1) Consider the constrained optimization problem (P)

$$\begin{cases} \min\ x_1^2 + x_2^2 - x_1x_2 + x_2x_3 - 3x_1 - 4x_2 \\ 2x_1 + x_2 + x_3 \leq 20 \\ x_1 \geq 0 \\ x_2 \geq 3 \\ x_3 \geq 4 \\ x \in \mathbb{R}^3 \end{cases}$$

(a) Is the problem (P) convex?

(b) Does (P) admit a global optimal solution?

(c) Apply the logarithmic barrier method with starting point $x^0 = (2, 4, 5)$, $\varepsilon^0 = 1$, $\tau = 0.5$ and tolerance 10^{-3}. How many iterations are needed by the algorithm? Write the vector x found at the last three iterations.

(d) Is the obtained solution a global minimum of the given problem? Justify the answer.

SOLUTION

(a)-(b) The problem is not convex since the objective function is a quadratic convex function $0.5x^TQx + c^Tx$ (the eigenvalues of the Hessian Q are: -0.4812, 1.3111, 3.1701 and are evaluated in the Matlab solution) and the constraints $Ax - b \leq 0$ are linear. Moreover the feasible set is compact, which implies that (P) admits a unique global optimal solution, being the objective function continuous. See the Matlab solution for the definition of the matrices Q and A and the vectors c and b.

(c) **Matlab solution**

```
global Q c A b eps;                               %% data

Q = [ 2 -1 0 ;-1 2 1; 0 1 0 ] ;
c = [ -3 ; -4; 0 ] ;
A = [2 1 1;-1 0 0; 0 -1 0 ;0 0 -1 ];
b = [ 20; 0 ; -3; -4  ];

eig(Q)

delta = 1e-3 ;
tau = 0.5 ;
eps1 = 1 ;
x0 = [2;4;5];


x = x0;                                     %% barrier method
eps = eps1 ;
m = size(A,1) ;

SOL=[]

while true
    [x,pval] = fminunc(@logbar,x);
    gap = m*eps;
    SOL=[SOL;eps,x',gap,pval];
    if gap < delta
        break
    else
        eps = eps*tau;
    end
end

fprintf('\t eps \t x(1) \t x(2) \t x(3) \t gap \t  pval \n\n');

SOL

function v = logbar(x)                               %% logarithmic barrier function
```

```
    global Q c A b eps

    v = 0.5*x'*Q*x + c'*x ;

    for i = 1 : length(b)
        v = v - eps*log(b(i)-A(i,:)*x) ;
    end

end
```

We obtain the following solution:

```
 iter    eps        x(1)       x(2)      x(3)        gap        pval

SOL =

11.     0.0010     3.0004     3.0003     4.0003     0.0039     0.0147
12.     0.0005     3.0004     3.0002     4.0002     0.0020     0.0080
13.     0.0002     3.0004     3.0001     4.0001     0.0010     0.0043
```

The iterations of the algorithm are 13 and the found solution is $x^* \approx (3, 3, 4)$ with optimal value $val(P) \approx 0$.

(d) The algorithm converges to a solution of the KKT conditions associated with (P), that in the present case are not sufficient for optimality. The KKT conditions for (P) are given by

$$\begin{cases} Qx + c + A^T\lambda = 0 \\ \lambda^T(Ax - b) = 0 \\ Ax - b \leq 0, \\ \lambda \geq 0 \\ \lambda \in \mathbb{R}^4, x \in \mathbb{R}^3 \end{cases}$$

It is easy to verify that $x^* = (3, 3, 4)$ with $\lambda^* = (0, 0, 3, 3)$ is a solution of the previous system.

Anyway, it is possible to show that the point $x^* = (3, 3, 4)$ is a global minimum. Indeed

$$f(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_1x_2 + x_2x_3 - 3x_1 - 4x_2 = (x_1 - x_2)^2 + x_1x_2 + x_2x_3 - 3x_1 - 4x_2 = (x_1 - x_2)^2 + x_1(x_2 - 3) + x_2(x_3 - 4) \geq 0$$

for every feasible solution of the given problem.

