Department of Information Engineering
MSc in Computer Engineering (a.y. 2024/2025)
University of Pisa

Quantum Computing and Quantum Internet

Luciano Lenzini
Full Professor
Department of Information Engineering
School of Engineering
University of Pisa, Italy
e-mail: lenzini44@gmail.com
http://www.iet.unipi.it/~lenzini/
http://www.originiinternetitalia.it/it/

Fidelity

Introduction

- One important thing to remember is that it's impossible to prepare the exact desired pure state with complete accuracy
- Sometimes, the prepared state is pure but doesn't quite match the target state we intended
- At other times, it may be affected by incoherent noise, resulting in a mixed state
- Therefore, it is essential to have a method for quantifying the difference between the actual state and the desired (or target) state
- A convenient tool that tells us how close the real state is to the target state is the **Fidelity**

Fidelity

- The fidelity of states ρ and σ is defined to be

$$F(\rho,\sigma) = \mathrm{tr}\sqrt{\rho^{1/2}\sigma\rho^{1/2}} \qquad (1)$$

- It is certainly not immediately obvious that this is a useful measure of distance between ρ and σ
- It doesn't even look symmetric!
- There is (at least) one important special case where it is possible to give more explicit formulae for the fidelity

Fidelity

- Specifically, let's calculate the fidelity between a pure state $|\psi\rangle$ (with density matrix $\sigma = |\psi\rangle\langle\psi|$) and an arbitrary state, ρ

- From Equation (1) we see that

$$F\left(|\psi\rangle, \rho\right) = \mathrm{tr}\sqrt{\sqrt{\sigma}\rho\sqrt{\sigma}} = \mathrm{tr}\sqrt{\sqrt{|\psi\rangle\langle\psi|}\rho\sqrt{|\psi\rangle\langle\psi|}} \qquad (6)$$

- However, from $\sigma = |\psi\rangle\langle\psi|$ it follows:

$$\sigma^2 = \left(|\psi\rangle\langle\psi|\right)^2 = \left(|\psi\rangle\langle\psi|\right)\left(|\psi\rangle\langle\psi|\right) = |\psi\rangle\langle\psi|\psi\rangle\langle\psi| = |\psi\rangle\langle\psi| = \sigma$$

- Thus

$$\sigma^2 = \sigma \qquad (7)$$

Fidelity

- Substituting (7) into (6) leads to the following result

$$F\left(|\psi\rangle,\rho\right)=\mathrm{tr}\sqrt{\sqrt{\sigma^2}\rho\sqrt{\sigma^2}}=\mathrm{tr}\sqrt{\sigma\rho\sigma}=\mathrm{tr}\sqrt{|\psi\rangle\langle\psi|\rho|\psi\rangle\langle\psi|} \tag{8}$$

- Since $\langle\psi|\rho|\psi\rangle$ is a scalar number and since $\left(|\psi\rangle\langle\psi|\right)^2=|\psi\rangle\langle\psi|$, (8) can be rewritten as follows:

$$F\left(|\psi\rangle,\rho\right)=\mathrm{tr}\sqrt{\langle\psi|\rho|\psi\rangle|\psi\rangle\langle\psi|}=\mathrm{tr}\sqrt{\langle\psi|\rho|\psi\rangle\left(|\psi\rangle\langle\psi|\right)^2}=\mathrm{tr}\left(\sqrt{\langle\psi|\rho|\psi\rangle}\left(|\psi\rangle\langle\psi|\right)\right)$$

$$=\sqrt{\langle\psi|\rho|\psi\rangle}\mathrm{tr}\left(\left(|\psi\rangle\langle\psi|\right)\right)$$

- Since $\mathrm{tr}\left(|\psi\rangle\langle\psi|\right)=1$, we can conclude that

$$F\left(|\psi\rangle,\rho\right)=\sqrt{\langle\psi|\rho|\psi\rangle} \tag{9}$$

Pure-State Fidelity

- The fidelity measure has a simple interpretation when both $\rho = |\phi\rangle\langle\phi|$ and $\sigma = |\psi\rangle\langle\psi|$ are pure states
- In this specific case, the pure-state fidelity $F(|\psi\rangle, |\phi\rangle)$ is a measure of how $|\psi\rangle$ is close to $|\phi\rangle$

$$F(|\psi\rangle, |\phi\rangle) = \sqrt{\langle\psi|\underbrace{(|\phi\rangle\langle\phi|)}_{\rho}|\psi\rangle} = \sqrt{\langle\psi|\phi\rangle\langle\phi|\psi\rangle} = \sqrt{\langle\psi|\phi\rangle\langle\psi|\phi\rangle^*} = \sqrt{|\langle\psi|\phi\rangle|^2} = |\langle\psi|\phi\rangle|$$

