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Quantum Computing and Quantum Internet

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Quantum Operations

Environments and Quantum Operations

- The mathematical formalism of **quantum operations** is the key tool for our description of the dynamics of **open quantum systems**
- This tool is very powerful, in that it simultaneously addresses a wide range of physical scenarios
- It can be used to describe not only nearly closed systems which are weakly coupled to their environments, but also systems which are strongly coupled to their environments, and closed systems that are opened suddenly and subject to measurement

Environments and Quantum Operations

- The dynamics of a **closed** quantum system are described by a unitary transform
- Conceptually, we can think of the unitary transform *U* as a box into which the input state enters and from which the output exits, as illustrated on the figure

- For our purposes, the interior workings of the box are not of concern to us; it could be implemented by a quantum circuit, or anything else

Environments and Quantum Operations

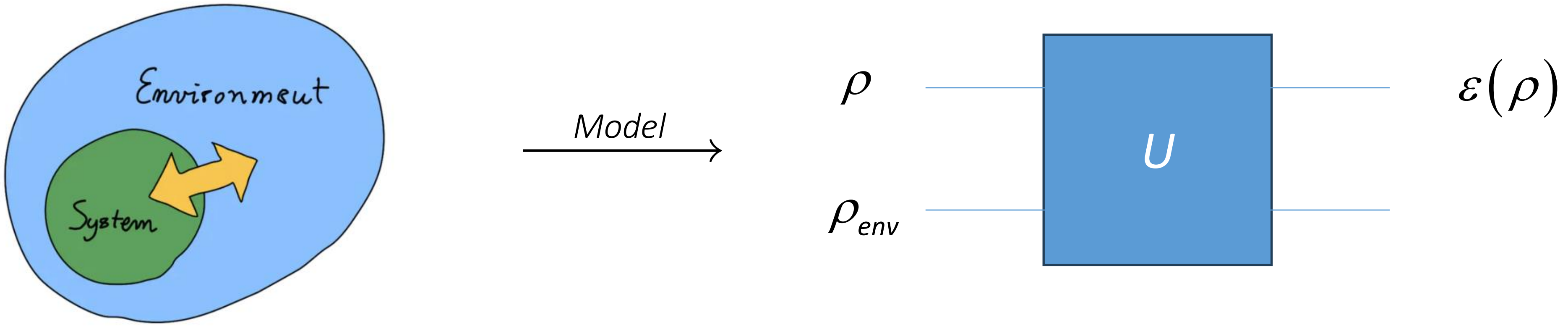
- The map ε in this equation is a quantum operation
- Another simple example of quantum operations which we have encountered previously is measurement, for which

$$\varepsilon_m(\rho) = M_m \rho M_m^\dagger$$

- The quantum operation captures the dynamic change to a state which occurs as the result of some physical process; ρ is the initial state before the process, and $\varepsilon(\rho)$ is the final state after the process occurs

Environments and Quantum Operations

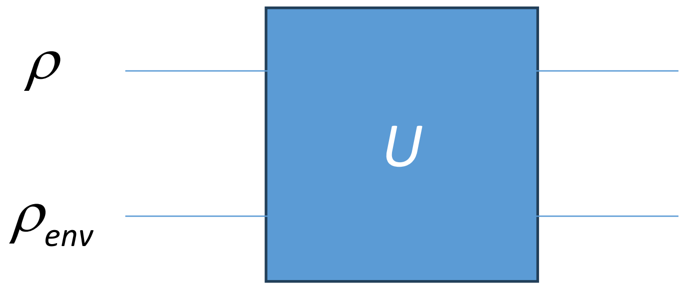
- A natural way to describe the dynamics of an **open** quantum system is to regard it as arising from an interaction between the system of interest, which we shall call the *principal system*, and an *environment*, which together form a *closed quantum system*, as illustrated on the figure



- In other words, suppose we have a system in state ρ, which is sent into a box which is coupled to an environment

Environments and Quantum Operations

- In general, the final state of the system, $\varepsilon(\rho)$, may not be related by a unitary transformation to the initial state ρ
- We assume (for now) that the system-environment input state is a product state, $\rho \otimes \rho_{env}$
- After the box's transformation U the system no longer interacts with the environment, and thus we perform a partial trace over the environment to obtain the reduced state of the system alone:



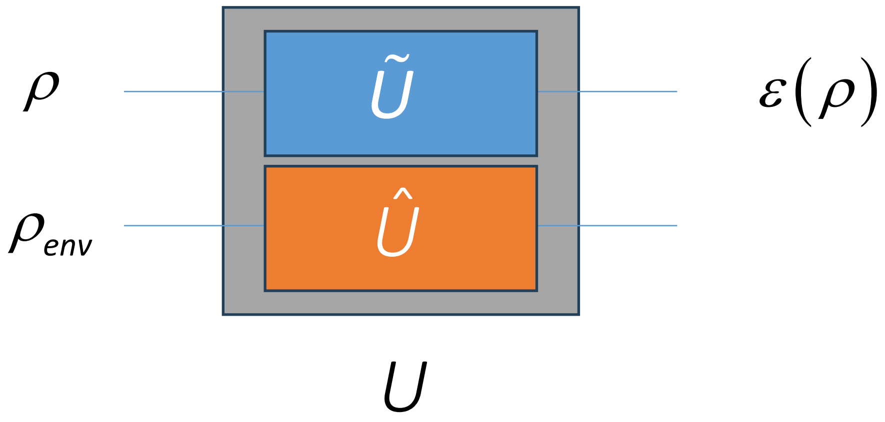
$$\varepsilon(\rho) \rightarrow \boxed{\varepsilon(\rho) = \mathrm{tr}_{env}\left[U(\rho \otimes \rho_{env})U^{\dagger}\right]} \quad (1)$$

Environments and Quantum Operations

- Of course, if U does not involve any interaction with the environment, then

$$\varepsilon(\rho) = \tilde{U}\rho\tilde{U}^{\dagger}$$

where $\tilde{U}$ is the part of U which acts on the system alone



Environments and Quantum Operations

- An important assumption is made in this definition - we assume that the system and the environment start in a product state
- In general, of course, this is not true since quantum systems interact constantly with their environments, building up correlations

Environments and Quantum Operations

- However, in many cases of practical interest it is reasonable to assume that the system and its environment start out in a product state
- When an experimentalist prepares a quantum system in a specified state they undo all the correlations between that system and the environment
- Ideally, the correlations will be completely destroyed, leaving the system in a pure state
- Even if this is not the case, the quantum operations formalism can even describe quantum dynamics when the system and environment do not start out in a product state

Environments and Quantum Operations

- Another issue one might raise is: how can *U* be specified if the environment has nearly infinite degrees of freedom?
- It turns out, very interestingly, that in order for this model to properly describe any possible transformation $\rho \to \varepsilon(\rho)$, if the ***principal system*** has a Hilbert space of ***d*** dimensions, then it suffices to model the ***environment*** as being in a Hilbert space of **no more** than $\boldsymbol{d^2}$ dimensions
- It also turns out **not** to be necessary for the environment to start out in a mixed state; a **pure state** will do

Operator-sum Representation

- Quantum operations can be represented in an elegant form known as the **operator-sum representation**, which is essentially a re-statement of Equation $\varepsilon(\rho)=\mathrm{tr}_{\mathrm{env}}\left[U(\rho\otimes\rho_{\mathrm{env}})U^{\dagger}\right]$ explicitly in terms of operators on the **principal** system's Hilbert space alone
- The main result is motivated by the following simple calculation
- Let $|e_k\rangle$ be an orthonormal basis for the (finite-dimensional) state space of the environment, and let $\rho_{\mathrm{env}}=|e_0\rangle\langle e_0|$ be the initial state of the environment

Operator-sum Representation

- There is no loss of generality in assuming that the environment starts in a pure state, since if it starts in a mixed state, we are free to introduce an extra system purifying the environment
- Although this extra system is *fictitious*, it makes no difference to the dynamics experienced by the principal system and thus can be used as an intermediate step in calculations

Operator-sum Representation

- Equation $\varepsilon(\rho) = \mathrm{tr}_{\mathrm{env}}\left[U(\rho \otimes \rho_{\mathrm{env}})U^\dagger\right]$ can thus be rewritten as

$$\varepsilon(\rho) = \sum_k \langle e_k | U\left(\rho \otimes |e_0\rangle\langle e_0|\right)U^\dagger | e_k \rangle \qquad (6)$$

$$= \sum_k E_k \rho E_k^\dagger, \qquad (7)$$

where $E_k \equiv \langle e_k | U | e_0 \rangle$ is an operator on the state space of the **principal system**

- Equation (7) is known as the **operator-sum representation of $\mathcal{E}$**

- The operators $\{E_k\}$ are known as **operation elements** for the quantum operation $\mathcal{E}$

Operator-sum Representation

Proof of (7)

- To rigorously formalize the partial trace, the following adjustment would be beneficial, while taking a partial trace solely on the environment basis e_k as mentioned above

$$\left|e_k\right\rangle \longrightarrow \mathrm{I}_{PS} \otimes \left|e_k\right\rangle, \quad \left\langle e_k\right| \longrightarrow \mathrm{I}_{PS} \otimes \left\langle e_k\right| \tag{8}$$

  where I_{PS} is the identity operator of the *Principal System (PS)*

- According to this notation, (6) needs to be written as

$$\varepsilon(\rho) = \sum_k \left(\mathrm{I}_{PS} \otimes \left\langle e_k\right|\right) U\left(\rho \otimes \left|e_0\right\rangle\left\langle e_0\right|\right) U^\dagger \left(\mathrm{I}_{PS} \otimes \left\langle e_k\right|\right) \tag{9}$$

For the sake of simplicity, we don't use the *PS* subscript to ρ

Operator-sum Representation

- Furthermore

$$E_k \equiv \langle e_k | U | e_0 \rangle \;\rightarrow\; \left(\mathrm{I}_{PS} \otimes \langle e_k | \right) U \left(\mathrm{I}_{PS} \otimes | e_0 \rangle \right) \tag{10}$$

- The trick allowing for suitably rearranging within each term of the partial trace lies in expanding

$$\rho \otimes | e_0 \rangle \langle e_0 |$$

into products as

$$\rho \otimes | e_0 \rangle \langle e_0 | = \underbrace{\left(\rho \otimes \mathrm{I}_E \right) \left(\mathrm{I}_{PS} \otimes | e_0 \rangle \right)}_{= \left(\mathrm{I}_{PS} \otimes \langle e_0 | \right) \rho} \left(\mathrm{I}_{PS} \otimes \langle e_0 | \right) = \left(\mathrm{I}_{PS} \otimes | e_0 \rangle \right) \rho \left(\mathrm{I}_{PS} \otimes \langle e_0 | \right)$$

Operator-sum Representation

- where the last equality can be seen through

$$(\rho\otimes \mathrm{I}_E)(\mathrm{I}_{PS}\otimes|e_0\rangle)=\underbrace{(\rho \mathrm{I}_{PS})}_{=\mathrm{I}_{PS}\rho}\otimes\underbrace{(\mathrm{I}_E|e_0\rangle)}_{=|e_0\rangle=|e_0\rangle\times 1}$$

$$=(\mathrm{I}_{PS}\rho)\otimes(|e_0\rangle\times 1)=(\mathrm{I}_{PS}\otimes|e_0\rangle)(\rho\otimes 1)=(\mathrm{I}_{PS}\otimes|e_0\rangle)\rho$$

- Thus

$$\rho\otimes|e_0\rangle\langle e_0|=(\mathrm{I}_{PS}\otimes|e_0\rangle)\rho(\mathrm{I}_{PS}\otimes\langle e_0|) \qquad (11)$$

- Note the usage of the so-called *mixed product property*

$$A\otimes B \quad C\otimes D \;=\; AC \;\otimes\; BD$$

Operator-sum Representation

$$\rho \otimes |e_0\rangle\langle e_0| = (\mathrm{I}_{PS} \otimes |e_0\rangle)\rho(\mathrm{I}_{PS} \otimes \langle e_0|)$$

- Then, considering the k^{th} term in the partial trace sum (9), by using (11) we have that

$$(\mathrm{I}_{PS} \otimes \langle e_k|)\left(U\left(\rho \otimes |e_0\rangle\langle e_0|\right)U^{\dagger}\right)(\mathrm{I}_{PS} \otimes |e_k\rangle)$$

$$= \underbrace{(\mathrm{I}_{PS} \otimes \langle e_k|)U(\mathrm{I}_{PS} \otimes |e_0\rangle)}_{=E_k}\rho\underbrace{(\mathrm{I}_{PS} \otimes \langle e_0|)U^{\dagger}(\mathrm{I}_{PS} \otimes |e_k\rangle)}_{=E_k^{\dagger}}$$

$$= E_k \rho E_k^{\dagger} \qquad (12)$$

$$\rightarrow \boxed{\varepsilon(\rho) = \sum_k E_k \rho E_k^{\dagger}}$$

■

Operator-sum Representation

- The operation elements satisfy an important constraint known as the **completeness relation** which arises from the requirement that the trace of $\varepsilon(\rho)$ be equal to one,

$$1 = \mathrm{tr}\left[\varepsilon(\rho)\right] = \mathrm{tr}\left(\sum_k E_k \rho E_k^\dagger\right) = \sum_k \mathrm{tr}\left(E_k \rho E_k^\dagger\right)$$

$$= \sum_k \mathrm{tr}\left(E_k^\dagger E_k \rho\right) = \mathrm{tr}\left(\sum_k E_k^\dagger E_k \rho\right)$$

- Since this relationship is true for all ρ it follows that we must have

$$\sum_k E_k^\dagger E_k = I \qquad (8)$$

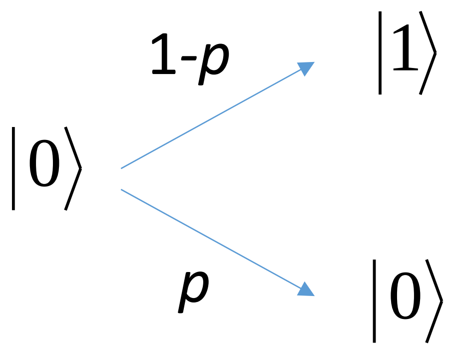
Examples of Quantum Noise

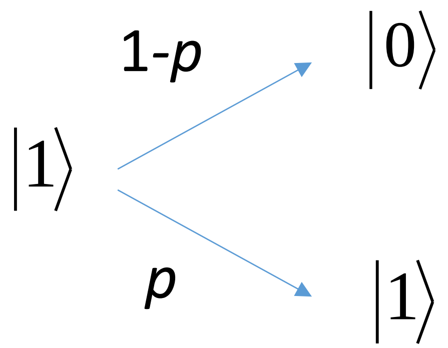
Examples of Quantum Noise

- Bit-Flip Channel
- Phase-Flip Channel
- Bit-Phase Flip Channel
- Depolarizing Channel
- Amplitude-Damping Channel
- Phase-Damping Channel

The Bit-Flip Channel

- The *bit flip* channel flips the state of a qubit from $|0\rangle$ to $|1\rangle$ (and vice versa) with probability 1 – *p*





