Definition 1 (cosimplicial object)**.** $X : \Delta \to \mathcal{C}$.

Denote the category of cosimplicial objects *in* $\mathcal{C}$ *as* $c\mathcal{C}$*.*

Theorem 2 (Dold-Kan again)**.** $c\mathcal{C} \simeq$ *cochain complexes over* $\mathcal{C}$

Remark 3. $c\mathcal{C} = s(\mathcal{C}^{op})$

Definition 4. *For* $X \in s\mathcal{C}$ *define* $kX \in C_*(\mathcal{C})$, $(kX)_n = X_n$, $d = \sum_{i=0}^{n}(-1)^i d_i$.

Theorem 5. *The natural embedding* $NX \hookrightarrow kX$ *is a chain homotopy equivalence.*

Remark 6.

- $NX = {kX}\big/{DX}$, where $(DX)_n = \bigcup_{i=0}^{n-1} \operatorname{im}(s_i : X_{n-1} \to X_n) = \operatorname{im}\left(\prod X_{n-1} \xrightarrow{\prod s_i} X_n\right)$
- $kX = NX \oplus DX$,
- DX is contractible.

Remark 7. One can get NX using $(-1)^n d_n$ instead of d_0.

Remark 8. Observe that if $\tau_n : [n] \to [n], i \mapsto n - i$, and $\alpha^* = \tau_n \alpha \tau_m$ for $\alpha : [m] \to [n]$, then we get an involution of the category Δ, $\alpha \to \alpha^*$.

Hence we get an involution of $s\mathcal{C}$, $X \to X^*$, $(X^*)_n = X_n$, $X^*(\alpha) = X(\alpha^*)$, d_i goes to d_{n-i}.

We can define $N^*X = N(X^*)$, $K^*C = (KC)^*$, getting $N^*K^* = NK$, $K^*N^* = \mathrm{Id}$.

Theorem 9.

1. *$f_1, f_2 : X \to Y$ homotopic in $s\mathcal{C}$ iff Nf_1, Nf_2 are chain homotopic in $C_*(\mathcal{C})$,*
2. *$\varphi_1, \varphi_2 : C \to D$ are chain homotopic in $C_*(\mathcal{C})$ iff $K\varphi_1, K\varphi_2$ are homotopic in $s\mathcal{C}$.*

Definition 10 (simplicial resolution)**.** *Let $T : \mathcal{C} \to \mathcal{C}'$ functor between abelian categories, $\mathcal{C}$ has enough projective objects, $A \in \mathrm{Ob}(\mathcal{C})$ and $n \in \mathbb{N}$.*

Then a pair $(X_\bullet, \xi)$ is called a siplicial resolution of A of degree n *(simplicial resolution of (A, n)) if $X_\bullet \in s\mathcal{C}$, $X_i = 0$ for $i < n$, $H_j(X) := H_j(kX) = 0$ for $j > n$ and $\xi : H_n(X) \to A$ is an isomorphism.*

If $\forall_i X_i$ is projective, then X is a projective resolution of (A, n).

Usually we will remove ξ from notation and say that $H_n(X) = A$.

Remark 11.

1. If $X_\bullet$ is a simplicial resolution of (A, n), then NX is a resolution of A shifted up by n. If $X_\bullet$ is projective, then NX is a projective resolution.
2. If $P \in C_*(\mathcal{C})$ is a projective resolution of A shifted by n, then KP is a simplicial projective resolution of (A, n).
3. If $\alpha : A \to B$ in $\mathcal{C}$ and X, Y are projective resolutions of (A, n) and (B, n), then there exists a simplicial morphism $f : X \to Y$ which induces $\alpha = H_n(f)$.

   Moreover, f is unique up to homotopy.

Definition 12 (derived functor). *Fuctor $L_qT(\bullet, n) : \mathcal{C} \to \mathcal{C}'$ defined below is called* q-th left derived functor of T of degree n*, where $L_qT(\bullet, n)(A) = H_q(T(X))$, where X is any simplicial resolution of A.*

Remark 13. If T is additive, then $k(T(X)) = T(kX)$, so $L_qT(A, n) = L_{q-n}T(A)$ (L_{q-n} from ordinary homotopy category).

Remark 14. When T is not additive, then $\sum_{i=0}^{n}(-1)^i T(d_i)$ is usually not equal $T(\sum(-1)^i d_i)$, so $k(TX)$ and $T(kX)$ may have different homology.

Let $T : \mathcal{C} \to \mathcal{C}'$ functor of abelian categories. Assume $T(0) = 0$ (if $T(0) = A$, then take $T' = \ker(T \to T(0) = A)$).

Definition 15 (cross effect). *For any $k \in \mathbb{N}$ we define the* k-th cross-effect of T *as a functor $T_k : \mathcal{C}^k \to \mathcal{C}'$ such that we get a functorial decomposition $T(A_1 \oplus \ldots \oplus A_k) = \bigoplus_{i=1}^{k} T(A_i) \oplus \bigoplus_{i_1 < i_2} T_2(A_{i_1}, A_{i_2}) \oplus \ldots \oplus T_k(A_1, \ldots, A_k)$.*

We can define T_k inductively,

- $T_1 = T$,
- $T_2(A_1, A_2) = \ker(T(A_1 \oplus A_2) \to T(A_1) \oplus T(A_2))$,
- $\ldots$,
- $T_k(A_1, \ldots, A_k) = \ker(T(A_1 \oplus \ldots \oplus A_n) \xrightarrow{\prod_i} \prod_i T(A_1 \oplus \ldots \oplus \hat{A}_i \oplus \ldots \oplus A_k))$.

Definition 16 (functor degree). *We say that T is of degree $\leqslant k$ if $T_{k+1} = 0$.*

We say that T is of degree k if T is of degree $\leqslant k$ and $T_k \neq 0$.

Theorem 17 (properties of cross-effects).

- *If for some i, $A_i = 0$, then $T_k(A_1, \ldots, A_n) = 0$.*
- *T_k is symmetric in its variables.*
- *If we define $s^{(1)}(A) = T_2(A, A_2)$, $s^{(2)}(A) = T_2(A_1, A)$, then $s_2^{(1)}(A_1, A_2; A_3) = s_2^{(2)}(A_1; A_2, A_3) = T_3(A_1, A_2, A_3)$.*

Example 18. $\deg T \leqslant 1$ iff T is additive.

Example 19. $T(A) = A^{\otimes 2}$, then $T_2(A, B) = (A \otimes B) \oplus (B \otimes A)$ and it is linear in A, B, so T is of degree 2.