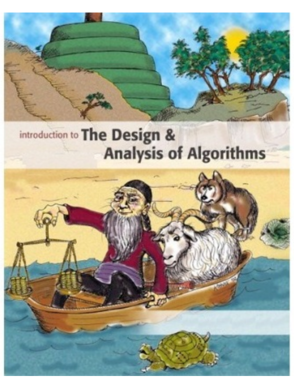
Introduction to

Algorithm Design and Analysis

[15] Path in Graph



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In the last class…

- **Optimization Problem**
  - o Greedy strategy

- **MST Problem**
  - o Prim algorithm
  - o Kruskal algorithm
- **Single-Source Shortest Path Problem**
  - o Dijkstra algorithm

Path in Graphs

- **Single-source shortest paths (SSSP)**
  - o Dijkstra algorithm by example
  - o Priority queue-based implementation
  - o Proof of correctness
- **All-pairs shortest paths (APSP)**
  - o Shortest path and transitive closure
  - o Warshall algorithm for transitive closure
    - BF1, BF2, BF3 => Warshall algorithm
    - Floyd algorithm for shortest paths

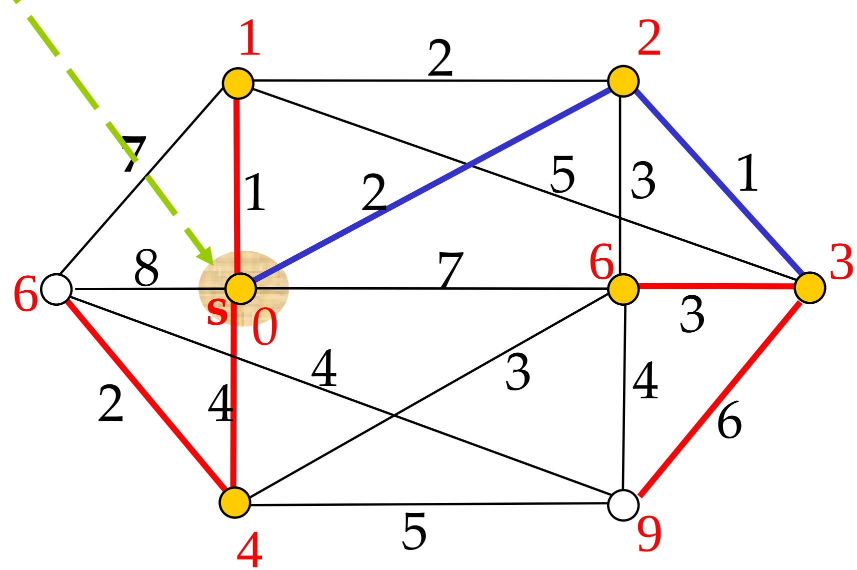
Single Source Shortest Paths

The single source

Red labels on each vertex is the length of the shortest path from s to the vertex.

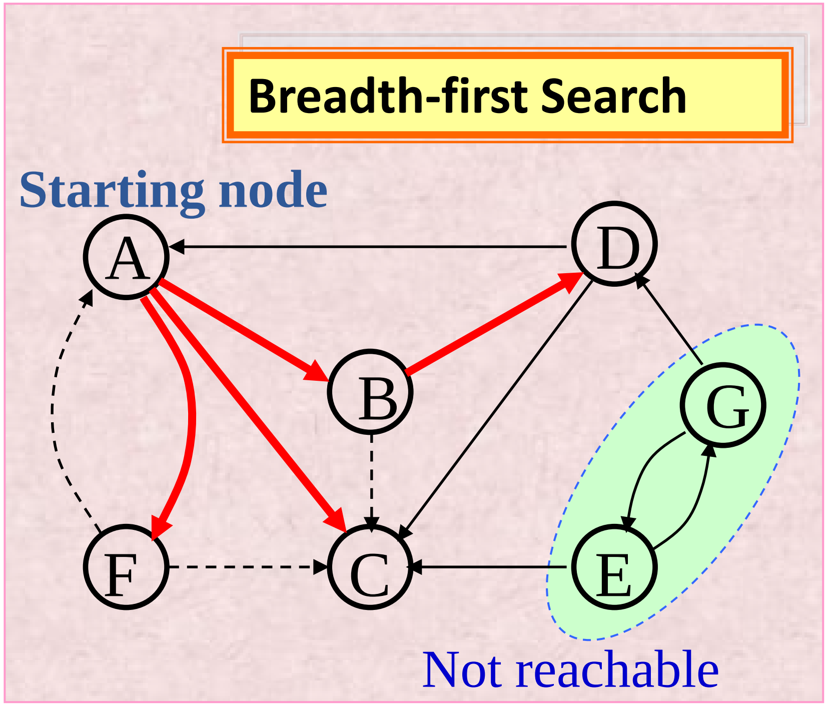
Note:

The shortest [0, 3]-path doesn't contain the shortest edge leaving s, the edge [0,1]

