}$

$\text{Kom}^-(\mathcal{C})$ obvious.

$\text{Kom}^b(\mathcal{C}) = \text{Kom}^+(\mathcal{C}) \cap \text{Kom}^-(\mathcal{C})$

Remark 3. T is well defined in each of these.

Definition 4 (cone). $f : X \rightarrow Y$, $\text{Cone}(f) = C(f) \in \text{Kom}(\mathcal{C})$ is the cone of f .

$C(f)_i = X[1]_i \oplus Y_i$, $d_{C(f)} = (-d_X \pi_1, f[1] \pi_1 + d_Y \pi_2)$.

Definition 5 (cylinder). $f : X \rightarrow Y$, $\text{Cyl}(f) \in \text{Kom}(\mathcal{C})$ is the cylinder of f .

$\text{Cyl}(f)_i = X_i \oplus X[1]_i \oplus Y_i$, $d_{\text{Cyl}(f)} = (d_X \pi_1 - \pi_2, -d_X \pi_2, f[1] \pi_2 + d_Y \pi_3)$.

Remark 6. Once in life it is worth to check that $d^2 = 0$ for the cone and the cylinder.

Fact 7. For any $f : X \rightarrow Y$ the following diagram has exact rows and is functorial in f :

$$\begin{array}{ccccccc}
 0 & \longrightarrow & Y & \xrightarrow{\pi} & C(f) & \longrightarrow & X[1] \longrightarrow 0 \\
 & & \downarrow \alpha & & \downarrow = & & \\
 0 & \longrightarrow & X & \xrightarrow{\bar{f}} & \text{Cyl}(f) & \longrightarrow & C(f) \longrightarrow 0 \\
 & & \downarrow = & & \downarrow \beta & & \\
 & & X & \xrightarrow{f} & Y & &
 \end{array}$$

where $\beta = f \pi_1 + \pi_3$ and other maps are obvious.

Also, α, β are quasi-isomorphism, with $\beta \alpha = \text{id}_Y$ (therefore $Y \sim \text{Cyl}(f)$ in $\text{D}(\mathcal{C})$).

Definition 8 (triangle). In the category $\text{Kom}(\mathcal{C})$ a triangle is any sequence of the form $X \rightarrow Y \rightarrow Z \rightarrow X[1]$.

A map of triangles is given by a commutative diagram

$$\begin{array}{ccccccc}
 X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\
 \downarrow f & & \downarrow g & & \downarrow h & & \downarrow f[1] \\
 X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1]
 \end{array}$$

A triangle $X \rightarrow Y \rightarrow Z \rightarrow X[1]$ is distinguished if it is isomorphic to a triangle $X' \rightarrow \text{Cyl}(f) \rightarrow C(f) \rightarrow X'[1]$ for some $f : X' \rightarrow Y'$.

Fact 9. Every exact sequence in $\text{Kom}(\mathcal{C})$ is quasi-isomorphic to a sequence

$0 \rightarrow X \rightarrow \text{Cyl}(f) \rightarrow C(f) \rightarrow 0$.

Fact 10. *If $X \rightarrow Y \rightarrow Z \rightarrow X[1]$ is distinguished, then it induces a long exact sequence of cohomology groups: $\dots \rightarrow H^i(X) \rightarrow H^i(Y) \rightarrow H^i(Z) \rightarrow H^{i+1}(X) \rightarrow \dots$*

Definition 11 (homotopy category). $K(\mathcal{C})$ is the homotopy category of $\text{Kom}(\mathcal{C})$, defined via $\text{Ob}(K(\mathcal{C})) = \text{Ob}(\text{Kom}(\mathcal{C}))$ and $\text{Mor}_{K(\mathcal{C})}(X, Y) = \text{Mor}_{\text{Kom}(\mathcal{C})}(X, Y) / \sim$, where \sim is a chain homotopy relation.

Theorem 12. *Let S be a class of quasi-isomorphisms in $K(\mathcal{C})$. Then $K(\mathcal{C})[S^{-1}]$ is isomorphic to $D(\mathcal{C})$ in a canonical way.*

This applies to any of $\text{Kom}^(\mathcal{C})$.*

Lemma 13. *Assume $f, g : X \rightarrow Y$ are chain homotopic in $\text{Kom}(\mathcal{C})$. Then $\mathcal{Q}(f) = \mathcal{Q}(g)$.*

Theorem 14. *In any of $K^*(\mathcal{C})$ the class of quasi-isomorphisms is localising.*

Theorem 15. $D(\mathcal{C})$ is an additive category.

Definition 1. X is an H^0 -complex if $H^i(X) \neq 0 \implies i = 0$.

Theorem 2. The precomposition of the localization functor $\mathcal{Q} : \text{Kom}(\mathcal{C}) \rightarrow \text{D}(\mathcal{C})$ with embedding $i_0 : \mathcal{C} \rightarrow \text{Kom}(\mathcal{C})$ defines an equivalence between \mathcal{C} and the full subcategory of $\text{D}(\mathcal{C})$ consisting of H^0 -complexes.

Definition 3. $X[i] = T^i([X])$ for $X \in \mathcal{C}$.

Definition 4. \mathcal{C} – abelian, then $\text{Ext}_{\mathcal{C}}^i(X, Y) = \text{Hom}_{\text{D}(\mathcal{C})}(X[0], Y[i])$.

Remark 5. One does not need projectives or injectives in this definition.

Remark 6. $\text{Ext}_{\mathcal{C}}^i(X, Y) = \text{Hom}_{\text{D}(\mathcal{C})}(X[k], Y[k+i])$ for any $k \in \mathbb{Z}$.

Definition 7 (multiplication). There is a multiplication

$$\text{Ext}_{\mathcal{C}}^i(X, Y) \times \text{Ext}_{\mathcal{C}}^j(Y, Z) \rightarrow \text{Ext}_{\mathcal{C}}^{i+j}(X, Z)$$

via composition $\text{Hom}_{\text{D}(\mathcal{C})}(X[0], Y[i]) \times \text{Hom}_{\text{D}(\mathcal{C})}(Y[i], Z[i+j]) \rightarrow \text{Hom}_{\text{D}(\mathcal{C})}(X[0], Z[i+j])$.

Fact 8. For an exact sequence $0 \rightarrow Y' \rightarrow Y \rightarrow Y'' \rightarrow 0$ there is an exact sequence

$$\dots \rightarrow \text{Ext}^i(X, Y') \rightarrow \text{Ext}^i(X, Y) \rightarrow \text{Ext}^i(X, Y'') \rightarrow \text{Ext}^{i+1}(X, Y') \rightarrow \dots$$

Exercise 9. Show that if $X \rightarrow Y \rightarrow Z \rightarrow X[1]$ is distinguished in $\text{D}(\mathcal{C})$, then we have an exact sequence of abelian groups

$$\dots \rightarrow \text{Hom}_{\text{D}(\mathcal{C})}(U, X[i]) \rightarrow \text{Hom}_{\text{D}(\mathcal{C})}(U, Y[i]) \rightarrow \text{Hom}_{\text{D}(\mathcal{C})}(U, Z[i]) \rightarrow \text{Hom}_{\text{D}(\mathcal{C})}(U, X[i+1]) \rightarrow \dots$$

Theorem 10. $\text{Ext}_{\mathcal{C}}^0(X, Y) = \text{Hom}_{\mathcal{C}}(X, Y)$

Theorem 11. $\text{Ext}_{\mathcal{C}}^i(X, Y) = 0$ for $i < 0$.

