MÄLARDALEN UNIVERSITY
School of Education, Culture and Communication
Department of Applied Mathematics
Examiner: Lars-Göran Larsson

EXAMINATION IN MATHEMATICS
MAA153 Linear Algebra
Date: 2016-06-08 *Write time:* 5 hours
Aid: Writing materials, ruler

This examination consists of eight RANDOMLY ORDERED problems each of which is worth at maximum 5 points. The maximum sum of points is thus 40. The PASS-marks 3, 4 and 5 require a minimum of 18, 26 and 34 points respectively. The minimum points for the ECTS-marks E, D, C, B and A are 18, 20, 26, 33 and 38 respectively. Solutions are supposed to include rigorous justifications and clear answers. All sheets with solutions must be sorted in the order the problems are given in. Especially, avoid to write on back pages of solution sheets.

1. The linear operator $F : \mathbb{R}^3 \to \mathbb{R}^3$ has relative to the standard basis the matrix

$$\begin{pmatrix} \beta & -1 & \beta \\ 1 & 3 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

where $\beta \in \mathbb{R}$. Find the numbers β for which the operator är diagonalizable, and state a basis of eigenvectors for each of these β.

2. Find a basis for the linear span of the vectors $(1, 3, -2, 1, 5)$, $(-1, 1, 0, 1, 1)$, $(-1, 5, -2, 3, 7)$, $(1, 2, -1, 1, 3)$ of $\mathbb{R}^5$. Then find, with respect to the chosen basis, the coordinates of the vector $(1, 2, 3, a, -1)$ for those a for which the vector belongs to the linear span.

3. Can the matrices $\begin{pmatrix} 2 & -5 & 1 \\ 3 & 1 & 2 \\ 2 & -3 & 1 \end{pmatrix}$ and $\begin{pmatrix} 4 & 1 & 3 \\ 2 & -3 & 1 \\ 5 & 2 & 4 \end{pmatrix}$ by a suitable choice of bases be the matrices of the same linear operator $F : \mathbb{R}^3 \to \mathbb{R}^3$?

4. Let the linear space $\mathcal{P}_2$, which is spanned by the real-valued polynomial functions p_0, p_1 and p_2 where $p_k(x) = x^k$ in the interval $[-1, 1]$, be equipped with the inner product $\langle p|q\rangle = \int_{-1}^{1} p(x)q(x)\,dx$. Find an ON-basis for the orthogonal complement of the subspace spanned by the functions p_0 and p_1.

5. The linear transformation $F : \mathbb{E}^4 \to \mathbb{E}^3$ is in the standard bases defined by

$$F(u) = (2x_1 + 6x_2 - 2x_3 + 4x_4, 3x_1 + 2x_2 + 4x_3 - x_4, -x_1 + 2x_2 - 4x_3 + 3x_4)$$

where $u = (x_1, x_2, x_3, x_4)$. Find an orthonormal basis for the kernel of F.

6. Classify the two quadric surfaces

$$\begin{cases} S_1 : (x - 2y + z)^2 + (y + z)^2 + (x - y + 2z)^2 = 1, \\ S_2 : 2xy + 2yz + 6xz + z^2 = 1, \end{cases}$$

i.e. find the geometric meaning of each equation. Motivate!

7. Let $\mathcal{H}$ denote the vector space spanned by the functions h_0, h_1 and h_2 definied according to $h_n(x) = x^n e^x$. Define the linear differential operator $D : \mathcal{H} \to \mathcal{H}$ by $D(h) = h'$. Find the matrix of D in the basis h_0, h_1, h_2. Also, prove that D is invertible and explain the meaning of $D^{-1}(h)$.

8. Prove that the relationships $\begin{cases} 2\tilde{x}_2 + \tilde{x}_3 &= x_1 - x_2 \\ \tilde{x}_2 &= 2x_1 + x_3 \\ \tilde{x}_1 - 2\tilde{x}_3 &= x_2 + x_3 \end{cases}$ defines a change-of-basis between two ordered bases $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and $\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3$, where x_1, x_2, x_3 and $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ are the coordinates of a vector $\mathbf{u}$ with respect to respectively of the bases. Also, find the coordinates of the vector $5\tilde{\mathbf{e}}_1 - \tilde{\mathbf{e}}_2 + 5\tilde{\mathbf{e}}_3$ with respect to the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

Om du föredrar uppgifterna formulerade på svenska, var god vänd på bladet.

