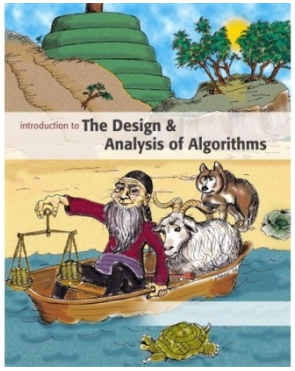
计算机科学与技术系
Department of Computer Science and Technology

南京大学
NANJING UNIVERSITY

Introduction to

Algorithm Design and Analysis

[11] Graph Traversal



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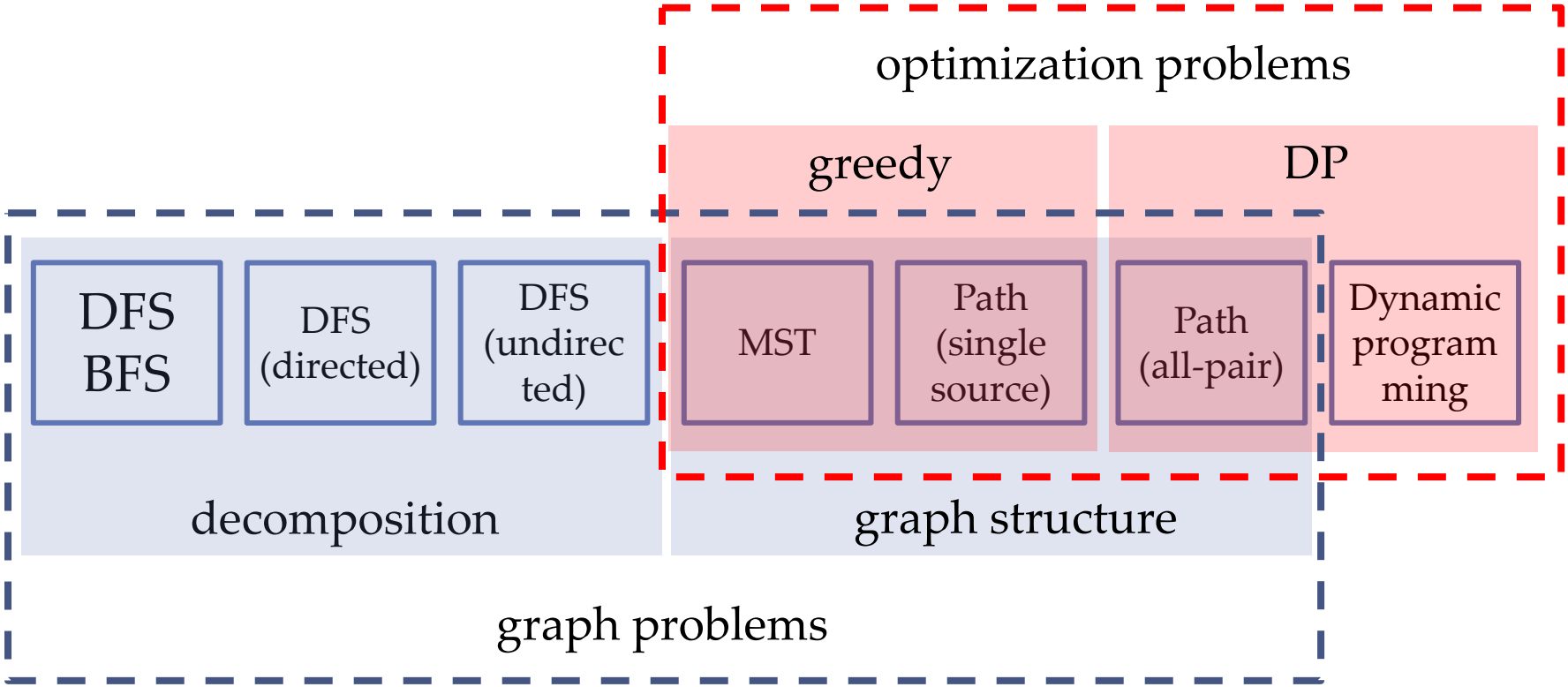
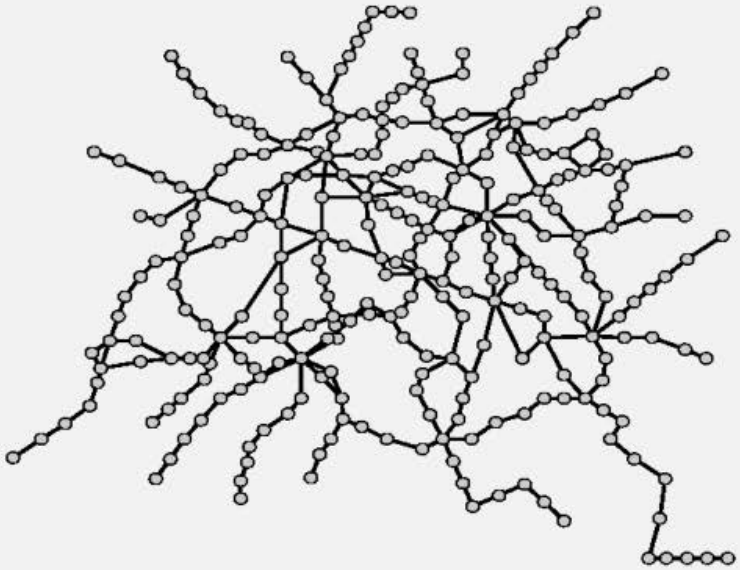
Course Contents

