1) Consider the unconstrained optimization problem

$$\begin{cases} \min\ x_1^2 + 4x_2^2 + x_3^2 - 4x_1x_2 - 2x_1x_3 + 4x_2x_3 - 2x_1 + 4x_2 + 2x_3 \\ x \in \mathbb{R}^3 \end{cases}$$

(a) Prove that the problem admits a global minimum;

(b) Apply the gradient method with an inexact line search, setting $\bar{t} = 1$,$\alpha = 0.2$, $\gamma = 0.7$, with starting point $x^0 = (-5, 8, 4)^T$ and using $\|\nabla f(x)\| < 10^{-4}$ as stopping criterion. How many iterations are needed by the algorithm? Write explicitly the vectors found at the last three iterations.

(c) Is the obtained solution a global minimum of the given problem? Justify the answer.

SOLUTION

(a) The objective function $f(x)$ is quadratic, i.e., $f(x) = (1/2)x^TQx + c^Tx$ with

$$Q = \begin{pmatrix} 2 & -4 & -2 \\ -4 & 8 & 4 \\ -2 & 4 & 2 \end{pmatrix} \qquad c^T = (-2, 4, 2)$$

Since $eig(Q) = [0; 0; 12]$ and $rank(Q) = rank([Q, -c]) = 1$, then f is convex and the system $Qx + c = 0$ admits a solution. Consequently, f admits a global minimum point.

(b) **Matlab solution**

```
%% Data
global Q,c

Q = [2 -4 -2;-4 8 4; -2 4 2];
c = [-2 4 2]';

disp('Eigenvalues of Q:')
eig(Q)


alpha = 0.2;
gamma = 0.7;
tbar = 1;
x0 = [-5;8;4];
tolerance = 10^(-4) ;

X=[ ];

ITER = 0 ;
x = x0;

while true
    [v, g] = f(x);

    X=[X;ITER,x(1),x(2),x(3),v,norm(g)];

    % stopping criterion
    if norm(g) < tolerance
        break
    end

    d = -g;                          % search direction


     t = tbar ;                       % Armijo inexact line search
    while f(x+t*d) > v + alpha*g'*d*t
        t = gamma*t ;
    end
```

```
    x = x + t*d ;                        % new point
    ITER = ITER + 1 ;
    end

disp('optimal solution')
x
v
norm(g)
ITER

function [v, g] = f(x)

global Q c

v = 0.5*x'*Q*x+ c'*x ;

g = Q*x+c;
end
```

We obtain the following solution:

```
x =

   -0.6667
   -0.6667
   -0.3333


v =

    -1


ITER =

    16
```

In particular, the gradient norm evaluated at the final point is:

```
ans =   0.0000870678769
```

The iterations of the algorithm are 16.

The vectors found at the last three iterations are:

```
14.0000   -0.6667   -0.6666   -0.3333
15.0000   -0.6667   -0.6667   -0.3333
 16.0000   -0.6667   -0.6667   -0.3333
```

(c) The found point $x = (-2/3, -2/3, -1/3)$ is a global minimum since the objective function is convex as shown in point (a) and $Qx + c = 0$.

2) Consider a regression problem with the following data set where the points $(x_i, y_i), i = 1, 26$, are given by the row vectors of the matrices:

$$\begin{pmatrix} -4.0000 & 4 \\ -3.6000 & 8 \\ -3.2000 & 16 \\ -3.0000 & 17.5 \\ -2.8000 & 11.48 \\ -2.4000 & 16.82 \\ -1.2000 & 16.15 \\ -1.0000 & 11.68 \\ -0.8000 & 6.00 \\ -0.6000 & 7.82 \\ -0.4000 & 2.82 \\ -0.2000 & 2.71 \\ 0 & 1.16 \end{pmatrix} \qquad \begin{pmatrix} 0.2000 & -1.42 \\ 0.4000 & -3.84 \\ 0.6000 & -4.71 \\ 0.8000 & -8.15 \\ 1.0000 & -7.33 \\ 1.2000 & -13.64 \\ 1.4000 & -15.26 \\ 1.6000 & -14 \\ 1.8000 & -13 \\ 2.0000 & -10.50 \\ 2.4000 & -11 \\ 3.6000 & -10 \\ 4.0000 & -2 \end{pmatrix}$$

(a) Write the dual formulation of a nonlinear ε-SV regression model with $C = 7$, $\varepsilon = 3$ and a Gaussian kernel $k(x, y) := e^{-\|x-y\|^2}$;

(b) Solve the problem in (a) and find the regression function;

(c) Find the support vectors;

(d) Find the points of the data set that are outside the ε-tube, by making use of the dual solution.

