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Quantum Computing and Quantum Internet

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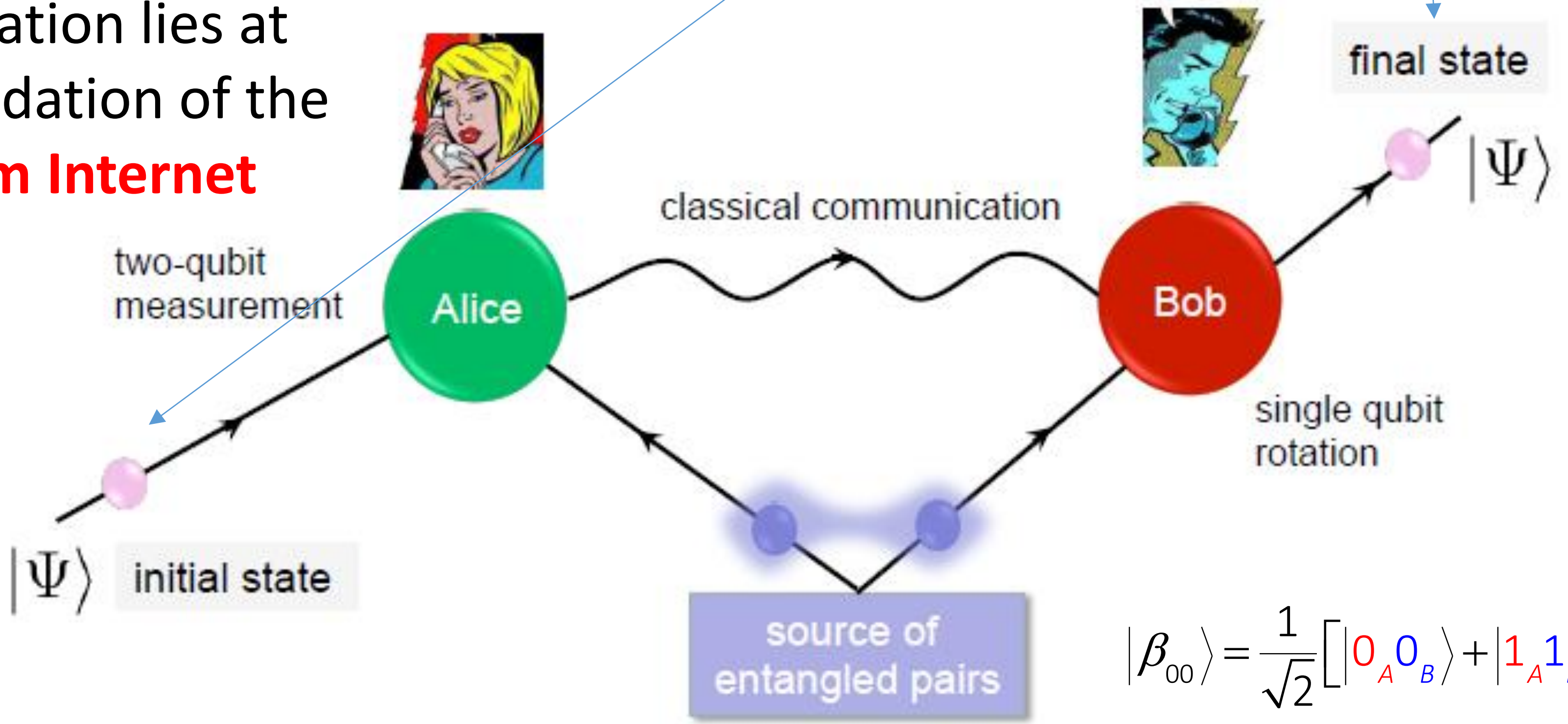
Quantum Teleportation

Quantum Teleportation

- Teleportation refers to an operation in which the quantum **state** of a qubit (confined single electron or photon) dissolves **here** and reappears ***there***, on a different qubit (electron or photon)

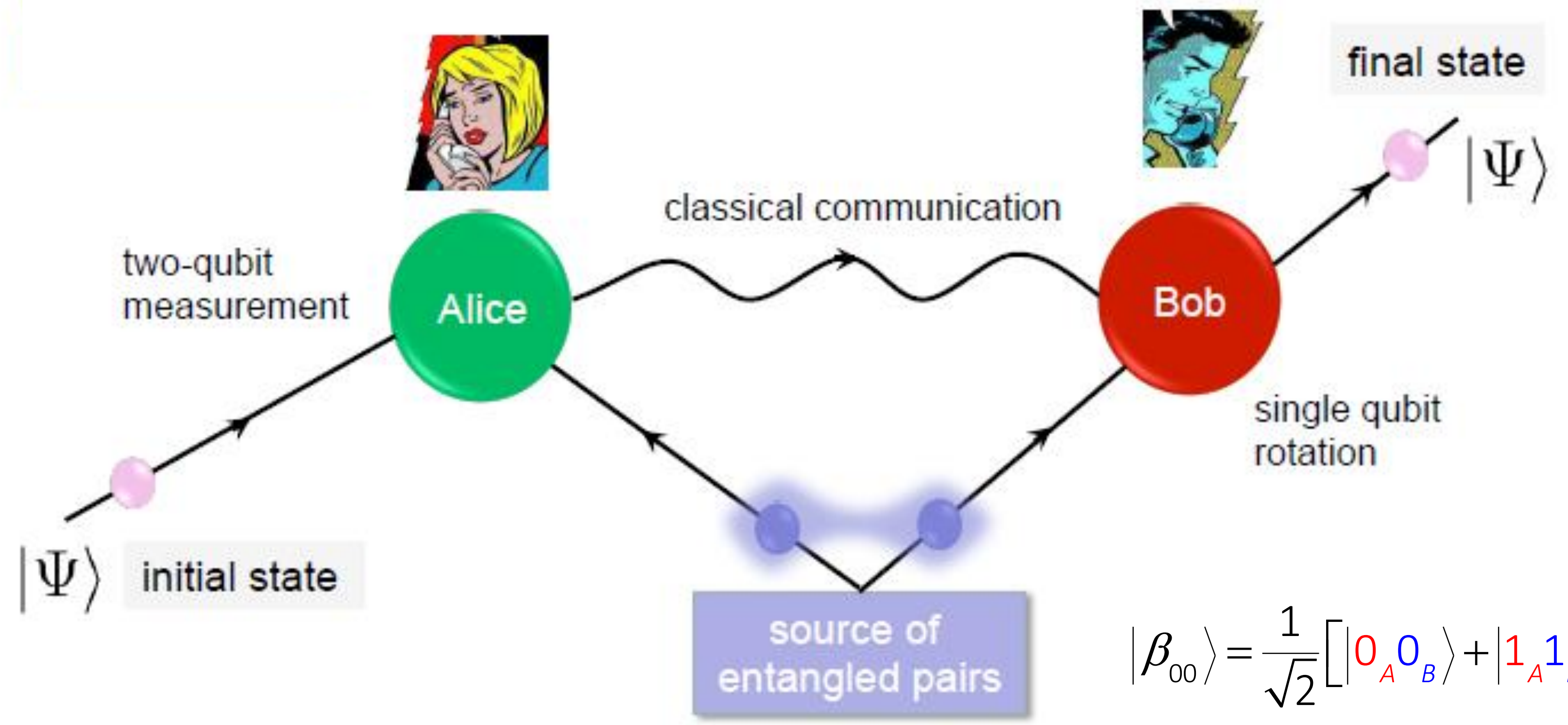
Teleportation lies at the foundation of the **Quantum Internet**



$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}\left[|0_A 0_B\rangle + |1_A 1_B\rangle\right]$$

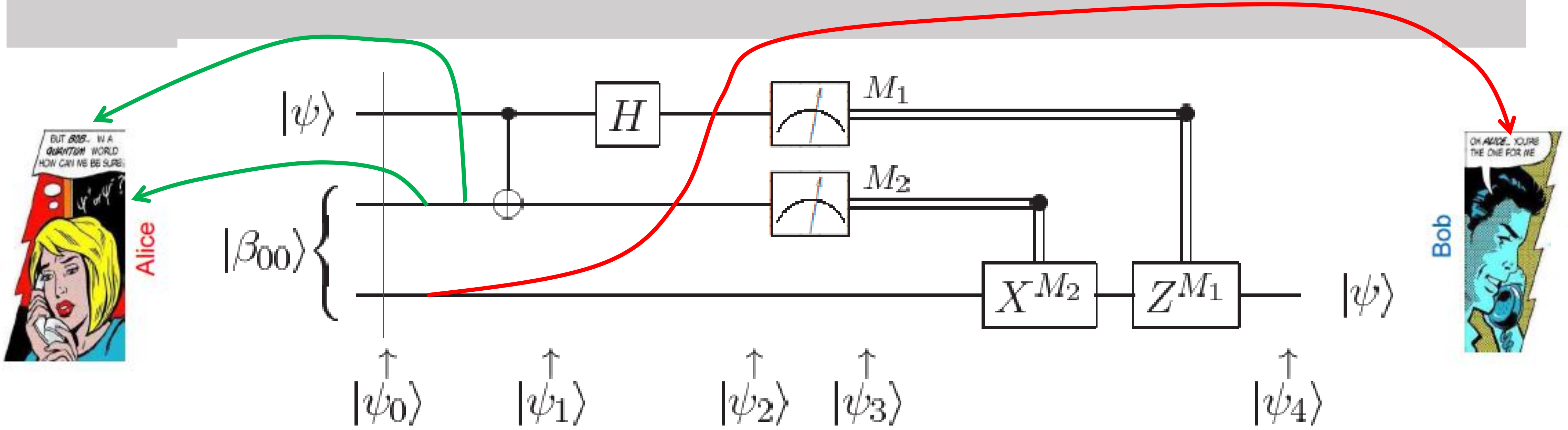
Quantum Teleportation

- Only the quantum *state* moves; the electron or other physical qubit remains where it was, and the receiver can in fact be a very different form of physical qubit than the sender



$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}\left[|0_A 0_B\rangle + |1_A 1_B\rangle\right]$$

