Name: **Surname:** **Roll number:**

1) Consider the unconstrained optimization problem

$$\begin{cases} \min \dfrac{e^{x_1^2-2x_2}}{e^{-x_2^2+x_1x_2}} \\ x \in \mathbb{R}^2 \end{cases}$$

(a) Prove that the problem admits a global minimum;

(b) Apply the gradient method with an inexact line search, setting $\bar{t} = 1$, $\alpha = 0.1$, $\gamma = 0.8$, with starting point $x^0 = (2, 5)$ and using $\|\nabla f(x)\| < 10^{-5}$ as stopping criterion. How many iterations are needed by the algorithm? Write explicitly the vectors found at the last three iterations.

(c) Is the obtained solution a global minimum of the given problem? Justify the answer.

SOLUTION

(a) Note that the objective function can be written as $f(x) = e^{h(x)}$, where $h(x) = x_1^2 + x_2^2 - x_1x_2 - 2x_2$. Since $h(x)$ is strongly convex and $\psi(y) = e^y$ is convex and increasing, then $f = \psi \circ h$ is convex and coercive . Consequently, f admits a global minimum point.

(b) We notice that

$$\nabla f(x_1, x_2) = \begin{pmatrix} e^{x_1^2+x_2^2-x_1x_2-2x_2}(2x_1 - x_2) \\ e^{x_1^2+x_2^2-x_1x_2-2x_2}(2x_2 - x_1 - 2) \end{pmatrix}$$

Matlab solution

```
%% Data

alpha = 0.1;
gamma = 0.8;
tbar = 1;
x0 = [2;5];
tolerance = 10^(-5) ;

X=[ ];

x = x0;

for ITER=1:100
    [v, g] = f(x);
  X=[X;ITER,x(1),x(2),v,norm(g)];
    % stopping criterion
    if norm(g) < tolerance
        break
    end

    % search direction
    d = -g;

    % Armijo inexact line search
    t = tbar ;
    while (f(x+t*d) > v + alpha*g'*d*t)
        t = gamma*t ;
    end

    % new point
    x = x + t*d;

end

x
v
norm(g)

function [v, g] = f(x)

v = exp(x(1)^2+x(2)^2-x(1)*x(2)-2*x(2))  ;
```

```
g = [exp(x(1)^2+x(2)^2-x(1)*x(2)-2*x(2))*(2*x(1)-x(2));
exp(x(1)^2+x(2)^2-x(1)*x(2)-2*x(2))*(2*x(2)-x(1)-2)];

end
```

We obtain the following solution:

```
x =

    0.6667
    1.3334


v =

    0.2636


ans =

   7.8948e-06


ITER =

    34
```

In particular, the gradient norm evaluated at the final point is:

```
ans =

   7.8948e-06
```

The effective iterations of the algorithm are 33.

The vectors found at the last three iterations are:

```
0.6667    1.3334
0.6667    1.3334
0.6667    1.3334
```

(c) The found point $x = (0.6667, 1.3334)$ is a good approximation of the global minimum since the norm of gradient of the objective function is close to zero and the objective function is convex and coercive as shown in point (a). Notice that the gradient is null for $x = (2/3, 4/3)$.

2) Consider a regression problem with the following data set where the points $(x_i, y_i), i = 1, .., 28$, are given by the row vectors of the matrices:

$$\begin{pmatrix} -3.0000 & 6 \\ -2.8000 & 8 \\ -2.6000 & 8.5 \\ -2.0000 & 17 \\ -1.8000 & 11.48 \\ -1.6000 & 14.10 \\ -1.4000 & 16.82 \\ -1.2000 & 16.15 \\ -1.0000 & 11.68 \\ -0.8000 & 6.00 \\ -0.6000 & 7.82 \\ -0.4000 & 2.82 \\ -0.2000 & 2.71 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0.2000 & -1 \\ 0.4000 & -3.84 \\ 0.6000 & -4.71 \\ 1.0000 & -7.33 \\ 1.2000 & -13.64 \\ 1.4000 & -15.26 \\ 1.6000 & -14.87 \\ 1.8000 & -9.92 \\ 2.0000 & -10.50 \\ 2.2000 & -7.72 \\ 2.4000 & -12.5 \\ 2.6000 & -10.26 \\ 2.8000 & -7 \\ 3.0000 & -2 \end{pmatrix}$$

