A briefing on NURBS curves and surfaces

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This note introduces NURBS curves and surfaces and summarizes their most important mathematical properties. Most of the material used to prepare this note is available in the NURBS book (Piegl and Tiller 1997, Chapter 4).

1 NURBS curves

1.1 Definition

A Non-Uniform Rational Basis Spline (NURBS) curve is a parametric curve defined by:

$$\mathbf{C}(u) = \sum_{i=0}^{n} R_{i,p}(u)\,\mathbf{P}_i \qquad 0 \leq u \leq 1 \tag{1}$$

where p is the order of the curve, the coefficients $\mathbf{P}_i$ are called control points, and $R_{i,p}$ are the rational basis functions given by:

$$R_{i,p}(u) = \frac{N_{i,p}(u)\,w_i}{\sum_{i=0}^{n} N_{i,p}(u)\,w_i} \tag{2}$$

in which w_i are the weights of the control points and $N_{i,p}$ are B-Spline basis functions defined on the non-decreasing knot vector U:

$$U = [u_0, \ldots, u_r] \in \mathbb{R}^{r+1} \quad \text{with} \quad r = n + p + 1 \tag{3}$$

The B-Spline basis functions are given by the recursive relation:

$$N_{i,0}(u) = \begin{cases} 1 & \text{if} \quad u_i \leq u < u_{i+1} \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

$$N_{i,p}(u) = \frac{u - u_i}{u_{i+p} - u_i} N_{i,p-1}(u) + \frac{u_{i+p+1} - u}{u_{i+p+1} - u_{i+1}} N_{i+1,p-1}(u) \tag{5}$$

This recursive relation can yield the $0/0$ quotient which is defined to be zero by convention.

1.2 Mathematical properties of rational basis functions

Here is a list of some important properties of the rational basis functions:

1. The relation $r = n + p + 1$ holds, where $n + 1$ is the number of basis functions and $r + 1$ is the number of elements of the knot vector U.
2. The numerator and denominator of $R_{i,p}(u)$ are, at most, polynomials of degree p.
3. Local support:
   (a) $R_{i,p}(u) = 0$ if u is outside the interval $[u_i, u_{i+p+1})$.
   (b) In any given knot interval, $[u_{i_0}, u_{i_0+1})$, at most $(p + 1)$ basis functions are nonzero, namely the $R_{i,p}(u)$ with $i_0 - p \leq i \leq i_0$.
4. Non-negativity:

   $$R_{i,p}(u) \geq 0 \quad \text{for all } i,\ p, \text{ and } u \in [0, 1]$$

5. Partition of unity:

$$\sum_{i=0}^{n} R_{i,n}(u) = 1 \text{ for all } u \in [0, 1]$$

In addition, for an arbitrary knot span, $[u_{i_0}, u_{i_0+1})$ we have that:

$$\sum_{i=i_0-p}^{i_0} R_{i,n}(u) = 1 \text{ for all } u \in [u_{i_0}, u_{i_0+1})$$

This means that the sum of the non-zero basis-functions of any knot span is unity.

6. Continuity and differentiability:
   (a) The basis functions are infinitely differentiable in the interior of the knot intervals.
   (b) The basis functions are $p - k$ continuously differentiable at a knot with multiplicity k.

7. Extrema:

   $R_{i,p}(u)$ attains exactly one maximum value in the interval $u \in [0, 1]$.

8. There is no compact and useful expression for the first derivative of the basis functions.

9. There is no compact and useful expression for the k-th order derivative of the basis functions.

10. When the first and last knots have multiplicity $p + 1$ the knot vector is given by:

$$U = [\underbrace{u_0,\, u_1,\, \ldots,\, u_p}_{p+1},\, \underbrace{u_{p+1},\, \ldots,\, u_n}_{n-p},\, \underbrace{u_{n+1},\, \ldots,\, u_{n+p},\, u_{n+p+1}}_{p+1}]$$

where:

$$u_0 = \cdots = u_p = 0$$
$$u_{n+1} = \cdots = u_{n+p+1} = 1$$

and is called a *clamped knot vector*. Basis functions of clamped knot vectors satisfy two additional properties:

   (a) $R_{0,p}(u = 0) = 1$ and $R_{i,p}(u = 0) = 0$ for $i \neq 0$
   (b) $R_{n,p}(u = 1) = 1$ and $R_{i,p}(u = 1) = 0$ for $i \neq n$

