Chapter 1

The z-transform

Lecture 14. *Thursday 12th November, 2020.*

Problems of DTFT and z-transform as alternative

We have seen that the DTFT provides a frequency-domain representation of discrete-time signals and LTI discrete-time systems. However, because of the convergence condition in many cases, the DTFT of a sequence may not exist. As a result, it is not possible to make use of such frequency-domain characterization in these cases.
A possible solution and alternative is a generalization of the DTFT, which leads to the z-transform. The ladder may exist for many sequences for which the DTFT does not exist. Moreover, the use of z-transform techniques permits simple but powerful algebraic manipulations. Consequently, the z-transform has become an important tool in the analysis and design of digital filters

1.1 Definition of z-transform

Definition of the z-transform

Let us begin the discussion on this topic with the definition of the main tool.

Definition 1: z-transform

For a given sequence $g[n]$, its **z-transform** $G(z)$ is defined as:

$$G(z) = \sum_{n=-\infty}^{\infty} g[n] z^{-n} \tag{1.1}$$

where $z = \mathrm{Re}[z] + j\,\mathrm{Im}[z]$ is a complex variable.

If we let $z = re^{j\omega}$, then the z-transform reduces to:

$$G(re^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n] r^{-n} e^{-j\omega n} \tag{1.2}$$

Connections to the DTFT

Eq. 1.2 can be interpreted as the DTFT of the modified sequence $\{g[n]r^{-n}\}$. For $r = 1$ (i.e., $|z| = 1$), the z-transform reduces to its DTFT, provided the ladder exists.
Like the DTFT, there are conditions on the convergence of the infinite series like:

$$\sum_{n=-\infty}^{\infty} g[n] z^{-n} \tag{1.3}$$

Region of convergence (ROC) of the z-transform

For a given sequence, the set R of values of z for which its z-transform converges is called the **region of convergence (ROC)**.
From our earlier discussion on the uniform convergence of the DTFT, it follows that the series:

$$G(re^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n] r^{-n} e^{-j\omega n} \tag{1.4}$$

converges if $\{g[n]r^{-n}\}$ is absolutely summable, i.e. if:

$$\sum_{n=-\infty}^{\infty} \left|g[n]r^{-n}\right| < \infty \tag{1.5}$$

In general, the ROC R of a z-transform of a sequence $g[n]$ is an annular region of the z-plane, namely:

$$R_{g^-} < |z| < R_{g^+} \tag{1.6}$$

where $0 \leq R_{g^-} < R_{g^+} < \infty$.

Examples of z-transform calculation

After having introduced the argument, we study some examples of z-transform calculation in order to understand how it works in practice.

Example 1: z-transform calculation

We determine the z-transform $X(z)$ of the causal sequence $x[n] = \alpha^n \mu[n]$ and its ROC. Now:

$$X(z) = \alpha^n \mu[n] z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n} \tag{1.7}$$

The above power series converges to:

$$X(z) = \frac{1}{1 - \alpha z^{-1}} \qquad \left|\alpha z^{-1}\right| < 1 \tag{1.8}$$

Therefore, the ROC is the annular region $|z| > |\alpha|$.

Example 2: z-transform calculation

The z-transform $\mu(z)$ of the unit step sequence $\mu[n]$ can be obtained from:

$$X(z) = \frac{1}{1 - \alpha z^{-1}} \qquad \left|\alpha z^{-1}\right| < 1 \tag{1.9}$$

By setting $\alpha = 1$:

$$\mu(z) = \frac{1}{1 - z^{-1}} \qquad \left|z^{-1}\right| < 1 \tag{1.10}$$

Therefore, the ROC is the annular region $1 < |z| < \infty$. Note that the unit step sequence $\mu[n]$ is not obsolutely summable, and hence its DTFT does not converge uniformly.

Example 3: z-transform calculation

Consider the anti-causal sequence:

$$y[n] = -\alpha^n \mu[-n-1] \tag{1.11}$$

Its z-transform is given by:

$$\begin{aligned} Y(z) &= -\sum_{n=-\infty}^{-1} \alpha^n z^{-n} = -\sum_{m=1}^{\infty} \alpha^{-m} z^m \\ &= -\alpha^{-1} z \sum_{m=0}^{\infty} \alpha^{-m} z^m = -\frac{-\alpha^{-1} z}{1-\alpha z^{-1}} \\ &= \frac{1}{1-\alpha z^{-1}} \end{aligned} \quad (1.12)$$

for $\left|\alpha^{-1} z\right| < 1$. Therefore, the ROC is the annular region $|z| < |\alpha|$.

