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Quantum Computing and Quantum Internet

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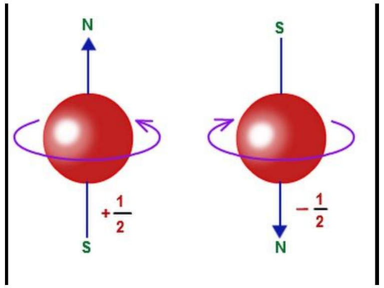
Composite Systems

Composite Systems

- Let's consider a composite quantum system made up of two distinct qubits, also referred to as a *bipartite* system



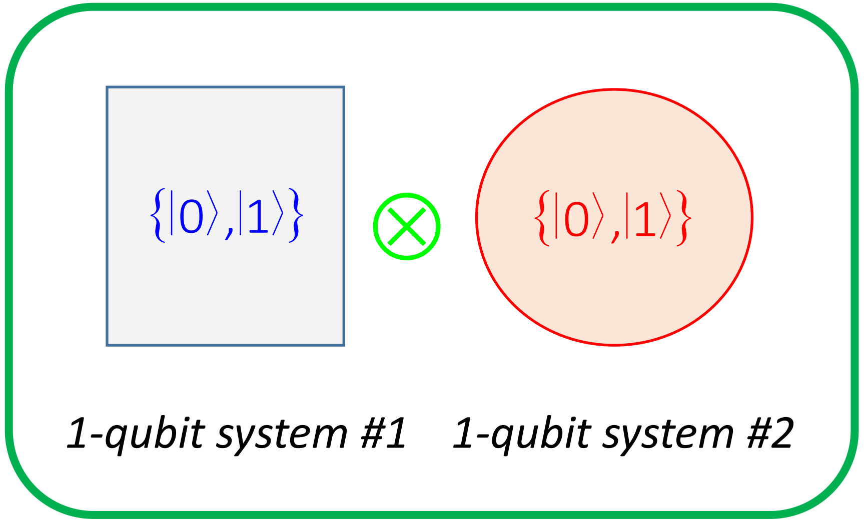
- How should we describe states of the composite system?

- *Postulate* 4 describes how the state space of a composite system is built up from the state spaces of the component systems

Composite Systems

Postulate 4:

- The state space of a *composite physical system* is the *tensor product* $(\otimes)$ of the state spaces of the *component physical systems*.



2-qubit system

Moreover, if we have systems numbered 1 through *n*, and system number ***i*** is prepared in the state $|\psi_i\rangle$, then the joint state of the total system is $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots\cdots \otimes |\psi_n\rangle$

$\rightarrow$

Thus, the tensor product is the mathematical structure used to describe the state space of a composite physical system

Mathematical Break

Tensor Products

- The *tensor product* is a way of putting vector spaces together to form larger vector spaces
- Suppose *V* and *W* are Hilbert spaces of dimension *m* and *n* respectively
- Then $V \otimes W$ (read "*V* tensor W") is an *mn* dimensional vector space
- Therefore, the **dimension** of the Hilbert space describing the states of the system composed of 2 qubits is $2 \times 2 = 2^2 = 4$
- If the number of qubits is *k*, the dimension is 2^k

Tensor Products

- The elements of $V \otimes W$ are linear combinations of *tensor products* $|v\rangle \otimes |w\rangle$ of elements $|v\rangle \in V$ and $|w\rangle \in W$

- In particular, if $|i\rangle$ and $|j\rangle$ are orthonormal bases for the spaces *V* and *W* then $|i\rangle \otimes |j\rangle$ is a basis for $|V\rangle \otimes |W\rangle$

- For example, if *V* is a two-dimensional vector space with basis vectors $|0\rangle$ and $|1\rangle$ then

$$\frac{1}{\sqrt{2}}\left(|0\rangle \otimes |0\rangle + |1\rangle \otimes |0\rangle\right)$$

is an element of $V \otimes V$

Tensor Products

- We often use the abbreviated notations $|v\rangle|w\rangle$ or even $|vw\rangle$ for the tensor product $|v\rangle\otimes|w\rangle$

- If $\{|0\rangle,|1\rangle\}$ is the **standard basis** for *V*, then the **standard basis** for $V\otimes V$ is

$$\left.\begin{array}{l}
\{|0\rangle\otimes|0\rangle,|0\rangle\otimes|1\rangle,|1\rangle\otimes|0\rangle,|1\rangle\otimes|1\rangle\} \\
\{|0\rangle|0\rangle,|0\rangle|1\rangle,|1\rangle|0\rangle,|1\rangle|1\rangle\} \\
\{|00\rangle,|01\rangle,|10\rangle,|11\rangle\}
\end{array}\right\} \textit{Equivalent notations}$$

Tensor Products

- In linear algebra, the *tensor product of two* matrices/vectors is simply the *Kronecker product*, which is obtained by multiplying each term of the *first* matrix/vector by the *entire second* matrix/vector

- Specifically, with two qubits $|\psi\rangle = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$ and $|\varphi\rangle = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$

$$|\psi\rangle \otimes |\varphi\rangle = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \otimes \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} a_0 \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} \\ a_1 \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} a_0 b_0 \\ a_0 b_1 \\ a_1 b_0 \\ a_1 b_1 \end{bmatrix} \xrightarrow{\textit{Example}} |0\rangle \otimes |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Tensor Products

- An additional example of tensor product between two matrices

$$X \otimes Y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 \times \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} & 1 \times \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\ 1 \times \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} & 0 \times \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}$$

Tensor Products Properties

By definition the tensor product satisfies the following basic properties:

1. For an arbitrary scalar *z (complex number)* and elements $|v\rangle$ of *V* and $|w\rangle$ of *W*

$$z\left(|v\rangle \otimes |w\rangle\right) = \left(z|v\rangle\right) \otimes |w\rangle = |v\rangle \otimes \left(z|w\rangle\right)$$

1. For arbitrary $|v_1\rangle$ and $|v_2\rangle$ in *V* and $|w\rangle$ in *W*,

$$\left(|v_1\rangle + |v_2\rangle\right) \otimes |w\rangle = |v_1\rangle \otimes |w\rangle + |v_2\rangle \otimes |w\rangle$$

2. For arbitrary $|v\rangle$ in *V* and $|w_1\rangle$ and $|w_2\rangle$ in *W*,

$$|v\rangle \otimes \left(|w_1\rangle + |w_2\rangle\right) = |v\rangle \otimes |w_1\rangle + |v\rangle \otimes |w_2\rangle$$

Tensor Products Properties (Example)

By exploiting the above properties, if:

$$\left(|v_1\rangle + |v_2\rangle\right) \otimes |w\rangle = |v_1\rangle \otimes |w\rangle + |v_2\rangle \otimes |w\rangle$$

$$|v\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \qquad |w\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

Then

$$|v\rangle \otimes |w\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) = \frac{1}{2}\left[|0\rangle \otimes \left(|0\rangle - |1\rangle\right) + |1\rangle \otimes \left(|0\rangle - |1\rangle\right)\right]$$

$$= \frac{1}{2}\left(|0\rangle \otimes |0\rangle - |0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle\right)$$

$$|v\rangle \otimes \left(|w_1\rangle + |w_2\rangle\right) = |v\rangle \otimes |w_1\rangle + |v\rangle \otimes |w_2\rangle$$

Orthonormal Basis

- If

$$\{|0\rangle, |1\rangle\} \equiv \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

  is the standard basis for *V*, then we are going to derive the vector form of the standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ for the composite system $V \otimes V$

$$\left|00\right\rangle = \left|0\right\rangle\left|0\right\rangle = \left|0\right\rangle \otimes \left|0\right\rangle = \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\cdot\begin{bmatrix}1\\0\end{bmatrix}\\ 0\cdot\begin{bmatrix}1\\0\end{bmatrix}\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

$$\left|01\right\rangle = \left|0\right\rangle\left|1\right\rangle = \left|0\right\rangle \otimes \left|1\right\rangle = \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\cdot\begin{bmatrix}0\\1\end{bmatrix}\\ 0\cdot\begin{bmatrix}0\\1\end{bmatrix}\end{bmatrix} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$$

$$\left|10\right\rangle = \left|1\right\rangle\left|0\right\rangle = \left|1\right\rangle \otimes \left|0\right\rangle = \begin{bmatrix}0\\1\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\cdot\begin{bmatrix}1\\0\end{bmatrix}\\ 1\cdot\begin{bmatrix}1\\0\end{bmatrix}\end{bmatrix} = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$$

$$\left|11\right\rangle = \left|1\right\rangle\left|1\right\rangle = \left|1\right\rangle \otimes \left|1\right\rangle = \begin{bmatrix}0\\1\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\cdot\begin{bmatrix}0\\1\end{bmatrix}\\ 1\cdot\begin{bmatrix}0\\1\end{bmatrix}\end{bmatrix} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

Orthonormal Basis for $V \otimes V$

- It's easy to show that $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ satisfies the completeness relation

$$|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11| = I$$

$$|00\rangle\langle 00| = \big(|0\rangle \otimes |0\rangle\big)\big(\langle 0| \otimes \langle 0|\big) = \left(\begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix}\right)\left(\begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix}\right)^{\dagger} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}$$

Orthonormal Basis for $V \otimes V$

$$|01\rangle\langle 01| = \left(|0\rangle \otimes |1\rangle\right)\left(\langle 0| \otimes \langle 1|\right) = \left(\begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix}\right)\left(\begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix}\right)^{\dagger} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}\begin{bmatrix}0 & 1 & 0 & 0\end{bmatrix} = \begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

$$|10\rangle\langle 10| = \left(|1\rangle \otimes |0\rangle\right)\left(\langle 1| \otimes \langle 0|\right) = \left(\begin{bmatrix}0\\1\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix}\right)\left(\begin{bmatrix}0\\1\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix}\right)^{\dagger} = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}\begin{bmatrix}0 & 0 & 1 & 0\end{bmatrix} = \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}$$

$$|11\rangle\langle 11| = \left(|1\rangle \otimes |1\rangle\right)\left(\langle 1| \otimes \langle 1|\right) = \left(\begin{bmatrix}0\\1\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix}\right)\left(\begin{bmatrix}0\\1\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix}\right)^{\dagger} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix}$$

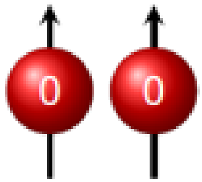
Orthonormal Basis for $V \otimes V$

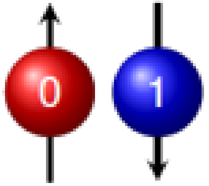
- By summing up the previous results we obtain

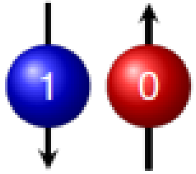
$$|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11| = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

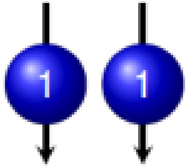
Superposition

- As any other *quantum system*, a system composed by a pair of qubits can exist in *superposition* of the four states belonging to the standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ **(Postulate 1)**

$|\psi\rangle =$ c_0 

$+ c_1$ 

$+ c_2$ 

$+ c_3$ 

$$|\psi\rangle = c_0|00\rangle + c_1|01\rangle + c_2|10\rangle + c_3|11\rangle \quad \text{where} \quad |c_0|^2 + |c_1|^2 + |c_2|^2 + |c_3|^2 = 1$$

$$\frac{1}{\sqrt{2}}\left(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle\right)$$ *Specific state with* $$\begin{cases} c_0 = c_3 = 0 \\ c_1 = c_2 = 1/\sqrt{2} \end{cases}$$

Notations

There are two ways of denoting the qubits

- **Binary Basis:** Sequence of 0 and 1, i.e. $\left|x_{n-1}x_{n-2}\ldots x_0\right\rangle$
- **Decimal Basis:** $\left|x\right\rangle$, with

$$x = x_{n-1}2^{n-1} + 2^{n-2}x_{n-2} + \cdots + x_0$$

Examples

$$\left|10\right\rangle = \left|1\times 2^1 + 0\times 2^0\right\rangle = \left|2\right\rangle$$

$$\left|101\right\rangle = \left|1\times 2^2 + 0\times 2^1 + 1\times 2^0\right\rangle = \left|5\right\rangle$$

Notations

- With three qubits, there are eight Z-basis states

$$\{|000\rangle,\ |001\rangle,\ |010\rangle,\ |011\rangle,\ |100\rangle,\ |101\rangle,\ |110\rangle,\ |111\rangle\}$$

- Sometimes, these binary strings are written as **decimal** numbers

$$\{|0\rangle,\ |1\rangle,\ |2\rangle,\ |3\rangle,\ |4\rangle,\ |5\rangle,\ |6\rangle,\ |7\rangle\}$$

Notations

- Inspired by this, let us call the right qubit the zeroth qubit, the middle qubit the first qubit, and the left qubit the second qubit, so a Z-basis state takes the form

$$|b_2 b_1 b_0\rangle$$

- Then, the decimal representation of this is

$$2^2 b_2 + 2^1 b_1 + 2^0 b_0$$

- In other words, we label qubits right-to-left, starting with zero
- This convention, where the rightmost qubit is the zeroth qubit, is called **little endian**

Notations

- For example, Qiskit (IBM) use **little endian**
- In contrast, the opposite convention, where the leftmost qubit is the zeroth qubit

$$\left|b_2 b_1 b_0\right\rangle \xrightarrow{\substack{decimal \\ representation}} 2^2 b_0 + 2^1 b_1 + 2^0 b_2$$

  is called **big endian**
- For example, Nielsen and Chuang’s standard advanced text book uses the **big endian** convention
- Disputes over which convention is “better” has raged classical computing for decades, and the same debates carry into quantum computing

