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EXAMINATION IN MATHEMATICS
MAA153 Linear Algebra
Date: 2017-06-09 *Write time:* 5 hours
Aid: Writing materials, ruler

This examination consists of eight RANDOMLY ORDERED problems each of which is worth at maximum 5 points. The maximum sum of points is thus 40. The PASS-marks 3, 4 and 5 require a minimum of 18, 26 and 34 points respectively. The minimum points for the ECTS-marks E, D, C, B and A are 18, 20, 26, 33 and 38 respectively. Solutions are supposed to include rigorous justifications and clear answers. All sheets with solutions must be sorted in the order the problems are given in. Especially, avoid to write on back pages of solution sheets.

1. Let q be a quadratic form on $\mathbb{E}^3$ defined by
$$q(u) = -2x^2 + 2y^2 - 2z^2 + 2xz\,,$$
where x, y, z are the coordinates of u relative to the standard basis. In which (closed) interval is $q(u)$ when $\|u\| = 1$, and for which vectors are the minimum and the maximum respectively attained?

2. Let the linear space which is spanned by the functions p_1, p_2, p_3, where $p_k(x) = x^k$, be equipped with the inner product $\langle p|q\rangle = \int_{-1}^{1} p(x)q(x)\,dx$. Find an ON-basis for the linear space.

3. Prove that the relationships $\begin{cases} x_1 = 7\tilde{x}_1 + 2\tilde{x}_2 - 6\tilde{x}_3 \\ x_2 = -3\tilde{x}_1 - \tilde{x}_2 + 3\tilde{x}_3 \\ x_3 = 2\tilde{x}_1 + \tilde{x}_2 - 2\tilde{x}_3 \end{cases}$ defines a change-of-basis between two ordered bases e_1, e_2, e_3 and $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$, where x_1, x_2, x_3 and $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ are the coordinates of a vector u with respect to respectively of the bases. Also, find the coordinates of the vector $5e_1 - e_2 + 3e_3$ with respect to the basis $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$.

4. Find, relative to the standard basis, the matrix of the linear operator $F : \mathbb{R}^3 \to \mathbb{R}^3$ whose kernel is equal to $\mathrm{span}\{(1, 0, 1)\}$, and where $\mathrm{span}\{(2, -1, 0), (1, 4, -1)\}$ is the eigenspace of F corresponding to the eigenvalue -2.

5. Find the length of the orthogonal projection of the vector $e_1 - 4e_2 + e_3$ on the vector $3e_1 + e_2 - 2e_3$ in the Euclidean space E for which the inner product is fixed as $\langle u|v\rangle = 4x_1y_1 + 5x_2y_2 + 11x_3y_3 + 4(x_1y_2 + x_2y_1) + 7(x_2y_3 + x_3y_2) + 6(x_3y_1 + x_1y_3)$, where x_1, x_2, x_3 and y_1, y_2, y_3 are the coordinates of u and v respectively relative to the basis e_1, e_2, e_3.

6. The linear operator $F : \mathbb{R}^3 \to \mathbb{R}^3$ has relative to the standard basis the matrix
$$\begin{pmatrix} 2 & 1 & 0 \\ 2 & 3 & 0 \\ 1 & \beta & \beta \end{pmatrix}$$
where $\beta \in \mathbb{R}$. Find the numbers β for which the operator är diagonalizable, and state a basis of eigenvectors for each of these β.

7. The linear transformation $F : \mathbb{R}^5 \to \mathbb{R}^4$ has the matrix
$$\begin{pmatrix} 6 & 4 & 16 & 2 & 2 \\ 2 & 5 & 9 & -3 & 8 \\ -3 & 4 & -2 & -7 & 11 \\ 4 & 3 & 11 & 1 & 2 \end{pmatrix}$$
relative to the standard bases for $\mathbb{R}^5$ and $\mathbb{R}^4$. Find a basis for the kernel of F and a basis for the image of F.

8. Find a basis for the subspace $\mathrm{span}\left\{ \begin{pmatrix} 6 & 3 \\ 2 & 5 \end{pmatrix}, \begin{pmatrix} 8 & 1 \\ 5 & 7 \end{pmatrix}, \begin{pmatrix} -1 & 4 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 8 & 16 \\ 5 & 10 \end{pmatrix} \right\}$ of the vector space of all real-valued 2×2-matrices. Also, find out whether the matrix $\begin{pmatrix} -1 & 9 \\ 2 & 3 \end{pmatrix}$ belong to the subspace or not.