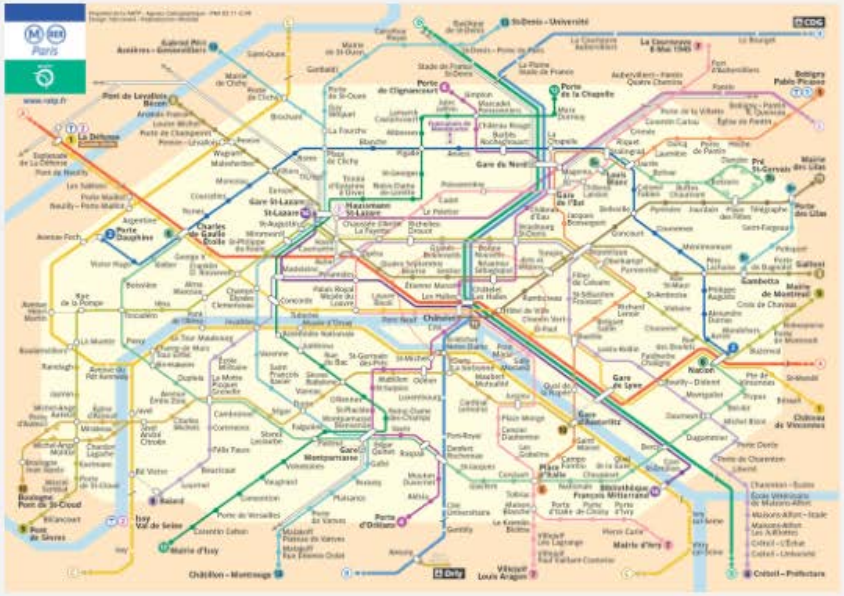


In the Last Class…

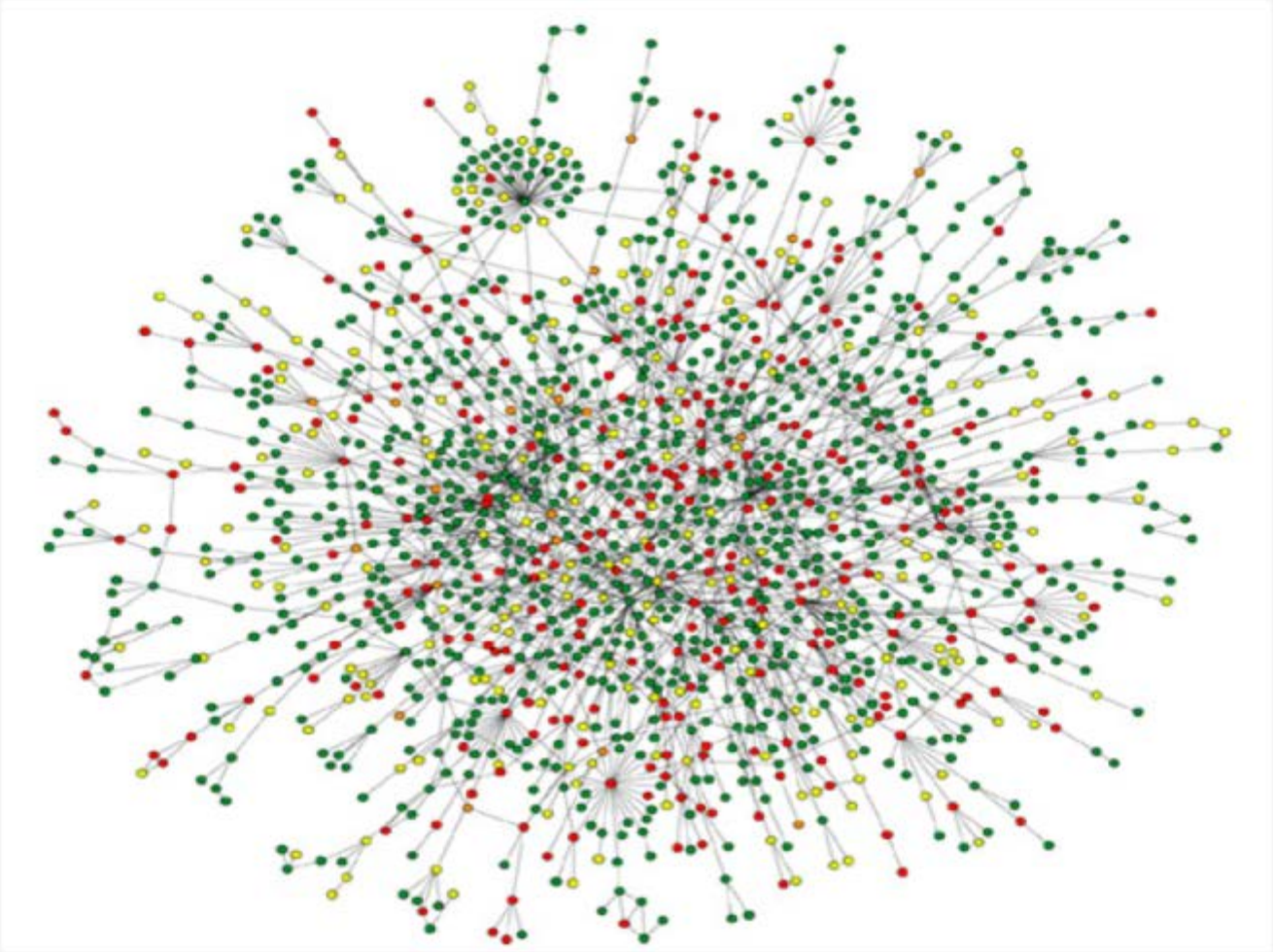
- **Dynamic Equivalence Relation**
- **Implementing *disjoint set* by Union-Find**
  - Straight Union-Find
  - Making Shorter Tree by Weighted Union
  - Compressing Path by Compressing Find
    - Amortized analysis of *wUnion-cFind*