which is the overlap of the states $|\psi\rangle$ and $|\phi\rangle$

Pure-State Fidelity

- The **pure-state** fidelity is *symmetric*

$$F\left(|\psi\rangle,|\phi\rangle\right)=\left|\langle\psi|\phi\rangle\right|^{2}=\left|\langle\phi|\psi\rangle\right|^{2}=F\left(|\phi\rangle,|\psi\rangle\right)$$

and it obeys the following bounds:

$$0\leq F\left(|\psi\rangle,|\phi\rangle\right)=\left|\langle\psi|\phi\rangle\right|^{2}\leq 1$$

- It is equal to *one* if and only if the two states are *the same*, and it is equal to *zero* if and only if the two states are *orthogonal to each other*
- The fidelity measure **is not** a **distance measure** in the strict mathematical sense because it is equal *to one when two states are equal*, whereas a distance measure should be *equal to zero when two states are equal*

How Well Does a Quantum Channel Preserve Information?

- So far, we have primarily considered qubits that evolve uncorrupted in time and space
- In other words, these, ideal, qubits navigate noisless communication channels
- In the real world, qubits and their classical cousins, bits, are regularly subjected to corrupting environmental perturbations

How Well Does a Quantum Channel Preserve Information?

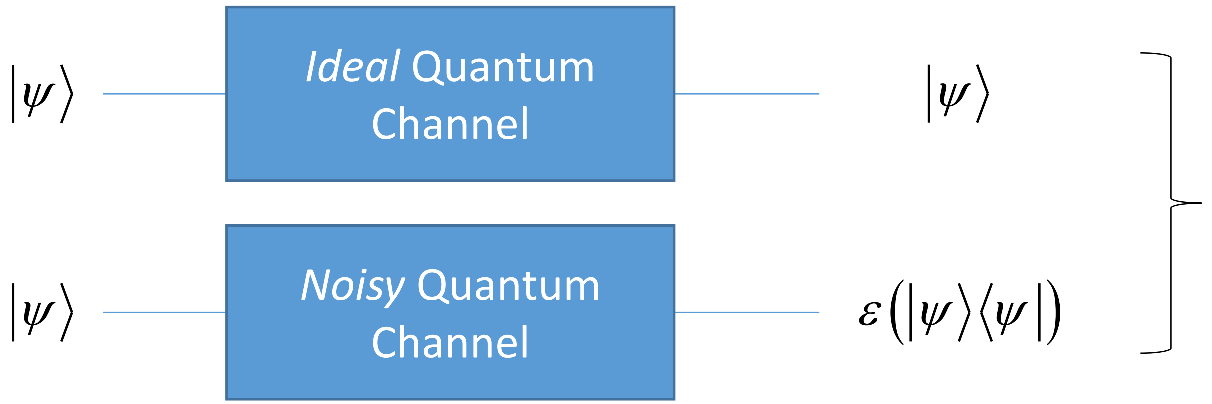
- Suppose a quantum system is in the state $|\psi\rangle$ and some physical process occurs, changing the quantum system to the state $\varepsilon(|\psi\rangle\langle\psi|)$
- How well has the physical ε process preserved the state $|\psi\rangle$ of the quantum system?
- The fidelity discussed previously can be used now to develop measures of how well a quantum system preserves information

How Well Does a Quantum Channel Preserve Information?

- This type of scenario occurs often in *quantum computation* and *quantum information*
- For example, in the memory of a quantum computer, $|\psi\rangle$ is the initial state of the memory, and $\mathcal{E}$ represents the dynamics that the memory undergoes, including noise processes arising from interaction with the environment

How Well Does a Quantum Channel Preserve Information?

- A second example is provided by a quantum communication channel for transmitting the state $|\psi\rangle$ from one location to another
- No channel is ever perfect, so the action of the channel is described by a quantum operation ε



The **Fidelity** measure the distance beteen $|\psi\rangle$ and $\varepsilon(|\psi\rangle\langle\psi|)$

How Well Does a Quantum Channel Preserve Information?

