Università di Pisa

Data Mining and Machine Learning

Bioinspired computational methods

Biological data mining

Clustering Graphs and Network Data

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Some slides belong to the collection

Jiawei Han, Micheline Kamber, and Jian Pei
University of Illinois at Urbana-Champaign
Simon Fraser University

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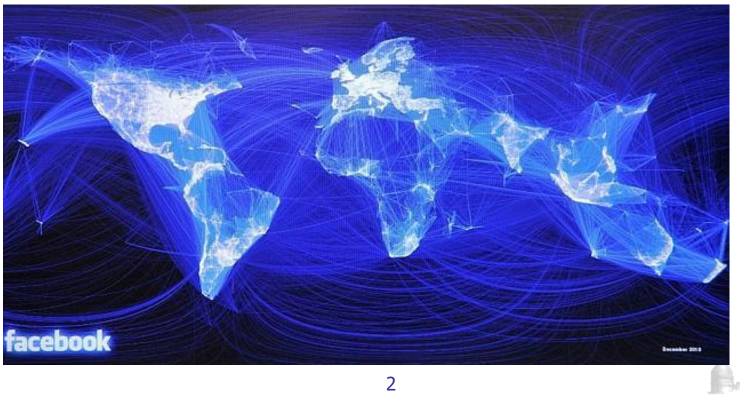
What is a network?

- Network: a collection of entities that are interconnected with links
  - Social networks
    - Entities: People
    - Links: Friendships



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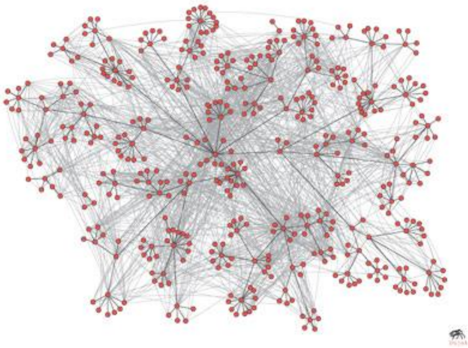
What is a network?

- Communication networks
  - Entities: People
  - Links: e-mail exchange



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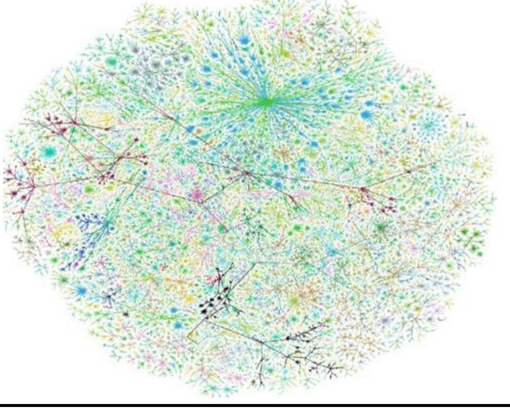
What is a network?

- Communication networks
  - Entities: Internet nodes
  - Links: Communication between nodes



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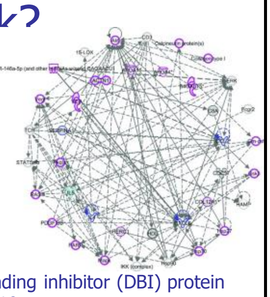
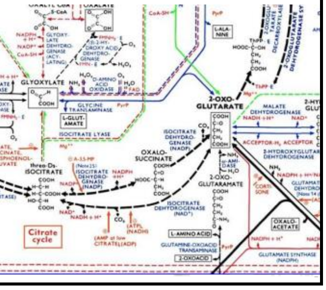
What is a network?



- Biological networks
  - Entities: Proteins
  - Links: Interactions

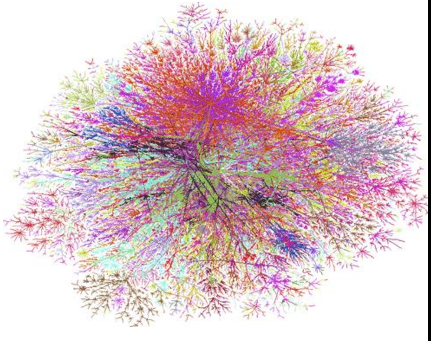
Phenotypic subgrouping and multi-omics analyses reveal reduced diazepam-binding inhibitor (DBI) protein levels in autism spectrum disorder with severe language impairment, March 2019
PLoS ONE 14(3):e0214198

- Entities: metabolites, enzymes
- Links: chemical reactions



What is a network?

- Information/Media networks
  - Entities: Web pages
  - Links: Links



- Entities: Twitter users
- Links: Follows/conversations

- Many other examples

Why networks are imporant?

- We cannot truly understand a complex system unless we understand the underlying network.
  - Everything is connected, studying individual entities gives only a partial view of a system
- Two main themes:
  - What are the structural properties of the network?
  - How do processes happen in the network?

Graphs and Networks

- In mathematics, networks are called graphs, the entities are nodes, and the links are edges
- Graph theory starts in the 18th century, with Leonhard Euler
- Graphs have been used in the past to model existing networks (e.g., networks of highways, social networks)
  - usually these networks were small
  - visual inspection can reveal a lot of information

Networks now

- More and larger networks appear
  - Products of technology
    - e.g., Internet, Web, Facebook, Twitter
  - Result of our ability to collect more, better, and more complex data
    - e.g., gene regulatory networks
  - Result of the willingness of users to contribute data
    - e.g., users making their relationships public online
- Networks of thousands, millions, or billions of nodes
  - Impossible to process visually
  - Problems become harder
  - Processes are more complex

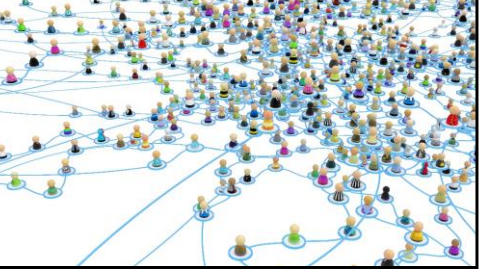
Current problems

- **Ranking of nodes on the web?**
  - Is my home page as important as the Google page?
  - We need algorithms to compute the importance of nodes in a graph
  - For instance, the PageRank algorithm in Google