11. If all the control point weights are equal, then rational basis functions reduce to B-Spline basis functions, that is, $R_{i,p}(u) = N_{i,p}(u)$

12. If all the control point weights are equal, the knot vector is clamped and $p = n$, then rational basis function reduce to Bernstein polynomials, that is, $R_{i,p}(u) = B_{i,n}(u)$

1.3 Mathematical properties of NURBS curves

Here is a list of some important properties of NURBS curves:

1. $\mathbf{C}(u)$ is a piecewise curve and its components are ratios of polynomials of, at most, degree p.

2. The order p, number of control points $n + 1$ and number of knots $r + 1$ are related according to $r = n + p + 1$.

3. NURBS curves contain B-Spline and rational/non-rational Bézier curves as special cases:
   (a) If all the control point weights are equal then the NURBS curve is reduced to a B-Spline curve.
   (b) If $n = p$ and the knot vector is clamped then the NURBS curve is reduced to a rational Bézier curve.
   (c) If all the control point weights are equal, $p = n$, and the knot vector is clamped then the NURBS curve is reduced to a polynomial Bézier curve.

4. Affine invariance:

   NURBS curves are invariant under affine transformations such as rotations, displacements and scalings. This means that one can apply an affine transformation to the NURBS curve by applying it to its set of control points.

   Given an affine transformation $\phi(\mathbf{v}) = \mathrm{A}\mathbf{v} + \mathbf{b}$ we have that:

$$
\begin{aligned}
\phi\big(\mathbf{C}(u)\big) &= \phi\Big(\sum_{i=0}^{n} R_{i,p}(u)\,\mathbf{P}_i\Big) = \\
&= \mathrm{A}\sum_{i=0}^{n} R_{i,p}(u)\,\mathbf{P}_i + \mathbf{b} = \\
&= \mathrm{A}\sum_{i=0}^{n} R_{i,p}(u)\,\mathbf{P}_i + \mathbf{b}\sum_{i=0}^{n} R_{i,p}(u) \qquad \text{(by partition of unity)} \\
&= \sum_{i=0}^{n} R_{i,p}(u)\,(\mathrm{A}\mathbf{P}_i + \mathbf{b}) \\
&= \sum_{i=0}^{n} R_{i,p}(u)\,\phi\big(\mathbf{P}_i\big)
\end{aligned}
$$

5. Convex hull property:

   All the points of a NURBS curve are contained in the *convex hull* of its control points. The convex hull of a set of points $P = \{\mathbf{P}_0, \ldots, \mathbf{P}_N\}$ is denoted by $\mathcal{CH}(P)$ and it is the set of all the convex combinations of points:

$$
\mathcal{CH}(P) = \Big\{\mathbf{C} = \sum_{k=0}^{N} a_k\,\mathbf{P}_k \text{ such that } \sum_{k=0}^{N} a_k = 1 \text{ and } a_k \geq 0 \text{ for } k = 0,\, 1,\, \ldots,\, N\Big\}
$$

   The convex hull property follows from the non-negativity and the partition-of-unity properties of the basis functions.

6. Strong convex hull property:

   If $u \in [u_{i_0},\, u_{i_0+1})$, then $\mathbf{C}(u)$ is contained in the convex hull of the control points $\boldsymbol{P}_i$ with $i_0 - p \leq i \leq i_0$.

7. Local modification scheme:

   Modifying the control point $\mathbf{P}_i$ or weight w_i affects $\mathbf{C}(u)$ only in the interval $[u_i, u_{i+p+1})$. This property follows from the fact that $R_{i,p}(u) = 0$ if u is outside the interval $[u_i, u_{i+p+1})$ and it implies that the shape of a NURBS can be modified locally without changing its shape globally.

8. The polygon formed by the set of control points is known as *control polygon.* The control polygon represents a piecewise linear approximation to the NURBS curve.

9. Variation diminishing property:

   No straight line (or plane in three dimensions) intersects the B-Spline curve more times than it intersects its control polygon. An intuitive explanation of this property is that the B-Spline curve does not wiggle more than its control polygon.

10. Continuity and differentiability:

    (a) NURBS curves are infinitely differentiable in the interior of the knot intervals.

