Mathematics Internal Assessment

Planar percolation: a seemingly simple model

1 Introduction

In addition to mathematical constructs and computer programs, I enjoy experimenting with electronic circuits. My first project, as would surprise no one, was a grid of wires:

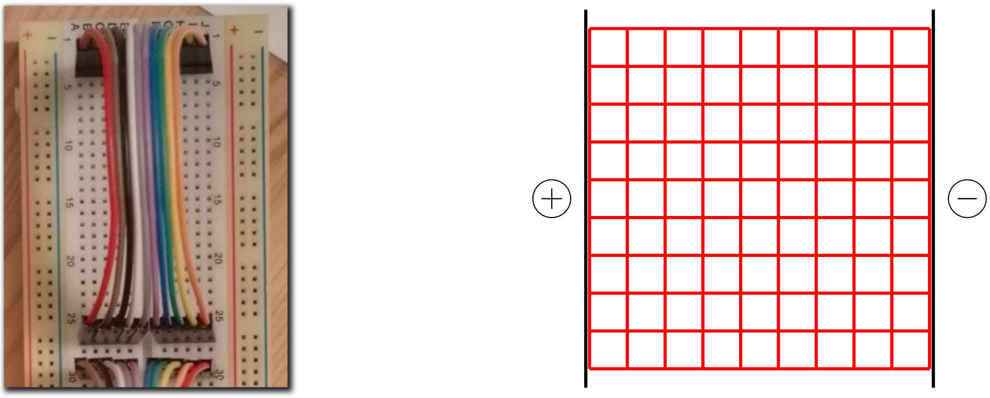


Figure 1: The circuit on a breadboard (left) and its schematic diagram (right)

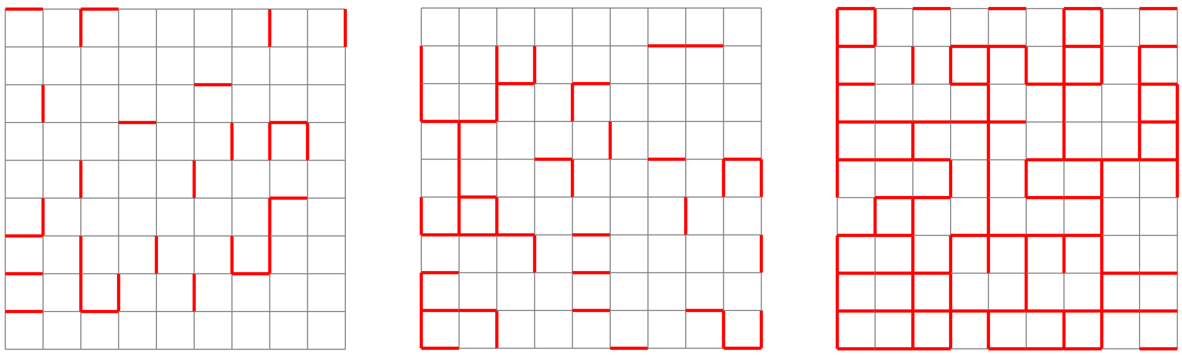
In a nutshell, every point in the grid is wired to its 4 immediate neighbours, forming a square lattice; an electric current is sent from the left side of the board to the right side, as shown in Fig. 1.

Something that bothers me about applied electronics is that, unlike maths, a creative approach that overlooks some careless mistakes can lead to physical damages. I, for one, managed to short-circuit the breadboard in Fig. 1 and burned many wires in the process. Fortunately, the circuit still served its purpose, as a current could still flow through the board. Assuming that the probability of each wire being undamaged is the same, I wondered what value it could take — to little avail.

