Definition 1 ("exactness")**.** *A functor* $\mathrm{D}^+(\mathcal{C}) \to \mathrm{D}^+(\mathcal{C})$ *is* "exact" *if it maps distinguished triangles to distinguished triangles.*

Proposition 2. *Let* $F : \mathcal{C} \to \mathcal{D}$ *be exact. Then it takes quasi-isomorphisms to quasi-isomorphisms.*

Therefore it defines a functor $\mathrm{D}\, F : \mathrm{D}^+(\mathcal{C}) \to \mathrm{D}^+(\mathcal{D})$ *and the functor is "exact".*

Definition 3 (adapted class)**.** *We say that a class* $\mathcal{R}$ *of objects in* $\mathcal{C}$ *is* adapted *to a functor* F *if it satisfies:*

- $\mathcal{R}$ *is closed under finite coproducts,*
- F *takes acyclic complexes from* $\mathrm{K}^+(\mathcal{R})$ *to acyclic ones,*
- *every object of* $\mathcal{C}$ *embeds into an object of* $\mathcal{R}$.

Proposition 4. *Let* $\mathcal{R}$ *be an adapted class of objects of* $\mathcal{C}$ *for a left exact additive functor* $F : \mathcal{C} \to \mathcal{D}$. *Let* $S_{\mathcal{R}}$ *be the class of quasi-isomorphisms in* $\mathrm{K}^+(\mathcal{R})$. *Then* $S_{\mathcal{R}}$ *is localizing and the canonical functor* $\mathrm{K}^+(\mathcal{R})[S_{\mathcal{R}}^{-1}] \to \mathrm{D}^+(\mathcal{C})$ *is an equivalence of categories.*

Definition 5. *Let* $F : \mathcal{C} \to \mathcal{D}$ *be an additive, left exact functor between abelian categories. Its derived functor* $\mathrm{D}^+ F$ *consists of a functor*

$$RF : \mathrm{D}^+(\mathcal{C}) \to \mathrm{D}^+(\mathcal{D})$$

which is exact and a morphism of functors

$$\varepsilon_F : \mathcal{Q}_{\mathcal{D}} \circ \mathrm{K}^+(F) \to \mathrm{D}^+(F) \circ \mathcal{Q}_{\mathcal{C}}$$

such that for any exact $G : \mathrm{D}^+(\mathcal{C}) \to \mathrm{D}^+(\mathcal{D})$ *and any natural* $\varepsilon : \mathcal{Q}_{\mathcal{D}} \circ \mathrm{K}^+(F) \to G \circ \mathcal{Q}_{\mathcal{C}}$ *there exists a unique morphism* $\eta : \mathrm{D}^+ F \to G$ *such that the diagram is commutative:*

$$\begin{tikzcd}
\mathcal{Q}_{\mathcal{D}} \circ \mathrm{K}^+(F) \arrow[r, "\varepsilon"] \arrow[dr, "\varepsilon_F"'] & G \circ \mathcal{Q}_{\mathcal{C}} \\
& \mathrm{D}^+ F \circ \mathcal{Q}_{\mathcal{C}} \arrow[u, "\eta \circ \mathrm{id}_{\mathcal{Q}_{\mathcal{C}}}"']
\end{tikzcd}$$

Theorem 6. *If for* F *there exists an adapted class* $\mathcal{R}$ *of objects in* $\mathcal{C}$, *then* $\mathrm{D}^+ F$ *exists and is unique up to an isomorphism.*

Theorem 7. RF *may be defined via the composition* $\mathrm{D}^+(\mathcal{C}) \to \mathrm{K}^+(\mathcal{R})[S_{\mathcal{R}}^{-1}] \xrightarrow{F} \mathrm{D}^+(\mathcal{D})$.

Theorem 8. *If* $\mathcal{C}$ *contains enough injective objects, then the class* $\mathcal{I}$ *of them is adapted to any left exact functor* $F : \mathcal{C} \to \mathcal{D}$.

Proposition 9. *When* $\mathcal{R} = \mathcal{I}$, *then* ε_F *may be defined in the following way.*

$T : \mathrm{K}^+(\mathcal{I}) \to \mathrm{D}^+(\mathcal{C})$ *is an equivalence of categories,* $U : \mathrm{D}^+(\mathcal{C}) \to \mathrm{K}^+(\mathcal{I})$ *inverse equivalence, we know that for* Y *injective* $\mathrm{Hom}_{\mathrm{D}^+(\mathcal{C})}(X, Y) \simeq \mathrm{Hom}_{\mathrm{K}^+(\mathcal{C})}(X, Y)$, *so take* $f : X \to I$ *a quasi-isomorphism in* $\mathrm{Hom}_{\mathrm{D}^+(\mathcal{C})}(\mathcal{Q}X, TU\mathcal{Q}X)$ *and denote by* f_X *its image under the bijection in* $\mathrm{Hom}_{\mathrm{K}^+(\mathcal{C})}(X, U\mathcal{Q}X)$, *and define* $(\varepsilon_F)_X = \mathcal{Q}F(f_X)$.