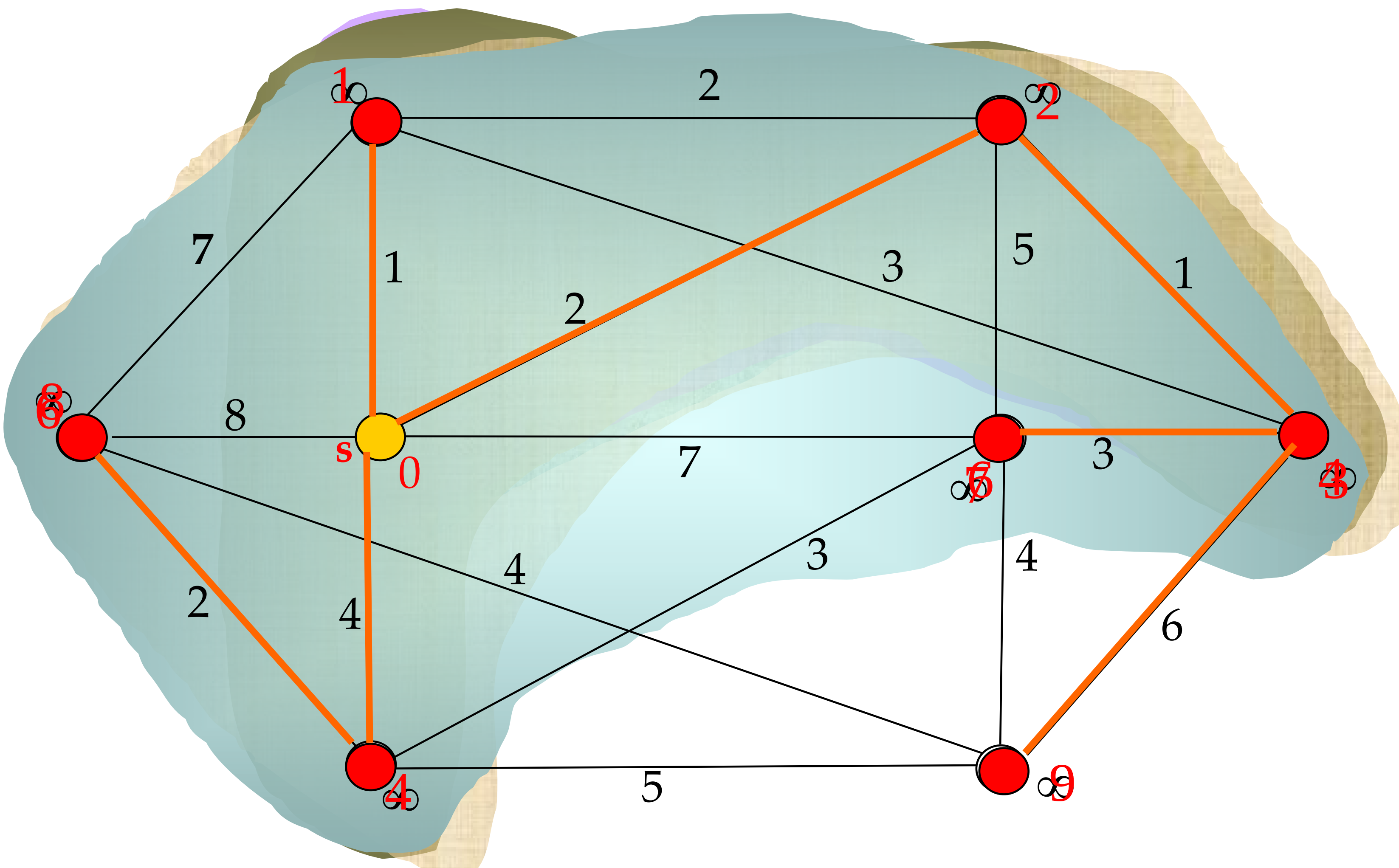


Warm Up

- Single-source shortest path over **uniformly weighted graph**
  - Just BFS



Dijkstra's Algorithm



Priority Queue-based Implementation

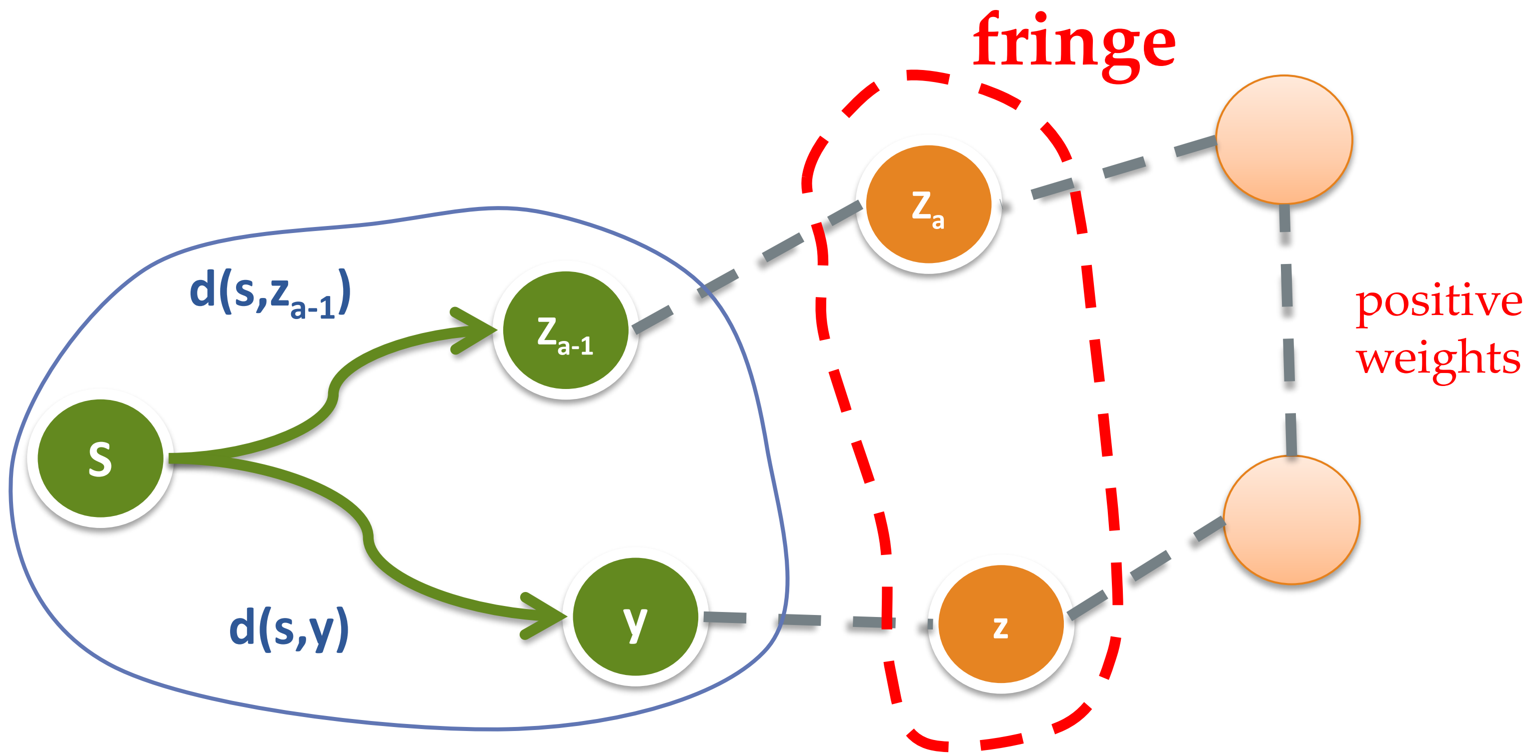
Shortest Paths

```
Void shortestPaths(EdgeList[] adjInfo, int n, int s,
int[] parent, float[]fringeWgt)

   int[] status = new int[n+1];
   MinPQ pq = create(n, status, parent, fringeWgt);

   insert(pq, s, -1, 0);
   while(isEmpty(pq)==false)
      int v = getMin(pq);
      deleteMin(pq);
      updateFringe(pq, adjInfo[v], v);
```

```
void updateFringe(MinPQ pq, EdgeList
adjInfoOfV, int v)

   float myDist = pq.fringeWgt[v];
   EdgeList remAdj;
   remAdj = adjInfoOfV;
   while{remAdj != nil}
      EdgeInfo wInfo = first(remAdj);
      int w = wInfo.to;
