We began with some motivation-discussion. Ain't got time for writing that nicely (in a motivating way).

Theorem 1 (definition of derived category)**.** *Let $\mathcal{C}$ be an abelian category and* $\mathrm{Kom}(\mathcal{C})$ *denote the category of cochain complexes over* $\mathcal{C}$*. Then there is a category* $\mathrm{D}(\mathcal{C})$ *(derived category of* $\mathcal{C}$*) and a functor* $\mathcal{Q} : \mathrm{Kom}(\mathcal{C}) \to \mathrm{D}(\mathcal{C})$ *such that*

1. *For every quasi-isomorphism* $f \in \mathrm{Mor}(\mathrm{Kom}(\mathcal{C}))$, $\mathcal{Q}(f)$ *is an isomorphism.*
2. $\mathcal{Q}$ *is universal with respect to* 1*, i.e. for every* $\mathcal{A}$ *and* $F : \mathrm{Kom}(\mathcal{C}) \to \mathcal{A}$*, such that for every quasi-isomorphism* f *the map* $F(f)$ *is invertible, there exists* $\mathcal{Q}F$ *making the diagram commutative:*

$$\begin{tikzcd} \mathrm{Kom}(\mathcal{C}) \arrow[r, "\mathcal{Q}"] \arrow[dr, "F"'] & \mathrm{D}(\mathcal{C}) \arrow[d, "\mathcal{Q}F"] \\ & \mathcal{A} \end{tikzcd}$$

$\mathrm{D}(\mathcal{C})$ *is called the* derived category *of* $\mathcal{C}$*.*

Definition 2 (localisation of a category)**.** $\mathcal{B}$ *is a category,* S *a class of morphisms in* $\mathcal{B}$*. We can find a new category* $\mathcal{B}[S^{-1}]$ *and a functor* $L : \mathcal{B} \to \mathcal{B}[S^{-1}]$ *such that for any functor* $F : \mathcal{B} \to \mathcal{B}'$ *which takes any* $s \in S$ *to an isomorphism there exists a functor* $LF : \mathcal{B}[S^{-1}] \to \mathcal{B}'$ *such that*

$$\begin{tikzcd} \mathcal{B} \arrow[r, "L"] \arrow[dr, "F"'] & \mathcal{B}[S^{-1}] \arrow[d, "LF"] \\ & \mathcal{B}' \end{tikzcd}$$

Fact 3. $\mathrm{D}(\mathcal{C}) = \mathrm{Kom}(\mathcal{C})[(\mathrm{q}-\mathrm{iso})^{-1}]$

Definition 4. *The class* $S \subset \mathrm{Mor}(\mathcal{B})$ *is localising if it satisfies*

- $\forall_{X \in \mathrm{Ob}(\mathcal{B})}\, \mathrm{id}_X \in S$,
- $s, t \in S \implies s \circ t \in S$,
- $\forall_{s \in S, f \text{ any}} \exists_{t \in S, g \text{ any}}$

$$\begin{array}{ccc} W & \xrightarrow{g} & Z \\ \downarrow t & & \downarrow s \\ X & \xrightarrow{f} & Y \end{array}$$

- $\forall_{t \in S, g \text{ any}} \exists_{s \in S, f \text{ any}}$ *as above,*
- $f, g : X \to Y$*, then* $\exists_{s \in S}\, sf = sg \iff \exists_{t \in S}\, ft = gt$.

Lemma 5. *If S is localizing in $\mathcal{B}$, then we can present any morphism in $\mathcal{B}[S^{-1}]$ as a triangle $X \xleftarrow{s} Z \xrightarrow{f} Y$ with equivalence $(s, f) \sim (t, g) \iff \exists r \in S, h$*

$$\begin{tikzcd}
& Z'' \arrow[dl, dotted, "r"'] \arrow[dr, dotted, "h"] & \\
Z \arrow[d, "s"'] \arrow[drr, "f"] & & Z' \arrow[d, "g"] \arrow[dll, "t"'] \\
X & & Y
\end{tikzcd}$$

Also, an equivalent statement with left fractions is true.

Lemma 6 (composition)**.** *Like that.*

$$\begin{tikzcd}
& & X'' \arrow[dl, dotted, "t'"'] \arrow[dr, dotted, "f'"] & & \\
& X' \arrow[dl, "s"'] \arrow[dr, "f"] & & Y' \arrow[dl, "t"'] \arrow[dr, "g"] & \\
X & & Y & & Z
\end{tikzcd}$$

Remark 7. Class of quasi-isomorphisms is not localising in $\mathrm{Kom}(\mathcal{C})$.

$\mathcal{C}$ – any abelian category.

Definition 1 (n-suspension functor)**.** $X \in \mathrm{Kom}(\mathcal{C})$, *then* $X[n] \in \mathrm{Kom}(\mathcal{C})$, $(X[n])_i = X_{n+i}$, $d_{X[n]} = (-1)^n d_X$.

$f : X \to Y$, $f[n] : X[n] \to Y[n]$ *defined in an obvious way.*

$T : \mathrm{Kom}(\mathcal{C}) \to \mathrm{Kom}(\mathcal{C})$, $T(X) = X[1]$, *is called a* translation / shift/ suspension *functor.*

Definition 2. $\mathrm{Kom}(\mathcal{C})$ *is already known.*

$\mathrm{Kom}^+(\mathcal{C}) = \{X \in \mathrm{Kom}(\mathcal{C}) : \exists_{i_0} \forall_{i \leqslant i_0} X_i = 0\}$

$\mathrm{Kom}^-(\mathcal{C})$ *obvious.*

$\mathrm{Kom}^b(\mathcal{C}) = \mathrm{Kom}^+(\mathcal{C}) \cap \mathrm{Kom}^-(\mathcal{C})$

Remark 3. T is well defined in each of these.

Definition 4 (cone)**.** $f : X \to Y$, $\mathrm{Cone}(f) = \mathrm{C}(f) \in \mathrm{Kom}(\mathcal{C})$ *is the* cone *of* f.

$\mathrm{C}(f)_i = X[1]_i \oplus Y_i$, $d_{\mathrm{C}(f)} = (-d_X \pi_1, f[1]\pi_1 + d_y \pi_2)$.

Definition 5 (cylinder)**.** $f : X \to Y$, $\mathrm{Cyl}(f) \in \mathrm{Kom}(\mathcal{C})$ *is the* cylinder *of* f.

$\mathrm{Cyl}(f)_i = X_i \oplus X[1]_i \oplus Y_i$, $d_{\mathrm{Cyl}(f)} = (d_x \pi_1 - \pi_2, -d_X \pi_2, f[1]\pi_2 + d_Y \pi_3)$.

Remark 6. Once in life it is worth to check that $d^2 = 0$ for the cone and the cylinder.

Fact 7. *For any* $f : X \to Y$ *the following diagram has exact rows and is functorial in* f*:*

$$\begin{tikzcd}
 & 0 \arrow[r] & Y \arrow[r, "\pi"] \arrow[d, "\alpha"] & \mathrm{C}(f) \arrow[r] \arrow[d, "="] & X[1] \arrow[r] & 0 \\
0 \arrow[r] & X \arrow[r, "\bar{f}"] \arrow[d, "="] & \mathrm{Cyl}(f) \arrow[r] \arrow[d, "\beta"] & \mathrm{C}(f) \arrow[r] & 0 & \\
 & X \arrow[r, "f"] & Y & & &
\end{tikzcd}$$

where $\beta = f\pi_1 + \pi_3$ *and other maps are obvious.*

Also, α, β *are quasi-isomorphism, with* $\beta\alpha = \mathrm{id}_Y$ *(therefore* $Y \sim \mathrm{Cyl}(f)$ *in* $\mathrm{D}(\mathcal{C})$.

Definition 8 (triangle)**.** *In the category* $\mathrm{Kom}(\mathcal{C})$ *a* triangle *is any sequence of the form* $X \to Y \to Z \to X[1]$.