Om du föredrar uppgifterna formulerade på svenska, var god vänd på bladet.

Denna tentamen består av åtta stycken om varannat SLUMPMÄSSIGT ORDNADE uppgifter som vardera kan ge maximalt 5 poäng. Den maximalt möjliga poängsumman är således 40. För GODKÄND-betygen 3, 4 och 5 krävs minst 18, 26 respektive 34 poäng. För ECTS-betygen E, D, C, B och A krävs 18, 20, 26, 33 respektive 38 poäng. Lösningar förutsätts innefatta ordentliga motiveringar och tydliga svar. Samtliga lösningsblad skall vid inlämning vara sorterade i den ordning som uppgifterna är givna i. Undvik speciellt att skriva på baksidor av lösningsblad.

1. Låt q vara en kvadratisk form på $\mathbb{E}^3$ definierad enligt

$$q(u) = -2x^2 + 2y^2 - 2z^2 + 2xz$$

där x, y, z är koordinaterna för u relativt standardbasen. I vilket (slutet) intervall ligger $q(u)$ då $\|u\| = 1$, och för vilka vektorer antas minimum respektive maximum?

2. Låt det linjära rum som spänns upp av funktionerna p_1, p_2, p_3, där $p_k(x) = x^k$, vara utrustat med skalärprodukten $\langle p|q\rangle = \int_{-1}^{1} p(x)q(x)\,dx$. Bestäm en ON-bas för det linjära rummet.

3. Bevisa att sambanden $\begin{cases} x_1 = 7\tilde{x}_1 + 2\tilde{x}_2 - 6\tilde{x}_3 \\ x_2 = -3\tilde{x}_1 - \tilde{x}_2 + 3\tilde{x}_3 \\ x_3 = 2\tilde{x}_1 + \tilde{x}_2 - 2\tilde{x}_3 \end{cases}$ definierar ett basbyte mellan två ordnade baser e_1, e_2, e_3 och $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$, där x_1, x_2, x_3 och $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ är koordinaterna för en vektor u med avseende på respektive av baserna. Bestäm även koordinaterna för vektorn $5e_1 - e_2 + 3e_3$ med avseende på basen $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$.

4. Bestäm, relativt standardbasen, matrisen för den linjära operator $F : \mathbb{R}^3 \to \mathbb{R}^3$ vars nollrum är lika med $\text{span}\{(1, 0, 1)\}$, och där $\text{span}\{(2, -1, 0), (1, 4, -1)\}$ är egenrummet till F motsvarande egenvärdet -2.

5. Bestäm längden av den ortogonala projektionen av vektorn $e_1 - 4e_2 + e_3$ på vektorn $3e_1 + e_2 - 2e_3$ i det euklidiska rum E för vilket skalärprodukten är fixerad till $\langle u|v\rangle = 4x_1y_1 + 5x_2y_2 + 11x_3y_3 + 4(x_1y_2 + x_2y_1) + 7(x_2y_3 + x_3y_2) + 6(x_3y_1 + x_1y_3)$, där x_1, x_2, x_3 och y_1, y_2, y_3 är koordinaterna för u respektive v relativt basen e_1, e_2, e_3.

6. Den linjära operatorn $F : \mathbb{R}^3 \to \mathbb{R}^3$ har relativt standardbasen matrisen

$$\begin{pmatrix} 2 & 1 & 0 \\ 2 & 3 & 0 \\ 1 & \beta & \beta \end{pmatrix}$$

där $\beta \in \mathbb{R}$. Bestäm de tal β för vilka operatorn är diagonaliserbar, och ange en bas av egenvektorer till F för var och en av dessa β.

7. Den linjära avbildningen $F : \mathbb{R}^5 \to \mathbb{R}^4$ har matrisen

$$\begin{pmatrix} 6 & 4 & 16 & 2 & 2 \\ 2 & 5 & 9 & -3 & 8 \\ -3 & 4 & -2 & -7 & 11 \\ 4 & 3 & 11 & 1 & 2 \end{pmatrix}$$

relativt standardbaserna för $\mathbb{R}^5$ och $\mathbb{R}^4$. Bestäm en bas för F:s nollrum och en bas för F:s värderum.