MÄLARDALENS HÖGSKOLA
Akademin för utbildning, kultur och kommunikation
Avdelningen för tillämpad matematik
Examinator: Lars-Göran Larsson

TENTAMEN I MATEMATIK
MAA153 Linjär algebra
Datum: 2016-06-08 *Skrivtid:* 5 timmar
Hjälpmedel: Skrivdon, linjal

Denna tentamen består av åtta stycken om varannat SLUMPMÄSSIGT ORDNADE uppgifter som vardera kan ge maximalt 5 poäng. Den maximalt möjliga poängsumman är således 40. För GODKÄND-betygen 3, 4 och 5 krävs minst 18, 26 respektive 34 poäng. För ECTS-betygen E, D, C, B och A krävs 18, 20, 26, 33 respektive 38 poäng. Lösningar förutsätts innefatta ordentliga motiveringar och tydliga svar. Samtliga lösningsblad skall vid inlämning vara sorterade i den ordning som uppgifterna är givna i. Undvik speciellt att skriva på baksidor av lösningsblad.

1. Den linjära operatorn $F : \mathbb{R}^3 \to \mathbb{R}^3$ har i standardbasen matrisen

$$\begin{pmatrix} \beta & -1 & \beta \\ 1 & 3 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

där $\beta \in \mathbb{R}$. Bestäm de tal β för vilka operatorn är diagonaliserbar, och ange en bas av egenvektorer till F för var och en av dessa β.

2. Bestäm en bas för det linjära höljet av vektorerna $(1, 3, -2, 1, 5)$, $(-1, 1, 0, 1, 1)$, $(-1, 5, -2, 3, 7)$, $(1, 2, -1, 1, 3)$ i $\mathbb{R}^5$. Bestäm sedan, med avseende på den valda basen, koordinaterna för vektorn $(1, 2, 3, a, -1)$ för de a för vilka vektorn tillhör det linjära höljet.

3. Kan matriserna $\begin{pmatrix} 2 & -5 & 1 \\ 3 & 1 & 2 \\ 2 & -3 & 1 \end{pmatrix}$ och $\begin{pmatrix} 4 & 1 & 3 \\ 2 & -3 & 1 \\ 5 & 2 & 4 \end{pmatrix}$ genom ett lämpligt val av baser vara matriserna till en och samma linjära operator $F : \mathbb{R}^3 \to \mathbb{R}^3$?

4. Låt det linjära rummet $\mathcal{P}_2$, som spänns upp av de reellvärda polynomfunktionerna p_0, p_1 och p_2 där $p_k(x) = x^k$ i intervallet $[-1, 1]$, vara utrustat med skalärprodukten $\langle p|q \rangle = \int_{-1}^{1} p(x)q(x)\,dx$. Bestäm en ON-bas för det ortogonala komplementet till underrummet som spänns upp av funktionerna p_0 och p_1.

5. Den linjära avbildningen $F : \mathbb{E}^4 \to \mathbb{E}^3$ är i standardbaserna definierad enligt

$$F(u) = (2x_1 + 6x_2 - 2x_3 + 4x_4, 3x_1 + 2x_2 + 4x_3 - x_4, -x_1 + 2x_2 - 4x_3 + 3x_4)$$

där $u = (x_1, x_2, x_3, x_4)$. Bestäm en ortonormerad bas för F:s nollrum.

6. Klassificera de två andragradsytorna

$$\begin{cases} S_1 : (x - 2y + z)^2 + (y + z)^2 + (x - y + 2z)^2 = 1, \\ S_2 : 2xy + 2yz + 6xz + z^2 = 1, \end{cases}$$

dvs bestäm den geometriska innebörden av varje ekvation. Motivera!