Note that the z-transforms of the two sequences $\alpha^n \mu[n]$ and $-\alpha^n \mu[-n-1]$ are identical even though the two parent sequences are different. The only way a unique sequence can be associated with a z-transform is by specifying its ROC.

Connection between uniform convergence of the DTFT and the ROC

Another important point is that the DTFT $G(e^{j\omega})$ of a sequence $g[n]$ converges uniformly if and only if the ROC of the z-transform $G(z)$ of $g[n]$ includes the unit circle. However, the existence of the DTFT does not always imply the existence of the z-transform.

Example 4: z-transform calculation

The finite energy sequence:

$$h_{LP}[n] = \frac{\sin(\omega_c n)}{\pi n} \qquad -\infty < n < \infty \quad (1.13)$$

has a DTFT given by:

$$H_{LP}(e^{j\omega}) = \begin{cases} 1 & 0 \le |\omega| \le \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases} \quad (1.14)$$

which converges in the mean-square sense. However, $h_{LP}[n]$ does not have a z-transform as it is not absolutely summable for any value of r.

Some commonly used z-transform pairs are listed in Figure 1.1.

Commonly used z-transform pairs

Sequence	z-Transform	ROC
$\delta[n]$	1	All values of z
$\mu[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$\alpha^n \mu[n]$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
$(r^n \cos\omega_o n)\mu[n]$	$\frac{1-(r\cos\omega_o)z^{-1}}{1-(2r\cos\omega_o)z^{-1}+r^2 z^{-2}}$	$\lvert z\rvert > r$
$(r^n \sin\omega_o n)\mu[n]$	$\frac{(r\sin\omega_o)z^{-1}}{1-(2r\cos\omega_o)z^{-1}+r^2 z^{-2}}$	$\lvert z\rvert > r$

Figure 1.1: Common z-transform pairs.

1.2 Rational z-transforms

Rational z-transforms

In the case of the LTI discrete-time systems we are concerned with in this course, all the pertinent z-transforms are **rational functions** of z^{-1}, that is, they are fractions of two polinomials in z^{-1}:

$$G(z) = \frac{P(z)}{D(z)} = \frac{p_0 + p_1 z^{-1} + \cdots + p_{M-1} z^{-(M-1)} + p_M z^{-M}}{d_0 + d_1 z^{-1} + \cdots + d_{N-1} z^{-(N-1)} + d_N z^{-N}} \tag{1.15}$$

The degree of the numerator polynomial $P(z)$ is M and the degree of the denominator polynomial $D(z)$ is N. An alternate representation of a rational z-transform is as a ratio of two polynomials in z:

$$G(z) = z^{(N-M)} \frac{p_0 z^M + \cdots + p_{M-1} z + p_M}{d_0 z^N + \cdots + d_{N-1} z + d_N} \tag{1.16}$$

Rational z-transform in factored form

Again, a rational z-transform can be alternately written in **factored form** as:

$$G(z) = \frac{p_0 \prod_{\ell=1}^{M} (1 - \xi_\ell z^{-1})}{d_0 \prod_{\ell=1}^{N} (1 - \lambda_\ell z^{-1})} = z^{(N-M)} \frac{p_0 \prod_{\ell=1}^{M} (z - \xi_\ell)}{d_0 \prod_{\ell=1}^{N} (z - \lambda_\ell)} \tag{1.17}$$

Zeros and poles of rational z-transform

In particular, we have the following quatities of interest:

- $z = \xi_\ell$, roots of the numerator polynomial. These values of z are known as the **zeros** of $G(z)$;
- $z = \lambda_\ell$, roots of the denominator polynomial. These values of z are known as the **poles** of $G(z)$.