8. Bestäm en bas för underrummet $\text{span}\left\{ \begin{pmatrix} 6 & 3 \\ 2 & 5 \end{pmatrix}, \begin{pmatrix} 8 & 1 \\ 5 & 7 \end{pmatrix}, \begin{pmatrix} -1 & 4 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 8 & 16 \\ 5 & 10 \end{pmatrix} \right\}$ till det linjära rummet av alla reellvärda 2×2-matriser. Utred även huruvida matrisen $\begin{pmatrix} -1 & 9 \\ 2 & 3 \end{pmatrix}$ tillhör underrummet eller inte.

If you prefer the problems formulated in English, please turn the page.

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(1) $q(u) = -2x^2+2y^2-2z^2+2xz = (x\ y\ z)\begin{pmatrix}-2&0&1\\0&2&0\\1&0&-2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$
$= X^T G X = (S\tilde{X})^T(S\tilde{G}S^{-1})(S\tilde{X}) = \tilde{X}^T S^T S\tilde{G}\tilde{X}$

where S is a change-of-basis matrix from the standard basis (here ON since E^3) to another basis. Since G is symmetric relative to an ON-basis, then the corresponding operator Γ is symmetric implying that there exists an ON-basis $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ of eigenvectors. The change-of-basis matrix from the ON standard basis to the ON-basis of eigenvectors is therefore orthogonal (i.e. $S^T = S^{-1}$) and the quadratic form becomes

$\tilde{X}^T I \tilde{G}\tilde{X} = \tilde{X}^T\tilde{G}\tilde{X} = \lambda_1\tilde{x}^2+\lambda_2\tilde{y}^2+\lambda_3\tilde{z}^2$

where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of Γ.

Eigenvalues: $0 = \det(G-\lambda I) = \det\begin{pmatrix}-2-\lambda&0&1\\0&2-\lambda&0\\1&0&-2-\lambda\end{pmatrix} = -(\lambda-2)(\lambda+2)^2+(\lambda-2)$
$= -(\lambda-2)[(\lambda+2)^2-1] = -(\lambda-2)(\lambda+2+1)(\lambda+2-1) = -(\lambda+3)(\lambda+1)(\lambda-2)$

$\lambda_1=-3$: $G-\lambda_1 I = \begin{pmatrix}1&0&1\\0&1&0\\1&0&1\end{pmatrix} \sim \begin{pmatrix}1&0&1\\0&1&0\\0&0&0\end{pmatrix}$ Eigenvectors: $t_1(1,0,-1)$, $t_1\neq 0$

$\lambda_2=-1$: $G-\lambda_2 I = \begin{pmatrix}-1&0&1\\0&1&0\\1&0&-1\end{pmatrix} \sim \begin{pmatrix}1&0&-1\\0&1&0\\0&0&0\end{pmatrix}$ Eigenvectors: $t_2(1,0,1)$, $t_2\neq 0$

$\lambda_3=2$: $G-\lambda_3 I = \begin{pmatrix}-4&0&1\\0&0&0\\1&0&-4\end{pmatrix} \sim \begin{pmatrix}1&0&-4\\0&0&-15\\0&0&0\end{pmatrix} \sim \begin{pmatrix}1&0&0\\0&0&1\\0&0&0\end{pmatrix}$ Eigenvectors: $t_3(0,1,0)$, $t_3\neq 0$

We note that $-3(\tilde{x}^2+\tilde{y}^2+\tilde{z}^2) \le -3\tilde{x}^2-\tilde{y}^2+2\tilde{z}^2 \le 2(\tilde{x}^2+\tilde{y}^2+\tilde{z}^2)$

and therefore that $-3\cdot 1 \le q(u)\big|_{\|u\|=1} \le 2\cdot 1$

where the optionally extreme values are attainable since

$q(\pm\tilde{e}_1) = q(\pm\frac{1}{\sqrt{2}}(1,0,-1)) = -3$ and $q(\pm\tilde{e}_3) = q(\pm(0,1,0)) = 2$

Furthermore, since $\|u\|=1$ defines a closed and bounded set, and since q is a continuous function, we conclude that q can take all values in the interval $[-3,2]$.