- Consider the input state

$$|\psi_1\rangle = \alpha|0\rangle + \beta|1\rangle \quad \rightarrow \quad \rho = |\psi_1\rangle\langle\psi_1|$$

The Bit-Flip Channel

- Now, with probability 1-*p*, the input state $\left|\psi_1\right\rangle$ flips to state

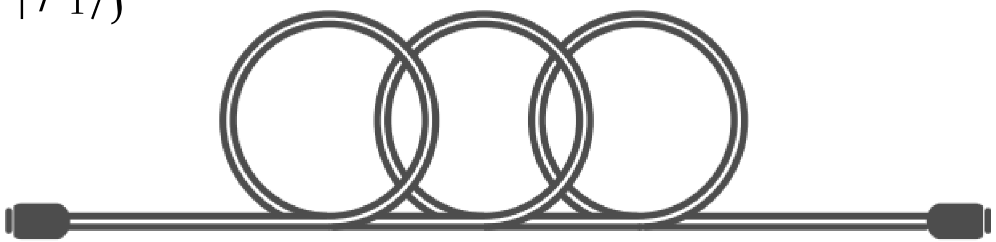
  $$\left|\psi_2\right\rangle = \alpha\left|1\right\rangle + \beta\left|0\right\rangle = \alpha X\left|0\right\rangle + \beta X\left|1\right\rangle = X\left(\alpha\left|0\right\rangle + \beta\left|1\right\rangle\right) = X\left|\psi_1\right\rangle$$

  and with probability *p*, the state $\left|\psi_1\right\rangle$ remains unchanged

- Thus, the final state (at the output of the quantum channel) can be represented by the ensemble

  $$\left\{p,\left|\psi_1\right\rangle;1-p,\left|\psi_2\right\rangle\right\} \equiv \left\{p,\left|\psi_1\right\rangle;1-p, X\left|\psi_1\right\rangle\right\}$$

$\left|\psi_1\right\rangle$



Bit-flip channel

$p,\ \left|\psi_1\right\rangle$

$1-p,\ X\left|\psi_1\right\rangle$

The Bit-Flip Channel

- Given the ensemble $\left\{p, |\psi_1\rangle; 1-p, X|\psi_1\rangle\right\}$ the output density matrix is

$$\rho' = \varepsilon(\rho) = p|\psi_1\rangle\langle\psi_1| + (1-p)X|\psi_1\rangle\langle\psi_1|X^\dagger$$

- Since, $\rho = |\psi_1\rangle\langle\psi_1|$ and $X^\dagger = X \quad \rightarrow$

$$\rho' = \varepsilon(\rho) = p\cdot\rho + (1-p)\cdot X\rho X^\dagger = \sqrt{p}I\rho I\sqrt{p} + \sqrt{1-p}X\rho X\sqrt{1-p}$$

$$= E_0\rho E_0 + E_1\rho E_1 = \sum_{k=0}^{1} E_k\rho E_k$$

The Bit-Flip Channel

- From

$$\rho' = \varepsilon(\rho) = E_0 \rho E_0 + E_1 \rho E_1 = \sum_{k=0}^{1} E_k \rho E_k$$

the *Kraus* operators are

$$E_0 = \left(\sqrt{p}\right) \cdot I = \sqrt{p}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad E_1 = \left(\sqrt{1-p}\right) \cdot X = \sqrt{1-p}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The Bit-Flip Channel (Stinespring Representation)

The Stinespring representation of the channel is:

- **System and Environment**: Let the system qubit be in state $|\psi_1\rangle$ and the environment qubit initially in $|0\rangle$
- **Unitary Evolution**: Define a unitary operation *U* on the combined system + environment, which entangles the system with the environment
- **Final State and Partial Trace**: After applying *U*, we obtain the combined state. To retrieve the output state of the system qubit, we take the **partial trace over the environment**

The Bit-Flip Channel (Stinespring Representation)

To construct the unitary *U*, consider the following actions:

- When the environment qubit is in |0⟩: apply $\sqrt{p}I$ to the system
- When the environment qubit is in |1⟩: apply $\sqrt{1-p}\ X$ to the system

Explicitly, let the initial state be:

$$\left|\psi_{in}\right\rangle = \left|\psi_1\right\rangle \otimes \left|0\right\rangle$$

then, applying *U* to $\left|\psi_{in}\right\rangle$ yields:

$$\left|\psi_{out}\right\rangle = U\left|\psi_{in}\right\rangle = U\ \left|\psi_1\right\rangle \otimes \left|0\right\rangle\ = \sqrt{p}\ I\left|\psi_1\right\rangle\ \otimes \left|0\right\rangle + \sqrt{1-p}\ X\left|\psi_1\right\rangle\ \otimes \left|1\right\rangle$$

$$\left\langle\psi_{out}\right| = \left\langle\psi_{in}\right|U^{\dagger} =\ \left\langle\psi_{in}\right| \otimes \left\langle 0\right|\ U^{\dagger} = \sqrt{p}\ \left\langle\psi_1\right|I\ \otimes \left\langle 0\right| + \sqrt{1-p}\ \left\langle\psi_1\right|X\ \otimes \left\langle 1\right|$$

The Bit-Flip Channel (Stinespring Representation)

Therefore

$$
\begin{aligned}
\rho_{out} &= |\psi_{out}\rangle\langle\psi_{out}| \\
&= \ \sqrt{p}|\psi_1\rangle \otimes |0\rangle + \sqrt{1-p}\ \ X|\psi_1\rangle\ \otimes|1\rangle\ \ \sqrt{p}\langle\psi_1| \otimes \langle 0| + \sqrt{1-p}\ \ \langle\psi_1|X\ \ \otimes\langle 1| \\
&= \left[\sqrt{p}|\psi_1\rangle \otimes |0\rangle\right]\left[\sqrt{p}\langle\psi_1| \otimes \langle 0|\right] + \left[\sqrt{p}|\psi_1\rangle \otimes |0\rangle\right]\left[\sqrt{1-p}\ \ \langle\psi_1|X\ \ \otimes\langle 1|\right] \\
&\quad + \left[\sqrt{1-p}\ \ X|\psi_1\rangle\ \otimes|1\rangle \sqrt{p}\langle\psi_1| \otimes \langle 0|\right] + \left[\sqrt{1-p}\ \ X|\psi_1\rangle\ \otimes|1\rangle \sqrt{1-p}\ \ \langle\psi_1|X\ \ \otimes\langle 1|\right] \\
&= p|\psi_1\rangle\langle\psi_1| \otimes |0\rangle\langle 0| + \sqrt{p}\sqrt{1-p}|\psi_1\rangle\langle\psi_1|X \otimes |0\rangle\langle 1| \\
&\quad + \sqrt{1-p}\sqrt{p}X|\psi_1\rangle\langle\psi_1| \otimes |1\rangle\langle 0| +\ 1-p\ \ X|\psi_1\rangle\langle\psi_1|X \otimes |1\rangle\langle 1|
\end{aligned}
$$

The Bit-Flip Channel (Stinespring Representation)

Then

$$
\begin{aligned}
\varepsilon\ \rho\ &= tr_{env}\ \ \rho_{out}\ \ = \langle 0|\rho_{out}|0\rangle + \langle 1|\rho_{out}|1\rangle \\
&= p\rho \otimes \langle 0|0\rangle\langle 0|0\rangle + \sqrt{p}\sqrt{1-p}\rho X \otimes \langle 0|0\rangle {\color{red}\langle 1|0\rangle} \\
&\quad + \sqrt{1-p}\sqrt{p}X\rho \otimes {\color{red}\langle 0|1\rangle}\langle 0|0\rangle + \ 1-p\ \ X\rho X \otimes {\color{red}\langle 0|1\rangle\langle 1|0\rangle} \\
&\quad\ \ p\rho \otimes {\color{red}\langle 1|0\rangle\langle 0|1\rangle} + \sqrt{p}\sqrt{1-p}\rho X \otimes {\color{red}\langle 1|0\rangle}\langle 1|1\rangle \\
&\quad + \sqrt{1-p}\sqrt{p}X\rho \otimes \langle 1|1\rangle{\color{red}\langle 0|1\rangle} + \ 1-p\ \ X\rho X \otimes \langle 1|1\rangle\langle 1|1\rangle
\end{aligned}
$$

The Bit-Flip Channel (Stinespring Representation)

Bearing in mind that that

$$\langle 0|0\rangle = \langle 1|1\rangle = 1$$

$$\langle 0|1\rangle = \langle 1|0\rangle = 0$$

we obtain

$$\varepsilon \quad \rho \quad = p\rho \otimes 1 + \quad 1 - p \quad X\rho X \otimes 1$$

Since the tensor product of a matrix for 1 equals the matrix, it goes like this

$$\varepsilon \quad \rho \quad = p\rho + \quad 1 - p \quad X\rho X$$

i.e., the expression for $\varepsilon \quad \rho$ that we were expecting

The Bit-Flip Channel (Stinespring Representation)

By making use of U we can obtain the operation elements E_0 and E_1, namely

$E_0 = \langle 0|U|0\rangle$, $E_1 = \langle 1|U|0\rangle$

The Bit-Flip Channel

- Let's now verify that the set $\{E_0, E_1\}$ of quantum operators is complete

$$\sum_{k=0}^{1} E_k E_k^\dagger = I$$

Proof

$$E_0 E_0^\dagger + E_1 E_1^\dagger = \left(\sqrt{p}\right)\cdot\left(\sqrt{p}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \left(\sqrt{1-p}\right)\cdot\left(\sqrt{1-p}\right)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= p\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (1-p)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

The Bit-Flip Channel

- Let's now show the effect of the bit flip channel on the Bloch ball
- Let's start from the expression

$$\varepsilon(\rho) = p \cdot \rho + (1-p) \cdot X \rho X^{\dagger}$$

where ρ can be written as

$$\rho = \frac{1}{2}(I + \mathbf{p}.\boldsymbol{\sigma}) = \frac{1}{2}\left(I + p_x X + p_y Y + p_z Z\right)$$

- Also, keeping in consideration that $X^{\dagger} = X \quad \longrightarrow$

$$\varepsilon(\rho) = p \cdot \frac{1}{2}\left(I + p_x X + p_y Y + p_z Z\right) + \left(\frac{1-p}{2}\right) \cdot \underbrace{X\left(I + p_x X + p_y Y + p_z Z\right)X}_{B}$$

The Bit-Flip Channel

- Let's develop *B*

$$B = X\left(I + p_x X + p_y Y + p_z Z\right)X = XIX + p_x XXX + p_y XYX + p_z XZX$$

- Keeping into consideration that

$$XIX = X^2 = I$$

$$XXX = XX^2 = XI = X \quad \longrightarrow \quad B = I + p_x X - p_y Y - p_z Z$$

$$XYX = -Y$$

$$XZX = -Z$$

The Bit-Flip Channel

- Thus

$$\varepsilon(\rho) = p \cdot \frac{1}{2}\left(I + p_x X + p_y Y + p_z Z\right) + \left(\frac{1-p}{2}\right) \cdot \left(I + p_x X - p_y Y - p_z Z\right)$$

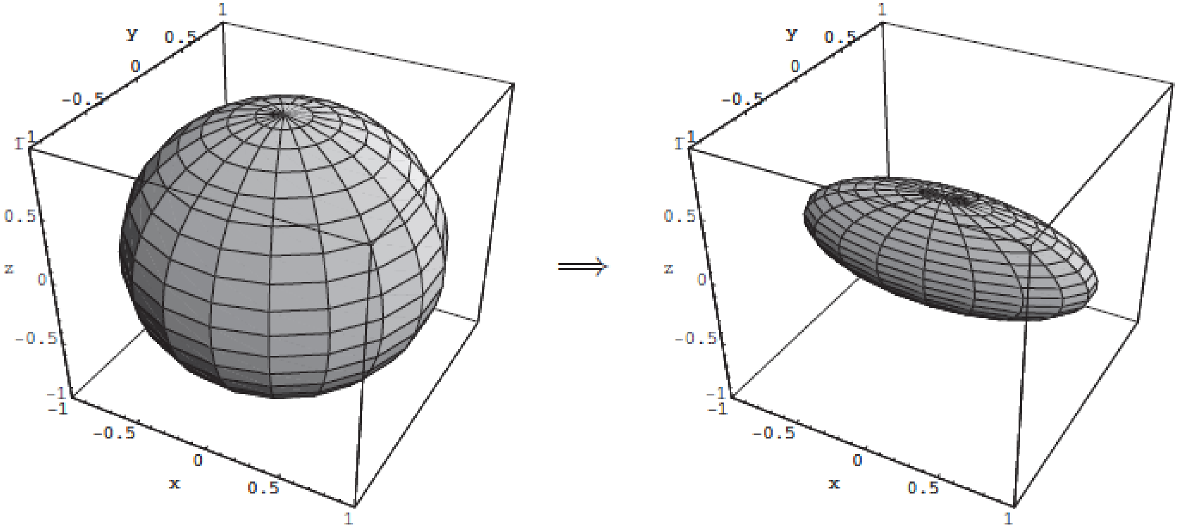
$$= \frac{1}{2}\left(\cancel{pI} + \cancel{pp_x X} + pp_y Y + pp_z Z + I + p_x X - p_y Y - p_z Z - \cancel{pI} - \cancel{pp_x X} + pp_y Y + pp_z Z\right)$$

$$= \frac{1}{2}\left[I + p_x X + p_y(2p-1)Y + p_z(2p-1)Z\right] \quad \longrightarrow$$

$$\varepsilon(\rho) = \frac{1}{2}\left[I + p_x X + p_y(2p-1)Y + p_z(2p-1)Z\right]$$

The Bit-Flip Channel

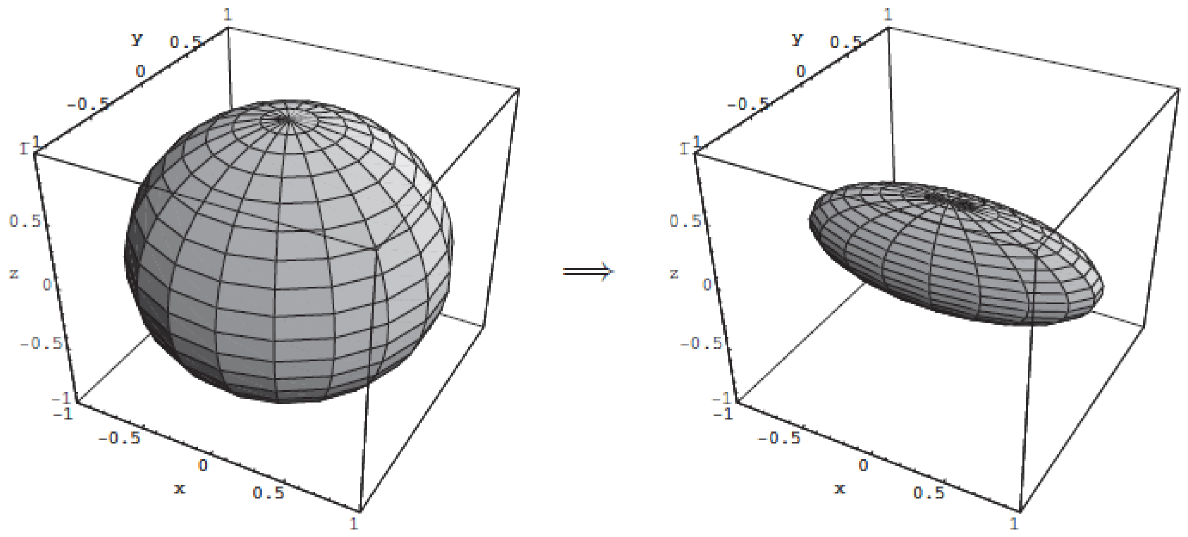
- The effect of the bit flip channel on the Bloch ball, for $p = 0.3$.
- The sphere on the left represents the set of all pure states, and the deformed ball on the right represents the states after going through the channel.



