Discrete Maths

19

Ex r-digit quintary (quinary) sequences

that is sequence made up of digits

$\{0, 1, 2, 3, 4\}$

How many such possible that contain an even number of 1's.

r-digit sequence possible any 5 digits unlimited (with repetition)

total number of ways 5^r ← length of sequence

← $\{0, 1, 2, 3, 4\}$ 5

Because, 0 itself is considered even,

sequences having none 1's also counts because even number of 1's that is zero times 1's.

r-digit
sequences containing only 2,3,4 are anyways counted to the final total

final_total $\leftarrow 0$

final_total $\leftarrow$ final_total $+ 3^r \leftarrow 2,3,4$

out of remaining

$5^r - (3^r)$

half will have even number of 1's

final_total

$$\boxed{3^r + \frac{1}{2}(5^r - 3^r)}$$

[$\because$ out of 2^r half will always have even number of 1's]

Circular Permutation

1
3 2

3
2 1

2
1 3

1
2 3

3
1 2

2
3 1

Given

n 3

1 2 3

can be arranged like ← 6 circular ways.

But, are these really different?

Do you care for clockwise order vs anti-clockwise order?

linearly $n^{\{1,2,3\}}_{3}$ can be arranged $n!$ ways.

~~First~~ i)

123
132
213
231
312
321

$3! \Rightarrow 3 \times 2 \times 1 \Rightarrow 6$

Similarly, if clock & anticlock matters (i.e. ~~different~~)

$$\begin{array}{ccc} & 1 & \\ 3 & & 2 \end{array} \qquad \begin{array}{ccc} & 1 & \\ 2 & & 3 \end{array}$$

(keep only one) (keep only one)

two ways possible. 2

that is ~~[illegible]~~ $= \frac{6}{3}$

$$= \frac{3 \times 2 \times 1}{3}$$

$$= \frac{3!}{3}$$

$$= \frac{n!}{n} = \frac{n \times (n-1)!}{n} = (n-1)!$$

$$\frac{\text{total linear ways}}{n}$$

$$\Rightarrow \frac{n!}{n}$$

$$\Rightarrow (n-1)!$$

n

$\frac{3}{3} \Rightarrow 1$ keep 1, out of three

$\frac{4}{4} \Rightarrow 1$ keep 1 out of three

You got to devide the number by itself, if you want '1'.

Ex $n : 4$ $\{1, 2, 3, 4\}$ permutation

~~linear $4!$~~

circular $3!$ $\Rightarrow 6$

(1) ✓

1
4 2
3

4
3 1
2

3
2 4
1

2
1 3
4

1 2 3 4   4 1 2 3   3 4 1 2   2 3 4 1 $\Rightarrow$ 1, 2, 3, 4

(2) ✓

1
3 2
4

3
4 1
2

4
2 3
1

2
1 4
3

$\Rightarrow$ 1 2 4 3

(3) ✓

1
4 3
2

4
2 1
3

2
3 4
1

3
1 2
4

$\Rightarrow$ 1 3 2 4

(4)

1 2 3 4 | 2 4 1 3 | 4 3 2 1 | 3 1 4 2 $\Rightarrow$ 1 3 4 2

(5)

1 3 4 2 | 3 2 1 4 | 2 4 3 1 | 4 1 2 3 $\Rightarrow$ 1 4 2 3

(6)

1 2 4 3 | 2 3 1 4 | 3 4 2 1 | 4 1 3 2 $\Rightarrow$ 1 4 3 2

$$\frac{4!}{4} = (n-1)! = 3! = 6$$

What if both clockwise and anti-clockwise pattern is of equal importance?

1
4 2
3

$\approx$

1
2 4
3

1 2 3 4

1 4 2 3

1
3 2
4

$\approx$

1
2 3
4

1 2 4 3

1 3 4 2

1
4 3
2

$\approx$

1
3 4
2

1 3 2 4 1 4 2 3

$\boxed{\text{Total number of ways } 3}$

$$\frac{(n-1)!}{2} \Rightarrow \frac{3!}{2} \Rightarrow \frac{3\times2\times1}{2} \Rightarrow 3$$

~~formula~~ when clockwise & anti are same $\frac{1}{2}(n-1)!$

other examples: necklaces, round shape dining table and chairs.

