

We began with some motivation-discussion. Ain't got time for writing that nicely (in a motivating way).

**Theorem 1** (definition of derived category). *Let  $\mathcal{C}$  be an abelian category and  $\text{Kom}(\mathcal{C})$  denote the category of cochain complexes over  $\mathcal{C}$ . Then there is a category  $D(\mathcal{C})$  (derived category of  $\mathcal{C}$ ) and a functor  $\mathcal{Q} : \text{Kom}(\mathcal{C}) \rightarrow D(\mathcal{C})$  such that*

1. *For every quasi-isomorphism  $f \in \text{Mor}(\text{Kom}(\mathcal{C}))$ ,  $\mathcal{Q}(f)$  is an isomorphism.*
2.  *$\mathcal{Q}$  is universal with respect to 1, i.e. for every  $\mathcal{A}$  and  $F : \text{Kom}(\mathcal{C}) \rightarrow \mathcal{A}$ , such that for every quasi-isomorphism  $f$  the map  $F(f)$  is invertible, there exists  $\mathcal{Q}F$  making the diagram commutative:*

$$\begin{array}{ccc} \text{Kom}(\mathcal{C}) & \xrightarrow{\mathcal{Q}} & D(\mathcal{C}) \\ F \searrow & & \swarrow \mathcal{Q}F \\ & \mathcal{A} & \end{array}$$

$D(\mathcal{C})$  is called the derived category of  $\mathcal{C}$ .

**Definition 2** (localisation of a category).  $\mathcal{B}$  is a category,  $S$  a class of morphisms in  $\mathcal{B}$ . We can find a new category  $\mathcal{B}[S^{-1}]$  and a functor  $L : \mathcal{B} \rightarrow \mathcal{B}[S^{-1}]$  such that for any functor  $F : \mathcal{B} \rightarrow \mathcal{B}'$  which takes any  $s \in S$  to an isomorphism there exists a functor  $LF : \mathcal{B}[S^{-1}] \rightarrow \mathcal{B}'$  such that

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{L} & \mathcal{B}[S^{-1}] \\ F \searrow & & \swarrow LF \\ & \mathcal{B}' & \end{array}$$

**Fact 3.**  $D(\mathcal{C}) = \text{Kom}(\mathcal{C})[(q - \text{iso})^{-1}]$

**Definition 4.** The class  $S \subset \text{Mor}(\mathcal{B})$  is localising if it satisfies

- $\forall_{X \in \text{Ob}(\mathcal{B})} \text{id}_X \in S$ ,
- $s, t \in S \implies s \circ t \in S$ ,
- $\forall_{s \in S, f \text{ any}} \exists_{t \in S, g \text{ any}}$

$$\begin{array}{ccc} W & \xrightarrow{g} & Z \\ \downarrow t & & \downarrow s \\ X & \xrightarrow{f} & Y \end{array}$$

- $\forall_{t \in S, g \text{ any}} \exists_{s \in S, f \text{ any}}$  as above,
- $f, g : X \rightarrow Y$ , then  $\exists_{s \in S} sf = sg \iff \exists_{t \in S} ft = gt$ .

**Lemma 5.** *If  $S$  is localizing in  $\mathcal{B}$ , then we can present any morphism in  $\mathcal{B}[S^{-1}]$  as a triangle  $X \xleftarrow{s} Z \xrightarrow{f} Y$  with equivalence  $(s, f) \sim (t, g) \iff \exists r \in S, h$*

$$\begin{array}{ccc}
 & Z'' & \\
 \swarrow \scriptstyle r & & \searrow \scriptstyle h \\
 Z & & Z' \\
 \downarrow \scriptstyle s & \xleftarrow{f} & \downarrow \scriptstyle g \\
 X & & Y
 \end{array}$$

*Also, an equivalent statement with left fractions is true.*

**Lemma 6** (composition). *Like that.*

$$\begin{array}{ccccc}
 & & X'' & & \\
 & \swarrow & \scriptstyle f' & \searrow & \\
 & X' & & Y' & \\
 \swarrow \scriptstyle s & \searrow \scriptstyle f & & \swarrow \scriptstyle t & \searrow \scriptstyle g \\
 X & & Y & & Z
 \end{array}$$

**Remark 7.** Class of quasi-isomorphisms is not localising in  $\text{Kom}(\mathcal{C})$ .

$\mathcal{C}$  – any abelian category.

**Definition 1** ( $n$ -suspension functor).  $X \in \text{Kom}(\mathcal{C})$ , then  $X[n] \in \text{Kom}(\mathcal{C})$ ,  $(X[n])_i = X_{n+i}$ ,  $d_{X[n]} = (-1)^n d_X$ .

$f : X \rightarrow Y$ ,  $f[n] : X[n] \rightarrow Y[n]$  defined in an obvious way.

$T : \text{Kom}(\mathcal{C}) \rightarrow \text{Kom}(\mathcal{C})$ ,  $T(X) = X[1]$ , is called a translation / shift / suspension functor.

**Definition 2.**  $\text{Kom}(\mathcal{C})$  is already known.

$\text{Kom}^+(\mathcal{C}) = \{X \in \text{Kom}(\mathcal{C}) : \exists_{i_0} \forall_{i \leq i_0} X_i = 0\}$

$\text{Kom}^-(\mathcal{C})$  obvious.

$\text{Kom}^b(\mathcal{C}) = \text{Kom}^+(\mathcal{C}) \cap \text{Kom}^-(\mathcal{C})$

**Remark 3.**  $T$  is well defined in each of these.

**Definition 4** (cone).  $f : X \rightarrow Y$ ,  $\text{Cone}(f) = C(f) \in \text{Kom}(\mathcal{C})$  is the cone of  $f$ .

$C(f)_i = X[1]_i \oplus Y_i$ ,  $d_{C(f)} = (-d_X \pi_1, f[1] \pi_1 + d_Y \pi_2)$ .

**Definition 5** (cylinder).  $f : X \rightarrow Y$ ,  $\text{Cyl}(f) \in \text{Kom}(\mathcal{C})$  is the cylinder of  $f$ .

$\text{Cyl}(f)_i = X_i \oplus X[1]_i \oplus Y_i$ ,  $d_{\text{Cyl}(f)} = (d_X \pi_1 - \pi_2, -d_X \pi_2, f[1] \pi_2 + d_Y \pi_3)$ .

**Remark 6.** Once in life it is worth to check that  $d^2 = 0$  for the cone and the cylinder.

**Fact 7.** For any  $f : X \rightarrow Y$  the following diagram has exact rows and is functorial in  $f$ :

$$\begin{array}{ccccccc}
 0 & \longrightarrow & Y & \xrightarrow{\pi} & C(f) & \longrightarrow & X[1] \longrightarrow 0 \\
 & & \downarrow \alpha & & \downarrow = & & \\
 0 & \longrightarrow & X & \xrightarrow{\bar{f}} & \text{Cyl}(f) & \longrightarrow & C(f) \longrightarrow 0 \\
 & & \downarrow = & & \downarrow \beta & & \\
 & & X & \xrightarrow{f} & Y & & 
 \end{array}$$

where  $\beta = f \pi_1 + \pi_3$  and other maps are obvious.

Also,  $\alpha, \beta$  are quasi-isomorphism, with  $\beta \alpha = \text{id}_Y$  (therefore  $Y \sim \text{Cyl}(f)$  in  $\text{D}(\mathcal{C})$ ).

**Definition 8** (triangle). In the category  $\text{Kom}(\mathcal{C})$  a triangle is any sequence of the form  $X \rightarrow Y \rightarrow Z \rightarrow X[1]$ .

A map of triangles is given by a commutative diagram

$$\begin{array}{ccccccc}
 X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\
 \downarrow f & & \downarrow g & & \downarrow h & & \downarrow f[1] \\
 X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1]
 \end{array}$$

A triangle  $X \rightarrow Y \rightarrow Z \rightarrow X[1]$  is distinguished if it is isomorphic to a triangle  $X' \rightarrow \text{Cyl}(f) \rightarrow C(f) \rightarrow X'[1]$  for some  $f : X' \rightarrow Y'$ .

**Fact 9.** Every exact sequence in  $\text{Kom}(\mathcal{C})$  is quasi-isomorphic to a sequence

$0 \rightarrow X \rightarrow \text{Cyl}(f) \rightarrow C(f) \rightarrow 0$ .

