

Lecture 20.
Thursday 10th
December, 2020.

*Linear-phase FIR
transfer functions*

0.1 Linear-phase FIR transfer functions

It is impossible to design an IIR transfer function with an exact linear-phase. However, it is always possible to design a FIR transfer function with an exact linear-phase response. We now develop the forms of the linear-phase FIR transfer function $H(z)$ with real impulse response $h[n]$. Let:

$$H(z) = \sum_{n=0}^N h[n]z^{-n} \quad (1)$$

If $H(z)$ is to have a linear-phase, its frequency response must be of the form:

$$H(e^{j\omega}) = e^{j(c\omega+\beta)} \check{H}(\omega) \quad (2)$$

where c and β are constants, and $\check{H}(\omega)$, called **amplitude response** or **zero-phase response**, is a real function of ω . For a real impulse response, the magnitude response $|H(e^{j\omega})|$ is an even function of ω , i.e.:

*Amplitude or
zero-phase
response*

$$|H(e^{j\omega})| = |H(e^{-j\omega})| \quad (3)$$

Since $|H(e^{j\omega})| = |\check{H}(\omega)|$, the amplitude response is then either an even function or an odd function of ω , i.e.:

$$\check{H}(-\omega) = \pm \check{H}(\omega) \quad (4)$$

The frequency response satisfies the relation:

$$H(e^{j\omega}) = H^*(e^{-j\omega}) \quad (5)$$

or, equivalently, the relation:

$$e^{j(c\omega+\beta)} \check{H}(\omega) = e^{-j(c(-\omega)+\beta)} \check{H}(\omega) \quad (6)$$

If $\check{H}(\omega)$ is an even function, then the relation in Eq. 6 leads to:

$$e^{j\beta} = e^{-j\beta} \quad (7)$$

implying that either $\beta = 0$ or $\beta = \pi$. From:

$$H(e^{j\omega}) = e^{j(c\omega+\beta)} \check{H}(\omega) \quad (8)$$

we have:

$$\check{H}(\omega) = e^{-j(c\omega+\beta)} H(e^{j\omega}) \quad (9)$$

Substituting the value of β in Eq. 9, we get:

$$\check{H}(\omega) = \pm e^{-jc\omega} H(e^{j\omega}) = \pm \sum_{n=0}^N h[n]e^{-j\omega(c+n)} \quad (10)$$

Replacing ω with $-\omega$ in Eq. 10, we get:

$$\check{H}(-\omega) = \pm \sum_{\ell=0}^N h[\ell]e^{j\omega(c+\ell)} \quad (11)$$

Making a change of variable $\ell = N - n$, we rewrite Eq. 11 as:

$$\check{H}(-\omega) = \pm \sum_{n=0}^N h[N-n] e^{j\omega(c+N-n)} \quad (12)$$

Now, as $\check{H}(\omega) = \check{H}(-\omega)$, we have:

$$h[n] e^{-j\omega(c+n)} = h[N-n] e^{j\omega(c+N-n)} \quad (13)$$

Eq. 13 leads to the condition:

$$h[n] = h[N-n] \quad 0 \leq n \leq N \quad (14)$$

with $c = -\frac{N}{2}$. Thus, the FIR filter with an even amplitude response will have a linear phase if it has a symmetric impulse response. If $\check{H}(\omega)$ is an odd function of ω , then from:

$$e^{j(c\omega+\beta)} \check{H}(\omega) = e^{-j(-c\omega+\beta)} \check{H}(-\omega) \quad (15)$$

we get $e^{j\beta} = -e^{-j\beta}$ as $\check{H}(-\omega) = -\check{H}(\omega)$. Eq. 15 is satisfied if $\beta = \pm\frac{\pi}{2}$. Then $H(e^{j\omega}) = e^{j(c\omega+\beta)}$ reduces to:

$$H(e^{j\omega}) = j e^{jc\omega} \check{H}(\omega) \quad (16)$$

Eq. 16 can be rewritten as:

$$\check{H}(\omega) = -j e^{-jc\omega} H(e^{j\omega}) = -j \sum_{m=0}^N h[m] e^{-j\omega(c+m)} \quad (17)$$

