Exact solutions of the Boltzmann equation

Max Krook and Tai Tsun Wu, The Physics of Fluids **20**, 1589 (1977)

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Introduction

In this notebook we reproduce some results from the paper.

In[•]:=
```
(* eq 20 *)
Clear[v, α, ϕ]
ϕ = (2 π α)^(-3/2) Exp[-v^2 / (2 α)] /. {α → 1}
4 π Integrate[ϕ v^(2 n+2), {v, 0, ∞}] // Normal // Simplify
```

Out[•]= $\dfrac{e^{-\frac{v^2}{2}}}{2\sqrt{2}\,\pi^{3/2}}$

Out[•]= $\dfrac{2^{1+n}\,\mathrm{Gamma}\left[\frac{3}{2}+n\right]}{\sqrt{\pi}}$

In[•]:=
```
(* eq 56 is solution to eq 55+  *)
Clear[z, Z, eq]
Z = 2 × (1 - z) × (1 - (1 - z)^(1/2));
Z D[Z, z] + 5 Z - 6 z (1 - z) // Simplify
(* boundary condition eq53 *)
Series[Z, {z, 0, 1}]
```

Out[•]= 0

Out[•]= $z + O[z]^2$

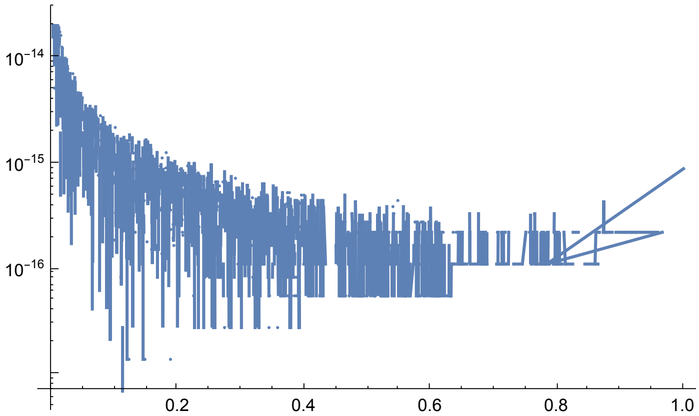
```
(* eq 57  *)
Clear[z, Z, η, dηdz, int, diff]
Z = 2 × (1 - z) × (1 - (1 - z)^(1/2));
dηdz = 1/Z
int = Refine[Integrate[dηdz, z] // Simplify, {z > 0, z < 1}]
diff = int - Log[(1 - z)^(-1/2) - 1]
(* same result *)
Series[diff, {z, 0, 13}]
LogPlot[Abs[diff], {z, 0, 1}]
```

Out[∘]= $\frac{1}{2 \times \left(1-\sqrt{1-z}\right) \times (1-z)}$

Out[∘]= $\mathrm{Log}\left[1-\sqrt{1-z}\right]-\frac{1}{2}\mathrm{Log}[1-z]$

Out[∘]= $-\mathrm{Log}\left[-1+\frac{1}{\sqrt{1-z}}\right]+\mathrm{Log}\left[1-\sqrt{1-z}\right]-\frac{1}{2}\mathrm{Log}[1-z]$

Out[∘]= $O[z]^{14}$

Out[∘]=


In[∘]:=
```
(* prove eq67+ *)
Clear[ξ, n, g]
g = 4/Sqrt[π] Sum[Gamma[n + 3/2]/(2 n)! ξ^n, {n, 0, ∞}]
D[g, ξ] // Simplify
```

