A briefing on B-Spline curves and surfaces

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This note introduces B-Spline curves and surfaces and summarizes their most important mathematical properties. Most of the material used to prepare this note is available in the NURBS book (Piegl and Tiller 1997, Chapters 2 and 3).

1 B-Spline curves

1.1 Definition

A B-Spline curve, shorthand for basis spline curve, is a parametric curve defined by:

$$\mathbf{C}(u) = \sum_{i=0}^{n} N_{i,p}(u)\,\mathbf{P}_i \qquad 0 \leq u \leq 1 \tag{1}$$

where p is the order of the curve, the coefficients $\mathbf{P}_i$ are called control points, and $N_{i,p}$ are the basis functions defined on the non-decreasing knot vector U:

$$U = [u_0, \ldots, u_r] \in \mathbb{R}^{r+1} \quad \text{with} \quad r = n + p + 1 \tag{2}$$

The B-Spline basis functions are given by the recursive relation:

$$N_{i,0}(u) = \begin{cases} 1 & \text{if} \quad u_i \leq u < u_{i+1} \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

$$N_{i,p}(u) = \frac{u - u_i}{u_{i+p} - u_i} N_{i,p-1}(u) + \frac{u_{i+p+1} - u}{u_{i+p+1} - u_{i+1}} N_{i+1,p-1}(u) \tag{4}$$

This recursive relation can yield the $0/0$ quotient which is defined to be zero by convention.

1.2 Mathematical properties of B-spline basis functions

Here is a list of some important properties of the B-Spline basis functions:

1. The relation $r = n + p + 1$ holds, where $n + 1$ is the number of basis functions and $r + 1$ is the number of elements of the knot vector U.
2. $N_{i,p}(u)$ is, at most, a polynomial of degree p.
3. Local support:
   (a) $N_{i,p}(u) = 0$ if u is outside the interval $[u_i, u_{i+p+1})$. This property is illustrated by the De Boor triangular computation.
   (b) In any given knot interval, $[u_{i_0}, u_{i_0+1})$, at most $(p + 1)$ basis functions are nonzero, namely the $N_{i,p}(u)$ with $i_0 - p \leq i \leq i_0$.
4. Non-negativity:

   $$N_{i,p}(u) \geq 0 \quad \text{for all } i,\ p, \text{ and } u \in [0, 1]$$

5. Partition of unity:

   $$\sum_{i=0}^{n} N_{i,n}(u) = 1 \text{ for all } u \in [0, 1]$$

In addition, for an arbitrary knot span, $[u_{i_0}, u_{i_0+1})$ we have that:

$$\sum_{i=i_0-p}^{i_0} N_{i,n}(u) = 1 \text{ for all } u \in [u_{i_0}, u_{i_0+1})$$

This means that the sum of the non-zero basis-functions of any knot span is unity.

6. Continuity and differentiability:
   (a) The basis functions are infinitely differentiable in the interior of the knot intervals.
   (b) The basis functions are $p-k$ continuously differentiable at a knot with multiplicity k.
7. Extrema:

   $N_{i,p}(u)$ attains exactly one maximum value in the interval $u \in [0,1]$.
8. The first derivative of the basis functions is given by:

   $$N'_{i,p}(u) = \frac{\mathrm{d}N_{i,p}}{\mathrm{d}u} = p\left(\frac{N_{i,p-1}(u)}{u_{i+p}-u_i} - \frac{N_{i+1,p-1}(u)}{u_{i+p+1}-u_{i+1}}\right)$$

   This is proven by induction after a good deal of algebra. This equation can yield the 0/0 quotient which is defined to be zero by convention.
9. The k-th order derivative of the basis functions is given by:

   $$N^{(k)}_{i,p}(u) = \frac{\mathrm{d}^{(k)}N_{i,p}}{\mathrm{d}u^{(k)}} = p\left(\frac{N^{(k-1)}_{i,p-1}(u)}{u_{i+p}-u_i} - \frac{N^{(k-1)}_{i+1,p-1}(u)}{u_{i+p+1}-u_{i+1}}\right)$$