7. Låt $\mathcal{H}$ beteckna det linjära rum som spänns upp av funktionerna h_0, h_1 och h_2 definierade enligt $h_n(x) = x^n e^x$. Definiera den linjära differentialoperatorn $D : \mathcal{H} \to \mathcal{H}$ genom $D(h) = h'$. Bestäm avbildningsmatrisen för D i basen h_0, h_1, h_2. Bevisa även att D har en invers och förklara innebörden av $D^{-1}(h)$.

8. Bevisa att sambanden $\begin{cases} 2\tilde{x}_2 + \tilde{x}_3 &= x_1 - x_2 \\ \tilde{x}_2 &= 2x_1 + x_3 \\ \tilde{x}_1 - 2\tilde{x}_3 &= x_2 + x_3 \end{cases}$ definierar ett basbyte mellan två ordnade baser $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ och $\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3$, där x_1, x_2, x_3 och $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ är koordinaterna för en vektor $\mathbf{u}$ med avseende på respektive av baserna. Bestäm även koordinaterna för vektorn $5\tilde{\mathbf{e}}_1 - \tilde{\mathbf{e}}_2 + 5\tilde{\mathbf{e}}_3$ med avseende på basen $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

If you prefer the problems formulated in English, please turn the page.

MAA153 & MMA129 / Solutions to the final exam 2016-06-08

1 $F: \mathbb{R}^3 \to \mathbb{R}^3$ has the matrix $\begin{pmatrix} \beta & -1 & \beta \\ 1 & 3 & 1 \\ 0 & 1 & 0 \end{pmatrix} = A$ relative to the standard basis. $(\beta \in \mathbb{R})$

Eigenvalues: $0 = \det(A - \lambda I) = \det\begin{pmatrix} \beta-\lambda & -1 & \beta \\ 1 & 3-\lambda & 1 \\ 0 & 1 & -\lambda \end{pmatrix} = \det\begin{pmatrix} \beta-\lambda & 0 & \beta-\lambda \\ 1 & 3-\lambda & 1 \\ 0 & 1 & -\lambda \end{pmatrix}$

$$= \det\begin{pmatrix} \beta-\lambda & 0 & 0 \\ 1 & 3-\lambda & 0 \\ 0 & 1 & -\lambda \end{pmatrix} = (\beta-\lambda)(3-\lambda)(-\lambda) = -\lambda(\lambda-\beta)(\lambda-3)$$

$\lambda = \beta$: $A - \beta I = \begin{pmatrix} 0 & -1 & \beta \\ 1 & 3-\beta & 1 \\ 0 & 1 & -\beta \end{pmatrix} \sim \begin{pmatrix} 1 & 3-\beta & 1 \\ 0 & 1 & -\beta \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1-\beta(\beta-3) \\ 0 & 1 & -\beta \\ 0 & 0 & 0 \end{pmatrix}$; Eigenvectors $= t_3(\beta^2-3\beta-1, \beta, 1)$

which implies that the dimension of the eigenspace for $\lambda = \beta$ is equal to one (1) irrespective of the value of β. This means that F is diagonalizable iff $\beta \neq 0, 3$ since a repeated eigenvalue 0 or 3 generates only a 1-dim. eigenspace.

Eigenvectors: $\lambda = 0$ $A - 0I = \begin{pmatrix} \beta & -1 & \beta \\ 1 & 3 & 1 \\ 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$; Eigenvectors $= t_1(1, 0, -1)$

$\lambda = 3$ $A - 3I = \begin{pmatrix} \beta-3 & -1 & \beta \\ 1 & 0 & 1 \\ 0 & 1 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}$; Eigenvectors $= t_2(-1, 3, 1)$

Thus F is diag. iff $\beta \neq 0, 3$. A basis of eigenvectors is e.g. $(1, 0, -1), (-1, 3, 1), (\beta^2-3\beta-1, \beta, 1)$

2 Let $\begin{cases} \text{span}\{\overset{v_1}{(1,3,-2,1,5)}, \overset{v_2}{(-1,1,0,1,1)}, \overset{v_3}{(-1,5,-2,3,7)}, \overset{v_4}{(1,2,-1,1,3)}\} = M \\ (1,2,3,a,-1) = u \end{cases}$

An analysis of the vector eq. $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = u$ will tell us which subsets of $\{v_1, v_2, v_3, v_4\}$ that can be used for bases of M, and whether $u \in M$ or not.