Graph Everywhere

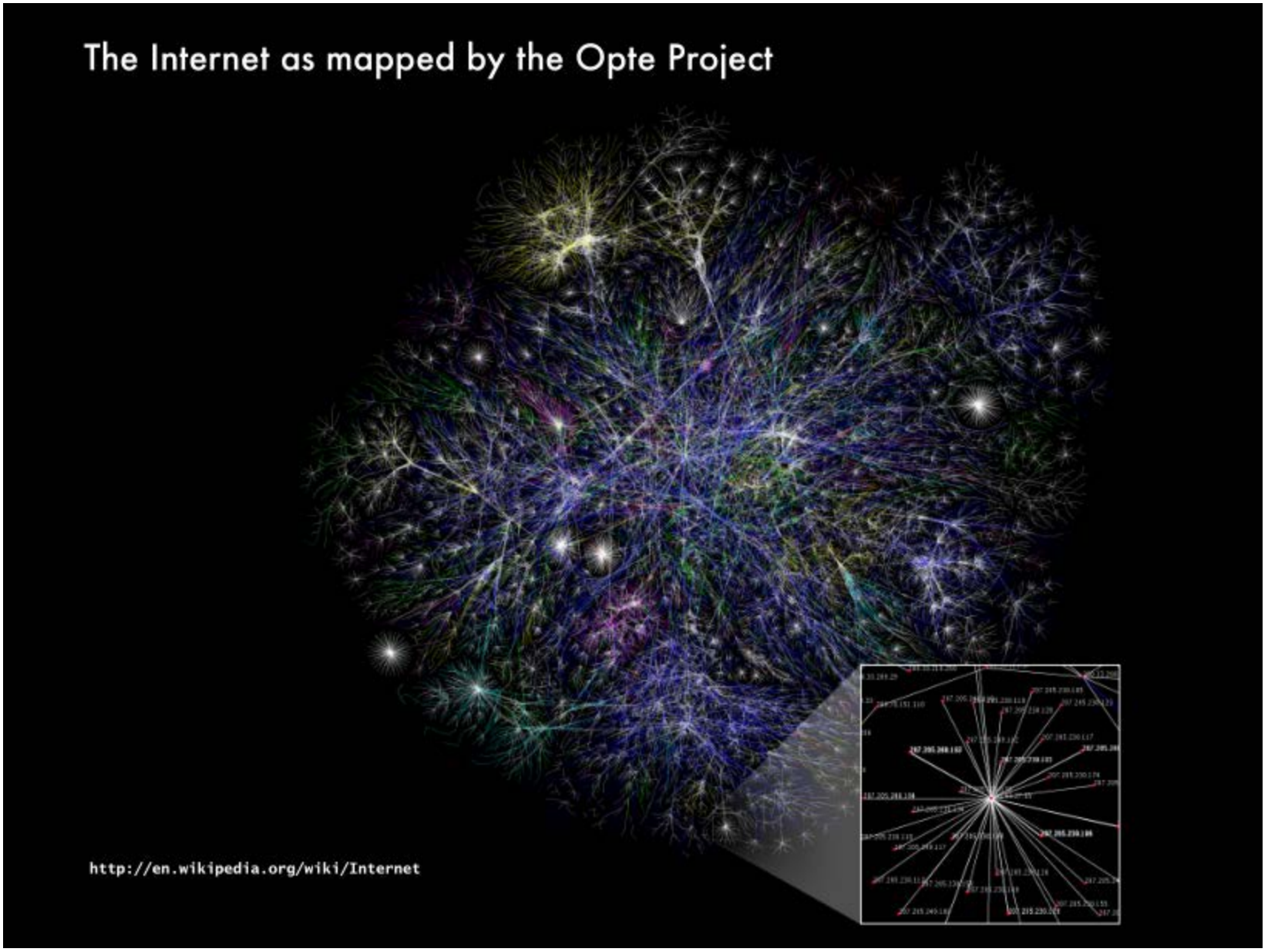




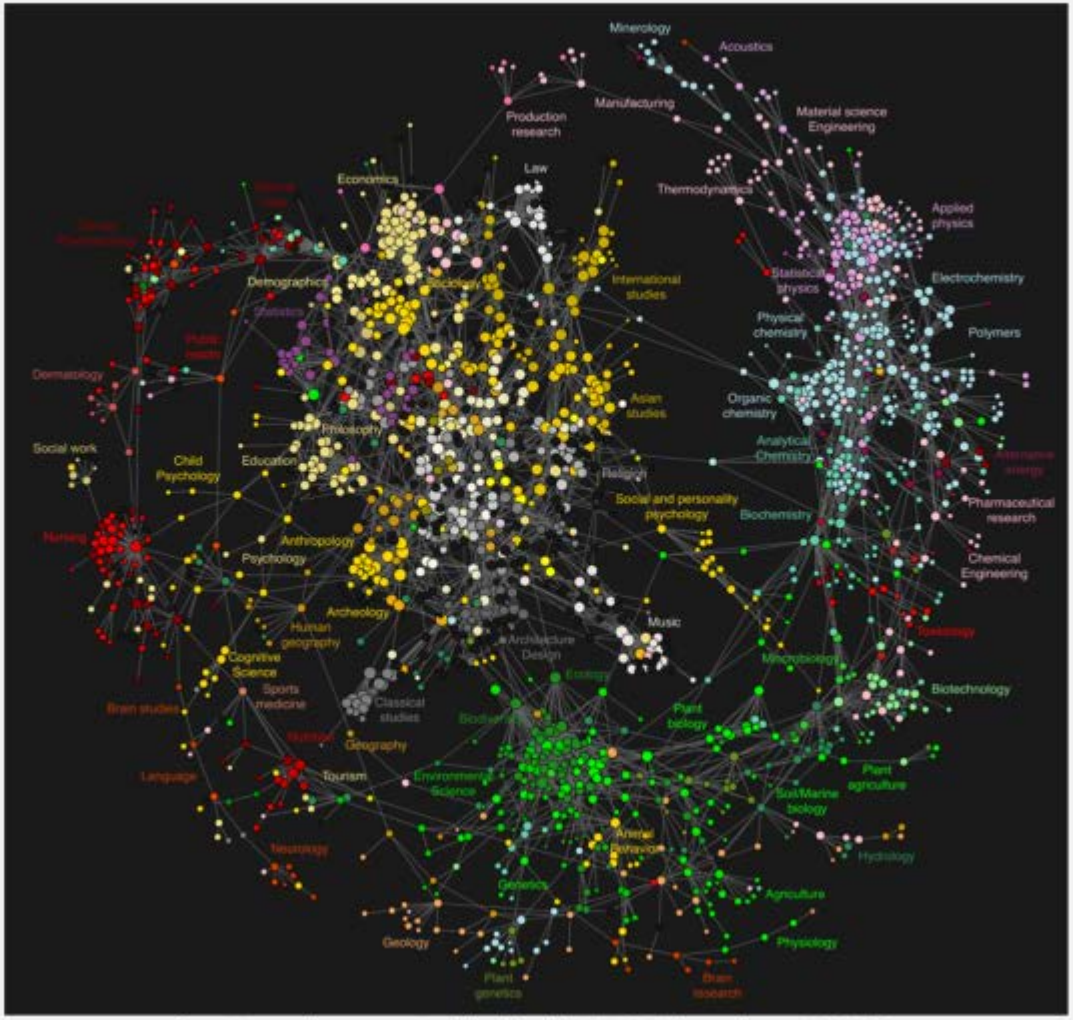
Protein-protein interaction network



Reference: Jeong et al, Nature Review | Genetics


The Internet as mapped by the Opte Project
http://en.wikipedia.org/wiki/Internet

Map of science clickstreams

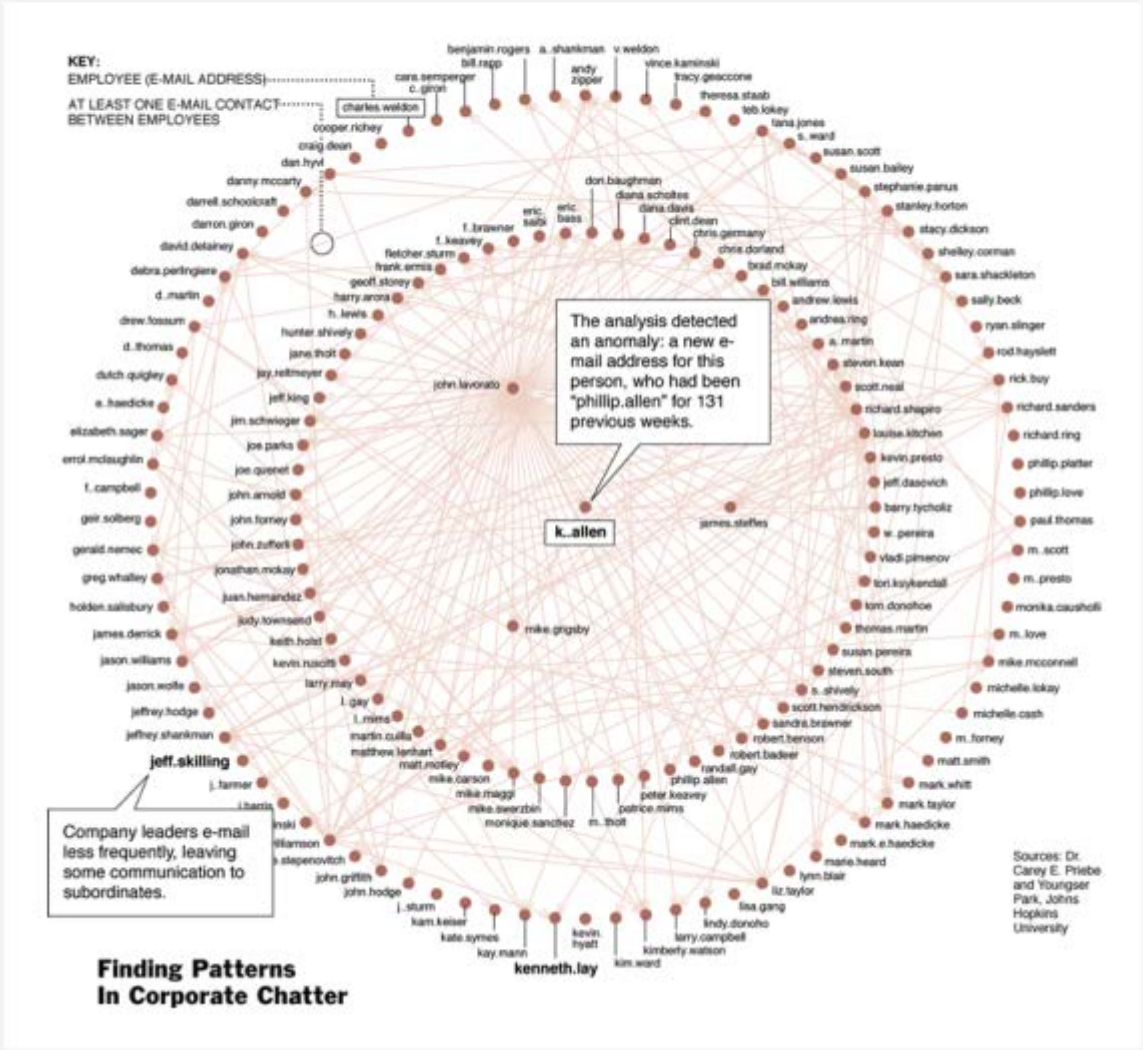


http://www.plosone.org/article/info:doi/10.1371/journal.pone.0004803

10 million Facebook friends

"Visualizing Friendships" by Paul Butler

One week of Enron emails



Graph Basics

- **Node**
  - Entities of interest
  - $V(G)$
- **Edge**
  - Relations of interest
  - $E(G) \subseteq V \times V$