Out[∘]= $e^{\xi/4}(2+\xi)$

Out[∘]= $\frac{1}{4}e^{\xi/4}(6+\xi)$

```
(* get eq 69 from Fourier transformed solution *)
Clear[hbar, p, τ, f, v, int, intp, intm, κ, h, f69]
hbar = (4 π)^-1 (2 + (κ - 3) p^2 + κ (1 - κ) p^4) Exp[-κ p^2 / 2];
(* there is a typo in the paper: the complex exponential in
   the integrand function must be Exp[I p v] and not Exp[I p τ],
otherwise h would never be a function of v *)
intp = Refine[1/(2 π) Integrate[hbar Exp[I p v], p] // Normal // Expand // Simplify,
   {p ∈ Reals, v > 0, κ > 0}] // Simplify
intm = intp /. {p → -p} // Simplify;
(* computing the limit would take too much time *)
(*h=((Limit[int,{p→∞}]//Normal//Simplify)-
     (Limit[int,{p→-∞}]//Normal//Simplify))//Simplify*)

(* int(p) - int(-p) *)
(intp - intm) // FullSimplify

(* the only term that survives p→
 ±∞ is the one containing (because exp(-p^2)→0 *)
I (Limit[Erfi[-ⅈ p], p → ∞] - Limit[Erfi[ⅈ p], p → ∞])
```

$$h = \frac{1}{16\,\pi^2\,\kappa^{7/2}}\,e^{-\frac{v^2}{2\kappa}}\,v\,\sqrt{2\,\pi}\,v\,\left(v^2 - \left(3 + v^2\right)\kappa + 5\,\kappa^2\right)\,2 \text{ // Simplify;}$$

$$f = \frac{1}{v^2}\,h$$

```
(* actual distribution *)
```

$$f69 = \frac{\text{Exp}\left[-\frac{v^2}{2\kappa}\right]}{2\,\kappa\,(2\,\pi\,\kappa)^{3/2}}\,\left((5\,\kappa - 3) + \left(\frac{1-\kappa}{\kappa}\right)v^2\right);$$

```
(* check *)
f - f69 // Simplify
```

Out[73]=

$$\frac{1}{16\,\pi^2\,\kappa^{7/2}}\,e^{-\frac{v^2+p^2\kappa^2}{2\kappa}}\left(2\,e^{\frac{v\,(v+2\,i\,p\,\kappa)}{2\kappa}}\,\sqrt{\kappa}\,\left(-i\,v^3\,(-1+\kappa) - p\,v^2\,(-1+\kappa)\,\kappa + p\,\left(2 + p^2\,(-1+\kappa)\right)\kappa^3 + i\,v\,\kappa\,\left(-2 - \left(-4+p^2\right)\kappa + p^2\,\kappa^2\right)\right) + i\,e^{\frac{p^2\kappa}{2}}\,\sqrt{2\,\pi}\,v^2\,\left(v^2\,(-1+\kappa) + (3 - 5\,\kappa)\,\kappa\right)\,\text{Erfi}\left[\frac{v + i\,p\,\kappa}{\sqrt{2}\,\sqrt{\kappa}}\right]\right)$$

Out[75]=
$$\frac{1}{16\,\pi^2\,K^{7/2}}\,e^{-\frac{v^2+p^2 K^2}{2K}}\left(4\,e^{\frac{v^2}{2K}}\,p\,K^{3/2}\left(-v^2\,(-1+K)+\left(2+p^2\,(-1+K)\right)K^2\right)\mathrm{Cos}[p\,v]+v\left(i\,e^{\frac{p^2 K}{2}}\,\sqrt{2\,\pi}\,v\left(v^2-\left(3+v^2\right)K+5\,K^2\right)\left(\mathrm{Erfi}\left[\frac{v-i\,p\,K}{\sqrt{2}\,\sqrt{K}}\right]-\mathrm{Erfi}\left[\frac{v+i\,p\,K}{\sqrt{2}\,\sqrt{K}}\right]\right)-4\,e^{\frac{v^2}{2K}}\,\sqrt{K}\left(-v^2\,(-1+K)+K\left(-2+\left(4+p^2\,(-1+K)\right)K\right)\right)\mathrm{Sin}[p\,v]\right)\right)$$

Out[76]= 2

Out[78]=
$$\frac{e^{-\frac{v^2}{2K}}\left(v^2-\left(3+v^2\right)K+5\,K^2\right)}{4\,\sqrt{2}\,\pi^{3/2}\,K^{7/2}}$$