$$\varepsilon(\rho) = \frac{1}{2}\left[I + p_x X + p_y (2p-1) Y + p_z (2p-1) Z\right]$$

The Bit-Flip Channel

Note that the states on the *x* axis are left alone, while the *y*-*z* plane is uniformly contracted by a factor of 1 − 2*p*.



$$\varepsilon(\rho) = \frac{1}{2}\left[I + p_x X + p_y (2p-1) Y + p_z (2p-1) Z \right]$$

The Phase-Flip Channel or Dephasing Channel

- The *phase flip* channel flips the state $|\psi_1\rangle$ of a qubit from $|1\rangle$ to $-|1\rangle$ with probability 1 – *p*

$$|0\rangle \longrightarrow |0\rangle, \quad |1\rangle \longrightarrow -|1\rangle$$

and with probability *p*, the state $|\psi_1\rangle$ remains unchanged

- Consider the input state

$$|\psi_1\rangle = \alpha|0\rangle + \beta|1\rangle \quad \longrightarrow \quad \rho = |\psi_1\rangle\langle\psi_1|$$

The Phase-Flip Channel or Dephasing Channel

- Now, with probability 1-*p*, the input state $\left|\psi_1\right\rangle$ flips to state

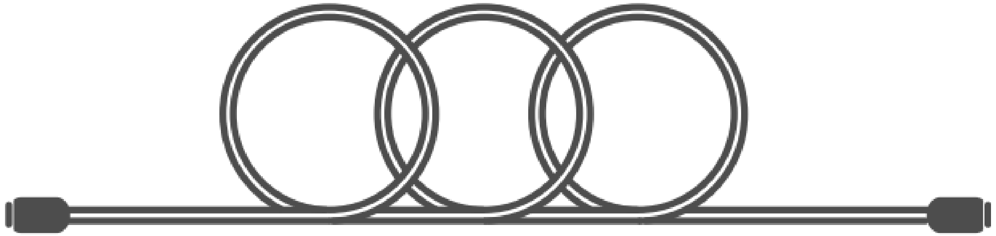
  $$\left|\psi_2\right\rangle = \alpha\left|0\right\rangle - \beta\left|1\right\rangle = \alpha Z\left|0\right\rangle + \beta Z\left|1\right\rangle = Z\left(\alpha\left|0\right\rangle + \beta\left|1\right\rangle\right) = Z\left|\psi_1\right\rangle$$

  and with probability *p*, the state $\left|\psi_1\right\rangle$ remains unchanged

- Thus, the final state (at the output of the quantum channel) can be represented by the ensemble

  $$\left\{p,\left|\psi_1\right\rangle;1-p,\left|\psi_2\right\rangle\right\} \equiv \left\{p,\left|\psi_1\right\rangle;1-p,Z\left|\psi_1\right\rangle\right\}$$

$\left|\psi_1\right\rangle$



$p,\ \left|\psi_1\right\rangle$

$1-p,\ Z\left|\psi_1\right\rangle$

Phase-flip channel

The Phase-Flip Channel or Dephasing Channel

- Given the ensamble $\left\{p, |\psi_1\rangle; 1-p, Z|\psi_1\rangle\right\}$ the output density matrix is

$$\rho' = \varepsilon(\rho) = p|\psi_1\rangle\langle\psi_1| + (1-p)Z|\psi_1\rangle\langle\psi_1|Z^\dagger$$
$$\varepsilon(\rho) = p\cdot\rho + (1-p)\cdot Z\rho Z^\dagger$$

- Since, $\rho = |\psi_1\rangle\langle\psi_1|$ and $Z^\dagger = Z \rightarrow$

$$\rho' = \varepsilon(\rho) = p\cdot\rho + (1-p)\cdot Z\rho Z^\dagger = \sqrt{p}(I\rho I)\sqrt{p} + \sqrt{1-p}(Z\rho Z)\sqrt{1-p}$$
$$= E_0\rho E_0 + E_1\rho E_1 = \sum_{k=0}^{1} E_k\rho E_k$$

The Phase-Flip Channel or Dephasing Channel

- From

$$\rho' = \varepsilon(\rho) = E_0 \rho E_0 + E_1 \rho E_1 = \sum_{k=0}^{1} E_k \rho E_k$$

the *Kraus* operators are

$$E_0 = \left(\sqrt{p}\right) \cdot I = \sqrt{p} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad E_1 = \left(\sqrt{1-p}\right) \cdot Z = \sqrt{1-p} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The Phase-Flip Channel or Dephasing Channel

- Let's now verify that the set $\{E_0, E_1\}$ of quantum operators is complete

$$\sum_{k=0}^{1} E_k E_k^\dagger = I$$

Proof

$$E_0 E_0^\dagger + E_1 E_1^\dagger = \left(\sqrt{p}\right)\cdot\left(\sqrt{p}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \left(\sqrt{1-p}\right)\cdot\left(\sqrt{1-p}\right)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= p\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (1-p)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

The Phase-Flip Channel or Dephasing Channel

- Let's now show the effect of the bit flip chennel on the Bloch sphere
- Let's start from the expression

$$\varepsilon(\rho) = p \cdot \rho + (1-p) \cdot Z \rho Z^{\dagger}$$

where ρ can be written as

$$\rho = \frac{1}{2}(I + \mathbf{p}.\boldsymbol{\sigma}) = \frac{1}{2}\left(I + p_x X + p_y Y + p_z Z\right)$$

- Also, keeping into consideration that $Z^{\dagger} = Z \quad \longrightarrow$

$$\varepsilon(\rho) = p \cdot \frac{1}{2}\left(I + p_x X + p_y Y + p_z Z\right) + \left(\frac{1-p}{2}\right) \cdot \underbrace{Z\left(I + p_x X + p_y Y + p_z Z\right)Z}_{B}$$

The Phase-Flip Channel or Dephasing Channel

- Let's develop *B*

$$B = Z\left(I + p_x X + p_y Y + p_z Z\right)Z = ZIZ + p_x ZXZ + p_y ZYZ + p_z ZZZ$$

- Keeping into consideration that

$$ZIZ = Z^2 = I$$

$$ZZZ = ZZ^2 = ZI = Z \qquad \longrightarrow \qquad B = I - p_x X - p_y Y + p_z Z$$

$$ZXZ = -X$$

$$ZYZ = -Y$$

The Phase-Flip Channel or Dephasing Channel

- Thus

$$\varepsilon(\rho) = p \cdot \frac{1}{2}\left(I + p_x X + p_y Y + p_z Z\right) + \left(\frac{1-p}{2}\right) \cdot \left(I - p_x X - p_y Y + p_z Z\right)$$

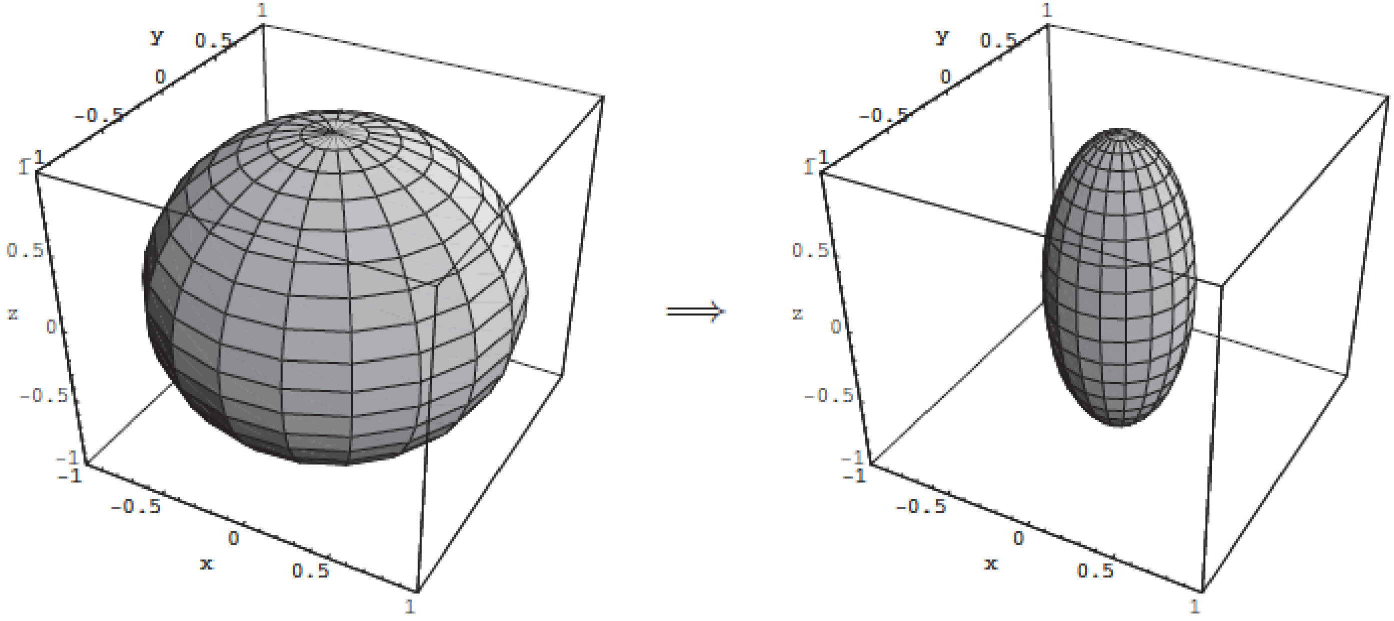
$$= \frac{1}{2}\left(\cancel{pI} + pp_x X + pp_y Y + \cancel{pp_z Z} + I - p_x X - p_y Y + p_z Z - \cancel{pI} + pp_x X + pp_y Y - \cancel{pp_z Z}\right)$$

$$= \frac{1}{2}\left[I + (2p-1)p_x X + (2p-1)p_y Y + p_z Z\right] \longrightarrow$$

$$\varepsilon(\rho) = \frac{1}{2}\left[I + (2p-1)p_x X + (2p-1)p_y Y + p_z Z\right]$$

The Phase-Flip Channel or Dephasing Channel

- The effect of the phase flip channel on the Bloch sphere, for p = 0.3
- Note that the states on the *z-axis* are left alone, while the plane is uniformly*xz*- contracted by a factor of 1 − 2*p*



$$\varepsilon(\rho) == \frac{1}{2}\left[I + (2p-1)\, p_x X + (2p-1)\, p_y Y + p_z Z\right]$$

The Phase-Flip Channel or Dephasing Channel

- As a special case of the **phase flip channel**, consider the quantum operation which arises when we choose p = 1/2.
- In that case

$$\varepsilon(\rho)=\frac{1}{2}\left[I+p_z Z\right]$$

- Geometrically, the Bloch vector is projected along the z axis, and the x and y components of the Bloch vector are lost.

The Bit-Phase Flip Channel

- Up to a global phase factor, which can be omitted due to its lack of physical significance, the *bit-phase flip* of a qubit state $|\psi\rangle$ can be represented by the Pauli *Y=iXZ* matrix, i.e.,

$$Y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

- It is straightforward to verify that $Y|0\rangle = i|1\rangle, \; Y|1\rangle = -i|0\rangle$.

- If $|\psi_1\rangle = a|0\rangle + b|1\rangle \longrightarrow Y|\psi_1\rangle = Y\left(a|0\rangle + b|1\rangle\right) = aY|0\rangle + bY|1\rangle = i\left(a|1\rangle - b|0\rangle\right)$

- Thus, *Y* induces a combined **bit-phase flip** on the state $|\psi_1\rangle$

The Bit-Phase Flip Channel

- Now, with probability 1-*p*, the input state $\left|\psi_1\right\rangle$ flips to state

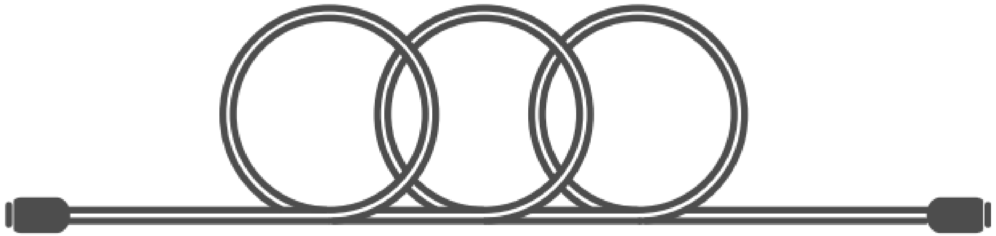
  $$\left|\psi_2\right\rangle = \alpha\left|1\right\rangle - \beta\left|0\right\rangle = \alpha XZ\left|0\right\rangle + \beta XZ\left|1\right\rangle = XZ\left(\alpha\left|0\right\rangle + \beta\left|1\right\rangle\right) \equiv iXZ\left|\psi_1\right\rangle = Y\left|\psi_1\right\rangle$$

  and with probability *p*, the state $\left|\psi_1\right\rangle$ remains unchanged

- Thus, the final state (at the output of the quantum channel) can be represented by the ensemble

  $$\left\{p,\left|\psi_1\right\rangle;1-p,\left|\psi_2\right\rangle\right\} \equiv \left\{p,\left|\psi_1\right\rangle;1-p,Y\left|\psi_1\right\rangle\right\}$$

$\left|\psi_1\right\rangle$



$p,\ \left|\psi_1\right\rangle$

$1-p,\ Y\left|\psi_1\right\rangle$

Bit-phase flip channel

The Bit-Phase Flip Channel

- Given the ensamble $\left\{p,|\psi_1\rangle;1-p,Y|\psi_1\rangle\right\}$ the output density matrix is

$$\rho' = \varepsilon(\rho) = p|\psi_1\rangle\langle\psi_1| + (1-p)Y|\psi_1\rangle\langle\psi_1|Y^\dagger$$

- Since, $\rho = |\psi_1\rangle\langle\psi_1|$ and $Y^\dagger = Y$ $\rightarrow$

$$\rho' = \varepsilon(\rho) = p\cdot\rho + (1-p)\cdot Y\rho Y^\dagger = \sqrt{p}\,(I\rho I)\sqrt{p} + \sqrt{1-p}\,(Y\rho Y)\sqrt{1-p}$$

$$= E_0\rho E_0 + E_1\rho E_1 = \sum_{k=0}^{1} E_k\rho E_k$$

The Bit-Phase Flip Channel

- From

$$\rho' = \varepsilon(\rho) = E_0 \rho E_0 + E_1 \rho E_1 = \sum_{k=0}^{1} E_k \rho E_k$$

the *Kraus* operators are

$$E_0 = \left(\sqrt{p}\right) \cdot I = \sqrt{p} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad E_1 = \left(\sqrt{1-p}\right) \cdot Y = \sqrt{1-p} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

