

Discrete Maths

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Ex Find the no. of solutions of
 $x_1 + x_2 + x_3 + x_4 + x_5 = 15$

$$1 \leq x_1 \leq 5, \quad 1 \leq x_2 \leq 5, \quad \overset{x_3 \geq 2}{2} \leq x_3, \quad \overset{x_4 \geq 2}{2} \leq x_4, \quad \overset{x_5 \geq 2}{2} \leq x_5$$

$$[1, 2, 3, 4, 5] \quad \text{for } a, b \quad [2, 3, \dots] \quad \text{for } c, d, e$$

$$(x^1 + x^2 + x^3 + x^4 + x^5) \cdot (x^2 + x^3 + \dots)^3$$

$$x^2 (1 + x + x^2 + x^3 + x^4) \cdot x^6 (1 + x + x^2 + \dots)^3$$

Find the co-efficient of x^{15} (series term)

Applying Geometric Progression $S_n = a \left(\frac{1 - r^n}{1 - r} \right)$

$$x^2 \left(\frac{1 - x^5}{1 - x} \right)^2 \cdot x^6 \left(\frac{1}{1 - x} \right)^3 \quad |x| < 1$$

$$= x^2 (1 - x^5)^2 (1 - x)^{-2} \cdot x^6 \cdot (1 - x)^{-3}$$

$$= x^8 \cdot (1 - x^5)^2 \cdot (1 - x)^{-5}$$

$$= x^8 \cdot (1 - 2x^5 + x^{10})$$

$$\sum_{r=0}^{\infty} \binom{-5}{r} (-x)^r$$

$$(a+b)^n = \sum_{r=0}^n \binom{n}{r} a^r b^{n-r}$$

$$= (x^8 - 2x^{13} + x^{18}) \cdot \sum_{r=0}^{\infty} \binom{-5}{r} (-1)^r x^r$$

$$= (x^8 - 2x^{13} + x^{18})$$

$$\cdot \sum_{r=0}^{\infty} \binom{-5}{r} (-1)^r x^r$$

$$\binom{-5+r-1}{r} \cdot (-1)^r \cdot x^r$$

convert
ext to ordinary

$$\binom{-n}{r} = (-1)^r \binom{n+r-1}{r}$$

$$= (x^8 - 2x^{13} + x^{18}) \cdot \sum_{k=0}^{\infty} \binom{r+k}{4} x^k \quad \text{where } n = r+k$$

Note that we only want coefficient of x^{15} .

$$\begin{aligned} & \therefore (-1)^k \cdot (-1)^r = (-1)^{r+k} = (-1)^{n-k} \\ & = (-1)^{r+k} = (-1)^{r+k} \end{aligned}$$

Let's do smart work

$$x^8 - 2x^{13} + x^{18}$$

↓

$$8 + 7 = 15$$

hence

$$\binom{r+k}{4} x^{15}$$

$$\Rightarrow \binom{11}{4} x^{15}$$

$$13 + 2 = 15$$

$$k = 2$$

$$-2 \cdot \binom{6}{4} x^{15}$$

$$\Rightarrow \binom{11}{4} - 2 \cdot \binom{6}{4} \Rightarrow 330 - 150 = 180$$

Ex In how many ~~different~~ ways can eight identical cookies be distributed amongst three distinct children, provided each child receives at least ~~two~~ cookies and no more than four cookies?

$$c_1 + c_2 + c_3 = 8$$

$$2 \leq c_1, c_2, c_3 \leq 4$$

$$[2, 3, 4]$$

$$[2, 3, 4]$$

$$[2, 3, 4]$$

$$(x^2 + x^3 + x^4)^3$$

We need to find coefficient of x^8 in the series.
 smartly.

$$= 6$$

Ex How many ways we choose a committee of 9 members from 3 political parties so that no party has absolute majority in committee?

Define: "Absolute majority"

3 Parties P_1, P_2, P_3

9 members $m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8, m_9$

when said 9 members from 3 political parties, do NOT assume

3 per each

~~3 members per party.~~

P_1, P_2, P_3

if 3, 3, 3 all same

2 3 4 P_3 has majority

4 4 1 No party has absolute majority

because both P_1 and P_2
are having max 4.

so, we have to

~~have limit party of 1 to 4 only.~~
members

$$p_1 + p_2 + p_3 = 9$$

$$1 \leq p_1, p_2, p_3 \leq 4$$

$$[1, 2, 3, 4] \quad [1, 2, 3, 4] \quad [1, 2, 3, 4]$$

$$(x + x^2 + x^3 + x^4)^3 \leftarrow 3 \text{ parties}$$

we need to find co-efficient of x^9 .

Answer 10

Exponential generating function

Background

Selection
Combination
order is of no importance

Arrangements
permutation

Order is important.

