Department of Information Engineering
MSc in Computer Engineering (a.y. 2024/2025)
University of Pisa

Quantum Computing and Quantum Internet

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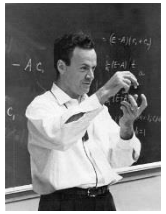
Introduction

A Brief History of Quantum Computing

- In **1979**, **Paul Benioff**, a young physicist at Argonne National Labs, submitted a paper entitled *"The computer as a physical system: A microscopic quantum mechanical Hamiltonian model of computers as represented by Turing machines"*

- **Yuri Manin** also laid out the core idea of quantum computing in his **1980** book *"Computable and Non-Computable".* The book was written in Russian, however, and only translated years later

- In **1982**, **Richard Feynman** pointed out that there seemed to be essential difficulties in simulating quantum mechanical systems on classical computers, and suggested that building computers based on the principles of quantum mechanics would allow us to avoid those difficulties



Yuri Manin Visited, as a Postdoc, the University of Pisa!!

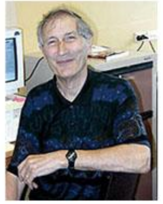
Caro Luciano,

....

*" In breve: ho conosciuto **Yuri Manin** almeno 50 anni fa, **visitatore a Pisa** perché come **giovane dottorato** aveva già pubblicato lavori importanti per Barsotti e Bombieri(Fields Medal). Puoi averlo incontrato per caso anche tu al CNUCE o all'IEI, perché o mi veniva a trovare o perfino lavorava al CNUCE. E' diventato uno dei maggiori matematici al livello mondiale. Io avevo perso contatti con lui (troppe altre cose stupide da fare). Da pioniere ha proposto il quantum computer. E' invitato ancora in tutto il mondo; ma pare lavori a Bonn.*

Può darsi si ricordi di me; a Pisa passeggiavamo spesso assieme; forse è stato ospite a casa mia ad Asciano. Scrivigli (in inglese o tedesco se lo sai; dubito che tu conosca il russo). Se non si dà ora troppe arie passagli i miei saluti. Tanti anni fa era timidissimo. "

Gianfranco Capriz



Pisa 26/06/2021

Excerpt from a private email between Capriz and me.

A Brief History of Quantum Computing

- Once *Benioff*, *Manin* and *Feynman* opened the doors, other researchers (e.g., Deutsch, Jozsa, and Shor) began to investigate the nature of the algorithms that could run on quantum computers

A Brief History of Quantum Computing

- **1994** - *Peter Shor* came up with a quantum algorithm to **factor** very large numbers in *polynomial time*
- On the other hand, to factor an *n*-bit number on a classic computer, the runtime is

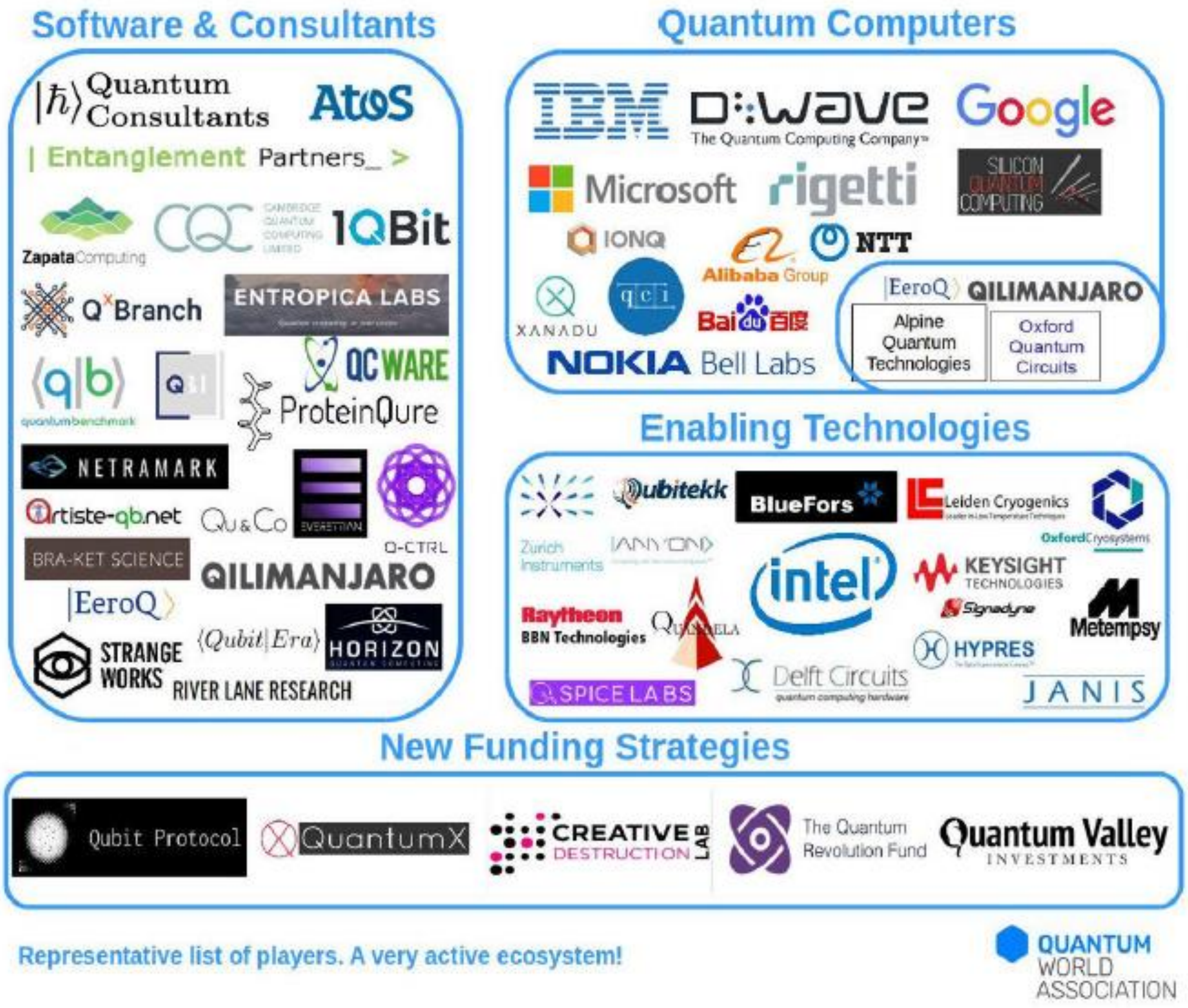
$$e^{\left(\sqrt[3]{64/9}+o(1)\right)(\ln n)^{1/3}(\ln\ln n)^{2/3}}$$

- This is an example of a *sub-exponential* function
- **1997** - *Lov Grover* developed a quantum **search algorithm** with query complexity $O\left(\sqrt{N}\right)$ where N is the number of entries in the database
- On the other hand, this type of search in a classical computer takes $O(N)$ queries

A Brief History of Quantum Computing

- In 1998, the following researchers
  > **Isaac Chuang** of the Los Alamos National Laboratory,
  > **Neil Gershenfeld** of the Massachusetts Institute of Technology (MIT), and
  > **Mark Kubinec** of the University of California at Berkeley
  created the first quantum computer (2-qubit) that could be **loaded** with data and **output** a solution
- Although the system was coherent for *only a few nanoseconds* and trivial from the perspective of solving meaningful problems, it demonstrated **the principles of quantum computation**

From that point forward, a thriving ecosystem began to emerge, encompassing both innovative startups and well-established companies



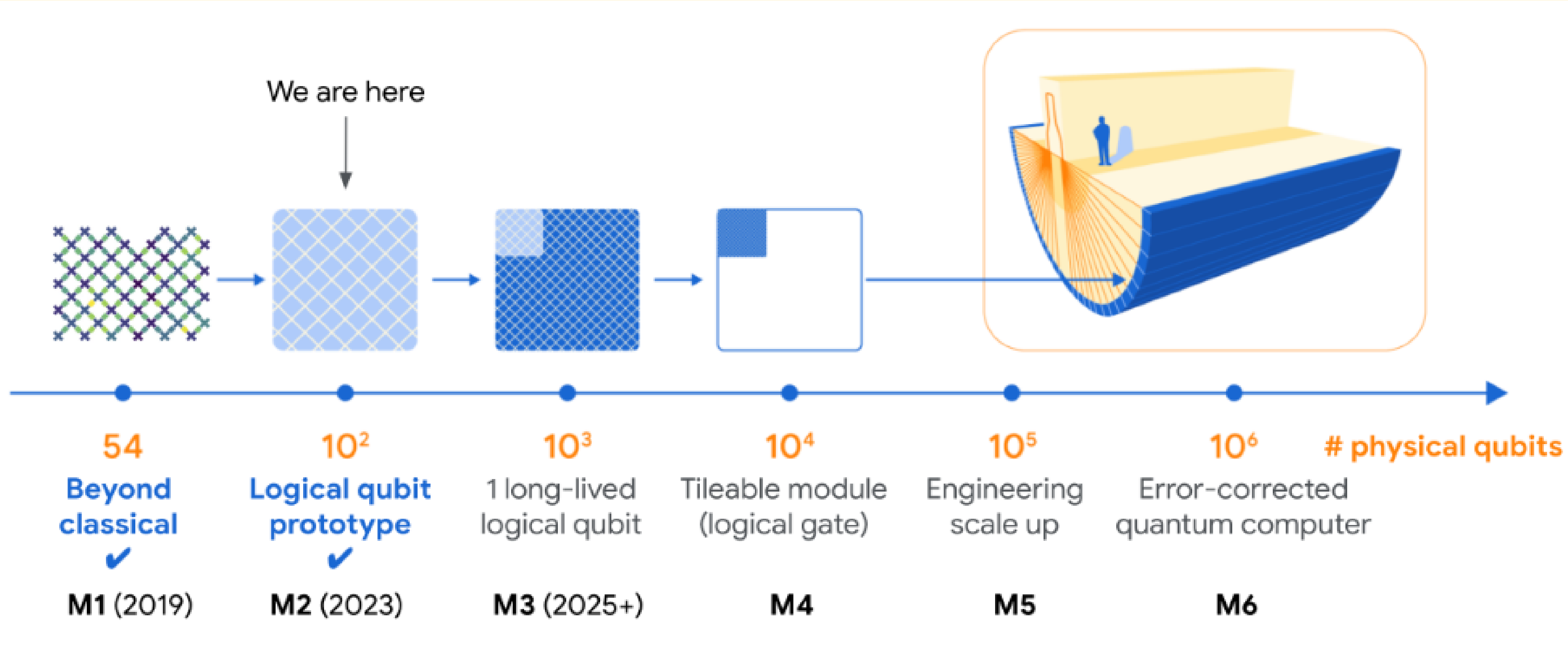
IBM Quantum Development Roadmap

- **2023:** IBM delivered the **1,121**-qubit **Condor process**.
- **2024:** IBM will demonstrate a quantum system comprising three Flamingo processors totaling **1,386** qubits.
- **2025:** IBM will demonstrate a quantum system of 3 Kookaburra processors totaling **4,158** qubits.
- This advancement will pave the way for a new era of scalability, establishing a clear trajectory toward achieving **100,000** qubits and beyond.

Google Quantum Development Roadmap

Building a useful quantum computer

Our roadmap for building a useful error-corrected quantum computer with key milestones. We are currently building one logical qubit that we will scale in the future.



Quantum Internet

- Alongside the efforts to build commercial quantum computers, a new initiative has emerged focused on designing and developing a **quantum network**, often referred to as the **quantum Internet**
- The course will cover also this topic

Physics Awards

We also want to highlight **two outstanding facts** that should further confirm, if any were needed, that *quantum computers* and *the quantum internet* will be a tangible reality in the years to come.

- *Nobel Prize for Physics*
- *Breakthrough Prize in Fundamental Physics*

The Nobel Prize in Physics 2022

The Nobel Prize in Physics 2022 was awarded jointly to

Alan Aspect

John F. Clauser

Anton Zeilinger

with the following motivation:

for experiments with entangled photons, establishing the violation of Bell inequalities and pioneering quantum information science

Breakthrough Prize in Fundamental Physics

Furthermore, the winners of the *2023 Breakthrough Prize in Fundamental Physics* also come from the science of quantum information

Charles H. Bennett

Gilled Brassard

Davide Deutsch

Peter Shor

for their contributions in the field of quantum information

These prizes are the most obvious proof of the paramount importance that quantum computing and the quantum Internet have now taken on

Actions Taken by UNIPI and DII

In response to the rapidly evolving fields of quantum computing and the quantum Internet, two significant actions have been recently taken by UNIPI and DII

Anna Grassellino, Director of the SQMS Center

Riccardo Zucchi, Rector of the University of Pisa

Action 1: UNIPI accelerates in the race towards the future quantum computer by signing an agreement with the **Fermi National Accelerator Laboratory in Chicago**, home to the **Superconducting Quantum Materials & Systems Center (SQMS Center)** led by **Anna Grassellino**, an electronic *engineering* student at the *University of Pisa*

As part of this collaboration with SMQS, the **Department of Information Engineering** conducts research focused on the experimental characterization and theoretical analysis of superconducting qubits used in the SMQS quantum computer. The cylindrical container shown in the photo holds a set of these qubits.

Actions Taken by DII

- **Action 2:** The Master’s Program in **Computer Engineering** for the 2024/2025 academic year will be offering students a **groundbreaking course** called **Quantum Computing and Quantum Internet** for the **first time ever**

Quantum Computing and Quantum Internet

- **QUESTION:** Since quantum computers are not yet available on the market, why should you take this course on Quantum Computing and the Quantum Internet?
- **ANSWER:** The answer is that companies want to have their software ready to run on quantum computers as soon as they become available
- Leading companies across various sectors, including finance, chemistry, and automotive, are **actively seeking** developers skilled in quantum programming
- To meet this **growing demand**, the Master's Degree in Computer Engineering at the University of Pisa offers this course to train highly qualified professionals

Quantum Computing and Quantum Internet

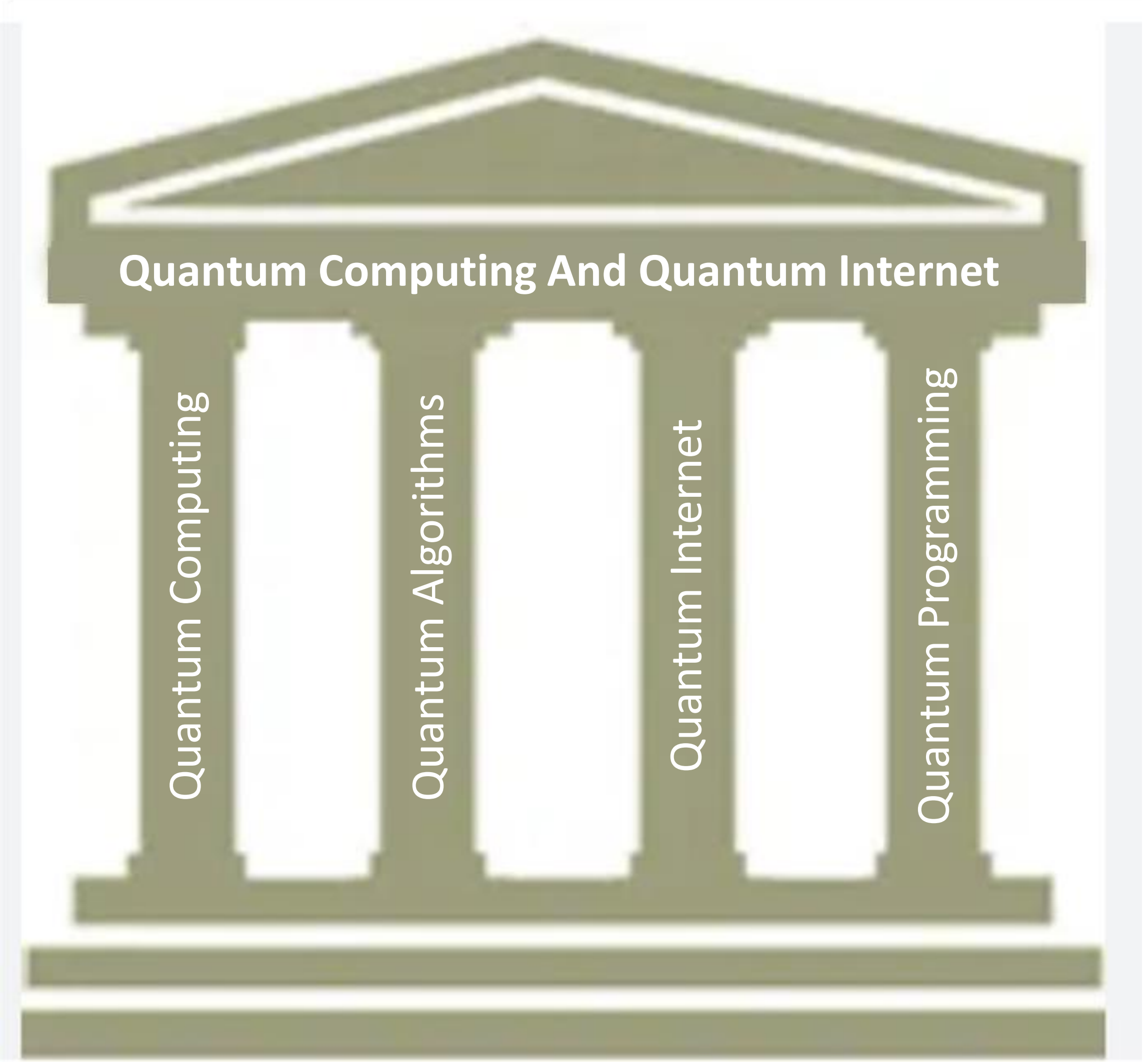
The key features of the course are summarized below:

- Positioned at the **forefront** of emerging technologies
- **Focuses on programming as the core approach**
- Offers an outstanding opportunity to explore **computation** and **networking** from completely new perspectives

On the other hand, the course:

- **DOES NOT** cover qubit and quantum gate technologies
- **DOES NOT** require prior knowledge of quantum mechanics

Quantum Computing and Quantum Internet



This course builds on four pillars:

- **Quantum Computing** (qubit paradigm, single and multiple qubits gates, circuit model of computation, entanglement, quantum teleportation, etc.)
- **Quantum Algorithms** (Deutsch, Grover, Shor, etc.) and their computational complexities
- **Quantum Internet** (architectures and protocols)
- **Quantum Programming** (Python, Qiskit, NetSquid)

Prerequisites

To succeed in this course, the following prerequisites are essential:

- A solid foundation in **linear algebra**, with deep insight into key concepts such as

  > *vector spaces,*
  > *linear operators,*
  > *change of bases, and*
  > *matrices.*

- Proficiency in Python programming

Examination

- The examination consists of
  > an **oral assessment,** and
  > the **successful completion of a project**
- Students can choose their projects from options provided by the professor, each requiring the writing of a Python program to implement quantum algorithms or quantum Internet protocols, among other possibilities

Class Schedule

- **Tuesday**, 11:30 - 1:30, Classroom ADII1
- **Wednesday**, 10:30 - 12:30, Classroom ADII1
- **Thursday**, 10:30 - 12:30, Classroom SI3
- **Thursday**, 16:30 - 18:30, Classroom SI3 (hands-on quantum software laboratory)

References on Quantum Computing and Quantum Information Textbooks

- Thomas G. Wong, *Introduction to Classical and Quantum Computing,* 2022.
- Phillip Kaye, Raymond Laflamme, Michele Mosca*, An Introduction to Quantum Computing,* Oxford University Press, 2007.
- Michael A. Nielsen and Isaac L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press, Cambridge, 2017.

- **The above books are listed in a progression of difficulty: basic, intermediate, and advanced**

References on Quantum Internet Textbooks

- Rodney Van Meter, *Quantum Networking*, ISTE Ltd and John Wiley & Sons, Inc., 2014.
- Peter P. Rohde, *The Quantum Internet-The Second Quantum Revolution,* Cambridge University Press, 2021.

Hands-On Quantum Software Laboratory Requirements

- Bring your own laptop (strongly suggested)
- Setup a new python 3 virtual environment (strongly suggested)
  > Either with venv, miniconda, or anaconda

Installation steps (suggest you do try beforehand)

1. activate virtual environment
2. pip install Qiskit[visualization] on Linux/Windows
3. pip install ‘Qiskit[visualization]’ on MacOS
4. pip install Qiskit-ibm-runtime

Check the official installation guide:
https://docs.quantum.ibm.com/guides/install-qiskit

Single-Qubit System and the Framework of Quantum Mechanics

What Is a Qubit?

- The **bit** is the fundamental concept of classical computation and classical information
- Quantum computation and quantum information are built upon an analogous concept, the **quantum bit**, or **qubit** for short
- We’re going to describe qubits as abstract **mathematical objects** with certain specific properties

What Is a Qubit?

- The beauty of treating qubits as abstract entities is that it gives us the freedom to construct a general theory of quantum computation and quantum information that does not depend upon a specific system for its realization

What Is a Qubit?

- Just as a classical bit has a state - either 0 or 1 - a qubit also has a state
- Two possible states for a qubit are the states $|0\rangle$ and $|1\rangle$, which correspond to the states 0 and 1 for a classical bit

classical bit states

qubit specific states

$$0 \rightarrow |0\rangle$$
$$1 \rightarrow |1\rangle$$

- Notation $|\ \rangle$ is called the *Dirac notation*, which is the standard notation for states in quantum mechanics

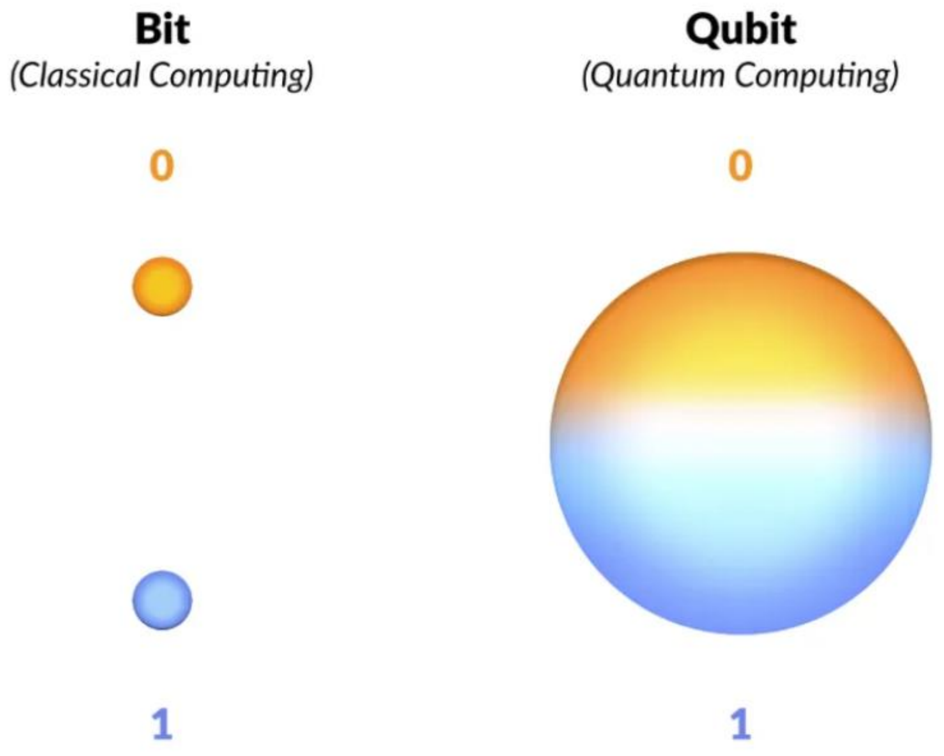
What Is a Qubit?

- The difference between bits and qubits is that a qubit can be in a state *other* than $|0\rangle$ or $|1\rangle$
- It is also possible to form linear combinations of states, often called *superpositions*:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

where α and β are complex numbers s.t.

$$|\alpha|^2 + |\beta|^2 = 1$$



- The coefficient α is called the *amplitude* of the $|0\rangle$ state, and the coefficient β is called the *amplitude* of the $|1\rangle$ state

Possible Realization of a Qubit

- Just as there are *many ways to realize classical bits physically* (two voltage levels, lights on or off in an array), there are *many ways to realize quantum bits physically*

- To get a concrete feel for how a qubit can be realized it may be helpful to list some of the ways this realization may occur:

  > as the electron spin;

  > as the two different polarizations of a photon;

  > as two states of an electron orbiting a single atom.

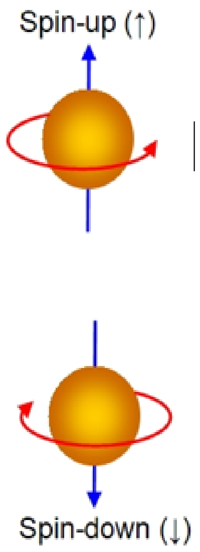
The Spin of an Electron

- When we measure the "spin" of an electron, for example, we always find it to have one of two possible values
- One value, called "spin up" or $|\uparrow\rangle$, means that the spin was found to be parallel to the axis along which the measurement was taken

classical bit states

qubit specific states

$$0 \rightarrow |0\rangle$$
$$1 \rightarrow |1\rangle$$

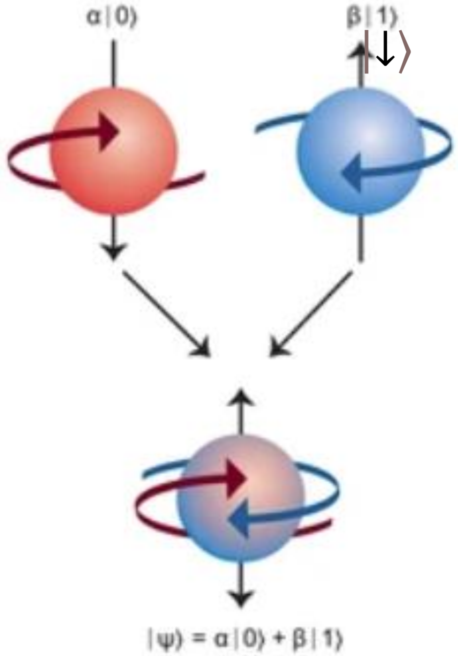


$$|0\rangle \equiv |\uparrow\rangle$$
$$|1\rangle \equiv |\downarrow\rangle$$

The other possibility, "spin-down" or $|\downarrow\rangle$, means that the spin was found to be anti-parallel to the axis along which the measurement was taken

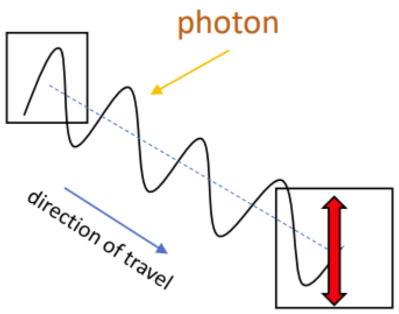
The Spin of an Electron

- The electron spin can also be in a superposition of the “spin up ” $|\uparrow\rangle \equiv |0\rangle$ and “spin down” $|\downarrow\rangle \equiv |1\rangle$ states
- A pictorial representation of this superposition is shown in the figure



The Photon Polarization

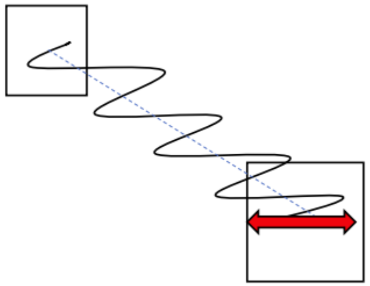
A photon with its plane of polarization (horizontal and vertical linear polarization) could implement a physical qubit



Vertical polarization

$$|0\rangle$$

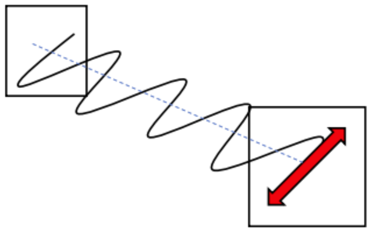
$$\rho = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$



Horizontal polarization

$$|1\rangle$$

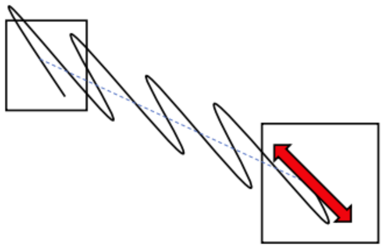
$$\rho = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$



Diagonal up polarization

$$|+\rangle = \sqrt{\tfrac{1}{2}}(|0\rangle + |1\rangle)$$

$$\rho_+ = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$



Diagonal down polarization

$$|-\rangle = \sqrt{\tfrac{1}{2}}(|0\rangle - |1\rangle)$$

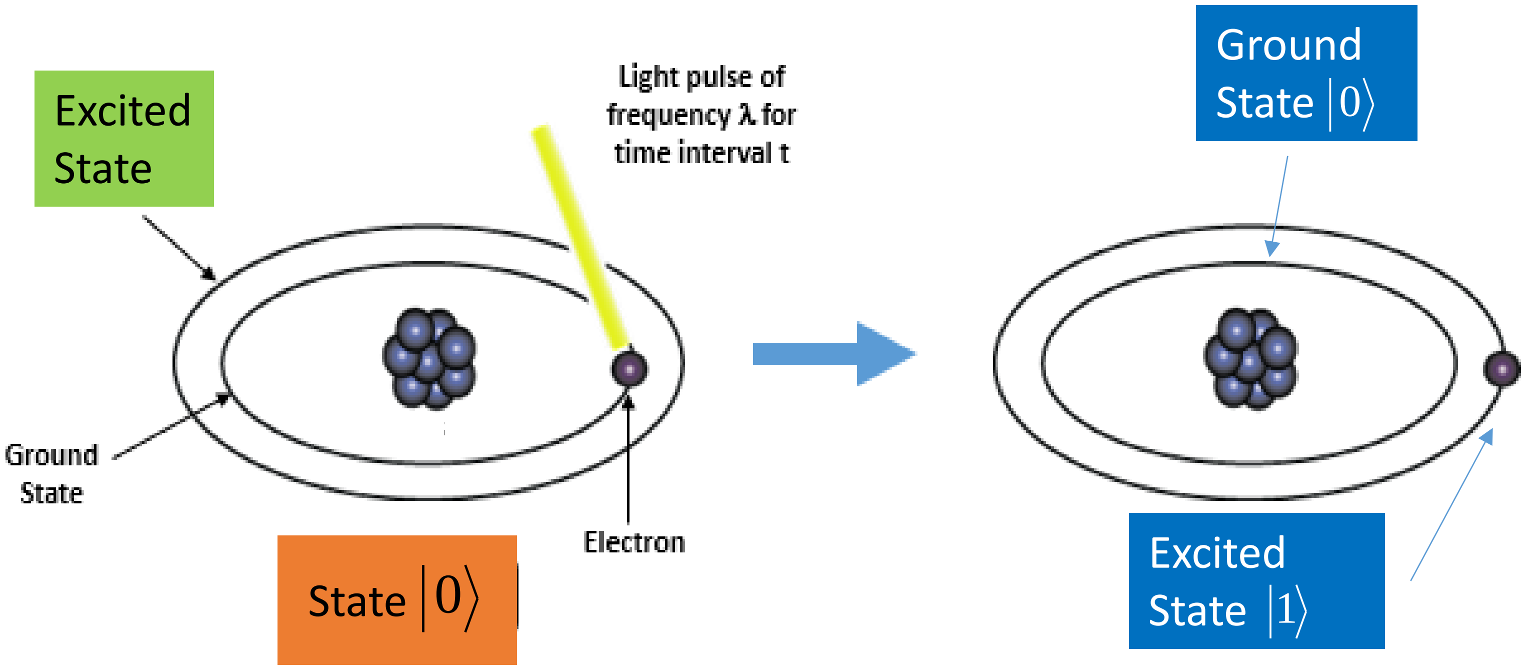
$$\rho_- = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

An Excited Atom

- In quantum mechanics, an electron in an atom can exist in different energy states which are quantized
- This quantization of energy levels in an atom is a key concept that underpins the behavior of qubits in quantum computing

An Excited Atom

- Natural choice for the state $|0\rangle$ is to use the **ground state** of the atom
- The other state of the qubit, $|1\rangle$, is then represented by one of the **excited states** of the atom