The Bit-Phase Flip Channel

- The bit–phase flip channel has operation elements

$$E_0 = \left(\sqrt{p}\right)\cdot I = \sqrt{p}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad E_1 = \left(\sqrt{1-p}\right)\cdot Y = \sqrt{1-p}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

- As the name indicates, this is a combination of a **phase flip** and a **bit flip**, since $Y = iXZ$

The Bit-Phase Flip Channel

- Let's now verify that the set $\{E_0, E_1\}$ of quantum operators is complete

$$\sum_{k=0}^{1} E_k E_k^{\dagger} = I$$

Proof

$$E_0 E_0^{\dagger} + E_1 E_1^{\dagger} = \left(\sqrt{p}\right)\cdot\left(\sqrt{p}\right)\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \left(\sqrt{1-p}\right)\cdot\left(\sqrt{1-p}\right)\begin{bmatrix}0 & -i\\i & 0\end{bmatrix}\begin{bmatrix}0 & -i\\i & 0\end{bmatrix}$$

$$= p\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + (1-p)\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = I$$

The Bit-Phase Flip Channel

- Let’s now show the effect of the bit-phase flip chennel on the Bloch sphere
- Let’s start from the expression

$$\varepsilon(\rho) = p \cdot \rho + (1-p) \cdot Y \rho Y^{\dagger}$$

where ρ can be written as

$$\rho = \frac{1}{2}(I + \mathbf{p}.\boldsymbol{\sigma}) = \frac{1}{2}\left(I + p_x X + p_y Y + p_z Z\right)$$

- Also, keeping into consideration that $Y^{\dagger} = Y \quad \longrightarrow$

$$\varepsilon(\rho) = p \cdot \frac{1}{2}\left(I + p_x X + p_y Y + p_z Z\right) + \left(\frac{1-p}{2}\right) \underbrace{\cdot Y\left(I + p_x X + p_y Y + p_z Z\right)Y}_{B}$$

The Bit-Phase Flip Channel

- Let's develop *B*

$$B = Y\left(I + p_x X + p_y Y + p_z Z\right)Y = YIY + p_x YXY + p_y YYY + p_z YZY$$

- Keeping into consideration that

$$YIY = Y^2 = I$$

$$YZY = -Z$$

$$YXY = -X$$

$$YYY = IY = Y$$

$$\longrightarrow \quad B = I - p_x X + p_y Y - p_z Z$$

The Bit-Phase Flip Channel

- Thus

$$\varepsilon(\rho) = p \cdot \frac{1}{2}\left(I + p_x X + p_y Y + p_z Z\right) + \left(\frac{1-p}{2}\right) \cdot \left(I - p_x X + p_y Y - p_z Z\right)$$

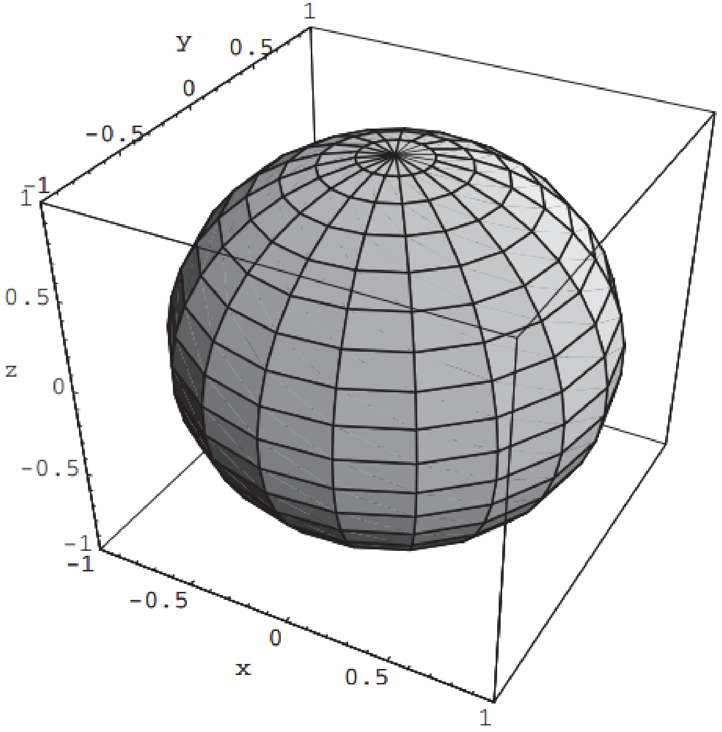
$$= \frac{1}{2}\left(\cancel{pI} + pp_x X + \cancel{pp_y Y} + pp_z Z + I - p_x X + p_y Y - p_z Z - \cancel{pI} + pp_x X - \cancel{pp_y Y} + pp_z Z\right)$$

$$= \frac{1}{2}\left[I + (2p-1) p_x X + p_y Y + (2p-1) p_z Z\right] \longrightarrow$$

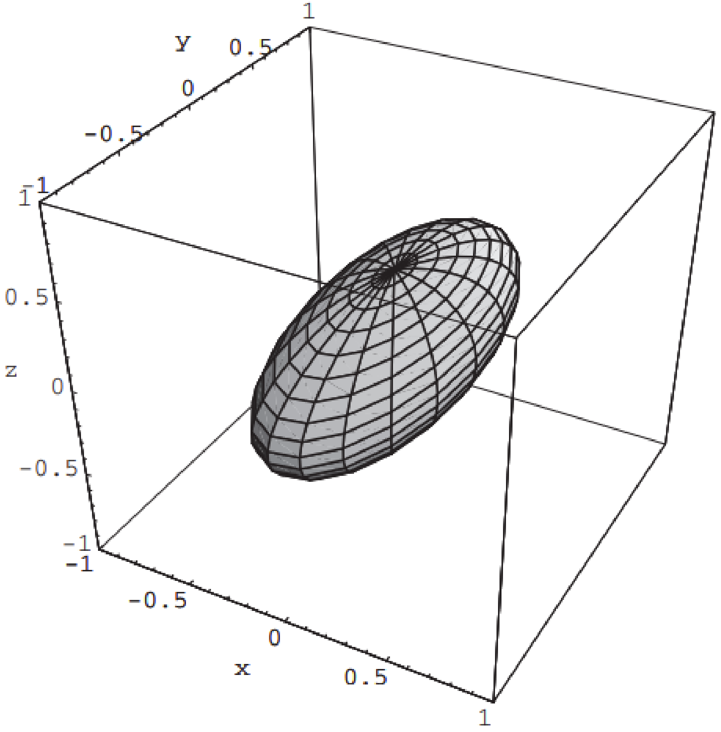
$$\varepsilon(\rho) = \frac{1}{2}\left[I + (2p-1) p_x X + p_y Y + (2p-1) p_z Z\right]$$

The Bit-Phase Flip Channel

- The effect of the bit - phase flip channel on the Bloch sphere, for $p = 0.3$
- Note that the states on the y-axis are left alone, while the x-z plane is uniformly contracted by a factor of $1 - 2p$



$\Longrightarrow$



Depolarizing Channel

- Imagine we take a single qubit, and
  - with probability *p* that qubit is **depolarized**, *i.e.* it is replaced by the completely mixed state, *I*/2
  - With probability 1−*p* the qubit is left untouched
- The state of the quantum system after this noise is

$$\varepsilon(\rho) = \left(\frac{I}{2}\right) p + (1-p)\rho$$

- This form **is not** in the **operator-sum representation**

Depolarizing Channel

- However, if we observe that for arbitrary ρ

$$\frac{I}{2} = \frac{\rho + X\rho X + Y\rho Y + Z\rho Z}{4} \qquad \rightarrow$$

$$\varepsilon(\rho) = \left(\frac{I}{2}\right)p + (1-p)\rho = (1 - \frac{3}{4}p)I\rho I + \frac{p}{4}(X\rho X + Y\rho Y + Z\rho Z)$$

showing that the depolarizing channel has operation elements

$$E_0 = \left(\sqrt{1-3p/4}\right)I \qquad E_1 = \sqrt{p}\,X/2 \qquad E_2 = \sqrt{p}\,Y/2 \qquad E_3 = \sqrt{p}\,Z/2$$

Depolarizing Channel

- Let’s now prove that the Kraus operators satisfy the completeness relation

$$\sum_{k=0}^{3} E_k^\dagger E_k = E_0^\dagger E_0 + E_1^\dagger E_1 + E_2^\dagger E_2 + E_3^\dagger E_3 = I$$

$$\sum_{k=0}^{3} E_k^\dagger E_k = (1-p)I + \frac{p}{3}X^\dagger X + \frac{p}{3}Y^\dagger Y + \frac{p}{3}Z^\dagger Z$$

Depolarizing Channel

- Since

$$X^\dagger X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \qquad Y^\dagger Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$Z^\dagger Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

- Then

$$\sum_{k=0}^{3} E_k^\dagger E_k = (1-p)I + \frac{p}{3}I + \frac{p}{3}I + \frac{p}{3}I = I$$

Depolarizing Channel

- Let's now investigate the effect of the depolarizing channel on the Bloch sphere

- Let's start from

$$\varepsilon(\rho) = \left(\frac{I}{2}\right)p + (1-p)\rho \qquad \text{where} \qquad \rho = \frac{1}{2}\left(I + p_x X + p_y Y + p_z Z\right)$$

- Then

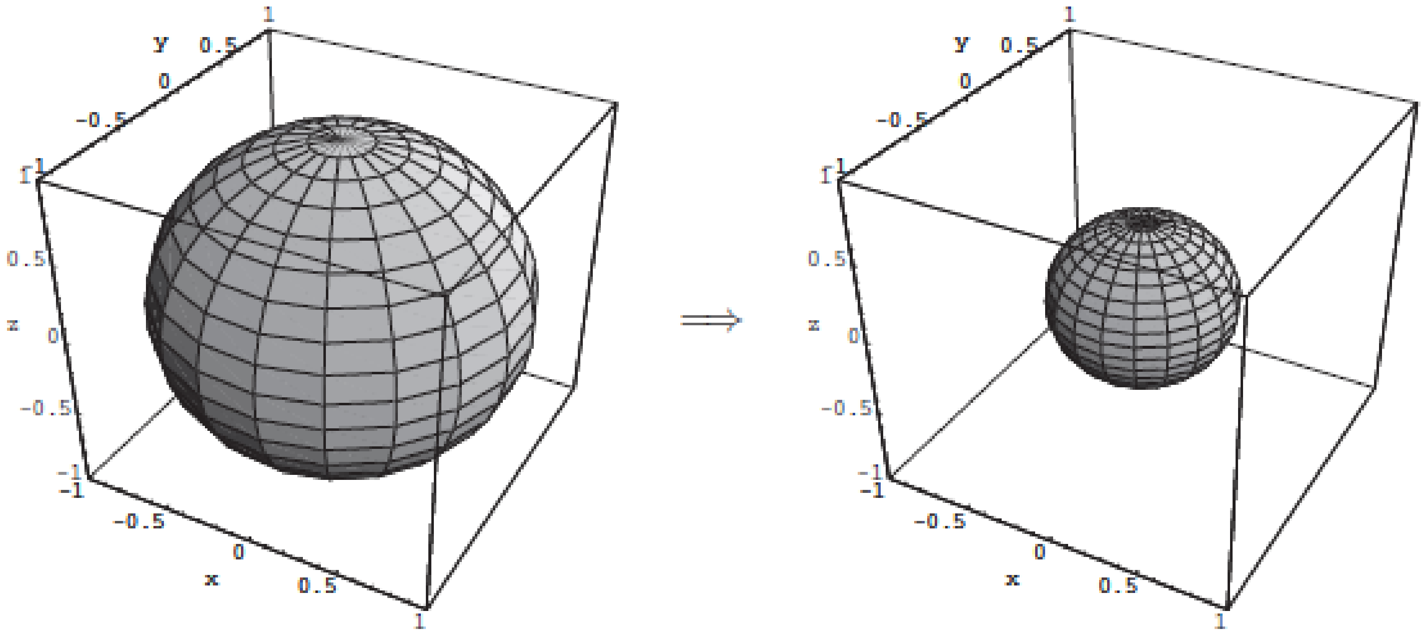
$$\varepsilon(\rho) = \left(\frac{I}{2}\right)p + (1-p)\rho = \left(\frac{I}{2}\right)p + \frac{1}{2}(1-p)\left(I + p_x X + p_y Y + p_z Z\right)$$

Depolarizing Channel

- After some algebraic manipulation

$$\varepsilon(\rho) = \left(\frac{I}{2}\right)p + (1-p)\rho = \frac{1}{2}\left[I + (1-p)p_x X + (1-p)p_y Y + (1-p)p_z Z\right]$$

- Then



The effect of the depolarizing channel on the Bloch sphere, for p= 0.5. Note how the entire sphere contracts uniformly as a function of p.

Depolarizing Channel

- The depolarizing channel can, of course, be generalized to quantum systems of dimension more than two
- For a *d*-dimensional quantum system the depolarizing channel again replaces the quantum system with the completely mixed state *I*/*d* with probability *p* and leaves the state untouched otherwise
- The corresponding quantum operation is

$$\varepsilon(\rho) = p\frac{I}{d} + (1-p)\rho$$

Exercise

- Let's prove that

$$\frac{I}{2} = \frac{\rho + X\rho X + Y\rho Y + Z\rho Z}{4}$$

- To start off, let's calculate $X\rho X$ by exploiting the general expression

$$\rho = \frac{1}{2}\left(I + p_x X + p_y Y + p_z Z\right)$$

- Thus

$$X\rho X = \frac{1}{2}\left(XIX + Xp_x XX + Xp_y YX + Xp_z ZX\right) = \frac{1}{2}\left(X^2 + p_x XX^2 + p_y XYX + p_z XZX\right)$$

Exercise

- Since

$$X^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I, \quad XX^2 = XI = X,$$

$$XYX = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = -\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} = -Y$$

$$XZX = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = -Z$$

Exercise

- Thus

$$X\rho X = \frac{1}{2}\left(I + p_x X - p_y Y - p_z Z\right)$$

- By taking the same approach, it can be easily veryfied that

$$Y\rho Y = \frac{1}{2}\left(I - p_x X + p_y Y - p_z Z\right)$$

$$Z\rho Z = \frac{1}{2}\left(I - p_x X - p_y Y + p_z Z\right)$$

Exercise

- By summing the previous results

$$X\rho X + Y\rho Y + Z\rho Z = \frac{1}{2}\left(3I - p_x X - p_y Y - p_z Z\right) = \frac{3}{2}I - \frac{p_x X + p_y Y + p_z Z}{2}$$

- Since

$$\rho = \frac{1}{2}\left(I + p_x X + p_y Y + p_z Z\right) \to \frac{p_x X + p_y Y + p_z Z}{2} = \rho - \frac{I}{2}$$

$$X\rho X + Y\rho Y + Z\rho Z = \frac{1}{2}\left(3I - p_x X - p_y Y - p_z Z\right) = \frac{3}{2}I - \rho + \frac{1}{2}I = 2I - \rho$$

