8 - Solution methods for constrained optimization problems

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Consider the constrained optimization problem defined by

$$\begin{cases} \min \ f(x) \\ g_i(x) \leq 0 & \forall\ i = 1, \dots, m \\ h_j(x) = 0 & \forall\ j = 1, \dots, p \end{cases} \tag{P}$$

Let $X = \{x \in \mathbb{R}^n : g(x) \leq 0,\ h(x) = 0\}$ be the feasible set of (P).

The methods for solving (P) are in general divided in the following classes:

- Primal methods that operate direct on the given problem (P) (e.g., methods of changing the variables, descent direction methods, as projected gradient method or Franke Wolfe method)
- Dual methods, that use the dual of (P), or related properties, (e.g., gradient methods for solving the dual problem, penalty methods)

Problems with linear equality constraints

As an example of a method of changing variables, we consider a problem with linear equality constraints only.

We observe that a constrained problem with linear equality constraints

$$\begin{cases} \min \ f(x) \\ Ax = b \end{cases}$$

where A is $p \times n$ matrix with $\text{rank}(A) = p$, is equivalent to an unconstrained problem:

indeed, write $A = (A_B, A_N)$ with $det(A_B) \neq 0$, where A_B is a $(p \times p)$ matrix. Setting $x = \begin{pmatrix} x_B \\ x_N \end{pmatrix}$, then $Ax = b$ is equivalent to

$$A_B x_B + A_N x_N = b \quad \Longrightarrow \quad x_B = A_B^{-1}(b - A_N x_N),$$

thus, eliminating the variables x_B,

$$\begin{cases} \min \ f(x) \\ Ax = b \end{cases} \quad \text{is equivalent to} \quad \begin{cases} \min \ f(A_B^{-1}(b - A_N x_N), x_N) \\ x_N \in \mathbb{R}^{n-p} \end{cases}$$

Note that, if f is convex then the previous unconstrained problem is an unconstrained convex problem.

Example. Consider

$$\begin{cases} \min\ x_1^2 + x_2^2 + x_3^2 \\ x_1 + x_3 = 1 \\ x_1 + x_2 - x_3 = 2 \end{cases}$$

Since $x_1 = 1 - x_3$ and $x_2 = 2 - x_1 + x_3 = 1 + 2x_3$, the original constrained problem is equivalent to the following unconstrained problem:

$$\begin{cases} \min\ (1 - x_3)^2 + (1 + 2x_3)^2 + x_3^2 = 6x_3^2 + 2x_3 + 2 \\ x_3 \in \mathbb{R} \end{cases}$$

Therefore, the optimal solution is $x_3 = -1/6$, $x_1 = 7/6$, $x_2 = 2/3$.

Consider a constrained optimization problem

$$\begin{cases} \min \ f(x) \\ g_i(x) \leq 0 \qquad \forall \ i = 1, \dots, m \end{cases} \tag{P}$$

Let $X = \{x \in \mathbb{R}^n : g(x) \leq 0\}$ the feasible set of (P).

Define the *quadratic* penalty function

$$p(x) = \sum_{i=1}^{m} (\max\{0, g_i(x)\})^2$$

and consider the unconstrained penalized problem

$$\begin{cases} \min \ f(x) + \frac{1}{\varepsilon} p(x) := p_\varepsilon(x) \\ x \in \mathbb{R}^n \end{cases} \tag{P_ε}$$

Note that

$$p_\varepsilon(x) \begin{cases} = f(x) & \text{if } x \in X \\ > f(x) & \text{if } x \notin X \end{cases}$$

Proposition 8.1

1. If f, g_i are continuously differentiable, then p_ε is continuously differentiable and $\nabla p_\varepsilon(x) = \nabla f(x) + \dfrac{2}{\varepsilon}\sum_{i=1}^{m} \max\{0, g_i(x)\}\nabla g_i(x)$
2. If f and g_i are convex, then p_ε is convex
3. Any (P_ε) is a relaxation of (P), i.e., $v(P_\varepsilon) \le v(P)$ for any $\varepsilon > 0$
4. If x^*_ε solves (P_ε) and $x^*_\varepsilon \in X$, then x^*_ε is optimal also for (P)
5. If $0 < \varepsilon_2 < \varepsilon_1$, then $v(P_{\varepsilon_1}) \le v(P_{\varepsilon_2})$

Penalty method

0. Set $\varepsilon_0 > 0$, $\tau \in (0,1)$, $k = 0$
1. Find an optimal solution x^k of the penalized problem (P_{ε_k})
2. **If** $x^k \in X$ **then** STOP
   else $\varepsilon_{k+1} = \tau\varepsilon_k$, $k = k + 1$ and go to step 1.