Theorem 12. Every element in $\text{Ext}_{\mathcal{C}}^i(X, Y)$ has a presentation $X[0] \xleftarrow{s} K \xrightarrow{f} Y[i]$, where $K_j = 0$ for $j < -i$ and for $j > 0$, $K_{-i} = Y$, $f_i = \text{id}$, and s is a quasi-isomorphism.

In other words, every such element comes from an exact sequence

$$0 \rightarrow Y = K^{-i} \rightarrow K^{-i+1} \rightarrow K^{-i+2} \rightarrow \dots \rightarrow K^1 \rightarrow K^0 \rightarrow X \rightarrow 0.$$

Definition 1 (“exactness”). A functor $D^+(\mathcal{C}) \rightarrow D^+(\mathcal{C})$ is “exact” if it maps distinguished triangles to distinguished triangles.

Proposition 2. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be exact. Then it takes quasi-isomorphisms to quasi-isomorphisms.

Therefore it defines a functor $D^+F : D^+(\mathcal{C}) \rightarrow D^+(\mathcal{D})$ and the functor is “exact”.

Definition 3 (adapted class). We say that a class \mathcal{R} of objects in \mathcal{C} is adapted to a functor F if it satisfies:

- \mathcal{R} is closed under finite coproducts,
- F takes acyclic complexes from $K^+(\mathcal{R})$ to acyclic ones,
- every object of \mathcal{C} embeds into an object of \mathcal{R} .

Proposition 4. Let \mathcal{R} be an adapted class of objects of \mathcal{C} for a left exact additive functor $F : \mathcal{C} \rightarrow \mathcal{D}$. Let $S_{\mathcal{R}}$ be the class of quasi-isomorphisms in $K^+(\mathcal{R})$. Then $S_{\mathcal{R}}$ is localizing and the canonical functor $K^+(\mathcal{R})[S_{\mathcal{R}}^{-1}] \rightarrow D^+(\mathcal{C})$ is an equivalence of categories.

Definition 5. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be an additive, left exact functor between abelian categories. Its derived functor D^+F consists of a functor

$$RF : D^+(\mathcal{C}) \rightarrow D^+(\mathcal{D})$$

which is exact and a morphism of functors

$$\varepsilon_F : \mathcal{Q}_{\mathcal{D}} \circ K^+(F) \rightarrow D^+(F) \circ \mathcal{Q}_{\mathcal{C}}$$

such that for any exact $G : D^+(\mathcal{C}) \rightarrow D^+(\mathcal{D})$ and any natural $\varepsilon : \mathcal{Q}_{\mathcal{D}} \circ K^+(F) \rightarrow G \circ \mathcal{Q}_{\mathcal{C}}$ there exists a unique morphism $\eta : D^+F \rightarrow G$ such that the diagram is commutative:

$$\begin{array}{ccc} \mathcal{Q}_{\mathcal{D}} \circ K^+(F) & \xrightarrow{\varepsilon} & G \circ \mathcal{Q}_{\mathcal{C}} \\ & \searrow \varepsilon_F & \uparrow \eta \circ \text{id}_{\mathcal{Q}_{\mathcal{C}}} \\ & D^+F \circ \mathcal{Q}_{\mathcal{C}} & \end{array}$$

Theorem 6. If for F there exists an adapted class \mathcal{R} of objects in \mathcal{C} , then D^+F exists and is unique up to an isomorphism.

Theorem 7. RF may be defined via the composition $D^+(\mathcal{C}) \rightarrow K^+(\mathcal{R})[S_{\mathcal{R}}^{-1}] \xrightarrow{F} D^+(\mathcal{D})$.

Theorem 8. If \mathcal{C} contains enough injective objects, then the class \mathcal{I} of them is adapted to any left exact functor $F : \mathcal{C} \rightarrow \mathcal{D}$.

Proposition 9. When $\mathcal{R} = \mathcal{I}$, then ε_F may be defined in the following way.

$T : K^+(\mathcal{I}) \rightarrow D^+(\mathcal{C})$ is an equivalence of categories, $U : D^+(\mathcal{C}) \rightarrow K^+(\mathcal{I})$ inverse equivalence, we know that for Y injective $\text{Hom}_{D^+(\mathcal{C})}(X, Y) \simeq \text{Hom}_{K^+(\mathcal{C})}(X, Y)$, so take $f : X \rightarrow I$ a quasi-isomorphism in $\text{Hom}_{D^+(\mathcal{C})}(\mathcal{Q}X, TU\mathcal{Q}X)$ and denote by f_X its image under the bijection in $\text{Hom}_{K^+(\mathcal{C})}(X, U\mathcal{Q}X)$, and define $(\varepsilon_F)_X = \mathcal{Q}F(f_X)$.

Remark 1. $\dots \rightarrow \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \rightarrow \dots$ is acyclic, so a 0 map is a quasi-isomorphism, but not a homotopy equivalence.

So one has 2 resolutions of $\dots \rightarrow 0 \rightarrow \dots$, which are not homotopy equivalent.

Definition 2 (K-injectivity, K-projectivity). *A complex A is K-injective (K-projective) if for any acyclic X , $\text{Hom}(X, A)$ ($\text{Hom}(A, X)$) is acyclic.*

Theorem 3 (Spatenstein). *In the category of chain complexes of R -modules every complex has a K-injective (K-projective) resolution.*

Definition 4 (F -acyclic). *Assume that $F : \mathcal{A} \rightarrow \mathcal{B}$ is left-exact, additive and $RF (= D^+ F)$ exists; then we can say that $A \in \mathcal{A}$ is F -acyclic if $RF(A)$ has only 0 cohomology group (i.e. $R^i F(A) = 0$ for $i > 0$).*

Theorem 5. *Let \mathcal{Z} be a class of F -acyclic objects.*

- *If \mathcal{Z} is sufficiently large, then there exists a class of objects adapted to F .*
- *If \mathcal{Z} is sufficiently large, then any class of objects adapted to F is contained in \mathcal{Z} .*
- *If \mathcal{Z} is sufficiently large, then it contains all injective objects of \mathcal{A} .*

$F : \mathcal{A} \rightarrow \mathcal{B}$, $G : \mathcal{B} \rightarrow \mathcal{C}$ additive left exact functors of abelian categories. Assume that there exists classes $\mathcal{R}_{\mathcal{A}}$ of objects adapted to F , $\mathcal{R}_{\mathcal{B}}$ adapted to G , and $F(\mathcal{R}_{\mathcal{A}}) \subset \mathcal{R}_{\mathcal{B}}$. These assumptions imply that $RF, RG, R(G \circ F)$ exist.

