Name: **Surname:** **Roll number:**

1) Consider the constrained optimization problem (P)

$$\begin{cases} \min\ (x_1+2x_2)\log(x_1+2x_2) \\ x_1 \geq 1 \\ 3x_1 - x_2 \leq 0 \\ x \in \mathbb{R}^2 \end{cases}$$

(a) Is the problem (P) convex?

(b) Does (P) admit a global optimal solution?

(c) Apply the logarithmic barrier method with starting point $x^0 = (2, 10)$, $\varepsilon^0 = 1$, $\tau = 0.5$ and tolerance 10^{-3}. How many iterations are needed by the algorithm? Write the vector x found at the last three iterations.

(d) Is the obtained solution a global minimum of the given problem? Justify the answer.

SOLUTION

(a) Note that the objective function can be written as $f(x) = \psi(h(x))$, where $h(x) = x_1 + 2x_2$, $\psi(y) = ylog(y)$. We note that $\psi(y)$ is convex on $Y = \{y : y > 0\}$, in fact $\psi''(y) = \frac{1}{y}$. Moreover h is linear so that $f = \psi \circ h$ is convex.

(b) We observe that f is coercive on the feasible set X. Indeed, $\|x\| \to +\infty$, $x \in X$ implies $h(x) \to +\infty$ and $\lim_{y\to+\infty} \psi(y) = +\infty$, so that

$$\lim_{\substack{\|x\|\to+\infty \\ x\in X}} f(x) = +\infty.$$

Consequently, being X closed and f continuous on X, f admits a global minimum point on X.

Matlab solution

```
global  A b eps;

A = [-1 0; 3 -1];
b = [-1; 0];

delta = 1e-3 ;
tau = 0.5 ;
eps1 = 1 ;
x0 = [2;10];

%% barrier method


x = x0;
eps = eps1 ;
m = size(A,1) ;

SOL=[];

while true
    [x,pval] = fminunc(@logbar,x);
    gap = m*eps;
    SOL=[SOL;eps,x',gap,pval];
    if gap < delta
        break
    else
        eps = eps*tau;
    end
end

fprintf('\t eps \t x(1) \t x(2) \t gap \t  pval \n\n');

SOL


%% logarithmic barrier function
```

```
function v = logbar(x)

    global A b eps

    v = (x(1)+2*x(2))*log(x(1)+2*x(2));

    for i = 1 : length(b)
        v = v - eps*log(b(i)-A(i,:)*x) ;
    end

end
```

The iterations of the algorithm are 12. The last iterations are:

```
 iter    eps       x(1)      x(2)        gap       pval

SOL =

10.    0.0020    1.0001    3.0006    0.0039   13.6590
11.    0.0010    1.0000    3.0003    0.0020   13.6415
12     0.0005    1.0000    3.0002    0.0009   13.6321
```

The found solution is $x^* \approx (1,3)$ with optimal value $val(P) \approx 13.63$.

(d) The algorithm converges to a solution of the KKT conditions associated with (P), that being (P) convex and the constraints linear are necessary and sufficient for optimality. The KKT conditions for (P) are given by

$$\begin{cases} \log(x_1+2x_2)+1-\lambda_1+3\lambda_2=0 \\ 2(\log(x_1+2x_2)+1)-\lambda_2=0 \\ \lambda_1(1-x_1)=\lambda_2(3x_1-x_2)=0 \\ 1-x_1\le 0,\ 3x_1-x_2\le 0, \\ \lambda\ge 0 \\ \lambda\in\mathbb{R}^2, x\in\mathbb{R}^2 \end{cases}$$

We observe that $x^*=(1,3)$ with $\lambda^*\approx(20.62137, 5.89182)$ is a solution of the previous system.

2) Consider a regression problem with the following data set where the points $(x_i, y_i), i = 1, .., 26$, are given by the row vectors of the matrices:

$$\begin{pmatrix} -3.0000 & 6 \\ -2.8000 & 8 \\ -2.6000 & 8.5 \\ -1.8000 & 11.48 \\ -1.6000 & 14.10 \\ -1.4000 & 16.82 \\ -1.2000 & 16.15 \\ -1.0000 & 11.68 \\ -0.8000 & 6.00 \\ -0.6000 & 7.82 \\ -0.4000 & 2.82 \\ -0.2000 & 2.71 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0.2000 & -1 \\ 0.4000 & -3.84 \\ 0.6000 & -4.71 \\ 1.0000 & -7.33 \\ 1.2000 & -13.64 \\ 1.6000 & -14.87 \\ 1.8000 & -9.92 \\ 2.0000 & -10.50 \\ 2.2000 & -7.72 \\ 2.4000 & -12.5 \\ 2.6000 & -10.26 \\ 2.8000 & -7 \\ 3.0000 & -2 \end{pmatrix}$$