Exercise

- After a simple algebraic manipulation

$$X\rho X + Y\rho Y + Z\rho Z + \rho = 2I$$

from where

$$\frac{\rho + X\rho X + Y\rho Y + Z\rho Z}{4} = \frac{I}{2} \qquad \blacksquare$$

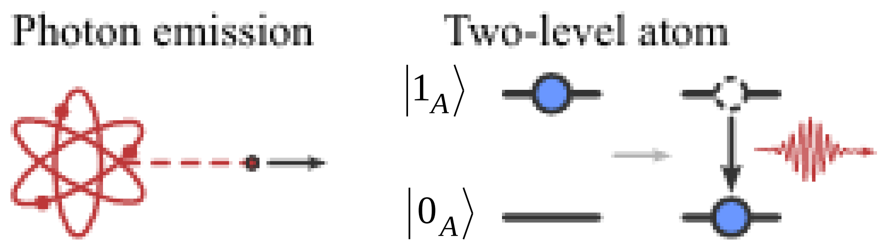
The Amplitude-Damping Channel

- An important application of quantum operations is the description of energy dissipation – effects due to loss of energy from a quantum system such as an atom which is **spontaneously** emitting a photon

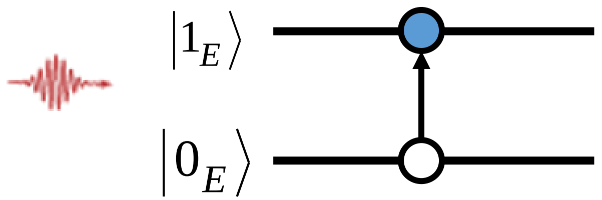
The Amplitude-Damping Channel

- Each of these type of processes which **dissipate energy,** has its own unique features, but the general behavior of all of them is well characterized by a quantum operation known as *amplitude damping*
- Amplitude damping describes a **decay process**
- The example we will use is an atom decaying from an excited energy state $|1_A\rangle$ to the ground state $|0_A\rangle$
- When an atom decays, it emits a photon, so the environment goes from $|0_E\rangle \rightarrow |1_E\rangle$

Principal System



Environment



The Amplitude-Damping Channel

- Suppose that there is a probability *p* for the process $|1_A\rangle \rightarrow |0_A\rangle$ to occur and a probability 1−*p* that the atom just stays in the state $|1_A\rangle$
- This quantum process can be described by a unitary operator *U* that acts as follows:

$$U|0_A\rangle|0_E\rangle = |0_A\rangle|0_E\rangle \qquad (1)$$

$$U|1_A\rangle|0_E\rangle = \sqrt{p}\,|0_A\rangle|1_E\rangle + \sqrt{1-p}\,|1_A\rangle|0_E\rangle \qquad (2)$$

The Amplitude-Damping Channel

- Equation (1) describes the situation where the atom and the environment are both in the ground state - nothing happens
- On the other hand, (2) describes the decay of the atom and excitation of the environment with probability *p* together with the atom remaining in the excited state with probability 1−*p*
- We can use this information to derive the Kraus operators for this damping process but first let's figure out what we can use for *U*

$$U|0_A\rangle|0_E\rangle = |0_A\rangle|0_E\rangle \quad (1)$$

$$U|1_A\rangle|0_E\rangle = \sqrt{p}\,|0_A\rangle|1_E\rangle + \sqrt{1-p}\,|1_A\rangle|0_E\rangle \quad (2)$$

The Amplitude-Damping Channel

- Let's introduce two new operators called the *ladder operators* $\sigma_\pm$
- The operator σ_+ (sometimes called the *raising operator*) takes the ground state to the excited state and **gives zero** when applied to the **excited state**:

$$\sigma_+|0\rangle = |1\rangle, \qquad \sigma_+|1\rangle = 0$$

- The operator σ_- takes the **excited state** to the **ground state** and eliminates or annihilates the ground state:

$$\sigma_-|0\rangle = 0 \qquad \sigma_-|1\rangle = |0\rangle$$

The Amplitude-Damping Channel

- We can build *U* by looking at (1) and (2) and using projection operators that project onto the appropriate state of the atom

- In the first case, (1), we want to project onto the $|0_A\rangle$ state while doing nothing to the environment

- We can do this with

$$U|0_A\rangle|0_E\rangle = |0_A\rangle|0_E\rangle$$

$$\Omega = |0_A\rangle\langle 0_A| \otimes I_E$$

The Amplitude-Damping Channel

- Looking at (2), we have two situations

$$U|1_A\rangle|0_E\rangle = \sqrt{p}\,|0_A\rangle|1_E\rangle + \sqrt{1-p}\,|1_A\rangle|0_E\rangle$$

- In the *first case*, we start with the atom in the $|1_A\rangle$ state and end in the $|0_A\rangle$ state

- This can be accomplished using σ_- which can be written as

$$\sigma_- = |0_A\rangle\langle 1_A|$$

The Amplitude-Damping Channel

$$U|1_A\rangle|0_E\rangle = \sqrt{p}|0_A\rangle|1_E\rangle + \sqrt{1-p}|1_A\rangle|0_E\rangle$$

- In the *second case* the environment goes from the ground state to the excited state, indicating the need to use $\sigma_+ = |1_E\rangle\langle 0_E|$
- This process occurs with probability *p*, so this piece of the operator is

$$\Delta = \sqrt{p}\,\big(\underbrace{|0_A\rangle\langle 1_A|}_{\sigma_-}\big)\otimes\big(\underbrace{|1_E\rangle\langle 0_E|}_{\sigma_+}\big) = \sqrt{p}\,(\sigma_- \otimes \sigma_+)$$

The Amplitude-Damping Channel

$$U|1_A\rangle|0_E\rangle = \sqrt{p}|0_A\rangle|1_E\rangle + \sqrt{1-p}|1_A\rangle|0_E\rangle$$

- Finally, with probability 1−*p*, the atom stays in the excited state and the environment stays in its initial state. This is described by

$$\Theta = \sqrt{1-p}|1_A\rangle\langle 1_A| \otimes I_E$$

- Putting Ω, Δ, and Θ together gives us the desired unitary operator for the amplitude-dumping channel

$$U = |0_A\rangle\langle 0_A| \otimes I_E + \sqrt{p}\left(\sigma_- \otimes \sigma_+\right) + \sqrt{1-p}|1_A\rangle\langle 1_A| \otimes I_E$$

The Amplitude-Damping Channel

- To figure out the Kraus operators we have to compute how *U* acts on the initial state of the environment as follows:

$$
\begin{aligned}
U|0_E\rangle &= |0_A\rangle\langle 0_A| \otimes \left(I_E|0_E\rangle\right) + \sqrt{p}\sigma_- \otimes \left(\sigma_+|0_E\rangle\right) + \sqrt{1-p}|1_A\rangle\langle 1_A| \otimes \left(I_E|0_E\rangle\right) \\
&= |0_A\rangle\langle 0_A| \otimes \left(|0_E\rangle\right) + \sqrt{p}\sigma_- \otimes \left(|1_E\rangle\right) + \sqrt{1-p}|1_A\rangle\langle 1_A| \otimes \left(|0_E\rangle\right)
\end{aligned}
$$

- So, the first Kraus operator is

$$
E_0 = \langle 0_E|U|0_E\rangle = |0_A\rangle\langle 0_A| \otimes \left(\langle 0_E|0_E\rangle\right)
$$

$$
+\sqrt{p}\sigma_- \otimes \left(\langle 0_E|1_E\rangle\right) + \sqrt{1-p}|1_A\rangle\langle 1_A| \otimes \left(\langle 0_E|0_E\rangle\right)
$$

The Amplitude-Damping Channel

- Since $\langle 0_E | 1_E \rangle = 0$ and $\langle 0_E | 0_E \rangle = 1$

$$E_0 = |0_A\rangle\langle 0_A| + \sqrt{1-p}\,|1_A\rangle\langle 1_A|$$

- In matrix form, using the $\{|0_A\rangle, |1_A\rangle\}$ basis results in

$$E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{bmatrix}$$

The Amplitude-Damping Channel

- The other Kraus operator is

$$E_1 = \langle 1_E | U | 0_E \rangle = |0_A\rangle\langle 0_A| \otimes \left(\langle 1_E | 0_E \rangle \right)$$

$$+\sqrt{p}\sigma_- \otimes \left(\langle 1_E | 1_E \rangle \right) + \sqrt{1-p}\, |1_A\rangle\langle 1_A| \otimes \left(\langle 1_E | 0_E \rangle \right)$$

$$= \sqrt{p}\sigma_- = |0_A\rangle\langle 1_A|$$

- The matrix representation is

$$E_1 = \begin{bmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{bmatrix}$$

The Amplitude-Damping Channel

- Let's now verify that the set $\{E_0, E_1\}$ of quantum operators is complete

$$\sum_{k=0}^{1} E_k E_k^\dagger = I$$

Proof

$$E_0 E_0^\dagger + E_1 E_1^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \sqrt{p} & 0 \end{bmatrix}\begin{bmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1-p \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & p \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \qquad \blacksquare$$

The Amplitude-Damping Channel
Time Evolution of a Qubit State

- The quantum operation $\varepsilon(\rho)$ for the amplitude-damping channel is

$$\varepsilon(\rho) = \sum_{k=0}^{1} E_k \rho E_k = E_0 \rho E_0 + E_1 \rho E_1$$

- For the general single qubit state $\rho = \begin{bmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{bmatrix}$ where $\rho_{00} + \rho_{11} = 1,\ \rho_{10}{}^* = \rho_{01}$

$$\varepsilon(\rho) = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{bmatrix} \begin{bmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{bmatrix} + \begin{bmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \sqrt{p} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{bmatrix} \begin{bmatrix} \rho_{00} & \rho_{01}\sqrt{1-p} \\ \rho_{10} & \rho_{11}\sqrt{1-p} \end{bmatrix} + \begin{bmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \rho_{01}\sqrt{p} & 0 \\ \rho_{11}\sqrt{p} & 0 \end{bmatrix}$$

The Amplitude-Damping Channel
Time Evolution of a Qubit State

- Continuing the algebraic manipulation

$$\varepsilon(\rho)=\begin{bmatrix}1 & 0\\ 0 & \sqrt{1-p}\end{bmatrix}\begin{bmatrix}\rho_{00} & \rho_{01}\sqrt{1-p}\\ \rho_{10} & \rho_{11}\sqrt{1-p}\end{bmatrix}+\begin{bmatrix}0 & \sqrt{p}\\ 0 & 0\end{bmatrix}\begin{bmatrix}\rho_{01}\sqrt{p} & 0\\ \rho_{11}\sqrt{p} & 0\end{bmatrix}$$

$$=\begin{bmatrix}\rho_{00} & \rho_{01}\sqrt{1-p}\\ \rho_{10}\sqrt{1-p} & \rho_{11}(1-p)\end{bmatrix}+\begin{bmatrix}\rho_{11}p & 0\\ 0 & 0\end{bmatrix}=\begin{bmatrix}\rho_{00}+\rho_{11}p & \rho_{01}\sqrt{1-p}\\ \rho_{10}\sqrt{1-p} & \rho_{11}(1-p)\end{bmatrix}$$

$$=\begin{bmatrix}\rho_{00}+\rho_{11}p+\rho_{11}-\rho_{11} & \rho_{01}\sqrt{1-p}\\ \rho_{10}\sqrt{1-p} & \rho_{11}(1-p)\end{bmatrix}=\begin{bmatrix}\rho_{00}+\rho_{11}p+\rho_{11}-\rho_{11} & \rho_{01}\sqrt{1-p}\\ \rho_{10}\sqrt{1-p} & \rho_{11}(1-p)\end{bmatrix}$$

The Amplitude-Damping Channel
Time Evolution of a Qubit State

- Thus

$$\varepsilon(\rho) = \begin{bmatrix} \rho_{00} + \rho_{11} - (1-p)\rho_{11} & \sqrt{1-p}\,\rho_{01} \\ \sqrt{1-p}\,\rho_{10} & (1-p)\rho_{11} \end{bmatrix} = \begin{bmatrix} \rho_{00} + p\rho_{11} & \sqrt{1-p}\,\rho_{01} \\ \sqrt{1-p}\,\rho_{10} & (1-p)\rho_{11} \end{bmatrix}$$

- If Γ is the *spontaneous decay rate* per *unit time*, then the decay occurs with probability $p = \Gamma \Delta t \ll 1$ in a small time-interval Δt
- We find the density operator after time $t = n\Delta t$ by applying the channel ***n*** times in succession

The Amplitude-Damping Channel
Time Evolution of a Qubit State

- Therefore, the evolution over a time $t = n\Delta t$ is governed by ε^n (ε repeated *n* times in succession)
- From linear algebra

$$\varepsilon^n(\rho) = \begin{bmatrix} \rho_{00} + \rho_{11} - \rho_{11}(1-p)^n & \rho_{01}(1-p)^{n/2} \\ \rho_{10}(1-p)^{n/2} & \rho_{11}(1-p)^n \end{bmatrix}$$

- Since $p = \Gamma\Delta t$ and $n = t/\Delta t$, with ***t* fixed** we have

$$\lim_{n\to\infty}(1-p)^n = \lim_{\Delta t\to 0}(1-\Gamma\Delta t)^{t/\Delta t} = e^{-\Gamma t}$$

The Amplitude-Damping Channel
Time Evolution of a Qubit State

- The ρ_{11} matrix element then decays as

$$\rho_{11} \mapsto (1-p)^n \rho_{11} = e^{-\Gamma t}\rho_{11}$$

which is the expected exponential decay law, while the off diagonal entries decay by the factor

$$\lim_{n\to\infty}(1-p)^{n/2} = \lim_{\Delta t\to 0}(1-\Gamma\Delta t)^{t/2\Delta t} = e^{-\Gamma t/2}$$

- Hence we find

$$\varepsilon(\rho(t)) = \begin{bmatrix} \rho_{00} + \left(1-e^{-\Gamma t}\right)\rho_{11} & e^{-\Gamma t/2}\rho_{01} \\ e^{-\Gamma t/2}\rho_{10} & e^{-\Gamma t}\rho_{11} \end{bmatrix}$$

The Amplitude-Damping Channel
Time Evolution of a Qubit State

- Letting *t* going to infinity

$$\lim_{t\to\infty}\varepsilon\left(\rho(t)\right)=\begin{bmatrix}\rho_{00}+\rho_{11} & 0\\ 0 & 0\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}=|0\rangle\langle 0|$$

the off-diagonal terms in the density operator become suppressed

$$\longrightarrow$$

- When applying a mixed state to an amplitude-damping channel, the output state is the pure state $|0\rangle\langle 0|$

Note On The Density Matrix Elements

- The terms ρ_{kk} on the main diagonal of a density matrix are referred to as the **population** of state $|k\rangle$
- The off-diagonal elements of a density matrix are called **coherence elements** or **coherences**, as they represent the correlations between different quantum states
- The physical process that leads from a coherently superposition state to a completely mixed state (which corresponds to the coherence elements of the density matrix transitioning from a value of 1 to 0) is a *decoherence process*

The Amplitude-Damping Channel

Time Evolution of a Pure State – Initial State $|1\rangle$

- The initial density matrix is

$$\rho(0) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = |1\rangle\langle 1|$$