Theorem 8.2

- If f is coercive, then the sequence $\{x^k\}$ is bounded and any of its cluster points is an optimal solution of (P).
- If $\{x^k\}$ converges to x^*, then x^* is an optimal solution of (P).
- If $\{x^k\}$ converges to x^* and the gradients of active constraints at x^* are linear independent, then x^* is an optimal solution of (P) and the sequence of vectors $\{\lambda^k\}$ defined as

$$\lambda_i^k := \frac{2}{\varepsilon_k} \max\{0, g_i(x^k)\}, \qquad i = 1, \dots, m$$

converges to a vector λ^* of KKT multipliers associated to x^*.

Remark

Notice that, by Proposition 8.1 (point 5), the sequence of the optimal values $v(P_{\varepsilon_k})$ generated by the penalty method, is nondecreasing.
In fact, if x_ε^* solves (P_ε) and $x_\varepsilon^* \notin X$, then $v(P_\varepsilon) \geq v(P_{\varepsilon'})$ for any $\varepsilon < \varepsilon'$.

Exercise 8.1

a) Implement in MATLAB the penalty method for solving the problem

$$\begin{cases} \min \ \frac{1}{2} x^{\mathsf{T}} Q x + c^{\mathsf{T}} x \\ Ax \leq b \end{cases}$$

where Q is a positive definite matrix.

b) Run the penalty method with $\tau = 0.1$ and $\varepsilon_0 = 5$ for solving the problem

$$\begin{cases} \min \ \frac{1}{2}(x_1 - 3)^2 + (x_2 - 2)^2 \\ -2x_1 + x_2 \leq 0 \\ x_1 + x_2 \leq 4 \\ -x_2 \leq 0 \end{cases}$$

[Use $\max(Ax - b) < 10^{-6}$ as stopping criterion.]

```
global Q c A b eps;

Q = [ 1 0 ; 0 2 ] ; c = [ -3 ; -4 ] ;   % data
A = [-2 1 ; 1 1 ; 0 -1 ]; b = [ 0 ; 4 ; 0 ];

tau = 0.1; eps0 = 5; tolerance = 1e-6 ;  % Penalty method
eps = eps0; x = [0;0]; iter = 0; SOL=[];

while true
[x,pval] = fminunc(@p_eps,x);
infeas = max(A*x-b);
SOL=[SOL;iter,eps,x',infeas,pval];
 if infeas < tolerance
  break
  else
  eps = tau*eps;
  iter = iter + 1 ;
 end
end

SOL
```

```
% The penalized function

function v= p_eps(x)

global Q c A b eps;

v = 0.5*x'*Q*x + c'*x ;

for i = 1 : size(A,1)
 v = v + (1/eps)*(max(0,A(i,:)*x-b(i)))^ 2 ;
end

end
```

The Matlab function 'fminunc' (from the Matlab help)

fminunc finds a local minimum of a function of several variables.
[X,FVAL] = fminunc(FUN,X0) starts at X0 and attempts to find a local minimizer X of the function FUN. FUN accepts input X and returns a scalar function value F evaluated at X. X0 can be a scalar, vector or matrix, FVAL is the optimal value of the function FUN.

FUN can be specified using @:

```
[X,FVAL] = fminunc(@myfun,X0)
```

where myfun is a MATLAB function defined as:

```
function F = myfun(x)
F = .........;
```

Consider a convex constrained problem

$$\begin{cases} \min \ f(x) \\ g_i(x) \leq 0 \qquad \forall \ i = 1, \ldots, m \end{cases} \tag{P}$$

and define the *linear* penalty function

$$\widetilde{p}(x) = \sum_{i=1}^{m} \max\{0, g_i(x)\}.$$

Consider the penalized problem

$$\begin{cases} \min \ \tilde{p}_\varepsilon(x) := f(x) + \dfrac{1}{\varepsilon}\,\widetilde{p}(x) \\ x \in \mathbb{R}^n \end{cases} \tag{$\widetilde{P}_\varepsilon$}$$

which is unconstrained, convex and nonsmooth.

Note that

$$\tilde{p}_\varepsilon(x) \begin{cases} = f(x) & \text{if } x \in X \\ > f(x) & \text{if } x \notin X \end{cases}$$

For such penalized problem we do not need a sequence $\varepsilon_k \to 0$ to approximate an optimal solution of (P) (which avoid numerical issues), in fact there exists a suitable ε such that the minimum of $(\widetilde{P}_\varepsilon)$ coincides with the minimum of (P).