Again, as $\check{H}(\omega) = \check{H}(-\omega)$, from Eq. 17 we get:

$$\check{H}(-\omega) = j \sum_{\ell=0}^N h[\ell] e^{j\omega(c+\ell)} \quad (18)$$

Making a change of variable $\ell = N - n$, we rewrite Eq. 18 as:

$$\check{H}(-\omega) = j \sum_{n=0}^N h[N-n] e^{j\omega(c+N-n)} \quad (19)$$

Equating the RHS of Eq. 19 with the RHS of Eq. 17, we arrive at the **condition for linear phase** as:

$$h[n] = h[N-n] \quad 0 \leq n \leq N \quad (20)$$

with $c = -\frac{N}{2}$. Therefore, a FIR filter with an odd amplitude response will have linear-phase response if it has an antisymmetric impulse response.

Since the length of the impulse response can be either even or odd, we can define **four types of linear-phase FIR transfer functions**. In particular, for an anti-symmetric FIR filter of odd length, namely N even, $h[\frac{N}{2}] = 0$. We examine in the following discussion each of the four cases, sketched in Figure 1.

Condition for
linear phase

Classification on
the length of the
impulse response

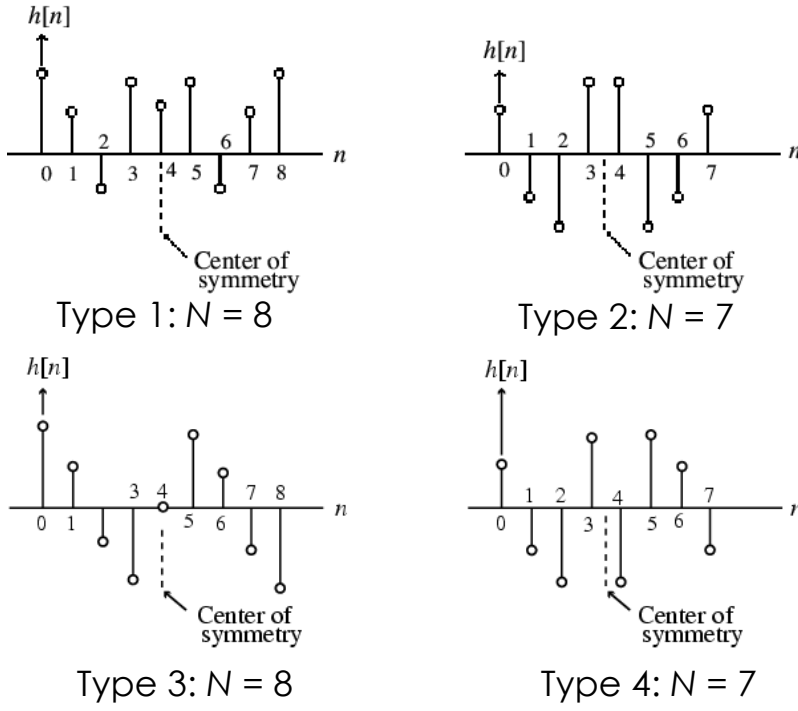


Figure 1: The four types of linear-phase FIR transfer functions.

0.1.1 Symmetric impulse response with odd length

In this case, the degree N is even. In the following discussion we assume also $N = 8$ for simplicity. Therefore, the transfer function $H(z)$ is given by:

Case of symmetric impulse response with odd length

$$H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[4]z^{-4} + h[5]z^{-5} + h[6]z^{-6} + h[7]z^{-7} + h[8]z^{-8} \quad (21)$$

Because of symmetry, we have:

$$h[0] = h[8] \quad (22)$$

$$h[1] = h[7] \quad (23)$$

$$h[2] = h[6] \quad (24)$$

$$h[3] = h[5] \quad (25)$$

Thus, we can write:

$$\begin{aligned} H(z) &= h[0](1 + z^{-8}) + h[1](z^{-1} + z^{-7}) + h[2](z^{-2} + z^{-6}) + h[3](z^{-3} + z^{-5}) + h[4]z^{-4} \\ &= z^{-4} \{ h[0](z^4 + z^{-4}) + h[1](z^3 + z^{-3}) + h[2](z^2 + z^{-2}) + h[3](z + z^{-1}) + h[4] \} \end{aligned} \quad (26)$$

The corresponding frequency response is then given by:

Frequency response

$$H(e^{j\omega}) = e^{-j4\omega} \{ 2h[0] \cos(4\omega) + 2h[1] \cos(3\omega) + 2h[2] \cos(2\omega) + 2h[3] \cos(\omega) + h[4] \} \quad (27)$$

The quantity inside the braces is a real function of ω and can assume positive or negative values in the range $0 \leq |\omega| \leq \pi$. The phase function is given by:

Phase function

$$\theta(\omega) = -4\omega + \beta \quad (28)$$

Group delay

where β is either 0 or π , and hence, it is a linear function of ω . The group delay is given by:

$$\tau(\omega) = -\frac{d\theta(\omega)}{d\omega} = 4 \quad (29)$$

indicating a constant group delay of 4 samples.

General case

In the general case for **Type 1 FIR filters**, the frequency response is of the form:

$$H(e^{j\omega}) = e^{-jN\frac{\omega}{2}} \check{H}(\omega) \quad (30)$$

where the amplitude response $\check{H}(\omega)$, also called the zero-phase response, is of the form:

$$\check{H}(\omega) = h\left[\frac{N}{2}\right] + 2 \sum_{n=1}^{\frac{N}{2}} h\left[\frac{N}{2} - n\right] \cos(\omega n) \quad (31)$$

Example of
symmetric impulse
response with odd
length

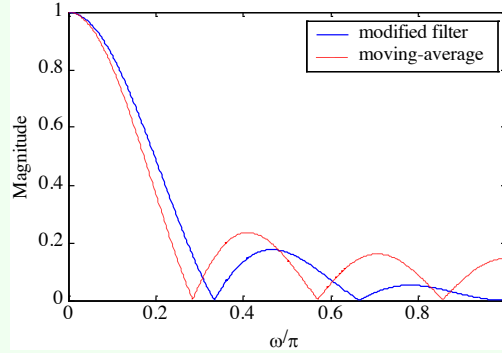
Example 1: Symmetric impulse response with odd length

We consider:

$$H_0(z) = \frac{1}{6} \left[\frac{1}{2} + z^{-1} + z^{-3} + z^{-4} + z^{-5} + \frac{1}{2} z^{-6} \right] \quad (32)$$

which is seen to be a slightly modified version of a length-7 moving-average FIR filter.

This transfer function has a symmetric impulse response and therefore a linear phase response. A plot of the magnitude response of $H_0(z)$ along with that of the 7-point moving-average filter is shown below.



Note the improved magnitude response obtained by simply changing the first and the last impulse response coefficients of a moving-average (MA) filter. It can be showed that we can express:

$$H_0(z) = \frac{1}{2}(1 + z^{-1}) \cdot \frac{1}{6}(1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}) \quad (33)$$

which is seen to be a cascade of a 2-point MA filter with a 6-point MA filter. Thus, $H_0(z)$ has a double zero at $z = -1$, i.e. $\omega = \pi$.

Case of symmetric
impulse response
with even length

0.1.2 Symmetric impulse response with even length

In this case, the degree N is odd and we assume $N = 7$ for simplicity for the following discussion. Therefore, the transfer function is of the form:

$$H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[4]z^{-4} + h[5]z^{-5} + h[6]z^{-6} + h[7]z^{-7} \quad (34)$$

Making use of the symmetry of the impulse response coefficients, the transfer function can be rewritten as:

$$\begin{aligned} H(z) &= h[0](1 + z^{-7}) + h[1](z^{-1} + z^{-6}) + h[2](z^{-2} + z^{-5}) + h[3](z^{-3} + z^{-4}) \\ &= z^{-\frac{7}{2}} \left\{ h[0](z^{\frac{7}{2}} + z^{-\frac{7}{2}}) + h[1](z^{\frac{5}{2}} + z^{-\frac{5}{2}}) + h[2](z^{\frac{3}{2}} + z^{-\frac{3}{2}}) + h[3](z^{\frac{1}{2}} + z^{-\frac{1}{2}}) \right\} \end{aligned} \quad (35)$$

The corresponding frequency response is given by:

Frequency response

$$H(e^{j\omega}) = e^{-j\frac{7\omega}{2}} \left\{ 2h[0] \cos\left(\frac{7\omega}{2}\right) + 2h[1] \cos\left(\frac{5\omega}{2}\right) + 2h[2] \cos\left(\frac{3\omega}{2}\right) + 2h[3] \cos\left(\frac{\omega}{2}\right) \right\} \quad (36)$$

