Quantum Gates

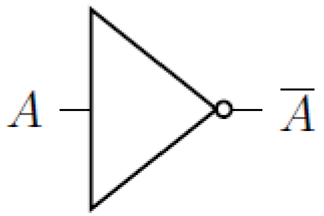
Quantum Gates

- A classical computer is built from an *electrical circuit* containing *wires* and *logic gates*
- The *wires* are used to carry information around the circuit, while the *logic gates* perform manipulations of the information, converting it from one form to another
- Similar to a classical computer, a quantum computer is built from a quantum circuit containing *wires* and elementary *quantum gates* to carry around and manipulate the quantum information
- Thus, quantum gates act on qubits, like logic gates act on bits
- Specifically, **a quantum gate transforms the state of a qubit into other states**

Single Qubit Gates

Classical Computer

- Consider, for example, the **NOT** classical single bit logic gate, whose operation is defined by its *truth table*



Input bit A	Output bit $\overline{\text{A}}$
0	1
1	0

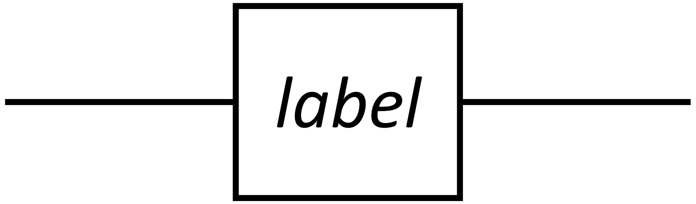
in which 0 → 1 and 1 → 0, that is, the 0 and 1 states are interchanged

Single Qubit Gates

Question: can an analogous quantum gate for qubits be defined?

- Imagine that we have a quantum gate which takes the state $|0\rangle$ to the state $|1\rangle$ and vice versa

- We denote a single-qubit gate with a box containing the label (straddling the line) representing the operation carried out by the gate

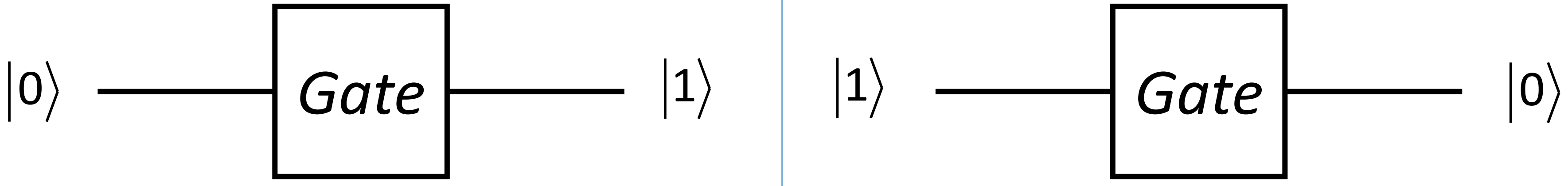


Single Qubit Gates

- Let's call *Gate* the operator which flips the state of a qubit from state $|0\rangle$ to state $|1\rangle$ and vice versa

$$Gate|0\rangle = |1\rangle$$
$$Gate|1\rangle = |0\rangle$$

Such a quantum operator would obviously be a good candidate for a quantum analogue to the **NOT** gate



Single Qubit Gates

- However, the specification given in the previous slide is not enough!
- **Question:** why is that?
- **Answer:** because specifying the action of the gate on the states $|0\rangle$ and $|1\rangle$ does not tell us what happens to **superpositions** of the states $|0\rangle$ and $|1\rangle$

Single Qubit Gates

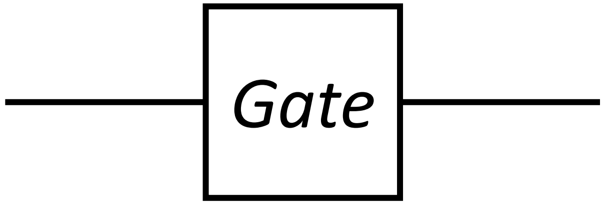
- In fact, the quantum gate acts *linearly*, that is, operating on a superposition it would do the following

$$|\psi\rangle_{out} = Gate|\psi\rangle_{in} = Gate\left(\alpha|0\rangle + \beta|1\rangle\right) = \alpha\left(Gate|0\rangle\right) + \beta\left(Gate|1\rangle\right)$$
$$= \alpha|1\rangle + \beta|0\rangle = \beta|0\rangle + \alpha|1\rangle$$

$$\rightarrow$$

$$|\psi\rangle_{out} = \beta|0\rangle + \alpha|1\rangle$$

$$|\psi\rangle_{in} = \alpha|0\rangle + \beta|1\rangle$$



$$|\psi\rangle_{out} = \beta|0\rangle + \alpha|1\rangle$$

As it should be, the output state is normalized.

NOTE

- Why the quantum gate *acts linearly* and not in some *nonlinear* fashion is a very interesting question, and the answer is not at all obvious
- It turns out that this *linear behavior* is a general property of quantum mechanics, and *very well motivated empirically*; moreover, nonlinear behavior can lead to apparent paradoxes such *as faster-than-light communication*, and *violations of the second laws of thermodynamics*

Single Qubit Gates

- Before doing an in-depth analysis of specific one-qubit quantum gates we extend the previous result to the case of the most general one-qubit gate

- As we said before a quantum gate *transforms* the state of a qubit into other states

Single Qubit Gates

$$|\psi\rangle \;—\boxed{U}—\; |\psi'\rangle = U|\psi\rangle$$

- Let's denote by *U* such a transformation which generally turns $|0\rangle$ and $|1\rangle$ into a superposition of $|0\rangle$ and $|1\rangle$

$$U|0\rangle = a|0\rangle + b|1\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$U|1\rangle = c|0\rangle + d|1\rangle = \begin{bmatrix} c \\ d \end{bmatrix}$$

We can arrange the resulting amplitudes side-by-side, resulting in the following 2x2 matrix

$$\rightarrow \quad U = \begin{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} & \begin{bmatrix} c \\ d \end{bmatrix} \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Single Qubit Gates

- Plugging this matrix into $U|0\rangle$ and $U|1\rangle$

$$U|0\rangle = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$U|1\rangle = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}$$

From the previous slide

$$U|0\rangle = a|0\rangle + b|1\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$U|1\rangle = c|0\rangle + d|1\rangle = \begin{bmatrix} c \\ d \end{bmatrix}$$

Single Qubit Gates

- Remember that the quantum gate (or *U*) acts *linearly*, meaning that if a qubit is in the state

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

From the previous slide

$$U|0\rangle = a|0\rangle + b|1\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$U|1\rangle = c|0\rangle + d|1\rangle = \begin{bmatrix} c \\ d \end{bmatrix}$$

then applying *U* transforms this to

$$U|\psi\rangle = \alpha U|0\rangle + \beta U|1\rangle = \alpha \begin{bmatrix} a \\ b \end{bmatrix} + \beta \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a\alpha \\ b\alpha \end{bmatrix} + \begin{bmatrix} c\beta \\ d\beta \end{bmatrix} = \begin{bmatrix} a\alpha + c\beta \\ b\alpha + d\beta \end{bmatrix}$$

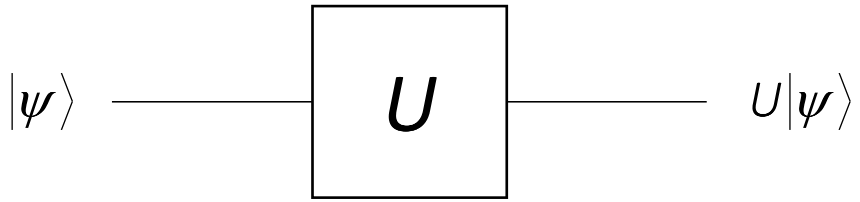
Single Qubit Gates

- From the above slide we have

$$U|\psi\rangle = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} a\alpha + c\beta \\ b\alpha + d\beta \end{bmatrix} = (a\alpha + c\beta)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (b\alpha + d\beta)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= (a\alpha + c\beta)|0\rangle + (b\alpha + d\beta)|1\rangle$$

- Assuming the original state was normalized, i.e. $|\alpha|^2 + |\beta|^2 = 1$, this must also be true for the quantum state after the gate has acted



Single Qubit Gates

- Of course, the matrix must ensure that the total probability remains 1, so in the previous example, we must have

$$|a\alpha + c\beta|^2 + |b\alpha + d\beta|^2 = 1$$

- This yields the following point: *quantum gates are (represented by) matrices that keep the total probability equal to 1*

$$\boxed{U|\psi\rangle = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} a\alpha + c\beta \\ b\alpha + d\beta \end{bmatrix}}$$

From the previous slide

Example

- For example, consider a quantum gate that performs the following transformation:

$$U|0\rangle = \frac{\sqrt{2}-i}{2}|0\rangle - \frac{1}{2}|1\rangle = \begin{bmatrix} \frac{\sqrt{2}-i}{2} \\ -\frac{1}{2} \end{bmatrix},$$

$$U|1\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{2}+i}{2}|1\rangle = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{2}+i}{2} \end{bmatrix}$$

$$\rightarrow \qquad U = \begin{bmatrix} \frac{\sqrt{2}-i}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{2}+i}{2} \end{bmatrix}$$

Example

A quantum gate must be *linear*, meaning we can distribute it across superpositions:

$$\begin{aligned} U(\alpha|0\rangle + \beta|1\rangle) &= \alpha U|0\rangle + \beta U|1\rangle \\ &= \alpha\left(\frac{\sqrt{2}-i}{2}|0\rangle - \frac{1}{2}|1\rangle\right) + \beta\left(\frac{1}{2}|0\rangle + \frac{\sqrt{2}+i}{2}|1\rangle\right) \\ &= \left(\alpha\frac{\sqrt{2}-i}{2} + \beta\frac{1}{2}\right)|0\rangle + \left(-\alpha\frac{1}{2} + \beta\frac{\sqrt{2}+i}{2}\right)|1\rangle. \end{aligned}$$

Example

For this to be a valid quantum gate, the total probability must remain 1. Assuming the original state was normalized, i.e., $|\alpha|^2 + |\beta|^2 = 1$, we can calculate the total probability by summing the norm-square of each amplitude to see if it is still 1:

$$
\begin{aligned}
&\left|\alpha\frac{\sqrt{2}-i}{2}+\beta\frac{1}{2}\right|^2+\left|-\alpha\frac{1}{2}+\beta\frac{\sqrt{2}+i}{2}\right|^2 \\
&=\left(\alpha\frac{\sqrt{2}-i}{2}+\beta\frac{1}{2}\right)\left(\alpha^*\frac{\sqrt{2}+i}{2}+\beta^*\frac{1}{2}\right) \\
&\quad+\left(-\alpha\frac{1}{2}+\beta\frac{\sqrt{2}+i}{2}\right)\left(-\alpha^*\frac{1}{2}+\beta^*\frac{\sqrt{2}-i}{2}\right) \\
&=|\alpha|^2\frac{(\sqrt{2}-i)(\sqrt{2}+i)}{4}+\alpha\beta^*\frac{\sqrt{2}-i}{2}+\beta\alpha^*\frac{\sqrt{2}+i}{4}+|\beta|^2\frac{1}{4} \\
&\quad+|\alpha|^2\frac{1}{4}-\alpha\beta^*\frac{\sqrt{2}-i}{4}-\beta\alpha^*\frac{\sqrt{2}+i}{4}+|\beta|^2\frac{(\sqrt{2}+i)(\sqrt{2}-i)}{4} \\
&=|\alpha|^2\frac{3}{4}+|\beta|^2\frac{1}{4}+|\alpha|^2\frac{1}{4}+|\beta|^2\frac{3}{4} \\
&=|\alpha|^2+|\beta|^2 \\
&=1.
\end{aligned}
$$

So, U is a valid quantum gate. Then,

Quantum gates are linear maps that keep the total probability equal to 1.