      float newDist = myDist + wInfo.weight;
      if(pq.status[w]==unseen)
         insert(pq,w,v,newDist);
      else if(pq.status[w] = fringe)
         if(newDist < getPriority(pq,w))
            decreaseKey(pq,w,v,newDist);
      remAdj = rest(remAdj);
return;
```

Correctness of the Dijkstra Algorithm

- W(s->y->z) < W(s->z_{a-1}->z_a->z)



The Dijkstra Skeleton

- **Single-source shortest path (SSSP)**
- **SSSP + node weight constraint**
    - o E.g. in routing
        - Each router has its cost (node cost)
        - Each route has its cost (edge cost)
- **SSSP + capacity constraint**
    - o The "pipe problem"
        - Maximize the min edge weight
    - o The "electric vehicle problem"
        - Minimize the max edge weight

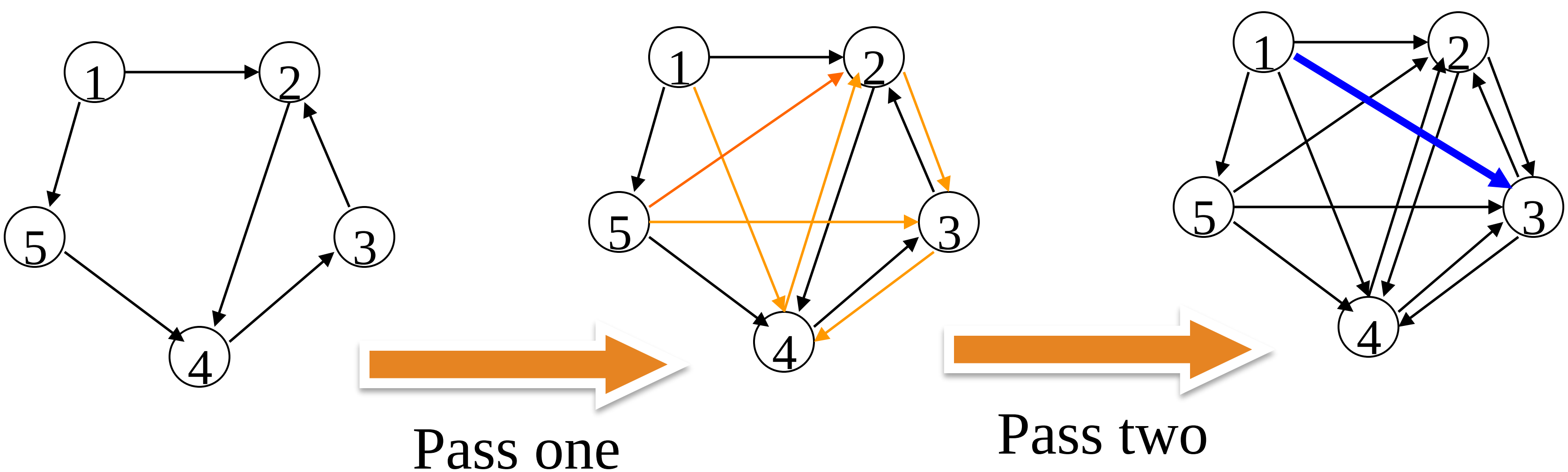
"Dijkstra Skeleton"