- By applying the same approach as previously, we end up with the following result

$$\varepsilon(\rho(t)) = \begin{bmatrix} 1 - e^{-\Gamma t} & 0 \\ 0 & e^{-\Gamma t} \end{bmatrix}$$

$$\lim_{t\to\infty} \varepsilon(\rho(t)) = \begin{bmatrix} 1 - e^{-\Gamma t} & 0 \\ 0 & e^{-\Gamma t} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = |0\rangle\langle 0|$$

This shows that the probability of remaining in state |1⟩ decays exponentially with time, while the probability of staying in |0⟩ increases correspondingly

The Amplitude-Damping Channel

Time Evolution of a Pure State – Initial State $|+\rangle$

- The initial density matrix is

$$\rho(0)=\frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$$

- Again, by applying the same approach as previously, we end up with the following result

$$\varepsilon\left(\rho(t)\right)=\frac{1}{2}\begin{bmatrix}2-e^{-\Gamma t} & e^{-\Gamma t/2}\\ e^{-\Gamma t/2} & e^{-\Gamma t}\end{bmatrix}$$

The Amplitude-Damping Channel

Time Evolution of a Pure State – Initial State $|+\rangle$

Taking the limit as $t \to \infty$

$$\lim_{t\to\infty} \varepsilon\left(\rho(t)\right) = \lim_{t\to\infty} \frac{1}{2}\begin{bmatrix} 2-e^{-\Gamma t} & 0 \\ 0 & e^{-\Gamma t} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = |0\rangle\langle 0|$$

This result shows that:

- The probability for the qubit to stay in $|1\rangle$ decays over time as $e^{-\Gamma t}$
- The coherence terms ρ_{01} and ρ_{10} decay by $e^{-\Gamma t/2}$ reflecting the loss of *quantum coherence* as the qubit interacts with its environment

The AD Channel On The Bloch Sphere

- The Kraus operators can also be written as a linear combination of Pauli matrices & Identity (*I*) matrix

$$E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \sqrt{1-p} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{2}(I+Z) + \frac{\sqrt{1-p}}{2}(I-Z)$$

$$E_1 = \begin{bmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{bmatrix} = \sqrt{p} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \frac{\sqrt{p}}{2}(X+iY)$$

where $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; Y = i \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

The AD Channel On The Bloch Sphere

- To find the effect of the amplitude-dumping channel on the Bloch sphere let's start from the quantum operation $\varepsilon(\rho)$ previously derived

$$\varepsilon(\rho)=\begin{bmatrix} \rho_{00}+\rho_{11}-(1-p)\rho_{11} & \sqrt{1-p}\rho_{01} \\ \sqrt{1-p}\rho_{10} & (1-p)\rho_{11} \end{bmatrix}=\begin{bmatrix} \rho_{00}+p\rho_{11} & \sqrt{1-p}\rho_{01} \\ \sqrt{1-p}\rho_{10} & (1-p)\rho_{11} \end{bmatrix}$$

- Since $\varepsilon(\rho)$ is a density operator it can be written as

$$\varepsilon(\rho)=\frac{1}{2}(I+\mathbf{p}.\boldsymbol{\sigma})=\frac{1}{2}\left(I+p_xX+p_yY+p_zZ\right)$$

The AD Channel On The Bloch Sphere

- Let's write $\varepsilon(\rho)$ in matrix form

$$\varepsilon(\rho)=\frac{1}{2}\left(I+p_xX+p_yY+p_zZ\right)=\frac{1}{2}\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}+p_x\begin{bmatrix}0&1\\1&0\end{bmatrix}+p_y\begin{bmatrix}0&-i\\i&0\end{bmatrix}+p_z\begin{bmatrix}1&0\\0&-1\end{bmatrix}\right)$$

$$=\frac{1}{2}\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}+\begin{bmatrix}0&p_x\\p_x&0\end{bmatrix}+\begin{bmatrix}0&-ip_y\\ip_y&0\end{bmatrix}+\begin{bmatrix}p_z&0\\0&-p_z\end{bmatrix}\right)=\frac{1}{2}\begin{bmatrix}1+p_z&p_x-ip_y\\p_x+ip_y&1-p_z\end{bmatrix}$$

$$\rightarrow$$

$$\varepsilon(\rho)=\frac{1}{2}\begin{bmatrix}1+p_z&p_x-ip_y\\p_x+ip_y&1-p_z\end{bmatrix}$$

The AD Channel On The Bloch Sphere

- Therefore

$$\varepsilon(\rho)=\begin{bmatrix} \rho_{00}+\rho_{11}-(1-p)\rho_{11} & \sqrt{1-p}\,\rho_{01} \\ \sqrt{1-p}\,\rho_{10} & (1-p)\rho_{11} \end{bmatrix}=\frac{1}{2}\begin{bmatrix} 1+p_z & p_x-ip_y \\ p_x+ip_y & 1-p_z \end{bmatrix}$$

- By equating element by element $\longrightarrow$
  - $1/2(1+p_z)=\rho_{00}+\rho_{11}-(1-p)\rho_{11}$
  - $1/2(p_x-ip_y)=\rho_{01}\sqrt{1-p}$
  - $1/2(p_x+ip_y)=\rho_{10}\sqrt{1-p}$
  - $1/2(1-p_z)=\rho_{11}(1-p)$

The AD Channel On The Bloch Sphere

- $\frac{1}{2}(1+p_z)=\rho_{00}+\rho_{11}-(1-p)\rho_{11}$
- $\frac{1}{2}(p_x-ip_y)=\rho_{01}\sqrt{1-p}$
- $\frac{1}{2}(p_x+ip_y)=\rho_{10}\sqrt{1-p}$
- $\frac{1}{2}(1-p_z)=\rho_{11}(1-p)$

- The first and fourth equations lead to the same result

$$p_z=\rho_{00}-(1-2p)\rho_{11}$$

- By summing and subtracting the second and third equations we obtain

$p_x=\sqrt{1-p}\left(\rho_{01}+\rho_{10}\right)$ and $p_y=i\sqrt{1-p}\left(\rho_{01}-\rho_{10}\right)$

The AD Channel On The Bloch Sphere

- As a conclusion, the Bloch vector becomes

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} \sqrt{1-p}\left(\rho_{01}+\rho_{10}\right) \\ i\sqrt{1-p}\left(\rho_{01}-\rho_{10}\right) \\ \rho_{00}-\left(1-2p\right)\rho_{11} \end{bmatrix} \quad \rightarrow$$

$$\begin{aligned} \varepsilon\left(\rho\right) &= \frac{1}{2}\left(I + p_x X + p_y Y + p_z Z\right) \\ &= \frac{1}{2}\left(I + \sqrt{1-p}\left(\rho_{01}+\rho_{10}\right)X + i\sqrt{1-p}\left(\rho_{01}-\rho_{10}\right)Y + \left(\rho_{00}-\left(1-2p\right)\rho_{11}\right)Z\right) \end{aligned}$$

The AD Channel On The Bloch Sphere

When *p* is replaced with a time-varying function like

$$p = 1 - e^{-t/T_1}$$

(t is time, and just T_1 some constant characterizing the speed of the process), as is often the case for real physical processes, we can visualize the effects of amplitude damping as a *flow* on the Bloch sphere, which moves every point in the unit ball towards a fixed point at the north pole, where $|0\rangle$ is located.

$$\varepsilon\left(\rho(t)\right) = \frac{1}{2}\left(I + e^{-t/2T_1}\left(\rho_{01} + \rho_{10}\right)X + ie^{-t/2T_1}\left(\rho_{01} - \rho_{10}\right)Y + \left(\rho_{00} - \left(2e^{-t/T_1} - 1\right)\rho_{11}\right)Z\right)$$

The AD Channel On The Bloch Sphere

Then

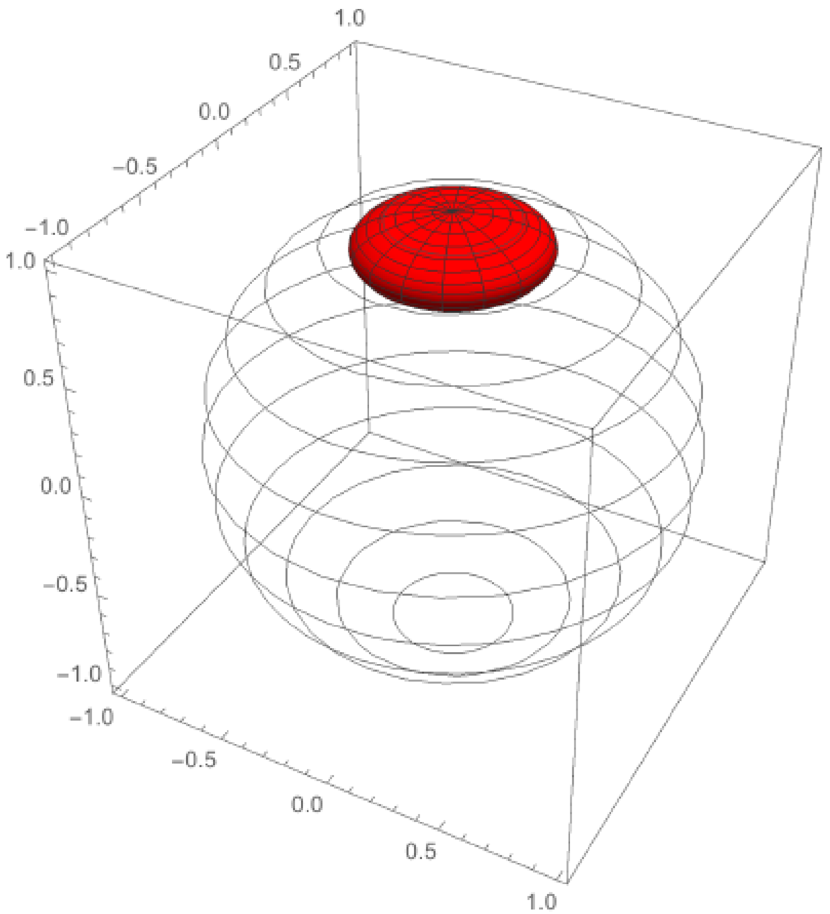
$$\varepsilon\left(\rho(t)\right)=\frac{1}{2}\left(I+e^{-t/2T_1}\left(\rho_{01}+\rho_{10}\right)X+ie^{-t/2T_1}\left(\rho_{01}-\rho_{10}\right)Y+\left(\rho_{00}-\left(2e^{-t/T_1}-1\right)\rho_{11}\right)Z\right)$$

from which

$$\lim_{t\to\infty}\varepsilon\left(\rho(t)\right)=\frac{1}{2}\left(I+\underbrace{\left(\rho_{00}+\rho_{11}\right)}_{=1}Z\right)=\frac{1}{2}\left(I+Z\right)=\frac{1}{2}\begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}=|0\rangle\langle 0|$$

The AD Channel On The Bloch Sphere

The map not only compresses the Bloch ball, but shifts the center as well



- Action of an amplitude damping channel on the Bloch ball
- The Bloch ball (outer ball of unit radius) is shown in white
- The image is plotted in red

The Phase-Dumping Channel

- *Phase damping*, also called *dephasing channel*, is a quantum process that involves **information loss**, but unlike amplitude damping, **it does not involve energy loss**
- During phase damping the principal quantum system becomes entangled with the environment
- Of course, this is undesirable if we are trying to use the quantum system to perform quantum computing

The Phase-Dumping Channel

- To illustrate phase damping, we consider a qubit *A* with a density matrix

$$\rho = \begin{bmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{bmatrix}$$

  interacting with the environment, which can assume three states: $|0_E\rangle, |1_E\rangle$, and $|2_E\rangle$

- We assume that the environment is initially in the state $|0_E\rangle$

- A unitary operator *U* generates entanglement between the principal system and the environment in the following way:

The Phase-Dumping Channel

- *U* is defined by

$$U|0\rangle|0_E\rangle = \sqrt{1-p}\,|0\rangle|0_E\rangle + \sqrt{p}\,|0\rangle|1_E\rangle$$
$$U|1\rangle|0_E\rangle = \sqrt{1-p}\,|1\rangle|0_E\rangle + \sqrt{p}\,|1\rangle|2_E\rangle$$

- Since there are three states of the environment, we need three Kraus operators, E_0, E_1, and E_2 which we are going to derive in the following slides

- Each Kraus operator is represented by a 2x2 matrix since the principal system is a qubit

The Phase-Dumping Channel

$$U|0\rangle|0_E\rangle = \sqrt{1-p}\,|0\rangle|0_E\rangle + \sqrt{p}\,|0\rangle|1_E\rangle$$
$$U|1\rangle|0_E\rangle = \sqrt{1-p}\,|1\rangle|0_E\rangle + \sqrt{p}\,|1\rangle|2_E\rangle$$

- In this case, unlike the depolarizing channel, qubit *A* does not make any transitions in the $\{|0\rangle,|1\rangle\}$ basis
- Instead, the environment "scatters" off of the qubit occasionally (with probability *p*), being kicked into the state $|1_E\rangle$, if *A* is in the state $|0\rangle$ and into the state $|2_E\rangle$ if *A* is in the state $|1\rangle$

The Phase-Dumping Channel

- Let's start with the calculation of the four elements $\left\{\left\|E_0\right\|_{00}, \left\|E_0\right\|_{01}, \left\|E_0\right\|_{10}, \left\|E_0\right\|_{11}\right\}$ of the 2x2 matrix which represents $E_0 = \left\langle 0_E \middle| U \middle| 0_E \right\rangle$

$$\left\|E_0\right\|_{00} = \left\langle 0_E \right| \left\langle 0 \right| U \left| 0 \right\rangle \left| 0_E \right\rangle = \left\langle 0_E \right| \left\langle 0 \right| \left(\sqrt{1-p} \left| 0 \right\rangle \left| 0_E \right\rangle + \sqrt{p} \left| 0 \right\rangle \left| 1_E \right\rangle \right)$$

$$= \sqrt{1-p} \left\langle 0_E \middle| 0_E \right\rangle \left\langle 0 \middle| 0 \right\rangle + \sqrt{p} \left\langle 0_E \middle| 1_E \right\rangle \left\langle 0 \middle| 0 \right\rangle = \sqrt{1-p}$$

$$\left\|E_0\right\|_{01} = \left\langle 0_E \right| \left\langle 0 \right| U \left| 1 \right\rangle \left| 0_E \right\rangle = \left\langle 0_E \right| \left\langle 0 \right| \left(\sqrt{1-p} \left| 1 \right\rangle \left| 0_E \right\rangle + \sqrt{p} \left| 0 \right\rangle \left| 2_E \right\rangle \right)$$

$$= \sqrt{1-p} \left\langle 0_E \middle| 0_E \right\rangle \left\langle 0 \middle| 1 \right\rangle + \sqrt{p} \left\langle 0_E \middle| 2_E \right\rangle \left\langle 0 \middle| 1 \right\rangle = 0$$