Unitariety

- Let's return to the previous example where

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

and

$$U|\psi\rangle = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} a\alpha + c\beta \\ b\alpha + d\beta \end{bmatrix}$$

- We see that $U|\psi\rangle$ is a column vector, so we can also write it as a ket $|\psi\rangle$

$$U|\psi\rangle = |\psi\rangle \quad \text{where} \quad |\psi\rangle = \begin{bmatrix} a\alpha + c\beta \\ b\alpha + d\beta \end{bmatrix}$$

Unitariety

- Now, consider the *conjugate transpose* of $U|\psi\rangle$

$$\langle\psi| = [(a\alpha + c\beta)^*, \quad (b\alpha + d\beta)^*] = [a^*\alpha^* + c^*\beta^*, \quad b^*\alpha^* + d^*\beta^*]$$

$$= [\alpha^*, \quad \beta^*]\begin{bmatrix} a^* & b^* \\ c^* & d^* \end{bmatrix} = [\alpha^*, \quad \beta^*]\begin{bmatrix} a & b \\ c & d \end{bmatrix}^*$$

$$= [\alpha^*, \quad \beta^*]\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}^*\right)^T = \underbrace{[\alpha^*, \quad \beta^*]}_{\langle\psi|}\underbrace{\begin{bmatrix} a & c \\ b & d \end{bmatrix}}_{U}{}^{\dagger} = \langle\psi|U^\dagger$$

- To summarize

$$|\psi\rangle = U|\psi\rangle \quad \rightarrow \quad \langle\psi| = \langle\psi|U^\dagger$$

Unitariety

- We can reach the same result $\langle\psi'| = \langle\psi|U^\dagger$ by considering that

$$\langle\psi'| = \left(U|\psi\rangle\right)^\dagger = |\psi\rangle^\dagger U^\dagger = \langle\psi|U^\dagger$$

$$|\psi\rangle^\dagger = \langle\psi|$$

$$(AB)^\dagger = B^\dagger A^\dagger$$

Unitariety

- Using this, we can come up with an easy way to determine whether a matrix keeps the total probability equal to 1

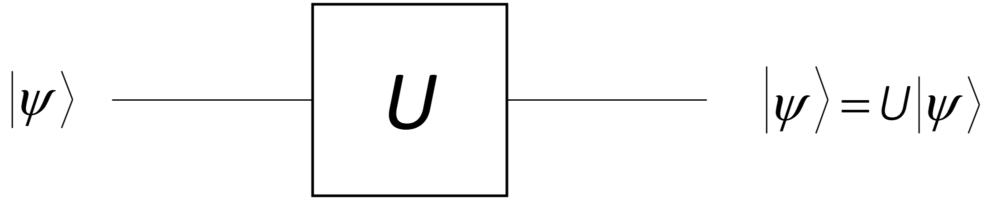
- Consider a quantum gate (matrix) *U*

- If it acts on $\left|\psi\right\rangle$, we have

$$U\left|\psi\right\rangle=\left|\psi\right\rangle$$

- For *U* to be a quantum gate, $\left|\psi\right\rangle$ must be normalized, that is, the inner product of $\left|\psi\right\rangle$ with itself must equal 1

Unitariety



$$\langle\psi|\psi\rangle = 1 \quad \rightarrow$$

$$\langle\psi|U^{\dagger}U|\psi\rangle = 1, \qquad \forall U \quad \rightarrow \quad U^{\dagger}U = I$$

A matrix that satisfies this property $U^{\dagger}U = I$ and $UU^{\dagger} = I$ is called *unitary*

Quantum gates **are unitary matrices**, and **unitary matrices** are quantum gates

Unitariety

This is why we typically use U to denote a quantum gate. It stands for unitary. As an example application of this, is the following matrix a quantum gate?

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

We can just check whether it is unitary, so whether $U^\dagger U = I$ or not.

$$U^\dagger U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \neq I.$$

So no, it's not a quantum gate.

Reversibility

- A matrix *M* is *reversible* or *invertible* if there exists a matrix denoted M^{-1} such that

$$MM^{-1} = M^{-1}M = I$$

- Now, since a quantum gate *U* must be unitary, i.e.

$$UU^{\dagger} = U^{\dagger}U = I$$

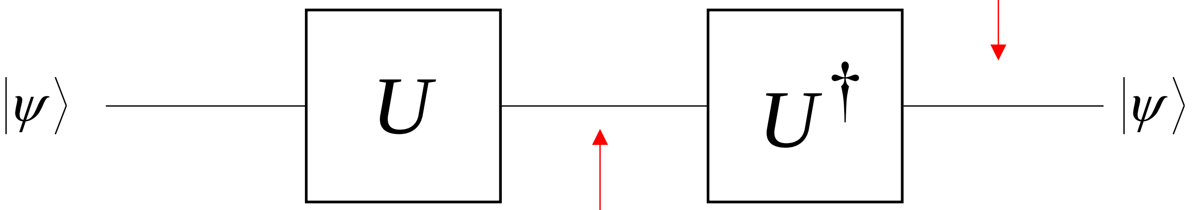
it follows that the inverse of the unitary matrix *U* is simply $U^{-1} = U^{\dagger}$

Reversibility

- As a consequence, a quantum gate *U* is *always reversible*

- If we have a qubit and we applied a quantum gate *U*, we can undo the operation by applying $U^\dagger$

$$U^\dagger\left(U|\psi\rangle\right) = U^\dagger U|\psi\rangle = I|\psi\rangle = |\psi\rangle$$



$$|\psi\rangle = U|\psi\rangle$$

One-Qubit Quantum Gates

Reference

To find out more about quantum gates, I recommend you read the paper

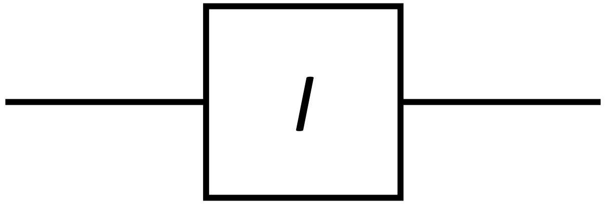
Quantum computers: registers, gates and algorithm
by Paul Isaac Hagouel

Identity Gate (I)

- The *identity gate* turns $|0\rangle$ into $|0\rangle$ and $|1\rangle$ into $|1\rangle$, hence doing nothing

- The *input* state is placed on the *left* of the gate symbol and the *output* state on the *right*

$|\psi\rangle_{in} = |\psi\rangle$



$|\psi\rangle_{out} = |\psi\rangle$

$$I|0\rangle = |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$I|1\rangle = |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\rightarrow \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \equiv |0\rangle\langle 0| + |1\rangle\langle 1|$$

The Pauli *X*-Gate, or NOT Gate

- This gate, turns $|0\rangle$ into $|1\rangle$, and $|1\rangle$ into $|0\rangle$

$$X|0\rangle = |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$X|1\rangle = |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

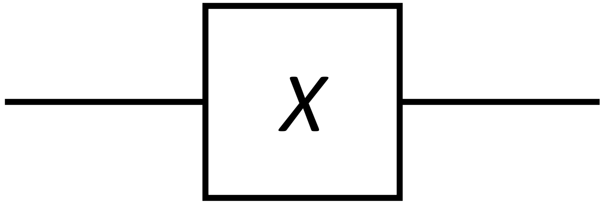
$$\rightarrow \quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \equiv |0\rangle\langle 1| + |1\rangle\langle 0| \quad \rightarrow \quad X = X^{\dagger}$$

The Pauli X-Gate, or NOT Gate

- When acting on $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where $|\alpha|^2 + |\beta|^2 = 1$, X swaps the amplitudes α and β
- The X operator is sometimes called the *bit flip operator*, because it "flips" the computational basis amplitudes, i.e., $\alpha \leftrightarrow \beta$

$$|\psi_{out}\rangle = X\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}\beta\\ \alpha\end{bmatrix}$$

$|\psi\rangle_{in} = \alpha|0\rangle + \beta|1\rangle$



$|\psi\rangle_{out} = \beta|0\rangle + \alpha|1\rangle$

The Pauli X-Gate, or NOT Gate

- Since

$$X = X^{\dagger} \quad \rightarrow \quad XX^{\dagger} = X^{\dagger}X = XX = X^{2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

- It turns out that the matrix X describing the single qubit gate is **unitary**

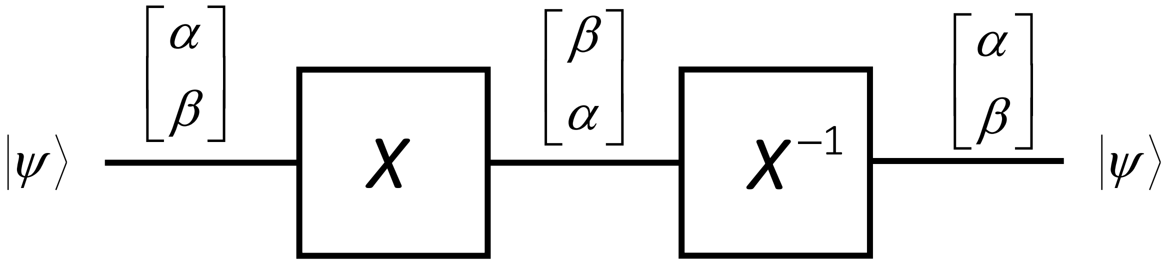
- We can use $X^{2} = 1$ to simplify consecutive applications of X. For example:

$$X^{1001} = X^{1000}X = \left(X^{2}\right)^{500} X = I^{500}X = X$$

The Pauli *X*-Gate, or NOT Gate

- Unitary quantum gates are *always* **invertible**, since the inverse of a unitary matrix is also a unitary matrix, and thus a quantum gate can always be inverted by another quantum gate

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \to X^{\dagger} = X^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = X$$



The Pauli *Z*-Gate or Phase Flip Gate

- This gate, keeps $|0\rangle$ as $|0\rangle$, and turns $|1\rangle$ into $-|1\rangle$

$$Z|0\rangle = |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$Z|1\rangle = -|1\rangle = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\rightarrow \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \equiv |0\rangle\langle 0| - |1\rangle\langle 1| \quad \rightarrow \quad Z^\dagger = Z$$