Out[80]= 0

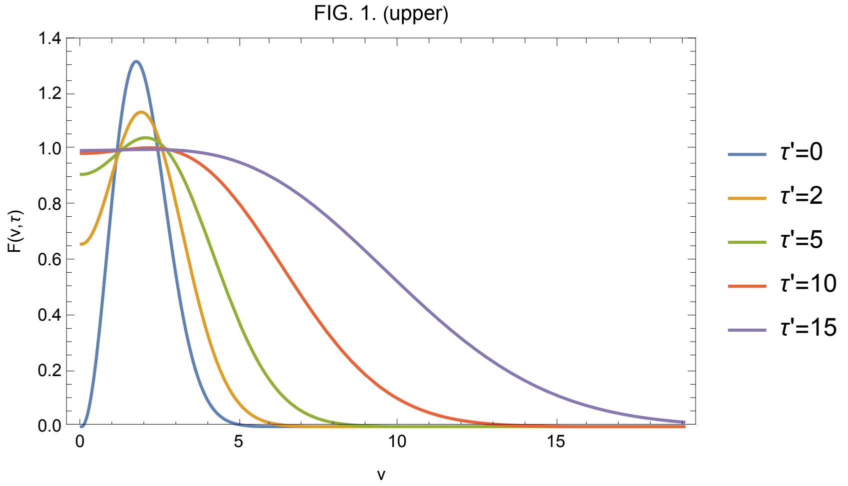
Fig 1 Analytical solution eq 71

In[*]:=
```
Clear[K, τ, F, v, τ]

(* actual distribution *)
```
$$f = \frac{\mathrm{Exp}\left[-\frac{v^2}{2K}\right]}{2K(2\pi K)^{3/2}}\left((5K-3)+\left(\frac{1-K}{K}\right)v^2\right) \text{ /. } \{K \to 1-\mathrm{Exp}[-\tau/6]\} \text{ // Simplify}$$
```
(* asymptotic time solution *)
Limit[f, τ → ∞]

(* eq 60 *)
K[τ_] := 1 - Exp[-τ / 6]

(* eq 71 F(v,τ) = f(v,τ)/f(v,∞) *)
```
$$F[v\_, \tau\_] := \left(\frac{5K[\tau]-3}{2K[\tau]\text{^}2.5}+\frac{1-K[\tau]}{2K[\tau]\text{^}3.5}v\text{^}2\right)\mathrm{Exp}\left[-\frac{1}{2}v\text{^}2\left(\frac{1}{K[\tau]}-1\right)\right]$$
```
(* fig 1 *)
Plot[{F[v, 0 + 6 Log[5 / 2]], F[v, 2 + 6 Log[5 / 2]], F[v, 5 + 6 Log[5 / 2]],
   F[v, 10 + 6 Log[5 / 2]], F[v, 15 + 6 Log[5 / 2]]}, {v, 0, 19}, PlotRange → {0, 1.4},
  PlotLabel → "FIG. 1. (upper)", Frame → True, FrameLabel → {"v", "F(v,τ)"},
  PlotLegends → {"τ'=0", "τ'=2", "τ'=5", "τ'=10", "τ'=15"}]
Plot[{F[v, 15 + 6 Log[5 / 2]], F[v, 30 + 6 Log[5 / 2]], F[v, 45 + 6 Log[5 / 2]],
   F[v, 60 + 6 Log[5 / 2]], F[v, 75 + 6 Log[5 / 2]]}, {v, 0, 750}, PlotRange → {0, 1.4},
  PlotLabel → "FIG. 1. (lower)", Frame → True, FrameLabel → {"v", "F(v,τ)"},
  PlotLegends → {"τ'=15", "τ'=30", "τ'=45", "τ'=60", "τ'=75"}]

(* f(v,τ), figure not included in paper *)
LogPlot[
  {(f /. {τ → 0 + 6 Log[5 / 2]}), (f /. {τ → 2 + 6 Log[5 / 2]}), (f /. {τ → 5 + 6 Log[5 / 2]}),
   (f /. {τ → 10 + 6 Log[5 / 2]}), (f /. {τ → 15 + 6 Log[5 / 2]})}, {v, 0, 6},
  PlotLabel → "FIG. X. (upper)", Frame → True, FrameLabel → {"v", "f(v,τ)"},
  PlotLegends → {"τ'=0", "τ'=2", "τ'=5", "τ'=10", "τ'=15"}, PlotRange → Automatic
]
LogPlot[{(f /. {τ → 15 + 6 Log[5 / 2]}),
   (f /. {τ → 30 + 6 Log[5 / 2]}), (f /. {τ → 45 + 6 Log[5 / 2]}),
   (f /. {τ → 60 + 6 Log[5 / 2]}), (f /. {τ → 55 + 6 Log[5 / 2]})}, {v, 0, 6},
  PlotLabel → "FIG. X. (lower)", Frame → True, FrameLabel → {"v", "f(v,τ)"},
  PlotLegends → {"τ'=15", "τ'=30", "τ'=45", "τ'=60", "τ'=75"},
  PlotRange → Automatic]
```