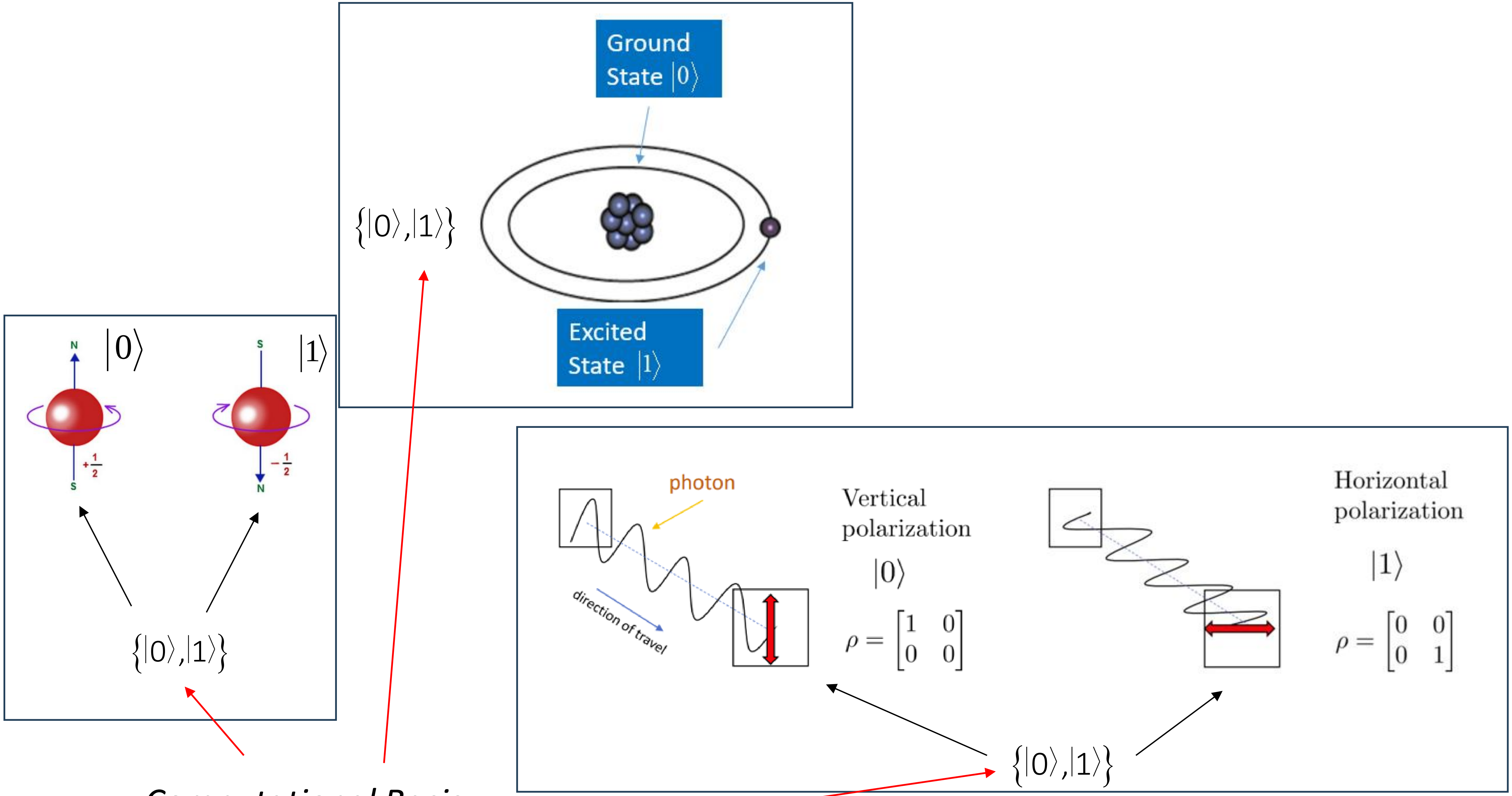
An Excited Atom

- By shining light on the atom, with appropriate energy and for an appropriate length of time, it is possible to move the electron from the $|0\rangle$ state to the $|1\rangle$ state and vice versa
- But more interestingly, by reducing the time we shine the light, an electron initially in the state $|0\rangle$ can be moved ‘halfway’ between $|0\rangle$ and $|1\rangle$ into the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

Possible Realization of a Qubit

- A physical qubit exists at the atomic/subatomic level
- As such, its behavior is described by the laws of *quantum mechanics*
- A qubit is our *basic quantum mechanical system*
- As we have seen there are many ways to implement a *physical* qubit
- Logical states $|0\rangle$ and $|1\rangle$ must be matched to the states of the physical phenomena in order to completely realize them



Computational Basis

or

Standard Basis

State Space

- **Postulate 1:** Associated to any ***isolated*** *physical system* is a complex vector space with inner product (that is, a **Hilbert space**) known as the **state space of the system**. The system is completely described by its state vector, which is a *unit vector in the system's state space*

- Quantum mechanics does not tell us, for a given physical system, what the *state space of that system is,* nor does it tell us what the *state vector of the system is*

State Space

- The system we will be starting to analyze is the **qubit**
- For our purposes it will be sufficient to make some very simple (and reasonable) assumptions about the *state spaces of the qubit* and stick with those assumptions. Specifically
- A qubit has a *two-dimensional state space*
- Suppose $|0\rangle$ and $|1\rangle$ form an orthonormal basis $\{|0\rangle, |1\rangle\}$, called *computation basis* or *standard basis* for that state space
- Then an *arbitrary state vector* in the state space can be written

$$|\psi\rangle = a|0\rangle + b|1\rangle \quad \text{where} \quad a, b \in \mathbb{C}$$

State Space

- For any *state vector* of our qubit

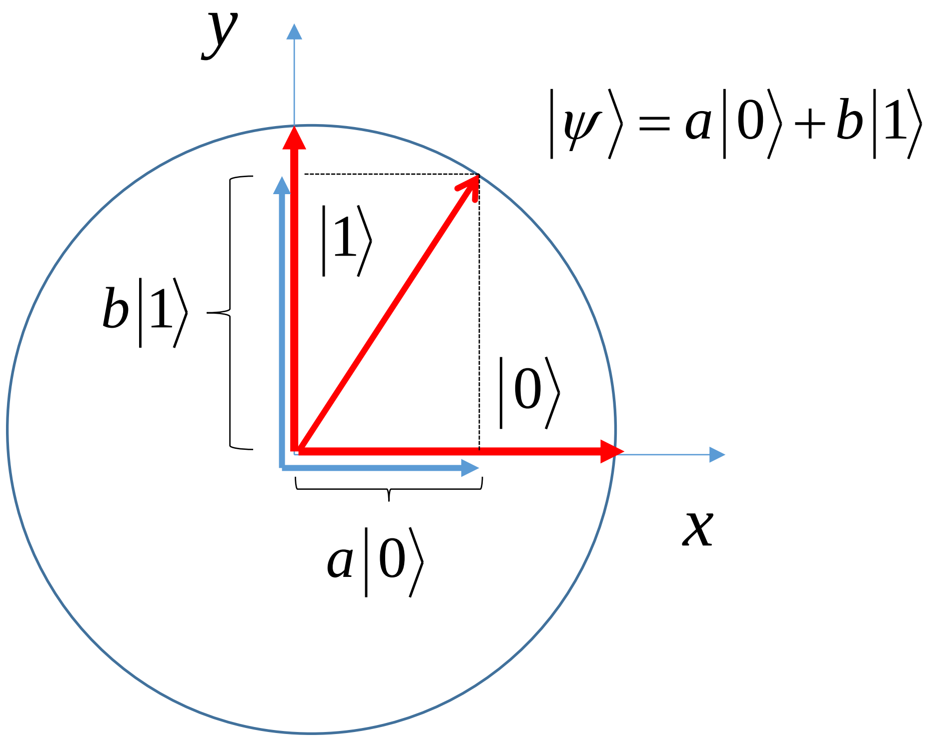
$$|\psi\rangle = a|0\rangle + b|1\rangle$$

- As we said before, the coefficient ***a*** is called the *amplitude* of the $|0\rangle$ state, and the coefficient ***b*** is called the *amplitude* of the $|1\rangle$ state

- The condition that $|\psi\rangle$ be a **unit vector,** $\langle\psi|\psi\rangle = 1,$ (we will see the definition of it very soon) is often known as the *normalization condition* for state vectors

Geometric Interpretation

- If *a* and *b* are real number, then $|\psi\rangle$ can geometrically be interpreted as a unit vector in the plane, where *a* and *b* can be regarded as the projection of $|\psi\rangle$ along the *x* and *y* axes



$$|0\rangle \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad |1\rangle \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \rightarrow$$

versors of the x and y axes

State Space

The previous discussion boils down to this:

- We will take the qubit as our fundamental quantum mechanical system
- There are real physical systems which may implement the qubit paradigm
- From now, though, we think of qubits in abstract terms, without reference to a specific realization

State Space

- **Postulate 1** can be "customized" for a qubit as follows: associated to any **isolated qubit** is a Hilbert space, known as the *state space* of the qubit. The qubit is completely described by its **state vector**, which is a **unit vector** in the **two-dimensional state space** (i.e., Hilbert space) of the qubit

State Space

- In our discussions of qubits, we have referred to a specific orthonormal set of basis vectors, $|0\rangle$ and $|1\rangle$, that were thought to have been fixed beforehand.
- Intuitively, the states $|0\rangle$ and $|1\rangle$ are analogous to the two values 0 and 1 which a bit may take
- The way a qubit differs from a bit is that superpositions of these two states, of the form $a|0\rangle + b|1\rangle$, can also exist, in which it is not possible to say that the qubit is definitely in the state $|0\rangle$, or definitely in the state $|1\rangle$
- During the course, other bases will be exploited

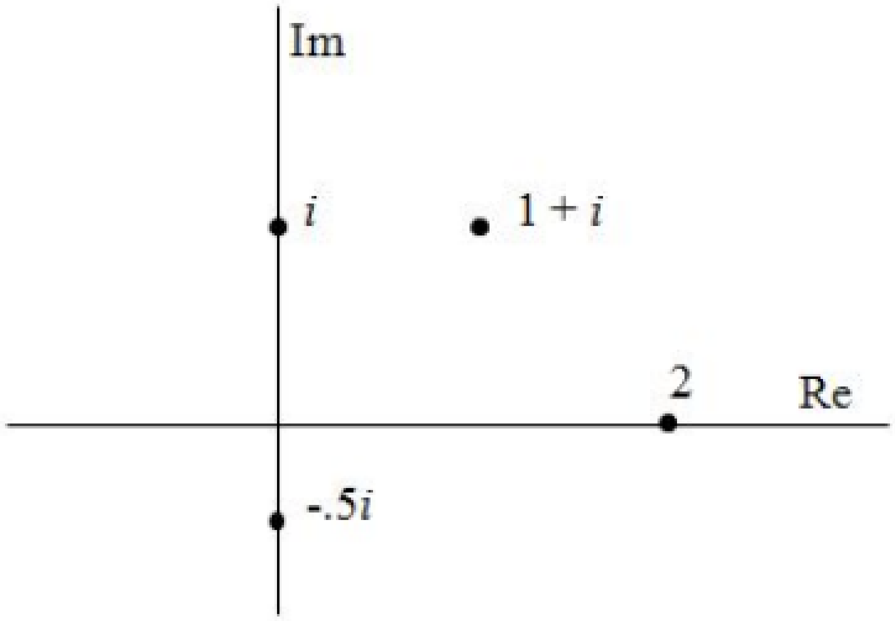
Complex Numbers

- Before moving on, let's recap the theory of complex numbers
- The complex numbers are the field $\mathbb{C}$ of numbers *z* of the form *a+ib*, where *a* and *b* are real numbers and *i* is the imaginary unit equal to the square root of -1, $\sqrt{-1}$, or

$$i^2 = -1$$

Complex Numbers

- Since every complex number is defined by an ordered pair of real numbers, (*a, b*), we have a natural way to represent each such number on a plane, whose *x*-axis is the real axis (which expresses the value *a*), and *y*-axis is the imaginary axis (which expresses the value *b*)

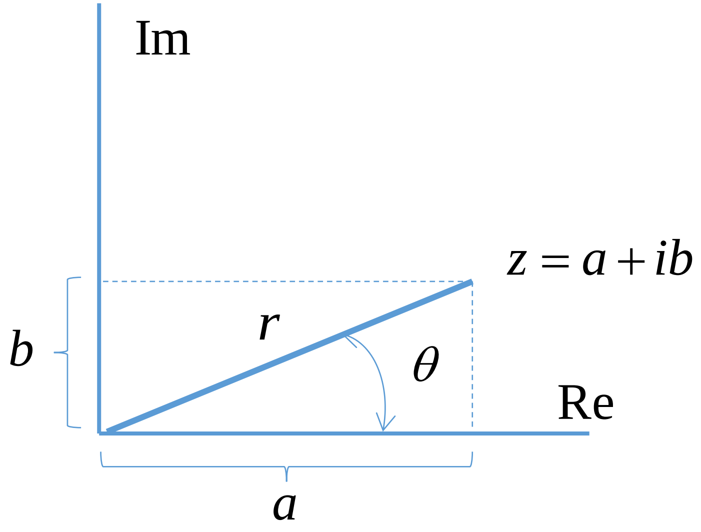


A few numbers $z=a+ib$ plotted in the complex plane

Polar Representation

Besides using (a, b) to describe z, we can use the polar representation suggested by polar coordinates (r,θ) of the complex plane

$$a+ib \leftrightarrow r\left(\cos\theta+i\sin\theta\right)$$



The connection between cartesian and polar coordinates of a complex number

Terminology

- ✓ a = Re(z), the *real part* of z
- ✓ b = Im(z), the *imaginary part* of z
- ✓ r = $|z|$, the *modulus* (*magnitude*) of z
- ✓ θ = arg z , the *argument* (*polar angle*) of z
- ✓ 3i, πi, 900i, etc. *purely imaginary* numbers

Complex Conjugate

- If

$$z = a + ib$$

then its **complex conjugate** (or just conjugate) is designated and defined as

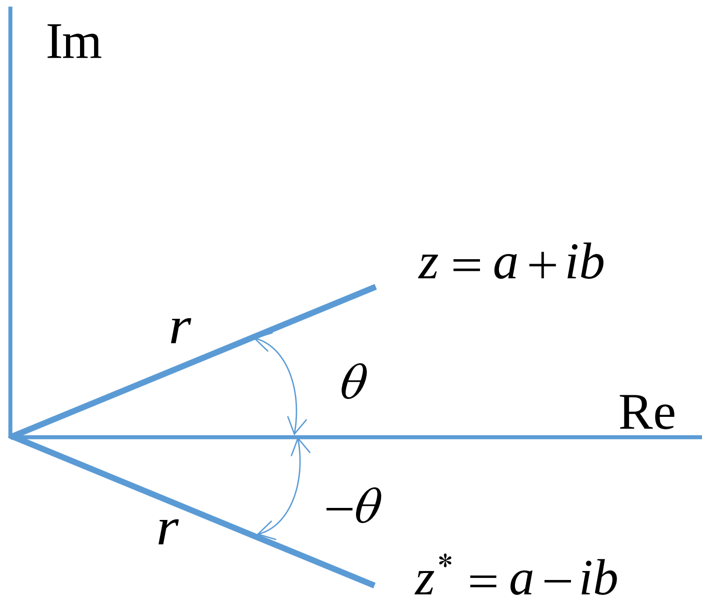
$$z^* = a - ib$$

the complex conjugate of *z*

- Furthermore

$$\left(z^*\right)^* = \left(\left(a+ib\right)^*\right)^* = \left(a-ib\right)^* = a + ib = z \longrightarrow$$

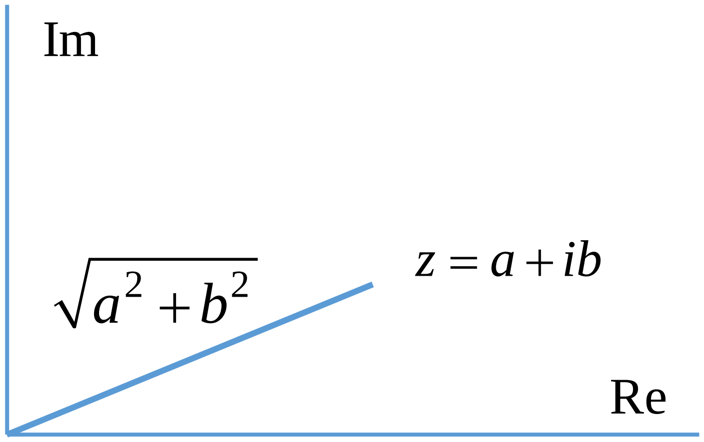
$$\left(z^*\right)^* = z$$



Conjugation as reflection

The Modulus of a Complex Number

The modulus of the complex z is the distance of the line segment (in the complex plane) between 0 and z, that is:



Modulus of a complex number

$$|z|=\sqrt{[\mathrm{Re}(z)]^2+[\mathrm{Im}(z)]^2}=\sqrt{a^2+b^2}$$

A short computation shows that multiplying z by its conjugate, z^*, results in a non-negative real number which is the square of the modulus of z

$$|z|^2=zz^*=z^*z=a^2+b^2 \quad \rightarrow$$

$$|z|=\sqrt{a^2+b^2}$$

Euler’s Formula

- Euler’s formula

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \rightarrow \quad \left|e^{i\theta}\right| = \sqrt{e^{i\theta}\cdot e^{-i\theta}} = 1, \; e^{i(\theta+\omega)} = e^{i\theta}e^{i\omega}$$

- Examples

$$e^{i\pi} = \cos\pi + i\sin\pi = -1, \qquad e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$$

- From

$$z = a + ib = r\left(\cos\vartheta + i\sin\vartheta\right) \;\rightarrow\; z = re^{i\theta}$$

Global Phase and Relative Phase

A general qubit state can be written

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

with complex numbers α and β, and the normalization constraint $\langle\psi|\psi\rangle = 1$ requires that:

$$|\alpha|^2 + |\beta|^2 = 1$$

We can express the state in polar coordinates as:

$$|\psi\rangle = r_\alpha e^{i\phi_\alpha}|0\rangle + r_\beta e^{i\phi_\beta}|1\rangle$$

with four real parameters r_α, ϕ_α, r_β and ϕ_β.