Years later, a lecture on percolation sparked my memory: the Bernoulli bond percolation model closely resembled the aforementioned problem. We consider the graph $G = (V, E)$, where the set of vertices V is the square lattice $\mathbb{Z}^2$, and the set of edges E comprises all the edges that connect two neighbouring points A and B on the square lattice, i.e. $|A_x - B_x| + |A_y - B_y| = 1$. We then choose a certain $p \in [0,1]$, and for each edge (alternatively called *bond*) e in E, we declare e to be *open* with probability p, and *closed* otherwise with probability $1-p$ independently of the states of other edges.

We dealt with the existence of a connected component of wires that allows electricity to course through the breadboard horizontally. Here, open edges refer to working wires, while closed edges refer to damaged ones. Whilst such components are of finite sizes realistically, for planar percolation, mathematicians prefer to deal with an arbitrarily large board, where these components of open edges become *infinite clusters*.

When $p = 0$, all edges are closed, and no connected component of open edges exists at all; when $p = 1$, all edges are open, and the entire $\mathbb{Z}^2$ is one single infinite cluster. It is quite difficult to fathom how the model behaves at other nontrivial values of p. Instead, we may resort to contemplating computer simulations of a finite section of the square lattice:



$p = 0.2$ $\qquad$ $p = 0.3$ $\qquad$ $p = 0.6$

Figure 2: Percolation realisations on a 10×10 section of $\mathbb{Z}^2$ for 3 different values of p

Readers with good eyesight may bother to verify that for $p = 0.6$, there is a path of open edges (in red) connecting the left and right sides, whereas such is not the case for $p = 0.2$ and $p = 0.3$ where the connected components are isolated. For which values of p do infinite clusters exist? As we increase p from 0 to 1, is there a critical value at which infinite clusters suddenly emerge? Can we compute this critical value of p?

What is described above forms the basis of percolation theory. It may appear to be an exotic crossover of probability and graph theory, both of which we have studied in class; however, since its birth in 1957 due to Broadbent and Hammersley (693), the problem has attracted pure mathematicians and scientists from myriad domains. Despite its simplicity, the percolation model brings insight into the behaviour of many natural systems ranging from the flow of fluids through a disordered porous medium to the spread of forest fires and epidemics (de Gennes 60). Nevertheless, the first major mathematical result — and the focus of this investigation — was not announced until only some twenty years later: in 1980, Kesten (41) completed Harris's 1960 work (13) to ascertain the value of the critical probability on the square lattice. Nowadays, percolation theory is a growing field that interconnects maths, physics, and computer science. Delighted to discover the links between my favourite disciplines, I happily delved deep into the mathematical details behind this topic.

Before answering the questions raised above on planar percolation, we will firstly acquaint ourselves with a few notions of probability in order to initiate a rigorous discussion of the subject.

2 Probabilistic Preliminaries

We shall not get burdened with the measure-theoretic details about the probability space, but to avoid juggling with infinities in the model, some formal definitions are required.

Definition 1. We call a realisation an *outcome*; e.g., the leftmost panel of Section 1 shows an outcome when $p = 0.2$. An *event* is a set of zero or more outcomes, and depends on finitely many edges. We say an event *occurs* if the outcome belongs to the event.

Example 2. Let n be a nonnegative integer. Recall that a *self-avoiding path* of length n is a sequence of vertices $v_1, v_2, \dots, v_{n+1}$ that are all distinct. Let C be the set of vertices that are connected to the origin O via open edges (with $O \in C$). We define the event

$$\{|C| \geq n+1\} = \{\text{there is a self-avoiding path of open edges of length } n \text{ from } O\}$$

where $|C|$ denotes the size of the cluster C.

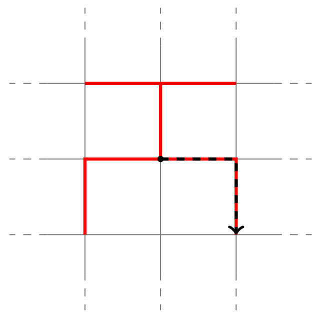


Figure 3: A self-avoiding path of length 2 from O — the event $|C| \geq 3$ occurs.