Example 5: ROC of a rational z-transform

The z-transform $H(z)$ of the sequence $h[n] = (-0.6)^n \mu[n]$ is given by:

$$H(z) = \frac{1}{1 + 0.6 z^{-1}} \qquad |z| > 0.6 \tag{1.18}$$

Here the ROC is just outside the circle going through the point $z = -0.6$.

Example 6: Zeros and poles

The z-transform:

$$\mu(z) = \frac{1}{1 - z^{-1}} \qquad |z| > 1 \tag{1.19}$$

has a zero at $z = 0$ and a pole at $z = 1$.

Log-magnitude plot of z-transform for physical interpretation

A physical interpretation of the concepts of poles and zeros can be given by plotting the **log-magnitude** $20 \log_{10} |G(z)|$ as showed in Figure 1.2 for:

$$G(z) = \frac{1 - 2.4 z^{-1} + 2.88 z^{-2}}{1 - 0.8 z^{-1} + 0.64 z^{-2}} \tag{1.20}$$

Observe that the **magnitude plot** exhibits very large peaks around the points $z = 0.4 \pm j0.6928$, which are the poles of $G(z)$. It also exhibits very narrow and deep wells around the location of the zeros at $z = 1.2 \pm j1.2$.

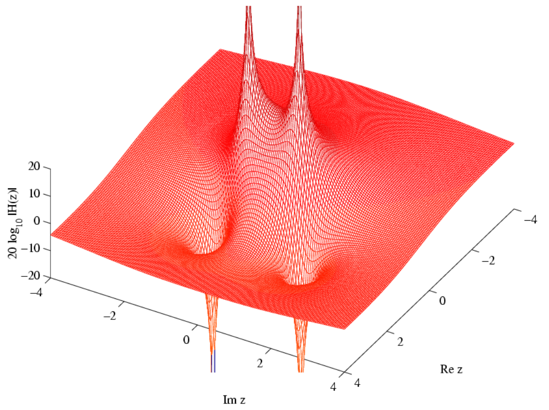

Figure 1.2: Log-magnitude plot for $G(z)$ in Eq. 1.20.

Importance of the ROC of z-transform

Now, we remark that the ROC of a z-transform is an important concept. Without its knowledge, there is no unique relationship between a sequence and its z-transform. Hence, **the z-transform must always be specified with its ROC**. Moreover, there is a relationship between the ROC of the z-transform of the impulse response of a causal LTI discrete-time system and its BIBO stability.

ROC dependency on the type of sequence

Another important distiction is that a sequence can be one of the following types:

- **finite-length**;
- **right-sided**;
- **left-sided**;
- **two-sided**.

In general, the ROC depends on the type of the sequence of interest and we show it in the following Subsections.

1.2.1 Finite-length sequence z-transform

z-transform of finite-length sequence

We consider a **finite-length sequence** $g[n]$ defined for $-M \leq n \leq N$, where M and N are non-negative integers and $|g[n]| < \infty$. Its z-transform is given by:

$$G(z) = \sum_{n=-M}^{N} g[n]z^{-n} = \frac{\sum_{n=0}^{N+M} g[n-M]z^{N+M-n}}{z^N} \tag{1.21}$$

Note that $G(z)$ has M zeros and N poles. As can be seen from the expression for $G(z)$, the z-transform of a finite-length bounded sequence converges everywhere in the z-plane except possibly at $z = 0$ and/or at $z = \infty$.

1.2.2 Right-sided sequence z-transform

z-transform of right-sided sequence

A **right-sided sequence** with nonzero sample values for $n \geq 0$ is sometimes called a **causal sequence**. So, we consider a causal sequence $u_1[n]$. Its z-transform is given by:

$$U_1(z) = \sum_{n=0}^{\infty} u_1[n]z^{-n} \tag{1.22}$$

It can be showed that $U_1(z)$ converges exterior to a circle with $|z| = R_1$, including the point $z = \infty$.

On the other hand, a right-sided sequence $u_2[n]$ with nonzero sample values only for $n \geq -M$ with M non-negative has a z-transform $U_2(z)$ with M poles at $z = \infty$. The ROC of $U_2(z)$ is exterior to a circle $|z| = R_2$, exluding the point $z = \infty$.

