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Quantum Computing and Quantum Internet

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Quantum Error-Correction
A General Framework

Introduction

- To protect quantum states against the effects of noise we would like to develop *quantum error-correcting codes* based on similar principles to those used for developing the *classical error-correcting codes*
- There are some important differences between classical information and quantum information that require new ideas to be introduced to make such quantum error-correcting codes possible
- In particular, at first glance we have three rather formidable difficulties to deal with:

Introduction

1. No Cloning

- One might try to implement the repetition code quantum mechanically by duplicating the quantum state three or more times
- This is forbidden by the no-cloning theorem
- Even if cloning were possible, it would not be possible to measure and compare the three quantum states output from the channel

2. Errors are Continuous

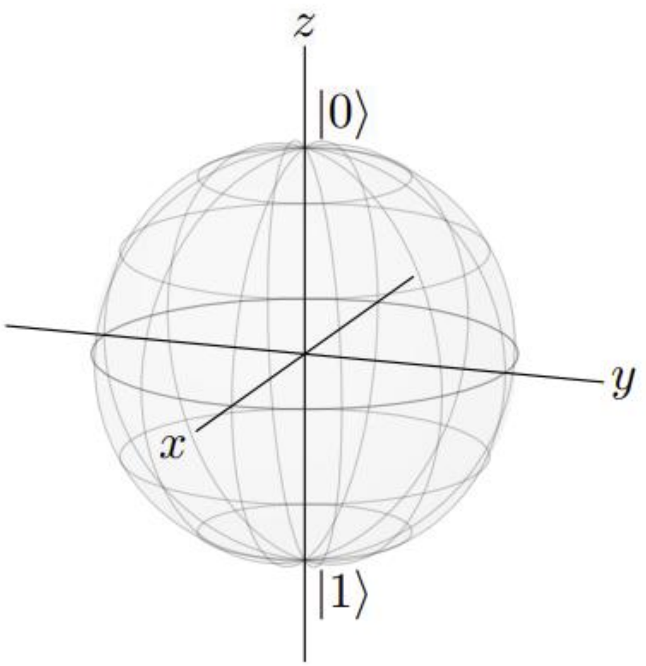
- A continuum of different errors may occur on a single qubit
- Determining which error occurred in order to correct it would appear to require infinite precision, and therefore infinite resources

Introduction

3. Measurement Destroys Quantum Information

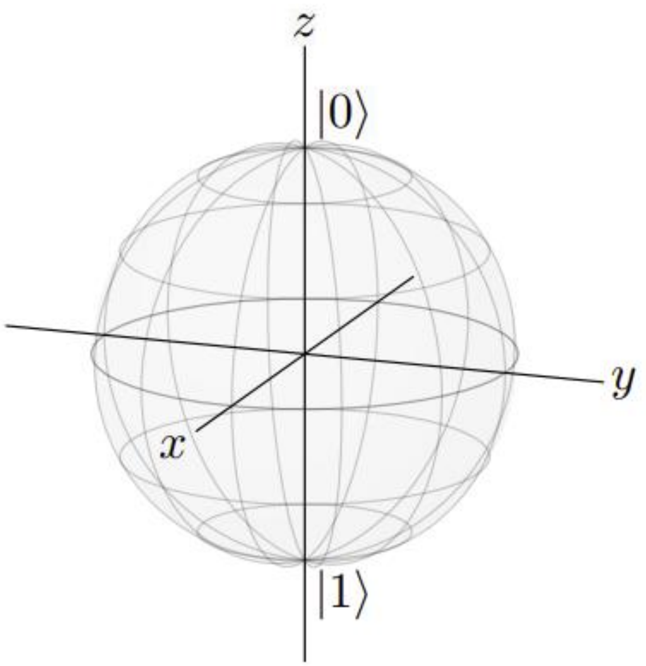
- Observation in quantum mechanics generally destroys the quantum state under observation and makes recovery impossible

Decoherence



- Recall a qubit can be represented by a point on the Bloch sphere
- The north pole corresponds to $|0\rangle$ and the south pole corresponds to $|1\rangle$
- For a classical bit, these would be the only possible states, and the only error is for the bit to completely flip between the north and south poles
- For a qubit, however, every location on the Bloch sphere is a different state

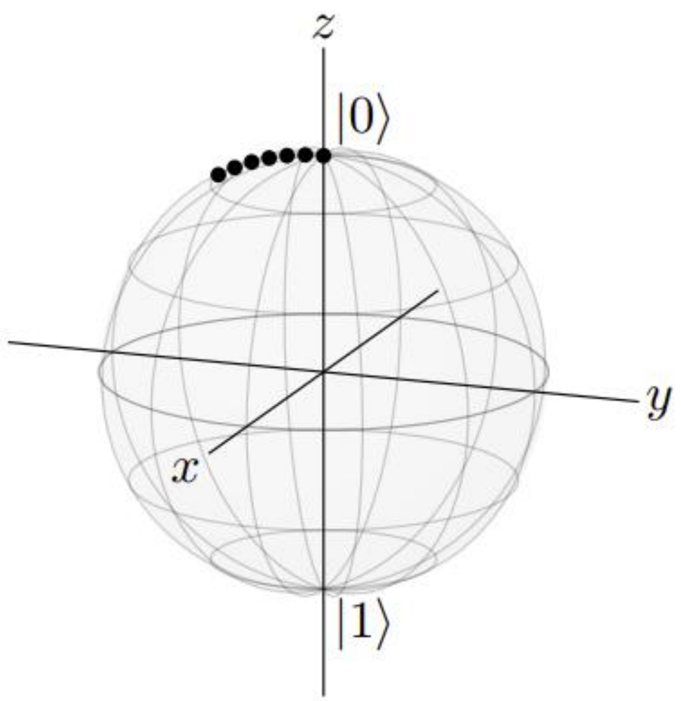
Decoherence



- To deepen the analysis of the second issue (**Errors are Continuous**) let's come back to the Bloch sphere
- Recall a qubit can be represented by a point on the Bloch sphere
- The north pole corresponds to $|0\rangle$ and the south pole corresponds to $|1\rangle$
- For a classical bit, these would be the only possible states, and the only error is for the bit to completely flip between the north and south poles
- For a qubit, however, every location on the Bloch sphere is a different state

Decoherence

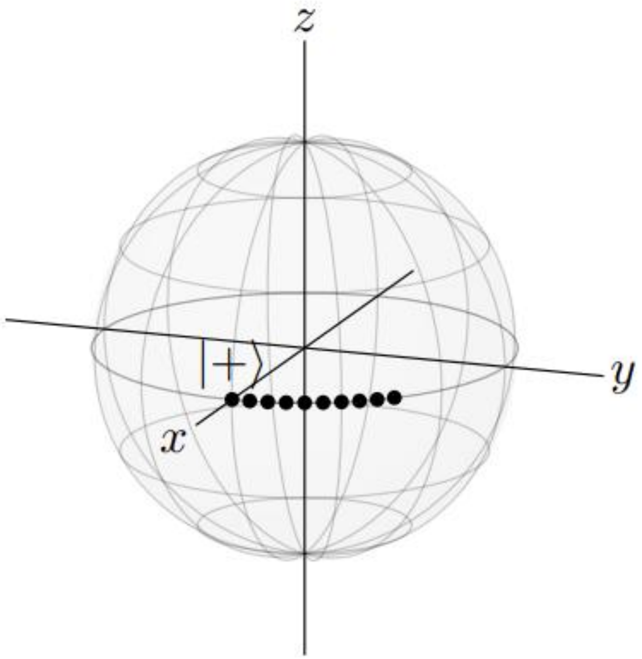
- For example, beginning at $|0\rangle$, instead of completely flipping to $|1\rangle$, a qubit could experience a partial bit flip error, where it only rotates a little toward $|1\rangle$



- Since a full *bit flip* corresponds to the *X* gate, and the *X* gate is a rotation about the *x*-axis by π = 180◦, a partial *bit flip* corresponds to rotating about the *x*-axis by some angle
- So, in the above figure, the state is moving leftward, down the Bloch sphere, in the *yz*-plane
- This small change is an error

Decoherence

- To further complicate matters, a qubit's state is not just its latitude up and down the Bloch sphere, but also its longitude around the Bloch sphere
- For example, if a qubit initially in the $|+\rangle$ state gets bumped to the side, we get a different state:



- This is called a *phase flip* error because rotations around the *z*-axis correspond to changes in the relative phase
- For example,

  $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and

  $|-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$

  lie on opposite sides of the equator

Decoherence

- Since qubits are more sensitive to errors than classical bits, small interactions with the environment can move the qubit to a different location on the Bloch sphere
- This process is called **decoherence**
- In practice, decoherence is the biggest obstacle to building large-scale quantum computers, since it is very difficult to isolate a qubit from its environment while making it accessible for quantum gates and measurements

Decoherence

- Next, we will see how to correct for *bit-flip* errors and then *phase-flip* errors
- Then, we will *combine* both types of error correction into what is known as the *Shor code*

Bit-Flip Errors

Bit-Flip Code

- To make it possible to correct *bit-flip errors,* we use **three physical qubits** to encode each logical qubit:

$$|0_L\rangle = |000\rangle, \quad |1_L\rangle = |111\rangle$$

where subscript ***L*** denotes a *logical qubit*

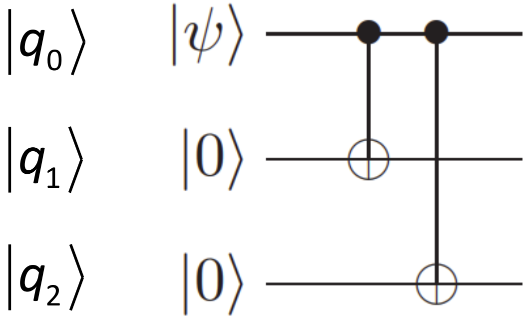
- A **logical qubit** is, in general, a superposition of $|0_L\rangle$ and $|1_L\rangle$

$$\alpha|0_L\rangle + \beta|1_L\rangle = \alpha|000\rangle + \beta|111\rangle$$

where it is understood that **superpositions of basis states are taken to corresponding superpositions of encoded states**

Bit-Flip Code

- A way to create this encoding is illustrated in the following figure

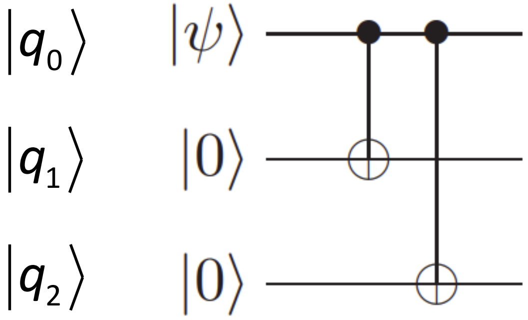


- Assume we have a single qubit in the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$
- To encode this state using the bit-flip code we add two more qubits to our system, all initially in $|0\rangle$, so our three qubits are in the state

$$|\psi\rangle|0\rangle|0\rangle = \left(\alpha|0\rangle + \beta|1\rangle\right)|0\rangle|0\rangle = \alpha|000\rangle + \beta|100\rangle$$

Bit-Flip Code

- Starting with this state



- Assume we have a single qubit in the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$
- To encode this state using the bit-flip code we add two more qubits to our system, all initially in $|0\rangle$, so our three qubits are in the state

$$|\psi\rangle|0\rangle|0\rangle = \alpha|000\rangle + \beta|100\rangle \xrightarrow{CNOT_{0,1}} \alpha|000\rangle + \beta|110\rangle \xrightarrow{CNOT_{0,2}} \alpha|000\rangle + \beta|111\rangle$$

Bit-Flip Code

- For the moment, let us first consider the case where a bit is completely flipped
- For example, say the **left qubit flips**, so

$$\begin{aligned}\alpha|000\rangle+\beta|111\rangle &\to \alpha|100\rangle+\beta|011\rangle \\ &=\beta|011\rangle+\alpha|100\rangle\end{aligned}$$

- We would like to *detect* this error and *correct* it
- Classically, we could just measure the bits, see which one disagrees with the others, and then flip it back to correct it

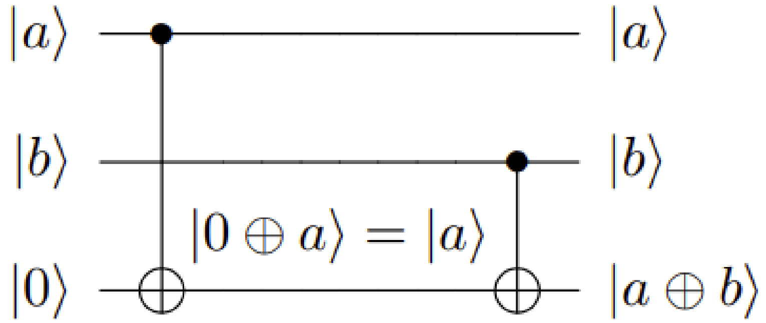
Bit-Flip Code

- Quantumly, however, if we measure the qubits (or even just a single qubit), the state collapses to $|100\rangle$ or $|011\rangle$, and we lose the superposition
- So, instead of measuring the qubits, we measure the **parity** of adjacent qubits
- Recall that the **parity** of two bits, ***a*** and ***b***, can be calculated using **Exclusive OR**
- That is,

$$\text{parity}(a,b) = a \oplus b$$

- Also recall that

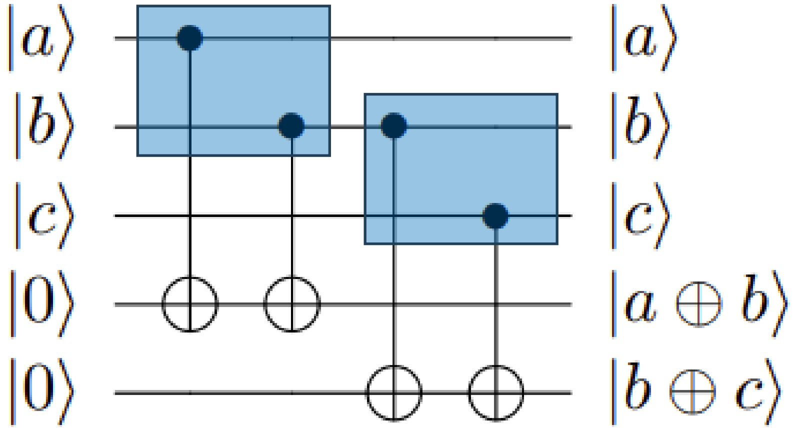
$$\text{CNOT}|a\rangle|b\rangle = |a\rangle|a \oplus b\rangle$$