Notations

- The reality is that you should be able to use both, therefore, we use both *little endian* and *big endian* throughout my lectures
- Next, the general state of three qubits is a superposition of these basis vectors:

$$\sum_{j=0}^{7} c_j \,|\, j\rangle = c_0 |0\rangle + c_1 |1\rangle + \cdots + c_7 |7\rangle$$

- And the probability of getting $|\, j\rangle$ when measuring in the *Z*-basis is $|c_j|^2$, so

$$\sum_{j=0}^{7} |c_j|^2 = 1$$

Notations

- With ***n*** qubits, there are ***N* =2*ⁿ*** (Z-)basis states, which we can label as *n*-bit strings or by the decimal numbers **0** through ***N* -1**
- As an *n*-bit string,

$$\left|b_{N-1}\dots b_1 b_0\right\rangle = \left|2^{N-1}b_{N-1} + 2^{N-2}b_{N-2} + \cdots + 2^1 b_1 + 2^0 b_0\right\rangle$$

- Of course, the general state of *n*-qubits is a superposition of these *Z*-basis states:

$$\sum_{j=0}^{N-1} c_j \left| j \right\rangle = c_0\left|0\right\rangle + c_1\left|1\right\rangle + \cdots + c_{N-1}\left|N-1\right\rangle$$

- This has *N* amplitudes c_0 through c_{N-1}

Notations

- We can also use powers to simplify the notation
- If we have *n* qubits, each in the state $|0\rangle$, we can write the state as

$$|0\rangle^{\otimes n} = \underbrace{|0\rangle \otimes |0\rangle \otimes \cdots \otimes |0\rangle}_{n} = \underbrace{|0\rangle|0\rangle\ldots|0\rangle}_{n} = \underbrace{|00\ldots0\rangle}_{n} = |0^n\rangle$$

- The notation admits the possibility of labeling each of the component kets with the vector space from whence it came, *V* or *W*
- For example:

$$|0_V\rangle \otimes |0_W\rangle = |0_V\rangle|0_W\rangle = |0_V 0_W\rangle$$

Bloch Sphere And General Multi-Qubit States

- We have already seen that with a single qubit, we could parameterize a state as

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

  with the coordinates (θ, ϕ) interpreted as a point on the Bloch sphere
- With two qubits, however, we have four complex amplitudes c_0, c_1, c_2, c_3 (although one can be made real by factoring out a global phase), and unfortunately, this is too many parameters to represent in three-dimensions
- There is **no Bloch sphere representation** for a **general multi-qubit state**

Tensor Product for Bra

- The tensor product also works for *bras*, so

$$\langle v| \otimes \langle w| = \langle v|\langle w| = \langle vw|$$

- Furthermore,

$$|vw\rangle^{\dagger} = \left(|v\rangle \otimes |w\rangle\right)^{\dagger} = |v\rangle^{\dagger} \otimes |w\rangle^{\dagger} = \langle v| \otimes \langle w| = \langle v|\langle w| = \langle vw|$$

$$\rightarrow \quad |vw\rangle^{\dagger} = \langle vw|$$

Linear Operators on the Space $V \otimes W$

- What sorts of linear operators act on the space $V \otimes W$?

- Suppose $|v\rangle$ and $|w\rangle$ are vectors in *V* and *W*, and *A* and *B* are linear operators on *V* and *W*, respectively, i.e.

$$A : V \to V, \qquad B : W \to W$$

- Then we can define a linear operator

$$A \otimes B : V \otimes W \to V \otimes W$$

by the equation

$$(A \otimes B)\big(|v\rangle \otimes |w\rangle\big) \equiv A|v\rangle \otimes B|w\rangle$$

Linear Operators on the Space $V \otimes W$

- The definition of $A \otimes B$ is then extended to all elements of $V \otimes W$ in the natural way to ensure **linearity** of $A \otimes B$ that is,

$$(A \otimes B)\left(\sum_i a_i |v_i\rangle \otimes |w_i\rangle\right) = \sum_i a_i A|v_i\rangle \otimes B|w_i\rangle$$

- It can be shown that $A \otimes B$ defined in this way is a well-defined linear operator on $V \otimes W$

- This notion of the tensor product of two operators extends in the obvious way to the case where $A : V \to V'$ and $B : W \to W'$ map between different vector spaces

Linear Operators on the Space $V \otimes W$

- Indeed, an arbitrary linear operator *C* mapping $V \otimes W$ to $V' \otimes W'$ can be represented as a linear combination of tensor products of operators mapping $V \to V'$ and $W \to W'$

$$C = \sum_i c_i A_i \otimes B_i$$

where by definition

$$\left(\sum_i c_i A_i \otimes B_i \right) \left(|v\rangle \otimes |w\rangle \right) = \sum_i c_i A_i |v\rangle \otimes B_i |w\rangle$$

Linear Operators on the Space $V \otimes W$

- The inner products on the spaces *V* and *W* can be used to define a natural inner product on $V \otimes W$
- If $|\psi\rangle = \sum_i a_i |v_i\rangle \otimes |w_i\rangle \in V \otimes W$ and $|\phi\rangle = \sum_i b_i |v_i'\rangle \otimes |w_i'\rangle \in V \otimes W$ then we define

$$\langle \psi | \phi \rangle = \sum_{i,j} a_i^* b_j \langle v_i | v_j' \rangle \langle w_i | w_j' \rangle$$

- It can be shown that the function so defined is a **well-defined inner product**
- From this inner product, the inner product space $V \otimes W$ inherits the other structure we are familiar with, such as notions of an **adjoint**, **unitarity**, **normality**, and **Hermiticity**

Tensor Product for Bra

$(A \otimes B)\cdot(C \otimes D) = (AC)\otimes(BD)$

- Then, the inner product of, say $\langle vw|$ and $|rs\rangle$ is defined by

$$\langle vw|rs\rangle = (\langle v| \otimes \langle w|)(|r\rangle \otimes |s\rangle) = \langle v|r\rangle\langle w|s\rangle,\ \forall |v\rangle, |r\rangle \in V \text{ and } \forall |w\rangle, |s\rangle \in W$$

- For qubit states $|00\rangle$ and $|01\rangle$

$$\langle 01|00\rangle = \underbrace{\langle 0|0\rangle}_{1} \cdot \underbrace{\langle 1|0\rangle}_{0} = 0 \quad \rightarrow \quad \text{So, } |00\rangle \text{ and } |01\rangle \text{ are orthogonal}$$

Notations

- We mentioned earlier the useful notation

$$|\psi\rangle^{\otimes k} = \underbrace{|\psi\rangle \otimes |\psi\rangle \otimes \cdots\cdots \otimes |\psi\rangle}_{k \text{ times}}$$

- For example

$$|\psi\rangle^{\otimes 2} = |\psi\rangle \otimes |\psi\rangle$$

- An analogous notation is also used for operators on tensor product spaces

$$H^{\otimes k} = \underbrace{H \otimes H \otimes \cdots\cdots \otimes H}_{k \text{ times}}$$

where H is the Hadamard operator

Useful Properties

$\left(A \otimes B\right)^{*} = A^{*} \otimes B^{*}$ *Property #1*

$\left(A \otimes B\right)^{T} = A^{T} \otimes B^{T}$ *Property #2*

$\left(A \otimes B\right)^{\dagger} = A^{\dagger} \otimes B^{\dagger}$ *Property #3*

$\left(A \otimes B\right) \cdot \left(C \otimes D\right) = \left(AC\right) \otimes \left(BD\right)$ *Property #4*

It follows the proofs under the assumption that *A, B, C* and *D* are $n \times n$ matrices

Useful Properties

$$\left(A \otimes B\right)^* = A^* \otimes B^*$$

Proof

definition of tensor product between two matrices

definition of conjugate matrix

$$\left(A \otimes B\right)^* = \left(\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix} \otimes B\right)^* = \left(\begin{bmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & \cdots & A_{2n}B \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1}B & A_{n2}B & \cdots & A_{nn}B \end{bmatrix}\right)^* = \begin{bmatrix} A_{11}^*B^* & A_{12}^*B^* & \cdots & A_{1n}^*B^* \\ A_{21}^*B^* & A_{22}^*B^* & \cdots & A_{2n}^*B^* \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1}^*B^* & A_{n2}^*B^* & \cdots & A_{nn}^*B^* \end{bmatrix}$$

$$= \begin{bmatrix} A_{11}^* & A_{12}^* & \cdots & A_{1n}^* \\ A_{21}^* & A_{22}^* & \cdots & A_{2n}^* \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1}^* & A_{n2}^* & \cdots & A_{nn}^* \end{bmatrix} \otimes B^* = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}^* \otimes B^* = A^* \otimes B^* \qquad \blacksquare$$

Useful Properties

$$\left(A\otimes B\right)^T = A^T \otimes B^T$$

definition of tensor product between two matrices

definition of transpose matrix

Proof

$$\left(A\otimes B\right)^T = \left(\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix} \otimes B\right)^T = \left(\begin{bmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & \cdots & A_{2n}B \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1}B & A_{n2}B & \cdots & A_{nn}B \end{bmatrix}\right)^T = \begin{bmatrix} A_{11}B^T & A_{21}B^T & \cdots & A_{n1}B^T \\ A_{12}B^T & A_{22}B^T & \cdots & A_{n2}B^T \\ \cdots & \cdots & \cdots & \cdots \\ A_{1n}B^T & A_{2n}B^T & \cdots & A_{nn}B^T \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix} \otimes B^T = A^T \otimes B^T$$

$\blacksquare$

Useful Properties

$$\left(A\otimes B\right)^{\dagger}=A^{\dagger}\otimes B^{\dagger}$$

Proof

Property #1 *Property #2*

$$\left(A\otimes B\right)^{\dagger}=\left(\left(A\otimes B\right)^{*}\right)^{T}=\left(A^{*}\otimes B^{*}\right)^{T}=\left(A^{*}\right)^{T}\otimes\left(B^{*}\right)^{T}=A^{\dagger}\otimes B^{\dagger} \qquad \blacksquare$$

definition of conjugate transpose (dagger)

definition of conjugate transpose (dagger)

Useful Properties

$$\left(A\otimes B\right)\cdot\left(C\otimes D\right)=\left(AC\right)\otimes\left(BD\right)$$

Proof

- We will use the notation $\left(A\otimes B\right)_{ij}=\left(a_{ij}B\right)$ to denote block matrices. Then

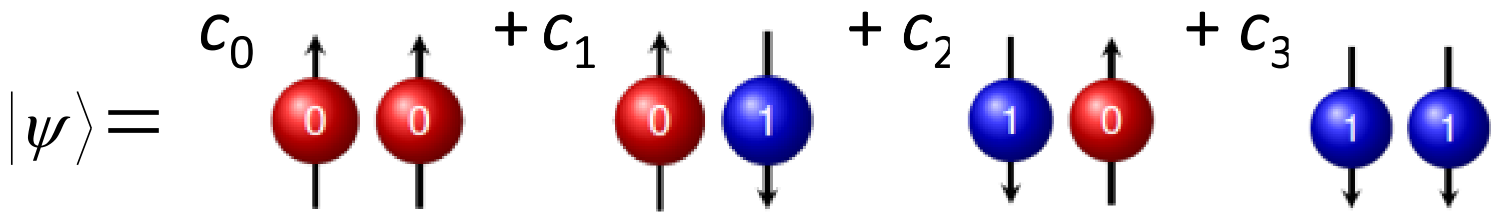
$$\left(\left(A\otimes B\right)\cdot\left(C\otimes D\right)\right)_{ij}=\sum_k\left(A\otimes B\right)_{ik}\left(C\otimes D\right)_{kj}=\sum_k\left(a_{ik}B\right)\left(c_{kj}D\right)$$

$$=\sum_k\left(a_{ik}c_{kj}\right)\left(BD\right)=\left(AC\right)_{ij}\left(BD\right)=\left(AC\otimes BD\right)_{ij} \qquad \blacksquare$$

Measurement

Superposition

- As any other *quantum system*, a system composed by a pair of qubits can exist in *superposition* of the four states belonging to the standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ **(Postulate 1)**



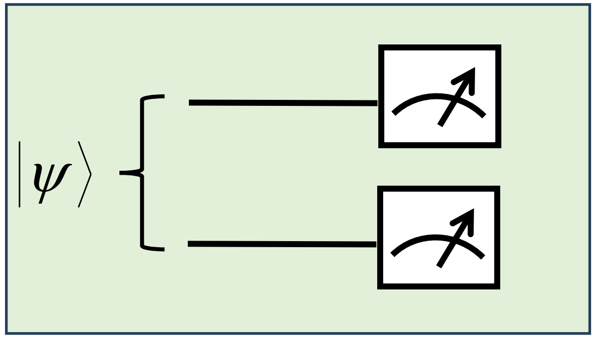
$$|\psi\rangle = c_0|00\rangle + c_1|01\rangle + c_2|10\rangle + c_3|11\rangle \quad \text{where} \quad |c_0|^2 + |c_1|^2 + |c_2|^2 + |c_3|^2 = 1$$

$$\frac{1}{\sqrt{2}}\left(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle\right) \quad \textit{Specific state with} \quad \begin{cases} c_0 = c_3 = 0 \\ c_1 = c_2 = 1/\sqrt{2} \end{cases}$$