**Fact 10.** *If  $X \rightarrow Y \rightarrow Z \rightarrow X[1]$  is distinguished, then it induces a long exact sequence of cohomology groups:  $\dots \rightarrow H^i(X) \rightarrow H^i(Y) \rightarrow H^i(Z) \rightarrow H^{i+1}(X) \rightarrow \dots$*

**Definition 11** (homotopy category).  $K(\mathcal{C})$  is the homotopy category of  $\text{Kom}(\mathcal{C})$ , defined via  $\text{Ob}(K(\mathcal{C})) = \text{Ob}(\text{Kom}(\mathcal{C}))$  and  $\text{Mor}_{K(\mathcal{C})}(X, Y) = \text{Mor}_{\text{Kom}(\mathcal{C})}(X, Y) / \sim$ , where  $\sim$  is a chain homotopy relation.

**Theorem 12.** *Let  $S$  be a class of quasi-isomorphisms in  $K(\mathcal{C})$ . Then  $K(\mathcal{C})[S^{-1}]$  is isomorphic to  $D(\mathcal{C})$  in a canonical way.*

*This applies to any of  $\text{Kom}^*(\mathcal{C})$ .*

**Lemma 13.** *Assume  $f, g : X \rightarrow Y$  are chain homotopic in  $\text{Kom}(\mathcal{C})$ . Then  $\mathcal{Q}(f) = \mathcal{Q}(g)$ .*

**Theorem 14.** *In any of  $K^*(\mathcal{C})$  the class of quasi-isomorphisms is localising.*

**Theorem 15.**  $D(\mathcal{C})$  is an additive category.

**Definition 1.**  $X$  is an  $H^0$ -complex if  $H^i(X) \neq 0 \implies i = 0$ .

**Theorem 2.** The precomposition of the localization functor  $\mathcal{Q} : \text{Kom}(\mathcal{C}) \rightarrow \text{D}(\mathcal{C})$  with embedding  $i_0 : \mathcal{C} \rightarrow \text{Kom}(\mathcal{C})$  defines an equivalence between  $\mathcal{C}$  and the full subcategory of  $\text{D}(\mathcal{C})$  consisting of  $H^0$ -complexes.

**Definition 3.**  $X[i] = T^i([X])$  for  $X \in \mathcal{C}$ .

**Definition 4.**  $\mathcal{C}$  – abelian, then  $\text{Ext}_{\mathcal{C}}^i(X, Y) = \text{Hom}_{\text{D}(\mathcal{C})}(X[0], Y[i])$ .

**Remark 5.** One does not need projectives or injectives in this definition.

**Remark 6.**  $\text{Ext}_{\mathcal{C}}^i(X, Y) = \text{Hom}_{\text{D}(\mathcal{C})}(X[k], Y[k+i])$  for any  $k \in \mathbb{Z}$ .

**Definition 7** (multiplication). There is a multiplication

$$\text{Ext}_{\mathcal{C}}^i(X, Y) \times \text{Ext}_{\mathcal{C}}^j(Y, Z) \rightarrow \text{Ext}_{\mathcal{C}}^{i+j}(X, Z)$$

via composition  $\text{Hom}_{\text{D}(\mathcal{C})}(X[0], Y[i]) \times \text{Hom}_{\text{D}(\mathcal{C})}(Y[i], Z[i+j]) \rightarrow \text{Hom}_{\text{D}(\mathcal{C})}(X[0], Z[i+j])$ .

**Fact 8.** For an exact sequence  $0 \rightarrow Y' \rightarrow Y \rightarrow Y'' \rightarrow 0$  there is an exact sequence

$$\dots \rightarrow \text{Ext}^i(X, Y') \rightarrow \text{Ext}^i(X, Y) \rightarrow \text{Ext}^i(X, Y'') \rightarrow \text{Ext}^{i+1}(X, Y') \rightarrow \dots$$

**Exercise 9.** Show that if  $X \rightarrow Y \rightarrow Z \rightarrow X[1]$  is distinguished in  $\text{D}(\mathcal{C})$ , then we have an exact sequence of abelian groups

$$\dots \rightarrow \text{Hom}_{\text{D}(\mathcal{C})}(U, X[i]) \rightarrow \text{Hom}_{\text{D}(\mathcal{C})}(U, Y[i]) \rightarrow \text{Hom}_{\text{D}(\mathcal{C})}(U, Z[i]) \rightarrow \text{Hom}_{\text{D}(\mathcal{C})}(U, X[i+1]) \rightarrow \dots$$

**Theorem 10.**  $\text{Ext}_{\mathcal{C}}^0(X, Y) = \text{Hom}_{\mathcal{C}}(X, Y)$

**Theorem 11.**  $\text{Ext}_{\mathcal{C}}^i(X, Y) = 0$  for  $i < 0$ .

**Theorem 12.** Every element in  $\text{Ext}_{\mathcal{C}}^i(X, Y)$  has a presentation  $X[0] \xleftarrow{s} K \xrightarrow{f} Y[i]$ , where  $K_j = 0$  for  $j < -i$  and for  $j > 0$ ,  $K_{-i} = Y$ ,  $f_i = \text{id}$ , and  $s$  is a quasi-isomorphism.

In other words, every such element comes from an exact sequence

$$0 \rightarrow Y = K^{-i} \rightarrow K^{-i+1} \rightarrow K^{-i+2} \rightarrow \dots \rightarrow K^1 \rightarrow K^0 \rightarrow X \rightarrow 0.$$

**Definition 1** (“exactness”). A functor  $D^+(\mathcal{C}) \rightarrow D^+(\mathcal{C})$  is “exact” if it maps distinguished triangles to distinguished triangles.

**Proposition 2.** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be exact. Then it takes quasi-isomorphisms to quasi-isomorphisms.

Therefore it defines a functor  $D^+F : D^+(\mathcal{C}) \rightarrow D^+(\mathcal{D})$  and the functor is “exact”.

**Definition 3** (adapted class). We say that a class  $\mathcal{R}$  of objects in  $\mathcal{C}$  is adapted to a functor  $F$  if it satisfies:

- $\mathcal{R}$  is closed under finite coproducts,
- $F$  takes acyclic complexes from  $K^+(\mathcal{R})$  to acyclic ones,
- every object of  $\mathcal{C}$  embeds into an object of  $\mathcal{R}$ .

**Proposition 4.** Let  $\mathcal{R}$  be an adapted class of objects of  $\mathcal{C}$  for a left exact additive functor  $F : \mathcal{C} \rightarrow \mathcal{D}$ . Let  $S_{\mathcal{R}}$  be the class of quasi-isomorphisms in  $K^+(\mathcal{R})$ . Then  $S_{\mathcal{R}}$  is localizing and the canonical functor  $K^+(\mathcal{R})[S_{\mathcal{R}}^{-1}] \rightarrow D^+(\mathcal{C})$  is an equivalence of categories.

**Definition 5.** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be an additive, left exact functor between abelian categories. Its derived functor  $D^+F$  consists of a functor

$$RF : D^+(\mathcal{C}) \rightarrow D^+(\mathcal{D})$$

which is exact and a morphism of functors

$$\varepsilon_F : \mathcal{Q}_{\mathcal{D}} \circ K^+(F) \rightarrow D^+(F) \circ \mathcal{Q}_{\mathcal{C}}$$

such that for any exact  $G : D^+(\mathcal{C}) \rightarrow D^+(\mathcal{D})$  and any natural  $\varepsilon : \mathcal{Q}_{\mathcal{D}} \circ K^+(F) \rightarrow G \circ \mathcal{Q}_{\mathcal{C}}$  there exists a unique morphism  $\eta : D^+F \rightarrow G$  such that the diagram is commutative:

$$\begin{array}{ccc} \mathcal{Q}_{\mathcal{D}} \circ K^+(F) & \xrightarrow{\varepsilon} & G \circ \mathcal{Q}_{\mathcal{C}} \\ & \searrow \varepsilon_F & \uparrow \eta \circ \text{id}_{\mathcal{Q}_{\mathcal{C}}} \\ & D^+F \circ \mathcal{Q}_{\mathcal{C}} & \end{array}$$

**Theorem 6.** If for  $F$  there exists an adapted class  $\mathcal{R}$  of objects in  $\mathcal{C}$ , then  $D^+F$  exists and is unique up to an isomorphism.

