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EXAMINATION IN MATHEMATICS
MMA129 Linear Algebra
Date: 2015-01-16 *Write time:* 5 hours
Aid: Writing materials

This examination consists of eight RANDOMLY ORDERED problems each of which is worth at maximum 5 points. The maximum sum of points is thus 40. The PASS-marks 3, 4 and 5 require a minimum of 18, 26 and 34 points respectively. Solutions are supposed to include rigorous justifications and clear answers. All sheets with solutions must be sorted in the order the problems are given in. Especially, avoid to write on back pages of solution sheets.

1. The linear span of the vectors $(1,1,2,-1),(2,3,1,1),(2,2,4,-2),(9,14,3,6)$ is a subspace M of $\mathbb{R}^4$. Find a basis for M, and find all real numbers a for which M contains the vector $(3a-5, 4a-9, a^2+5a+2, a-5)$.

2. The linear transformation $F:\mathbb{R}^4 \to \mathbb{R}^3$ is defined by

$$F(x) = (4x_1+x_2+7x_3+7x_4,\, 3x_1+x_2+5x_3+7x_4,\, 2x_1+x_2+3x_3+2x_4),$$

where $x=(x_1,x_2,x_3,x_4)$. Find a basis for each of the image of F and the kernel of F, and state the dimensions of the two spaces.

3. Let $L = \operatorname{span}\{(1,0,-1,1),(1,1,0,1),(1,-2,-3,1),(3,1,-2,3)\}$ be equipped with the standard scalar product, i.e. $L \subset \mathbb{E}^4$. Find the orthogonal projection of the vector $w=(2,1,2,1)$ on the orthogonal complement $L^\perp$ of L (where $L^\perp = \{u \in \mathbb{E}^4 : \langle u|v\rangle = 0 \text{ for all } v \in L\}$).

4. The linear operator $F:\mathbb{R}^3 \to \mathbb{R}^3$ has the matrix

$$\begin{pmatrix} 2 & 1 & 2 \\ a & 2 & a-3 \\ 1 & -1 & 1 \end{pmatrix}$$

relative to the basis $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$ and where $a \in \mathbb{R}$. Find those a for which the operator är diagonalizable, and state for each of these a a basis of eigenvectors.

5. In the two-dimensional linear space L, the vectors $\mathbf{f}_1$ and $\mathbf{f}_2$ have the coordinates $(1,2)$ and $(1,3)$ respectively relative to the basis $\mathbf{e}_1,\mathbf{e}_2$. Define a scalar product in L, such that $\mathbf{f}_1,\mathbf{f}_2$ is an orthonormal basis for L, and specify the scalar product expanded in the basis $\mathbf{e}_1,\mathbf{e}_2$. Finally, determine the length of $\mathbf{e}_1+4\mathbf{e}_2$.

6. A linear operator F reflects every vector $\mathbf{u} \in \mathbb{E}^3$ about the linear space

$$M = \{(x_1,x_2,x_3) \in \mathbb{E}^3 : x_1+2x_2-x_3=0,\ 2x_1+x_2+x_3=0\}.$$

Find the matrix of F in the standard basis.

7. It can be proven relatively easy that the functions f_1,f_2,f_3, where $f_n(x)=e^{(2-n)x}$, are linearly independent in any interval $[a,b]$ with $a<b$. Let now the three functions g_1,g_2,g_3 be defined by

$$g_1(x) = 3e^x+2+2e^{-x}\,,\quad g_2(x)=e^x+e^{-x}\,,\quad g_3(x)=e^x+3\,.$$

Prove that g_1,g_2,g_3 is a basis for the linear space spanned by f_1,f_2,f_3, and find in that basis the coordinates of the function sinh (where $\sinh(x)=\frac{1}{2}(e^x-e^{-x})$).

8. Let (x,y,z) denote the coordinates of a point in an orthonormal system. Prove that the equation $3+2xy=2yz+2zx$ describes a one-sheeted rotational hyperboloid, and find relative the given coordinate system an equation for the points on the rotational axis. Also, determine the shortest distance a particle on the surface must travel one lap around the axis of rotation, returning to the starting point.

Om du föredrar uppgifterna formulerade på svenska, var god vänd på bladet.

Denna tentamen består av åtta om varannat SLUMPMÄSSIGT ORDNADE uppgifter som vardera kan ge maximalt 5 poäng. Den maximalt möjliga poängsumman är således 40. För betygen 3, 4 och 5 krävs minst 18, 26 respektive 34 poäng. Lösningar förutsätts innefatta ordentliga motiveringar och tydliga svar. Samtliga lösningsblad skall vid inlämning vara sorterade i den ordning som uppgifterna är givna i. Undvik speciellt att skriva på baksidor av lösningsblad.

