Discrete Maths

16

Ex Find the no. of solutions of

$$x_1+x_2+x_3+x_4+x_5=15$$

$x_3 \geq 2$ $x_4 \geq 2$ $x_5 \geq 2$

$1 \leq x_1 \leq 5$, $1 \leq x_2 \leq 5$, $2 \leq x_3$, $2 \leq x_4$, $2 \leq x_5$

a b c d e

$[1,2,3,4,5]$ for a, b $\to 2$ $[2,3,\dots\dots]$

$$(x^1+x^2+x^3+x^4+x^5)^2 \cdot (x^2+x^3+\dots)^3 \quad \text{for } c, d, e$$

$$x^2(1+x+x^2+x^3+x^4) \cdot x^6(1+x+x^2+\dots)$$

Find the co-efficient of x^{15} (series term.)

Applying Geometric Progression $S_n = a\left(\frac{1-r^n}{1-r}\right)$

$$x^2\left(\frac{1-x^5}{1-x}\right)^2 \cdot x^6\left(\frac{1}{1-x}\right)^3 \qquad |x|<1$$

$$= x^2(1-x^5)^2(1-x)^{-2} \cdot x^6 \cdot (1-x)^{-3}$$

$$= x^8 \cdot (1-x^5)^2 \cdot (1-x)^{-5}$$

$$= x^8 \cdot (1-2x^5+x^{10}) \cdot \sum_{r=0}^{\infty} \binom{-5}{r} (-x)^r$$

$$\because (a+b)^{-n} = \sum_{r=0}^{\infty} \binom{-n}{r} \cdot a^r b^{n-r}$$

$$= (x^8 - 2x^{13} + x^{18}) \cdot \sum_{r=0}^{\infty} \binom{-5}{r} \underline{(-1)^r}\, x^r$$

Convert ext to ordinary

$$\binom{-n}{r} = (-1)^r \cdot {}^{n+r-1}C_r$$

$$= (x^8 - 2x^{13} + x^{18}) \cdot \sum_{r=0}^{\infty} \underline{(-1)^r \; {}^{+5+r-1}C_r} \cdot (-1)^r \cdot x^r$$

$= (x^8 - 2x^{13} + x^{18}) \cdot \sum_{r=0}^{\infty} \binom{r+4}{4} x^r$ $\left(\because {}^nC_r = {}^nC_{n-r} \right)$

$\binom{r+4}{r} = \binom{r+4}{r+4-r} = \binom{r+4}{4}$

~~Note that we only want~~

coefficient of x^{15}.

$\therefore (-1)^r \cdot (-1)^r = (-1)^{2r} \Rightarrow 1$

Let's do smart work

$x^8 \cancel{-2x^{13}} + x^{18}$

$8 + 7 = 15$

hence

$\binom{7+4}{4} x^{15}$

$\Rightarrow \binom{11}{4} x^{15}$

$13 + 2 = 15$

$r = 2$

$-2 \binom{6}{4} x^{15}$

N/A as $18 + r > 15$, $r \geq 0$.

$\Rightarrow \binom{11}{4} - 2 \cdot \binom{6}{4} \Rightarrow 330 - 30$

$\Rightarrow 300$

Ex In how many different ways can eight identical cookies be distributed amongst three distinct children, provided each child receives at least two cookies and no more than four cookies.?

$$c_1 + c_2 + c_3 = 8$$

$$2 \leq c_1, c_2, c_3 \leq 4$$

$[2, 3, 4]$ $[2, 3, 4]$ $[2, 3, 4]$

$$(x^2 + x^3 + x^4)^3$$

We need to find coefficient of $\cancel{x}^8$ in the series.

Smartify.

$= 6$

Ex How many ways we choose a committee of 9 members from 3 political parties so that no parti has absolute majority in committee?