**Theorem 7.**  $RF$  may be defined via the composition  $D^+(\mathcal{C}) \rightarrow K^+(\mathcal{R})[S_{\mathcal{R}}^{-1}] \xrightarrow{F} D^+(\mathcal{D})$ .

**Theorem 8.** If  $\mathcal{C}$  contains enough injective objects, then the class  $\mathcal{I}$  of them is adapted to any left exact functor  $F : \mathcal{C} \rightarrow \mathcal{D}$ .

**Proposition 9.** When  $\mathcal{R} = \mathcal{I}$ , then  $\varepsilon_F$  may be defined in the following way.

$T : K^+(\mathcal{I}) \rightarrow D^+(\mathcal{C})$  is an equivalence of categories,  $U : D^+(\mathcal{C}) \rightarrow K^+(\mathcal{I})$  inverse equivalence, we know that for  $Y$  injective  $\text{Hom}_{D^+(\mathcal{C})}(X, Y) \simeq \text{Hom}_{K^+(\mathcal{C})}(X, Y)$ , so take  $f : X \rightarrow I$  a quasi-isomorphism in  $\text{Hom}_{D^+(\mathcal{C})}(\mathcal{Q}X, TU\mathcal{Q}X)$  and denote by  $f_X$  its image under the bijection in  $\text{Hom}_{K^+(\mathcal{C})}(X, U\mathcal{Q}X)$ , and define  $(\varepsilon_F)_X = \mathcal{Q}F(f_X)$ .

**Remark 1.**  $\dots \rightarrow \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \rightarrow \dots$  is acyclic, so a 0 map is a quasi-isomorphism, but not a homotopy equivalence.

So one has 2 resolutions of  $\dots \rightarrow 0 \rightarrow \dots$ , which are not homotopy equivalent.

**Definition 2** (K-injectivity, K-projectivity). *A complex  $A$  is K-injective (K-projective) if for any acyclic  $X$ ,  $\text{Hom}(X, A)$  ( $\text{Hom}(A, X)$ ) is acyclic.*

**Theorem 3** (Spatenstein). *In the category of chain complexes of  $R$ -modules every complex has a K-injective (K-projective) resolution.*

**Definition 4** ( $F$ -acyclic). *Assume that  $F : \mathcal{A} \rightarrow \mathcal{B}$  is left-exact, additive and  $RF (= D^+ F)$  exists; then we can say that  $A \in \mathcal{A}$  is  $F$ -acyclic if  $RF(A)$  has only 0 cohomology group (i.e.  $R^i F(A) = 0$  for  $i > 0$ ).*

**Theorem 5.** *Let  $\mathcal{Z}$  be a class of  $F$ -acyclic objects.*

- *If  $\mathcal{Z}$  is sufficiently large, then there exists a class of objects adapted to  $F$ .*
- *If  $\mathcal{Z}$  is sufficiently large, then any class of objects adapted to  $F$  is contained in  $\mathcal{Z}$ .*
- *If  $\mathcal{Z}$  is sufficiently large, then it contains all injective objects of  $\mathcal{A}$ .*

$F : \mathcal{A} \rightarrow \mathcal{B}$ ,  $G : \mathcal{B} \rightarrow \mathcal{C}$  additive left exact functors of abelian categories. Assume that there exists classes  $\mathcal{R}_{\mathcal{A}}$  of objects adapted to  $F$ ,  $\mathcal{R}_{\mathcal{B}}$  adapted to  $G$ , and  $F(\mathcal{R}_{\mathcal{A}}) \subset \mathcal{R}_{\mathcal{B}}$ . These assumptions imply that  $RF, RG, R(G \circ F)$  exist.

**Theorem 6.** *The functors  $RG \circ RF$  and  $R(G \circ F)$  are isomorphic as functors  $\mathcal{D}(\mathcal{A}) \rightarrow \mathcal{D}(\mathcal{C})$ .*

**Remark 7.** Assume  $X$  is of the type that  $R^i F(X) = 0$  for  $i \neq k$  for  $k$ -a fixed integer. Then  $RG(RF(X)) = RG(R^k F(X)[-k])$ ,  $R^n(G \circ F)(X) = R^{n-k}G(R^k F(X))$ .

### Triangulated categories

Assume that  $\mathcal{C}$  is an additive category with an automorphism  $T : \mathcal{C} \rightarrow \mathcal{C}$  (called the translation functor).

**Definition 8.**  $X[1] = T(X)$ ,  $X[n] = T(X[n-1])$

**Definition 9** (triangle). *A triangle in  $\mathcal{C}$  is a sequence of maps  $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ .*

*A map of triangles is a commutative diagram*

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & T(X) \\ \downarrow f & & \downarrow & & \downarrow & & \downarrow T(f) \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & T(X') \end{array}$$

**Definition 10** (triangulated category). *An additive category  $\mathcal{C}$  with  $T$  on it is called a triangulated category if it is equipped with a class of distinguished triangles  $(u, v, w)$ , which satisfy the following conditions:*

- *TR1. Every morphism  $v$  can be embedded into distinguished triangle  $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ .*

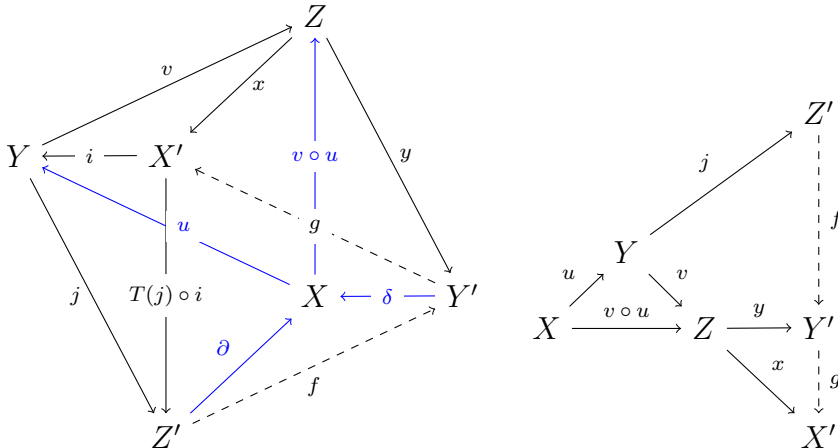
*Moreover, if  $X = Y$  and  $Z = 0$  and  $u = \text{id}$ , then  $X \xrightarrow{\text{id}} X \rightarrow 0 \rightarrow T(X)$  is distinguished.*

- *TR2.  $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$  is distinguished iff  $Y \xrightarrow{v} Z \xrightarrow{w} T(X) \xrightarrow{-T(u)} T(Y)$  is distinguished.*
- *TR3. Assume that in the diagram*

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & T(X) \\ f \downarrow & * & \downarrow & & \downarrow h & & \downarrow T(f) \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & T(X') \end{array}$$

*rows are distinguished and  $*$  commutes. Then there exists  $h : Z \rightarrow Z'$  such that  $(f, g, h)$  is a morphism of triangles.*

- *TR4. [Octahedron axiom] Assume that we have  $X, Y, Z, X', Y', Z'$  in  $\mathcal{C}$ . Assume that  $X \xrightarrow{u} Y \xrightarrow{j} Z' \xrightarrow{\partial} T(X)$ ,  $Y \xrightarrow{v} Z \xrightarrow{x} X' \xrightarrow{i} T(Y)$ ,  $X \xrightarrow{v \circ u} Z \xrightarrow{y} Y' \xrightarrow{\delta} T(X)$  are distinguished. Then there exists distinguished  $Z' \xrightarrow{f} Y' \xrightarrow{g} X' \xrightarrow{T(j) \circ i} T(Z')$  such that*
  - 1. the four distinguished triangles form faces of octahedron,*
  - 2. the remaining faces commute,*
  - 3.  $yv = fj : Y \rightarrow Y'$ ,*
  - 4.  $u\delta = ig : Y' \rightarrow Y$ .*





**Theorem 1.** *Let  $\mathcal{C}$  be an abelian category. Then  $K(\mathcal{C})$  (also  $K^+, K^-, K^b$ ) with standard translation functor and distinguished triangles is triangulated.*

**Remark 2.**  $C(U)$  fits as  $Z$  in TR1.

**Definition 3** (cohomological functor). *Assume  $\mathcal{C}$  is triangulated,  $\mathcal{A}$  is abelian. Let  $F : \mathcal{C} \rightarrow \mathcal{A}$  be an additive functor. We call it cohomological if for any distinguished triangle  $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$  we have an exact sequence*

$$\dots \rightarrow F(T^i(X)) \rightarrow F(T^i(Y)) \rightarrow F(T^i(Z)) \rightarrow F(T^{i+1}(X)) \rightarrow \dots$$

**Definition 4.** *Let  $\mathcal{C}$  be a triangulated category,  $S$  a localizing class of morphisms in  $\mathcal{C}$ . We say that  $S$  is compatible with triangulation if*

- $s \in S \iff T(s) \in S$ ,
- in TR3,  $f, g \in S \implies h \in S$  for any  $h$ .

