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Quantum Computing and Quantum Internet

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Linear Algebra

- The set of **natural** numbers $\{1,2,3,\ldots\}$ is denoted by $\mathbb{N}$
- The set of integers $\{\ldots,-2,-1,0,1,2,\ldots\}$ is denoted by $\mathbb{Z}$
- $\mathbb{Q}$ denotes the set of **rational** numbers
- Finally, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of **real** numbers and **complex** numbers, respectively
- Observe that

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

Hilbert Space Definition

- A **Hilbert Space** is a real or complex *vector space* that
  - has an *inner product* and
  - is *complete*

- *Completeness here means that any Cauchy sequence of vectors in the space converges to some vector also in the space (this property will not be used in the course)*

Linear Algebra

- The vector spaces encountered in physics are mostly **real** vector spaces and **complex** vector spaces
- **Classical mechanics** and **electrodynamics** are formulated mainly in **real vector spaces** while **quantum mechanics** (and hence this course) is founded on **complex vector spaces**

Linear Algebra

- A good understanding of quantum mechanics is based upon a solid grasp of elementary linear algebra
- *In the next two lectures, we review some basic concepts from linear algebra and describe the standard notations that are used for these concepts in the study of quantum mechanics*
- These notations are summarized in the table reported in the next slide, with the quantum notation in the left column, and the linear-algebraic description in the right column
- You may like to glance at the table and see how many of the concepts in the right column you recognize

Linear Algebra

Notation	Description
z^*	Complex conjugate of the complex number z.
	$(1+i)^* = 1 - i$
$\|\psi\rangle$	Vector. Also known as a *ket*.
$\langle\psi\|$	Vector dual to $\|\psi\rangle$. Also known as a *bra*.
$\langle\varphi\|\psi\rangle$	Inner product between the vectors $\|\varphi\rangle$ and $\|\psi\rangle$.
$\|\varphi\rangle \otimes \|\psi\rangle$	Tensor product of $\|\varphi\rangle$ and $\|\psi\rangle$.
$\|\varphi\rangle\|\psi\rangle$	Abbreviated notation for tensor product of $\|\varphi\rangle$ and $\|\psi\rangle$.
A^*	Complex conjugate of the A matrix.
A^T	Transpose of the A matrix.
$A^\dagger$	Hermitian conjugate or adjoint of the A matrix, $A^\dagger = (A^T)^*$.
	$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^\dagger = \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix}$.
$\langle\varphi\|A\|\psi\rangle$	Inner product between $\|\varphi\rangle$ and $A\|\psi\rangle$.
	Equivalently, inner product between $A^\dagger\|\varphi\rangle$ and $\|\psi\rangle$.

Vector Spaces

- Let K be a **field**, which is a set where ordinary **addition**, **subtraction**, **multi plication** and **division** are defined
- The sets $\mathbb{R}$ and $\mathbb{C}$ are **the only fields** which we will be concerned with in this course
- A vector space is a set where the addition of two vectors and a multiplication by an element of K, so-called a scalar, are defined

Vector Spaces

Definition: A vector space V is a set with the following properties

1. For any $u, v \in V$, their sum $u + v \in V$.
2. For any $u \in V$ and $c \in K$, their scalar multiple $cu \in V$.
3. $(u + v) + w = u + (v + w)$ for any $u, v, w \in V$.
4. $u + v = v + u$ for any $u, v \in V$.
5. There exists an element $0 \in V$ such that $u + 0 = u$ for any $u \in V$. This element **0** is called the **zero-vector**.
6. For any element $u \in V$, there exists an element $v \in V$ such that $u + v = 0$. The vector v is called the **inverse** of u and denoted by $-u$.

continue to the next slide $\rightarrow$

Vector Spaces

7. $c(x + y)=cx + cy$ for any $c \in K$, $u, v \in V$.
8. $(c + d)u = cu + du$ for any $c, d \in K$, $u \in V$.
9. $(cd)u = c(du)$ for any $c, d \in K$, $u \in V$.
10. Let 1 be the unit element of K. Then $1u = u$, for any $u \in V$.

Vector Spaces

- In our lectures, we will be concerned mostly with the complex vector space $\mathbb{C}^n$
- There are occasional instances where the real vector space $\mathbb{R}^n$ is considered.
- An element of $V = \mathbb{C}^n$ will be denoted by $|z\rangle$, instead of *z*, and expressed as a column of *n* complex numbers z_i $(1 \leq i \leq n)$ as

$$|z\rangle = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}, \qquad z_i \in \mathbb{C}$$

Vector Spaces

- It is often written as a **transpose** of a **row vector**, as $|z\rangle = [z_1, z_2, \ldots, z_n]^T$, to save space
- The integer $n \in \mathbb{N}$ is called the **dimension** of the vector space
- An element $|z\rangle$ is also called a **ket vector** or simply a **ket**
- From now on we will use the letter ***V***, instead of ***H***, to denote a Hilbert space

Vector Spaces

- For $|z\rangle, |z'\rangle \in \mathbb{C}^n$ and $z \in \mathbb{C}$, vector **addition** and **scalar multiplication** are defined as

$$|z\rangle = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}, \quad |z'\rangle = \begin{bmatrix} z_1' \\ z_2' \\ \vdots \\ z_n' \end{bmatrix} \quad \rightarrow \quad |z\rangle + |z'\rangle = \begin{bmatrix} z_1 + z_1' \\ z_2 + z_2' \\ \vdots \\ z_n + z_n' \end{bmatrix}, \quad z|z\rangle = \begin{bmatrix} zz_1 \\ zz_2 \\ \vdots \\ zz_n \end{bmatrix},$$

respectively

- All the components of the zero-vector 0 are zero
- We can verify that these definitions satisfy all the axioms in the definition of a vector space

Vector Spaces

- Note, in particular, that any **linear combination** $c|z\rangle + c^{'}|z^{'}\rangle$ of vectors $|z\rangle$, $|z^{'}\rangle \in \mathbb{C}^n$ with $c, c^{'} \in \mathbb{C}$ is also an element of $\mathbb{C}^n$

Bases and Linear Independence

- A *spanning set* for a vector space is a set of vectors $\{v_1,\ldots,v_n\}$ such that any vector $|v\rangle$ in the vector space can be written as a **linear combination** of vectors in that set.

