A briefing on Bézier curves and surfaces

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This note introduces Bézier curves and surfaces and summarizes their most important mathematical properties. Most of the material used to prepare this note is available in the NURBS book (Piegl and Tiller 1997, Chapter 1).

1 Bézier curves

1.1 Definition

A Bézier curve is a parametric curve defined by:

$$\mathbf{C}(u) = \sum_{i=0}^{n} B_{i,n}(u)\,\mathbf{P}_i \qquad 0 \leq u \leq 1 \tag{1}$$

where p is the order of the curve, the coefficients $\mathbf{P}_i$ are called control points, and $B_{i,n}$ are the basis functions, which are n-th degree Bernstein polynomials given by the explicit formula:

$$B_{i,n}(u) = \binom{n}{i} (1-u)^{n-i}\, u^i = \frac{n!}{i!\,(n-i)!} (1-u)^{n-i}\, u^i \tag{2}$$

Equivalently, the Bernstein polynomials of degree n can be defined in a recursive way blending together two Bernstein polynomials of degree $n-1$:

$$B_{i,n}(u) = (1-u)\, B_{i,n-1}(u) + u\, B_{i-1,n-1}(u) \tag{3}$$

1.2 Mathematical properties of Bernstein polynomials

Here is a list of some important properties of Bernstein polynomials:

1. $B_{i,n}(u)$ is a polynomial of degree n.
2. Symmetry of the basis functions:

   $$B_{i,n}(u) = B_{n-i,n}(1-u)$$

3. Global support:

   $$B_{i,n}(u) > 0 \text{ for } u \in (0,1)$$

4. Non-negativity:

   $$B_{i,n}(u) \geq 0 \quad \text{for all } i,\, p, \text{ and } u \in [0,1]$$

   In addition, $B_{i,n}(u) = 0$ for $i < 0$ or $i > n$ by convention (this is used in some proofs).
5. Partition of unity:

   $$\sum_{i=0}^{n} B_{i,n}(u) = (1-t+t)^n = 1 \text{ for all } u \in [0,1]$$

6. Continuity and differentiability:

   Bernstein polynomials are continuous and infinitely differentiable.
7. Extrema:

   $B_{i,n}(u)$ attains exactly one maximum value in the interval $u \in [0,1]$.
8. The first derivative of the Bernstein polynomials is given by:

   $$B'_{i,n}(u) = \frac{\mathrm{d}B_{i,n}}{\mathrm{d}u} = n\left(B_{i-1,n-1}(u) - B_{i,n-1}(u)\right)$$

9. $B_{0,n}(u=0) = 1$ and $B_{i,n}(u=0) = 0$ for $i \neq 0$.
10. $B_{n,n}(u=1) = 1$ and $B_{i,n}(u=1) = 0$ for $i \neq n$.

1.3 Mathematical properties of Bézier curves

Here is a list of some important properties of Bézier curves:

1. A Bézier curve of degree n is defined by $n+1$ control points and the components of $\mathbf{C}(u)$ are polynomials of degree n.
2. Affine invariance:

   Bézier curves are invariant under affine transformations such as rotations, displacements and scalings. This means that one can apply an affine transformation to the Bézier curve by applying it to its set of control points.
3. Convex hull property:

   All the points of a Bézier curve are contained in the *convex hull* of its control points. The convex hull of a set of points $P = \{\mathbf{P}_0, \ldots, \mathbf{P}_N\}$ is denoted by $\mathcal{CH}(P)$ and it is the set of all the convex combinations of points:

$$\mathcal{CH}(P) = \left\{ \mathbf{C} = \sum_{k=0}^{N} a_k \, \mathbf{P}_k \text{ such that } \sum_{k=0}^{N} a_k = 1 \text{ and } a_k \geq 0 \text{ for } k = 0, 1, \ldots, N \right\}$$

   The convex hull property follows from the non-negativity and the partition-of-unity properties of the basis functions.
4. Global modification scheme:

   Modifying any of the interior control points will affect the location of all the points of the Bézier curve except at $u = 0$ and $u = 1$.
5. The polygon formed by the set of control points is known as *control polygon*. The control polygon represents a piecewise linear approximation to the Bézier curve.
6. Variation diminishing property:

   No straight line (or plane in three dimensions) intersects the Bézier curve more times than it intersects its control polygon. An intuitive explanation of this property is that the Bézier curve does not wiggle more than its control polygon.
7. Continuity and differentiability:

   Bézier curves are continuous and infinitely differentiable.
8. First and higher order derivatives:

   The first derivative of a Bézier curve is given by:

$$\frac{\mathrm{d}\mathbf{C}}{\mathrm{d}u} = \sum_{i=0}^{n} B'_{i,n}(u) \, \mathbf{P}_i$$

$$\frac{\mathrm{d}\mathbf{C}}{\mathrm{d}u} = \sum_{i=0}^{n} n \left(B_{i-1,n-1}(u) - B_{i,n-1}(u) \right) \mathbf{P}_i$$

$$\frac{\mathrm{d}\mathbf{C}}{\mathrm{d}u} = n \sum_{i=0}^{n-1} B_{i,n-1}(u) \, (\mathbf{P}_{i+1} - \mathbf{P}_i)$$