    (b) NURBS curves are *at least* $p - k$ continuously differentiable at a knot with multiplicity k.

11. Homogeneous coordinates:

N-dimensional rational functions with a common denominator can be represented as polynomial functions in $(N+1)$-dimensional space using *homogeneous coordinates*.

Consider a three-dimensional point $\mathbf{P} = (x,\, y,\, z)$. $\mathbf{P}$ can be written in four-dimensional space as $\mathbf{P}^w = (wx,\, wy,\, wz,\, w) = (X,\, Y,\, Z,\, W)$, where $w \neq 0$. The point $\mathbf{P}$ can be obtained from $\mathbf{P}^w$ using the mapping $\mathcal{H}$ given by:

$$\mathbf{P} = \mathcal{H}\{\mathbf{P}^w\} = \mathcal{H}\{(X,\, Y,\, Z,\, W)\} = \left(\frac{X}{W},\, \frac{Y}{W},\, \frac{Z}{W}\right)$$

This theory can be used to represent a N-dimensional NURBS curve as a $(N+1)$-dimensional B-Spline curve and then retrieve the coordinates of the NURBS curve using the $\mathcal{H}$ mapping.

Indeed, consider a NURBS curve $\mathbf{C}(u)$ with control points $\mathbf{P}_i$ and weights w_i. It is possible to construct a set of weighted control points $\mathbf{P}_i^w = (w_i x_i,\, w_i y_i,\, w_i z_i,\, w_i)$ and define the corresponding non-rational B-Spline curve in four-dimensional space as:

$$\mathbf{C}^w(u) = \sum_{i=0}^{n} N_{i,p}(u)\, \mathbf{P}_i^w \qquad 0 \leq u \leq 1$$

In order to compute the coordinates of the original NURBS curve it is possible to use standard B-Spline algorithms to evaluate the coordinates of $\mathbf{C}^w(u)$ in homogeneous space and then apply the $\mathcal{H}$ mapping:

$$\begin{aligned}
\mathbf{C}(u) &= \mathcal{H}\{\mathbf{C}^w\} \\
&= \mathcal{H}\left\{ \sum_{i=0}^{n} N_{i,p}(u)\, \mathbf{P}_i^w \right\} \\
&= \frac{\sum_{i=0}^{n} N_{i,p}(u)\, w_i\, \mathbf{P}_i}{\sum_{i=0}^{n} N_{i,p}(u)\, w_i} \\
&= \sum_{i=0}^{n} R_{i,p}(u)\, \mathbf{P}_i
\end{aligned}$$

This property is important to evaluate the derivatives of NURBS curves.

12. First and higher order derivatives:

The derivatives of a NURBS curve $\mathbf{C}(u)$ can be expressed in terms of the derivatives of the corresponding B-Spline in homogeneous space $\mathbf{C}^w(u)$. Indeed, let:

$$\mathbf{C}(u) = \frac{\sum_{i=0}^{n} N_{i,p}(u)\, w_i\, \mathbf{P}_i}{\sum_{i=0}^{n} N_{i,p}(u)\, w_i} = \frac{\mathbf{A}(u)}{w(u)}$$

where $\mathbf{A}(u)$ is the vector function whose coordinates are the three first elements of $\mathbf{C}^w(u)$ and $w(u)$ is the scalar function with the fourth element of $\mathbf{C}^w(u)$.

To compute the first derivatives, first rearrange the previous expression to avoid the denominator, then differentiate both sides of the equation and, finally, solve for $\mathbf{C}'(u)$:

$$\begin{aligned}
&w\, \mathbf{C} = \mathbf{A} \\
&\big[w\, \mathbf{C}\big]' = \mathbf{A}' \\
&w'\, \mathbf{C} + w\, \mathbf{C}' = \mathbf{A}' \\
&\mathbf{C}' = \frac{1}{w}\left(\mathbf{A}' - w'\, \mathbf{C}\right)
\end{aligned}$$