  - Theoretically, it is impossible to develop a web search engine without understanding the web graph

Current problems

- **Information/Virus Cascade?**
  - How do viruses spread between individuals? How can we stop them?
  - How does information propagate in social and information networks? What items become viral? Who are the influencers and trend-setters?
  - We need models and algorithms to answer these questions



Current problems

- **Link Prediction**
  - Given a snapshot of a social network at time t, we seek to accurately predict the edges that will be added to the network during the interval from time t to a given future time t'.
  - Applications
    - Accelerate the growth of a social network (e.g., Facebook, LinkedIn, Twitter) that would otherwise take longer to form.
    - Identify suspect relationships

Current problems

- **Network content**
  - Users on online social networks generate content.
  - Mining the content in conjunction with the network can be useful
    - Do friends post similar content on Facebook?
    - Can we understand a user's interests by looking at those of their friends?
    - Social recommendations: Can we predict a movie rating using the social network?

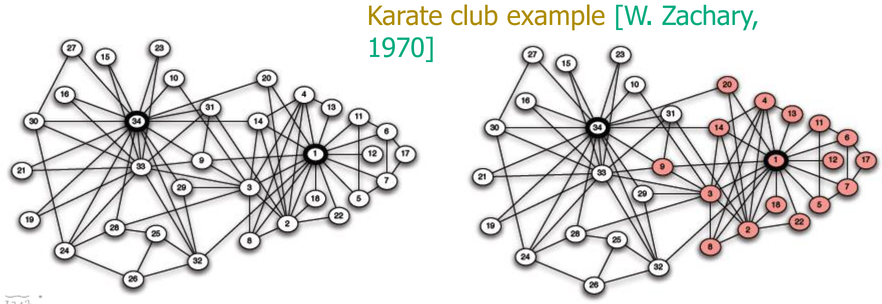
Current problems

- **Social Media**
  - Today Social Media (Twitter, Facebook, Instagram) have supplanted the traditional media sources
    - Information is generated and disseminated mostly online by users
      - E.g., the assassination of Bin Laden appeared first on Twitter
    - Twitter has become a global "sensor" detecting and reporting everything
  - Interesting problems:
    - Automatically detect events using Twitter
      - Earthquake news propagation
      - Crisis detection and management
    - Sentiment mining
    - Track the evolution of events: socially, geographically, over time.

Current problems

- **Clustering and Finding Communities?**
  - A community: "Cohesive subgroups are subsets of actors among whom there are relatively strong, direct, intense, frequent, or positive ties." [Wasserman & Faust '97]

Karate club example [W. Zachary, 1970]

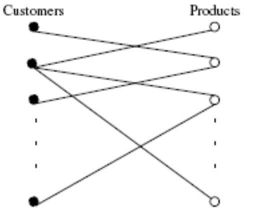


Current problems

- **Community Evolution**
  - Homophily:"Birds of a feather flock together"
  - Caused by two related social forces [Friedkin98, Lazarsfeld54]
    - Social influence: People become similar to those they interact with
    - Selection: People seek out similar people to interact with
  - Both processes contribute to homophily, but
    - Social influence leads to community-wide homogeneity
    - Selection leads to fragmentation of the community
  - Applications in online marketing
    - viral marketing relies upon social influence affecting behavior
    - recommender systems predict behavior based on similarity

Clustering Graphs and Network Data

- Applications
  - Bi-partite graphs, e.g., customers and products, authors and conferences
    - Clustering customers buying similar products
    - Identify customers out of the clusters



Clustering Graphs and Network Data

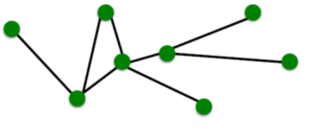
- Applications
  - Web search engines, e.g., click through graphs and Web graphs
    - Click-through information
      - An edge links a query to a web page if a user clicks the web page when asking the query.
      - Valuable information can be obtained by cluster analyses on the query–web page bipartite graph.
    - web graph: each web page is a vertex, and each hyperlink is an edge pointing from a source page to a destination page.

Clustering Graphs and Network Data

- Applications
  - Social networks, friendship/coauthor graphs
    - the vertices are individuals or organizations, and the links are interdependencies between the vertices, representing friendship, common interests, or collaborative activities
    - For instance, customers of a company form a social network, where each customer is a vertex, and an edge links two customers if they know each other.
    - Customers within a cluster may influence one another regarding purchase decision making.
    - As another example, the authors of scientific publications form a social network.
    - The **network** is, in general, a **weighted graph** because an edge between two authors can carry a weight representing the strength of the collaboration such as how many publications the two authors (as the end vertices) coauthored.

Basics on a Network

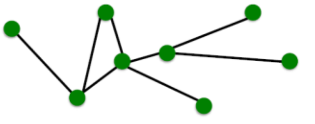
- Objects: nodes, vertices **N**
- Interactions: links, edges **E**
- System: network, graph **G(N,E)**



- **Network often refers to real systems**
  - Web, Social network, Metabolic network
  - **Language: Network, node, link**
- **Graph is a mathematical representation of a network**
  - Web graph, Social graph, Knowledge Graph
  - **Language: Graph, vertex, edge**

Basics on a Network

- How to build a graph:
  - What are nodes?
  - What are edges?