1.2.3 Left-sided sequence z-transform

z-transform of left-sided sequence

A **left-sided sequence** with nonzero sample values for $n \leq 0$ is sometimes called **anticausal sequence**. So, we consider an anticausal sequence $v_1[n]$. Its z-transform is given by:

$$V_1(z) = \sum_{n=-\infty}^{0} v_1[n]z^{-n} \tag{1.23}$$

It can be showed that $V_1(z)$ converges interior to a circle $|z| = R_3$, including the point $z = 0$.
On the other hand, a left-sided sequence with nonzero sample values only for $n \leq N$ with N non-negative has a z-transform $V_2(z)$ with N poles at $z = 0$. The ROC of $V_2(z)$ is interior to a circle $|z| = R_4$, excluding the point $z = 0$.

1.2.4 Two-sided sequence z-transform

z-transform of two-sided sequence

The z-transform of a **two-sided sequence** $w[n]$ can be expressed as:

$$W(z) = \sum_{n=-\infty}^{\infty} w[n]z^{-n} = \sum_{n=0}^{\infty} w[n]z^{-n} + \sum_{n=-\infty}^{-1} w[n]z^{-n} \tag{1.24}$$

The first term on the RHS can be interpreted as the z-transform of a right-sided sequence and it thus converges exterior to the circle $|z| = R_5$. The second term of the RHS can be interpreted as the z-transform of a left-sided sequence and it thus converges interior to the circle $|z| = R_6$. If $R_5 < R_6$, there is an overlapping ROC given by $R_5 < |z| < R_6$. If $R_5 > R_6$, there is no overlap and the z-transform does not exist.
In particular, let us consider as example the two-sided sequence:

$$u[n] = \alpha^n \tag{1.25}$$

where α can be either real or complex. Its z-transform is given by:

$$U(z) = \sum_{n=-\infty}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n} + \sum_{n=-\infty}^{-1} \alpha^n z^{-n} \tag{1.26}$$

The first term on the RHS converges for $|z| > |\alpha|$, whereas the second term converges for $|z| < |\alpha|$. There is no overlap between these two regions, hence the z-transform of $u[n] = \alpha^n$ does not exist.

The ROC of a rational z-transform cannot contain any pole (since it is infinite at a pole) and is bounded by the poles. To show the latter statement, we assume that the z-transform $X(z)$ has simple poles at $z = \alpha$ and $z = \beta$. We also assume that the corresponding sequence $x[n]$ is a right-sided sequence. Then, $x[n]$ has the form:

$$x[n] = (r_1\alpha^n + r_2\beta^n)\mu[n - N_0] \qquad |\alpha| < |\beta| \tag{1.27}$$

where N_0 is a positive or negative integer. Now, the z-transform of the right-sided sequence $\gamma^n \mu[n - N_0]$ exists if:

$$\sum_{n=N_0}^{\infty} \left|\gamma^n z^{-n}\right| < \infty \tag{1.28}$$

for some z. The condition in Eq. 1.28 holds for $|z| > |\gamma|$, but not for $|z| \leq |\gamma|$. Therefore, the z-transform of Eq. 1.27 has a ROC defined by $|\beta| < |z| \leq \infty$.
Likewise, the z-transform of a left-sided sequence:

$$x[n] = (r_1\alpha^n + r_2\beta^n)\mu[-n - N_0] \qquad |\alpha| < |\beta| \tag{1.29}$$

has a ROC defined by $0 \leq |z| < |\alpha|$.

1.3 Inverse z-transform

Inverse z-transform using DTFT analogy

Firstly, we recall that, for $z = re^{j\omega}$, the z-transform $G(z)$ given by:

$$G(z) = \sum_{n=-\infty}^{\infty} g[n]z^{-n} = \sum_{n=-\infty}^{\infty} g[n]r^{-n}e^{-j\omega n} \tag{1.30}$$

is the DTFT of the modified sequence $g[n]r^{-n}$. Accordingly, the inverse DTFT is thus given by:

$$g[n]r^{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(re^{j\omega})e^{j\omega n}\mathrm{d}\omega \tag{1.31}$$