To compute the k-th order derivatives, differentiate k-times using Leibniz' rule for product differentiation and then solve for $\mathbf{C}^{(k)}(u)$:

$$
\begin{aligned}
& w\,\mathbf{C} = \mathbf{A} \\
& \left[w\,\mathbf{C}\right]^{(k)} = \mathbf{A}^{(k)} \\
& \sum_{i=0}^{k} \binom{k}{i} w^{(i)}\,\mathbf{C}^{(k-i)} = \mathbf{A}^{(k)} \\
& \mathbf{C}^{(k)} = \frac{1}{w}\left(\mathbf{A}^{(k)} - \sum_{i=1}^{k} \binom{k}{i} w^{(i)}\,\mathbf{C}^{(k-i)}\right)
\end{aligned}
$$

The derivatives of $\mathbf{A}(u)$ and $w(u)$ can be easily computed by differentiating $\mathbf{C}^w(u)$:

$$
\left[\mathbf{C}^w\right]^{(k)}(u) = \sum_{i=0}^{n} N_{i,p}^{(k)}(u)\,\mathbf{P}_i^w
$$

13. End point interpolation:

    The start and end points of a clamped NURBS curve coincide with the first and last control points, respectively.

$$
\begin{aligned}
& \mathbf{C}(u=0) = \mathbf{P}_0 \\
& \mathbf{C}(u=1) = \mathbf{P}_n
\end{aligned}
$$

14. End point tangency:

    A clamped NURBS curve is tangent to the control polygon at the endpoints.

$$
\begin{aligned}
& \left.\frac{\mathrm{d}\mathbf{C}}{\mathrm{d}u}\right|_{u=0} = \left(\frac{p}{u_{p+1}}\right)\left(\frac{w_1}{w_0}\right)(\mathbf{P}_1 - \mathbf{P}_0) \\
& \left.\frac{\mathrm{d}\mathbf{C}}{\mathrm{d}u}\right|_{u=1} = \left(\frac{p}{1-u_n}\right)\left(\frac{w_{n-1}}{w_n}\right)(\mathbf{P}_n - \mathbf{P}_{n-1})
\end{aligned}
$$

15. End point curvature:

    The second derivative of NURBS curve at its endpoints in given by:

$$
\begin{aligned}
& \left.\frac{\mathrm{d}^2\mathbf{C}}{\mathrm{d}u^2}\right|_{u=0} = \frac{p(p-1)}{u_{p+1}}\left[\left(\frac{1}{u_{p+2}}\right)\left(\frac{w_2}{w_0}\right)(\mathbf{P}_2 - \mathbf{P}_0) - \left(\frac{1}{u_{p+1}} + \frac{1}{u_{p+2}}\right)\left(\frac{w_1}{w_0}\right)(\mathbf{P}_1 - \mathbf{P}_0)\right] + \frac{2p^2}{u_{p+1}^2}\left(\frac{w_1}{w_0}\right)\left(1 - \frac{w_1}{w_0}\right)(\mathbf{P}_1 - \mathbf{P}_0) \\
& \left.\frac{\mathrm{d}^2\mathbf{C}}{\mathrm{d}u^2}\right|_{u=1} = \frac{p(p-1)}{1-u_n}\left[\left(\frac{1}{1-u_{n-1}}\right)\left(\frac{w_{n-2}}{w_n}\right)(\mathbf{P}_{n-2} - \mathbf{P}_n) - \left(\frac{1}{1-u_n} + \frac{1}{1-u_{n-1}}\right)\left(\frac{w_{n-1}}{w_n}\right)(\mathbf{P}_{n-1} - \mathbf{P}_n)\right] + \frac{2p^2}{(1-u_n)^2}\left(\frac{w_{n-1}}{w_n}\right)\left(1 - \frac{w_{n-1}}{w_n}\right)(\mathbf{P}_{n-1} - \mathbf{P}_n)
\end{aligned}
$$

    The curvature of a clamped NURBS curve at its endpoints is given by:

$$
\begin{aligned}
& \kappa(u=0) = \left(\frac{p-1}{p}\right)\left(\frac{u_{p+1}}{u_{p+2}}\right)\left(\frac{w_0\,w_2}{w_1^2}\right)\frac{\left\|(\mathbf{P}_2 - \mathbf{P}_0)\times(\mathbf{P}_1 - \mathbf{P}_0)\right\|}{\left\|\mathbf{P}_1 - \mathbf{P}_0\right\|^3} \\
& \kappa(u=1) = \left(\frac{p-1}{p}\right)\left(\frac{1-u_n}{1-u_{n-1}}\right)\left(\frac{w_n\,w_{n-2}}{w_{n-1}^2}\right)\frac{\left\|(\mathbf{P}_{n-2} - \mathbf{P}_n)\times(\mathbf{P}_{n-1} - \mathbf{P}_n)\right\|}{\left\|\mathbf{P}_{n-1} - \mathbf{P}_n\right\|^3}
\end{aligned}
$$