SOLUTION

(a) Let $\ell = 26$, $(x_i, y_i))$, $i = 1, .., \ell$ be the i-th element of the data set, $C = 7$, $\varepsilon = 3$, $k(x, y) := e^{-\|x-y\|^2}$. The dual formulation of a nonlinear ε-SV regression model is

$$\begin{cases} \max\limits_{(\lambda^+,\lambda^-)} \ -\dfrac{1}{2}\sum\limits_{i=1}^{\ell}\sum\limits_{j=1}^{\ell}(\lambda_i^+ - \lambda_i^-)(\lambda_j^+ - \lambda_j^-)k(x_i, x_j) \\ \qquad\qquad -\varepsilon\sum\limits_{i=1}^{\ell}(\lambda_i^+ + \lambda_i^-) + \sum\limits_{i=1}^{\ell} y_i(\lambda_i^+ - \lambda_i^-) \\ \sum\limits_{i=1}^{\ell}(\lambda_i^+ - \lambda_i^-) = 0 \\ \lambda_i^+, \lambda_i^- \in [0, C], i = 1, .., \ell \end{cases} \tag{1}$$

(b) **Matlab solution**

```
data = [
   -4.0000    4
   -3.6000     8
      -3.2000    16
   -3.0000    17.5
   -2.8000    11.48
   -2.4000    16.82
   -1.2000    16.15
   -1.0000    11.68
   -0.8000     6.00
   -0.6000     7.82
   -0.4000     2.82
   -0.2000     2.71
         0     1.16
    0.2000    -1.42
    0.4000    -3.84
    0.6000    -4.71
    0.8000    -8.15
    1.0000    -7.33
    1.2000   -13.64
    1.4000   -15.26
    1.6000   -14
    1.8000    -13
    2.0000   -10.50
    2.4000   -11
    3.6000   -10
    4.0000    -2
```

```
       ];

x = data(:,1) ;
y = data(:,2) ;
l = length(x) ;

epsilon = 3 ;
C = 7;

X = zeros(l,l);
for i = 1 : l
    for j = 1 : l
        X(i,j) = kernel(x(i),x(j)) ;
    end
end
Q = [ X -X ; -X X ];
c = epsilon*ones(2*l,1) + [-y;y];

sol = quadprog(Q,c,[],[],[ones(1,l) -ones(1,l)],0,zeros(2*l,1),C*ones(2*l,1));
lap = sol(1:l);
lam = sol(l+1:2*l);

% compute b
ind = find(lap > 1e-3 & lap < C-1e-3);
if isempty(ind)==0
    i = ind(1);
    b = y(i) - epsilon;
    for j = 1 : l
        b = b - (lap(j)-lam(j))*kernel(x(i),x(j));
    end
else
    ind = find(lam > 1e-3 & lam < C-1e-3);
    i = ind(1);
    b = y(i) + epsilon ;
    for j = 1 : l
        b = b - (lap(j)-lam(j))*kernel(x(i),x(j));
    end
end


z = zeros(l,1);                                                   % find regression and epsilon-tube
for i = 1 : l
    z(i) = b ;
    for j = 1 : l
        z(i) = z(i) + (lap(j)-lam(j))*kernel(x(i),x(j));
    end
end
zp = z + epsilon ;
zm = z - epsilon ;

sv = [find(lap > 1e-3);find(lam > 1e-3)];                          % find support vectors
sv = sort(sv);

plot(x,y,'b.',x(sv),y(sv),'ro',x,z,'k-',x,zp,'r-',x,zm,'r-');     % plot the solution

    disp('Support vectors')

[sv,x(sv),y(sv),lam(sv),lap(sv)]    % Indexes of support vectors, support vectors, lambda_-,lambda_+

function v = kernel(x,y)
v = exp(-norm(x-y)^2);
end
```

Let λ_- and λ_+ be the vectors given by the Matlab solutions lam, lap. In particular we find, $b = 1.0703$.