Global Phase and Relative Phase

Thus

$$|\psi\rangle = e^{i\phi_\alpha}\left(r_\alpha|0\rangle + r_\beta e^{i(\phi_\beta-\phi_\alpha)}|1\rangle\right)$$

or

$$|\psi\rangle = e^{i\gamma}\left(r_\alpha|0\rangle + r_\beta e^{i\phi}|1\rangle\right)$$

where $\gamma = \phi_\alpha$ *Global Phase*

$\phi = \phi_\beta - \phi_\alpha$ *Relative Phase*

Global Phase and Relative Phase

- However, the only measurement quantities are the probabilities $|\alpha|^2$ and $|\beta|^2$, so multiplying the state by an arbitrary factor $e^{i\gamma}$ (*a global phase*) has no observable consequence, because

$$\left|e^{i\gamma}\alpha\right|^2 = \left(e^{i\gamma}\alpha\right)^* \left(e^{-i\gamma}\alpha^*\right) = \alpha^*\alpha = |\alpha|^2$$

and similarly for $|\beta|^2$

Conclusion on Global Phase

- Therefore, from an observational point of view, $e^{i\gamma}|\psi\rangle$ and $|\psi\rangle$ are identical

- For this reason, we may ignore global phase factors as being irrelevant to the observed properties of the physical system

Relative Phase

- There is another kind of phase known as the *relative phase,* which has quite a different meaning
- Consider the states

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \quad \text{and} \quad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Relative Phase

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \quad \text{and} \quad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = \frac{|0\rangle + e^{i\pi}|1\rangle}{\sqrt{2}}$$

- For $|+\rangle$ the amplitude of $|1\rangle$ is $1/\sqrt{2}$
- For $|-\rangle$ the amplitude of $|1\rangle$ is $-1/\sqrt{2}$
- In each case the *magnitude* of the amplitudes is the same, but they differ in sign

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Relative Phase

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \quad \text{and} \quad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = \frac{|0\rangle + e^{i\pi}|1\rangle}{\sqrt{2}}$$

- More generally still, *two states* are said to *differ by a relative phase* in some basis if each of the amplitudes in that basis is related by such a phase factor
- For example, the two states displayed above are the same up to a relative phase shift because the $|0\rangle$ amplitudes are identical (a relative phase factor of 1), and the $|1\rangle$ amplitudes differ only by a relative phase factor of −1

Relative Phase

- The difference between *relative phase factors* and *global phase factors* is that for relative phase the phase factors may vary from basis to basis

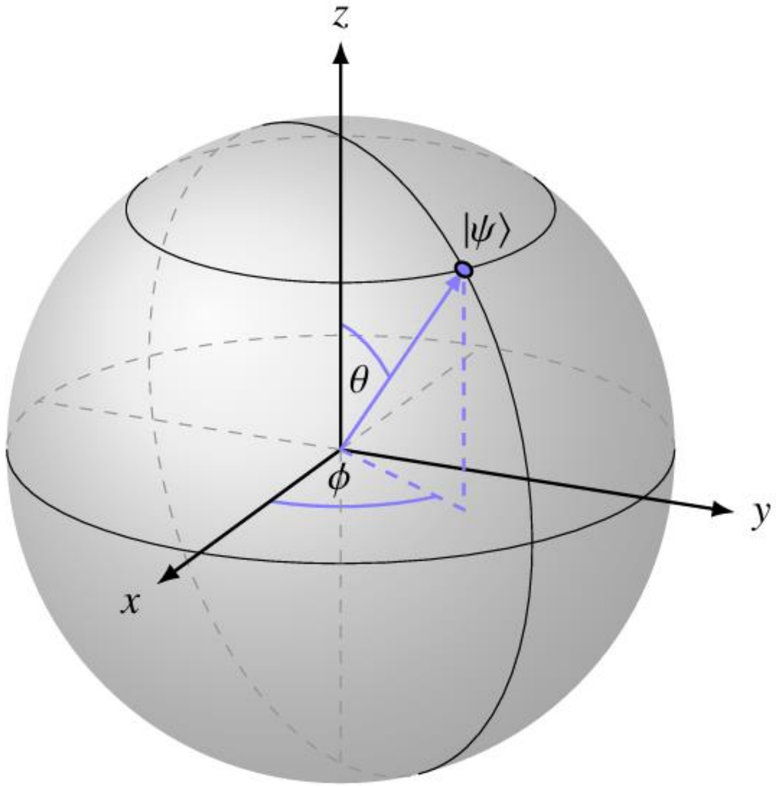
- *This makes the relative phase a basis-dependent concept unlike global phase*

- As a result, states which differ only by relative phases in some basis give rise to physically observable differences in measurement statistics, and it is NOT possible to regard these states as physically equivalent, as we do with states differing by a global phase factor

- We will reconsider this issue later on in the course

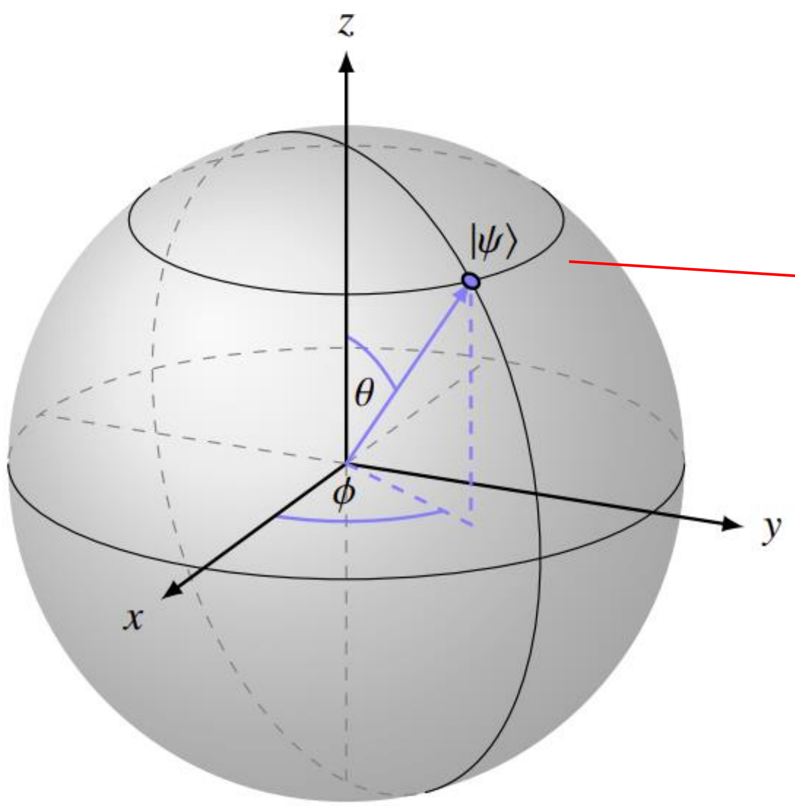
Bloch Sphere Picture of a Qubit

- A way to visualize the quantum state $|\psi\rangle$ of a *single qubit* is to picture it as a *unit vector* inside a bounding sphere, called the *Bloch sphere*



The parameters defining the quantum state are related to the *azimuth* (ϕ) and *elevation* (θ) angles that determine where the tip of this vector touches the surface of the Bloch sphere

Bloch Sphere Picture of a Qubit



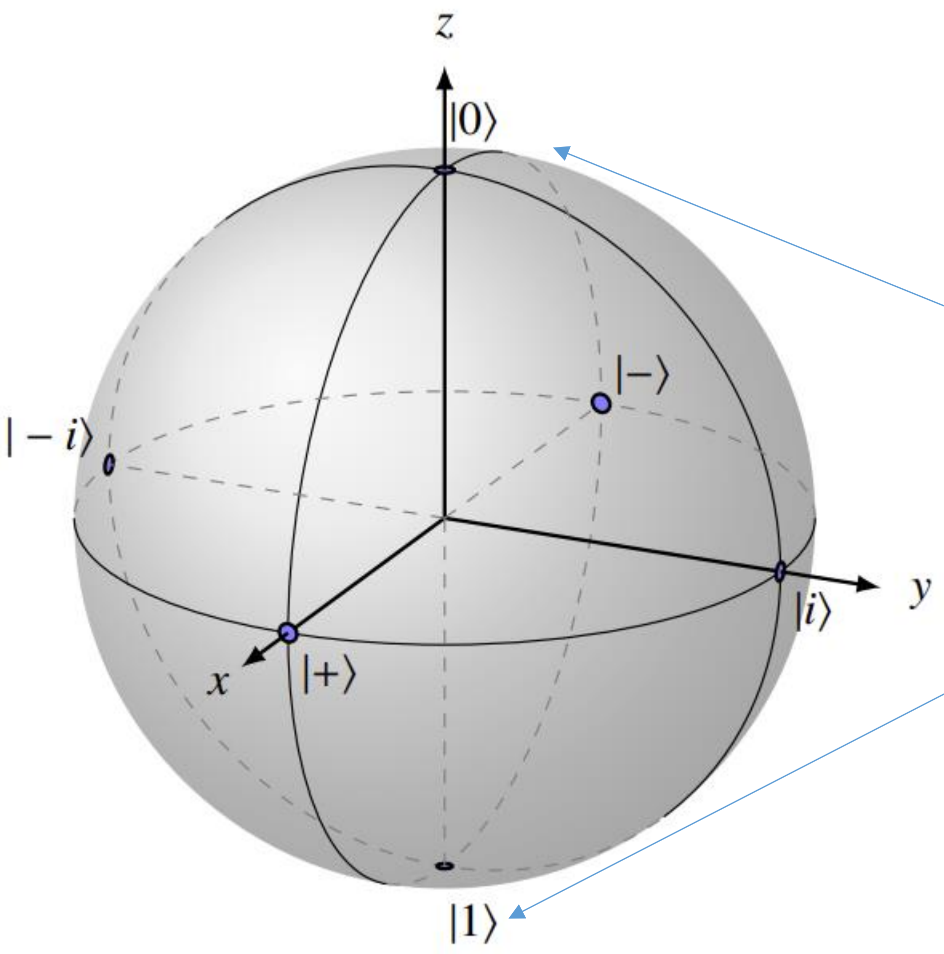
- An arbitrary state of a single qubit $|\psi\rangle = a|0\rangle + b|1\rangle$ such that $|a|^2 + |b|^2 = 1$ can be written in terms of these *azimuth* and *elevation* angles as:

$$|\psi\rangle = e^{i\gamma}\left(\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle\right)$$

  where γ, θ, and ϕ are all real number

- A pair of elevation and azimuth angles (θ, ϕ) in the range $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$ pick out a point on the Bloch sphere
- Qubit states corresponding to different values of γ are indistinguishable and are all represented by the same point on the Bloch sphere

Bloch Sphere Picture of a Qubit



$$|\psi\rangle = e^{i\gamma}\left(\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle\right)$$

The *North pole* $(\theta = 0)$ corresponds to the state $|0\rangle$

The *South pole* $(\theta = \pi)$ corresponds to the *(orthogonal)* state $|1\rangle$

Bloch Sphere Picture of a Qubit

Two questions:

- First, how come the *azimuth* and *elevation* angles are expressed in *half-angles?* Why not use the following expression which seems the more appropriate one?

$$|\psi\rangle = e^{i\gamma}\left(\cos\theta|0\rangle + e^{i\phi}\sin\theta|1\rangle\right)$$

- Second, how come orthogonal states are not at right angles on the Bloch sphere? Instead, they are 180° apart

Bloch Sphere Picture of a Qubit

- Consider the general quantum state

$$|\psi\rangle = a|0\rangle + b|1\rangle, \quad a,b \in \mathbb{C}, \quad |a|^2 + |b|^2 = 1$$

- Since *a* and *b* are complex numbers, they can be written in either Cartesian or Polar coordinates:

$$a = x_a + iy_a = r_a e^{i\phi_a}, \quad b = x_b + iy_b = r_b e^{i\phi_b}$$

with $i = \sqrt{-1}$ and the *x*'s, *y*'s, *r*'s, and ϕ's are all real numbers

Bloch Sphere Picture of a Qubit

- Write the general state of a qubit $|\varphi\rangle = a|0\rangle + b|1\rangle$, as

$$|\psi\rangle = r_a e^{i\phi_a}|0\rangle + r_b e^{i\phi_b}|1\rangle = e^{i\phi_a}\left(r_a|0\rangle + r_b e^{i\left(\phi_b - \phi_a\right)}|1\rangle\right) = e^{i\phi_a}\left(r_a|0\rangle + r_b e^{i\phi}|1\rangle\right)$$

where: $\phi = \phi_b - \phi_a$

- Since an overall phase factor has no observable consequence, the above state is equivalent to

$$|\psi'\rangle = r_a|0\rangle + r_b e^{i\phi}|1\rangle$$

Bloch Sphere Picture of a Qubit

- Switching back to Cartesian coordinates for the amplitude of the $|1\rangle$ component we can write this state as

$$|\psi'\rangle = r_a|0\rangle + (x+iy)|1\rangle$$

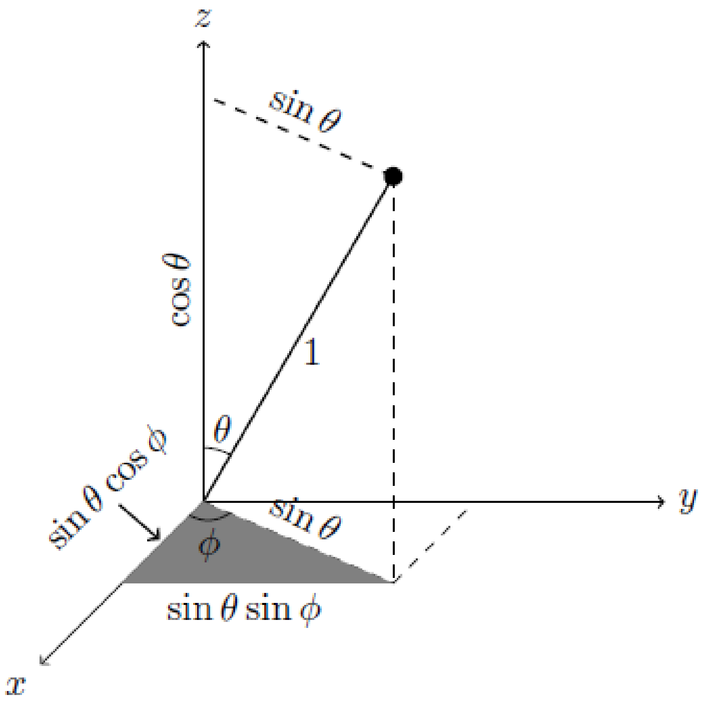
- Applying normalization, we have $|r_a|^2 + |x+iy|^2 = 1$ or equivalently

$$|r_a|^2 + (x+iy)(x+iy)^* = |r_a|^2 + (x+iy)(x-iy) = r_a{}^2 + x^2 + y^2 = 1$$

which is the equation of a sphere in coordinates r_a, *x*, and *y*

- We can rename $r_a = z$ for aesthetic reasons, and it doesn't change anything but now we have the equation of a sphere in coordinates *x*, *y*, and *z,* i.e. $z^2 + x^2 + y^2 = 1$

Bloch Sphere Picture of a Qubit



- Ok so let's switch from these Cartesian coordinates to spherical coordinates. We have,

$$x = r\sin(\theta)\cos(\phi)$$
$$y = r\sin(\theta)\sin(\phi)$$
$$z = r\cos(\theta)$$

- Since r=1, the position on the surface of the sphere is specified using only two parameters, θ and ϕ, and the general qubit state can be written

Bloch Sphere Picture of a Qubit

$$\left|\psi'\right\rangle = z\left|0\right\rangle + \left(x+iy\right)\left|1\right\rangle = \cos(\theta)\left|0\right\rangle + \left(\sin(\theta)\cos(\phi) + i\sin(\theta)\sin(\phi)\right)\left|1\right\rangle$$

$$= \cos(\theta)\left|0\right\rangle + \sin(\theta)\underbrace{\left(\cos(\phi) + i\sin(\phi)\right)}_{e^{i\phi}}\left|1\right\rangle = \cos(\theta)\left|0\right\rangle + \sin(\theta)e^{i\phi}\left|1\right\rangle$$

$$\Longrightarrow$$

$$\left|\psi'\right\rangle = \cos(\theta)\left|0\right\rangle + \sin(\theta)e^{i\phi}\left|1\right\rangle$$

Why Half Angles

- But this is still not the Bloch sphere
- What about the half angles?
- Let

$$|\psi\rangle = \cos\theta' |0\rangle + e^{i\phi} \sin\theta' |1\rangle$$

and notice that

$$\theta' = 0 \rightarrow |\psi\rangle = |0\rangle$$
$$\theta' = \pi/2 \rightarrow |\psi\rangle = e^{i\phi} |1\rangle$$

which suggests that $0 \leq \theta' \leq \pi/2$ may generate all points on the Bloch sphere