Theorem 6. *The functors $RG \circ RF$ and $R(G \circ F)$ are isomorphic as functors $\mathcal{D}(\mathcal{A}) \rightarrow \mathcal{D}(\mathcal{C})$.*

Remark 7. Assume X is of the type that $R^i F(X) = 0$ for $i \neq k$ for k -a fixed integer. Then $RG(RF(X)) = RG(R^k F(X)[-k])$, $R^n(G \circ F)(X) = R^{n-k}G(R^k F(X))$.

Triangulated categories

Assume that \mathcal{C} is an additive category with an automorphism $T : \mathcal{C} \rightarrow \mathcal{C}$ (called the *translation functor*).

Definition 8. $X[1] = T(X)$, $X[n] = T(X[n-1])$

Definition 9 (triangle). *A triangle in \mathcal{C} is a sequence of maps $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$.*

A map of triangles is a commutative diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & T(X) \\ \downarrow f & & \downarrow & & \downarrow & & \downarrow T(f) \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & T(X') \end{array}$$

Definition 10 (triangulated category). *An additive category \mathcal{C} with T on it is called a triangulated category if it is equipped with a class of distinguished triangles (u, v, w) , which satisfy the following conditions:*

- *TR1. Every morphism v can be embedded into distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$.*

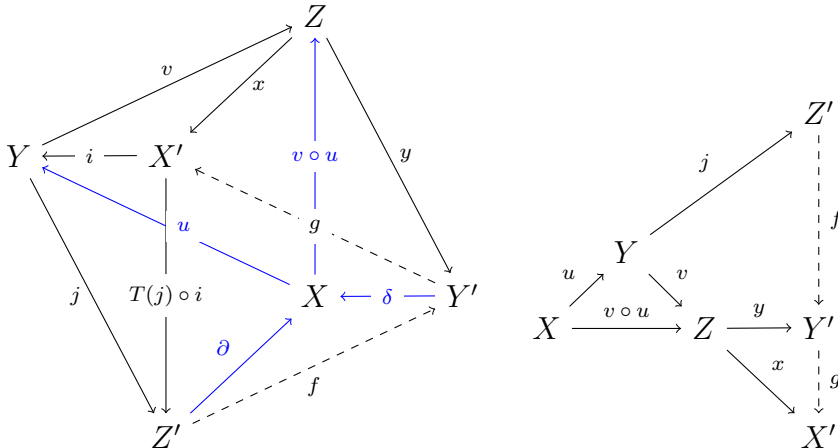
Moreover, if $X = Y$ and $Z = 0$ and $u = \text{id}$, then $X \xrightarrow{\text{id}} X \rightarrow 0 \rightarrow T(X)$ is distinguished.

- *TR2. $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ is distinguished iff $Y \xrightarrow{v} Z \xrightarrow{w} T(X) \xrightarrow{-T(u)} T(Y)$ is distinguished.*
- *TR3. Assume that in the diagram*

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & T(X) \\ f \downarrow & * & \downarrow & & \downarrow h & & \downarrow T(f) \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & T(X') \end{array}$$

rows are distinguished and $$ commutes. Then there exists $h : Z \rightarrow Z'$ such that (f, g, h) is a morphism of triangles.*

- *TR4. [Octahedron axiom] Assume that we have X, Y, Z, X', Y', Z' in \mathcal{C} . Assume that $X \xrightarrow{u} Y \xrightarrow{j} Z' \xrightarrow{\partial} T(X)$, $Y \xrightarrow{v} Z \xrightarrow{x} X' \xrightarrow{i} T(Y)$, $X \xrightarrow{v \circ u} Z \xrightarrow{y} Y' \xrightarrow{\delta} T(X)$ are distinguished. Then there exists distinguished $Z' \xrightarrow{f} Y' \xrightarrow{g} X' \xrightarrow{T(j) \circ i} T(Z')$ such that*
 - the four distinguished triangles form faces of octahedron,*
 - the remaining faces commute,*
 - $yv = fj : Y \rightarrow Y'$,*
 - $u\delta = ig : Y' \rightarrow Y$.*



Theorem 1. *Let \mathcal{C} be an abelian category. Then $K(\mathcal{C})$ (also K^+, K^-, K^b) with standard translation functor and distinguished triangles is triangulated.*

Remark 2. $C(U)$ fits as Z in TR1.

Definition 3 (cohomological functor). *Assume \mathcal{C} is triangulated, \mathcal{A} is abelian. Let $F : \mathcal{C} \rightarrow \mathcal{A}$ be an additive functor. We call it cohomological if for any distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ we have an exact sequence*

$$\dots \rightarrow F(T^i(X)) \rightarrow F(T^i(Y)) \rightarrow F(T^i(Z)) \rightarrow F(T^{i+1}(X)) \rightarrow \dots$$

Definition 4. *Let \mathcal{C} be a triangulated category, S a localizing class of morphisms in \mathcal{C} . We say that S is compatible with triangulation if*

- $s \in S \iff T(s) \in S$,
- in TR3, $f, g \in S \implies h \in S$ for any h .

Theorem 5. *Let \mathcal{C} and S be as above. On $\mathcal{C}[S^{-1}]$ we can define*

- $T_S : \mathcal{C}[S^{-1}] \rightarrow \mathcal{C}[S^{-1}]$, $T_S = T$ on objects and morphisms, i.e. $T(X \xleftarrow{s} Z \xrightarrow{f} Y) = T(X) \xleftarrow{T(s)} T(Z) \xrightarrow{T(f)} T(Y)$.
- $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ is distinguished in $\mathcal{C}[S^{-1}]$ if it is isomorphic to a distinguished triangle coming from \mathcal{C} .

Then $\mathcal{C}[S^{-1}]$ with the structure defined above is triangulated.

Corollary 6. *Derived category of an abelian category inherits the triangulated structure from the homotopy category of complexes.*

Definition 1. $K^{\leq n}(\mathcal{A}), D^{\leq n}(\mathcal{A})$ – full subcategories of objects X for which $H^i(X) = 0$ for $i > n$.

Analogously one defines $K^{\geq n}(\mathcal{A}), D^{\geq n}(\mathcal{A})$.

We know $\mathcal{A} = D^{\leq 0}(\mathcal{A}) \cap D^{\geq 0}(\mathcal{A})$.

Definition 2. Let \mathcal{C} be a triangulated category. Assume that D is a subcategory of \mathcal{C} . Then we write $D^{\leq n} = D^{\leq 0}[-n]$, $D^{\geq n} = D^{\geq 0}[-n]$.

Definition 3 (t-structure). A t-structure on \mathcal{C} consists of a pair of full subcategories $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ in \mathcal{C} which satisfy the following properties:

- $\mathcal{C}^{\leq 0} \subset \mathcal{C}^{\leq 1}, \mathcal{C}^{\geq 0} \supset \mathcal{C}^{\geq 1}$,
- $X \in \mathcal{C}^{\leq 0} \wedge Y \in \mathcal{C}^{\geq 1} \implies \text{Hom}(X, Y) = 0$,
- for all $X \in \mathcal{C}$ there is a distinguished triangle $A \rightarrow X \rightarrow B \rightarrow A[1]$ such that $A \in \mathcal{C}^{\leq 0}, B \in \mathcal{C}^{\geq 1}$.