As before, the quantity inside the braces is a real function of ω and can assume positive or negative values in the range $0 \leq |\omega| \leq \pi$. Here, the phase function is given by:

Phase function

$$\theta(\omega) = -\frac{7}{2}\omega + \beta \quad (37)$$

where β is either 0 or π . As a result, the phase is also a linear function of ω and the corresponding group delay is:

Group delay

$$\tau(\omega) = \frac{7}{2} \quad (38)$$

indicating a group delay of $\frac{7}{2}$ samples.

The expression for the frequency response in the general case for **Type 2 FIR filters** is of the form:

General case

$$H(e^{j\omega}) = e^{-jN\frac{\omega}{2}} \check{H}(\omega) \quad (39)$$

where the amplitude response is given by:

$$\check{H}(\omega) = 2 \sum_{n=1}^{\frac{N+1}{2}} h \left[\frac{N+1}{2} - n \right] \cos \left(\omega \left(n - \frac{1}{2} \right) \right) \quad (40)$$

0.1.3 Antisymmetric impulse response with odd length

In this case, the degree N is even and we assume $N = 8$ for simplicity for the following discussion. Therefore, applying the symmetry condition we get:

Case of antisymmetric impulse response with odd length

$$H(z) = z^{-4} \{ h[0](z^4 - z^{-4}) + h[1](z^3 - z^{-3}) + h[2](z^2 - z^{-2}) + h[3](z - z^{-1}) \} \quad (41)$$

The corresponding frequency response is given by:

Frequency response

$$H(e^{j\omega}) = e^{-j4\omega} e^{j\frac{\pi}{2}} \{ 2h[0] \sin(4\omega) + 2h[1] \sin(3\omega) + 2h[2] \sin(2\omega) + 2h[3] \sin(\omega) \} \quad (42)$$

It also exhibits a linear phase response given by:

Phase function

$$\theta(\omega) = -4\omega + \frac{\pi}{2} + \beta \quad (43)$$

where β is either 0 or π . The group delay here is:

Group delay

$$\tau(\omega) = 4 \quad (44)$$

indicating a constant group delay of 4 samples.

The expression for the frequency response in the general case for **Type 3 FIR filters** is of the form:

$$H(e^{j\omega}) = e^{-jN\frac{\omega}{2}} \check{H}(\omega) \quad (45)$$

where the amplitude response is given by:

$$\check{H}(\omega) = 2 \sum_{n=1}^{\frac{N}{2}} h \left[\frac{N}{2} - n \right] \sin(\omega n) \quad (46)$$

0.1.4 Antisymmetric impulse response with even length

In this case, the degree N is even and we assume $N = 7$ for simplicity for the following discussion. Therefore, applying the symmetry condition we get:

$$H(z) = z^{\frac{7}{2}} \left\{ h[0](z^{\frac{7}{2}} - z^{-\frac{7}{2}}) + h[1](z^{\frac{5}{2}} - z^{-\frac{5}{2}}) + h[2](z^{\frac{3}{2}} - z^{-\frac{3}{2}}) + h[3](z^{\frac{1}{2}} - z^{-\frac{1}{2}}) \right\} \quad (47)$$

The corresponding frequency response is given by:

$$H(e^{j\omega}) = e^{-j\frac{7\omega}{2}} e^{j\frac{\pi}{2}} \left\{ 2h[0] \sin\left(\frac{7\omega}{2}\right) + 2h[1] \sin\left(\frac{5\omega}{2}\right) + 2h[2] \sin\left(\frac{3\omega}{2}\right) + 2h[3] \sin\left(\frac{\omega}{2}\right) \right\} \quad (48)$$

It again exhibits a linear phase response given by:

$$\theta(\omega) = -\frac{7}{2}\omega + \frac{\pi}{2} + \beta \quad (49)$$

where β is either 0 or π . The group delay is constant and is given by:

$$\tau(\omega) = \frac{7}{2} \quad (50)$$

The expression for the frequency response in the general case for **Type 4 FIR filters** is of the form:

$$H(e^{j\omega}) = e^{-jN\frac{\omega}{2}} \check{H}(\omega) \quad (51)$$

where the amplitude response is given by:

$$\check{H}(\omega) = 2 \sum_{n=1}^{\frac{N+1}{2}} h \left[\frac{N+1}{2} - n \right] \sin \left(\omega \left(n - \frac{1}{2} \right) \right) \quad (52)$$