1. Det linjära höljet av vektorerna $(1,1,2,-1),(2,3,1,1),(2,2,4,-2),(9,14,3,6)$ är ett underrum M till $\mathbb{R}^4$. Bestäm en bas för M, och bestäm alla reella tal a för vilka M innehåller vektorn $(3a-5,4a-9,a^2+5a+2,a-5)$.

2. Den linjära avbildningen $F:\mathbb{R}^4\to\mathbb{R}^3$ är definierad enligt

$$F(x)=(4x_1+x_2+7x_3+7x_4,3x_1+x_2+5x_3+7x_4,2x_1+x_2+3x_3+2x_4),$$

där $x=(x_1,x_2,x_3,x_4)$. Bestäm en bas för vardera av F:s värderum och F:s nollrum, och ange dimensionerna av de två rummen.

3. Låt L vara det linjära höljet $[(1,0,-1,1),(1,1,0,1),(1,-2,-3,1),(3,1,-2,3)]$ utrustat med standardskalärprodukten, dvs $L\subset\mathbb{E}^4$. Bestäm den ortogonala projektionen av vektorn $w=(2,1,2,1)$ på det ortogonala komplementet $L^\perp$ till L (där $L^\perp=\{u\in\mathbb{E}^4:\langle u|v\rangle=0 \text{ för alla } v\in L\}$).

4. Den linjära operatorn $F:\mathbb{R}^3\to\mathbb{R}^3$ har i basen $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$ matrisen

$$\begin{pmatrix}2&1&2\\a&2&a-3\\1&-1&1\end{pmatrix}$$

där $a\in\mathbb{R}$. Bestäm de a för vilka operatorn är diagonaliserbar, och ange för var och en av dessa a en bas av egenvektorer till F.

5. I det tvådimensionella, linjära rummet L har vektorerna $\mathbf{f}_1$ och $\mathbf{f}_2$ koordinaterna $(1,2)$ respektive $(1,3)$ i basen $\mathbf{e}_1,\mathbf{e}_2$. Definiera en skalärprodukt i L, sådan att $\mathbf{f}_1,\mathbf{f}_2$ utgör en ON-bas för L, och specificera skalärprodukten utvecklad i basen $\mathbf{e}_1,\mathbf{e}_2$. Bestäm sedan längden av vektorn $\mathbf{e}_1+4\mathbf{e}_2$.

6. En linjär operator F speglar varje vektor $\mathbf{u}\in\mathbb{E}^3$ i det linjära rummet

$$M=\{(x_1,x_2,x_3)\in\mathbb{E}^3: x_1+2x_2-x_3=0,\ 2x_1+x_2+x_3=0\}.$$

Bestäm avbildningsmatrisen för F i standardbasen.

7. Det kan visas relativt enkelt att funktionerna f_1,f_2,f_3, där $f_n(x)=e^{(2-n)x}$, är linjärt oberoende på vilket intervall $[a,b]$ som helst med $a<b$. Låt nu de tre funktionerna g_1,g_2,g_3 vara definierade genom

$$g_1(x)=3e^x+2+2e^{-x}\,,\quad g_2(x)=e^x+e^{-x}\,,\quad g_3(x)=e^x+3\,.$$

Visa att g_1,g_2,g_3 är en bas för det linjära rum som f_1,f_2,f_3 spänner upp, och bestäm i denna bas koordinaterna för funktionen sinh (där $\sinh(x)=\frac{1}{2}(e^x-e^{-x})$).

8. Låt (x,y,z) beteckna en punkts koordinater i ett ON-system. Visa att ekvationen $3+2xy=2yz+2zx$ beskriver en enmantlad rotationshyperboloid, och bestäm i det givna koordinatsystemet en ekvation för punkterna på rotationsaxeln. Bestäm även den kortaste sträcka en partikel måste tillryggalägga för att på ytan ta sig ett varv runt rotationsaxeln och komma tillbaka till utgångspunkten.

If you prefer the problems formulated in English, please turn the page.

MMA129 / Solutions to the final exam 2015-01-16

(1) $\begin{cases} M = \text{span}\{\overset{u_1}{(1,1,2,-1)}, \overset{u_2}{(2,3,1,1)}, \overset{u_3}{(2,2,4,-2)}, \overset{u_4}{(9,14,3,6)}\} \subset R^4 \\ (3a-5,\ 4a-9,\ a^2+5a+2,\ a-5) = v \end{cases}$

Study the vector equation $c_1u_1 + c_2u_2 + c_3u_3 + c_4u_4 = v$.