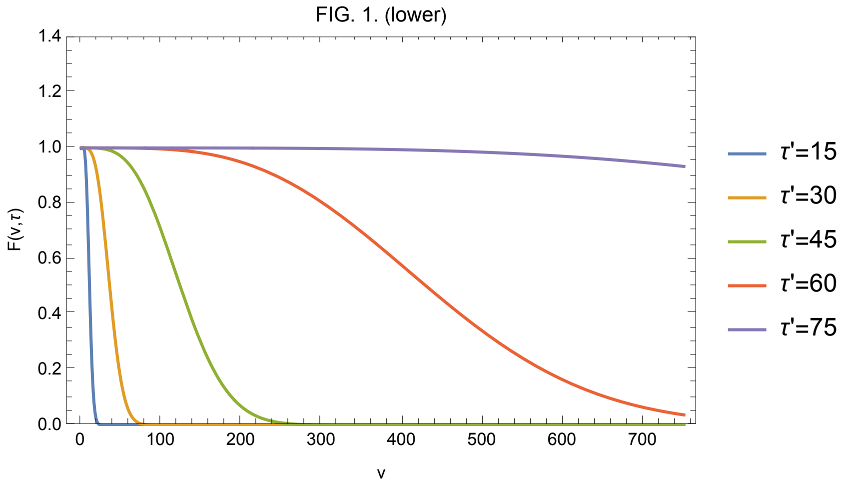
Out[*]=
$$\frac{e^{\frac{1}{6}\left(-\frac{3v^2}{1-e^{-\tau/6}}+\tau\right)}\left(5+2e^{\tau/3}+e^{\tau/6}\left(-7+v^2\right)\right)}{4\sqrt{2-2e^{-\tau/6}}\left(-1+e^{\tau/6}\right)^3\pi^{3/2}}$$

Out[∘]= $\frac{e^{-\frac{v^2}{2}}}{2\sqrt{2}\,\pi^{3/2}}$

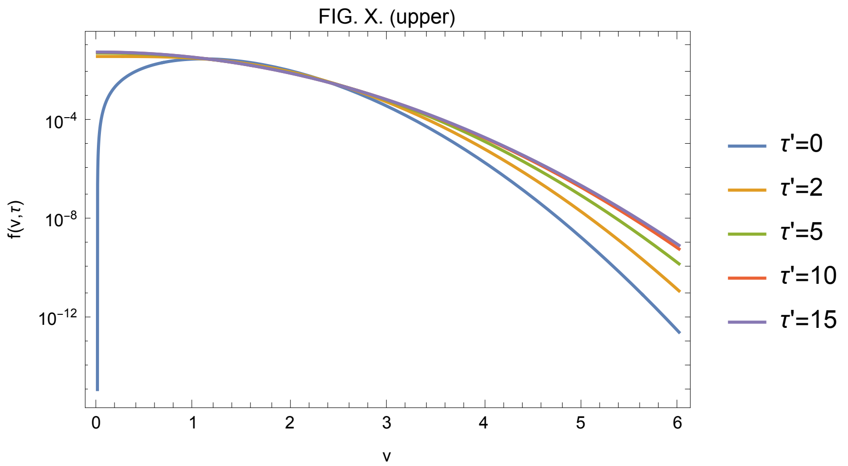
Out[∘]=



Out[∘]=



Out[∘]=



Out[ ]=

FIG. X. (lower)