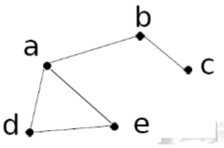
- **Choice of the proper network representation of a given domain/problem determines our ability to use networks successfully:**
  - In some cases there is a unique, unambiguous representation
  - In other cases, the representation is by no means unique
  - The way you assign links will determine the nature of the question you can study

Clustering Graphs and Network Data

- We can apply standard clustering algorithms by introducing a specific definition of similarity measures
  - Geodesic distances
  - Distance based on random walk (SimRank)
- Graph clustering methods
  - Minimum cuts: FastModularity (Clauset, Newman & Moore, 2004)
  - Density-based clustering: SCAN (Xu et al., KDD'2007)

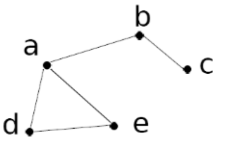
Similarity Measure (I) Geodesic Distance

- Distance between two vertices in a graph: the shortest path between the vertices:
  - Geodesic distance (A, B): length (i.e., # of edges) of the shortest path between A and B (if not connected, defined as infinite)
- Eccentricity of v, eccen(v): The largest geodesic distance between v and any other vertex $u \in V - \{v\}$.
  E.g., eccen(a) = eccen(b) = 2; eccen(c) = eccen(d) = eccen(e) = 3
- Radius of graph G: The minimum eccentricity of all vertices, i.e., the distance between the "most central point" and the "farthest border"
  $r = \min_{v \in V} \text{eccen}(v)$
  E.g., radius (g) = 2



Similarity Measure (II) Geodesic Distance

- Diameter of graph G: The maximum eccentricity of all vertices, i.e., the largest distance between any pair of vertices in G
  $d = \max_{v \in V} \text{eccen}(v)$
  E.g., diameter (g) = 3
- A peripheral vertex is a vertex that achieves the diameter.
  E.g., Vertices c, d, and e are peripheral vertices



Similarity Measure (III) Geodesic Distance

- Let us consider the similarity between two vertices in a customer social network.
- How well can geodesic distance measure similarity and closeness in a network?
  - Suppose that Ada and Bob are two customers in the network
  - The geodesic distance (i.e., the length of the shortest path between Ada and Bob) is the shortest path that a message can be passed from Ada to Bob and vice versa.
  - Is this information useful?
    - Typically, the company is not interested in how a message is passed from Ada to Bob.
  - We need to define what does similarity mean in a social network

Similarity Measure (IV) Similarity in a social network

Two different meanings

- Structural context-based similarity
  - Two customers are considered similar to one another if they have similar neighbors in the social network.
  - two people receiving recommendations from a good number of common friends often make similar decisions: intuitive!
- Similarity based on random walk
  - the company sends promotional information to both Ada and Bob in the social network.
  - Ada and Bob may randomly forward such information to their friends (or neighbors) in the network.
  - The closeness between Ada and Bob can then be measured by the likelihood that other customers simultaneously receive the promotional information that was originally sent to Ada and Bob.

SimRank: Similarity Based on Random Walk and Structural Context

- SimRank: structural-context similarity, i.e., based on the similarity of its neighbors

- In a directed graph G = (V,E),
  - individual in-neighborhood of v: $I(v) = \{u \mid (u, v) \in E\}$
  - individual out-neighborhood of v: $O(v) = \{w \mid (v, w) \in E\}$

- Similarity in SimRank:

$$s(u, v) = \frac{C}{|I(u)||I(v)|} \sum_{x \in I(u)} \sum_{y \in I(v)} s(x, y)$$

  where C is a constant between 0 and 1.
- If a vertex does not have any neighbor, we define s(u,v) = 0

SimRank: Similarity Based on Random Walk and Structural Context

- How can compute SimRank?
  - Iteratively compute the previous equation until a fixed point is reached.

  - Let n be the number of nodes in graph G.
  - For each iteration i we can keep n^2 entries $s_i(*,*)$, where $s_i(u,v)$ gives the score between u and v on iteration i.
  - We start with $s_0(*,*)$ where each $s_0(u,v)$ is a lower bound on the actual SimRank score $s(u,v)$:

$$s_0(u, v) = \begin{cases} 0 & \text{if } u \neq v \\ 1 & \text{if } u = v. \end{cases}$$

SimRank: Similarity Based on Random Walk and Structural Context

- To compute $s_{i+1}(u,v)$ from $s_i(*,*)$ we use

$$s_{i+1}(u,v) = \frac{C}{|I(u)||I(v)|} \sum_{x \in I(u)} \sum_{y \in I(v)} s_i(x,y) \quad \text{if } u \neq v$$

$$s_{i+1}(u,v) = 1 \quad \text{if } u = v.$$

The values $s_i(*,*)$ are non-decreasing as i increases.

Complexity: $O(Kn^2d_2)$ where d_2 is the average of $|I(u)||I(v)|$

K is the number of iterations and typically is equal to 5

SimRank: Similarity Based on Random Walk and Structural Context

- Similarity based on random walk: in a strongly connected graph a path exists between every two nodes).
- Expected distance from u to v:

$$d(u,v) = \sum_{t: u \rightsquigarrow v} P[t]\, l(t)$$

- The sum is computed over all tours t which start at u and end at v, and do not touch v except at the end.
- For a tour $t = \langle w_1, \ldots, w_k \rangle$ the length $l(t)$ of t is k-1.
- The probability $P(t)$ of travelling t is

$$P[t] = \begin{cases} \prod_{i=1}^{k-1} \frac{1}{\boxed{|O(w_i)|}} & \text{if } l(t) > 0 \\ 0 & \text{if } l(t) = 0 \end{cases}$$

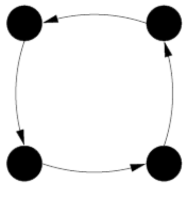
Out-neighbors of w_i

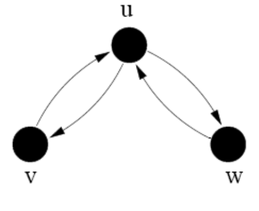
SimRank: Similarity Based on Random Walk and Structural Context

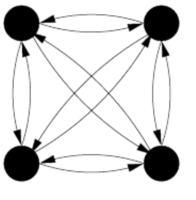
- Note that the case where $u = v$, for which $d(u,v) = 0$ is a special case of the formula of the distance: only one tour is in the summation and it has length 0.
- The expected distance from u to v is exactly the expected number of steps a random surfer, who at each step follows a random out-edge, would take before he first reaches v, starting from u.
- Expected meeting distance (EMD): the expected meeting distance $m(u,v)$ between u and v is the expected number of steps required before two surfers, one starting at u and the other at v, would meet if they walked (randomly) in lock-step.
- The EMD is symmetric by definition