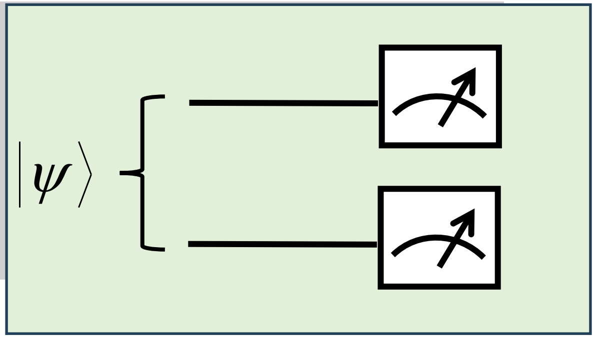
Multiple Qubits Measurement

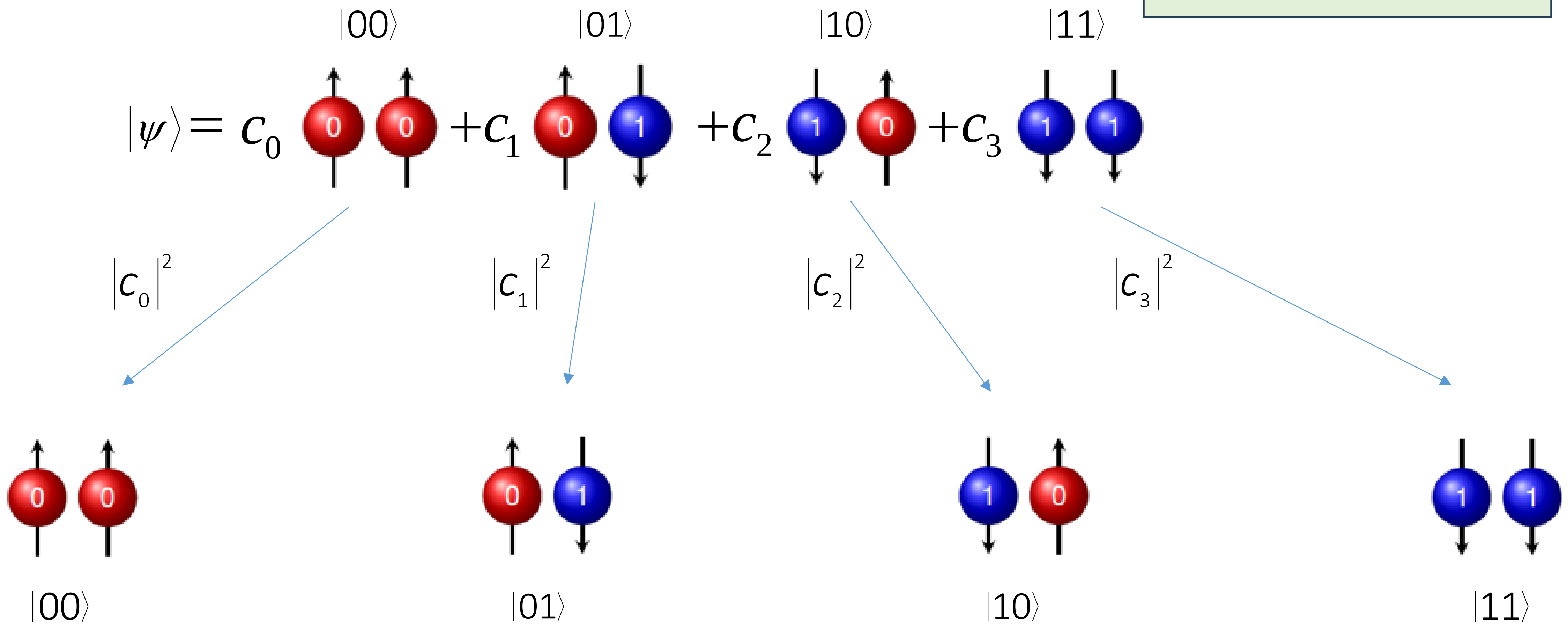
- If we measure these two qubits in the standard basis, we get $|00\rangle$ with probability $|c_0|^2$, $|01\rangle$ with probability $|c_1|^2$, $|10\rangle$ with probability $|c_2|^2$, or $|11\rangle$ with probability $|c_3|^2$
- Thus, the total probability is $|c_0|^2 + |c_1|^2 + |c_2|^2 + |c_3|^2$ and it should equal 1

$$|c_0|^2 + |c_1|^2 + |c_2|^2 + |c_3|^2 = 1$$



Multiple Qubits Measurement





$$|c_0|^2 + |c_1|^2 + |c_2|^2 + |c_3|^2 = 1$$

Multiple Qubits Measurement (Example)

- For example, if we have two qubits in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|01\rangle + \frac{\sqrt{3}}{4}|10\rangle + \frac{1}{4}|11\rangle$$

and we measured *both qubits*, we would get

$|00\rangle$ with probability 1/2

$|01\rangle$ with probability 1/4

$|10\rangle$ with probability 3/16

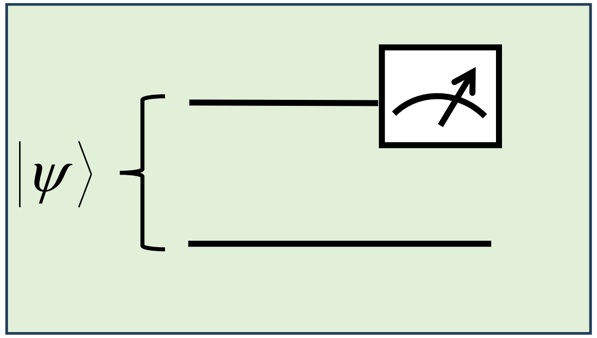
$|11\rangle$ with probability 1/16

$\rightarrow$ $|\psi\rangle$ is **projected** onto a basis vector associated with the measurement device

Measuring Individual Qubits

- For a two-qubit system, we could measure, for example, the first qubit

$$|\psi\rangle = c_0|00\rangle + c_1|01\rangle + c_2|10\rangle + c_3|11\rangle$$



- Measuring the **first** qubit alone gives **0** with probability

$$|c_0|^2 + |c_1|^2$$

leaving the post-measurement state

$$|\psi\rangle = \frac{c_0|00\rangle + c_1|01\rangle}{\sqrt{|c_0|^2 + |c_1|^2}}$$

$|\psi\rangle$ is **projected** onto a subspace of V generated by a linear combination of two vectors $(\{|00\rangle, |01\rangle\})$ belonging to the basis vectors associated with the measurement device

- Note how the post-measurement state is **re-normalized**

Measuring Individual Qubits

- Similarly, measuring the **first** qubit alone gives **1** with probability

$$|c_2|^2 + |c_3|^2$$

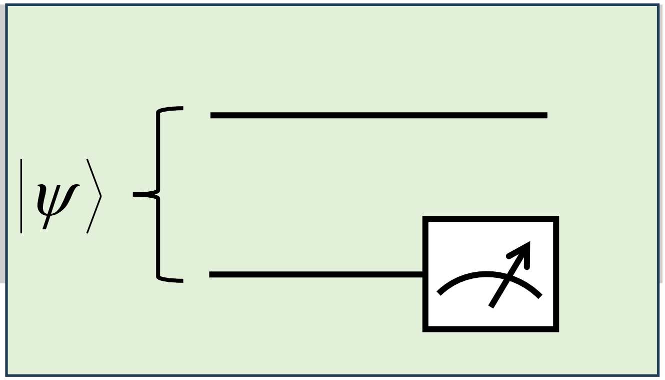
leaving the post-measurement state

$$|\psi\rangle = \frac{c_2|10\rangle + c_3|11\rangle}{\sqrt{|c_2|^2 + |c_3|^2}}$$

$|\psi\rangle$ is **projected** onto a subspace of V generated by a linear combination of two vectors $(\{|10\rangle, |11\rangle\})$ belonging to the basis vectors associated with the measurement device

- Again, note how the post-measurement state is **re-normalized**

Measuring Individual Qubits



- The same reasoning can be applied if we measure the second qubit

$$|\psi\rangle = c_0|00\rangle + c_1|01\rangle + c_2|10\rangle + c_3|11\rangle$$

- Measuring the **second** qubit alone gives **0** with probability

$$|c_0|^2 + |c_2|^2$$

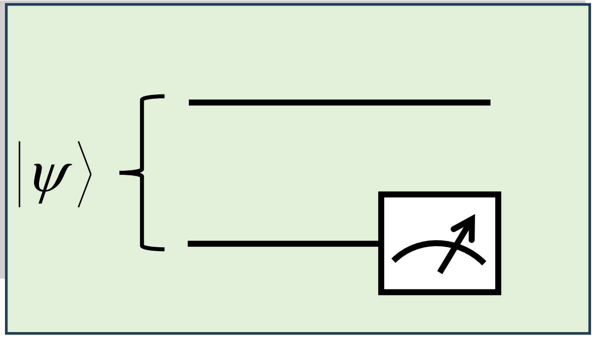
leaving the post-measurement state

$$|\psi_1\rangle = \frac{c_0|00\rangle + c_2|10\rangle}{\sqrt{|c_0|^2 + |c_2|^2}}$$

$|\psi\rangle$ is **projected** onto a subspace of V generated by a linear combination of two vectors $(\{|00\rangle, |10\rangle\})$ belonging to the basis vectors associated with the measurement device

- Note how the post-measurement state is **re-normalized**

Measuring Individual Qubits



- Similarly, measuring the **second** qubit alone gives **1** with probability

$$|c_1|^2 + |c_3|^2$$

leaving the post-measurement state

$$|\psi_2\rangle = \frac{c_1|01\rangle + c_3|11\rangle}{\sqrt{|c_1|^2 + |c_3|^2}}$$

$|\psi\rangle$ is **projected** onto a subspace of V generated by a linear combination of two vectors $(\{|01\rangle, |11\rangle\})$ belonging to the basis vectors associated with the measurement device

- Again, note how the post-measurement state is **re-normalized**

Measuring Individual Qubits

- **Example 1:** Let's focus on the state analyzed before by measuring its *left qubit*

$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|01\rangle + \frac{\sqrt{3}}{4}|10\rangle + \frac{1}{4}|11\rangle$$

Measuring Individual Qubits

- The probability of getting $|0\rangle$ when measuring the left qubit is given by the sum of the norm-squares of the amplitudes of $|00\rangle$ and $|01\rangle$, since those both have the left qubit as $|0\rangle$

$$|\psi\rangle = \boxed{\frac{1}{\sqrt{2}}}|00\rangle + \boxed{\frac{1}{2}}|01\rangle + \frac{\sqrt{3}}{4}|10\rangle + \frac{1}{4}|11\rangle \quad \rightarrow \quad \left|\frac{1}{\sqrt{2}}\right|^2 + \left|\frac{1}{2}\right|^2 = \frac{3}{4}$$

- Then the state collapses to the parts where the left qubit is $|0\rangle$, so it becomes

$$\left|\psi_{|0\rangle}\right\rangle = A\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|01\rangle\right)$$

Measuring Individual Qubits

- Let's calculate *A*

$$\left|A\frac{1}{\sqrt{2}}\right|^2+\left|A\frac{1}{2}\right|^2=1 \quad\rightarrow\quad A^2\left(\frac{1}{2}+\frac{1}{4}\right)=1 \quad\rightarrow\quad A^2\left(\frac{3}{4}\right)=1$$

- Thus, the normalization constant is

$$A=\frac{2}{\sqrt{3}}$$

- Therefore

$$\left|\psi_{|0\rangle}\right\rangle=\sqrt{\frac{2}{3}}\,|00\rangle+\sqrt{\frac{1}{3}}\,|01\rangle$$

Measuring Individual Qubits

- Similarly, if the outcome is $|1\rangle$, then from the $|10\rangle$ and $|11\rangle$ states, the probability is

$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|01\rangle + \boxed{\frac{\sqrt{3}}{4}}|10\rangle + \boxed{\frac{1}{4}}|11\rangle \qquad \rightarrow \qquad \left|\frac{\sqrt{3}}{4}\right|^2 + \left|\frac{1}{4}\right|^2 = \frac{1}{4}$$

- Similarly, if the outcome is $|1\rangle$ then the state collapses to the terms where the left qubit is $|1\rangle$, so it becomes

$$\left|\psi_{|1\rangle}\right\rangle = B\left(\frac{\sqrt{3}}{4}|10\rangle + \frac{1}{4}|11\rangle\right),$$

where *B* is the normalization constant

Measuring Individual Qubits

- Let’s calculate *B*

$$\left|B\frac{\sqrt{3}}{4}\right|^2+\left|B\frac{1}{4}\right|^2=1 \;\longrightarrow\; B^2\left(\frac{3}{16}+\frac{1}{16}\right)=1 \;\longrightarrow\; B^2\left(\frac{4}{16}\right)=1 \;\longrightarrow\; B^2\left(\frac{1}{4}\right)=1$$

- Thus, the normalization constant is

B =2

- Therefore

$$\left|\psi_{|1\rangle}\right\rangle=\frac{\sqrt{3}}{2}|10\rangle+\frac{1}{2}|11\rangle$$

Measuring Individual Qubits

So, measuring the left qubit yelds:

- $|0\rangle$ with probability $\frac{3}{4}$, and the state collapses to $\sqrt{\frac{2}{3}}|00\rangle + \sqrt{\frac{1}{3}}|01\rangle$

- $|1\rangle$ with probability $\frac{1}{4}$, and the state collapses to $\frac{\sqrt{3}}{2}|10\rangle + \frac{1}{2}|11\rangle$

Projection Operators For Measurement

- So far, we have described measurement of a **single qubit** and of a **pair of qubits** in terms of **projection** onto a basis vector associated with the measurement device
- This notion generalizes to measurement in multiple-qubit systems

