

Theorem 1. Assume T is of degree $\leq k$, $A \in \text{Ob}(\mathcal{C})$ is of projective dimension $\leq n$, then $L_q T(A, n) = 0$ for $q > k(r + n)$.

Lemma 2. Let T be as above and $X \in s\mathcal{C}$ such that $(NX)_i = 0$ for $i > m$. Then $N(TX)_i = 0$ for $i > km$.

Definition 3 (suspension). $SA = \text{coker}(A \rightarrow CA)$, or $(SA)_q = A_{q-1}$ and $d^{SA} = -d^A$.

Corollary 4. We have an exact sequence $0 \rightarrow A \xrightarrow{\kappa} CA \xrightarrow{\pi} SA \rightarrow 0$.

Definition 5. Let $X \in s\mathcal{C}$. Define cone and suspension of X by the formulas $CX = KCNX$, $SX = KSNX$.

Remark 6. We have an exact sequence (exact on each level) $0 \rightarrow X \xrightarrow{\kappa} CX \xrightarrow{\pi} SX \rightarrow 0$.

Applying T we get (not necessarily exact) $0 \rightarrow TX \xrightarrow{T(\kappa)} T(CX) \xrightarrow{T(\pi)} T(SX) \rightarrow 0$.

Remark 7. Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a sequence in $C_*(\mathcal{C})$ such that $g \circ f = 0$ and B is contractible, i.e. we have $s_q : B_q \rightarrow B_{q+1}$ such that $d^B s + s d^B = \text{id}$. Then $gsf : A \rightarrow C$ gives a chain map $SA \rightarrow C$ and hence a map $H_q(A) \rightarrow H_{q+1}(C)$.

Theorem 8. $H(gsf)$ does not depend on the choice of s .

Definition 9 (suspension homomorphism). The map $\sigma : H_q(TX) \rightarrow H_{q+1}(TSX)$ induced by κ and π is called a suspension homomorphism.

Proposition 10. σ defines a natural transformation of functors.

Proposition 11. Assume T additive, then $0 \rightarrow T(X) \rightarrow T(CX) \rightarrow T(SX) \rightarrow 0$ exact and we have a long exact sequence of homology groups: $\dots \rightarrow 0 \rightarrow H_{q+1}(TSX) \rightarrow H_q(TX) \rightarrow 0 \rightarrow \dots$, and σ is the inverse of the map in the middle.

Definition 12. Let $T_p^d(A) = T_p(A, \dots, A)$ (d means diagonal).

Definition 13. Define $d_i = \rho \circ T(\alpha'_i) \circ \lambda : T_p^d(A) \rightarrow T_{p-1}^d(A)$, where λ monomorphism $T_p^d(A) \rightarrow T(A \oplus \dots \oplus A)$, ρ epimorphism $T(A \oplus \dots \oplus A) \rightarrow T_{p-1}^d(A)$, and $d'_j : \bigoplus_{i=1}^p A \rightarrow \bigoplus_{i=1}^{p-1} A$, equal to $(\text{id}, \dots, \text{id}, (\text{id} + \text{id})_j, \text{id}, \dots, \text{id})$.

Definition 14. Let $X \in s\mathcal{C}$. Define a sequence of simplicial objects in \mathcal{C}' :

$$\mathcal{T}X = \left(T_1^d(X) \xleftarrow{\partial'} T_2^d(X) \leftarrow T_3^d(X) \leftarrow \dots \right), \quad \partial' = \sum_{i=1}^{p-1} (-1)^i d_i.$$

Remark 15. $\partial' \circ \partial' = 0$.

Corollary 16. Therefore $\mathcal{T}X$ gives a bicomplex

$$(\mathcal{T}X)_{p,q} = T_p^d(X_q)$$

with horizontal differentials ∂' and vertical differentials from kX .

Proposition 17. *We have an embedding $i : kTX = (TX)_{1,*} \hookrightarrow \text{Tot}(TX)$ and it is a chain map of degree 1.*

Theorem 18. *There is a natural isomorphism $\omega : H\text{tot}(TSX) \simeq H(TSX)$ such that for*

$$\begin{array}{ccc} & H_{q+1}(TSX) & \\ \sigma \nearrow & \uparrow \omega & \\ H_q TX & & \\ \searrow i & & \\ & H_q(TX) & \end{array}$$

any q the diagram commutes:

Definition 19 (bar construction). TX is called the bar construction for T .

Corollary 20. *If T is additive, then σ is an isomorphism.*

Corollary 21. *If T is of degree 2, then there exists a morphism β such that the sequence is exact: $\dots \rightarrow H_q T_2(X, X) \xrightarrow{\alpha} H_q(TX) \xrightarrow{\sigma} H_{q+1}(TSX) \rightarrow H_{q+1} T_2(X, X) \rightarrow H_{q+1}(TX) \rightarrow \dots$*

Corollary 22. *There exists a spectral sequence which converges to $H_* TSX$ and which satisfies*

- E'_{pq} is equal to the complex $H_q TX \xleftarrow{H_q(\partial')} H_q T_2(X, X) \xleftarrow{H_q(\partial')} H_q T_3(X, X, X) \leftarrow \dots$,
- the homomorphism $H_q TX = E'_{pq} \rightarrow H_{q+1} TSX$ is the same as σ .

Definition 23. We say that $X \in s\mathcal{C}$ is trivial below n if there exists $X' \in s\mathcal{C}$ which is homotopy equivalent to X and satisfies $X'_i = 0$ for $i < n$.

Lemma 24. *If X is projective and $H_q(X) = 0$ for $q < n$, then X is trivial below n .*

Remark 25 (digression). A bisimplicial object is $X_{p,q}$ with $X_{p,q} \rightarrow X_{r,s}$ for any $\alpha : [r] \rightarrow [p], \beta : [s] \rightarrow [q]$, which satisfy simplicial identities in both directions.

Every bisimplicial object gives us a bicomplex kX .

If X is bisimplicial, then it comes with a diagonal simplicial object $X_{k,k} \xrightarrow{(\alpha, \alpha)} X_{l,l}$ (where $\alpha : [l] \rightarrow [k]$).

Theorem 26 (Eilenberg-Zilber(-Cantier)). *There is a chain homotopy equivalence $k(X_{p,p}) \simeq \text{tot}(kX_{p,q})$.*

Remark 27. Observe that $X_{p,p}$ is in degree p to the left and $p + p$ to the right.

Proposition 28. *Let $T : \mathcal{C}^l \rightarrow \mathcal{C}'$ be such that $T(\dots, 0_j, \dots) = 0$. Let, for $j = 1, \dots, l$, $X^j \in s\mathcal{C}$ be trivial below n_j . Then $T(X^1, \dots, X^l)$ is trivial below $n_1 + \dots + n_l = n$ (therefore $H_q T(X^1, \dots, X^l) = 0$ for $q < n$).*

Corollary 29. *If X is trivial below n , then the suspension homomorphism $\sigma : H_q(TX) \rightarrow H_{q+1}(TSX)$ is an isomorphism for $q < 2n$ and epimorphism for $q = 2n$.*

Definition 30 (stable derived functors). $L_{q+n}(T\bullet, n)$ for $n > q$ is called the q -th stable derived functor of T , denoted $L_q^s T(\bullet)$.