2 NURBS surfaces

2.1 Definition

A Non-Uniform Rational Basis Spline (NURBS) surface is a parametric surface defined by:

$$\mathbf{S}(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{m} R_{i,j}^{p,q}(u,v)\,\mathbf{P}_{i,j} \qquad 0 \le (u,\, v) \le 1 \tag{6}$$

where p and q are the orders of the surface in the u- and v-directions, the coefficients $\mathbf{P}_{i,j}$ are a bidirectional net of control points and $R_{i,j}^{p,q}(u,v)$ are the rational basis functions given by:

$$R_{i,j}^{p,q}(u,v) = \frac{N_{i,p}(u)N_{j,q}(v)\,w_{i,j}}{\sum\limits_{i=0}^{n}\sum\limits_{j=0}^{m} N_{i,p}(u)N_{j,p}(v)\,w_{i,j}} \tag{7}$$

in which $w_{i,j}$ are the weights of the control points and $N_{i,p}(u)N_{j,q}(v)$ are the product of univariate B-Spline basis functions defined on the non-decreasing knot vectors U and V:

$$U = [u_0,\, \ldots,\, u_r] \in \mathbb{R}^{r+1} \quad \text{with} \quad r = n + p + 1 \tag{8}$$

$$V = [v_0,\, \ldots,\, v_s] \in \mathbb{R}^{s+1} \quad \text{with} \quad s = m + q + 1 \tag{9}$$

The u-direction basis functions are given by the recursive relation:

$$N_{i,0}(u) = \begin{cases} 1 & \text{if} \quad u_i \le u < u_{i+1} \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

$$N_{i,p}(u) = \frac{u - u_i}{u_{i+p} - u_i} N_{i,p-1}(u) + \frac{u_{i+p+1} - u}{u_{i+p+1} - u_{i+1}} N_{i+1,p-1}(u) \tag{11}$$

whereas the v-direction basis functions are defined in an analogous way replacing the variable u by v and the indices i and p by j and q, respectively.

2.2 Mathematical properties of bivariate rational basis functions

Here is a list of some important properties of the bivariate rational basis functions:

1. The relation $r = n+p+1$ holds, where $n+1$ is the number of basis functions in the u-direction and $r + 1$ is the number of elements of the knot vector U.

   The relation $s = m + q + 1$ holds, where $m + 1$ is the number of basis functions in the v-direction and $s + 1$ is the number of elements of the knot vector V.

2. The numerator and denominator of $R_{i,j}^{p,q}(u,v)$ are, at most, polynomials of degree $p \times q$.

3. Local support:

   (a) $R_{i,j}^{p,q}(u,v) = 0$ if (u,v) is outside the rectangle $[u_i, u_{i+p+1}) \times [v_j, v_{j+q+1})$.

   (b) In any given knot rectangle $[u_{i_0}, u_{i_0+1}) \times [v_{j_0}, v_{j_0+1})$, at most $(p+1)(q+1)$ basis functions are nonzero, namely the $R_{i,j}^{p,q}(u,v)$ with $i_0 - p \le i \le i_0$ and $j_0 - q \le j \le j_0$.

4. Non-negativity:

   $$R_{i,j}^{p,q}(u,v) \ge 0 \quad \text{for all } i,\, j,\, p,\, q, \text{ and } (u,v) \in [0,1] \times [0,1]$$

5. Partition of unity:

   $$\sum_{i=0}^{n}\sum_{j=0}^{m} R_{i,j}^{p,q}(u,v) = 1 \text{ for all } (u,v) \in [0,1] \times [0,1]$$

In addition, for an arbitrary knot rectangle, $[u_{i_0}, u_{i_0+1}) \times [v_{j_0}, v_{j_0+1})$ we have that:

$$\sum_{i=i_0-p}^{i_0} \sum_{j=j_0-q}^{j_0} R_{i,j}^{p,q}(u,v) = 1 \text{ for all } (u,v) \in [u_{i_0}, u_{i_0+1}) \times [v_{j_0}, v_{j_0+1})$$

This means that the sum of the non-zero basis-functions of any knot rectangle is unity.