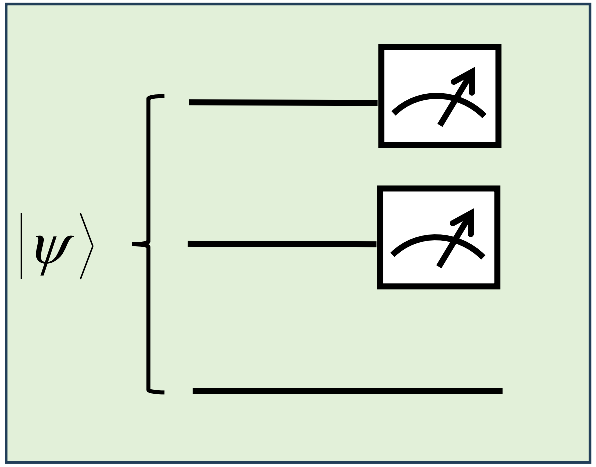
Measuring Individual Qubits

For example, if we have three qubits

$$c_0|000\rangle + c_1|001\rangle + c_2|010\rangle + c_3|011\rangle + c_4|100\rangle + c_5|101\rangle + c_6|110\rangle + c_7|111\rangle,$$

and we measure the left and middle qubits, the possible outcomes are

$$\frac{c_0|000\rangle + c_1|001\rangle}{\sqrt{|c_0|^2+|c_1|^2}} \text{ with probability } |c_0|^2+|c_1|^2,$$
$$\frac{c_2|010\rangle + c_3|011\rangle}{\sqrt{|c_2|^2+|c_3|^2}} \text{ with probability } |c_2|^2+|c_3|^2,$$
$$\frac{c_4|100\rangle + c_5|101\rangle}{\sqrt{|c_4|^2+|c_5|^2}} \text{ with probability } |c_4|^2+|c_5|^2,$$
$$\frac{c_6|110\rangle + c_7|111\rangle}{\sqrt{|c_6|^2+|c_7|^2}} \text{ with probability } |c_6|^2+|c_7|^2.$$



Projection Operators For Measurement

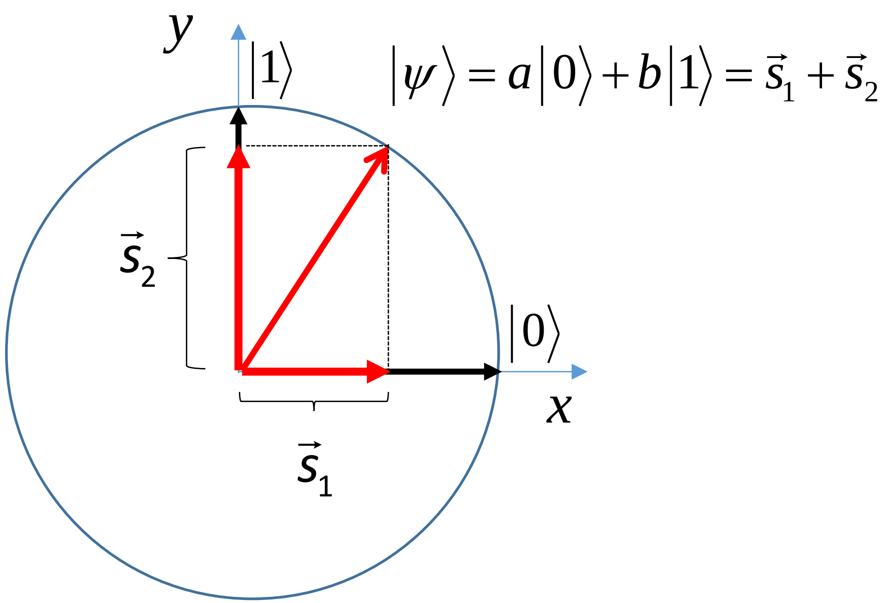
- For any subspace S of V, the subspace $S^{\perp}$ consists of all vectors that are perpendicular to all vectors in S
- The subspaces S and $S^{\perp}$ satisfy $V = S \oplus S^{\perp}$; thus, any vector $|\psi\rangle \in V$ can be written uniquely as the sum of vectors

$$\vec{s}_1 \in S \quad \text{and} \quad \vec{s}_2 \in S^{\perp}$$

i.e. $|\psi\rangle = \vec{s}_1 + \vec{s}_2$

Geometric Interpretation

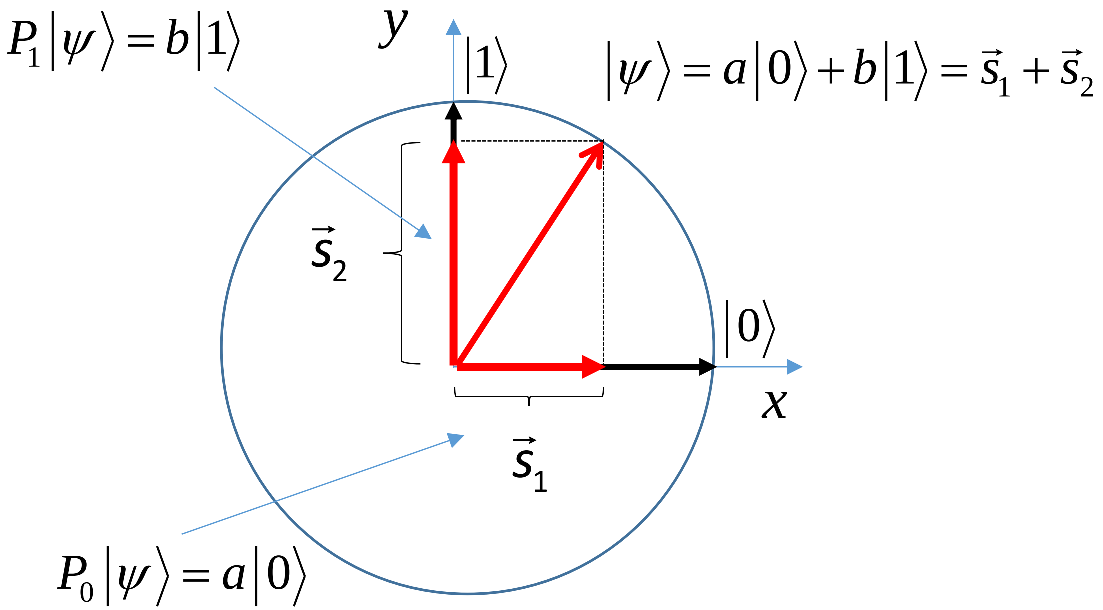
- If $|\psi\rangle$ is a unit vector in the plane (i.e., $V \equiv \mathbb{R}^2$), then S can be the subspace spanned by $|0\rangle$ and $S^{\perp}$ the subspace generated by $|1\rangle$. Obviously, $V = S \oplus S^{\perp}$



- From now on we change slightly the notations: we use S_1 and S_2 instead of S and $S^{\perp}$ respectively. Therefore $V = S_1 \oplus S_2$

Projection Operators For Measurement

- For subspaces S_1 and S_2, the **projection** operators P_1 and P_2 are the linear operators $P_1 : V \rightarrow S_1$, and $P_2 : V \rightarrow S_2$ that sends $|\psi\rangle \rightarrow \vec{s}_1$ and $|\psi\rangle \rightarrow \vec{s}_2$ where $|\psi\rangle = \vec{s}_1 + \vec{s}_2$ with $\vec{s}_1 \in S_1$ and $\vec{s}_2 \in S_2$
- We use the notation $\vec{s}_i$ because $\vec{s}_1$ and $\vec{s}_2$ are generally not unit vectors
- Projection operators are sometimes called **projectors** for short



Projection Operators For Measurement

- The above considerations can be extended to multi-qubit systems
- For any direct sum decomposition of $V = S_1 \oplus S_2 \oplus \cdots \oplus S_k$ into orthogonal subspaces S_i, there are k related projection operators $P_i : V \to S_i$ where $P_i|v\rangle \to \vec{s}_i$ where $|\psi\rangle = \vec{s}_1 + \vec{s}_2 + \cdots + \vec{s}_k$ with $\vec{s}_i \in S_i$

Projection Operators For Measurement

- Using the terminology provided, the essence of **Postulate 3**, also known as the measurement postulate, can be formulated as follows.
- **Postulate 3:** A measuring device with associated decomposition

$$V = S_1 \oplus S_2 \oplus \cdots \oplus S_k$$

acting on a state $|\psi\rangle$,outputs label $i \in \{1,2,....,k\}$, with probability

$$p(i) = \langle\psi|P_i^{\dagger}P_i|\psi\rangle = \langle\psi|PP_i|\psi\rangle = \langle\psi|P_i^2|\psi\rangle = \langle\psi|P_i|\psi\rangle = \left|P_i|\psi\rangle\right|^2$$

and leaves the system in the normalized state

$$|\phi_i\rangle = \frac{P_i|\psi\rangle}{\left|P_i|\psi\rangle\right|}$$

Projection Operators For Measurement

- To solidify our understanding of projection operators and Dirac's notation, let us describe single-qubit measurement in the standard basis in terms of this formalism
- **Example 2:** Let V be the vector space associated with a **single-qubit system**
- The direct sum decomposition for V associated with measurement in the standard basis is $V = S_0 \oplus S_1$, where S_0 is the subspace generated by $|0\rangle$ and S_1 is the subspace generated by $|1\rangle$

Projection Operators For Measurement

- The related projection operators are

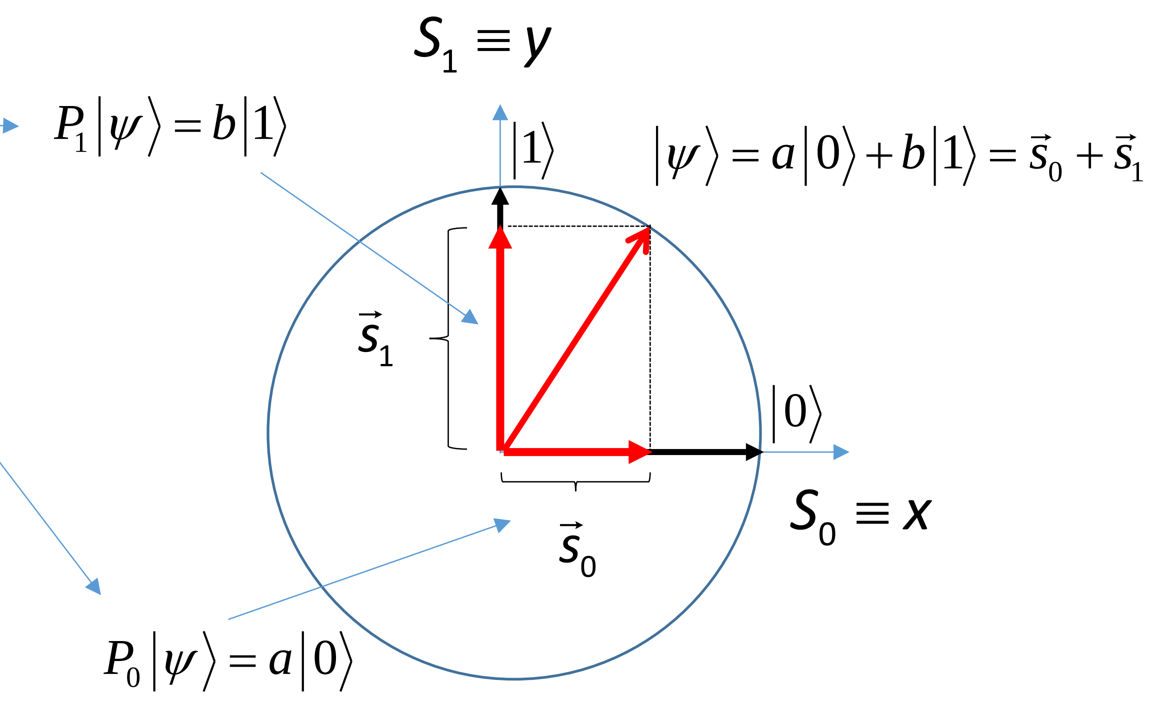
  $P_0 : V \to S_0$, and

  $P_1 : V \to S_1$,

  where

  $P_0 = |0\rangle\langle 0|$, and

  $P_1 = |1\rangle\langle 1|$



Projection Operators For Measurement

- The **projector** $P_0 = |0\rangle\langle 0|$ acts on a single-qubit state $|\psi\rangle$ and obtains the component of $|\psi\rangle$ in the subspace generated by $|0\rangle$
- Let's check it on the state $|\psi\rangle = a|0\rangle + b|1\rangle$

$$P_0|\psi\rangle = \left(|0\rangle\langle 0|\right)|\psi\rangle = \left(|0\rangle\langle 0|\right)\left(a|0\rangle + b|1\rangle\right) = a|0\rangle\langle 0|0\rangle + b|0\rangle\langle 0|1\rangle = a|0\rangle \;\rightarrow$$

$$P_0|\psi\rangle = a|0\rangle$$

- Similarly, the projector $P_1 = |1\rangle\langle 1|$ acts on a single-qubit state $|\psi\rangle$

$$P_1|\psi\rangle = \left(|1\rangle\langle 1|\right)|\psi\rangle = \left(|1\rangle\langle 1|\right)\left(a|0\rangle + b|1\rangle\right) = a|1\rangle\langle 1|0\rangle + b|1\rangle\langle 1|1\rangle = b|1\rangle \;\rightarrow$$

$$P_1|\psi\rangle = b|1\rangle$$

Projection Operators For Measurement

- Measurement of the state $|\psi\rangle = a|0\rangle + b|1\rangle$ results in the state $\dfrac{P_i|\psi\rangle}{\left|P_i|\psi\rangle\right|}, \quad i \in \{0,1\}$ with probability $\left|P_i|\psi\rangle\right|^2$

- Since

$$P_0|\psi\rangle = \left(|0\rangle\langle 0|\right)|\psi\rangle = a|0\rangle$$

and

$$P_0 = P_0^{\dagger}$$

$$\left|P_0|\psi\rangle\right|^2 = \langle\psi|P_0^{\dagger}P_0|\psi\rangle = \langle\psi|P_0P_0|\psi\rangle = \langle\psi|P_0|\psi\rangle = \langle\psi|\left(|0\rangle\langle 0|\right)|\psi\rangle = \langle\psi|0\rangle\langle 0|\psi\rangle = a^*a = |a|^2$$

the result of the measurement is $\dfrac{a|0\rangle}{|a|}$ with probability $|a|^2$