(2) $\mathrm{span}\{p_1,p_2,p_3\}$ is equipped with the inner product $\langle p|q\rangle = \int_{-1}^{1} p(x)q(x)\,dx$

The Gram-Schmidt procedure gives

$f_1 = p_1$ and $\|f_1\|^2 = \int_{-1}^{1} x^2\,dx = \frac{2}{3}$ giving $e_1 = \frac{1}{\|f_1\|}f_1 = \sqrt{\frac{3}{2}}\,p_1$

$f_2 = p_2 - \langle p_2|e_1\rangle e_1 = p_2 - \left(\frac{3}{2}\int_{-1}^{1}x^2\cdot x\,dx\right)p_1 = p_2 - 0\,p_1 = p_2$

and $\|f_2\|^2 = \int_{-1}^{1}x^4\,dx = \frac{2}{5}$ giving $e_2 = \frac{1}{\|f_2\|}f_2 = \sqrt{\frac{5}{2}}\,p_2$

$f_3 = p_3 - \langle p_3|e_1\rangle e_1 - \langle p_3|e_2\rangle e_2 = p_3 - \frac{3}{2}\cdot\frac{2}{5}p_1 - \frac{5}{2}\cdot 0\,p_2 = p_3 - \frac{3}{5}p_1$

and $\|f_3\|^2 = \int_{-1}^{1}(x^3-\frac{3}{5}x)^2\,dx = 2\left(\frac{1}{7}-\frac{6}{5}\frac{1}{5}+\frac{9}{25}\frac{1}{3}\right) = 2\left(\frac{1}{7}-\frac{3}{25}\right) = \frac{8}{7\cdot 25}$

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(2) cont. giving $e_3 = \frac{1}{\|f_3\|} f_3 = \frac{5}{2}\sqrt{\frac{7}{2}}\left(p_3 - \frac{3}{5}p_1\right) = \frac{1}{2}\sqrt{\frac{7}{2}}\left(5p_3 - 3p_1\right)$

Answer: An ON-basis for span$\{p_1, p_2, p_3\}$ is e.g.

$$\sqrt{\frac{3}{2}}\, p_1 ,\ \sqrt{\frac{5}{2}}\, p_2 ,\ \frac{1}{2}\sqrt{\frac{7}{2}}\left(5p_3 - 3p_1\right)$$

(3) $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 & 2 & -6 \\ -3 & -1 & 3 \\ 2 & 1 & -2 \end{pmatrix}\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix}$ i.e. $X = S\tilde{X}$

where $S \sim \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 2 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

i.e. the matrix S has full rank and is therefore a change-of-basis matrix from e_1, e_2, e_3 to $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ where $(\tilde{e}_1\ \tilde{e}_2\ \tilde{e}_3) = (e_1\ e_2\ e_3)\, S$ q.e.d.

Moreover $5e_1 - e_2 + 3e_3 = (e_1\ e_2\ e_3)\begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix} = (\tilde{e}_1\ \tilde{e}_2\ \tilde{e}_3)\, S^{-1}\begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$

$$= \tilde{e}\begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 1 & 3 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix} = \tilde{e}\begin{pmatrix} 3 \\ 7 \\ 5 \end{pmatrix} = 3\tilde{e}_1 + 7\tilde{e}_2 + 5\tilde{e}_3$$

i.e. $\mathrm{coord}_{\tilde{e}}(5e_1 - e_2 + 3e_3) = (3, 7, 5)$

(4) $F: \mathbb{R}^3 \to \mathbb{R}^3$ where $\begin{cases} \ker(F) = \mathrm{span}\{(1,0,1)\} \\ E_{-2} = \mathrm{span}\{(2,-1,0), (1,4,-1)\} \end{cases}$

The conditions mean that $\begin{cases} F((1,0,1)) = (0,0,0) \\ F((2,-1,0)) = -2\,(2,-1,0) \\ F((1,4,-1)) = -2\,(1,4,-1) \end{cases}$

If A is the matrix of F relative to the standard basis, then $A\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $A\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix}$, $A\begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ -8 \\ 2 \end{pmatrix}$

i.e. $A\begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 4 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -4 & -2 \\ 0 & 2 & -8 \\ 0 & 0 & 2 \end{pmatrix}$

Thus $A = \begin{pmatrix} 0 & -4 & -2 \\ 0 & 2 & -8 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 4 \\ 1 & 0 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -4 & -2 \\ 0 & 2 & -8 \\ 0 & 0 & 2 \end{pmatrix}\frac{1}{(1+8+0)-(-1+0+0)}\begin{pmatrix} 1 & 2 & 9 \\ 4 & -2 & -4 \\ 1 & 2 & -1 \end{pmatrix}$

$$= \frac{1}{10}\begin{pmatrix} -18 & 4 & 18 \\ 0 & -20 & 0 \\ 2 & 4 & -2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -9 & 2 & 9 \\ 0 & -10 & 0 \\ 1 & 2 & -1 \end{pmatrix}$$