Note that we use $\mathbb{P}_p(A)$ instead of the habitual $P(A)$ to denote the probability of an event A occurring, because it depends on the value of the parameter p. Nonetheless, the probabilities of any two events A and B satisfy the following as usual:

- $0 \leq \mathbb{P}_p(A) \leq 1$ — in particular, $\mathbb{P}_p(\varnothing) = 0$;
- $\mathbb{P}_p(A \cup B) = \mathbb{P}_p(A) + \mathbb{P}_p(B) - \mathbb{P}_p(A \cap B)$;
- A is the *complement* of B if and only if $\mathbb{P}_p(A) = 1 - \mathbb{P}_p(B)$;
- A and B are *independent* if and only if $\mathbb{P}_p(A \cap B) = \mathbb{P}_p(A)\mathbb{P}_p(B)$.

Lemma 3 (Monotonicity)**.** *If the occurrence of event A implies the occurrence of event B, then*

$$\mathbb{P}_p(A) \leq \mathbb{P}_p(B).$$

Proof. By Definition 1, we see that A and B are two sets of outcomes where, for all $\omega \in A$, we also have $\omega \in B$. Hence $A \subseteq B$, and

$$\begin{aligned} \mathbb{P}_p(B) &= \mathbb{P}_p(A \cup (B \setminus A)) \\ &= \mathbb{P}_p(A) + \mathbb{P}_p(B \setminus A) - \mathbb{P}_p(A \cap (B \setminus A)) \\ \iff \mathbb{P}_p(B) - \mathbb{P}_p(A) &= \mathbb{P}_p(B \setminus A) - \mathbb{P}_p(A \cap (B \setminus A)) \end{aligned}$$

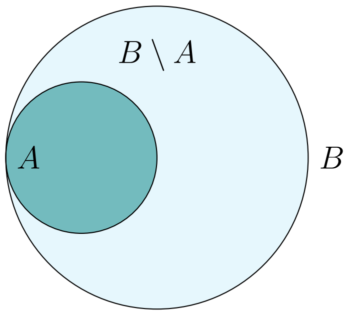


Figure 4: Venn diagram of the event sets A and B

As shown in Fig. 4 above, we have $A \cap (B \setminus A) = \varnothing$, and as such $\mathbb{P}_p(A \cap (B \setminus A)) = 0$. Then, since $\mathbb{P}_p(B \setminus A) \geq 0$ by definition,

$$\begin{aligned} \mathbb{P}_p(B) - \mathbb{P}_p(A) &= \mathbb{P}_p(B \setminus A) - 0 \geq 0 \\ \iff \mathbb{P}_p(B) &\geq \mathbb{P}_p(A). \end{aligned}$$

□

Corollary 4. *Using the definition from Example 2, for integers m and n, if $n > m \geq 0$, we have $\{|C| \geq n\} \subseteq \{|C| \geq m\}$ and hence $\mathbb{P}_p(|C| \geq n) \leq \mathbb{P}_p(|C| \geq m)$. We then define*

$$\mathbb{P}_p(|C| = \infty) = \lim_{n \to \infty} \mathbb{P}_p(\textit{there is a self-avoiding path of open edges of length } n \textit{ from } O).$$

We also use $\theta(p)$ to refer to $\mathbb{P}_p(|C| = \infty)$, the probability that there is an infinite cluster at the origin, as a function of p. Of course, we know that

$$\begin{cases} \theta(0) &= 0, \\ \theta(1) &= 1; \end{cases}$$

we are more concerned with the behaviour of $\theta(p)$ when $0 < p < 1$.

3 Percolation on the Square Lattice

3.1 No Infinite Clusters

To start with, we want to know for which values of p (apart from $p = 0$, more specifically) there is no infinite cluster on the square lattice, i.e., $\theta(p) = 0$.