- *Z* is unitary and thus reversible

$$ZZ^\dagger = Z^\dagger Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

The Pauli Z-Gate or Phase Flip Gate

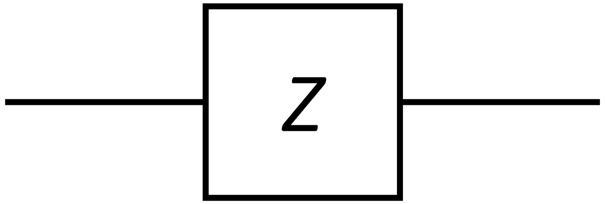
When acting on $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where $|\alpha|^2 + |\beta|^2 = 1$

$$Z\begin{bmatrix}\alpha \\ \beta\end{bmatrix} = \begin{bmatrix}1 & 0 \\ 0 & -1\end{bmatrix}\begin{bmatrix}\alpha \\ \beta\end{bmatrix} = \begin{bmatrix}\alpha \\ -\beta\end{bmatrix} = \alpha|0\rangle - \beta|1\rangle$$

Z leaves $|0\rangle$ unchanged, and flips the sign of $|1\rangle$ to give $-|1\rangle$

$$-1 = e^{i\pi}$$

$|\psi\rangle_{in} = \alpha|0\rangle + \beta|1\rangle$

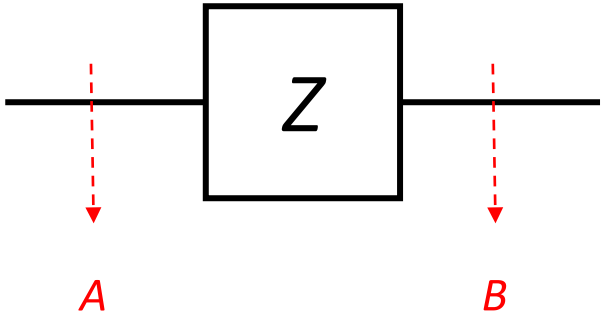


$|\psi\rangle_{out} = \alpha|0\rangle - \beta|1\rangle = \alpha|0\rangle + e^{i\pi}\beta|1\rangle$

The Pauli Z-Gate or Phase Flip Gate

- Z is also called a *phase flip*, because it changes the sign of the amplitude of the state $|1\rangle$

$$|\psi\rangle_{in} = \alpha|0\rangle + \beta|1\rangle$$



$$|\psi\rangle_{out} = \alpha|0\rangle - \beta|1\rangle = \alpha|0\rangle + e^{i\pi}\beta|1\rangle$$

- The circuit produces the A-to-B trasition $\begin{bmatrix}\alpha \\ \beta\end{bmatrix} \rightarrow \begin{bmatrix}\alpha \\ -\beta\end{bmatrix}$ yielding the same probabilities, $|\alpha|^2$ and $|\beta|^2$, at both access point

The Pauli *Y*-Gate

- This gate, turns $|0\rangle$ into $i|1\rangle$, and $|1\rangle$ into $-i|0\rangle$

$$Y|0\rangle = i|1\rangle = \begin{bmatrix} 0 \\ i \end{bmatrix}$$

$$Y|1\rangle = -i|0\rangle = \begin{bmatrix} -i \\ 0 \end{bmatrix}$$

$$\rightarrow \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$$

The Pauli *Y*-Gate

- *Y* is *unitary* and thus *reversible*

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \rightarrow \quad Y^{\dagger} = \left(\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}^{*} \right)^{T} = \left(\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \right)^{T} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = Y$$

$$\rightarrow \quad Y^{\dagger} = Y$$

- Thus

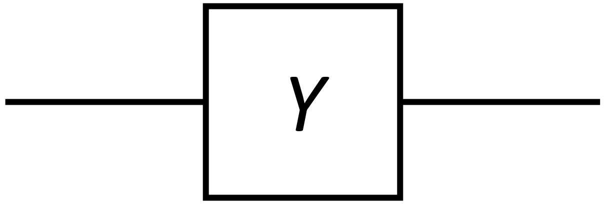
$$YY^{\dagger} = Y^{\dagger}Y = YY = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

The Pauli *Y*-Gate

- When acting on $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where $|\alpha|^2 + |\beta|^2 = 1$

$$Y\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}-i\beta\\ i\alpha\end{bmatrix} = -i\begin{bmatrix}\beta\\ -\alpha\end{bmatrix} \equiv \beta|0\rangle - \alpha|1\rangle$$

$|\psi\rangle_{in} = \alpha|0\rangle + \beta|1\rangle$



$|\psi\rangle_{out} = -i\beta|0\rangle + i\alpha|1\rangle \equiv \beta|0\rangle - \alpha|1\rangle$

- Although *Y* has no official name, it could be called the *bit-and-phase flip*, because it *flips* both the *bits* and *the relative phase*, simultaneously

The Phase-Shift Gates, S and T

- We have already seen that he phase-flip gate, *Z*, when operates on a qubit state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ *flips* the relative phase, i.e. *shifts* it, by π radians

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \rightarrow \begin{bmatrix} \alpha \\ -\beta \end{bmatrix} = \begin{bmatrix} \alpha \\ e^{i\pi}\beta \end{bmatrix}$$

- There are two other common shift amounts, $\pi/2$ (the *S* operator) and $\pi/4$ (the *T* operator)

The Phase S-Gate

- This gate, keeps $|0\rangle$ as $|0\rangle$, and turns $|1\rangle$ into $i|1\rangle$

$$S|0\rangle = |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\rightarrow \quad S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \equiv |0\rangle\langle 0| + i|1\rangle\langle 1|$$

$$S|1\rangle = i|1\rangle = \begin{bmatrix} 0 \\ i \end{bmatrix}$$

The Phase *S*-Gate

- *S* is unitary

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \quad \rightarrow \quad S^\dagger = \left(\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}^* \right)^T = \left(\begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} \right)^T = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$$

$$SS^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$S^\dagger S = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

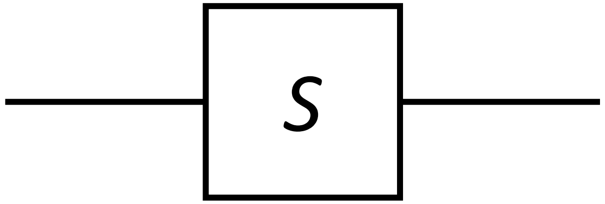
$$\rightarrow \quad SS^\dagger = S^\dagger S = I$$

The Phase S-Gate

- When acting on $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where $|\alpha|^2 + |\beta|^2 = 1$

$$S\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}\alpha\\ i\beta\end{bmatrix} = \alpha|0\rangle + i\beta|1\rangle = \alpha|0\rangle + e^{i\pi/2}\beta|1\rangle$$

$|\psi\rangle_{in} = \alpha|0\rangle + \beta|1\rangle$



$|\psi\rangle_{out} = \alpha|0\rangle + i\beta|1\rangle = \alpha|0\rangle + e^{i\pi/2}\beta|1\rangle$

The π/8 or *T*-Gate

- This gate, keeps $|0\rangle$ as $|0\rangle$, and turns $|1\rangle$ into $e^{i\pi/4}|1\rangle$

$$T|0\rangle = |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\rightarrow \qquad T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} \equiv |0\rangle\langle 0| + e^{i\pi/4}|1\rangle\langle 1|$$

$$T|1\rangle = e^{i\pi/4}|1\rangle = \begin{bmatrix} 0 \\ e^{i\pi/4} \end{bmatrix}$$

The T-Gate

T is unitary

$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} \quad \rightarrow \quad T^{\dagger} = \left(\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}^{*} \right)^{T} = \left(\begin{bmatrix} 1 & 0 \\ 0 & e^{-i\pi/4} \end{bmatrix} \right)^{T} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\pi/4} \end{bmatrix}$$

$$TT^{\dagger} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\pi/4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$T^{\dagger}T = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\pi/4} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

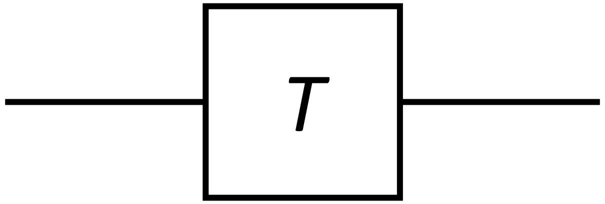
$$\rightarrow \quad TT^{\dagger} = T^{\dagger}T = I$$

The *T*-Gate

- When acting on $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where $|\alpha|^2 + |\beta|^2 = 1$

$$T\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \mathrm{T} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ e^{i\pi/4}\beta \end{bmatrix} = \alpha|0\rangle + e^{i\pi/4}\beta|1\rangle$$

$|\psi\rangle_{in} = \alpha|0\rangle + \beta|1\rangle$



$|\psi\rangle_{out} = \alpha|0\rangle + e^{i\pi/4}\beta|1\rangle$

The *T*-Gate

- You might wonder why the *T* gate is called the $\pi/8$ gate when it is $\pi/4$ that appears in the definition
- The reason is that the gate has historically often been referred to as the $\pi/8$ gate, simply because up to an unimportant global phase, *T* is equal to a gate which has $\exp(\pm i\pi/8)$ appearing on its diagonals

$$T = e^{i\pi/8} \begin{bmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{bmatrix}$$

- Nevertheless, the nomenclature is in some respects rather unfortunate, and we often refer to this gate as the *T* gate

The Hadamard *H*-Gate

- This gate, turns $|0\rangle$ into $|+\rangle$, and $|1\rangle$ into $|-\rangle$

$$H|0\rangle = |+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$

$$H|1\rangle = |-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

$$\rightarrow \qquad H|x\rangle = \frac{|0\rangle + (-1)^x|1\rangle}{\sqrt{2}} \qquad \forall x \in \{0,1\}$$

compact form

$$\downarrow$$

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} \equiv \frac{1}{\sqrt{2}}\left(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|\right)$$

The Hadamard *H*-Gate

- *H* is unitary

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \rightarrow \quad H^\dagger = \left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^* \right)^T = \left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right)^T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = H$$

$$HH^\dagger = H^\dagger H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \quad \rightarrow \quad HH^\dagger = H^\dagger H = I$$

The Hadamard *H*-Gate

- The application of a Hadamard gate to an arbitrary qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ gives the following output

$$H\begin{bmatrix}\alpha \\ \beta\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1 \\ 1 & -1\end{bmatrix}\begin{bmatrix}\alpha \\ \beta\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}\alpha+\beta \\ \alpha-\beta\end{bmatrix} = \left(\frac{\alpha+\beta}{\sqrt{2}}\right)|0\rangle + \left(\frac{\alpha-\beta}{\sqrt{2}}\right)|1\rangle$$

$$= \alpha\frac{|0\rangle+|1\rangle}{\sqrt{2}} + \beta\frac{|0\rangle-|1\rangle}{\sqrt{2}} = \alpha|+\rangle + \beta|-\rangle$$

Quantum Interference

- This is an example of *quantum interference*

- In the previous slide we have shown that

$$H\begin{bmatrix}\alpha\\ \beta\end{bmatrix}=\begin{bmatrix}\dfrac{\alpha+\beta}{\sqrt{2}}\\ \dfrac{\alpha-\beta}{\sqrt{2}}\end{bmatrix}$$

Quantum Interference

- Thus

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix}=\alpha|0\rangle+\beta|1\rangle \quad \text{———} \boxed{H} \text{———} \quad \begin{bmatrix}\frac{\alpha+\beta}{\sqrt{2}}\\ \frac{\alpha-\beta}{\sqrt{2}}\end{bmatrix}=\frac{\alpha+\beta}{\sqrt{2}}|0\rangle+\frac{\alpha-\beta}{\sqrt{2}}|1\rangle$$

- Notice that the probability to obtain $|0\rangle$ upon measurement has been changed as the amplitude

$$\alpha\rightarrow\frac{\alpha+\beta}{\sqrt{2}}$$

while the probability to find $|1\rangle$ has been changed as the amplitude

$$\beta\rightarrow\frac{\alpha-\beta}{\sqrt{2}}$$

Quantum Interference

- Now, let’s look at the following scenario with $\alpha = \beta = 1/\sqrt{2}$

$$|+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \quad \text{---}\boxed{H}\text{---} \quad \begin{bmatrix}1\\0\end{bmatrix} = |0\rangle$$

- With the Hadamard transformation of the state $|+\rangle$ we have the following:
    - *Positive interference* with regard to the basis state $|0\rangle$. The two amplitudes add to increase the probability of finding $|0\rangle$ upon measurement. In fact, in this case, it goes to unity meaning we are certain to find $|0\rangle$.
    - *Negative interference* with regard to the basis state $|1\rangle$. We go from a state where there was a *50% chance of finding* 1 upon measurement to one where there is *no chance of finding* 1 upon measurement.