Ex (A)

In How many ways
we can choose 3 letters when the letters
are to be chosen from unlimited supply of

a's
 e_1

a's and b's

b's
 e_2

$[0, 1, 2, 3]$

$[0, 1, 2, 3]$

$$e_1 + e_2 = 3$$

$$(x^0 + x^1 + x^2 + x^3)^2$$

aaa aab abb bbb

$$= \dots + 4x^3 + \dots$$

↑
4 ways.

Note that

aab is no
different than
aba.

Hence,

combination

but

~~how about~~

repetition allowed or
not?

Q 3

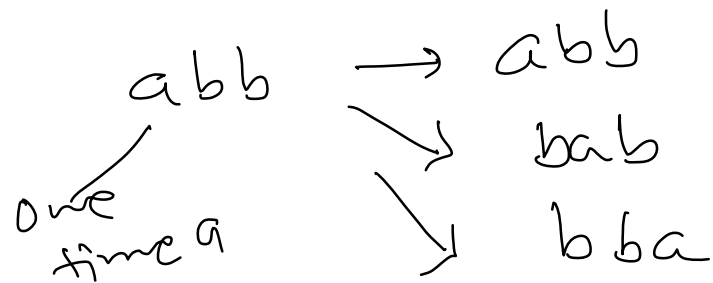
Find the number of ~~different~~ words
of three letters when the letters are
to be chosen from an unlimited supply of
a's and b's.

Word

↳ order of letter matters

i.e. saw vs war

$$\begin{array}{lcl}
 aaa \rightarrow aay & x^3 y^0 & \\
 bbb \rightarrow bbb & x^0 y^3 &
 \end{array}$$

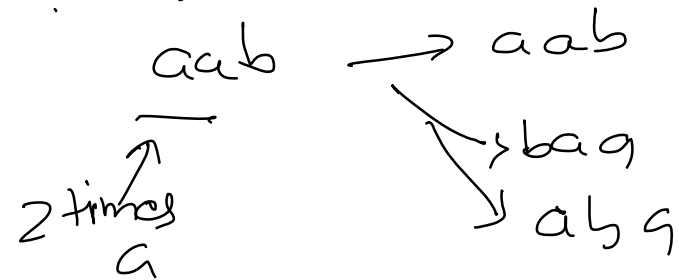


$$x^1 y^2$$

$$\begin{array}{r}
 3! \\
 \hline
 1! \cdot 2!
 \end{array}
 \Rightarrow 3$$

$$\begin{array}{r}
 3! \\
 \hline
 3! \cdot 0!
 \end{array}
 \Rightarrow 1$$

$$\begin{array}{r}
 3! \\
 \hline
 0! \cdot 3!
 \end{array}
 \Rightarrow 1$$



$$x^2 y^1$$

$$\begin{array}{r}
 3! \\
 \hline
 2! \cdot 1!
 \end{array}
 \Rightarrow 3$$

$$\frac{3!}{0! 3!} + \frac{3!}{1! 2!} + \frac{3!}{2! 1!} + \frac{3!}{3! 0!} =$$

$$1 + 3 + 3 + 1 = 8$$

$$3!_0 \left(\frac{1}{0! 3!_0} + \frac{1}{1! 2!} + \frac{\cancel{1}}{2! 1!} + \frac{\cancel{1}}{3! 0!} \right)$$

$$e_1 + e_2 = 3$$

$$[0, 1, 2, 3] \quad [0, 1, 2, 3]$$

$$\left(\frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} \right) \left(\frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} \right)$$

co-efficient of x^3

$$\frac{x^0 x^3}{0! 3!} + \frac{x^1 x^2}{1! 2!} + \frac{x^2 x^1}{2! 1!} + \frac{x^3 x^0}{3! 0!}$$

$$\Rightarrow x^3 \left(\frac{1}{0! 3!} + \frac{1}{1! 2!} + \frac{1}{2! 1!} + \frac{1}{3! 0!} \right)$$

$$\Rightarrow x^3 A_3$$

In general

$$\begin{aligned} \text{EGF}(x) &= A_0 x^0 + A_1 x^1 + A_2 x^2 + \dots \\ &= \sum_{r=0}^n A_r x^r \end{aligned}$$

$$= \sum_{r=0}^n r! A_r \frac{x^r}{r!}$$

(\therefore multiply and divide by $r!$)

$$= \sum_{r=0}^n a_r \frac{x^r}{r!}$$

Ans for counting problems

in the case of

EGF formula
categorical

permutations.

$$\text{EGF}(x) = \sum_{r=0}^n a_r \frac{x^r}{r!}$$

Definition EGF

Let $(a_0, a_1, a_2, \dots, a_n)$ be a symbolic representation of a sequence of a event, or let it be a sequence of numbers,

The function

$$f(x) = a_0 \frac{x^0}{0!} + a_1 \frac{x^1}{1!} + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \dots + \underbrace{a_n \frac{x^n}{n!}}_{\text{is}}$$

called Exponential Generating Function of the sequence $(a_0, a_1, a_2, \dots, a_n)$

$$f(x) = \sum_{r=0}^n a_r \frac{x^r}{r!} \quad \text{Note that there are } n+1 \text{ terms.}$$

Ex Find exponential ~~generating~~ function
for the number of r arrangements without
repetition of n objects.