All-pairs Shortest Paths

- **For all pair of vertices in a graph, say, *u*, *v*:**
  - Is there a path from *u* to *v*?
  - What is the shortest path from *u* to *v*?
- **Reachability as a (reflexive) transitive closure of the adjacency relation**
  - Which can be represented as a bit matrix

Transitive Closure by Shortcuts

- The idea: if there are edges $s_i s_k$, $s_k s_j$, then an edge $s_i s_j$, the "shortcut" is inserted.



Shortcut Algorithm

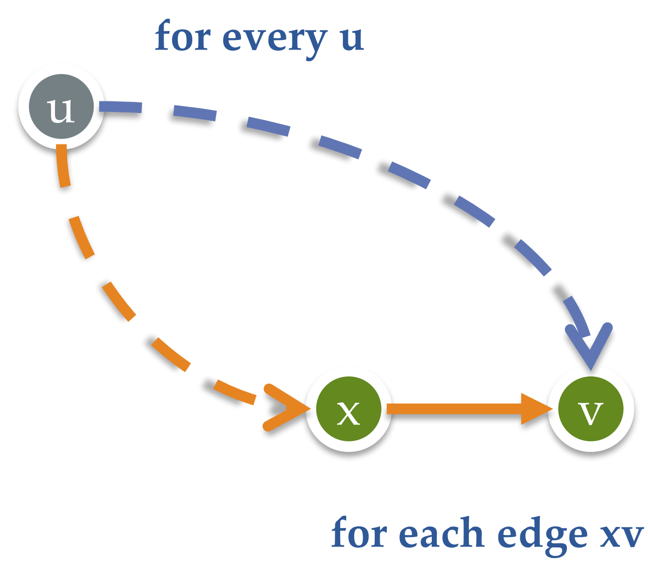
- **Input:** A**, an** $n \times n$ **boolean matrix that represents a binary relation**
- **Output:** R**, the boolean matrix for the transitive closure of** A
- **Procedure**
  - **void** simpleTransitiveClosure(**boolean**[ ][ ] A, **int** n, **boolean**[ ][ ] R)
  - **int** i,j,k;
  - Copy A to R;
  - Set all main diagonal entries, r_{ii}, to *true*;
  - **while** (any entry of R changed during one complete pass)
  - **for** (i=1; i≤n; i++)
  - **for** (j=1; j≤n; j++)
  - **for** (k=1; k≤n; k++)
  - $r_{ij} = r_{ij} \vee (r_{ik} \wedge r_{kj})$

$O(n^4)$

The order of (i,j,k) matters

Another Way to Add Shortcuts

- **Enumerate all edges (x,v)**
  - o v as the destination
  - o Enumerate all possible sources u



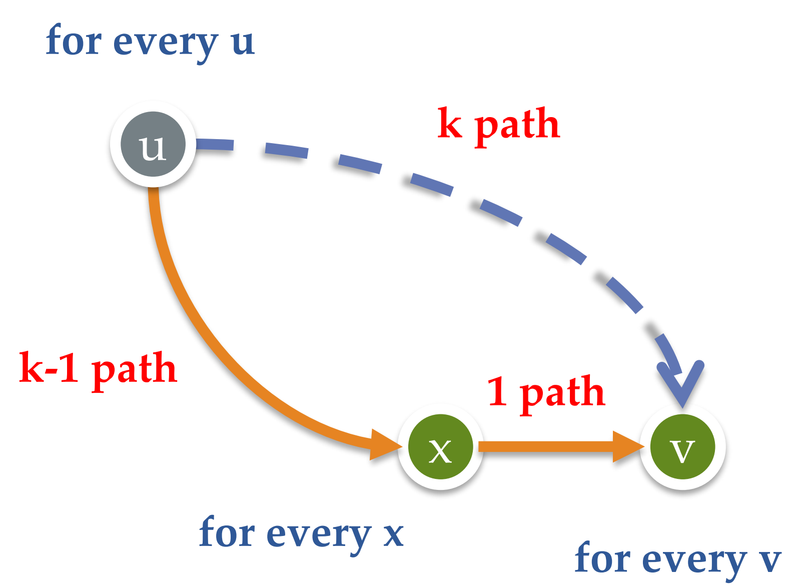
While any entry of R changed
  for all vertices u
    for every edge (x,v)
      $r_{uv} = r_{uv} \vee (r_{ux} \wedge r_{xv})$