- Then, we can use two CNOTs to calculate the parity of **two** qubits, putting the answer in an ancilla qubit

Bit-Flip Code

- With **three** qubits, we can calculate the parities of adjacent qubits by doing this twice:



Bit-Flip Code

- In this example, the parity of the left two qubits (01 and 10, in red) is 1, and the parity of the right two qubits (11 and 00, in blue) is 0

$$\beta|011\rangle + \alpha|100\rangle$$

$$\beta|011\rangle + \alpha|100\rangle$$

- This tells us that the left two qubits differ, and the right two qubits are the same
- Then, we know the left qubit has flipped, and we inferred this without directly measuring and collapsing the state
- This is called an *error syndrome*

Bit-Flip Code

- To correct the error, we can simply apply $(X \otimes I \otimes I)$ to $\beta|011\rangle + \alpha|100\rangle$

$$
\begin{aligned}
(X \otimes I \otimes I)\big(\beta|011\rangle + \alpha|100\rangle\big) &= \beta(X \otimes I \otimes I)|011\rangle + \alpha(X \otimes I \otimes I)|100\rangle \\
&= \beta\big(X|0\rangle \otimes I|1\rangle \otimes I|1\rangle\big) + \alpha\big(X|1\rangle \otimes I|0\rangle \otimes I|0\rangle\big) \\
&= \beta\big(|1\rangle \otimes |1\rangle \otimes |1\rangle\big) + \alpha\big(|0\rangle \otimes |0\rangle \otimes |0\rangle\big) \\
&= \alpha|000\rangle + \beta|111\rangle
\end{aligned}
$$

thus, correcting the error

NOTE

- One must pay close attention to the order of the qubits to avoid algebraic mistakes that can impact the output of the circuit
- If the order of the qubits is $\left|q_2 q_1 q_0\right\rangle$, then the operator $\left(A \otimes B \otimes C\right)$, when applied to $\left|q_2 q_1 q_0\right\rangle$ behaves as follows:

$$\left(A \otimes B \otimes C\right)\left|q_2 q_1 q_0\right\rangle = A\left|q_2\right\rangle \otimes B\left|q_1\right\rangle \otimes C\left|q_0\right\rangle$$

- In this way, the operator $\left(X \otimes I \otimes I\right)$ is able to correct a bit flip on the leftmost qubit $\left|q_2\right\rangle$, as shown in the previous slide

Bit-Flip Code

- Now, assume there is a **partial flip**
- We have already seen that on the Bloch sphere, a rotation by angle θ about the axis $n = \left[n_x, n_y, n_z\right]$ is given by:

$$R_n(\theta) = e^{i\alpha}\left[\cos\left(\frac{\theta}{2}\right)I - i\sin\left(\frac{\theta}{2}\right)\left(n_x X + n_y Y + n_z Z\right)\right],$$

  where $e^{i\alpha}$ is a global phase, so α can be chosen as we please
- Now, a partial bit flip corresponds to a rotation about the *x*-axis by some angle θ, so we have $n = [1,0,0]$
- We also **choose** $\alpha = \pi/2$

Bit-Flip Code

- Then, the rotation corresponds to

$$i\left[\cos\left(\frac{\theta}{2}\right)I - i\sin\left(\frac{\theta}{2}\right)X\right] = i\cos\left(\frac{\theta}{2}\right)I + \sin\left(\frac{\theta}{2}\right)X$$

- Letting $\varepsilon = \sin(\theta/2)$, we get $\cos(\theta/2) = \sqrt{1-\sin^2(\theta/2)} = \sqrt{1-\varepsilon^2}$, so the rotation

$$R_x(\theta) = i\sqrt{1-\varepsilon^2}I + \varepsilon X = i\sqrt{1-\varepsilon^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \varepsilon\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} i\sqrt{1-\varepsilon^2} & \varepsilon \\ \varepsilon & i\sqrt{1-\varepsilon^2} \end{bmatrix}$$

Bit-Flip Code

- Since

$$R_x(\theta)|0\rangle = \begin{bmatrix} i\sqrt{1-\varepsilon^2} & \varepsilon \\ \varepsilon & i\sqrt{1-\varepsilon^2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} i\sqrt{1-\varepsilon^2} \\ \varepsilon \end{bmatrix}$$

$$R_x(\theta)|1\rangle = \begin{bmatrix} i\sqrt{1-\varepsilon^2} & \varepsilon \\ \varepsilon & i\sqrt{1-\varepsilon^2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \varepsilon \\ i\sqrt{1-\varepsilon^2} \end{bmatrix}$$

- The partial bit flip maps

$$|0\rangle \to i\sqrt{1-\varepsilon^2}\,|0\rangle + \varepsilon|1\rangle$$
$$|1\rangle \to \varepsilon|0\rangle + i\sqrt{1-\varepsilon^2}\,|1\rangle$$

Bit-Flip Code

- When $\theta = \pi \;\to\; \varepsilon = 1$, we get,

$$|0\rangle \to |1\rangle$$
$$|1\rangle \to |0\rangle$$

  which is a complete bit flip, or the *X* gate

- For example, if the left qubit partially flips,

$$\alpha|0\rangle|00\rangle + \beta|1\rangle|11\rangle \to \alpha\left(i\sqrt{1-\varepsilon^2}\,|0\rangle + \varepsilon|1\rangle\right)|00\rangle + \beta\left(\varepsilon|0\rangle + i\sqrt{1-\varepsilon^2}\,|1\rangle\right)|11\rangle$$
$$= \left(i\alpha\sqrt{1-\varepsilon^2}\,|000\rangle + \varepsilon\alpha|100\rangle\right) + \left(\beta\varepsilon|011\rangle + i\beta\sqrt{1-\varepsilon^2}\,|111\rangle\right)$$
$$= \alpha i\sqrt{1-\varepsilon^2}\,|000\rangle + \beta\varepsilon|011\rangle + \alpha\varepsilon|100\rangle + \beta i\sqrt{1-\varepsilon^2}\,|111\rangle$$

Bit-Flip Code

$$|q_2 q_1 q_0\rangle$$

$$\alpha|0\rangle|00\rangle+\beta|1\rangle|11\rangle \to \alpha i\sqrt{1-\varepsilon^2}\,|000\rangle+\beta\varepsilon|011\rangle+\alpha\varepsilon|100\rangle+\beta i\sqrt{1-\varepsilon^2}\,|111\rangle$$

- Now, we **measure** the parity of adjacent qubits
- Labeling the qubits $|q_2 q_1 q_0\rangle$, we get the following possible outcomes with corresponding probabilities:
- Outcomes #1: parity$(q_2, q_1)=0$ and parity$(q_1, q_0)=0$ with probability

$$\left|\alpha i\sqrt{1-\varepsilon^2}\right|^2+\left|\beta i\sqrt{1-\varepsilon^2}\right|^2=|\alpha|^2\left(1-\varepsilon^2\right)+|\beta|^2\left(1-\varepsilon^2\right)$$

$$=\left(|\alpha|^2+|\beta|^2\right)\left(1-\varepsilon^2\right)$$

$$=\left(1-\varepsilon^2\right)$$

Bit-Flip Code

$$|q_2 q_1 q_0\rangle$$

$$\alpha|0\rangle|00\rangle+\beta|1\rangle|11\rangle \rightarrow \alpha i\sqrt{1-\varepsilon^2}\,|000\rangle+\beta\varepsilon|011\rangle+\alpha\varepsilon|100\rangle+\beta i\sqrt{1-\varepsilon^2}\,|111\rangle$$

- After the measurements the state collapses to

$$A\left(\alpha i\sqrt{1-\varepsilon^2}\,|000\rangle+\beta i\sqrt{1-\varepsilon^2}\,|111\rangle\right)=\alpha|000\rangle+\beta|111\rangle$$

  where $A=1\big/ i\sqrt{1-\varepsilon^2}$ is a normalization constant

- We see that the resulting state is already corrected, so we do not need to do anything further to correct the error
- That is, *the measurement fixed the error*

Bit-Flip Code

$$|q_2 q_1 q_0\rangle$$

$$\alpha|0\rangle|00\rangle+\beta|1\rangle|11\rangle \rightarrow \alpha i\sqrt{1-\varepsilon^2}\,|000\rangle+\beta\varepsilon|011\rangle+\alpha\varepsilon|100\rangle+\beta i\sqrt{1-\varepsilon^2}\,|111\rangle$$

- Outcomes #2: parity$(q_2,q_1)=1$ and parity$(q_1,q_0)=0$ with probability

$$|\beta\varepsilon|^2+|\alpha\varepsilon|^2=\left(|\beta|^2+|\alpha|^2\right)\varepsilon^2=\varepsilon^2$$

  and the state collapses to

$$B\left(\beta\varepsilon|011\rangle+\alpha\varepsilon|100\rangle\right)=\beta|011\rangle+\alpha|100\rangle$$

  where $B=1/\varepsilon$ is a normalization constant

- To correct this state, we apply $(X\otimes I\otimes I)$ so that it becomes

$$\beta|111\rangle+\alpha|000\rangle=\alpha|000\rangle+\beta|111\rangle$$

  so we have corrected the error

Bit-Flip Code

$$|q_2 q_1 q_0\rangle$$

$$\alpha|0\rangle|00\rangle + \beta|1\rangle|11\rangle \to \alpha i\sqrt{1-\varepsilon^2}\,|000\rangle + \beta\varepsilon|011\rangle + \alpha\varepsilon|100\rangle + \beta i\sqrt{1-\varepsilon^2}\,|111\rangle$$

- Furthermore, in the partial bitflip scenario we are analyzing, the following patterns:

$$\text{parity}(q_2, q_1) = 0 \text{ and parity}(q_1, q_0) = 1 \to 001 \text{ or } 110$$

$$\text{parity}(q_2, q_1) = 1 \text{ and parity}(q_1, q_0) = 1 \to 010 \text{ or } 101$$

  will occur with probability 0

- Finally, we need to reset the ancilla qubits to $|0\rangle$ so that we can reuse them, since we want to repeatedly do error correction to fix any errors that appear

Bit-Flip Code

- We can do this by conditionally applying an *X* gate

- If we measured a parity to be 0, we know that the ancilla qubit is $|0\rangle$, so we leave it alone

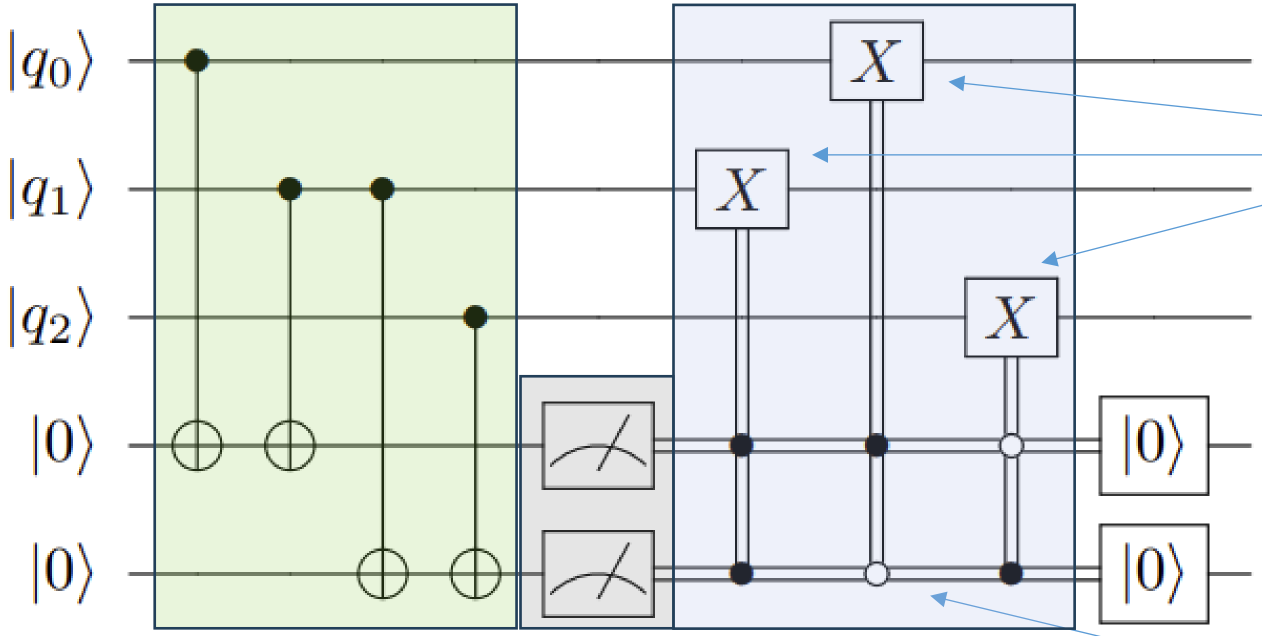
- If we measured the parity to be 1, we know that the ancilla qubit is $|1\rangle$, and so we apply an *X* gate to it, turning it into a $|0\rangle$

Bit-Flip Code

- To summarize, when we have a *partial bit flip*, the measurement forces it to be corrected or to become a *complete bit flip*, which we can correct by using an X gate

Bit-Flip Code

- Here is the circuit for the bit flip error



We end with three X gates conditioned on these classical bits/parities.