**Theorem 5.** *Let  $\mathcal{C}$  and  $S$  be as above. On  $\mathcal{C}[S^{-1}]$  we can define*

- $T_S : \mathcal{C}[S^{-1}] \rightarrow \mathcal{C}[S^{-1}]$ ,  $T_S = T$  on objects and morphisms, i.e.  $T(X \xleftarrow{s} Z \xrightarrow{f} Y) = T(X) \xleftarrow{T(s)} T(Z) \xrightarrow{T(f)} T(Y)$ .
- $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$  is distinguished in  $\mathcal{C}[S^{-1}]$  if it is isomorphic to a distinguished triangle coming from  $\mathcal{C}$ .

*Then  $\mathcal{C}[S^{-1}]$  with the structure defined above is triangulated.*

**Corollary 6.** *Derived category of an abelian category inherits the triangulated structure from the homotopy category of complexes.*

**Definition 1.**  $K^{\leq n}(\mathcal{A}), D^{\leq n}(\mathcal{A})$  – full subcategories of objects  $X$  for which  $H^i(X) = 0$  for  $i > n$ .

Analogously one defines  $K^{\geq n}(\mathcal{A}), D^{\geq n}(\mathcal{A})$ .

We know  $\mathcal{A} = D^{\leq 0}(\mathcal{A}) \cap D^{\geq 0}(\mathcal{A})$ .

**Definition 2.** Let  $\mathcal{C}$  be a triangulated category. Assume that  $D$  is a subcategory of  $\mathcal{C}$ . Then we write  $D^{\leq n} = D^{\leq 0}[-n]$ ,  $D^{\geq n} = D^{\geq 0}[-n]$ .

**Definition 3** (t-structure). A t-structure on  $\mathcal{C}$  consists of a pair of full subcategories  $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$  in  $\mathcal{C}$  which satisfy the following properties:

- $\mathcal{C}^{\leq 0} \subset \mathcal{C}^{\leq 1}, \mathcal{C}^{\geq 0} \supset \mathcal{C}^{\geq 1}$ ,
- $X \in \mathcal{C}^{\leq 0} \wedge Y \in \mathcal{C}^{\geq 1} \implies \text{Hom}(X, Y) = 0$ ,
- for all  $X \in \mathcal{C}$  there is a distinguished triangle  $A \rightarrow X \rightarrow B \rightarrow A[1]$  such that  $A \in \mathcal{C}^{\leq 0}, B \in \mathcal{C}^{\geq 1}$ .

**Definition 4** (core).  $\mathcal{A} = \mathcal{C}^{\leq 0} \cap \mathcal{C}^{\geq 0}$  is called the core of the t-structure on  $\mathcal{C}$ .

**Proposition 5.** In  $D(\mathcal{A})$ , the subcategories  $D^{\leq 0}(\mathcal{A})$  and  $D^{\geq 0}(\mathcal{A})$  define a t-structure on  $D(\mathcal{A})$ .

**Theorem 6.** The core of any t-structure on triangulated  $\mathcal{C}$  is an additive category.

**Lemma 7.** There exist functors  $\tau_{\leq n} : \mathcal{C} \rightarrow \mathcal{C}^{\leq n}$  ( $\tau_{\geq n} : \mathcal{C} \rightarrow \mathcal{C}^{\geq n}$ ) which are right (left) adjoint to the inclusion functors.

Moreover, for any  $X \in \text{Ob}(\mathcal{C})$  there exists a distinguished triangle

$$\tau_{\leq 0}X \rightarrow X \rightarrow \tau_{\geq 1}X \rightarrow (\tau_{\leq 0}X)[1].$$

Moreover, any distinguished triangle  $A \rightarrow X \rightarrow B \rightarrow A[1]$  such that  $A \in \mathcal{C}^{\leq 0}, B \in \mathcal{C}^{\geq 1}$  is isomorphic to the one above.

**Lemma 8.** •  $\tau_{\leq n}X = 0 \iff X \rightarrow \tau_{\geq n+1}X$  is an isomorphism,

- if  $m \leq n$ , then there are maps

$$\tau_{\leq m}X \rightarrow \tau_{\leq m}\tau_{\leq n}X,$$

$$\tau_{\geq n}X \rightarrow \tau_{\geq n}\tau_{\geq m}X,$$

which are isomorphisms, if  $m \leq n$ , then there is a unique isomorphism

$$\tau_{\geq m}\tau_{\leq n}X \rightarrow \tau_{\leq n}\tau_{\geq m}X (= \tau_{[m,n]}X).$$

**Theorem 1.** *The core  $\mathcal{A} = \mathcal{C}^{\geq 0} \cap \mathcal{C}^{\leq 0} \subset \mathcal{C}$  is an abelian category.*

**Definition 2** (cohomology object). *The  $i$ -th cohomology object of  $X \in \mathcal{C}$  is defined as*

$$H^0(X) = \tau_{[0,0]}(X) \in \mathcal{A},$$

$$H^i(X) = H^0(X[i]) \in \mathcal{A}.$$

**Definition 3** (nondegenerate t-structure). *A t-structure on  $\mathcal{C}$  is nondegenerate if  $\bigcap_n \text{Ob } \mathcal{C}^{\geq n} = \bigcap_n \text{Ob } \mathcal{C}^{\leq n} = \{0\}$ .*

**Theorem 4.**  *$H^0$  is a cohomological functor.*

*If additionally the t-structure is nondegenerate, then*

- $f : X \rightarrow Y$  in  $\mathcal{C}$  is an isomorphism iff  $\forall_i H^i(f)$  is an isomorphism,
- $\text{Ob}(\mathcal{C}^{\leq n}) = \{X \in \text{Ob } \mathcal{C} : \forall_{i > n} H^i(X) = 0\}$ ,
- $\text{Ob}(\mathcal{C}^{\geq n}) = \{X \in \text{Ob } \mathcal{C} : \forall_{i < n} H^i(X) = 0\}$ .

**Definition 5** (bounded t-structure). *A t-structure is bounded if it is nondegenerate and for any  $X \in \mathcal{C}$ ,  $H^i(X) \neq 0$  only for a finite number of  $i$ .*

**Definition 6** (Ext).  $\text{Ext}_{\mathcal{C}}^i(X, Y) = \text{Hom}_{\mathcal{C}}(X, Y[i])$

**Definition 7** (multiplication on Ext). *Notice that  $\text{Hom}_{\mathcal{C}}(X, Y[i]) = \text{Hom}_{\mathcal{C}}(X[k], Y[i+k])$  and define multiplication  $\text{Ext}_{\mathcal{C}}^i(X, Y) \times \text{Ext}_{\mathcal{C}}^j(Y, Z) \rightarrow \text{Ext}_{\mathcal{C}}^{i+j}(X, Z)$  in the most obvious way.*

**Theorem 8.** *Let  $\mathcal{A}$  be a core of a bounded t-structure on  $\mathcal{C}$ . Assume  $F : \text{D}^b(\mathcal{A}) \rightarrow \mathcal{C}$  satisfies*

$$F(\text{D}^b(\mathcal{A})^{\geq 0}) \subset \mathcal{C}^{\geq 0},$$

$$F(\text{D}^b(\mathcal{A})^{\leq 0}) \subset \mathcal{C}^{\leq 0},$$

*then  $F$  is an equivalence of categories iff  $\text{Ext}_{\mathcal{C}}$  is generated by  $\text{Ext}_{\mathcal{C}}^1$  under Yoneda multiplication.*

**Remark 9.** In chain complexes  $\text{Ext}_{\mathcal{C}}^i(X, Y) : 0 \rightarrow Y \rightarrow E_i \xrightarrow{d^i} \dots \xrightarrow{d^2} E_1 \xrightarrow{d^1} X \rightarrow 0$ .