Permutation involving (Regular) Non-circular

~~General formula for~~
the number of ways to place r colored balls in n boxes, where

$$\begin{array}{l} q_1 \text{ of these are one color} \\ + \; q_2 \text{ of these are second color} \\ + \; \vdots \\ + \; q_t \text{ of these are } t^{th} \text{ color} \\ \hline r \end{array}$$

Observe ~~Note~~ that
placement of r balls overall is not affected when q_1 internally ~~exchanged~~ permuted,
q_2 internally ~~exchanged~~ permuted
$\vdots$
q_t internally permuted.

That's why
the actual permutation ways will be

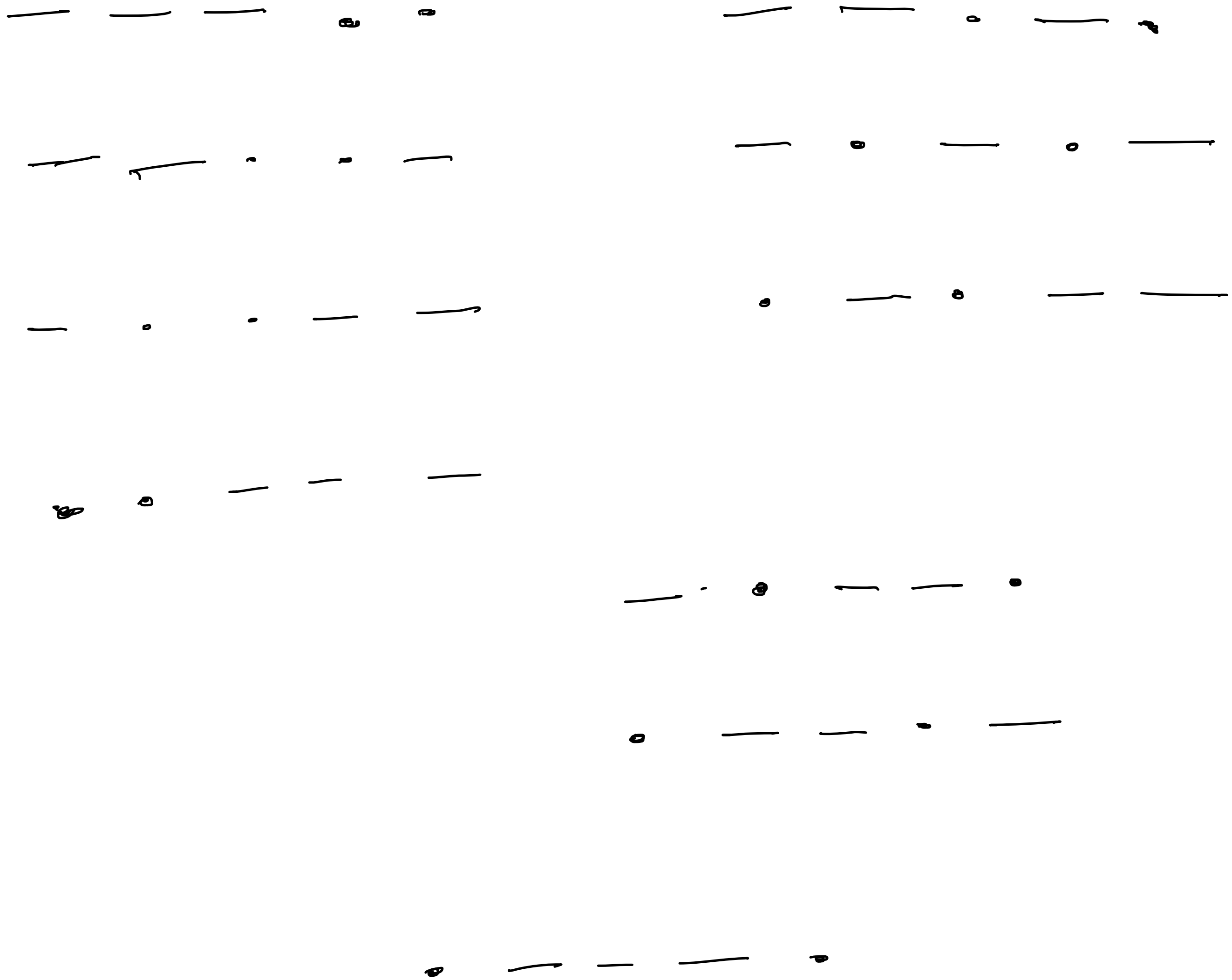
$$\frac{p(n, r)}{q_1!\, q_2!\, q_3! \ldots q_t!}$$

Example

Number of different messages that can be represented by 3 dashes and 2 dots

$$\frac{5!}{3!\, 2!} \Rightarrow 10$$

and not $5!$ itself.