SimRank: Similarity Based on Random Walk and Structural Context

- Some examples of EMD







EMD between two distinct nodes is infinite

$m(u,v) = m(u,w) = \infty$
$m(v,w) = 1$

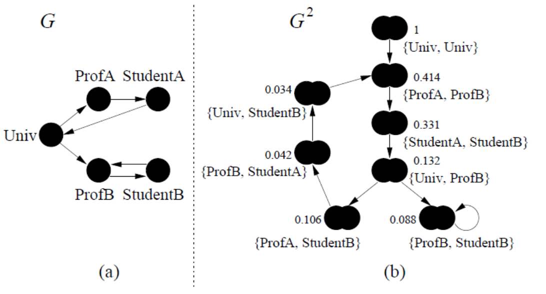
$EMD = 3$

SimRank: Similarity Based on Random Walk and Structural Context

- To define EMD formally in G, we use the derived graph G^2 of node-pairs.
  - Each node (u, v) of V^2 can be thought of as the present state of a pair of surfers in V, where an edge from (u, v) to (c, d) in G^2 says that in the original graph G, one surfer can move from u to c while the other moves from v to d.
  - A tour in G^2 of length n represents a pair of tours in G also having length n.

SimRank: Similarity Based on Random Walk and Structural Context

- G^2 represents an ordered pair of nodes of G. A node (a,b) of G^2 points to a node (c,d) if, in G, a points to c and b points to d. The example represents the Web pages of two professors ProfA and ProfB, their students StudentA and StudentB, and the home page of their university Univ



SimRank: Similarity Based on Random Walk and Structural Context

- Formally, the EMD $m(u, v)$ is simply the expected distance in G^2 from (u, v) to any singleton node $(x, x) \in V^2$, since singleton nodes in G^2 represent states where both surfers are at the same node. More precisely,

$$m(u,v) = \sum_{t:(u,v) \rightsquigarrow (x,x)} P[t]\, l(t)$$

- The sum is taken over all tours t starting from (u,v) which touch a singleton node at the end and only at the end.
- Unfortunately, G^2 may not always be strongly connected (even if G is), and in such cases there may be no tours t for (u,v) in the summation. In this case, $m(u,v) = \infty$.
  - this definition would cause problems in defining distances for nodes from which some tours lead to singleton nodes while others lead to (u, v).

SimRank: Similarity Based on Random Walk and Structural Context

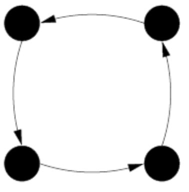
- Solution: Expected-f Meeting distance
  - Map all distances to a finite interval: instead of computing expected length l(t) of a tour, we can compute the expected f(l(t)), for a nonnegative, monotonic function which is bounded on the domain $[0,\infty)$.

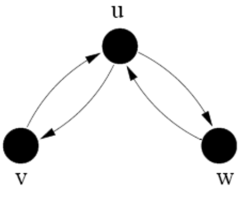
$$s'(u,v) = \sum_{t:(u,v) \rightsquigarrow (x,x)} P[t]\, C^{l(t)} \qquad C \in (0,1)$$

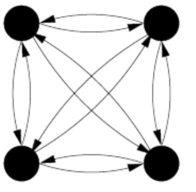
  - Close nodes have a lower score (meeting distances of 0 go to 1 and distances of ∞ go to 0), matching our intuition of similarity.
    - s′(a,b)=0 -> No tour from (a,b) to any singleton nodes
    - s′(a,b)=1 -> a=b
    - s′(a,b)$\in$[0,1] -> for all a,b

SimRank: Similarity Based on Random Walk and Structural Context

- Some examples of expected-f meeting distance with C=0.8.







s′(a,b) = 0

$s'(u,v) = s'(u,w) = 0$
$s'(v,w) = 0.8$

$s'(a,b) = 0.47$

SimRank: Similarity Based on Random Walk and Structural Context

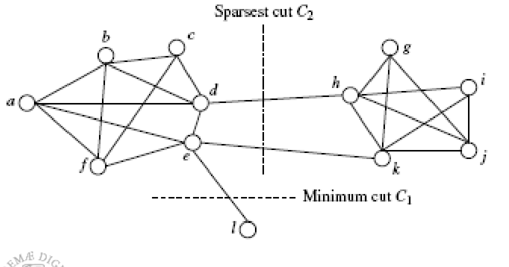
- It has been proved that the SimRank score, with parameter C, between two nodes is their expected-f meeting distance traveling back-edges, for $f(z) = C^z$
- In other words, s(u,v) = s′(u,v) for any two vertices u and v. That is, SimRank is based on both structural context and random walk.

Graph Clustering: Sparsest Cut

- How should we conduct clustering in a graph?
  - Intuitively, we should cut the graph into pieces, each piece being a cluster, such that the vertices within a cluster are well connected and the vertices in different clusters are connected in a much weaker way.
- Let G = (V,E) be a direct graph.
  - A cut C(S,T) is a partitioning of the set of vertices V in G, that is, V= $S \cup T$ and $S \cap T = \emptyset$.
  - The cut set of a cut is the set of edges {(u, v) ∈ E | u ∈ S, v ∈ T}
  - Size of the cut: number of edges in the cut set. If the edges are weighted, the value of the cut is the sum of weights.

Graph Clustering: Sparsest Cut

- What kinds of cuts are good for deriving clusters in graphs?
  - Minimum cut: cut's size is not greater than any other cut's size.
    - Polynomial time algorithms to compute minimum cuts of graphs (Edmonds-Karp algorithm)



Cut C_2 = ({*a*, *b*, *c*, *d*, *e*, *f* , *l* }, {*g*, *h*, *i*, *j*, *k*}) leads to a much better clustering than C_1. The edges in the cut set of C_2 are those connecting the two "natural clusters" in the graph. Specifically, for edges (*d*,*h*) and (*e*,*k*) that are in the cut set, most of the edges connecting *d*, *h*, *e*, and *k* belong to one cluster.