(a) Write the dual formulation of a nonlinear ε-SV regression model with $C = 8$, $\varepsilon = 2.5$ and a Gaussian kernel $k(x, y) := e^{-\|x-y\|^2}$;

(b) Solve the problem in (a) and find the regression function;

(c) Find the support vectors;

(d) Find the points of the data set that are outside the ε-tube, by making use of the dual solution.

SOLUTION

(a) Let $\ell = 26$, (x_i, y_i), $i = 1, .., \ell$ be the i-th element of the data set, $C = 8$, $\varepsilon = 2.5$, $k(x, y) := e^{-\|x-y\|^2}$. The dual formulation of a nonlinear ε-SV regression model is

$$\begin{cases} \max\limits_{(\lambda^+,\lambda^-)} & -\dfrac{1}{2}\sum\limits_{i=1}^{26}\sum\limits_{j=1}^{26}(\lambda_i^+ - \lambda_i^-)(\lambda_j^+ - \lambda_j^-)e^{-\|x_i-x_j\|^2} \\ & -2.5\sum\limits_{i=1}^{26}(\lambda_i^+ + \lambda_i^-) + \sum\limits_{i=1}^{26} y_i(\lambda_i^+ - \lambda_i^-) \\ \sum\limits_{i=1}^{26}(\lambda_i^+ - \lambda_i^-) = 0 \\ \lambda_i^+, \lambda_i^- \in [0, 8],\ i = 1, .., 26 \end{cases} \tag{1}$$

(b) **Matlab solution**

```
data = [
   -3.0000     6
   -2.8000     8
   -2.6000     8.5
   -1.8000    11.48
   -1.6000    14.10
   -1.4000    16.82
   -1.2000    16.15
   -1.0000    11.68
   -0.8000     6.00
   -0.6000     7.82
   -0.4000     2.82
   -0.2000     2.71
         0        1
    0.2000    -1
    0.4000    -3.84
    0.6000    -4.71
    1.0000    -7.33
    1.2000   -13.64
    1.6000   -14.87
    1.8000    -9.92
    2.0000   -10.50
    2.2000    -7.72
    2.4000   -12.5
    2.6000   -10.26
    2.8000    -7
    3.0000    -2
        ];

x = data(:,1) ;
y = data(:,2) ;
l = length(x) ;
```

```
epsilon = 2.5 ;
C = 8;

X = zeros(l,l);
for i = 1 : l
    for j = 1 : l
        X(i,j) = kernel(x(i),x(j)) ;
    end
end
Q = [ X -X ; -X X ];
c = epsilon*ones(2*l,1) + [-y;y];

sol = quadprog(Q,c,[],[],[ones(1,l) -ones(1,l)],0,zeros(2*l,1),C*ones(2*l,1));
lap = sol(1:l);
lam = sol(l+1:2*l);

ind = find(lap > 1e-3 & lap < C-1e-3);              % compute b
if isempty(ind)==0
    i = ind(1);
    b = y(i) - epsilon;
    for j = 1 : l
        b = b - (lap(j)-lam(j))*kernel(x(i),x(j));
    end
else
    ind = find(lam > 1e-3 & lam < C-1e-3);
    i = ind(1);
    b = y(i) + epsilon ;
    for j = 1 : l
        b = b - (lap(j)-lam(j))*kernel(x(i),x(j));
    end
end


z = zeros(l,1);                                              % find regression and epsilon-tube
for i = 1 : l
    z(i) = b ;
    for j = 1 : l
        z(i) = z(i) + (lap(j)-lam(j))*kernel(x(i),x(j));
    end
end
zp = z + epsilon ;
zm = z - epsilon ;

sv = [find(lap > 1e-3);find(lam > 1e-3)];                     % find support vectors
sv = sort(sv);

 plot(x,y,'b.',x(sv),y(sv),'ro',x,z,'k-',x,zp,'r-',x,zm,'r-');     % plot the solution

    disp('Support vectors')

[sv,x(sv),y(sv),lam(sv),lap(sv)]    % Indexes of support vectors, support vectors, lambda_-,lambda_+

function v = kernel(x,y)

v = exp(-norm(x-y)^2)

end
```

Let λ_- and λ_+ be the vectors given by the Matlab solutions lam, lap. In particular we find, $b = 0.1595$.