The augmented matrix for the system of linear eq:s is

$$\left(\begin{array}{cccc|c} 1 & 2 & 2 & 9 & 3a-5 \\ 1 & 3 & 2 & 14 & 4a-9 \\ 2 & 1 & 4 & 3 & a^2+5a+2 \\ -1 & 1 & -2 & 6 & a-5 \end{array}\right) \overset{(-1)\,(-2)\,(1)}{\sim} \left(\begin{array}{cccc|c} 1 & 2 & 2 & 9 & 3a-5 \\ 0 & 1 & 0 & 5 & a-4 \\ 0 & -3 & 0 & -15 & a^2-a+12 \\ 0 & 3 & 0 & 15 & 4a-10 \end{array}\right) \overset{(-2)\,(3)\,(-3)}{\sim} \left(\begin{array}{cccc|c} 1 & 0 & 2 & -1 & a+3 \\ 0 & 1 & 0 & 5 & a-4 \\ 0 & 0 & 0 & 0 & a(a+2) \\ 0 & 0 & 0 & 0 & a+2 \end{array}\right)$$

where the reduced row-echelon form of the matrix implies that u_1, u_2 are linearly independent and a basis for M, that $\begin{cases} u_3 = 2u_1 \\ u_4 = -u_1 + 5u_2 \end{cases}$ and that $v \in M$ iff $a = -2$ (v is then equal to $u_1 - 6u_2$)

(2) $F: R^4 \to R^3$ is in the standard bases defined by

$$F(x) = (4x_1 + x_2 + 7x_3 + 7x_4,\ 3x_1 + x_2 + 5x_3 + 7x_4,\ 2x_1 + x_2 + 3x_3 + 2x_4)$$

$$= (f_1\ f_2\ f_3)\begin{pmatrix} 4 & 1 & 7 & 7 \\ 3 & 1 & 5 & 7 \\ 2 & 1 & 3 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \underline{f}\, A\, \underline{X}$$

where f_1, f_2, f_3 is the standard basis for R^3

$$A \sim \begin{pmatrix} 0 & -1 & 1 & 3 \\ 1 & 0 & 2 & 5 \\ 2 & 1 & 3 & 2 \end{pmatrix} \sim \begin{pmatrix} 0 & -1 & 1 & 3 \\ 1 & 0 & 2 & 5 \\ 0 & 1 & -1 & -8 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & 5 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

i.e.
- a basis for $\text{im}(F)$ is e.g. $(4,3,2), (1,1,1), (7,7,2)$
- (or simpler the standard basis since $\dim(\text{im}(F)) = \dim(R^3)$)
- a basis for $\ker(F)$ is e.g. $(-2,1,1,0)$
- (since $A\underline{X}_{ker} = 0 \Leftrightarrow x_{ker} = (-2t, t, t, 0) = t(-2,1,1,0)$)
- $\dim(\text{im}(F)) = 3$
- $\dim(\ker(F)) = 1$

(confirming that $\dim(\text{im}(F)) + \dim(\ker(F)) = \dim(\text{pre-image space}) \overset{\text{here}}{=} 4$)

③ $\begin{cases} L = \text{span}\{\overset{u_1}{(1,0,-1,1)}, \overset{u_2}{(1,1,0,1)}, \overset{u_3}{(1,-2,-3,1)}, \overset{u_4}{(3,1,-2,3)}\} \subset E^4 \\ w = (2,1,2,1) \end{cases}$

The coordinate matrix of the vectors u_1, u_2, u_3, u_4 is

$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & -2 & 1 \\ -1 & 0 & -3 & -2 \\ 1 & 1 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where the reduced row-echelon form of the matrix implies that e.g. $(1,0,-1,1), (1,1,0,1)$ is a basis for L.

In order to find the orthogonal projection of w on the orthogonal complement $L^\perp$ of L, we may either (1) project orthogonally on L followed by a subtraction, i.e. $w_{L^\perp} = w - w_L$, or (2) find a basis for $L^\perp$ and project ortogonally on $L^\perp$.

To simplify the determination of the projection, we start (scenario (1)) by finding an ON-basis e_1, e_2 for L.

