1) Consider the following optimization problem

$$\begin{cases} \min\ x_1^2 + x_2^2 - 2x_1x_2 + \frac{1}{x_1+1} \\ x \in X := \{x \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\} \end{cases}$$

(a) Is the problem convex?

(b) Apply the gradient method with an inexact line search, setting $\bar{t} = 1$,$\alpha = 0.5$, $\gamma = 0.8$, with starting point $x^0 = (1, 2)$ and using $\|\nabla f(x)\| < 10^{-3}$ as stopping criterion. How many iterations are needed by the algorithm? Write explicitly the vectors found at the last three iterations.

(c) Is the obtained solution a global minimum of the given problem? Justify the answer.

SOLUTION (a) The objective function $f(x) = f_1(x) + f_2(x)$, where $f_1 = x_1^2 + x_2^2 - 2x_1x_2$ is convex and $f_2(x) = \frac{1}{x_1+1}$ is convex on X, being independent on x_2 and $\frac{\partial^2 f_2}{\partial x_1^2}(x) = \frac{2}{(x_1+1)^3} > 0$, $\forall x \in X$. Therefore $f_1 + f_2$ is convex being the sum of two convex functions. Moreover X is an open convex set so that the problem is convex.

(b) We notice that

$$\nabla f(x_1, x_2) = \begin{pmatrix} 2x_1 - 2x_2 - \frac{1}{(x_1+1)^2} \\ 2x_2 - 2x_1 \end{pmatrix}$$

Matlab solution

```
%% Data

alpha = 0.5;
gamma = 0.8;
tbar = 1;
x0 = [1;2];
tolerance = 10^(-3) ;

X=[ ];

ITER = 0 ;
x = x0;

while true
    [v, g] = f(x);

    X=[X;ITER,x(1),x(2),v,norm(g)];

    % stopping criterion
    if norm(g) < tolerance
        break
    end

    d = -g;                        % search direction


     t = tbar ;                     % Armijo inexact line search
    while f(x+t*d) > v + alpha*g'*d*t
        t = gamma*t ;
    end


    x = x + t*d ;                  % new point
    ITER = ITER + 1 ;
    end

disp('optimal solution')
x
v
norm(g)
ITER
```

```
function [v, g] = f(x)

v = x(1)^2 + x(2)^2 - 2*x(1)*x(2)+ 1/(x(1)+1) ;

g = [2*x(1)-2*x(2)-1/(x(1)+1)^2;
      -2*x(1)+2*x(2)];

end
```

We obtain the following solution:

```
x =

   26.5982
   26.5980

v =

    0.0362

ITER =

       26038
```

In particular, the gradient norm evaluated at the final point is:

```
ans =

   9.9974e-04
```

The iterations of the algorithm are 26038.

The vectors found at the last three iterations are:

```
26036     26.5974     26.5967
26037     26.5974     26.5972
26038     26.5982     26.5980
```

(c) The found point $x = (26.5982, 26.5980)$ is not a global minimum, in fact X is an open set and

$$\nabla f(x_1, x_2) = \begin{pmatrix} 2x_1 - 2x_2 - \frac{1}{(x_1+1)^2} \\ 2x_2 - 2x_1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \forall x \in X,$$

so that f does not admit any minimum point on X.

2) Consider a regression problem with the following data set where the points $(x_i, y_i), i = 1, .., 30$, are given by the row vectors of the matrices:

$$\begin{pmatrix} -3.0000 & 6 \\ -2.8000 & 7.5 \\ -2.6000 & 8.5 \\ -2.2000 & 16.42 \\ -2.0000 & 17.53 \\ -1.8000 & 11.48 \\ -1.6000 & 14.10 \\ -1.4000 & 16.82 \\ -1.2000 & 16.15 \\ -1.0000 & 11.68 \\ -0.8000 & 6.00 \\ -0.6000 & 7.82 \\ -0.4000 & 2.82 \\ -0.2000 & 2.71 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0.2000 & -1 \\ 0.4000 & -3.84 \\ 0.6000 & -4.71 \\ 0.8000 & -8.15 \\ 1.0000 & -7.33 \\ 1.2000 & -13.64 \\ 1.4000 & -15.26 \\ 1.6000 & -14.87 \\ 1.8000 & -9.92 \\ 2.0000 & -10.50 \\ 2.2000 & -7.72 \\ 2.4000 & -12 \\ 2.6000 & -10.26 \\ 2.8000 & -7 \\ 3.0000 & -2 \end{pmatrix}$$

(a) Write the dual formulation of a nonlinear ε-SV regression model with $C = 5$, $\varepsilon = 3.5$ and a Gaussian kernel $k(x, y) := e^{-\|x-y\|^2}$;

(b) Solve the problem in (a) and find the regression function;

(c) Find the support vectors;

(d) Find the points of the data set that are outside the ε-tube, by making use of the dual solution.