Quantum Teleportation

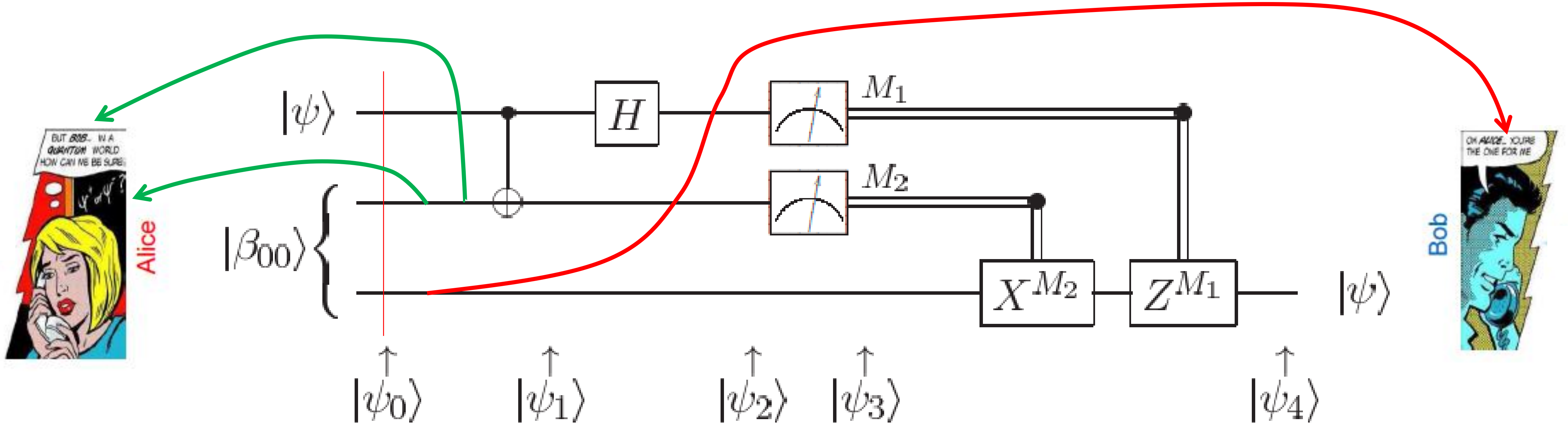


- The state to be teleported is $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where α and β are unknown amplitudes
- The state input into the circuit

$$|\psi_0\rangle = |\psi\rangle|\beta_{00}\rangle = \left(\alpha|0_A\rangle + \beta|1_A\rangle\right)\left(\frac{|0_A 0_B\rangle + |1_A 1_B\rangle}{\sqrt{2}}\right)$$

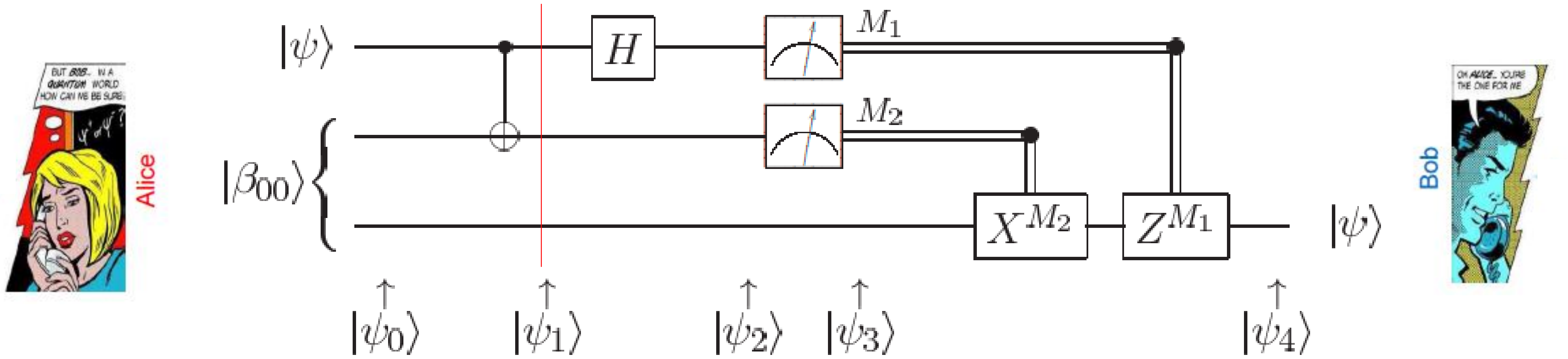
Quantum Teleportation

$$|\psi_0\rangle = |\psi\rangle|\beta_{00}\rangle = (\alpha|0\rangle + \beta|1\rangle)\left(\frac{|0_A0_B\rangle + |1_A1_B\rangle}{\sqrt{2}}\right)$$



Thus $|\psi_0\rangle = \frac{1}{\sqrt{2}}\left[\alpha|0_A\rangle(|0_A0_B\rangle + |1_A1_B\rangle) + \beta|1_A\rangle(|0_A0_B\rangle + |1_A1_B\rangle)\right]$

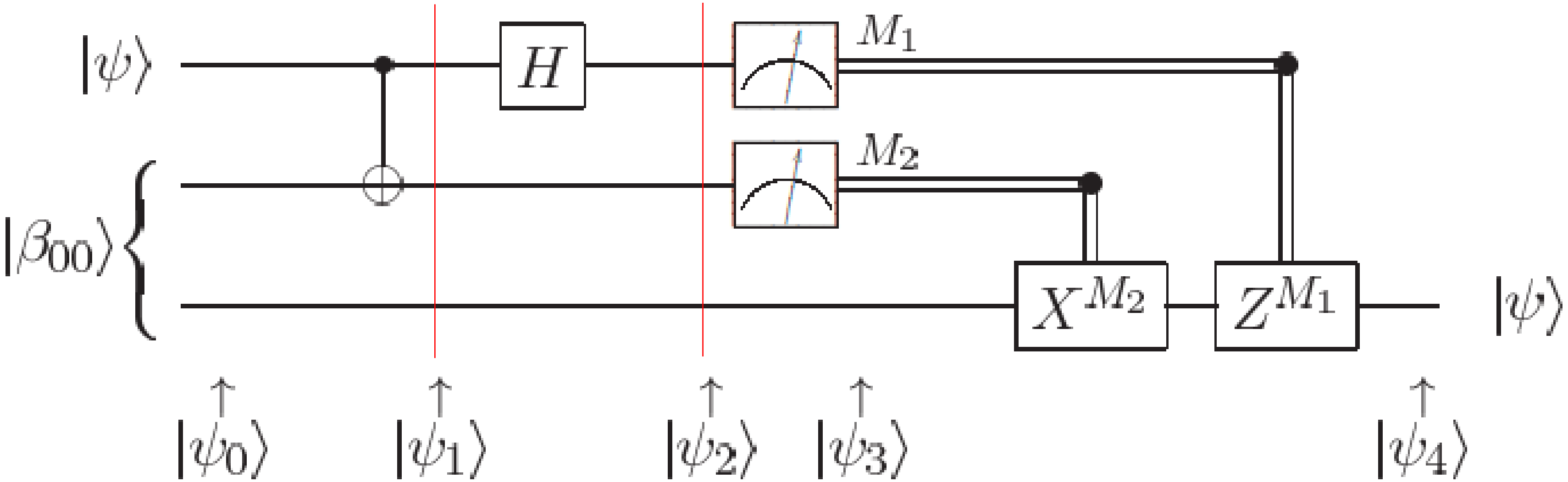
Quantum Teleportation



- Alice sends her qubits through a CNOT gate, obtaining

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}\left[\alpha|0_A\rangle\left(|0_A0_B\rangle+|1_A1_B\rangle\right)+\beta|1_A\rangle\left(|0_A0_B\rangle+|1_A1_B\rangle\right)\right]$$

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\left[\alpha|0_A\rangle\left(|0_A0_B\rangle+|1_A1_B\rangle\right)+\beta|1_A\rangle\left(\overset{\text{CNOT}\downarrow}{|1_A0_B\rangle}+\overset{\downarrow\text{CNOT}}{|0_A1_B\rangle}\right)\right]$$



- Alice then sends the first qubit through a Hadamard gate, obtaining

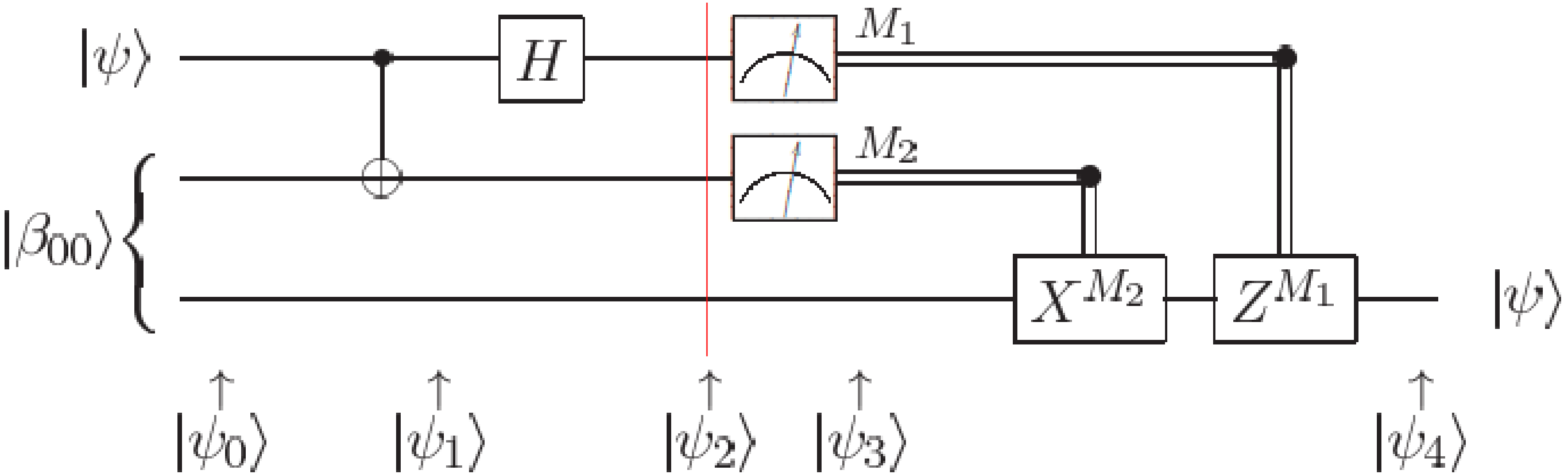
$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\left[\alpha|0_A\rangle\left(|0_A 0_B\rangle + |1_A 1_B\rangle\right) + \beta|1_A\rangle\left(|1_A 0_B\rangle + |0_A 1_B\rangle\right)\right]$$