- An obvious way of quantifying how well the state $|\psi\rangle$ is preserved by the channel is to make use of the **Fidelity** measure introduced earlier
- For example, we can compute the fidelity between the starting state $|\psi\rangle$ and the ending state $\varepsilon\left(|\psi\rangle\langle\psi|\right)$
- For the case of the **depolarizing channel**, we obtain

$$
\begin{aligned}
F\left(|\psi\rangle, \varepsilon\left(|\psi\rangle\langle\psi|\right)\right) &= \sqrt{\langle\psi|\left(p\frac{I}{2}+(1-p)|\psi\rangle\langle\psi|\right)|\psi\rangle} = \sqrt{\langle\psi|\,p\frac{I}{2}|\psi\rangle+\langle\psi|(1-p)|\psi\rangle\langle\psi|\psi\rangle} \\
&= \sqrt{p\frac{1}{2}\langle\psi|\psi\rangle+\langle\psi|\psi\rangle\langle\psi|\psi\rangle-p\langle\psi|\psi\rangle\langle\psi|\psi\rangle} = \sqrt{p\frac{1}{2}+1-p} \\
&= \sqrt{1-\frac{p}{2}}
\end{aligned}
$$

How Well Does a Quantum Channel Preserve Information?

- Thus

$$F\left(|\psi\rangle, \varepsilon\left(|\psi\rangle\langle\psi|\right)\right) = \sqrt{1 - \frac{p}{2}}$$

- This result agrees well with our intuition - the higher the probability *p* of depolarizing, the lower the fidelity of the final state with the initial state
- Provided *p* is very small the **Fidelity** is close to one, and the state $\varepsilon\left(|\psi\rangle\langle\psi|\right)$ is practically indistinguishable from the initial state $|\psi\rangle$

How Well Does a Quantum Channel Preserve Information?

- Nevertheless, the fidelity $F\left(|\psi\rangle, \varepsilon\left(|\psi\rangle\langle\psi|\right)\right)$ has some drawbacks which need to be remedied
- In a real quantum memory or quantum communications channel, we don't know in advance what the initial state $|\psi\rangle$ of the system will be
- However, we can quantify the worst-case behavior of the system by minimizing over all possible initial states

$$F_{\min}\left(\varepsilon\right) = \min_{|\psi\rangle} F\left(|\psi\rangle, \varepsilon\left(|\psi\rangle\langle\psi|\right)\right)$$

How Well Does a Quantum Channel Preserve Information?

- For example, for the *p*-depolarizing channel

$$F_{\min} = \sqrt{1 - \frac{p}{2}}$$

as the fidelity of the channel is the same for all input states $|\psi\rangle$

- A more interesting example is the **phase damping channel**

$$\varepsilon(\rho) = p\rho + (1-p)Z\rho Z$$

How Well Does a Quantum Channel Preserve Information?

$\varepsilon(\rho) = p\rho + (1-p)Z\rho Z$

- For the phase damping channel, the fidelity is given by

$$
\begin{aligned}
F\left(|\psi\rangle, \varepsilon(|\psi\rangle\langle\psi|)\right) &= \sqrt{\langle\psi|\left(p|\psi\rangle\langle\psi| + (1-p)Z|\psi\rangle\langle\psi|Z\right)|\psi\rangle} \\
&= \sqrt{\langle\psi|p|\psi\rangle\langle\psi|\psi\rangle + \langle\psi|(1-p)Z|\psi\rangle\langle\psi|Z|\psi\rangle} \\
&= \sqrt{p\langle\psi|\psi\rangle\langle\psi|\psi\rangle + (1-p)\langle\psi|Z|\psi\rangle\langle\psi|Z|\psi\rangle} \\
&= \sqrt{p + (1-p)\langle\psi|Z|\psi\rangle^2}
\end{aligned}
$$

How Well Does a Quantum Channel Preserve Information?

$\varepsilon(\rho) = p\rho + (1-p)Z\rho Z$

- The second term under the square root sign is non-negative, and equal to zero when

$$|\psi\rangle = (|0\rangle + |1\rangle)/\sqrt{2} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- In fact,

$$\langle\psi|Z|\psi\rangle = \frac{1}{2}\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{2}(1-1) = 0$$

- Thus, for the phase damping channel the minimum fidelity is

$$F_{\min} = \sqrt{p}$$

How Well Does a Quantum Channel Preserve Information?

- You might wonder why we minimized over pure states in the definition of F_{min}
- After all, might not the quantum system of interest start in a **mixed state *ρ***?
- For example, a quantum memory might be entangled with the rest of the quantum computer, and therefore would start out in a mixed state
- Fortunately, the **joint concavity of the fidelity** (not proved in the lecture) can be used to show that allowing mixed states does not change F_{min}
- Suppose that $\rho = \sum_i \lambda_i |i\rangle\langle i|$ is the initial state of the quantum system, it can be proved that

$$F \quad \rho, \varepsilon \quad \rho \quad \geq F_{\min}$$

How Well Does a Quantum Channel Preserve Information?

