Discrete Maths

14

Ex.

$0, 1, 3, 7, 15, 31, \ldots$

$$OGF(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + \ldots$$

$$= 0 \cdot x^0 + 1 \cdot x^1 + 3x^2 + 7x^3 + 15x^4 + 31x^5 + \ldots$$

$$= (2^0 - 1)x^0 +$$

$$(2^1 - 1)x^1 +$$

$$(2^2 - 1)x^2 +$$

$$(2^3 - 1)x^3 +$$

$$(2^4 - 1)x^4 +$$

$$(2^5 - 1)x^5 + \ldots$$

$$= \left(2^0 x^0 + 2^1 x^1 + 2^2 x^2 + \cdots\right)$$

$$=$$

$$-\left(1 \cdot x^0 + 1 \cdot x^1 + 1 \cdot x^2 + 1 \cdot x^3 + \cdots\right)$$

$$= \left(1 + 2x + 2^2 x^2 + 2^3 x^3 + \cdots\right)$$

$$-\left(1 + x + x^2 + x^3 + x^4 + \cdots\right)$$

$$a_n = k^n$$

$$2^n \quad 2$$

$$= \left(2x + 2^2 x^2 + 2^3 x^3 + 2^4 x^4 + \cdots\right)$$

$$-\left(x + x^2 + x^3 + x^4 + \cdots\right)$$

$$2 \cdot 2^2$$

$$= x\left(\left(2^1 + 2^2 x + 2^3 x^2 + 2^4 x^3 + \cdots\right) - \left(1 + x^1 + x^2 + x^3 + \cdots\right)\right)$$

$$2 \cdot 2^3$$

$$a_1 = 1$$

$$= x\left(\frac{2}{1-2x} - \frac{1}{1-x}\right) = \text{Jono}\ \frac{x}{(1-x)(1-2x)}$$

$$\frac{k}{1-kx}$$

Ex LHS RHS

$$a_{n+1} = 2a_n + 1$$

$n \geq 0$, $a_0 = 0$: Given.

Let's work it out

1. LHS multiply by x^n and sum over for valid n.

2. RHS multiply by x^n and sum over for valid n

3. Compare Ans. from 1 & 2 come to result.

LHS

a_{n+1}

$$\sum_{n\geq 0} a_{n+1} x^n$$

$$= a_1 x^0 + a_2 x^1 + a_3 x^2 + a_4 x^3 + a_5 x^4 + \ldots$$

$\because n+1$ for $n=0$, a_0 but $a_0 \Rightarrow a_{n+1} \Rightarrow a_1$

$$= \frac{a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots}{x}$$

$$= \frac{1}{x}\left(\boxed{a_0 x^0 + a_1 x + a_2 x^2 + \ldots} - a_0 x^0\right)$$

$\because a_0 = 0$ given

$$= \frac{1}{x}\left(\underset{\text{OGF}}{A(x)}\right)$$

~~LHS work~~ (1)

RHS $2a_n + 1$

multiply by x^n and sum over

$$\sum_{n\geq 0}^{\infty} (2a_n + 1) x^n$$

$$= 2 \underbrace{\sum_{n\geq 0}^{\infty} a_n x^n} + \underbrace{\sum_{n\geq 0} 1 \cdot x^n}$$

$$= 2 \underset{\text{OGF}}{A(x)} + \frac{1}{1-x} \quad \text{— (2)}$$

$$a_n = 2a_n + 1$$

∴ from (1) & (2) LHS & RHS

$$\frac{1}{x} A(x) = 2A(x) + \frac{1}{1-x}$$

$$\frac{A(x) - 2x\,A(x)}{x} = \frac{1}{1-x}$$

$$\frac{A(x)(1-2x)}{x} = \frac{1}{1-x}$$

$$A(x) = \frac{1}{1-x} \cdot \frac{x}{1-2x}$$

MIB

$a_{n+1} = 2a_n + 1$ given $a_0 = 0$

Numeric function way

0	0	$\Rightarrow 0$	$1-1$	2^0-1
1	$2*0+1$	$\Rightarrow 1$	$2-1$	2^1-1
2	$2*1+1$	$\Rightarrow 3$	$4-1$	2^2-1
3	$2*3+1$	$\Rightarrow 7$	$8-1$	2^3-1
4	$2*7+1$	$\Rightarrow 15$	$16-1$	2^4-1
5	$2*15+1$	$\Rightarrow 31$	$32-1$	2^5-1

$\vdots$

$2^n - 1$ $\quad O(2^n)$

Tower of Hanoi

~~Given recurrence & get OGF closed form~~
solve it.

Show OGF closed form $\longrightarrow$ Numeric Function

$\downarrow$

O(__)

Ex2

$$\underbrace{a_{n+1}}_{LHS} = \underbrace{2a_n + n}_{RHS} \qquad n \geq 0,\ a_0 = 1 \text{ given.}$$

LHS is a_{n+1}

$$\sum_{n \geq 0} a_{n+1} x^n$$

$$\Rightarrow a_1 x^0 + a_2 x + a_3 x^2 + \cdots$$

$$\frac{1}{x}\left(A(x) - a_0 \right)$$

given $a_0 \Rightarrow 1$

$$= \frac{1}{x}\left(A(x) - 1 \right)$$

RHS $\quad 2a_n + n$

$$\sum_{n \geq 0} (2a_n + n) x^n$$

$$= (2a_0 + 0) x^0 + (2a_1 + 1) x^1 + (2a_2 + 2) x^2 + \cdots$$

$$= 2a_0 x^0 + 2a_1 x^1 \cancel{+ 2a_2 x^2 + 2a_3 x^3} + \cdots$$

$$+ \left(\underline{0 \cdot x^0 + 1 \cdot x^1 + 2 \cdot x^2 + \cdots} \right)$$

$$= 2 \left(a_0 x^0 + a_1 x^1 + a_2 x^2 + \cdots \right)$$

$$+ \left(\sum_{n \geq 0} n x^n \right)$$

$$= 2A(x) + \sum_{n \geq 0} n x^n$$

0, 1, 2, 3, ... lookup

$$\frac{x}{(1-x)^2}$$

$$= 2A(x) + \frac{x}{(1-x)^2}$$

A

By method of partial fractions to get series.

$$a_r \Rightarrow 2^{r+1} - r - 1 \Rightarrow O(\_\_)$$

Ex. 3

$$F_{n+1} = F_n + F_{n-1}$$

Given

$n \geq 1$, $F_0 = 0$, $F_1 = 1$

LHS

$$F_2 x + F_3 x^2 + F_4 x^3 + \cdots = \frac{F(x) - x}{x}$$

RHS $(F_1 x + F_2 x^2 + \cdots) + (F_0 x + F_1 x^2 + \cdots)$

$$= F(x) + x F(x)$$

$$\frac{F(x) - x}{x} = F + x F(x) \Rightarrow$$

$$F = \frac{x}{1 - x + x^2}$$

~~Ex~~

Given $\frac{x}{1-x+x^2}$, Find series.

stirling second . of number

$S(n, r)$