SOLUTION

(a) Let $\ell = 30$, (x_i, y_i), $i = 1, .., \ell$ be the i-th element of the data set, $C = 5$, $\varepsilon = 3.5$, $k(x, y) := e^{-\|x-y\|^2}$. The dual formulation of a nonlinear ε-SV regression model is

$$\begin{cases} \max\limits_{(\lambda^+,\lambda^-)} & -\dfrac{1}{2}\sum\limits_{i=1}^{30}\sum\limits_{j=1}^{30}(\lambda_i^+ - \lambda_i^-)(\lambda_j^+ - \lambda_j^-)e^{-\|x_i-x_j\|^2} \\ & -3.5\sum\limits_{i=1}^{30}(\lambda_i^+ + \lambda_i^-) + \sum\limits_{i=1}^{30} y_i(\lambda_i^+ - \lambda_i^-) \\ \sum\limits_{i=1}^{30}(\lambda_i^+ - \lambda_i^-) = 0 \\ \lambda_i^+, \lambda_i^- \in [0, 5], \ i = 1, .., 30 \end{cases} \tag{1}$$

(b) **Matlab solution**

```
data = [
   -3.0000     6
   -2.8000     7.5
   -2.6000     8.5
   -2.2000    16.42
   -2.0000    17.53
   -1.8000    11.48
   -1.6000    14.10
   -1.4000    16.82
   -1.2000    16.15
   -1.0000    11.68
   -0.8000     6.00
   -0.6000     7.82
   -0.4000     2.82
   -0.2000     2.71
         0        1
    0.2000    -1
    0.4000    -3.84
    0.6000    -4.71
    0.8000    -8.15
    1.0000    -7.33
    1.2000   -13.64
    1.4000   -15.26
    1.6000   -14.87
    1.8000    -9.92
    2.0000   -10.50
```

```
    2.2000   -7.72
    2.4000  -12
    2.6000  -10.26
    2.8000   -7
    3.0000   -2
        ];

x = data(:,1) ;
y = data(:,2) ;
l = length(x) ;

epsilon = 3.5 ;
C = 5;

X = zeros(l,l);
for i = 1 : l
    for j = 1 : l
        X(i,j) = kernel(x(i),x(j)) ;
    end
end
Q = [ X -X ; -X X ];
c = epsilon*ones(2*l,1) + [-y;y];

sol = quadprog(Q,c,[],[],[ones(1,l) -ones(1,l)],0,zeros(2*l,1),C*ones(2*l,1));
lap = sol(1:l);
lam = sol(l+1:2*l);

% compute b
ind = find(lap > 1e-3 & lap < C-1e-3);
if isempty(ind)==0
    i = ind(1);
    b = y(i) - epsilon;
    for j = 1 : l
        b = b - (lap(j)-lam(j))*kernel(x(i),x(j));
    end
else
    ind = find(lam > 1e-3 & lam < C-1e-3);
    i = ind(1);
    b = y(i) + epsilon ;
    for j = 1 : l
        b = b - (lap(j)-lam(j))*kernel(x(i),x(j));
    end
end


z = zeros(l,1);                                               % find regression and epsilon-tube
for i = 1 : l
    z(i) = b ;
    for j = 1 : l
        z(i) = z(i) + (lap(j)-lam(j))*kernel(x(i),x(j));
    end
end
zp = z + epsilon ;
zm = z - epsilon ;

sv = [find(lap > 1e-3);find(lam > 1e-3)];                      % find support vectors
sv = sort(sv);

 plot(x,y,'b.',x(sv),y(sv),'ro',x,z,'k-',x,zp,'r-',x,zm,'r-');     % plot the solution

    disp('Support vectors')

[sv,x(sv),y(sv),lam(sv),lap(sv)]    % Indexes of support vectors, support vectors, lambda_-,lambda_+
```

```
function v = kernel(x,y)

v = exp(-norm(x-y)^2)

end
```

Let λ_- and λ_+ be the vectors given by the Matlab solutions lam, lap. In particular we find, $b = 1.0366$.

