Discrete Maths

15

Fundamental counting principle

- If there are m ways to perform a task and n ways to perform another task

~~then there are~~

$m \times n$ ways ~~of~~ doing both of them.

Ex. Find the numbers of solutions to

$$x_1 + x_2 + x_3 + x_4 = 17$$

~~where~~, $0 \le x_1 \le 2$

$0 \le x_2 \le 5$

$0 \le x_3 \le 5$

$2 \le x_4 \le 6$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ [0,1,2] \times & [0,1,2,3,4,5] \times & [0,1,2,3,4,5] \times & [2,3,4,5,6] \end{array}$$

$$\cdots + x^{17} + \cdots$$

$(1+x+x^2)$ $\quad 0 \le x_1 \le 2$

x^0

$(1+x+x^2+x^3+x^4+x^5) \quad 0 \le x_2 \le 5$

x^0

$0 \le x_3 \le 5$

1	5	5	6
2	5	5	5
2	4	5	6
2	5	4	6

$x^2+x^3+x^4+x^5+x^6 \quad 2 \le x_4 \le 6$

$$(1+x+x^2)\cdot(1+x+x^2+x^3+x^4+x^5)^2(x^2+x^3+x^4+x^5+x^6)$$

$$= x^{18}+4x^{17}+10x^{16}+19x^{15}+\underline{\quad}4x^3\quad+x^2$$

Co-efficient of $(4)x^{17}$ term is answer $\underline{4}$.

Ex

Find the number of solutions of

$$e_1 + e_2 + e_3 = 17$$

$$2 \le e_1 \le 5, \quad 3 \le e_2 \le 6, \quad 4 \le e_3 \le 7$$

$$(x^2 + x^3 + x^4 + x^5) \cdot (x^3 + x^4 + x^5 + x^6) \cdot (x^4 + x^5 + x^6 + x^7)$$

$$x^2(1 + x + x^2 + x^3) \cdot x^3(1 + x + x^2 + x^3) \cdot x^4(1 + x + x^2 + x^3)$$

$P(x)$

$$= x^9 (1 + x + x^2 + x^3)^3$$

$$= x^9 \left(\frac{1 - x^{4}}{1 - x}\right)^3 \quad \text{\# terms } 4$$

$$= x^9 (1 - x^4)^3 (1 - x)^{-3}$$

~~G.P~~ common ratio r

a first term 1

n How many terms 4

$$\Rightarrow \frac{a(1 - r^n)}{1 - r}$$

The purpose here is to avoid exhaustive product.

We are only interested in

co-efficient of x^{17} all remaining can be ~~ignored.~~

P_1		P_2
0	$\binom{3}{0}(-1)^0 x^9$	$(-1)^0\binom{2}{2}(-1)^0 x^0$
	$+$	$+$
1	$\binom{3}{1}(-1)^1 x^{13}$	$(-1)^1\, {}^3C_2\, (-1)^1 x^1$
	$+$	$+$
2	$\binom{3}{2}(-1)^2 x^{17}$	$(-1)^2\binom{4}{2}(-1)^2 x^2$
	$+$	$+$
3	$\binom{3}{3}(-1)^3 x^{21}$	$(-1)^3\binom{5}{2}(-1)^3 x^3$

$$(x^9 - 3x^{13} + 3x^{17} - x^{21}) \times (1 + 3x + 6x^2 + 10x^3)$$

$$\cdots + 3x^{17} + \cdots$$

~~But we still~~ can do better $\Longrightarrow$

$(a+b)^n$

~~relation of~~

a and b

$\left|\frac{a}{b}\right| < 1$

$\Rightarrow$

choose a, b as per above rule.

$b^{-n}\left(\frac{a}{b}+1\right)^{-n}$

$\Rightarrow b^{-n}(x+1)^{-n}$

Remember

diverge

converge

series will never converge in $x > 1$ ∞.

$|x| < 1$

$\therefore \left|\frac{a}{b}\right| < 1$ $\quad \frac{a}{b} \Rightarrow x$

$(1-x^4)^3$

$= x^9 \cdot \sum_{r=0}^{3} \binom{3}{r} (1)^{n} (-x^4)^{r}$

a b

$a^n b^{n-r}$

$(1-x)^{-3}$

$|x|<1$

$$\boxed{\cdot \sum_{r=0}^{\infty} \binom{-3}{r} (-x)^r}$$

a	b
1	$-x$
$-x$	1

$= x^9 \cdot \sum_{r=0}^{3} \binom{3}{r} (-1)^r x^{4r+9}$ ($\because$ -1 took out)

$$\boxed{\sum_{r=0}^{\infty} (-1)^r \, {}^{r+2}C_r \, (-1)^r x^r}$$

$|a| < |b|$

b is 1

a is $-x$

$\therefore |x|<1$

$|x|<1$

$|a| < 1, |x| < 1$

$$\boxed{|b| > |a|}$$

$\therefore$ the Bi-exp is (extended) written in form of the Bi- ordinary/general