Let n be a nonnegative integer, and let Ω_n be the set of all self-avoiding paths of length n starting from the origin. Corollary 4 tells us that

$$\theta(p) = \lim_{n\to\infty} \mathbb{P}_p(\bigcup_{\gamma\in\Omega_n} \text{all edges of the path } \gamma \text{ are open}).$$

We can then find an upper bound on $\theta(p)$ with the help of the following lemma.

Lemma 5 (Boole's inequality)**.** $\mathbb{P}_p(A_1 \cup A_2 \cup \cdots \cup A_n) \leq \mathbb{P}_p(A_1) + \mathbb{P}_p(A_2) + \cdots + \mathbb{P}_p(A_n)$

Proof. We proceed by induction on n. Let $\mathcal{P}(n) : \mathbb{P}_p(\bigcup_{i=1}^n A_i) \leq \sum_{i=1}^n \mathbb{P}_p(A_i)$ for $n \in \mathbb{N}$.

Base case. We have $\mathbb{P}_p(A_1) \leq \mathbb{P}_p(A_1)$, and hence $\mathcal{P}(1)$ is true.

Inductive step. Assume that $\mathcal{P}(k)$ is true for a certain $k \in \mathbb{N}$; show that $\mathcal{P}(k+1)$ is true. We use the associative property of the union operation:

$$\begin{aligned}
\mathbb{P}_p(\bigcup_{i=1}^{k+1} A_i) &= \mathbb{P}_p(\bigcup_{i=1}^{k} A_i \cup A_{k+1}) \\
&= \mathbb{P}_p(\bigcup_{i=1}^{k} A_i) + \mathbb{P}_p(A_{k+1}) - \mathbb{P}_p(\bigcup_{i=1}^{k} A_i \cap A_{k+1}) \\
&\leq \sum_{i=1}^{k} \mathbb{P}_p(A_i) + \mathbb{P}_p(A_{k+1}) - \mathbb{P}_p(\bigcup_{i=1}^{k} A_i \cap A_{k+1})
\end{aligned}$$

Since $\mathbb{P}_p(\bigcup_{i=1}^k A_i \cap A_{k+1}) \geq 0$, we can conclude that $\mathbb{P}_p(\bigcup_{i=1}^{k+1} A_i) \leq \sum_{i=1}^{k+1} \mathbb{P}_p(A_i)$.

By the principle of mathematical induction, $\mathcal{P}(n)$ is true for any $n \in \mathbb{N}$. $\square$

It follows that

$$\theta(p) \leq \lim_{n\to\infty} \sum_{\gamma\in\Omega_n} \mathbb{P}_p(\text{all edges of the path } \gamma \text{ are open}).$$

Proposition 6. *$\mathbb{P}_p$(all edges of γ are open) $= p^n$ for all path $\gamma \in \Omega_n$.*

Proof. By the definition of the set Ω_n, the self-avoiding path γ contains n distinct edges. Furthermore, each edge e in γ is defined to independently be open with probability p. Thus, the probability that all edges of the given path is open is $\prod_{e\in\gamma} p = p^n$. $\square$

Figure 5: Enumeration of self-avoiding paths in $\mathbb{Z}^2$

4 choices for the first step 3 choices for any step that follows

Proposition 7. *There are at most* $4 \cdot 3^{n-1}$ *self-avoiding paths in* Ω_n*.*

Proof. Using the reasoning from Fig. 5 above, we can deduce that $|\Omega_n| \leq 4 \times 3^{n-1}$. □

Theorem 8. *If* $p < \dfrac{1}{3}$*, then* $\theta(p) = 0$*.*

Proof. Although we have only obtained a very crude bound on the number of paths in Ω_n, we can still conclude that

$$\begin{aligned} \theta(p) &\leq \lim_{n\to\infty} \sum_{\gamma\in\Omega_n} p^n \\ &\leq \lim_{n\to\infty} \left(4 \cdot 3^{n-1}\right) p^n \\ &= \frac{4}{3} \lim_{n\to\infty} (3p)^n \end{aligned}$$

The limit tends toward 0 if $|3p| < 1$. Then, $p < \frac{1}{3}$ implies $\theta(p) \leq 0$. But since $\theta(p) \geq 0$ as a probability, we have $\theta(p) = 0$. □

As a result, there is no infinite cluster on the square lattice if $p < \frac{1}{3}$.