$obj_1 \quad obj_2 \quad obj_3 \quad \dots \quad obj_n \quad n \text{ objects}$

$[0, 1] \quad [0, 1] \quad [0, 1] \dots [0, 1]$
~~not selected~~ ~~selected~~

$\#$ arrangements of n objects

$$[0, 1] \Rightarrow \frac{x^0}{0!} + \frac{x^1}{1!} \Rightarrow 1 + x$$

n objects

$$(1+x) \cdot (1+x) \cdot (1+x) \dots (1+x) \quad n \text{ times}$$

$$EGF(x) = (1+x)^n$$

$$\therefore (a+b)^n = \sum_{r=0}^n \binom{n}{r} a^r b^{n-r}$$

$$(1+x)^n = \sum_{r=0}^n \binom{n}{r} x^r 1^{(n-r)}$$

$$= \sum_{r=0}^n \binom{n}{r} x^r$$

$$= \sum_{r=0}^n \frac{n!}{(n-r)! r!} x^r$$

$$= \sum_{r=0}^n \left(\frac{n!}{(n-r)!} \right) \frac{x^r}{r!}$$

numeric
function $q_r \rightarrow$

$$|x| < 1$$

$$\left| \frac{a}{b} \right| < 1$$

$$|a| < |b|$$

$$\frac{a}{b} \Rightarrow \frac{x}{1} \Rightarrow x$$

$$|x| < 1$$

ilk permutation

$$n P_2 \Rightarrow \frac{n!}{(n-2)!}$$



Ex Find the exponential generating function for the number of different arrangements of x objects from 4 different types of objects with each type of object appearing at least 2 and no more than 5 times.

$$\sum_{j \in [2, 3, 4, 5]} \alpha_j = \sum_{j \in [2, 3, 4, 5]} \alpha_j$$

$$p(x) = \left(\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} \right) 4^{\text{types}}$$

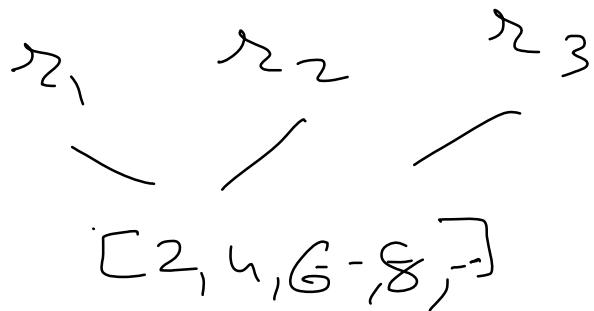
may find further closed form:

Ex: Find the exponential generating function for the no. of ways to place n distinct people into three rooms with at least 1 person in each room.

$$\begin{cases} x_1 + x_2 + x_3 = n \\ [1, 2, 3, \dots] \end{cases}$$

$$P(x) = \left(\frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right)^3$$

What if we want an even number of people
in each room?



$$p(x) = \left(\frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots \right)^3$$

Ex Find number of 2-digit quaternary sequences
(whose digits are 0, 1, 2, 3) with an
even no. of 0's and odd number of 1's.

$$e_0 + e_1 + e_2 + e_3 = 2$$

\hookrightarrow '0' digit happening even times

[0 times, 2 times, 4 times, 6 times, ...]

\hookrightarrow '1' digit happening odd times

[1 time, 3 times, 5 times, 7 times, ...]

e_2



repeat twice anything
will not change

total no. of
odds or
evens

Hence

[1, 2, 3, 4, 5, 6, 7, ...]

Any number of
times digit 0 or 1

But repeat twice the action.

$$p(x) = e_0 \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \right)$$

$$e_1 \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \right)$$

e_2 and e_3

$$\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right)^2$$

$$= \frac{1}{2} (e^x + e^{-x}) \frac{1}{2} (e^x - e^{-x}) (e^x)^2$$

$$= \frac{1}{4} (e^{2x} - e^{-2x}) e^{2x}$$

$$=$$

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Topics from maths to learn :

(1) partial fraction method

(2) Division method

(3) Stirling numbers of second kind

(4) Taylor series

(5) Maclaurian series