The augmented matrix for the system of equations is

$$\left(\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 1 \\ 3 & 1 & 5 & 2 & 2 \\ -2 & 0 & -2 & -1 & 3 \\ 1 & 1 & 3 & 1 & a \\ 5 & 1 & 7 & 3 & -1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 1 \\ 0 & 4 & 8 & -1 & -1 \\ 0 & -2 & -4 & 1 & 5 \\ 0 & 2 & 4 & 0 & a-1 \\ 0 & 6 & 12 & -2 & -6 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 & -2a+1 \\ 0 & 0 & 0 & 1 & a+4 \\ 0 & 2 & 4 & 0 & a-1 \\ 0 & 0 & 0 & -2 & -3a-3 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -\frac{1}{2}(a+7) \\ 0 & 1 & 2 & 0 & \frac{1}{2}(a-1) \\ 0 & 0 & 0 & 1 & a+4 \\ 0 & 0 & 0 & 0 & -a+5 \\ 0 & 0 & 0 & 0 & -a+5 \end{array}\right)$$

which implies that e.g. v_1, v_2, v_4 is a basis for M and that $u \in M$ iff $a = 5$. If $a = 5$ and v_1, v_2, v_4 is the chosen basis, the vector u is equal to $-6v_1 + 2v_2 + 9v_4$ i.e. the coordinates of $u_{a=5}$ in the basis v_1, v_2, v_4 are $-6, 2, 9$

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(3) Let $\begin{pmatrix} 2 & -5 & 1 \\ 3 & 1 & 2 \\ 2 & -3 & 1 \end{pmatrix} = A$ and $\begin{pmatrix} 4 & 1 & 3 \\ 2 & -3 & 1 \\ 5 & 2 & 4 \end{pmatrix} = B$

If A and B are the matrices of the same linear operator $F: \mathbb{R}^3 \to \mathbb{R}^3$, but relative two different bases, then there must exist a change-of-basis matrix S such that $A = S^{-1}BS$ (where S transforms from the basis in which B represents F to the basis in which A represents F). Then

$$\det(A) = \det(S^{-1}BS) \overset{\text{product rule for determinants}}{=} \det(S^{-1})\det(B)\det(S)$$
$$= \frac{1}{\cancel{\det(S)}} \det(B) \cancel{\det(S)} = \det(B)$$

i.e. a necessary condition for A and B to both represent F is that the determinants of the matrices A and B equals.

Since $\begin{cases} \det(A) = \det\begin{pmatrix} 2 & -5 & 1 \\ 3 & 1 & 2 \\ 2 & -3 & 1 \end{pmatrix} = \det\begin{pmatrix} 2 & -5 & 1 \\ -1 & 11 & 0 \\ 0 & 2 & 0 \end{pmatrix} = (-1)^{1+3}\cdot 1\,(-2) + 0 + 0 = -2 \\ \det(B) = \det\begin{pmatrix} 4 & 1 & 3 \\ 2 & -3 & 1 \\ 5 & 2 & 4 \end{pmatrix} = \det\begin{pmatrix} 4 & 1 & 3 \\ 14 & 0 & 10 \\ -3 & 0 & -2 \end{pmatrix} = (-1)^{1+2}\cdot 1 \begin{vmatrix} -1 & 0 \\ -3 & -2 \end{vmatrix} + 0 + 0 = -2 \end{cases}$

we conclude that A and B may be the matrices of the same linear operator F but it is not for sure. A final answer can be found only by solving $SA = BS$ for S (but that is beyond the scope of this exam)

(4) $\mathcal{P}_2 = \text{span}\{p_0, p_1, p_2\}$ on $[-1,1]$ and equipped with the inner product $\langle p|q\rangle = \int_{-1}^{1} p(x)q(x)\,dx$

Let $\text{span}\{p_0, p_1\} = M$. Then the orthogonal complement of M (in $\mathcal{P}_2$) is $M^\perp = \{p \in \mathcal{P}_2 : \langle p|q\rangle = 0 \text{ for all } q \in M\}$.