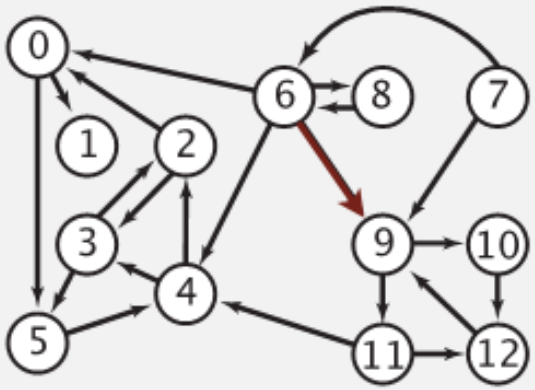
Graph Traversals

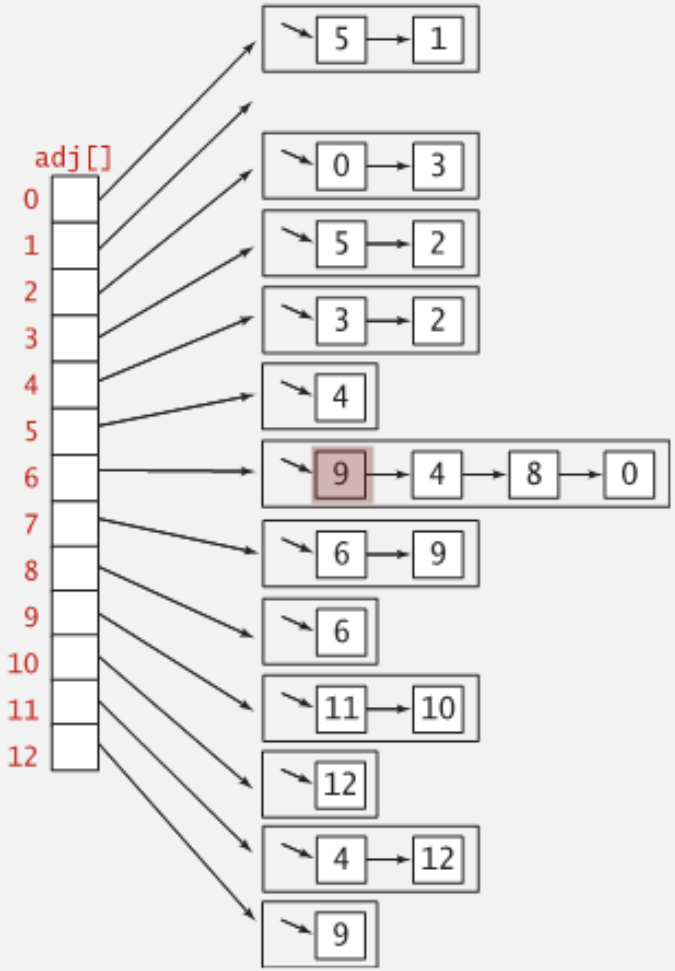
- Depth-First and Breadth-First Search
- Finding Connected Components

- **General DFS/BFS Skeleton**
- **Depth-First Search Trace**

Adjacency-lists digraph representation

Maintain vertex-indexed array of lists.

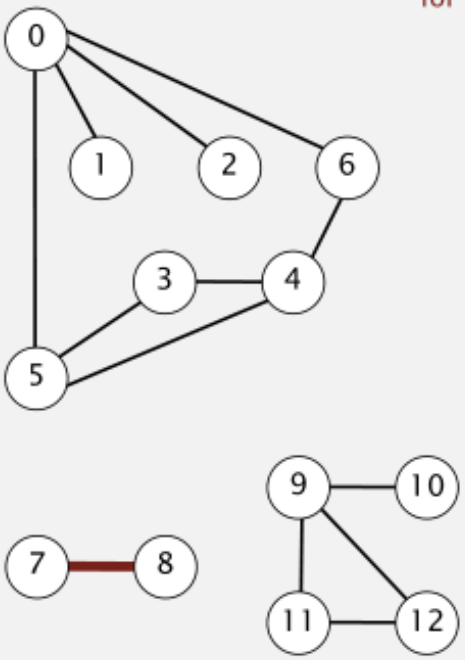




Adjacency-matrix graph representation

Maintain a two-dimensional V-by-V boolean array;
for each edge v–w in graph: `adj[v][w] = adj[w][v] = true.`