Define: "Absolute majority"

3 Parties P_1, P_2, P_3

9 members $m_1, m_2, 3 \cdot m_4, m_5, m_6$ m_7, m_8 m_9

when said 9 members from 3 political parties, do NOT assume 3 per each

~~3 members per party.~~

	P_1	P_2	P_3	
if	3,	3,	3	all same
	2	3	4	P_3 has majority
	4	4	1	No parti has absolute majority because both P_1 and P_2 are having max 4.

so, we have to ~~have limit party of 1 to 4 only~~ members.

$P_1 + P_2 + P_3 = 9$

$1 \le P_1, P_2, P_3 \le 4$

[1, 2, 3, 4]

$[1, 2, 3, 4]^3 \leftarrow$ 3 parties [1, 2, 3, 4]

$(x + x^2 + x^3 + x^4)^3$

we need to find co-efficient of x^9.

Answer 10

Exponential generating function

Background

Selection
Combination
order is of no importance

Arrangements
permutation
order is important.

Ex (A)

In How many ways we can choose 3 letters when the letters are to be chosen from unlimited supply of a's and b's.

a's
e_1

b's
e_2

$[0, 1, 2, 3]$ $[0, 1, 2, 3]$

$e_1 + e_2 = 3$

$(x^0 + x^1 + x^2 + x^3)^2$

aaa aab abb bbb

$= \cdots + 4x^3 + \cdots$

4 ways. Note that aab is no different than aba.

Hence,

combination

but

how ~~about~~

repeatation allowed or not?

Ex 3

Find the number of different words of three letters when the letters are to be chosen from an unlimited supply of a's and b's.

Word

↳ order of letter matters

i.e. raw vs war

aaa → aaa $\quad x^3 y^0 \quad \frac{3!}{3!\,0!} \quad 1$

bbb → bbb $\quad x^0 y^3 \quad \frac{3!}{0!\cdot 3!} \quad 1$

abb → abb

→ bab

→ bba

one time a

aab → aab

→ baa

→ aba

2 times a

$x^1 y^2$

$$\frac{3!}{1!\,2!} \Rightarrow 3$$

$x^2 y^1$

$$\frac{3!}{2!\,1!} \Rightarrow 3$$

$$\frac{3!}{0!\,3!} + \frac{3!}{1!\,2!} + \frac{3!}{2!\,1!} + \frac{3!}{3!\,0!} =$$

$$1 + 3 + 3 + 1 = 8$$

$$3!\left(\frac{1}{0!\,3!} + \frac{1}{1!\,2!} + \frac{1}{2!\,1!} + \frac{1}{3!\,0!} \right)$$

$$e_1 + e_2 = 3$$

$[0, 1, 2, 3]$ $[0, 1, 2, 3]$

$$\left(\frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}\right)\left(\frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}\right)$$

co-efficient of x^3

$$\frac{\overbrace{x^0 x^3}}{0!\,3!} + \frac{\overbrace{x^1 x^2}}{1!\,2!} + \frac{\overbrace{x^2 x^1}}{2!\,1!} + \frac{\overbrace{x^3 x^0}}{3!\,0!}$$

$$\Rightarrow x^3\left(\frac{1}{0!\,3!} + \frac{1}{1!\,2!} + \frac{1}{2!\,1!} + \frac{1}{3!\,0!}\right)$$

$$\Rightarrow x^3 A_3$$

In general

$$EGF(x) = A_0 x^0 + A_1 x^1 + x_2 x^2 + \cdots$$

$$= \sum_{r=0}^{n} A_r x^r$$

$$= \sum_{r=0}^{n} r! \, A_r \frac{x^r}{r!}$$ ~~(∵ multiply and devide by r!)~~

$$= \sum_{r=0}^{n} a_r \frac{x^r}{r!}$$

Ans for counting ~~problems~~.

in the case of ~~permutations.~~

EGF formula

c[illegible]

$$EGF(x) = \sum_{r=0}^{n} a_r \frac{x^r}{r!}$$

Definition EGF

Let $(a_0, a_1, a_2, \ldots, a_n)$ be a symbolic representation of a sequence of a event, or let it be a sequence of numbers,

The function

$$f(x) = a_0 \frac{x^0}{0!} + a_1 \frac{x^1}{1!} + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \ldots + a_n \frac{x^n}{n!}$$ is

called Exponential Generating Function of the sequence $(a_0, a_1, a_2, \ldots, a_n)$

$$f(x) = \sum_{r=0}^{n} a_r \frac{x^r}{r!}$$ Note that there are $n+1$ terms.