A map of triangles is given by a commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "h"] & X[1] \arrow[d, "f[1]"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

A triangle $X \to Y \to Z \to X[1]$ *is distinguished if it is isomorphic to a triangle* $X' \to \mathrm{Cyl}(f) \to \mathrm{C}(f) \to X'[1]$ *for some* $f : X' \to Y'$.

Fact 9. *Every exact sequence in* $\mathrm{Kom}(\mathcal{C})$ *is quasi-isomorphic to a sequence* $0 \to X \to \mathrm{Cyl}(f) \to C(f) \to 0$.

Fact 10. *If $X \to Y \to Z \to X[1]$ is distinguished, then it induces a long exact sequence of cohomology groups: $\ldots \to H^i(X) \to H^i(Y) \to H^i(Z) \to H^{i+1}(X) \to \ldots$*

Definition 11 (homotopy category)**.** $\mathrm{K}(\mathcal{C})$ *is the* homotopy category *of* $\mathrm{Kom}(\mathcal{C})$*, defined via* $\mathrm{Ob}(\mathrm{K}(\mathcal{C})) = \mathrm{Ob}(\mathrm{Kom}(\mathcal{C}))$ *and* $\mathrm{Mor}_{\mathrm{K}(\mathcal{C})}(X,Y) = \mathrm{Mor}_{\mathrm{Kom}(\mathcal{C})}(X,Y)/\sim$*, where* $\sim$ *is a chain homotopy relation.*

Theorem 12. *Let S be a class of quasi-isomorphisms in* $\mathrm{K}(\mathcal{C})$*. Then* $\mathrm{K}(\mathcal{C})[S^{-1}]$ *is isomorphic to* $\mathrm{D}(\mathcal{C})$ *in a canonical way.*

This applies to any of $\mathrm{Kom}^*(\mathcal{C})$*.*

Lemma 13. *Assume $f, g : X \to Y$ are chain homotopic in* $\mathrm{Kom}(\mathcal{C})$*. Then $\mathcal{Q}(f) = \mathcal{Q}(g)$.*

Theorem 14. *In any of* $\mathrm{K}^*(\mathcal{C})$ *the class of quasi-isomorphisms is localising.*

Theorem 15. $\mathrm{D}(\mathcal{C})$ *is an additive category.*

Definition 1. *X is an H^0-complex if $H^i(X) \neq 0 \implies i = 0$.*

Theorem 2. *The precomposition of the localization functor $\mathcal{Q} : \mathrm{Kom}(\mathcal{C}) \to \mathrm{D}(\mathcal{C})$ with embedding $i_0 : \mathcal{C} \to \mathrm{Kom}(\mathcal{C})$ defines an equivalence between $\mathcal{C}$ and the full subcategory of $\mathrm{D}(\mathcal{C})$ consisting of H^0-complexes.*

Definition 3. *$X[i] = T^i([X])$ for $X \in \mathcal{C}$.*

Definition 4. *$\mathcal{C}$ – abelian, then $\mathrm{Ext}^i_{\mathcal{C}}(X, Y) = \mathrm{Hom}_{\mathrm{D}(\mathcal{C})}(X[0], Y[i])$.*

Remark 5. One does not need projectives or injectives in this definition.

Remark 6. $\mathrm{Ext}^i_{\mathcal{C}}(X, Y) = \mathrm{Hom}_{\mathrm{D}(\mathcal{C})}(X[k], Y[k+i])$ for any $k \in \mathbb{Z}$.

Definition 7 (multiplication)**.** *There is a multiplication*

$$\mathrm{Ext}^i_{\mathcal{C}}(X, Y) \times \mathrm{Ext}^j_{\mathcal{C}}(Y, Z) \to \mathrm{Ext}^{i+j}_{\mathcal{C}}(X, Z)$$

via composition $\mathrm{Hom}_{\mathrm{D}(\mathcal{C})}(X[0], Y[i]) \times \mathrm{Hom}_{\mathrm{D}(\mathcal{C})}(Y[i], Z[i+j]) \to \mathrm{Hom}_{\mathrm{D}(\mathcal{C})}(X[0], Z[i+j])$.

Fact 8. *For an exact sequence $0 \to Y' \to Y \to Y'' \to 0$ there is an exact sequence*

$$\ldots \to \mathrm{Ext}^i(X, Y') \to \mathrm{Ext}^i(X, Y) \to \mathrm{Ext}^i(X, Y'') \to \mathrm{Ext}^{i+1}(X, Y') \to \ldots$$

Exercise 9. Show that if $X \to Y \to Z \to X[1]$ is distinguished in $\mathrm{D}(\mathcal{C})$, then we have an exact sequence of abelian groups

$$\ldots \to \mathrm{Hom}_{\mathrm{D}(\mathcal{C})}(U, X[i]) \to \mathrm{Hom}_{\mathrm{D}(\mathcal{C})}(U, Y[i]) \to \mathrm{Hom}_{\mathrm{D}(\mathcal{C})}(U, Z[i]) \to \mathrm{Hom}_{\mathrm{D}(\mathcal{C})}(U, X[i+1]) \to \ldots$$

Theorem 10. $\mathrm{Ext}^0_{\mathcal{C}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$

Theorem 11. *$\mathrm{Ext}^i_{\mathcal{C}}(X, Y) = 0$ for $i < 0$.*

Theorem 12. *Every element in $\mathrm{Ext}^i_{\mathcal{C}}(X, Y)$ has a presentation $X[0] \xleftarrow{s} K \xrightarrow{f} Y[i]$, where $K_j = 0$ for $j < -i$ and for $j > 0$, $K_{-i} = Y$, $f_i = \mathrm{id}$, and s is a quasi-isomorphism.*

In other words, every such element comes from an exact sequence

$$0 \to Y = K^{-i} \to K^{-i+1} \to K^{-i+2} \to \ldots \to K^1 \to K^0 \to X \to 0.$$

Definition 1 ("exactness")**.** *A functor* $\mathrm{D}^+(\mathcal{C}) \to \mathrm{D}^+(\mathcal{C})$ *is* "exact" *if it maps distinguished triangles to distinguished triangles.*

Proposition 2. *Let $F : \mathcal{C} \to \mathcal{D}$ be exact. Then it takes quasi-isomorphisms to quasi-isomorphisms.*

Therefore it defines a functor $\mathrm{D}\, F : \mathrm{D}^+(\mathcal{C}) \to \mathrm{D}^+(\mathcal{D})$ and the functor is "exact".

Definition 3 (adapted class)**.** *We say that a class $\mathcal{R}$ of objects in $\mathcal{C}$ is* adapted *to a functor F if it satisfies:*

- *$\mathcal{R}$ is closed under finite coproducts,*
- *F takes acyclic complexes from $\mathrm{K}^+(\mathcal{R})$ to acyclic ones,*
- *every object of $\mathcal{C}$ embeds into an object of $\mathcal{R}$.*

Proposition 4. *Let $\mathcal{R}$ be an adapted class of objects of $\mathcal{C}$ for a left exact additive functor $F : \mathcal{C} \to \mathcal{D}$. Let $S_\mathcal{R}$ be the class of quasi-isomorphisms in $\mathrm{K}^+(\mathcal{R})$. Then $S_\mathcal{R}$ is localizing and the canonical functor $\mathrm{K}^+(\mathcal{R})[S_\mathcal{R}^{-1}] \to \mathrm{D}^+(\mathcal{C})$ is an equivalence of categories.*