Let
$$\begin{cases} e_1 = \dfrac{1}{\|u_1\|} u_1 = \dfrac{1}{\sqrt{1+0+1+1}} u_1 = \dfrac{1}{\sqrt{3}}(1,0,-1,1) \\ f_2 = u_2 - \langle u_2 | e_1 \rangle e_1 \\ \quad = (1,1,0,1) - \langle (1,1,0,1) | \frac{1}{\sqrt{3}}(1,0,-1,1) \rangle \frac{1}{\sqrt{3}}(1,0,-1,1) \\ \quad = (1,1,0,1) - \frac{1}{3}(1+0+0+1)(1,0,-1,1) \\ \quad = \frac{1}{3}[(3,3,0,3) - (2,0,-2,2)] = \frac{1}{3}(1,3,2,1) \\ e_2 = \dfrac{1}{\|f_2\|} f_2 = \dfrac{1}{\sqrt{1+9+4+1}}(1,3,2,1) = \dfrac{1}{\sqrt{15}}(1,3,2,1) \end{cases}$$

We get

$$\begin{aligned} w_L &= \langle w | e_1 \rangle e_1 + \langle w | e_2 \rangle e_2 \\ &= \langle (2,1,2,1) | \tfrac{1}{\sqrt{3}}(1,0,-1,1) \rangle \tfrac{1}{\sqrt{3}}(1,0,-1,1) + \langle (2,1,2,1) | \tfrac{1}{\sqrt{15}}(1,3,2,1) \rangle \tfrac{1}{\sqrt{15}}(1,3,2,1) \\ &= \tfrac{1}{3}(2+0-2+1)(1,0,-1,1) + \tfrac{1}{15}(2+3+4+1)(1,3,2,1) \\ &= \tfrac{1}{3}(1,0,-1,1) + \tfrac{2}{3}(1,3,2,1) = \tfrac{1}{3}(3,6,3,3) = (1,2,1,1) \end{aligned}$$

and finally

$$w_{L^\perp} = w - w_L = (2,1,2,1) - (1,2,1,1) = \underline{(1,-1,1,0)}$$

④ $F: \mathbb{R}^3 \to \mathbb{R}^3$ has the matrix $A = \begin{pmatrix} 2 & 1 & 2 \\ a & 2 & a-3 \\ 1 & -1 & 1 \end{pmatrix}$ relative to a basis e_1, e_2, e_3 and where $a \in \mathbb{R}$.

To diagonalize F we need to find a basis of eigenvectors. The corresponding eigenvalues are given by $0 = \det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & 1 & 2 \\ a & 2-\lambda & a-3 \\ 1 & -1 & 1-\lambda \end{pmatrix}$

$$= \det \begin{pmatrix} 2-\lambda & 1 & 2 \\ a & 2-\lambda & a-3 \\ 3-\lambda & 0 & 3-\lambda \end{pmatrix} = (3-\lambda) \det \begin{pmatrix} 2-\lambda & 1 & 2 \\ a & 2-\lambda & a-3 \\ 1 & 0 & 1 \end{pmatrix} = (3-\lambda) \det \begin{pmatrix} -\lambda & 1 & 2 \\ 3 & 2-\lambda & a-3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= -(\lambda-3) \det \begin{pmatrix} -\lambda & 1 \\ 3 & 2-\lambda \end{pmatrix} = -(\lambda-3)(\lambda^2 - 2\lambda - 3) = -(\lambda+1)(\lambda-3)^2$$

$\lambda_1 = -1$: $A - \lambda_1 I = \begin{pmatrix} 3 & 1 & 2 \\ a & 3 & a-3 \\ 1 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 2 \\ 0 & 4 & -4 \\ 0 & a+3 & -a-3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & a+3 & -a-3 \end{pmatrix}$

$$\sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{i.e. } X_1 = t_1 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad t_1 \neq 0$$

$\lambda_{2,3} = 3$: $A - \lambda_{2,3} I = \begin{pmatrix} -1 & 1 & 2 \\ a & -1 & a-3 \\ 1 & -1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -2 \\ 0 & a-1 & 3a-3 \\ 0 & 0 & 0 \end{pmatrix} \overset{\text{if } a \neq 1}{\sim} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$

If $a = 1$ then $X_{2,3} = t_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ t_2, t_3 not both eq. to 0

If $a \neq 1$ then $X_{2,3} = t_{2,3} \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$, $t_{2,3} \neq 0$

i.e. F is diagonalizable iff $a = 1$. A basis of eigenvectors is e.g. $(-1,1,1), (1,1,0), (1,-1,1)$.

⑤ e_1, e_2 is a basis for the linear space L. L is equipped with a scalar product $\langle \, | \, \rangle$ such that f_1, f_2 is an ON-basis for L. Let $u, v \in L$. Then

$$\langle u | v \rangle = g_{11} x_1 y_1 + g_{12} x_1 y_2 + g_{21} x_2 y_1 + g_{22} x_2 y_2$$

(making no distinction between scalars and 1x1-matrices) $= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = X^T G Y$

when expanded in the basis e_1, e_2. $\left(\text{where } \begin{cases} u = x_1 e_1 + x_2 e_2 \\ v = y_1 e_1 + y_2 e_2 \end{cases} \right)$

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Cont. of (5) We know that $\begin{cases} f_1 = e_1 + 2e_2 \\ f_2 = e_1 + 3e_2 \end{cases}$ i.e. that $(f_1\ f_2) = (e_1\ e_2)\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$

where $\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} = S$ is the change-of-basis matrix from e_1, e_2 to f_1, f_2.

