Discrete Maths

16

Ex Find the no. of solutions of

$$x_1 + x_2 + x_3 + x_4 + x_5 = 15$$

$x_3 \geq 2$ $x_4 \geq 2$ $x_5 \geq 2$

$1 \leq x_1 \leq 5$, $1 \leq x_2 \leq 5$, $2 \leq x_3$, $2 \leq x_4$, $2 \leq x_5$

a b c d e

$[1, 2, 3, 4, 5]$ for a, b $\rightarrow 2$ $[2, 3, \ldots]$

$(x^1 + x^2 + x^3 + x^4 + x^5)^2 \cdot (x^2 + x^3 + \ldots)^3$ for c, d, e

$x^2 (1 + x + x^2 + x^3 + x^4) \cdot x^6 (1 + x + x^2 + \ldots)^3$

Find the co-efficient of x^{15} (series term.)

Applying Geometric Progression $S_n = a \left(\frac{1 - r^n}{1 - r} \right)$

$$x^2 \left(\frac{1 - x^5}{1 - x} \right)^2 \cdot x^6 \left(\frac{1}{1 - x} \right)^3 \qquad |x| < 1$$

$$= x^2 (1 - x^5)^2 (1 - x)^{-2} \cdot x^6 \cdot (1 - x)^{-3}$$

$$= x^8 \cdot (1-x^5)^2 \cdot (1-x)^{-5}$$

$$= x^8 \cdot (1-2x^5+x^{10}) \cdot \sum_{r=0}^{\infty} \binom{-5}{r} (-x)^r$$

$\because$ $(a+b)^{-n}$

$$\sum_{r=0}^{\infty} \binom{-n}{r} \cdot a^r b^{n-r}$$

$$= (x^8 - 2x^{13} + x^{18}) \cdot \sum_{r=0}^{\infty} \binom{-5}{r} \underline{(-1)^r} \; x^r$$

Convert ext to ordinary

$$\binom{-n}{r} = (-1)^r \cdot {}^{n+r-1}C_r$$

$$= (x^8 - 2x^{13} + x^{18}) \cdot \sum_{r=0}^{\infty} \underline{(-1)^r \; {}^{+5+r-1}C_r \cdot (-1)^r \cdot x^r}$$

$$= (x^8 - 2x^{13} + x^{18}) \cdot \sum_{r=0}^{\infty} \binom{r+4}{4} x^r$$

$\because$ $^nC_r = {}^nC_{n-r}$

$\binom{r+4}{r} = \binom{r+4}{r+4-r} = \binom{r+4}{4}$

$\therefore$ $(-1)^r \cdot (-1)^r = (-1)^{2r} \Rightarrow 1$

~~Note that we only want~~

coefficient of x^{15}.

Let's do smart work

$x^8 - 2x^{13} + x^{18}$

$8 + 7 = 15$

hence

$\binom{7+4}{4} x^{15}$

$\Rightarrow \binom{11}{4} x^{15}$

$13 + 2 = 15$

$r = 2$

$-2 \binom{6}{4} x^{15}$

N/A as $18 + r > 15$, $r \geq 0$.

$\Rightarrow \binom{11}{4} - 2 \cdot \binom{6}{4} \Rightarrow 330 - 30$

$\Rightarrow 300$

Ex

In how many ~~different~~ ways can eight identical cookies be distributed amongst three distinct children, provided each child receives at least ~~two cookies~~ and no more than four cookies.?

$$c_1 + c_2 + c_3 = 8$$

$$2 \le c_1, c_2, c_3 \le 4$$

$[2, 3, 4]$ $[2, 3, 4]$ $[2, 3, 4]$

$$(x^2 + x^3 + x^4)^3$$

We need to find coefficient of $\cancel{x}^8$ in the series.

Smartify.

$= 6$

Ex How many ways we choose a committee of 9 members from 3 political parties so that no party has absolute majority in committee?