$$|v\rangle = \sum_{k=1}^{n} a_k |v_k\rangle$$

- For example, a *spanning set* for the vector space $\mathbb{C}^2$ is the set

$$|v_1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |v_2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Bases and Linear Independence

$$|v_1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |v_2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- This is because any vector

$$|v\rangle = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

in $\mathbb{C}^2$ can be written as a linear combination

$$|v\rangle = a_1|v_1\rangle + a_2|v_2\rangle$$

of vectors $|v_1\rangle$ and $|v_2\rangle$

- *Proof*

$$a_1|v_1\rangle + a_2|v_2\rangle = a_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + a_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1 + 0 \\ 0 + a_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = |v\rangle \qquad \blacksquare$$

Bases and Linear Independence

- Generally, a vector space may have many different spanning sets
- A second spanning set for the vector space $\mathbb{C}^2$ is the set

$$|v_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad |v_2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

since an arbitrary vector

$$|v\rangle = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

can be written as a linear combination of $|v_1\rangle$ and $|v_2\rangle$

$$|v\rangle = \frac{a_1 + a_2}{\sqrt{2}}|v_1\rangle + \frac{a_1 - a_2}{\sqrt{2}}|v_2\rangle$$

Bases and Linear Independence

- A set of non-zero vectors $\{v_1,\ldots,v_n\}$ are *linearly dependent* if there exists a set of complex numbers $\{a_1,\ldots,a_n\}$ with $a_i \neq 0$ for at least one value of *i*, such that

$$a_1|v_1\rangle + a_2|v_2\rangle + \cdots + a_n|v_n\rangle = 0$$

- A set of vectors is *linearly independent* if it is not linearly dependent.
- It can be shown that any *two sets* of linearly independent vectors which **span** a vector space V contain *the same number of elements*
- We call such a set a **basis** for V. Furthermore, such a basis set always exists
- The number of elements in the basis is defined to be the *dimension* of V

Bases and Linear Independence

- For example the following vectors:

$$|v_1\rangle = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad |v_2\rangle = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad |v_3\rangle = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

  are **linearly dependent** since

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 0$$

  for $a_1 = a_2 = 1,\ a_3 = -1$, i.e., all coefficients differ from zero

Bases and Linear Independence

- On the other hand, if we take the elements of the standard basis

$$|v_1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |v_2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

then, the only way for the following equality to hold

$$a_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$$

is $a_1 = a_2 = 0$

Bases and Linear Independence

- In this course we will only be interested in **finite dimensional vector spaces**
- There are many interesting and often difficult questions associated with **infinite-dimensional vector spaces**
- These questions are not covered in the course, so you won't have to worry about them

Bras, Kets, Inner and Outer Products

- For every *ket* $|\psi\rangle$, which can be thought of as a shorthand notation for a *column vector,* there is a corresponding *bra* $\langle\psi|$, the **conjugate transpose** of $|\psi\rangle$
- $\langle\psi|$ can be though of as shorthand for a **complex conjugate** of each component of the transpose of $|\psi\rangle$
- The conjugate transpose operation is denoted by † (**dagger**):

$$\underbrace{|\psi\rangle = a|0\rangle + b|1\rangle}_{\text{ket}} = \begin{bmatrix} a \\ b \end{bmatrix} \qquad \underbrace{|\psi\rangle^\dagger = \begin{bmatrix} a \\ b \end{bmatrix}^\dagger \equiv \langle\psi| = a^*\langle 0| + b^*\langle 1| = \begin{bmatrix} a^*, & b^* \end{bmatrix}}_{\text{bra}}$$

Note
If z=x+iy is a complex number with real part x and part y, then the complex conjugate of z is z*=x-iy

Examples

Here are some kets (not necessarily normalized) and their associated bras.

$$|\psi\rangle = \begin{pmatrix} 1 + i \\ \sqrt{2} - 2i \end{pmatrix} \longrightarrow \langle\psi| = \left(1 - i\,,\ \sqrt{2} + 2i\right)$$

$$|\psi\rangle = \begin{pmatrix} \sqrt{3}/2 \\ -i \end{pmatrix} \longrightarrow \langle\psi| = \left(\sqrt{3}/2\,,\ i\right)$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 5i \\ 0 \end{pmatrix} \longrightarrow \langle\psi| = \frac{1}{\sqrt{2}}\left(-5i\,,\ 0\right)$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \longrightarrow \langle\psi| = \frac{1}{\sqrt{2}}\left(1\,,\ 1\right)$$

$$|+\rangle \longrightarrow \langle +|$$

Bras, Kets, Inner and Outer Products

- The *ket* and the *bra* contain *equivalent information* about the quantum state in question
- What is the purpose of introducing *bra* vectors into the discussion if they don't contain any new information about the quantum state?
- It turns out that **products** of *bras* and *kets* give us insight into similarities between two quantum states

Inner Product

- For a pair of qubits in states

  $$|\psi\rangle = a|0\rangle + b|1\rangle, \quad |\phi\rangle = c|0\rangle + b|1\rangle$$

  we can define their *inner product*

  $$\langle\psi|\phi\rangle = (\langle\psi|)\cdot(|\phi\rangle) = \begin{bmatrix} a^* & b^* \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} = a^*c + b^*d$$

- Note that the result is just a number, or *scalar*
- So, an inner product is also called *scalar product*
- We call a vector space equipped with an *inner product* an **inner product space**

Inner Product

- Discussions of quantum mechanics often refer to **Hilbert space**
- In the **finite dimensional** complex vector spaces that come up in quantum computation and quantum information, *a Hilbert space is exactly the same thing as an inner product space*
- From now on we use the two terms interchangeably, preferring the term **Hilbert space**
- In **infinite dimensions** Hilbert spaces satisfy additional technical restrictions above and beyond inner product spaces, which we will not need to worry about

Norm of a Vector

- We define the *norm* of a vector $|\psi\rangle$ by

$$\||\psi\rangle\| \equiv \sqrt{\langle\psi|\psi\rangle}$$

- A *unit vector* is a vector $|\psi\rangle$ such that $\||\psi\rangle\| = 1$

- We also say that $|\psi\rangle$ is *normalized* if $\||\psi\rangle\| = 1$

- It is convenient to talk of *normalizing* a vector by dividing by its norm; thus $|\psi\rangle / \||\psi\rangle\|$ is the *normalized* form of $|\psi\rangle$, for any *non-zero* vector $|\psi\rangle$

Exercise

Prove that $\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^*$ where $|\psi\rangle = a|0\rangle + b|1\rangle$ and $|\phi\rangle = c|0\rangle + d|1\rangle$