H H

$$|\psi_2\rangle = \frac{1}{2}\left[\alpha\left(|0_A\rangle + |1_A\rangle\right)\left(|0_A 0_B\rangle + |1_A 1_B\rangle\right) + \beta\left(|0_A\rangle - |1_A\rangle\right)\left(|1_A 0_B\rangle + |0_A 1_B\rangle\right)\right]$$

Quantum Teleportation

Alice

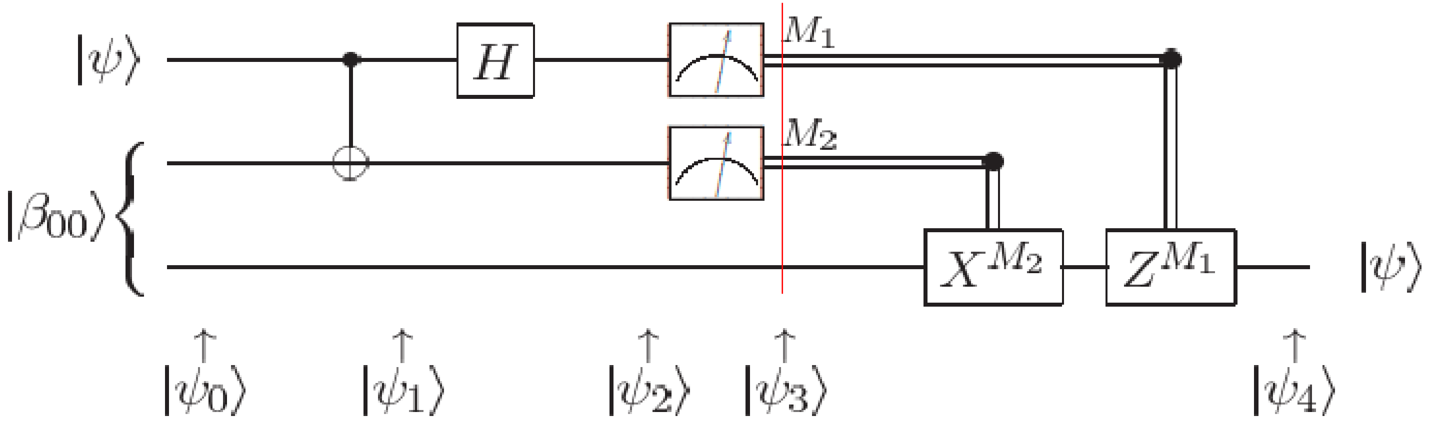


Bob

- $|\psi_2\rangle$ may be re-written in the following way, simply by regrouping terms:

$$|\psi_2\rangle = \frac{1}{2}\Big[|0_A 0_A\rangle\big(\alpha|0_B\rangle + \beta|1_B\rangle\big) + |0_A 1_A\rangle\big(\alpha|1_B\rangle + \beta|0_B\rangle\big)$$
$$+|1_A 0_A\rangle\big(\alpha|0_B\rangle - \beta|1_B\rangle\big) + |1_A 1_A\rangle\big(\alpha|1_B\rangle - \beta|0_B\rangle\big)\Big]$$

Quantum Teleportation



- Depending on Alice's measurement outcome, Bob's qubit will end up in one of the following four possible states

$$|\psi_2\rangle = \frac{1}{2}\Big[|0_A 0_A\rangle\big(\alpha|0_B\rangle + \beta|1_B\rangle\big) + |0_A 1_A\rangle\big(\alpha|1_B\rangle + \beta|0_B\rangle\big)$$
$$+ |1_A 0_A\rangle\big(\alpha|0_B\rangle - \beta|1_B\rangle\big) + |1_A 1_A\rangle\big(\alpha|1_B\rangle - \beta|0_B\rangle\big)\Big]$$

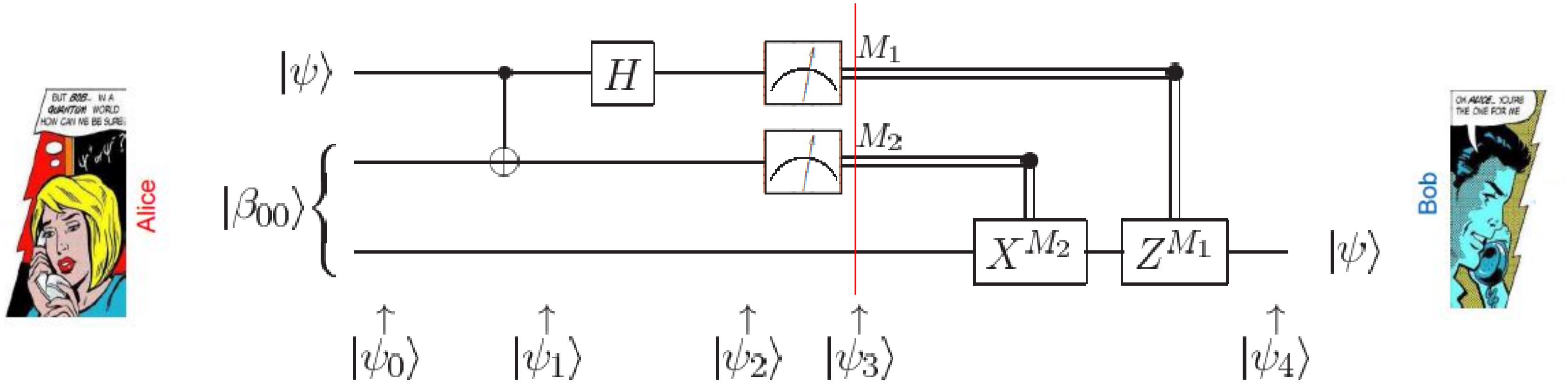
Measurement by Alice

Post Measurement state $|\psi_3\rangle$

Alice *Bob*

$$|0_A 0_A\rangle\big(\alpha|0_B\rangle + \beta|1_B\rangle\big)$$
$$|0_A 1_A\rangle\big(\alpha|1_B\rangle + \beta|0_B\rangle\big)$$
$$|1_A 0_A\rangle\big(\alpha|0_B\rangle - \beta|1_B\rangle\big)$$
$$|1_A 1_A\rangle\big(\alpha|1_B\rangle - \beta|0_B\rangle\big)$$