Definition 5. *Let $F : \mathcal{C} \to \mathcal{D}$ be an additive, left exact functor between abelian categories. Its derived functor $\mathrm{D}^+ F$ consists of a functor*

$$RF : \mathrm{D}^+(\mathcal{C}) \to \mathrm{D}^+(\mathcal{D})$$

which is exact and a morphism of functors

$$\varepsilon_F : \mathcal{Q}_\mathcal{D} \circ \mathrm{K}^+(F) \to \mathrm{D}^+(F) \circ \mathcal{Q}_\mathcal{C}$$

such that for any exact $G : \mathrm{D}^+(\mathcal{C}) \to \mathrm{D}^+(\mathcal{D})$ and any natural $\varepsilon : \mathcal{Q}_\mathcal{D} \circ \mathrm{K}^+(F) \to G \circ \mathcal{Q}_\mathcal{C}$ there exists a unique morphism $\eta : \mathrm{D}^+ F \to G$ such that the diagram is commutative:

$$\begin{tikzcd}
\mathcal{Q}_\mathcal{D} \circ \mathrm{K}^+(F) \arrow[r, "\varepsilon"] \arrow[dr, "\varepsilon_F"'] & G \circ \mathcal{Q}_\mathcal{C} \\
& \mathrm{D}^+ F \circ \mathcal{Q}_\mathcal{C} \arrow[u, "\eta \circ \mathrm{id}_{\mathcal{Q}_\mathcal{C}}"']
\end{tikzcd}$$

Theorem 6. *If for F there exists an adapted class $\mathcal{R}$ of objects in $\mathcal{C}$, then $\mathrm{D}^+ F$ exists and is unique up to an isomorphism.*

Theorem 7. *RF may be defined via the composition $\mathrm{D}^+(\mathcal{C}) \to \mathrm{K}^+(\mathcal{R})[S_\mathcal{R}^{-1}] \xrightarrow{F} \mathrm{D}^+(\mathcal{D})$.*

Theorem 8. *If $\mathcal{C}$ contains enough injective objects, then the class $\mathcal{I}$ of them is adapted to any left exact functor $F : \mathcal{C} \to \mathcal{D}$.*

Proposition 9. *When $\mathcal{R} = \mathcal{I}$, then ε_F may be defined in the following way.*

$T : \mathrm{K}^+(\mathcal{I}) \to \mathrm{D}^+(\mathcal{C})$ is an equivalence of categories, $U : \mathrm{D}^+(\mathcal{C}) \to \mathrm{K}^+(\mathcal{I})$ inverse equivalence, we know that for Y injective $\mathrm{Hom}_{\mathrm{D}^+(\mathcal{C})}(X, Y) \simeq \mathrm{Hom}_{\mathrm{K}^+(\mathcal{C})}(X, Y)$, so take $f : X \to I$ a quasi-isomorphism in $\mathrm{Hom}_{\mathrm{D}^+(\mathcal{C})}(\mathcal{Q}X, TU\mathcal{Q}X)$ and denote by f_X its image under the bijection in $\mathrm{Hom}_{\mathrm{K}^+(\mathcal{C})}(X, U\mathcal{Q}X)$, and define $(\varepsilon_F)_X = \mathcal{Q}F(f_X)$.

Remark 1. $\ldots \to \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \to \ldots$ is acyclic, so a 0 map is a quasi-isomorphism, but not a homotopy equivalence.

So one has 2 resolutions of $\ldots \to 0 \to \ldots$, which are not homotopy equivalent.

Definition 2 (K-injectivity, K-projectivity)**.** *A complex A is* K-injective (K-projective) *if for any acyclic X,* $\mathrm{Hom}(X, A)$ *(*$\mathrm{Hom}(A, X)$*) is acyclic.*

Theorem 3 (Spatenstein)**.** *In the category of chain complexes of R-modules every complex has a K-injective (K-projective) resolution.*

Definition 4 (F-acyclic)**.** *Assume that $F : \mathcal{A} \to \mathcal{B}$ is left-exact, additive and RF $(= \mathrm{D}^+ F)$ exists; then we can say that $A \in \mathcal{A}$ is* F-acyclic *if $RF(A)$ has only* 0 *cohomology group (i.e. $R^iF(A) = 0$ for $i > 0$).*

Theorem 5. *Let $\mathcal{Z}$ be a class of F-acyclic objects.*

- *If $\mathcal{Z}$ is sufficiently large, then there exists a class of objects adapted to F.*
- *If $\mathcal{Z}$ is sufficiently large, then any class of objects adapted to F is contained in $\mathcal{Z}$.*
- *If $\mathcal{Z}$ is sufficiently large, then it contains all injective objects of $\mathcal{A}$.*

$F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{C}$ additive left exact functors of abelian categories. Assume that there exists classes $\mathcal{R}_{\mathcal{A}}$ of objects adapted to F, $\mathcal{R}_{\mathcal{B}}$ adapted to G, and $F(\mathcal{R}_{\mathcal{A}}) \subset \mathcal{R}_{\mathcal{B}}$. These assumptions imply that $RF, RG, R(G \circ F)$ exist.

Theorem 6. *The functors $RG \circ RF$ and $R(G \circ F)$ are isomorphic as functors $\mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{C})$.*

Remark 7. Assume X is of the type that $R^iF(X) = 0$ for $i \neq k$ for k-a fixed integer. Then $RG(RF(X)) = RG(R^kF(X)[-k])$, $R^n(G \circ F)(X) = R^{n-k}G(R^kF(X))$.

Triangulated categories

Assume that $\mathcal{C}$ is an additive category with an automorphism $T : \mathcal{C} \to \mathcal{C}$ (called the *translation functor*).

Definition 8. $X[1] = T(X), X[n] = T(X[n-1])$

Definition 9 (triangle)**.** *A* triangle *in $\mathcal{C}$ is a sequence of maps $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$.*

A map of triangles is a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & T(X) \arrow[d, "T(f)"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & T(X') \end{tikzcd}$$

Definition 10 (triangulated category)**.** *An additive category $\mathcal{C}$ with T on it is called a* triangulated category *if it is equipped with a class of distinguished triangles (u, v, w), which satisfy the following conditions:*

- *TR1. Every morphism v can be embedded into distinguished triangle* $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$.

  Moreover, if $X = Y$ and $Z = 0$ and $u = \mathrm{id}$, then $X \xrightarrow{\mathrm{id}} X \to 0 \to T(X)$ is distinguished.

- *TR2. $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ is distinguished iff $Y \xrightarrow{v} Z \xrightarrow{w} T(X) \xrightarrow{-T(u)} T(U)$ is distinguished.*

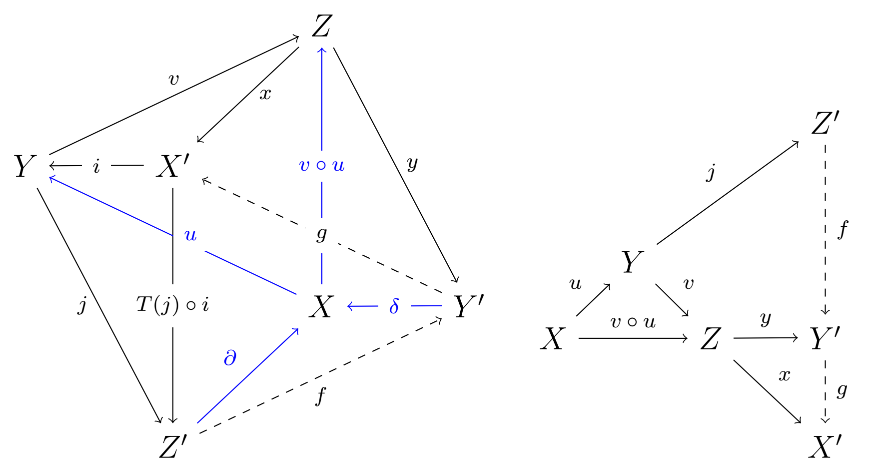
- *TR3. Assume that in the diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "*"] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, dotted, "h"] & T(X) \arrow[d, "T(f)"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & T(X') \end{tikzcd}$$

  rows are distinguished and $*$ commutes. Then there exists $h : Z \to Z'$ such that (f, g, h) is a morphism of triangles.