0.1.5 General form of frequency response

In each of the four types of linear-phase FIR filters, the **frequency response** is of the form:

$$H(e^{j\omega}) = e^{-jN\frac{\omega}{2}} e^{j\beta} \check{H}(\omega) \quad (53)$$

The amplitude response $\check{H}(\omega)$ for each type can become negative over certain frequency ranges, typically in the stopband.

Example 2: General form of frequency response

We consider the causal Type 1 FIR transfer function:

$$H_1(z) = -1 + 2z^{-1} - 3z^{-2} + 6z^{-3} - 3z^{-4} + 2z^{-5} - z^{-6} \quad (54)$$

Its amplitude and phase responses are given by:

$$\check{H}_1(\omega) = 6 - 6\cos(\omega) + 4\cos(2\omega) - 2\cos(3\omega) \quad (55)$$

$$\theta_1(\omega) = -3\omega \quad (56)$$

Next, we consider the causal Type 1 FIR transfer function:

$$H_2(z) = 1 - 2z^{-1} + 3z^{-2} - 6z^{-3} + 3z^{-4} - 2z^{-5} + z^{-6} \quad (57)$$

Its amplitude and phase responses are given by:

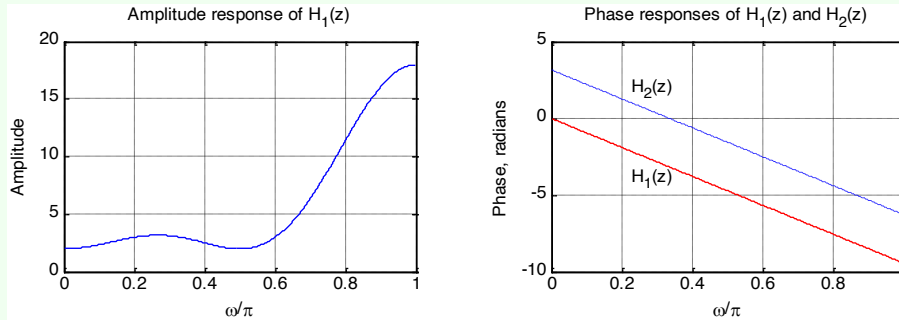
$$\check{H}_2(\omega) = -\check{H}_1(\omega) \quad (58)$$

$$\theta_2(\omega) = -3\omega + \pi \quad (59)$$

Note that:

$$|H_1(e^{j\omega})| = |H_2(e^{j\omega})| \quad (60)$$

Hence, $H_1(z)$ and $H_2(z)$ have identical magnitude responses but phase responses differing by π , as showed in the figure below.

**Example 3: General form of frequency response**

We consider the causal Type 1 FIR transfer function:

$$H_3(z) = 1 - 2z^{-1} + 3z^{-2} - 3z^{-4} + 2z^{-5} - z^{-6} \quad (61)$$

Its amplitude and phase responses are given by:

$$\check{H}_3(\omega) = -6\sin(\omega) + 4\sin(2\omega) + 2\sin(3\omega) \quad (62)$$

$$\theta_3(\omega) = -3\omega + \frac{\pi}{2} \quad (63)$$

Next, we consider the causal Type 1 FIR transfer function:

$$H_4(z) = -1 + 2z^{-1} - 3z^{-2} + 3z^{-4} - 2z^{-5} + z^{-6} \quad (64)$$

Its amplitude and phase responses are given by:

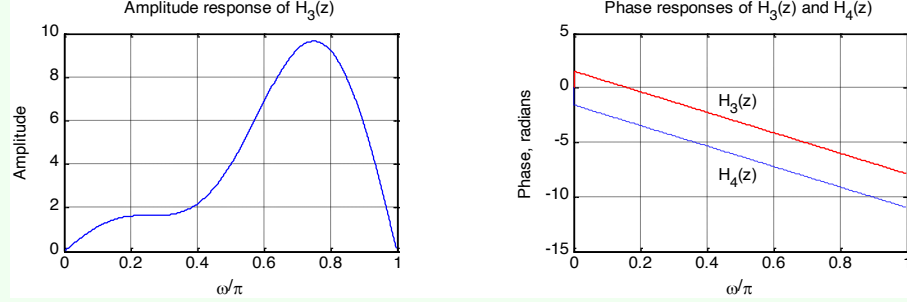
$$\check{H}_4(\omega) = -\check{H}_3(\omega) \quad (65)$$

$$\theta_4(\omega) = -3\omega - \frac{\pi}{2} \quad (66)$$

Note that:

$$|H_3(e^{j\omega})| = |H_4(e^{j\omega})| \quad (67)$$

Hence, $H_3(z)$ and $H_4(z)$ have identical magnitude responses but phase responses differing by π , as showed in the figure below.



Now, in general, the **magnitude and phase responses** of the linear-phase FIR are given by:

$$|H(e^{j\omega})| = |\check{H}(\omega)| \quad (68)$$

$$\theta(\omega) = \begin{cases} -\frac{N\omega}{2} + \beta & \check{H}(\omega) \geq 0 \\ -\frac{N\omega}{2} + \beta - \pi & \check{H}(\omega) < 0 \end{cases} \quad (69)$$

The **group delay** in each case is:

$$\tau(\omega) = \frac{N}{2} \quad (70)$$

Note that, even though the group delay is constant, since in general $|H(e^{j\omega})|$ is not a constant, the output waveform is not a replica of the input waveform.

A FIR filter with a frequency response that is a real function of ω is often called a zero-phase filter. Such a filter must have a non-causal impulse response. In fact, it needs to have a purely real-valued frequency response, and, consequently, it must have an impulse response that is even with respect to the time index $n = 0$. Therefore, it must be non-causal.

0.1.6 Zero locations

In this context, let us consider first a FIR filter with a symmetric impulse response:

$$h[n] = h[N - n] \quad (71)$$

Its transfer function can be written as:

$$H(z) = \sum_{n=0}^N h[n]z^{-n} = \sum_{n=0}^N h[N - n]z^{-n} \quad (72)$$

By making a change of variable $m = N - n$, we can rewrite:

$$H(z) = \sum_{n=0}^N h[N - n]z^{-n} = \sum_{m=0}^N h[m]z^{-N+m} = z^{-N} \underbrace{\sum_{m=0}^N h[m]z^m}_{H(z^{-1})} \quad (73)$$

General form of
magnitude and
phase responses

General form of
group delay

Zero locations

Hence, for a FIR filter with a symmetric impulse response of length $N + 1$ we have:

$$H(z) = z^{-N}H(z^{-1}) \quad (74)$$

A real-coefficient polynomial $H(z)$ satisfying the above condition is called a **mirror-image polynomial (MIP)**.

Mirror image polynomial

Now, let us consider first an FIR filter with an antisymmetric impulse response:

$$h[n] = -h[N - n] \quad (75)$$

Its transfer function can be written as:

$$H(z) = \sum_{n=0}^N h[n]z^{-n} = -\sum_{n=0}^N h[N - n]z^{-n} \quad (76)$$

By making a change of variable $m = N - n$, we can write:

$$H(z) = -\sum_{n=0}^N h[N - n]z^{-n} = -\sum_{m=0}^N h[m]z^{-N+m} = -z^{-N}H(z^{-1}) \quad (77)$$

Hence, the transfer function $H(z)$ of an FIR filter with an antisymmetric impulse response satisfies the condition:

$$H(z) = z^{-N}H(z^{-1}) \quad (78)$$

A real-coefficient polynomial $H(z)$ satisfying the above condition is called **antimirror-image polynomial (AIP)**.

antimirror image polynomial

Now, it follows from the relation $H(z) = \pm z^{-N}H(z^{-1})$ that if $z = \xi_0$ is a zero of $H(z)$, so is $z = \frac{1}{\xi_0}$. Moreover, for an FIR filter with a real impulse response, the zeros of $H(z)$ occur in complex conjugate pairs. Hence, a zero at $z = \xi_0$ is associated with a zero at $z = \xi_0^*$. Thus, a complex zero that is not on the unit circle is associated with a set of 4 zeros given by:

$$z = re^{\pm j\varphi}, \quad \frac{1}{r}e^{\pm j\varphi} \quad (79)$$

Zeros (not) on the unit circle

A zero on the unit circle appear as a pair:

$$z = e^{\pm j\varphi} \quad (80)$$

as its reciprocal is also its complex conjugate. Since a zero at $z = \pm 1$ is its own reciprocal, it can appear only singly.