Projection Operators For Measurement

- Since we know that

$$\frac{a|0\rangle}{|a|} = \frac{|a|e^{i\theta}|0\rangle}{|a|} = e^{i\theta}|0\rangle \equiv |0\rangle$$

  an overall phase factor is physically meaningless, it turns out that the state represented by $|0\rangle$ has been obtained with probability $|a|^2$

- A similar calculation shows that the state represented by $|1\rangle$ is obtained with probability $|b|^2$
- The above consideration can be extended to a multi-qubit system

Projection Operators For Measurement

- Before giving examples of more interesting measurements, we describe measurement of a **two-qubit state** with respect to the full decomposition associated with the standard basis
- **Example 3:** Let V be the vector space associated with a two-qubit system and

$$|\psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$$

  an arbitrary two-qubit state
- Consider a measurement with decomposition $V = S_{00} \oplus S_{01} \oplus S_{10} \oplus S_{11}$, where S_{ij} is the *one-dimensional complex subspace* spanned by $|ij\rangle,\ \forall ij \in \{0,1\}$

Projection Operators For Measurement

$$V = S_{00} \oplus S_{01} \oplus S_{10} \oplus S_{11}$$

- In detail, this means that

  S_{00} is the one-dimensional complex subspace spanned by $|00\rangle$

  S_{01} is the one-dimensional complex subspace spanned by $|01\rangle$

  S_{10} is the one-dimensional complex subspace spanned by $|10\rangle$

  S_{11} is the one-dimensional complex subspace spanned by $|11\rangle$

-

Projection Operators For Measurement

- The related projection operators $P_{ij}: V \to S_{ij}$ are

$$P_{00} = |00\rangle\langle 00|, \quad P_{01} = |01\rangle\langle 01|, \quad P_{10} = |10\rangle\langle 10|, \quad P_{11} = |11\rangle\langle 11|$$

- The state after measurement will be

$$\frac{P_{ij}|\psi\rangle}{\left|P_{ij}|\psi\rangle\right|}, \quad i, j \in \{0,1\}$$

with probability

$$\left|P_{ij}|\psi\rangle\right|^2, \quad i, j \in \{0,1\}$$

Projection Operators For Measurement

- Projectors $P_{00} = |00\rangle\langle 00|,\ P_{01} = |01\rangle\langle 01|,\ P_{10} = |10\rangle\langle 10|,\ P_{11} = |11\rangle\langle 11|$ act on two-qubit states

$$|\psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$$

as follows

$$P_{00}|\psi\rangle = \left(|00\rangle\langle 00|\right)|\psi\rangle = a_{00}|00\rangle$$
$$P_{01}|\psi\rangle = \left(|01\rangle\langle 01|\right)|\psi\rangle = a_{01}|01\rangle$$
$$P_{10}|\psi\rangle = \left(|10\rangle\langle 10|\right)|\psi\rangle = a_{10}|10\rangle$$
$$P_{11}|\psi\rangle = \left(|11\rangle\langle 11|\right)|\psi\rangle = a_{11}|11\rangle$$

Projection Operators For Measurement

- Recall that two unit vectors $|v\rangle$ and $|w\rangle$ represent the same quantum state if $|v\rangle = e^{i\theta}|w\rangle$ for some θ, and that $|v\rangle \sim |w\rangle$ indicates that $|v\rangle$ and $|w\rangle$ represent the same quantum state
- The state after measurement is either

$$\frac{P_{00}|\psi\rangle}{\left|P_{00}|\psi\rangle\right|} = \frac{a_{00}|00\rangle}{|a_{00}|} \sim |00\rangle \xrightarrow[\text{}]{\textit{with probability}} |a_{00}|^2, \text{ or} \qquad \frac{P_{10}|\psi\rangle}{\left|P_{10}|\psi\rangle\right|} = \frac{a_{10}|10\rangle}{|a_{10}|} \sim |10\rangle \xrightarrow{\textit{with probability}} |a_{10}|^2, \text{ or}$$

$$\frac{P_{01}|\psi\rangle}{\left|P_{01}|\psi\rangle\right|} = \frac{a_{01}|01\rangle}{|a_{01}|} \sim |01\rangle \xrightarrow{\textit{with probability}} |a_{01}|^2, \text{ or} \qquad \frac{P_{11}|\psi\rangle}{\left|P_{11}|\psi\rangle\right|} = \frac{a_{11}|11\rangle}{|a_{11}|} \sim |11\rangle \xrightarrow{\textit{with probability}} |a_{11}|^2$$

Projection Operators For Measurement

- More interesting are measurements that give information about the **relations** between **qubit values** without giving any information about the qubit values themselves
- For example, we can measure two qubits for **bit equality** without determining the **actual value** of the bits
- Such measurements will be used heavily in quantum error correction schemes

Projection Operators For Measurement

- **Example 4:** Let V be the vector space associated with a **two-qubit system**
- Consider a measurement with associated direct sum decomposition

$$V = S_1 \oplus S_2,$$

where

> S_1 is the subspace generated by $\{|00\rangle, |11\rangle\}$, the subspace in which the two bits **are equal**, and

> S_2 is the subspace generated by $\{|10\rangle, |01\rangle\}$, the subspace in which the two bits **are not equal**

Projection Operators For Measurement

- Let P_1 and P_2 be the projection operators onto S_1 and S_2 respectively, i.e.,

$$P_1 = |00\rangle\langle 00| + |11\rangle\langle 11| \qquad P_2 = |10\rangle\langle 10| + |01\rangle\langle 01|$$

- When a system in state

$$|\psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$$

is measured in this way, the state after measurement becomes

$$\frac{P_i|\psi\rangle}{\left|P_i|\psi\rangle\right|} \xrightarrow[\text{}]{\text{with probability}} \left|P_i|\psi\rangle\right|^2, \; i \in \{1, 2\}$$

Projection Operators For Measurement

- We'll begin by assessing $P_1|\psi\rangle$ and $P_2|\psi\rangle$

$$P_1|\psi\rangle = \left(|00\rangle\langle 00| + |11\rangle\langle 11|\right)\left(a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle\right) = a_{00}|00\rangle + a_{11}|11\rangle$$

$$P_2|\psi\rangle = \left(|10\rangle\langle 10| + |01\rangle\langle 01|\right)\left(a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle\right) = a_{01}|01\rangle + a_{10}|10\rangle$$

and then move on to evaluating $\left|P_1|\psi\rangle\right|^2$ and $\left|P_2|\psi\rangle\right|^2$

$$|c_1|^2 = \left|P_1|\psi\rangle\right|^2 = \left(P_1|\psi\rangle\right)^\dagger\left(P_1|\psi\rangle\right) = \langle\psi|P_1^\dagger P_1|\psi\rangle$$

$$= \left(\langle 00|a_{00}^* + \langle 11|a_{11}^*\right)\left(a_{00}|00\rangle + a_{11}|11\rangle\right) = |a_{00}|^2 + |a_{11}|^2$$

$$|c_2|^2 = \left|P_2|\psi\rangle\right|^2 = \left(P_2|\psi\rangle\right)^\dagger\left(P_2|\psi\rangle\right) = \langle\psi|P_2^\dagger P_2|\psi\rangle$$

$$= \left(\langle 01|a_{01}^* + \langle 10|a_{10}^*\right)\left(a_{01}|01\rangle + a_{10}|10\rangle\right) = |a_{01}|^2 + |a_{10}|^2$$

Projection Operators For Measurement

- After measurement the state will be

$$\frac{1}{|c_1|}\left(a_{00}|00\rangle + a_{11}|11\rangle\right) = \frac{a_{00}|00\rangle + a_{11}|11\rangle}{\sqrt{|a_{00}|^2 + |a_{11}|^2}} \xrightarrow{\text{with probability}} |c_1|^2 = |a_{00}|^2 + |a_{11}|^2$$

and

$$\frac{1}{|c_2|}\left(a_{01}|01\rangle + a_{10}|10\rangle\right) = \frac{a_{01}|01\rangle + a_{10}|10\rangle}{\sqrt{|a_{01}|^2 + |a_{10}|^2}} \xrightarrow{\text{with probability}} |c_2|^2 = |a_{01}|^2 + |a_{10}|^2$$

Projection Operators For Measurement

- If the first outcome happens, then we know that the two-bit values are equal, but we do not know whether they are 0 or 1.
- If the second case happens, we know that the two bit values are not equal, but we do not know which one is 0 and which one is 1.
- Thus, the measurement does not determine the value of the two bits, only whether the two bits are equal

Projection Operators For Measurement

- As in the case of single-qubit states, most states are a superposition with respect to a measurement's subspace decomposition
- In the previous example, a state that is a superposition containing components with both **equal** and **unequal** bit values is transformed by measurement either to a state (generally still a superposition of standard basis elements), in which in all components the bit values are equal, or to a state in which the bit values are not equal in all of the components
- Before further developing the formalism used to describe quantum measurement, we give an additional example, one in which the associated subspaces are not generated by subsets of the standard basis elements

Projection Operators For Measurement

- **Example 5:** Measuring a two-qubit state with respect to the Bell basis decomposition
- Recall from an earlier lecture the four **Bell states**

$$\left|\Phi^{+}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|00\right\rangle + \left|11\right\rangle\right),\ \left|\Psi^{+}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|01\right\rangle + \left|10\right\rangle\right),$$

$$\left|\Phi^{-}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|00\right\rangle - \left|11\right\rangle\right),\ \left|\Psi^{-}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|01\right\rangle - \left|10\right\rangle\right)$$

- Let

$$V = S_{\Phi^{+}} \oplus S_{\Phi^{-}} \oplus S_{\Psi^{+}} \oplus S_{\Psi^{-}}$$

  be the direct sum decomposition into the subspaces generated by the Bell states

Projection Operators For Measurement

- Measurement of the state $|00\rangle$ with respect to this decomposition yields
  - > $|\Phi^+\rangle$ with probability ½, and
  - > $|\Phi^-\rangle$ with probability ½,

  because

$$|00\rangle = \frac{1}{\sqrt{2}}\left(|\Phi^+\rangle + |\Phi^-\rangle\right)$$

- It is easy to determine the outcomes and their probabilities for the three other standard basis elements, and a general two-qubit state

Hermitian Operator Formalism For Measurement

- Instead of explicitly writing out the subspace decomposition associated with a measurement, including the definition of each subspace of the decomposition in terms of a generating set, a mathematical shorthand is used
- **Hermitian operators,** define a **unique orthogonal subspace** decomposition, their **eigenspace decomposition**
- Moreover, for every such decomposition, there exists a Hermitian operator whose eigenspace decomposition is this decomposition
- Given this correspondence, Hermitian operators can be used to describe measurements

Hermitian Operator Formalism For Measurement

- To establish the connection between Hermitian operators and orthogonal subspace decompositions, we exploit (without proving) the following result
- Let V be an N-dimensional vector space, and let

$$\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_k$$

  be the $k \leq N$ **distinct eigenvalues** of a Hermitian operator $O : V \to V$
- Then

$$V = S_{\lambda_1} \oplus S_{\lambda_2} \oplus \cdots \oplus S_{\lambda_k}$$

  where S_{λ_i} is the **eigenspace** of O with **eigenvalue** λ_i

Hermitian Operator Formalism For Measurement

- To establish the connection between Hermitian operators and orthogonal subspace decompositions, we exploit (without proving) the following result

$O : V \rightarrow V$ (Hermitian operator in an *N*-dimensional Hilbert space)

$O\left|\lambda_i\right\rangle = \lambda_i\left|\lambda_i\right\rangle, \quad i \in \{1,2,\dots,k\}, \quad k \le N$ (distinct eigenvalues)

- For the sake of simplicity, we assume *k* = *N*

$$O \quad \longleftrightarrow \quad \left\{ \begin{array}{l} \lambda_1 \rightarrow \left|\lambda_1\right\rangle \rightarrow S_{\lambda_1}, \quad \text{eigenspace of } O \text{ with eigenvalue } \lambda_1 \\ \lambda_2 \rightarrow \left|\lambda_2\right\rangle \rightarrow S_{\lambda_2}, \quad \text{eigenspace of } O \text{ with eigenvalue } \lambda_2 \\ \qquad \vdots \\ \lambda_N \rightarrow \left|\lambda_N\right\rangle \rightarrow S_{\lambda_N}, \quad \text{eigenspace of } O \text{ with eigenvalue } \lambda_N \end{array} \right\rangle$$

eigenspace decomposition of *V* for the **Hermitian** operator *O*

$$\downarrow$$

$$V = S_{\lambda_1} \oplus S_{\lambda_2} \oplus \cdots \oplus S_{\lambda_N}$$

Hermitian Operator Formalism For Measurement

$$O \;\longleftrightarrow\; \left\{ \begin{array}{l} \lambda_1 \to |\lambda_1\rangle \to S_{\lambda_1}, \quad \text{eigenspace of } O \text{ with eigenvalue } \lambda_1 \\ \lambda_2 \to |\lambda_2\rangle \to S_{\lambda_2}, \quad \text{eigenspace of } O \text{ with eigenvalue } \lambda_2 \\ \qquad\qquad\qquad\qquad \vdots \\ \lambda_N \to |\lambda_N\rangle \to S_{\lambda_N}, \quad \text{eigenspace of } O \text{ with eigenvalue } \lambda_N \end{array} \right\rangle \quad V = S_{\lambda_1} \oplus S_{\lambda_2} \oplus \cdots \oplus S_{\lambda_N}$$

$$O = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{bmatrix} = \sum_{i=1}^{N} \lambda_i |\lambda_i\rangle\langle\lambda_i| = \sum_{i=1}^{N} \lambda_i P_i$$