$O(n^2m)$

n-1 round iteration

Length of the Path

- **Recursion**
  - Reachable via at most k edges
- **Enumeration**
  - Enumerate all path length
  - Enumerate all sources and destinations



for k=1 to n-1
    for all vertices u
        for all vertices v
            for all vertices x pointing to v
                $r^{k}_{uv} = r^{k-1}_{uv} \vee (r^{k-1}_{ux} \wedge r_{xv})$

$O(n^4)$

$r_{xv} = 1$

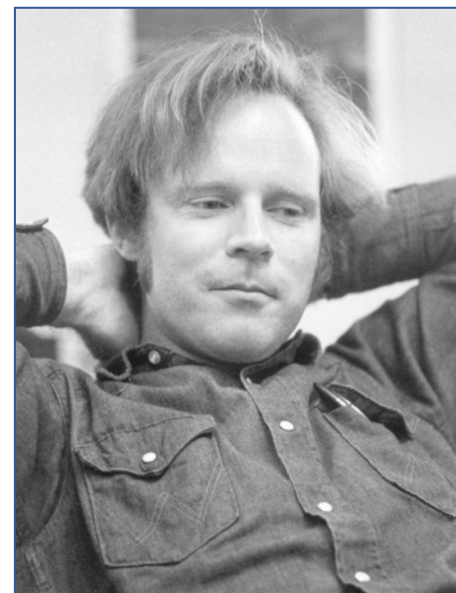
Floyd's Lemma

组合问题的优良算法具有巨大回报，这个事实激励了技术水平的突飞猛进。……大约从1970年起，计算机科学家们经历了所谓的‘**Floyd引理**’现象：看似需用n^3次运算的问题实际上可能用$O(n^2)$次运算就能求解，看似需用n^2次运算的问题实际上可能用$O(nlogn)$次运算就能处理，而且nlogn通常还可以减少到$O(n)$。一些更难的问题的运行时间也从$O(2^n)$减少到$O(1.5^n)$，再减少到$O(1.3^n)$，等等。

- Knuth, Volumn4A, TAOCP

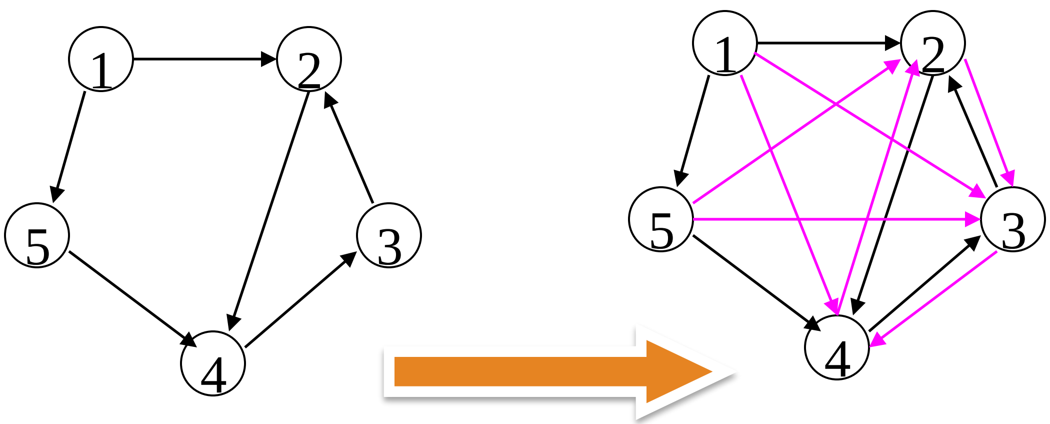
Robert W Floyd, In Memoriam

by Donald E. Knuth, Stanford University



Shortcuts in Different Order

- **Duplicated checking may be deleted by changing the order of the vertices.**



Pass one

Check the vertices in decreasing order.

No edge is added in Pass two. End.

Change the Order: the Warshall Algorithm

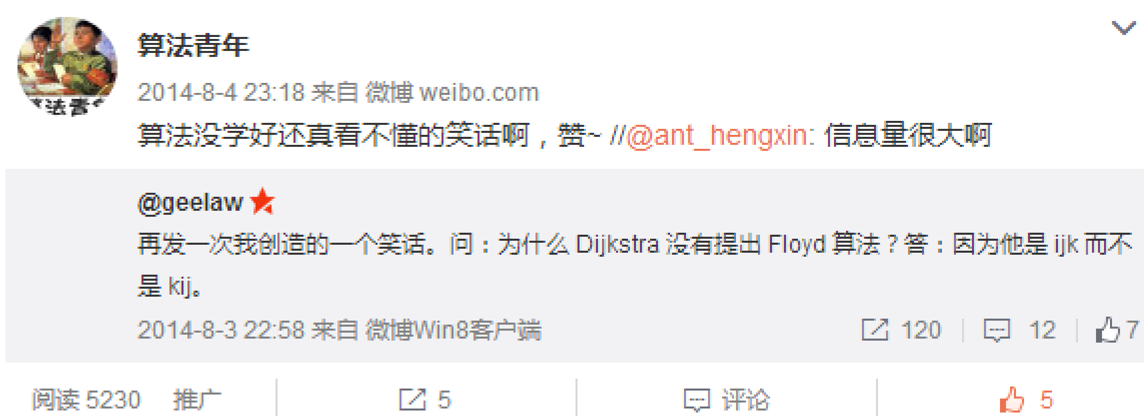
- **void** simpleTransitiveClosure(**boolean**[][] A, **int** n, **boolean**[][] R)
- **int** i,j,k;
- Copy A to R;
- Set all main diagonal entries, r_{ii}, to *true*;
- ~~**while** (any entry of R changed during one complete pass)~~
- **for** (k=1; k≤n; k++)
- **for** (i=1; i≤n; i++)
- **for** (j=1; j≤n; j++)
- $r_{ij}=r_{ij}\vee(r_{ik}\wedge r_{kj})$