6. Continuity and differentiability:
   (a) The basis functions are infinitely differentiable in the interior of the knot rectangles formed by the U and V vectors.
   (b) The basis functions are $p-k$ $(q-k)$ continuously differentiable in the u-direction (v-direction) at a u-knot (v-knot) with multiplicity k.
7. Extrema:

   $R_{i,j}^{p,q}(u,v)$ attains exactly one maximum value in the region $(u,v) \in [0,1] \times [0,1]$.
8. There is no compact and useful expression for the first partial derivatives of the basis functions.
9. There is no compact and useful expression for the (k,l)-th order partial derivatives of the basis functions.
10. If all the control point weights are equal, then rational basis functions reduce to B-Spline basis functions, that is, $R_{i,j}^{p,q}(u,v) = N_{i,p}(u)N_{j,q}(v)$.
11. If all the control point weights are equal, the knot vectors are clamped, and $(p,q) = (n,m)$, then rational basis function reduce to Bernstein polynomials, that is, $R_{i,j}^{p,q}(u,v) = B_{i,n}(u)B_{j,m}(v)$.

2.3 Mathematical properties of NURBS surface

Here is a list of some important properties of NURBS surfaces:

1. $\mathbf{S}(u,v)$ is a piecewise surface and its components are ratios of bivariate polynomials of, at most, degree $p \times q$.
2. In the u-direction: the degree p, the number of control points $n+1$ and the number of knots $r+1$ are related according to $r = n+p+1$

   In the v-direction: the degree q, the number of control points $m+1$ and the number of knots $s+1$ are related according to $s = m+q+1$
3. NURBS surfaces contain B-Spline and rational/non-rational Bézier surfaces as special cases:
   (a) If all the control point weights are equal then the NURBS surface is reduced to a B-Spline surface.
   (b) If $(p,q) = (n,m)$ and the knot vectors are clamped then the NURBS surface is reduced to a rational Bézier surface.
   (c) If all the control point weights are equal, $(p,q) = (n,m)$, and the knot vectors are clamped then the NURBS surface is reduced to a polynomial Bézier surface.
4. Affine invariance:

   NURBS surfaces are invariant under affine transformations such as rotations, displacements and scalings. This means that one can apply an affine transformation to the NURBS surface by applying it to its set of control points.

Given an affine transformation $\phi(\mathbf{v}) = \mathrm{A}\mathbf{v} + \mathbf{b}$ we have that:

$$\begin{aligned}
\phi\big(\mathbf{S}(u,v)\big) &= \phi\big(\sum_{i=0}^{n}\sum_{j=0}^{m} R_{i,j}^{p,q}(u,v)\,\mathbf{P}_{i,j}\big) = \\
&= \mathrm{A}\sum_{i=0}^{n}\sum_{j=0}^{m} R_{i,j}^{p,q}(u,v)\,\mathbf{P}_{i,j} + \mathbf{b} = \\
&= \mathrm{A}\sum_{i=0}^{n}\sum_{j=0}^{m} R_{i,j}^{p,q}(u,v)\,\mathbf{P}_{i,j} + \mathbf{b}\sum_{i=0}^{n}\sum_{j=0}^{m} R_{i,j}^{p,q}(u,v) \qquad \text{(by partition of unity)} \\
&= \sum_{i=0}^{n}\sum_{j=0}^{m} R_{i,j}^{p,q}(u,v)\,(\mathrm{A}\mathbf{P}_{i,j} + \mathbf{b}) \\
&= \sum_{i=0}^{n}\sum_{j=0}^{m} R_{i,j}^{p,q}(u,v)\,\phi\big(\mathbf{P}_{i,j}\big)
\end{aligned}$$

5. Convex hull property:

   All the points of a NURBS surface are contained in the *convex hull* of its control points. The convex hull of a set of points $P = \{\mathbf{P}_0, \ldots, \mathbf{P}_N\}$ is denoted by $\mathcal{CH}(P)$ and it is the set of all the convex combinations of points:

   $$\mathcal{CH}(P) = \Big\{\mathbf{C} = \sum_{k=0}^{N} a_k\,\mathbf{P}_k \text{ such that } \sum_{k=0}^{N} a_k = 1 \text{ and } a_k \geq 0 \text{ for } k = 0,\, 1,\, \ldots,\, N\Big\}$$

   The convex hull property follows from the non-negativity and the partition-of-unity properties of the basis functions.