Quantum Interference

- Similarly, with $\alpha = 1/\sqrt{2}$, $\beta = -1/\sqrt{2}$

$$|-\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} = \frac{|0\rangle - |1\rangle}{\sqrt{2}} \;\longrightarrow\; \boxed{H} \;\longrightarrow\; \begin{bmatrix}0\\1\end{bmatrix} = |1\rangle$$

- With the Hadamard transformation of the state $|-\rangle$ we have the following:
  - *Positive interference* with regard to the basis state $|1\rangle$. The two amplitudes add to increase the probability of finding $|1\rangle$ upon measurement. In fact, in this case, it goes to unity meaning we are certain to find $|1\rangle$.
  - *Negative interference* with regard to the basis state $|0\rangle$. We go from a state where there was a *50% chance of finding* 1 upon measurement to one where there is *no chance of finding* 1 upon measurement.

Relative Phase

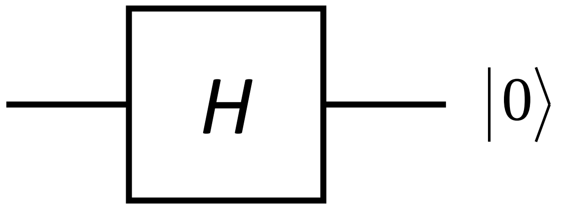
- Obviously, both $|+\rangle$ and $|-\rangle$ will have identical measurement probabilities; if we have 1000 electrons in spin state $|+\rangle$ and 1000 in state $|-\rangle$, a measurement of all of them will throw about $\left|1/\sqrt{2}\right|^2 \times 1000 = 500$ into state $|0\rangle$ and $\left|1/\sqrt{2}\right|^2 \times 1000 = 500$ into state $|1\rangle$

- *However, two states that have the same measurement probabilities are not necessarily the same state*

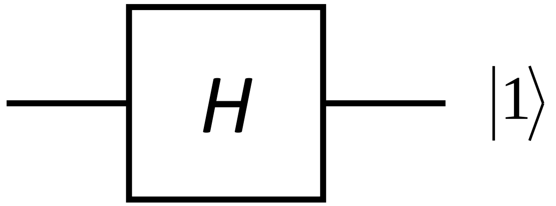
$$|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \qquad |-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

Relative Phase

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$



$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

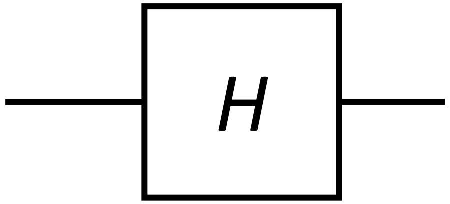


- If we give $|+\rangle$ and $|-\rangle$ as input to the *H* gate we obtain as output $|0\rangle$ and $|1\rangle$ states respectively, i.e., states which give rise to physically observable differences in measurement statistics
- Therefore, it is not possible to regard these states ($|+\rangle$ and $|-\rangle$) as physically equivalent, as we do with states differing by a global phase factor

Global Phase

The input states $|+\rangle$ and $e^{i\theta}|+\rangle$ are physically equivalent and so are the output states $|0\rangle$ and $e^{i\theta}|0\rangle$

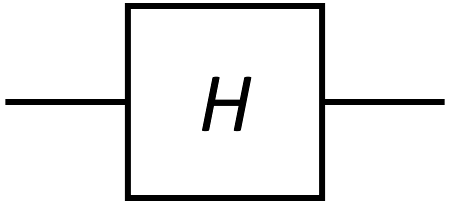
$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$



$$|0\rangle$$

$$H|+\rangle = |0\rangle$$

$$e^{i\theta}|+\rangle = e^{i\theta}\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)$$



$$e^{i\theta}|0\rangle$$

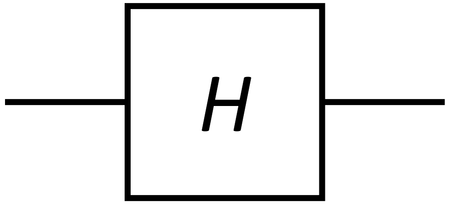
$$H\left(e^{i\theta}|+\rangle\right) = e^{i\theta}H\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) = e^{i\theta}|0\rangle$$

It follows from the H linearity

Relative Phase

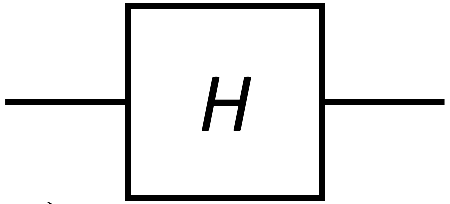
The input states $|-\rangle$ and $e^{i\theta}|-\rangle$ are physically equivalent and so are the output states $|1\rangle$ and $e^{i\theta}|1\rangle$

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$



$$|1\rangle$$

$$H|-\rangle = |1\rangle$$



$$e^{i\theta}|1\rangle$$

$$e^{i\theta}|-\rangle = e^{i\theta}\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

$$H\left(e^{i\theta}|-\rangle\right) = e^{i\theta} H\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) = e^{i\theta}|1\rangle$$

It follows from the H linearity

Special 1-Qubit Gates

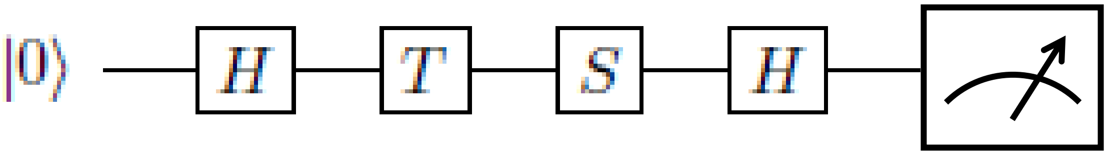
Gate	Action on Computational Basis	Matrix Representation
Identity	$I\|0\rangle = \|0\rangle$ $I\|1\rangle = \|1\rangle$	$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
Pauli X	$X\|0\rangle = \|1\rangle$ $X\|1\rangle = \|0\rangle$	$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
Pauli Y	$Y\|0\rangle = i\|1\rangle$ $Y\|1\rangle = -i\|0\rangle$	$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$
Pauli Z	$Z\|0\rangle = \|0\rangle$ $Z\|1\rangle = -\|1\rangle$	$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
Phase S	$S\|0\rangle = \|0\rangle$ $S\|1\rangle = i\|1\rangle$	$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$
T	$T\|0\rangle = \|0\rangle$ $T\|1\rangle = e^{i\pi/4}\|1\rangle$	$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$
Hadamard H	$H\|0\rangle = \frac{1}{\sqrt{2}}(\|0\rangle + \|1\rangle)$ $H\|1\rangle = \frac{1}{\sqrt{2}}(\|0\rangle - \|1\rangle)$	$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

Combination of Quantum Gates

- We can combine these quantum gates to create all sorts of states

$$
\begin{aligned}
HSTH|0\rangle &= HST\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\
&= HS\frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\pi/4}|1\rangle\right) \\
&= H\frac{1}{\sqrt{2}}\left(|0\rangle + e^{i3\pi/4}|1\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) + e^{i3\pi/4}\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right] \\
&= \frac{1}{2}\left[\left(1 + e^{i3\pi/4}\right)|0\rangle + \left(1 - e^{i3\pi/4}\right)|1\rangle\right],
\end{aligned}
$$

where in the third line, we used $ie^{i\pi/4} = e^{i\pi/2}e^{i\pi/4} = e^{i3\pi/4}$.



Gates operate from *left-to-right*, but operator algebra moves from *right to-left*. When translating a circuit into a product of matrices, we must reverse the order. So, *stay alert*!

Combination of Quantum Gates

- Then, if we measure this qubit in the Z-basis $\{|0\rangle, |1\rangle\}$ the probability of getting $|0\rangle$ is

$$\begin{aligned}
\left|\frac{1}{2}\left(1+e^{i3\pi/4}\right)\right|^2 &= \frac{1}{2}\left(1+e^{i3\pi/4}\right)\frac{1}{2}\left(1+e^{-i3\pi/4}\right) \\
&= \frac{1}{4}\left(1+e^{-i3\pi/4}+e^{i3\pi/4}+e^{0}\right) \\
&= \frac{1}{4}\left(2+2\cos\frac{3\pi}{\cancel{2}}\right) \\
&= \frac{1}{2}\left(1-\frac{\sqrt{2}}{2}\right) \\
&\approx 0.146,
\end{aligned}$$

4

where to go from the second to the third line, we used Euler's formula

$$e^{i\vartheta}+e^{-i\vartheta}=2\cos\vartheta$$

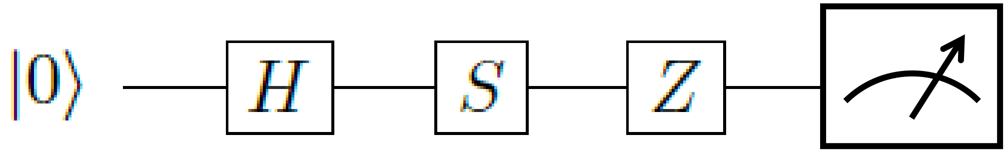
Combination of Quantum Gates

- Similarly, the probability of getting $|1\rangle$ is

$$\left|\frac{1}{2}\left(1-e^{i3\pi/4}\right)\right|^2=\frac{1}{4}\left(2-2\cos\frac{3\pi}{2}\right)=\frac{1}{2}\left(1+\frac{\sqrt{2}}{2}\right)\approx 0.854.$$

Combination of Quantum Gates

- Another combination of quantum gates.