Inverse z-transform through contour integral

By making a change of variable $z = re^{j\omega}$, the previous equation can be converted into a **contour integral** given by:

$$g[n] = \frac{1}{2\pi j}\oint_{C'} G(z)z^{n-1}\mathrm{d}z \tag{1.32}$$

where C' is a counterclockwise contour of integration defined by $|z| = r$. But the integral remains unchanged when it is replaced with any contour C encircling the point $z = 0$ in the ROC of $G(z)$. The contour integral can be evaluated using the **Cauchy's residue theorem** resulting in:

Cauchy's residue theorem

$$g[n] = \sum \operatorname*{Res}_{C}\left[G(z)z^{n-1}\right] \tag{1.33}$$

Eq. 1.33 needs to be evaluated at all the values of n, but this is not pursued here.

z-transform in partial fraction expansion form

Now, a rational z-transform $G(z)$ with a causal inverse transform $g[n]$ has an ROC that is exterior to a circle. Here, it is more convenient to express $G(z)$ in a **partial-fraction expansion form** and then determine $g[n]$ by summing the inverse transforms of the individual simpler terms in the expansion. Therefore, a rational $G(z)$ can be expressed as:

$$G(z) = \frac{P(z)}{D(z)} = \frac{\displaystyle\sum_{i=0}^{M} p_i z^{-i}}{\displaystyle\sum_{i=0}^{N} d_i z^{-i}} \tag{1.34}$$

If $M \geq N$, then $G(z)$ can be re-expressed as:

$$G(z) = \sum_{\ell=0}^{M-N} \eta_\ell z^{-\ell} + \frac{P_1(z)}{D(z)} \tag{1.35}$$

Proper fractions and long division technique

where the degree of $P_1(z)$ is less than N. The rational function $\frac{P_1(z)}{D(z)}$ is called a **proper fraction**. To develop the proper fraction part $\frac{P_1(z)}{D(z)}$ from $G(z)$, a **long division** of $P(z)$ by $D(z)$ should be carried out in a reverse order until the remainder polynomial $P_1(z)$ is of lower degree that that of the denominator $D(z)$.

Example 7: Inverse transform by partial-fraction expansion

Consider:

$$G(z) = \frac{2 + 0.8z^{-1} + 0.5z^{-2} + 0.3z^{-3}}{1 + 0.8z^{-1} + 0.2z^{-2}} \tag{1.36}$$

By long division in reverse order we arrive at:

$$G(z) = -3.5 + 1.5z^{-1} + \underbrace{\frac{5.5 + 2.1z^{-1}}{1 + 0.8z^{-1} + 0.2z^{-2}}}_{\text{Proper fraction}} \tag{1.37}$$

In most practical cases, the rational z-transform of interest $G(z)$ is a proper fraction with simple poles. Let the poles of $G(z)$ be at $z = \lambda_k$, with $1 \le k \le N$. A partial-fraction expansion of $G(z)$ is then of the form:

$$G(z) = \sum_{\ell=1}^{N} \left(\frac{\rho_\ell}{1 - \lambda_\ell z^{-1}} \right) \tag{1.38}$$

Residues in partial-fraction expansion

The constants ρ_ℓ in the partial-fraction expansion are called the **residues** and are given by:

$$\rho_\ell = \left[(1 - \lambda_\ell z^{-1}) G(z) \right]_{z=\lambda_\ell} \tag{1.39}$$

Each term of the sum in partial-fraction expansion has a ROC given by $|z| > |\lambda_\ell|$ and thus has an inverse transform of the form $\rho_\ell(\lambda_\ell)^n \mu[n]$. Therefore, the inverse transform $g[n]$ of $G(z)$ is given by:

$$g[n] = \sum_{\ell=1}^{N} \rho_\ell(\lambda_\ell)^n \mu[n] \tag{1.40}$$

Note that the approach in Eq. 1.40 with a slight modification can also be used to determine the inverse of a rational z-transform of a non-causal sequence.