Let $p \in M^\perp$ Then $p = c_0p_0 + c_1p_1 + c_2p_2$ and $\begin{cases} \langle p|p_0\rangle = 0 \\ \langle p|p_1\rangle = 0 \end{cases}$

The conditions are explicitly

$$\begin{cases} 0 = \int_{-1}^{1} (c_0 + c_1x + c_2x^2)\cdot 1\,dx = c_0\cdot 2 + c_1\cdot 0 + c_2\cdot\frac{2}{3} \\ 0 = \int_{-1}^{1} (c_0 + c_1x + c_2x^2)\cdot x\,dx = c_0\cdot 0 + c_1\cdot\frac{2}{3} + c_2\cdot 0 \end{cases} \Leftrightarrow \begin{cases} c_2 = -3c_0 \\ c_1 = 0 \end{cases}$$

i.e. $M^\perp$ is spanned by $p_0 - 3p_2$ i.e. $M^\perp = \text{span}\{p_0 - 3p_2\}$

Furthermore $\|p_0 - 3p_2\|^2 = \langle p_0 - 3p_2 | p_0 - 3p_2\rangle = \int_{-1}^{1} (1-3x^2)^2\,dx$

$$= 2\int_0^1 (1 - 6x^2 + 9x^4)\,dx = 2\left(1 - 6\cdot\frac{1}{3} + 9\cdot\frac{1}{5}\right) = 2\left(-1 + \frac{9}{5}\right) = \frac{8}{5}$$

Thus $\frac{5}{2\sqrt{2}}(p_0 - 3p_2)$ is an ON-basis for $M^\perp$

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5 $F: E^4 \to E^3$ where

$$F(u) = (2x_1+6x_2-2x_3+4x_4,\ 3x_1+2x_2+4x_3-x_4,\ -x_1+2x_2-4x_3+3x_4)$$

The kernel of F is given by $F(u)=0$, i.e. by

$$\underbrace{\begin{pmatrix} 2 & 6 & -2 & 4 \\ 3 & 2 & 4 & -1 \\ -1 & 2 & -4 & 3 \end{pmatrix}}_{A}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{where} \quad A \sim \begin{pmatrix} 0 & 10 & -10 & 10 \\ 0 & 8 & -8 & 8 \\ -1 & 2 & -4 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 4 & -3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

i.e. a basis for $\ker(F)$ is e.g. $\underbrace{(-2,1,1,0)}_{u_1}, \underbrace{(1,-1,0,1)}_{u_2}$

The Gram-Schmidt procedure gives

$$\begin{cases} e_1 = \dfrac{1}{\|u_1\|}u_1 \overset{E^4}{=} \dfrac{1}{\sqrt{4+1+1+0}}u_1 = \dfrac{1}{\sqrt{6}}(-2,1,1,0) \\[2ex] f_2 = u_2 - \langle u_2|e_1\rangle e_1 = (1,-1,0,1) - \frac{1}{6}(-2-1+0+0)(-2,1,1,0) \\ \quad = \frac{1}{2}[(2,-2,0,2)+(-2,1,1,0)] = \frac{1}{2}(0,-1,1,2) \\[2ex] e_2 = \dfrac{1}{\|(0,-1,1,2)\|}(0,-1,1,2) = \dfrac{1}{\sqrt{0+1+1+2}}(0,-1,1,2) = \dfrac{1}{\sqrt{6}}(0,-1,1,2) \end{cases}$$

i.e. $\frac{1}{\sqrt{6}}(-2,1,1,0), \frac{1}{\sqrt{6}}(0,-1,1,2)$ is an ON-basis for $\ker(F)$

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$$\begin{cases} S_1: (x-2y+z)^2+(y+z)^2+(x-y+2z)^2 = 1 \\ S_2: 2xy+2yz+6xz+z^2 = 1 \end{cases}$$

$\boxed{S_1}$ It may be tempting to interpret the LHS of the equation as a quadratic form of rank 3 and with the signature (1,1,1) but $\begin{cases} x-2y+z = \tilde{x} \\ y+z = \tilde{y} \\ x-y+2z = \tilde{z} \end{cases}$ are ~~not~~ the relationships for the coordinates in a change of basis.