- *TR4. [Octahedron axiom] Assume that we have X, Y, Z, X', Y', Z' in $\mathcal{C}$. Assume that $X \xrightarrow{u} Y \xrightarrow{j} Z' \xrightarrow{\partial} T(X)$, $Y \xrightarrow{v} Z \xrightarrow{x} X' \xrightarrow{i} T(Y)$, $X \xrightarrow{v \circ u} Z \xrightarrow{y} Y' \xrightarrow{\delta} T(X)$ are distinguished. Then there exists distinguished $Z' \xrightarrow{f} Y' \xrightarrow{g} X' \xrightarrow{T(j)\circ i} T(Z')$ such that*

  1. *the four distinguished triangles form faces of octahedron,*
  2. *the remaining faces commute,*
  3. *$yv = fj : Y \to Y'$,*
  4. *$u\delta = ig : Y' \to Y$.*



Theorem 1. *Let $\mathcal{C}$ be an abelian category. Then* $\mathrm{K}(\mathcal{C})$ *(also* $\mathrm{K}^+, \mathrm{K}^-, \mathrm{K}^b$*) with standard translation functor and distinguished triangles is triangulated.*

Remark 2. $\mathrm{C}(U)$ fits as Z in TR1.

Definition 3 (cohomological functor)**.** *Assume $\mathcal{C}$ is triangulated, $\mathcal{A}$ is abelian. Let $F : \mathcal{C} \to \mathcal{A}$ be an additive functor. We call it* cohomological *if for any distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ we have an exact sequence*
$\ldots \to F(T^i(X)) \to F(T^i(Y)) \to F(T^i(Z)) \to F(T^{i+1}(X)) \to \ldots.$

Definition 4. *Let $\mathcal{C}$ be a triangulated category, S a localizing class of morphisms in $\mathcal{C}$. We say that S is* compatible *with triangulation if*

- $s \in S \iff T(s) \in S$,
- *in TR3, $f, g \in S \implies h \in S$ for any h.*

Theorem 5. *Let $\mathcal{C}$ and S be as above. On $\mathcal{C}[S^{-1}]$ we can define*

- $T_S : \mathcal{C}[S^{-1}] \to \mathcal{C}[S^{-1}]$, $T_S = T$ *on objects and morphisms, i.e.* $T(X \xleftarrow{s} Z \xrightarrow{f} Y) = T(X) \xleftarrow{T(s)} T(Z) \xrightarrow{T(f)} T(Y)$.
- $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ *is distinguished in $\mathcal{C}[S^{-1}]$ if it is isomorphic to a distinguished triangle coming from $\mathcal{C}$.*

Then $\mathcal{C}[S^{-1}]$ with the structure defined above is triangulated.

Corollary 6. *Derived category of an abelian category inherits the triangulated structure from the homotopy category of complexes.*

Definition 1. $\mathrm{K}^{\leqslant n}(\mathcal{A}), \mathrm{D}^{\leqslant n}(\mathcal{A})$ – *full subcategories of objects X for which $H^i(X) = 0$ for $i > n$.*

Analogously one defines $\mathrm{K}^{\geqslant n}(\mathcal{A}), \mathrm{D}^{\geqslant n}(\mathcal{A})$.

We know $\mathcal{A} = \mathrm{D}^{\leqslant 0}(\mathcal{A}) \cap \mathrm{D}^{\geqslant 0}(\mathcal{A})$.

Definition 2. *Let $\mathcal{C}$ be a triangulated category. Assume that D is a subcategory of $\mathcal{C}$. Then we write $D^{\leqslant n} = D^{\leqslant 0}[-n]$, $D^{\geqslant n} = D^{\geqslant 0}[-n]$.*

Definition 3 (t-structure)**.** *A t-structure on $\mathcal{C}$ consists of a pair of full subcategories $(\mathcal{C}^{\leqslant 0}, \mathcal{C}^{\geqslant 0})$ in $\mathcal{C}$ which satisfy the following properties:*

- $\mathcal{C}^{\leqslant 0} \subset \mathcal{C}^{\leqslant 1}$, $\mathcal{C}^{\geqslant 0} \supset \mathcal{C}^{\geqslant 1}$,
- $X \in \mathcal{C}^{\leqslant 0} \wedge Y \in \mathcal{C}^{\geqslant 1} \implies \mathrm{Hom}(X, Y) = 0$,
- *for all $X \in \mathcal{C}$ there is a distinguished triangle $A \to X \to B \to A[1]$ such that $A \in \mathcal{C}^{\leqslant 0}, B \in \mathcal{C}^{\geqslant 1}$.*

Definition 4 (core)**.** $\mathcal{A} = \mathcal{C}^{\leqslant 0} \cap \mathcal{C}^{\geqslant 0}$ *is called the* core *of the t-structure on $\mathcal{C}$.*

Proposition 5. *In $\mathrm{D}(\mathcal{A})$, the subcategories $\mathrm{D}^{\leqslant 0}(\mathcal{A})$ and $\mathrm{D}^{\geqslant 0}(\mathcal{A})$ define a t-structure on $\mathrm{D}(\mathcal{A})$.*

Theorem 6. *The core of any t-structure on triangulated $\mathcal{C}$ is an additive category.*

Lemma 7. *There exist functors $\tau_{\leqslant n} : \mathcal{C} \to \mathcal{C}^{\leqslant n}$ ($\tau_{\geqslant n} : \mathcal{C} \to \mathcal{C}^{\geqslant n}$) which are right (left) adjoint to the inclusion functors.*

Moreover, for any $X \in \mathrm{Ob}(\mathcal{C})$ there exists a distinguished triangle

$$\tau_{\leqslant 0}X \to X \to \tau_{\geqslant 1}X \to (\tau_{\leqslant 0}X)[1].$$

Moreover, any distinguished triangle $A \to X \to B \to A[1]$ such that $A \in \mathcal{C}^{\leqslant 0}, B \in \mathcal{C}^{\geqslant 1}$ is isomorphic to the one above.

Lemma 8.

- $\tau_{\leqslant n}X = 0 \iff X \to \tau_{\geqslant n+1}X$ *is an isomorphism,*
- *if $m \leqslant n$, then there are maps*

$$\tau_{\leqslant m}X \to \tau_{\leqslant m}\tau_{\leqslant n}X,$$

$$\tau_{\geqslant n}X \to \tau_{\geqslant n}\tau_{\geqslant m}X,$$

which are isomorphisms, if $m \leqslant n$, then there is a unique isomorphism

$$\tau_{\geqslant m}\tau_{\leqslant n}X \to \tau_{\leqslant n}\tau_{\geqslant m}X (= \tau_{[m,n]}X).$$

Theorem 1. *The core* $\mathcal{A} = \mathcal{C}^{\geqslant 0} \cap \mathcal{C}^{\leqslant 0} \subset \mathcal{C}$ *is an abelian category.*

Definition 2 (cohomology object)**.** *The* i*-th* cohomology object *of* $X \in \mathcal{C}$ *is defined as*

$$H^0(X) = \tau_{[0,0]}(X) \in \mathcal{A},$$

$$H^i(X) = H^0(X[i]) \in \mathcal{A}.$$

Definition 3 (nondegenerate t-structure)**.** *A t-structure on* $\mathcal{C}$ *is* nondegenerate *if* $\bigcap_n \mathrm{Ob}\, \mathcal{C}^{\geqslant n} = \bigcap_n \mathrm{Ob}\, \mathcal{C}^{\leqslant n} = \{0\}$.

Theorem 4. H^0 *is a cohomological functor.*

If additionally the t-structure is nondegenerate, then

- $f : X \to Y$ *in* $\mathcal{C}$ *is an isomorphism iff* $\forall_i H^i(f)$ *is an isomorphism,*
- $\mathrm{Ob}(\mathcal{C}^{\leqslant n}) = \{X \in \mathrm{Ob}\, \mathcal{C} : \forall_{i>n} H^i(X) = 0\}$,
- $\mathrm{Ob}(\mathcal{C}^{\geqslant n}) = \{X \in \mathrm{Ob}\, \mathcal{C} : \forall_{i<n} H^i(X) = 0\}$.

Definition 5 (bounded t-structure)**.** *A t-structure is* bounded *if it is nondegenerate and for any* $X \in \mathcal{C}$, $H^i(X) \neq 0$ *only for a finite number of* i.