3.2 Infinite Clusters

Then, we are interested in knowing for which values of p (other than $p = 1$) there exists an infinite cluster, i.e. $\theta(p) > 0$.

This time, we will be employing a more elaborated method that is often called a *Peierls argument*. It is named after the British physicist who proved a phase transition — the change of macroscopic behaviour as the parameter varies, e.g. the appearance of an infinite cluster — for the Ising model, another probabilistic model in statistical physics that explains the phenomenon of ferromagnetism. The gist is that, instead of trying to directly find a lower bound on $\theta(p)$, we establish an upper bound on $1-\theta(p)$ using a variant of the technique from the previous section. The method revolves around the *dual graph* $(\mathbb{Z}^2)^*$, a key notion that we will introduce now.

Definition 9. Given a planar graph G, we define its *dual graph* G^* in such a way that:

- G^* has a vertex for each face delimited by edges of G;
- an edge connects two vertices of G^* if the corresponding faces are separated by an edge of G.

Example 10. Consider $(\mathbb{Z}^2)^*$, the dual graph of the square lattice $\mathbb{Z}^2$ and its edges.

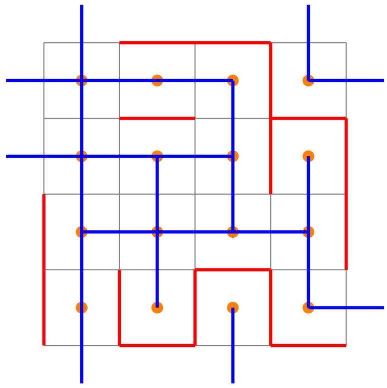


Figure 6: A percolation realisation on a 5×5 section of the square lattice and its dual

We observe that the graph $\mathbb{Z}^2$ is *self-dual*: by a vector transformation of $(\frac{1}{2}, \frac{1}{2})$, each vertex (resp. edge) of $\mathbb{Z}^2$ uniquely corresponds to a vertex (resp. edge) in $(\mathbb{Z}^2)^*$. Moreover, concerning percolation, we declare each edge e^* of $(\mathbb{Z}^2)^*$ to be open (resp. closed) if the edge e of $\mathbb{Z}^2$ that e^* crosses is closed (resp. open). In other words, the event that e^* is open is the complement of the event that e is open. Since e is open with probability p, the probability of e^* being open is $1 - p$. As such, we consider percolation is realised in $(\mathbb{Z}^2)^*$ with parameter $1 - p$.

Figure 6 above shows an example of how from the open edges (red) in $\mathbb{Z}^2$ (grey grid) we can create the dual open edges (blue) in $(\mathbb{Z}^2)^*$ (orange dots). Notice that, in the top-left corner, the cluster of 2 vertices in $\mathbb{Z}^2$ is surrounded by a cycle of open edges in $(\mathbb{Z}^2)^*$. We can then formulate a condition for the cluster at the origin to be finite in a similar fashion.

Let $\{|C| < \infty\}$ denote the event that the cluster C at the origin contains finitely many vertices.

Proposition 11. *$|C| < \infty$ if and only if there exists a cycle of open edges in $(\mathbb{Z}^2)^*$ surrounding O. Also, the cycle passes by the point $(n + \frac{1}{2}, \frac{1}{2})$ for some nonnegative n.*

This is a result from pure graph theory. We will not present the rigorous proof due to Grimmett (14–15) here, but the reader should be able to convince him/herself that this proposition is very believable simply by drawing some pictures. Indeed, a cycle of open edges in $(\mathbb{Z}^2)^*$ blocks all open edges of $\mathbb{Z}^2$ connecting the interior and the exterior of the cycle. The cluster C is then confined inside the cycle, and therefore can only contain finitely many vertices. In addition, for the cycle to surround the origin, it must intersect the line $y = \frac{1}{2}$ (which lies right above the origin) on the right, i.e., at a nonnegative x.