This, since the matrix M in $M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix}$ doesn't have rank 3.

By expanding the LHS of the eq. for S_1, we get

$$1 = (x^2+4y^2+z^2-4xy+2xz-4yz)+(y^2+z^2+2yz)+(x^2+y^2+4z^2-2xy+4xz-4yz)$$
$$= 2x^2+6y^2+6z^2-6xy+6xz-6yz = 2\left(x-\tfrac{3}{2}y+\tfrac{3}{2}z\right)^2+\tfrac{3}{2}(y+z)^2$$

i.e. S_1 is an elliptic cylinder

$\boxed{S_2}$

$$1 = 2xy+2yz+6xz+z^2 = (z+y+3x)^2-y^2-9x^2-4xy$$
$$= (z+y+3x)^2-(y+2x)^2-5x^2 = \tilde{x}^2-\tilde{y}^2-5\tilde{z}^2$$

with $\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ where $\begin{pmatrix} 3 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ is invertible and therefore is the change-of-basis matrix (from the basis $\tilde{e}_1\tilde{e}_2\tilde{e}_3$ to the basis $e_1e_2e_3$)

i.e. S_2 is a two-sheeted hyperboloid

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7 $\mathcal{H} = \text{span}\{h_0, h_1, h_2\}$ where $h_n(x) = x^n e^x$

$D: \mathcal{H} \to \mathcal{H}$ where $D(h) = h'$

We have that
$$\begin{cases} h_0'(x) = \frac{d}{dx} e^x = e^x = h_0(x) \\ h_1'(x) = \frac{d}{dx}(xe^x) = 1\cdot e^x + x\cdot e^x = h_0(x) + h_1(x) \\ h_2'(x) = \frac{d}{dx}(x^2e^x) = 2x\cdot e^x + x^2 e^x = 2h_1(x) + h_2(x) \end{cases}$$

i.e. $\left(D(h_0)\ D(h_1)\ D(h_2)\right) = (h_0\ h_1\ h_2)\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = (h_0\ h_1\ h_2)\,\Delta$

i.e. the matrix of D in the basis h_0, h_1, h_2 is $\underline{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Since the matrix is invertible (the rank equals 3), the operator D is invertible, q.e.d. Furthermore $D^{-1}(h)$ means the anti-derivative of h for which the "integration constant" equals 0, i.e. $D^{-1}(0) = 0$.

We have that $\left(D^{-1}(h_0)\ D^{-1}(h_1)\ D^{-1}(h_2)\right) = (h_0\ h_1\ h_2)\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} = (h_0\ h_1\ h_2)\,\Delta^{-1}$

Ex.
$$\begin{cases} D^{-1}(h_2)(x) = (h_0(x)\ h_1(x)\ h_2(x))\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 2h_0(x) - 2h_1(x) + h_2(x) \\ \int h_2(x)\,dx \Big|_{const=0} = \int x^2 e^x\,dx \Big|_{const=0} = x^2e^x - 2xe^x + 2e^x = h_2(x) - 2h_1(x) + 2h_0(x) \end{cases}$$

i.e. OK

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$$\begin{cases} 2\tilde{x}_2 + \tilde{x}_3 = x_1 - x_2 \\ \tilde{x}_2 = 2x_1 + x_3 \\ \tilde{x}_1 - 2\tilde{x}_3 = x_2 + x_3 \end{cases}$$
i.e. $\underbrace{\begin{pmatrix} 0 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \end{pmatrix}}_{A}\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}}_{B}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

If $A^{-1}, B^{-1}\ \exists$ then $B^{-1}A$ is the change-of-basis matrix from the basis e_1, e_2, e_3 to $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$. We have that
$$\begin{cases} \det(A) = 0 + 0 + 1(-1)^{3+1}(0-1) = -1 \neq 0, \text{ i.e. } A^{-1}\ \exists \\ \det(B) = \det\begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} = 1(-1)^{1+1}(2-1) + 0 + 0 = 1 \neq 0, \text{ i.e. } B^{-1}\ \exists \end{cases}$$

i.e. the relationships defines a change-of-basis matrix. q.e.d.