   This is derived by repeated differentiation. This equation can yield the 0/0 quotient which is defined to be zero by convention.
10. When the first and last knots have multiplicity $p+1$ the knot vector is given by:

    $$U = [\underbrace{u_0,\, u_1,\, \ldots,\, u_p}_{p+1},\, \underbrace{u_{p+1},\, \ldots,\, u_n}_{n-p},\, \underbrace{u_{n+1},\, \ldots,\, u_{n+p},\, u_{n+p+1}}_{p+1}]$$

    where:

    $$u_0 = \cdots = u_p = 0$$
    $$u_{n+1} = \cdots = u_{n+p+1} = 1$$

    and is called a *clamped knot vector*. Basis functions of clamped knot vectors satisfy two additional properties:
    (a) $N_{0,p}(u=0) = 1$ and $N_{i,p}(u=0) = 0$ for $i \neq 0$
    (b) $N_{n,p}(u=1) = 1$ and $N_{i,p}(u=1) = 0$ for $i \neq n$

    In the remainder of this note, all knot vectors are understood to be clamped.
11. If the knot vector is clamped and $p = n$, then the B-Spline basis function reduces to Bernstein polynomials, that is, $N_{i,p}(u) = B_{i,n}(u)$. This is true because the recursive definition of the B-Spline basis functions is reduced to the recursive definition of the Bernstein polynomials when the knot vector given by

    $$U = [\underbrace{0,\, \ldots,\, 0}_{p+1},\, \underbrace{1,\, \ldots,\, 1}_{p+1}]$$

1.3 Mathematical properties of B-Spline curves

Here is a list of some important properties of B-Spline curves:

1. $\mathbf{C}(u)$ is a piecewise curve and its components are polynomials of, at most, degree p.

2. The order p, number of control points $n+1$ and number of knots $r+1$ are related according to $r = n + p + 1$.

3. If $n = p$ and the knot vector is clamped then $\mathbf{C}(u)$ is a Bézier curve.

4. Affine invariance:

   B-Spline curves are invariant under affine transformations such as rotations, displacements and scalings. This means that one can apply an affine transformation to the B-Spline curve by applying it to its set of control points.

5. Convex hull property:

   All the points of a B-Spline curve are contained in the *convex hull* of its control points. The convex hull of a set of points $P = \{\mathbf{P}_0, \ldots, \mathbf{P}_N\}$ is denoted by $\mathcal{CH}(P)$ and it is the set of all the convex combinations of points:

   $$\mathcal{CH}(P) = \Big\{ \mathbf{C} = \sum_{k=0}^{N} a_k \, \mathbf{P}_k \text{ such that } \sum_{k=0}^{N} a_k = 1 \text{ and } a_k \geq 0 \text{ for } k = 0, 1, \ldots, N \Big\}$$

   The convex hull property follows from the non-negativity and the partition-of-unity properties of the basis functions.

6. Strong convex hull property:

   If $u \in [u_{i_0}, u_{i_0+1})$, then $\mathbf{C}(u)$ is contained in the convex hull of the control points $\boldsymbol{P}_i$ with $i_0 - p \leq i \leq i_0$.

7. Local modification scheme:

   Modifying the control point $\mathbf{P}_i$ affects $\mathbf{C}(u)$ only in the interval $[u_i, u_{i+p+1})$. This property follows from the fact that $N_{i,p}(u) = 0$ if u is outside the interval $[u_i, u_{i+p+1})$ and it implies that the shape of a B-Spline can be modified locally without changing its shape globally (nice motivation in my handwritten notes)

8. The polygon formed by the set of control points is known as *control polygon*. The control polygon represents a piecewise linear approximation to the B-Spline curve.

9. Variation diminishing property:

   No straight line (or plane in three dimensions) intersects the B-Spline curve more times than it intersects its control polygon. An intuitive explanation of this property is that the B-Spline curve does not wiggle more than its control polygon.

10. Continuity and differentiability:

    (a) B-Spline curves are infinitely differentiable in the interior of the knot intervals.

    (b) B-Spline curves are *at least* $p - k$ continuously differentiable at a knot with multiplicity k.

11. The first and higher order derivatives of a B-Spline curve are given by:

    $$\frac{\mathrm{d}\mathbf{C}}{\mathrm{d}u} = \sum_{i=0}^{n} N'_{i,p}(u) \, \mathbf{P}_i$$

    $$\frac{\mathrm{d}^k\mathbf{C}}{\mathrm{d}u^k} = \sum_{i=0}^{n} N^{(k)}_{i,p}(u) \, \mathbf{P}_i$$

    The first derivative of a B-Spline curve is also given by:

    $$\frac{\mathrm{d}\mathbf{C}}{\mathrm{d}u} = \sum_{i=0}^{n} N'_{i,p}(u) \, \mathbf{P}_i$$

evaluated at a new knot vector U' given by:

$$U' = [\underbrace{0, \ldots, 0}_{p}, \underbrace{u_{p+1}, \ldots, u_n}_{n-p}, \underbrace{1, \ldots, 1}_{p}]$$

Note that the derivative of a B-Spline curve of degree p is a new B-Spline curve of degree $p-1$ with a new set of control points and knot vector.