Define: "Absolute majority"

3 Parties P_1, P_2, P_3

9 members $m_1, m_2, 3 \cdot$ ~~m~~ m_4, m_5, m_6 m_7, m_8, m_9

when said 9 members from 3 political parties, do NOT assume

3 per each

~~3 member~~ per party.

	P_1	P_2	P_3	
if	3,	3,	3	all same
	2	3	4	P3 has majority
	4	4	1	No parti has absolute majority because both P1 and P2 are having max 4.

so, we have to ~~have~~ limit party of 1 to 4 members only.

$P_1 + P_2 + P_3 = 9$

$1 \leq P_1, P_2, P_3 \leq 4$

[1, 2, 3, 4] [1, 2, 3, 4] [1, 2, 3, 4]

$(x + x^2 + x^3 + x^4)^3$ ← 3 parties

We need to find co-efficient of x^9.

Answer 10

Exponential generating function

Background

Selection
Combination
order is of no importance

Arrangements
permutation
order is important.

Ex (A)

In How many ways we can choose 3 letters when the letters are to be chosen from unlimited supply of a's and b's.

a's e_1 b's e_2

$[0, 1, 2, 3]$ $[0, 1, 2, 3]$

$$e_1 + e_2 = 3$$

$$(x^0 + x^1 + x^2 + x^3)^2$$

aaa aab abb bbb

$= \dots + 4x^3 + \dots$

4 ways. Note that aab is no different than aba.

Hence,

combination

but

~~How about~~

repeatation allowed or not?

Ex 13

Find the number of different words of three letters when the letters are to be chosen from an unlimited supply of a's and b's.

Word

↳ order of letter matters

i.e raw vs war

aaa → aaa $\quad x^3 y^0 \quad \frac{3!}{3!\,0!} \quad 1$

bbb → bbb $\quad x^0 y^3 \quad \frac{3!}{0!\cdot 3!} \quad 1$

abb → abb

→ bab

→ bba

one time a

aab → aab

→ baa

→ aba

2 times a

$x^1 y^2$

$$\frac{3!}{1!\,2!} \Rightarrow 3$$

$x^2 y^1$

$$\frac{3!}{2!\,1!} \Rightarrow 3$$

$$\frac{3!}{0!\,3!}+\frac{3!}{1!\,2!}+\frac{3!}{2!\,1!}+\frac{3!}{3!\,0!}=$$

$$1+3+3+1=8$$

$$3!\left(\frac{1}{0!\,3!}+\frac{1}{1!\,2!}+\frac{1}{2!\,1!}+\frac{1}{3!\,0!}\right)$$

$$e_1 + e_2 = 3$$

$[0, 1, 2, 3]$ $\quad$ $[0, 1, 2, 3]$

$$\left(\frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}\right)\left(\frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}\right)$$

co-efficient of x^3

$$\frac{x^0 \, x^3}{0! \, 3!} + \frac{x^1 \, x^2}{1! \, 2!} + \frac{x^2 \, x^1}{2! \, 1!} + \frac{x^3 \, x^0}{3! \, 0!}$$

$$\Rightarrow x^3 \left(\frac{1}{0!3!} + \frac{1}{1!2!} + \frac{1}{2!\,1!} + \frac{1}{3!\,0!} \right)$$

$$\Rightarrow x^3 A_3$$

In general

$$EGF(x) = A_0 x^0 + A_1 x^1 + x_2 x^2 + \cdots$$

$$= \sum_{r=0}^{n} A_r x^r$$

$$= \sum_{r=0}^{n} r!\, A_r \frac{x^r}{r!}$$

(∵ multiply and devide by $r!$)

$$= \sum_{r=0}^{n} a_r \frac{x^r}{r!}$$

Ans for counting ~~problems~~.

in the case of ~~permutations~~

EGF formula

ckotenelive

$$EGF(x) = \sum_{r=0}^{n} a_r \frac{x^r}{r!}$$

Definition EGF

Let $(a_0, a_1, a_2, \cdots, a_n)$ be a symbolic representation of a sequence of a event, or let it be a sequence of numbers,

The function

$$f(x) = a_0 \frac{x^0}{0!} + a_1 \frac{x^1}{1!} + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \cdots + a_n \frac{x^n}{n!}$$ is

called Exponential Generating Function of the sequence $(a_0, a_1, a_2, \cdots, a_n)$

$$f(x) = \sum_{r=0}^{n} a_r \frac{x^r}{r!}$$

Note that there are $n+1$ terms.