Proof

$$\langle\psi|\phi\rangle = \begin{bmatrix} a^* & b^* \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} = a^*c + b^*d$$

$$\langle\phi|\psi\rangle = \begin{bmatrix} c^* & d^* \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = c^*a + d^*b$$

$$\langle\phi|\psi\rangle^* = (c^*a + d^*b)^* = ca^* + db^* = a^*c + b^*d = \langle\psi|\phi\rangle \qquad \blacksquare$$

Inner Product

- $\langle\psi|\phi\rangle$ is a scalar which varies from *zero* for *orthogonal states* to *one* for *identical normalized states*
- For this reason, $\langle\psi|\phi\rangle$ is called the *overlap* between (normalized) states $|\psi\rangle$ and $|\phi\rangle$

Inner Product for Identical States

- When

$$|\psi\rangle = |\phi\rangle = a|0\rangle + b|1\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$$

their *inner product* is

$$\langle\psi|\psi\rangle = (\langle\psi|)\cdot(|\psi\rangle) = [a^* \quad b^*]\cdot\begin{bmatrix} a \\ b \end{bmatrix} = a^*a + b^*b = |a|^2 + |b|^2 = 1$$

Inner Product for Orthogonal States

- If $|\psi\rangle$ and $|\phi\rangle$ are $|0\rangle$ and $|1\rangle$

$$|\psi\rangle = |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad |\phi\rangle = |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \rightarrow$$

their inner product is

$$\langle 0|1\rangle = (\langle 0|)\cdot(|1\rangle) = [1 \quad 0]\cdot\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1\cdot 0 + 0\cdot 1 = 0 \quad \rightarrow \quad |0\rangle \text{ and } |1\rangle \text{ are orthogonal}$$

Second Question/Orthogonality of Opposite Points

Consider a general qubit state $|\psi\rangle$

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

and $|\chi\rangle$ corresponding to the opposite point on the Bloch sphere

$$\begin{aligned}|\chi\rangle &= \cos\left(\frac{\pi-\theta}{2}\right)|0\rangle + e^{i(\phi+\pi)}\sin\left(\frac{\pi-\theta}{2}\right)|1\rangle \\ &= \cos\left(\frac{\pi-\theta}{2}\right)|0\rangle - e^{i\phi}\sin\left(\frac{\pi-\theta}{2}\right)|1\rangle\end{aligned}$$

$$e^{i\ \phi+\pi} = e^{i\phi}e^{i\pi} = -e^{i\phi}$$

$$e^{i\pi} = -1$$

So

$$\langle\chi|\psi\rangle = \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\pi-\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\pi-\theta}{2}\right)$$

Orthogonality of Opposite Points

$$\sin(\alpha + \beta) = \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta$$
$$\sin(\alpha - \beta) = \sin\alpha \cdot \cos\beta - \cos\alpha \cdot \sin\beta$$
$$\cos(\alpha + \beta) = \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta$$
$$\cos(\alpha - \beta) = \cos\alpha \cdot \cos\beta + \sin\alpha \cdot \sin\beta$$
$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$
$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$
$$\cot(\alpha + \beta) = \frac{1 - \tan\alpha\tan\beta}{\tan\alpha + \tan\beta}$$
$$\cot(\alpha - \beta) = \frac{1 + \tan\alpha\tan\beta}{\tan\alpha - \tan\beta}$$

$$\langle\chi|\psi\rangle = \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\pi-\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\pi-\theta}{2}\right)$$

But $\cos(a+b) = \cos a \cos b - \sin a \sin b$, so

$$\langle\chi|\psi\rangle = \cos\frac{\pi}{2} = 0$$

and opposite points correspond to orthogonal qubit states.

Note that in the coordinate system we used in the derivation of the Bloch sphere, with $\theta' = \theta/2$, the two points are also orthogonal - $90°$ apart.

Orthonormality

- The two properties, *normalized* and *orthogonal*, can be combined as a single word, *orthonormal*

- So $\{|0\rangle, |1\rangle\}$ are *orthonormal* because each state is individually normalized, and they are orthogonal to each other, i.e.

$$\langle i | j \rangle = \delta_{ij}, \qquad \forall i, j \in \{0,1\}$$

Gram–Schmidt Procedure

- Suppose $\{w_1,\ldots,w_d\}$ is a basis set for some vector space *V* with an inner product
- There is a useful method, the **Gram–Schmidt procedure**, which can be used to produce an orthonormal basis set $\{v_1,\ldots,v_d\}$ for the vector space *V*
- Define $|v_1\rangle = |w_1\rangle / \||w_1\rangle\|$, and for 1 ≤ *k* ≤ *d*−1 define $|v_{k+1}\rangle$ inductively by

$$|v_{k+1}\rangle \equiv \frac{|w_{k+1}\rangle - \sum_{i=1}^{k} \langle v_i | w_{k+1}\rangle |v_i\rangle}{\left\| |w_{k+1}\rangle - \sum_{i=1}^{k} \langle v_i | w_{k+1}\rangle |v_i\rangle \right\|}$$

Gram–Schmidt Procedure

- It is not difficult to verify that the vectors $\{v_1,\ldots,v_d\}$ form an orthonormal set which is also a basis for V
- Thus, any **finite** dimensional vector space of dimension ***d*** has an orthonormal basis, $\{v_1,\ldots,v_d\}$

Outer Product

- Consider two states

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad |\phi\rangle = \gamma|0\rangle + \delta|1\rangle$$

- Instead of multiplying $|\psi\rangle$ and $|\phi\rangle$ as an inner product $\langle\psi|\phi\rangle$, where the bra is on the left and the ket is on the right, another way to multiply them is by having the ket on the left and the bra on the right, which is called an **outer product**

$$|\psi\rangle\langle\phi| = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \begin{bmatrix} \gamma^* & \delta^* \end{bmatrix} = \begin{bmatrix} \alpha\gamma^* & \alpha\delta^* \\ \beta\gamma^* & \beta\delta^* \end{bmatrix}$$

- So, the outer product of two qubit states is a 2X2 matrix

Amplitude and Inner Product

- Given a state

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

we have

$$\langle 0|\psi\rangle = a\underbrace{\langle 0|0\rangle}_{=1} + b\underbrace{\langle 0|1\rangle}_{=0} = a$$

$$\langle 1|\psi\rangle = a\underbrace{\langle 1|0\rangle}_{=0} + b\underbrace{\langle 1|1\rangle}_{=1} = b$$

$$\rightarrow \quad |\psi\rangle = \langle 0|\psi\rangle|0\rangle + \langle 1|\psi\rangle|1\rangle$$