2) Consider a clustering problem for the following set of 18 patterns given by the columns of the matrix:

$$D = \begin{pmatrix} 3 & 2 & 1 & 4 & 3 & 4.8 & 4.0 & 3.7 & 3.5 & 4 & 1.8 & 2.4 & 2.6 & 6 & 5 & 7.5 & 8 & 6.5 \\ 4 & 8 & 7 & 8 & 6.5 & 8 & 6.7 & 5.1 & 7.8 & 5.7 & 6.5 & 6.2 & 7.2 & 8 & 7 & 7 & 4 & 7.5 \end{pmatrix}$$

(a) Write the optimization model with 2-norm and $k = 3$ clusters;

(b) Solve the problem using the k-means algorithm with $k = 3$ and starting from centroids $x^1 = (3, 4), x^2 = (2, 5), x^3 = (4, 1)$. Write explicitly the vector of the obtained centroids and the value of the objective function;

(c) Solve the problem using the k-means algorithm with $k = 3$ and starting from centroids $x^1 = (5, 3), x^2 = (4, 4), x^3 = (7, 4)$. Write explicitly the vector of the obtained centroids, the value of the objective function and the clusters;

(d) Is any of the solutions found at points (b) and (c) a global optimum? Justify the answer.

SOLUTION

(a) Let $\ell = 18$, $(p^i)^T$ be the i-th row of the matrix D^T, $i = 1, .., 18$, the problem can be formulated by

$$\begin{cases} \min\limits_{x} \sum\limits_{i=1}^{18} \min\limits_{j=1,2,3} \|p^i - x^j\|_2^2 \\ x^j \in \mathbb{R}^2 \qquad \forall\, j = 1, 2, 3 \end{cases} \tag{1}$$

which is equivalent to

$$\begin{cases} \min\limits_{x,\alpha} \ f(x, \alpha) := \sum\limits_{i=1}^{18} \sum\limits_{j=1}^{3} \alpha_{ij} \|p^i - x^j\|_2^2 \\ \sum\limits_{j=1}^{3} \alpha_{ij} = 1 \quad \forall\, i = 1, \dots, 18 \\ \alpha_{ij} \ge 0 \qquad \forall\, i = 1, \dots, 18,\ j = 1, 2, 3 \\ x^j \in \mathbb{R}^2 \qquad \forall\, j = 1, 2, 3. \end{cases} \tag{2}$$

provided that we look for an optimal solution with $\alpha \in \{0, 1\}^{18\times 3}$, which can be done by the k-means algorithm.

(b)-(d) **Matlab solution**

```
data = [ 3    4
         2     8
         1    7
         4    8
         3    6.5
         4.8    8
         4.0    6.7
         3.7    5.1
         3.5    7.8
         4    5.7
         1.8    6.5
         2.4    6.2
         2.6    7.2
          6     8
         5    7
         7.5    7.0
          8      4
          6 .5      7.5
                      ];

l = size(data,1); % number of patterns

k=3;

InitialCentroids=[3,4;2,5;4,1];
 % InitialCentroids=[5,3;442;7,4];

[x,cluster,v] = kmeans1(data,k,InitialCentroids)
```

```
function [x,cluster,v] = kmeans1(data,k,InitialCentroids)

l = size(data,1); % number of patterns

x = InitialCentroids;

cluster = zeros(l,1);                              % initialize clusters
for i = 1 : l
    d = inf;
    for j = 1 : k
        if norm(data(i,:)-x(j,:)) < d
            d = norm(data(i,:)-x(j,:));
            cluster(i) = j;
        end
    end
end

vold = 0;                                 % compute the objective function value
for i = 1 : l
    vold = vold + norm(data(i,:)-x(cluster(i),:))^2 ;
end

while true
          for j = 1 : k                                      % update centroids
        ind = find(cluster == j);
        if isempty(ind)==0
            x(j,:) = mean(data(ind,:),1);
        end
    end


    for i = 1 : l                             % update clusters
        d = inf;
        for j = 1 : k
            if norm(data(i,:)-x(j,:)) < d
                d = norm(data(i,:)-x(j,:));
                cluster(i) = j;
            end
        end
    end

    v = 0;                                % update objective function
    for i = 1 : l
        v = v + norm(data(i,:)-x(cluster(i),:))^2 ;
    end

    if vold - v < 1e-5                        % stopping criterion
        break
    else
        vold = v;
    end
end
end
```

