Wysocki2019MassModel Normalizing Constant Derivation

Meesum Qazalbash

December 26, 2023

Power law in paper is defined as:

$$p(m_1, m_2 \mid \alpha, k, m_{\min}, m_{\max}, M_{\max}) = \frac{m_1^{-\alpha-k} m_2^k}{m_1 - m_{\min}} \frac{1}{\mathcal{Z}} \tag{1}$$

where, $m_{\min} \leq m_2 \leq m_1 \leq m_{\max}$, $m_1 + m_2 \leq M_{\max}$ and $k \in \mathbb{W}$. Lets find the normalizing constant $\mathcal{Z}$. First we will find the marginal distribution of m_1.

$$\begin{aligned}
p(m_1 \mid \alpha, k, m_{\min}, m_{\max}, M_{\max}) &= \int\limits_{m_2=m_{\min}}^{m_1} \frac{m_1^{-\alpha-k} m_2^k}{m_1 - m_{\min}} \frac{1}{\mathcal{Z}} \mathrm{d}m_2 && (2)\\
&= \frac{1}{\mathcal{Z}} \frac{m_1^{-\alpha-k}}{m_1 - m_{\min}} \int\limits_{m_2=m_{\min}}^{m_1} m_2^k \mathrm{d}m_2 && (3)\\
&= \frac{1}{\mathcal{Z}} \frac{m_1^{-\alpha-k}}{m_1 - m_{\min}} \frac{{m_1}^{k+1} - {m_{\min}}^{k+1}}{k+1} && (4)\\
&= \frac{m_1^{-\alpha-k}}{(k+1)\mathcal{Z}} \frac{{m_1}^{k+1} - {m_{\min}}^{k+1}}{m_1 - m_{\min}} && (5)
\end{aligned}$$

As k is a positive integer we will open the fraction in 5 using the Geometric series.

$$\begin{aligned}
p(m_1 \mid \alpha, k, m_{\min}, m_{\max}, M_{\max}) &= \frac{m_1^{-\alpha-k}}{(k+1)\mathcal{Z}} \frac{{m_1}^{k+1} - {m_{\min}}^{k+1}}{m_1 - m_{\min}} && (6)\\
&= \frac{m_1^{-\alpha}}{(k+1)\mathcal{Z}} \frac{1 - \left(\frac{m_{\min}}{m_1}\right)^{k+1}}{1 - \frac{m_{\min}}{m_1}} && (7)\\
&= \frac{m_1^{-\alpha}}{(k+1)\mathcal{Z}} \sum_{i=0}^{k} \left(\frac{m_{\min}}{m_1}\right)^i && (8)\\
&= \frac{1}{(k+1)\mathcal{Z}} \sum_{i=0}^{k} m_{\min}^i {m_1}^{-\alpha-i} && (9)
\end{aligned}$$

Depending on the value of α summation would contain $\frac{1}{m_1}$. Lets device conditions for α. If α is not an integer then summation would not contain any $\frac{1}{m_1}$. If α is an integer then summation would contain $\frac{1}{m_1}$ if $\alpha + i = 1$. Now we know that $0 \leq i \leq k$, therefore $\alpha \leq \alpha + i \leq \alpha + k$ which makes it, $\alpha \leq 1 \leq \alpha + k$. From here we can see that the bounds on α are $1 - k \leq \alpha \leq 1$. Lets consider the case when α follows the bounds and conditions we have just derived.

$$p(m_1 \mid \alpha, k, m_{\min}, m_{\max}, M_{\max}) = \frac{1}{(k+1)\mathcal{Z}} \left(\frac{m_{\min}^{1-\alpha}}{m_1} + \sum_{i=0, i\neq 1-\alpha}^{k} m_{\min}^{i} {m_1}^{-\alpha-i} \right) \tag{10}$$

$$\implies \mathcal{Z} = \int\limits_{m_1=m_{\min}}^{m_{\max}} \frac{1}{k+1} \left(\frac{m_{\min}^{1-\alpha}}{m_1} + \sum_{i=0, i\neq 1-\alpha}^{k} m_{\min}^{i} {m_1}^{-\alpha-i} \right) \mathrm{d}m_1 \tag{11}$$

$$= \frac{1}{k+1} \int\limits_{m_1=m_{\min}}^{m_{\max}} \frac{m_{\min}^{1-\alpha}}{m_1} \mathrm{d}m_1 + \frac{1}{k+1} \sum_{i=0, i\neq 1-\alpha}^{k} \int\limits_{m_1=m_{\min}}^{m_{\max}} m_{\min}^{i} {m_1}^{-\alpha-i} \mathrm{d}m_1 \tag{12}$$

$$= \frac{m_{\min}^{1-\alpha}}{k+1} \log\left(\frac{m_{\max}}{m_{\min}}\right) + \frac{1}{k+1} \sum_{i=0, i\neq 1-\alpha}^{k} \frac{m_{\min}^{i}}{1-\alpha-i} \left({m_{\max}}^{1-\alpha-i} - {m_{\min}}^{1-\alpha-i} \right) \tag{13}$$

$$= \frac{m_{\min}^{1-\alpha}}{k+1} \log\left(\frac{m_{\max}}{m_{\min}}\right) + \frac{1}{k+1} \sum_{i=0, i\neq 1-\alpha}^{k} \left(\left(\frac{m_{\min}}{m_{\max}}\right)^{i} \frac{m_{\max}^{1-\alpha}}{1-\alpha-i} - \frac{m_{\min}^{1-\alpha}}{1-\alpha-i} \right) \tag{14}$$

$$= \frac{m_{\min}^{1-\alpha}}{k+1} \log\left(\frac{m_{\max}}{m_{\min}}\right) + \frac{m_{\min}^{1-\alpha}}{k+1} \sum_{i=0, i\neq 1-\alpha}^{k} \frac{\left(\frac{m_{\min}}{m_{\max}}\right)^{i+\alpha-1} - 1}{1-\alpha-i} \tag{15}$$

And when α does not follow the bounds and conditions we have derived, then the summation would not contain $\dfrac{1}{m_1}$ and the normalizing constant would be:

$$\mathcal{Z} = \frac{m_{\min}^{1-\alpha}}{k+1} \sum_{i=0}^{k} \frac{\left(\frac{m_{\min}}{m_{\max}}\right)^{i+\alpha-1} - 1}{1-\alpha-i} \tag{16}$$

We can write the normalizing constant as a piecewise function,

$$\mathcal{Z}(\alpha, k, m_{\min}, m_{\max}, M_{\max}) = \begin{cases} \frac{m_{\min}^{1-\alpha}}{k+1} \log\left(\frac{m_{\max}}{m_{\min}}\right) + \frac{m_{\min}^{1-\alpha}}{k+1} \sum\limits_{i=0, i\neq 1-\alpha}^{k} \frac{\left(\frac{m_{\min}}{m_{\max}}\right)^{i+\alpha-1} - 1}{1-\alpha-i} & \text{if } 1-k \leq \alpha \leq 1 \text{ and } \alpha \in \mathbb{W} \\ \frac{m_{\min}^{1-\alpha}}{k+1} \sum\limits_{i=0}^{k} \frac{\left(\frac{m_{\min}}{m_{\max}}\right)^{i+\alpha-1} - 1}{1-\alpha-i} & \text{otherwise} \end{cases} \tag{17}$$