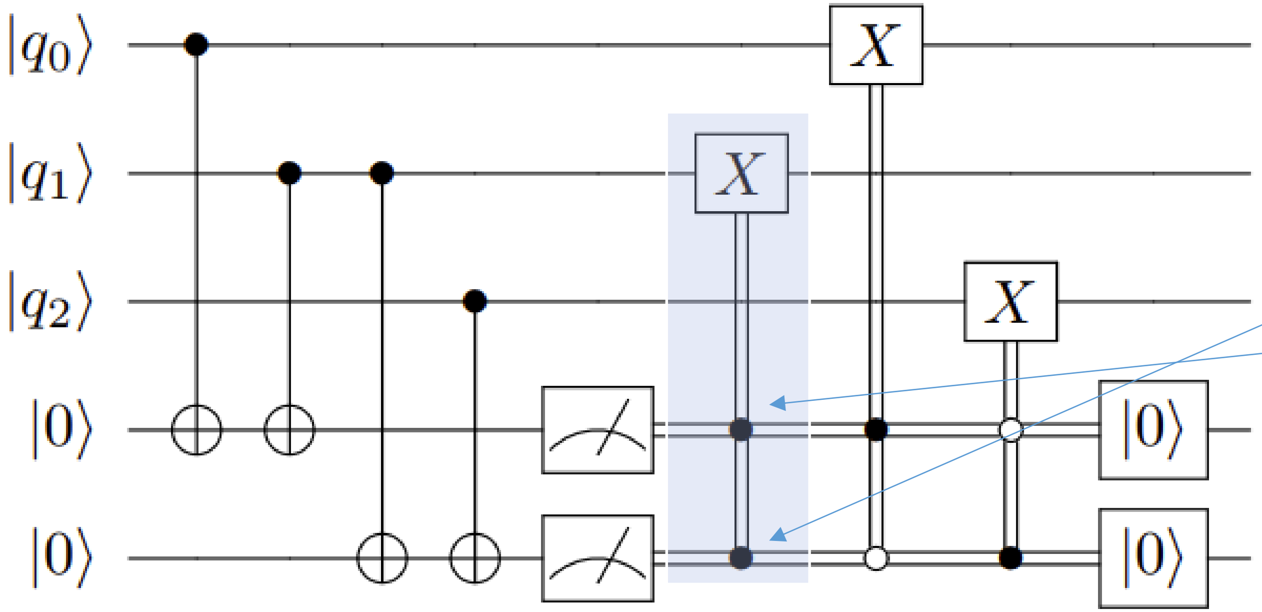
The first four columns are the CNOTs that calculate the parities of adjacent qubits.

Then, measure these parities, as shown by the meter symbols, which results in classical bits.

We denote these classical bits/wires using double lines

Bit-Flip Code

parity$(q_2, q_1) = 1$ and parity$(q_1, q_0) = 1 \rightarrow 010$ or 101

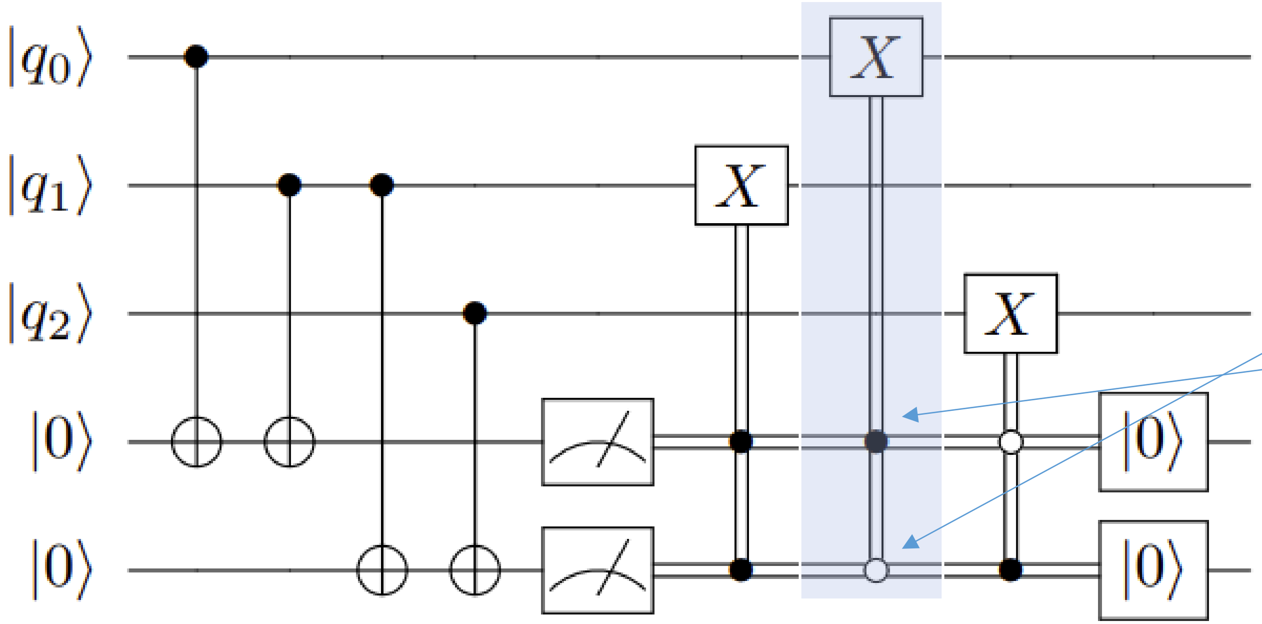


parity$(q_2, q_1) = 1$
parity$(q_1, q_0) = 1$

If parity(q_2,q_1) = 1 and parity(q_1,q_0) = 1, then q_1 flipped, so we apply an X gate to q_1 to correct it.

Bit-Flip Code

$\text{parity}(q_2, q_1) = 0$ and $\text{parity}(q_1, q_0) = 1 \rightarrow 001$ or 110



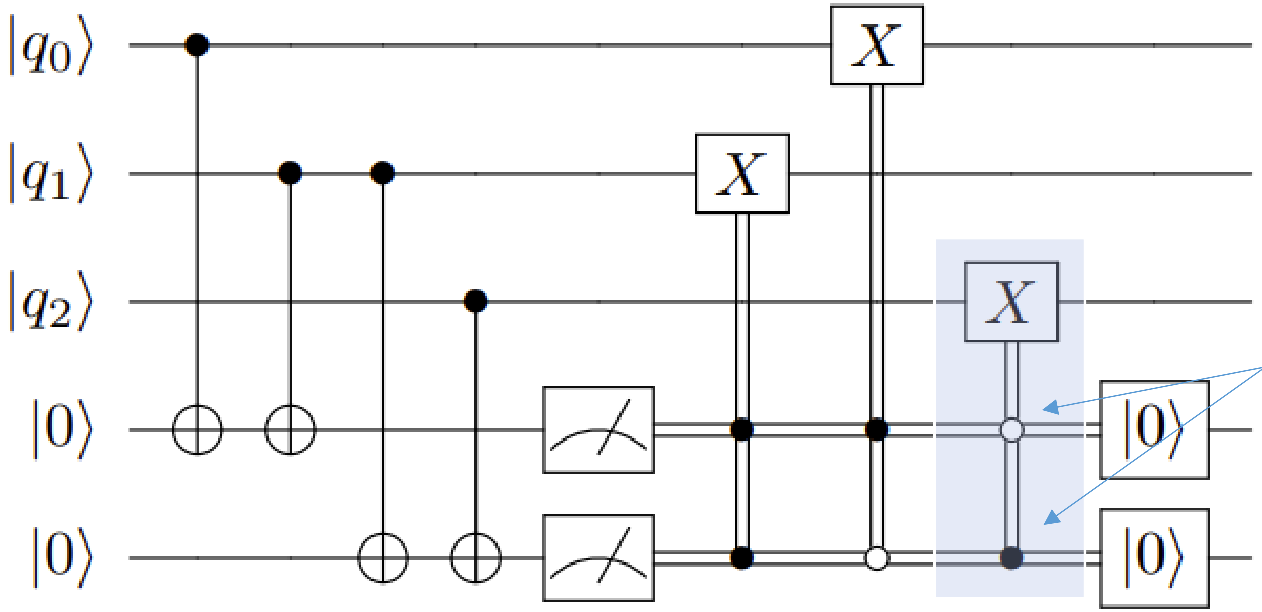
$\text{parity}(q_2, q_1) = 0$
$\text{parity}(q_1, q_0) = 1$

If parity(q_2,q_1) = 0 and parity(q_1,q_0) = 1, then q_0 flipped, so we apply an X gate to q_0 to correct it.

Bit-Flip Code

$$|q_2 q_1 q_0\rangle$$

$$\alpha|0\rangle|00\rangle + \beta|1\rangle|11\rangle \rightarrow \alpha i\sqrt{1-\varepsilon^2}\,|000\rangle + \beta\varepsilon|011\rangle + \alpha\varepsilon|100\rangle + \beta i\sqrt{1-\varepsilon^2}\,|111\rangle$$



$\text{parity}(q_2, q_1) = 1$

$\text{parity}(q_1, q_0) = 0$

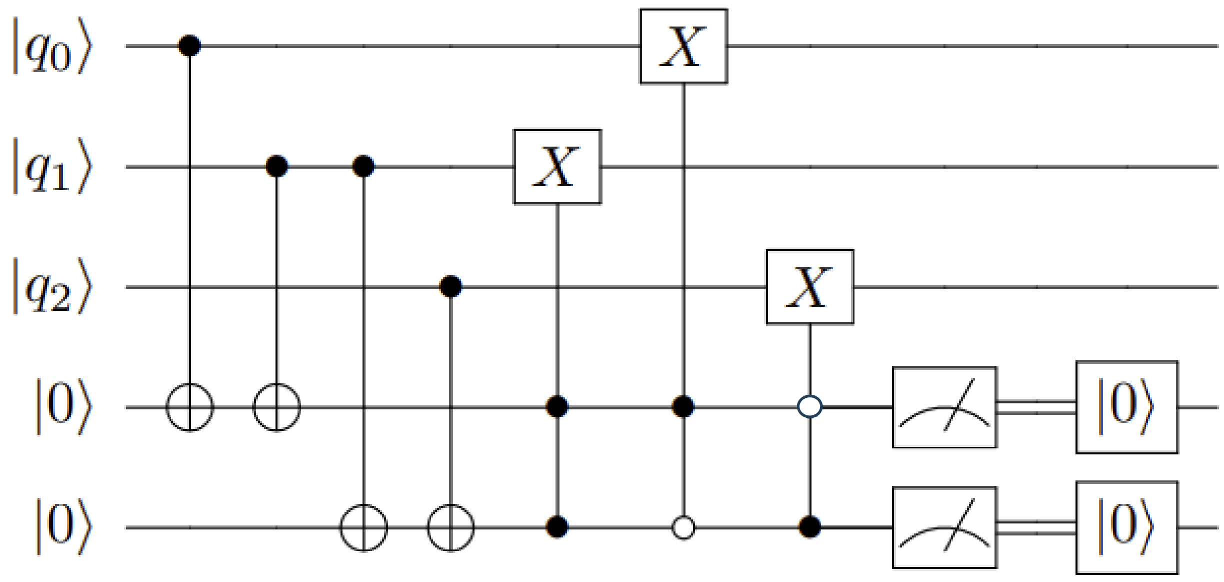
If parity(q_2,q_1) = 1 and parity(q_1,q_0) = 0, then q_2 flipped, so we apply an X gate to q_2 to correct it.

Bit-Flip Code

- We can modify the above circuit using *the principle of deferred measurement*, which says that *intermediate measurements that are used to control operations can be moved after the operations, and the controls can be replaced by quantum controls.*

Bit-Flip Code

- Then, the previous quantum circuit to correct bit flips is equivalent to



- Phrased another way, we can collapse and then do the controlled operations, or we can do the controlled operations in superposition, and then collapse.

Bit-Flip Code

- **Proof:** Let us prove this for our previous example, where the qubits started in the state $\alpha|000\rangle+\beta|111\rangle$, but then the left qubit partially flips with amplitude ε
- From earlier, the state after the first four CNOTs

$$\alpha i\sqrt{1-\varepsilon^2}\,|000\rangle+\beta\varepsilon|011\rangle+\alpha\varepsilon|100\rangle+\beta i\sqrt{1-\varepsilon^2}\,|111\rangle$$

- If we include the ancilla qubits (in red), the above state becomes

$$\alpha i\sqrt{1-\varepsilon^2}\,|{\color{red}00}000\rangle+\beta\varepsilon|{\color{red}10}011\rangle+\alpha\varepsilon|{\color{red}10}100\rangle+\beta i\sqrt{1-\varepsilon^2}\,|{\color{red}00}111\rangle$$

- Recall the qubits are ordered as

$$|\text{parity}(q_2,q_1)\rangle|\text{parity}(q_1,q_0)\rangle|q_2\rangle|q_1\rangle|q_0\rangle$$

Bit-Flip Code $\alpha i\sqrt{1-\varepsilon^2}\,|00000\rangle+\beta\varepsilon|10011\rangle+\alpha\varepsilon|10100\rangle+\beta i\sqrt{1-\varepsilon^2}\,|00111\rangle$

- Now, if we apply the controlled and anti-controlled (in red) *X* gates to correct the answers (in black), the state becomes

$$\alpha i\sqrt{1-\varepsilon^2}\,|00000\rangle+\beta\varepsilon|10111\rangle+\alpha\varepsilon|10000\rangle+\beta i\sqrt{1-\varepsilon^2}\,|00111\rangle$$

$$=i\sqrt{1-\varepsilon^2}\,|00\rangle\big(\alpha|000\rangle+\beta|111\rangle\big)+\varepsilon|10\rangle\big(\alpha|000\rangle+\beta|111\rangle\big)$$

$$=\left(i\sqrt{1-\varepsilon^2}\,|00\rangle+\varepsilon|10\rangle\right)\big(\alpha|000\rangle+\beta|111\rangle\big)$$

$$=i\sqrt{1-\varepsilon^2}\,|00\rangle\big(\alpha|000\rangle+\beta|111\rangle\big)+\varepsilon|10\rangle\big(\alpha|000\rangle+\beta|111\rangle\big)$$

Bit-Flip Code

$$i\sqrt{1-\varepsilon^2}\,|00\rangle\left(\alpha|000\rangle+\beta|111\rangle\right)+\varepsilon|10\rangle\left(\alpha|000\rangle+\beta|111\rangle\right)$$

Measuring the ancilla qubits now, we get:

- parity(q_2,q_1) = 0 and parity(q_1,q_0) = 0 with probability $1 - \varepsilon^2$, and the state collapses to

$$|00\rangle\left(\alpha|000\rangle+\beta|111\rangle\right)$$

- parity(q_2,q_1) = 1 and parity(q_1,q_0) = 0 with probability ε^2 , and the state collapses to

$$|10\rangle\left(\alpha|000\rangle+\beta|111\rangle\right)$$

Now, we apply an X gate to the left ancilla to reset it to 0, yielding

$$|00\rangle\left(\alpha|000\rangle+\beta|111\rangle\right) \qquad \square$$

Phase-Flip Errors

Phase-Flip Code

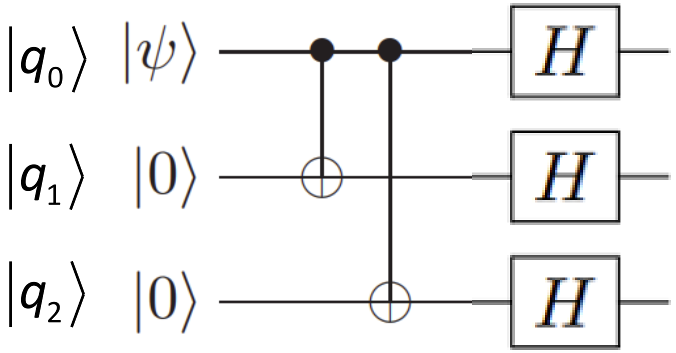
- We can similarly correct phase-flip errors by using **three physical qubits** to encode each **logical qubit**, but instead of using three $|0\rangle$'s and $|1\rangle$'s, we use three $|+\rangle$'s and $|-\rangle$'s, i.e.,

$$|0\rangle_L = |+++\rangle, \quad |1\rangle_L = |---\rangle,$$

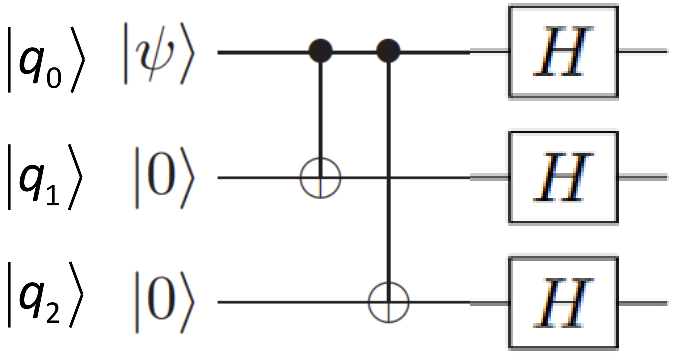
so, a general superposition is

$$\alpha|0\rangle_L + \beta|1\rangle_L = \alpha|+++\rangle + \beta|---\rangle$$

which can be created by the following circuit



Phase-Flip Code



- Let's prove it
- **Proof:** Assume that we have a single qubit in the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$
- We want to encode this state using the bit-flip code
- We add two more qubits to our system, all initially in $|0\rangle$, so our three qubits are in the initial state

$$|\psi 00\rangle \equiv |\psi\rangle|00\rangle = \left(\alpha|0\rangle + \beta|1\rangle\right)|00\rangle = \alpha|000\rangle + \beta|100\rangle$$

- Starting with this state, we apply the quantum circuit:

$$|\psi 00\rangle = \alpha|000\rangle + \beta|100\rangle \xrightarrow{CNOT_{0,1}} \alpha|000\rangle + \beta|110\rangle \xrightarrow{CNOT_{0,2}} \alpha|000\rangle + \beta|111\rangle$$

$$\xrightarrow{H^{\otimes 3}} \alpha|+++\rangle + \beta|---\rangle = \alpha|0\rangle_L + \beta|1\rangle_L \qquad \square$$

Phase-Flip Code

- The reason why we use $|+\rangle$ and $|-\rangle$ is because a **complete** phase flip (the *Z* gate) switches between these states:

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \xrightarrow{Z} \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) = |-\rangle$$

$$|-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \xrightarrow{Z} \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) = |+\rangle$$

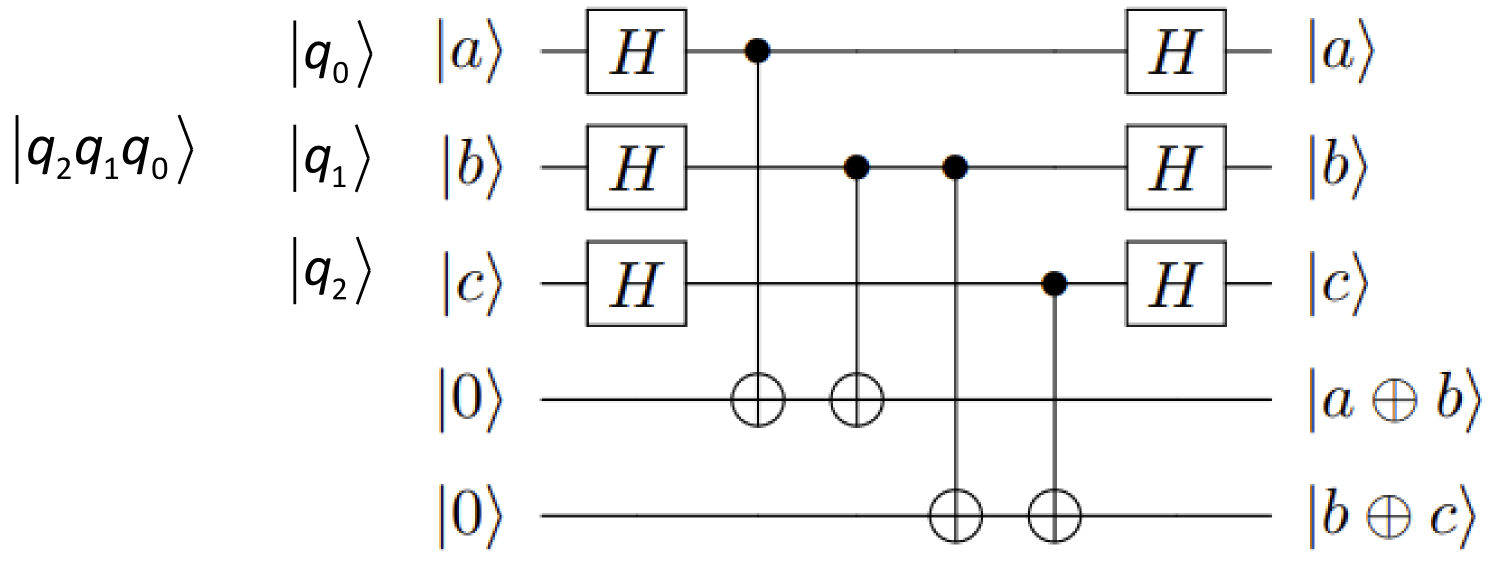
- Assume that the left qubit experiences a complete phase flip:

$$\alpha|+++\rangle + \beta|---\rangle \to \alpha|-++\rangle + \beta|+--\rangle$$

then we detect and correct this just like we did for the bit-flip error, except working in the *X*-basis

Phase-Flip Code

- So, we measure the parity of consecutive qubits in the *X*-basis, which is 0 if the number of minuses/plus is **even** and 1 if the number of minuses/plus is odd
- We will see later that the **parities** can be calculated using the following circuit



Phase-Flip Code

- In our example where the left qubit $(|q_2\rangle)$ experienced a phase flip,

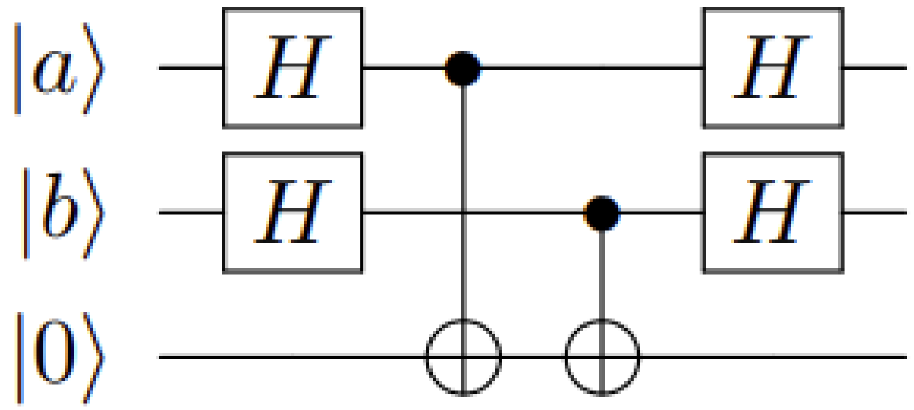
$$\alpha|+++\rangle+\beta|---\rangle \to \alpha|-++\rangle+\beta|+--\rangle$$

- Thus, we get **parity 1** for the left two qubits and **parity 0** for the right two qubits, implying that the first qubit is flipped
- So, we apply $(Z\otimes I\otimes I)$, restoring $\alpha|+++\rangle+\beta|---\rangle$

$$\begin{aligned}(Z\otimes I\otimes I)(\alpha|-++\rangle+\beta|+--\rangle) &= \alpha(Z\otimes I\otimes I)|-\rangle|+\rangle|+\rangle+\beta(Z\otimes I\otimes I)|+\rangle|-\rangle|-\rangle \\ &= \alpha(Z|-\rangle)\otimes(I|+\rangle)\otimes(I|+\rangle)+\beta(Z|+\rangle)\otimes(I|-\rangle)\otimes(I|-\rangle) \\ &= \alpha|+\rangle\otimes|+\rangle\otimes|+\rangle+\beta|-\rangle\otimes|-\rangle\otimes|-\rangle \\ &= \alpha|+++\rangle+\beta|---\rangle\end{aligned}$$

Phase-Flip Code

- Before delving into a more comprehensive analysis of phase-flip error correction, we want to demonstrate how to measure the **parity** of consecutive qubits in the *X*-basis
- Let’s start with the following basic quantum circuit

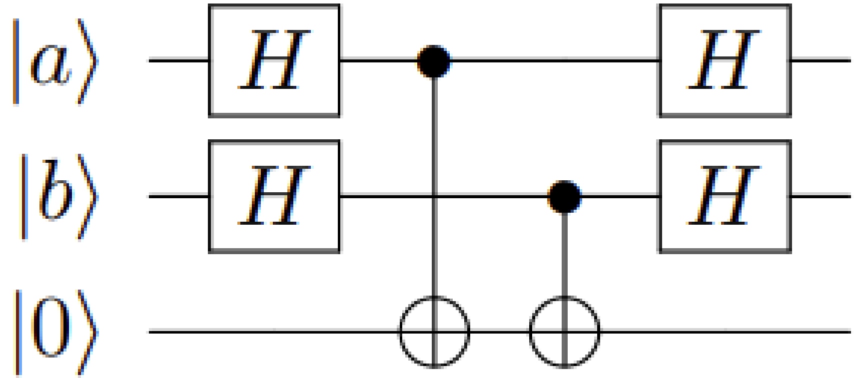


Phase-Flip Code

- We want to derive the resulting states at the end of the circuit in the following cases:

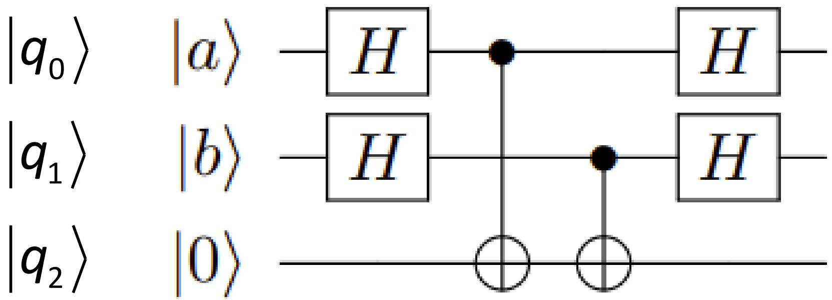
$(a)\ |a\rangle = |+\rangle, \quad |b\rangle = |+\rangle \qquad (b)\ |a\rangle = |+\rangle, \quad |b\rangle = |-\rangle$

$(c)\ |a\rangle = |-\rangle, \quad |b\rangle = |+\rangle \qquad (d)\ |a\rangle = |-\rangle, \quad |b\rangle = |-\rangle$



Phase-Flip Code

If we apply the quantum circuit to the above states, we obtain



$$|q_2 q_1 q_0\rangle = |0ba\rangle$$

(a) $|0++\rangle \xrightarrow{I\otimes H\otimes H} |000\rangle \xrightarrow{CNOT_{0,2}} |000\rangle \xrightarrow{CNOT_{1,2}} |000\rangle \xrightarrow{I\otimes H\otimes H} {\color{blue}|0++\rangle}$

(b) $|0-+\rangle \xrightarrow{I\otimes H\otimes H} |010\rangle \xrightarrow{CNOT_{0,2}} |010\rangle \xrightarrow{CNOT_{1,2}} |110\rangle \xrightarrow{I\otimes H\otimes H} {\color{red}|1-+\rangle}$

(c) $|0+-\rangle \xrightarrow{I\otimes H\otimes H} |001\rangle \xrightarrow{CNOT_{0,2}} |101\rangle \xrightarrow{CNOT_{1,2}} |101\rangle \xrightarrow{I\otimes H\otimes H} {\color{red}|1+-\rangle}$

(d) $|0--\rangle \xrightarrow{I\otimes H\otimes H} |011\rangle \xrightarrow{CNOT_{0,2}} |111\rangle \xrightarrow{CNOT_{1,2}} |011\rangle \xrightarrow{I\otimes H\otimes H} {\color{blue}|0--\rangle}$

Phase-Flip Code

As a conclusion the quantum circuit outputs 0 when there is an even number of $\left|-\right\rangle / \left|+\right\rangle$ and 1 when there is an odd number of $\left|-\right\rangle / \left|+\right\rangle$

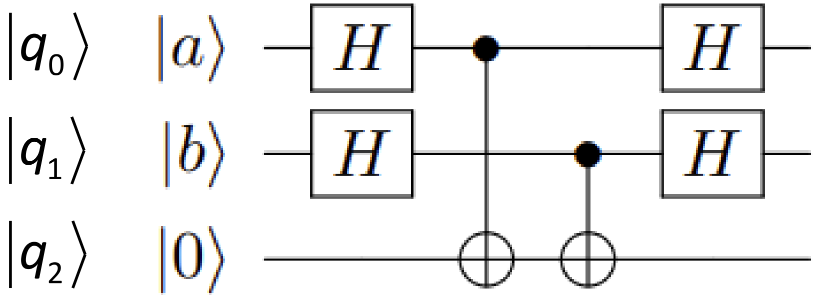
Therefore, we have proved that the quantum circuit analyzed provides the measure of the parity of two consecutive qubits, i.e., $\left|a\right\rangle$ and $\left|b\right\rangle$

(a) $\left|0++\right\rangle \xrightarrow{\text{Quantum Circuit}} \left|0++\right\rangle$

(b) $\left|0-+\right\rangle \xrightarrow{\text{Quantum Circuit}} \left|1-+\right\rangle$

(c) $\left|0+-\right\rangle \xrightarrow{\text{Quantum Circuit}} \left|1+-\right\rangle$