Graph Clustering: Sparsest Cut

- A better measure: Sparsity
- Intuition: choose a cut where, for each vertex *u* that is involved in an edge in the cut set, most of the edges connecting to *u* belong to one cluster.
- The sparsity of a cut C = (S,T) is defined as:

$$\Phi = \frac{\text{cut size}}{\min\{|S|, \boxed{|T|}\}}$$

Number of vertices

- A cut is sparsest if its sparsity is not greater than that of any other cut.
  - Favors solutions that are both sparse (few edges crossing the cut) and balanced (close to a bisection).
  - The problem is known to be NP-Hard, and the best known algorithm is an $O(\sqrt{\log n})$ approximation due to Arora, Rao & Vazirani (2009)
- Ex. Cut C_2 = ({a, b, c, d, e, f, l}, {g, h, i, j, k}) is the sparsest cut

Graph Clustering: Sparsest Cut

- For k clusters, the modularity of a clustering assesses the quality of the clustering:

$$Q = \sum_{i=1}^{k}\left(\boxed{\frac{l_i}{|E|}} - \boxed{\left(\frac{d_i}{2|E|}\right)^2}\right)$$

probability edge is in cluster i

probability a random edge would fall into cluster i

l_i: number of edges between vertices in the i-th cluster

d_i: the sum of the degrees of the vertices in the i-th cluster

where degree of a vertex u: number of edges connecting to u

- The modularity of a clustering of a graph is the difference between the fraction of all edges that fall into individual clusters and the fraction that would do so if the graph vertices were randomly connected

- The optimal clustering of graphs maximizes the modularity

Graph Clustering: Challenges of Finding Good Cuts

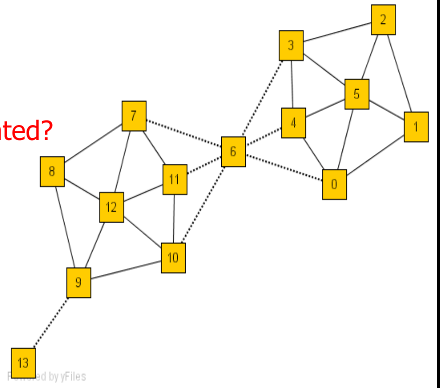
- High computational cost
  - Many graph cut problems are computationally expensive
  - The sparsest cut problem is NP-hard
  - Need to tradeoff between efficiency/scalability and quality
- Sophisticated graphs
  - May involve weights and/or cycles.
- High dimensionality
  - A graph can have many vertices. In a similarity matrix, a vertex is represented as a vector (a row in the matrix) whose dimensionality is the number of vertices in the graph
- Sparsity
  - A large graph is often sparse, meaning each vertex on average connects to only a small number of other vertices
  - A similarity matrix from a large sparse graph can also be sparse

Graph Clustering: Methods

- There exist two kinds of methods
  - Clustering methods for high-dimensional data
  - Clustering methods designed specifically for clustering graphs
- Clustering methods for high-dimensional data
  - Extract a similarity matrix from a graph using a similarity measure
  - A clustering algorithm for high-dimensional data is therefore applied
- Clustering methods designed specifically for clustering graphs
  - Exploit the peculiarities of the graph for performing the clustering process

SCAN: Density-based clustering of Networks

- How many clusters?
- What size should they be?
- What is the best partitioning?
- Should some points be segregated?



- Application: Given simply information of who associates with whom, could one identify clusters of individuals with common interests or special relationships (families, cliques, terrorist cells)?

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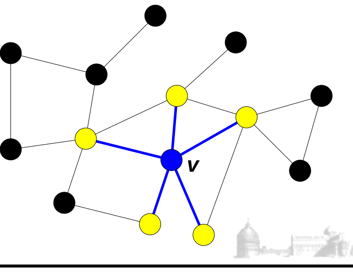
SCAN: Density-based clustering of Networks

- Cliques, hubs and outliers
- Individuals in a tight social group, or clique, know many of the same people, regardless of the size of the group
- Individuals who are hubs know many people in different groups but belong to no single group. Politicians, for example bridge multiple groups
- Individuals who are outliers reside at the margins of society. Hermits, for example, know few people and belong to no group
- The Neighborhood of a Vertex

Define $\Gamma(v)$ as the immediate neighborhood of a vertex (i.e. the set of people that an individual knows)

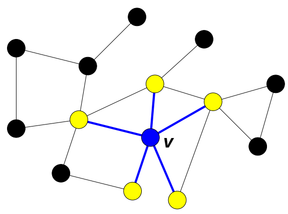


Structure Similarity

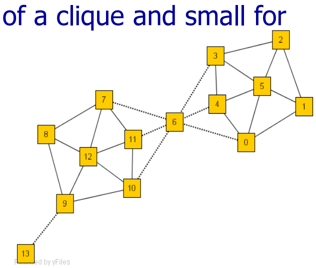
- The desired features tend to be captured by a measure we call Structural Similarity

$$\sigma(v,w) = \frac{|\Gamma(v) \cap \Gamma(w)|}{\sqrt{|\Gamma(v)||\Gamma(w)|}}$$

$$\Gamma(u) = \{v | (u,v) \in E\} \cup \{u\}$$



- Structural similarity is large for members of a clique and small for hubs and outliers



Structural Connectivity

- SCAN uses a similarity threshold ε to define the cluster membership
- For a vertex $v \in V$, the ε-Neighborhood of v is defined as:

$$N_\varepsilon(v) = \{w \in \Gamma(v) | \sigma(v,w) \geq \varepsilon\}$$

- A core vertex is a vertex inside of a cluster. v is a core vertex if and only if:

$$CORE_{\varepsilon,\mu}(v) \Leftrightarrow |N_\varepsilon(v)| \geq \mu$$

where μ is a popularity threshold.

- SCAN grows cluster from core vertices (similar to DBSCAN)
  - If a vertex v is in the ε-Neighborhood of *a core u*, then v is assigned to the same cluster as u
  - The growing process continues until no cluster can be further grown.