k varys in the outmost loop

Note: "false to true" can not be reversed

Why the Floyd-Warshall Algorithm Works

- <k,i,j> or <i,j,k>
  - The order matters
  - *That's why Dijkstra fails*



Correctness of the Warshall Algorithm

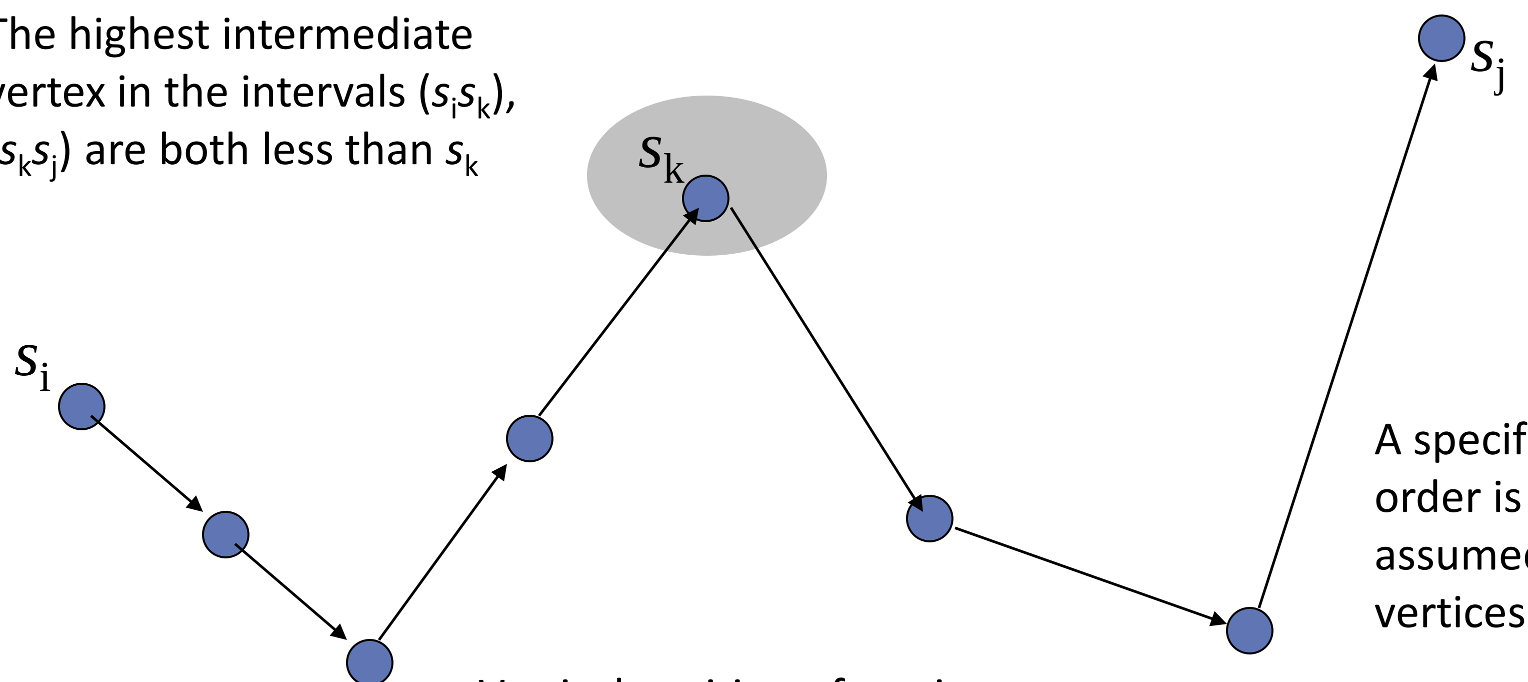
- **Notation:**
  - The value of r_{ij} changes during the execution of the body of the "**for** *k*…" loop
    - After initializations: $\boldsymbol{r_{ij}^{(0)}}$
    - After the k^{th} time of execution: $\boldsymbol{r_{ij}^{(k)}}$

Correctness of the Warshall Algorithm

- **If there is a simple path from s_i to $s_j (i \neq j)$ for which the highest-numbered intermediate vertex is s_k, then $r_{ij}^{(k)}$=true.**
- **Proof by induction:**
  - Base case: $r_{ij}^{(0)}$=true if and only if $s_i s_j \in E$
  - Hypothesis: the conclusion holds for $h < k (h \geq 0)$
  - Induction: the simple $s_i s_j$-path can be looked as $s_i s_k$-path+$s_k s_j$-path, with the indices h_1, h_2 of the highest-numbered intermediate vertices of both segment **strictly**(simple path) less than k. So, $r_{ik}^{(h1)}$=true, $r_{kj}^{(h2)}$=true, then $r_{ik}^{(k-1)}$=true, $r_{kj}^{(k-1)}$=true(Remember, false to true can not be reversed). So, $r_{ij}^{(k)}$=true .