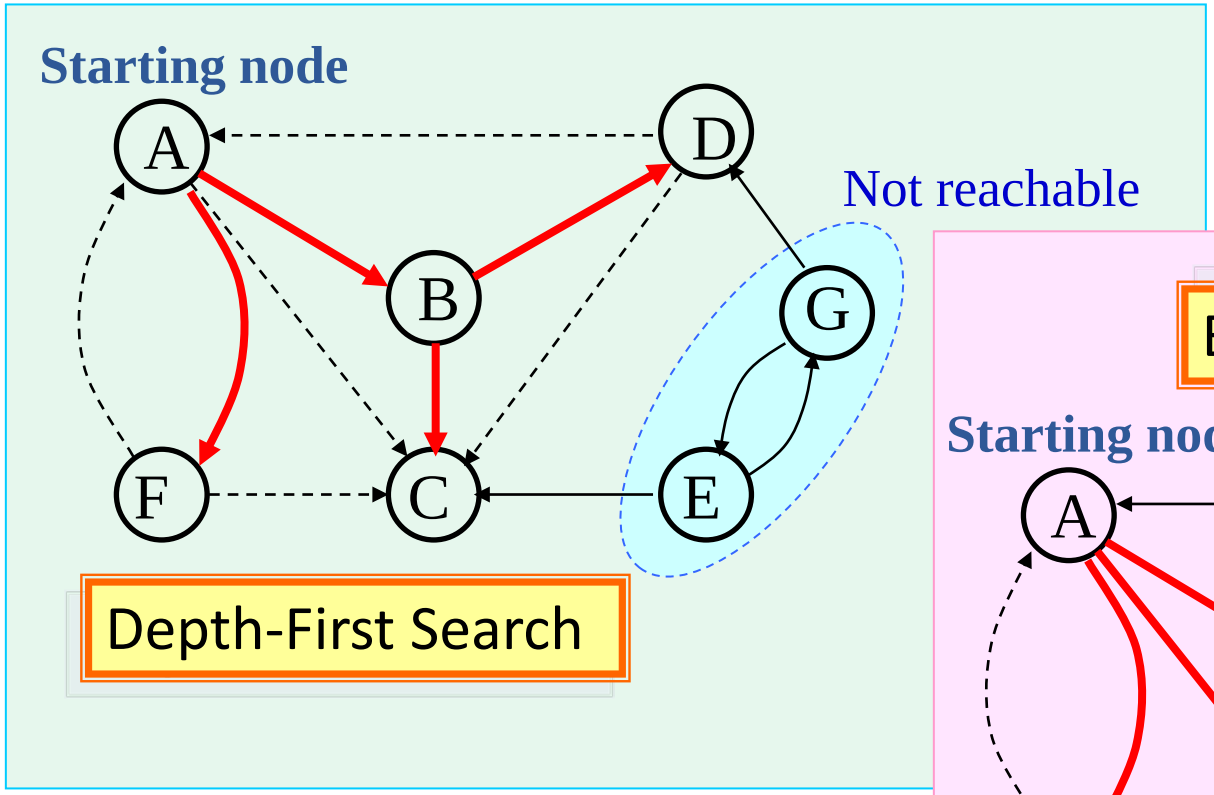
Directed vs. **Undirected** graphs

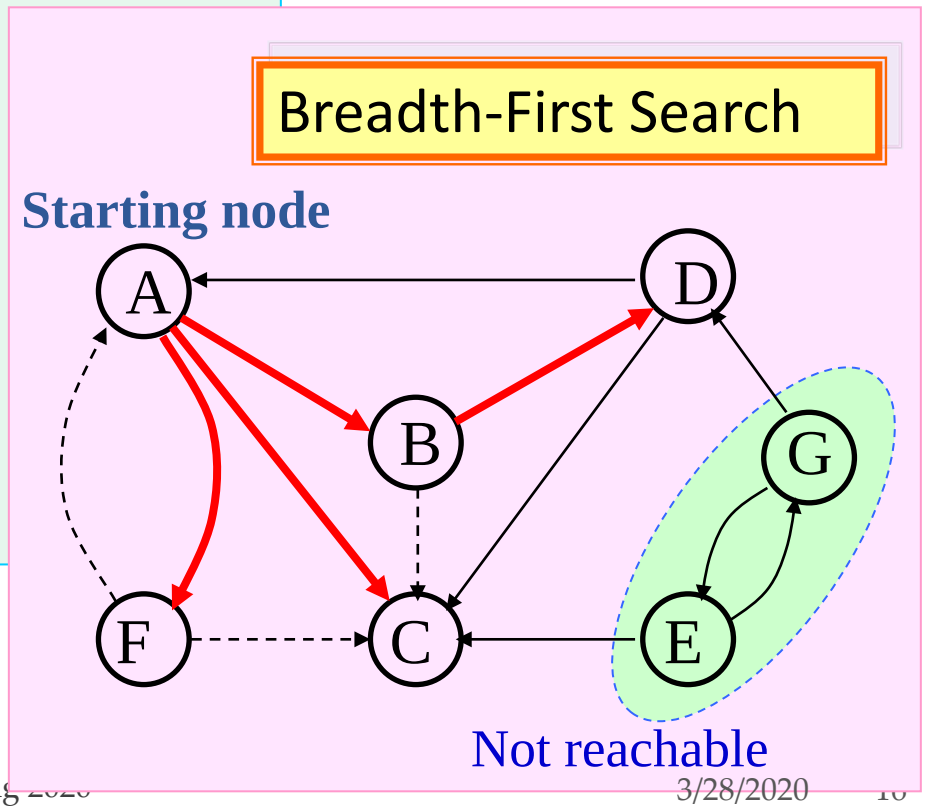


two entries for each edge

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	1	1	0	0	1	1	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	1	1	0	0	0	0	0	0	0
4	0	0	0	1	0	1	1	0	0	0	0	0	0
5	1	0	0	1	1	0	0	0	0	0	0	0	0
6	1	0	0	0	1	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	1	0	0	0	0
8	0	0	0	0	0	0	0	1	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	1	1	1
10	0	0	0	0	0	0	0	0	0	1	0	0	0
11	0	0	0	0	0	0	0	0	0	1	0	0	1
12	0	0	0	0	0	0	0	0	0	1	0	1	0

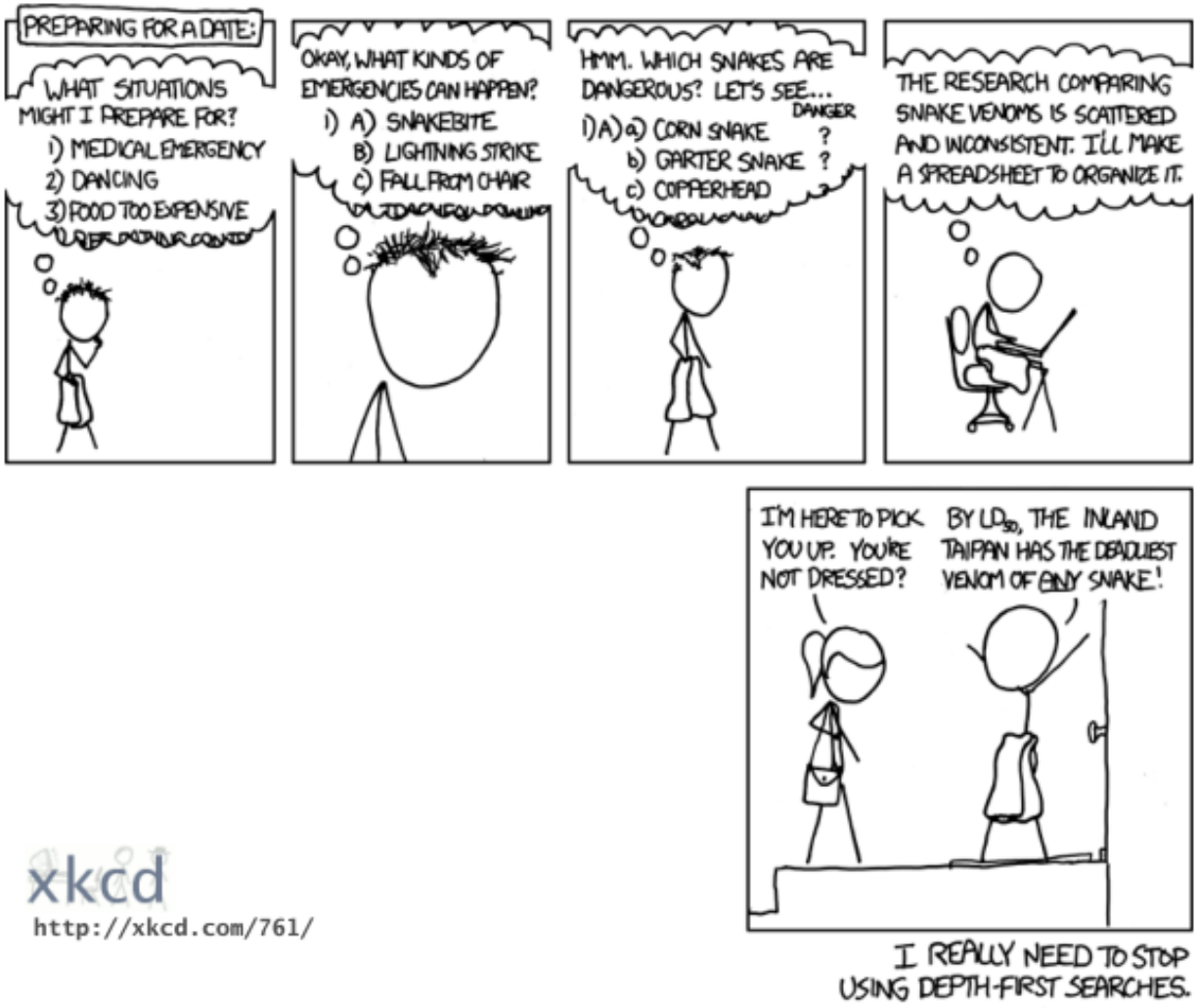
Graph Traversal





---------- *Edges only "checked"*

Depth-first search application: preparing for a date



Outline of Depth-First Search

- dfs(G,v)
- Mark v as "discovered".
- For each vertex w that edge vw is in G:
- If w is undiscovered:
- dfs(G,w)
- Otherwise:
- "Check" vw without visiting w.
- Mark v as "finished".