Structural Connectivity

- Formally, a vertex *w* can be directly reached from a core *v* if

$$DirRECH_{\varepsilon,\mu}(v,w) \Leftrightarrow CORE_{\varepsilon,\mu}(v) \wedge w \in N_{\varepsilon}(v)$$

- Structure reachable: transitive closure of direct structure reachability. A vertex *v* can be reached from a core vertex *u* if there exist vertices w_1, …, w_n such that w_1 can be reached from *u*, w_i can be reached from w_{i-1}, for $1<i<=n$, and *v* can be reached from w_n.

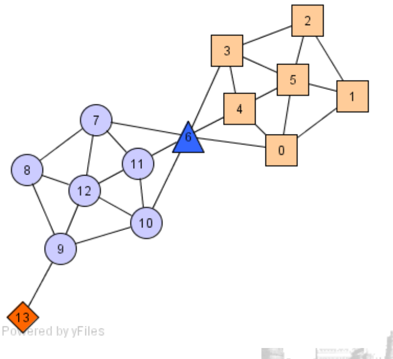
- Structure connected: two vertices v and w, which may or may not be cores, are said connected there exists a core *u such that v and w can be reached from u.*

$$CONNECT_{\varepsilon,\mu}(v,w) \Leftrightarrow \exists u \in V : RECH_{\varepsilon,\mu}(u,v) \wedge RECH_{\varepsilon,\mu}(u,w)$$

M. Ester, H. P. Kriegel, J. Sander, & X. Xu (KDD'96) "A Density-Based Algorithm for Discovering Clusters in Large Spatial Databases

Structure-connected clusters

- Structure-connected cluster C
  - Connectivity: $\forall v,w \in C : CONNECT_{\varepsilon,\mu}(v,w)$
  - Maximality: $\forall v,w \in V : v \in C \wedge REACH_{\varepsilon,\mu}(v,w) \Rightarrow w \in C$
- Hubs:
  - Not belong to any cluster
  - Bridge to many clusters
- Outliers:
  - Not belong to any cluster
  - Connect to less clusters



SCAN Algorithm

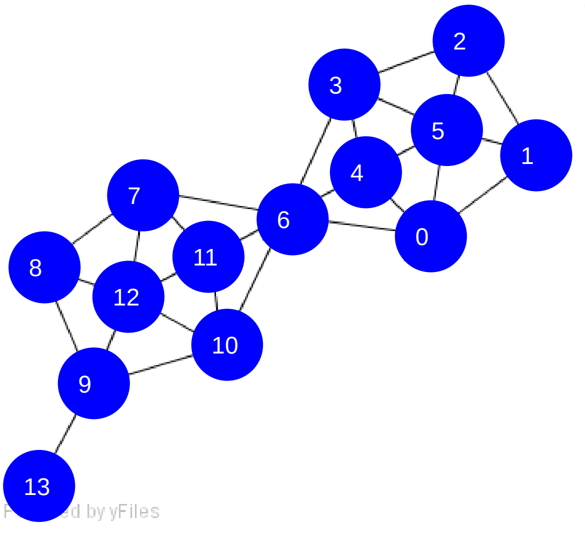
Algorithm: SCAN for clusters on graph data.
Input: a graph $G = (V, E)$, a similarity threshold ε, and a
population threshold μ
Output: a set of clusters
Method: set all vertices in V unlabeled
for all unlabeled vertex u **do**
if u is a core **then**
generate a new cluster-id c
insert all $v \in N_\varepsilon(u)$ into a queue Q
while $Q \neq$ **do**
$w \leftarrow$ the first vertex in Q
$R \leftarrow$ the set of vertices that can be directly reached from w
for all $s \in R$ **do**
if s is not unlabeled or labeled as nonmember **then**
assign the current cluster-id c to s
endif
if s is unlabeled **then**
insert s into queue Q
endif
endfor

SCAN Algorithm

remove w from Q
end while
else
label u as nonmember
endif
endfor
for all vertex u labeled nonmember **do**
if $\exists x, y \in \Gamma(u)$: x and y have different cluster-ids **then**
label u as hub
else
label u as outlier
endif
endfor

Algorithm

$\mu = 2$
$\varepsilon = 0.7$



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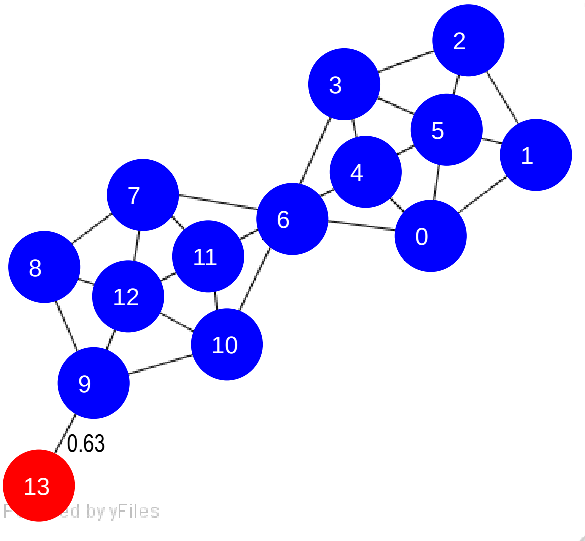
Algorithm

$\mu = 2$
$\varepsilon = 0.7$



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Algorithm

$\mu = 2$
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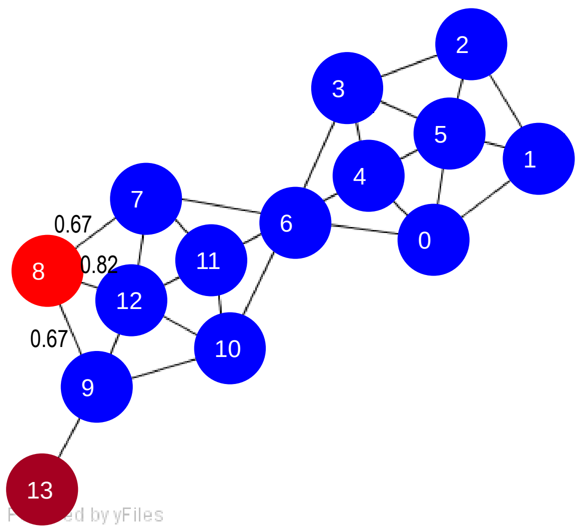