Demanding that f_1, f_2 is an ON-basis gives

$$\delta_{ij} = \langle f_i | f_j \rangle = \left\langle \sum_{m=1}^{2} e_m s_{mi} \,\middle|\, \sum_{n=1}^{2} e_n s_{nj} \right\rangle$$

$\langle\,|\,\rangle$ bilinear

$$= \sum_{m,n=1}^{2} s_{mi} s_{nj} \underbrace{\langle e_m | e_n \rangle}_{g_{mn}} = \sum_{m,n=1}^{2} (S^T)_{im} (G)_{mn} (S)_{nj}$$

$$= (S^T G S)_{ij} \quad \text{i.e.} \quad I = S^T G S$$

since $\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$ i.e. δ_{ij} are the elements of the identity matrix.

Solving for G gives

$$G = (S^T)^{-1} I S^{-1} = (S^T)^{-1} S^{-1} = (S S^T)^{-1}$$

$$= \left[\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\right]^{-1} = \begin{pmatrix} 2 & 5 \\ 5 & 13 \end{pmatrix}^{-1} = \frac{1}{26-25}\begin{pmatrix} 13 & -5 \\ -5 & 2 \end{pmatrix} = \begin{pmatrix} 13 & -5 \\ -5 & 2 \end{pmatrix}$$

i.e. $\langle u | v \rangle = 13 x_1 y_1 - 5(x_1 y_2 + x_2 y_1) + 2 x_2 y_2$ when expanded in the basis e_1, e_2.

The length of $e_1 + 4e_2$ is equal to

$$\|e_1 + 4e_2\| = \sqrt{\langle e_1 + 4e_2 | e_1 + 4e_2 \rangle} = \sqrt{(1\ 4)\begin{pmatrix} 13 & -5 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix}} \text{ l.u.}$$

$$= \sqrt{13 \cdot 1^2 - 5(1 \cdot 4 + 4 \cdot 1) + 2 \cdot 4^2} \text{ l.u.}$$

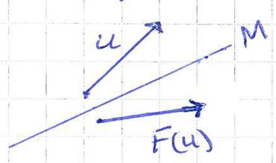
$$= \sqrt{13 - 40 + 32} \text{ l.u.}$$

$$= \sqrt{5} \text{ l.u.}$$

length units

6 $M = \{ (x_1, x_2, x_3) \in R^3 : \; x_1 + 2x_2 - x_3 = 0, \; 2x_1 + x_2 + x_3 = 0 \}$

F is a linear operator which reflects every vector $u \in E^3$ about M.



We have that $M : \underbrace{\begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

where $A \sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & -3 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$

i.e. the vector $(-1, 1, 1)$ span the subspace M of R^3.

Also, we note that the elements of the rows of A are the coordinates of vectors in the ortogonal complement $M^\perp$ of M, i.e. $M^\perp = \operatorname{span}\{(1,2,-1), (2,1,1)\}$.

From the specification of F, we have that

$$\begin{cases} F((-1,1,1)) = (-1,1,1) & \text{the reflection of } u \in M \text{ is } u \text{ itself} \\ F((1,2,-1)) = -(1,2,-1) & \text{— // — of } u \in M^\perp \text{ is } -u \\ F((2,1,1)) = -(2,1,1) & \text{—— // ——} \end{cases}$$

Let A be the matrix of F relative to the standard basis.

Then $A \begin{pmatrix} -1 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 & -2 \\ 1 & -2 & -1 \\ 1 & 1 & -1 \end{pmatrix}$

$$A = \begin{pmatrix} -1 & -1 & -2 \\ 1 & -2 & -1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & -1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & -1 & -2 \\ 1 & -2 & -1 \\ 1 & 1 & -1 \end{pmatrix} \frac{1}{(-2+1-2)-(4+1+1)} \begin{pmatrix} 3 & -3 & -3 \\ 0 & -3 & 3 \\ -3 & 0 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -1 & -2 \\ 1 & -2 & -1 \\ 1 & 1 & -1 \end{pmatrix} \frac{1}{3} \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -1 & -2 & -2 \\ -2 & -1 & 2 \\ -2 & 2 & -1 \end{pmatrix}$$

7 $f_n(x) = e^{(2-n)x}, \; n = 1, 2, 3$ and $\begin{cases} g_1(x) = 3e^x + 2 + 2e^{-x} \\ g_2(x) = e^x + e^{-x} \\ g_3(x) = e^x + 3 \end{cases}$

We have that

$$(g_1 \; g_2 \; g_3) = (3f_1 + 2f_2 + 2f_3 \quad f_1 + f_3 \quad f_1 + 3f_2) = (f_1 \; f_2 \; f_3) \underbrace{\begin{pmatrix} 3 & 1 & 1 \\ 2 & 0 & 3 \\ 2 & 1 & 0 \end{pmatrix}}_{S}$$

where $S \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 3 \\ 2 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$

i.e. S has full rank and therefore g_1, g_2, g_3 is linear independent since f_1, f_2, f_3 is linear independent.

i.e. g_1, g_2, g_3 is a basis for $\operatorname{span}\{f_1, f_2, f_3\}$ q.e.d.