Completeness Relation

- In general, for any orthonormal basis $\{|a\rangle, |b\rangle\}$, the state of a qubit can be written as

$$|\psi\rangle = \alpha|a\rangle + \beta|b\rangle \quad \text{where} \quad \alpha = \langle a|\psi\rangle,\ \beta = \langle b|\psi\rangle$$

- Substituting these values

$$|\psi\rangle = \underbrace{\langle a|\psi\rangle}_{\textit{scalar}}|a\rangle + \underbrace{\langle b|\psi\rangle}_{\textit{scalar}}|b\rangle$$

Completeness Relation

$$|\psi\rangle = \langle a|\psi\rangle|a\rangle + \langle b|\psi\rangle|b\rangle$$

- As show above, the inner products are just scalars/numbers, so instead of multiply them onto the vectors $|a\rangle$ and $|b\rangle$ on the left, we can equivalently multiply them on the right

$$|\psi\rangle = |a\rangle\underbrace{\langle a|\psi\rangle}_{scalar} + |b\rangle\underbrace{\langle b|\psi\rangle}_{scalar}$$

- Both of these terms are a *ket* times a *bra* times a *ket*

Completeness Relation

- To make this clearer we can write them as

$$\left|\psi\right\rangle = \left|a\right\rangle\left\langle a\right|\left|\psi\right\rangle + \left|b\right\rangle\left\langle b\right|\left|\psi\right\rangle$$

- Now, notice we have two outer products, $\left|a\right\rangle\left\langle a\right|$ and $\left|b\right\rangle\left\langle b\right|$

- Since they are both multiplying $\left|\psi\right\rangle$, we can factor to get

$$\left|\psi\right\rangle = \left(\left|a\right\rangle\left\langle a\right| + \left|b\right\rangle\left\langle b\right|\right)\left|\psi\right\rangle$$

Completeness Relation

- For this to be true for all $|\psi\rangle$, we must have

$$|a\rangle\langle a| + |b\rangle\langle b| = I$$

- This is called the *completeness relation*, and it indicates the state of any qubit can be expressed in terms of $|a\rangle$ and $|b\rangle$, a property we call *completeness*

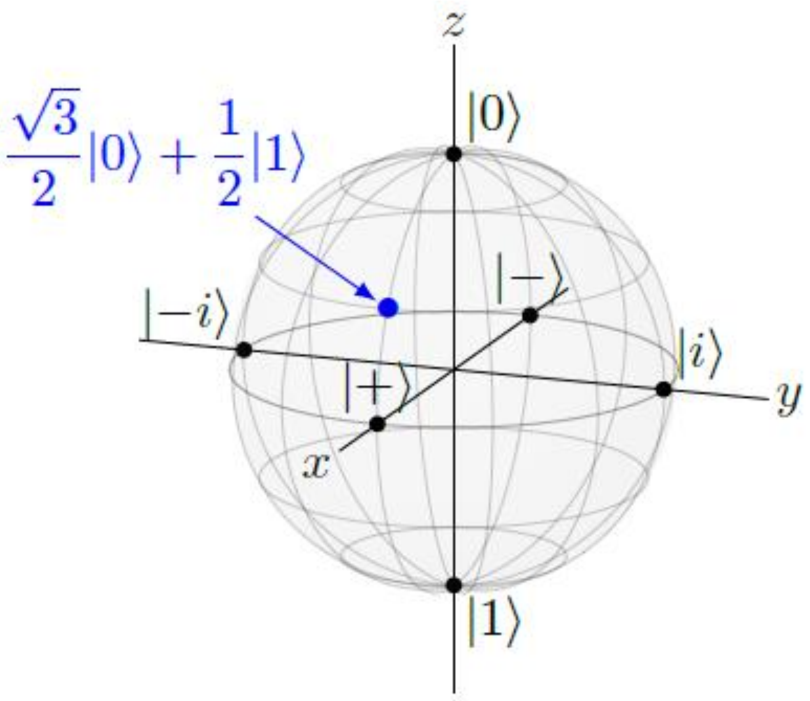
$$\downarrow$$

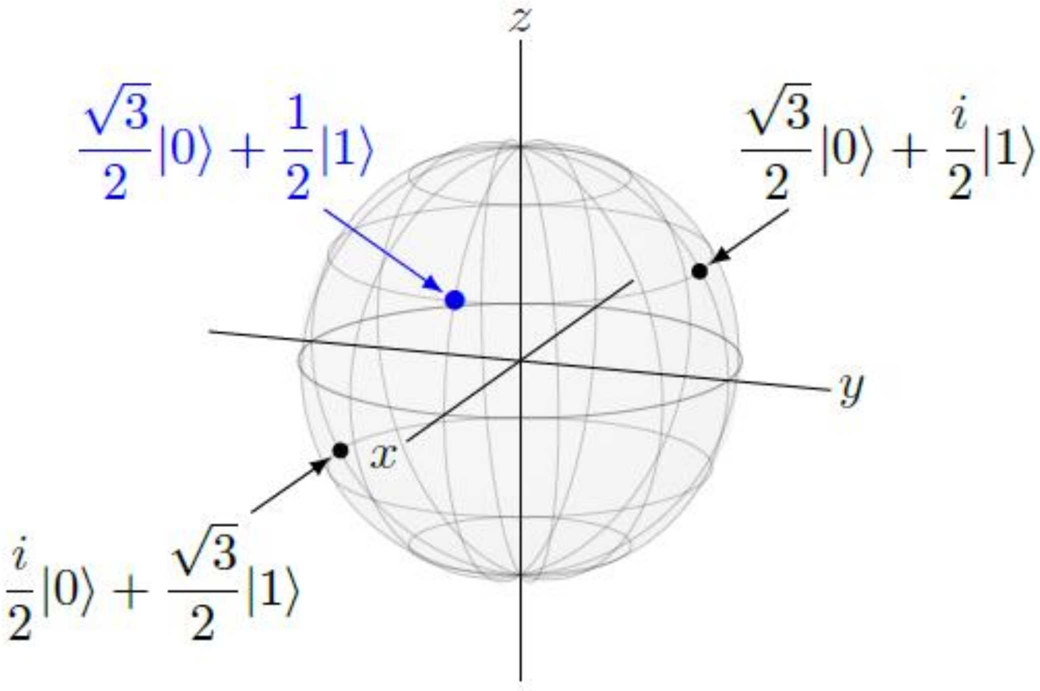
A complete *orthonormal basis* $\{|a\rangle, |b\rangle\}$ satisfies the *completeness relation*

$$|a\rangle\langle a| + |b\rangle\langle b| = I$$

Completeness Relation

- All the bases we have discussed (any two states on opposite sides on the Bloch sphere) are complete