Definition 4 (core). $\mathcal{A} = \mathcal{C}^{\leq 0} \cap \mathcal{C}^{\geq 0}$ is called the core of the t-structure on \mathcal{C} .

Proposition 5. In $D(\mathcal{A})$, the subcategories $D^{\leq 0}(\mathcal{A})$ and $D^{\geq 0}(\mathcal{A})$ define a t-structure on $D(\mathcal{A})$.

Theorem 6. The core of any t-structure on triangulated \mathcal{C} is an additive category.

Lemma 7. There exist functors $\tau_{\leq n} : \mathcal{C} \rightarrow \mathcal{C}^{\leq n}$ ($\tau_{\geq n} : \mathcal{C} \rightarrow \mathcal{C}^{\geq n}$) which are right (left) adjoint to the inclusion functors.

Moreover, for any $X \in \text{Ob}(\mathcal{C})$ there exists a distinguished triangle

$$\tau_{\leq 0}X \rightarrow X \rightarrow \tau_{\geq 1}X \rightarrow (\tau_{\leq 0}X)[1].$$

Moreover, any distinguished triangle $A \rightarrow X \rightarrow B \rightarrow A[1]$ such that $A \in \mathcal{C}^{\leq 0}, B \in \mathcal{C}^{\geq 1}$ is isomorphic to the one above.

Lemma 8. • $\tau_{\leq n}X = 0 \iff X \rightarrow \tau_{\geq n+1}X$ is an isomorphism,

- if $m \leq n$, then there are maps

$$\tau_{\leq m}X \rightarrow \tau_{\leq m}\tau_{\leq n}X,$$

$$\tau_{\geq n}X \rightarrow \tau_{\geq n}\tau_{\geq m}X,$$

which are isomorphisms, if $m \leq n$, then there is a unique isomorphism

$$\tau_{\geq m}\tau_{\leq n}X \rightarrow \tau_{\leq n}\tau_{\geq m}X (= \tau_{[m,n]}X).$$

Theorem 1. *The core $\mathcal{A} = \mathcal{C}^{\geq 0} \cap \mathcal{C}^{\leq 0} \subset \mathcal{C}$ is an abelian category.*

Definition 2 (cohomology object). *The i -th cohomology object of $X \in \mathcal{C}$ is defined as*

$$H^0(X) = \tau_{[0,0]}(X) \in \mathcal{A},$$

$$H^i(X) = H^0(X[i]) \in \mathcal{A}.$$

Definition 3 (nondegenerate t-structure). *A t-structure on \mathcal{C} is nondegenerate if $\bigcap_n \text{Ob } \mathcal{C}^{\geq n} = \bigcap_n \text{Ob } \mathcal{C}^{\leq n} = \{0\}$.*

Theorem 4. *H^0 is a cohomological functor.*

If additionally the t-structure is nondegenerate, then

- *$f : X \rightarrow Y$ in \mathcal{C} is an isomorphism iff $\forall_i H^i(f)$ is an isomorphism,*
- *$\text{Ob}(\mathcal{C}^{\leq n}) = \{X \in \text{Ob } \mathcal{C} : \forall_{i > n} H^i(X) = 0\}$,*
- *$\text{Ob}(\mathcal{C}^{\geq n}) = \{X \in \text{Ob } \mathcal{C} : \forall_{i < n} H^i(X) = 0\}$.*

Definition 5 (bounded t-structure). *A t-structure is bounded if it is nondegenerate and for any $X \in \mathcal{C}$, $H^i(X) \neq 0$ only for a finite number of i .*

Definition 6 (Ext). $\text{Ext}_{\mathcal{C}}^i(X, Y) = \text{Hom}_{\mathcal{C}}(X, Y[i])$

Definition 7 (multiplication on Ext). *Notice that $\text{Hom}_{\mathcal{C}}(X, Y[i]) = \text{Hom}_{\mathcal{C}}(X[k], Y[i+k])$ and define multiplication $\text{Ext}_{\mathcal{C}}^i(X, Y) \times \text{Ext}_{\mathcal{C}}^j(Y, Z) \rightarrow \text{Ext}_{\mathcal{C}}^{i+j}(X, Z)$ in the most obvious way.*

Theorem 8. *Let \mathcal{A} be a core of a bounded t-structure on \mathcal{C} . Assume $F : \mathcal{D}^b(\mathcal{A}) \rightarrow \mathcal{C}$ satisfies*

$$F(\mathcal{D}^b(\mathcal{A})^{\geq 0}) \subset \mathcal{C}^{\geq 0},$$

$$F(\mathcal{D}^b(\mathcal{A})^{\leq 0}) \subset \mathcal{C}^{\leq 0},$$

then F is an equivalence of categories iff $\text{Ext}_{\mathcal{C}}$ is generated by $\text{Ext}_{\mathcal{C}}^1$ under Yoneda multiplication.

Remark 9. In chain complexes $\text{Ext}_{\mathcal{C}}^i(X, Y) : 0 \rightarrow Y \rightarrow E_i \xrightarrow{d^i} \dots \xrightarrow{d^2} E_1 \xrightarrow{d^1} X \rightarrow 0$.

Theorem 10. *Assume that \mathcal{C} satisfies additionally:*

- *TR5. Arbitrary coproducts and products exist in \mathcal{C} .*
- *There is a generating set Λ of objects in \mathcal{C} , i.e. set Λ such that*
 - *$T(\Lambda) \subset \Lambda$, T – translation functor,*
 - *if $X \in \mathcal{C}$ and $\forall_{\lambda \in \Lambda} \text{Hom}(\lambda, X) = 0$, then $X \simeq 0$.*

Then any homological functor $H : \mathcal{C} \rightarrow \mathcal{A}$ (where \mathcal{A} is abelian) which sends coproducts to products is representable, i.e. $H = \mathcal{C}(\cdot, h)$.

Simplicial objects in categories

Definition 1 (simplicial object). A simplicial object X in \mathcal{C} consists of:

- $\forall_{n \geq 0} X_n \in \text{Ob } \mathcal{C}$ – n -simplices of X ,
- $\forall_{n \geq 0} \forall_{0 \leq i \leq n} d_i : X_n \rightarrow X_{n-1}$ – boundaries (faces),
- $\forall_{n \geq 0} \forall_{0 \leq i \leq n} s_i : X_n \rightarrow X_{n+1}$ – degeneracies,

such that

- $\forall_{i < j} d_i d_j = d_{j-1} d_i$,
- $\forall_{i > j} s_i s_j = s_j s_{i-1}$,
- $d_i s_j = \begin{cases} s_{j-1} d_i & \forall_{i < j} \\ \text{id} & \forall_{i=j} \vee i=j+1 \\ s_i d_{i-1} & \forall_{i > j+1} \end{cases}$.

Definition 2 (simplicial map). A simplicial map between simplicial objects $X \rightarrow Y$ consists of the sequence of $f_n : X_n \rightarrow Y_n$ which commute with boundaries and degeneracies.