The regression function is:

$$f(x) = \sum_{i=1}^{\ell}(\lambda_i^+ - \lambda_i^-)k(x_i, x) + b = \sum_{i=1}^{30}(\lambda_i^+ - \lambda_i^-)e^{-\|x_i - x\|^2} + 1.0366$$

(c) We obtain the support vectors (columns 2-3) and corresponding λ^- and λ^+ (columns 4-5) :

```
ans =

    4.0000   -2.2000   16.4200    0.0000    4.4719
    5.0000   -2.0000   17.5300    0.0000    4.8373
    8.0000   -1.4000   16.8200    0.0000    1.7534
    9.0000   -1.2000   16.1500    0.0000    5.0000
   21.0000    1.2000  -13.6400    4.0449    0.0000
   22.0000    1.4000  -15.2600    5.0000    0.0000
   23.0000    1.6000  -14.8700    1.2329    0.0000
   27.0000    2.4000  -12.0000    5.0000    0.0000
   28.0000    2.6000  -10.2600    0.7847    0.0000
```

(d) Consider the feasibility condition of the primal formulation of the regression problem:

$$y_i - f(x_i) - \varepsilon - \xi_i^+ \leq 0, \quad y_i - f(x_i) + \varepsilon + \xi_i^- \geq 0, \quad i = 1, .., \ell$$

If a point (x_i, y_i) is outside the ε-tube then $\xi_i^+ > 0$ or $\xi_i^- > 0$.

Given the dual optimal solution (λ^+, λ^-) of (1), we can find the errors ξ_i^+ and ξ_i^- associated with the point (x_i, y_i) by the complementarity conditions:

$$\begin{cases} \lambda_i^+ \left[y_i - f(x_i) - \varepsilon - \xi_i^+\right] = 0, & i = 1, .., \ell \\ \lambda_i^- \left[y_i - f(x_i) + \varepsilon + \xi_i^-\right] = 0, & i = 1, .., \ell \\ \xi_i^+(C - \lambda_i^+) = 0, & i = 1, .., \ell \\ \xi_i^-(C - \lambda_i^-) = 0, & i = 1, .., \ell \end{cases} \tag{2}$$

it follows that a necessary condition for a point (x_i, y_i) to be outside the ε-tube is that $\lambda_i^+ = C = 5$ or $\lambda_i^- = C = 5$. We find that $\lambda_i^- = 5$, for $i = 22, 27$, $\lambda_i^+ = 5$, for $i = 9$ which correspond to the points

$$(x_9, y_9) = (-1.2, 16.15), \quad (x_{22}, y_{22}) = (1.4, -15.26), \quad (x_{27}, y_{27}) = (2.4, -12)$$

3) Consider the following constrained multiobjective optimization problem (P):

$$\begin{cases} \min\ f(x_1,x_2) = (x_1^3 + x_2,\ x_1 - x_2) \\ 1 - x_1 \le 0 \\ x \in \mathbb{R}^2 \end{cases}$$

(a) Is the given problem convex?

(b) Prove that the problem admits a Pareto minimum point.

(c) Find a suitable subset of Pareto and weak Pareto minima, by means of the scalarization method.

SOLUTION

(a) We observe that the function f_1 is convex on the feasible set $X := \{x \in \mathbb{R}^2 : x_1 \ge 1\}$ which is obviously convex. Indeed the Hessian

$$\nabla^2 f_1(x_1,x_2) = \begin{pmatrix} 6x_1 & 0 \\ 0 & 0 \end{pmatrix}$$

is positive semidefinite on the set $\{x \in \mathbb{R}^2 : x_1 > 0\} \supseteq X$ and therefore f_1 is convex on X. Since f_2 is linear and therefore convex, then the given problem is convex.

(b) Consider the scalarized problem (P_{α_1}), where $0 \le \alpha_1 \le 1$, i.e.

$$\begin{cases} \min\ \alpha_1(x_1^3 + x_2) + (1-\alpha_1)(x_1 - x_2) =: \psi_{\alpha_1}(x) \\ 1 - x_1 \le 0 \\ x \in \mathbb{R}^2 \end{cases}$$

For $\alpha_1 = \frac{1}{2}$ the scalarized problem becomes

$$\begin{cases} \min\ \frac{1}{2}(x_1^3 + x_1) \\ 1 - x_1 \le 0 \\ x \in \mathbb{R}^2 \end{cases}$$

that admits as global minima the set $A := \{(x_1,x_2) : x_1 = 1, x_2 \in \mathbb{R}\}$. A is therefore a subset of Pareto minima.