The k-th derivative of a B-Spline curve is also given by:

$$\frac{\mathrm{d}^k \mathbf{C}}{\mathrm{d}u^k} = \sum_{i=0}^{n-k} N_{i,p-k}(u)\, \mathbf{P}_i^{(k)}$$

where:

$$\mathbf{P}_i^{(k)} = \begin{cases} \mathbf{P}_i & \text{if } k = 0 \\ \frac{p-k+1}{u_{i+p+1}-u_{i+k}} \left(\mathbf{P}_{i+1}^{(k-1)} - \mathbf{P}_i^{(k)}\right) & \text{if } k > 0 \end{cases}$$

evaluated at the knot vector $U^{(k)}$ given by:

$$U^{(k)} = [\underbrace{0, \ldots, 0}_{p-k+1}, \underbrace{u_{p+1}, \ldots, u_n}_{n-p}, \underbrace{1, \ldots, 1}_{p-k+1}]$$

12. End point interpolation:

    The start and end points of a clampled B-Spline curve coincide with the first and last control points, respectively.

    $$\mathbf{C}(u = 0) = \mathbf{P}_0$$
    $$\mathbf{C}(u = 1) = \mathbf{P}_n$$

13. End point tangency:

    A clamped B-Spline curve is tangent to the control polygon at the endpoints.

    $$\left.\frac{\mathrm{d}\mathbf{C}}{\mathrm{d}u}\right|_{u=0} = \left(\frac{p}{u_{p+1}}\right)(\mathbf{P}_1 - \mathbf{P}_0)$$
    $$\left.\frac{\mathrm{d}\mathbf{C}}{\mathrm{d}u}\right|_{u=1} = \left(\frac{p}{1-u_n}\right)(\mathbf{P}_n - \mathbf{P}_{n-1})$$

14. End point curvature:

    The second derivative of B-Spline curve at its endpoints in given by:

    $$\left.\frac{\mathrm{d}^2\mathbf{C}}{\mathrm{d}u^2}\right|_{u=0} = \frac{p\,(p-1)}{u_{p+1}} \left[\left(\frac{1}{u_{p+2}}\right)(\mathbf{P}_2 - \mathbf{P}_0) - \left(\frac{1}{u_{p+1}} + \frac{1}{u_{p+2}}\right)(\mathbf{P}_1 - \mathbf{P}_0)\right]$$
    $$\left.\frac{\mathrm{d}^2\mathbf{C}}{\mathrm{d}u^2}\right|_{u=1} = \frac{p\,(p-1)}{1-u_n} \left[\left(\frac{1}{1-u_{n-1}}\right)(\mathbf{P}_{n-2} - \mathbf{P}_n) - \left(\frac{1}{1-u_n} + \frac{1}{1-u_{n-1}}\right)(\mathbf{P}_{n-1} - \mathbf{P}_n)\right]$$

    The curvature of a clamped B-Spline curve at its endpoints is given by:

    $$\kappa(u = 0) = \left(\frac{p-1}{p}\right)\left(\frac{u_{p+1}}{u_{p+2}}\right) \frac{\left\|(\mathbf{P}_2 - \mathbf{P}_0) \times (\mathbf{P}_1 - \mathbf{P}_0)\right\|}{\left\|\mathbf{P}_1 - \mathbf{P}_0\right\|^3}$$
    $$\kappa(u = 1) = \left(\frac{p-1}{p}\right)\left(\frac{1-u_n}{1-u_{n-1}}\right) \frac{\left\|(\mathbf{P}_{n-2} - \mathbf{P}_n) \times (\mathbf{P}_{n-1} - \mathbf{P}_n)\right\|}{\left\|\mathbf{P}_{n-1} - \mathbf{P}_n\right\|^3}$$

2 B-Spline surfaces

2.1 Definition

A B-Spline surface, shorthand for basis spline surface, is a parametric surface defined by:

$$\mathbf{S}(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{m} N_{i,p}(u)\, N_{j,q}(v)\, \mathbf{P}_{i,j} \qquad 0 \leq (u,\, v) \leq 1 \tag{5}$$

where p and q are the orders of the surface in the u- and v-directions, the coefficients $\mathbf{P}_{i,j}$ are a bidirectional net of control points and $N_{i,p}(u)N_{j,q}(v)$ are the product of univariate B-Spline basis functions defined on the non-decreasing knot vectors U and V:

$$U = [u_0,\, \ldots,\, u_r] \in \mathbb{R}^{r+1} \quad \text{with} \quad r = n + p + 1 \tag{6}$$

$$V = [v_0,\, \ldots,\, v_s] \in \mathbb{R}^{s+1} \quad \text{with} \quad s = m + q + 1 \tag{7}$$