The regression function is:

$$f(x) = \sum_{i=1}^{\ell}(\lambda_i^+ - \lambda_i^-)k(x_i, x) + b = \sum_{i=1}^{26}(\lambda_i^+ - \lambda_i^-)e^{-\|x_i - x\|^2} + 1.0703$$

(c) We obtain the support vectors (columns 2-3) and corresponding λ_- and λ_+ (columns 4-5) :

```
ans =

    3.0000   -3.2000   16.0000    0.0000    4.2299
    4.0000   -3.0000   17.5000    0.0000    7.0000
    5.0000   -2.8000   11.4800    3.1057    0.0000
    6.0000   -2.4000   16.8200    0.0000    6.6236
    7.0000   -1.2000   16.1500    0.0000    7.0000
   19.0000    1.2000  -13.6400    4.0632    0.0000
   20.0000    1.4000  -15.2600    7.0000    0.0000
   24.0000    2.4000  -11.0000    3.8898    0.0000
   25.0000    3.6000  -10.0000    7.0000    0.0000
   26.0000    4.0000   -2.0000    0.0000    0.2052
```

(d) Consider the feasibility condition of the primal formulation of the regression problem:

$$y_i - f(x_i) - \varepsilon - \xi_i^+ \leq 0, \quad y_i - f(x_i) + \varepsilon + \xi_i^- \geq 0, \quad i = 1, .., \ell$$

If a point (x_i, y_i) is outside the ε-tube then $\xi_i^+ > 0$ or $\xi_i^- > 0$.

Given the dual optimal solution (λ_+, λ_-) of (1), we can find the errors ξ_i^+ and ξ_i^- associated with the point (x_i, y_i) by the complementarity conditions:

$$\begin{cases} \lambda_i^+ \left[y_i - f(x_i) - \varepsilon - \xi_i^+\right] = 0, & i = 1, .., \ell \\ \lambda_i^- \left[y_i - f(x_i) + \varepsilon + \xi_i^-\right] = 0, & i = 1, .., \ell \\ \xi_i^+(C - \lambda_i^+) = 0, & i = 1, .., \ell \\ \xi_i^-(C - \lambda_i^-) = 0, & i = 1, .., \ell \end{cases} \tag{2}$$

It follows that a necessary condition for a point (x_i, y_i) to be outside the ε-tube is that $\lambda_i^+ = C = 7$ or $\lambda_i^- = C = 7$. We find that $\lambda_i^- = 7$, for $i = 20, 25$, $\lambda_i^+ = 7$, for $i = 4, 7$ which correspond to the points

$$(x_4, y_4) = (-3, 17.5) \quad (x_7, y_7) = (-1.2, 16.15), \quad (x_{20}, y_{20}) = (1.4, -15.26), \quad (x_{25}, y_{25}) = (3.6, -10)$$

3) Consider the following constrained multiobjective optimization problem (P):

$$\begin{cases} \min \ f(x_1, x_2) = (x_1^2 + 3x_2 - 2, \ -x_2) \\ x_2 \leq 2 \end{cases}$$

(a) Is the problem convex?

(b) Prove that the problem admits a Pareto minimum point.

(c) Find the set of all weak Pareto minima.

(d) Find a suitable subset of Pareto minima, by means of the scalarization method.

(e) Do ideal Pareto minima exist?

SOLUTION

(a) We observe that the function f_1 is convex, with hessian $Q = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$, while f_2 and the constraints are linear so that (P) is convex.

(b)-(d) Since the objective and the constraint functions are convex, the set of weak minima coincides with the set of solutions of the scalarized problem (P_{α_1}), where $0 \leq \alpha_1 \leq 1$, i.e.

$$\begin{cases} \min \ \alpha_1(x_1^2 + 3x_2 - 2) + (1 - \alpha_1)(-x_2) = \alpha_1 x_1^2 + x_2(4\alpha_1 - 1) - 2\alpha_1 =: \psi_{\alpha_1}(x) \\ x_2 \leq 2 \end{cases}$$

Moreover for $0 < \alpha_1 < 1$ the optimal solutions of (P_{α_1}) are Pareto minima.