Let's Play with the Outer Product

- Let *V* be a vector space associated with the state of a single-qubit system

- The outer products $|i\rangle\langle j|,\ \forall i, j \in \{0,1\}$ with respect to the standard basis $\{|0\rangle, |1\rangle\}$ are

$$|0\rangle\langle 0| = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$|0\rangle\langle 1| = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$|1\rangle\langle 0| = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$|1\rangle\langle 1| = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Completeness Relation

- Based on the above results, the *completeness relation* can be easily checked

$$|0\rangle\langle 0| + |1\rangle\langle 1| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

- Since $\{|+\rangle, |-\rangle\}$ is an *orthonormal basis*, following the same approach as before we could show that

$$|+\rangle\langle +| + |-\rangle\langle -| = I$$

i.e. $\{|+\rangle, |-\rangle\}$ is a *complete orthonormal basis*

Completeness Relation

- Let's prove that $\{|+\rangle, |-\rangle\}$ is a *complete orthonormal basis*, i.e.

$$|+\rangle\langle+| + |-\rangle\langle-| = I$$

- Proof

$$|+\rangle\langle+| = \left(\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}\right)\left(\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} \qquad |-\rangle\langle-| = \left(\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}\right)\left(\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$$

$$\downarrow$$

$$|+\rangle\langle+| + |-\rangle\langle-| = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = I$$

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

Completeness Relation

- On the other hand, $\{|0\rangle, |+\rangle\}$ is not a *complete orthonormal basis* due to

$$|0\rangle\langle 0| + |+\rangle\langle +| \neq I$$

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Proof

$$|0\rangle\langle 0| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad |+\rangle\langle +| = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\downarrow$$

$$|0\rangle\langle 0| + |+\rangle\langle +| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Outer Product

- The outer product of $|\phi\rangle$ and $|\psi\rangle$ is just the conjugate transpose of the outer product of $|\psi\rangle$ and $|\phi\rangle$

$$|\phi\rangle\langle\psi| = \left(|\psi\rangle\langle\phi|\right)^{\dagger}$$

- *Proof*

$$\left(|\psi\rangle\langle\phi|\right)^{\dagger} = \left(\langle\phi|\right)^{\dagger}\left(|\psi\rangle\right)^{\dagger} = |\phi\rangle\langle\psi| \qquad \blacksquare$$

$$(AB)^{\dagger} = B^{\dagger}A^{\dagger}$$

$$\left\{\begin{array}{l} \langle\phi|^{\dagger} = |\phi\rangle \\ |\psi\rangle^{\dagger} = \langle\psi| \end{array}\right.$$

Outer Product

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad |\phi\rangle = \gamma|0\rangle + \delta|1\rangle$$

- This is an alternative proof that does not take advantage of the properties of matrices

$$|\psi\rangle\langle\phi| = \begin{bmatrix}\alpha \\ \beta\end{bmatrix}\begin{bmatrix}\gamma * & \delta *\end{bmatrix} = \begin{bmatrix}\alpha\gamma * & \alpha\delta * \\ \beta\gamma * & \beta\delta *\end{bmatrix}$$

$$|\psi\rangle\langle\phi|^{\dagger} = \left(\begin{bmatrix}\alpha\gamma * & \alpha\delta * \\ \beta\gamma * & \beta\delta *\end{bmatrix}^{*}\right)^{T} = \left(\begin{bmatrix}\alpha * \gamma & \alpha * \delta \\ \beta * \gamma & \beta * \delta\end{bmatrix}\right)^{T} = \begin{bmatrix}\alpha * \gamma & \beta * \gamma \\ \alpha * \delta & \beta * \delta\end{bmatrix} = \begin{bmatrix}\gamma \\ \delta\end{bmatrix}\begin{bmatrix}\alpha * & \beta *\end{bmatrix} = |\phi\rangle\langle\psi|$$

$\blacksquare$

Linear Operators and Matrices

- A **linear operator** between vector spaces *V* and *W* is defined to be any function $A : V \to W$ that is linear in its inputs

$$A\left(\sum_i a_i |v_i\rangle\right) = \sum_i a_i A\left(|v_i\rangle\right)$$

- Usually, we just write $A|v\rangle$ to denote $A\left(|v\rangle\right)$

- It is clear that once the action of a linear operator *A* on a basis is specified, the action of *A* is completely determined on all inputs

- When we say that a linear operator *A* is defined on a vector space, *V*, we mean that *A* is a linear operator from *V* to *V*

Linear Operators and Matrices

- An important linear operator on any vector space V is the **identity operator**, I_V, defined by the equation $I_V|v\rangle = |v\rangle$ for all vectors $|v\rangle$

- Where no chance of confusion arises, we drop the subscript V and just write I to denote the identity operator

- Another important linear operator is the **zero operator**, which we denote 0

- The zero operator maps all vectors to the zero vector, $0|v\rangle = 0$

Linear Operators and Matrices

- Suppose *V*, *W*, and *X* are vector spaces, and $A : V \to W$ and $B : W \to X$ are linear operators $\left(V \xrightarrow{A} W \xrightarrow{B} X \right)$

- Then we use the notation *BA* to denote the composition of *B* with *A*, defined by

$$(BA)(|v\rangle) = B\left(A(|v\rangle)\right)$$

- Once again, we write $BA|v\rangle$ as an abbreviation for $(BA)(|v\rangle)$

Linear Operators and Matrices

- Assume that $A: V \to V$ is a linear operator defined on vector space V

- Let's choose an arbitrary **orthonormal basis** $\{v_1, \ldots, v_n\}$

- Let $|v\rangle = \sum_{k=1}^{n} a_k |v_k\rangle$ be an arbitrary vector in V

- Linearity implies that $A|v\rangle = \sum_{k=1}^{n} a_k A|v_k\rangle$

- Since $A|v_k\rangle \in V$, it can be expanded as

$$A|v_k\rangle = \sum_{i=1}^{n} |v_i\rangle A_{ik}$$

Linear Operators and Matrices

- By taking the inner product between $\langle v_j |$ and the above equation, we obtain

$$\langle v_j | A | v_k \rangle = \sum_{i=1}^{n} \langle v_j | v_i \rangle A_{ik} = \sum_{i=1}^{n} \delta_{ji} A_{ik} = A_{jk}$$

- The matrix whose entries are the values

$$A_{jk} = \langle v_j | A | v_k \rangle$$

is said to form a **matrix representation** of the operator *A*

- This matrix representation of *A* is **completely equivalent** to the operator *A*, and we will use the **matrix representation** and **abstract operator** viewpoints interchangeably