Now, a Type 2 FIR filter satisfies:

$$H(z) = z^{-N}H(z^{-1}) \quad (81)$$

with degree N odd. Hence, $H(-1) = (-1)^{-N}H(-1) = -H(-1)$, implying $H(-1) = 0$, i.e., $H(z)$ must have a zero at $z = -1$.

Likewise, a Type 3 or 4 FIR filter satisfies:

$$H(z) = -z^{-N}H(z^{-1}) \quad (82)$$

Thus:

$$H(1) = (-1)^{-N}H(1) = -H(1) \quad (83)$$

implying that $H(z)$ must have a zero at $z = 1$. On the other hand, only the Type 3 FIR filter is restricted to have a zero at $z = -1$ since here the degree N is even and hence:

$$H(-1) = -(-1)^{-N}H(-1) = -H(-1) \quad (84)$$

Typical zero locations are showed in Figure 2.

So, to summarize:

Summary on zero locations depending on the type of filter

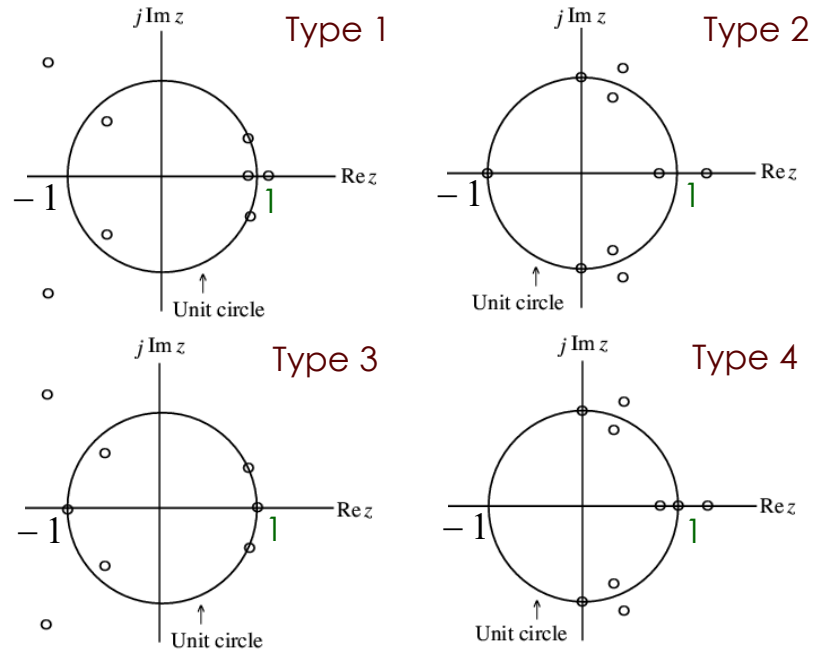


Figure 2: Typical zero locations.

- **Type 1 FIR filter:** either an even number or no zeros at $z = 1$ and $z = -1$;
- **Type 2 FIR filter:** either an even number or no zeros at $z = 1$ and an odd number of zeros at $z = -1$;
- **Type 3 FIR filter:** an odd number of zeros at $z = 1$ and $z = -1$;
- **Type 4 FIR filter:** an odd number of zeros at $z = 1$ and either an even number or no zeros at $z = -1$.

The presence of zeros at $z = \pm 1$ leads to the following limitations on the use of these linear-phase transfer functions for designing frequency-selective filters:

Limitations due to zero locations

- a Type 2 FIR filter cannot be used to design a highpass filter since it always has a zero;
- a Type 3 FIR filter has zeros at both $z = 1$ and $z = -1$, and hence cannot be used to design either a lowpass or a highpass or a bandstop filter;
- a Type 4 FIR filter is not appropriate to design lowpass and bandstop filters due to the presence of a zero at $z = 1$;
- a Type 1 FIR filter has no such restrictions and can be used to design almost any type of filter.