Definition 6 (Ext)**.** $\mathrm{Ext}^i_{\mathcal{C}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y[i])$

Definition 7 (multiplication on Ext)**.** *Notice that* $\mathrm{Hom}_{\mathcal{C}}(X, Y[i]) = \mathrm{Hom}_{\mathcal{C}}(X[k], Y[i+k])$ *and define multiplication* $\mathrm{Ext}^i_{\mathcal{C}}(X,Y) \times \mathrm{Ext}^j_{\mathcal{C}}(Y,Z) \to \mathrm{Ext}^{i+j}_{\mathcal{C}}(X,Z)$ *in the most obvious way.*

Theorem 8. *Let* $\mathcal{A}$ *be a core of a bounded t-structure on* $\mathcal{C}$*. Assume* $F : \mathrm{D}^b(\mathcal{A}) \to \mathcal{C}$ *satisfies*

$$F(\mathrm{D}^b(\mathcal{A})^{\geqslant 0}) \subset \mathcal{C}^{\geqslant 0},$$

$$F(\mathrm{D}^b(\mathcal{A})^{\leqslant 0}) \subset \mathcal{C}^{\leqslant 0},$$

then F *is an equivalence of categories iff* $\mathrm{Ext}_{\mathcal{C}}$ *is generated by* $\mathrm{Ext}^1_{\mathcal{C}}$ *under Yoneda multiplication.*

Remark 9. In chain complexes $\mathrm{Ext}^i_{\mathcal{C}}(X,Y) : 0 \to Y \to E_i \xrightarrow{d^i} \ldots \xrightarrow{d^2} E_1 \xrightarrow{d^1} X \to 0$.

Theorem 10. *Assume that* $\mathcal{C}$ *satisfies additionally:*

- *TR5. Arbitrary coproducts and products exist in* $\mathcal{C}$.
- *There is a generating set* Λ *of objects in* $\mathcal{C}$*, i.e. set* Λ *such that*
  - $T(\Lambda) \subset \Lambda$, T *– translation functor,*
  - *if* $X \in \mathcal{C}$ *and* $\forall_{\lambda \in \Lambda} \mathrm{Hom}(\lambda, X) = 0$*, then* $X \simeq 0$.

Then any homological functor $H : \mathcal{C} \to \mathcal{A}$ *(where* $\mathcal{A}$ *is abelian) which sends coproducts to products is representable, i.e.* $H = \mathcal{C}(\cdot, h)$.

Simpliecial objects in categories

Definition 1 (simplicial object)**.** *A* simplicial object *X in $\mathcal{C}$ consists of:*

- $\forall_{n\geqslant 0}\, X_n \in \mathrm{Ob}\,\mathcal{C}$ – n-simplices *of X,*
- $\forall_{n\geqslant 0}\forall_{0\leqslant i\leqslant n}\, d_i : X_n \to X_{n-1}$ – boundaries (faces)*,*
- $\forall_{n\geqslant 0}\forall_{0\leqslant i\leqslant n}\, s_i : X_n \to X_{n+1}$ – degeneracies*,*

such that

- $\forall_{i<j}\, d_i d_j = d_{j-1} d_i$*,*
- $\forall_{i>j}\, s_i s_j = s_j s_{i-1}$*,*
- $d_i s_j = \begin{cases} s_{j-1} d_i & \forall_{i<j} \\ \mathrm{id} & \forall_{i=j \vee i=j+1} \\ s_i d_{i-1} & \forall_{i>j+1} \end{cases}$.

Definition 2 (simplicial map)**.** *A* simplicial map *between simplicial objects $X \to Y$ consists of the sequence of $f_n : X_n \to Y_n$ which commute with boundaries and degeneracies.*

Definition 3 (simplicial category)**.** *Denote by $s\mathcal{C}$ the category of simplicial objects in $\mathcal{C}$.*

Remark 4. Any functor $F : \mathcal{C} \to \mathcal{C}'$ extends to $F : s\mathcal{C} \to s\mathcal{C}'$

Definition 5. *Let Δ denote the subcategory of sets* $\mathrm{Ob}(\Delta) = \{[n]\} = \{\{0, 1, \ldots, n\} : n \geqslant 0\}$*,* $\mathrm{Mor}_\Delta([n], [m]) =$ *nondecreasing maps* $[n] \to [m]$*.*

Definition 6 (simplicial object again)**.** *Any functor $X : \Delta^{op} \to \mathcal{C}$ is called a* simplicial object *in $\mathcal{C}$.*

Definition 7 (simplicial maps)**.** *For X, Y simplicial objects,* $\mathrm{Mor}_{s\mathcal{C}}(X, Y) = \mathrm{Mor}_{F(\Delta^{op},\mathcal{C})}(X, Y)$*.*

Definition 8. *Let $\varepsilon^i : [n-1] \to [n]$ be defined as* $\varepsilon^i(j) = \begin{cases} j & \forall_{j<i} \\ j+1 & \forall_{j\geqslant i} \end{cases}$ *and $\eta^i : [n+1] \to [n]$ be defined as* $\eta^i(j) = \begin{cases} j & \forall_{j\leqslant i} \\ j-1 & \forall_{j>i} \end{cases}$.

Remark 9. These correspond to d_i, s_i respectively.

Proposition 10. *Any morphism $\alpha \in \Delta$ can be uniquely expressed as $\varepsilon \circ \eta$, where ε is a composition of ε^i's, and η is a composition of η^i's.*

Remark 11. A bunch of examples appear:

- $\tilde{K}$ – simplicial set of a geometric simplicial complex K,
- Δ_n – topological simplices, and $S : \mathrm{Top} \to s\mathrm{Set}$ singular simplicial set functor,
- $\Delta[n] = \mathrm{Hom}_\Delta(\cdot, [n])$,
- nerve of a small category $N(\mathcal{C})$,
- functor $s\mathrm{Set} \to s\mathrm{R}-\mathrm{mod}$ induced by a functor $\mathrm{Set} \to \mathrm{R}-\mathrm{mod}$ mapping $X \mapsto R[X]$.

Remark 12. If $X, Y \in s\mathrm{Set}$, then there is a simplicial product $(X \times Y)_n = X_n \times Y_n$, $d_i = d_i^X \times d_i^Y$ and $s_i = s_i^X \times s_i^Y$.

Definition 13 (geometric realization)**.** *Define*
$\sigma_i : \Delta_n \to \Delta_{n-1}$, $\sigma_i(t_0, \ldots, t_n) = (t_0, \ldots, t_i + t_{i+1}, \ldots, t_n)$ *and*
$\delta_i : \Delta_n \to \Delta_{n+1}$, $\delta_i(t_0, \ldots, t_n) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_n)$.

Assume $X \in s\mathrm{Set}$. *We can define a geometric realization of* X

$$|X_\bullet| = \bigsqcup X_n \times \Delta_n \Big/ \sim,$$

where $(d_i(x), s) \sim (x, \delta_i(s))$ *for* $(x, s) \in X_n \times \Delta_{n-1}$
and $(s_i(x), s) \sim (x, \sigma_i(s))$ *for* $(x, s) \in X_n \times \Delta_{n+1}$.

Remark 14. If a category $\mathcal{C}$ has a faithful functor to Set, then for $X_\bullet \in s\mathcal{C}$ we define its $|X_\bullet|$.

Theorem 15 (properties of $|\bullet| : s\mathrm{Set} \to \mathrm{Top}$)**.**

1. $|X \times Y| \simeq |X| \times |Y|$ *homeomorphism (in CW topology),*
2. K *– geometric simplicial complex, then* $|\tilde{K}_\bullet| \simeq K$ *homeomorphic,*
3. $\mathcal{C}$ *is group* G, *i.e.* $\mathrm{Ob}(\mathcal{C}) = *, \mathrm{Mor}_\mathcal{C}(*, *) = G$, *then* $|N(\mathcal{C})| = K(G, 1)$,
4. *Functors* $S : \mathrm{Top} \to s\mathrm{Set}$ *and* $|\bullet| : s\mathrm{Set} \to \mathrm{Top}$ *are adjoint.*

Let $\mathcal{C}$ be abelian, remind $s\mathcal{C}$ – simplicial objects in $\mathcal{C}$, $C_*(\mathcal{C})$ – chain complexes over $\mathcal{C}$.