Hermitian Operator Formalism For Measurement

- This direct sum decomposition of V is called the **eigenspace decomposition** of V for the **Hermitian** operator O
- Thus, any Hermitian operator $O: V \to V$ uniquely determines a subspace decomposition for V
- Furthermore, any decomposition of a vector space V into the direct sum of subspaces $S_1, S_2, \cdots, S_N$ can be realized as the eigenspace decomposition of a Hermitian operator $O: V \to V$
- Let P_i be the projectors onto the subspaces S_i, and let $S_1, S_2, \cdots, S_N$ be **any set of distinct real values**; then $O = \sum_{i=1}^{N} \lambda_i P_i$ is a Hermitian operator with the desired direct sum decomposition

Hermitian Operator Formalism For Measurement

- Thus, when describing a measurement, instead of **directly** specifying the associated subspace decomposition, we can specify a Hermitian operator whose eigenspace decomposition is that decomposition
- Any Hermitian operator with the appropriate direct sum decomposition can be used to specify a given measurement; in particular, the values of the λ_i are **irrelevant** as long as they are **distinct**
- The λ_i should be thought of simply as labels for the corresponding subspaces, or equivalently as **labels for the measurement outcomes**
- For our purposes, we do not need to assign labels with meaning; any distinct set of eigenvalues will do

Hermitian Operator Formalism For Measurement

- *Specifying a measurement in terms of a Hermitian operator is standard practice throughout the quantum-mechanics and quantum-information-processing literature*
- It is important to recognize, however, that quantum measurement is not modeled by the action of a Hermitian operator on a state
- The projectors P_j associated with a Hermitian operator *O*, not *O* itself, act on a state.
- Which projector acts on the state depends on the probabilities $p_j = \langle \psi | P_j | \psi \rangle$

Hermitian Operator Formalism For Measurement

- For example, the result of measuring $|\psi\rangle = a|0\rangle + b|1\rangle$ according to the Hermitian operator $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$ does not result in the state $a|0\rangle - b|1\rangle$, even though

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ -b \end{bmatrix}$$

- Direct multiplication by a Hermitian operator generally does not even result in a well-defined state; for example,

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}|0\rangle = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hermitian Operator Formalism For Measurement

- The Hermitian operator is only a convenient **bookkeeping trick**, a concise way of specifying he subspace decomposition associated with the measurement

The Measurement Postulate (Postulate 3)

- Many aspects of our model of quantum mechanics are not directly observable by experiment
- For example, as we saw in a previous lecture, given a single instance of an unknown single-qubits state $|\psi\rangle = a|0\rangle + b|1\rangle$, there is no way to determine experimentally what state it is in; we cannot directly observe the quantum state
- It is only the results of measurements that we can directly observe
- For this reason, the Hermitian operators we use to specify measurements are called **observables**

The Measurement Postulate (Postulate 3)

Postulate 3:

- Any quantum measurement can be specified by a **Hermitian** operator *O* called an **observable**
- The possible outcomes of measuring a state $|\psi\rangle$ with an observable *O* are **labeled by the eigenvalues** of *O*
- Measurement of state $|\psi\rangle$ results in the **outcome labeled by the eigenvalue** $\boldsymbol{\lambda_i}$ of *O* with probability $\left|P_i|\psi\rangle\right|^2$ where P_i is the projector onto the $\boldsymbol{\lambda_i}$**-eigenspace**

The Measurement Postulate (Postulate 3)

- (*Projection*) The **state after measurement** is the normalized projection

$$\frac{P_i|\psi\rangle}{\left|P_i|\psi\rangle\right|} = \frac{P_i|\psi\rangle}{\sqrt{\langle\psi|P_i|\psi\rangle}}$$

onto the $\boldsymbol{\lambda_i}$**-eigenspace** $\boldsymbol{S_i}$. Thus, the state after measurement is a unit length eigenvector of O with eigenvalue λ_i

- **NOTE:** $$\frac{P_i|\psi\rangle}{\left|P_i|\psi\rangle\right|} = \frac{P_i|\psi\rangle}{\sqrt{\langle\psi|P_i^\dagger P_i|\psi\rangle}} = \frac{P_i|\psi\rangle}{\sqrt{\langle\psi|P_iP_i|\psi\rangle}}$$

$$= \frac{P_i|\psi\rangle}{\sqrt{\langle\psi|P_i^2|\psi\rangle}} = \frac{P_i|\psi\rangle}{\sqrt{\langle\psi|P_i|\psi\rangle}}$$

$P_i^\dagger = P_i$ (P_i is Hermitian)

$P_i^2 = P_i$ (P_i is a Projector)

The Measurement Postulate (Postulate 3)

Hermitian Operator A
Eigenvectors: $|i\rangle$
Eigenvalues: λ_i

Observable A

Quantum system
in state $|\psi\rangle \in V$

dim(V)=N

N Projectors

$P_1, P_2, \ldots, P_N$

$P_i = |i\rangle\langle i|$

$\sum_{i=1}^{N} P_i = I$

$$A = \sum_{i=1}^{N} \lambda_i P_i \longleftarrow$$

P_1

P_i

P_N

New state $P_1|\psi\rangle \in V$
Outcome of measurement: λ_1
Probability of the outcome: $|P_1|\psi\rangle|^2$

New state $P_i|\psi\rangle \in V$
Outcome of measurement: λ_i
Probability of the outcome: $|P_i|\psi\rangle|^2$

New state $P_N|\psi\rangle \in V$
Outcome of measurement: λ_N
Probability of the outcome: $|P_N|\psi\rangle|^2$

The Measurement Postulate (Postulate 3)

- We should make clear that what we have described here is a mathematical formalism for measurement
- It does not tell us what measurements can be done in practice, or with what efficiency
- Some measurements that may be mathematically simple to state may not be easy to implement
- Furthermore, the eigenvalues of physically realizable measurements may have meaning - for example, as the position or energy of a particle - but **for us the eigenvalues are just arbitrary labels**

The Measurement Postulate (Postulate 3)

Example 6: *Hermitian operator formalism for measurement of a single qubit in the standard basis*

- Using the description in **Example 2**, of measurement of a single-qubit system in the standard basis, let us build up a **Hermitian operator** that specifies this measurement

- We will generally use

$$Z = |0\rangle\langle 0| - |1\rangle\langle 1| = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

  to specify single-qubit measurements in the standard basis

- The eigenvalues of Z are λ_0 =+1 and λ_1 =-1 and the corresponding eigenvectors are $|0\rangle$ and $|1\rangle$

The Measurement Postulate (Postulate 3)

- The subspace decomposition corresponding to this measurement is

$$V = S_0 \oplus S_1$$

  where:
  > S_0 is the subspace generated by $|0\rangle$, and
  > S_1 is the subspace generated by $|1\rangle$

- The projectors associated with S_0 and S_1 are $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$ respectively.

The Measurement Postulate (Postulate 3)

- Let λ_0 and λ_1 be any two distinct real values, say $\lambda_0 = 2$ and $\lambda_1 = -3$
- Then the operator

$$O = 2|0\rangle\langle 0| - 3|1\rangle\langle 1| = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$$

  is a Hermitian operator specifying the measurement of a single-qubit state in the standard basis.

- Any other distinct values λ_0 and λ_1 could have been used

The Measurement Postulate (Postulate 3)

Example 7: *Hermitian operator formalism for measurement of a single qubit in the Hadamard basis*

- We wish to construct a Hermitian operator corresponding to measurement of a single qubit in the Hadamard basis $\{|+\rangle, |-\rangle\}$
- The Hermitian operator generally used is

$$X = |0\rangle\langle 1| + |1\rangle\langle 0| = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- The eigenvalues of X are λ_+ =+1 and λ_- =-1 and the corresponding eigenvectors are $|+\rangle$ and $|-\rangle$,i.e,

$$X|+\rangle = |+\rangle$$
$$X|-\rangle = -|-\rangle$$

The Measurement Postulate (Postulate 3)

- The subspace decomposition corresponding to this measurement is

$$V = S_+ \oplus S_-$$

  where:

  > S_+ is the subspace generated by $|+\rangle$, and
  > S_- is the subspace generated by $|-\rangle$

- The projectors associated with S_+ and S_- are

$$P_+ = |+\rangle\langle+| \text{ and}$$

$$P_- = |-\rangle\langle-|$$

  respectively

- Thus

$$O = \lambda_+ P_+ + \lambda_- P_- = |+\rangle\langle+| - |-\rangle\langle-|$$

The Measurement Postulate (Postulate 3)

- However, we are free to choose λ_+ and λ_- any way we like as long as they are distinct

$$O = \lambda_+ P_+ + \lambda_- P_-$$

The Measurement Postulate (Postulate 3)

Example 8: The Hermitian operator

$$A = 1|00\rangle\langle 00| + 2|01\rangle\langle 01| + 3|10\rangle\langle 10| + 5|11\rangle\langle 11|$$

has matrix representation

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

with respect to the standard basis in the standard order $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$

The eigenspace decomposition for A consists of four subspaces, each generated by one of the standard basis vectors $|00\rangle, |01\rangle, |10\rangle, |11\rangle$

The Measurement Postulate (Postulate 3)

- The operator A is one of many Hermitian operators that specify measurement with respect to the full standard basis decomposition described in **Example 3**
- The Hermitian operator

$$\widehat{A} = 73|00\rangle\langle 00| + 50|01\rangle\langle 01| - 3|10\rangle\langle 10| + 23|11\rangle\langle 11|$$

is another

The Measurement Postulate (Postulate 3)

Example 9: The Hermitian operator

$$B = \left(|00\rangle\langle 00| + |01\rangle\langle 01|\right) + \pi\left(|10\rangle\langle 10| + |11\rangle\langle 11|\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \pi & 0 \\ 0 & 0 & 0 & \pi \end{bmatrix}$$

specifies measurement of a two-qubit system with respect to the subspace decomposition $V = S_0 \oplus S_1$, where S_0 is generated by $\{|00\rangle,\ |01\rangle\}$ and S_1 is generated by $\{|10\rangle,\ |11\rangle\}$, so B specifies measurement of the first qubit in the standard basis as described in **Example 1**

The Measurement Postulate (Postulate 3)

Example 10: The Hermitian operator

$$C = 2\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right) + 3\left(|01\rangle\langle 01| + |10\rangle\langle 10|\right) = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

specifies measurement of a two-qubit system with respect to the subspace decomposition $V = S_2 \oplus S_3$, where S_2 is generated by $\{|00\rangle, |11\rangle\}$ and S_3 is generated by $\{|01\rangle, |10\rangle\}$, so C specifies the measurement for bit equality described in **Example 4**

Product States & Entangled States

Product States

- Some quantum states can be factored into (the tensor product of) individual qubit states. For example,

$$\begin{aligned}\frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle) &= \underbrace{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)}_{|+\rangle} \otimes \underbrace{\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)}_{|-\rangle} \\ &= |+\rangle \otimes |-\rangle \\ &= |+\rangle|-\rangle.\end{aligned}$$

- Such factorizable states are called *product states* or simply *separable states*

Product States

Let us work through an example of how to factor a state. Say two qubits are the state

$$\frac{1}{2\sqrt{2}}\left(\sqrt{3}|00\rangle - \sqrt{3}|01\rangle + |10\rangle - |11\rangle\right).$$

We want to write this as the product of two single-qubit states,

$$|\psi_1\rangle|\psi_0\rangle,$$

where

$$|\psi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle, \quad |\psi_0\rangle = \alpha_0|0\rangle + \beta_0|1\rangle.$$

Then,

$$\begin{aligned}|\psi_1\rangle|\psi_0\rangle &= (\alpha_1|0\rangle + \beta_1|1\rangle)(\alpha_0|0\rangle + \beta_0|1\rangle) \\ &= \alpha_1\alpha_0|00\rangle + \alpha_1\beta_0|01\rangle + \beta_1\alpha_0|10\rangle + \beta_1\beta_0|11\rangle.\end{aligned}$$

Matching up the coefficients with our original state,

$$\alpha_1\alpha_0 = \frac{\sqrt{3}}{2\sqrt{2}}, \quad \alpha_1\beta_0 = \frac{-\sqrt{3}}{2\sqrt{2}}, \quad \beta_1\alpha_0 = \frac{1}{2\sqrt{2}}|10\rangle, \quad \beta_1\beta_0 = \frac{-1}{2\sqrt{2}}.$$

$$\boxed{\frac{1}{2\sqrt{2}}\left(\sqrt{3}|00\rangle - \sqrt{3}|01\rangle + |10\rangle - |11\rangle\right)}$$

$$\boxed{\alpha_1\alpha_0|00\rangle + \alpha_1\beta_0|01\rangle + \beta_1\alpha_0|10\rangle + \beta_1\beta_0|11\rangle}$$

Using these equations, let us solve for the variables in terms of one of them. Starting with the first equation, we can solve for α_1 in terms of α_0:

$$\alpha_1 = \frac{\sqrt{3}}{2\sqrt{2}\alpha_0}.$$

Plugging this into the second equation, we can solve for β_0 in terms of α_0:

$$\beta_0 = -\alpha_0.$$

For the third equation, we can solve for β_1 in terms of α_0:

$$\beta_1 = \frac{1}{2\sqrt{2}\alpha_0}.$$

Finally, plugging in $\beta_1 = \frac{1}{2\sqrt{2}}\frac{1}{\alpha_0}$ and $\beta_0 = \alpha_0$ into the fourth equation, we get

$$\frac{-1}{2\sqrt{2}} = \frac{-1}{2\sqrt{2}},$$

which is a true statement, so it is satisfied, although it does not tell us anything new. So, we have solve for α_1, β_1, and β_0 in terms of α_0, and this is actually sufficient.