A vertex must be exact one of three different status:

- undiscovered
- discovered but not finished
- finished

That is: exploring vw, visiting w, exploring from there as much as possible, and backtrack from w to v.

Outline of Breadth-First Search

- Bfs(G,*s*)
- Mark *s* as "discovered";
- enqueue(pending,*s*);
- while (pending is nonempty)
- dequeue(pending, *v*);
- For each vertex *w* that edge *vw* is in G:
- If *w* is "undiscovered"
- Mark *w* as "discovered" and enqueue(pending, *w*)
- Mark *v* as "finished";

Finding Connected Components

- Input: a symmetric digraph G, with n nodes and $2m$ edges(interpreted as an undirected graph), implemented as a array *adjVertices*[1,…n] of adjacency lists.
- Output: an array *cc*[1..n] of component number for each node v_i

```
void connectedComponents(Intlist[ ] adjVertices, int n,
        int[ ] cc) // This is a wrapper procedure
    int[ ] color=new int[n+1];
    int v;
    <Initialize color array to white for all vertices>
    for (v=1; v≤n; v++)
        if (color[v]==white)
            ccDFS(adjVertices, color, v, v, cc);
    return
```

Depth-first search

ccDFS: the procedure

- void ccDFS(IntList[ ] *adjVertices*, int[ ] *color*, int *v*, int *ccNum*, int [ ] *cc*)//*v* as the code of current connected component
- int *w*;
- IntList remAdj;
- color[v]=gray;
- cc[v]=ccNum;
- remAdj=adjVertices[v];
- while (remAdj≠nil)
- w=first(remAdj);
- if (color[w]==white)
- ccDFS(adjVertices, color, w, ccNum, cc);
- remAdj=rest(remAdj);
- color[v]=black;
- return

The elements of *remAdj* are neighbors of *v*

Processing the next neighbor, if existing, another depth-first search to be incurred

v finished

Analysis of CC Algorithm

- **connectedComponents, the wrapper**
  - Linear in n (color array initialization+for loop on *adjVertices*)
- **ccDFS, the depth-first searcher**
  - In one execution of ccDFS on v, the number of instructions(rest(*remAdj*)) executed is proportional to the size of *adjVertices*[v].
  - Note: Σ(size of *adjVertices*[v]) is $2m$, and the adjacency lists are traveresed **only once**.

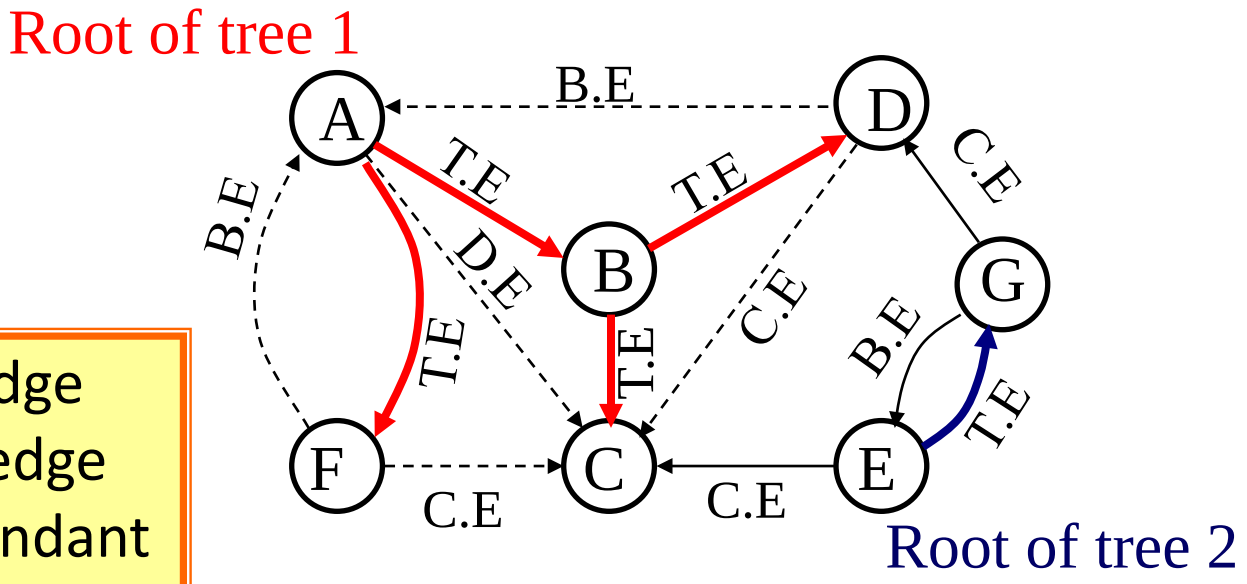
- **So, the time complexity is in Θ(m+n)**
  - Extra space requirements:
    - Color array
    - Activation frame stack for recursion

Visits On a Vertex

- **Classification for the visits on a vertex**
  - First visit(exploring): status: white→gray
  - (Possibly) multi-visits by backtracking to: status keeps gray
  - Last visit(no more branch-finished): status: gray→black
- **Different operations can be done, during the different visits on a specific vertex**
  - On the vertex
  - On (selected) incident edges

Depth-first Search Trees

DFS forest={(DFS tree1), (DFS tree2)}



T.E: tree edge
B.E: back edge
D.E: descendant edge
C.E: cross edge

A finished vertex is never revisited, such as C

Depth-First Search – Generalized Skeleton

- Input: Array *adjVertices* for graph G
- Output: Return value depends on application.

```
int dfsSweep(IntList[] adjVertices,int n, …)
    int ans;
    <Allocate color array and initialize to white>
    For each vertex v of G, in some order
        if  (color[v]==white)
            int vAns=dfs(adjVertices, color, v, …);
            <Process vAns>
        // Continue loop
    return ans;
```

Depth-First Search – Generalized Skeleton

```
int dfs(IntList[] adjVertices, int[] color, int v, …)
    int w;
    IntList remAdj;
    int ans;
    color[v]=gray;
    <Preorder processing of vertex v>
    remAdj=adjVertices[v];
    while (remAdj≠nil)
       w=first(remAdj);
       if (color[w]==white)
          <Exploratory processing for tree edge vw>
          int wAns=dfs(adjVertices, color, w, …);
          < Backtrack processing for tree edge vw , using wAns>
       else
          <Checking for nontree edge vw>
       remAdj=rest(remAdj);
    <Postorder processing of vertex v, including final computation of ans>
    color[v]=black;
    return ans;
```

If partial search is used for a application, tests for termination may be inserted here.