The Phase-Dumping Channel

- Let's continue with the calculation of the other elements

$$\left\|E_0\right\|_{10} = \langle 0_E | \langle 1 | U | 0 \rangle | 0_E \rangle = \langle 0_E | \langle 1 | \left(\sqrt{1-p} | 0 \rangle | 0_E \rangle + \sqrt{p} | 0 \rangle | 1_E \rangle \right)$$

$$= \sqrt{1-p} \langle 0_E | 0_E \rangle \langle 1 | 0 \rangle + \sqrt{p} \langle 0_E | 1_E \rangle \langle 1 | 0 \rangle = 0$$

$$\left\|E_0\right\|_{11} = \langle 0_E | \langle 1 | U | 1 \rangle | 0_E \rangle = \langle 0_E | \langle 1 | \left(\sqrt{1-p} | 1 \rangle | 0_E \rangle + \sqrt{p} | 0 \rangle | 2_E \rangle \right)$$

$$= \sqrt{1-p} \langle 0_E | 0_E \rangle \langle 1 | 1 \rangle + \sqrt{p} \langle 0_E | 2_E \rangle \langle 1 | 1 \rangle = \sqrt{1-p}$$

The Phase-Dumping Channel

- As a conclusion

$$E_0 = \begin{bmatrix} \sqrt{1-p} & 0 \\ 0 & \sqrt{1-p} \end{bmatrix} = \sqrt{1-p} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \left(\sqrt{1-p} \right) I$$

The Phase-Dumping Channel

- Using the same approach as above, we can calculate $E_1 = \langle 1_E | U | 0_E \rangle$ and $E_2 = \langle 2_E | U | 0_E \rangle$ $\rightarrow$

$$E_0 = \left(\sqrt{1-p}\right) I = \sqrt{1-p}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$E_1 = \left(\sqrt{p}\right) |0\rangle\langle 0| = \sqrt{p}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$E_2 = \left(\sqrt{p}\right) |1\rangle\langle 1| = \sqrt{p}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The Phase-Dumping Channel

- The quantum operation is then

$$\varepsilon(\rho)=\sum_{k=0}^{2}E_k\rho E_k=E_0\rho E_0+E_1\rho E_1+E_2\rho E_2$$

$$=(1-p)\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}\rho_{00}&\rho_{01}\\\rho_{10}&\rho_{11}\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix}+p\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}\rho_{00}&\rho_{01}\\\rho_{10}&\rho_{11}\end{bmatrix}\begin{bmatrix}1&0\\0&0\end{bmatrix}$$

$$+p\begin{bmatrix}0&0\\0&1\end{bmatrix}\begin{bmatrix}\rho_{00}&\rho_{01}\\\rho_{10}&\rho_{11}\end{bmatrix}\begin{bmatrix}0&0\\0&1\end{bmatrix}\quad\rightarrow$$

$$\varepsilon(\rho)=\begin{bmatrix}\rho_{00}&(1-p)\rho_{01}\\(1-p)\rho_{10}&\rho_{11}\end{bmatrix}$$

Thus, an initial density matrix ρ evolves to $\varepsilon(\rho)$ where:

- the diagonal terms are unchanged;
- the non-diagonal terms decay with the factor (1 − p).

The Phase-Dumping Channel

- If this operation is carried out *n* times, then

$$\rho \rightarrow \begin{bmatrix} \rho_{00} & (1-p)^n \rho_{01} \\ (1-p)^n \rho_{10} & \rho_{11} \end{bmatrix}$$

which allows us to study *continuous dephasing*, *i.e.*, dephasing that occurs continuously in time

The Phase-Dumping Channel

- Suppose that the probability of **phase changing** per unit time is Γ, so that $p = \Gamma \Delta t \ll 1$ when a brief time interval Δt elapses
- The evolution over a time $t = n\Delta t$ is governed by ε^n (ε repeated *n* times in succession) so that the off-diagonal terms in the density operator become suppressed by

$$\lim_{n\to\infty}(1-p)^n = \lim_{\Delta t\to 0}(1-\Gamma\Delta t)^{t/\Delta t} = e^{-\Gamma t}$$

taking the limit $n \to \infty$ with *t* fixed

$$\rho \to \rho_{out}(t) = \begin{bmatrix} \rho_{00} & e^{-\Gamma t}\rho_{01} \\ e^{-\Gamma t}\rho_{10} & \rho_{11} \end{bmatrix}$$

The Phase-Dumping Channel

- Now, if $t \to \infty$

$$\begin{bmatrix} \rho_{00} & e^{-\Gamma t}\rho_{01} \\ e^{-\Gamma t}\rho_{10} & \rho_{11} \end{bmatrix} \quad \text{exponentially decays to} \underset{t \to \infty}{} \quad \begin{bmatrix} \rho_{00} & 0 \\ 0 & \rho_{11} \end{bmatrix}$$

- The off-diagonal terms go to zero, leaving only the terms along the diagonal
- All information about the relative phases in the original state of the principal system is lost
- This is *phase damping*

The Phase-Dumping Channel

- Thus, if we prepare an initial pure state $\alpha|0\rangle + \beta|1\rangle$ then after a time $t \gg \Gamma^{-1}$, the density operator evolves as

$$\begin{bmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{bmatrix} \mapsto \begin{bmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{bmatrix}$$

- The state decoheres, in the preferred basis $|0\rangle, |1\rangle$

The Phase-Dumping Channel

The Kraus operators of the phase-damping channel can be re-written as follows

$$E_0 = \left(\sqrt{1-p}\right) I = \sqrt{1-p} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad\qquad E_0 = \left(\sqrt{1-p}\right) I$$

$$E_1 = \left(\sqrt{p}\right) |0\rangle\langle 0| = \sqrt{p} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \rightarrow \qquad E_1 = \frac{\sqrt{p}}{2}(I+Z)$$

$$E_2 = \left(\sqrt{p}\right) |1\rangle\langle 1| = \sqrt{p} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad\qquad E_2 = \frac{\sqrt{p}}{2}(I-Z)$$

where

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \rightarrow Z = Z^{\dagger}$$

The Phase-Dumping Channel

Then

$$\varepsilon(\rho) = \sum_{k=0}^{2} E_k \rho E_k = E_0 \rho E_0 + E_1 \rho E_1 + E_2 \rho E_2$$

$$= (1-p) I \rho I + \frac{p}{4}(I+Z)\rho(I+Z) + \frac{p}{4}(I-Z)\rho(I-Z)$$

$$= (1-p) I \rho I + \frac{p}{4}\left[(I+Z)(\rho+\rho Z) + (I-Z)(\rho-\rho Z)\right]$$

$$= (1-p) I \rho I + \frac{p}{4}\left(\rho + \cancel{\rho Z} + \cancel{Z\rho} + Z\rho Z + \rho - \cancel{\rho Z} - \cancel{Z\rho} + Z\rho Z\right)$$

$$= (1-p) I \rho I + \frac{p}{2}(\rho + Z\rho Z) = \rho - p\rho + \frac{p}{2}\rho + \frac{p}{2} Z\rho Z = \left(1-\frac{p}{2}\right)\rho + \frac{p}{2} Z\rho Z$$

The Phase-Dumping Channel

- Thus

$$\varepsilon(\rho) = \left(1 - \frac{p}{2}\right) I \rho I + \frac{p}{2} Z \rho Z$$

- Due to the unitary freedom of quantum operations, we can redefine a new set of operation elements for the phase-damping channel

$$E_0^{'} = \left(\sqrt{1 - \frac{p}{2}}\right) I, \quad E_1^{'} = \left(\sqrt{\frac{p}{2}}\right) Z, \quad \text{where} \quad \left\{E_0^{'}, E_1^{'}\right\} \cong \left\{E_0, E_1, E_2\right\}$$

- It is worth noting that the *phase damping* quantum operation is the same as the *phase flip* channel

The Phase-Dumping Channel

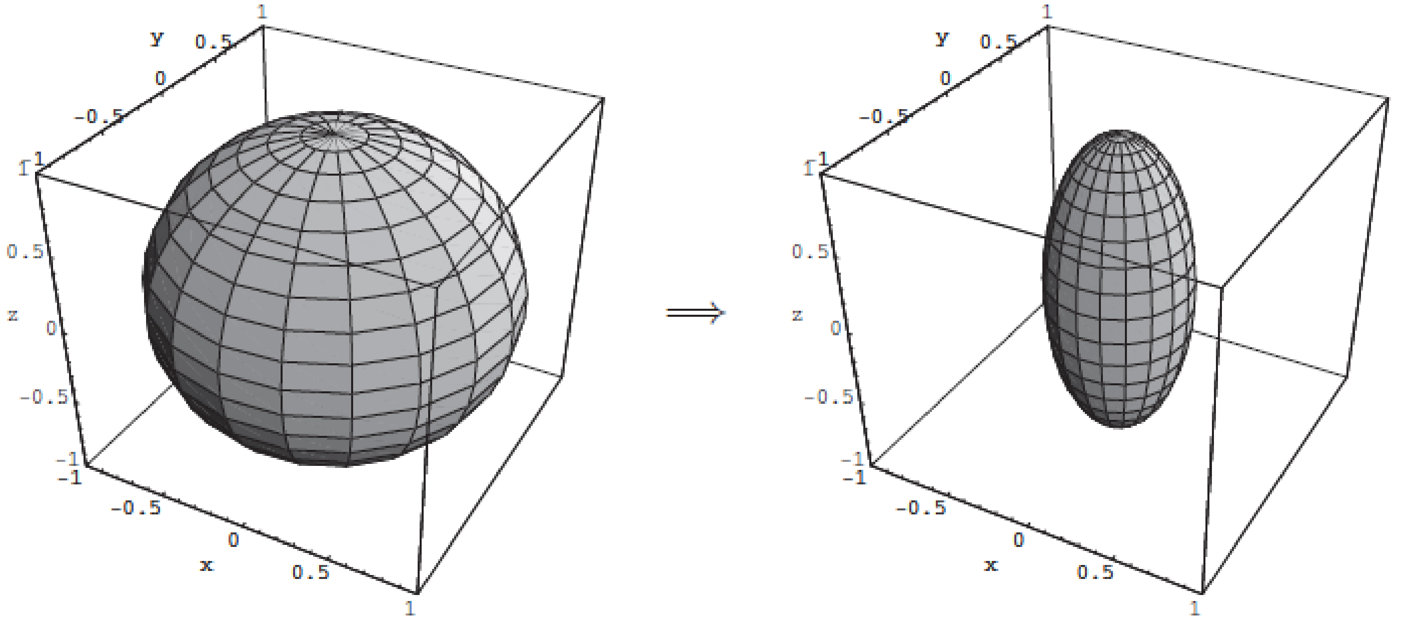
- In the textbook entitled *Quantum Computation and Quantum Information* by M. A. Nielsen & I. L. Chuang, the Kraus operators are given by the following expressions

$$E_0 = \sqrt{\alpha}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad E_1 = \sqrt{1-\alpha}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- These operators are the same as $\left\{E_0^{'}, E_1^{'}\right\}$ by setting $\alpha = 1 - \dfrac{p}{2}$
- Since *phase damping* is the same as the *phase flip channel*, we have already seen how it is visualized on the Bloch sphere, i.e., by shrinking the sphere into ellipsoids

The Phase-Dumping Channel

- Thus



$$\varepsilon(\rho) == \frac{1}{2}\left[I + (2\alpha - 1)\,p_x X + (2\alpha - 1)\,p_y Y + p_z Z\right]$$

T_1 and T_2 Time Scales

- What processes degrade the quantum memories?
- The two main processes are **energy relaxation** and **dephasing**
- They are characterized by two different time scales, referred to as the T_1 time scale and the T_2 time scale, respectively

T_1 and T_2 Time Scales

- Let's consider the **energy relaxation time**, given by T_1 , first
- The two states of the physical system encoding the qubit often have different energies
- All physical systems have the tendency to try and seek the lowest energy state
- So, a system initialized in the excited state will not stay there indefinitely, eventually it will lose energy and transition to a state of lower energy

T_1 and T_2 Time Scales

- Typically, the lower energy state is chosen to encode a $|0\rangle$, while the higher-energy state encodes a $|1\rangle$
- Of course, that choice is by convention, not an immutable fact of physics
- The relaxation time T_1 measures at what time scales the qubit relaxes and undergoes the transition $|1\rangle \rightarrow |0\rangle$
- It tells the probability that a qubit initialized in $|1\rangle$ is still in that state after time t

$$\text{Prob}\left(|1\rangle\right) = e^{-t/T_1}$$

- The probability that after $t = T_1$ seconds we still find our state in $|1\rangle$ is given by $1/e$

T_1 and T_2 Time Scales

- The **dephasing time** T_2 tells us about the time scale of a different decoherence process, namely the loss of phase coherence in the qubit
- Unlike the relaxation process, the qubit **does not necessarily lose energy**, but **its phase** between the states $|0\rangle$ and $|1\rangle$ **becomes uncertain over time**
- *The T_2 time tells us how quickly superpositions are washed out and the qubit decoheres and loses its quantum properties*

Amplitude-Damping vs. Phase-Damping

- The **amplitude damping process** can introduce **dephasing** as a secondary effect
- However, its **primary** function is **energy dissipation or relaxation**, particularly the loss of excitation (e.g., decay from an excited state to a ground state in quantum systems)

Amplitude-Damping vs. Phase-Damping

Let’s look at how dephasing might be involved in this process:

1. **Amplitude Damping and Energy Loss**: In the amplitude damping process, a quantum system loses energy by decaying from an excited state $|1\rangle$ to a ground state $|0\rangle$

   This leads to a gradual reduction in the probability amplitude of the excited state

Amplitude-Damping vs. Phase-Damping

2. **Indirect Dephasing Effect**: Although amplitude damping primarily affects the populations (the diagonal elements of the density matrix), it also affects the **coherence terms** (off-diagonal elements)

   As the system transitions between states, coherence is gradually lost because of the fluctuating phase relationship between the $|0\rangle$ and $|1\rangle$ states

   This results in partial dephasing as a byproduct of the damping process.

Amplitude-Damping vs. Phase-Damping

3. **Distinction from Pure Dephasing**: In pure dephasing (often called phase damping), there is no energy loss

   Only the off-diagonal elements decay, meaning coherence is lost without affecting the population distribution

   In contrast, amplitude damping typically affects both populations and coherences, leading to a more complex decoherence pattern

Amplitude-Damping vs. Phase-Damping

- In summary, while amplitude damping's main function is to decrease energy (affecting populations), it indirectly causes dephasing by reducing coherence between states
- This means amplitude damping often leads to both relaxation and some degree of dephasing in quantum systems

Quantum State Tomography

Quantum State Tomography

- Before tackling this topic, we need to provide the following mathematical results

Math Break/1

The Hilbert–Schmidt inner product on operators

- The set L_V of linear operators on a Hilbert space V is a vector space
- The sum of two linear operators is a linear operator
- zA is a linear operator if A is a linear operator and z is a complex number, and
- There is a zero element 0
- An important additional result is that the vector space L_V can be given a natural inner product structure, turning it into a Hilbert space

Math Break/1

1) Show that the function $(\cdot,\cdot)$ on $L_V \times L_V$ defined by

$$A,B \equiv \mathrm{tr}\; A^{\dagger},B$$

   is an inner product function. This inner product is known as the Hilbert-Schmidt or trace inner product.