Example 8: Inverse transfrom of a causal sequence

Let the z-transform $H(z)$ of a causal sequence $h[n]$ be given by:

$$H(z) = \frac{z(z+2)}{(z-0.2)(z+0.6)} = \frac{1 + 2z^{-1}}{(1 - 0.2z^{-1})(1 + 0.6z^{-1})} \tag{1.41}$$

A partial-fraction expansion of $H(z)$ is then of the form:

$$H(z) = \frac{\rho_1}{1 - 0.2z^{-1}} + \frac{\rho_2}{1 - 0.6z^{-1}} \tag{1.42}$$

Now:

$$\rho_1 = \left[(1 - 0.2z^{-1}) H(z) \right]_{z=0.2} = \left[\frac{1 + 2z^{-1}}{1 + 0.6z^{-1}} \right]_{z=0.2} = 2.75 \tag{1.43}$$

$$\rho_2 = \left[(1 + 0.6z^{-1}) H(z) \right]_{z=-0.6} = \left[\frac{1 + 2z^{-1}}{1 - 0.2z^{-1}} \right]_{z=-0.6} = -1.75 \tag{1.44}$$

Hence:

$$H(z) = \frac{2.75}{1 - 0.2z^{-1}} - \frac{1.75}{1 + 0.6z^{-1}} \tag{1.45}$$

The inverse transform of Eq. 1.45 is therefore given by:

$$h[n] = 2.75(0.2)^n \mu[n] - 1.75(-0.6)^n \mu[n] \tag{1.46}$$

In case $G(z)$ has multiple poles, the partial-fraction expansion is of slightly different form. Let the pole at $z = v$ be of multiplicity L and the remaining $N - L$ poles be simple and at $z = \lambda_\ell$, for $1 \le \ell \le N - L$. Then, the partial-fraction expansion of $G(z)$ is of the form:

Partial-fraction expansion with poles of higher multiplicity

$$G(z) = \sum_{\ell=0}^{M-N} \eta_\ell z^{-\ell} + \sum_{\ell=1}^{N-L} \frac{\rho_\ell}{1 - \lambda_\ell z^{-1}} + \sum_{i=1}^{L} \frac{\gamma_i}{(1 - vz^{-1})^i} \tag{1.47}$$

where the constants γ_i are computed using:

$$\gamma_i = \frac{1}{(L-i)!(-v)^{L-i}} \frac{\mathrm{d}^{L-i}}{\mathrm{d}(z^{-1})^{L-i}} \big[(1 - vz^{-1})G(z)\big]_{z=v} \qquad 1 \le i \le L \tag{1.48}$$

The residues ρ_ℓ are calculated as before.

1.4 z-transform properties

A list of properties of the z-transform is showed in Figure 1.3.

Properties of z-transform

Property	Sequence	z-Transform	ROC
	$g[n]$ $h[n]$	$G(z)$ $H(z)$	$\mathcal{R}_g$ $\mathcal{R}_h$
Conjugation	$g^*[n]$	$G^*(z^*)$	$\mathcal{R}_g$
Time-reversal	$g[-n]$	$G(1/z)$	$1/\mathcal{R}_g$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G(z) + \beta H(z)$	Includes $\mathcal{R}_g \cap \mathcal{R}_h$
Time-shifting	$g[n - n_o]$	$z^{-n_o} G(z)$	$\mathcal{R}_g$, except possibly the point $z = 0$ or ∞
Multiplication by an exponential sequence	$\alpha^n g[n]$	$G(z/\alpha)$	$\lvert\alpha\rvert \mathcal{R}_g$
Differentiation of $G(z)$	$ng[n]$	$-z \dfrac{dG(z)}{dz}$	$\mathcal{R}_g$, except possibly the point $z = 0$ or ∞
Convolution	$g[n] \circledast h[n]$	$G(z)H(z)$	Includes $\mathcal{R}_g \cap \mathcal{R}_h$
Modulation	$g[n]h[n]$	$\frac{1}{2\pi j} \oint_C G(v)H(z/v)v^{-1}\,dv$	Includes $\mathcal{R}_g \mathcal{R}_h$
Parseval's relation	$\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi j} \oint_C G(v)H^*(1/v^*)v^{-1}\,dv$		

Note: If $\mathcal{R}_g$ denotes the region $R_{g^-} < |z| < R_{g^+}$ and $\mathcal{R}_h$ denotes the region $R_{h^-} < |z| < R_{h^+}$, then $1/\mathcal{R}_g$ denotes the region $1/R_{g^+} < |z| < 1/R_{g^-}$ and $\mathcal{R}_g\mathcal{R}_h$ denotes the region $R_{g^-}R_{h^-} < |z| < R_{g^+}R_{h^+}$.