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5 $\langle u|v\rangle = 4x_1y_1 + 5x_2y_2 + 11x_3y_3 + 4(x_1y_2+x_2y_1) + 7(x_2y_3+x_3y_2) + 6(x_3y_1+x_1y_3)$

$$= (x_1\ x_2\ x_3)\begin{pmatrix}4&4&6\\4&5&7\\6&7&11\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}$$

where x_1, x_2, x_3 and y_1, y_2, y_3 are the coordinates of u and v respectively relative to the basis e_1, e_2, e_3

The orthogonal projection of w_1 on w_2 is equal to $\dfrac{\langle w_1|w_2\rangle}{\langle w_2|w_2\rangle} w_2$ and the length equals $\dfrac{|\langle w_1|w_2\rangle|}{\sqrt{\langle w_2|w_2\rangle}}$.

With $w_1 = e_1 - 4e_2 + e_3$ and $w_2 = 3e_1 + e_2 - 2e_3$, we get

$$\begin{cases}\langle w_1|w_2\rangle = (1\ -4\ 1)\begin{pmatrix}4&4&6\\4&5&7\\6&7&11\end{pmatrix}\begin{pmatrix}3\\1\\-2\end{pmatrix} = (1\ -4\ 1)\begin{pmatrix}4\\3\\3\end{pmatrix} = -5\\ \langle w_2|w_2\rangle = (3\ 1\ -2)\begin{pmatrix}4\\3\\3\end{pmatrix} = 9\end{cases}$$

Thus, the length of the orthog. proj. equals $\dfrac{|-5|}{\sqrt{9}} = \dfrac{5}{3}$

6 $F: R^3 \to R^3$ has the matrix $A = \begin{pmatrix}2&1&0\\2&3&0\\1&\beta&\beta\end{pmatrix}$, $\beta \in R$, relative to the standard basis for R^3.

Eigenvalues: $0 = \det(A-\lambda I) = \det\begin{pmatrix}2-\lambda&1&0\\2&3-\lambda&0\\1&\beta&\beta-\lambda\end{pmatrix} = -(\lambda-\beta)(\lambda-2)(\lambda-3) + 2(\lambda-\beta)$

$= -(\lambda-\beta)(\lambda^2-5\lambda+6-2) = -(\lambda-1)(\lambda-\beta)(\lambda-4)$

F is definitely diagonalizable if all eigenvalues are distinct, i.e. if $\beta \neq 1, 4$. For these β, we get (since $\beta\neq 1$)

Case 1

$\lambda_1 = 1$: $A - \lambda_1 I = \begin{pmatrix}1&1&0\\2&2&0\\1&\beta&\beta-1\end{pmatrix} \sim \begin{pmatrix}1&1&0\\0&\beta-1&\beta-1\\0&0&0\end{pmatrix} \sim \begin{pmatrix}1&1&0\\0&1&1\\0&0&0\end{pmatrix} \sim \begin{pmatrix}1&0&-1\\0&1&1\\0&0&0\end{pmatrix}$ Eigenvectors are $t_1(1,-1,1)$, $t_1 \neq 0$

$\lambda_2 = \beta$: $A - \lambda_2 I = \begin{pmatrix}2-\beta&1&0\\2&3-\beta&0\\1&\beta&0\end{pmatrix} \sim \begin{pmatrix}1&\beta&0\\0&3-3\beta&0\\0&\beta^2-2\beta+1&0\end{pmatrix} \sim \begin{pmatrix}1&\beta&0\\0&1&0\\0&1&0\end{pmatrix} \sim \begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$ Eigenvectors are $t_2(0,0,1)$, $t_2 \neq 0$

$\lambda_3 = 4$: $A - \lambda_3 I = \begin{pmatrix}-2&1&0\\2&-1&0\\1&\beta&\beta-4\end{pmatrix} \sim \begin{pmatrix}2&-1&0\\0&2\beta+1&2(\beta-4)\\0&0&0\end{pmatrix}$ Eigenvectors are $t_3(4-\beta, 2(4-\beta), 1+2\beta)$, $t_3 \neq 0$