   Note that the first derivative of a Bézier curve of degree n is another Bézier curve of degree $n-1$ and it is called the *hodograph* of the original Bézier curve:

$$\frac{\mathrm{d}\mathbf{C}}{\mathrm{d}u} = \sum_{i=0}^{n-1} B_{i,n-1}(u) \, \mathbf{P}_i^{(1)} \quad \text{with} \quad \mathbf{P}_i^{(1)} = n \, (\mathbf{P}_{i+1} - \mathbf{P}_i)$$

Since the derivative of a Bézier curve is another Bézier curve, we can differentiate the original curve recursively to compute its k-th derivative:

$$\frac{\mathrm{d}^k \mathbf{C}}{\mathrm{d}u^k} = \sum_{i=0}^{n-k} B_{i,n-k}(u)\, \mathbf{P}_i^{(k)}$$

where:

$$\mathbf{P}_i^{(k)} = \begin{cases} \mathbf{P}_i & \text{if } k = 0 \\ (n-k+1)\,(\mathbf{P}_{i+1}^{(k-1)} - \mathbf{P}_i^{(k)}) & \text{if } k > 0 \end{cases}$$

9. Endpoint interpolation:

   The start and end points of a Bézier curve coincide with the first and last control points, respectively:

$$\mathbf{C}(u=0) = \mathbf{P}_0$$
$$\mathbf{C}(u=1) = \mathbf{P}_n$$

10. End point tangency:

    The Bézier curve is tangent to the control polygon at the endpoints:

$$\left.\frac{\mathrm{d}\mathbf{C}}{\mathrm{d}u}\right|_{u=0} = n\,(\mathbf{P}_1 - \mathbf{P}_0)$$
$$\left.\frac{\mathrm{d}\mathbf{C}}{\mathrm{d}u}\right|_{u=1} = n\,(\mathbf{P}_n - \mathbf{P}_{n-1})$$

11. End point curvature:

    The second derivative of Bézier curve at its endpoints in given by:

$$\left.\frac{\mathrm{d}^2\mathbf{C}}{\mathrm{d}u^2}\right|_{u=0} = n\,(n-1)\;\big[(\mathbf{P}_2 - \mathbf{P}_0) - 2\,(\mathbf{P}_1 - \mathbf{P}_0)\big]$$
$$\left.\frac{\mathrm{d}^2\mathbf{C}}{\mathrm{d}u^2}\right|_{u=1} = n\,(n-1)\;\big[(\mathbf{P}_{n-2} - \mathbf{P}_n) - 2\,(\mathbf{P}_{n-1} - \mathbf{P}_n)\big]$$

    The curvature of a Bézier curve at its endpoints is given by:

$$\kappa(u=0) = \left(\frac{n-1}{n}\right) \frac{\big\|(\mathbf{P}_2 - \mathbf{P}_0) \times (\mathbf{P}_1 - \mathbf{P}_0)\big\|}{\|\mathbf{P}_1 - \mathbf{P}_0\|^3}$$
$$\kappa(u=1) = \left(\frac{n-1}{n}\right) \frac{\big\|(\mathbf{P}_{n-2} - \mathbf{P}_n) \times (\mathbf{P}_{n-1} - \mathbf{P}_n)\big\|}{\|\mathbf{P}_{n-1} - \mathbf{P}_n\|^3}$$

2 Bézier surfaces

2.1 Definition

A Bézier surface is a parametric surface defined by:

$$\mathbf{S}(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{m} B_{i,n}(u)\, B_{j,m}(v)\, \mathbf{P}_{i,j} \qquad 0 \le (u,\, v) \le 1 \tag{4}$$

where p and q are the orders of the surface in the u- and v-directions, the coefficients $\mathbf{P}_{i,j}$ are a bidirectional net of control points and $B_{i,n}(u)B_{j,m}(v)$ are the product of univariate Bernstein polynomials.

The u-direction Bernstein polynomials are given by the explicit formula:

$$B_{i,n}(u) = \binom{n}{i} (1-u)^{n-i}\, u^i = \frac{n!}{i!\,(n-i)!} (1-u)^{n-i}\, u^i \tag{5}$$

or by the recursive relation:

$$B_{i,n}(u) = (1-u)\, B_{i,n-1}(u) + u\, B_{i-1,n-1}(u) \tag{6}$$

whereas the v-direction basis functions are defined in an analogous way replacing the variable u by v and the indices i and n by j and m, respectively.

2.2 Mathematical properties of tensor product Bernstein polynomials

Here is a list of some important properties of the tensor product Bernstein polynomials:

1. $B_{i,n}(u)$ is a polynomial of degree p.