Ex Find exponential ~~generating~~ function for the number of r arrangements without repetition of n objects.

$Obj_1 \quad Obj_2 \quad Obj_3 \quad \ldots\ldots \quad Obj_n$ $\quad$ n objects

$[0, 1] \quad [0, 1] \quad [0, 1] \ldots\ldots \quad [0, 1]$

not selected $\quad$ selected

r arrangments of n objects

$$[0, 1] \Rightarrow \frac{x^0}{0!} + \frac{x^1}{1!} \Rightarrow 1 + x$$

n objects

$$(1+x) \cdot (1+x) \cdot (1+x) \ldots \cdot (1+x) \quad \text{n times}$$

$$EGF(x) = (1+x)^n$$

$\because (a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}$

$|x| < 1$

$\left|\frac{a}{b}\right| < 1$

$|a| < |b|$

$(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r 1^{(n-r)}$

$\frac{a}{b} \Rightarrow \frac{x}{1} \Rightarrow x$

$|x| < 1$

$= \sum_{r=0}^{n} \binom{n}{r} x^r$

$= \sum_{r=0}^{n} \frac{n!}{(n-r)!\, r!} \cdot x^r$

$= \sum_{r=0}^{n} \left(\frac{n!}{(n-r)!} \right) \frac{x^r}{r!}$

mnmoric function a_r $\longrightarrow$

it's permutation

$$ {}^{n}P_{r} \Rightarrow \frac{n!}{(n-r)!} $$

Ex Find ~~text~~ exponential generating function for the ~~[illegible]~~ different ~~arrangements~~ of r objects from 4 different types of objects with each type of object appearing at least 2 and no more than 5 times.

type # of obj_1 + # of obj_2 + # of obj_3 + # of obj_4 = k

[2, 3, 4, 5]

$$P(x) = \left(\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} \right)^{4 \text{ types}}$$

may find ~~further closed~~ form.

Ex: Find the exponential generating function for the no. of ways to place r distinct ~~people~~ into three rooms with at least 1 person in each room.

$$r_1 + r_2 + r_3 = r$$

$$[1, 2, 3, \dots]$$

$$P(x) = \left(\frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right)^3$$

What if we want an even number of people in each room?

r_1 $\quad$ r_2 $\quad$ r_3

$[2, 4, 6, 8, \ldots]$

$$P(x) = \left(\frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \right)^3$$

Ex Find number of r-digit quaternary sequences (whose digits are 0, 1, 2, 3) with an even no. of 0's and odd number of 1s.

$e_0 + e_1 + e_2 + e_3 = r$

'y' digit happening

even times

[0 times, 2 times, 4 time, 6 times, ...]

'y' digit happening odd times

[1 time, 3 time, 5 time, 7 time, ...]

e_2

e_3

repeat twice anything will not change total no. of odds or evens

Hence

[1, 2, 3, 4, 5, 6, 7, ...]

Any number of times digit 0 or 1

But repeat twice times the action.

e_0

$$P(x) = \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)$$

e_1

$$\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

e_2 and e_3

$$\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^2$$

$$= \frac{1}{2}\left(e^{x} + e^{-x}\right) \frac{1}{2}\left(e^{x} - e^{-x}\right) \left(e^{x}\right)^{2}$$

$$= \frac{1}{4}\left(e^{2x} - e^{-2x}\right) e^{2x}$$

$$=$$

Topics from maths to learn :

1. partial fraction method
2. Division method
3. stirling numbers of second kind
4. Taylor series
5. Maclaurian series