The u-direction basis functions are given by the recursive relation:

$$N_{i,0}(u) = \begin{cases} 1 & \text{if} \quad u_i \leq u < u_{i+1} \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

$$N_{i,p}(u) = \frac{u - u_i}{u_{i+p} - u_i} N_{i,p-1}(u) + \frac{u_{i+p+1} - u}{u_{i+p+1} - u_{i+1}} N_{i+1,p-1}(u) \tag{9}$$

whereas the v-direction basis functions are defined in an analogous way replacing the variable u by v and the indices i and p by j and q, respectively.

2.2 Mathematical properties of tensor product B-spline basis functions

Here is a list of some important properties of the tensor product B-Spline basis functions:

1. The relation $r = n+p+1$ holds, where $n+1$ is the number of basis functions in the u-direction and $r + 1$ is the number of elements of the knot vector U.

   The relation $s = m + q + 1$ holds, where $m + 1$ is the number of basis functions in the v-direction and $s + 1$ is the number of elements of the knot vector V.

2. $N_{i,p}(u)$ is, at most, a polynomial of degree p.

   $N_{j,q}(v)$ is, at most, a polynomial of degree q.

3. Local support:

   (a) $N_{i,p}(u)N_{j,q}(v) = 0$ if (u, v) is outside the rectangle $[u_i, u_{i+p+1}) \times [v_j, v_{j+q+1})$.

   (b) In any given knot rectangle $[u_{i_0}, u_{i_0+1}) \times [v_{j_0}, v_{j_0+1})$, at most $(p+1)(q+1)$ basis functions are nonzero, namely the $N_{i,p}(u)N_{j,q}(v)$ with $i_0 - p \leq i \leq i_0$ and $j_0 - q \leq j \leq j_0$.

4. Non-negativity:

   $$N_{i,p}(u)N_{j,q}(v) \geq 0 \quad \text{for all } i,\, j,\, p,\, q, \text{ and } (u, v) \in [0, 1] \times [0, 1]$$

5. Partition of unity:

   $$\sum_{i=0}^{n}\sum_{j=0}^{m} N_{i,p}(u)N_{j,q}(v) = 1 \text{ for all } (u, v) \in [0, 1] \times [0, 1]$$

   In addition, for an arbitrary knot rectangle, $[u_{i_0}, u_{i_0+1}) \times [v_{j_0}, v_{j_0+1})$ we have that:

   $$\sum_{i=i_0-p}^{i_0}\sum_{j=j_0-q}^{j_0} N_{i,p}(u)N_{j,q}(v) = 1 \text{ for all } (u, v) \in [u_{i_0}, u_{i_0+1}) \times [v_{j_0}, v_{j_0+1})$$

   This means that the sum of the non-zero basis-functions of any knot rectangle is unity.

6. Continuity and differentiability:
   (a) The basis functions are infinitely differentiable in the interior of the knot rectangles formed by the U and V vectors.
   (b) The basis functions are $p-k$ $(q-k)$ continuously differentiable in the u-direction (v-direction) at a u-knot (v-knot) with multiplicity k.
7. Extrema:

   $N_{i,p}(u)N_{j,q}(v)$ attains exactly one maximum value in the region $(u,v) \in [0,1] \times [0,1]$.
8. The first partial derivatives of the tensor product basis functions are given by:
$$\frac{\partial}{\partial u}\big(N_{i,p}(u)N_{j,q}(v)\big) = N_{j,q}(v)\,\frac{\partial}{\partial u}\big(N_{i,p}(u)\big)$$
$$\frac{\partial}{\partial v}\big(N_{i,p}(u)N_{j,q}(v)\big) = N_{i,p}(u)\,\frac{\partial}{\partial v}\big(N_{j,q}(v)\big)$$
9. The (k,l)-th order derivatives of the tensor product basis functions are given by:
$$\frac{\partial^{k+l}}{\partial u^k \partial v^l}\big(N_{i,p}(u)N_{j,q}(v)\big) = \frac{\partial^k}{\partial u^k}\big(N_{i,p}(u)\big)\,\frac{\partial^l}{\partial v^l}\big(N_{j,q}(v)\big)$$
10. If the knot vectors are clamped and $(p,q) = (n,m)$, then the product of B-Spline basis functions reduces to the product of Bernstein polynomials, that is, $N_{i,p}(u)N_{j,q}(u) = B_{i,p}(u)B_{i,q}(u)$.