- So we start with a single qubit in the $|0\rangle$ state and apply a Hadamard gate *H* to it, followed by a phase gate *S*, and finally a *Z* gate
- The state at the output of the Z gate, will be $ZSH|0\rangle$

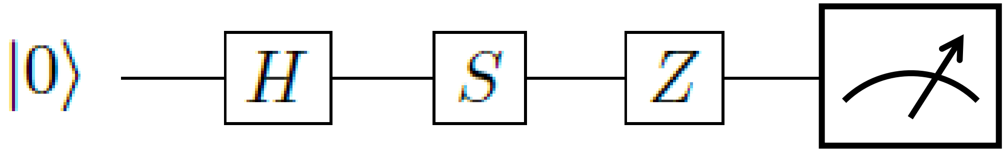
Combination of Quantum Gates

- $ZSH\left|0\right\rangle$ results in the state,

$$ZSH\left|0\right\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1\\ i\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -i\end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -i\end{bmatrix} = \frac{1}{\sqrt{2}}\left[\left|0\right\rangle - i\left|1\right\rangle\right]$$

and if we measure the qubit in the Z-basis, we get $\left|0\right\rangle$ or $\left|1\right\rangle$ with equal probability



Rotation About x-, y-, and z-Axes

- Question: **what is the most general kind of quantum gate for a single qubit?**

- To address this, we must first introduce the family of quantum gates that perform *rotations* about *the three mutually perpendicular axes* of the *Bloch sphere*

- A single qubit state is represented by a point on the surface of the Bloch sphere

Rotation About x-, y-, and z-Axes

- The effect of a single qubit gate that acts in this state is to map it to some other point on the Bloch sphere
- The gates that rotate states around the *x*-, *y*-, and *z*-axes are of special significance since we will be able *to decompose an **arbitrary** 1-qubit quantum gate into sequences of such rotation gates*
- Any point on the surface of the Bloch sphere can be specified using its (*x, y, z*) coordinates or, equivalently, its r, θ, ϕ coordinates (let's ignore the global phase for now)

Rotation About x-, y-, and z-Axes

- These two coordinate systems are related via the equations:

$$x = r\sin(\theta)\cos(\phi)$$
$$y = r\sin(\theta)\sin(\phi)$$
$$z = r\cos(\theta)$$

- So, what are the quantum gates that rotate this state about the *x*-, *y*-, or *z*-axes?

- We claim that these gates can be built from the Pauli *X*, *Y* , *Z*, matrices, and the fourth Pauli matrix, *I*, can be used to achieve a *global overall phase shift*

Rotation About x-, y-, and z-Axes

Let's define the following unitary matrices

$$R_x(\alpha) = \exp(-i\alpha\, X/2) = \begin{bmatrix} \cos\left(\frac{\alpha}{2}\right) & -i\sin\left(\frac{\alpha}{2}\right) \\ -i\sin\left(\frac{\alpha}{2}\right) & \cos\left(\frac{\alpha}{2}\right) \end{bmatrix}$$

$$R_z(\alpha) = \exp(-i\alpha\, Z/2) = \begin{bmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{bmatrix}$$

$$R_y(\alpha) = \exp(-i\alpha\, Y/2) = \begin{bmatrix} \cos\left(\frac{\alpha}{2}\right) & -\sin\left(\frac{\alpha}{2}\right) \\ \sin\left(\frac{\alpha}{2}\right) & \cos\left(\frac{\alpha}{2}\right) \end{bmatrix}$$

$$Ph(\delta) = e^{i\delta}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Rotation About x-, y-, and z-Axes

To prove (in the above expressions for the rotations) that the exponentials are equivalent to the matrices, you have to prove first that, given a real number *x* and an *A* matrix such that $A^2=I$ the following relation holds

$$e^{iAx} = \cos(x)I + i\sin(x)A$$

This condition holds true for the rotation gates as we have already proved that $X^2=Y^2=Z^2=I$

Rotation About x-, y-, and z-Axes

- Consider the gate $R_z(\alpha)$

- Let's see how this gate transforms an arbitrary single qubit

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle \qquad \rightarrow$$

$$R_z(\alpha)|\psi\rangle = \begin{bmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{bmatrix} = \begin{bmatrix} e^{-i\alpha/2}\cos\left(\frac{\theta}{2}\right) \\ e^{i\alpha/2}e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{bmatrix} = e^{-i\alpha/2}\cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\alpha/2}e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

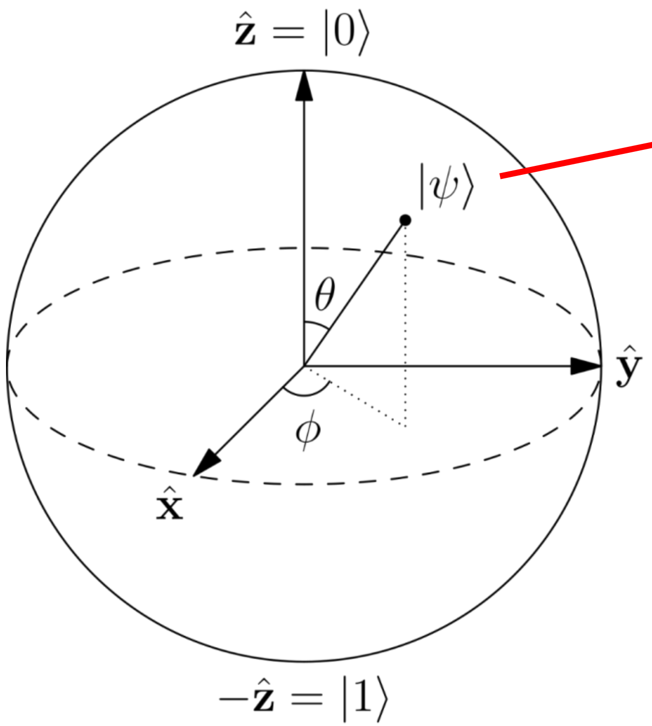
Rotation About x-, y-, and z-Axes

From the above slide $\rightarrow$

$$R_z(\alpha)|\psi\rangle = e^{-i\alpha/2}\left(\cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\alpha/2}e^{i\alpha/2}e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle\right) \equiv \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\alpha}e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

$$= \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i(\phi+\alpha)}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

where ≡ is to be read as "equal up to an unimportant arbitrary *global phase factor*"

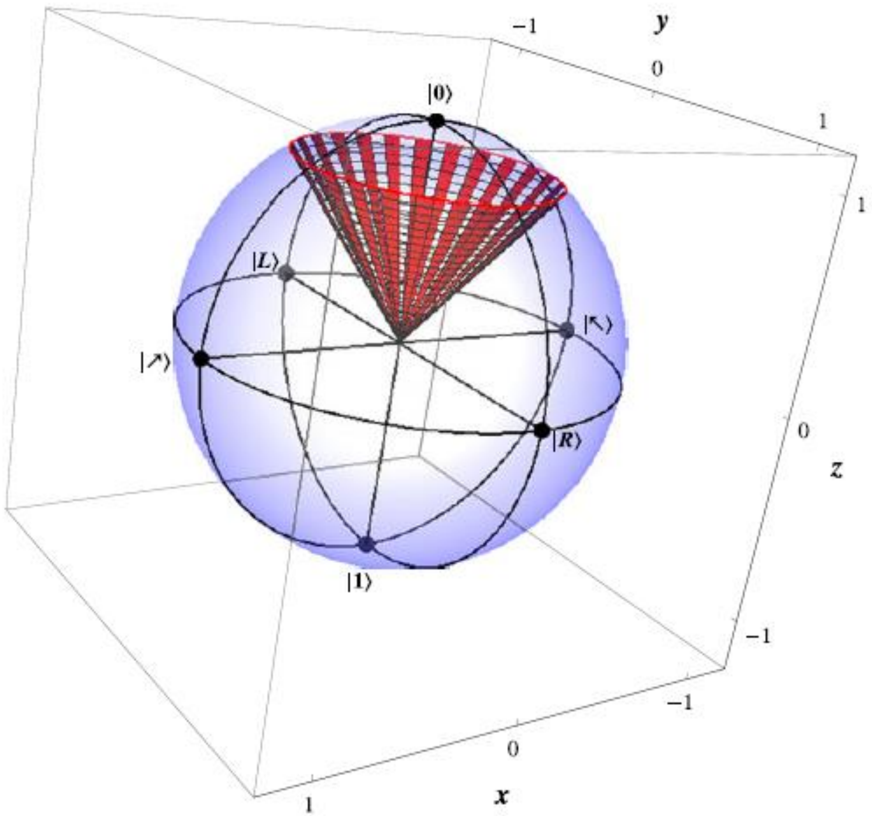


$$|\psi\rangle = e^{i\gamma}\left(\cos\frac{\vartheta}{2}|0\rangle + e^{i\phi}\sin\frac{\vartheta}{2}|1\rangle\right)$$

Rotation About x-, y-, and z-Axes

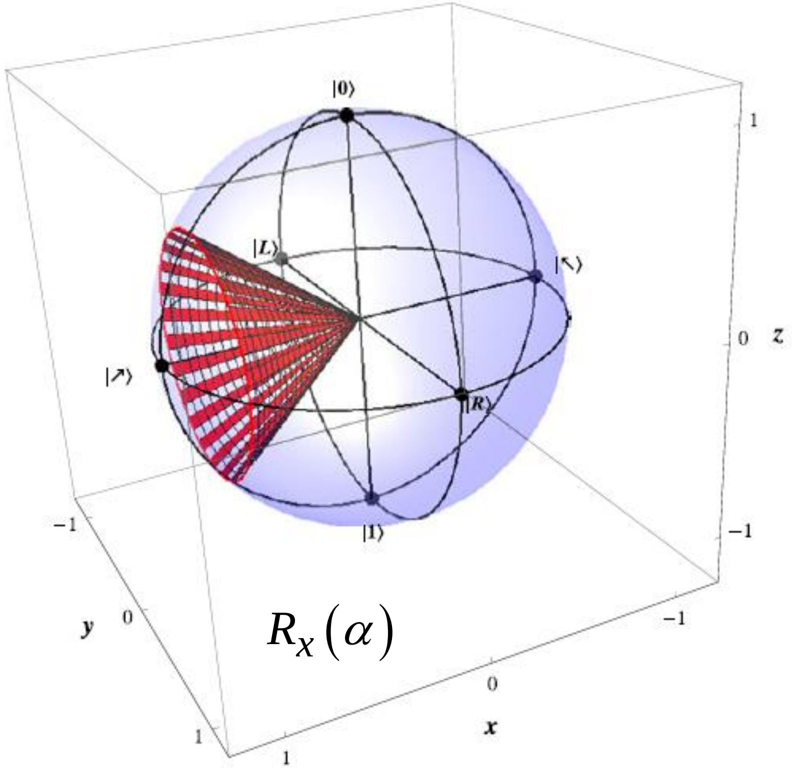
- Hence the action of the $R_z(\alpha)$ gate on $|\psi\rangle$ has been to advance the angle ϕ by α and hence rotate the state about the *z*-axis through angle α