**Theorem 10.** *Assume that  $\mathcal{C}$  satisfies additionally:*

- TR5. Arbitrary coproducts and products exist in  $\mathcal{C}$ .
- There is a generating set  $\Lambda$  of objects in  $\mathcal{C}$ , i.e. set  $\Lambda$  such that
  - $T(\Lambda) \subset \Lambda$ ,  $T$  – translation functor,
  - if  $X \in \mathcal{C}$  and  $\forall_{\lambda \in \Lambda} \text{Hom}(\lambda, X) = 0$ , then  $X \simeq 0$ .

*Then any homological functor  $H : \mathcal{C} \rightarrow \mathcal{A}$  (where  $\mathcal{A}$  is abelian) which sends coproducts to products is representable, i.e.  $H = \mathcal{C}(\cdot, h)$ .*

### Simplicial objects in categories

**Definition 1** (simplicial object). A simplicial object  $X$  in  $\mathcal{C}$  consists of:

- $\forall_{n \geq 0} X_n \in \text{Ob } \mathcal{C}$  –  $n$ -simplices of  $X$ ,
- $\forall_{n \geq 0} \forall_{0 \leq i \leq n} d_i : X_n \rightarrow X_{n-1}$  – boundaries (faces),
- $\forall_{n \geq 0} \forall_{0 \leq i \leq n} s_i : X_n \rightarrow X_{n+1}$  – degeneracies,

such that

- $\forall_{i < j} d_i d_j = d_{j-1} d_i$ ,
- $\forall_{i > j} s_i s_j = s_j s_{i-1}$ ,
- $d_i s_j = \begin{cases} s_{j-1} d_i & \forall_{i < j} \\ \text{id} & \forall_{i=j} \vee i=j+1 \\ s_i d_{i-1} & \forall_{i > j+1} \end{cases}$ .

**Definition 2** (simplicial map). A simplicial map between simplicial objects  $X \rightarrow Y$  consists of the sequence of  $f_n : X_n \rightarrow Y_n$  which commute with boundaries and degeneracies.

**Definition 3** (simplicial category). Denote by  $s\mathcal{C}$  the category of simplicial objects in  $\mathcal{C}$ .

**Remark 4.** Any functor  $F : \mathcal{C} \rightarrow \mathcal{C}'$  extends to  $F : s\mathcal{C} \rightarrow s\mathcal{C}'$

**Definition 5.** Let  $\Delta$  denote the subcategory of sets  $\text{Ob}(\Delta) = \{[n]\} = \{\{0, 1, \dots, n\} : n \geq 0\}$ ,  $\text{Mor}_\Delta([n], [m]) = \text{nondecreasing maps } [n] \rightarrow [m]$ .

**Definition 6** (simplicial object again). Any functor  $X : \Delta^{op} \rightarrow \mathcal{C}$  is called a simplicial object in  $\mathcal{C}$ .

**Definition 7** (simplicial maps). For  $X, Y$  simplicial objects,  $\text{Mor}_{s\mathcal{C}}(X, Y) = \text{Mor}_{F(\Delta^{op}, \mathcal{C})}(X, Y)$ .

**Definition 8.** Let  $\varepsilon^i : [n-1] \rightarrow [n]$  be defined as  $\varepsilon^i(j) = \begin{cases} j & \forall_{j < i} \\ j+1 & \forall_{j \geq i} \end{cases}$  and  $\eta^i : [n+1] \rightarrow [n]$

be defined as  $\eta^i(j) = \begin{cases} j & \forall_{j \leq i} \\ j-1 & \forall_{j > i} \end{cases}$ .

**Remark 9.** These correspond to  $d_i, s_i$  respectively.

**Proposition 10.** Any morphism  $\alpha \in \Delta$  can be uniquely expressed as  $\varepsilon \circ \eta$ , where  $\varepsilon$  is a composition of  $\varepsilon^i$ 's, and  $\eta$  is a composition of  $\eta^i$ 's.

**Remark 11.** A bunch of examples appear:

- $\tilde{K}$  – simplicial set of a geometric simplicial complex  $K$ ,
- $\Delta_n$  – topological simplices, and  $S : \text{Top} \rightarrow s\text{Set}$  singular simplicial set functor,
- $\Delta[n] = \text{Hom}_\Delta(\cdot, [n])$ ,
- nerve of a small category  $N(\mathcal{C})$ ,
- functor  $s\text{Set} \rightarrow s\mathbf{R}\text{-mod}$  induced by a functor  $\text{Set} \rightarrow \mathbf{R}\text{-mod}$  mapping  $X \mapsto R[X]$ .

**Remark 12.** If  $X, Y \in s\text{Set}$ , then there is a simplicial product  $(X \times Y)_n = X_n \times Y_n$ ,  $d_i = d_i^X \times d_i^Y$  and  $s_i = s_i^X \times s_i^Y$ .

**Definition 13** (geometric realization). *Define*

$\sigma_i : \Delta_n \rightarrow \Delta_{n-1}$ ,  $\sigma_i(t_0, \dots, t_n) = (t_0, \dots, t_i + t_{i+1}, \dots, t_n)$  and

$\delta_i : \Delta_n \rightarrow \Delta_{n+1}$ ,  $\delta_i(t_0, \dots, t_n) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_n)$ .

Assume  $X \in s\text{Set}$ . We can define a geometric realization of  $X$

$$|X_\bullet| = \sqcup X_n \times \Delta_n / \sim,$$

where  $(d_i(x), s) \sim (x, \delta_i(s))$  for  $(x, s) \in X_n \times \Delta_{n-1}$

and  $(s_i(x), s) \sim (x, \sigma_i(s))$  for  $(x, s) \in X_n \times \Delta_{n+1}$ .

**Remark 14.** If a category  $\mathcal{C}$  has a faithful functor to  $\text{Set}$ , then for  $X_\bullet \in s\mathcal{C}$  we define its  $|X_\bullet|$ .

**Theorem 15** (properties of  $|\bullet| : s\text{Set} \rightarrow \text{Top}$ ). 1.  $|X \times Y| \simeq |X| \times |Y|$  homeomorphism (in CW topology),

2.  $K$  – geometric simplicial complex, then  $|\tilde{K}_\bullet| \simeq K$  homeomorphic,

3.  $\mathcal{C}$  is group  $G$ , i.e.  $\text{Ob}(\mathcal{C}) = *$ ,  $\text{Mor}_{\mathcal{C}}(*, *) = G$ , then  $|N(\mathcal{C})| = K(G, 1)$ ,

4. Functors  $S : \text{Top} \rightarrow s\text{Set}$  and  $|\bullet| : s\text{Set} \rightarrow \text{Top}$  are adjoint.

Let  $\mathcal{C}$  be abelian, remind  $s\mathcal{C}$  – simplicial objects in  $\mathcal{C}$ ,  $C_*(\mathcal{C})$  – chain complexes over  $\mathcal{C}$ .

**Definition 1** (*s-morphism*).  $X_\bullet, Y_\bullet \in s\mathcal{C}$ . For a simplicial set  $K \in s\text{Set}$  a map which associates  $F(\sigma) : X_n \rightarrow Y_n$  to any  $\sigma \in K_n$  is called *s-morphism* (denote  $F : K \times X_\bullet \rightarrow Y_\bullet$ ) if for any  $\alpha : [m] \rightarrow [n]$  in  $\Delta$  we have  $F(K(\alpha)(\sigma)) \circ X(\alpha) = Y(\alpha)F(\sigma)$ .

Observe that when  $X_\bullet, Y_\bullet$  are in  $s\text{Set}$  then *s-morphisms* are simplicial maps  $K_\bullet \times X_\bullet \rightarrow Y_\bullet$ .

**Example 2.** If  $K = \Delta[0]$ , then *s-morphism* is just a simplicial morphism  $X_\bullet \rightarrow Y_\bullet$ .

**Example 3.** If  $K = \Delta[1]$  then *s-morphism* is called a *homotopy* between  $F(0)$  and  $F(1)$ .

**Remark 4.**  $T : \mathcal{C} \rightarrow \mathcal{C}'$  functor induces  $T : s\mathcal{C} \rightarrow s\mathcal{C}'$ , if  $X_\bullet \in s\mathcal{C}$ , then  $T(X)_n = T(X_n), T(X)(\alpha) = T(X(\alpha))$ .