Highest-numbered Intermediate Vertex

The highest intermediate vertex in the intervals $(s_i s_k)$, $(s_k s_j)$ are both less than s_k



A specific order is assumed for all vertices

Vertical position of vertices reflect their vertex numbers

$$r_{ij} = r_{ij} \vee (r_{ik} \wedge r_{kj})$$

Correctness of the Warshall Algorithm

- **If $r_{ij}^{(k)}$=true, then there is a $(s_i, s_j)^{(k)}$ path**
- **Proof**
  - o If $r_{ij}^{(0)}$=true, then there is $(s_i,s_j)^{(0)}$ path
  - o If r_{ij} first becomes true in round k, then
    - $r_{ik}^{(k-1)}$ = true, $r_{kj}^{(k-1)}$ = true
  - o We have a "s_i->s_k->s_j" path
    - Intermediate nodes in $\{1, 2, \ldots, k\text{-}1\} \cup \{k\}$

All-pairs Shortest Paths

- **Shortest path property**
  - If a shortest path from x to z consisting of path P from x to y followed by path Q from y to z. Then P is a shortest xy-path, and Q, a shortest yz-path.
- **The regular matrix representing a graph can easily be transformed into a (minimum) distance matrix D**

(just replacing 1 by edge weight, 0 by infinity, and setting main diagonal elements as 0)

Computing the Distance Matrix

- **Basic formula:**
  - $D^{(0)}[i][j]=w_{ij}$
  - $D^{(k)}[i][j]=\min(D^{(k-1)}[i][j], D^{(k-1)}[i][k]+ D^{(k-1)}[k][j])$

- **Basic property:**
  - $D^{(k)}[i][j] = d_{ij}^{(k)}$

    where $d_{ij}^{(k)}$ is the weight of a shortest path from v_i to v_j with highest numbered intermediate vertex v_k.

All-pairs Shortest Paths

- **Floyd algorithm**
  - Only slight changes on Warshall's algorithm.

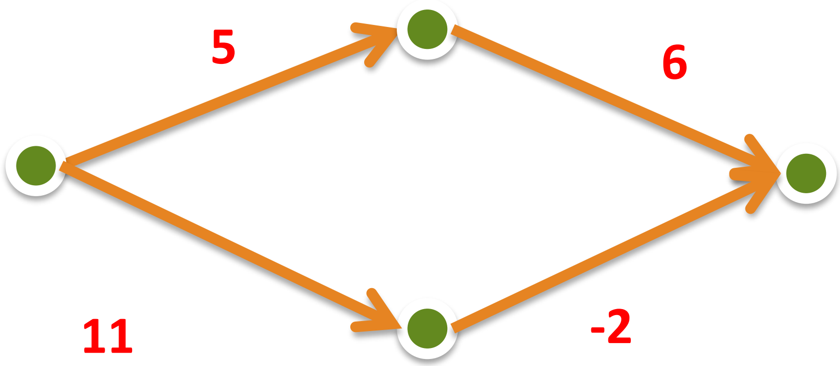
```
Void allPairsShortestPaths(float [][] W, int n, float [][] D)
   int i, j, k;
   Copy W into D;
   for (k=1; k≤n; k++)
      for (i=1; i≤n; i++)
         for (j=1; j≤n; j++)
            D[i][j] = min (D[i][j], D[i][k]+D[k][j]);
```

All-pairs Shortest Paths

- **Construction of the routing table**
  - Forward, backward
- **APSP + capacity constraints**
  - The pipeline problem
  - The electric vehicle problem