(a) Write the dual formulation of a nonlinear ε-SV regression model with $C = 8$, $\varepsilon = 3.5$ and a Gaussian kernel $k(x, y) := e^{-\|x-y\|^2}$;

(b) Solve the problem in (a) and find the regression function;

(c) Find the support vectors;

(d) Find the points of the data set that are outside the ε-tube, by making use of the dual solution.

SOLUTION

(a) Let $\ell = 28$, (x_i, y_i), $i = 1, .., \ell$ be the i-th element of the data set, $C = 8$, $\varepsilon = 3.5$, $k(x, y) := e^{-\|x-y\|^2}$. The dual formulation of a nonlinear ε-SV regression model is

$$\begin{cases} \max\limits_{(\lambda^+,\lambda^-)} \; -\dfrac{1}{2}\sum\limits_{i=1}^{28}\sum\limits_{j=1}^{28}(\lambda_i^+ - \lambda_i^-)(\lambda_j^+ - \lambda_j^-)e^{-\|x_i-x_j\|^2} \\ \qquad\qquad -3.5\sum\limits_{i=1}^{28}(\lambda_i^+ + \lambda_i^-) + \sum\limits_{i=1}^{28} y_i(\lambda_i^+ - \lambda_i^-) \\ \sum\limits_{i=1}^{28}(\lambda_i^+ - \lambda_i^-) = 0 \\ \lambda_i^+, \lambda_i^- \in [0, 8], \; i = 1, .., 28 \end{cases} \tag{1}$$

(b) **Matlab solution**

```
data = [
   -3.0000    6
   -2.8000    8
   -2.6000    8.5
   -2.0000   17
   -1.8000   11.48
   -1.6000   14.10
   -1.4000   16.82
   -1.2000   16.15
   -1.0000   11.68
   -0.8000    6.00
   -0.6000    7.82
   -0.4000    2.82
   -0.2000    2.71
         0       1
    0.2000   -1
    0.4000   -3.84
    0.6000   -4.71
    1.0000   -7.33
    1.2000  -13.64
    1.4000  -15.26
    1.6000  -14.87
    1.8000   -9.92
    2.0000  -10.50
    2.2000   -7.72
    2.4000  -12.5
    2.6000  -10.26
    2.8000   -7
    3.0000   -2
       ];
```

```
x = data(:,1) ;
y = data(:,2) ;
l = length(x) ;

epsilon = 3.5 ;
C = 8;

X = zeros(l,l);
for i = 1 : l
    for j = 1 : l
        X(i,j) = kernel(x(i),x(j)) ;
    end
end
Q = [ X -X ; -X X ];
c = epsilon*ones(2*l,1) + [-y;y];

sol = quadprog(Q,c,[],[],[ones(1,l) -ones(1,l)],0,zeros(2*l,1),C*ones(2*l,1));
lap = sol(1:l);
lam = sol(l+1:2*l);

ind = find(lap > 1e-3 & lap < C-1e-3);              % compute b
if isempty(ind)==0
    i = ind(1);
    b = y(i) - epsilon;
    for j = 1 : l
        b = b - (lap(j)-lam(j))*kernel(x(i),x(j));
    end
else
    ind = find(lam > 1e-3 & lam < C-1e-3);
    i = ind(1);
    b = y(i) + epsilon ;
    for j = 1 : l
        b = b - (lap(j)-lam(j))*kernel(x(i),x(j));
    end
end


z = zeros(l,1);                                              % find regression and epsilon-tube
for i = 1 : l
    z(i) = b ;
    for j = 1 : l
        z(i) = z(i) + (lap(j)-lam(j))*kernel(x(i),x(j));
    end
end
zp = z + epsilon ;
zm = z - epsilon ;

sv = [find(lap > 1e-3);find(lam > 1e-3)];                     % find support vectors
sv = sort(sv);

 plot(x,y,'b.',x(sv),y(sv),'ro',x,z,'k-',x,zp,'r-',x,zm,'r-');     % plot the solution

    disp('Support vectors')

[sv,x(sv),y(sv),lam(sv),lap(sv)]    % Indexes of support vectors, support vectors, lambda_-,lambda_+

function v = kernel(x,y)

v = exp(-norm(x-y)^2)

end
```

Let λ_- and λ_+ be the vectors given by the Matlab solutions lam, lap. In particular we find, $b = 0.5905$.