6. Strong convex hull property:

   If $(u,v) \in [u_{i_0},\, u_{i_0+1}) \times [v_{j_0},\, v_{j_0+1})$, then $\mathbf{S}(u,v)$ is contained in the convex hull of the control points $\boldsymbol{P}_{i,j}$ with $i_0 - p \leq i \leq i_0$ and $j_0 - q \leq j \leq j_0$.

7. Local modification scheme:

   Modifying the control point $\mathbf{P}_{i,j}$ or weight $w_{i,j}$ affects $\mathbf{S}(u,v)$ only in the rectangle $[u_i, u_{i+p+1}) \times [v_j, v_{j+q+1})$. This property follows from the fact that $R_{i,j}^{p,q}(u,v) = 0$ if (u,v) is outside the rectangle $[u_i, u_{i+p+1}) \times [v_j, v_{j+q+1})$ and it implies that the shape of a NURBS can be modified locally without changing its shape globally.

8. If triangulated, the net of control points represents a piecewise planar approximation to the NURBS surface.

9. No known variation diminishing property.

10. Continuity and differentiability:

    (a) NURBS surfaces are infinitely differentiable in the interior of the knot rectangles formed by the U and V vectors.

    (b) NURBS surfaces are $p - k$ $(q - k)$ continuously differentiable in the u-direction (v-direction) at a u-knot (v-knot) with multiplicity k.

11. Homogeneous coordinates:

    N-dimensional rational functions with a common denominator can be represented as polynomial functions in $(N+1)$-dimensional space using *homogeneous coordinates.*

    Consider a three-dimensional point $\mathbf{P} = (x,\, y,\, z)$. $\mathbf{P}$ can be written in four-dimensional space as $\mathbf{P}^w = (wx,\, wy,\, wz,\, w) = (X,\, Y,\, Z,\, W)$, where $w \neq 0$. The point $\mathbf{P}$ can be obtained from $\mathbf{P}^w$ using the mapping $\mathcal{H}$ given by:

    $$\mathbf{P} = \mathcal{H}\{\mathbf{P}^w\} = \mathcal{H}\{(X,\, Y,\, Z,\, W)\} = \left(\frac{X}{W},\, \frac{Y}{W},\, \frac{Z}{W}\right)$$

This theory can be used to represent a N-dimensional NURBS surface as a $(N+1)$-dimensional B-Spline surface and then retrieve the coordinates of the NURBS surface using the $\mathcal{H}$ mapping.

Indeed, consider a NURBS surface $\mathbf{S}(u,v)$ with control points $\mathbf{P}_{i,j}$ and weights $w_{i,j}$. It is possible to construct a set of weighted control points $\mathbf{P}^w_{i,j} = (w_{i,j}x_{i,j},\, w_{i,j}y_{i,j},\, w_{i,j}z_{i,j},\, w_{i,j})$ and define the corresponding non-rational B-Spline surface in four-dimensional space as:

$$\mathbf{S}^w(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{m} R^{p,q}_{i,j}(u,v)\,\mathbf{P}^w_{i,j} \qquad 0 \le (u,v) \le 1$$

In order to compute the coordinates of the original NURBS surface it is possible to use standard B-Spline algorithms to evaluate the coordinates of $\mathbf{S}^w(u,v)$ in homogeneous space and then apply the $\mathcal{H}$ mapping:

$$\begin{aligned}
\mathbf{S}(u,v) &= \mathcal{H}\{\mathbf{S}^w\} \\
&= \mathcal{H}\left\{ \sum_{i=0}^{n}\sum_{j=0}^{m} N_{i,p}(u)N_{j,q}(u,v)\,\mathbf{P}^w_{i,j} \right\} \\
&= \frac{\sum_{i=0}^{n}\sum_{j=0}^{m} N_{i,p}(u)N_{j,q}(u)\,w_{i,j}\,\mathbf{P}_{i,j}}{\sum_{i=0}^{n}\sum_{j=0}^{m} N_{i,p}(u)N_{j,q}(v)\,w_{i,j}} \\
&= \sum_{i=0}^{n}\sum_{j=0}^{m} R^{p,q}_{i,j}(u,v)\,\mathbf{P}_{i,j}
\end{aligned}$$

This property is important to evaluate the derivatives of NURBS surfaces.