**Definition 5.** If  $F$  is an *s-morphism*  $F : K_\bullet \times X_\bullet \rightarrow Y_\bullet$ , then it defines  $TF : K_\bullet \times T(X_\bullet) \rightarrow T(Y_\bullet)$  defined by  $TF(\sigma) = T(F(\sigma))$ .

**Remark 6.** Any functor  $T : \mathcal{C} \rightarrow \mathcal{C}'$  sends homotopic maps to homotopic ones.

**Definition 7.** Let  $\mathcal{C}$  be abelian, then there are functors

$$s\mathcal{C} \xrightleftharpoons[K]{N} C_*(\mathcal{C})$$

defined as follows.

Normalization  $N$  is defined, for  $X_\bullet \in s\mathcal{C}$ , as

$$N(X)_n = \bigcap_{i=1}^n \ker(d_i : X_n \rightarrow X_{n-1})$$

(e.g.  $\ker \left( X_n \xrightarrow{\prod d_i} \prod X_{n-1} \right)$ ), with

$$d : N(X)_n \rightarrow N(X)_{n-1}$$

induced by  $d_0$ .

$K$  is defined in such a way. If  $\alpha : [n] \rightarrow [q]$ , then  $d(\alpha) = n$  and  $r(\alpha) = q$ . Notice for any  $\alpha$  there is unique  $\alpha = \varepsilon \circ \eta$ , where  $\varepsilon$  is an injection and  $\eta$  is a surjection. For  $C \in C_*(\mathcal{C})$ , take

$$K(C)_n = \bigoplus_{\eta: d(\eta)=n} C_{r(\eta)},$$

Now for  $\alpha : [m] \rightarrow [n]$  define

$$KC(\alpha) : K(C)_n \rightarrow K(C)_m$$

on every  $C_{r(\eta)}$  in such a way:  $\eta\alpha = \varepsilon'\eta'$ , let  $KC(\alpha)$  map  $C_{r(\eta)}$  into  $C_{r(\eta')}$  via the formula

$$K(\eta, \alpha) = \begin{cases} \text{id}_{C_{r(\eta)}} & \text{for } \varepsilon' = \text{id}_{[r(\eta)]} \\ d : C_{r(\eta)} \rightarrow C_{r(\eta)-1} = C_{r(\eta')} & \text{for } \varepsilon' = \varepsilon^0 \\ 0 & \text{otherwise} \end{cases}.$$

**Remark 8.** Observe that if  $f : C \rightarrow D$  in  $C_*(\mathcal{C})$ , then the induced map  $KC \rightarrow KD$  is simplicial.

**Theorem 9** (Dold-Kan). The functors  $N$  and  $K$  give an equivalence of  $s\mathcal{C}$  and  $C_*(\mathcal{C})$ .

**Remark 10.** It was somehow convenient to define, for  $X \in s\mathcal{C}$ ,  $\bar{X} \in s\mathcal{C}$  via  $\bar{X}_n = \ker(d_{n+1} : X_{n+1} \rightarrow X_n)$ .

**Lemma 11.** Let  $f : X_\bullet \rightarrow Y_\bullet$  be a simplicial morphism which satisfies  $(Nf)_i$  is mono(epi) for  $i \leq n$ . Then  $f_n : X_n \rightarrow Y_n$  is mono(epi) for  $i \leq n$ .

**Definition 1** (cosimplicial object).  $X : \Delta \rightarrow \mathcal{C}$ .

Denote the category of cosimplicial objects in  $\mathcal{C}$  as  $c\mathcal{C}$ .

**Theorem 2** (Dold-Kan again).  $c\mathcal{C} \simeq \text{cochain complexes over } \mathcal{C}$

**Remark 3.**  $c\mathcal{C} = s(\mathcal{C}^{op})$

**Definition 4.** For  $X \in s\mathcal{C}$  define  $kX \in C_*(\mathcal{C})$ ,  $(kX)_n = X_n$ ,  $d = \sum_{i=0}^n (-1)^i d_i$ .

**Theorem 5.** The natural embedding  $NX \hookrightarrow kX$  is a chain homotopy equivalence.

**Remark 6.** •  $NX = kX / DX$ , where  $(DX)_n = \bigcup_{i=0}^{n-1} \text{im}(s_i : X_{n-1} \rightarrow X_n)$   
 $= \text{im} \left( \prod X_{n-1} \xrightarrow{\prod s_i} X_n \right)$ .

- $kX = NX \oplus DX$ .
- $DX$  is contractible.

**Remark 7.** One can get  $NX$  using  $(-1)^n d_n$  instead of  $d_0$ .

**Remark 8.** Observe that if  $\tau_n : [n] \rightarrow [n], i \mapsto n - i$ , and  $\alpha^* = \tau_n \alpha \tau_m$  for  $\alpha : [m] \rightarrow [n]$ , then we get an involution of the category  $\Delta$ ,  $\alpha \rightarrow \alpha^*$ .

Hence we get an involution of  $s\mathcal{C}$ ,  $X \rightarrow X^*$ ,  $(X^*)_n = X_n$ ,  $X^*(\alpha) = X(\alpha^*)$ ,  $d_i$  goes to  $d_{n-i}$ .

We can define  $N^*X = N(X^*)$ ,  $K^*C = (KC)^*$ , getting  $N^*K^* = NK$ ,  $K^*N^* = \text{Id}$ .

**Theorem 9.** 1.  $f_1, f_2 : X \rightarrow Y$  homotopic in  $s\mathcal{C}$  iff  $Nf_1, Nf_2$  are chain homotopic in  $C_*(\mathcal{C})$ ,

2.  $\varphi_1, \varphi_2 : C \rightarrow D$  are chain homotopic in  $C_*(\mathcal{C})$  iff  $K\varphi_1, K\varphi_2$  are homotopic in  $s\mathcal{C}$ .

**Definition 10** (simplicial resolution). Let  $T : \mathcal{C} \rightarrow \mathcal{C}'$  functor between abelian categories,  $\mathcal{C}$  has enough projective objects,  $A \in \text{Ob}(\mathcal{C})$  and  $n \in \mathbb{N}$ .

Then a pair  $(X_\bullet, \xi)$  is called a simplicial resolution of  $A$  of degree  $n$  (simplicial resolution of  $(A, n)$ ) if  $X_\bullet \in s\mathcal{C}$ ,  $X_i = 0$  for  $i < n$ ,  $H_j(X) := H_j(kX) = 0$  for  $j > n$  and  $\xi : H_n(X) \rightarrow A$  is an isomorphism.

If  $\forall_i X_i$  is projective, then  $X$  is a projective resolution of  $(A, n)$ .

Usually we will remove  $\xi$  from notation and say that  $H_n(X) = A$ .

**Remark 11.** 1. If  $X_\bullet$  is a simplicial resolution of  $(A, n)$ , then  $NX$  is a resolution of  $A$  shifted up by  $n$ . If  $X_\bullet$  is projective, then  $NX$  is a projective resolution.

2. If  $P \in C_*(\mathcal{C})$  is a projective resolution of  $A$  shifted by  $n$ , then  $KP$  is a simplicial projective resolution of  $(A, n)$ .

3. If  $\alpha : A \rightarrow B$  in  $\mathcal{C}$  and  $X, Y$  are projective resolutions of  $(A, n)$  and  $(B, n)$ , then there exists a simplicial morphism  $f : X \rightarrow Y$  which induces  $\alpha = H_n(f)$ .

Moreover,  $f$  is unique up to homotopy.



**Definition 12** (derived functor). *Functor  $L_q T(\bullet, n) : \mathcal{C} \rightarrow \mathcal{C}'$  defined below is called  $q$ -th left derived functor of  $T$  of degree  $n$ , where  $L_q T(\bullet, n)(A) = H_q(T(X))$ , where  $X$  is any simplicial resolution of  $A$ .*

**Remark 13.** If  $T$  is additive, then  $k(T(X)) = T(kX)$ , so  $L_q T(A, n) = L_{q-n} T(A)$  ( $L_{q-n}$  from ordinary homotopy category).