The regression function is:

$$f(x) = \sum_{i=1}^{28} (\lambda_i^+ - \lambda_i^-) k(x_i, x) + b = \sum_{i=1}^{28} (\lambda_i^+ - \lambda_i^-) e^{-\|x_i - x\|^2} + 0.5905$$

(c) We obtain the support vectors (columns 2-3) and corresponding λ^- and λ^+ (columns 4-5) :

```
ans =

    4.0000   -2.0000   17.0000    0.0000    8.0000
    8.0000   -1.2000   16.1500    0.0000    7.8555
   19.0000    1.2000  -13.6400    1.5831    0.0000
   20.0000    1.4000  -15.2600    8.0000    0.0000
   25.0000    2.4000  -12.5000    6.2724    0.0000
```

(d) Consider the feasibility condition of the primal formulation of the regression problem:

$$y_i - f(x_i) - \varepsilon - \xi_i^+ \le 0, \quad y_i - f(x_i) + \varepsilon + \xi_i^- \ge 0, \quad i = 1, .., \ell = 28$$

If a point (x_i, y_i) is outside the ε-tube then $\xi_i^+ > 0$ or $\xi_i^- > 0$.

Given the dual optimal solution (λ^+, λ^-) of (1), we can find the errors ξ_i^+ and ξ_i^- associated with the point (x_i, y_i) by the complementarity conditions:

$$\begin{cases} \lambda_i^+ \left[y_i - f(x_i) - \varepsilon - \xi_i^+\right] = 0, & i = 1, .., \ell \\ \lambda_i^- \left[y_i - f(x_i) + \varepsilon + \xi_i^-\right] = 0, & i = 1, .., \ell \\ \xi_i^+ (C - \lambda_i^+) = 0, & i = 1, .., \ell \\ \xi_i^- (C - \lambda_i^-) = 0, & i = 1, .., \ell \end{cases} \tag{2}$$

it follows that a necessary condition for a point (x_i, y_i) to be outside the ε-tube is that $\lambda_i^+ = C = 8$ or $\lambda_i^- = C = 8$. We find that $\lambda_i^- = 8$, for $i = 20$, $\lambda_i^+ = 8$, for $i = 4$ which correspond to the points

$$(x_4, y_4) = (-2, 17), \quad (x_{20}, y_{20}) = (1.4, -15.26).$$

3) Consider the following multiobjective optimization problem (P):

$$\begin{cases} \min\ (x_1 + x_2 - x_3,\ x_1 + x_2) \\ x_1 + x_2 + x_3 \le 4 \\ -x_1 - x_2 \le 0 \\ \quad -x_2 \le 2 \\ (x_1, x_2, x_3) \in \mathbb{R}^3 \end{cases}$$

(a) Prove that the problem admits a Pareto minimum point.

(b) Find the set of all weak Pareto minima.

(c) Find a suitable subset of Pareto minima.

(d) Does the problem admit any ideal minimum?

SOLUTION

We preliminarly observe that the problem is linear, since the objective and the constraint functions are linear. Therefore the set of weak minima coincides with the set of solutions of the scalarized problems (P_{α_1}), i.e.

$$\begin{cases} \min\ \psi_{\alpha_1}(x) := \alpha_1(x_1 + x_2 - x_3) + (1 - \alpha_1)(x_1 + x_2) = x_1 + x_2 - \alpha_1 x_3 \\ x_1 + x_2 + x_3 \le 4 \\ -x_1 - x_2 \le 0 \\ \quad -x_2 \le 2 \\ (x_1, x_2, x_3) \in \mathbb{R}^3 \end{cases}$$

where $0 \le \alpha_1 \le 1$, while the set of minima coincides with the set of solutions of the scalarized problems (P_{α_1}), where $0 < \alpha_1 < 1$.