To explore and learn

→ The ~~total~~ number of arrangements (permutations) of n different thing taken all at a time is $n!$

→ The ~~total~~ number of arrangements of n different things taken r at a time, in which a particular thing ALWAYS occurs

$$r \times {}^{n-1}P_{r-1}$$

→ The total number of permutations of n different things taken r at a time in which a particular thing never occurs

$$^{n-1}P_{r}$$

- The total number of permutation of n dissimilar things taken r at a time with~~out~~ repetitions

$$n^r$$

- the number of permutations of n things taken all at a time when p of them are alike and of one kind, q of them are alike and of second kind, all other things being different,

is

$$\frac{n!}{p! \times q!}$$

Combinations

Without repetitions

~~Number of way~~s

^nP_r (permutations)

comprises of two steps

1. Selecting/choosing r out of n
2. Arranging r

So, note that

~~Selec~~ting/choosing is combination

$$^nP_r = {}^nC_r \times r!$$

$$\frac{n!}{(n-r)!} = {}^nC_r \times r!$$

$$\therefore \quad {}^nC_r = \frac{n!}{(n-r)!\,r!}$$

$${}^nC_r = {}^nC_{n-r}$$

$${}^nC_0 = 1$$ choose / Select none (only one way)

Nothing at all

$${}^nC_n = 1$$ choose / Select all (only one way)

All of them

$${}^nC_1 = n$$ choose / select 1 out of n

n options

Order is of not importance

very much like group of things.

Explore and learn

- Number of combinations of n different things taken r at a time in which p particular things will ALWAYS occur is

$$^{n-p}C_{r-p}$$

- Number of ~~combinations~~ of n different things taken r at a time in which p particular things will NEVER occur

$$^{n-p}C_{r}$$

$$-\ {}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n$$

The number of ways in which $(m+n)$ things can be divided into two groups containing m & n things respectively

is $$\frac{(m+n)!}{m! \times n!}$$

Example

~~Suppose a housekeeper wants~~
to schedule spaghetti dinners
three times each week. How many ways?

3 dinners $\leftrightarrow$ 7 days

~~7 choose 3~~

$$\Rightarrow \; {}^{7}C_{3}$$

$$\Rightarrow \; \frac{7!}{(7-3)!\,3!}$$

$$\Rightarrow \; \frac{7 \times 6 \times 5 \times \cancel{4!}}{\cancel{4!} \times 3 \times 2 \times 1}$$

$$= 35$$

Example

How many diagonals does an octagon have?

$$^8C_2 - 8$$

$$\Rightarrow \frac{8 \times 7 \times 6!}{(8-2)! \times 2!} - 8$$

$$= \frac{\cancel{8}^4 \times 7}{\cancel{2}} - 8$$

$$= 28 - 8$$

$$= 20$$

$5+5+4+3+2+1+\cancel{0}+0$

p1 to except 2, 8 and self $8-3$ 5
3, 4, 5, 6, 7

p2 to except ~~p1, p3 and~~ self 5
4, 5, 6, 7, 8 $8-3$

p3 to except 2, 4 and self $8-3-1$
5, 6, 7, 8 4
Also, not to 1
as 1 to 3 done already

p4 to except 3, 5 and self $8-3-2 \Rightarrow 3$
6, 7, 8 4→1, 2 → done 4-2

- P_5 to except 4, 6, self

  7, 8

  5 to 1, 2, 3 done already

$$8 - 3 - (5 - 2)$$
$$= 8 - 3 - 3$$
$$= 2$$

- P_6 to except 5, 7, self

  8

  6 to 1, 2, 3, 4 done already

$$8 - 3 - (6 - 2)$$
$$= 8 - 3 - 4$$
$$= 8 - 7$$
$$= 1$$

- P_7 to ~~except 6, 8, self~~

  None

  7 to 1, 2, 3, 4, 5 done already

$$8 - 3 - (7 - 2)$$
$$= 8 - 3 - 5 = 0$$

- P_8 to except ~~7~~, self

  None!

  8 to 1, 2, 3, 4, 5, 6 done.

$$8 - 2 - (8 - 2)$$
$$= 8 - 2 - 8 + 2$$
$$= 0$$

Example

How many diagonals a decagon has?