Algorithm

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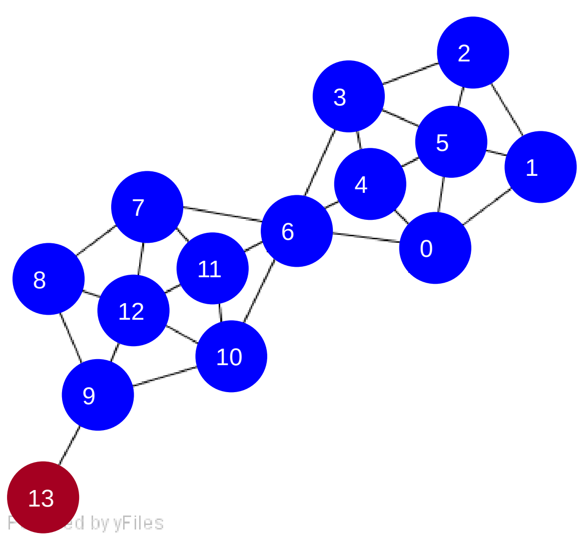

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Algorithm

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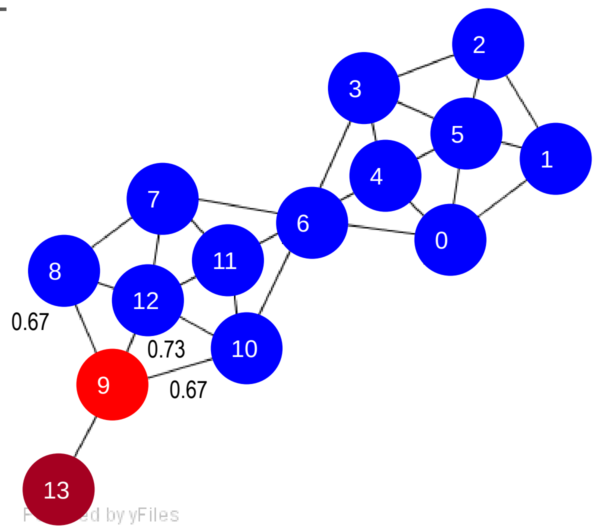

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Algorithm

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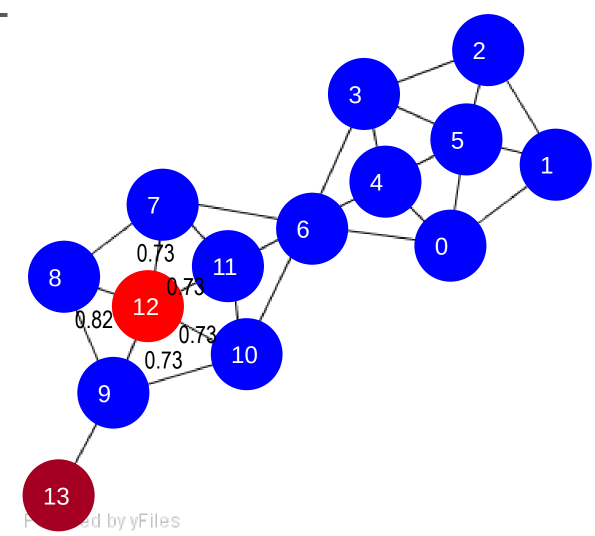
$\varepsilon = 0.7$



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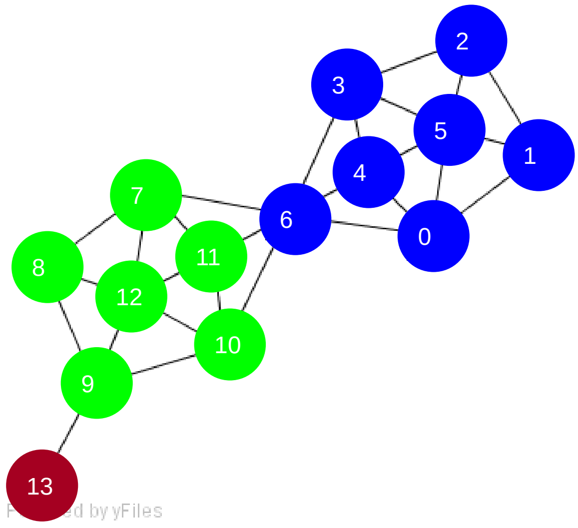
Algorithm

$\mu = 2$
$\varepsilon = 0.7$



Algorithm

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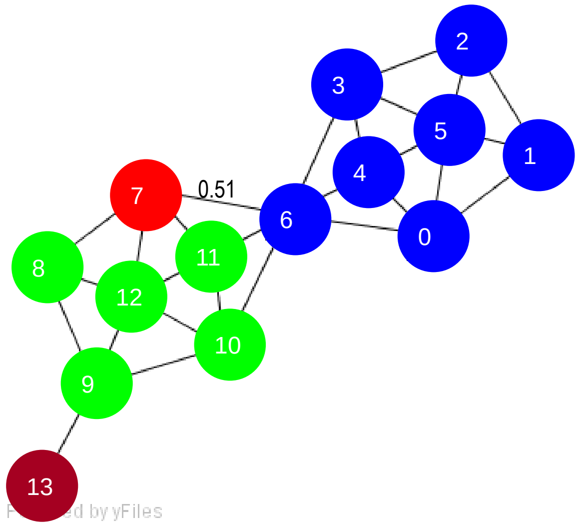

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Algorithm

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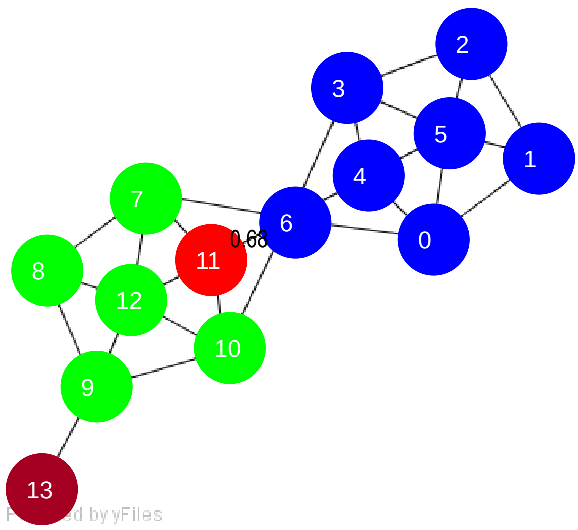

