

We began with some motivation-discussion. Ain't got time to write that nicely (in a motivating way).

Theorem 1 (definition of derived category). *Let \mathcal{C} be an abelian category and $\text{Kom}(\mathcal{C})$ denote the category of cochain complexes over \mathcal{C} . Then there is a category $D(\mathcal{C})$ (derived category of \mathcal{C}) and a functor $\mathcal{Q} : \text{Kom}(\mathcal{C}) \rightarrow D(\mathcal{C})$ such that*

1. *For every quasi-isomorphism $f \in \text{Mor}(\text{Kom}(\mathcal{C}))$, $\mathcal{Q}(f)$ is an isomorphism.*
2. *\mathcal{Q} is universal with respect to 1, i.e. for every \mathcal{A} and $F : \text{Kom}(\mathcal{C}) \rightarrow \mathcal{A}$, such that for every quasi-isomorphism f the map $F(f)$ is invertible, there exists $\mathcal{Q}F$ making the diagram commutative:*

$$\begin{array}{ccc} \text{Kom}(\mathcal{C}) & \xrightarrow{\mathcal{Q}} & D(\mathcal{C}) \\ F \searrow & & \swarrow \mathcal{Q}F \\ & \mathcal{A} & \end{array}$$

$D(\mathcal{C})$ is called the derived category of \mathcal{C} .

Definition 2 (localisation of a category). \mathcal{B} is a category, S a class of morphisms in \mathcal{B} . We can find a new category $\mathcal{B}[S^{-1}]$ and a functor $L : \mathcal{B} \rightarrow \mathcal{B}[S^{-1}]$ such that for any functor $F : \mathcal{B} \rightarrow \mathcal{B}'$ which takes any $s \in S$ to an isomorphism there exists a functor $LF : \mathcal{B}[S^{-1}] \rightarrow \mathcal{B}'$ such that

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{L} & \mathcal{B}[S^{-1}] \\ F \searrow & & \swarrow LF \\ & \mathcal{B}' & \end{array}$$

Fact 3. $D(\mathcal{C}) = \text{Kom}(\mathcal{C})[(q - \text{iso})^{-1}]$

Definition 4. The class $S \subset \text{Mor}(\mathcal{B})$ is localising if it satisfies

- $\forall_{X \in \text{Ob}(\mathcal{B})} \text{id}_X \in S$
- $s, t \in S \implies s \circ t \in S$
- $\forall_{s \in S, f} \exists_{t \in S, g}$

$$\begin{array}{ccc} W & \xrightarrow{g} & Z \\ \downarrow t & & \downarrow s \\ X & \xrightarrow{f} & Y \end{array}$$

- $\forall_{t \in S, g} \exists_{s \in S, f}$ as above
- $f, g : X \rightarrow Y$, then $\exists_{s \in S} sf = sg \iff \exists_{t \in S} ft = gt$.

Lemma 5. *If S is localizing in \mathcal{B} , then we can present any morphism in $\mathcal{B}[S^{-1}]$ as a triangle $X \xleftarrow{s} Z \xrightarrow{Y} Y$ with equivalence $(s, f) \sim (t, g) \iff \exists r \in S, h$*

$$\begin{array}{ccc}
 & Z'' & \\
 \swarrow \scriptstyle r & & \searrow \scriptstyle h \\
 Z & & Z' \\
 \swarrow \scriptstyle f & & \searrow \scriptstyle t \\
 \downarrow \scriptstyle s & & \downarrow \scriptstyle g \\
 X & & Y
 \end{array}$$

Also, an equivalent statement with left fractions is true.

Lemma 6 (composition). *Like that.*

$$\begin{array}{ccccc}
 & & X'' & & \\
 & \swarrow \scriptstyle t' & & \searrow \scriptstyle f' & \\
 X' & & & & Y' \\
 \swarrow \scriptstyle s & & \searrow \scriptstyle f & & \swarrow \scriptstyle t \\
 X & & Y & & Z
 \end{array}$$

Remark 7. Class of quasi-isomorphisms not localising in $\text{Kom}(\mathcal{C})$.