Definition 1 (s-morphism)**.** *$X_\bullet, Y_\bullet \in s\mathcal{C}$. For a simplicial set $K \in s\mathrm{Set}$ a map which associates $F(\sigma) : X_n \to Y_n$ to any $\sigma \in K_n$ is called* s-morphism *(denote $F : K \times X_\bullet \to Y_\bullet$) if for any $\alpha : [m] \to [n]$ in Δ we have $F(K(\alpha)(\sigma)) \circ X(\alpha) = Y(\alpha)F(\sigma)$.*

Observe that when $X_\bullet, Y_\bullet$ are in $s\mathrm{Set}$ then s-morphisms are simplicial maps $K_\bullet \times X_\bullet \to Y_\bullet$.

Example 2. If $K = \Delta[0]$, then s-morphism is just a simplicial morphism $X_\bullet \to Y_\bullet$.

Example 3. If $K = \Delta[1]$ then s-morphism is called *a homotopy* between $F(0)$ and $F(1)$.

Remark 4. $T : \mathcal{C} \to \mathcal{C}'$ functor induces $T : s\mathcal{C} \to s\mathcal{C}'$, if $X_\bullet \in s\mathcal{C}$, then $T(X)_n = T(X_n), T(X)(\alpha) = T(X(\alpha))$.

Definition 5. *If F is an s-morphism $F : K_\bullet \times X_\bullet \to Y_\bullet$, then it defines $TF : K_\bullet \times T(X_\bullet) \to T(Y_\bullet)$ defined by $TF(\sigma) = T(F(\sigma))$.*

Remark 6. Any functor $T : \mathcal{C} \to \mathcal{C}'$ sends homotopic maps to homotopic ones.

Definition 7. *Let $\mathcal{C}$ be abelian, then there are functors*

$$s\mathcal{C} \underset{K}{\overset{N}{\rightleftharpoons}} C_*(\mathcal{C})$$

defined as follows.

Normalization N is defined, for $X_\bullet \in s\mathcal{C}$, as

$$N(X)_n = \bigcap_{i=1}^{n} \ker(d_i : X_n \to X_{n-1})$$

(e.g. $\ker\left(X_n \xrightarrow{\prod d_i} \prod X_{n-1}\right)$*), with*

$$d : N(X)_n \to N(X)_{n-1}$$

induced by d_0.

K is defined in such a way. If $\alpha : [n] \to [q]$, then $d(\alpha) = n$ and $r(\alpha) = q$. Notice for any α there is unique $\alpha = \varepsilon \circ \eta$, where ε is an injection and η is a surjection. For $C \in C_*(\mathcal{C})$, take

$$K(C)_n = \bigoplus_{\eta : d(\eta) = n} C_{r(\eta)},$$

Now for $\alpha : [m] \to [n]$ define

$$KC(\alpha) : K(C)_n \to K(C)_m$$

on every $C_{r(\eta)}$ in such a way: $\eta\alpha = \varepsilon'\eta'$, let $KC(\alpha)$ map $C_{r(\eta)}$ into $C_{r(\eta')}$ via the formula

$$K(\eta, \alpha) = \begin{cases} \mathrm{id}_{C_{r(\eta)}} & \text{for } \varepsilon' = \mathrm{id}_{[r(\eta)]} \\ d : C_{r(\eta)} \to C_{r(\eta)-1} = C_{r(\eta')} & \text{for } \varepsilon' = \varepsilon^0 \\ 0 & \text{otherwise} \end{cases} .$$

Remark 8. Observe that if $f : C \to D$ in $C_*(\mathcal{C})$, then the induced map $KC \to KD$ is simplicial.

Theorem 9 (Dold-Kan)**.** *The functors N and K give an equivalence of $s\mathcal{C}$ and $C_*(\mathcal{C})$.*

Remark 10. It was somehow convenient to define, for $X \in s\mathcal{C}$, $\bar{X} \in s\mathcal{C}$ via
$\bar{X}_n = \ker(d_{n+1} : X_{n+1} \to X_n)$.

Lemma 11. *Let $f : X_\bullet \to Y_\bullet$ be a simplicial morphism which satisfies $(Nf)_i$ is mono(epi) for $i \leqslant n$. Then $f_n : X_n \to Y_n$ is mono(epi) for $i \leqslant n$.*

Definition 1 (cosimplicial object)**.** $X : \Delta \to \mathcal{C}$.

Denote the category of cosimplicial objects *in* $\mathcal{C}$ *as* $c\mathcal{C}$*.*

Theorem 2 (Dold-Kan again)**.** $c\mathcal{C} \simeq$ *cochain complexes over* $\mathcal{C}$

Remark 3. $c\mathcal{C} = s(\mathcal{C}^{op})$

Definition 4. *For* $X \in s\mathcal{C}$ *define* $kX \in C_*(\mathcal{C})$, $(kX)_n = X_n$, $d = \sum_{i=0}^{n}(-1)^i d_i$.

Theorem 5. *The natural embedding* $NX \hookrightarrow kX$ *is a chain homotopy equivalence.*

Remark 6.

- $NX = {}^{kX}\big/_{DX}$, where $(DX)_n = \bigcup_{i=0}^{n-1} \operatorname{im}(s_i : X_{n-1} \to X_n)$ $= \operatorname{im}\left(\prod X_{n-1} \xrightarrow{\prod s_i} X_n\right)$.
- $kX = NX \oplus DX$.
- DX is contractible.

Remark 7. One can get NX using $(-1)^n d_n$ instead of d_0.

Remark 8. Observe that if $\tau_n : [n] \to [n], i \mapsto n-i$, and $\alpha^* = \tau_n \alpha \tau_m$ for $\alpha : [m] \to [n]$, then we get an involution of the category Δ, $\alpha \to \alpha^*$.

Hence we get an involution of $s\mathcal{C}$, $X \to X^*$, $(X^*)_n = X_n$, $X^*(\alpha) = X(\alpha^*)$, d_i goes to d_{n-i}.

We can define $N^*X = N(X^*)$, $K^*C = (KC)^*$, getting $N^*K^* = NK$, $K^*N^* = \mathrm{Id}$.

Theorem 9.

1. *$f_1, f_2 : X \to Y$ homotopic in $s\mathcal{C}$ iff Nf_1, Nf_2 are chain homotopic in $C_*(\mathcal{C})$,*
2. *$\varphi_1, \varphi_2 : C \to D$ are chain homotopic in $C_*(\mathcal{C})$ iff $K\varphi_1, K\varphi_2$ are homotopic in $s\mathcal{C}$.*

Definition 10 (simplicial resolution)**.** *Let $T : \mathcal{C} \to \mathcal{C}'$ functor between abelian categories, $\mathcal{C}$ has enough projective objects, $A \in \mathrm{Ob}(\mathcal{C})$ and $n \in \mathbb{N}$.*

Then a pair $(X_\bullet, \xi)$ is called a siplicial resolution of A of degree n *(simplicial resolution of (A, n)) if $X_\bullet \in s\mathcal{C}$, $X_i = 0$ for $i < n$, $H_j(X) := H_j(kX) = 0$ for $j > n$ and $\xi : H_n(X) \to A$ is an isomorphism.*

If $\forall_i X_i$ is projective, then X is a projective resolution of (A, n).

Usually we will remove ξ from notation and say that $H_n(X) = A$.

Remark 11.

1. If $X_\bullet$ is a simplicial resolution of (A, n), then NX is a resolution of A shifted up by n. If $X_\bullet$ is projective, then NX is a projective resolution.
2. If $P \in C_*(\mathcal{C})$ is a projective resolution of A shifted by n, then KP is a simplicial projective resolution of (A, n).
3. If $\alpha : A \to B$ in $\mathcal{C}$ and X, Y are projective resolutions of (A, n) and (B, n), then there exists a simplicial morphism $f : X \to Y$ which induces $\alpha = H_n(f)$.