2) If V has d dimensions show that L_V has dimension d^2
3) Find an orthonormal basis of Hermitian matrices for the Hilbert space L_V

Math Break/1

We should verify if this definition meets the properties of an inner product

a) $\cdot,\cdot$ is linear in the second argument

$$\left(A, \sum_i \lambda_i B_i\right) = \mathrm{tr}\left(A^\dagger \left(\sum_i \lambda_i B_i\right)\right) = \mathrm{tr}\;\; A^\dagger \lambda_1 B_1 \;\; + \cdots + \mathrm{tr}\;\; A^\dagger \lambda_n B_n$$

$$= \lambda_1 \mathrm{tr}\;\; A^\dagger B_1 \;\; + \cdots + \lambda_n \mathrm{tr}\;\; A^\dagger B_n$$

$$= \sum_i \lambda_i \mathrm{tr}\;\; A^\dagger B_i$$

Math Break/1

b) $A,B^{\;*} = B,A$

$$A,B^{\;*} = \text{tr}\; A^\dagger B^{\;\;*} = \left(\sum_{i,j}\langle i|A^\dagger|j\rangle\langle j|B|i\rangle\right)^*$$

$$= \sum_{i,j}\langle i|A^\dagger|j\rangle^*\langle j|B|i\rangle^*$$

$$= \sum_{i,j}\langle j|B|i\rangle^*\langle i|A^\dagger|j\rangle^*$$

$$= \sum_{i,j}\langle i|B^\dagger|j\rangle\langle j|A|i\rangle$$

$$= \sum_{i}\langle i|B^\dagger A|i\rangle = \text{tr}\; B^\dagger A$$

$$= B,A$$

Math Break/1

c) $A,A \geq 0$ with equality if and only if A = 0

$$A,A = \mathrm{tr}\ A^{\dagger}A = \sum_i \langle i|A^{\dagger}A|i\rangle$$

Since $A^{\dagger}A$ is positive, $\langle i|A^{\dagger}A|i\rangle \geq 0$ for all $|i\rangle$

Let a_i the *i-th* column of A. If $\langle i|A^{\dagger}A|i\rangle = 0$, then

$$\langle i|A^{\dagger}A|i\rangle = a_i^{\dagger}a_i = \|a_i\|^2 = 0 \ \ iff \ \ a_i = 0$$

Therefore

$$A,A = 0 \ \ iff \ \ A = 0$$

Math Break/1

(2)

- A linear transformation T: $V \rightarrow V$ (dim$V=d$) can be represented as a $d \times d$ matrix

- Since there are $d \times d = d^2$ matrices that are linearly independent, the dimension of L_V is d^2 $\blacksquare$

Math Break/2

- Suppose we have a single qubit density matrix, ρ, the set

$$\left\{ I/\sqrt{2},\ X/\sqrt{2},\ Y/\sqrt{2},\ Z/\sqrt{2} \right\} \tag{1}$$

  forms an orthonormal set of matrices with respect to the *Hilbert– Schmidt* inner product, so ρ may be expanded as

$$\rho = \frac{\mathrm{tr}(\rho)I + \mathrm{tr}(X\rho)X + \mathrm{tr}(Y\rho)Y + \mathrm{tr}(Z\rho)Z}{2} \tag{2}$$

- It's easy to show that the elements of the basis are orthonormal

Math Break/2

- As an example, let's show that $X/\sqrt{2}$ and $Z/\sqrt{2}$ are orthogonal

$$\left(\frac{X}{\sqrt{2}},\frac{Z}{\sqrt{2}}\right)=\mathrm{tr}\left(\frac{X^{\dagger}}{\sqrt{2}}\frac{Z}{\sqrt{2}}\right)=\frac{1}{2}\mathrm{tr}\;XZ\;=\frac{1}{2}\mathrm{tr}\left(\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}\right)=\frac{1}{2}\mathrm{tr}\begin{bmatrix}0&-1\\1&0\end{bmatrix}=0$$

- Furthermore, let's show that the norm of the element $X/\sqrt{2}$ is equal to one. Since

$$\left(\frac{X}{\sqrt{2}},\frac{X}{\sqrt{2}}\right)=\mathrm{tr}\left(\frac{X^{\dagger}}{\sqrt{2}}\frac{X}{\sqrt{2}}\right)=\frac{1}{2}\mathrm{tr}\;XX\;=\frac{1}{2}\mathrm{tr}\;I\;=\frac{1}{2}\mathrm{tr}\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)=1$$

- then, the norm of $X/\sqrt{2}$ is

$$\left\|\frac{X}{\sqrt{2}}\right\|=\sqrt{\left(\frac{X}{\sqrt{2}},\frac{X}{\sqrt{2}}\right)}=1$$

Math Break/2

- It is easy to demonstrate the orthonormality of the other combinations of the elements of the basis.
- Now, by using the $\left\{I/\sqrt{2},\ X/\sqrt{2},\ Y/\sqrt{2},\ Z/\sqrt{2}\right\}$ basis, we can write

$$\rho = \frac{aI + bX + cY + dZ}{\sqrt{2}} \tag{3}$$

- To calculate *a*, *b*, *c*, and *d* we form the scalar product of
- $I/\sqrt{2},\ X/\sqrt{2},\ Y/\sqrt{2},$ and $Z/\sqrt{2}$ with $\rho,$ i.e.

$$\begin{gathered} a = \left(I/\sqrt{2},\ \rho\right) = \frac{1}{\sqrt{2}}\mathrm{tr}(\rho),\ b = \left(X/\sqrt{2},\rho\right) = \frac{1}{\sqrt{2}}\mathrm{tr}(X\rho) \\ c = \left(Y/\sqrt{2},\rho\right) = \frac{1}{\sqrt{2}}\mathrm{tr}(Y\rho),\ d = \left(Z/\sqrt{2},\rho\right) = \frac{1}{\sqrt{2}}\mathrm{tr}(Z\rho) \end{gathered} \tag{4}$$

Math Break/2

- By plugging (4) into (3) we obtain (2), i.e.

$$\rho = \frac{\mathrm{tr}(\rho)I + \mathrm{tr}(X\rho)X + \mathrm{tr}(Y\rho)Y + \mathrm{tr}(Z\rho)Z}{2} \qquad \blacksquare$$

Math Break/3

- Let's consider the following ensemble

$$\left|\psi_i\right\rangle, p_i \mid i=1,\ldots.,n \ , \ \ with \ \ p_i \in 0,1 \quad and \quad \sum_{i=1}^{n} p_i = 1$$

- For any observable *A*, and state $\rho = \sum_{i=1}^{n} p_i \left|\psi_i\right\rangle\left\langle\psi_i\right|$

$$\mathrm{tr}\ \ A\rho \ = \sum_{k=1}^{d}\langle k|A\rho|k\rangle = \sum_{k=1}^{d}\langle k|A\left(\sum_{i=1}^{n} p_i\left|\psi_i\right\rangle\left\langle\psi_i\right|\right)|k\rangle$$

$$= \sum_{k=1}^{d}\sum_{i=1}^{n} p_i\langle k|A|\psi_i\rangle\langle\psi_i|k\rangle = \sum_{k=1}^{d}\sum_{i=1}^{n} p_i\langle\psi_i|k\rangle\langle k|A|\psi_i\rangle$$

$$= \sum_{i=1}^{n} p_i\langle\psi_i|\left(\sum_{k=1}^{d}|k\rangle\langle k|\right)A|\psi_i\rangle = \sum_{i=1}^{n} p_i\langle\psi_i|IA|\psi_i\rangle = \sum_{i=1}^{n} p_i\langle\psi_i|A|\psi_i\rangle = \langle A\rangle$$

Math Break/3

- Thus

$$\langle A\rangle = \sum_{i=1}^{n} p_i \langle \psi_i | A | \psi_i \rangle = \mathrm{tr}\ A\rho$$

where $|k\rangle \mid k = 1,2,\ldots,d$ is an orthonormal basis for the Hilbert space over which A is defined

Quantum State Tomography

- *State tomography* is the procedure of *experimentally* determining an unknown quantum state
- Suppose we are given an *unknown* state, ρ, of a single qubit
- How can we experimentally determine what the state of ρ is?
- If we are given just a single copy of ρ then it turns out to be impossible to characterize ρ
- The basic problem is that there is no quantum measurement which can distinguish non-orthogonal quantum states like $|0\rangle$ and $\left(|0\rangle+|1\rangle\right)/\sqrt{2}$ with certainty

Quantum State Tomography

- However, it is possible to estimate ρ if we have a large number of copies of ρ
- For instance, if ρ is the quantum state produced by some experiment, then we simply repeat the experiment many times to produce many copies of the state ρ
- Suppose we have many copies of a single qubit density matrix, ρ, the set

$$\left\{ I/\sqrt{2},\ X/\sqrt{2},\ Y/\sqrt{2},\ Z/\sqrt{2} \right\}$$

forms an orthonormal set of matrices with respect to the *Hilbert– Schmidt* inner product, so ρ may be expanded as

$$\rho = \frac{\mathrm{tr}(\rho)I + \mathrm{tr}(X\rho)X + \mathrm{tr}(Y\rho)Y + \mathrm{tr}(Z\rho)Z}{2}$$

Quantum State Tomography

- Recall, however, that expressions like tr($A\rho$) have an interpretation as the average value of observables, i.e.,

$$\langle A \rangle = \mathrm{tr}(A\rho)$$

- For example, to estimate tr($Z\rho$) we measure the observable Z a large number of times, m, obtaining outcomes $z_1, z_2, \ldots\ldots, z_m$, all equal to +1 or −1
- The empirical average of these quantities,

$$\sum_i z_i / m$$

  is an estimate for the true value of tr($Z\rho$)

Quantum State Tomography

- We can use the central limit theorem to determine how well this estimate behaves for large *m*, where it becomes approximately Gaussian with
  - > **mean** equal to tr($Z\rho$) and with
  - > **standard deviation** $\Delta(Z)/\sqrt{m}$, where $\Delta(Z)$ is the standard deviation for a single measurement of *Z*, which is upper bounded by 1, so the standard deviation in our estimate

$$\sum_i z_i/m$$

is at most $1/\sqrt{m}$

- In a similar way we can estimate the quantities tr($X\rho$) and tr($Y\rho$) with a high degree of confidence in the limit of a large sample size, and thus obtain a good estimate for ρ

Quantum State Tomography

$$\rho = \frac{\mathrm{tr}(\rho)I + \mathrm{tr}(X\rho)X + \mathrm{tr}(Y\rho)Y + \mathrm{tr}(Z\rho)Z}{2}$$

- Generalizing this procedure to the case of more than one qubit is not difficult, at least in principle!
- Similar to the single qubit case, an arbitrary density matrix on *n* qubits can be expanded as

$$\rho = \sum_{\vec{v}} \frac{\mathrm{tr}\left[\left(\sigma_{v_1} \otimes \sigma_{v_2} \otimes \cdots \otimes \sigma_{v_n}\right)\rho\right]\left(\sigma_{v_1} \otimes \sigma_{v_2} \otimes \cdots \otimes \sigma_{v_n}\right)}{2^n}$$

  where the sum is over vectors $\vec{v} = (v_1, v_2, \ldots, v_n)$ with entries v_i chosen from the set 0, 1, 2, 3

- By performing measurements of observables which are products of Pauli matrices we can estimate each term in this sum, and thus obtaining an estimate for ρ

Backup Slides

Operator-sum Representation

Proof of (7)

- Suppose V and W are Hilbert spaces. An arbitrary linear operator

$$C : V \otimes W \rightarrow V \otimes W$$

can be represented as a linear combination of tensor products of operators A_i and B_j acting on V and W respectively

$$C = \sum_{ij} c_{ij} A_i \otimes B_j, \quad c_{ij} \in \mathbb{C}$$

where by definition

$$\left(\sum_{ij} c_{ij} A_i \otimes B_j \right) |v\rangle \otimes |w\rangle = \sum_{ij} c_{ij} A_i |v\rangle \otimes B_j |w\rangle$$

Operator-sum Representation

- In the proof we use the following notations
  - *V* is the Hilbert space associated to the principal system
  - W is the Hilbert space associated to the environment, and

$$U = \sum_{ij} c_{ij} A_i \otimes B_j, \quad c_{ij} \in \mathbb{C}$$

$$|e_k\rangle \rightarrow I_V \otimes |e_k\rangle$$

$$\langle e_k| \rightarrow I_V \otimes \langle e_k|$$

- Furthermore, we exploit the following property

$$(A \otimes B) \cdot (C \otimes D) = (AC) \otimes (BD)$$

Operator-sum Representation

$$\begin{aligned}
\varepsilon(\rho) &= \sum_k \langle e_k | U\left(\rho \otimes |e_0\rangle\langle e_0|\right) U^\dagger |e_k\rangle \\
&= \sum_k \langle e_k | \left(\sum_{ij} c_{ij} A_i \otimes B_j\right)\left(\rho \otimes |e_0\rangle\langle e_0|\right)\left(\sum_{lm} c_{lm} A_l \otimes B_m\right)^\dagger |e_k\rangle \\
&= \sum_k \left(I_V \otimes \langle e_k|\right)\left(\sum_{ij} c_{ij} A_i \otimes B_j\right)\left(\rho \otimes |e_0\rangle\langle e_0|\right)\left(\sum_{lm} c_{lm} A_l^\dagger \otimes B_m^\dagger\right)\left(I_V \otimes |e_k\rangle\right) \\
&= \sum_k \left(\sum_{ij} c_{ij} I_V A_i \otimes \langle e_k | B_j\right)\left(\rho \otimes |e_0\rangle\langle e_0|\right)\left(\sum_{lm} c_{lm} A_l^\dagger I_V \otimes B_m^\dagger |e_k\rangle\right) \\
&= \sum_k \sum_{ijlm} c_{ij} c_{lm} A_i \rho A_l^\dagger \otimes \underbrace{\langle e_k | B_j | e_0\rangle}_{scalar} \underbrace{\langle e_0 | B_m^\dagger | e_k\rangle}_{scalar}
\end{aligned}$$

Operator-sum Representation

- By taking into consideration that the tensor product with a scalar is the same as multiplication, we obtain

$$\varepsilon(\rho) = \sum_{k} \sum_{ijlm} c_{ij} c_{lm} A_i \rho A_l^\dagger \times \langle e_k | B_j | e_0 \rangle \langle e_0 | B_m^\dagger | e_k \rangle$$

$$= \sum_{k} \left(\sum_{ij} c_{ij} A_i \langle e_k | B_j | e_0 \rangle \right) \rho \left(\sum_{lm} c_{lm} A_l^\dagger \langle e_0 | B_m^\dagger | e_k \rangle \right) \qquad (10)$$

Operator-sum Representation

- Since

$$E_k = \langle e_k | U | e_0 \rangle = \left(I_V \otimes \langle e_k | \right) \left(\sum_{ij} c_{ij} A_i \otimes B_j \right) \left(I_V \otimes | e_k \rangle \right)$$

$$= \sum_{ij} c_{ij} A_i \otimes \langle e_k | B_j | e_0 \rangle$$

$$= \sum_{ij} c_{ij} A_i \langle e_k | B_j | e_0 \rangle \qquad (11)$$

- By plugging (11) into (10) we obtain

$$\boxed{\varepsilon(\rho) = \sum_k E_k \rho E_k^\dagger}$$

$\blacksquare$