The regression function is:

$$f(x) = \sum_{i=1}^{26}(\lambda_i^+ - \lambda_i^-)k(x_i, x) + b = \sum_{i=1}^{26}(\lambda_i^+ - \lambda_i^-)e^{-\|x_i - x\|^2} + 0.1595$$

(c) We obtain the support vectors (columns 2-3) and corresponding λ^- and λ^+ (columns 4-5) :

```
ans =
```

```
 2.0000   -2.8000    8.0000   0.0000   3.6687
 6.0000   -1.4000   16.8200   0.0000   8.0000
 7.0000   -1.2000   16.1500   0.0000   8.0000
 9.0000   -0.8000    6.0000   4.0137   0.0000
18.0000    1.2000  -13.6400   4.7592   0.0000
19.0000    1.6000  -14.8700   7.9578   0.0000
22.0000    2.2000   -7.7200   0.0000   8.0000
23.0000    2.4000  -12.5000   8.0000   0.0000
24.0000    2.6000  -10.2600   6.4174   0.0000
26.0000    3.0000   -2.0000   0.0000   3.4794
```

(d) Consider the feasibility condition of the primal formulation of the regression problem:

$$y_i - f(x_i) - \varepsilon - \xi_i^+ \leq 0, \quad y_i - f(x_i) + \varepsilon + \xi_i^- \geq 0, \quad i = 1, .., \ell = 26$$

If a point (x_i, y_i) is outside the ε-tube then $\xi_i^+ > 0$ or $\xi_i^- > 0$.

Given the dual optimal solution (λ^+, λ^-) of (1), we can find the errors ξ_i^+ and ξ_i^- associated with the point (x_i, y_i) by the complementarity conditions:

$$\begin{cases} \lambda_i^+ \left[y_i - f(x_i) - \varepsilon - \xi_i^+\right] = 0, & i = 1, .., \ell \\ \lambda_i^- \left[y_i - f(x_i) + \varepsilon + \xi_i^-\right] = 0, & i = 1, .., \ell \\ \xi_i^+ (C - \lambda_i^+) = 0, & i = 1, .., \ell \\ \xi_i^- (C - \lambda_i^-) = 0, & i = 1, .., \ell \end{cases} \tag{2}$$

it follows that a necessary condition for a point (x_i, y_i) to be outside the ε-tube is that $\lambda_i^+ = C = 8$ or $\lambda_i^- = C = 8$. We find that $\lambda_i^- = 8$, for $i = 23$, $\lambda_i^+ = 8$, for $i = 6, 7, 22$, which correspond to the points

$$(x_6, y_6) = (-1.4, 16.82), \quad (x_7, y_7) = (-1.2, 16.15), \quad (x_{22}, y_{22}) = (2.2, -7.72), \quad (x_{23}, y_{23}) = (2.4, -12.5).$$

3) Consider the following multiobjective optimization problem (P):

$$\begin{cases} \min\ (x_1 - 2x_2,\ x_1 - x_3) \\ \quad -x_1 \leq 5 \\ x_1 + x_2 \leq 2 \\ x_1 + x_3 \leq 0 \\ (x_1, x_2, x_3) \in \mathbb{R}^3 \end{cases}$$

(a) Prove that the problem admits a Pareto minimum point.

(b) Find the set of all weak Pareto minima.

(c) Find a suitable subset of Pareto minima.

(d) Does the problem admit any ideal minimum?

SOLUTION

We preliminarly observe that the problem is linear, since the objective and the constraint functions are linear. Therefore the set of weak minima coincides with the set of solutions of the scalarized problems (P_{α_1}), i.e.

$$\begin{cases} \min\ \psi_{\alpha_1}(x) := \alpha_1(x_1 - 2x_2) + (1 - \alpha_1)(x_1 - x_3) = x_1 - 2\alpha_1 x_2 - (1 - \alpha_1)x_3 \\ \quad -x_1 \leq 5 \\ x_1 + x_2 \leq 2 \\ x_1 + x_3 \leq 0 \\ (x_1, x_2, x_3) \in \mathbb{R}^3 \end{cases}$$

where $0 \leq \alpha_1 \leq 1$, while the set of minima coincides with the set of solutions of the scalarized problems (P_{α_1}), where $0 < \alpha_1 < 1$.