Furthermore, $5\tilde{e}_1 - \tilde{e}_2 + 5\tilde{e}_3 = (\tilde{e}_1\ \tilde{e}_2\ \tilde{e}_3)\begin{pmatrix} 5 \\ -1 \\ 5 \end{pmatrix}$

$$= (e_1\ e_2\ e_3)\,B^{-1}A\begin{pmatrix} 5 \\ -1 \\ 5 \end{pmatrix} = (e_1\ e_2\ e_3)\,\frac{1}{1}\begin{pmatrix} -1 & 1 & -1 \\ -2 & 1 & -1 \\ 2 & -1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \end{pmatrix}\begin{pmatrix} 5 \\ -1 \\ 5 \end{pmatrix}$$

$$= (e_1\ e_2\ e_3)\begin{pmatrix} -1 & 1 & -1 \\ -2 & 1 & -1 \\ 2 & -1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \\ -5 \end{pmatrix} = (e_1\ e_2\ e_3)\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} = e_1 - 2e_2 - 3e_3$$

i.e. the coordinates of the vector $5\tilde{e}_1 - \tilde{e}_2 + 5\tilde{e}_3$ with respect to the basis e_1, e_2, e_3 are $\underline{1, -2, -3}$

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8 Alternative solution to part II

The vector $5\tilde{e}_1 - \tilde{e}_2 + 5\tilde{e}_3$ has the coordinates $5, -1, 5$ with respect to the basis $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$. The relationships of the coordinates are then

$$\begin{cases} 2(-1)+5 = x_1 - x_2 \\ -1 = 2x_1 + x_3 \\ 5 - 2\cdot 5 = x_2 + x_3 \end{cases} \Leftrightarrow \begin{cases} x_1 - x_2 = 3 \\ 2x_1 + x_3 = -1 \\ x_2 + x_3 = -5 \end{cases} \Leftrightarrow \begin{cases} x_1 - x_2 = 3 \\ 2x_2 + x_3 = -7 \\ x_2 + x_3 = -5 \end{cases}$$

$$\Leftrightarrow \begin{cases} x_1 - x_2 = 3 \\ -x_3 = 3 \\ x_2 + x_3 = -5 \end{cases} \Leftrightarrow \begin{cases} x_1 = 1 \\ x_2 = -2 \\ x_3 = -3 \end{cases}$$

i.e. the coordinates of the vector $5\tilde{e} - \tilde{e}_2 + 5\tilde{e}_3$ with respect to the basis e_1, e_2, e_3 are $\underline{1, -2, -3}$

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EXAMINATION IN MATHEMATICS
MAA153 Linear algebra & MMA129 Linear algebra
EVALUATION PRINCIPLES with POINT RANGES
Academic year: 2015/16