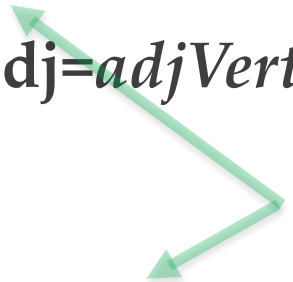
Specialized for connected components:
- parameter added
- preorder processing inserted – cc[v]=ccNum

Breadth-First Search - Skeleton

- Input: Array *adjVertices* for graph G
- Output: Return value depends on application.
- void bfsSweep(IntList[] *adjVertices*,int n, …)
- int ans;
- <Allocate color array and initialize to white>
- For each vertex *v* of G, in some order
- if (color[v]==white)
- void bfs(*adjVertices*, color, v, …);
- // Continue loop
- return;

Breadth-First Search - Skeleton

```
void bfs(IntList[] adjVertices, int[] color, int v, …)
    int w; IntList remAdj; Queue pending;
    color[v]=gray;    enqueue(pending, v);
    while (pending is nonempty)
        w=dequeue(pending); remAdj=adjVertices[w];
        while (remAdj≠nil)
            x=first(remAdj);
            if (color[x]==white)
                color[x]=gray; enqueue(pending, x);
            remAdj=rest(remAdj);
        <processing of vertex w>
        color[w]=black;
    return ;
```



can be further generalized

DFS vs. BFS Search

- **Processing opportunities for a node**
  - Depth-first: 2
    - At discovering
    - At finishing
  - Breadth-first: only 1, when de-queued
  - At the second processing opportunity for the DFS, the algorithm can make use of information about the descendants of the current node.

Time Relation on Changing Color

- **Keeping the order in which vertices are encountered for the first or last time**
  - A global integer time: 0 as the initial value, incremented with each color changing for *any* vertex, and the final value is $2n$
  - Array *discoverTime*: the i th element records the time vertex v_i turns into gray
  - Array *finishTime*: the i th element records the time vertex v_i turns into black
  - The active interval for vertex v, denoted as $active(v)$, is the duration while v is gray, that is:

$$discoverTime[v], \ldots, finishTime[v]$$

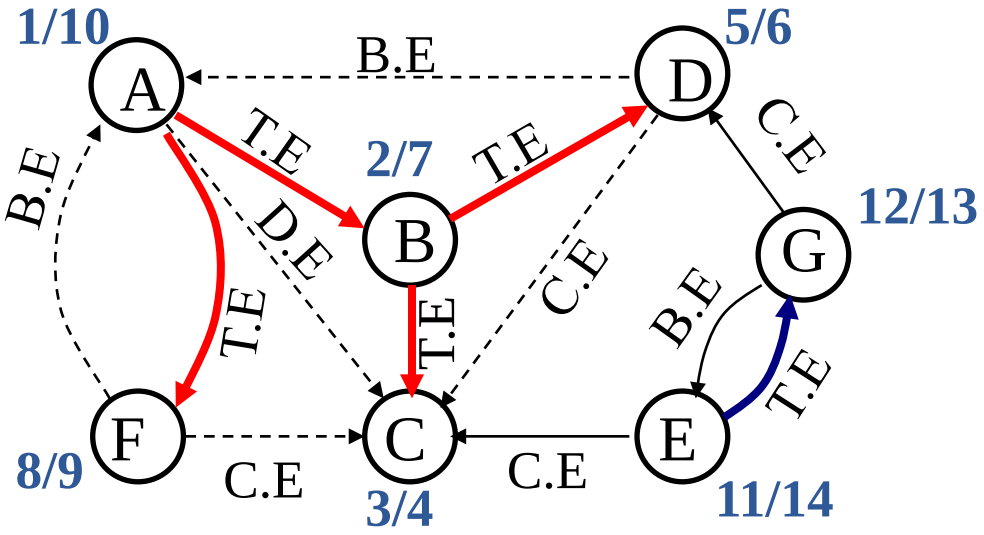
Depth-First Search Trace

- General DFS skeleton modified to compute discovery and finishing times and "construct" the depth-first search forest.
- int dfsTraceSweep(IntList[ ] *adjVertices*,int n, int[ ] *discoverTime*, int[ ] *finishTime*, int[ ] *parent*)
- int ans; int *time*=0
- <Allocate color array and initialize to white>
- For each vertex *v* of G, in some order
- if (color[v]==white)
- parent[v]=-1
- int vAns=dfsTrace(*adnVertices*, color, v, *discoverTime*, *finishTime*, *parent*, *time*);
- // Continue loop
- return ans;

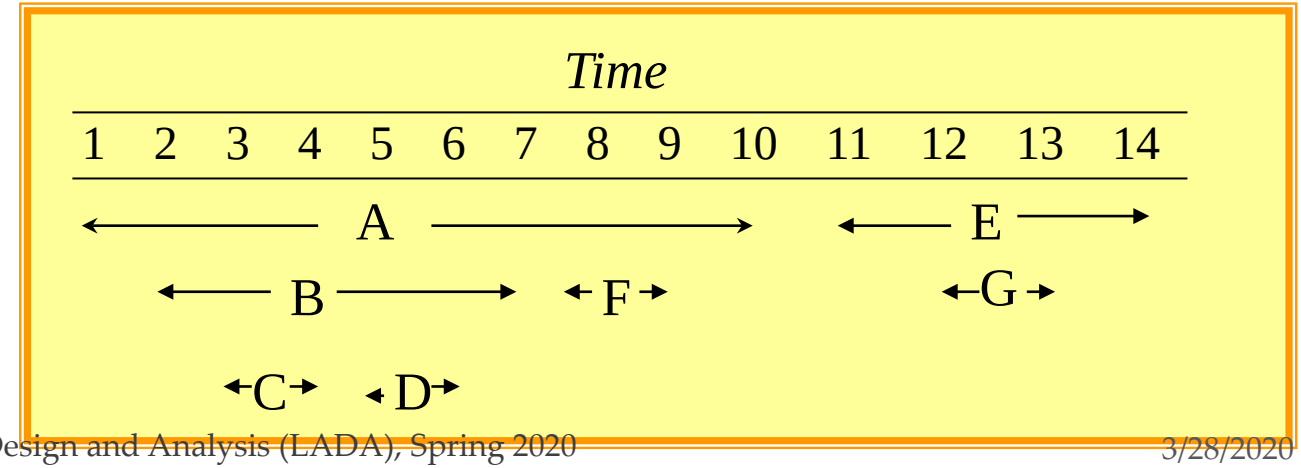
Depth-First Search Trace

```
int dfsTrace(intList[ ] adjVertices, int[ ] color, int v,  int[ ] discoverTime,
             int[ ] finishTime, int[ ] parent  int time)
   int w; IntList remAdj; int ans;
   color[v]=gray; time++; discoverTime[v]=time;
   remAdj=adjVertices[v];
   while (remAdj≠nil)
      w=first(remAdj);
      if (color[w]==white)
         parent[w]=v;
         int wAns=dfsTrace(adjVertices, color, w, discoverTime, finishTime,
                                                  parent, time);
      else  <Checking for nontree edge vw>
      remAdj=rest(remAdj);
    time++; finishTime[v]=time; color[v]=black;
   return ans;
```

Active Interval



The relations are summarized in the next frame



Properties of Active Intervals(1)

- **If w is a descendant of v in the DFS forest, then $active(w) \subseteq active(v)$, and the inclusion is proper if w≠v.**
- **Proof:**
  - Define a partial order <: w<v iff. w is a proper descendants of v in its DFS tree. The proof is by induction on <.
  - If v is minimal. The only descendant of v is itself. Trivial.
  - Assume that for all x<v, if w is a descendant of x, then active(w)⊆active(x).
  - Let w be any proper descendant of v in the DFS tree, there must be some x such that vx is a tree edge on the tree path to w, so w is a descendant of x. According to dfsTrace, we have $active(x) \subset active(v)$, by inductive hypothesis, $active(w) \subset active(v)$.