(c) Let us find all the weak Pareto minima of P by the KKT conditions for (P_{α_1}) which are necessary and sufficient for a weak minimum point.

$$\begin{cases} 3\alpha_1 x_1^2 + (1-\alpha_1) - \lambda = 0 \\ 2\alpha_1 - 1 = 0 \\ \lambda(1 - x_1) = 0 \\ \lambda \ge 0,\ x_1 \ge 1,\ 0 \le \alpha_1 \le 1, \end{cases}$$

We obtain:

$$\begin{cases} \frac{3}{2}x_1^2 + \frac{1}{2} = \lambda \\ \alpha_1 = \frac{1}{2} \\ \lambda(1 - x_1) = 0 \\ \lambda \ge 0,\ x_1 \ge 1,\ 0 \le \alpha_1 \le 1, \end{cases} \tag{3}$$

Not that for $\lambda = 0$ the previous system is impossible, so that the only solutions are given by

$$(x_1,x_2) \in A, \quad \lambda = 2$$

where A has been defined at point (b). In conclusion:

$$Weak\ Min(P) = Min(P) = \{(x_1,x_2) : x_1 = 1,\ x_2 \in \mathbb{R}\}.$$

4) Consider the following matrix game:

$$C = \begin{pmatrix} 2 & 4 & 1 & 3 & -2 \\ 2 & 1 & 3 & 2 & 5 \\ 3 & 3 & -2 & 3 & 1 \\ 3 & 5 & 5 & 4 & 3 \end{pmatrix}$$

(a) Find the dominated strategies and reduce the cost matrix accordingly;

(b) Find the set of pure strategies Nash equilibria, if any. Alternatively, show that no pure strategies Nash equilibrium exists.

(c) Find a mixed strategies Nash equilibrium.

SOLUTION

(a) Strategy 4 of Player 1 is dominated by Strategy 1, so that row 4 can be deleted. The reduced game is given by the matrix

$$C_{R1} = \begin{pmatrix} 2 & 4 & 1 & 3 & -2 \\ 2 & 1 & 3 & 2 & 5 \\ 3 & 3 & -2 & 3 & 1 \end{pmatrix}$$

(b) We observe that no minimum component on the columns of the reduced matrix is a maximum on the respective row, so that no pure strategies Nash equilibrium exists.

(c) The optimization problem associated with Player 1 is

$$\begin{cases} \min\ v \\ v \geq 2x_1 + 2x_2 + 3x_3 + 3x_4 \\ v \geq 4x_1 + x_2 + 3x_3 + 5x_4 \\ v \geq x_1 + 3x_2 - 2x_3 + 5x_4 \\ v \geq 3x_1 + 2x_2 + 3x_3 + 4x_4 \\ v \geq -2x_1 + 5x_2 + x_3 + 3x_4 \\ x_1 + x_2 + x_3 + x_4 = 1 \\ x \geq 0 \end{cases} \tag{4}$$

The previous problem can be solved by Matlab.

Matlab solution

```
C=[2,4,1,3, -2; 2 1 3 2 5; 3 3 -2 3 1;3 5  5 4 3]

m = size(C,1);
n = size(C,2);
c=[zeros(m,1);1];
A= [C', -ones(n,1)]; b=zeros(n,1); Aeq=[ones(1,m),0]; beq=1;
lb= [zeros(m,1);-inf]; ub=[ ];
[sol,Val,exitflag,output,lambda] = linprog(c, A,b, Aeq, beq, lb, ub);
x = sol(1:m)
y = lambda.ineqlin
```

We obtain the optimal solution $(x, v) = (\frac{3}{8}, \frac{5}{8}, 0, 0, 2.375)$. The optimal solution of the dual of (4) is given by $(y, w) = (0, 0, 0, \frac{7}{8}, \frac{1}{8}, 2.375)$. y can be found in the vector *lambda.ineqlin* given by the Matlab function linprog.

Therefore,

$$(x_1, x_2, x_3, x_4) = (\frac{3}{8}, \frac{5}{8}, 0, 0), \quad (y_1, y_2, y_3, y_4, y_5) = (0, 0, 0, \frac{7}{8}, \frac{1}{8}),$$

is a mixed strategies Nash equilibrium.