(a) Let X be the feasible set of (P). By the first two inequality constraints, we obtain that $-x_2 \geq x_1 - 2 \geq -5 - 2 = -7$. By the first and the third inequality , we obtain that $-x_3 \geq x_1 \geq -5$.

Note that, for every $\alpha_1 \in [0, 1]$,

$$\psi_{\alpha_1}(x) = x_1 - 2\alpha_1 x_2 - (1 - \alpha_1)x_3 \geq -5 - 14\alpha_1 - 5(1 - \alpha_1) = -10 - 9\alpha_1, \quad \forall x \in X,$$

Therefore P_{α_1} admits finite optimum, for every $\alpha_1 \in [0, 1]$ and the optimal solutions, obtained for $0 < \alpha_1 < 1$, are Pareto minima for the given problem.

(b) - (c) By solving P_{α_1} with Matlab we obtain:

```
C = [1 -2 0;  1  0 -1 ] ;

A =[ -1  0 0;   1  1 0; 1 0 1 ];

 b = [5 2 0]';

 % solve the scalarized problem with 0 =< alfa =< 1

MINIMA=[Inf,Inf,Inf, Inf];

lambda=[Inf,Inf,Inf,Inf];

 for alfa = 0 : 0.01 : 1
 [x,fval,exitflag,output,Lambda] = linprog(alfa*C(1,:)+(1-alfa)*C(2,:),A,b) ;

      MINIMA=[MINIMA; alfa x'];
     lambda=[lambda;alfa,Lambda.ineqlin'];

 end
```

We obtain

$$x(\alpha_1) = (-5, 7, 5) \quad \lambda(\alpha_1) = (2 + \alpha_1, 2\alpha_1, 1 - \alpha_1), \quad for\ 0 \le \alpha_1 \le 1,$$

Since the problem is linear then the KKT conditions provide a necessary and sufficient condition for an optimal solution of (P_{α_1}):

$$\begin{cases} 1 - \lambda_1 + \lambda_2 + \lambda_3 = 0 \\ -2\alpha_1 + \lambda_2 = 0 \\ \alpha_1 - 1 + \lambda_3 = 0 \\ \lambda_1(-x_1 - 5) = 0 \\ \lambda_2(x_1 - x_2 - 2) = 0 \\ \lambda_3(x_1 + x_3) = 0 \\ -x_1 \le 5 \\ x_1 + x_2 \le 2 \\ x_1 + x_3 \le 0 \\ \lambda \ge 0 \\ 0 \le \alpha_1 \le 1, \end{cases}$$

Note that by the first three equations, we obtain

$$\lambda(\alpha_1) = (2 + \alpha_1, 2\alpha_1, 1 - \alpha_1), \quad for\ 0 \le \alpha_1 \le 1,$$

as previously found. Exploiting the complementarity conditions, we obtain the following solutions:

(i) For $0 < \alpha_1 < 1$, being $\lambda > 0$, the set of optimal solutions of $P(\alpha_1)$ is given by the system

$$\begin{cases} -x_1 = 5 \\ x_1 + x_2 - 2 = 0 \\ x_1 + x_3 = 0 \\ (x_1, x_2, x_3) \in \mathbb{R}^3 \end{cases}$$

so that $\bar{x} = (-5, 7, 5)$ is the unique minimum point of P_{α_1}.

(ii) For $\alpha_1 = 0$, being $\lambda = (2, 0, 1)$, the set of optimal solutions of P_0 is given by the following system

$$\begin{cases} -x_1 = 5 \\ x_1 + x_2 - 2 \le 0 \\ x_1 + x_3 = 0 \\ (x_1, x_2, x_3) \in \mathbb{R}^3 \end{cases}$$

(iii) For $\alpha_1 = 1$, being $\lambda = (3, 2, 0)$, the set of optimal solutions of P_1 is given by the following system

$$\begin{cases} -x_1 = 5 \\ x_1 + x_2 - 2 = 0 \\ x_1 + x_3 \le 0 \\ (x_1, x_2, x_3) \in \mathbb{R}^3 \end{cases}$$

Then, $Weak\ Min(P) = \{(x_1, x_2, x_3) : x_1 = -5, x_2 \leq 7,\ x_3 = 5\} \ \cup\ \{(x_1, x_2, x_3) : x_1 = -5, x_2 = 7,\ x_3 \leq 5\}$,

$$Min(P) = \{(x_1, x_2, x_3) = (-5, 7, 5)\}.$$

(d) Since $(-5, 7, 5)$ is the (only) simultaneous optimal solution of P_0 and P_1, then it is an ideal minimum, indeed it minimizes both the objective functions.