In case (b) we obtain the solution:

```
x =

    6.3000    6.9167       % centroids
    2.7000    7.1000
    3.5667    4.9333
```

```
v =     33.8417       % value of the objective function

cluster'=  3   2   2   2   2   1   2   3   2   3   2   2   2   1   1   1   1   1

% The j-th component is the cluster assigned to pattern j
```

In case (c) we obtain the solution:

```
x =

    3.5667    4.9333
    3.1000    7.1727
    7.0000    6.6250


cluster' =  1   2   2   2   2   2   2   1   2   1   2   2   2   3   2   3   3   3

% The j-th component is the cluster assigned to pattern j

v =     35.1227
```

(d) It is possible to improve the previous solutions, with a multistart approach. For example, starting from centroids $x^1 = (1,4), x^2 = (3,4), x^3 = (4,2)$, we obtain the solution

```
x =
        2.6111    6.2444
       4.8286    7.5714
       7.7500    5.5000

v=   32.0847

cluster'=  1   1   1   2   1   2   2   1   2   1   1   1   1   2   2   3   3   2
```

3) Consider the following multiobjective optimization problem (P):

$$\begin{cases} \min\ (x_1^2 + x_2^2,\ x_1x_2 - x_1) \\ x \in \mathbb{R}^2 \end{cases}$$

(a) Is the given problem (P) convex?

(b) Prove that (P) admits a Pareto minimum point.

(c) Find a suitable subset of Pareto minima by means of the scalarization method.

SOLUTION

(a) The problem is not convex, since the objective function $f_2(x_1, x_2) = x_1x_2 - x_1$ is not convex.

(b) The function $f_1(x_1, x_2)$ admits the unique minimum point $(0, 0)$ which is a Pareto minimum.

(c) Consider the scalarized problem (P_{α_1}), where $0 \le \alpha_1 \le 1$, i.e.

$$\begin{cases} \min\ \alpha_1(x_1^2 + x_2^2) + (1-\alpha_1)(x_1x_2 - x_1) =: \psi_{\alpha_1}(x) \\ x \in \mathbb{R}^2 \end{cases}$$

The Hessian matrix of P_{α_1} is given by

$$H(\alpha_1) = \begin{pmatrix} 2\alpha_1 & 1-\alpha_1 \\ 1-\alpha_1 & 2\alpha_1 \end{pmatrix}$$

Since $det(H(\alpha_1)) = 3\alpha_1^2 + 2\alpha_1 - 1 \ge 0$, for $\frac{1}{3} \le \alpha_1 \le 1$, then for such values of α_1, the Hessian matrix is positive semidefinite and P_{α_1} is convex and any of its optimal solutions is a Pareto minimum for $\frac{1}{3} \le \alpha_1 \le 1$, noticing that for $\alpha_1 = 1$, $(0, 0)$ is the unique optimal solution, as already observed.

Consider the necessary optimality conditions for (P_{α_1}):

$$\begin{cases} 2\alpha_1 x_1 + (1-\alpha_1)(x_2-1) = 0 \\ 2\alpha_1 x_2 + (1-\alpha_1)x_1 = 0 \\ 0 \le \alpha_1 \le 1, \end{cases}$$

Such conditions are also sufficient for $\frac{1}{3} \le \alpha_1 \le 1$ being the problem convex. The solutions of the previous system are:

$$\begin{cases} x_1 = \frac{2\alpha_1(1-\alpha_1)}{3\alpha_1^2+2\alpha_1-1} \\ x_2 = -\frac{(1-\alpha_1)^2}{3\alpha_1^2+2\alpha_1-1} \\ \\ 0 \le \alpha_1 \le 1, \alpha_1 \neq \frac{1}{3} \end{cases}$$