Properties of Active Intervals(2)

- **If $active(w) \subseteq active(v)$, then w is a descendant of v. And if $active(w) \subset active(v)$, then w is a proper descendant of v.**

  That is: *w* is discovered while *v* is active.

- **Proof:**
  - If w is **not** a descendant of v, there are two cases:
    - v is a proper descendant of w, then $active(v) \subset active(w)$, so, it is impossible that $active(w) \subseteq active(v)$, contradiction.
    - There is no ancestor/descendant relationship between v and w, then $active(w)$ and $active(v)$ are disjoint, contradiction.

Properties of Active Intervals(3)

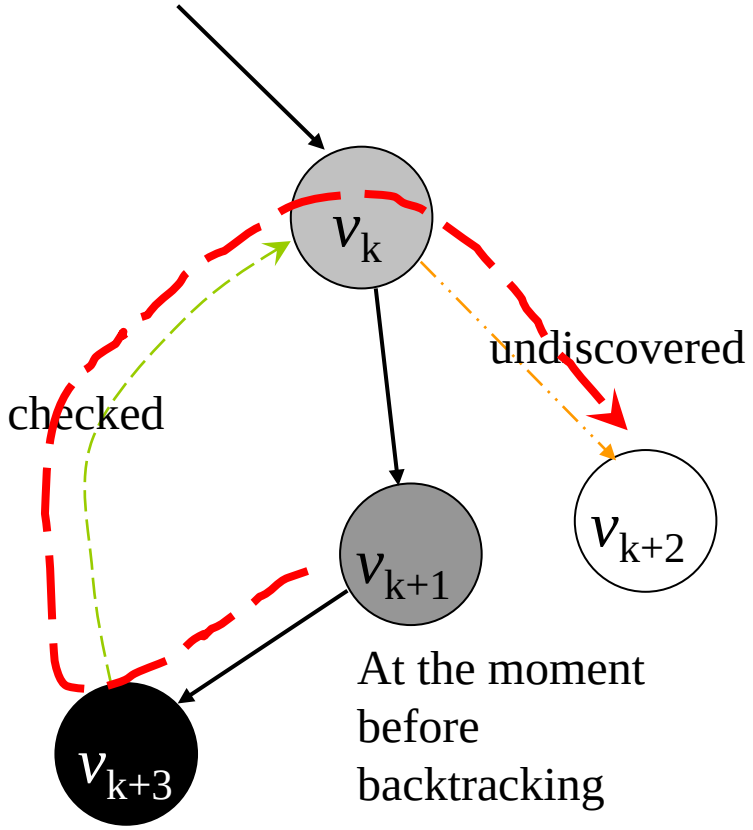
- **If v and w have no ancestor/descendant relationship in the DFS forest, then their active intervals are disjoint.**
- **Proof:**
  - If v and w are in different DFS tree, it is trivially true, since the trees are processed one by one.
  - Otherwise, there must be a vertex c, satisfying that there are tree paths c to v, and c to w, without edges in common. Let the leading edges of the two tree path are cy, cz, respectively. According to dfsTrace, $active(y)$ and $active(z)$ are disjoint.
  - We have $active(v) \subseteq active(y)$, $active(w) \subseteq active(z)$. So, $active(v)$ and $active(w)$ are disjoint.

Properties of Active Intervals(4)

- **If edge vw$\in E_G$, then**
  - vw is a **cross edge** iff. *active*(w) entirely precedes *active*(v).
  - vw is a **descendant edge** iff. there is some third vertex x, such that *active*(w)$\subset$*active*(x)$\subset$*active*(v),
  - vw is a **tree edge** iff. *active*(w)$\subset$*active*(v), and there is no third vertex x, such that *active*(w)$\subset$*active*(x) $\subset$*active*(v),
  - vw is a **back edge** iff. *active*(v)$\subset$*active*(w),

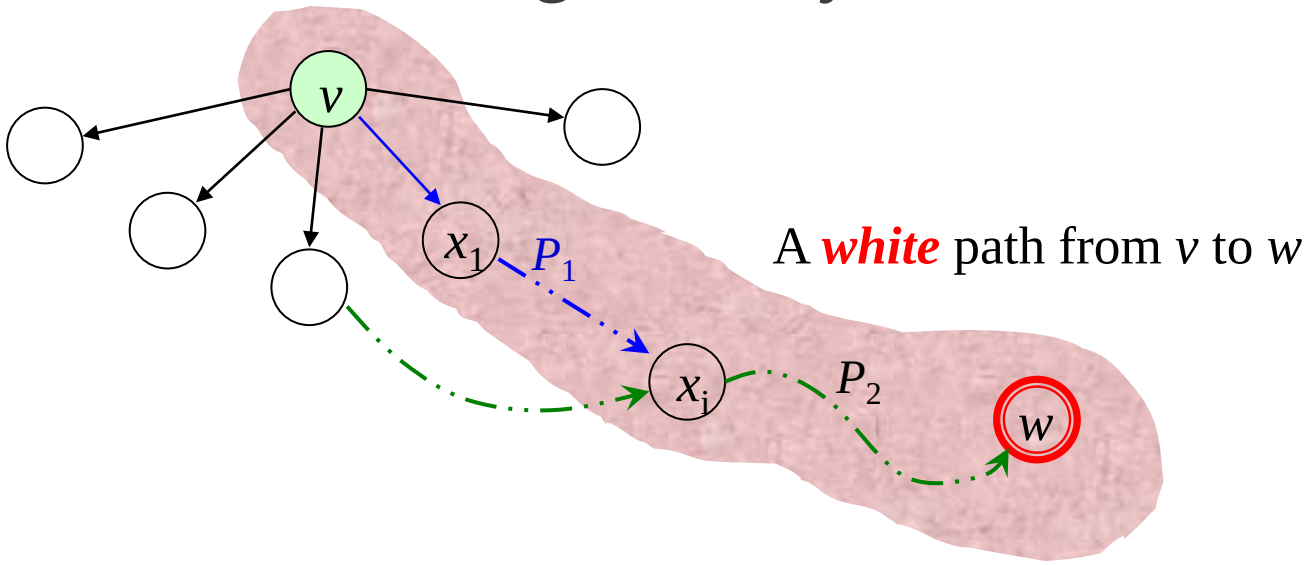
Ancestor and Descendant

- That w is a descendant of v in the DFS forest means that there is a direct path from v to w in some DFS tree.
- The path is also a path in G.
- **However, if there is a direct path from v to w in G, is w necessarily a descendant of v in *the* DFS forest?**



DFS Tree Path

- **[White Path Theorem] w is a descendant of v in a DFS tree iff. at the time v is discovered(just to be changing color into gray), there is a path in G from v to w consisting entirely of white vertices.**



Proof of White Path Theorem

- **Proof**
  - $\Rightarrow$ All the vertices in the path are descendants of v.
  - $\Leftarrow$ by induction on the length k of a white path from v to w.
    - When k=0, v=w.
    - For k>0, let P=(v, x_1,x_2,…x_k=w). There must be some vertex on P which is discovered during the active interval of v, e.g. x_1, Let x_i is earliest discovered among them. Divide P into P_1 from v to x_i, and P_2 from x_i to w. P_2 is a white path with length less than k, so, by inductive hypothesis, w is a descendant of x_i. Note: *active*(x_i)$\subseteq$*active*(v), so x_i is a descendant of v. By transitivity, w is a descendant of v.

Thank you!

Q & A

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