   $B_{j,m}(v)$ is a polynomial of degree q.
2. Global support:

   $$B_{i,n}(u)B_{j,m}(v) > 0 \text{ for } (u,v) \in (0,1) \times (0,1)$$
3. Non-negativity:

   $$B_{i,n}(u)B_{j,m}(v) \ge 0 \quad \text{for all } i,\, j,\, p,\, q, \text{ and } (u,v) \in [0,1] \times [0,1]$$
4. Partition of unity:

   $$\sum_{i=0}^{n}\sum_{j=0}^{m} B_{i,n}(u)B_{j,m}(u) = 1 \text{ for all } (u,v) \in [0,1] \times [0,1]$$
5. Continuity and differentiability:

   Tensor product Bernstein polynomials are continuous and infinitely differentiable.
6. Extrema:

   $B_{i,n}(u)B_{j,m}(v)$ attains exactly one maximum value in the region $(u,v) \in [0,1] \times [0,1]$.
7. The first partial derivatives of the tensor product Bernstein polynomials are given by:

   $$\frac{\partial}{\partial u}\big(B_{i,n}(u)B_{j,m}(v)\big) = B_{j,m}(v)\,\frac{\partial}{\partial u}\big(B_{i,n}(u)\big)$$
   $$\frac{\partial}{\partial v}\big(B_{i,n}(u)B_{j,m}(v)\big) = B_{i,n}(u)\,\frac{\partial}{\partial v}\big(B_{j,m}(v)\big)$$

2.3 Mathematical properties of Bézier surfaces

Here is a list of some important properties of Bézier surfaces:

1. A Bézier surface of degree $n \times q$ is defined by $(n+1) \times (m+1)$ control points and the components of $\mathbf{S}(u, v)$ are bivariate polynomials of degree $n \times m$.
2. Affine invariance:

   Bézier surfaces are invariant under affine transformations such as rotations, displacements and scalings. This means that one can apply an affine transformation to the Bézier surface by applying it to its set of control points.
3. Convex hull property:

   All the points of a Bézier surface are contained in the *convex hull* of its control points. The convex hull of a set of points $P = \{\mathbf{P}_0, \ldots, \mathbf{P}_N\}$ is denoted by $\mathcal{CH}(P)$ and it is the set of all the convex combinations of points:

   $$\mathcal{CH}(P) = \left\{ \mathbf{C} = \sum_{k=0}^{N} a_k \, \mathbf{P}_k \text{ such that } \sum_{k=0}^{N} a_k = 1 \text{ and } a_k \geq 0 \text{ for } k = 1, 2, \ldots, N \right\}$$

   The convex hull property follows from the non-negativity and the partition-of-unity properties of the basis functions.
4. Global modification scheme:

   Modifying any of the interior control points will affect the location of all the points of the Bézier surface except at $(u, v) = (0, 0)$, $(u, v) = (0, 1)$, $(u, v) = (1, 0)$, and $(u, v) = (1, 1)$.
5. If triangulated, the net of control points represents a piecewise planar approximation to the Bézier surface.
6. No known variation diminishing property.
7. Continuity and differentiability:

   Bézier surfaces are continuous and infinitely differentiable.
8. First and higher order derivatives:

   The first partial derivatives of a Bézier surface are given by:

   $$\frac{\partial \mathbf{S}}{\partial u} = n \sum_{i=0}^{n-1} \sum_{j=0}^{m} B_{i,n-1}(u) B_{j,m}(v) \, \left(\mathbf{P}_{i+1,j} - \mathbf{P}_{i,j} \right)$$
   $$\frac{\partial \mathbf{S}}{\partial v} = m \sum_{i=0}^{m-1} \sum_{j=0}^{m} B_{i,n}(u) B_{j,m-1}(v) \, \left(\mathbf{P}_{i,j+1} - \mathbf{P}_{i,j} \right)$$

   Since the derivative of a Bézier surface is a new Bézier surfaces, we can differentiate the original surface recursively to compute its (k, l)-th partial derivative:

   $$\frac{\partial^{k+l} \mathbf{S}}{\partial u^k \partial v^l} = \sum_{i=0}^{n-k} \sum_{j=0}^{m-l} B_{i,n-k}(u) B_{j,m-l}(v) \, \mathbf{P}_{i,j}^{(k,l)}$$

   where:

   $$\mathbf{P}_{i,j}^{(k,l)} = \begin{cases} \mathbf{P}_{i,j} & \text{if } k = 0 \\ (m - l + 1) \, (\mathbf{P}_{i,j+1}^{(0,l-1)} - \mathbf{P}_{i,j}^{(0,l-1)}) & \text{if } k > 0 \text{ and } l = 0 \\ (n - k + 1) \, (\mathbf{P}_{i+1,j}^{(k-1,l)} - \mathbf{P}_{i,j}^{(k-1,l)}) & \text{if } k > 0 \text{ and } l > 0 \end{cases}$$
9. Corner point interpolation:

The corners of a clamped B-Spline surface coincide with the corner points of its control net:

$$\mathbf{S}(u = 0,\, v = 0) = \mathbf{P}_{0,0}$$
$$\mathbf{S}(u = 1,\, v = 0) = \mathbf{P}_{n,0}$$
$$\mathbf{S}(u = 0,\, v = 1) = \mathbf{P}_{0,m}$$
$$\mathbf{S}(u = 1,\, v = 1) = \mathbf{P}_{n,m}$$

References

Piegl, L. and W. Tiller (1997). *The NURBS book*. Springer.