Definition 3 (simplicial category). Denote by $s\mathcal{C}$ the category of simplicial objects in \mathcal{C} .

Remark 4. Any functor $F : \mathcal{C} \rightarrow \mathcal{C}'$ extends to $F : s\mathcal{C} \rightarrow s\mathcal{C}'$

Definition 5. Let Δ denote the subcategory of sets $\text{Ob}(\Delta) = \{[n]\} = \{\{0, 1, \dots, n\} : n \geq 0\}$, $\text{Mor}_\Delta([n], [m]) = \text{nondecreasing maps } [n] \rightarrow [m]$.

Definition 6 (simplicial object again). Any functor $X : \Delta^{op} \rightarrow \mathcal{C}$ is called a simplicial object in \mathcal{C} .

Definition 7 (simplicial maps). For X, Y simplicial objects, $\text{Mor}_{s\mathcal{C}}(X, Y) = \text{Mor}_{F(\Delta^{op}, \mathcal{C})}(X, Y)$.

Definition 8. Let $\varepsilon^i : [n-1] \rightarrow [n]$ be defined as $\varepsilon^i(j) = \begin{cases} j & \forall_{j < i} \\ j+1 & \forall_{j \geq i} \end{cases}$ and $\eta^i : [n+1] \rightarrow [n]$

be defined as $\eta^i(j) = \begin{cases} j & \forall_{j \leq i} \\ j-1 & \forall_{j > i} \end{cases}$.

Remark 9. These correspond to d_i, s_i respectively.

Proposition 10. Any morphism $\alpha \in \Delta$ can be uniquely expressed as $\varepsilon \circ \eta$, where ε is a composition of ε^i 's, and η is a composition of η^i 's.

Remark 11. A bunch of examples appear:

- \tilde{K} – simplicial set of a geometric simplicial complex K ,
- Δ_n – topological simplices, and $S : \text{Top} \rightarrow s\text{Set}$ singular simplicial set functor,
- $\Delta[n] = \text{Hom}_\Delta(\cdot, [n])$,
- nerve of a small category $N(\mathcal{C})$,
- functor $s\text{Set} \rightarrow s\mathbf{R}\text{-mod}$ induced by a functor $\text{Set} \rightarrow \mathbf{R}\text{-mod}$ mapping $X \mapsto R[X]$.

Remark 12. If $X, Y \in s\text{Set}$, then there is a simplicial product $(X \times Y)_n = X_n \times Y_n$, $d_i = d_i^X \times d_i^Y$ and $s_i = s_i^X \times s_i^Y$.

Definition 13 (geometric realization). *Define*

$\sigma_i : \Delta_n \rightarrow \Delta_{n-1}$, $\sigma_i(t_0, \dots, t_n) = (t_0, \dots, t_i + t_{i+1}, \dots, t_n)$ and

$\delta_i : \Delta_n \rightarrow \Delta_{n+1}$, $\delta_i(t_0, \dots, t_n) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_n)$.

Assume $X \in s\text{Set}$. We can define a geometric realization of X

$$|X_\bullet| = \sqcup X_n \times \Delta_n / \sim,$$

where $(d_i(x), s) \sim (x, \delta_i(s))$ for $(x, s) \in X_n \times \Delta_{n-1}$

and $(s_i(x), s) \sim (x, \sigma_i(s))$ for $(x, s) \in X_n \times \Delta_{n+1}$.

Remark 14. If a category \mathcal{C} has a faithful functor to Set , then for $X_\bullet \in s\mathcal{C}$ we define its $|X_\bullet|$.

Theorem 15 (properties of $|\bullet| : s\text{Set} \rightarrow \text{Top}$). 1. $|X \times Y| \simeq |X| \times |Y|$ homeomorphism (in CW topology),

2. K – geometric simplicial complex, then $|\tilde{K}_\bullet| \simeq K$ homeomorphic,

3. \mathcal{C} is group G , i.e. $\text{Ob}(\mathcal{C}) = *$, $\text{Mor}_{\mathcal{C}}(*, *) = G$, then $|N(\mathcal{C})| = K(G, 1)$,

4. Functors $S : \text{Top} \rightarrow s\text{Set}$ and $|\bullet| : s\text{Set} \rightarrow \text{Top}$ are adjoint.

Let \mathcal{C} be abelian, remind $s\mathcal{C}$ – simplicial objects in \mathcal{C} , $C_*(\mathcal{C})$ – chain complexes over \mathcal{C} .

Definition 1 (*s-morphism*). $X_\bullet, Y_\bullet \in s\mathcal{C}$. For a simplicial set $K \in s\text{Set}$ a map which associates $F(\sigma) : X_n \rightarrow Y_n$ to any $\sigma \in K_n$ is called *s-morphism* (denote $F : K \times X_\bullet \rightarrow Y_\bullet$) if for any $\alpha : [m] \rightarrow [n]$ in Δ we have $F(K(\alpha)(\sigma)) \circ X(\alpha) = Y(\alpha)F(\sigma)$.

Observe that when X_\bullet, Y_\bullet are in $s\text{Set}$ then *s-morphisms* are simplicial maps $K_\bullet \times X_\bullet \rightarrow Y_\bullet$.

Example 2. If $K = \Delta[0]$, then *s-morphism* is just a simplicial morphism $X_\bullet \rightarrow Y_\bullet$.

Example 3. If $K = \Delta[1]$ then *s-morphism* is called a *homotopy* between $F(0)$ and $F(1)$.

Remark 4. $T : \mathcal{C} \rightarrow \mathcal{C}'$ functor induces $T : s\mathcal{C} \rightarrow s\mathcal{C}'$, if $X_\bullet \in s\mathcal{C}$, then $T(X)_n = T(X_n), T(X)(\alpha) = T(X(\alpha))$.

Definition 5. If F is an *s-morphism* $F : K_\bullet \times X_\bullet \rightarrow Y_\bullet$, then it defines $TF : K_\bullet \times T(X_\bullet) \rightarrow T(Y_\bullet)$ defined by $TF(\sigma) = T(F(\sigma))$.

Remark 6. Any functor $T : \mathcal{C} \rightarrow \mathcal{C}'$ sends homotopic maps to homotopic ones.

Definition 7. Let \mathcal{C} be abelian, then there are functors

$$s\mathcal{C} \xrightleftharpoons[K]{N} C_*(\mathcal{C})$$

defined as follows.

Normalization N is defined, for $X_\bullet \in s\mathcal{C}$, as

$$N(X)_n = \bigcap_{i=1}^n \ker(d_i : X_n \rightarrow X_{n-1})$$

(e.g. $\ker \left(X_n \xrightarrow{\prod d_i} \prod X_{n-1} \right)$), with

$$d : N(X)_n \rightarrow N(X)_{n-1}$$

induced by d_0 .