   Moreover, f is unique up to homotopy.

Definition 12 (derived functor)**.** *Fuctor $L_qT(\bullet, n) : \mathcal{C} \to \mathcal{C}'$ defined below is called q-*th left derived functor of T of degree n*, where $L_qT(\bullet, n)(A) = H_q(T(X))$, where X is any simplicial resolution of A.*

Remark 13. If T is additive, then $k(T(X)) = T(kX)$, so $L_qT(A, n) = L_{q-n}T(A)$ (L_{q-n} from ordinary homotopy category).

Remark 14. When T is not additive, then $\sum_{i=0}^{n}(-1)^i T(d_i)$ is usually not equal $T(\sum(-1)^i d_i)$, so $k(TX)$ and $T(kX)$ may have different homology.

Let $T : \mathcal{C} \to \mathcal{C}'$ functor of abelian categories. Assume $T(0) = 0$ (if $T(0) = A$, then take $T' = \ker(T \to T(0) = A)$).

Definition 15 (cross effect)**.** *For any $k \in \mathbb{N}$ we define the k-*th cross-effect of T *as a functor $T_k : \mathcal{C}^k \to \mathcal{C}'$ such that we get a functorial decomposition*
$T(A_1 \oplus \ldots \oplus A_k) = \bigoplus_{i=1}^{k} T(A_i) \oplus \bigoplus_{i_1 < i_2} T_2(A_{i_1}, A_{i_2}) \oplus \ldots \oplus T_k(A_1, \ldots, A_k)$.

We can define T_k inductively,

- $T_1 = T$,
- $T_2(A_1, A_2) = \ker(T(A_1 \oplus A_2) \to T(A_1) \oplus T(A_2))$,
- $\ldots$,
- $T_k(A_1, \ldots, A_k) = \ker(T(A_1 \oplus \ldots \oplus A_n) \xrightarrow{\prod_i} \prod_i T(A_1 \oplus \ldots \oplus \hat{A}_i \oplus \ldots \oplus A_k))$.

Definition 16 (functor degree)**.** *We say that T is of degree $\leqslant k$ if $T_{k+1} = 0$.*

We say that T is of degree k if T is of degree $\leqslant k$ and $T_k \neq 0$.

Theorem 17. *Cross-effects have the following properties:*

- *if for some i, $A_i = 0$, then $T_k(A_1, \ldots, A_n) = 0$,*
- *T_k is symmetric in its variables,*
- *if we define $s^{(1)}(A) = T_2(A, A_2)$, $s^{(2)}(A) = T_2(A_1, A)$, then $s_2^{(1)}(A_1, A_2; A_3) = s_2^{(2)}(A_1; A_2, A_3) = T_3(A_1, A_2, A_3)$.*

Example 18. $\deg T \leqslant 1$ iff T is additive.

Example 19. $T(A) = A^{\otimes 2}$, then $T_2(A, B) = (A \otimes B) \oplus (B \otimes A)$ and it is linear in A, B, so T is of degree 2.

Theorem 1. *Assume T is of degree $\leqslant k$, $A \in \mathrm{Ob}(\mathcal{C})$ is of projective dimension $\leqslant r$, then $L_qT(A, n) = 0$ for $q > k(r + n)$.*

Lemma 2. *Let T be as above and $X \in s\mathcal{C}$ such that $(NX)_i = 0$ for $i > m$. Then $N(TX)_i = 0$ for $i > km$.*

Definition 3 (suspension)**.** *$SA = \mathrm{coker}(A \to CA)$, or $(SA)_q = A_{q-1}$ and $d^{SA} = -d^A$.*

Corollary 4. *We have an exact sequence $0 \to A \xrightarrow{\kappa} CA \xrightarrow{\pi} SA \to 0$.*

Definition 5. *Let $X \in s\mathcal{C}$. Define* cone and suspension *of X by the formulas $CX = KCNX, SX = KSNX$.*

Remark 6. We have an exact sequence (exact on each level) $0 \to X \xrightarrow{\kappa} CX \xrightarrow{\pi} SX \to 0$.

Applying T we get (not necessarily exact) $0 \to TX \xrightarrow{T(\kappa)} T(CX) \xrightarrow{T(\pi)} T(SX) \to 0$.

Remark 7. Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a sequence in $C_*(\mathcal{C})$ such that $g \circ f = 0$ and B is contractible, i.e. we have $s_q : B_q \to B_{q+1}$ such that $d^B s + sd^B = \mathrm{id}$. Then $gsf : A \to C$ gives a chain map $SA \to C$ and hence a map $H_q(A) \to H_{q+1}(C)$.

Theorem 8. *$H(gsf)$ does not depend on the choice of s.*

Definition 9 (suspension homomorphism)**.** *The map $\sigma : H_q(TX) \to H_{q+1}(TSX)$ induced by κ and π is called a* suspension homomorphism*.*

Proposition 10. *σ defines a natural transformation of functors.*

Proposition 11. *Assume T additive, then $0 \to T(X) \to T(CX) \to T(SX) \to 0$ exact and we have a long exact sequence of homology groups: $\ldots \to 0 \to H_{q+1}(TSX) \to H_q(TX) \to 0 \to \ldots$, and σ is the inverse of the map in the middle.*

Definition 12. *Let $T^d_p(A) = T_p(A, \ldots, A)$ (d means diagonal).*

Definition 13. *Define $d_i = \rho \circ T(\alpha'_i) \circ \lambda : T^d_p(A) \to T^d_{p-1}(A)$, where λ monomorphism $T^d_p(A) \to T(A \oplus \ldots \oplus A)$, ρ epimorphism $T(A \oplus \ldots \oplus A) \to T^d_{p-1}(A)$, and $d'_j : \bigoplus_{i=1}^p A \to \bigoplus_{i=1}^{p-1} A$, equal to $(\mathrm{id}, \ldots, \mathrm{id}, (\mathrm{id} + \mathrm{id})_j, \mathrm{id}, \ldots, \mathrm{id})$.*

Definition 14. *Let $X \in s\mathcal{C}$. Define a sequence of simplicial objects in $\mathcal{C}'$:*

$$\mathcal{T}X = \left(T^d_1(X) \xleftarrow{\partial'} T^d_2(X) \leftarrow T^d_3(X) \leftarrow \ldots \right), \qquad \partial' = \sum_{i=1}^{p-1} (-1)^i d_i.$$

Remark 15. $\partial' \circ \partial' = 0$.

Corollary 16. *Therefore $\mathcal{T}X$ gives a bicomplex*

$$(\mathcal{T}X)_{p,q} = T^d_p(X_q)$$

with horizontal differentials ∂' and vertical differentials from kX.

Proposition 17. *We have an embedding $i : kTX = (\mathcal{T}X)_{1,*} \hookrightarrow \mathrm{Tot}(\mathcal{T}X)$ and it is a chain map of degree 1.*

Theorem 18. *There is a natural isomorphism $\omega : Htot(\mathcal{T}SX) \simeq H(TSX)$ such that for*

$$\begin{tikzcd} & H_{q+1}(TSX) \\ H_qTX \arrow[ur, "\sigma"] \arrow[dr, "i"'] & \\ & H_q(\mathcal{T}X) \arrow[uu, "\omega"'] \end{tikzcd}$$

any q the diagram commutes:

Definition 19 (bar construction)**.** *$\mathcal{T}X$ is called the* bar construction *for T.*

Corollary 20. *If T is additive, then σ is an isomorphism.*

Corollary 21. *If T is of degree 2, then there exists a morphism β such that the sequence is exact: $\ldots \to H_qT_2(X,X) \xrightarrow{\alpha} H_q(TX) \xrightarrow{\sigma} H_{q+1}(TSX) \to H_{q+1}T_2(X,X) \to H_{q+1}(TX) \to \ldots$.*

Corollary 22. *There exists a spectral sequence wchich converges to H_*TSX and which satisfies*

- *E'_{pq} is equal to the complex $H_qTX \xleftarrow{H_q(\partial')} H_qT_2(X,X) \xleftarrow{H_q(\partial')} H_qT_3(X,X,X) \leftarrow \ldots$,*
- *the homomorphism $H_qTX = E'_{pq} \to H_{q+1}TSX$ is the same as σ.*

Definition 23. *We say that $X \in s\mathcal{C}$ is* trivial below n *if there exists $X' \in s\mathcal{C}$ which is homotopy equivalent to X and satisfies $X'_i = 0$ for $i < n$.*

Lemma 24. *If X is projective and $H_q(X) = 0$ for $q < n$, then X is trivial below n.*

Remark 25 (digression)**.** A *bisimplicial object* is $X_{p,q}$ with $X_{p,q} \to X_{r,s}$
for any $\alpha : [r] \to [p], \beta : [s] \to [q]$, which satisfy simplicial identities in both directions.