Case 2

If $\beta = 1$, then
$$\begin{cases}A - \lambda_{1,2} I = \begin{pmatrix}1&1&0\\2&2&0\\1&1&0\end{pmatrix} \sim \begin{pmatrix}1&1&0\\0&0&0\\0&0&0\end{pmatrix}\\ A - \lambda_3 I = \begin{pmatrix}-2&1&0\\2&-1&0\\1&1&-3\end{pmatrix} \sim \begin{pmatrix}1&1&-3\\0&3&-6\\0&0&0\end{pmatrix} \sim \begin{pmatrix}1&0&-1\\0&1&-2\\0&0&0\end{pmatrix}\end{cases}$$

Eigenvectors are $t_1(1,-1,0) + t_2(0,0,1) = \tilde{t}_1(1,-1,1) + \tilde{t}_2(0,0,1)$ where at least one of t_1, t_2 is non zero

Eigenvectors are $t_3(1,2,1)$

Case 3

If $\beta = 4$, then $A - \lambda_{2,3} I = \begin{pmatrix}-2&1&0\\2&-1&0\\1&4&0\end{pmatrix} \sim \begin{pmatrix}1&4&0\\0&9&0\\0&0&0\end{pmatrix} \sim \begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$

i.e. $\dim(\lambda_{2,3}\text{-eigenspace}) = 1 < \text{multiplicity}(\lambda_{2,3}) = 2$

i.e. F is not diagonalizable if $\beta = 4$

Answer: F is diagonalizable iff $\beta \neq 4$, and a basis of eigenvectors is then e.g. $(1,-1,1), (0,0,1), (4-\beta, 2(4-\beta), 1+2\beta)$.

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7. $F: R^5 \to R^4$ has, relative to the standard bases, the matrix

$$\begin{pmatrix} 6 & 4 & 16 & 2 & 2 \\ 2 & 5 & 9 & -3 & 8 \\ -3 & 4 & -2 & -7 & 11 \\ 4 & 3 & 11 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 0 & -11 & -11 & 11 & -22 \\ 2 & 5 & 9 & -3 & 8 \\ -1 & 9 & 7 & -10 & 19 \\ 0 & -7 & -7 & 7 & -14 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 1 & -1 & 2 \\ 0 & 23 & 23 & -23 & 46 \\ 1 & -9 & -7 & 10 & -19 \\ 0 & 1 & 1 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & 1 & -1 \\ 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

From the row-reduced echelon form, we conclude that only two of possible five vectors span $\operatorname{im}(F)$, and that a basis for $\operatorname{im}(F)$ is e.g. $(6,2,-3,4)$, $(4,5,4,3)$. We also conclude that kernel of F is given by the equations $x_1+2x_3+x_4-x_5=0$ and $x_2+x_3-x_4+2x_5=0$

i.e. $$u_{\ker(F)} = (-2x_3-x_4+x_5, -x_3+x_4-2x_5, x_3, x_4, x_5)$$
$$= x_3(-2,-1,1,0,0)+x_4(-1,1,0,1,0)+x_5(1,-2,0,0,1)$$

Thus $(-2,-1,1,0,0)$, $(-1,1,0,1,0)$, $(1,-2,0,0,1)$ is e.g. a basis for $\ker(F)$.

8. $\operatorname{span}\left\{\underbrace{\begin{pmatrix} 6 & 3 \\ 2 & 5 \end{pmatrix}}_{m_1}, \underbrace{\begin{pmatrix} 8 & 1 \\ 5 & 7 \end{pmatrix}}_{m_2}, \underbrace{\begin{pmatrix} -1 & 4 \\ 2 & 1 \end{pmatrix}}_{m_3}, \underbrace{\begin{pmatrix} 8 & 16 \\ 5 & 10 \end{pmatrix}}_{m_4}\right\}$ is a subspace of $\operatorname{span}\left\{\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{e_1}, \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{e_2}, \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{e_3}, \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{e_4}\right\}$

We have that
$$\begin{cases} m_1 = 6e_1+2e_2+3e_3+5e_4 \\ m_2 = 8e_1+5e_2+e_3+7e_4 \\ m_3 = -e_1+2e_2+4e_3+e_4 \\ m_4 = 8e_1+5e_2+16e_3+10e_4 \end{cases}$$