Why Half Angles

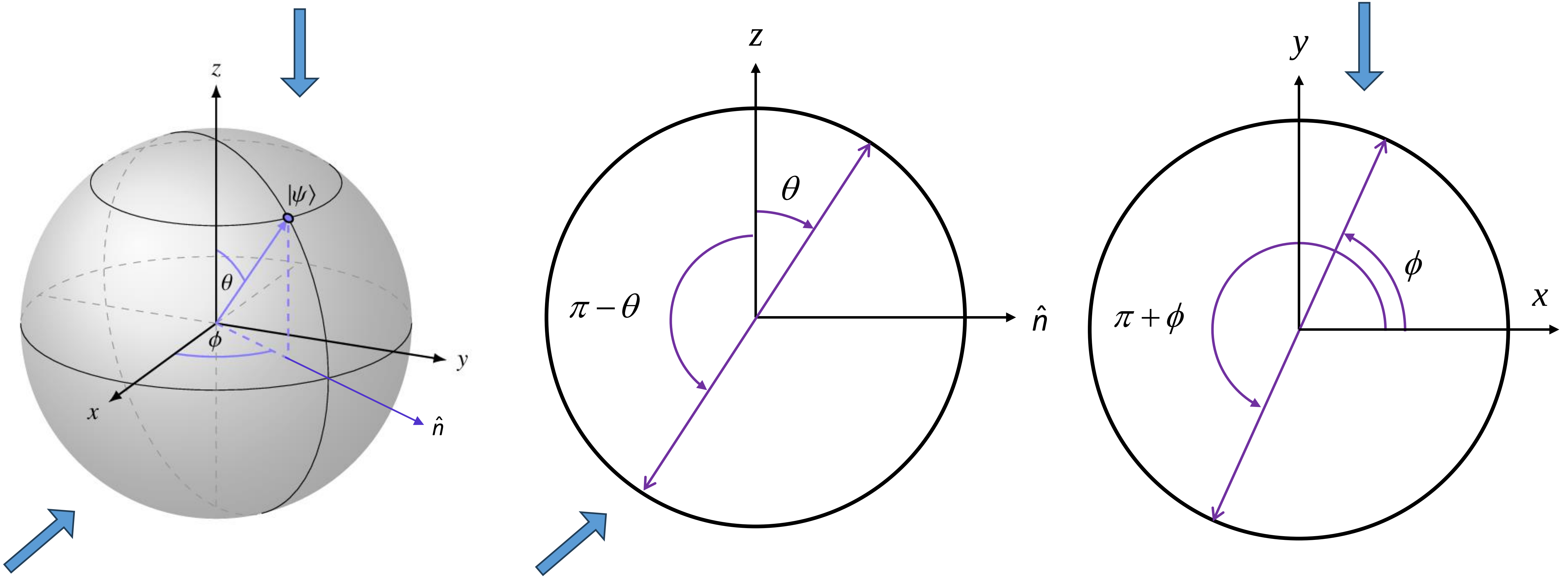
- Consider a state $|\psi'\rangle$ corresponding to the opposite point on the sphere, which has polar coordinates $1, \pi-\theta', \phi+\pi$

$$
\begin{aligned}
|\psi'\rangle &= cos(\pi-\theta')|0\rangle + e^{i\,(\phi+\pi)}\, sin(\pi-\theta')|1\rangle \\
&= -cos(\theta')|0\rangle + e^{i\phi}e^{i\pi} sin(\theta')|1\rangle \\
&= -cos(\theta')|0\rangle - e^{i\phi} sin(\theta')|1\rangle \\
&= -\left(cos(\theta')|0\rangle + e^{i\phi} sin(\theta')|1\rangle \right) \\
&= -|\psi\rangle
\end{aligned}
$$

$$e^{i\pi} = \cos\pi + i\sin\pi = -1$$

- So, it is only necessary to consider the upper hemisphere $0 \le \theta' \le \pi/2$, as opposite points in the lower hemisphere differ only by a phase factor of −1 and so are equivalent in the Bloch sphere representation

Why Half Angles



Why Half Angles

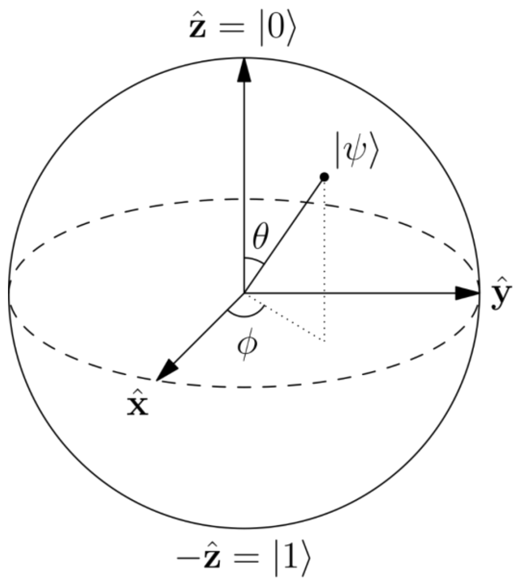
- We can map points on the upper hemisphere onto points on a sphere by defining

$$\theta = 2\theta' \;\rightarrow\; \theta' = \theta/2$$

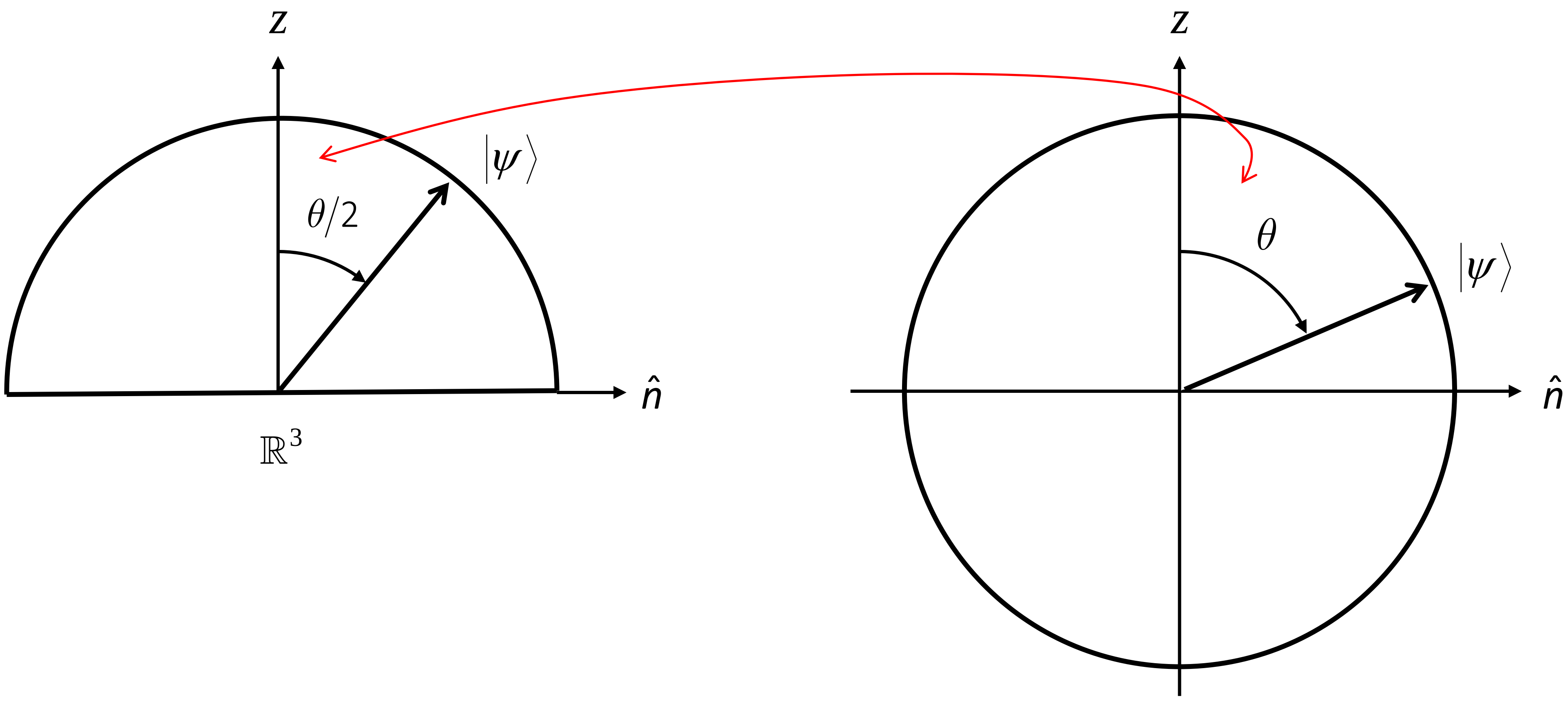
and we now have

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

- where $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$ are the coordinates of points on the *Bloch sphere*



Why Half Angles



Bloch Sphere

Why Half Angles

- Notice that $\theta = 2\theta'$ is a one-to-one mapping except at $\theta' = \pi/2$, where all the points on the θ' *equator* are mapped to the single point $\theta = \pi$, the *south pole* on the Bloch sphere
- This is okay, since at the south pole $|\psi\rangle = e^{i\phi}|1\rangle$ and ϕ is a global phase with no significance. (Longitude is meaningless at a pole!)
- Notice also that as we cross the θ'-equator going south, we effectively start going north again on the other side of the Bloch sphere, because opposite points are equivalent on the θ' sphere

Second Question/Orthogonality of Opposite Points

Consider a general qubit state $|\psi\rangle$

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

and $|\chi\rangle$ corresponding to the opposite point on the Bloch sphere

$$\begin{aligned}
|\chi\rangle &= \cos\left(\frac{\pi-\theta}{2}\right)|0\rangle + e^{i(\phi+\pi)}\sin\left(\frac{\pi-\theta}{2}\right)|1\rangle \\
&= \cos\left(\frac{\pi-\theta}{2}\right)|0\rangle - e^{i\phi}\sin\left(\frac{\pi-\theta}{2}\right)|1\rangle
\end{aligned}$$

So

$$\langle\chi|\psi\rangle = \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\pi-\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\pi-\theta}{2}\right)$$

Orthogonality of Opposite Points

$$\langle\chi|\psi\rangle = \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\pi-\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\pi-\theta}{2}\right)$$

But $\cos(a+b) = \cos a \cos b - \sin a \sin b$, so

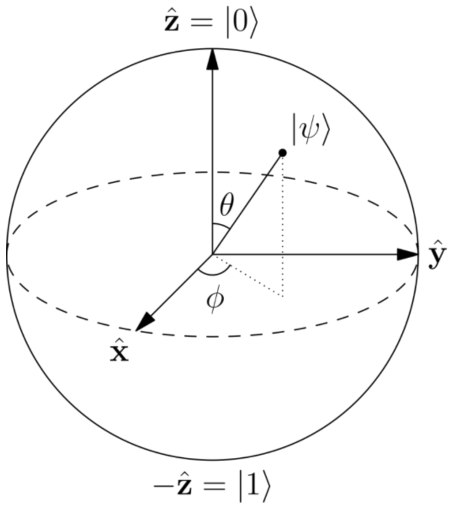
$$\langle\chi|\psi\rangle = \cos\frac{\pi}{2} = 0$$

and opposite points correspond to orthogonal qubit states.

Example #1

For example, let’s represent on the Bloch sphere the state

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

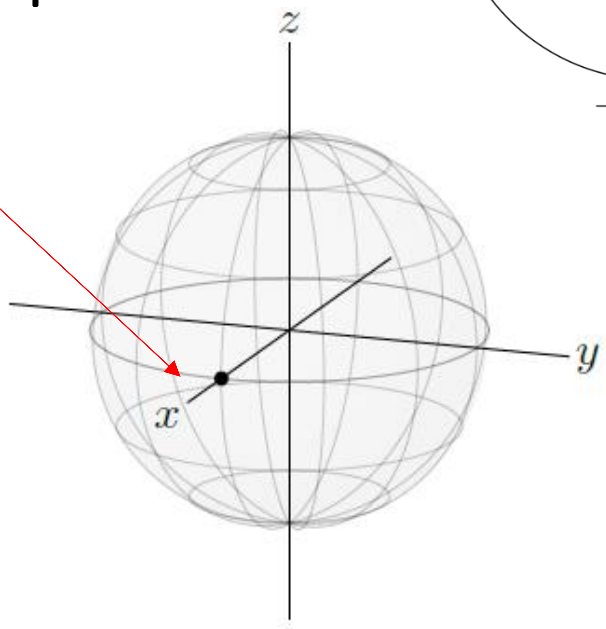


We saw erlier that a generic state on the Bloch sphere can be expressed as

$$|+\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

From which it follows

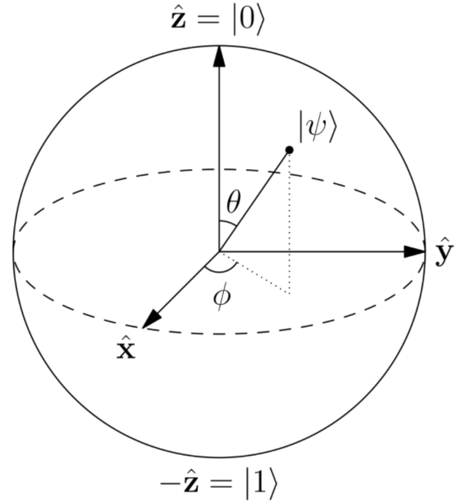
$$\boxed{\begin{aligned} &\cos\frac{\theta}{2} = \sin\frac{\theta}{2} = \frac{1}{\sqrt{2}} \\ &e^{i\phi} = 1 \end{aligned}} \longrightarrow \boxed{\begin{aligned} &\frac{\theta}{2} = \frac{\pi}{4} \rightarrow \theta = \frac{\pi}{2} \\ &\phi = 0 \end{aligned}} \longrightarrow$$



Example #2

For example, let's represent on the Bloch sphere the state

$$\underbrace{|-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)}$$

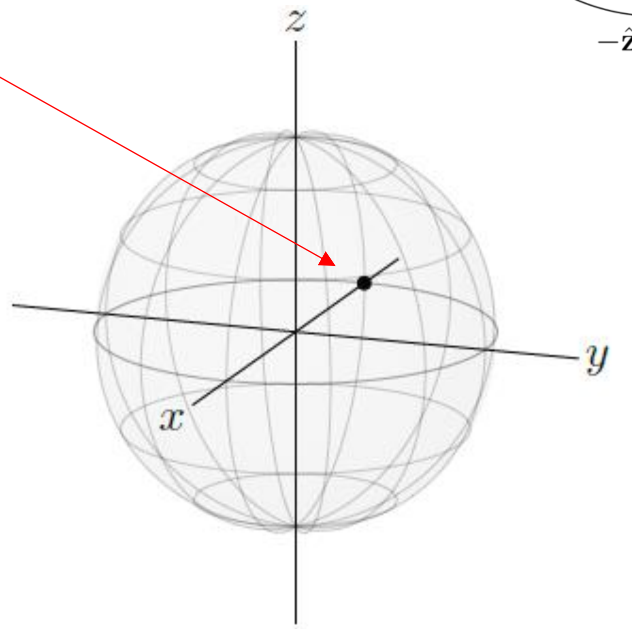


Let's compare the above state with

$$|-\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

from which it follows

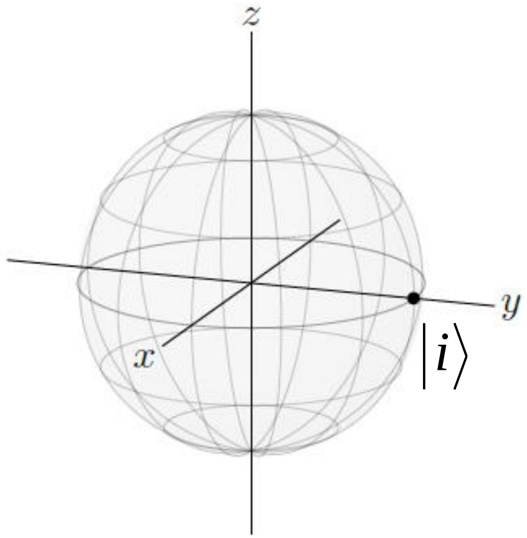
$$\boxed{\begin{array}{l}\cos\frac{\theta}{2} = \sin\frac{\theta}{2} = \frac{1}{\sqrt{2}} \\ e^{i\phi} = -1\end{array}} \rightarrow \boxed{\begin{array}{l}\frac{\theta}{2} = \frac{\pi}{4} \rightarrow \theta = \frac{\pi}{2} \\ \phi = \pi\end{array}} \rightarrow$$