Proposition 8.3

Suppose that there exists an optimal solution x^* of (P) and λ^* is a KKT multipliers vector associated to x^*. Then, the sets of optimal solutions of (P) and $(\widetilde{P}_\varepsilon)$ coincide provided that $\varepsilon \in (0, 1/\|\lambda^*\|_\infty)$.

Exact penalty method

0. Set $\varepsilon_0 > 0$, $\tau \in (0,1)$, $k = 0$
1. Find an optimal solution x^k of the penalized problem $(\widetilde{P}_{\varepsilon_k})$
2. **If** $x^k \in X$ **then** STOP
   else $\varepsilon_{k+1} = \tau\varepsilon_k$, $k = k+1$ and go to step 1.

Theorem 8.4

The exact penalty method stops after a finite number of iterations at an optimal solution of (P).

Notice that penalty methods generate a sequence of unfeasible points that approximate an optimal solution of (P).

Exercise 8.2

Run the exact penalty method with $\tau = 0.5$ and $\varepsilon_0 = 4$ for solving the problem

$$\begin{cases} \min \ \frac{1}{2}(x_1 - 3)^2 + (x_2 - 2)^2 \\ -2x_1 + x_2 \leq 0 \\ x_1 + x_2 \leq 4 \\ -x_2 \leq 0 \end{cases}$$

[Use $\max(Ax - b) < 10^{-6}$ as stopping criterion.]

Unlike penalty methods, barrier methods generate a sequence of feasible points that approximate an optimal solution of (P).

Consider

$$\begin{cases} \min \ f(x) \\ g_i(x) \leq 0 \quad i = 1, .., m \end{cases} \tag{P}$$

under the following assumptions:

- f, g_i convex and twice continuously differentiable (on $int(X) \neq \emptyset$)
- there exists an optimal solution (e.g. f is coercive or X is bounded)
- Slater constraint qualification holds: there exists $\bar{x}$ such that

$$g_i(\bar{x}) < 0, \ \forall \ i = 1, \dots, m$$

Hence strong duality holds.

Special cases: linear programming, convex quadratic programming

Logarithmic barrier

On the interior $\text{int}(X)$ of the feasible set X, we can approximate the given problem (P) with

$$\begin{cases} \min\ \psi_\varepsilon(x) := f(x) - \varepsilon \sum_{i=1}^{m} log(-g_i(x)) \\ x \in \text{int}(X) \end{cases}$$

We define

$$B(x) := -\sum_{i=1}^{m} log(-g_i(x))$$

$B(x)$ is called logarithmic barrier function.

Then

$$\psi_\varepsilon(x) := f(x) + \varepsilon B(x)$$

Note that, as x tends to the boundary of X, then $\psi_\varepsilon(x) \to +\infty$.

As $\varepsilon \to 0$, ψ_ε tends to f.

Logarithmic barrier function

The function $B(x)$ has the following properties:

- $\text{dom}(B) = \text{int}(X)$
- B is convex
- B is smooth with

$$\nabla B(x) = -\sum_{i=1}^{m} \frac{1}{g_i(x)} \nabla g_i(x)$$

$$\nabla^2 B(x) = \sum_{i=1}^{m} \frac{1}{g_i(x)^2} \nabla g_i(x) \nabla g_i(x)^{\mathsf{T}} + \sum_{i=1}^{m} \frac{1}{-g_i(x)} \nabla^2 g_i(x)$$

If x^*_ε is an optimal solution of

$$\begin{cases} \min\ f(x) - \varepsilon \sum_{i=1}^{m} log(-g_i(x)) \\ x \in \text{int}(X) \end{cases}$$

then

$$\nabla f(x^*_\varepsilon) + \sum_{i=1}^{m} \frac{\varepsilon}{-g_i(x^*_\varepsilon)} \nabla g_i(x^*_\varepsilon) = 0.$$

Define $\lambda^*_\varepsilon = \left(\frac{\varepsilon}{-g_1(x^*_\varepsilon)}, \dots, \frac{\varepsilon}{-g_m(x^*_\varepsilon)} \right) > 0.$

Consider the Lagrangian function L associated with the given problem (P), $L(x,\lambda) := f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)$. Then

$$L(x, \lambda^*_\varepsilon) = f(x) + \sum_{i=1}^{m} (\lambda^*_\varepsilon)_i g_i(x)$$

is convex and $\nabla_x L(x^*_\varepsilon, \lambda^*_\varepsilon) = 0$.