- This is why we call $R_z(\alpha)$ a *z*-rotation gate

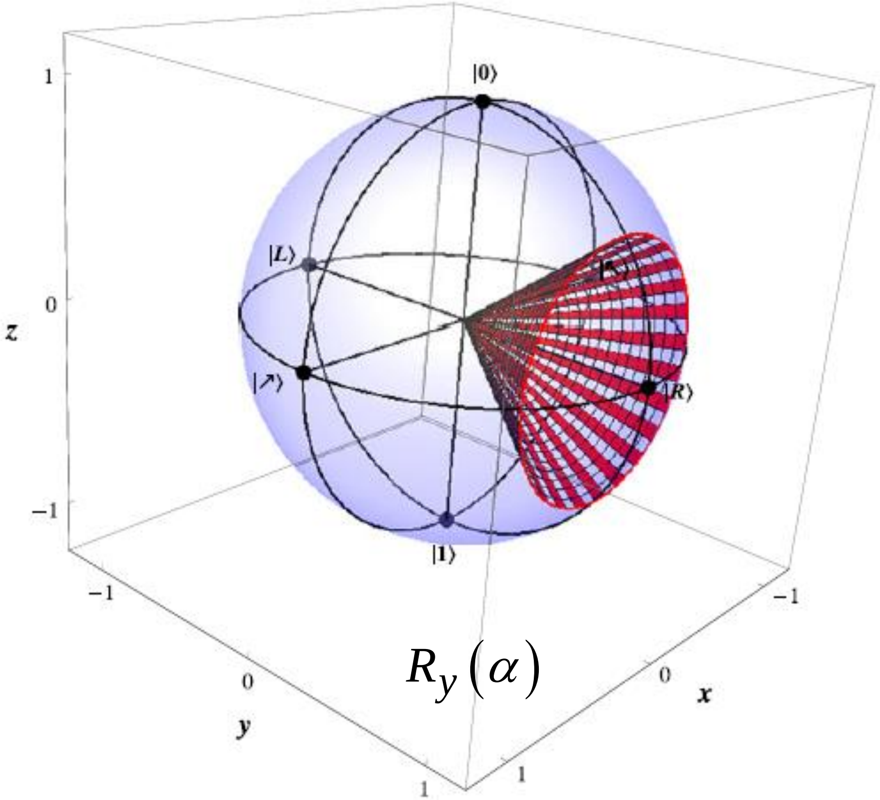


Rotation About x-, y-, and z-Axes

- We leave it to the exercises for you to prove that $R_x(\alpha)$ and $R_y(\alpha)$ rotate the state about the *x*- and *y*-axes respectively



$R_x(\alpha)$



$R_y(\alpha)$

Rotation About x-, y-, and z-Axes

- Rotations on the Bloch sphere do not conform to commonsense intuitions about rotations that we have learned from our experience of the everyday world

- In particular, usually, a rotation of 2π radians (i.e., 360 degrees) of a solid object about any axis, restores that object to its initial orientation

Rotation About x-, y-, and z-Axes

- However, this is not true of rotations on the Bloch sphere! When we rotate a quantum state through 2π on the Bloch sphere we don't return it to its initial state

- Instead, we pick up a phase factor

Rotation About x-, y-, and z-Axes

- To see this, let's compute the effect of rotating our arbitrary single qubit state

$$\left|\psi\right\rangle = \cos\left(\frac{\theta}{2}\right)\left|0\right\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)\left|1\right\rangle$$

about the *z*-axis through 2*π* radians

$$R_z(2\pi)\left|\psi\right\rangle = \begin{bmatrix} e^{-i\pi} & 0 \\ 0 & e^{i\pi} \end{bmatrix}\begin{bmatrix} \cos\left(\frac{\theta+2\pi}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta+2\pi}{2}\right) \end{bmatrix} = \begin{bmatrix} -\cos\left(\frac{\theta}{2}\right) \\ -e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{bmatrix} = -\left|\psi\right\rangle$$

which has an extra overall phase of −1

Rotation About x-, y-, and z-Axes

- To see this, let's compute the effect of rotating our arbitrary single qubit state

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

$$e^{-i\pi} = e^{i\pi} = -1$$

about the *z*-axis through 2π radians

$$R_z(2\pi)|\psi\rangle = \begin{bmatrix} e^{-i\pi} & 0 \\ 0 & e^{i\pi} \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{bmatrix} = \begin{bmatrix} e^{-i\pi}\cos\left(\frac{\theta}{2}\right) \\ e^{i\pi}e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{bmatrix} = \begin{bmatrix} -\cos\left(\frac{\theta}{2}\right) \\ -e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{bmatrix} = -|\psi\rangle$$

which has an extra overall phase of −1

Rotation About x-, y-, and z-Axes

- To restore a state back to its original form we need to rotate it through 4π on the Bloch sphere

- Have you ever encountered anything like this in your everyday world? You probably think not, but you'd be wrong!

- See the "Dirac's Belt" or the "Belt Trick" video https://www.youtube.com/watch?v=Vfh21o-JW9Q

X From a Rotation Gate

- On the Bloch sphere, it can be shown that *X* is a rotation of 180^0 about the *x*-axis together with *global phase shifts* of 90^0

$$R_x(\pi)\cdot Ph\left(\frac{\pi}{2}\right)=\begin{bmatrix}\cos\frac{\pi}{2} & -i\sin\frac{\pi}{2}\\ -i\sin\frac{\pi}{2} & \cos\frac{\pi}{2}\end{bmatrix}e^{i\pi/2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}0 & -i\\ -i & 0\end{bmatrix}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}0 & -i\\ -i & 0\end{bmatrix}(i)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}=X$$

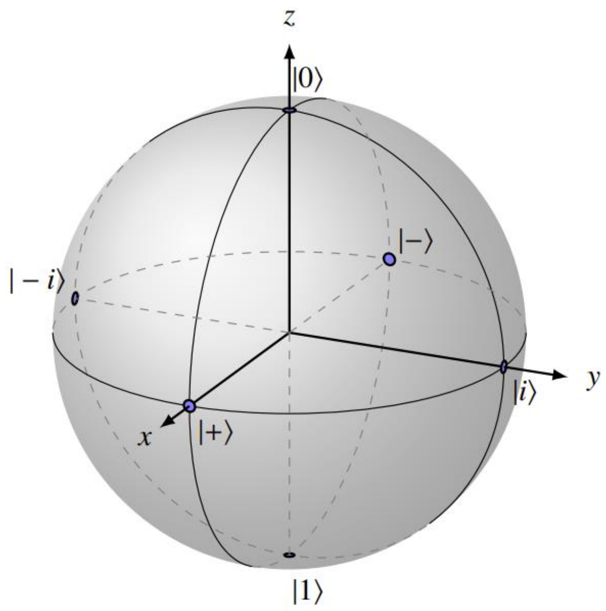
X From a Rotation Gate

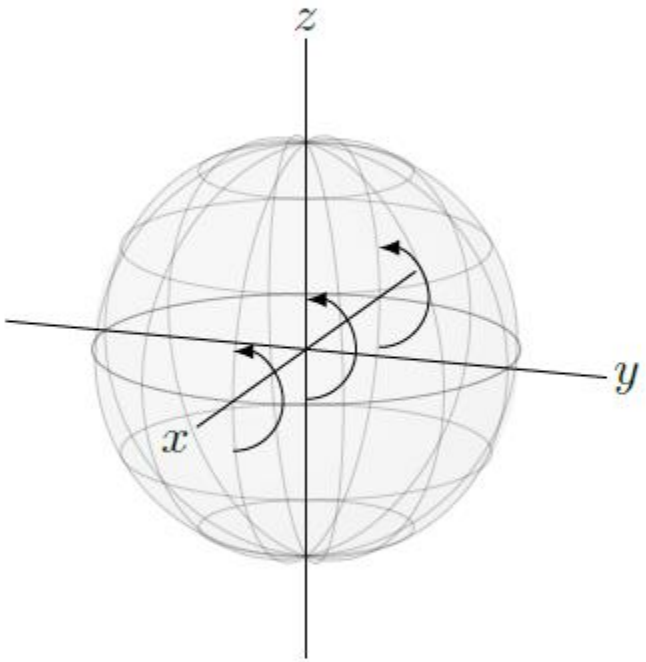
- With this rotation in mind, we geometrically see that X causes $|0\rangle$ (the north pole) to rotate to $|1\rangle$ (the south pole), and vice versa
- We also see that $|i\rangle$ and $|{-i}\rangle$ rotate to each other, whereas $|+\rangle$ and $|-\rangle$ are unchanged
- Note, however, that mathematically

$$X|-\rangle = -|-\rangle \equiv |-\rangle$$

since the global phase does not matter

- If we apply the X-gate twice, we rotate around the x-axis of the Bloch sphere by 360^0, which does nothing. Then, $X^2 = I$



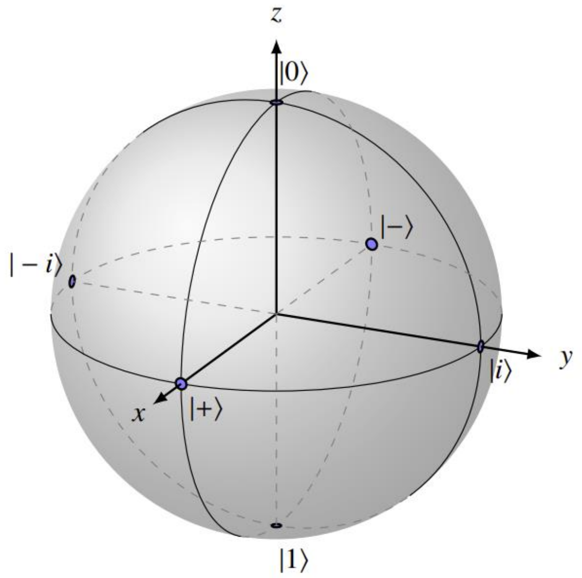


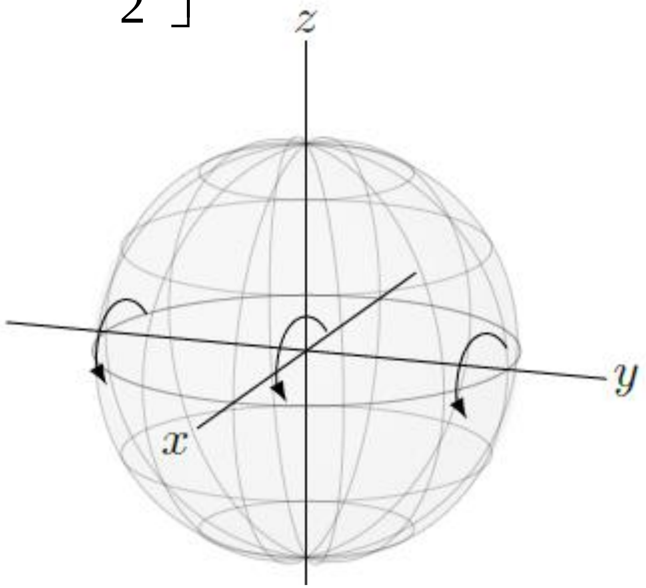
Y From a Rotation Gate

- On the Bloch sphere, it can be shown that Y is a rotation of 180^0 about the y-axis together with *global phase shifts* of 90^0

$$R_y(\pi)\cdot Ph\left(\frac{\pi}{2}\right)=\begin{bmatrix}\cos\frac{\pi}{2} & -\sin\frac{\pi}{2}\\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2}\end{bmatrix}e^{i\pi/2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}(i)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}=Y$$