Linear Operators and Matrices

- It is easy to show that

$$A = \sum_{j,k} A_{jk} \left| v_j \right\rangle \left\langle v_k \right|$$

- By multiplying the **completeness** relation $I = \sum_{i=1}^{n} \left| v_i \right\rangle \left\langle v_i \right|$ from the left and the right on *A* simultaneously, we obtain

$$A = IAI = \sum_{j,k} \left| v_j \right\rangle \underbrace{\left\langle v_j \middle| A \middle| v_k \right\rangle}_{\text{This is a scalar}} \left\langle v_k \right| = \sum_{j,k} \underbrace{\left\langle v_j \middle| A \middle| v_k \right\rangle}_{A_{jk}} \left| v_j \right\rangle \left\langle v_k \right| = \sum_{j,k} A_{jk} \left| v_j \right\rangle \left\langle v_k \right|$$

Let's Play with the Outer Product

$$A_{jk} = \langle v_j | A | v_k \rangle$$

- All two-dimensional linear transformations on *V* can be written down using

$$A = \sum_{j,k} \langle v_j | A | v_k \rangle |v_j\rangle\langle v_k| = \sum_{j,k} A_{jk} |v_j\rangle\langle v_k|$$

- Let's denote by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix} = \begin{bmatrix} \langle 0|A|0\rangle & \langle 0|A|1\rangle \\ \langle 1|A|0\rangle & \langle 1|A|1\rangle \end{bmatrix}$$

the operator *A* represented on the basis $\{|0\rangle, |1\rangle\}$, then

$$A = a|0\rangle\langle 0| + b|0\rangle\langle 1| + c|1\rangle\langle 0| + d|1\rangle\langle 1|$$

Let’s Play with the Outer Product

- Let’s take the matrix representing a linear transformation that exchange $|0\rangle$ and $|1\rangle$

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- The operator represented by the above matrix on the basis $\{|0\rangle, |1\rangle\}$ can be written as follows

$$X = |0\rangle\langle 1| + |1\rangle\langle 0|$$

- As expected, the matrix representing the *X* operator is

$$X = |0\rangle\langle 1| + |1\rangle\langle 0| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Let’s Play with the Outer Product

- By using the Dirac’s notation let’s find how *X* acts on $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$$X|\psi\rangle = \left(|0\rangle\langle 1| + |1\rangle\langle 0|\right)\left(\alpha|0\rangle + \beta|1\rangle\right)$$

$$= \alpha|0\rangle\underbrace{\langle 1|0\rangle}_{=0} + \alpha|1\rangle\underbrace{\langle 0|0\rangle}_{=1} + \beta|0\rangle\underbrace{\langle 1|1\rangle}_{=1} + \beta|1\rangle\underbrace{\langle 0|1\rangle}_{=0} = \alpha|1\rangle + \beta|0\rangle$$

and this proves that *X* exchange $|0\rangle$ with $|1\rangle$

Eigenvectors and Eigenvalues

- An *eigenvector* of a linear operator *A* on a vector space is a **non-zero** vector $|v\rangle$ such that $A|v\rangle = v|v\rangle$, where *v* is a complex number known as the *eigenvalue* of *A* corresponding to $|v\rangle$
- It will often be convenient to use the notation *v* both as a label for the eigenvector and to represent the eigenvalue
- We assume that you are familiar with the elementary properties of eigenvalues and eigenvectors - in particular, how to find them, via the characteristic equation

Eigenvectors and Eigenvalues

- The *characteristic function* is defined to be

$$c(\lambda) = \det|A - \lambda I|,$$

where det is the determinant function for matrices; it can be shown that the characteristic function **depends only** upon the operator A, **and not** on the specific matrix representation used for A

Eigenvectors and Eigenvalues

- The solutions of the *characteristic equation* $\boldsymbol{c(\lambda)=0}$ are the eigenvalues of the operator *A*
- By the fundamental theorem of algebra, every polynomial has at least one complex root, so every operator *A* has at least one eigenvalue and corresponding eigenvector
- The *eigenspace* corresponding to an eigenvalue *v* is the set of vectors that have eigenvalues *v*
- An *eigenspace* is a **vector subspace** of the vector space on which *A* acts

Eigenvectors and Eigenvalues

- A diagonal representation for an operator *A* on a vector space *V* is a representation

$$A = \sum_i \lambda_i |i\rangle\langle i|$$

  where the vectors $|i\rangle$ form an **orthonormal set of eigenvectors** for *A*, with corresponding **eigenvalues** λ_i
- An operator is said to be diagonalizable if it has a diagonal representation
- In a few slides we will find a simple set of **necessary and sufficient conditions for an operator on a Hilbert space to be diagonalizable**

Eigenvectors and Eigenvalues

- As an example of a diagonal representation, note that the Pauli Z matrix may be written

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|,$$

  where the matrix representation is with respect to orthonormal vectors $|0\rangle$ and $|1\rangle$ respectively

- Diagonal representations are sometimes also known as *orthonormal decompositions*

Eigenvectors and Eigenvalues

- When an eigenspace is more than one dimensional we say that it is degenerate
- For example, the matrix A defined by

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

has a **two-dimensional eigenspace** corresponding to the **eigenvalue 2**

- The eigenvectors $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ and $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$ are said to be degenerate because they are linearly independent eigenvectors of *A* with the same eigenvalue

Adjoints and Hermitian Operators

- Given a matrix *A*

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$ where *a, b, c,* and *d* are complex numbers

- The *adjoint* or *Hermitian conjugate* $A^\dagger$ of *A* is the *transpose* of its *complex conjugate*

$$A^\dagger = \begin{bmatrix} a & c \\ b & d \end{bmatrix}^\dagger = \left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}^*\right)^T = \left(\begin{bmatrix} a* & c* \\ b* & d* \end{bmatrix}\right)^T = \begin{bmatrix} a* & b* \\ c* & d* \end{bmatrix} \longrightarrow \left[A^\dagger\right]_{ij} = \left[A^*\right]_{ji}$$

- Sometimes $A^\dagger$ is pronounced "*A* dagger"

Adjoints and Hermitian Operators

- We would have achieved the same result if we had *first transposed* the matrix *A* and then taken the *conjugate of the elements*

$$A^{\dagger} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}^{\dagger} = \left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}^{T} \right)^{*} = \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)^{*} = \begin{bmatrix} a* & b* \\ c* & d* \end{bmatrix}$$