Proposition 12. *Let n be a nonnegative integer. If there exists a cycle in $(\mathbb{Z}^2)^*$ surrounding O and passing by $(n+\frac{1}{2}, \frac{1}{2})$, then there exists a self-avoiding path in $(\mathbb{Z}^2)^*$ starting from $(n+\frac{1}{2}, \frac{1}{2})$ of length $2n+3$.*

Proof. As shown in Fig. 7 below, the shortest cycle passing by $(n+\frac{1}{2}, \frac{1}{2})$ that surrounds the origin has

$$2\left((n+\frac{1}{2}) - (-\frac{1}{2})\right) + 2 = 2n+4$$

distinct edges.

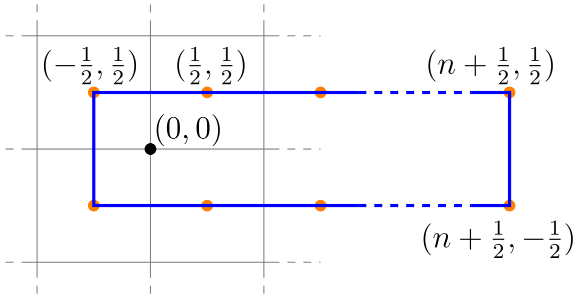


Figure 7: The shortest cycle in $(\mathbb{Z}^2)^*$ passing by $(n+\frac{1}{2}, \frac{1}{2})$ and surrounding O

Recall from graph theory that a cycle has no repeated edges or repeated vertices except for the starting and ending ones. By definition, a cycle without the last vertex is a self-avoiding path. Since all cycles passing by $(n+\frac{1}{2}, \frac{1}{2})$ contain at least $2n+4$ edges, it must also contain a self-avoiding path starting from $(n+\frac{1}{2}, \frac{1}{2})$ that is of length $2n+3$. □

Theorem 13. *If $p > 0.762$, then $\theta(p) > 0$.*

Proof. We begin by finding an upper bound on the probability that $|C|$ is finite.

$$\mathbb{P}_p(|C| < \infty) = \lim_{N\to\infty} \mathbb{P}_p\left(\bigcup_{n=0}^{N} \exists \text{ cycle surrounding the origin passing by } (n+\frac{1}{2}, \frac{1}{2})\right) \quad (1)$$

$$\leq \lim_{N\to\infty} \sum_{n=0}^{N} \mathbb{P}_p\left(\exists \text{ cycle surrounding the origin passing by } (n+\frac{1}{2}, \frac{1}{2})\right) \quad (2)$$

$$\leq \lim_{N\to\infty} \sum_{n=0}^{N} \mathbb{P}_p\Big(\bigcup_{\gamma\in\Omega_{2n+3}} \text{all edges of the path } \gamma \text{ are open}\Big) \quad (3)$$

$$\leq \lim_{N\to\infty} \sum_{n=0}^{N} \sum_{\gamma\in\Omega_{2n+3}} \mathbb{P}_p(\text{all edges of the path } \gamma \text{ are open}) \quad (4)$$

$$\leq \lim_{N\to\infty} \sum_{n=0}^{N} (4\cdot 3^{2n+2})(1-p)^{2n+3} \quad (5)$$