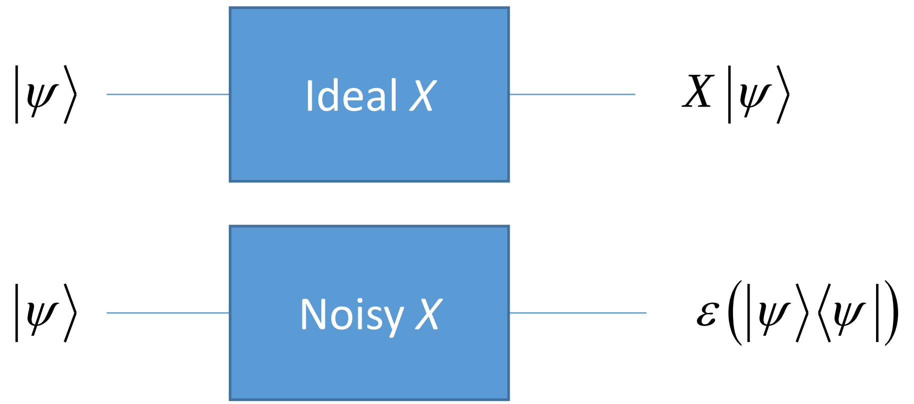
- Of course, we are interested not only in protecting quantum states as they are transmitted through a quantum communications channel, but also as they dynamically undergo computation
- Suppose, for example, that as part of a quantum computation we attempt to implement a quantum gate described by the unitary operator U
- As described in a previous lecture, any such attempt will inevitably encounter some (hopefully not too severe) noise, so the correct description of the gate is using a quantum operation $\mathcal{E}$

How Well Does a Quantum Channel Preserve Information?

- A natural measure of how successful our gate has been is the *gate fidelity*,

$$F(U,\varepsilon) \equiv \min_{|\psi\rangle} F(U|\psi\rangle, \varepsilon(|\psi\rangle\langle\psi|))$$

- Suppose, for example, that we try to implement an *X* gate on a single qubit, but instead implement the noisy operation $\varepsilon(\rho) = (1-p)X\rho X + pZ\rho Z$, for some small noise parameter *p*



How Well Does a Quantum Channel Preserve Information?

- Then the gate fidelity for this operation is given by

$$F(X,\varepsilon) \equiv \min_{|\psi\rangle} \sqrt{\langle\psi|X\left((1-p)X\rho X + pZ\rho Z\right)X|\psi\rangle}$$

$$= \min_{|\psi\rangle} \sqrt{\langle\psi|X\left((1-p)X|\psi\rangle\langle\psi|X + pZ|\psi\rangle\langle\psi|Z\right)X|\psi\rangle}$$

$$= \min_{|\psi\rangle} \sqrt{\langle\psi|X\left((1-p)X|\psi\rangle\langle\psi|XX|\psi\rangle + pZ|\psi\rangle\langle\psi|ZX|\psi\rangle\right)}$$

$$= \min_{|\psi\rangle} \sqrt{(1-p)\langle\psi|XX|\psi\rangle\langle\psi|XX|\psi\rangle + p\langle\psi|XZ|\psi\rangle\langle\psi|ZX|\psi\rangle}$$

- Since $X^2 = I \rightarrow \langle\psi|XX|\psi\rangle = 1$

How Well Does a Quantum Channel Preserve Information?

- Keeping also into consideration that $ZX = -XZ$ the gate fidelity for this operation is given by

$$F\ \ X,\varepsilon\ \ = \min_{|\psi\rangle} \sqrt{1-p\ \ \langle\psi|XX|\psi\rangle\langle\psi|XX|\psi\rangle + p\langle\psi|XZ|\psi\rangle\langle\psi|ZX|\psi\rangle}$$

$$= \min_{|\psi\rangle} \sqrt{1-p\ \ -p\langle\psi|XZ|\psi\rangle^{2}}$$

- Since $XZ = -iY$

$$F\ \ X,\varepsilon\ \ = \min_{|\psi\rangle} \sqrt{1-p\ \ \langle\psi|XX|\psi\rangle\langle\psi|XX|\psi\rangle + p\langle\psi|XZ|\psi\rangle\langle\psi|ZX|\psi\rangle}$$

$$= \min_{|\psi\rangle} \sqrt{1-p\ \ +p\langle\psi|Y|\psi\rangle^{2}} = \sqrt{1-p}$$