Remark 1. $\ldots \to \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \to \ldots$ is acyclic, so a 0 map is a quasi-isomorphism, but not a homotopy equivalence.

So one has 2 resolutions of $\ldots \to 0 \to \ldots$, which are not homotopy equivalent.

Definition 2 (K-injectivity, K-projectivity)**.** *A complex A is* K-injective (K-projective) *if for any acyclic X,* $\mathrm{Hom}(X, A)$ *(*$\mathrm{Hom}(A, X)$*) is acyclic.*

Theorem 3 (Spatenstein)**.** *In the category of chain complexes of R-modules every complex has a K-injective (K-projective) resolution.*

Definition 4 (F-acyclic)**.** *Assume that $F : \mathcal{A} \to \mathcal{B}$ is left-exact, additive and RF $(= \mathrm{D}^+ F)$ exists; then we can say that $A \in \mathcal{A}$ is* F-acyclic *if $RF(A)$ has only* 0 *cohomology group (i.e. $R^i F(A) = 0$ for $i > 0$).*

Theorem 5. *Let $\mathcal{Z}$ be a class of F-acyclic objects.*

- *If $\mathcal{Z}$ is sufficiently large, then there exists a class of objects adapted to F.*
- *If $\mathcal{Z}$ is sufficiently large, then any class of objects adapted to F is contained in $\mathcal{Z}$.*
- *If $\mathcal{Z}$ is sufficiently large, then it contains all injective objects of $\mathcal{A}$.*

$F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{C}$ additive left exact functors of abelian categories. Assume that there exists classes $\mathcal{R}_{\mathcal{A}}$ of objects adapted to F, $\mathcal{R}_{\mathcal{B}}$ adapted to G, and $F(\mathcal{R}_{\mathcal{A}}) \subset \mathcal{R}_{\mathcal{B}}$. These assumptions imply that $RF, RG, R(G \circ F)$ exist.

Theorem 6. *The functors $RG \circ RF$ and $R(G \circ F)$ are isomorphic as functors $\mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{C})$.*

Remark 7. Assume X is of the type that $R^i F(X) = 0$ for $i \neq k$ for k-a fixed integer. Then $RG(RF(X)) = RG(R^k F(X)[-k])$, $R^n(G \circ F)(X) = R^{n-k} G(R^k F(X))$.

Triangulated categories

Assume that $\mathcal{C}$ is an additive category with an automorphism $T : \mathcal{C} \to \mathcal{C}$ (called the *translation functor*).

Definition 8. $X[1] = T(X), X[n] = T(X[n-1])$

Definition 9 (triangle)**.** *A* triangle *in $\mathcal{C}$ is a sequence of maps $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$.*

A map of triangles is a commutative diagram

$$\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & T(X) \\
\big\downarrow{\scriptstyle f} & & \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle T(f)} \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & T(X')
\end{array}$$

Definition 10 (triangulated category)**.** *An additive category $\mathcal{C}$ with T on it is called a* triangulated category *if it is equipped with a class of distinguished triangles (u, v, w), which satisfy the following conditions:*

- *TR1. Every morphism v can be embedded into distinguished triangle* $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$.

  Moreover, if $X = Y$ and $Z = 0$ and $u = \mathrm{id}$, then $X \xrightarrow{\mathrm{id}} X \to 0 \to T(X)$ is distinguished.

- *TR2. $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ is distinguished iff $Y \xrightarrow{v} Z \xrightarrow{w} T(X) \xrightarrow{-T(u)} T(U)$ is distinguished.*

- *TR3. Assume that in the diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "*"] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, dotted, "h"] & T(X) \arrow[d, "T(f)"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & T(X') \end{tikzcd}$$

  rows are distinguished and $*$ commutes. Then there exists $h : Z \to Z'$ such that (f, g, h) is a morphism of triangles.

- *TR4. [Octahedron axiom] Assume that we have X, Y, Z, X', Y', Z' in $\mathcal{C}$. Assume that $X \xrightarrow{u} Y \xrightarrow{j} Z' \xrightarrow{\partial} T(X)$, $Y \xrightarrow{v} Z \xrightarrow{x} X' \xrightarrow{i} T(Y)$, $X \xrightarrow{v \circ u} Z \xrightarrow{y} Y' \xrightarrow{\delta} T(X)$ are distinguished. Then there exists distinguished $Z' \xrightarrow{f} Y' \xrightarrow{g} X' \xrightarrow{T(j) \circ i} T(Z')$ such that*

  1. *the four distinguished triangles form faces of octahedron,*
  2. *the remaining faces commute,*
  3. *$yv = fj : Y \to Y'$,*
  4. *$u\delta = ig : Y' \to Y$.*