Also $m = \begin{pmatrix} -1 & 9 \\ 2 & 3 \end{pmatrix} = -e_1+2e_2+9e_3+3e_4$

A test equation for linear dependence of m_1, m_2, m_3, m_4 and for the question whether $m \in \operatorname{span}\{m_1,m_2,m_3,m_4\}$ or not is $\lambda_1 m_1+\lambda_2 m_2+\lambda_3 m_3+\lambda_4 m_4 = m$

i.e. $$\begin{pmatrix} 6 & 8 & -1 & 8 \\ 2 & 5 & 2 & 5 \\ 3 & 1 & 4 & 16 \\ 5 & 7 & 1 & 10 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 9 \\ 3 \end{pmatrix} \quad \text{i.e. } M\lambda = \begin{pmatrix} -1 \\ 2 \\ 9 \\ 3 \end{pmatrix}$$

where $$\left(M \,\middle|\, \begin{matrix} -1 \\ 2 \\ 9 \\ 3 \end{matrix}\right) \sim \left(\begin{array}{cccc|c} 0 & -7 & -7 & -7 & -7 \\ 2 & 5 & 2 & 5 & 2 \\ 1 & -4 & 2 & 11 & 7 \\ 1 & -3 & -3 & 0 & -1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -3 & -3 & 0 & -1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 11 & 8 & 5 & 4 \\ 0 & -1 & 5 & 11 & 8 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 2 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & -3 & -6 & -7 \\ 0 & 0 & 6 & 12 & 9 \end{array}\right)$$
$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 2 \\ 0 & 3 & 0 & -3 & -4 \\ 0 & 0 & 3 & 6 & 7 \\ 0 & 0 & 0 & 0 & -5 \end{array}\right)$$

Thus, a basis for $\operatorname{span}\{m_1,m_2,m_3,m_4\}$ is e.g. m_1, m_2, m_3 and $m \notin \operatorname{span}\{m_1,m_2,m_3,m_4\} = \operatorname{span}\{m_1,m_2,m_3\}$

MÄLARDALEN UNIVERSITY
School of Education, Culture and Communication
Department of Applied Mathematics
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EXAMINATION IN MATHEMATICS
MAA153 Linear algebra
EVALUATION PRINCIPLES with POINT RANGES
Academic year: 2016/17

Examination 2017-06-09	Maximum points for subparts of the problems in the final examination
1. The quadratic form $q(u)$ is in the interval $[-3,2]$. The minimum is attained for $\pm\frac{1}{\sqrt{2}}(1,0,-1)$ and the maximum for $\pm(0,1,0)$	**1p**: Correctly found the eigenvalues of the symmetric operator corresponding to the quadratic form (QF) **1p**: Correctly identified the diagonal form of the QF **1p**: Correctly, from the diagonalized QF, identified the interval in which $q(u)$ lies as $\lVert u \rVert = 1$ **1p**: Correctly found the vectors for which q attains minimum as $\lVert u \rVert = 1$ **1p**: Correctly found the vectors for which q attains maximum as $\lVert u \rVert = 1$
2. An ON-basis for the linear space is e.g. $\sqrt{\frac{3}{2}}\,p_1,\ \sqrt{\frac{5}{2}}\,p_2,\ \frac{1}{2}\sqrt{\frac{7}{2}}(5p_3-3p_1)$	**1p**: Correctly normalized one of the three functions, e.g. p_1, into e_1 **1p**: Correctly found and explicitly evaluated a nonzero function f_2 which is orthogonal to the function e_1, e.g. the function $p_2 - e_1\langle e_1 \mid p_2\rangle$ **1p**: Correctly normalized the function f_2 into e_2 **1p**: Correctly found and explicitly evaluated a nonzero function f_3 which is orthogonal to the two functions e_1 and e_2, e.g. the function $p_3 - e_1\langle e_1 \mid p_3\rangle - e_2\langle e_2 \mid p_3\rangle$ **1p**: Correctly normalized the function f_3 into e_3, and correctly summarized e_1, e_2, e_3 as an ON-basis for the given linear space
3. Proof: The matrix S in the matrix version $\mathsf{X} = \mathsf{S}\widetilde{\mathsf{X}}$ of the given relationships is invertible and thus S is a change-of-basis matrix in changing from the basis e_1, e_2, e_3 to $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$. The coordinates of $5e_1 - e_2 + 3e_3$ relative to the basis $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ are $3, 7, 5$, i.e. $5e_1 - e_2 + 3e_3 = 3\tilde{e}_1 + 7\tilde{e}_2 + 5\tilde{e}_3$	**2p**: Correctly proved why the relationships define a change-of-basis matrix S between two ordered bases e_1, e_2, e_3 and $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ **1p**: Correctly found that the coordinates of $5e_1 - e_2 + 3e_3$ with respect to the basis $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ are given by the coordinate matrix $\mathsf{S}^{-1}\mathsf{X}$, where X is equal to the coordinate matrix $(5\ \ {-1}\ \ 3)^T$ **1p**: Correctly found the matrix S^{-1} **1p**: Correctly found the coordinates of $5e_1 - e_2 + 3e_3$ with respect to the basis $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$
4. F has the matrix $\frac{1}{5}\begin{pmatrix} -9 & 2 & 9 \\ 0 & -10 & 0 \\ 1 & 2 & 1 \end{pmatrix}$ relative to the standard basis	**1p**: Correctly interpreted the vector spanning the kernel, i.e. that $\mathsf{A}(1\ \ 0\ \ 1)^T = (0\ \ 0\ \ 0)^T$ for the matrix A of F relative the standard basis **1p**: Correctly interpreted the vectors spanning the eigenspace, i.e. that $\mathsf{A}(2\ \ {-1}\ \ 0)^T = -2\,(2\ \ {-1}\ \ 0)^T$ and $\mathsf{A}(1\ \ 4\ \ {-1})^T = -2\,(1\ \ 4\ \ {-1})^T$ for the matrix A of F **1p**: Correctly, based on the three given conditions, formulated an equation for the matrix A **2p**: Correctly solved the equation for the matrix A