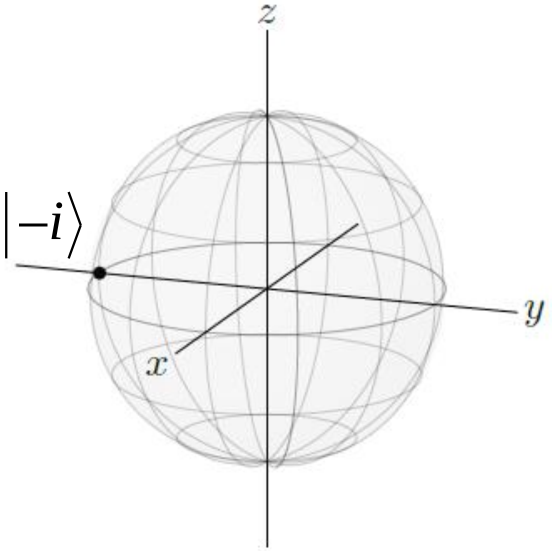
Examples #3 & #4

Two more examples

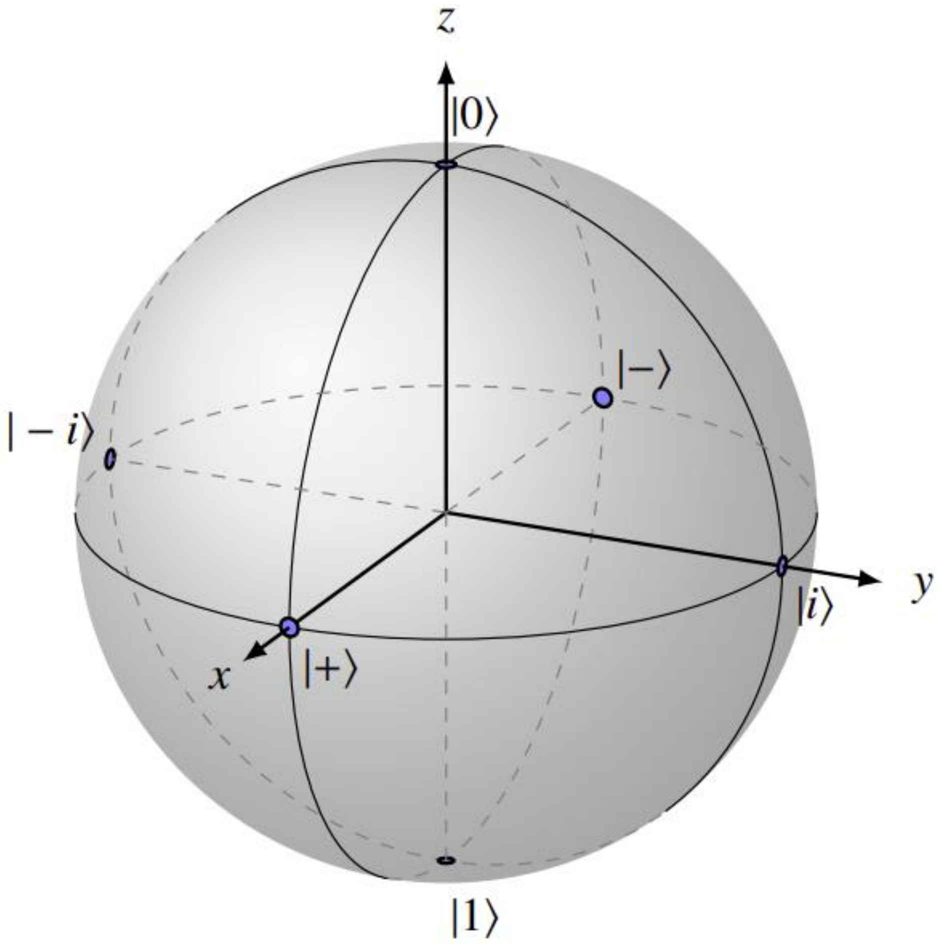
$$|i\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right)$$



$$|-i\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - i|1\rangle\right)$$



Let’s Put All the Examples Together



$$|0\rangle$$
$$|1\rangle$$

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$
$$|-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

$$|i\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right)$$
$$|-i\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - i|1\rangle\right)$$

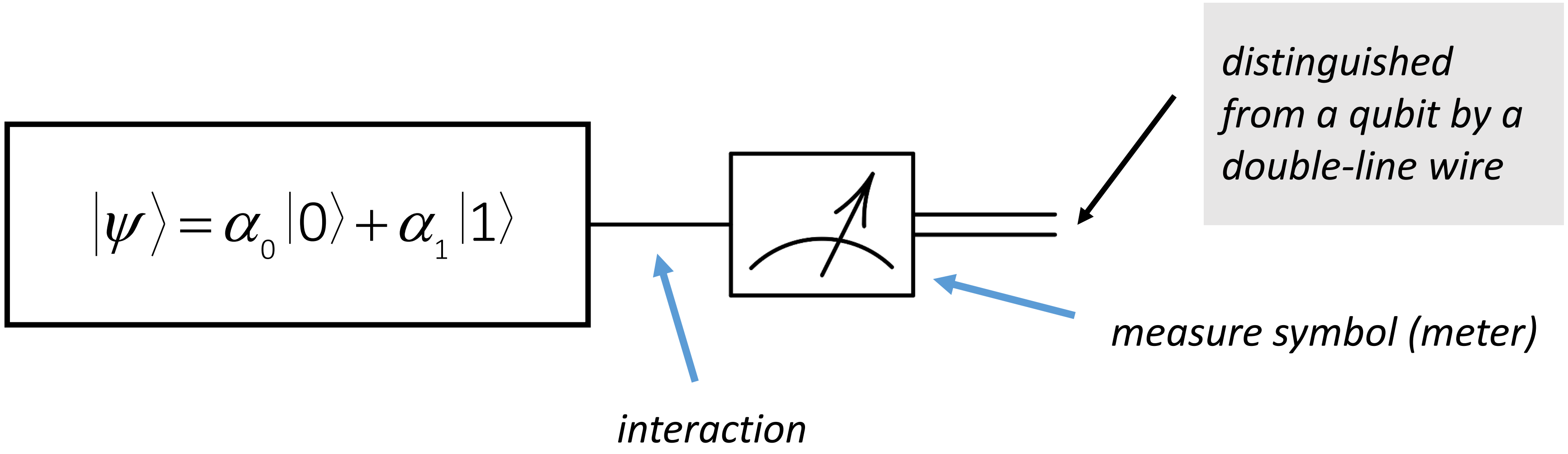
Measurement

Measurement

- We have seen how the state of a single-qubit system is represented in quantum mechanics as a vector in a *Hilbert space*
- Ultimately, we will be interested in *measuring some properties of a system*, and so at some point we must allow the system to interact with a *measurement device* (that we also call *measurement system* or *MS*) of an observer
- In this lecture we consider only measurements of **single-qubit systems**
- We will address the measurement of general quantum systems later on

Single Qubit Measurement

- In quantum circuits, a *measurement* of a *single qubit* in the *computational basis* is denoted by the *meter* symbol

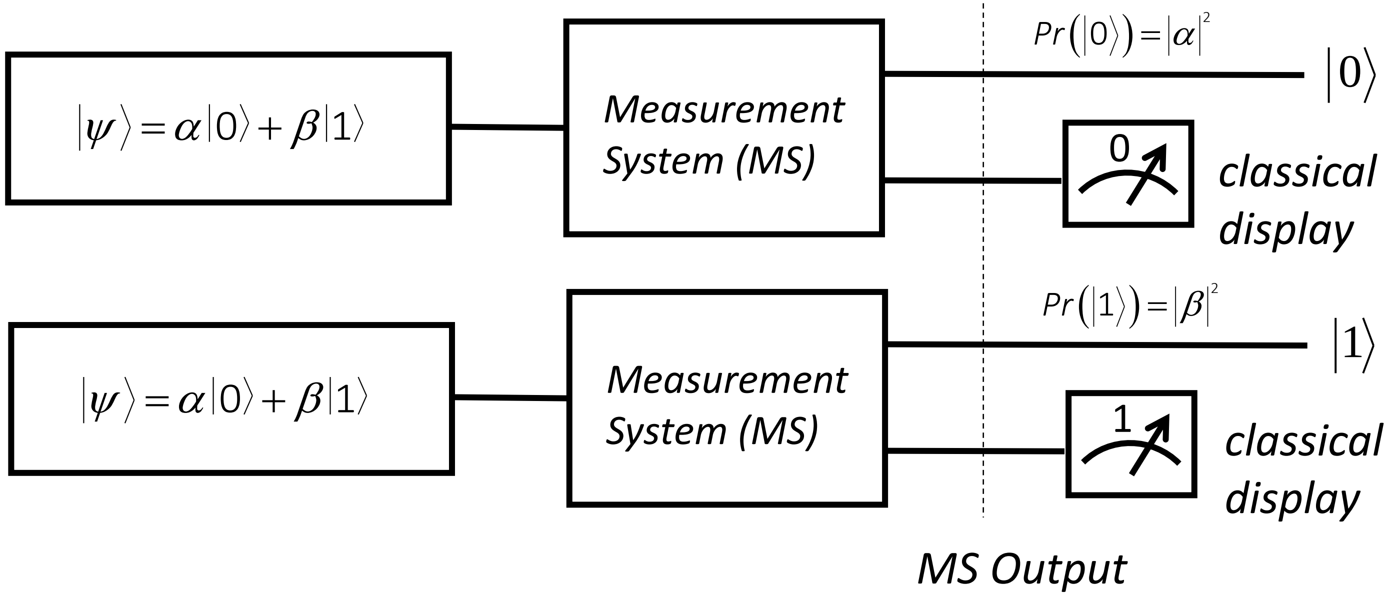


Single Qubit Measurement

- Let's consider the state of a qubit on the computational basis
- Rather remarkably, we **cannot** examine a **qubit** to determine its quantum state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, that is, the values of α and β
- Instead, **quantum theory postulates** that we can only acquire much more restricted information about the quantum state
- Let me express **Postulate 3** for a single qubit in a very colloquial way

Single Qubit Measurement

- When we measure a qubit in the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, we get (on the classical display) either the result 0, with probability $|\alpha|^2$, or the result 1, with probability $|\beta|^2$
- Furthermore, measurement changes the state of a qubit, collapsing it from its superposition of $|0\rangle$ and $|1\rangle$ to the specific state consistent with the measurement result appearing on the classical display



Naturally, $|\alpha|^2 + |\beta|^2 = 1$, since the probabilities must sum to one

Single qubit measurement

- Moreover, if we *rapidly* and *repeatedly* keep measuring the same state we can suppress its evolution and effectively freeze it in a fixed quantum state

$$|\psi\rangle \xrightarrow[Pr(|0\rangle)=|\alpha|^2]{\text{meas.}} |0\rangle \xrightarrow[Pr(|0\rangle)=1]{\text{meas.}} |0\rangle \xrightarrow[Pr(|0\rangle)=1]{\text{meas.}} |0\rangle \cdots$$

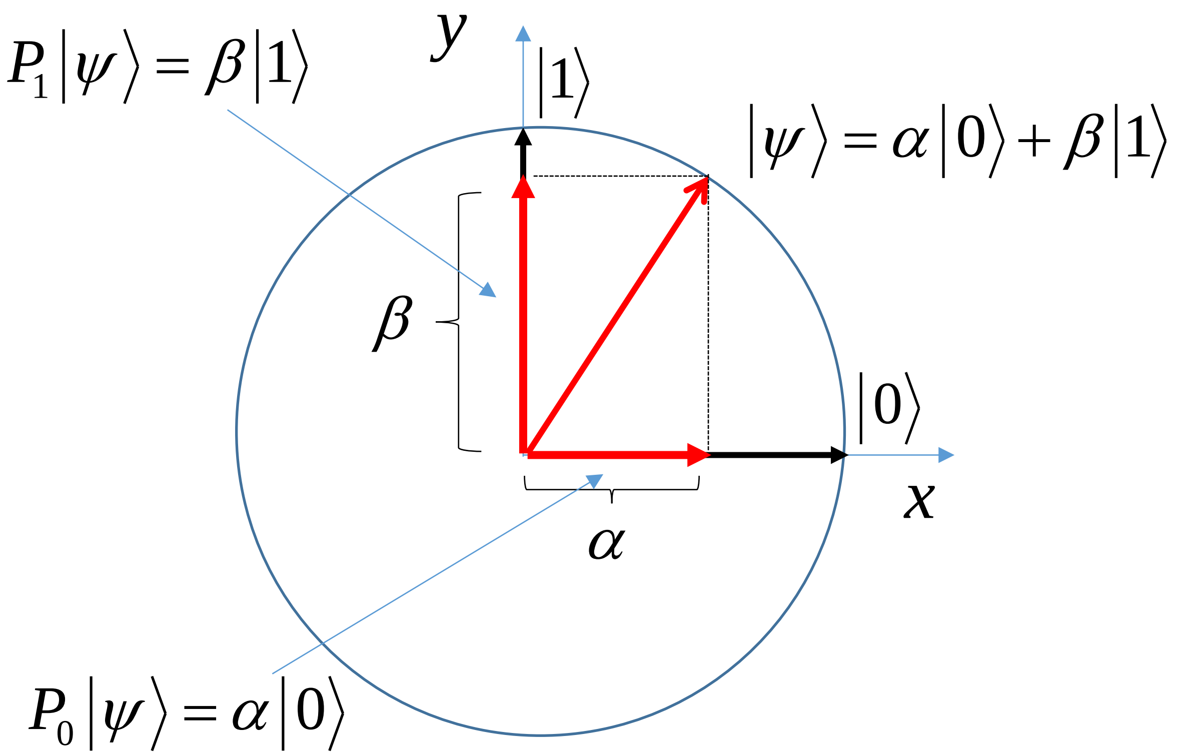
or

$$|\psi\rangle \xrightarrow[Pr(|1\rangle)=|\beta|^2]{\text{meas.}} |1\rangle \xrightarrow[Pr(|1\rangle)=1]{\text{meas.}} |1\rangle \xrightarrow[Pr(|1\rangle)=1]{\text{meas.}} |1\rangle \cdots$$

- We said *rapidly* because if we allow time to elapse between measurements the state will, in general, evolve, or “drift off”, in accordance with Schrödinger’s equation

Geometric Interpretation

- If $|\psi\rangle$ is a unit vector in the plane, then α and β can be interpreted as the projection of $|\psi\rangle$ along the *x* and *y* axes



$$|0\rangle \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad |1\rangle \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow$$

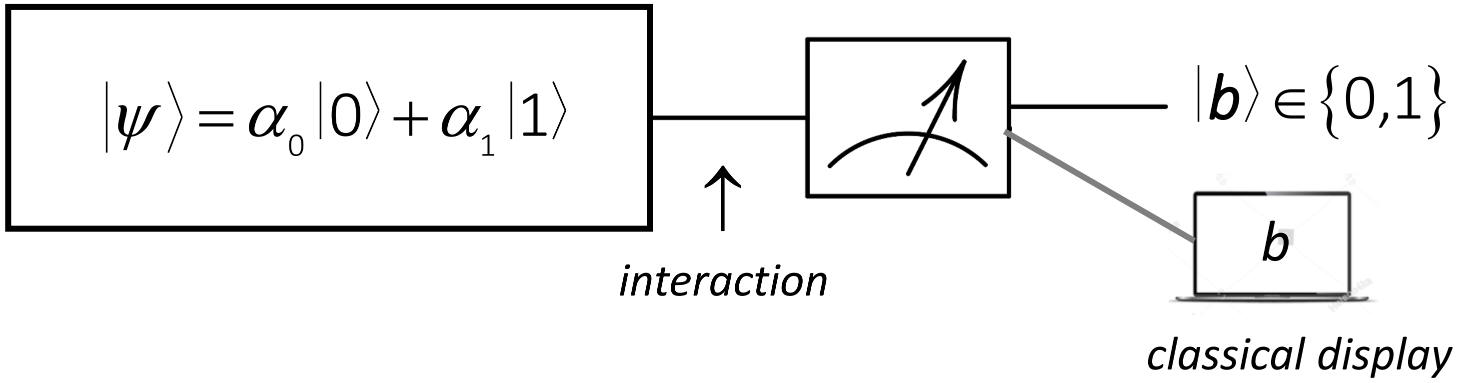
versors of the x and y axes

Single Qubit Measurement

- Geometrically, we can interpret $|\alpha|^2 + |\beta|^2 = 1$, as the condition that the qubit's state be normalized to length 1
- Thus, in general, a qubit's state is a unit vector in a two-dimensional complex vector space

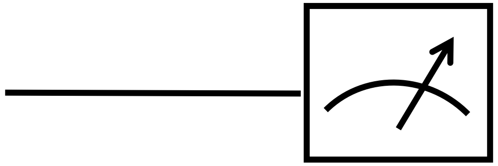
Single Qubit Measurement

- Thus, we could in general draw our measurement symbol with a ‘quantum’ wire carrying the quantum state resulting from the measurement, together with a classical wire carrying the classical label, as depicted in figure



Single Qubit Measurement

- Quite often, the **quantum outcome** is discarded or ignored, and we are only interested in the classical information telling us which outcome occurred
- In such cases, we will not draw the quantum wire coming out of the measurement symbol
- We will usually omit the classical wire from circuit diagrams as well and this is the resulting symbol



