We began with some motivation-discussion. Ain't got time for writing that nicely (in a motivating way).

Theorem 1 (definition of derived category). Let C be an abelian category and Kom(C) denote the category of cochain complexes over C. Then there is a category D(C) (derived category of C) and a functor $Q : Kom(C) \to D(C)$ such that

- 1. For every quasi-isomorphism $f \in \text{Mor}(\text{Kom}(\mathcal{C})), \mathcal{Q}(f)$ is an isomorphism.
- 2. Q is universal with respect to 1, i.e. for every A and F: $Kom(C) \to A$, such that for every quasi-isomorphism f the map F(f) is invertible, there exists QF making the diagram commutative:

$$\operatorname{Kom}(\mathcal{C}) \xrightarrow{\mathcal{Q}} \operatorname{D}(\mathcal{C})$$

$$f \searrow \operatorname{\mathcal{Q}F}$$

$$\mathcal{A}$$

D(C) is called the derived category of C.

Definition 2 (localisation of a category). \mathcal{B} is a category, S a class of morphisms in \mathcal{B} . We can find a new category $\mathcal{B}[S^{-1}]$ and a functor $L: \mathcal{B} \to \mathcal{B}[S^{-1}]$ such that for any functor $F: \mathcal{B} \to \mathcal{B}'$ which takes any $s \in S$ to an isomorphism there exists a functor $LF: \mathcal{B}[S^{-1}] \to \mathcal{B}'$ such that

$$\mathcal{B} \xrightarrow{L} \mathcal{B}[S^{-1}]$$

$$F \searrow \swarrow_{LF}$$

$$\mathcal{B}'$$

Fact 3. $D(C) = Kom(C)[(q - iso)^{-1}]$

Definition 4. The class $S \subset \text{Mor}(\mathcal{B})$ is localising if it satisfies

- $\forall_{X \in \mathrm{Ob}(\mathcal{B})} \mathrm{id}_X \in S$,
- $s, t \in S \implies s \circ t \in S$,
- $\bullet \ \forall_{s \in S, f \ any} \, \exists_{t \in S, g \ any}$

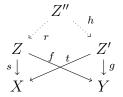
$$W \xrightarrow{g} Z$$

$$\downarrow t \qquad \downarrow s$$

$$X \xrightarrow{f} Y$$

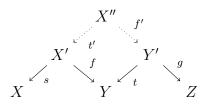
- $\forall_{t \in S, g \ any} \, \exists_{s \in S, f \ any} \ as \ above,$
- $f, g: X \to Y$, then $\exists_{s \in S} sf = sg \iff \exists_{t \in S} ft = gt$.

Lemma 5. If S is localizing in \mathcal{B} , then we can present any morphism in $\mathcal{B}[S^{-1}]$ as a triangle $X \stackrel{s}{\leftarrow} Z \xrightarrow{f} Y$ with equivalence $(s, f) \sim (t, g) \iff \exists r \in S, h$



Also, an equivalent statement with left fractions is true.

Lemma 6 (composition). Like that.



Remark 7. Class of quasi-isomorphisms is not localising in $Kom(\mathcal{C})$.