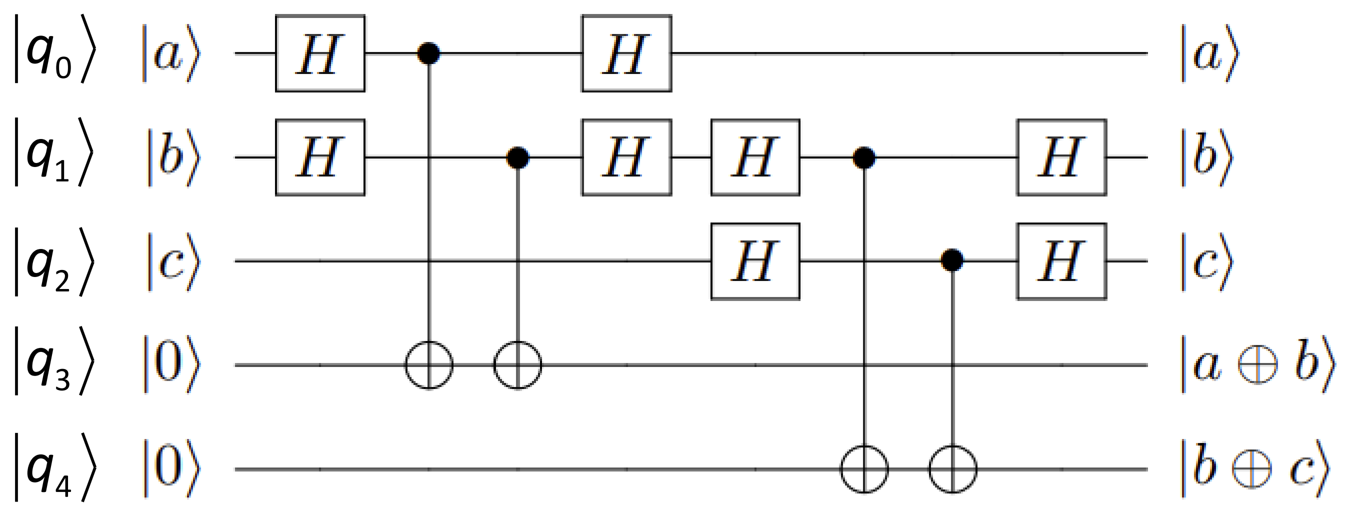
(d) $\left|0--\right\rangle \xrightarrow{\text{Quantum Circuit}} \left|0--\right\rangle$



$\left|q_2 q_1 q_0\right\rangle = \left|0ba\right\rangle$

Phase-Flip Code

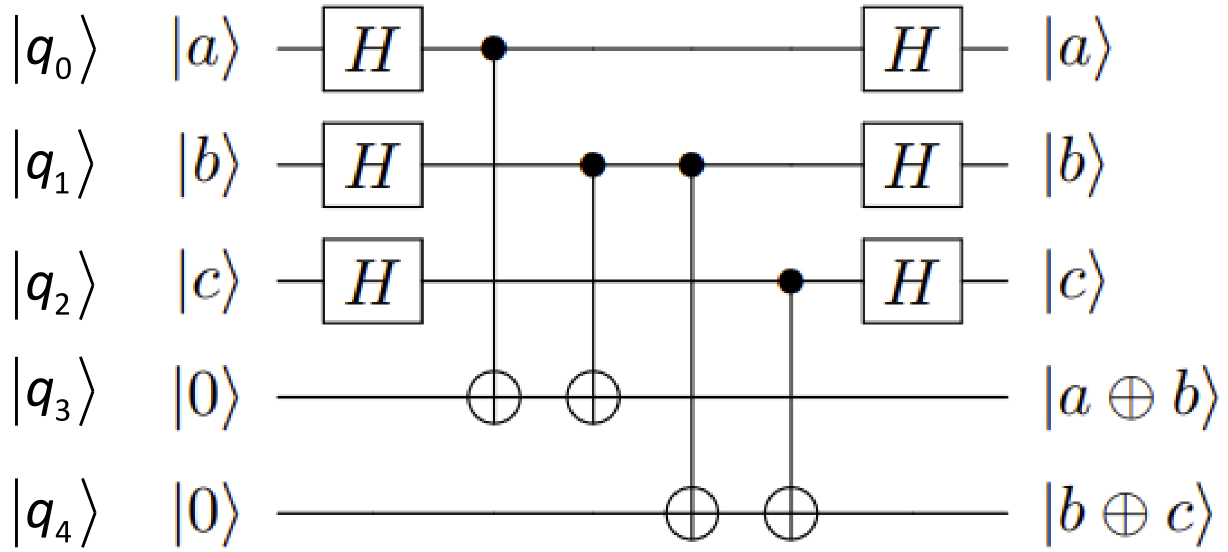
Using two copies of the above circuit, we can calculate the parity of adjacent qubits in the *X*-basis using the following circuit:



$$|q_4 q_3 q_2 q_1 q_0\rangle = |00cba\rangle$$

Phase-Flip Code

Considering that $H^2 = I$, the quantum circuit of the previous slide is equivalent to



$|q_4 q_3 q_2 q_1 q_0\rangle = |00cba\rangle$

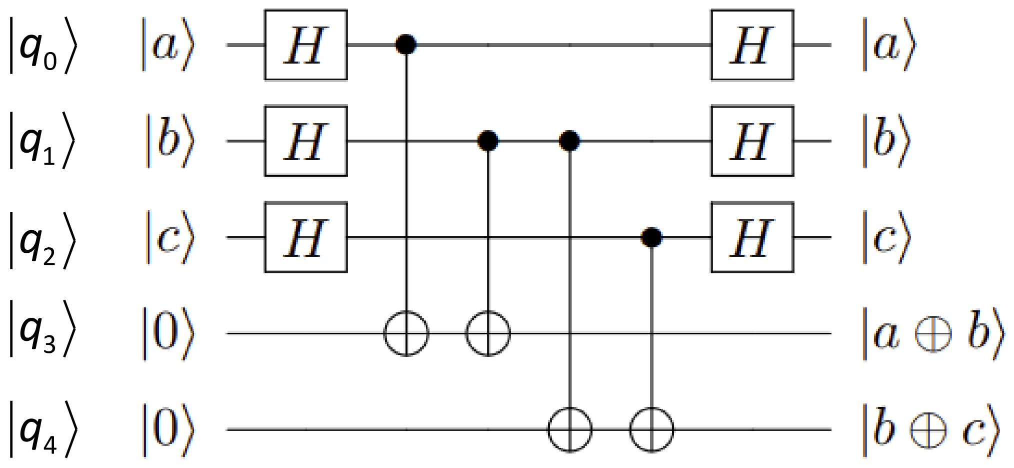
$(a)\ \ |a\rangle = |+\rangle, \quad |b\rangle = |+\rangle, \quad |c\rangle = |+\rangle$

$(b)\ \ |a\rangle = |+\rangle, \quad |b\rangle = |-\rangle, \quad |b\rangle = |+\rangle$

Phase-Flip Code

$\left|q_4q_3q_2q_1q_0\right\rangle = \left|00cba\right\rangle$

Let's check a few cases



$(a)\ \ |a\rangle = |+\rangle, \quad |b\rangle = |+\rangle, \quad |c\rangle = |+\rangle$

$(b)\ \ |a\rangle = |+\rangle, \quad |b\rangle = |-\rangle, \quad |b\rangle = |+\rangle$

$$(a)\ \ |00cba\rangle = |00+++\rangle \xrightarrow{I^{\otimes 2}H^{\otimes 3}} |00000\rangle \xrightarrow{CNOT_{0,3}} |00000\rangle \xrightarrow{CNOT_{1,3}} |00000\rangle$$
$$\xrightarrow{CNOT_{1,4}} |00000\rangle \xrightarrow{CNOT_{2,4}} |00000\rangle \xrightarrow{I^{\otimes 2}H^{\otimes 3}} |{\color{red}00}+++\rangle$$

$$(b)\ \ |00cba\rangle = |00+-+\rangle \xrightarrow{I^{\otimes 2}H^{\otimes 3}} |00010\rangle \xrightarrow{CNOT_{0,3}} |00010\rangle \xrightarrow{CNOT_{1,3}} |01010\rangle$$
$$\xrightarrow{CNOT_{1,4}} |11010\rangle \xrightarrow{CNOT_{2,4}} |11010\rangle \xrightarrow{I^{\otimes 2}H^{\otimes 3}} |{\color{red}11}+-+\rangle$$

Phase-Flip Code

$\left|q_4 q_3 q_2 q_1 q_0\right\rangle = \left|00cba\right\rangle$

From the above partial results, we can observe that:

- Qubits q_3 and q_4 hold the parities of the couples q_0, q_1 and q_1, q_2 respectively. In case a) the parities are both 0 since q_0, q_1, and q_2 are in the same state. In case b) the parities are both 1 since q_0, q_1, and q_1, q_2 are couples with qubits in different states.
- The states of qubits q_0, q_1, and q_2 **remain unchanged**

$$(a)\quad \left|00cba\right\rangle = \left|00+++\right\rangle \xrightarrow{I^{\otimes 2}H^{\otimes 3}} \left|00000\right\rangle \xrightarrow{CNOT_{0,3}} \left|00000\right\rangle \xrightarrow{CNOT_{1,3}} \left|00000\right\rangle \xrightarrow{CNOT_{1,4}} \left|00000\right\rangle \xrightarrow{CNOT_{2,4}} \left|00000\right\rangle \xrightarrow{I^{\otimes 2}H^{\otimes 3}} \left|{\color{red}00}+++\right\rangle$$

$$(b)\quad \left|00cba\right\rangle = \left|00+-+\right\rangle \xrightarrow{I^{\otimes 2}H^{\otimes 3}} \left|00010\right\rangle \xrightarrow{CNOT_{0,3}} \left|00010\right\rangle \xrightarrow{CNOT_{1,3}} \left|01010\right\rangle \xrightarrow{CNOT_{1,4}} \left|11010\right\rangle \xrightarrow{CNOT_{2,4}} \left|11010\right\rangle \xrightarrow{I^{\otimes 2}H^{\otimes 3}} \left|{\color{red}11}+-+\right\rangle$$

Phase-Flip Code

$|q_4q_3q_2q_1q_0\rangle = |00cba\rangle$

Let's analyze two more cases

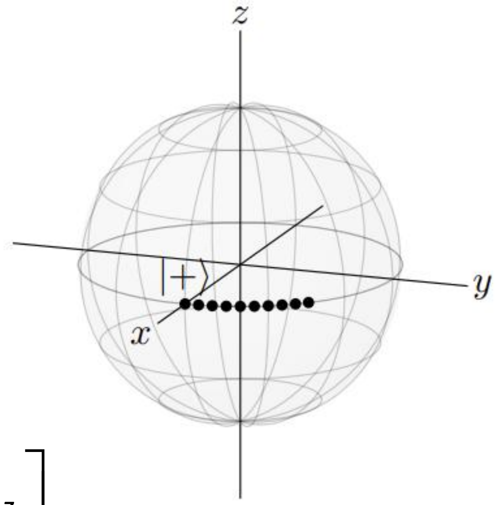
$$(c)\quad |00cba\rangle = |00+--\rangle \xrightarrow{I^{\otimes 2}H^{\otimes 3}} |00011\rangle \xrightarrow{CNOT_{0,3}} |01011\rangle \xrightarrow{CNOT_{1,3}} |00011\rangle$$
$$\xrightarrow{CNOT_{1,4}} |10011\rangle \xrightarrow{CNOT_{2,4}} |10011\rangle \xrightarrow{I^{\otimes 2}H^{\otimes 3}} |\color{red}{10}+--\rangle$$

$$(d)\quad |00cba\rangle = |00--+\rangle \xrightarrow{I^{\otimes 2}H^{\otimes 3}} |00110\rangle \xrightarrow{CNOT_{0,3}} |00110\rangle \xrightarrow{CNOT_{1,3}} |01110\rangle$$
$$\xrightarrow{CNOT_{1,4}} |11110\rangle \xrightarrow{CNOT_{2,4}} |01110\rangle \xrightarrow{I^{\otimes 2}H^{\otimes 3}} |\color{red}{01}--+\rangle$$

- Verifying that the rules that were observed in the previous slide still apply is an easy task

Phase-Flip Code

- As we saw earlier, a **partial phase flip** corresponds to a rotation about the *z*-axis by some angle θ



- Using

$$R_n(\theta) = e^{i\alpha}\left[\cos\left(\frac{\theta}{2}\right)I - i\sin\left(\frac{\theta}{2}\right)\left(n_xX + n_yY + n_zZ\right)\right], \qquad n = \left[n_x, n_y, n_z\right]$$

  - with $\alpha = \pi/2$ and $\varepsilon = \sin(\vartheta/2)$, but now with $n = [0,0,1]$, we get that the rotation is

$$R_z(\theta) = e^{i\frac{\pi}{2}}\left(\sqrt{1-\varepsilon^2}I - i\varepsilon Z\right) = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\left(\sqrt{1-\varepsilon^2}I - i\varepsilon Z\right)$$

$$= i\left(\sqrt{1-\varepsilon^2}I - i\varepsilon Z\right) = i\sqrt{1-\varepsilon^2}I + \varepsilon Z \qquad \rightarrow$$

Phase-Flip Code

$$= i\sqrt{1-\varepsilon^2}I + \varepsilon Z = i\sqrt{1-\varepsilon^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \varepsilon\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} i\sqrt{1-\varepsilon^2}+\varepsilon & 0 \\ 0 & i\sqrt{1-\varepsilon^2}-\varepsilon \end{bmatrix}$$

- Thus

$$R_z(\theta) == \begin{bmatrix} i\sqrt{1-\varepsilon^2}+\varepsilon & 0 \\ 0 & i\sqrt{1-\varepsilon^2}-\varepsilon \end{bmatrix}, \quad \varepsilon = \sin\left(\frac{\theta}{2}\right)$$

Phase-Flip Code

- We can now derive

$$R_z(\theta)|0\rangle = \begin{bmatrix} i\sqrt{1-\varepsilon^2}+\varepsilon & 0 \\ 0 & i\sqrt{1-\varepsilon^2}-\varepsilon \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} i\sqrt{1-\varepsilon^2}+\varepsilon \\ 0 \end{bmatrix} = \left(i\sqrt{1-\varepsilon^2}+\varepsilon\right)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \left(i\sqrt{1-\varepsilon^2}+\varepsilon\right)|0\rangle$$

$$R_z(\theta)|1\rangle = \begin{bmatrix} i\sqrt{1-\varepsilon^2}+\varepsilon & 0 \\ 0 & i\sqrt{1-\varepsilon^2}-\varepsilon \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ i\sqrt{1-\varepsilon^2}-\varepsilon \end{bmatrix} = \left(i\sqrt{1-\varepsilon^2}-\varepsilon\right)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \left(i\sqrt{1-\varepsilon^2}-\varepsilon\right)|1\rangle$$