- Therefore, the generic element of $A^{\dagger}$ can be written as

$$\left[A^{\dagger} \right]_{ij} = \left(\left[A_{ji} \right] \right)^{*}$$

Adjoints and Hermitian Operators

- For example, if

$$A=\begin{bmatrix}1+3i & 2i \\ 1+i & 1-4i\end{bmatrix}$$

the *Hermitian conjugate* $A^\dagger$ is

$$A^\dagger=\begin{bmatrix}1-3i & 1-i \\ -2i & 1+4i\end{bmatrix}$$

Adjoints and Hermitian Operators

- Let A and B be $n \times n$ matrices and $c \in \mathbb{C}$. We can *prove* the following properties

  1) Property #1: $(cA)^\dagger = c^* A^\dagger$
  2) Property #2: $(A+B)^\dagger = A^\dagger + B^\dagger$
  3) Property #3: $\left(\sum_i a_i A_i\right)^\dagger = \sum_i a_i^* A_i^\dagger$
  4) Property #4: $(AB)^\dagger = B^\dagger A^\dagger$
  5) Property #5: $(A|v\rangle)^\dagger = \langle v| A^\dagger$
  6) Property #6: $(|w\rangle\langle v|)^\dagger = |v\rangle\langle w|$ for any two vectors $|w\rangle$ and $|v\rangle$
  7) Property #7: $(A^\dagger)^\dagger = A$

Adjoints and Hermitian Operators

Property #1

$$(cA)^{\dagger}_{i,j} = (cA)^{*}_{j,i} = \left(c^{*}A^{*}\right)_{j,i} = c^{*}\left(A^{*}\right)_{j,i} = c^{*}A^{\dagger}{}_{i,j}$$

Property #2

$$(A+B)^{\dagger}_{i,j} = (A+B)^{*}_{j,i} = \left(A^{*}+B^{*}\right)_{j,i} = A^{*}_{j,i} + B^{*}_{j,i} = A^{\dagger}_{i,j} + B^{\dagger}_{i,j}$$

Property #3

- From *Property* #1 and *Property* #2 easily follow

$$\left(\sum_i a_i A_i\right)^{\dagger} = \sum_i a_i^{*} A_i^{\dagger}$$

and this indicates that the *adjoint* operation is *anti-linear*

Adjoints and Hermitian Operators

Property #4

$$\left(\left[AB\right]^{\dagger}\right)_{i,j} = \left(\left[AB\right]_{j,i}\right)^{*} = \sum_{k}\left(\left[A\right]_{j,k}\left[B\right]_{k,i}\right)^{*} = \sum_{k}\left(\left[A\right]_{j,k}\right)^{*}\left(\left[B\right]_{k,i}\right)^{*}$$

$$= \sum_{k}\left[B^{\dagger}\right]_{ik}\left[A^{\dagger}\right]_{kj} = \left[B^{\dagger}A^{\dagger}\right]_{ij}$$

Property #5

- By convention, if $|v\rangle$ is a vector, then $|v\rangle^{\dagger} = \langle v|$. With this definition, by exploiting the above property

$$\left(A|v\rangle\right)^{\dagger} = |v\rangle^{\dagger}A^{\dagger} = \langle v|A^{\dagger}$$

Adjoints and Hermitian Operators

$$\left(A\left|v\right\rangle\right)^{\dagger} = \left|v\right\rangle^{\dagger} A^{\dagger} = \left\langle v\right|A^{\dagger}$$

Property #6

- Let's take the element *l,m* of the matrix $\left(\left|w\right\rangle\left\langle v\right|\right)^{\dagger}$,i.e., $\left(\left|w\right\rangle\left\langle v\right|\right)^{\dagger}_{l,m}$

$$\left\langle l\right|\left(\left|w\right\rangle\left\langle v\right|\right)^{\dagger}\left|m\right\rangle = \left(\left(\left|w\right\rangle\left\langle v\right|\right)\left|l\right\rangle\right)^{\dagger}\left|m\right\rangle = \left(\left|w\right\rangle\left\langle v\middle|l\right\rangle\right)^{\dagger}\left|m\right\rangle = \left|w\right\rangle^{\dagger}\left\langle v\middle|l\right\rangle^{*}\left|m\right\rangle$$
$$= \left\langle w\right|\left\langle l\middle|v\right\rangle\left|m\right\rangle = \left\langle l\middle|v\right\rangle\left\langle w\middle|m\right\rangle = \left\langle l\right|\left(\left|v\right\rangle\left\langle w\right|\right)\left|m\right\rangle$$

- Since the previous equality holds for any *l* and *m*, it turns out that Property #6 is proved

Adjoints and Hermitian Operators

Property #7

- Since

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \qquad A^{\dagger} = \begin{bmatrix} a^* & b^* \\ c^* & d^* \end{bmatrix}$$

it follows

$$\left(A^{\dagger}\right)^{\dagger} = \begin{bmatrix} a^* & b^* \\ c^* & d^* \end{bmatrix}^{\dagger} = \left(\begin{bmatrix} a^* & b^* \\ c^* & d^* \end{bmatrix}^{*}\right)^{T} = \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)^{T} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = A$$

Hermitian Operators

Definition

- An operator A is said to be *Hermitian* or *self-adjoint* operator if it satisfy

$$A^{\dagger} = A$$

- The eigenvalue problems of Hermitian matrices are particularly important in practical applications

Hermitian Operators

- **Theorem**: All the eigenvalues of a **Hermitian** matrix are **real** numbers. Moreover, two eigenvectors corresponding to different eigenvalues are orthogonal
- *Proof.* Let *A* be a Hermitian matrix and let $A|\lambda\rangle = \lambda|\lambda\rangle$
- The Hermitian conjugate of this equation is $\langle\lambda|A^\dagger = \lambda^*\langle\lambda| \xrightarrow{A^\dagger = A} \langle\lambda|A = \lambda^*\langle\lambda|$
- From these equations we obtain $\langle\lambda|A|\lambda\rangle = \lambda\langle\lambda|\lambda\rangle = \lambda^*\langle\lambda|\lambda\rangle$, which proves $\lambda = \lambda^*$ since $\langle\lambda|\lambda\rangle \neq 0$