We note that for $\alpha_1 = 1/4$, the objective function $\psi_{\alpha_1}(x) = \frac{1}{4}x_1^2 - \frac{1}{2}$ and admits the global optimal solutions

$$\{(x_1, x_2) : x_1 = 0, x_2 \leq 2\}$$

which are all Pareto minima.

P_{α_1} can be solved by Matlab for $0 \leq \alpha_1 \leq 1$:

```
clear all;

Q1 = [2 0; 0 0];
Q2 = [0 0; 0 0] ;
c1=[0 3]';
c2=[0 -1]';

A =[ 0 1];
b = 2;

% solve the scalarized problem with  alfa1 in [0,1]

MINIMA=[ ]; % First column: value of alfa1

LAMBDA=[ ]; % First column: value of alfa1

for alfa1 = 0 : 0.001 : 1
[x,fval,exitflag,output,lambda] = quadprog(alfa1*Q1+(1-alfa1)*Q2,alfa1*c1+(1-alfa1)*c2,A,b) ;
if exitflag == -3
    continue
end
MINIMA=[MINIMA; alfa1 x'];
LAMBDA=[LAMBDA;alfa1,lambda.ineqlin'];
end
```

%%%%%%%%%%%%%%%%%%

For every $0 \leq \alpha_1 \leq 0.25$, the problem admits the solution $(0, 2)$ which is a Pareto minimum. In particular,

- For $0 \leq \alpha_1 < 0.25$, the multiplier λ is strictly positive and $x_2 = 2$ necessarily.
- For $\alpha_1 > 0.25$ the problem P_{α_1} is unbounded.

By the KKT conditions we will be able to obtain further solutions of P_{α_1}, for $0 \leq \alpha_1 \leq 0.25$.

We note that the previous solutions can also be obtained by the KKT conditions for (P_{α_1}) which are necessary and sufficient for a weak minimum point.

$$\begin{cases} 2\alpha_1 x_1 = 0 \\ 4\alpha_1 - 1 + \lambda = 0 \\ \lambda(x_2 - 2) = 0 \\ \lambda \geq 0, \ x_2 \leq 2 \\ 0 \leq \alpha_1 \leq 1, \end{cases}$$

By the second equation and $\lambda \geq 0$ we obtain:

$$\lambda = 1 - 4\alpha_1 \geq 0, \quad \Rightarrow \quad 0 \leq \alpha_1 \leq \frac{1}{4}$$

Then we have the following cases:
I) $\alpha_1 = 0$ which leads to the following solutions (x_1, x_2, λ), such that (x_1, x_2) are weak Pareto minima:

$$\begin{cases} \alpha_1 = 0 \\ \lambda = 1 \\ x_2 = 2 \\ x_1 \in \mathbb{R}, \end{cases}$$

II) $0 < \alpha_1 < \frac{1}{4}$ which leads to following the solutions (x_1, x_2, λ), such that (x_1, x_2) are Pareto minima:

$$\begin{cases} x_1 = 0 \\ \lambda = 1 - 4\alpha_1 \\ x_2 = 2 \\ 0 < \alpha_1 < \frac{1}{4}, \end{cases}$$

III) $\alpha_1 = \frac{1}{4}$ which leads to following the solutions (x_1, x_2, λ), such that (x_1, x_2) are Pareto minima:

$$\begin{cases} x_1 = 0 \\ \lambda = 0 \\ x_2 \leq 2 \\ \alpha_1 = \frac{1}{4}, \end{cases}$$

In conclusion,

$$Weak\ Min(P) = \{(x_1, x_2) : x_1 \in \mathbb{R}, \ x_2 = 2, \} \cup \{(x_1, x_2) : x_1 = 0, \ x_2 \leq 2, \}$$

$$Min(P) \supseteq \{(x_1, x_2) : x_1 = 0, \ x_2 \leq 2, \}$$

(e) No ideal minima exist since there are no simultaneous optimal solutions of P_{α_1} for $\alpha_1 = 0$ and $\alpha_1 = 1$, i.e., points that minimize simultaneously f_1 and f_2 on the feasible set of (P).