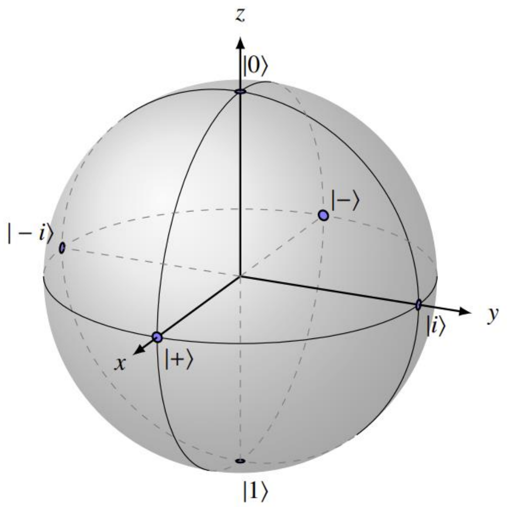


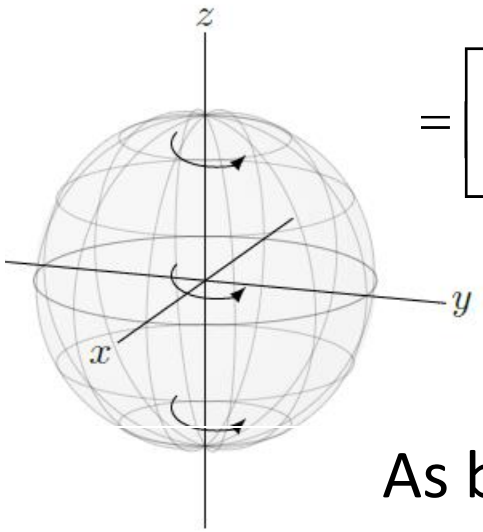
If we apply the Y-gate twice, we rotate around the y-axis of the Bloch sphere by 360^0, which does nothing. Then, $Y^2=I$

Z From a Rotation Gate

- On the Bloch sphere, it can be shown that *Z* is a rotation of 180^0 about the z-axis together with *global phase shifts* of 90^0

$$R_z(\pi)\cdot Ph\left(\frac{\pi}{2}\right)=\begin{bmatrix} e^{-i\pi/2} & 0 \\ 0 & e^{i\pi/2}\end{bmatrix} e^{i\pi/2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}\cos\frac{\pi}{2}-i\sin\frac{\pi}{2} & 0 \\ 0 & \cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\end{bmatrix}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$





$$=\begin{bmatrix}-i & 0\\ 0 & i\end{bmatrix}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}-i & 0\\ 0 & i\end{bmatrix}(i)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}=Z$$

As before $Z^2=I$

S From a Rotation Gate

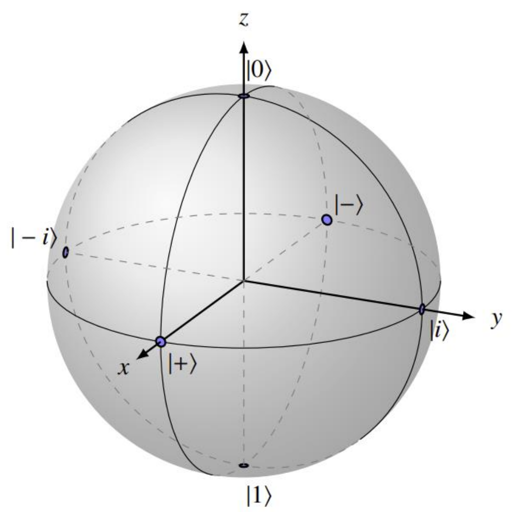
- On the Bloch sphere, it can be shown that *S* is a rotation of 90^0 about the *z*-axis together with *global phase shifts* of 45^0

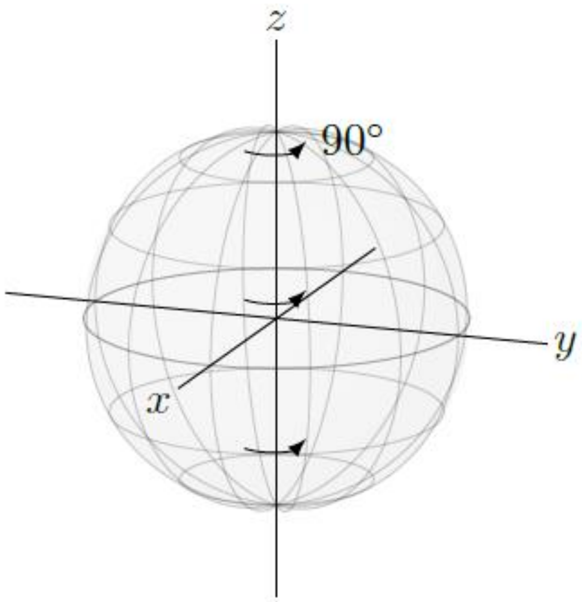
$$R_z\left(\frac{\pi}{2}\right)\cdot Ph\left(\frac{\pi}{4}\right)=\begin{bmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4}\end{bmatrix} e^{i\pi/8}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}\cos\frac{\pi}{4}-i\sin\frac{\pi}{4} & 0 \\ 0 & \cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\end{bmatrix}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$=\frac{1}{\sqrt{2}}\begin{bmatrix}1-i & 0\\ 0 & 1+i\end{bmatrix}\left(\frac{1}{\sqrt{2}}(1+i)\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}1-i & 0\\ 0 & 1+i\end{bmatrix}\begin{bmatrix}1+i & 0\\ 0 & 1+i\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1-i^2 & 0\\ 0 & (1+i)^2\end{bmatrix}=\frac{1}{2}\begin{bmatrix}2 & 0\\ 0 & 2i\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix}=S$$

S From a Rotation Gate





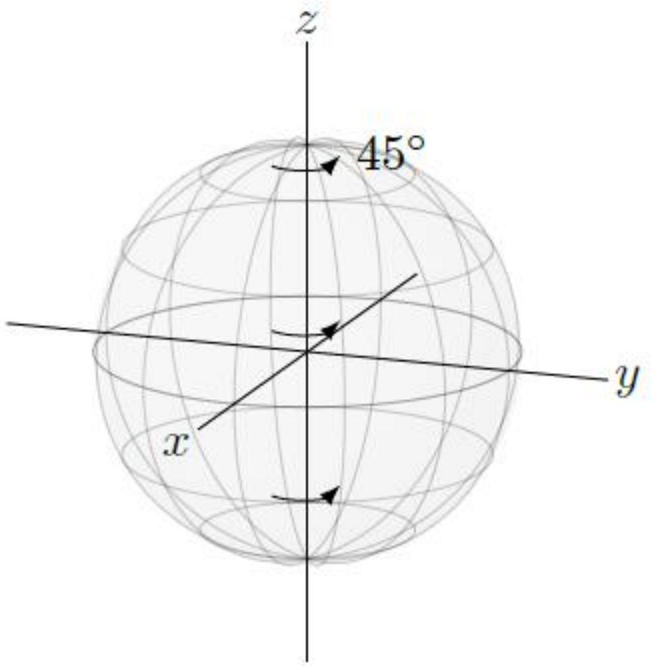
- Now, S^2 rotates by 90^0 twice, so it is equivalent to rotating by 180^0
- Then, $S^2 = Z$
- We would need to apply S four times in order to return to the same point on the Bloch sphere, so $S^4 = I$

T From a Rotation Gate

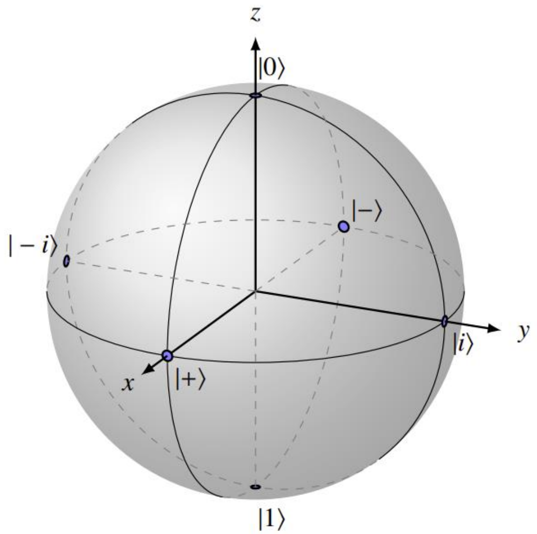
- On the Bloch sphere, it can be shown that T (also called $\pi/8$ gate) is a rotation of 45^0 about the z-axis together with *global phase shifts* of $\pi/8$ radiants

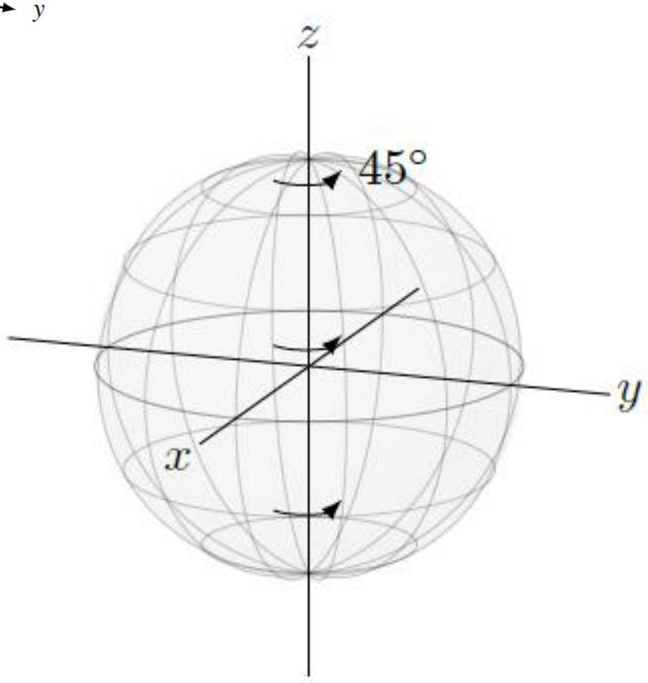
$$R_z\left(\frac{\pi}{4}\right).Ph\left(\frac{\pi}{8}\right)=\begin{bmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{bmatrix} e^{i\pi/8}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}=T$$