Hermitian Operators

- Assume now that $A|\mu\rangle = \mu|\mu\rangle \; (\mu \neq \lambda)$
- Then $\langle\mu|A = \mu\langle\mu|$ since $A = A^{\dagger}, \quad \mu \in \mathbb{R}$

$$A|\lambda\rangle = \lambda|\lambda\rangle$$

- From $\langle\mu|A|\lambda\rangle = \lambda\langle\mu|\lambda\rangle$

$$\langle\mu|A|\lambda\rangle = \mu\langle\mu|\lambda\rangle$$

  we obtain $0 = (\lambda - \mu)\langle\mu|\lambda\rangle$
- Since $\mu \neq \lambda$, we must have $\langle\mu|\lambda\rangle = 0$ $\blacksquare$

Hermitian Operators

- *Property* #2 asserts that $(A+B)^\dagger = A^\dagger + B^\dagger$
- Therefore, if $A = A^\dagger$ and $B = B^\dagger$, i.e., *A* and *B* are Hermitian, then *A*+*B* is Hermitian as well

$$(A+B)^\dagger = A^\dagger + B^\dagger = A + B$$

- On the other hand, their product is not necessarily Hermitian
- The necessary and sufficient condition for the **product** of two Hermitian operators to be Hermitian is

$$AB = BA \text{ or } AB - BA = 0$$

Hermitian Operators

- The expression

$$\left[A, B\right] = AB - BA$$

  is called the **commutator** of *A* and *B*

- Thus, *the product of two Hermitian operators is a Hermitian operator if and only if their commutator Is equal to zero*

Hermitian Operators

For a Hermitian operator the following property holds

$$\langle v|A|v\rangle = \langle v|A|v\rangle^* \rightarrow \langle v|A|v\rangle \in \mathbb{R}$$

Proof

- Assume that $|v\rangle = \sum_i a_i |i\rangle$ where $\{|i\rangle\}$ is an orthonormal basis. Then

$$\langle v|A|v\rangle = \sum_{i,h} a_i^* a_h \langle i|A|h\rangle = \sum_{i,h} a_i^* a_h \langle i|A^\dagger|h\rangle = \sum_{i,h} a_i^* a_h \langle h|A|i\rangle^*$$

$$= \sum_{i,h} \left(a_h^* a_i \langle h|A|i\rangle\right)^* = \left(\sum_{i,h} a_h^* a_i \langle h|A|i\rangle\right)^* = \langle v|A|v\rangle^* \qquad \blacksquare$$

Hermitian Operators

- An important class of Hermitian operators is the *projectors*
- Suppose W is a k-dimensional vector **subspace** of the d-dimensional vector space V

Projector Operators

- Using the *Gram Schmidt* procedure, it is possible to construct an orthonormal basis $\{|1\rangle, \ldots, |d\rangle\}$ for V such that $\{|1\rangle, \ldots, |k\rangle\}$ is an orthonormal basis for W. By definition

$$P = \sum_{i=1}^{k} |i\rangle\langle i|$$

  is the projector onto the subspace W

- It is easy to check that this definition is independent of the orthonormal basis $\{|1\rangle, \ldots, |k\rangle\}$ used for W

Projector Operators

- Since $|i\rangle\langle i|$ is Hermitian (*Property* #6), it turns out that *P* is Hermitian (*Property* #3), i.e. $P^{\dagger} = P$
- We will often refer to the **'vector space' *P***, as shorthand for the vector space onto which P is a projector
- The *orthogonal complement* of *P* is the operator *Q* ≡ *I* − *P*
- *Q* is clearly a projector onto the vector space spanned by $\{|k+1\rangle,\ldots,|d\rangle\}$, which we also refer to as the *orthogonal complement* of *P*, and may denote by *Q*

Projector Operators

- We will now prove that $P^2 = P$

- *Proof*

$$P^2 = \left(\sum_{i=1}^{k} |i\rangle\langle i| \right)\left(\sum_{j=1}^{k} |j\rangle\langle j| \right) = \sum_{i,j=1}^{k} |i\rangle\langle i|j\rangle\langle j| = \sum_{i,j=1}^{k} |i\rangle\delta_{i,j}\langle j|$$

$$= \sum_{i,j=1}^{k} |i\rangle\langle i|j\rangle\langle j| = \sum_{i=1}^{k} |i\rangle\langle i| = P \qquad \blacksquare$$

Projector Operators

- It is easy to show that the eigenvalues of a projector *P* are all either 0 or 1

Proof

- Suppose *P* is projector and $|\lambda\rangle$ are eigenvectors of P with eigenvalues λ
- Then

$$P|\lambda\rangle = \lambda|\lambda\rangle$$

$$P^2|\lambda\rangle = P\big(P|\lambda\rangle\big) = \lambda P|\lambda\rangle = \lambda^2|\lambda\rangle$$

- From $P^2 = P$ it follows $P^2|\lambda\rangle = P|\lambda\rangle$ and therefore

$$\lambda^2 = \lambda \rightarrow \lambda(\lambda - 1) = 0 \rightarrow \lambda = 0,\ \lambda = 1 \qquad \blacksquare$$

Projector Operators

- The outer product of a state vector $|\varphi\rangle$ with itself is a projector:

$$P_\varphi = |\varphi\rangle\langle\varphi|$$

- P_φ is Hermitian, i.e., $P_\varphi^\dagger = \left(|\varphi\rangle\langle\varphi|\right)^\dagger = |\varphi\rangle\langle\varphi| = P_\varphi$, and

$$P_\varphi^2 = \left(|\varphi\rangle\langle\varphi|\right)\left(|\varphi\rangle\langle\varphi|\right) = |\varphi\rangle\langle\varphi|\varphi\rangle\langle\varphi| = |\varphi\rangle\langle\varphi| = P_\varphi$$

- Two projectors P_i, P_j are orthogonal if, for every state $|\psi\rangle$ the following equality holds

$$P_i P_j |\psi\rangle = 0$$

- This condition is often written as

$$P_i P_j = 0$$

Projectors Operators

- A set of orthogonal projectors $\{P_0, P_1, P_2 \ldots\}$ is *complete/exhaustive* if

$$\sum_i P_i = I$$

Normal Operators

- An operator A is said to be **normal** if

$$AA^{\dagger} = A^{\dagger}A$$

- Clearly, an operator A which is *Hermitian* $\left(A = A^{\dagger}\right)$ is also *normal*

$$A = A^{\dagger} \xrightarrow[\text{on the right}]{\text{Multiply by } A^{\dagger}} AA^{\dagger} = A^{\dagger}A^{\dagger} \xrightarrow{A^{\dagger}=A} A^{\dagger}A$$

- On the other hand, a *normal operator* is not necessarily *Hermitian*

$$A = A^{\dagger} \rightarrow A^{\dagger}A = AA^{\dagger}$$

A Hermitian *A Normal*