Quantum Teleportation



To know which state it is in, Bob must be told the result of Alice's measurement

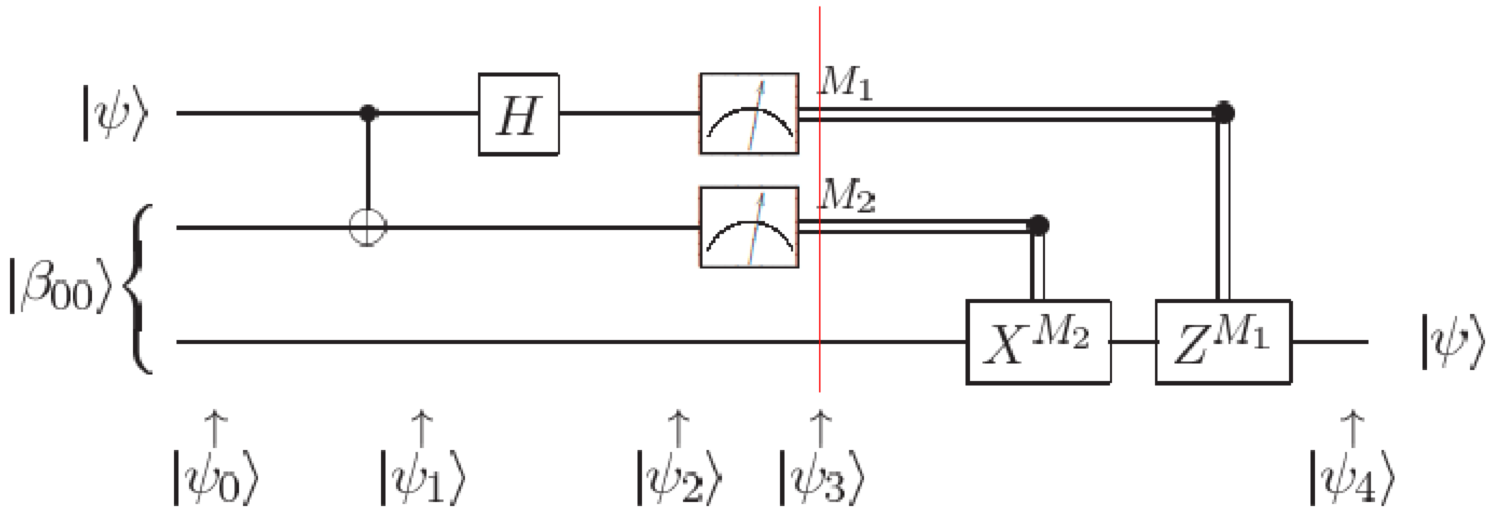
Alice Bob

$$M_1 = 0_A,\ M_2 = 0_A \rightarrow \alpha|0_B\rangle + \beta|1_B\rangle$$

$$M_1 = 0_A,\ M_2 = 1_A \rightarrow \alpha|1_B\rangle + \beta|0_B\rangle$$

$$M_1 = 1_A,\ M_2 = 0_A \rightarrow \alpha|0_B\rangle - \beta|1_B\rangle$$

$$M_1 = 1_A,\ M_2 = 1_A \rightarrow \alpha|1_B\rangle - \beta|0_B\rangle$$



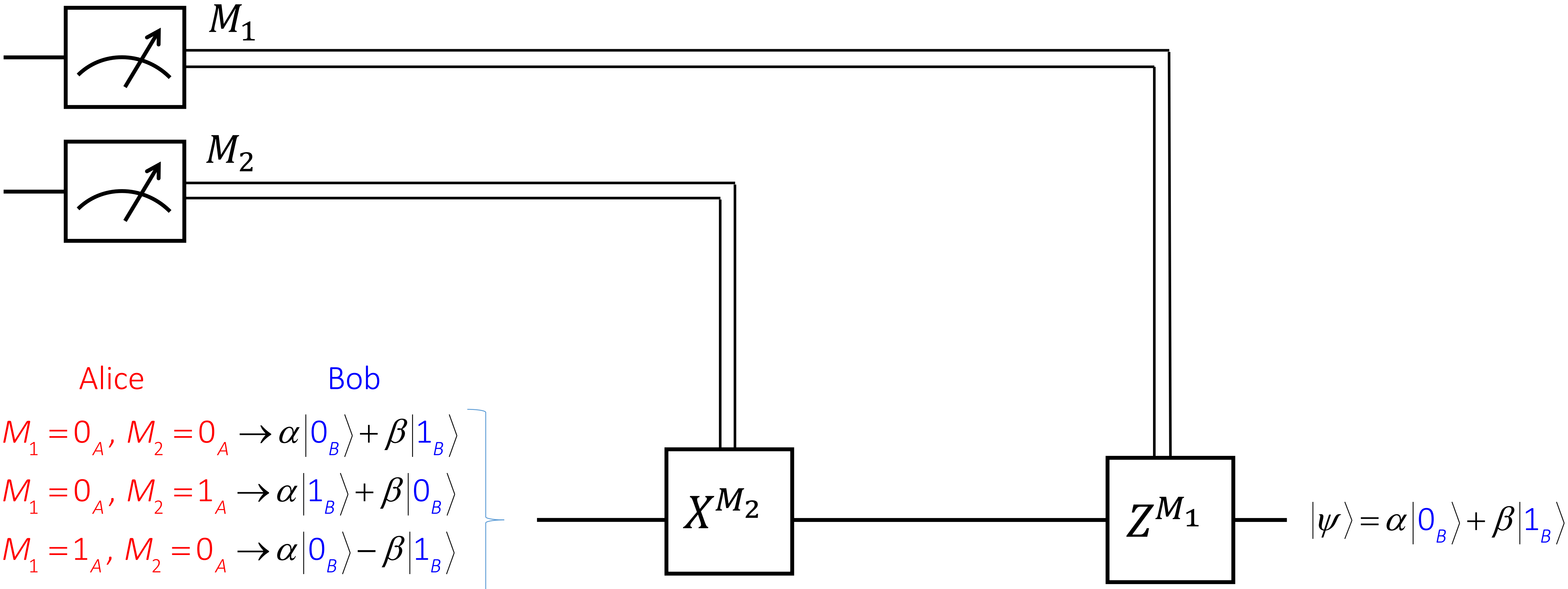
Once Bob has learned the measurement outcome, Bob can ‘fix up’ his state, recovering $|\psi\rangle$, by applying the appropriate quantum gate

Alice		Bob
$M_1 = 0_A,\ M_2 = 0_A$	$\rightarrow$	$\alpha\lvert 0_B\rangle + \beta\lvert 1_B\rangle$
$M_1 = 0_A,\ M_2 = 1_A$	$\rightarrow$	$\alpha\lvert 1_B\rangle + \beta\lvert 0_B\rangle$
$M_1 = 1_A,\ M_2 = 0_A$	$\rightarrow$	$\alpha\lvert 0_B\rangle - \beta\lvert 1_B\rangle$
$M_1 = 1_A,\ M_2 = 1_A$	$\rightarrow$	$\alpha\lvert 1_B\rangle - \beta\lvert 0_B\rangle$

$$X^{M_2} = \begin{cases} X & \text{for } M_2 = 1 \\ I & \text{for } M_2 = 0 \end{cases}$$

$$Z^{M_1} = \begin{cases} Z & \text{for } M_1 = 1 \\ I & \text{for } M_1 = 0 \end{cases}$$

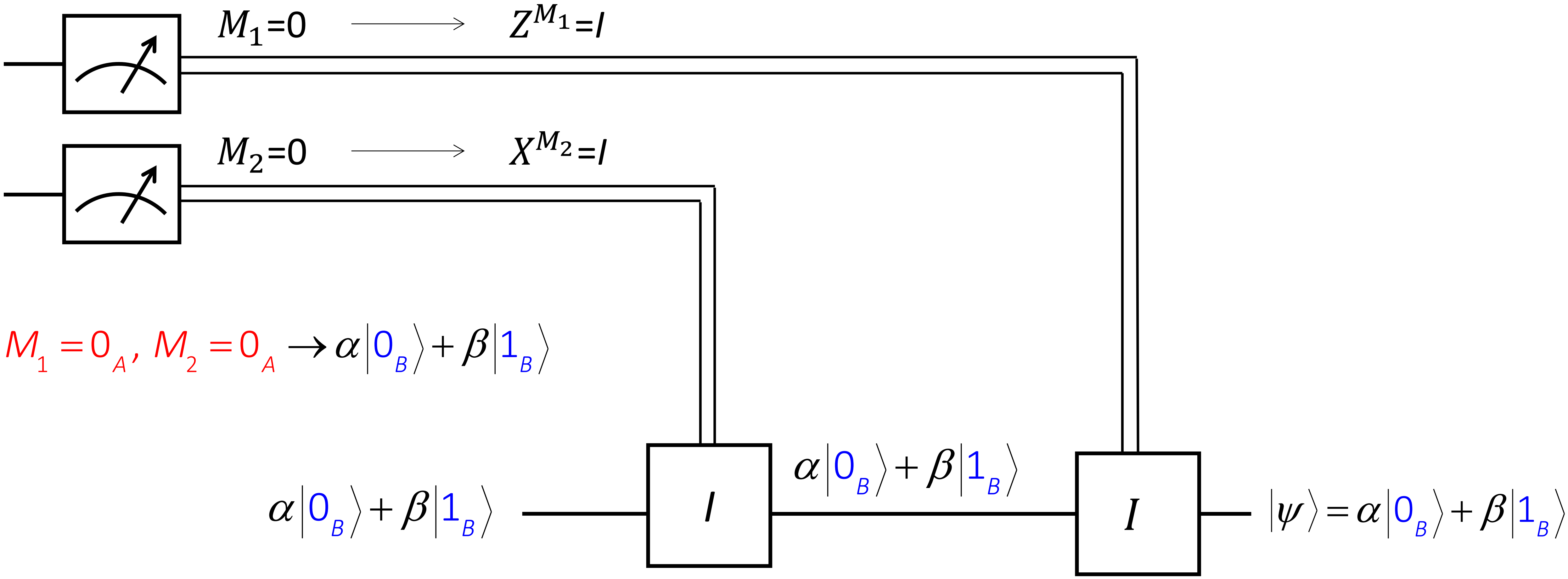
Quantum Teleportation



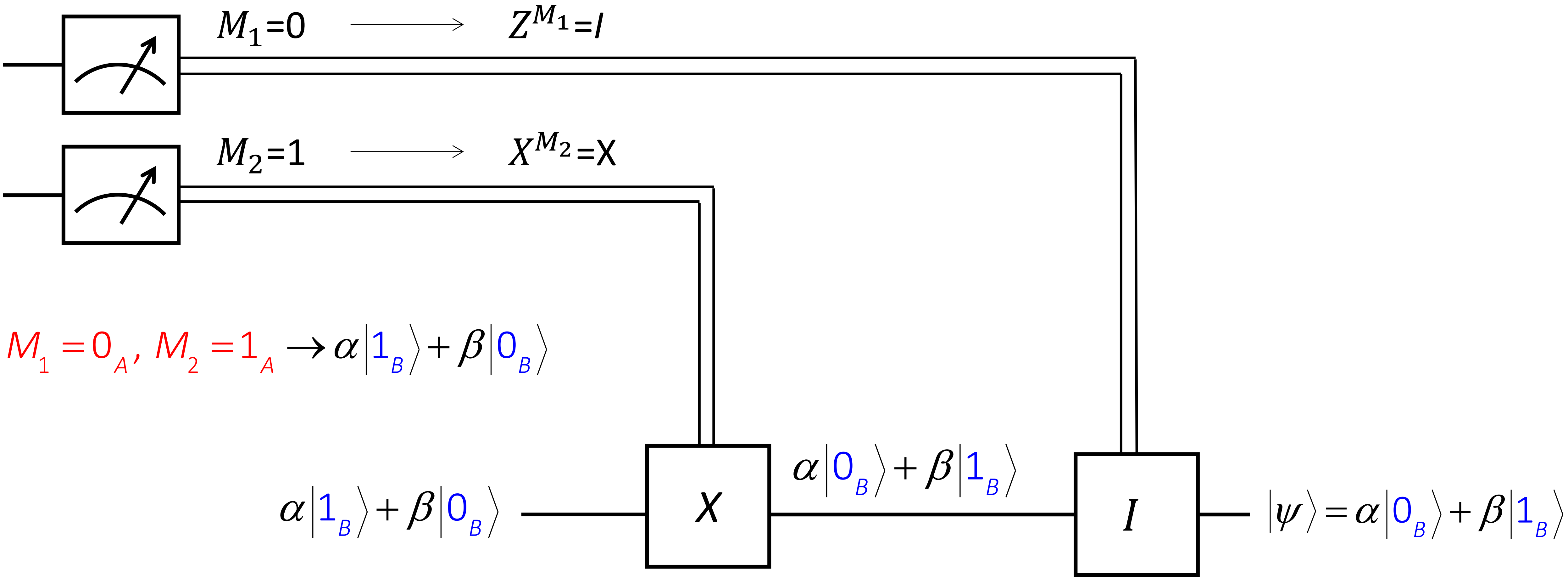
Alice Bob

$$
\left.
\begin{aligned}
&M_1 = 0_A,\ M_2 = 0_A \rightarrow \alpha|0_B\rangle + \beta|1_B\rangle \\
&M_1 = 0_A,\ M_2 = 1_A \rightarrow \alpha|1_B\rangle + \beta|0_B\rangle \\
&M_1 = 1_A,\ M_2 = 0_A \rightarrow \alpha|0_B\rangle - \beta|1_B\rangle \\
&M_1 = 1_A,\ M_2 = 1_A \rightarrow \alpha|1_B\rangle - \beta|0_B\rangle
\end{aligned}
\right\}
$$