K is defined in such a way. If $\alpha : [n] \rightarrow [q]$, then $d(\alpha) = n$ and $r(\alpha) = q$. Notice for any α there is unique $\alpha = \varepsilon \circ \eta$, where ε is an injection and η is a surjection. For $C \in C_*(\mathcal{C})$, take

$$K(C)_n = \bigoplus_{\eta: d(\eta)=n} C_{r(\eta)},$$

Now for $\alpha : [m] \rightarrow [n]$ define

$$KC(\alpha) : K(C)_n \rightarrow K(C)_m$$

on every $C_{r(\eta)}$ in such a way: $\eta\alpha = \varepsilon'\eta'$, let $KC(\alpha)$ map $C_{r(\eta)}$ into $C_{r(\eta')}$ via the formula

$$K(\eta, \alpha) = \begin{cases} \text{id}_{C_{r(\eta)}} & \text{for } \varepsilon' = \text{id}_{[r(\eta)]} \\ d : C_{r(\eta)} \rightarrow C_{r(\eta)-1} = C_{r(\eta')} & \text{for } \varepsilon' = \varepsilon^0 \\ 0 & \text{otherwise} \end{cases}.$$

Remark 8. Observe that if $f : C \rightarrow D$ in $C_*(\mathcal{C})$, then the induced map $KC \rightarrow KD$ is simplicial.

Theorem 9 (Dold-Kan). The functors N and K give an equivalence of $s\mathcal{C}$ and $C_*(\mathcal{C})$.

Remark 10. It was somehow convenient to define, for $X \in s\mathcal{C}$, $\bar{X} \in s\mathcal{C}$ via $\bar{X}_n = \ker(d_{n+1} : X_{n+1} \rightarrow X_n)$.

Lemma 11. Let $f : X_\bullet \rightarrow Y_\bullet$ be a simplicial morphism which satisfies $(Nf)_i$ is mono(epi) for $i \leq n$. Then $f_n : X_n \rightarrow Y_n$ is mono(epi) for $i \leq n$.

Definition 1 (cosimplicial object). $X : \Delta \rightarrow \mathcal{C}$.

Denote the category of cosimplicial objects in \mathcal{C} as $c\mathcal{C}$.

Theorem 2 (Dold-Kan again). $c\mathcal{C} \simeq \text{cochain complexes over } \mathcal{C}$

Remark 3. $c\mathcal{C} = s(\mathcal{C}^{op})$

Definition 4. For $X \in s\mathcal{C}$ define $kX \in C_*(\mathcal{C})$, $(kX)_n = X_n$, $d = \sum_{i=0}^n (-1)^i d_i$.

Theorem 5. The natural embedding $NX \hookrightarrow kX$ is a chain homotopy equivalence.

Remark 6. • $NX = kX / DX$, where $(DX)_n = \bigcup_{i=0}^{n-1} \text{im}(s_i : X_{n-1} \rightarrow X_n)$
 $= \text{im} \left(\prod X_{n-1} \xrightarrow{\prod s_i} X_n \right)$.

- $kX = NX \oplus DX$.
- DX is contractible.

Remark 7. One can get NX using $(-1)^n d_n$ instead of d_0 .

Remark 8. Observe that if $\tau_n : [n] \rightarrow [n], i \mapsto n - i$, and $\alpha^* = \tau_n \alpha \tau_m$ for $\alpha : [m] \rightarrow [n]$, then we get an involution of the category Δ , $\alpha \rightarrow \alpha^*$.

Hence we get an involution of $s\mathcal{C}$, $X \rightarrow X^*$, $(X^*)_n = X_n$, $X^*(\alpha) = X(\alpha^*)$, d_i goes to d_{n-i} .

We can define $N^*X = N(X^*)$, $K^*C = (KC)^*$, getting $N^*K^* = NK$, $K^*N^* = \text{Id}$.

Theorem 9. 1. $f_1, f_2 : X \rightarrow Y$ homotopic in $s\mathcal{C}$ iff Nf_1, Nf_2 are chain homotopic in $C_*(\mathcal{C})$,

2. $\varphi_1, \varphi_2 : C \rightarrow D$ are chain homotopic in $C_*(\mathcal{C})$ iff $K\varphi_1, K\varphi_2$ are homotopic in $s\mathcal{C}$.

Definition 10 (simplicial resolution). Let $T : \mathcal{C} \rightarrow \mathcal{C}'$ functor between abelian categories, \mathcal{C} has enough projective objects, $A \in \text{Ob}(\mathcal{C})$ and $n \in \mathbb{N}$.

Then a pair (X_\bullet, ξ) is called a simplicial resolution of A of degree n (simplicial resolution of (A, n)) if $X_\bullet \in s\mathcal{C}$, $X_i = 0$ for $i < n$, $H_j(X) := H_j(kX) = 0$ for $j > n$ and $\xi : H_n(X) \rightarrow A$ is an isomorphism.

If $\forall_i X_i$ is projective, then X is a projective resolution of (A, n) .

Usually we will remove ξ from notation and say that $H_n(X) = A$.

Remark 11. 1. If X_\bullet is a simplicial resolution of (A, n) , then NX is a resolution of A shifted up by n . If X_\bullet is projective, then NX is a projective resolution.

2. If $P \in C_*(\mathcal{C})$ is a projective resolution of A shifted by n , then KP is a simplicial projective resolution of (A, n) .

3. If $\alpha : A \rightarrow B$ in \mathcal{C} and X, Y are projective resolutions of (A, n) and (B, n) , then there exists a simplicial morphism $f : X \rightarrow Y$ which induces $\alpha = H_n(f)$.

Moreover, f is unique up to homotopy.

Definition 12 (derived functor). *Functor $L_q T(\bullet, n) : \mathcal{C} \rightarrow \mathcal{C}'$ defined below is called q -th left derived functor of T of degree n , where $L_q T(\bullet, n)(A) = H_q(T(X))$, where X is any simplicial resolution of A .*

Remark 13. If T is additive, then $k(T(X)) = T(kX)$, so $L_q T(A, n) = L_{q-n} T(A)$ (L_{q-n} from ordinary homotopy category).

Remark 14. When T is not additive, then $\sum_{i=0}^n (-1)^i T(d_i)$ is usually not equal $T(\sum (-1)^i d_i)$, so $k(TX)$ and $T(kX)$ may have different homology.

Let $T : \mathcal{C} \rightarrow \mathcal{C}'$ functor of abelian categories. Assume $T(0) = 0$ (if $T(0) = A$, then take $T' = \ker(T \rightarrow T(0) = A)$).

Definition 15 (cross effect). *For any $k \in \mathbb{N}$ we define the k -th cross-effect of T as a functor $T_k : \mathcal{C}^k \rightarrow \mathcal{C}'$ such that we get a functorial decomposition $T(A_1 \oplus \dots \oplus A_k) = \bigoplus_{i=1}^k T(A_i) \oplus \bigoplus_{i_1 < i_2} T_2(A_{i_1}, A_{i_2}) \oplus \dots \oplus T_k(A_1, \dots, A_k)$. We can define T_k inductively,*

- $T_1 = T$,
- $T_2(A_1, A_2) = \ker(T(A_1 \oplus A_2) \rightarrow T(A_1) \oplus T(A_2))$,
- \dots ,
- $T_k(A_1, \dots, A_k) = \ker(T(A_1 \oplus \dots \oplus A_n) \xrightarrow{\prod_i} \prod_i T(A_1 \oplus \dots \oplus \hat{A}_i \oplus \dots \oplus A_k))$.