2.3 Mathematical properties of B-Spline surfaces

Here is a list of some important properties of B-Spline surfaces:

1. $\mathbf{S}(u,v)$ is a piecewise surface and its components are bivariate polynomials of, at most, degree $p \times q$.
2. In the u-direction: the degree p, the number of control points $n+1$ and the number of knots $r+1$ are related according to $r = n+p+1$

   In the v-direction: the degree q, the number of control points $m+1$ and the number of knots $s+1$ are related according to $s = m+q+1$
3. If $n = p$, $m = q$ and the knot vectors are clamped then $\mathbf{S}(u,v)$ is a Bézier surface.
4. Affine invariance:

   B-Spline surfaces are invariant under affine transformations such as rotations, displacements and scalings. This means that one can apply an affine transformation to the B-Spline surface by applying it to its set of control points.
5. Convex hull property:

   All the points of a B-Spline surface are contained in the *convex hull* of its control points. The convex hull of a set of points $P = \{\mathbf{P}_0, \ldots, \mathbf{P}_N\}$ is denoted by $\mathcal{CH}(P)$ and it is the set of all the convex combinations of points:
$$\mathcal{CH}(P) = \Big\{\mathbf{C} = \sum_{k=0}^{N} a_k\,\mathbf{P}_k \text{ such that } \sum_{k=0}^{N} a_k = 1 \text{ and } a_k \geq 0 \text{ for } k = 1,\, 2,\, \ldots,\, N\Big\}$$
   The convex hull property follows from the non-negativity and the partition-of-unity properties of the basis functions.
6. Strong convex hull property:

   If $(u,v) \in [u_{i_0},\, u_{i_0+1}) \times [v_{j_0},\, v_{j_0+1})$, then $\mathbf{S}(u,v)$ is contained in the convex hull of the control points $\boldsymbol{P}_{i,j}$ with $i_0 - p \leq i \leq i_0$ and $j_0 - q \leq j \leq j_0$.

7. Local modification scheme:

   Modifying the control point $\mathbf{P}_{i,j}$ affects $\mathbf{S}(u,v)$ only in the rectangle $[u_i, u_{i+p+1}) \times [v_j, v_{j+q+1})$. This property follows from the fact that $N_{i,p}(u)N_{j,q}(u) = 0$ if (u,v) is outside the interval $[u_i, u_{i+p+1}) \times [v_j, v_{j+q+1})$ and it implies that the shape of a B-Spline can be modified locally without changing its shape globally.

8. If triangulated, the net of control points represents a piecewise planar approximation to the B-Spline surface.

9. No known variation diminishing property.

10. Continuity and differentiability:

    (a) B-Spline surfaces are infinitely differentiable in the interior of the knot rectangles formed by the U and V vectors.

    (b) B-Spline surfaces are $p-k$ $(q-k)$ continuously differentiable in the u-direction (v-direction) at a u-knot (v-knot) with multiplicity k.

11. The first and higher order derivatives of a B-Spline surface are given by:

$$\frac{\partial \mathbf{S}}{\partial u} = \sum_{i=0}^{n}\sum_{j=0}^{m} N'_{i,p}(u)N_{j,q}(v)\,\mathbf{P}_{i,j}$$

$$\frac{\partial \mathbf{S}}{\partial v} = \sum_{i=0}^{n}\sum_{j=0}^{m} N_{i,p}(u)N'_{j,q}(v)\,\mathbf{P}_{i,j}$$

$$\frac{\partial^{k+l} \mathbf{S}}{\partial u^k \partial v^l} = \sum_{i=0}^{n}\sum_{j=0}^{m} N^{(k)}_{i,p}(u)N^{(l)}_{j,q}(v)\,\mathbf{P}_{i,j}$$

12. Corner point interpolation:

    The corners of a clamped B-Spline surface coincide with the corner points of its control net:

$$\mathbf{S}(u=0,\, v=0) = \mathbf{P}_{0,0}$$
$$\mathbf{S}(u=1,\, v=0) = \mathbf{P}_{n,0}$$
$$\mathbf{S}(u=0,\, v=1) = \mathbf{P}_{0,m}$$
$$\mathbf{S}(u=1,\, v=1) = \mathbf{P}_{n,m}$$

References

Piegl, L. and W. Tiller (1997). *The NURBS book*. Springer.