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Algorithm

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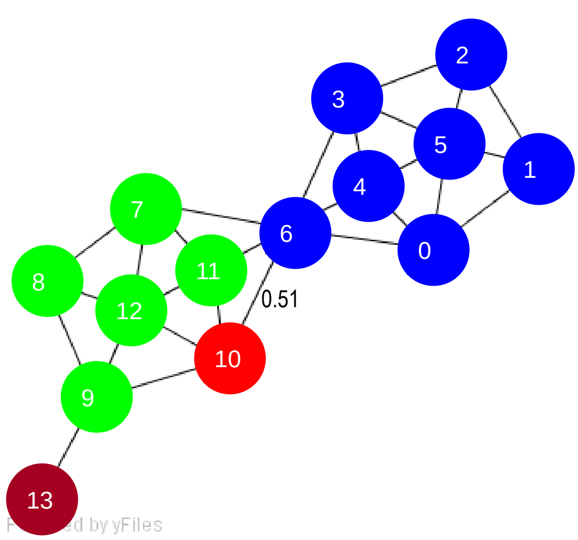

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Algorithm

$\mu = 2$
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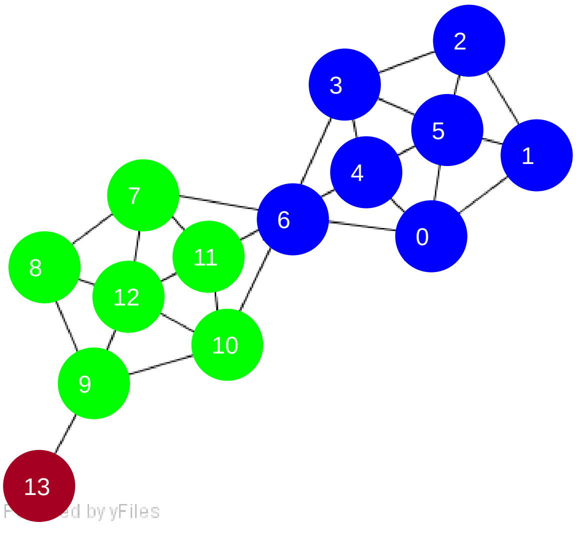

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Algorithm

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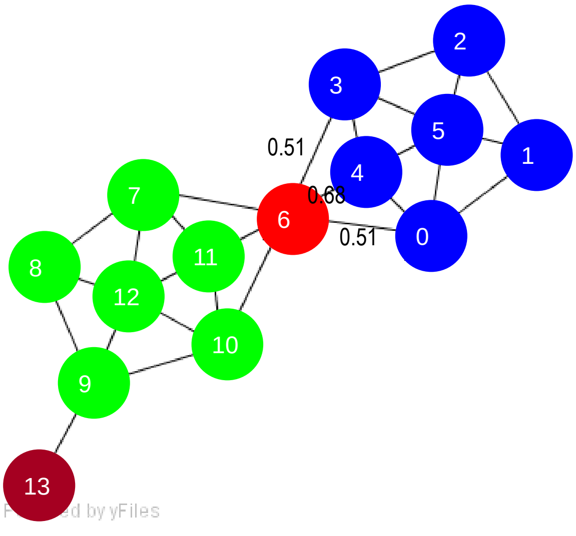

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Algorithm

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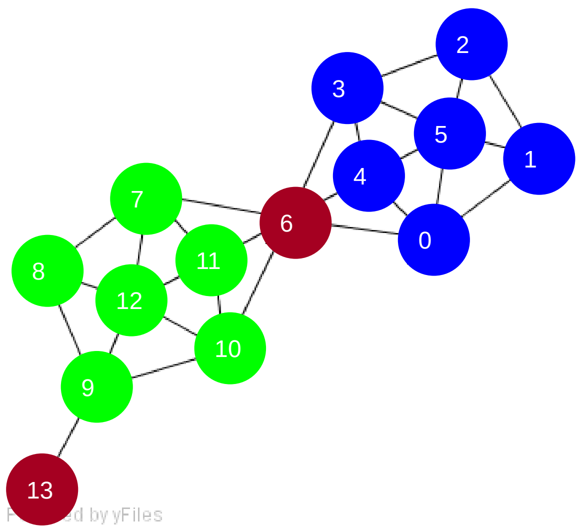
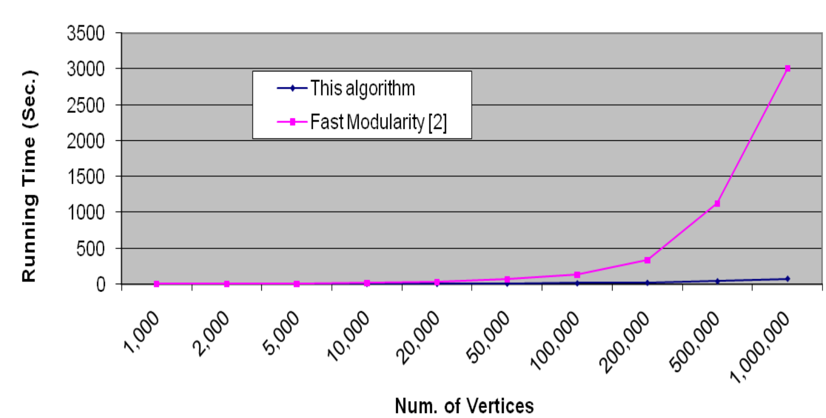


Running time

- Running time = O(|E|)
- For sparse networks = O(|V|)



A. Clauset, M. E. J. Newman, & C. Moore, *Phys. Rev. E* **70**, 066111 (2004).