Plugging into the product state,

$$|\psi_1\rangle|\psi_0\rangle = (\alpha_1|0\rangle + \beta_1|1\rangle)(\alpha_0|0\rangle + \beta_0|1\rangle)$$
$$= \left(\frac{\sqrt{3}}{2\sqrt{2}\alpha_0}|0\rangle + \frac{1}{2\sqrt{2}}\frac{1}{\alpha_0}|1\rangle\right)(\alpha_0|0\rangle - \alpha_0|1\rangle).$$

We see that α_0 cancels, yielding

$$|\psi_1\rangle|\psi_0\rangle = \left(\frac{\sqrt{3}}{2\sqrt{2}}|0\rangle + \frac{1}{2\sqrt{2}}|1\rangle\right)(|0\rangle - |1\rangle).$$

Moving the factor of $1/\sqrt{2}$ to the right qubit so that both qubits are normalized,

$$|\psi_1\rangle|\psi_0\rangle = \left(\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right).$$

Thus, the left qubit is in the state $\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$, and the right qubit is in the state $|-\rangle$.

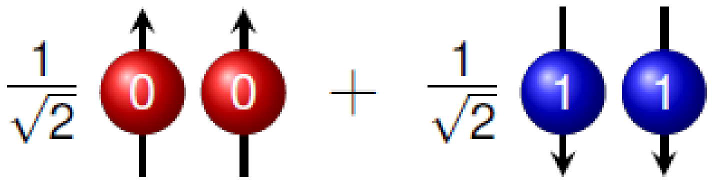
In general, a product state of n qubits can be written

$$(\alpha_{n-1}|0\rangle + \beta_{n-1}|0\rangle) \otimes \cdots \otimes (\alpha_1|0\rangle + \beta_1|1\rangle) \otimes (\alpha_0|0\rangle + \beta_0|1\rangle).$$

Entangled States

- There exist quantum states that *cannot be factored into product states*
- These are called *entangled states*
- The *entanglement* is one of the most interesting and puzzling ideas associated with composite quantum systems
- Consider the two qubit state named the *Bell State* or *EPR pair*

$$\left|\Phi^+\right\rangle = \frac{\left|00\right\rangle + \left|11\right\rangle}{\sqrt{2}} \rightarrow$$



Entangled States

- $|\Phi^+\rangle$ cannot be written as $|\psi_1\rangle|\psi_0\rangle$

- As a proof, let us try writing it in a product using the same procedure as in the previous slides

$$|\psi_1\rangle|\psi_0\rangle = (\alpha_1|0\rangle + \beta_1|1\rangle)(\alpha_0|0\rangle + \beta_0|1\rangle) = \underbrace{\alpha_1\alpha_0|00\rangle + \alpha_1\beta_0|01\rangle + \beta_1\alpha_0|10\rangle + \beta_1\beta_0|11\rangle}$$

- Matching the coefficients, we get

$$\alpha_1\alpha_0 = \frac{1}{\sqrt{2}}, \quad \alpha_1\beta_0 = 0, \quad \beta_1\alpha_0 = 0, \quad \beta_1\beta_0 = \frac{1}{\sqrt{2}}$$

$$\boxed{|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}}$$

Entangled States

$$\alpha_1\alpha_0 = \frac{1}{\sqrt{2}}, \quad \alpha_1\beta_0 = 0, \quad \beta_1\alpha_0 = 0, \quad \beta_1\beta_0 = \frac{1}{\sqrt{2}}$$

- The second equation requires $\alpha_1 = 0$ or $\beta_0 = 0$
- If $\alpha_1 = 0$, then the *first equation* gives $0 = \frac{1}{\sqrt{2}}$, which is false
- If $\beta_0 = 0$, then the *fourth equation* gives $0 = \frac{1}{\sqrt{2}}$, which is false
- Thus, there is no solution to these four equations, so $|\Phi^+\rangle$ cannot be written as a product state
- Therefore, $|\Phi^+\rangle$ is an *entangled state*

Bell States or EPR Pair

physicists
use this alternative symbols

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \rightarrow \boxed{|\Phi^+\rangle}$$

$$|\beta_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \rightarrow \boxed{|\Psi^+\rangle}$$

$$|\beta_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \rightarrow \boxed{|\Phi^-\rangle}$$

$$|\beta_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \rightarrow \boxed{|\Psi^-\rangle}$$

These are the **only** four **maximally** entangled pairs, and they are known as the *Bell states* or *EPR states* or *EPR pairs* (for the physicists Einstein, Podolsky, and Rosen)

Bell States or EPR Pair

- The mnemonic notation,

$$|\beta_{00}\rangle, |\beta_{01}\rangle, |\beta_{10}\rangle, |\beta_{11}\rangle$$

can be easily understood via the equations

$$|\beta_{xy}\rangle = \frac{1}{\sqrt{2}}\left(|0, y\rangle + (-1)^x |1, \bar{y}\rangle\right), \quad x, y \in \{0,1\}$$

where $\bar{y}$ is the negation of y

Entanglement Measure

- Given the importance of entanglement in quantum computing, we need a way to quantify the *degree of entanglement within a state*, i.e., we need an *entanglement measure*, to characterize the degree of entanglement within a *2-qubit quantum state*
- There *are many* of such performance measures
- Fortunately, in the case of 2-qubit states, all the different entanglement measures turn out to be equivalent to one another
- However, *no such equivalence* is found for entanglement measures of n-qubit states ($n > 2$)

Entanglement Measure

- Specifically, the *tangle* provides a *quantitative measure* of *the degree of entanglement* within a quantum state

- Formally, the *tangle is the square of the concurrence*, which for a 2-qubit pure state, $|\psi\rangle$, is given by:

$$\text{Concurrence}|\psi\rangle = |\langle\psi|\tilde{\psi}\rangle|$$

where $|\tilde{\psi}\rangle$ is the **spin-flipped version** of $|\psi\rangle$ which is defined as

$$|\tilde{\psi}\rangle = (Y \otimes Y)|\psi^*\rangle$$

Entanglement Measure

- If

$$|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle \quad \rightarrow \quad |\psi^*\rangle = a^*|00\rangle + b^*|01\rangle + c^*|10\rangle + d^*|11\rangle$$

and therefore

$$|\tilde{\psi}\rangle = (Y \otimes Y)|\psi^*\rangle = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} a^* \\ b^* \\ c^* \\ d^* \end{bmatrix} = \begin{bmatrix} 0 \cdot \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} & -i \cdot \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\ i \cdot \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} & 0 \cdot \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} a^* \\ b^* \\ c^* \\ d^* \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a^* \\ b^* \\ c^* \\ d^* \end{bmatrix} = \begin{bmatrix} -d^* \\ c^* \\ b^* \\ -a^* \end{bmatrix} = -d^*|00\rangle + c^*|01\rangle + b^*|10\rangle - a^*|11\rangle$$

Entanglement Measure

$$|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$$

- Thus

$$|\tilde{\psi}\rangle = (Y \otimes Y)|\psi^*\rangle = -d^*|00\rangle + c^*|01\rangle + b^*|10\rangle - a^*|11\rangle$$

and therefore the concurrence of a general 2-qubit state $|\psi\rangle$ is given by:

$$\text{Concurrence}|\psi\rangle = |\langle\psi|\tilde{\psi}\rangle| = |2b^*c^* - 2a^*d^*| = 2|b^*c^* - a^*d^*|$$

Entanglement Measure $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$

- If $|\psi\rangle$ is *separable*

$$|\psi\rangle = |\psi_1\rangle|\psi_0\rangle = (\alpha_1|0\rangle + \beta_1|1\rangle)(\alpha_0|0\rangle + \beta_0|1\rangle) = \alpha_1\alpha_0|00\rangle + \alpha_1\beta_0|01\rangle + \beta_1\alpha_0|10\rangle + \beta_1\beta_0|11\rangle$$

then

$$a = \alpha_1\alpha_0 \quad b = \alpha_1\beta_0 \quad c = \beta_1\alpha_0 \quad d = \beta_1\beta_0$$

- Hence, if *a*, *b*, *c* and *d* are different from zero, the *concurrence* of a general 2-qubit state $|\psi\rangle$ is given by:

$$\text{Concurrence}|\psi\rangle = |\langle\psi|\tilde{\psi}\rangle| = |2b^*c^* - 2a^*d^*| = 0 \quad \rightarrow$$

Unentangled states have a concurrence of zero

Entanglement Measure

- At the other extreme, under the spin-flip transformation *maximally entangled states*, such as Bell states, remain invariant up to an unimportant overall phase

- To see this, the four Bell states are given by

$$\left|\beta_{00}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|00\right\rangle + \left|11\right\rangle\right) \qquad -\left|\beta_{00}\right\rangle$$

$$\left|\beta_{01}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|01\right\rangle + \left|10\right\rangle\right) \qquad \left|\beta_{xy}\right\rangle = \left(Y \otimes Y\right)\left|\beta_{xy}^{*}\right\rangle \qquad +\left|\beta_{01}\right\rangle$$

$$\rightarrow \qquad \rightarrow \qquad \text{Concurrence}\left|\beta_{xy}\right\rangle = \left|\left\langle \beta_{xy} \middle| \beta_{xy} \right\rangle\right| = 1$$

$$\left|\beta_{10}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|00\right\rangle - \left|11\right\rangle\right) \qquad +\left|\beta_{10}\right\rangle$$

$$\left|\beta_{11}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|01\right\rangle - \left|10\right\rangle\right) \qquad -\left|\beta_{11}\right\rangle$$

Entanglement Measure

- Hence, the overlap between a maximally entangled state and its spin-flipped counterpart is unity, which is the most it can be, implying that maximally entangled states have a concurrence of one

- Thus, the *tangle*, as defined above, provides a quantitative measure for the *degree of entanglement* within a pure 2-qubit state

- Generalizations of tangle to mixed states (will be defined later) and multi-partite states are beyond the scope of these seminars

Entanglement Measure

- Example

$$|\psi\rangle = \sqrt{\frac{1}{3}}|00\rangle + \sqrt{\frac{2}{3}}|11\rangle \quad \rightarrow \quad \begin{cases} a = a^* = \sqrt{\frac{1}{3}} \\ b = b^* = 0 \\ c = c^* = 0 \\ d = d^* = \sqrt{\frac{2}{3}} \end{cases}$$

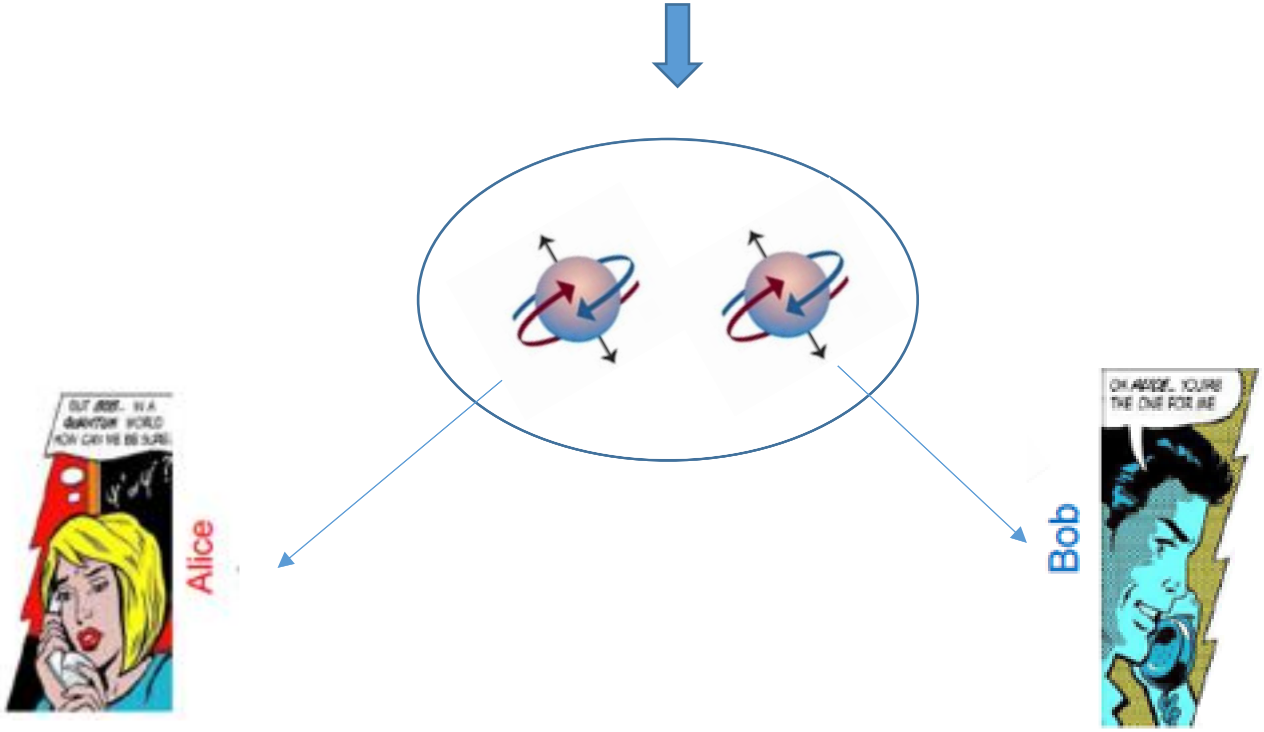
- Thus

$$\text{Concurrence}|\psi\rangle = \left|\langle\psi|\tilde{\psi}\rangle\right| = |2b^*c^* - 2a^*d^*| = \left|0 - 2\frac{\sqrt{2}}{3}\right| \cong 0.93$$