T From a Rotation Gate





It is obvious that $T^2 = S$ and $T^4 = Z$, since rotating by 45^0 twice is equivalent to rotating by 90^0, and rotating by 45^0 four times is equivalent to rotating by 180^0

$$T^2 = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = S$$

H From Rotation Gates

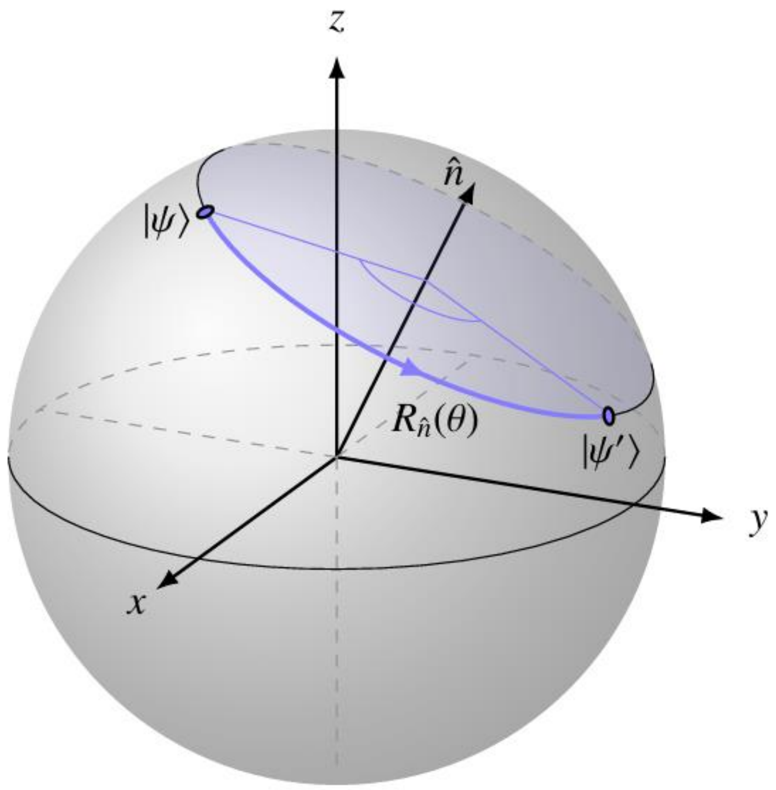
- On the Bloch sphere, it can be shown that *H* is a rotation of 180^0 about the x + z-axis

- Before showing it, we need to consider the Rotation About an Arbitrary Axis

Rotation About an Arbitrary Axis

- If $n = \left[n_x, n_y, n_z \right]$ is a real unit vector in three dimensions, then it can be shown that the operator $R_n(\theta)$ rotates the Bloch vector by an angle θ about the $\hat{n}$ axis, where

$$R_n(\theta) = \exp\left(-i\theta \left(\hat{n} \cdot \frac{\hat{\sigma}}{2} \right) \right)$$



and $\hat{\sigma}$ denotes the three-component vector (*X,Y,Z*) of Pauli matrices

Rotation About an Arbitrary Axis

- Furthermore, it is not hard to show that $\left(n\cdot\vec{\sigma}\right)^2 = I$, and therefore we can use the special case operator exponential and write

$$R_n(\theta) = \exp\left(-i\theta\left(n\cdot\frac{\vec{\sigma}}{2}\right)\right) = \exp\left(-i\frac{\theta}{2}\left(n\cdot\vec{\sigma}\right)\right) = \cos\left(\frac{\theta}{2}\right)I - i\sin\left(\frac{\theta}{2}\right)n\cdot\vec{\sigma}$$

$$= \cos\left(\frac{\theta}{2}\right)I - i\sin\left(\frac{\theta}{2}\right)\left(n_x X + n_y Y + n_z Z\right)$$

Given a real number *x* and an *A* matrix such that $A^2=I$ the following relation holds

$$e^{iAx} = \cos(x)I + i\sin(x)A$$

Arbitrary Unitary Operator

- It can be shown that *an arbitrary single qubit unitary operator* *U* can be written in the form

$$U = \exp(i\alpha) R_n(\theta)$$

for some real number α and θ, and a real three-dimensional unit vector $n = [n_x, n_y, n_z] \quad \rightarrow$

$$U = \exp(i\alpha) R_n(\theta) = \exp(i\alpha)\left[\cos\left(\frac{\theta}{2}\right) I - i \sin\left(\frac{\theta}{2}\right)\left(n_x X + n_y Y + n_z Z\right)\right]$$

Arbitrary Unitary Operator

$$U = \exp(i\alpha)\left[\cos\left(\frac{\theta}{2}\right)I - i\sin\left(\frac{\theta}{2}\right)\left(n_x X + n_y Y + n_z Z\right)\right]$$

- For example, consider

$\alpha = \frac{\pi}{2}$, $\theta = \pi$, and $n = \left[\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right] \quad \rightarrow \quad n \cdot \vec{\sigma} = \left(n_x X + n_y Y + n_z Z\right) = \frac{X+Z}{\sqrt{2}}$

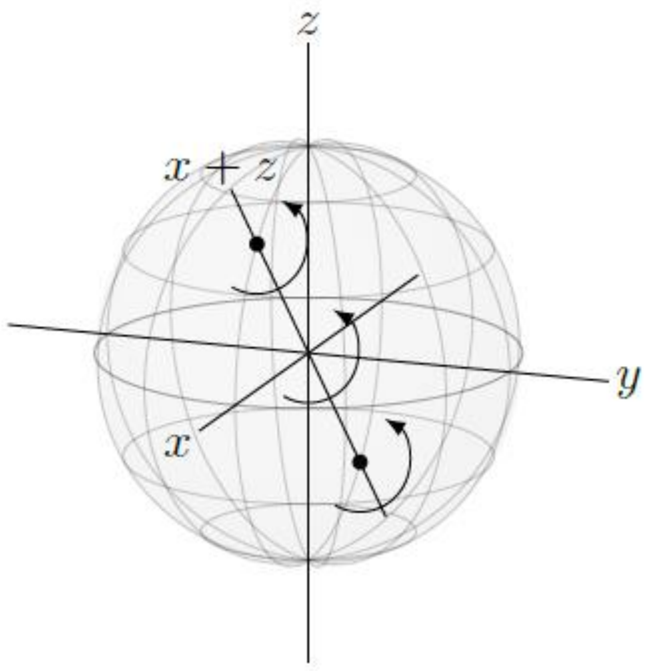
$$U = \exp(i\,\pi/2)R_n(\pi) = \exp(i\,\pi/2)\left[\cos\left(\frac{\pi}{2}\right)I - i\sin\left(\frac{\pi}{2}\right)\frac{1}{\sqrt{2}}(X+Z)\right]$$

$$= i\left[\cos\left(\frac{\pi}{2}\right)I - i\sin\left(\frac{\pi}{2}\right)\frac{1}{\sqrt{2}}(X+Z)\right] = \frac{1}{\sqrt{2}}(X+Z)$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = H$$

H From a Rotation Gate

Thus, on the Bloch sphere, *H* is a rotation of 180^0 about the x+z-axis together with *global phase shifts* of 90^0



H From Rotation Gates

- However, we can also demonstrate that H can be obtained by the following combination of rotation

$$R_x(\pi)\cdot R_y\left(\frac{\pi}{2}\right)\cdot Ph\left(\frac{\pi}{2}\right)=\begin{bmatrix}\cos\frac{\pi}{2} & -i\sin\frac{\pi}{2}\\ -i\sin\frac{\pi}{2} & \cos\frac{\pi}{2}\end{bmatrix}\begin{bmatrix}\cos\frac{\pi}{4} & -\sin\frac{\pi}{4}\\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4}\end{bmatrix}e^{i\pi/2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}0 & -i\\ -i & 0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\frac{i}{\sqrt{2}}\begin{bmatrix}0 & -i\\ -i & 0\end{bmatrix}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$=\frac{1}{\sqrt{2}}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}=H$$

Arbitrary Unitary Operator

- An arbitrary unitary operator on a single qubit can be written in many ways as a combination of rotations, together with *global phase shifts* on the qubit

- For example, we have already seen that

$$X = R_x(\pi) \cdot Ph\left(\frac{\pi}{2}\right) \qquad H = R_x(\pi) \cdot R_y\left(\frac{\pi}{2}\right) \cdot Ph\left(\frac{\pi}{2}\right)$$

- However, we can easily show that

$$X = R_y(\pi) \cdot R_z(\pi) \cdot Ph\left(\frac{\pi}{2}\right) \qquad H = R_y\left(\frac{\pi}{2}\right) \cdot R_z(\pi) \cdot Ph\left(\frac{\pi}{2}\right)$$

Arbitrary Unitary Operator

- The following theorem will be particularly useful in later applications to controlled operations.

- *Theorem*: (*Z-Y* **decomposition for a single qubit**) Suppose *U* is a unitary operation on a single qubit. Then there exist real numbers α, β, γ and δ such that

$$U = \exp(i\alpha) R_z(\beta) \cdot R_y(\gamma) \cdot R_z(\delta)$$

Evolution

Evolution

Question

- How does the state, $|\psi\rangle$, of a quantum mechanical system change with time?

Answer

- **Postulate 2:** The evolution of a *closed* quantum system is described by a *unitary transformation*. That is, the state $|\psi\rangle$ of the system at time t_1 is related to the state $|\psi'\rangle$ of the system at time t_2 by a unitary operator U which depends only on the times t_1 and t_2,

$$|\psi'\rangle = |\psi(t_2)\rangle = U(t_1, t_2)|\psi(t_1)\rangle$$

Evolution

- Postulate 2 requires that the system being described be closed

- That is, it is not interacting in any way with other systems

- In reality, all systems (except the Universe as a whole) interact at least somewhat with other systems

- Nevertheless, there are interesting systems which can be described to a good approximation as being closed, and which are described by unitary evolution to some good approximation

Evolution

- Postulate 2 describes how the quantum states of a closed quantum system at two different times are related

- A more refined version of this postulate can be given which describes the evolution of a quantum system in ***continuous time***

- From this more refined postulate we will recover Postulate 2

Evolution

- **Postulate 2':** The time evolution of the state of a closed quantum system is described by the *Schrodinger equation*,

$$i\hbar \frac{d\left|\psi(t)\right\rangle}{dt} = H\left|\psi(t)\right\rangle$$

- In this equation, $\hbar$ is a physical constant known as *Planck's constant*
- The exact value is not important to us. In practice, it is common to absorb the factor $\hbar$ into *H*, effectively setting $\hbar = 1$
- *H* is a fixed Hermitian $\left(H = H^{\dagger}\right)$ operator known as the *Hamiltonian (Not Hadamard!!)* of the closed system

Evolution

- **Question:** What is the connection between the Hamiltonian picture of dynamics, Postulate 2', and the unitary operator picture, Postulate 2?

$$\left|\psi\left(t_2\right)\right\rangle = \exp\left[\frac{-iH\left(t_2 - t_1\right)}{\hbar}\right]\left|\psi\left(t_1\right)\right\rangle = U\left(t_1, t_2\right)\left|\psi\left(t_1\right)\right\rangle$$

  where we define

$$U\left(t_1, t_2\right) = \exp\left[\frac{-iH\left(t_2 - t_1\right)}{\hbar}\right]$$

- There is therefore a one-to-one correspondence between the discrete-time description of dynamics using unitary operators, and the continuous time description using Hamiltonians