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$$\sin h = \tfrac{1}{2}f_1 - \tfrac{1}{2}f_3 = (f_1\, f_2\, f_3)\begin{pmatrix}1/2\\0\\-1/2\end{pmatrix} = (g_1\, g_2\, g_3)\, S^{-1}\begin{pmatrix}1/2\\0\\-1/2\end{pmatrix}$$

where $S^{-1} = \dfrac{1}{(0+6+2)-(0+9+0)}\begin{pmatrix}-3 & 1 & 3\\ 6 & -2 & -7\\ 2 & -1 & -2\end{pmatrix} = \begin{pmatrix}3 & -1 & -3\\ -6 & 2 & 7\\ -2 & 1 & 2\end{pmatrix}$

i.e. $\sin h = (g_1\, g_2\, g_3)\begin{pmatrix}3\\-13/2\\-2\end{pmatrix}$ i.e. $\underset{g_1g_2g_3}{\text{coord}}(\sin h) = \left(3, -\frac{13}{2}, -2\right)$

8 Surface $3+2xy = 2yz+2zx$ (ON-system)

The equation involves a quadratic form h such that

$$h(u) = -2xy+2yz+2zx = (x\ y\ z)^T\begin{pmatrix}0 & -1 & 1\\ -1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = X^T G X$$

where G is the matrix of a symmetric linear transformation Γ (since G is symmetric in an ON-basis).

Refering to the spectral theorem, we know that Γ is diagonalizable in an ON-basis of eigenvectors. By choosing such a basis, the change-of-basis matrix S from the given ON-basis to that of eigenvectors is orthogonal, i.e. $S^{-1} = S^T$ (giving that also $h(u)$ becomes diagonal)

Eigenvalues $0 = \det(G-\lambda I) = \det\begin{pmatrix}-\lambda & -1 & 1\\ -1 & -\lambda & 1\\ 1 & 1 & -\lambda\end{pmatrix} \overset{(-1)\ (1)}{=} \det\begin{pmatrix}-\lambda & -1 & 1\\ \lambda-1 & 1-\lambda & 0\\ 1-\lambda & 0 & 1-\lambda\end{pmatrix}$

$$= (\lambda-1)(1-\lambda)\det\begin{pmatrix}-\lambda & -1 & 1\\ 1 & -1 & 0\\ 1 & 0 & 1\end{pmatrix} = -(\lambda-1)^2(\lambda+2)$$

$\lambda_1 = -2$: $A-\lambda_1 I = \begin{pmatrix}2 & -1 & 1\\ -1 & 2 & 1\\ 1 & 1 & 2\end{pmatrix} \sim \begin{pmatrix}1 & 1 & 2\\ 0 & 3 & 3\\ 0 & -3 & -3\end{pmatrix} \sim \begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 0 & 0 & 0\end{pmatrix}$ i.e. $X_1 = t_1\begin{pmatrix}1\\1\\-1\end{pmatrix}$, $t_1 \neq 0$

$\lambda_{2,3} = 1$: $A-\lambda_{2,3} I = \begin{pmatrix}-1 & -1 & 1\\ -1 & -1 & 1\\ 1 & 1 & -1\end{pmatrix} \sim \begin{pmatrix}1 & 1 & -1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$ i.e. $X_{2,3} = t_2\begin{pmatrix}1\\-1\\0\end{pmatrix} + t_3\begin{pmatrix}1\\1\\2\end{pmatrix}$

t_2, t_3 not both equal to 0

We get with $(\tilde e_1\, \tilde e_2\, \tilde e_3) = (e_1\, e_2\, e_3)\begin{pmatrix}\frac{1}{\sqrt3} & \frac{1}{\sqrt2} & \frac{1}{\sqrt6}\\ \frac{1}{\sqrt3} & -\frac{1}{\sqrt2} & \frac{1}{\sqrt6}\\ -\frac{1}{\sqrt3} & 0 & \frac{2}{\sqrt6}\end{pmatrix}$ (i.e. $\tilde e = eS$)