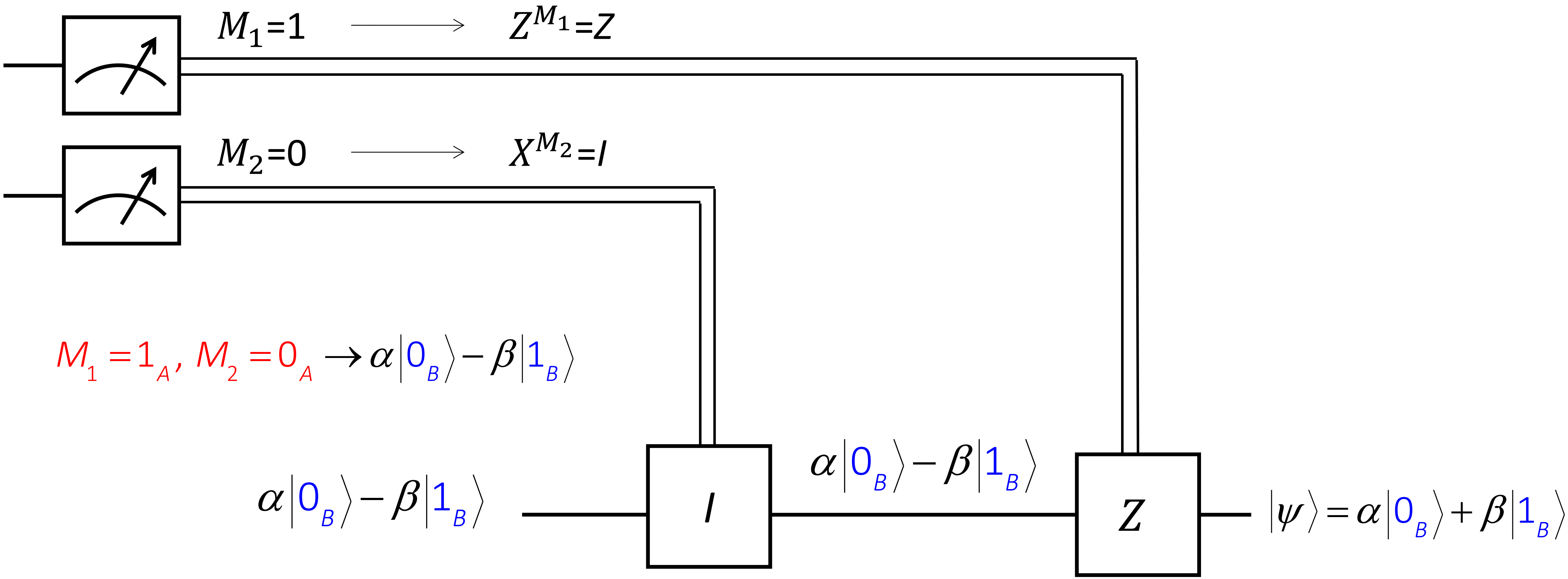
Quantum Teleportation



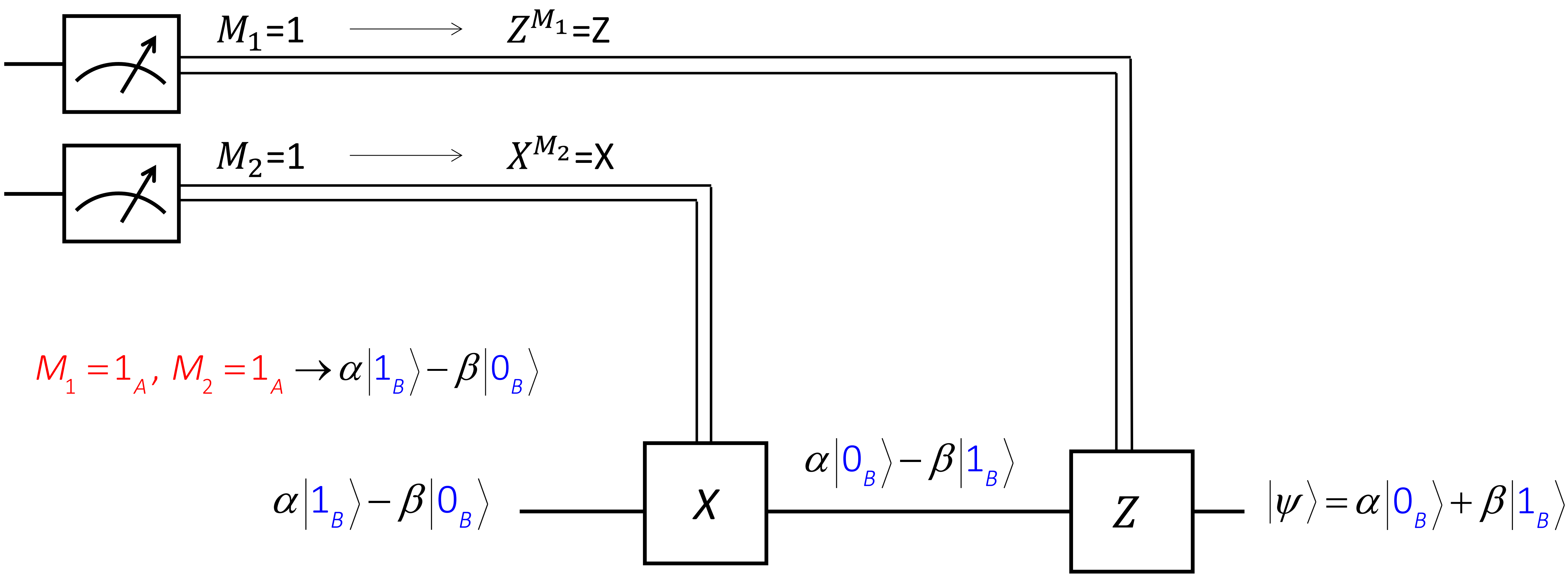
Quantum Teleportation



Quantum Teleportation



Quantum Teleportation



Quantum Teleportation Questions

- **First,** does teleportation allow one to transmit quantum states faster than light?

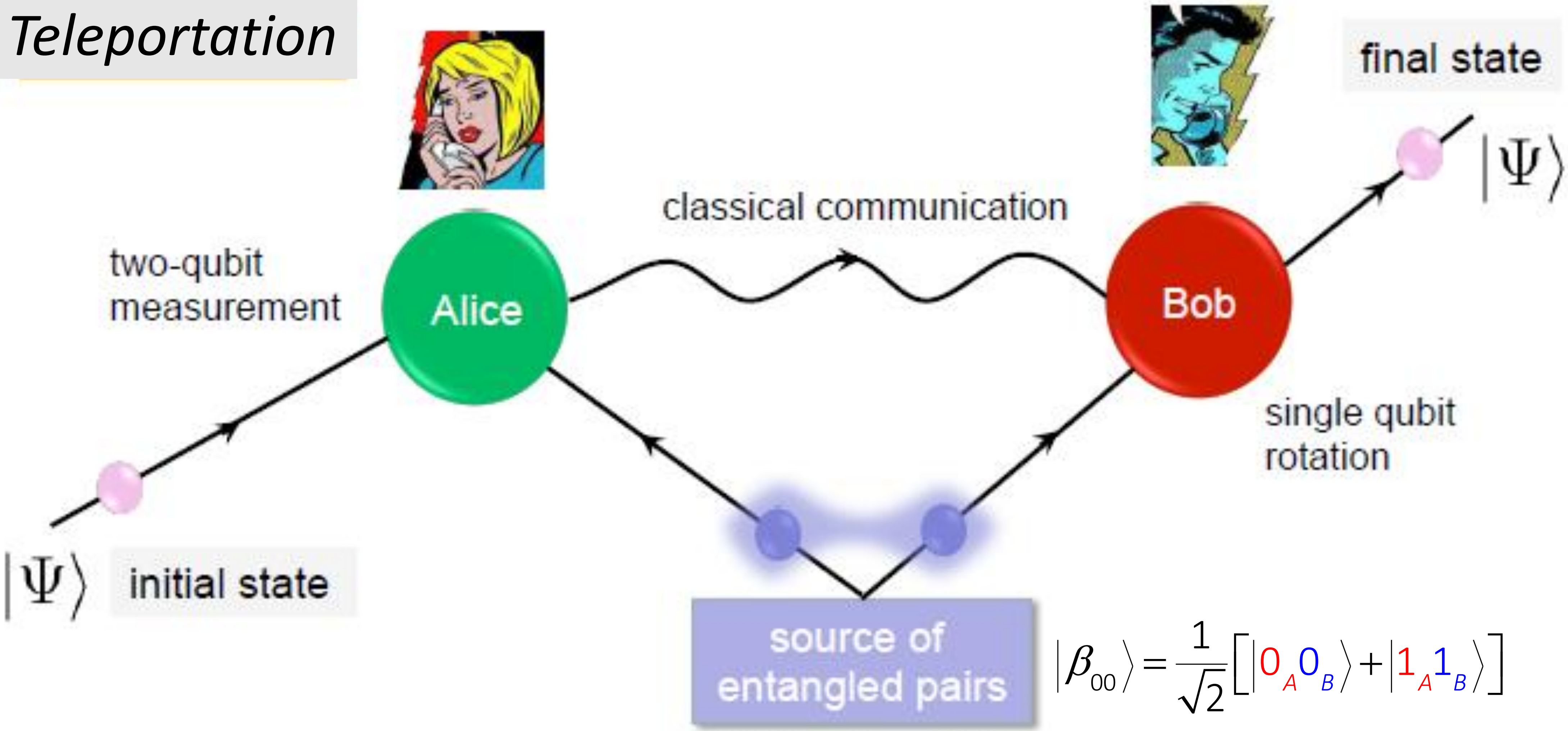
- No, because to complete the teleportation Alice must transmit her measurement result to Bob over a **classical communications channel**

- The classical channel is limited by the speed of light, resolving the apparent paradox

Quantum Teleportation Questions

- **Second,** teleportation appears to create a copy of the quantum state being teleported, in apparent violation of the no-cloning theorem
- This violation is only illusory since after the teleportation process only the target qubit is left in the state $|\psi\rangle$ and the original data qubit ends up in one of the computational basis states $|0\rangle$ or $|1\rangle$, depending upon the measurement result on the first qubit

Teleportation



$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}\left[|0_A 0_B\rangle + |1_A 1_B\rangle\right]$$

Superdense Coding

- Superdense coding is a protocol that, in some sense, achieves a complementary aim to teleportation
- Rather than allowing for the transmission of **one qubit** using **two classical bits** of communication (at the cost of one *EPR* pair), it allows for the transmission of **two classical bits** using **one qubit** of quantum communication (again, at the cost of one of one *EPR* pair)

Superdense Coding

- In greater detail, we have a sender (Alice) and a receiver (Bob) that share one pair of qubit in a maximally entangled state
- According to the conventions in place for the lesson, this means that Alice holds a qubit ***A***, Bob holds a qubit ***B***, and together the pair (***A***,***B***) is in the state