Recall that (P) is a convex problem and strong duality holds, hence

$$v(P) = \max_{\lambda \in \mathbb{R}^m_+} \inf_{x \in \mathbb{R}^n} L(x, \lambda)$$

Consequently,

$$v(P) \geq \min_x L(x, \lambda^*_\varepsilon) = L(x^*_\varepsilon, \lambda^*_\varepsilon).$$

Finally

$$f(x^*_\varepsilon) \geq v(P) \geq L(x^*_\varepsilon, \lambda^*_\varepsilon) = f(x^*_\varepsilon) + \sum_{i=1}^{m} (\lambda^*_\varepsilon)_i g_i(x^*_\varepsilon) = f(x^*_\varepsilon) - \underbrace{m\,\varepsilon}_{\text{optimality gap}}$$

Remark

Note that:

$$\text{As } \ \varepsilon \to 0, \qquad f(x^*_\varepsilon) \to v(P).$$

The KKT system of the original problem is

$$\begin{cases} \nabla f(x) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x) = 0 \\ -\lambda_i g_i(x) = 0 \quad i = 1,..,m \\ \lambda \geq 0 \\ g(x) \leq 0 \end{cases}$$

Notice that $(x^*_\varepsilon, \lambda^*_\varepsilon)$ solves the system

$$\begin{cases} \nabla f(x) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x) = 0 \\ -\lambda_i g_i(x) = \varepsilon, \quad i = 1,..,m \\ \lambda \geq 0 \\ g(x) \leq 0 \end{cases}$$

which is an approximation of the above KKT system.

Logarithmic barrier method

0. Set tolerance $\delta > 0$, $\tau \in (0,1)$ and $\varepsilon_1 > 0$. Choose $x^0 \in \text{int}(X)$, set $k = 1$

1. Find the optimal solution x^k of

$$\begin{cases} \min \ f(x) - \varepsilon_k \sum_{i=1}^{m} log(-g_i(x)) \\ x \in \text{int}(X) \end{cases}$$

using x^{k-1} as starting point

2. **If** $m\,\varepsilon_k < \delta$ **then** STOP
else $\varepsilon_{k+1} = \tau\varepsilon_k$, $k = k+1$ and go to step 1

Choice of starting point

In order to find an initial point $x^0 \in \text{int}(X)$ we can consider the auxiliary problem

$$\begin{cases} \min\limits_{x,s} \; s \\ g_i(x) \leq s, \quad i = 1, .., m \end{cases}$$

- Take any $\tilde{x} \in \mathbb{R}^n$, find $\tilde{s} > \max\limits_{i=1,\dots,m} g_i(\tilde{x})$
  [$(\tilde{x}, \tilde{s})$ is in the interior of the feasible region of the auxiliary problem]
- Find an optimal solution (x^*, s^*) of the auxiliary problem using a barrier method starting from $(\tilde{x}, \tilde{s})$
- If $s^* < 0$ then $x^* \in \text{int}(X)$
  else $\text{int}(X) = \emptyset$

Exercise 8.3.

a) Implement in MATLAB the logarithmic barrier method for solving the problem

$$\begin{cases} \min \ \frac{1}{2} x^{\mathsf{T}} Q x + c^{\mathsf{T}} x \\ Ax \leq b \end{cases}$$

where Q is a positive definite matrix.

b) Run the logarithmic barrier method with $\delta = 10^{-3}$, $\tau = 0.5$, $\varepsilon_1 = 1$ and $x^0 = (1, 1)$ for solving the problem

$$\begin{cases} \min \ \frac{1}{2}(x_1 - 3)^2 + (x_2 - 2)^2 \\ -2x_1 + x_2 \leq 0 \\ x_1 + x_2 \leq 4 \\ -x_2 \leq 0 \end{cases}$$

Matlab commands

```
global Q c A b eps;

Q = [ 1 0 ; 0 2 ] ; c = [ -3 ; -4 ] ;   % data
A = [-2 1 ; 1 1 ; 0 -1 ]; b = [ 0 ; 4 ; 0 ];

tau = 0.5; eps1 = 5; delta = 1e-3 ; x0=[1,1];   % barrier method

eps = eps1 ; m = size(A,1) ;
SOL=[];

while true
 [x,pval] = fminunc(@logbar,x);
  gap = m*eps;
  SOL=[SOL;eps,x',gap,pval];
   if gap < delta
    break
   else
    eps = eps*tau;
   end
end

SOL
```

```
% The penalized function

function v= logbar(x)

global Q c A b eps;

v = 0.5*x'*Q*x + c'*x ;

for i = 1 : length(b)
 v = v - eps*log(b(i)-A(i,:)*x) ;
end

end
```