Ex Find exponential ~~generating~~ function for the number of r arrangements without repetition of n objects.

$Obj_1 \quad Obj_2 \quad Obj_3 \quad \text{----} \quad Obj_n$ n objects

$[0, 1] \quad [0,1] \quad [0,1] \text{----} \quad [0,1]$

not selected selected

r arrangments of n objects

$$[0,1] \Rightarrow \frac{x^0}{0!} + \frac{x^1}{1!} \Rightarrow 1 + x$$

n objects

$$(1+x) \cdot (1+x) \cdot (1+x) \cdots \cdot (1+x) \quad \text{n times}$$

$$EGF(x) = (1+x)^n$$

$$\because (a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}$$

$$(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r 1^{(n-r)}$$

$$= \sum_{r=0}^{n} \binom{n}{r} x^r$$

$$= \sum_{r=0}^{n} \frac{n!}{(n-r)!\, r!} \cdot x^r$$

$$= \sum_{r=0}^{n} \left(\frac{n!}{(n-r)!}\right) \frac{x^r}{r!}$$

mnmoric function a_r

$|x| < 1$

$\left|\frac{a}{b}\right| < 1$

$|a| < |b|$

$\frac{a}{b} \Rightarrow \frac{x}{1} \Rightarrow x$

$|x| < 1$

its permutation

$$nP_r \Rightarrow \frac{n!}{(n-r)!}$$

Ex Find the exponential generating function for the number of different arrangements of r objects from 4 different types of objects with each type of object appearing at least 2 and no more than 5 times.

$$\underbrace{\text{type}^{\#\text{ of}}\ obj_1}_{} + \overset{\#\text{ of}}{obj_2} + \overset{\#\text{ of}}{obj_3} + \overset{\#\text{ of}}{obj_4} = 8$$

$$[2, 3, 4, 5]$$

$$P(x) = \left(\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} \right)^{4 \text{ types}}$$

may find ~~further closed form.~~

Ex: Find the exponential generating function for the no. of ways to place r distinct people into three rooms with at least 1 person in each room.

$$r_1 + r_2 + r_3 = r$$

$$[1, 2, 3, \ldots]$$

$$P(x) = \left(\frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right)^3$$

What if we want an even number of people in each room?

r_1 r_2 r_3

$[2, 4, 6, 8, \ldots]$

$$P(x) = \left(\frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\right)^3$$

Ex Find number of r-digit quaternary sequences (whose digits are $0, 1, 2, 3$) with an even no. of 0's and odd number of 1s.

$e_0 + e_1 + e_2 + e_3 = r$

'0' digit happening even times

[0 times, 2 times, 4 time, 6 times, ...]

'1' digit happening odd times

[1 time, 3 time, 5 time, 7 time, ...]

e_2

e_3

repeat twice anything will not change total no. of odds as evens

Hence

[1, 2, 3, 4, 5, 6, 7, ...]

Any number of times digit 0 or 1

But repeat twice times the action.

e_0

$$p(x) = \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)$$

e_1

$$\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

e_2 and e_3

$$\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^2$$

$$= \frac{1}{2}\left(e^{x} + e^{-x}\right) \frac{1}{2}\left(e^{x} - e^{-x}\right) \left(e^{x}\right)^{2}$$

$$= \frac{1}{4}\left(e^{2x} - e^{-2x}\right) e^{2x}$$

$$=$$

Topics from maths to learn :

1. partial fraction method
2. Division method
3. stirling numbers of second kind
4. Taylor series
5. Maclaurian series