4) Consider the following bimatrix game:

$$C_1 = \begin{pmatrix} 2 & 2 & 1 & 3 \\ 3 & 1 & 2 & 2 \\ 4 & 3 & 1 & 4 \end{pmatrix} \quad C_2 = \begin{pmatrix} 1 & 0 & 2 & 1 \\ -1 & 1 & 0 & 2 \\ 2 & 3 & 3 & 1 \end{pmatrix}$$

(a) Find the strictly dominated strategies, if any, and reduce the cost matrices accordingly.

(b) Find the set of pure strategies Nash equilibria, if any. Alternatively, show that no pure strategies Nash equilibrium exists.

(c) Find a mixed strategies Nash equilibrium.

SOLUTION

(a) Strategy 3 of Player 2 is dominated by Strategy 1, so that column 3 in the two matrices can be deleted. Now, in the reduced matrix of player 1, Strategy 3 of Player 1 is dominated by Strategy 1 and row 3 in the two matrices can be deleted. Finally, in the second reduced matrix, Strategy 4 of Player 2 is dominated by Strategy 2. The reduced game is given by the matrices

$$C_1^R = \begin{pmatrix} 2 & 2 \\ 3 & 1 \end{pmatrix} \quad C_2^R = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

The minima on the columns of C_1^R are the elements of the principal diagonal, and the corresponding elements on the principal diagonal of C_2^R are maximaon the rows of C_2^R, which implies that no pure strategies Nash equilibria exist.

This will also be shown in part (b) in the wider context of mixed strategies Nash equilibria.

(b) Consider the reduced game obtained in (a). The optimization problem associated with Player 1 is

$$\begin{cases} \min\ x^T C_1^R y = (2x_1 + 3x_2)y_1 + (2x_1 + x_2)y_2 \\ x_1 + x_2 = 1 \\ x_1, x_2 \geq 0 \end{cases} \equiv \begin{cases} \min\ (1 - 2y_1)x_1 + 2y_1 + 1 \\ 0 \leq x_1 \leq 1 \end{cases} \qquad (P_1(y_1))$$

where, we have eliminated the variables x_2 and y_2, since $x_2 = 1-x_1$ and $y_2 = 1-y_1$,taking into account that $x_3 = 0, y_3 = y_4 = 0$. Then, the best response mapping associated with $P_1(y_1)$ is:

$$B_1(y_1) = \begin{cases} 0 & \text{if } y_1 \in [0, 1/2) \\ [0,1] & \text{if } y_1 = 1/2 \\ 1 & \text{if } y_1 \in (1/2, 1) \end{cases}$$

Similarly, the optimization problem associated with Player 2 is

$$\begin{cases} \min\ x^T C_2^R y = x_1 y_1 + (-y_1 + y_2)x_2 \\ y_1 + y_2 = 1 \\ y_1, y_2 \geq 0 \end{cases} \equiv \begin{cases} \min\ (3x_1 - 2)y_1 - x_1 + 1 \\ 0 \leq y_1 \leq 1 \end{cases} \qquad (P_2(x_1))$$

Then, the best response mapping associated with $P_2(x_1)$ is:

$$B_2(x_1) = \begin{cases} 0 & \text{if } x_1 \in (2/3, 1] \\ [0,1] & \text{if } x_1 = 2/3 \\ 1 & \text{if } x_1 \in [0, 2/3) \end{cases}$$

The couples (x_1, y_1) such that $x_1 \in B_1(y_1)$ and $y_1 \in B_2(x_1)$ are

1. $x_1 = \frac{2}{3}, y_1 = \frac{1}{2}$,

so that, recalling that $x_3 = 0, y_3 = y_4 = 0$,

- $(x_1, x_2, x_3) = (\frac{2}{3}, \frac{1}{3}, 0)$, $\quad (y_1, y_2, y_3, y_4) = (\frac{1}{2}, \frac{1}{2}, 0, 0)$, is a mixed strategies Nash equilibrium.