Measurement Implications

1. The outcome we obtain from reading the state of a qubit in a computational basis *is nondeterministic,* i.e. sometimes we will find it in state $|0\rangle$ and sometimes we will find it in state $|1\rangle$
2. Even though a quantum bit can be in **infinitely** many different **superposition** states, it is possible to extract only *a single classical bit's worth of information* from a single quantum bit

Single Qubit Measurement

- For example, if measurement of

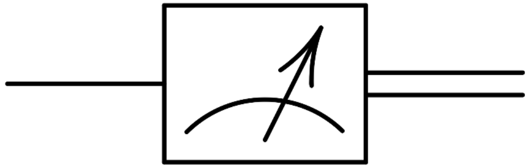
$$|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

  gives 0, then the post-measurement state of the qubit will be $|0\rangle$
- Why does this type of collapse occur? **Nobody knows!**
- This behavior of measurement is a postulate (**Postulate 3**) of quantum mechanics
- It is not derivable from other physical principles; rather, it is derived from the *empirical observation of experiments with measuring devices*

Single Qubit Measurement

- What is relevant for our purposes is that from a single measurement, one obtains only a single bit of information about the state of the qubit
- It turns out that only if infinitely many identically prepared qubits were measured would one be able to determine $|\alpha|$ and $|\beta|$ for a qubit in the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$\cdots|\psi\rangle|\psi\rangle|\psi\rangle|\psi\rangle\cdots\rightarrow$

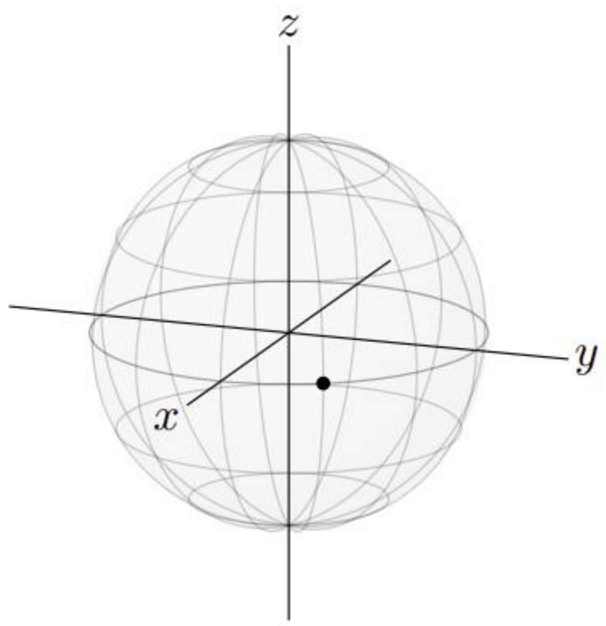


$\cdots 0,1,1,0\cdots$

Single Qubit Measurement: Example #1

Let's consider a qubit in the following state:

$$\left|\psi\right\rangle = \frac{1}{\sqrt{2}}\left(\left|0\right\rangle + e^{i\pi/6}\left|1\right\rangle\right) = \frac{1}{\sqrt{2}}\left|0\right\rangle + \frac{e^{i\pi/6}}{\sqrt{2}}\left|1\right\rangle$$



Geometrically, this particular qubit lies on the equator, halfway between the north and south poles, so if we measure it, we get $\left|0\right\rangle$ with probability 1/2 or $\left|1\right\rangle$ with probability 1/2

Single Qubit Measurement: Example #1

- In fact, for this qubit

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\pi/6}|1\rangle\right) = \frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\pi/6}}{\sqrt{2}}|1\rangle$$

to calculate these probabilities, we take the *norm-square* of the *amplitude* of $|0\rangle$ or $|1\rangle$

- That is, the probability of getting $|0\rangle$ is $\longrightarrow$ $\left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$

while the probability of getting $|1\rangle$ is $\longrightarrow$ $\left|\frac{e^{i\pi/6}}{\sqrt{2}}\right|^2 = \frac{e^{i\pi/6}}{\sqrt{2}}\frac{e^{-i\pi/6}}{\sqrt{2}} = \frac{e^0}{2} = \frac{1}{2}$

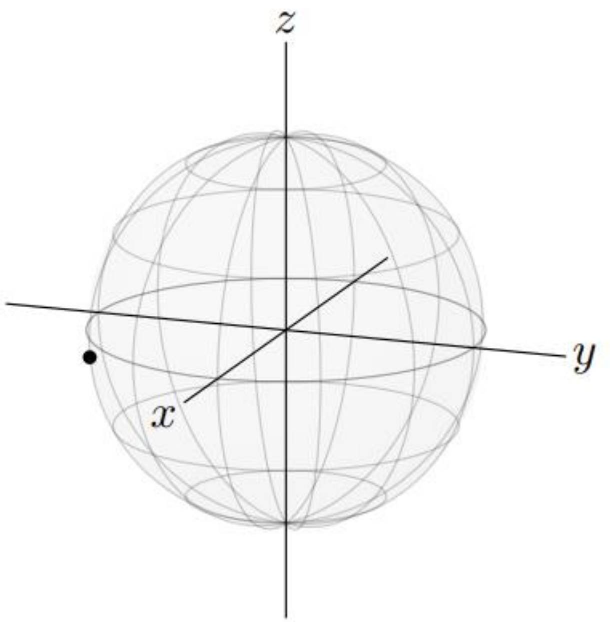
Single Qubit Measurement: Example #2

- Let us look at another example:
- Geometrically, since the qubit is closer to the south pole, we expect that the probability of getting $|1\rangle$ is greater than the probability of getting $|0\rangle$
- To get the exact probabilities, we calculate the norm-square of each amplitude:

$$\left|\frac{2}{3}\right|^2 = \frac{4}{9}$$

$$\left|\frac{1-2i}{3}\right|^2 = \frac{1-2i}{3}\frac{1+2i}{3} = \frac{1+2i-2i-4i^2}{3} = \frac{5}{9}$$

$$\frac{2}{3}|0\rangle + \frac{1-2i}{3}|1\rangle$$

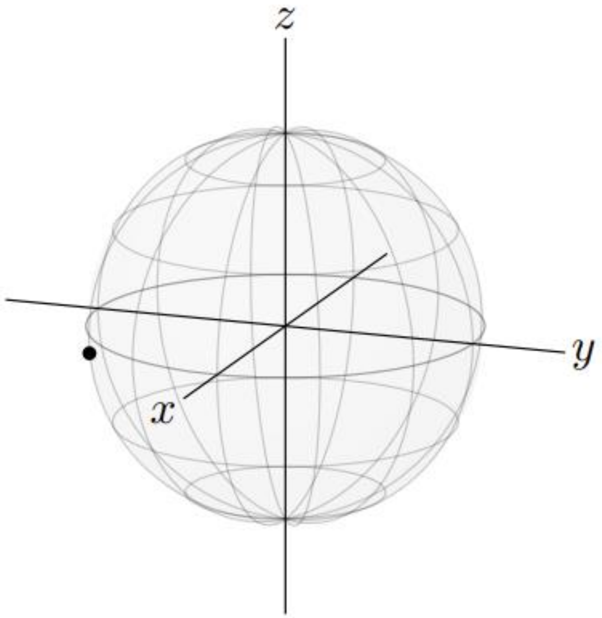


Single Qubit Measurement: Example #2

- So, if we measure the qubit, the probability of getting $|0\rangle$ is 4/9, and the probability of getting $|1\rangle$ is 5/9

- As expected, the probability of getting $|1\rangle$ is greater than the probability of getting $|0\rangle$, and also note that the total probability is 4/9 + 5/9 = 1, as it must

$$\frac{2}{3}|0\rangle + \frac{1-2i}{3}|1\rangle$$



Single Qubit Measurement: Additional Examples

Premeasurement State (Wave function) of Qubit	Measurement Outcome	Probability of Outcome	Postmeasurement State of Qubit
$\vert\psi\rangle = \vert 0\rangle$	0	100%	$\vert\psi\rangle = \vert 0\rangle$
$\vert\psi\rangle = \vert 1\rangle$	1	100%	$\vert\psi\rangle = \vert 1\rangle$
$\vert\psi\rangle = \frac{1}{\sqrt{2}}\vert 0\rangle + \frac{1}{\sqrt{2}}\vert 1\rangle$	0	50%	$\vert\psi\rangle = \vert 0\rangle$
	1	50%	$\vert\psi\rangle = \vert 1\rangle$
$\vert\psi\rangle = \frac{1}{2}\vert 0\rangle + \frac{\sqrt{3}}{2}\vert 1\rangle$	0	25%	$\vert\psi\rangle = \vert 0\rangle$
	1	75%	$\vert\psi\rangle = \vert 1\rangle$
$\vert\psi\rangle = \frac{1}{2}\vert 0\rangle + \frac{\sqrt{3}e^{-i\pi/4}}{2}\vert 1\rangle$	0	25%	$\vert\psi\rangle = \vert 0\rangle$
	1	75%	$\vert\psi\rangle = \vert 1\rangle$

Postulate 3 for a Two-state Quantum System

- Up until now, we have focused only on a device that measures the state

$$|v\rangle = \alpha|0\rangle + \beta|1\rangle$$

  of a qubit on the basis $\{|0\rangle, |1\rangle\}$,

- However, **Postulate 3** of quantum mechanics doesn't address any specific basis

Postulate 3 for a Two-state Quantum System

- **Postulate 3**: Any device (or measurement System) that measures a **two-state quantum system**, e.g. a qubit, must have **two preferred states** whose representative vectors, $\{|u\rangle, |u^{\perp}\rangle\}$ form an orthonormal basis for the associated vector space
- $\{|u\rangle, |u^{\perp}\rangle\}$ can be any basis, e.g., $\{|0\rangle, |1\rangle\}$, $\{|+\rangle, |-\rangle\}$, $\{|i\rangle, |-i\rangle\}$
- Measurement of a state transforms the state into one of the measuring device's associated basis vectors $|u\rangle$ or $|u^{\perp}\rangle$
- The probability that the state is measured as basis vector $|u\rangle$ $(|u^{\perp}\rangle)$ is the square of the magnitude of the amplitude of the component of the state in the direction of the basis vector $|u\rangle$ $(|u^{\perp}\rangle)$

Postulate 3 for a Two-state Quantum System

- For example, given a device for measuring with associated basis $\{|u\rangle, |u^{\perp}\rangle\}$, the state $|v\rangle = a|u\rangle + b|u^{\perp}\rangle$ is measured as $|u\rangle$ with probability $|a|^2$ and as $|u^{\perp}\rangle$ with probability $|b|^2$
- Measurement of a quantum state changes the state

Postulate 3 for a Two-state Quantum System

- If a state $|v\rangle = a|u\rangle + b|u^{\perp}\rangle$ is measured as $|u\rangle$ then the state $|v\rangle$ changes to $|u\rangle$
- A second measurement with respect to the same basis will return $|u\rangle$ with probability 1
- Similarly, if a state $|v\rangle = a|u\rangle + b|u^{\perp}\rangle$ is measured as $|u^{\perp}\rangle$ then the state $|v\rangle$ changes to $|u^{\perp}\rangle$
- A second measurement with respect to the same basis will return $|u^{\perp}\rangle$ with probability 1
- Thus, unless the original state happens to be one of the basis states, i.e., $|u\rangle$ or $|u^{\perp}\rangle$, a single measurement will change that state, making it impossible to determine the original state from any sequence of measurements

Measurements in Other Basis

- Measurement brings up questions as to the meaning of a *superposition*
- In fact, the notion of **superposition is basis-dependent**; all states **are superpositions** with respect to some bases and **not** with respect to others
- For instance, $a|0\rangle + b|1\rangle$ is a superposition with respect to the basis $\{|0\rangle, |1\rangle\}$, but not with respect to $\{a|0\rangle + b|1\rangle, b^*|0\rangle - a^*|1\rangle\}$

$\uparrow$

Check that this is a basis

Measurements in Other Basis

- Then, when a state $|v\rangle$ of a qubit is measured in certain bases, gives **deterministic** results, while in others it gives **random** results
- For example, a qubit in state

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

behaves deterministically when measured with respect to the Hadamard basis $\{|+\rangle, |-\rangle\}$, but it gives random results when measured with respect to the standard basis $\{|0\rangle, |1\rangle\}$

Measurements in Other Basis

- It is okay to think of a superposition $|v\rangle = a|0\rangle + b|1\rangle$ as in some sense being in both state $|0\rangle$ and state $|1\rangle$ at the same time, as long as that statement is not taken too literally
- States that are combinations of $|0\rangle$ and $|1\rangle$ in similar proportions but with different amplitudes, such as

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}}, \ \frac{|0\rangle - |1\rangle}{\sqrt{2}}, \text{ and } \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$$

represent distinct states that behave differently in many situations

Measurements in Other Basis

- The states $|0\rangle$ and $|1\rangle$ correspond to the north and south poles of the Bloch sphere respectively, and the axis passing through these points is the *z*-axis

- However, any two *opposite points* such as

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}; \ |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} \quad \text{or} \quad |i\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}; |-i\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}}$$

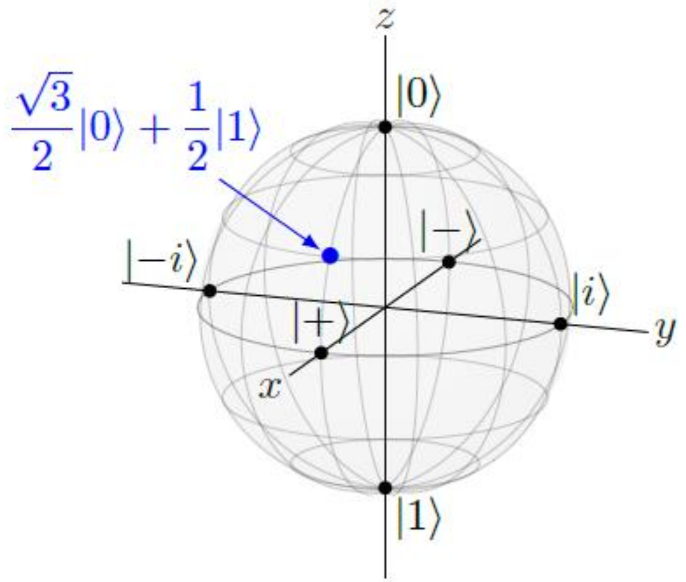
could be the *north* and *south poles*, or *any opposite points*

Measurements in Other Basis

- A set of distinct measurement outcomes is called a *basis*, and $\{|0\rangle, |1\rangle\}$ is called the *Z-basis* because they lie on the *z-axis* of the Bloch sphere

- Similarly, $\{|+\rangle, |-\rangle\}$ is called the *X-basis* because they lie on the *x-axis* of the Bloch sphere, and $\{|i\rangle, |-i\rangle\}$ is called the *Y-basis* because they lie on the *y-axis* of the Bloch sphere

- We can measure with respect to any of these bases, or with respect to any two states on opposite sides of the Bloch sphere

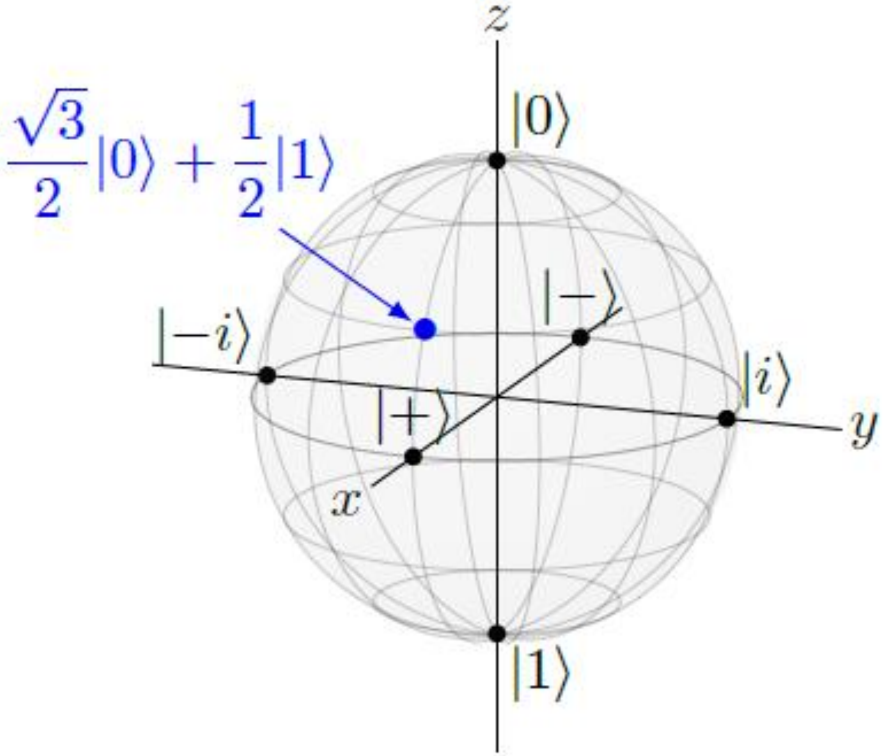


Measurements in Other Basis

- Let's consider a qubit on the state $|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$