We start by using the equivalence between the event $\{|C| < \infty\}$ and the existence of a cycle in $(\mathbb{Z}^2)^*$ around the origin as stated in Proposition 11 to establish Eq. (1). Next, Lemma 5 allows us to express the probability of a union of events as the sum of the probabilities of the events in Eq. (2). Before passing to Eq. (3), we need to make several observations. Firstly, we can apply Lemma 3, which allows the probability of an event to be bounded from above by the probability of another event that it implies, to find an upper bound on the probability of such cycles existing using the probability of that self-avoiding paths of a certain length exist by Proposition 12. In addition, the number of such paths does not depend on the starting vertex, whether it is the origin or an arbitrary $(n+\frac{1}{2}, \frac{1}{2})$; we can thus use the set Ω_{2n+3} from Section 3.1. Now, the reader should be familiar with the remaining steps, since we almost replicated them from the proof of Theorem 8. To derive Eq. (4), we again leverage Lemma 5 to rid of the union. Finally, in Eq. (5), we expand the sum with the upper bound on $|\Omega_{2n+3}|$ that we established in Proposition 7. The only difference is that in $(\mathbb{Z}^2)^*$, each edge is open with probability $1-p$ as defined in Example 10; we simply need to adapt Proposition 6 so that $1-p$ is considered as the parameter instead.

We continue by considering the limit

$$\begin{aligned} &\lim_{N\to\infty}\sum_{n=0}^{N}(4\cdot 3^{2n+2})(1-p)^{2n+3} \\ &= 4\cdot 3^2(1-p)^3 \lim_{N\to\infty}\sum_{n=0}^{N}(3(1-p))^{2n} \\ &= 36(1-p)^3 \lim_{N\to\infty}\sum_{n=0}^{N}\left(9(1-p)^2\right)^n . \end{aligned}$$

The geometric series only converges when $|9(1-p)^2| < 1$, i.e., when $-\frac{1}{3} < (1-p) < \frac{1}{3}$. Since we specified that $p \in [0,1]$, when $p > \frac{2}{3}$, the limit evaluates to

$$\frac{36(1-p)^3}{1-9(1-p)^2}.$$

As shown in Fig. 8 opposite, graphically we can determine that for $p > 0.762$,

$$\mathbb{P}_p(|C| < \infty) \le \frac{36(1-p)^3}{1-9(1-p)^2} < 1.$$

Note that the event $\{|C| < \infty\}$ is the complement of the event $\{|C| = \infty\}$, and hence

$$\mathbb{P}_p(|C| < \infty) = 1 - \mathbb{P}_p(|C| = \infty) = 1 - \theta(p).$$

We can now conclude that

$$\begin{aligned} 1 - \theta(p) &< 1 \\ \iff \theta(p) &> 0 \end{aligned}$$

when $p > 0.762$. □

In consequence, there exists an infinite cluster on the square lattice if $p > 0.762$.

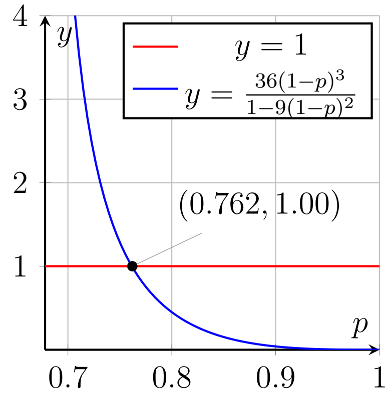


Figure 8: The upper bound on $\mathbb{P}_p(|C| < \infty)$

3.3 The Critical Value

Having seen how there is no infinite cluster on the square lattice when p is close to 0 and how there exists an infinite cluster when p is close to 1, via both computer simulations and mathematical proofs, it is now natural to define the critical probability p_c.

Definition 14. The critical probability p_c is a value in $[0, 1]$ such that

$$\begin{cases} \theta(p) = 0 & \text{if } p < p_c, \\ \theta(p) > 0 & \text{if } p > p_c. \end{cases}$$

Moreover, in Sections 3.1 and 3.2, we have shown that $\theta(p) = 0$ if $p < \frac{1}{3}$ and $\theta(p) > 0$ if $p > 0.762$. It follows that $\frac{1}{3} \leq p_c \leq 0.762$.

Granted, it was quite an effort. That said, we are still left with an interval of values for p_c, and one endpoint is even numerically approximated. Dispirited and depressed, the reader makes a random guess before throwing in the towel: $p_c = \frac{1}{2}$ ... Bingo!