Every bisimplicial object gives us a bicomplex kX.

If X is bisimplicial, then it comes with a diagonal simplicial object $X_{k,k} \xrightarrow{(\alpha,\alpha)} X_{l,l}$ (where $\alpha : [l] \to [k]$).

Theorem 26 (Eilenberg-Zilber(-Cantier))**.** *There is a chain homotopy equivalence $k(X_{p,p}) \simeq \mathrm{tot}(kX_{p,q})$.*

Remark 27. Observe that $X_{p,p}$ is in degree p to the left and $p+p$ to the right.

Proposition 28. *Let $T : \mathcal{C}^l \to \mathcal{C}'$ be such that $T(\ldots, 0_j, \ldots) = 0$. Let, for $j = 1, \ldots, l$, $X^j \in s\mathcal{C}$ be trivial below n_j. Then $T(X^1, \ldots, X^l)$ is trivial below $n_1 + \ldots + n_l = n$ (therefore $H_qT(X^1, \ldots, X^l) = 0$ for $q < n$).*

Corollary 29. *If X is trivial below n, then the suspension homomorphism $\sigma : H_q(TX) \to H_{q+1}(TSX)$ is an isomorphism for $q < 2n$ and epimorphism for $q = 2n$.*

Remark 30. Observe that if X is a projective resolution of (A, n), then SX is a projective resolution of $(A, n+1)$.

Definition 31 (stable derived functors)**.** *$L_{q+n}(T\bullet, n)$ for $n > q$ is called the q-th stable derived functor of T, denoted $L^s_q T(\bullet)$.*

Remark 1. $C_*(\mathcal{C})$ does not have enough projective objects.

Theorem 2. *The sequence of functors* $\{H_i\}_{i=0}^{\infty}$ *gives us a* universal δ-functor *(takes short exact sequences to long exact sequences), i.e. if we have another sequence* T_i *such that* $T_0 = H_0$*, then* $\forall_i H_i^* = T_i^*$*.*

Lemma 3. *For a given* $C_* \in C_*(\mathcal{C})$ *there exists* $P_* \twoheadrightarrow C_*$ *such that* $H_i(P_*) = 0$ *for* $i > 0$*.*

Remark 4. If $p + q = n$, let $f_{pq} : X_{nn} \to X_{pq}$ be defined as $d^h_{p+1} \circ \ldots \circ d^h_n \circ d^v_0 \circ \ldots \circ d^v_0$, and then the Alexander-Whitney map $\sum_{p+q=n} f_{pq} : X_{nn} \to \bigoplus_{p+q=n} X_{pq}$ gives a chain homotopy equivalence of $k(X_{pp})$ and $\mathrm{tot}(kX_{pq})$.

Remark 5. We may take a projective simplicial resolution P_* of A of degree $n > i$, then $L^s_i T(A) = H_{n+i}(T(P_*))$.

Theorem 6. $\deg L_i T(\bullet, n) \leqslant \lfloor \frac{i}{n} \rfloor$.

Remark 7. Or theorem? Or proof? It is written that
$T((A, n) \oplus (B, n)) = T(A, n) \oplus T(B, n) \oplus V$ where V is trivial below $2n$.

Proposition 8. $\forall_i\, L^s_i T$ *is an additive functor.*

Proposition 9. *Let* $0 \to A \to B \to C \to 0$ *be exact in* $\mathcal{C}$*. Then we have a long exact sequence* $\ldots \to L^s_{q+1}T(C) \to L^s_q T(A) \to L^s_q T(B) \to L^s_q T(C) \to \ldots$*.*

Proposition 10. *If* $0 \to T' \to T \to T'' \to 0$ *is an exact sequence of functors, then we have a long exact sequence of functors* $\ldots \to L^s_{i+1}T'' \to L^s_i T' \to L^s_i T \to L^s_i T'' \to \ldots$*.*

Proposition 11. *Let* U *be an additive functor, then for any functor* T *we have*
$\mathrm{Hom}_{sth}(T, U) \simeq \mathrm{Hom}_{sth2}(L^s_0 T, U)$*.*

Applications of stable derived functors

Theorem 1. $T : \mathrm{R-mod} \to \mathrm{R-mod}$, *then*

$$L_i^s T(A) = \lim_n \pi_{i+n}(T(\tilde{R}[S^n] \otimes P_*)) \lim_n H_{i+n}(T(\tilde{R}[S^n] \otimes P_*)),$$

where S^n *is any simplicial model of* n*-sphere,* $\tilde{R}[\gamma] = {}^{R[\gamma]}\big/_{R[*]}$ *a simplicial set,* P_* *is any projective resolution of* A.

The limit is taken via suspension
$\pi_{i+n}(T(\tilde{R}[S^n] \otimes P_*)) \to \pi_{i+n+1}(S^1 \wedge T(\tilde{R}[S^n] \otimes P_*)) \to \pi_{i+n+1}(T(\tilde{R}[S^{n+1}] \otimes P_*))$.

In general for $S^1 \wedge F(X) \to F(S^1 \wedge X)$ one has to have for any $z \in S^1$, $F(X) \to F(S^1 \wedge X)$, $X \to S^1 \wedge X$, $x \to z \wedge x$.

One takes $R = \mathbb{Z}/p$ or $R = \mathbb{Z}$.

$L_i^s T(\mathbb{Z}/p) = \lim \pi_{i+n} T(\mathbb{Z}/p[S^n])$, but $\widetilde{\mathbb{Z}/p}[S^n] = K(\mathbb{Z}/p, n)$, $\tilde{\mathbb{Z}}[S^n] = K(\mathbb{Z}, n)$.

Stalk skewed gra..itions on $H^*(\bullet, \mathbb{Z}/p)$ is
$H_*^s(K(\mathbb{Z}/p), \mathbb{Z}/p) = H_*^s(K(\mathbb{Z}/p, n), \mathbb{Z}/p) = L_*^s \mathbb{Z}_p[.](\mathbb{Z}/p)$. (?)

Theorem 2. *Let* SP^i *be the* i*-th symmetric power functor, and* SP_p^i *the* p*-reduced* i*-th symmetric power, and* $SP_p^* = {}^{\bigoplus SP^i}\big/_{\langle x^p - 1 \rangle}$.

Then $L_*^s SP^*(\mathbb{Z}/p) = H_*^s(K(\mathbb{Z}), \mathbb{Z}/p)$, $L_*^s SP_p^*(\mathbb{Z}/p) = H_*^s(K(\mathbb{Z}/p), \mathbb{Z}/p)$.

Calculations: Let Γ be a category of functors T : finite pointed sets $\to \mathbb{Z}/p-$vect, $T(*) = 0$. $L \in \Gamma$ is defined as $L(X) = \widetilde{\mathbb{Z}/p}[X]$.

Lemma 3. *Let* $T : \mathbb{Z}/p-\mathrm{vect} \to \mathbb{Z}/p-\mathrm{vect}$. *Then* $L_i^s T(\mathbb{Z}/p) = \mathrm{Tor}_i^\Gamma(L^*, T \circ L)$, *where* $L^*(X) = L(X)^*$, *and*

I have found these notes useful in understanding derived functors.

Some detailed constructions are here, and some worked out examples are here in section 2.2.