$\left|\frac{a}{b}\right| < 1$

$|a| < |b|$

P.S. (side note)

Proof Relation between ordinary binomial expansion (+ve n) to extended binomial expansion (-ve n)

$$\binom{n}{r} \Rightarrow \frac{n!}{(n-r)!\, r!} \qquad +ve\ n$$

$$\binom{-n}{r} = \frac{(-n)(-n-1)(-n-2)\cdots(-n-(r-1))}{r!}$$

$$= (-1)^r \cdot \frac{n(n+1)(n+2)\cdots(n+(r-1))}{r!}$$

$$= (-1)^r \frac{(n-1)!\; n(n+1)(n+2)\cdots \times (n-(r-2)) \times (n+(r-1))}{(n-1)!\; r!}$$

see right to left

$$(-1)^r = (-1)^r \frac{1\cdot 2\cdot 3 \cdots (n-1)\cdot n\cdot(n+1)(n+2)\cdots \times(n-(r-2))\times(n-(r-1))}{(n-1)!\; r!}$$

$$= \frac{(n-(r-1))!}{(n-1)!\, r!}$$

$$\binom{-n}{r} = (-1)^r \binom{n+r-1}{r}$$

Extended binomial expansion

Ordinary/regular binomial ~~expansion~~.

P1

$x^m \times x^n \Rightarrow x^{m+n}$

$$= \sum_{r=0}^{3} \binom{3}{r} (-1)^r \cdot x^{4r+9} \times$$

P2

$$\sum_{r=0}^{3} (1)^r \; {}^{r+2}C_2 \; (-1)^{\cancel{r}} \; x^r$$

Ignoring all other terms than x^{17} with special r ~~range~~ values.

	P1	P2
r	(0 ,	8)
r	(1 ,	4)
r	(2 ,	0)

$\Rightarrow (1)^{10}C_2$

$\Rightarrow \left((3)(-1) \cdot {}^6C_2 \right)$

$\Rightarrow \left({}^3C_2 \cdot {}^2C_2 \right)$

P1	P2	
$x^9 \times$	$\cdot\, x^{8}$	$\Rightarrow x^{17}$
x^{4+9}, x^{13}	$\times\, x^4$	x^{17}
$x^{4\times 2+9}$, x^{17}	$\times\, x^0$	x^{17}

$${}^{10}C_2 - 3 \cdot {}^6C_2 + {}^3C_2$$

of term x^{17}

$= 45 - 45 + 3 \Rightarrow 3.$

OGF (combination)

ordinary $\displaystyle\sum_{r=0}^{n} a_r x^r$

$^nP_r = {}^nC_r \cdot r!$

EGF (Permutation)

exponential

$\displaystyle \cancel{\sum_{r=0}^{n} A_r x^r} \Rightarrow \sum_{r=0}^{n} \frac{r! \cdot a_r \, x^r}{r!}$

[multiply]
[divide by $r!$]

$\cancel{r! \cdot a_r} = \cancel{A_r}$

$\cancel{a_r} \; r!$ $\Rightarrow \displaystyle\sum_{r=0}^{n} A_r \frac{x^r}{r!}$

Permutation

$$e^x = \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \text{—} \quad \sim$$

Exponen

```
/*
Find the number of solutions to
x1+x2+x3+x4=17 where 0<=x1<=2;0<=x2<=5;0<=x3<=5;2<=x4<=6;
1 5 5 6
2 4 5 6
2 5 4 6
2 5 5 5
 Total solutions 4 . Improve below program for efficiency.
*/
#include <stdio.h>
int main()
{
 int x1,x2,x3,x4;
 int count=0;
 int requiredSum=17;
 for(x1=0;x1<=2;x1++)
 {
        for(x2=0;x2<=5;x2++)
         {
                for(x3=0;x3<=5;x3++)
                {
                        for(x4=2;x4<=6;x4++)
                        {
                                if((x1+x2+x3+x4)==requiredSum)
                                {
                                        count++;
                                        printf("\n%d %d %d %d", x1,x2,x3,x4);
                                }


                        }

                }

         }
 }
 printf("\n Total solutions %d ",count);
}
```

You can use your brain to solve this via summing up the highest and follow subtracting from first x1 range and forward increase back and forth.

i.e. 2+5+5+6 = 18

So, first solution can be
1 5 5 6 = 17
Next increase in x1 and decrease from x2, x3, x4 one by one
2 4 5 6
2 5 4 6
2 5 5 5