Notice that, for $\alpha_1 = \frac{1}{3}$, the system is impossible. For $\frac{1}{3} < \alpha_1 \le 1$ all the solutions of the previous system are Pareto minima of (P).

$$Min(P) \supseteq \{(x_1, x_2) : x_1 = \frac{2\alpha_1(1-\alpha_1)}{3\alpha_1^2+2\alpha_1-1}, x_2 = -\frac{(1-\alpha_1)^2}{3\alpha_1^2+2\alpha_1-1},\ \frac{1}{3} < \alpha_1 \le 1\}.$$

4) Consider the following matrix game:

$$C = \begin{pmatrix} -1 & 0 & 2 & 3 \\ 1 & -1 & 0 & 1 \\ 2 & 1 & 1 & 0 \\ 4 & 2 & 2 & -1 \end{pmatrix}$$

(a) Find the strictly dominated strategies, if any, and reduce the cost matrix accordingly.

(b) Find the set of pure strategies Nash equilibria, if any. Alternatively, show that no pure strategies Nash equilibrium exists.

(c) Find a mixed strategies Nash equilibrium and the related optimal cost of each player.

(d) Is the mixed strategies Nash equilibrium unique?

SOLUTION

(a) No strictly dominated strategy exists since no row is strictly (componentwise) greater than another and no columns is strictly (componentwise) minor than another.

(b) We observe that $c_{11}, c_{22}, c_{23}, c_{44}$, are the minima on the columns of the matrix C, while $c_{14}, c_{21}, c_{24}, c_{31}, c_{41}$ are the maxima on the rows.

Since no minimum on on the columns correspond to a maximum on the rows , no pure strategies Nash equilibria exist.

(c) The optimization problem associated with Player 1 is

$$\begin{cases} \min\ v \\ v \geq -x_1 + x_2 + 2x_3 + 4x_4 \\ v \geq -x_2 + x_3 + 2x_4 \\ v \geq 2x_1 + x_3 + 2x_4 \\ v \geq 3x_1 + x_2 - x_4 \\ x_1 + x_2 + x_3 + x_4 = 1 \\ x \geq 0 \end{cases} \tag{3}$$

The previous problem can be solved by Matlab.

Matlab solution

```
C= [-1,0,2,3; 1 -1 0 1; 2 1 1 0; 4 2 2 -1]


m = size(C,1);
n = size(C,2);
c=[zeros(m,1);1];
A= [C', -ones(n,1)]; b=zeros(n,1); Aeq=[ones(1,m),0]; beq=1;
lb= [zeros(m,1);-inf]; ub=[ ];
[sol,Val,exitflag,output,lambda] = linprog(c, A,b, Aeq, beq, lb, ub);
x = sol(1:m)
y = lambda.ineqlin
```

We obtain the optimal solution $(\bar{x}, \bar{v}) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0, 1)$. The optimal solution of the dual of (3) is given by $(\bar{y}, \bar{w}) = (\frac{1}{2}, 0, 0, \frac{1}{2}, 1)$. $\bar{y}$ can be found in the vector *lambda.ineqlin* given by the Matlab function linprog.

Therefore,

$$(x_1, x_2, x_3, x_4) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0), \quad (y_1, y_2, y_3, y_4) = (\frac{1}{2}, 0, 0, \frac{1}{2}),$$

is a mixed strategies Nash equilibrium.

The optimal cost of each player is $v = 1$.

(d) We note that the optimal solution of (3) is not unique. For example $(\hat{x}, \hat{v}) = (\frac{1}{5}, \frac{2}{5}, \frac{2}{5}, 0, 1)$ is an alternative optimal solution of (3). Therefore, let $\hat{y} = (\frac{1}{2}, 0, 0, \frac{1}{2})$, $(\hat{x}, \hat{y})$ is a further mixed strategies Nash equilibrium.