that (using $S^TS = S^{-1}S = I$)

$$3 = X^TGX = (S\tilde X)^T(S\tilde G S^{-1})(S\tilde X) = \tilde X^T S^T S \tilde G \tilde X$$
$$= \tilde X^T I \tilde G \tilde X = \tilde X^T \tilde G \tilde X = -2\tilde x_1^2 + \tilde x_2^2 + \tilde x_3^2$$

where the signature of $h(u)$ clearly is $(1,1,-1)$ q.e.d

* Implying that $3 = h(u)$ describes a one-sheeted hyperboloid.
* It is 'rotational' since $\tilde x_2^2 + \tilde x_3^2$ is invariant rotating in the $\tilde x_2\tilde x_3$-plane.
* The points on the axis of rotation are given by $(x, y, z) = t(1,1,-1) \parallel \tilde e_1$
* The shortest path around the axis of rot. has the length $2\pi\sqrt3$ l.u. (since it is the circle $3 = \tilde x_2^2 + \tilde x_3^2$, $\tilde x_1 = 0$)

MÄLARDALEN UNIVERSITY
School of Education, Culture and Communication
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EXAMINATION IN MATHEMATICS
MMA129 Linear algebra
EVALUATION PRINCIPLES with POINT RANGES
Academic year: 2014/15

Examination 2015-01-16 — Maximum points for subparts of the problems in the final examination

1. A basis for M is e.g.

$(1,1,2,-1)$, $(2,3,1,1)$

M contains the vector $(3a-5, 4a-9, a^2+3a+2, a-5)$ iff $a=-2$.

1p: Correctly initiated an analysis of the relation between the five vectors in R^4, and correctly found the reduced echelon form of the augmented coordinate matrix of the vectors in the span for M together with the fifth vector
2p: Correctly from the reduced echelon form identified a basis for the subspace M of R^4
2p: Correctly from the reduced echelon form identified the value of a for which M contains the fifth vector

---

2. A basis for the image of F is e.g. $(4,3,2)$, $(1,1,1)$, $(7,7,2)$, or simpler the standard basis since $\dim(\mathrm{im}(F)) = \dim(R^3) = 3$.

A basis for the kernel of F is e.g. $(-2,1,1,0)$, and $\dim(\ker(F)) = 1$

1p: Correctly identified the matrix A of F, and correctly found the reduced row-echelon form of A
1p: Correctly stated the dimension of the image of F
1p: Correctly found a basis for the image of F
1p: Correctly stated the dimension of the kernel of F
1p: Correctly found a basis for the kernel of F

---

3. $(1,-1,1,0)$

-------------------- **One scenario** -----------------------

2p: Correctly found a two-dimensional basis for $L^\perp$
1p: Correctly orthonormalized the basis for $L^\perp$
2p: Correctly determined the orthogonal projection $\mathbf{w}_L$ of the vector $\mathbf{w}$ on $L^\perp$

---------------------------- **Another scenario** --------------------------------------

1p: Correctly found a two-dimensional basis for L
1p: Correctly orthonormalized the basis for L
1p: Correctly determined the orthogonal projection $\mathbf{w}_L$ of the vector $\mathbf{w}$ on L
2p: Correctly determined the orthogonal projection $\mathbf{w}_{L^\perp}$ of the vector $\mathbf{w}$ on the orthogonal complement of L, all by subtracting $\mathbf{w}_L$ from $\mathbf{w}$

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4. The linear operator is diagonalizable iff $a=1$.

A basis of eigenvectors is e.g.

$(-1,1,1)$, $(1,1,0)$, $(1,-1,1)$

1p: Correctly found the two eigenvalues of the operator
1p: Correctly found the set of eigenvectors corresponding to the (single) eigenvalue -1
2p: Correctly found that the linear operator is diagonalizable iff $a=1$
1p: Correctly for $a=1$ found the two linear independent set of eigenvectors corresp. to the (double) eigenvalue 3

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5. $\langle \mathbf{u} | \mathbf{v} \rangle = \mathsf{X}^T \mathsf{A} \mathsf{Y}$ where

$$\mathsf{A} = \begin{pmatrix} 13 & -5 \\ -5 & 2 \end{pmatrix} \text{ in the basis } \mathbf{e}_1, \mathbf{e}_2$$