- Thus, the *partial phase flip* maps

$$|0\rangle \to \left(i\sqrt{1-\varepsilon^2}+\varepsilon\right)|0\rangle$$

$$|1\rangle \to \left(i\sqrt{1-\varepsilon^2}-\varepsilon\right)|1\rangle$$

Note: When $\theta=\pi \to \varepsilon=1$, we get $|0\rangle \to |0\rangle$ and $|1\rangle \to -|1\rangle$, which is a complete phase flip, or the Z gate

Phase-Flip Code

- Let us see how a partial phase flip transforms $|+\rangle$ and $|-\rangle$

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \rightarrow \frac{1}{\sqrt{2}}\left[\left(i\sqrt{1-\varepsilon^2} + \varepsilon\right)|0\rangle + \left(i\sqrt{1-\varepsilon^2} - \varepsilon\right)|1\rangle\right]$$

$$= i\sqrt{1-\varepsilon^2}\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) + \varepsilon\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

$$= i\sqrt{1-\varepsilon^2}\,|+\rangle + \varepsilon|-\rangle$$

$$|-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \rightarrow \frac{1}{\sqrt{2}}\left[\left(i\sqrt{1-\varepsilon^2} + \varepsilon\right)|0\rangle - \left(i\sqrt{1-\varepsilon^2} - \varepsilon\right)|1\rangle\right]$$

$$= \varepsilon\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) + i\sqrt{1-\varepsilon^2}\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

$$= \varepsilon|+\rangle + i\sqrt{1-\varepsilon^2}\,|-\rangle$$

Phase-Flip Code

- Using this, if we have a logical qubit in the state

$$\alpha|0\rangle_L + \beta|1\rangle_L = \alpha|+++\rangle + \beta|---\rangle = \alpha|+\rangle|++\rangle + \beta|-\rangle|--\rangle$$

a partial phase flip on the left qubit transforms this to

$$\alpha\left(i\sqrt{1-\varepsilon^2}\,|+\rangle + \varepsilon|-\rangle\right)|++\rangle + \beta\left(\varepsilon|+\rangle + i\sqrt{1-\varepsilon^2}\,|-\rangle\right)|--\rangle$$

$$= \alpha i\sqrt{1-\varepsilon^2}\,|+++\rangle + \alpha\varepsilon|-++\rangle + \beta\varepsilon|+--\rangle + \beta i\sqrt{1-\varepsilon^2}\,|---\rangle$$

$$= \alpha i\sqrt{1-\varepsilon^2}\,|+++\rangle + \beta\varepsilon|+--\rangle + \alpha\varepsilon|-++\rangle + \beta i\sqrt{1-\varepsilon^2}\,|---\rangle$$

Phase-Flip Code

$$\alpha i\sqrt{1-\varepsilon^2}\left|+++\right\rangle+\beta\varepsilon\left|+--\right\rangle+\alpha\varepsilon\left|-++\right\rangle+\beta i\sqrt{1-\varepsilon^2}\left|---\right\rangle$$

$\left|q_2q_1q_0\right\rangle$

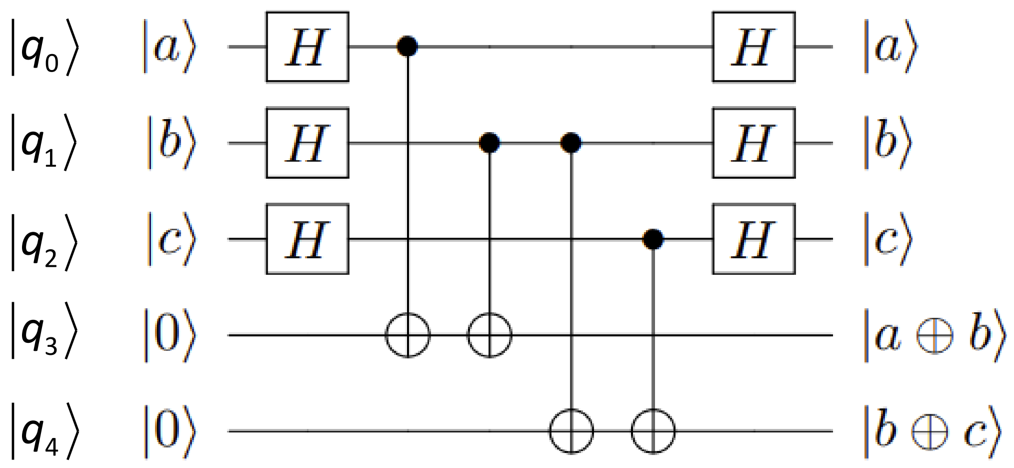
Now, we measure the parity of adjacent qubits in the *X*-basis (i.e., whether the number of $\left|-\right\rangle$'s is even or odd). We get

- parity(q_2,q_1)=0 and parity (q_1,q_0)=0 with probability

$$\left|\alpha i\sqrt{1-\varepsilon^2}\right|^2+\left|\beta i\sqrt{1-\varepsilon^2}\right|^2=\left|\alpha\right|^2\left(1-\varepsilon^2\right)+\left|\beta\right|^2\left(1-\varepsilon^2\right)$$

$$=\left(\left|\alpha\right|^2+\left|\beta\right|^2\right)\left(1-\varepsilon^2\right)$$

$$=\left(1-\varepsilon^2\right)$$



Phase-Flip Code

$$\alpha i\sqrt{1-\varepsilon^2}\,|{+}{+}{+}\rangle + \beta\varepsilon|{+}{-}{-}\rangle + \alpha\varepsilon|{-}{+}{+}\rangle + \beta i\sqrt{1-\varepsilon^2}\,|{-}{-}{-}\rangle$$

$$|q_2q_1q_0\rangle$$

and the state collapses to

$$A\left(\alpha i\sqrt{1-\varepsilon^2}\,|{+}{+}{+}\rangle + \beta i\sqrt{1-\varepsilon^2}\,|{-}{-}{-}\rangle\right) = \alpha|000\rangle + \beta|111\rangle$$

where, $A = 1\big/ i\sqrt{1-\varepsilon^2}$ is a normalization constant

- We see that the resulting state is already corrected, so we do not need to do anything further to correct the error

- That is, the measurement fixed the error

Phase-Flip Code

$$\alpha i\sqrt{1-\varepsilon^2}\left|+++\right\rangle+\beta\varepsilon\left|+--\right\rangle+\alpha\varepsilon\left|-++\right\rangle+\beta i\sqrt{1-\varepsilon^2}\left|---\right\rangle$$

$$\left|q_2q_1q_0\right\rangle$$

- parity(q_2,q_1)=1 and parity (q_1,q_0)=0 with probability

$$|\beta\varepsilon|^2+|\alpha\varepsilon|^2=\left(|\beta|^2+|\alpha|^2\right)\varepsilon^2=\varepsilon^2$$

and the state collapses to

$$B\left(\beta\varepsilon\left|+--\right\rangle+\alpha\varepsilon\left|-++\right\rangle\right)=\beta\left|+--\right\rangle+\alpha\left|-++\right\rangle$$

where, $B=1/\varepsilon$ is a normalization constant

To correct this state, we apply $Z\otimes I\otimes I$ so that it becomes

$$\begin{aligned}(Z\otimes I\otimes I)\left(\beta\left|+--\right\rangle+\alpha\left|-++\right\rangle\right)&=\beta(Z\otimes I\otimes I)\left|+--\right\rangle+\alpha(Z\otimes I\otimes I)\left|-++\right\rangle\\&=\beta\left|---\right\rangle+\alpha\left|+++\right\rangle=\alpha\left|+++\right\rangle+\beta\left|---\right\rangle\end{aligned}$$

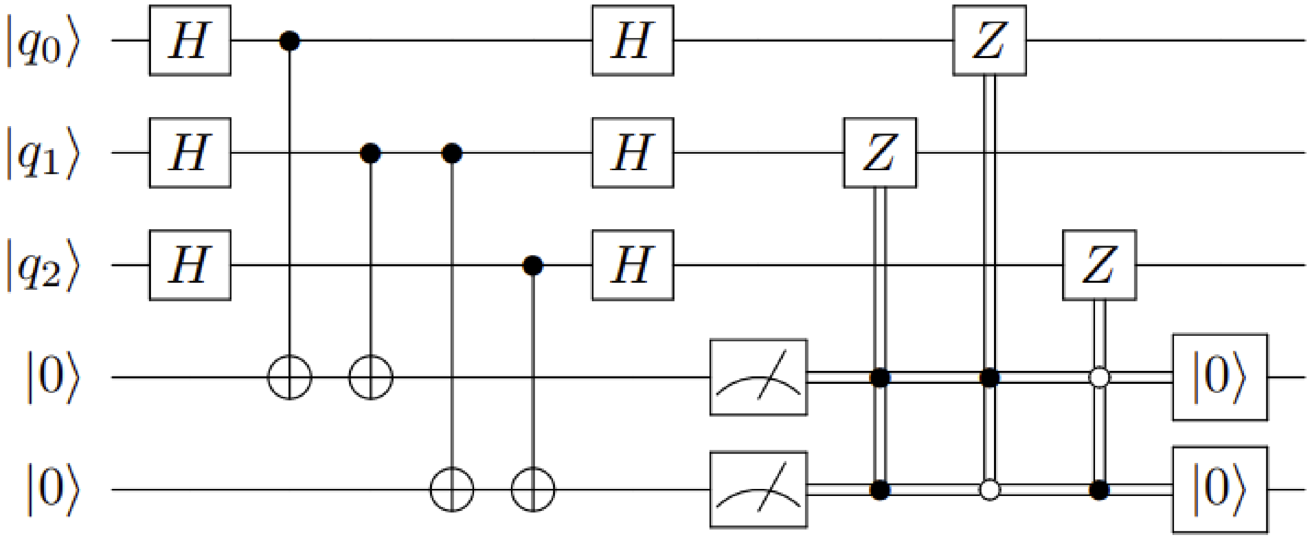
so, we have corrected the error.

Phase-Flip Code

- parity(q_2,q_1)=0 and parity (q_1,q_0)=1 with probability 0
- parity(q_2,q_1)=1 and parity (q_1,q_0)=1 with probability 0

Phase-Flip Code

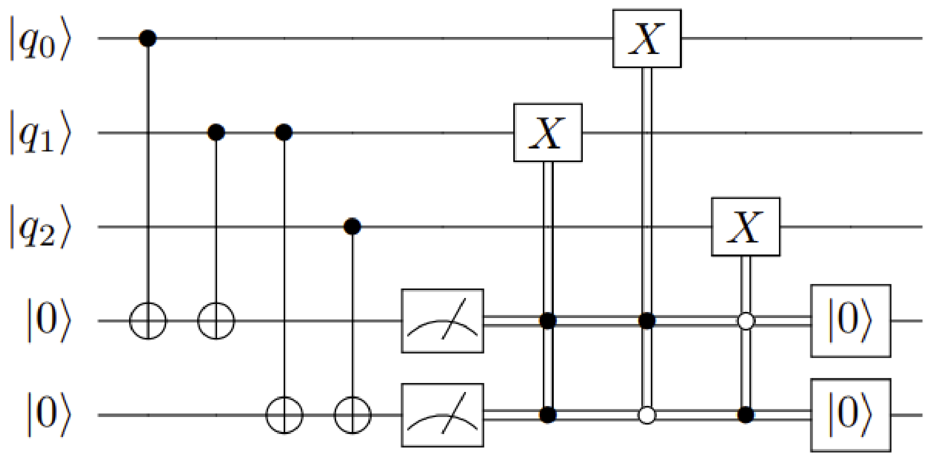
- To summarize, when we have a partial phase flip, the measurement forces it to be corrected or to become a complete bit flip, which we can correct by applying an *Z* gate
- The quantum circuit for this procedure is shown below:



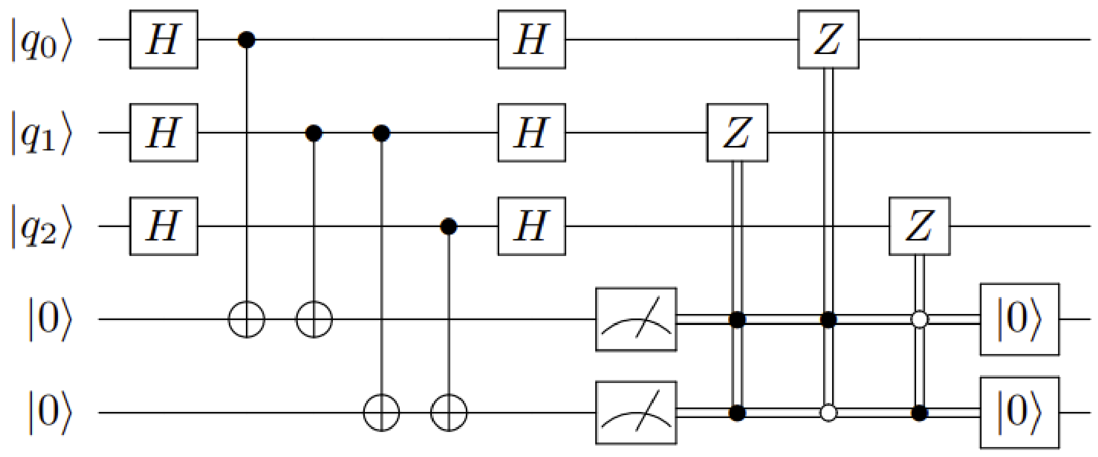
$|q_2q_1q_0\rangle$

Phase-Flip Code

- The circuit is the same as the bit-flip circuit, except we apply Hadamard gates before and after the four CNOTs that calculate the parity of consecutive qubits. This because we work in the *X*-basis