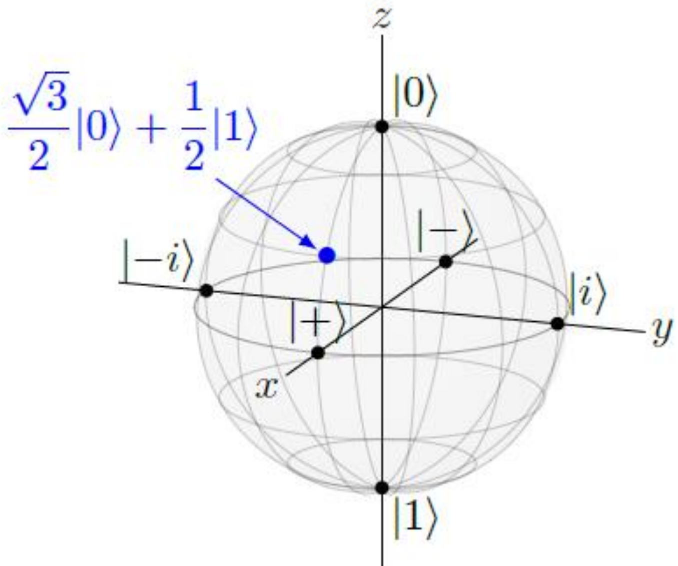
- Let us measure this with respect to four different bases
  - → the *Z*-basis $\{|0\rangle, |1\rangle\}$
  - → the *X*-basis $\{|+\rangle, |-\rangle\}$
  - → the *Y*-basis $\{|i\rangle, |-i\rangle\}$, and
  - → a fourth basis

$$\left\{|a\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle, |b\rangle = \frac{i}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle\right\}$$



Measurements in Other Basis

- If we measure the qubit in the *Z*-basis $\{|0\rangle, |1\rangle\}$, then we get $|0\rangle$ with probability 3/4 or $|1\rangle$ with probability 1/4
- What if we measure in the *X*-basis $\{|+\rangle, |-\rangle\}$ instead?
- To calculate the probabilities precisely, we need to express the state $|\psi\rangle$ in terms of $|+\rangle$ and $|-\rangle$ so that we can identify the amplitudes and then find their norm-squares



Measurements in Other Basis

- From the definitions of $|+\rangle$ and $|-\rangle$, we have

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}; \; |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

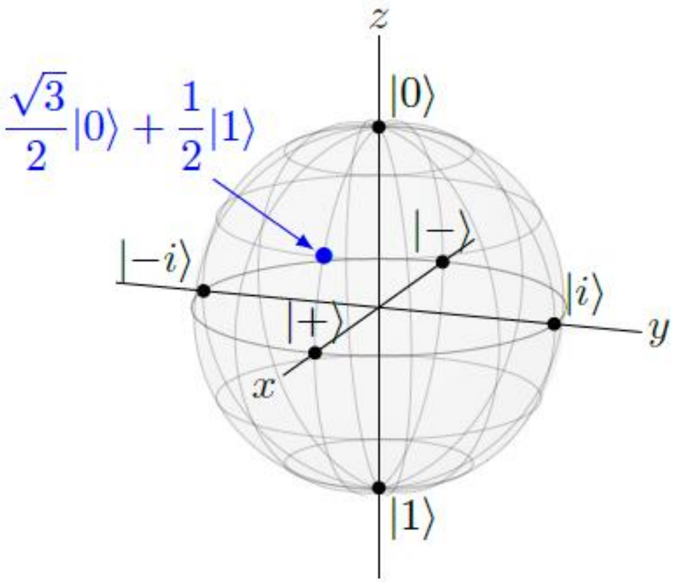
$$|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle), \quad |1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle).$$

Substituting into the state of our qubit,

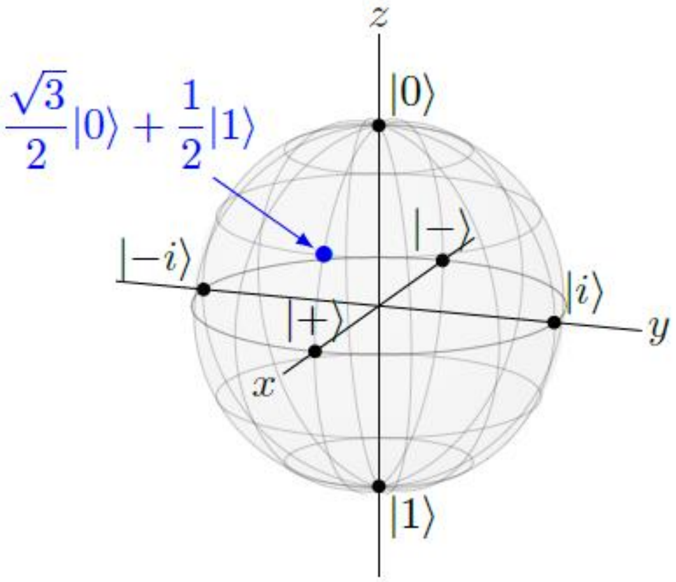
$$\begin{aligned} \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle &= \frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) + \frac{1}{2}\frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) \\ &= \frac{\sqrt{3}+1}{2\sqrt{2}}|+\rangle + \frac{\sqrt{3}-1}{2\sqrt{2}}|-\rangle. \end{aligned}$$

Now the amplitudes are easy to identify, and we can find the probabilities by taking their norm-squares. The probability of measuring $|+\rangle$ is

$$\left|\frac{\sqrt{3}+1}{2\sqrt{2}}\right|^2 = \frac{\sqrt{3}+2}{4} \approx 0.93,$$



Measurements in Other Basis



- And the probability of measuring $|-\rangle$

$$\left|\frac{\sqrt{3}-1}{2\sqrt{2}}\right|^2 = \frac{-\sqrt{3}+2}{4} \approx 0.07$$

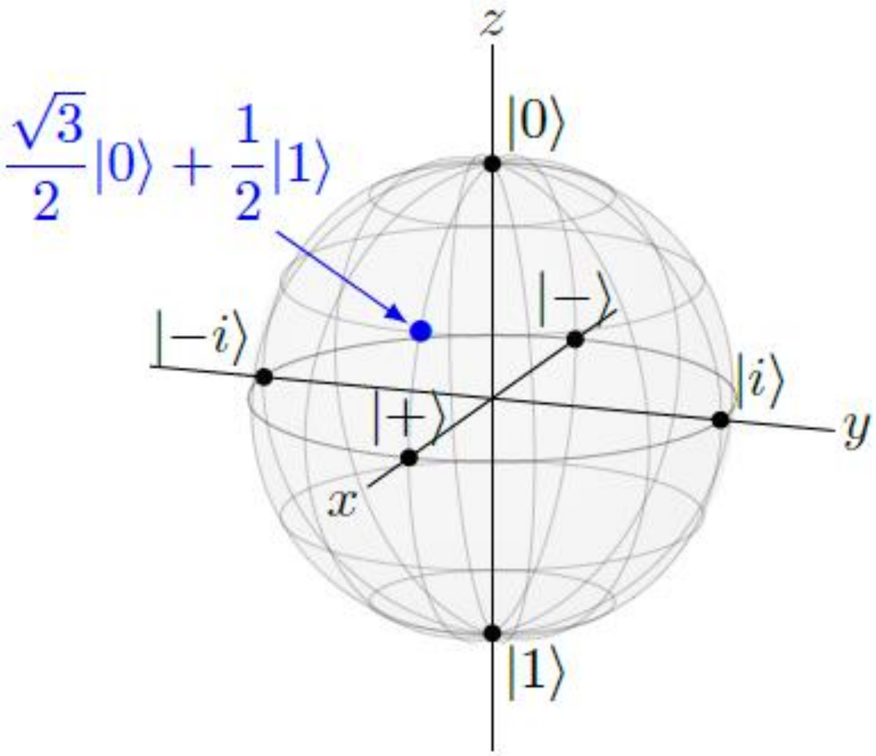
- This is consistent with the Bloch sphere, since the state is much closer to $|+\rangle$ than it is to $|-\rangle$

- Now what if we measure in the $\{|i\rangle, |-i\rangle\}$ basis?
- As we did before

$$|0\rangle = \frac{1}{\sqrt{2}}\left(|i\rangle + |-i\rangle\right), \quad |1\rangle = \frac{-i}{\sqrt{2}}\left(|i\rangle - |-i\rangle\right).$$

Substituting,

$$\begin{aligned}\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle &= \frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}}\left(|i\rangle + |-i\rangle\right) + \frac{-i}{\sqrt{2}}\left(|i\rangle - |-i\rangle\right) \\ &= \frac{\sqrt{3}-i}{2\sqrt{2}}|i\rangle + \frac{\sqrt{3}+i}{2\sqrt{2}}|-i\rangle.\end{aligned}$$



So, the probability of getting $|i\rangle$ is

$$\left|\frac{\sqrt{3}-i}{2\sqrt{2}}\right|^2 = \frac{3+1}{8} = \frac{1}{2},$$

and the probability of getting $|-i\rangle$ is

$$\left|\frac{\sqrt{3}+i}{2\sqrt{2}}\right|^2 = \frac{3+1}{8} = \frac{1}{2},$$

as expected.

Measurements in Other Basis

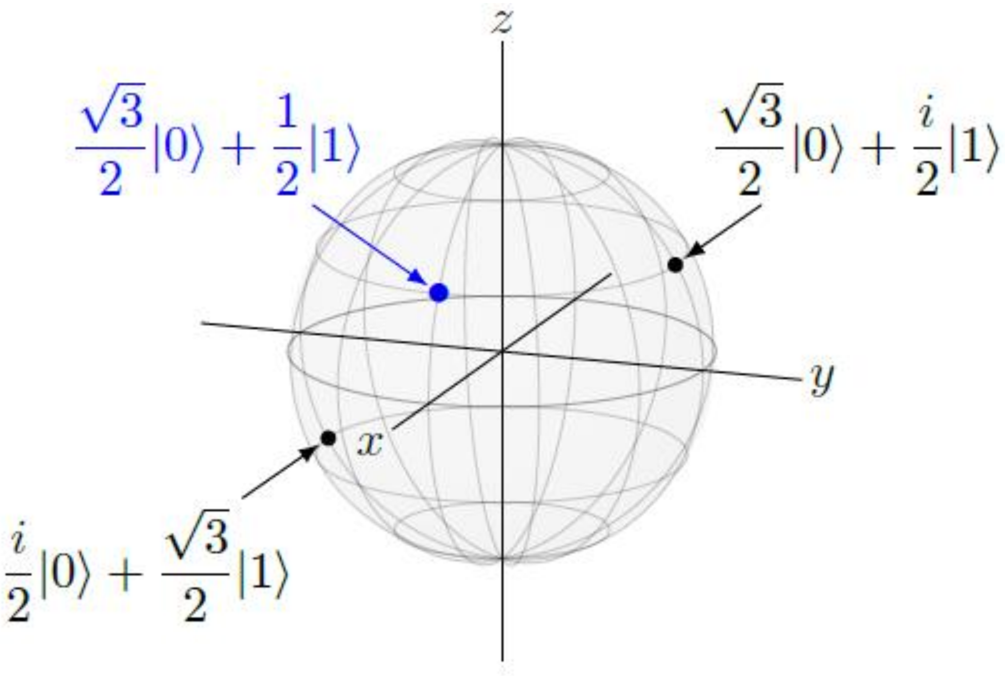
- Consider the following two states, which we will call $|a\rangle$ and $|b\rangle$

$$|a\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle \qquad |b\rangle = \frac{i}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$

Since they are located on opposite points of the Bloch sphere, they are a basis.

Let us measure in this $\{|a\rangle, |b\rangle\}$ basis.

To calculate the precise numbers, we used the approach already seen before.



$$|0\rangle = \frac{\sqrt{3}}{2}|a\rangle - \frac{i}{2}|b\rangle, \quad |1\rangle = \frac{-i}{2}|a\rangle + \frac{\sqrt{3}}{2}|b\rangle.$$

Substituting, the state of our qubit is

$$\begin{aligned}\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle &= \frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}}{2}|a\rangle - \frac{i}{2}|b\rangle\right) + \frac{1}{2}\left(\frac{-i}{2}|a\rangle + \frac{\sqrt{3}}{2}|b\rangle\right)\\ &= \frac{3-i}{4}|a\rangle + \frac{\sqrt{3}(1-i)}{4}|b\rangle.\end{aligned}$$

Taking the norm-square of each amplitude, the probability of getting $|a\rangle$ is

$$\left|\frac{3-i}{4}\right|^2 = \frac{9+1}{16} = \frac{5}{8},$$

and the probability of getting $|b\rangle$ is

$$\left|\frac{\sqrt{3}(1-i)}{4}\right|^2 = \frac{3}{8}.$$

Later, when we describe qubits in the mathematics of linear algebra, we will see another way to convert between bases.

Measurement

To sum up:

- *All measurement devices have associated bases*, and the measurement outcome is always one of the two basis vectors
- For this reason, whenever anyone says, *measure a qubit*, they must specify with respect to which basis the measurement takes place
- However, when we say, *measure a qubit*, without any further specification, we implicitly mean that the measurement is with respect to the computational basis, i.e., $\{|0\rangle, |1\rangle\}$

Single Qubit Measurement Implication

- Given that qubits can take on any **one** of **infinitely many states**, one might hope that a single qubit could store lots of classical information
- However, the properties of quantum measurement severely restrict the amount of information that can be extracted from a qubit
- Information about a quantum bit can be obtained only by measurement, and any measurement results in one of only two states, the two basis states associated with the measuring device; thus, a single measurement yields at most a single classical bit of information
- Because measurement changes the state, one cannot make two measurements on the original state of a qubit

Single Qubit Measurement Implication

- Moreover, at a later point in the course, we will show that an unknown quantum state cannot be cloned, which means it is not possible to measure a qubit's state in two ways, even indirectly by copying the qubit's state and measuring the copy
- Thus, even though a quantum bit can be in infinitely many different superposition states, it is possible to extract only a single classical bit's worth of information from a single quantum bit
- Two implications of **Postulate 3** are described below

Single Qubit Measurement Implication

Implication 1: Quantum Distinguishability

- Quantum mechanically it is not always possible to distinguish between arbitrary states
- For example, there is no process allowed by quantum mechanics that will reliably distinguish between the states $|0\rangle$ and $|0\rangle + |1\rangle \big/ \sqrt{2}$
- It's pretty easy to convince oneself that it is not possible.
- Suppose, for example, that we try to distinguish the two states by measuring in the computational basis

Single Qubit Measurement Implication

- Then, if we have been given the state $|0\rangle$, the measurement will yield 0 with probability 1
- However, when we measure $|0\rangle+|1\rangle\ /\sqrt{2}$ the measurement yields 0 with probability 1/2 and 1 with probability 1/2
- Thus, while a measurement result of 1 implies that state must have been $|0\rangle+|1\rangle\ /\sqrt{2}$, since it couldn't have been $|0\rangle$, we can't infer anything about the identity of the quantum state from a measurement result of 0

Single Qubit Measurement Implication

Implication 2

- Imagine, for example, that we translate each book of a big library into a bit string and concatenate them together
- Then the entire library is equivalent to some (very long) **binary string** that we call $\Sigma \equiv j_1 j_2 ....j_n$
- Now let's transform this string into a binary fraction $0.j_1 j_2 ....j_n$
- This represents a real number between 0 and 1 specifically

$$0 \leq \phi = j_1 2^{-1} + j_2 2^{-2} + j_3 2^{-3} + \cdots + + j_n 2^{-n} \leq 1$$

Single Qubit Measurement Implication

- Thus, in principle, we could imagine creating a single qubit state of the form

$$|\psi\rangle_{\Sigma} = |0\rangle + \exp(i\phi)|0\rangle$$

  and so, all the information in all the books in the library can be stored in this single quantum state!

- So, it may appear possible to compress information into a single qubit by an exponential factor

Single Qubit Measurement Implication

- Unfortunately, this is not possible
- To encode all the bits needed to specify the complete contents of the library would require a **physically unrealistic precision** in setting the angle ϕ
- Moreover, any single attempt to perform a measurement on $|\psi\rangle_\Sigma$, or any transformed version thereof, will only reveal at most one bit of information
- It is **neither practically possible** to **squeeze** the library into a single qubit, **nor to extract** more than one bit of information from a single qubit state

Single Qubit Measurement Implication

- But an even more interesting question to ask might be: *how much information is represented by a qubit if we do not measure it*?
- This is a trick question, because how can one quantify information if it cannot be measured?
- Nevertheless, there is something conceptually important here, because when Nature evolves a closed quantum system of qubits, not performing any 'measurements', she does keep track of all the continuous variables describing the state, like α and β