12. First and higher order derivatives:

The derivatives of a NURBS surface $\mathbf{S}(u,v)$ can be expressed in terms of the derivatives of the corresponding B-Spline in homogeneous space $\mathbf{S}^w(u,v)$. Indeed, let:

$$\mathbf{S}(u,v) = \frac{\sum_{i=0}^{n}\sum_{j=0}^{m} N_{i,p}(u)N_{j,q}(v)\,w_{i,j}\,\mathbf{P}_{i,j}}{\sum_{i=0}^{n}\sum_{j=0}^{m} N_{i,p}(u)N_{j,q}(u)\,w_{i,j}} = \frac{\mathbf{A}(u,v)}{w(u,v)}$$

where $\mathbf{A}(u,v)$ is the vector function whose coordinates are the three first elements of $\mathbf{S}^w(u,v)$ and $w(u,v)$ is the scalar function with the fourth element of $\mathbf{S}^w(u,v)$.

To compute the first partial derivatives, first rearrange the previous expression to avoid the denominator, then differentiate both sides of the equation and, finally, solve for $\frac{\partial \mathbf{S}}{\partial \alpha}$, where α is either u or v:

$$\begin{aligned}
&w\,\mathbf{S} = \mathbf{A} \\
&\big[w\,\mathbf{S}\big]_\alpha = \mathbf{A}_\alpha \\
&w_\alpha\,\mathbf{S} + w\,\mathbf{S}_\alpha = \mathbf{A}_\alpha \\
&\frac{\partial \mathbf{S}}{\partial \alpha} = \mathbf{S}_\alpha = \frac{1}{w}\left(\mathbf{A}_\alpha - w_\alpha\,\mathbf{S}\right)
\end{aligned}$$

To compute the (k,l)-th order partial derivatives, differentiate (k,l)-times using Leibniz' rule for product differentiation and then solve for $\mathbf{S}^{(k)}(u,v)$:

$$
\begin{aligned}
& w\,\mathbf{S} = \mathbf{A} \\
& \big[w\,\mathbf{S}\big]^{(k,l)} = \mathbf{A}^{(k,l)} \\
& \sum_{i=0}^{k}\sum_{j=0}^{l}\binom{k}{i}\binom{l}{j} w^{(i,j)}\,\mathbf{S}^{(k-i,l-j)} = \mathbf{A}^{(k,l)} \\
& \mathbf{S}^{(k,l)} = \frac{1}{w}\left[\mathbf{A}^{(k,l)} - \sum_{i=1}^{k}\binom{k}{i} w^{(i,0)}\,\mathbf{S}^{(k-i,l)} - \sum_{j=1}^{l}\binom{l}{j} w^{(0,j)}\,\mathbf{S}^{(k,l-j)} - \sum_{i=1}^{k}\sum_{j=1}^{l}\binom{k}{i}\binom{l}{j} w^{(i,j)}\,\mathbf{S}^{(k-i,l-j)}\right]
\end{aligned}
$$

The derivatives of $\mathbf{A}(u,v)$ and $w(u,v)$ can be easily computed by differentiating $\mathbf{S}^w(u,v)$:

$$
[\mathbf{S}^w]^{(k,l)}(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{m} N_{i,p}^{(k)}(u) N_{j,q}^{(l)}(u)\,\mathbf{P}_{i,j}^{w}
$$

13. Corner point interpolation:

The corners of a clamped NURBS surface coincide with the corner points of its control net:

$$
\begin{aligned}
& \mathbf{S}(u=0,\ v=0) = \mathbf{P}_{0,0} \\
& \mathbf{S}(u=1,\ v=0) = \mathbf{P}_{n,0} \\
& \mathbf{S}(u=0,\ v=1) = \mathbf{P}_{0,m} \\
& \mathbf{S}(u=1,\ v=1) = \mathbf{P}_{n,m}
\end{aligned}
$$

References

Piegl, L. and W. Tiller (1997). *The NURBS book*. Springer.