**Remark 14.** When  $T$  is not additive, then  $\sum_{i=0}^n (-1)^i T(d_i)$  is usually not equal  $T(\sum (-1)^i d_i)$ , so  $k(TX)$  and  $T(kX)$  may have different homology.

Let  $T : \mathcal{C} \rightarrow \mathcal{C}'$  functor of abelian categories. Assume  $T(0) = 0$  (if  $T(0) = A$ , then take  $T' = \ker(T \rightarrow T(0) = A)$ ).

**Definition 15** (cross effect). *For any  $k \in \mathbb{N}$  we define the  $k$ -th cross-effect of  $T$  as a functor  $T_k : \mathcal{C}^k \rightarrow \mathcal{C}'$  such that we get a functorial decomposition  $T(A_1 \oplus \dots \oplus A_k) = \bigoplus_{i=1}^k T(A_i) \oplus \bigoplus_{i_1 < i_2} T_2(A_{i_1}, A_{i_2}) \oplus \dots \oplus T_k(A_1, \dots, A_k)$ . We can define  $T_k$  inductively,*

- $T_1 = T$ ,
- $T_2(A_1, A_2) = \ker(T(A_1 \oplus A_2) \rightarrow T(A_1) \oplus T(A_2))$ ,
- $\dots$ ,
- $T_k(A_1, \dots, A_k) = \ker(T(A_1 \oplus \dots \oplus A_n) \xrightarrow{\prod_i} \prod_i T(A_1 \oplus \dots \oplus \hat{A}_i \oplus \dots \oplus A_k))$ .

**Definition 16** (functor degree). *We say that  $T$  is of degree  $\leq k$  if  $T_{k+1} = 0$ .*

*We say that  $T$  is of degree  $k$  if  $T$  is of degree  $\leq k$  and  $T_k \neq 0$ .*

**Theorem 17.** *Cross-effects have the following properties:*

- if for some  $i$ ,  $A_i = 0$ , then  $T_k(A_1, \dots, A_n) = 0$ ,
- $T_k$  is symmetric in its variables,
- if we define  $s^{(1)}(A) = T_2(A, A_2)$ ,  $s^{(2)}(A) = T_2(A_1, A)$ , then  $s_2^{(1)}(A_1, A_2; A_3) = s_2^{(2)}(A_1; A_2, A_3) = T_3(A_1, A_2, A_3)$ .

**Example 18.**  $\deg T \leq 1$  iff  $T$  is additive.

**Example 19.**  $T(A) = A^{\otimes 2}$ , then  $T_2(A, B) = (A \otimes B) \oplus (B \otimes A)$  and it is linear in  $A, B$ , so  $T$  is of degree 2.

**Theorem 1.** Assume  $T$  is of degree  $\leq k$ ,  $A \in \text{Ob}(\mathcal{C})$  is of projective dimension  $\leq n$ , then  $L_q T(A, n) = 0$  for  $q > k(r + n)$ .

**Lemma 2.** Let  $T$  be as above and  $X \in s\mathcal{C}$  such that  $(NX)_i = 0$  for  $i > m$ . Then  $N(TX)_i = 0$  for  $i > km$ .

**Definition 3** (suspension).  $SA = \text{coker}(A \rightarrow CA)$ , or  $(SA)_q = A_{q-1}$  and  $d^{SA} = -d^A$ .

**Corollary 4.** We have an exact sequence  $0 \rightarrow A \xrightarrow{\kappa} CA \xrightarrow{\pi} SA \rightarrow 0$ .

**Definition 5.** Let  $X \in s\mathcal{C}$ . Define cone and suspension of  $X$  by the formulas  $CX = KCN X$ ,  $SX = KSN X$ .

**Remark 6.** We have an exact sequence (exact on each level)  $0 \rightarrow X \xrightarrow{\kappa} CX \xrightarrow{\pi} SX \rightarrow 0$ .

Applying  $T$  we get (not necessarily exact)  $0 \rightarrow TX \xrightarrow{T(\kappa)} T(CX) \xrightarrow{T(\pi)} T(SX) \rightarrow 0$ .

**Remark 7.** Let  $A \xrightarrow{f} B \xrightarrow{g} C$  be a sequence in  $C_*(\mathcal{C})$  such that  $g \circ f = 0$  and  $B$  is contractible, i.e. we have  $s_q : B_q \rightarrow B_{q+1}$  such that  $d^B s + s d^B = \text{id}$ . Then  $gsf : A \rightarrow C$  gives a chain map  $SA \rightarrow C$  and hence a map  $H_q(A) \rightarrow H_{q+1}(C)$ .

**Theorem 8.**  $H(gsf)$  does not depend on the choice of  $s$ .

**Definition 9** (suspension homomorphism). The map  $\sigma : H_q(TX) \rightarrow H_{q+1}(TSX)$  induced by  $\kappa$  and  $\pi$  is called a suspension homomorphism.

**Proposition 10.**  $\sigma$  defines a natural transformation of functors.

**Proposition 11.** Assume  $T$  additive, then  $0 \rightarrow T(X) \rightarrow T(CX) \rightarrow T(SX) \rightarrow 0$  exact and we have a long exact sequence of homology groups:  $\dots \rightarrow 0 \rightarrow H_{q+1}(TSX) \rightarrow H_q(TX) \rightarrow 0 \rightarrow \dots$ , and  $\sigma$  is the inverse of the map in the middle.

**Definition 12.** Let  $T_p^d(A) = T_p(A, \dots, A)$  ( $d$  means diagonal).

**Definition 13.** Define  $d_i = \rho \circ T(\alpha'_i) \circ \lambda : T_p^d(A) \rightarrow T_{p-1}^d(A)$ , where  $\lambda$  monomorphism  $T_p^d(A) \rightarrow T(A \oplus \dots \oplus A)$ ,  $\rho$  epimorphism  $T(A \oplus \dots \oplus A) \rightarrow T_{p-1}^d(A)$ , and  $d'_j : \bigoplus_{i=1}^p A \rightarrow \bigoplus_{i=1}^{p-1} A$ , equal to  $(\text{id}, \dots, \text{id}, (\text{id} + \text{id})_j, \text{id}, \dots, \text{id})$ .

**Definition 14.** Let  $X \in s\mathcal{C}$ . Define a sequence of simplicial objects in  $\mathcal{C}'$ :

$$\mathcal{T}X = \left( T_1^d(X) \xleftarrow{\partial'} T_2^d(X) \leftarrow T_3^d(X) \leftarrow \dots \right), \quad \partial' = \sum_{i=1}^{p-1} (-1)^i d_i.$$

**Remark 15.**  $\partial' \circ \partial' = 0$ .

**Corollary 16.** Therefore  $\mathcal{T}X$  gives a bicomplex

$$(\mathcal{T}X)_{p,q} = T_p^d(X_q)$$

with horizontal differentials  $\partial'$  and vertical differentials from  $kX$ .

**Proposition 17.** *We have an embedding  $i : kTX = (TX)_{1,*} \hookrightarrow \text{Tot}(TX)$  and it is a chain map of degree 1.*

**Theorem 18.** *There is a natural isomorphism  $\omega : H\text{tot}(TSX) \simeq H(TSX)$  such that for*

$$\begin{array}{ccc} & H_{q+1}(TSX) & \\ \sigma \nearrow & \uparrow \omega & \\ H_q TX & & \\ \searrow i & \downarrow & \\ & H_q(TX) & \end{array}$$

any  $q$  the diagram commutes:

**Definition 19** (bar construction).  $TX$  is called the bar construction for  $T$ .

**Corollary 20.** *If  $T$  is additive, then  $\sigma$  is an isomorphism.*

**Corollary 21.** *If  $T$  is of degree 2, then there exists a morphism  $\beta$  such that the sequence is exact:  $\dots \rightarrow H_q T_2(X, X) \xrightarrow{\alpha} H_q(TX) \xrightarrow{\sigma} H_{q+1}(TSX) \rightarrow H_{q+1} T_2(X, X) \rightarrow H_{q+1}(TX) \rightarrow \dots$*

**Corollary 22.** *There exists a spectral sequence which converges to  $H_* TSX$  and which satisfies*

- $E'_{pq}$  is equal to the complex  $H_q TX \xleftarrow{H_q(\partial')} H_q T_2(X, X) \xleftarrow{H_q(\partial')} H_q T_3(X, X, X) \leftarrow \dots$ ,
- the homomorphism  $H_q TX = E'_{pq} \rightarrow H_{q+1} TSX$  is the same as  $\sigma$ .