Definition 16 (functor degree). *We say that T is of degree $\leq k$ if $T_{k+1} = 0$.*

We say that T is of degree k if T is of degree $\leq k$ and $T_k \neq 0$.

Theorem 17. *Cross-effects have the following properties:*

- if for some i , $A_i = 0$, then $T_k(A_1, \dots, A_n) = 0$,
- T_k is symmetric in its variables,
- if we define $s^{(1)}(A) = T_2(A, A_2)$, $s^{(2)}(A) = T_2(A_1, A)$, then $s_2^{(1)}(A_1, A_2; A_3) = s_2^{(2)}(A_1; A_2, A_3) = T_3(A_1, A_2, A_3)$.

Example 18. $\deg T \leq 1$ iff T is additive.

Example 19. $T(A) = A^{\otimes 2}$, then $T_2(A, B) = (A \otimes B) \oplus (B \otimes A)$ and it is linear in A, B , so T is of degree 2.

Theorem 1. Assume T is of degree $\leq k$, $A \in \text{Ob}(\mathcal{C})$ is of projective dimension $\leq r$, then $L_q T(A, n) = 0$ for $q > k(r + n)$.

Lemma 2. Let T be as above and $X \in s\mathcal{C}$ such that $(NX)_i = 0$ for $i > m$. Then $N(TX)_i = 0$ for $i > km$.

Definition 3 (suspension). $SA = \text{coker}(A \rightarrow CA)$, or $(SA)_q = A_{q-1}$ and $d^{SA} = -d^A$.

Corollary 4. We have an exact sequence $0 \rightarrow A \xrightarrow{\kappa} CA \xrightarrow{\pi} SA \rightarrow 0$.

Definition 5. Let $X \in s\mathcal{C}$. Define cone and suspension of X by the formulas $CX = KCN X$, $SX = KSN X$.

Remark 6. We have an exact sequence (exact on each level) $0 \rightarrow X \xrightarrow{\kappa} CX \xrightarrow{\pi} SX \rightarrow 0$.

Applying T we get (not necessarily exact) $0 \rightarrow TX \xrightarrow{T(\kappa)} T(CX) \xrightarrow{T(\pi)} T(SX) \rightarrow 0$.

Remark 7. Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a sequence in $C_*(\mathcal{C})$ such that $g \circ f = 0$ and B is contractible, i.e. we have $s_q : B_q \rightarrow B_{q+1}$ such that $d^B s + s d^B = \text{id}$. Then $gsf : A \rightarrow C$ gives a chain map $SA \rightarrow C$ and hence a map $H_q(A) \rightarrow H_{q+1}(C)$.

Theorem 8. $H(gsf)$ does not depend on the choice of s .

Definition 9 (suspension homomorphism). The map $\sigma : H_q(TX) \rightarrow H_{q+1}(TSX)$ induced by κ and π is called a suspension homomorphism.

Proposition 10. σ defines a natural transformation of functors.

Proposition 11. Assume T additive, then $0 \rightarrow T(X) \rightarrow T(CX) \rightarrow T(SX) \rightarrow 0$ exact and we have a long exact sequence of homology groups: $\dots \rightarrow 0 \rightarrow H_{q+1}(TSX) \rightarrow H_q(TX) \rightarrow 0 \rightarrow \dots$, and σ is the inverse of the map in the middle.

Definition 12. Let $T_p^d(A) = T_p(A, \dots, A)$ (d means diagonal).

Definition 13. Define $d_i = \rho \circ T(\alpha'_i) \circ \lambda : T_p^d(A) \rightarrow T_{p-1}^d(A)$, where λ monomorphism $T_p^d(A) \rightarrow T(A \oplus \dots \oplus A)$, ρ epimorphism $T(A \oplus \dots \oplus A) \rightarrow T_{p-1}^d(A)$, and $d'_j : \bigoplus_{i=1}^p A \rightarrow \bigoplus_{i=1}^{p-1} A$, equal to $(\text{id}, \dots, \text{id}, (\text{id} + \text{id})_j, \text{id}, \dots, \text{id})$.

Definition 14. Let $X \in s\mathcal{C}$. Define a sequence of simplicial objects in \mathcal{C}' :

$$\mathcal{T}X = \left(T_1^d(X) \xleftarrow{\partial'} T_2^d(X) \leftarrow T_3^d(X) \leftarrow \dots \right), \quad \partial' = \sum_{i=1}^{p-1} (-1)^i d_i.$$

Remark 15. $\partial' \circ \partial' = 0$.

Corollary 16. Therefore $\mathcal{T}X$ gives a bicomplex

$$(\mathcal{T}X)_{p,q} = T_p^d(X_q)$$

with horizontal differentials ∂' and vertical differentials from kX .

Proposition 17. *We have an embedding $i : kTX = (\mathcal{T}X)_{1,*} \hookrightarrow \text{Tot}(\mathcal{T}X)$ and it is a chain map of degree 1.*

Theorem 18. *There is a natural isomorphism $\omega : H\text{tot}(\mathcal{T}SX) \simeq H(\mathcal{T}SX)$ such that for*

$$\begin{array}{ccc} & H_{q+1}(\mathcal{T}SX) & \\ \sigma \nearrow & \uparrow \omega & \\ H_q \mathcal{T}X & & \\ \searrow i & & \\ & H_q(\mathcal{T}X) & \end{array}$$

any q the diagram commutes:

Definition 19 (bar construction). $\mathcal{T}X$ is called the bar construction for T .

Corollary 20. *If T is additive, then σ is an isomorphism.*

Corollary 21. *If T is of degree 2, then there exists a morphism β such that the sequence is exact: $\dots \rightarrow H_q T_2(X, X) \xrightarrow{\alpha} H_q(\mathcal{T}X) \xrightarrow{\sigma} H_{q+1}(\mathcal{T}SX) \rightarrow H_{q+1} T_2(X, X) \rightarrow H_{q+1}(\mathcal{T}X) \rightarrow \dots$*

Corollary 22. *There exists a spectral sequence which converges to $H_* \mathcal{T}SX$ and which satisfies*

- E'_{pq} is equal to the complex $H_q \mathcal{T}X \xleftarrow{H_q(\partial')} H_q T_2(X, X) \xleftarrow{H_q(\partial')} H_q T_3(X, X, X) \leftarrow \dots$,
- the homomorphism $H_q \mathcal{T}X = E'_{pq} \rightarrow H_{q+1} \mathcal{T}SX$ is the same as σ .

Definition 23. We say that $X \in s\mathcal{C}$ is trivial below n if there exists $X' \in s\mathcal{C}$ which is homotopy equivalent to X and satisfies $X'_i = 0$ for $i < n$.

Lemma 24. *If X is projective and $H_q(X) = 0$ for $q < n$, then X is trivial below n .*

Remark 25 (digression). A bisimplicial object is $X_{p,q}$ with $X_{p,q} \rightarrow X_{r,s}$ for any $\alpha : [r] \rightarrow [p], \beta : [s] \rightarrow [q]$, which satisfy simplicial identities in both directions.

