Theorem 1. *Assume T is of degree $\leqslant k$, $A \in \mathrm{Ob}(\mathcal{C})$ is of projective dimension $\leqslant r$, then $L_q T(A, n) = 0$ for $q > k(r+n)$.*

Lemma 2. *Let T be as above and $X \in s\mathcal{C}$ such that $(NX)_i = 0$ for $i > m$. Then $N(TX)_i = 0$ for $i > km$.*

Definition 3 (suspension)**.** *$SA = \mathrm{coker}(A \to CA)$, or $(SA)_q = A_{q-1}$ and $d^{SA} = -d^A$.*

Corollary 4. *We have an exact sequence $0 \to A \xrightarrow{\kappa} CA \xrightarrow{\pi} SA \to 0$.*

Definition 5. *Let $X \in s\mathcal{C}$. Define* cone and suspension *of X by the formulas $CX = KCNX, SX = KSNX$.*

Remark 6. We have an exact sequence (exact on each level) $0 \to X \xrightarrow{\kappa} CX \xrightarrow{\pi} SX \to 0$.

Applying T we get (not necessarily exact) $0 \to TX \xrightarrow{T(\kappa)} T(CX) \xrightarrow{T(\pi)} T(SX) \to 0$.

Remark 7. Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a sequence in $C_*(\mathcal{C})$ such that $g \circ f = 0$ and B is contractible, i.e. we have $s_q : B_q \to B_{q+1}$ such that $d^B s + s d^B = \mathrm{id}$. Then $gsf : A \to C$ gives a chain map $SA \to C$ and hence a map $H_q(A) \to H_{q+1}(C)$.

Theorem 8. *$H(gsf)$ does not depend on the choice of s.*

Definition 9 (suspension homomorphism)**.** *The map $\sigma : H_q(TX) \to H_{q+1}(TSX)$ induced by κ and π is called a* suspension homomorphism*.*

Proposition 10. *σ defines a natural transformation of functors.*

Proposition 11. *Assume T additive, then $0 \to T(X) \to T(CX) \to T(SX) \to 0$ exact and we have a long exact sequence of homology groups: $\ldots \to 0 \to H_{q+1}(TSX) \to H_q(TX) \to 0 \to \ldots$, and σ is the inverse of the map in the middle.*

Definition 12. *Let $T_p^d(A) = T_p(A, \ldots, A)$ (d means diagonal).*

Definition 13. *Define $d_i = \rho \circ T(\alpha_i') \circ \lambda : T_p^d(A) \to T_{p-1}^d(A)$, where λ monomorphism $T_p^d(A) \to T(A \oplus \ldots \oplus A)$, ρ epimorphism $T(A \oplus \ldots \oplus A) \to T_{p-1}^d(A)$, and $d_j' : \bigoplus_{i=1}^p A \to \bigoplus_{i=1}^{p-1} A$, equal to $(\mathrm{id}, \ldots, \mathrm{id}, (\mathrm{id} + \mathrm{id})_j, \mathrm{id}, \ldots, \mathrm{id})$.*

Definition 14. *Let $X \in s\mathcal{C}$. Define a sequence of simplicial objects in $\mathcal{C}'$:*

$$\mathcal{T}X = \left(T_1^d(X) \xleftarrow{\partial'} T_2^d(X) \leftarrow T_3^d(X) \leftarrow \ldots \right), \qquad \partial' = \sum_{i=1}^{p-1} (-1)^i d_i.$$

Remark 15. $\partial' \circ \partial' = 0$.

Corollary 16. *Therefore $\mathcal{T}X$ gives a bicomplex*

$$(\mathcal{T}X)_{p,q} = T_p^d(X_q)$$

with horizontal differentials ∂' and vertical differentials from kX.

Proposition 17. *We have an embedding $i : kTX = (\mathcal{T}X)_{1,*} \hookrightarrow \mathrm{Tot}(\mathcal{T}X)$ and it is a chain map of degree 1.*

Theorem 18. *There is a natural isomorphism $\omega : Htot(\mathcal{T}SX) \simeq H(TSX)$ such that for*

$$\begin{tikzcd} & H_{q+1}(TSX) \\ H_qTX \arrow[ur, "\sigma"] \arrow[dr, "i"'] & \\ & H_q(\mathcal{T}X) \arrow[uu, "\omega"'] \end{tikzcd}$$

any q the diagram commutes:

Definition 19 (bar construction)**.** *$\mathcal{T}X$ is called the* bar construction *for T.*

Corollary 20. *If T is additive, then σ is an isomorphism.*

Corollary 21. *If T is of degree 2, then there exists a morphism β such that the sequence is exact: $\ldots \to H_qT_2(X,X) \xrightarrow{\alpha} H_q(TX) \xrightarrow{\sigma} H_{q+1}(TSX) \to H_{q+1}T_2(X,X) \to H_{q+1}(TX) \to \ldots$.*

Corollary 22. *There exists a spectral sequence wchich converges to H_*TSX and which satisfies*

- *E'_{pq} is equal to the complex $H_qTX \xleftarrow{H_q(\partial')} H_qT_2(X,X) \xleftarrow{H_q(\partial')} H_qT_3(X,X,X) \leftarrow \ldots$,*
- *the homomorphism $H_qTX = E'_{pq} \to H_{q+1}TSX$ is the same as σ.*

Definition 23. *We say that $X \in s\mathcal{C}$ is* trivial below n *if there exists $X' \in s\mathcal{C}$ which is homotopy equivalent to X and satisfies $X'_i = 0$ for $i < n$.*

Lemma 24. *If X is projective and $H_q(X) = 0$ for $q < n$, then X is trivial below n.*

Remark 25 (digression)**.** A *bisimplicial object* is $X_{p,q}$ with $X_{p,q} \to X_{r,s}$
for any $\alpha : [r] \to [p], \beta : [s] \to [q]$, which satisfy simplicial identities in both directions.

Every bisimplicial object gives us a bicomplex kX.

If X is bisimplicial, then it comes with a diagonal simplicial object $X_{k,k} \xrightarrow{(\alpha,\alpha)} X_{l,l}$ (where $\alpha : [l] \to [k]$).

Theorem 26 (Eilenberg-Zilber(-Cantier))**.** *There is a chain homotopy equivalence*
$k(X_{p,p}) \simeq \mathrm{tot}(kX_{p,q})$.

Remark 27. Observe that $X_{p,p}$ is in degree p to the left and $p+p$ to the right.

Proposition 28. *Let $T : \mathcal{C}^l \to \mathcal{C}'$ be such that $T(\ldots, 0_j, \ldots) = 0$. Let, for $j = 1, \ldots, l$, $X^j \in s\mathcal{C}$ be trivial below n_j. Then $T(X^1, \ldots, X^l)$ is trivial below $n_1 + \ldots + n_l = n$ (therefore $H_qT(X^1, \ldots, X^l) = 0$ for $q < n$).*

Corollary 29. *If X is trivial below n, then the suspension homomorphism*
$\sigma : H_q(TX) \to H_{q+1}(TSX)$ is an isomorphism for $q < 2n$ and epimorphism for $q = 2n$.

Definition 30 (stable derived functors)**.** *$L_{q+n}(T\bullet, n)$ for $n > q$ is called the q-th stable derived functor of T, denoted $L^s_qT(\bullet)$.*