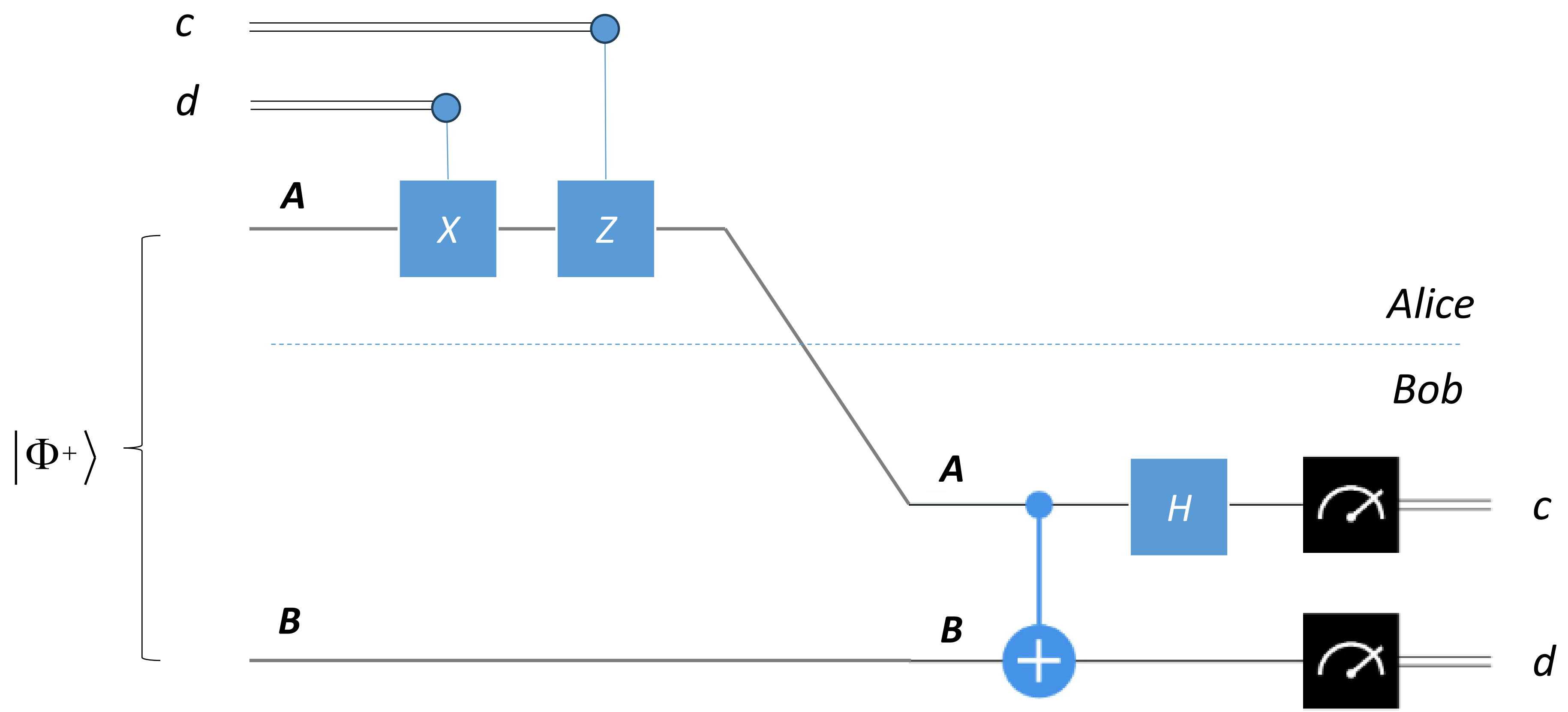
$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|0_A 0_B\rangle + |1_A 1_B\rangle\right)$$

- Alice wishes to transmit *two classical bits* to Bob, which we'll denoted by $\boldsymbol{c}$ and $\boldsymbol{d}$, and she will accomplish this by sending him one qubit

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|0_A 0_B\rangle + |1_A 1_B\rangle\right)$$

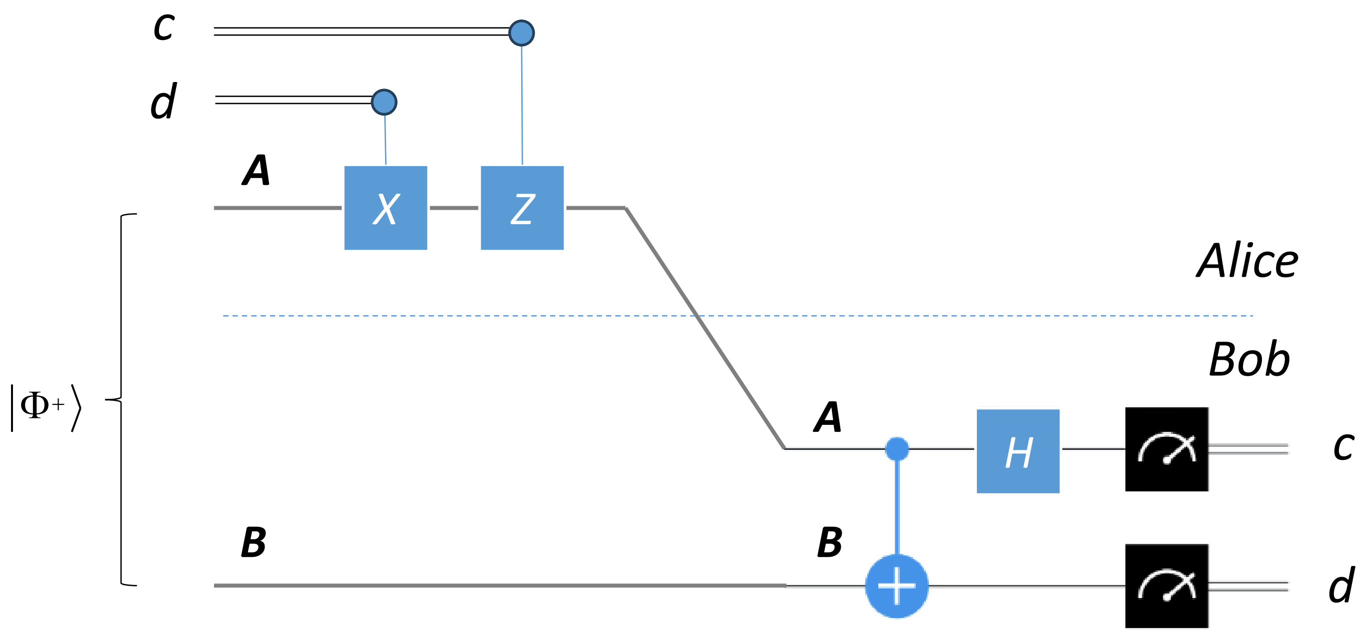
Superdense Coding

The following quantum circuit diagram describes the superdense coding protocol



Superdense Coding

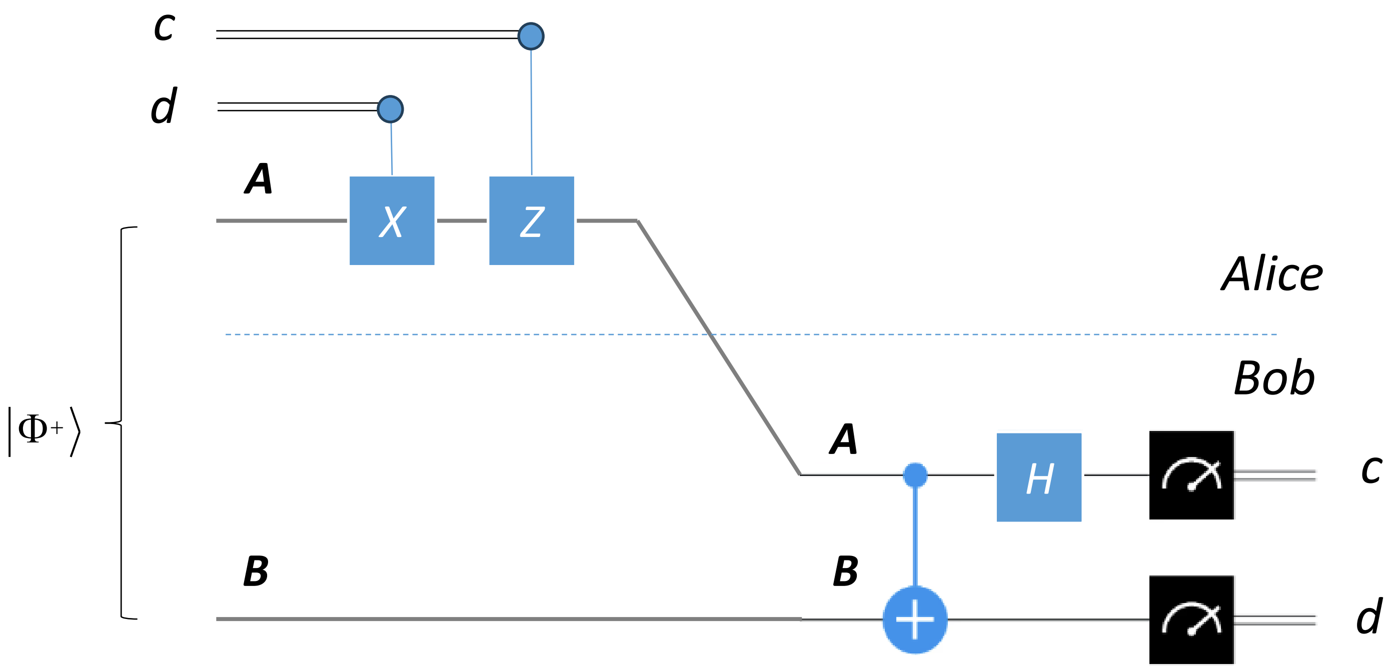
- In words, here is what Alice does:
  1. If c = 1, Alice performs a Z gate on her qubit $\boldsymbol{A}$ (and if c = 0 she does not).
  2. If d = 1, Alice performs an X gate on her qubit $\boldsymbol{A}$ (and if d = 0 she does not).
- Alice then sends her qubit $\boldsymbol{A}$ to Bob.