	Examination 2016-06-08	Maximum points for subparts of the problems in the final examination
1.	The linear operator is diagonalizable iff $\beta \neq 0,3$. A basis of eigenvectors is then e.g. $(1,0,-1)$, $(-1,3,1)$, $(\beta^2-3\beta-1,\beta,1)$.	**3p**: Correctly found that the linear operator is diagonalizable iff $\beta \neq 0,3$ **2p**: Correctly for $\beta \neq 0,3$ found a basis of eigenvectors
2.	A basis for the span is e.g. $(1,3,-2,1,5)$, $(-1,1,0,1,1)$, $(1,2,-1,1,3)$. The vector $(1,2,3,a,-1)$ belong to the span iff $a=5$. The coordinates of the vector in the chosen (ordered) basis are $-6,2,9$	**2p**: Correctly found a basis for $\mathcal{U}$ **1p**: Correctly found that the fifth vector belong to $\mathcal{U}$ iff $a=5$ **2p**: Correctly found the coordinates of the fifth vector relative to the chosen basis
3.	Since the determinants of the matrices A and B are equal, it may be the case that the matrices represents the same linear operator F, but it is not for sure. A final answer can only be found by solving $\mathsf{SA}=\mathsf{BS}$ for S.	**1p**: Correctly formulated the necessary condition for the two matrices to represents the same linear operator but in different bases, i.e. stated that A must be equal to $\mathsf{S}^{-1}\mathsf{BS}$ where S is the change-of-basis matrix from a basis where F has the matrix B to a basis where F has the matrix A **2p**: Correctly found that $\mathsf{A}=\mathsf{S}^{-1}\mathsf{BS}$ imply that the determinants of A and B must be equal **2p**: Correctly concluded that the determinants are equal, and therefore that A and B may (but not for sure) be the matrices of the linear operator F but relative different bases
4.	An ON-basis for the orthogonal complement of $\operatorname{span}\{p_0,p_1\}$ is e.g. $\frac{5}{2\sqrt{2}}(p_0-3p_2)$	**1p**: Correctly formulated the conditions for the polynomial functions to belong to the ortogonal complement of $\operatorname{span}\{p_0,p_1\}$ **3p**: Correctly evaluated the conditions for the polynomial functions to belong to the ortogonal complement of $\operatorname{span}\{p_0,p_1\}$ **1p**: Correctly normalized p_0-3p_2 to become an ON-basis for the orthogonal complement of $\operatorname{span}\{p_0,p_1\}$
5.	An ON-basis for the kernel of F is e.g. $\frac{1}{\sqrt{6}}(-2,1,1,0)$, $\frac{1}{\sqrt{6}}(0,-1,1,2)$.	**2p**: Correctly found a basis for the kernel of F **3p**: Correctly found an ON-basis for the kernel of F
6.	S_1 is an elliptic cylinder S_2 is a two-sheeted hyperboloid	**3p**: Correctly classified the first surface **2p**: Correctly classified the second surface

7.	The matrix of D in the basis of h_0,h_1,h_2 is $\begin{pmatrix}1&1&0\\0&1&2\\0&0&1\end{pmatrix}$. D^{-1} exists since the matrix is invertible. $D^{-1}(h)$ means the anti-derivative of h which has the value 0 at the point 0	**1p**: Correctly found how D maps the three functions, i.e. that $D(h_0)=h_0$, $D(h_1)=h_0+h_1$, $D(h_2)=2h_1+h_2$ **2p**: Correctly found the matrix of D in the basis h_0,h_1,h_2 **1p**: Correctly found that D is invertible exists since the matrix of D is invertible **1p**: Correctly explained that the meaning of $D^{-1}(h)$ for $h\in\mathcal{H}$ is the antiderivative of h without any added constant, the latter since D^{-1} is linear, i.e. $D^{-1}(0)=0$
8.	Proof. The coordinates of the vector $5\tilde{\mathbf{e}}_1-\tilde{\mathbf{e}}_2+5\tilde{\mathbf{e}}_3$ in the basis $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$ are $1,-2,-3$	**2p**: Correctly explained why the relationships define a change-of-basis matrix S between two ordered bases $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$ and $\tilde{\mathbf{e}}_1,\tilde{\mathbf{e}}_2,\tilde{\mathbf{e}}_3$ ($\tilde{\mathsf{e}}=\mathsf{eS}$), where $\mathsf{S}=\mathsf{B}^{-1}\mathsf{A}$ and $\mathsf{A}\tilde{\mathsf{X}}=\mathsf{BX}$ are the relationships on a matrix form with $\tilde{\mathsf{X}}$ and X as the coordinate column matrices **1p**: Correctly found that the coordinates of $5\tilde{\mathbf{e}}_1-\tilde{\mathbf{e}}_2+5\tilde{\mathbf{e}}_3$ with respect to the basis $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$ are given by the coordinate matrix $\mathsf{B}^{-1}\mathsf{A}\tilde{\mathsf{X}}$, where $\tilde{\mathsf{X}}$ is equal to the coordinate matrix $(5\ \ -1\ \ 5)^T$ **1p**: Correctly found the matrix B^{-1} **1p**: Correctly found the coordinates of $5\tilde{\mathbf{e}}_1-\tilde{\mathbf{e}}_2+5\tilde{\mathbf{e}}_3$ with respect to the basis $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$