(a) Let X be the feasible set of (P). By summing up the first two inequality constraints, we obtain that $x_3 \le 4$, or $-x_3 \ge -4$. Note that

$$\psi_{\alpha_1}(x) = x_1 + x_2 - \alpha_1 x_3 \ge 0 - 4\alpha_1 \ge -4, \quad \forall x \in X, \quad \forall \alpha_1 \in [0, 1]$$

Therefore P_{α_1} admits finite optimum, for every $\alpha_1 \in (0, 1)$ and the related optimal solutions are Pareto minima for the given problem.

(b) - (c) By solving P_{α_1} with Matlab we obtain:

```
C = [1 1 -1;  1 1 0 ] ;

A =[ 1  1 1;   -1 -1 0; 0 -1 0 ];

 b = [4 0 2]';

 % solve the scalarized problem with 0 =< alfa =< 1

MINIMA=[Inf,Inf,Inf, Inf];

lambda=[Inf,Inf,Inf,Inf];

 for alfa = 0 : 0.01 : 1
 [x,fval,exitflag,output,Lambda] = linprog(alfa*C(1,:)+(1-alfa)*C(2,:),A,b) ;

      MINIMA=[MINIMA; alfa x'];
     lambda=[lambda;alfa,Lambda.ineqlin'];

 end
```

We obtain

$$x(\alpha_1) = (2, -2, 4) \quad \lambda(\alpha_1) = (\alpha_1, 1+\alpha_1, 0), \quad for\ 0 \le \alpha_1 \le 1,$$

Since the problem is linear then the KKT conditions provide a necessary and sufficient condition for an optimal solution of (P_{α_1}):

$$\begin{cases} 1+\lambda_1-\lambda_2 = 0 \\ 1+\lambda_1-\lambda_2-\lambda_3 = 0 \\ -\alpha_1+\lambda_1 = 0 \\ \lambda_1(x_1+x_2+x_3-4) = 0 \\ \lambda_2(-x_1-x_2) = 0 \\ \quad \lambda_3(-x_2-2) = 0 \\ x_1+x_2+x_3 \le 4 \\ -x_1-x_2 \le 0 \\ \quad -x_2 \le 2 \\ \lambda \ge 0 \\ 0 \le \alpha_1 \le 1, \end{cases}$$

Note that by the first three equations, we obtain

$$\lambda(\alpha_1) = (\alpha_1, 1+\alpha_1, 0), \quad for\ 0 \le \alpha_1 \le 1,$$

as previously found. Exploiting the complementarity conditions, we obtain the following solutions:

(i) For $0 < \alpha_1 \le 1$, the set of optimal solutions of $P(\alpha_1)$ is given by the system

$$\begin{cases} x_1+x_2+x_3 = 4 \\ -x_1-x_2 = 0 \\ \quad -x_2 \le 2 \\ (x_1,x_2,x_3) \in \mathbb{R}^3 \end{cases}$$

(ii) For $\alpha_1 = 0$ the set of optimal solutions of P_0 is given by the following system

$$\begin{cases} x_1+x_2+x_3 \le 4 \\ -x_1-x_2 = 0 \\ \quad -x_2 \le 2 \\ (x_1,x_2,x_3) \in \mathbb{R}^3 \end{cases}$$

Then, $Weak\ Min(P) = \{(x_1,x_2,x_3) : x_1+x_2 = 0,\ x_2 \ge -2, x_3 \le 4\}$,

$$Min(P) = \{(x_1,x_2,x_3) : x_1+x_2 = 0,\ x_2 \ge -2, x_3 = 4\}.$$

(d) Since $Min(P)$ are optimal solutions of P_0 and P_1 simultaneously, then all the points in $Min(P)$ are ideal minima, indeed they minimize both the objective functions.

