

$\mathcal{C}$  – any abelian category.

**Definition 1** ( $n$ -suspension functor).  $X \in \text{Kom}(\mathcal{C})$ , then  $X[n] \in \text{Kom}(\mathcal{C})$ ,  $(X[n])_i = X_{n+i}$ ,  $d_{X[n]} = (-1)^n d_X$ .

$f : X \rightarrow Y$ ,  $f[n] : X[n] \rightarrow Y[n]$  defined in an obvious way.

$T : \text{Kom}(\mathcal{C}) \rightarrow \text{Kom}(\mathcal{C})$ ,  $T(X) = X[1]$ , is called a translation / shift / suspension functor.

**Definition 2.**  $\text{Kom}(\mathcal{C})$  is known already.

$\text{Kom}^+(\mathcal{C}) = \{X \in \text{Kom}(\mathcal{C}) : \exists_{i_0} \forall_{i \leq i_0} X_i = 0\}$

$\text{Kom}^-(\mathcal{C})$  obvious.

$\text{Kom}^b(\mathcal{C}) = \text{Kom}^+(\mathcal{C}) \cap \text{Kom}^-(\mathcal{C})$

**Remark 3.**  $T$  is well defined in each of these.

**Definition 4** (cone).  $f : X \rightarrow Y$ ,  $\text{Cone}(f) = C(f) \in \text{Kom}(\mathcal{C})$  is the cone of  $f$ .

$C(f)_i = X[1]_i \oplus Y_i$ ,  $d_{C(f)} = (-d_X \pi_1, f[1] \pi_1 + d_Y \pi_2)$ .

**Definition 5** (cylinder).  $f : X \rightarrow Y$ ,  $\text{Cyl}(f) \in \text{Kom}(\mathcal{C})$  is the cylinder of  $f$ .

$\text{Cyl}(f)_i = X_i \oplus X[1]_i \oplus Y_i$ ,  $d_{\text{Cyl}(f)} = (d_X \pi_1 - \pi_2, -d_X \pi_2, f[1] \pi_2 + d_Y \pi_3)$ .

**Remark 6.** Once in life it is worth to check that  $d^2 = 0$  for the cone and the cylinder.

**Fact 7.** For any  $f : X \rightarrow Y$  the following diagram has exact rows and is functorial in  $f$ :

$$\begin{array}{ccccccc}
 0 & \longrightarrow & Y & \xrightarrow{\pi} & C(f) & \longrightarrow & X[1] \longrightarrow 0 \\
 & & \downarrow \alpha & & \downarrow = & & \\
 0 & \longrightarrow & X & \xrightarrow{\bar{f}} & \text{Cyl}(f) & \longrightarrow & C(f) \longrightarrow 0 \\
 & & \downarrow = & & \downarrow \beta & & \\
 & & X & \xrightarrow{f} & Y & & 
 \end{array}$$

where  $\beta = f \pi_1 + \pi_3$  and other maps are obvious.

Also,  $\alpha, \beta$  are quasi-isomorphism, with  $\beta \alpha = \text{id}_Y$  (therefore  $Y \sim \text{Cyl}(f)$  in  $\text{D}(\mathcal{C})$ ).

**Definition 8** (triangle). In the category  $\text{Kom}(\mathcal{C})$  a triangle is any sequence of the form  $X \rightarrow Y \rightarrow Z \rightarrow X[1]$ .

A map of triangles is given by a commutative diagram

$$\begin{array}{ccccccc}
 X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\
 \downarrow f & & \downarrow g & & \downarrow h & & \downarrow f[1] \\
 X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1]
 \end{array}$$

A triangle  $X \rightarrow Y \rightarrow Z \rightarrow X[1]$  is distinguished if it is isomorphic to a triangle  $X' \rightarrow \text{Cyl}(f) \rightarrow C(f) \rightarrow X'[1]$  for some  $f : X' \rightarrow Y'$ .

**Fact 9.** Every exact sequence in  $\text{Kom}(\mathcal{C})$  is quasi-isomorphic to a sequence  $0 \rightarrow X \rightarrow \text{Cyl}(f) \rightarrow C(f) \rightarrow 0$ .

**Fact 10.** *If  $X \rightarrow Y \rightarrow Z \rightarrow X[1]$  is distinguished, then it induces a long exact sequence of cohomology groups:  $\dots \rightarrow H^i(X) \rightarrow H^i(Y) \rightarrow H^i(Z) \rightarrow H^{i+1}(X) \rightarrow \dots$*

**Definition 11** (homotopy category).  $K(\mathcal{C})$  is the homotopy category of  $\text{Kom}(\mathcal{C})$ , defined via  $\text{Ob}(K(\mathcal{C})) = \text{Ob}(\text{Kom}(\mathcal{C}))$  and  $\text{Mor}_{K(\mathcal{C})}(X, Y) = \text{Mor}_{\text{Kom}(\mathcal{C})}(X, Y) / \sim$ , where  $\sim$  is a chain homotopy relation.

**Theorem 12.** *Let  $S$  be a class of quasi-isomorphisms in  $K(\mathcal{C})$ . Then  $K(\mathcal{C})[S^{-1}]$  is isomorphic to  $D(\mathcal{C})$  in a canonical way.*

*This applies to any of  $\text{Kom}^*(\mathcal{C})$ .*

**Lemma 13.** *Assume  $f, g : X \rightarrow Y$  are chain homotopic in  $\text{Kom}(\mathcal{C})$ . Then  $\mathcal{Q}(f) = \mathcal{Q}(g)$ .*

**Theorem 14.** *In any of  $K^*(\mathcal{C})$  the class of quasi-isomorphisms is localising.*

**Theorem 15.**  $D(\mathcal{C})$  is an additive category.