Superdense Coding

- When Bob receives the qubit $\boldsymbol{A}$:
    1. He first perform a controlled-*NOT* gate, with $\boldsymbol{A}$ being the control and $\boldsymbol{B}$ being the target, and then
    2. He applies a Hadamard gate to $\boldsymbol{A}$

- He then measures ***A*** to obtain c and ***B*** to obtain d, with standard basis measurements in both cases



Superdense Coding

To verify that the protocol works correctly is a matter of checking each case:

- If Alice wants to send ***cd* = 00**, gates *X* and *Z* are not performed on her qubit

- Alice does nothing on **A** and sends it to Bob so that he has both qubits

- Formally

$$\left(I_A \otimes I_B\right)\left|\Phi^+\right\rangle = \frac{1}{\sqrt{2}}\left(I_A\left|0_A\right\rangle \otimes I_B\left|0_B\right\rangle + I_A\left|1_A\right\rangle \otimes I_B\left|1_B\right\rangle\right) = \frac{1}{\sqrt{2}}\left(\left|0_A\right\rangle \otimes \left|0_B\right\rangle + \left|1_A\right\rangle \otimes \left|1_B\right\rangle\right) = \left|\Phi^+\right\rangle$$

Superdense Coding

- If Alice wants to send ***cd* = 01**, she applies the *X* gate to her qubit, which transform $|\Phi^{+}\rangle$ to

$$|\Psi^{+}\rangle = \frac{1}{\sqrt{2}}\left(|1_A 0_B\rangle + |0_A 1_B\rangle\right)$$

- Formally

$$\left(X_A \otimes I_B\right)|\Phi^{+}\rangle = \frac{1}{\sqrt{2}}\left(X_A|0_A\rangle \otimes I_B|0_B\rangle + X_A|1_A\rangle \otimes I_B|1_B\rangle\right) = \frac{1}{\sqrt{2}}\left(|1_A\rangle \otimes |0_B\rangle + |0_A\rangle \otimes |1_B\rangle\right) = |\Psi^{+}\rangle$$

- Then Alice sends her qubit to Bob, so that he has both qubits

Superdense Coding

- If Alice wants to send ***cd* = 10**, she applies the *Z* gate to her qubit, which transforms $|\Phi^+\rangle$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}\left(|0_A 0_B\rangle - |1_A 1_B\rangle\right)$$

- Formally

$$\left(Z_A \otimes I_B\right)|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(Z_A|0_A\rangle \otimes I_B|0_B\rangle + Z_A|1_A\rangle \otimes I_B|1_B\rangle\right) = \frac{1}{\sqrt{2}}\left(|1_A\rangle \otimes |0_B\rangle - |0_A\rangle \otimes |1_B\rangle\right) = |\Phi^-\rangle$$

- Then Alice sends her qubit to Bob, so that he has both qubits

Superdense Coding

- If Alice wants to send ***cd* = 11**, she applies both *X* and Z gate to her qubit

- Applying *X* transforms $|\Phi^+\rangle$ to $|\Psi^+\rangle$, and applying *Z* transforms $|\Psi^+\rangle$, to

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left(|0_A 1_B\rangle - |1_A 0_B\rangle\right)$$

- Formally

$$\left(Z_A X_A \otimes I_B\right)|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(Z_A X_A |0_A\rangle \otimes I_B |0_B\rangle + Z_A X_A |1_A\rangle \otimes I_B |1_B\rangle\right) = \frac{1}{\sqrt{2}}\left(Z_A |1_A\rangle \otimes |0_B\rangle + Z_A |0_A\rangle \otimes |1_B\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(-|1_A\rangle \otimes |0_B\rangle + |0_A\rangle \otimes |1_B\rangle\right) = \frac{1}{\sqrt{2}}\left(|0_A\rangle \otimes |1_B\rangle - |1_A\rangle \otimes |0_B\rangle\right) = |\Phi^-\rangle$$

- Then Alice sends her qubit to Bob, so that he has both qubits

- Note: *ZX* = *iY*

Superdense Coding

- Now Bob has both qubits, and they are in one of four states:

$$\left|\Phi^{+}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|0_A 0_B\right\rangle + \left|1_A 1_B\right\rangle\right)$$

$$\left|\Psi^{+}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|1_A 0_B\right\rangle + \left|0_A 1_B\right\rangle\right)$$

$$\left|\Phi^{-}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|0_A 0_B\right\rangle - \left|1_A 1_B\right\rangle\right)$$

$$\left|\Psi^{-}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|0_A 1_B\right\rangle - \left|1_A 0_B\right\rangle\right)$$

- Since these four states are orthonormal, they form a measurement basis.
- Bob can measure the two qubits in this *Bell basis* to distinguish them, thus determining what Alice wanted to send.
- This is called a *Bell measurement*

Superdense Coding

- Another way to understand the *Bell measurement* is to apply *CNOT* and then $H \otimes I,$ then measuring in the Z-basis. That is

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\left(|00\rangle + |10\rangle\right) = |+\rangle|0\rangle \xrightarrow{H\otimes I} |00\rangle$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\left(|01\rangle + |11\rangle\right) = |+\rangle|1\rangle \xrightarrow{H\otimes I} |01\rangle$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\left(|00\rangle - |10\rangle\right) = |-\rangle|0\rangle \xrightarrow{H\otimes I} |10\rangle$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\left(|01\rangle - |11\rangle\right) = |-\rangle|1\rangle \xrightarrow{H\otimes I} |11\rangle$$

Note: For the sake of simplicity, from now on we will leave out the subscripts

Superdense Coding

- Concluding

$$|\Phi^{+}\rangle \rightarrow |00\rangle \xrightarrow{Measurement} 00$$

$$|\Psi^{+}\rangle \rightarrow |01\rangle \xrightarrow{Measurement} 01$$

$$|\Phi^{-}\rangle \rightarrow |10\rangle \xrightarrow{Measurement} 10$$

$$|\Psi^{-}\rangle \rightarrow |11\rangle \xrightarrow{Measurement} 11$$

Superdense Coding

- Computationally, this protocol still requires two qubits, as it must because *Holevo's theorem* says that n qubits can only store n bits of classical information
- Yet as a communication protocol, it only requires one qubit to be sent
- Stated differently, Alice, interacting with only a single qubit, is able to transmit two bits of information to Bob
- Of course, two qubits are involved in the protocol, but *Alice never need interact with the second qubit*

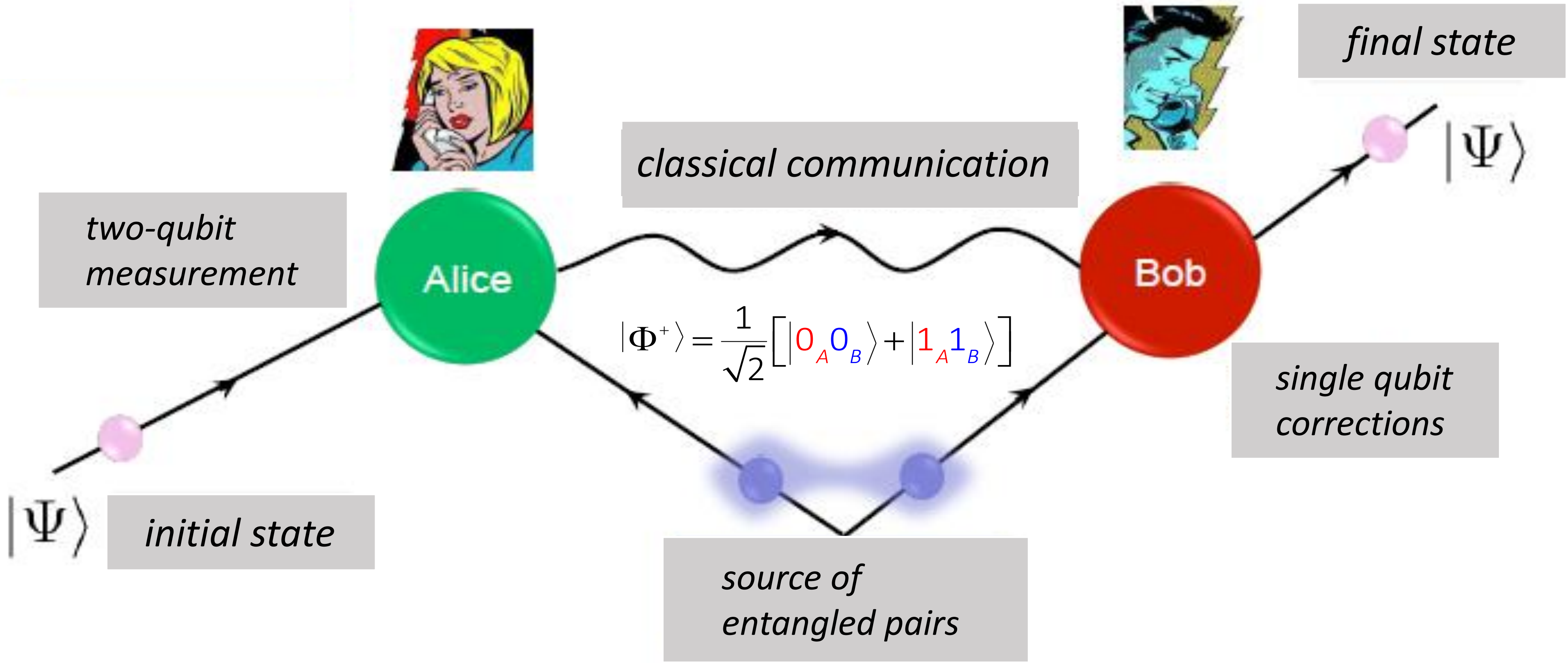
Principle of Deferred Measurement

- Often, quantum measurements are performed as an intermediate step in a quantum circuit, and the measurement results are used to conditionally control subsequent quantum gates
- This is the case, for example, in the teleportation circuit
- However, such measurements can always be moved to the end of the circuit