$\|\mathbf{e}_1 + 4\mathbf{e}_2\| = \sqrt{5}$

------------------------------- **One scenario** ---

1p: Correctly noted that the scalar product of the vectors $\mathbf{u}$ and $\mathbf{v}$ in a given basis i equal to the matrix product $\mathsf{X}^T\mathsf{A}\mathsf{Y}$, where X and Y are the coordinate matrices for $\mathbf{u}$ and $\mathbf{v}$ respectively, where A is representing the symmetric, bilinear and positive definite function which the scalar product is equal to
2p: Correctly noted that the matrix of the scalar product relative to the ON-basis $\mathbf{f}_1, \mathbf{f}_2$ is equal to the identity matrix, and that the matrix relative to the basis $\mathbf{e}_1, \mathbf{e}_2$ is

equal to the matrix $(\mathsf{S}\mathsf{S}^T)^{-1}$ where S is the change-of-basis matrix from $\mathbf{e}_1, \mathbf{e}_2$ to $\mathbf{f}_1, \mathbf{f}_2$

1p: Correctly found the matrix $(\mathsf{S}\mathsf{S}^T)^{-1}$

1p: Correctly found the length of the vector $\mathbf{e}_1 + 4\mathbf{e}_2$

---------------------------- **Another scenario** --------------------------------------

2p: Introduced a_{mn} as the elements of a matrix A representing the scalar product in the basis $\mathbf{e}_1, \mathbf{e}_2$, and correctly developed the orthogonality and norm conditions for $\mathbf{f}_1, \mathbf{f}_2$, i.e.

$$1 = \langle \mathbf{f}_1 | \mathbf{f}_1 \rangle = 1^2 a_{11} + 1\cdot 2a_{12} + 2\cdot 1a_{21} + 2^2 a_{22}$$
$$0 = \langle \mathbf{f}_1 | \mathbf{f}_2 \rangle = 1^2 a_{11} + 1\cdot 3a_{12} + 2\cdot 1a_{21} + 2\cdot 3a_{22}$$
$$1 = \langle \mathbf{f}_2 | \mathbf{f}_2 \rangle = 1^2 a_{11} + 1\cdot 3a_{12} + 3\cdot 1a_{21} + 3^2 a_{22}$$

2p: Correctly solved the system of equations, where the number of unknowns are $4-1=3$ since the matrix A is symmetric (implying that $a_{21} = a_{12}$)

1p: Correctly found the length of the vector $\mathbf{e}_1 + 4\mathbf{e}_2$

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6. $\frac{1}{3}\begin{pmatrix} -1 & -2 & -2 \\ -2 & -1 & 2 \\ -2 & 2 & -1 \end{pmatrix}$

1p: Correctly from the condition for M identified a vector which spans M and two linearly independent vectors which span the orthogonal complement $M^\perp$ of M

1p: Correctly noted that F maps vectors in M on themselves, i.e. $F(u) = u$ for each $u \in M$

1p: Correctly noted that F maps vectors in $M^\perp$ on 'minus themselves', i.e. $F(v) = -v$ for each $v \in M^\perp$

1p: Correctly on the form $\mathsf{C}\mathsf{B}^{-1}$, and in the standard basis, found the matrix A of the linear transformation F

1p: Correctly found the explicit expression for the matrix A

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7. Since the matrix S in the change-of-basis-relation $(g_1\ g_2\ g_3) = (f_1\ f_2\ f_3)\mathsf{S}$ is invertible, the functions g_1, g_2, g_3 constitute a basis for $\text{span}\{f_1, f_2, f_3\}$

The function sinh has the coordinates $(3, -\frac{13}{2}, -2)$ relative to the basis g_1, g_2, g_3

1p: Correctly found the matrix (or corresponding) which describes the relation between the functions g_1, g_2, g_3 and f_1, f_2, f_3

1p: Correctly proved that the matrix in question is invertible (or equivalent has full rank or equivalent has a determinant different from zero), <u>and</u> from this correctly drawn the conclusion that also the functions g_1, g_2, g_3 constitute a basis for $\text{span}\{f_1, f_2, f_3\}$

1p: Correctly expressed the sinh as the matrix product $(g_1\ \ g_2\ \ g_3)\,\mathsf{S}^{-1}(\frac{1}{2}\ 0\ -\frac{1}{2})^T$

1p: Correctly found the inverse of the matrix S

1p: Correctly relative the basis g_1, g_2, g_3 found the coordinates of the function sinh

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8. <u>Part 1</u>: A proof

<u>Part 2</u>: Points at the axis of rotation satisfies the equation $(x, y, z) = t(1,1,-1)$, $t \in R$

The shortest distance one lap around the axis of rotation is equal to $2\pi\sqrt{3}$ l.u.

2p: Correctly found that the quadratic form $-2xy + 2yz + 2zx$ has the signature $(1,1,-1)$, meaning that the equation $3 = h(u)$ describes a one-sheeted rotational hyperboloid

2p: Correctly found an equation for the axis-of-rotation

1p: Correctly found the shortest distance for one lap around the axis of rotation, not leaving the surface