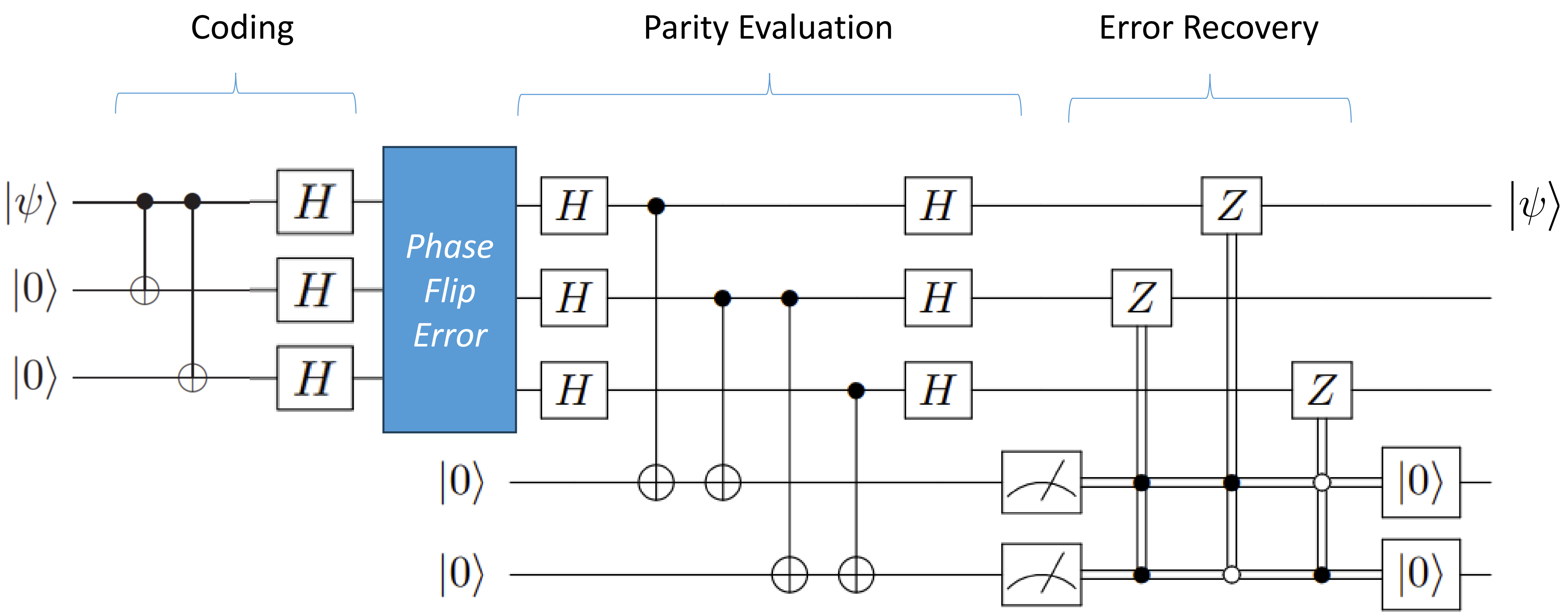


Bit-Flip quantum circuit



Phase-Flip quantum circuit

Phase-Flip Code



Phase-Flip quantum circuit: the full chain

Shor Code

Shor Code

- There is a simple quantum code that can protect against the effects of an *arbitrary error* on a **single qubit**!

- The code is known as the **Shor code**, after its inventor

- The code combines the three-qubit *phase-flip* and *bit-flip codes*

Shor Code

- We **begin** with the *phase-flip* code:

$$|0_L\rangle = |+++\rangle$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$= \left(\frac{1}{\sqrt{2}}\right)^3 \left(|0\rangle + |1\rangle\right)\left(|0\rangle + |1\rangle\right)\left(|0\rangle + |1\rangle\right)$$

Shor Code

- Next, we replace each qubit with three qubits using the **bit-flip encoding**:

$$|0\rangle \to |000\rangle, \quad |1\rangle \to |111\rangle$$

- Thus, each logical qubit is encoded using **nine** physical qubits:

$$|0_L\rangle = \left(\frac{1}{\sqrt{2}}\right)^3 \left(|000\rangle + |111\rangle\right)\left(|000\rangle + |111\rangle\right)\left(|000\rangle + |111\rangle\right)$$

Shor Code

- Similarly, we begin with $|1_L\rangle = |---\rangle$ and replace $|0\rangle \rightarrow |000\rangle, \quad |1\rangle \rightarrow |111\rangle$

$$|1_L\rangle = \left(\frac{1}{\sqrt{2}}\right)^3 \left(|000\rangle - |111\rangle\right)\left(|000\rangle - |111\rangle\right)\left(|000\rangle - |111\rangle\right)$$

- The nine-qubit Hilbert space is of dimension $2^9 = 512$, but the **logical qubits** reside in a sub-space of dimension 2

Shor Code

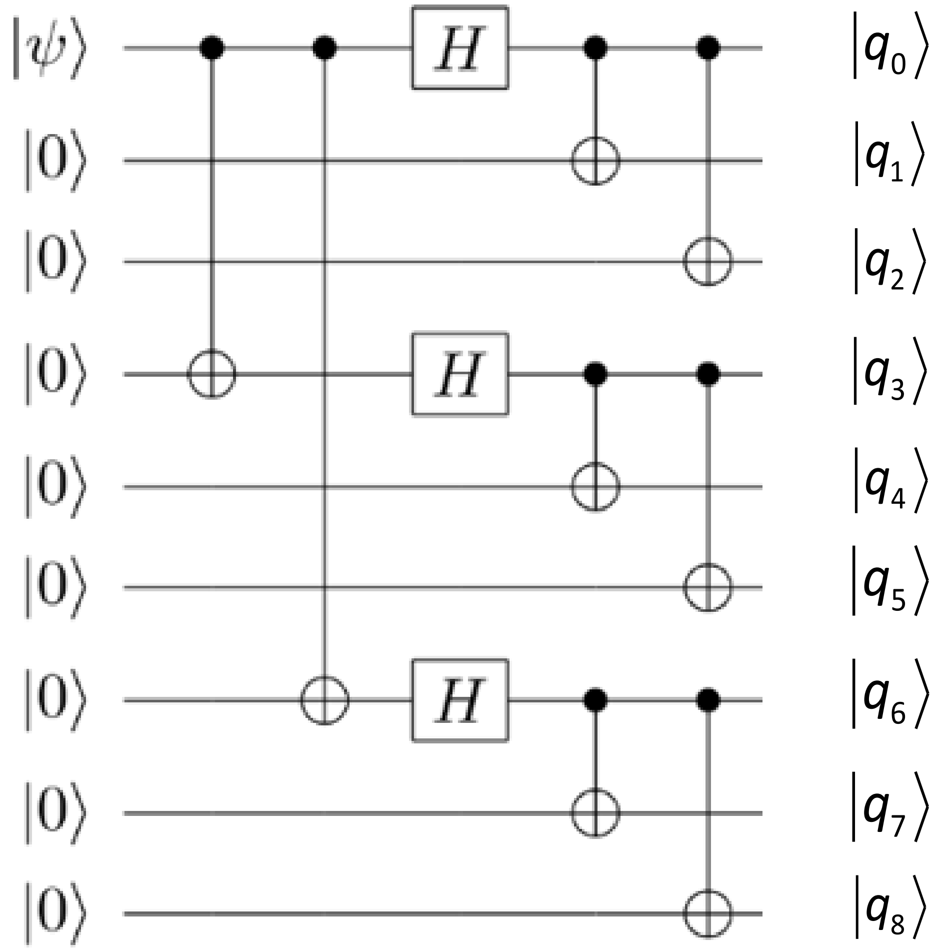
- Then, the state of a general logical qubit is

$$\alpha|0\rangle_L + \beta|1\rangle_L = \frac{\alpha}{2^{3/2}}\left(|000\rangle + |111\rangle\right)\left(|000\rangle + |111\rangle\right)\left(|000\rangle + |111\rangle\right)$$
$$+\frac{\beta}{2^{3/2}}\left(|000\rangle - |111\rangle\right)\left(|000\rangle - |111\rangle\right)\left(|000\rangle - |111\rangle\right)$$

- This encoding is called the *Shor code*, and it is named after its inventor, Peter Shor, who proposed it in 1995 and, by doing so, invented quantum error correction

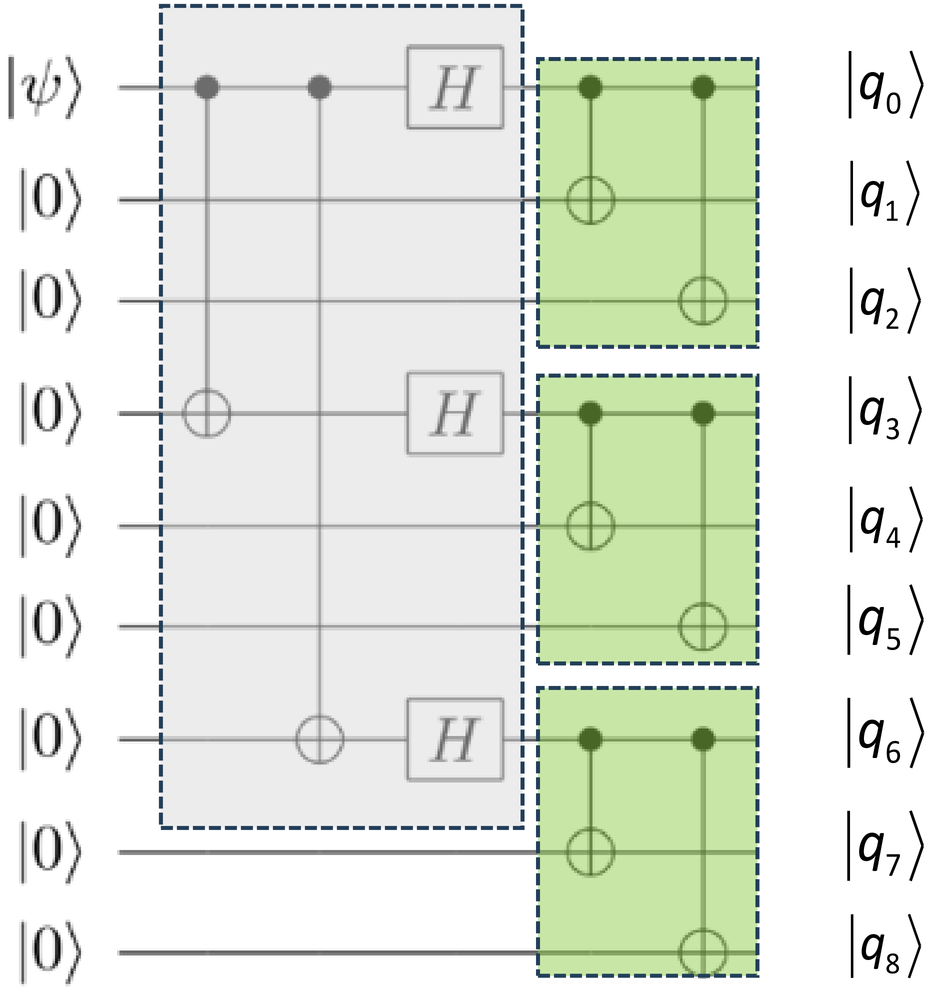
Shor Code

- The Shor code uses nine **physical** qubits to encode one **logical** qubit
- A way to create this encoding is shown in the figure



Shor Code/Proof

- The first part of the circuit (the large dashed box in gray to the left) encodes the qubit using the three-qubit *phase flip code*
- The second part of the circuit (the three dashed boxes in green) encodes each of these three qubits using the *bit flip code*, using three copies of the *bit flip code* encoding circuit
- This method of encoding using a hierarchy of levels is known as *concatenation*



Shor Code/Proof

- The initial state is

$$|\psi\rangle|00\rangle|000\rangle|000\rangle, \quad |\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

- Therefore

$$\begin{aligned} |\psi\rangle|00\rangle|000\rangle|000\rangle &= \left(\alpha|0\rangle + \beta|1\rangle\right)|00\rangle|000\rangle|000\rangle \\ &= \alpha|000\rangle|000\rangle|000\rangle + \beta|100\rangle|000\rangle|000\rangle \end{aligned}$$

- The state of the circuit before the Hadamard gates can be obtained as follows:

$$\alpha|000\rangle|000\rangle|000\rangle + \beta|100\rangle|000\rangle|000\rangle \xrightarrow{CNOT_{0;3,6}} \alpha|\underset{\uparrow\, q_0}{0}00\rangle|\underset{\uparrow\, q_3}{0}00\rangle|\underset{\uparrow\, q_6}{0}00\rangle + \beta|\underset{\uparrow\, q_0}{1}00\rangle|\underset{\uparrow\, q_3}{1}00\rangle|\underset{\uparrow\, q_6}{1}00\rangle$$

NOTE: The CNOT gate has a subscript of three numbers. The first number indicates the control qubit, followed by a semicolon and two numbers separated by a comma, which indicate the target qubits.

Shor Code/Proof

- After the Hadamard gates on the 0th, 3rd, and 6th qubits

$$\xrightarrow{H_0 H_3 H_6} \alpha H|0\rangle|00\rangle H|0\rangle|00\rangle H|0\rangle|00\rangle + \beta H|1\rangle|00\rangle H|1\rangle|00\rangle H|1\rangle|00\rangle$$

NOTE: The *H* gate has a subscript indicating the qubit to which it is applied.

$$= \alpha|+\rangle|00\rangle|+\rangle|00\rangle|+\rangle|00\rangle + \beta|-\rangle|00\rangle|-\rangle|00\rangle|-\rangle|00\rangle$$

$$= \alpha \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)|00\rangle \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)|00\rangle \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)|00\rangle$$

$$+ \beta \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)|00\rangle \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)|00\rangle \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)|00\rangle$$

$$= \alpha \frac{1}{\sqrt{2}}\left(|000\rangle + |100\rangle\right) \frac{1}{\sqrt{2}}\left(|000\rangle + |100\rangle\right) \frac{1}{\sqrt{2}}\left(|000\rangle + |100\rangle\right)$$

$$+ \beta \frac{1}{\sqrt{2}}\left(|000\rangle - |100\rangle\right) \frac{1}{\sqrt{2}}\left(|000\rangle - |100\rangle\right) \frac{1}{\sqrt{2}}\left(|000\rangle - |100\rangle\right)$$