Principle of Deferred Measurement

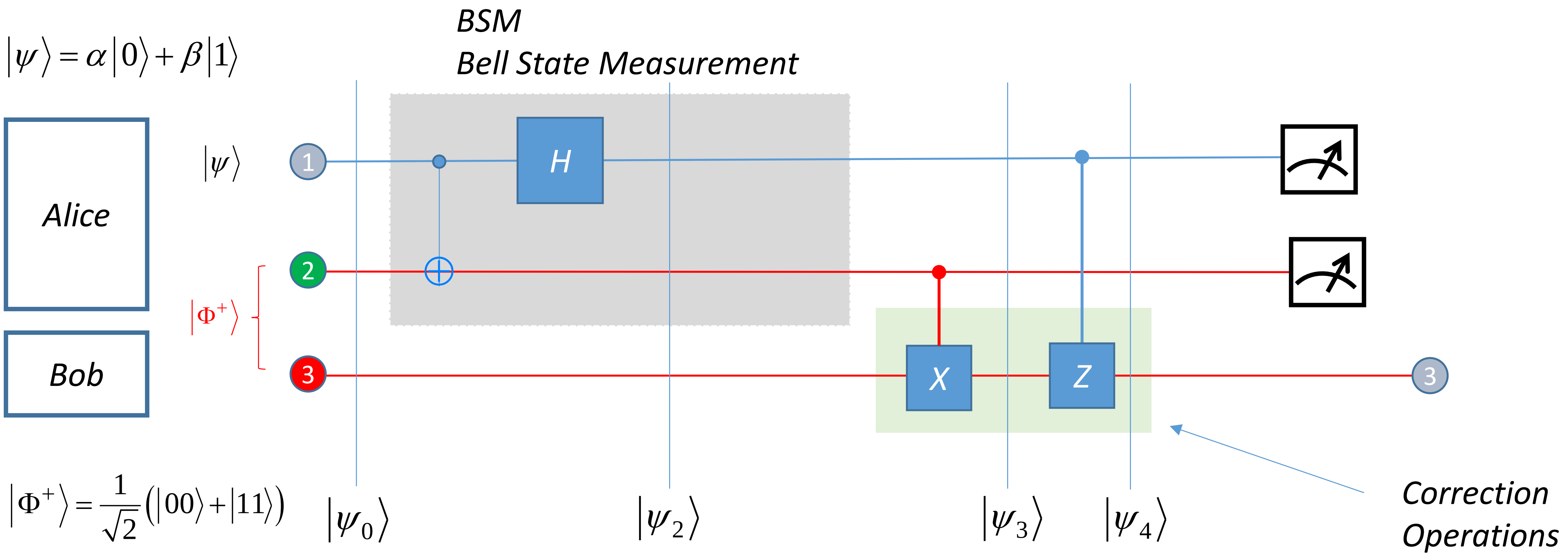
- The next slides illustrate how this may be done by replacing all the classical conditional operations by corresponding quantum conditional operations
- Of course, some of the interpretation of this circuit as performing ‘teleportation’ is lost, because no classical information is transmitted from Alice to Bob, but it is clear that the overall action of the two quantum circuits is the same, which is the key point

Principle of Deferred Measurement



Principle of Deferred Measurement

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$



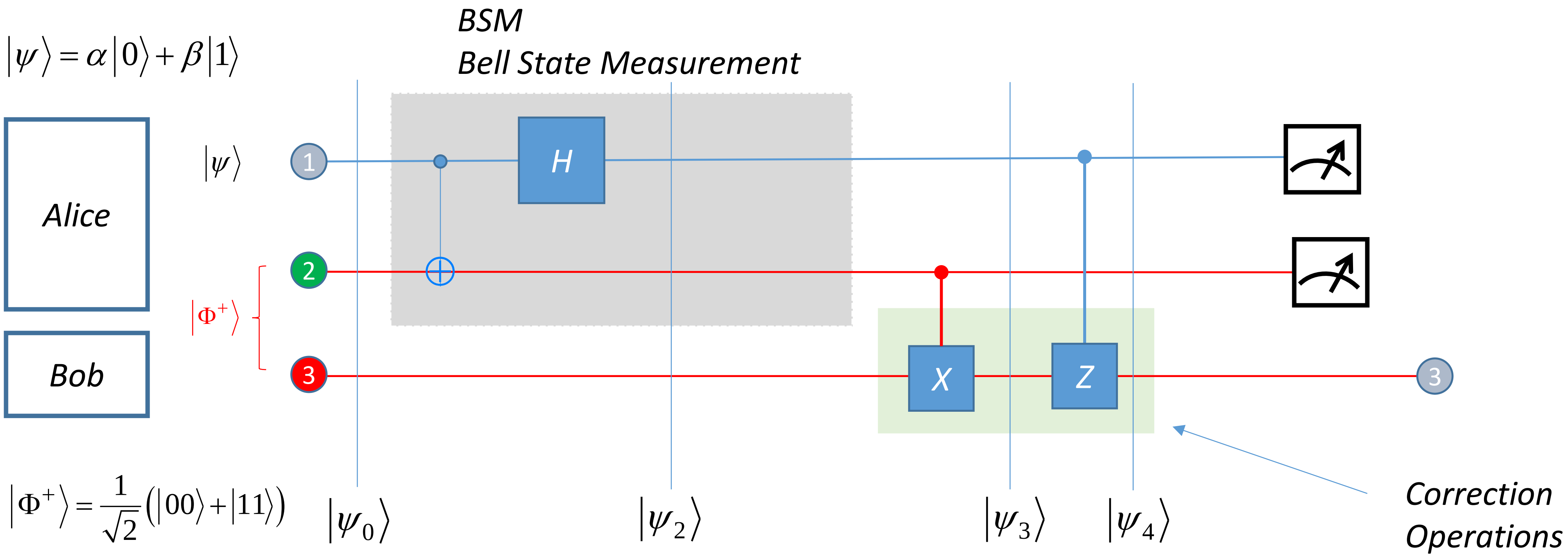
$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|\psi_0\rangle = |\psi\rangle \otimes |\Phi^+\rangle = (\alpha|0_1\rangle + \beta|1_1\rangle) \otimes \left(\frac{|0_2 0_3\rangle + |1_2 1_3\rangle}{\sqrt{2}}\right)$$

Principle of Deferred Measurement

$$|\psi_2\rangle = \frac{1}{2}\Big[\underbrace{|0_1 0_2\rangle}_{\textit{Alice}}\underbrace{\big(\alpha|0_3\rangle + \beta|1_3\rangle\big)}_{\textit{Bob}} + |0_1 1_2\rangle\big(\alpha|1_3\rangle + \beta|0_3\rangle\big) + |1_1 0_2\rangle\big(\alpha|0_3\rangle - \beta|1_3\rangle\big) + |1_1 1_2\rangle\big(\alpha|1_3\rangle - \beta|0_3\rangle\big)\Big]$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$



$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big)$$

Principle of Deferred Measurement

$$|\psi_2\rangle = \frac{1}{2}\Big[\underbrace{|0_1 0_2\rangle}_{\color{red}{Alice}}\underbrace{(\alpha|0_3\rangle + \beta|1_3\rangle)}_{\color{blue}{Bob}} + |0_1 1_2\rangle(\alpha|1_3\rangle + \beta|0_3\rangle) + |1_1 0_2\rangle(\alpha|0_3\rangle - \beta|1_3\rangle) + |1_1 1_2\rangle(\alpha|1_3\rangle - \beta|0_3\rangle)\Big]$$

– By applying *c-X* to $|\psi_2\rangle$ we obtain

$$|\psi_3\rangle = \frac{1}{2}\Big[|0_1 0_2\rangle(\alpha|0_3\rangle + \beta|1_3\rangle) + |0_1 1_2\rangle(\alpha|0_3\rangle + \beta|1_3\rangle) + |1_1 0_2\rangle(\alpha|0_3\rangle - \beta|1_3\rangle) + |1_1 1_2\rangle(\alpha|0_3\rangle - \beta|1_3\rangle)\Big]$$

– Finally, by applying *c-Z* to $|\psi_3\rangle$

$$|\psi_4\rangle = \frac{1}{2}\Big[|0_1 0_2\rangle(\alpha|0_3\rangle + \beta|1_3\rangle) + |0_1 1_2\rangle(\alpha|0_3\rangle + \beta|1_3\rangle) + |1_1 0_2\rangle(\alpha|0_3\rangle + \beta|1_3\rangle) + |1_1 1_2\rangle(\alpha|0_3\rangle + \beta|1_3\rangle)\Big]$$

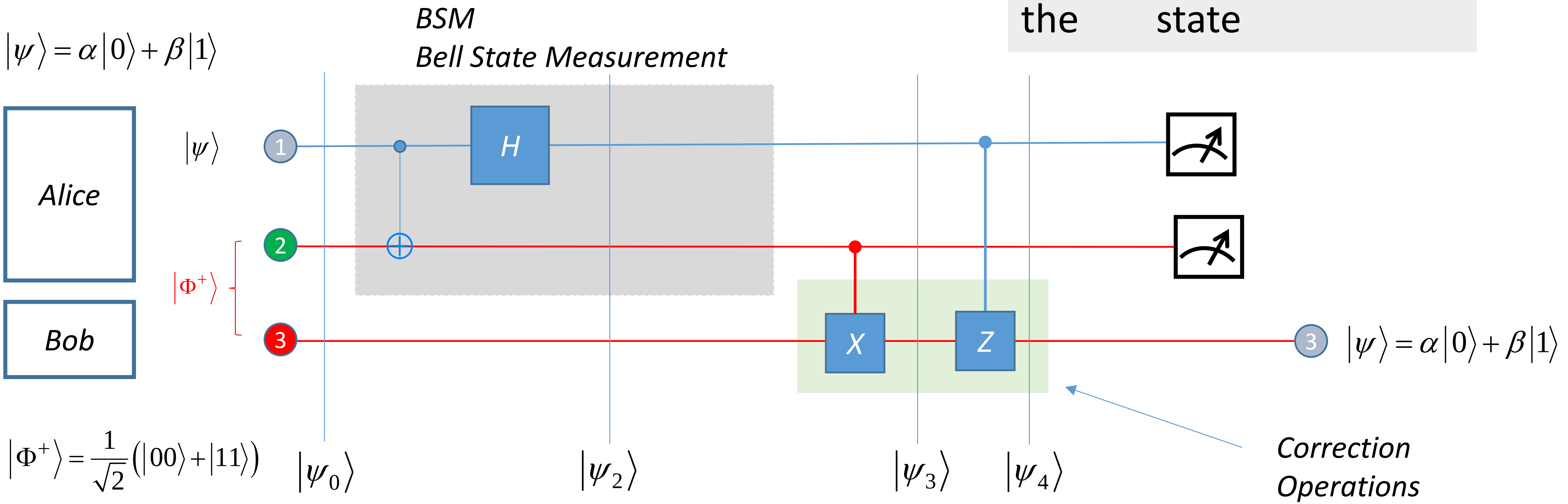
$$= \frac{1}{2}\Big[|0_1 0_2\rangle|\psi\rangle + |0_1 1_2\rangle|\psi\rangle + |1_1 0_2\rangle|\psi\rangle + |1_1 1_2\rangle|\psi\rangle\Big]$$

Principle of Deferred Measurement

$$|\psi_4\rangle = \frac{1}{2}\left[|0_1 0_2\rangle|\psi\rangle + |0_1 1_2\rangle|\psi\rangle + |1_1 0_2\rangle|\psi\rangle + |1_1 1_2\rangle|\psi\rangle \right]$$

Regardless of Alice's measurement outcome, Bob's qubit will end up in the state

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$



$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

Principle of Deferred Measurement

Principle of deferred measurement: Measurements can always be moved from an **intermediate** stage of a quantum circuit to the **end** of the circuit; if the measurement results are used at any stage of the circuit then the classically controlled operations can be replaced by conditional quantum operations