4) Consider the following bimatrix game:

$$C_1 = \begin{pmatrix} 4 & 3 & 2 \\ -1 & 4 & 3 \\ 2 & 5 & -1 \end{pmatrix} \qquad C_2 = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 3 & 2 \\ 0 & 2 & -1 \end{pmatrix}$$

(a) Find the strictly dominated strategies, if any, and reduce the cost matrices accordingly.

(b) Find the set of pure strategies Nash equilibria, if any. Alternatively, show that no pure strategies Nash equilibrium exists.

(c) Find a mixed strategies Nash equilibrium.

SOLUTION

(a) Strategy 1 of Player 2 is dominated by Strategy 3, so that column 1 in the two matrices can be deleted. Now, in the reduced matrix of player 1, Strategy 2 of Player 1 is dominated by Strategy 1 and row 2 in the two matrices can be deleted. The reduced game is given by the matrices

$$C_1^R = \begin{pmatrix} 3 & 2 \\ 5 & -1 \end{pmatrix} \quad C_2^R = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$$

The minima on the columns of C_1^R are the elements of the principal diagonal, and the corresponding elements on the principal diagonal of C_2^R are minima on the rows of C_2^R, which implies that the related couples of strategies, namely (1,2) and (3,3), are pure strategies Nash equilibria.

This will also be shown in part (b) in the wider context of mixed strategies Nash equilibria.

(b) Consider the reduced game obtained in (a). The optimization problem associated with Player 1 is

$$\begin{cases} \min\ x^T C_1^R y = (3x_1 + 5x_3)y_2 + (2x_1 - x_3)y_3 \\ x_1 + x_3 = 1 \\ x_1, x_3 \geq 0 \end{cases} \equiv \begin{cases} \min\ (3 - 5y_2)x_1 + 6y_2 - 1 \\ 0 \leq x_1 \leq 1 \end{cases} \qquad (P_1(y_2))$$

where, we have eliminated the variables x_3 and y_3, since $x_3 = 1 - x_1$ and $y_3 = 1 - y_2$,taking into account that $x_2 = 0, y_1 = 0$. Then, the best response mapping associated with $P_1(y_2)$ is:

$$B_1(y_2) = \begin{cases} 0 & \text{if } y_3 \in [0, 3/5) \\ [0,1] & \text{if } y_3 = 3/5 \\ 1 & \text{if } y_3 \in (3/5, 1) \end{cases}$$

Similarly, the optimization problem associated with Player 2 is

$$\begin{cases} \min\ x^T C_2^R y = (-x_1 + 2x_3)y_2 + (x_1 - x_3)y_3 \\ y_2 + y_3 = 1 \\ y_2, y_3 \geq 0 \end{cases} \equiv \begin{cases} \min\ (-5x_1 + 3)y_2 + 2x_1 - 1 \\ 0 \leq y_2 \leq 1 \end{cases} \qquad (P_2(x_1))$$

Then, the best response mapping associated with $P_2(x_1)$ is:

$$B_2(x_1) = \begin{cases} 0 & \text{if } x_1 \in [0, 3/5) \\ [0,1] & \text{if } x_1 = 3/5 \\ 1 & \text{if } x_2 \in (3/5, 1] \end{cases}$$

The couples (x_1, y_2) such that $x_1 \in B_1(y_2)$ and $y_2 \in B_2(x_1)$ are

1. $x_1 = 0, y_2 = 0$,
2. $x_1 = 1, y_2 = 1$,
3. $x_1 = \frac{3}{5}, y_2 = \frac{3}{5}$,

so that, recalling that $x_2 = 0, y_1 = 0$,

- $(x_1, x_2, x_3) = (0, 0, 1), \quad (y_1, y_2, y_3) = (0, 0, 1)$, is a pure strategies Nash equilibrium.
- $(x_1, x_2, x_3) = (1, 0, 0), \quad (y_1, y_2, y_3) = (0, 1, 0)$, is a pure strategies Nash equilibrium.
- $(x_1, x_2, x_3) = (\frac{3}{5}, 0, \frac{2}{5}), \quad (y_1, y_2, y_3) = (0, \frac{3}{5}, \frac{2}{5})$, is a mixed strategies Nash equilibrium.