4) Consider the following bimatrix game:

$$C_1 = \begin{pmatrix} 2 & 1 & 4 \\ 3 & 2 & 1 \\ -1 & 3 & 2 \end{pmatrix} \qquad C_2 = \begin{pmatrix} 1 & 5 & -1 \\ 4 & 6 & 3 \\ 1 & 2 & 2 \end{pmatrix}$$

(a) Find strictly dominated strategies and reduce the game accordingly.

(b) Find the set of pure strategies Nash equilibria, if any. Alternatively, show that no pure strategies Nash equilibrium exists.

(c) Find a mixed strategies Nash equilibrium.

SOLUTION

(a) Strategy 2 of Player 2 is dominated by Strategy 3, so that column 2 in the two matrices can be deleted. Now Strategy 1 of Player 1 is dominated by Strategy 1 and row 1 in the two matrices can be deleted. The reduced game is given by the matrices

$$C_1^R = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix} \qquad C_2^R = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$$

(b) Now, it is simple to show that (3,1) and (2,3) are pure strategies Nash equilibria. Indeed, the minima on the columns of C_1^R, (i.e., -1 and 1), are obtained in correspondence of the minima on the rows of C_2^R, (i.e., 1 and 3) and are related to the components $(3,1)$ and $(2,3)$ of the given matrices C_1 and C_2.

This will also be shown in part (c) in the wider context of mixed strategies Nash equilibria.

(b) Consider the reduced game obtained in (a). The optimization problem associated with Player 1 is

$$\left\{ \begin{array}{l} \min\ x^T C_1^R y = (3x_2 - x_3)y_1 + (x_2 + 2x_3)y_3 \\ x_2 + x_3 = 1 \\ x_2, x_3 \geq 0 \end{array} \right. \equiv \left\{ \begin{array}{l} \min\ (5y_1 - 1)x_2 - 3y_1 + 2 \\ 0 \leq x_2 \leq 1 \end{array} \right. \qquad (P_1(y_1))$$

Then, the best response mapping associated with $P_1(y_1)$ is:

$$B_1(y_1) = \left\{ \begin{array}{cl} 0 & \text{if } y_1 \in (1/5, 1] \\ [0,1] & \text{if } y_1 = 1/5 \\ 1 & \text{if } y_1 \in [0, 1/5) \end{array} \right.$$

Similarly, the optimization problem associated with Player 2 is

$$\left\{ \begin{array}{l} \min\ x^T C_2^R y = (4x_2 + x_3)y_1 + (3x_2 + 2x_3)y_3 \\ y_1 + y_3 = 1 \\ y_2, y_3 \geq 0 \end{array} \right. \equiv \left\{ \begin{array}{l} \min\ (2x_2 - 1)y_1 + 2 + x_2 \\ 0 \leq y_1 \leq 1 \end{array} \right. \qquad (P_2(x_2))$$

Then, the best response mapping associated with $P_2(x_2)$ is:

$$B_2(x_2) = \left\{ \begin{array}{cl} 0 & \text{if } x_2 \in (1/2, 1] \\ [0,1] & \text{if } x_2 = 1/2 \\ 1 & \text{if } x_2 \in [0, 1/2) \end{array} \right.$$

The couples (x_2, y_1) such that $x_2 \in B_1(y_1)$ and $y_1 \in B_2(x_2)$ are

1. $x_2 = 0, y_1 = 1$,
2. $x_2 = \frac{1}{2}, y_1 = \frac{1}{5}$,
3. $x_2 = 1, y_1 = 0$,

so that, recalling that $x_1 = 0$ and $y_2 = 0$,

- $(x_1, x_2, x_3) = (0, 0, 1), \quad (y_1, y_2, y_3) = (1, 0, 0)$, is a pure strategies Nash equilibrium,
- $(x_1, x_2, x_3) = (0, \frac{1}{2}, \frac{1}{2}), \quad (y_1, y_2, y_3) = (\frac{1}{5}, 0, \frac{4}{5})$, is a mixed strategies Nash equilibrium,
- $(x_1, x_2, x_3) = (0, 1, 0), \quad (y_1, y_2, y_3) = (0, 0, 1)$, is a pure strategies Nash equilibrium.