Wysocki2019SpinModel Normalizing Constant Derivation

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Spin model in paper is defined as:

$$p(\chi \mid \alpha, \beta, \chi_{\max}) = \frac{1}{\mathcal{Z}} \chi^{\alpha-1} \left(\chi_{\max} - \chi\right)^{\beta-1} \tag{1}$$

where χ is the spin and α, β are the shape parameters. $\chi_{\max}$ is the maximum spin value. $\mathcal{Z}$ is the normalizing constant. Spin is in the range $[0, \chi_{\max}]$.

$$\int_0^{\chi_{\max}} p(\chi \mid \alpha, \beta, \chi_{\max}) \mathrm{d}\chi = 1 \iff \mathcal{Z} = \int_0^{\chi_{\max}} \chi^{\alpha-1} \left(\chi_{\max} - \chi\right)^{\beta-1} \mathrm{d}\chi \tag{2}$$

Let,

$$\chi = \chi_{\max} t \iff \mathrm{d}\chi = \chi_{\max} \mathrm{d}t \tag{3}$$

$$\chi \in [0, \chi_{\max}] \iff t \in [0, 1] \tag{4}$$

$$\mathcal{Z} = \int_0^1 \left(\chi_{\max} t\right)^{\alpha-1} \left(\chi_{\max} - \chi_{\max} t\right)^{\beta-1} \chi_{\max} \mathrm{d}t \tag{5}$$

$$\mathcal{Z} = \chi_{\max}^{\alpha+\beta-1} \int_0^1 t^{\alpha-1} \left(1 - t\right)^{\beta-1} \mathrm{d}t \tag{6}$$

$$\mathcal{Z} = \chi_{\max}^{\alpha+\beta-1} \mathrm{B}\left(\alpha, \beta\right) \tag{7}$$