Every bisimplicial object gives us a bicomplex kX .

If X is bisimplicial, then it comes with a diagonal simplicial object $X_{k,k} \xrightarrow{(\alpha, \alpha)} X_{l,l}$ (where $\alpha : [l] \rightarrow [k]$).

Theorem 26 (Eilenberg-Zilber(-Cantier)). *There is a chain homotopy equivalence $k(X_{p,p}) \simeq \text{tot}(kX_{p,q})$.*

Remark 27. Observe that $X_{p,p}$ is in degree p to the left and $p + p$ to the right.

Proposition 28. *Let $T : \mathcal{C}^l \rightarrow \mathcal{C}'$ be such that $T(\dots, 0_j, \dots) = 0$. Let, for $j = 1, \dots, l$, $X^j \in s\mathcal{C}$ be trivial below n_j . Then $T(X^1, \dots, X^l)$ is trivial below $n_1 + \dots + n_l = n$ (therefore $H_q T(X^1, \dots, X^l) = 0$ for $q < n$).*

Corollary 29. *If X is trivial below n , then the suspension homomorphism $\sigma : H_q(\mathcal{T}X) \rightarrow H_{q+1}(\mathcal{T}SX)$ is an isomorphism for $q < 2n$ and epimorphism for $q = 2n$.*

Definition 30 (stable derived functors). $L_{q+n}(T\bullet, n)$ for $n > q$ is called the q -th stable derived functor of T , denoted $L_q^s T(\bullet)$.

Remark 1. $C_*(\mathcal{C})$ does not have enough projective objects.

Theorem 2. The sequence of functors $\{H_i\}_{i=0}^\infty$ gives us a universal δ -functor (takes short exact sequences to long exact sequences), i.e. if we have another sequence T_i such that $T_0 = H_0$, then $\forall_i H_i^* = T_i^*$.

Lemma 3. For a given $C_* \in C_*(\mathcal{C})$ there exists $P_* \rightarrow C_*$ such that $H_i(P_*) = 0$ for $i > 0$.

Remark 4. If $p + q = n$, let $f_{pq} : X_{nn} \rightarrow X_{pq}$ be defined as $d_{p+1}^h \circ \dots \circ d_n^h \circ d_0^v \circ \dots \circ d_0^v$, and then the Alexander-Whitney map $\sum_{p+q=n} f_{pq} : X_{nn} \rightarrow \bigoplus_{p+q=n} X_{pq}$ gives a chain homotopy equivalence of $k(X_{pp})$ and $\text{tot}(kX_{pq})$.

Remark 5. We may take a projective simplicial resolution P_* of A of degree $n > i$, then $L_i^s T(A) = H_{n+i}(T(P_*))$.

Theorem 6. $\deg L_i T(\bullet, n) \leq \lfloor \frac{i}{n} \rfloor$.

Remark 7. Or theorem? Or proof? It is written that $T((A, n) \oplus (B, n)) = T(A, n) \oplus T(B, n) \oplus V$ where V is trivial below $2n$.

Proposition 8. $\forall_i L_i^s T$ is an additive functor.

Proposition 9. Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be exact in \mathcal{C} . Then we have a long exact sequence $\dots \rightarrow L_{q+1}^s T(C) \rightarrow L_q^s T(A) \rightarrow L_q^s T(B) \rightarrow L_q^s T(C) \rightarrow \dots$

Proposition 10. If $0 \rightarrow T' \rightarrow T \rightarrow T'' \rightarrow 0$ is an exact sequence of functors, then we have a long exact sequence of functors $\dots \rightarrow L_{i+1}^s T'' \rightarrow L_i^s T' \rightarrow L_i^s T \rightarrow L_i^s T'' \rightarrow \dots$

Proposition 11. Let U be an additive functor, then for any functor T we have $\text{Hom}_{sth}(T, U) \simeq \text{Hom}_{sth2}(L_0^s T, U)$.

Applications of stable derived functors

Theorem 1. $T : R\text{-mod} \rightarrow R\text{-mod}$, then

$$L_i^s T(A) = \lim_n \pi_{i+n}(T(\tilde{R}[S^n] \otimes P_*)) \lim_n H_{i+n}(T(\tilde{R}[S^n] \otimes P_*)),$$

where S^n is any simplicial model of n -sphere, $\tilde{R}[\gamma] = R[\gamma]/R[*]$ a simplicial set, P_* is any projective resolution of A .

The limit is taken via suspension

$$\pi_{i+n}(T(\tilde{R}[S^n] \otimes P_*)) \rightarrow \pi_{i+n+1}(S^1 \wedge T(\tilde{R}[S^n] \otimes P_*)) \rightarrow \pi_{i+n+1}(T(\tilde{R}[S^{n+1}] \otimes P_*)).$$

In general for $S^1 \wedge F(X) \rightarrow F(S^1 \wedge X)$ one has to have for any $z \in S^1$, $F(X) \rightarrow F(S^1 \wedge X)$, $X \rightarrow S^1 \wedge X$, $x \rightarrow z \wedge x$.

One takes $R = \mathbb{Z}/p$ or $R = \mathbb{Z}$.

$$L_i^s T(\mathbb{Z}/p) = \lim \pi_{i+n} T(\mathbb{Z}/p[S^n]), \text{ but } \widetilde{\mathbb{Z}/p}[S^n] = K(\mathbb{Z}/p, n), \tilde{\mathbb{Z}}[S^n] = K(\mathbb{Z}, n).$$

Stalk skewed gra..itions on $H^*(\bullet, \mathbb{Z}/p)$ is

$$H_*^s(K(\mathbb{Z}/p), \mathbb{Z}/p) = H_*^s(K(\mathbb{Z}/p, n), \mathbb{Z}/p) = L_*^s \mathbb{Z}_p[.] (\mathbb{Z}/p). (?)$$

Theorem 2. Let SP^i be the i -th symmetric power functor, and SP_p^i the p -reduced i -th symmetric power, and $SP_p^* = \bigoplus SP_p^i / \langle x^p - 1 \rangle$.

$$\text{Then } L_*^s SP^*(\mathbb{Z}/p) = H_*^s(K(\mathbb{Z}), \mathbb{Z}/p), L_*^s SP_p^*(\mathbb{Z}/p) = H_*^s(K(\mathbb{Z}/p), \mathbb{Z}/p).$$

Calculations: Let Γ be a category of functors $T : \text{finite pointed sets} \rightarrow \mathbb{Z}/p\text{-vect}$, $T(*) = 0$. $L \in \Gamma$ is defined as $L(X) = \widetilde{\mathbb{Z}/p}[X]$.

Lemma 3. Let $T : \mathbb{Z}/p\text{-vect} \rightarrow \mathbb{Z}/p\text{-vect}$. Then $L_i^s T(\mathbb{Z}/p) = \text{Tor}_i^\Gamma(L^*, T \circ L)$, where $L^*(X) = L(X)^*$, and

I have found these notes useful in understanding derived functors.

Some detailed constructions are here, and some worked out examples are here in section 2.2.