Entanglement

qubits *A* and *B*

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|0_A 0_B\rangle + |1_A 1_B\rangle\right]$$

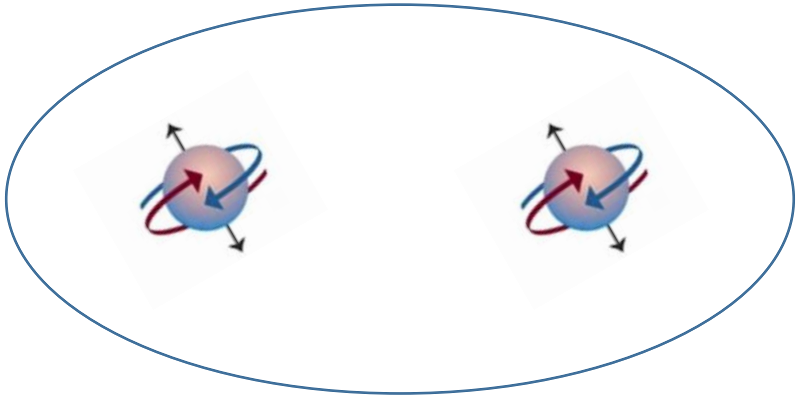


Entanglement

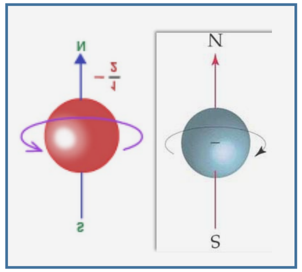
- Upon measuring the first qubit, Alice obtains two possible results
  - → 0 with probability ½, leaving the post-measurement state $|\varphi\rangle = |{\color{red}0}_{\color{red}A}{\color{blue}0}_{\color{blue}B}\rangle$
  - → 1 with probability ½, leaving the post-measurement state $|\psi\rangle = |{\color{red}1}_{\color{red}A}{\color{blue}1}_{\color{blue}B}\rangle$

qubits A and B

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|0_A 0_B\rangle + |1_A 1_B\rangle\right]$$



$|\varphi\rangle = |0_A 0_B\rangle$

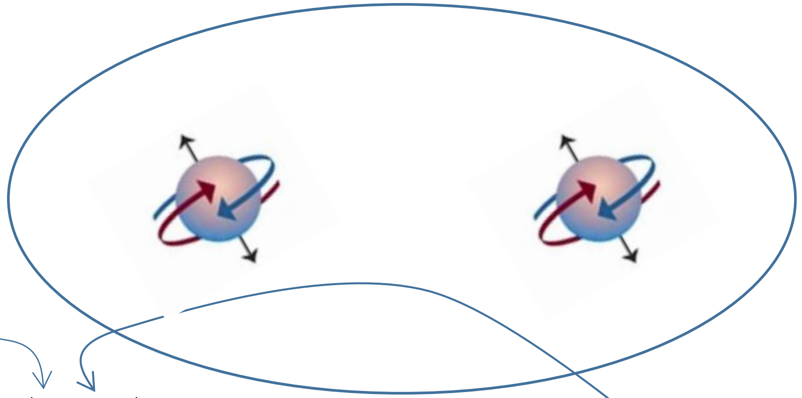


Assume that the Alice outcome is 0

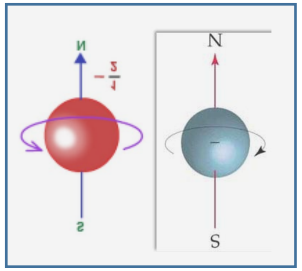
Entanglement

qubits *A* and *B*

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|0_A 0_B\rangle + |1_A 1_B\rangle\right]$$



$$|\varphi\rangle = |0_A 0_B\rangle$$



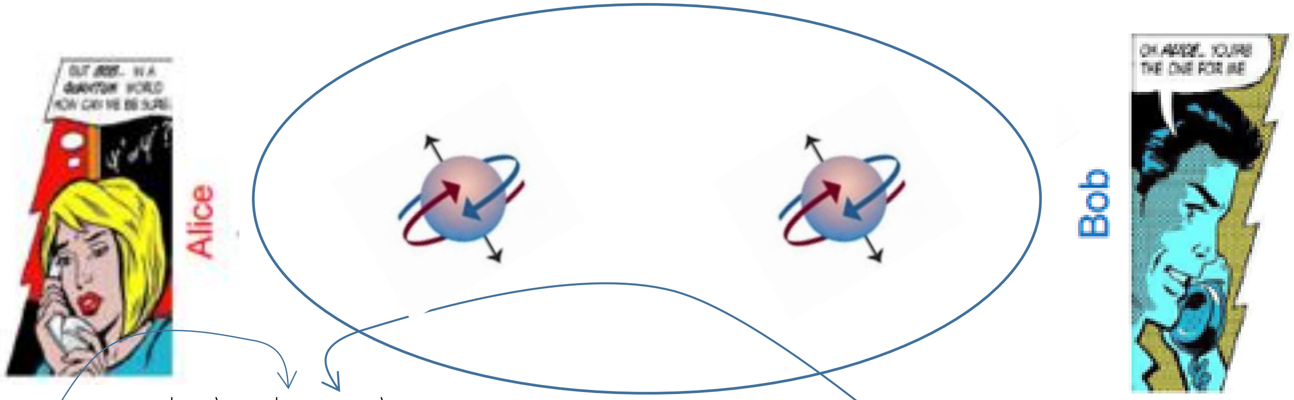
A measurement by Bob of the **second qubit** always gives the same result as the measurement by Alice of the **first qubit** (outcomes are *correlated*)

Einstein's "Spooky Action at a Distance"

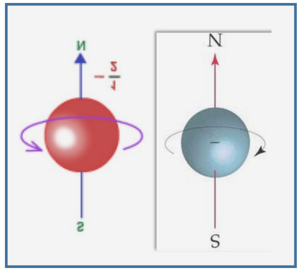
Entanglement

qubits *A* and *B*

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|0_A 0_B\rangle + |1_A 1_B\rangle\right]$$



$$|\varphi\rangle = |0_A 0_B\rangle$$



A measurement by Bob of the **second qubit** always gives the same result as the measurement by Alice of the **first qubit** (outcomes are *correlated*)

Einstein's "Spooky Action at a Distance"

Non-Locality

- *Entangled states* are also said to be *non-local*, meaning that if you are in a room with only one of the two qubit you do not have *full control* over what happens to the data there; the owner of the other qubit in a different room may measure his qubit and affect your data (i.e. qubit state) even though you have done nothing

- Furthermore, if you measure the state of your qubit, your efforts are not confined to your room but extend to the outside world where the other qubit is located

Locality

- Likewise, *separable states* are considered *local*, since they do allow full segregation of the actions on separate qubit states

- Each observer has total control of the destiny of his qubit, and his actions does not affect the other observer

Monogamy of Maximal Entanglement

- Consider the three-qubit state $|\psi\rangle$ that is associated with qubits A, B, and C held by Alice, Bob, and Charlie
- Suppose that *A* and *B* form a **(maximally entangled)** *EPR* pair
- Then, we will show that $|\psi\rangle$ can only exist in a state with the following structure:

$$|\psi\rangle = |EPR\rangle_{AB} \otimes |\phi\rangle_C = \frac{|0_A 0_B\rangle + |1_A 1_B\rangle}{\sqrt{2}} \otimes |\phi\rangle_C$$

for some valid quantum state $|\phi\rangle_C$

$$|EPR\rangle = \begin{cases} \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \\ \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right) \\ \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right) \\ \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right) \end{cases}$$

- By the definition of entanglement, this implies that *C* must be completely disentangled from *A* and *B*

Monogamy of Maximal Entanglement

- To maintain maximal entanglement between *A* and *B*, and to ensure that *C* is also entangled with both *A* and *B*, the state structure of $|\psi\rangle$ must be

$$|\psi\rangle = |00\rangle \otimes \left(\alpha_0|0\rangle + \alpha_1|1\rangle\right) + |11\rangle \otimes \left(\beta_0|0\rangle + \beta_1|1\rangle\right)$$

  i.e., the *EPR* state and the state of *C* cannot be separable

Monogamy of Maximal Entanglement

- When measured in the standard basis, *A* and *B* collapse to the states $|00\rangle$ and $|11\rangle$ with probability 1/2 each if they must remain **maximally entangled**
- It follows that:

$$|\psi\rangle = |00\rangle \otimes \left(\alpha_0|0\rangle + \alpha_1|1\rangle\right) + |11\rangle \otimes \left(\beta_0|0\rangle + \beta_1|1\rangle\right) \qquad (1)$$

for some $\alpha_0,\ \alpha_1,\ \beta_0,\ \beta_1 \in \mathbb{C}$ such that

$$|\alpha_0|^2 + |\alpha_1|^2 = |\beta_0|^2 + |\beta_1|^2 = \frac{1}{2} \qquad (2)$$

Monogamy of Maximal Entanglement

- Let's justify (2) by rewriting (1) as follows:

$$|\psi\rangle = \left(\alpha_0|00\rangle \otimes |0\rangle + \alpha_1|00\rangle \otimes |1\rangle\right) + \left(\beta_0|11\rangle \otimes |0\rangle + \beta_1|11\rangle \otimes |1\rangle\right)$$

- If Alice measure $|0\rangle$, the state $|\psi\rangle$ collapses to state

$$\alpha_0|00\rangle \otimes |0\rangle + \alpha_1|00\rangle \otimes |1\rangle$$

  with probability $|\alpha_0|^2 + |\alpha_1|^2$

- If Alice measure $|1\rangle$ the state $|\psi\rangle$ collapses to state

$$\beta_0|11\rangle \otimes |0\rangle + \beta_1|11\rangle \otimes |1\rangle$$

  with probability $|\beta_0|^2 + |\beta_1|^2$

Monogamy of Maximal Entanglement

- If Alice and Bob should remain maximally entangled, after Charlie got entangled with Alice and Bob, then the following equalities should hold

$$|\alpha_0|^2 + |\alpha_1|^2 = |\beta_0|^2 + |\beta_1|^2 = \frac{1}{2}$$

Monogamy of Maximal Entanglement

- We can rewrite the states of A and B in terms of diagonal basis vectors $|+\rangle$ and $|-\rangle$ by keeping into consideration that

$$|0\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}, \quad |1\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}} \qquad \rightarrow$$

$$|00\rangle = \left(\frac{|+\rangle + |-\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|+\rangle + |-\rangle}{\sqrt{2}}\right) = \frac{1}{2}\left(|++\rangle + |+-\rangle + |-+\rangle + |--\rangle\right)$$

$$|11\rangle = \left(\frac{|+\rangle - |-\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|+\rangle - |-\rangle}{\sqrt{2}}\right) = \frac{1}{2}\left(|++\rangle - |+-\rangle - |-+\rangle + |--\rangle\right) \qquad (3)$$

$$\rightarrow \quad \frac{|0_A 0_B\rangle + |1_A 1_B\rangle}{\sqrt{2}} = |+_A +_B\rangle + |-_A -_B\rangle$$

Monogamy of Maximal Entanglement

- By substituting (3) in (1) it follows:

$$
\begin{aligned}
|\psi\rangle = & \frac{1}{2}\big(|++\rangle + |+-\rangle + |-+\rangle + |--\rangle\big) \otimes \big(\alpha_0 |0\rangle + \alpha_1 |1\rangle\big) \\
& + \frac{1}{2}\big(|++\rangle - |+-\rangle - |-+\rangle + |--\rangle\big) \otimes \big(\beta_0 |0\rangle + \beta_1 |1\rangle\big)
\end{aligned}
$$

$$
\begin{aligned}
= & \frac{1}{2}\big(|++\rangle + |--\rangle\big) \otimes \big((\alpha_0 + \beta_0)|0\rangle + (\alpha_1 + \beta_1)|1\rangle\big) \\
& + \frac{1}{2}\big(|+-\rangle + |-+\rangle\big) \otimes \big((\alpha_0 - \beta_0)|0\rangle + (\alpha_1 - \beta_1)|1\rangle\big)
\end{aligned}
$$

Monogamy of Maximal Entanglement

- Being maximally entangled, *A* and *B* collapse to one of the two states $|++\rangle$ or $|--\rangle$ when measured in the diagonal basis
- The probability of observing outcomes $|+-\rangle$ or $|-+\rangle$ is zero
- Therefore, according to the equation above, it must be the case that $\alpha_0 - \beta_0 = 0$ and $\alpha_1 - \beta_1 = 0$
- It follows immediately that $\alpha_0 = \beta_0$ and $\alpha_1 = \beta_1$

Monogamy of Maximal Entanglement

- We can rewrite our expression for $|\psi\rangle$ accordingly:

$$
\begin{aligned}
|\psi\rangle &= \left(|++\rangle + |--\rangle\right) \otimes \left(\alpha_0 |0\rangle + \alpha_1 |1\rangle\right) \\
&= |EPR\rangle_{AB} \otimes \left(\sqrt{2}\alpha_0 |0\rangle + \sqrt{2}\alpha_1 |1\rangle\right) \\
&= |EPR\rangle_{AB} \otimes |\phi\rangle_C
\end{aligned}
$$

Monogamy of Maximal Entanglement

- This shows that the original state can be written as a product of a pure state in *AB* and a pure state in *C*, which means that the *EPR* state in qubits *A* and *B* is not entangled with the qubit *C*

Monogamy of Maximal Entanglement

- Finally, note if Alice and Bob are partially entangled, then it is possible for there to be some entanglement with Charlie
- A proper treatment of this involves expressing quantum states, not as kets or vectors, but as **density matrices**, which also allow probabilistic mixtures of kets, but that is beyond the scope of this course