**Definition 23.** We say that  $X \in s\mathcal{C}$  is trivial below  $n$  if there exists  $X' \in s\mathcal{C}$  which is homotopy equivalent to  $X$  and satisfies  $X'_i = 0$  for  $i < n$ .

**Lemma 24.** *If  $X$  is projective and  $H_q(X) = 0$  for  $q < n$ , then  $X$  is trivial below  $n$ .*

**Remark 25** (digression). A bisimplicial object is  $X_{p,q}$  with  $X_{p,q} \rightarrow X_{r,s}$  for any  $\alpha : [r] \rightarrow [p], \beta : [s] \rightarrow [q]$ , which satisfy simplicial identities in both directions.

Every bisimplicial object gives us a bicomplex  $kX$ .

If  $X$  is bisimplicial, then it comes with a diagonal simplicial object  $X_{k,k} \xrightarrow{(\alpha, \alpha)} X_{l,l}$  (where  $\alpha : [l] \rightarrow [k]$ ).

**Theorem 26** (Eilenberg-Zilber(-Cantier)). *There is a chain homotopy equivalence  $k(X_{p,p}) \simeq \text{tot}(kX_{p,q})$ .*

**Remark 27.** Observe that  $X_{p,p}$  is in degree  $p$  to the left and  $p + p$  to the right.

**Proposition 28.** *Let  $T : \mathcal{C}^l \rightarrow \mathcal{C}'$  be such that  $T(\dots, 0_j, \dots) = 0$ . Let, for  $j = 1, \dots, l$ ,  $X^j \in s\mathcal{C}$  be trivial below  $n_j$ . Then  $T(X^1, \dots, X^l)$  is trivial below  $n_1 + \dots + n_l = n$  (therefore  $H_q T(X^1, \dots, X^l) = 0$  for  $q < n$ ).*

**Corollary 29.** *If  $X$  is trivial below  $n$ , then the suspension homomorphism  $\sigma : H_q(TX) \rightarrow H_{q+1}(TSX)$  is an isomorphism for  $q < 2n$  and epimorphism for  $q = 2n$ .*

**Definition 30** (stable derived functors).  $L_{q+n}(T\bullet, n)$  for  $n > q$  is called the  $q$ -th stable derived functor of  $T$ , denoted  $L_q^s T(\bullet)$ .

**Remark 1.**  $C_*(\mathcal{C})$  does not have enough projective objects.

**Theorem 2.** The sequence of functors  $\{H_i\}_{i=0}^\infty$  gives us a universal  $\delta$ -functor (takes short exact sequences to long exact sequences), i.e. if we have another sequence  $T_i$  such that  $T_0 = H_0$ , then  $\forall_i H_i^* = T_i^*$ .

**Lemma 3.** For a given  $C_* \in C_*(\mathcal{C})$  there exists  $P_* \rightarrow C_*$  such that  $H_i(P_*) = 0$  for  $i > 0$ .

**Remark 4.** If  $p + q = n$ , let  $f_{pq} : X_{nn} \rightarrow X_{pq}$  be defined as  $d_{p+1}^h \circ \dots \circ d_n^h \circ d_0^v \circ \dots \circ d_0^v$ , and then the Alexander-Whitney map  $\sum_{p+q=n} f_{pq} : X_{nn} \rightarrow \bigoplus_{p+q=n} X_{pq}$  gives a chain homotopy equivalence of  $k(X_{pp})$  and  $\text{tot}(kX_{pq})$ .

**Remark 5.** We may take a projective simplicial resolution  $P_*$  of  $A$  of degree  $n > i$ , then  $L_i^s T(A) = H_{n+i}(T(P_*))$ .

**Theorem 6.**  $\deg L_i T(\bullet, n) \leq \lfloor \frac{i}{n} \rfloor$ .

**Remark 7.** Or theorem? Or proof? It is written that  $T((A, n) \oplus (B, n)) = T(A, n) \oplus T(B, n) \oplus V$  where  $V$  is trivial below  $2n$ .

**Proposition 8.**  $\forall_i L_i^s T$  is an additive functor.

**Proposition 9.** Let  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  be exact in  $\mathcal{C}$ . Then we have a long exact sequence  $\dots \rightarrow L_{q+1}^s T(C) \rightarrow L_q^s T(A) \rightarrow L_q^s T(B) \rightarrow L_q^s T(C) \rightarrow \dots$

**Proposition 10.** If  $0 \rightarrow T' \rightarrow T \rightarrow T'' \rightarrow 0$  is an exact sequence of functors, then we have a long exact sequence of functors  $\dots \rightarrow L_{i+1}^s T'' \rightarrow L_i^s T' \rightarrow L_i^s T \rightarrow L_i^s T'' \rightarrow \dots$

**Proposition 11.** Let  $U$  be an additive functor, then for any functor  $T$  we have  $\text{Hom}_{sth}(T, U) \simeq \text{Hom}_{sth2}(L_0^s T, U)$ .

### Applications of stable derived functors

**Theorem 1.**  $T : R\text{-mod} \rightarrow R\text{-mod}$ , then

$$L_i^s T(A) = \lim_n \pi_{i+n}(T(\tilde{R}[S^n] \otimes P_*)) \lim_n H_{i+n}(T(\tilde{R}[S^n] \otimes P_*)),$$

where  $S^n$  is any simplicial model of  $n$ -sphere,  $\tilde{R}[\gamma] = R[\gamma]/R[*]$  a simplicial set,  $P_*$  is any projective resolution of  $A$ .

The limit is taken via suspension

$$\pi_{i+n}(T(\tilde{R}[S^n] \otimes P_*)) \rightarrow \pi_{i+n+1}(S^1 \wedge T(\tilde{R}[S^n] \otimes P_*)) \rightarrow \pi_{i+n+1}(T(\tilde{R}[S^{n+1}] \otimes P_*)).$$

In general for  $S^1 \wedge F(X) \rightarrow F(S^1 \wedge X)$  one has to have for any  $z \in S^1$ ,  $F(X) \rightarrow F(S^1 \wedge X)$ ,  $X \rightarrow S^1 \wedge X$ ,  $x \rightarrow z \wedge x$ .

One takes  $R = \mathbb{Z}/p$  or  $R = \mathbb{Z}$ .

$$L_i^s T(\mathbb{Z}/p) = \lim_n \pi_{i+n} T(\mathbb{Z}/p[S^n]), \text{ but } \widetilde{\mathbb{Z}/p}[S^n] = K(\mathbb{Z}/p, n), \tilde{\mathbb{Z}}[S^n] = K(\mathbb{Z}, n).$$

Stalk skewed gra..itions on  $H^*(\bullet, \mathbb{Z}/p)$  is

$$H_*^s(K(\mathbb{Z}/p), \mathbb{Z}/p) = H_*^s(K(\mathbb{Z}/p, n), \mathbb{Z}/p) = L_*^s \mathbb{Z}_p[.](\mathbb{Z}/p). (?)$$

**Theorem 2.** Let  $SP^i$  be the  $i$ -th symmetric power functor, and  $SP_p^i$  the  $p$ -reduced  $i$ -th symmetric power, and  $SP_p^* = \bigoplus SP_p^i / \langle x^p - 1 \rangle$ .

$$\text{Then } L_*^s SP^*(\mathbb{Z}/p) = H_*^s(K(\mathbb{Z}), \mathbb{Z}/p), L_*^s SP_p^*(\mathbb{Z}/p) = H_*^s(K(\mathbb{Z}/p), \mathbb{Z}/p).$$

Calculations: Let  $\Gamma$  be a category of functors  $T : \text{finite pointed sets} \rightarrow \mathbb{Z}/p\text{-vect}$ ,  $T(*) = 0$ .  $L \in \Gamma$  is defined as  $L(X) = \widetilde{\mathbb{Z}/p}[X]$ .

**Lemma 3.** Let  $T : \mathbb{Z}/p\text{-vect} \rightarrow \mathbb{Z}/p\text{-vect}$ . Then  $L_i^s T(\mathbb{Z}/p) = \text{Tor}_i^\Gamma(L^*, T \circ L)$ , where  $L^*(X) = L(X)^*$ , and

I have found these notes useful in understanding derived functors.

Some detailed constructions are here, and some worked out examples are here in section 2.2.