5.	$\left\| (e_1 - 4e_2 + e_3)_{3e_1 + e_2 - 2e_3} \right\| = \frac{5}{3}$	**1p**: Correctly stated the (formula) expression for the orthogonal projection of $e_1 - 4e_2 + e_3$ on $3e_1 + e_2 - 2e_3$ **1p**: Correctly interpreted how the given inner product is applied **1p**: Correctly found the inner product of the vectors $e_1 - 4e_2 + e_3$ and $3e_1 + e_2 - 2e_3$ **1p**: Correctly found the length of the vector $3e_1 + e_2 - 2e_3$ **1p**: Correctly found the length of the orthogonal projection
6.	The linear operator F is diagonalizable iff $\beta \neq 4$, and a basis of eigenvectors is e.g. $(1,-1,1)$, $(0,0,1)$, $(4-\beta, 8-2\beta, 1+2\beta)$	**1p**: Correctly found that the linear operator (LO) is diagonalizable if $\beta \neq 1,4$ **1p**: Correctly found that the LO is diagonalizable if $\beta = 1$ **1p**: Correctly found that the LO is not diagonalizable if $\beta = 4$ **1p**: Correctly for $\beta = 1$ found a basis of eigenvectors **1p**: Correctly for $\beta \neq 1,4$ found a basis of eigenvectors
7.	A basis for the image of F is e.g. $(6,2,-3,4)$, $(4,5,4,3)$. A basis for the kernel of F is e.g. $(-2,-1,1,0,0)$, $(-1,1,0,1,0)$, $(1,-2,0,0,1)$	**2p**: Correctly found a basis for the image of F **3p**: Correctly found a basis for the kernel of F
8.	A basis for the given subspace of $\mathcal{M}(2,\mathcal{R})$ is e.g. $\begin{pmatrix} 6 & 3 \\ 2 & 5 \end{pmatrix}, \begin{pmatrix} 8 & 1 \\ 5 & 7 \end{pmatrix}, \begin{pmatrix} -1 & 4 \\ 2 & 1 \end{pmatrix}$. The matrix $\begin{pmatrix} -1 & 9 \\ 2 & 3 \end{pmatrix}$ does not belong to the subspace	**2p**: Correctly initiated an analysis of the relation between the five vectors in $\mathcal{M}(2,\mathcal{R})$, <u>and</u> correctly found the row-echelon form of the augmented coordinate matrix of the vectors spanning the subspace of $\mathcal{M}(2,\mathcal{R})$ together with the fifth vector **2p**: Correctly from the row-echelon form identified a basis for the subspace of $\mathcal{M}(2,\mathcal{R})$ **1p**: Correctly from the row-echelon form concluded that the fifth vector does not belong to the subspace of $\mathcal{M}(2,\mathcal{R})$