Two points choosen a time creates one diagonal.

out of 10 points

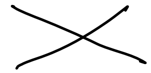
10 choose 2 except the border count

$$^{10}C_2 - 10 \Rightarrow \frac{10 \times 9}{2} - 10 \Rightarrow 45 - 10 \Rightarrow 35$$

$$\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 7 + & 7 + & 6 + & 5 + & 4 + & 3 + & 2 + & 1 + & 0 + & 0 \end{array} \Rightarrow 35$$

Example

How many intersection patterns are there



in decagon

given that no three (or more) diagonals meet at a single point.

Four points together prepares one time pattern

Hence, $^{10}C_4 \Rightarrow \dfrac{10 \times \cancel{9}^{\,3} \times \cancel{8} \times 7}{\cancel{4} \times \cancel{3} \times \cancel{2} \times 1} = 210$

Example: For a decagon, given no 3 ~~obi~~diagonals meet at single point, into how many line segments are the diagonals divided by their intersections?

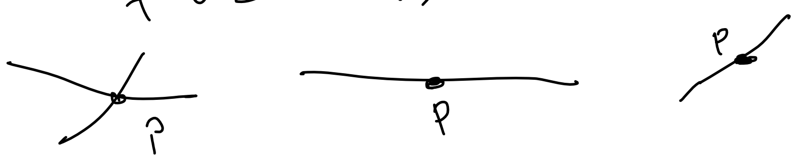
~~Total number of diagonals~~

$^{10}C_2 - 10 \Rightarrow 35$

Total ~~x~~ intersections $^{10}C_4 \Rightarrow 210$

but, ~~note that~~ ~~one~~ intesection point is falling on ~~2 diagonals.~~

that is



Thats why actually ~~both~~ points counted twice.

~~that~~ is $210 \times 2 \Rightarrow 420$

Also, observe that

if on a given line there is one point

then divides into two segments.

that is K points divides into K+1 segments

$$35 \left(1 + \underset{\uparrow}{12} \right) \Rightarrow 455$$

diagonals number of points per diagonal

35 di — 420 point

1 ?

$$\frac{420}{35} \Rightarrow 12$$

Book says

$35 + 2 * 210$

$= 455$

$35(1 + 2*6)$

$= 35(1+12)$

455 segments

Example

~~Given 11 senators~~

→ To select committee of 5 members

11 choose 5 $\quad {}^{11}C_5 \Rightarrow 462$

→ To select committee of 5 members such that a particular senator, i.e A is always ~~selected~~

solution:

Then actual selection is of only 4 members from 10 senators

10 choose 4 $\quad {}^{10}C_4 \Rightarrow 210$

→ i.e. a particular senator A is always EXCLUDED

that means

out of 11 only 10 are really considered.

we still have to choose 5

10 choose 5

$^{10}C_5 = 252$

How many ways, we can select a committee

→ of five members so that

at least one ~~of the senator A~~

and senator B will be included?

3 various solutions. ⇒⇒⇒

~~Solution~~ 1

Including both A as well as B

2 fixed.

11 − 2 ⇒ 9   9 choose 3 ⇒ 84

5 − 2 ⇒ 3

Including A and excluding B

11 − 2 ⇒ 9   9 choose 4 ⇒ 126

5 − 1 ⇒ 4

~~Including B and excluding A~~

9 choos 4 ⇒ 126

Total: 84 + 126 + 126 = 336

Solution #2

Total number of Committees excluding both A AND B

9 choose 5 $\Rightarrow$ 126

same as 9 choose 4

Total

11 choose 5 $\Rightarrow$ 462

$$
\begin{array}{r}
462 \\
-126 \\
\hline
336
\end{array}
$$

~~Solution~~ 3

Apply principle of inclusion and exclusion

A_1 set A → include senators A

A_2 set B → includes senator B

$$|A_1| = C(10, 4) = 210$$

11−1

$$|A_2| = C(10, 4) = 210$$

$$|A_1 \cap A_2| = C(9, 3) = 84$$

$$|A_1 \cup A_2| = 210 + 210 - 84 = 336$$

~~Suppose~~

we are to place r balls of the same color in a n numbered boxes.

Allowing as many balls in a box as we wish.

The number of ways to place all r balls is

$$\frac{(n+r-1)!}{r!\,(n-1)!} = C(n+r-1, r)$$

Example

Choose 3 out of 7 days with repetitions necessarily allowed

$C(7+3-1, 3)$ $\qquad$ $C(9,3) = 84$

~~Ex~~ The number of ways to choose ~~seven out~~ of 3 days with repetitions necessarily allowed is

$$C(3+7-1, 7) \Rightarrow 36$$

$$C(9,7)$$