Theorem 15 (Kesten 42)**.** *For Bernoulli bond percolation on $\mathbb{Z}^2$, the value of p_c is $\dfrac{1}{2}$.*

It is worth mentioning that the actual proof for the above theorem is far beyond the scope of this investigation. Instead, we will present a simple heuristic argument here.

Heuristic Proof. It makes intuitive sense that as p varies, the macroscopic behaviour of the percolation model should only change once. This observation implies that p_c is necessarily equal to $1 - p_c$. Otherwise, an infinite cluster will appear in $\mathbb{Z}^2$ at $p = p_c$, and an infinite cluster will disappear in $(\mathbb{Z}^2)^*$ at $p = 1 - p_c$. Furthermore, if we assume that $p_c < \frac{1}{2}$, for any $p \in \,]p_c, 1 - p_c[$, there is an infinite cluster both on $\mathbb{Z}^2$ and on $(\mathbb{Z}^2)^*$. This contradicts our intuition that the coexistence of an infinite cluster on the square lattice and on its dual is impossible. A similar contradiction arises if we assume instead that $p_c > \frac{1}{2}$. Thus, $p_c = \frac{1}{2}$. $\square$

In the case of the electronic circuit from the introduction, if the breadboard is to be sufficiently large, then as long as more than half of the wires continue to function, an electric current will still be able to flow through the board. Indeed, after some painstaking testing and counting, it was determined that 81 out of 100 wires were intact.

4 Conclusion

In this investigation, we have studied the critical probability for percolation on $\mathbb{Z}^2$.

One possible way to further the investigation is to consider the values of p_c on other kinds of graphs. It is natural to consider $\mathbb{Z}^3$ as a more accurate physical model. The critical probability can be bounded from below using the same method from Section 3.1, and from above with the trifling observation that a copy of $\mathbb{Z}^2$ is embedded in $\mathbb{Z}^3$ — the exact details are left as an exercise to the reader. Also of interest are trees of a fixed number of branches and triangular lattices. The former is linked to genealogical trees, and it has been mathematically shown that the probability of extinction of family names after infinitely many generations is 92% (de Gennes) for instance. In addition, for large values of d, the lattice $\mathbb{Z}^d$ often behaves in a similar manner to trees concerning infinite clusters (de Gennes). Studies of the latter have taken this idea of similarity between different graphs even further to suggest that some properties of percolation are conserved under certain transformations; Smirnov's proof of Cardy's formula (1), earning him a Fields medal, has sparked a recent interest in this area. At the same time, many questions and a few answers have been conceived in considering more complex systems of percolation, such as directed edges where the state of each edge depends on its neighbours.

Another direction to pursue is a closer study of how the percolation model behaves. While the notion of the critical probability allows us to understand the macroscopic behaviour concerning the existence of infinite clusters, more questions arise: when $p > p_c$, is the infinite cluster unique? (*Spoiler alert:* it almost surely is.) When $p = p_c$, does an infinite cluster exist? Kesten (42) has shown that for bond percolation on $\mathbb{Z}^2$, it is true that $\theta(p) = 0$; it remains an open question whether such is the case for lattices in higher dimensions.

At last, we shall return to the mathematical model itself, whose intricacy is concealed behind its simplistic formulation. Moreover, we saw that the intuitively obvious explanations, in spite of how they may serve as a beacon for formal proofs, are often far from rigorous reasonings. This sheds light on the role of intuition in mathematics, and calls into question whether accepting conclusions that go beyond evidence is required in the production of mathematical knowledge. These ideas are perhaps best summed up in Kesten's words:

> "Quite apart from the fact that percolation theory had its origin in an honest applied problem, it is a source of fascinating problems of the best kind a mathematician can wish for: problems which are easy to state with a minimum of preparation, but whose solutions are (apparently) difficult and require new methods."

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