Shor Code/Proof

- The final state is obtained by performing $CNOT_{0;1,2}$, $CNOT_{3;4,5}$, $CNOT_{6;7,8}$

$$\xrightarrow{CNOT_{0;1,2}\otimes CNOT_{3;4,5}\otimes CNOT_{6;7,8}} \alpha\frac{1}{\sqrt{2}}CNOT_{0;1,2}\left(|000\rangle+|100\rangle\right)\frac{1}{\sqrt{2}}CNOT_{3;4,5}\left(|000\rangle+|100\rangle\right)\frac{1}{\sqrt{2}}CNOT_{6;7,8}\left(|000\rangle+|100\rangle\right)$$

$$+\beta\frac{1}{\sqrt{2}}CNOT_{0;1,2}\left(|000\rangle-|100\rangle\right)\frac{1}{\sqrt{2}}CNOT_{3;4,5}\left(|000\rangle-|100\rangle\right)\frac{1}{\sqrt{2}}CNOT_{6;7,8}\left(|000\rangle-|100\rangle\right)$$

$$=\alpha\left(\frac{1}{\sqrt{2}}\right)^3\left(|000\rangle+|111\rangle\right)\left(|000\rangle+|111\rangle\right)\left(|000\rangle+|111\rangle\right)$$

$$+\beta\left(\frac{1}{\sqrt{2}}\right)^3\left(|000\rangle-|111\rangle\right)\left(|000\rangle-|111\rangle\right)\left(|000\rangle-|111\rangle\right)$$

$$=\alpha|0_L\rangle+\beta|1_L\rangle \qquad \square$$

Shor Code

- Let us see how to correct bit flips and phase flips using the Shor code, *beginning* with *bit flips*
- First, remember that the qubits are ordered $q_8 q_7 .... \, q_0$, where q_0, is the top qubit while q_8 is the bottom qubit
- Let's assume that q_0 and q_5 both experience complete bit flips
- Then, the state of the system is

$$\frac{\alpha}{2^{3/2}}\left(|\overset{q_0}{1}00\rangle+|\overset{q_0}{0}11\rangle\right)\left(|00\overset{q_5}{1}\rangle+|11\overset{q_5}{0}\rangle\right)\left(|000\rangle+|111\rangle\right)$$

$$+\frac{\beta}{2^{3/2}}\left(|\underset{q_0}{1}00\rangle-|\underset{q_0}{0}11\rangle\right)\left(|00\underset{q_5}{1}\rangle-|11\underset{q_5}{0}\rangle\right)\left(|000\rangle-|111\rangle\right)$$

Shor Code

- To detect this, we measure the *parities of adjacent qubits* within each triplet
- In this example, we would get:

left triplet	parity(q_0,q_1)=1	parity(q_1,q_2)=0
middle triplet	parity(q_3,q_4)=0	parity(q_4,q_5)=1
right triplet	parity(q_6,q_7)=0	parity(q_7,q_8)=0

$$\begin{array}{ccc} q_0q_1q_2 & q_3q_4q_5 & q_6q_7q_8 \end{array}$$

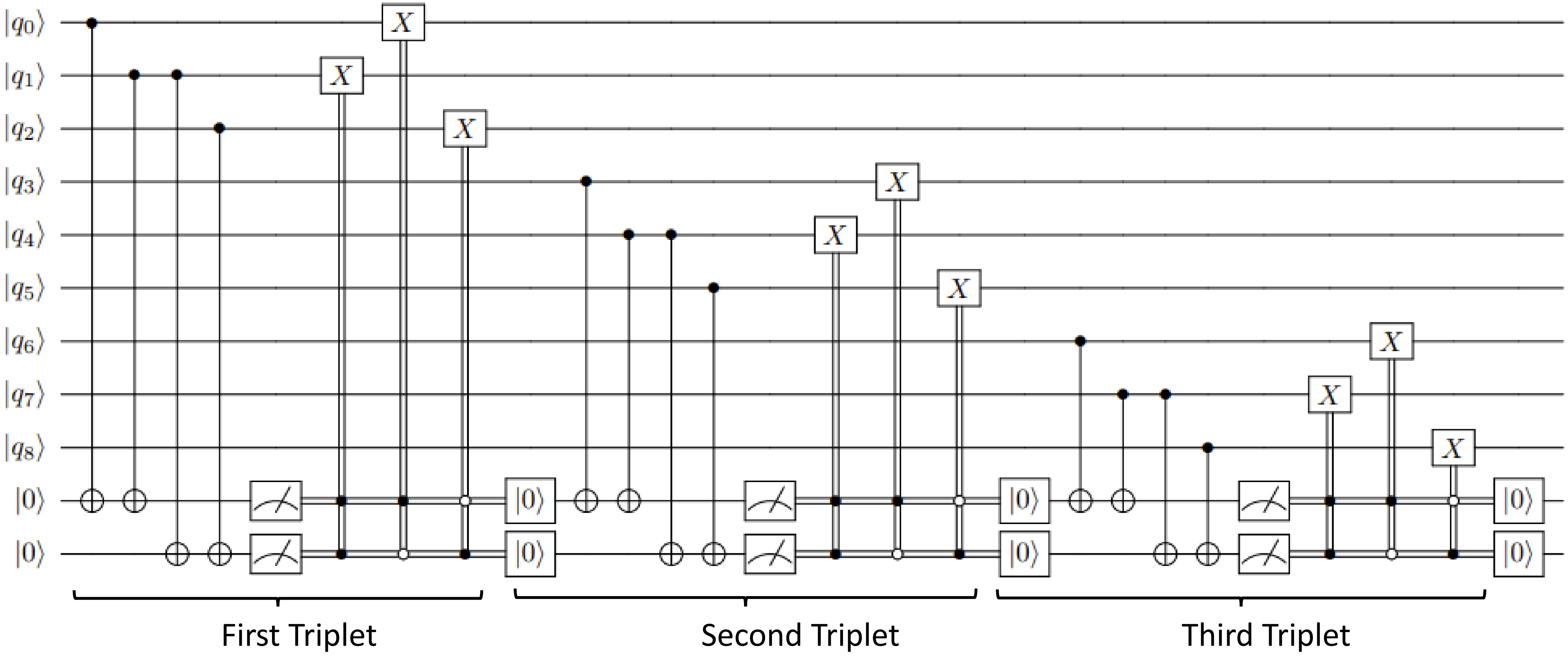
$$\frac{\alpha}{2^{3/2}}\left(|100\rangle+|011\rangle\right)\left(|001\rangle+|110\rangle\right)\left(|000\rangle+|111\rangle\right)$$
$$+\frac{\beta}{2^{3/2}}\left(|100\rangle-|011\rangle\right)\left(|001\rangle-|110\rangle\right)\left(|000\rangle-|111\rangle\right)$$

Shor Code

- This tells us that the 0^{th} *qubit* and 5^{th} *qubit* have *flipped*, so we can apply X gates to those two qubits to correct them
- Similarly, if there is a *partial bit flip*, measuring all the parities to be zero collapses the state and automatically corrects it, or if there is a discrepancy, we apply X to the appropriate bit to correct it
- This also works with ***partial bit flips***
- Measuring the parities of adjacent qubits might collapse the state and correct the errors, or it might collapse the state into a full bit flip, which we correct as previously described

Shor Code

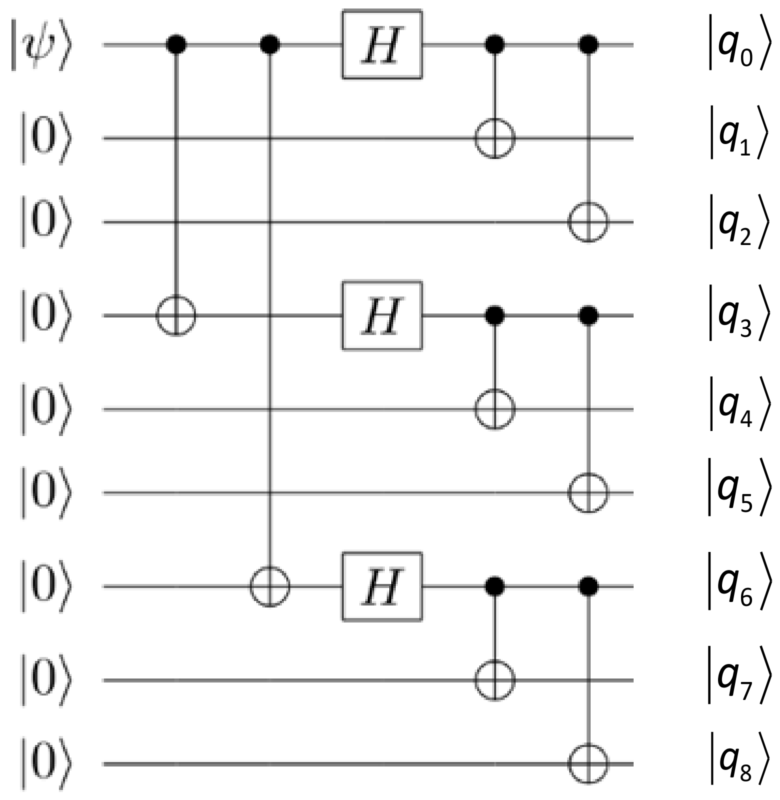
Bit flips can be corrected in the Shor code using the following quantum circuit



Shor Code

- The *first third* of the circuit measures the parities of adjacent qubits in the top three qubits, correct any errors, and reset the ancillas
- The *middle third* of the circuit calculates the parities of adjacent qubits in the next triplet, correcting any errors
- Finally, it does the same for the *last triplet of qubits*

Shor Code



Complete Phase Flip

Shor Code

- Next, let us see how the Shor code also allows us to correct *phase flips*
- Say q_0 experiences a *complete phase flip*
- The system is initialized in the following state

$$\alpha|0\rangle_L + \beta|1\rangle_L = \frac{\alpha}{2^{3/2}}\left(|000\rangle+|111\rangle\right)\left(|000\rangle+|111\rangle\right)\left(|000\rangle+|111\rangle\right)$$

$$+\frac{\beta}{2^{3/2}}\left(|000\rangle-|111\rangle\right)\left(|000\rangle-|111\rangle\right)\left(|000\rangle-|111\rangle\right)$$

- Then, due to the phase shift of q_5 the state of the system becomes

$$\frac{\alpha}{2^{3/2}}\left(|000\rangle-|111\rangle\right)\left(|000\rangle+|111\rangle\right)\left(|000\rangle+|111\rangle\right)$$

$$+\frac{\beta}{2^{3/2}}\left(|000\rangle+|111\rangle\right)\left(|000\rangle-|111\rangle\right)\left(|000\rangle-|111\rangle\right)$$

Shor Code

- Then, we can measure the *phase parity* of adjacent triplets, i.e., whether the number of $(|000\rangle - |111\rangle)/\sqrt{2}$ triplets is even or odd
- This is similar to the *phase flip code*, where we measured the parity in the X basis, which whether the number of $|-\rangle$ was even or odd
- In our example, we would get

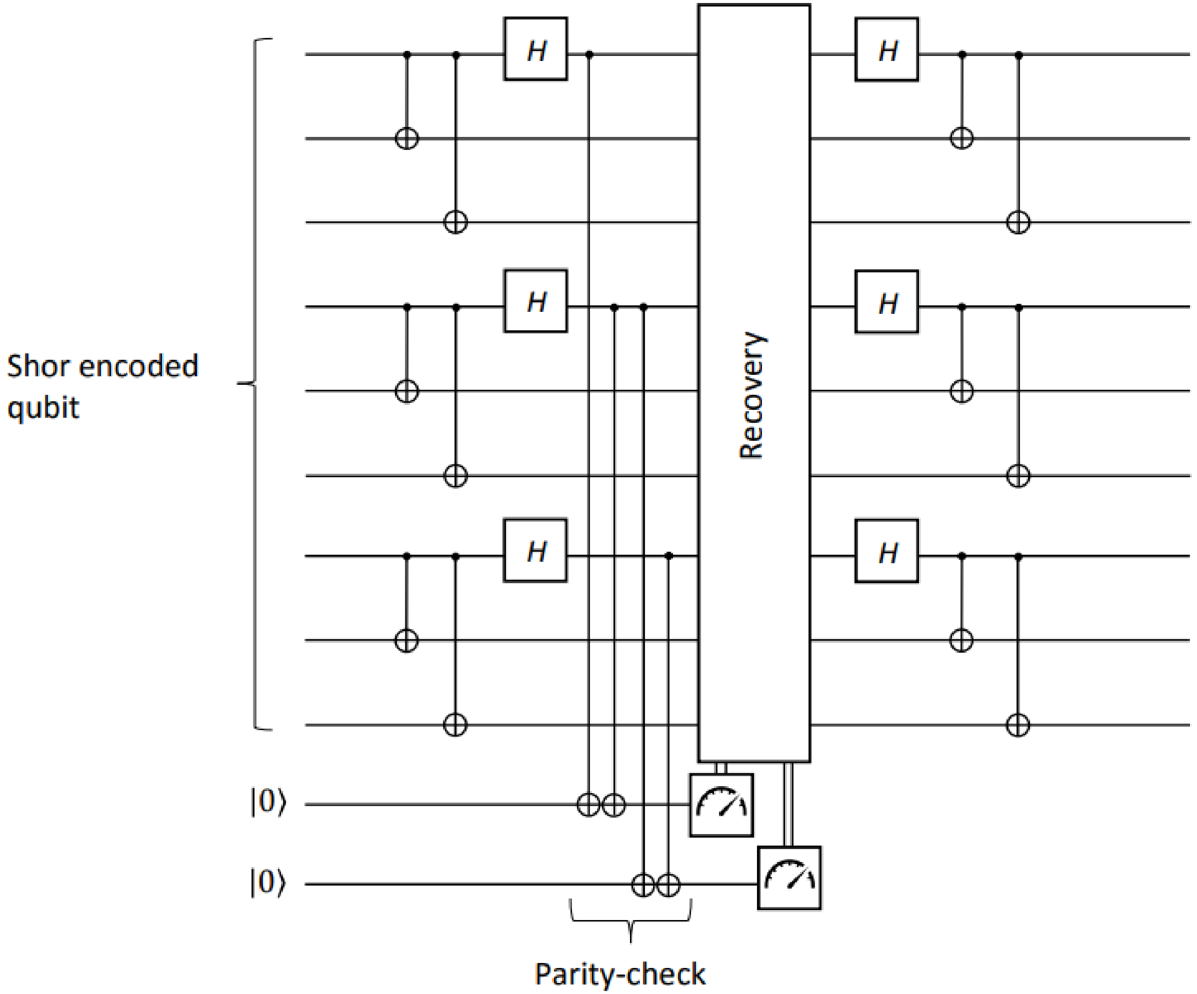
  parity(triplet$_0$, triplet$_1$) = 1, parity(triplet$_1$, triplet$_2$) = 0

- This indicates that the first triplet needs to be flipped, so we can apply the Z gate to either q_0, q_1, or q_2, correcting the error

Shor Code

The key idea here is to detect which of the three blocks of three qubits has experienced a change of sign.

This is achieved using the circuit shown on the figure.



Shor Code

- Similarly, when there is a partial phase flip, if we measure all the phase parities and get zero, the state collapsed and corrected the error, and if there was a discrepancy in phase parities, we apply a Z gate to the appropriate triplet to correct it
- By alternating between correcting bit-flip errors and phase-flip errors, the Shor code corrects all quantum errors, assuming *each triplet experiences **at most** one bitflip error, and **at most** one triplet experiences a phase-flip error*