Figure 1.3: Properties of z-transform.

Examples for z-transform properties

Now, we present some examples with cases where the properties can be usefully applied.

Example 9: z-transform properties

Consider the two-sided sequence:

$$v[n] = \alpha^n \mu[n] - \beta^n \mu[-n-1] \tag{1.49}$$

Let $x[n] = \alpha^n \mu[n]$ and $y = -\beta^n \mu[-n-1]$ with $X(z)$ and $Y(z)$ denoting, respectively, their z-transforms. Now:

$$X(z) = \frac{1}{1 - \alpha z^{-1}} \qquad |z| > |\alpha| \tag{1.50}$$

$$Y(z) = \frac{1}{1 - \beta z^{-1}} \qquad |z| < |\beta| \tag{1.51}$$

Using the linearity property we arrive at:

$$V(z) = X(z) + Y(z) = \frac{1}{1 - \alpha z^{-1}} + \frac{1}{1 - \beta z^{-1}} \tag{1.52}$$

The ROC of $V(z)$ is given by the overlap regions of $|z| > |\alpha|$ and $|z| < |\beta|$. We have that:

- if $|\alpha| < |\beta|$, then there is an overlap and the ROC is an annular region $|\alpha| < |z| < |\beta|$;
- if $|\alpha| > |\beta|$, then there is no overlap and $V(z)$ does not exist.

Example 10: *z*-transform properties

We determine the z-transform and its ROC of the causal sequence:

$$x[n] = r^n (\cos(\omega_0 n)) \mu[n] \tag{1.53}$$

We can express $x[n] = v[n] + v^*[n]$, where:

$$v[n] = \frac{1}{2} r^n e^{j\omega_0 n} \mu[n] = \frac{1}{2} \alpha^n \mu[n] \tag{1.54}$$

The z-transform of $v[n]$ is given by:

$$V(z) = \frac{1}{2} \frac{1}{1 - \alpha z^{-1}} = \frac{1}{2} \frac{1}{1 - r e^{j\omega_0 z^{-1}}} \qquad |z| > |\alpha| = r \tag{1.55}$$

Using the conjugation property, we obtain the z-transform of $v^*[n]$ as:

$$V^*(z^*) = \frac{1}{2} \frac{1}{1 - \alpha^* z^{-1}} = \frac{1}{2} \frac{1}{1 - r e^{-j\omega_0 z^{-1}}} \qquad |z| > |\alpha| \tag{1.56}$$

Finally, using the linearity property we get:

$$X(z) = V(z) + V^*(z^*) = \frac{1}{2} \left(\frac{1}{1 - r e^{j\omega_0 z^{-1}}} + \frac{1}{1 - r e^{-j\omega_0 z^{-1}}} \right) \tag{1.57}$$

or:

$$X(z) = \frac{1 - (r \cos\omega_0) z^{-1}}{1 - (2r \cos\omega_0) z^{-1} + r^2 z^{-2}} \qquad |z| > r \tag{1.58}$$

Example 11: *z*-transform properties

We determine the z-transform $Y(z)$ and the ROC of the sequence:

$$y[n] = (n+1) \alpha^n \mu[n] \tag{1.59}$$

We can write $y[n] = nx[n] + x[n]$ where:

$$x[n] = \alpha^n \mu[n] \tag{1.60}$$

Now, the z-transform $X(z)$ of $x[n] = \alpha^n \mu[n]$ is given by:

$$X(z) = \frac{1}{1 - \alpha z^{-1}} \qquad |z| > |\alpha| \tag{1.61}$$

Using the differentiation property, we arrive at the z-transform of $nx[n]$ as:

$$-z\frac{\mathrm{d}X(z)}{\mathrm{d}z} = \frac{\alpha z^{-1}}{1 - \alpha z^{-1}} \qquad |z| > |\alpha| \tag{1.62}$$

Using the linearity property we finally obtain:

$$Y(z) = \frac{1}{1 - \alpha z^{-1}} + \frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2} = \frac{1}{(1 - \alpha z^{-1})^2} \qquad |z| > |\alpha| \tag{1.63}$$