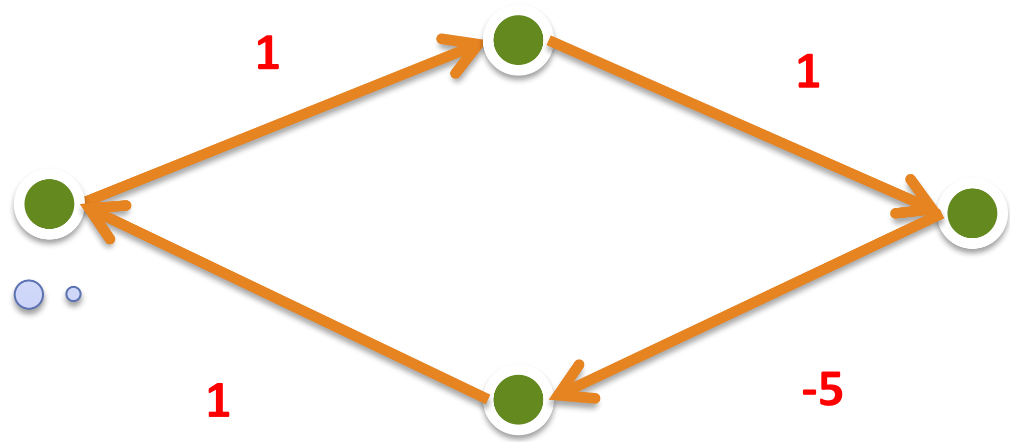
Floyd algorithm => Floyd skeleton

Negative Weight



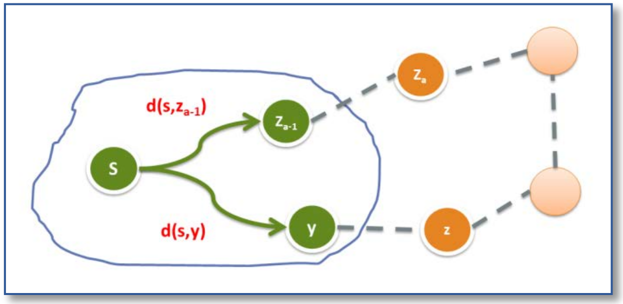
"shortest path" is not well defined

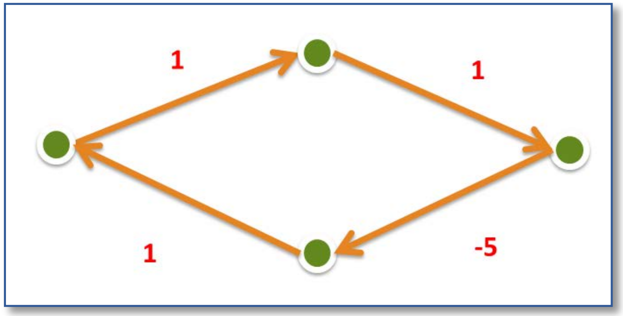
Negative weight cycle



Negative Weight

- **Can the shortest path algorithm work correctly?**
  - o Dijkstra algorithm
    - No negative weight edge
  - o Floyd algorithm
    - No negative weight cycle
  - o Bellman-Ford algorithm
    - Solves SSSP and detects negative cycles





Matrix Representation

- **Define family of matrix $A^{(p)}$:**
  - $a_{ij}^{(p)}$=*true* if and only if there is a path of length p from s_i to s_j.
- **$A^{(0)}$ is specified as identity matrix. $A^{(1)}$ is exactly the adjacency matrix.**
- **Note that $a_{ij}^{(2)}$=*true* if and only if exists some s_k, such that both $a_{ik}^{(1)}$ and $a_{kj}^{(1)}$ are *true*. So, $a_{ij}^{(2)}=\bigvee_{k=1,2,\ldots,n} (a_{ik}^{(1)} \wedge a_{kj}^{(1)})$, which is an entry in the *Boolean matrix product*.**

Boolean Matrix Operations

- **Boolean matrix product *C*=*AB* as:**
  - $c_{ij}=\vee_{k=1,2,\ldots,n}(a_{ik}\wedge b_{kj})$
- **Boolean matrix sum *D*=*A*+*B* as:**
  - $d_{ij}=a_{ij}\vee b_{ij}$
- ***R*, the transitive closure matrix of *A*, is the sum of all A^p, *p* is a non-negative integer.**
- **For a digraph with *n* vertices, the length of the longest simple path is no larger than *n*-1.**

Bit Matrix

- A **bit string** of length n is a sequence of n bits occupying contiguous storage(word boundary) (usually, n is larger than the word length of a computer)
- If A is a **bit matrix** of $n \times n$, then $A[i]$ denotes the ith row of A which is a bit string of length n. a_{ij} is the jth bit of $A[i]$.
- The **procedure** **bitwiseOR(a,b,n)** compute a$\vee$b bitwise for n bits, leaving the result in a.

Straightforward Multiplication of Bit Matrix

- **Computing** $C=AB$
  - <Initialize C to the zero matrix>
  - **for** (i=1; i≤n, i++)
  - **for** (k=1; k≤n, k++)
  - **if** (a_{ik}==*true*) bitwiseOR($C[i]$, $B[k]$, n)

Thought as a union of sets (row union), n^2 unions are done at most

In the case of a_{ik} is *true*, $c_{ij}=a_{ik}b_{kj}$ is true iff. b_{kj} is true. As a result: $C[i]=\cup_{k\in A[i]}B[k]$, ($A[i]=\{k\,|\,a_{ik}=true\}$)

Union for B[k] is **repeated each time** when the kth bit is *true* in a different row of A is encountered.

Reducing the Duplicates by Grouping

- **Multiplication of *A*, *B*, two 12×12 matrices**

$$
\begin{array}{c}
A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \\ A_8 \\ A_9 \\ A_{10} \\ A_{11} \\ A_{12}
\end{array}
\begin{pmatrix}
1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
 & & & & & & & & & & & \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
 & & & & & & & & & & & \\
 & & & & & & & & & & & \\
 & & & & 0 & 1 & 0 & 1 & & & & \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
 & & & & & & & & & & & \\
 & & & & & & & & & & & \\
 & & & & & & & & & & & \\
 & & & & & & & & & & & \\
 & & & & & & & & & & &
\end{pmatrix}
$$

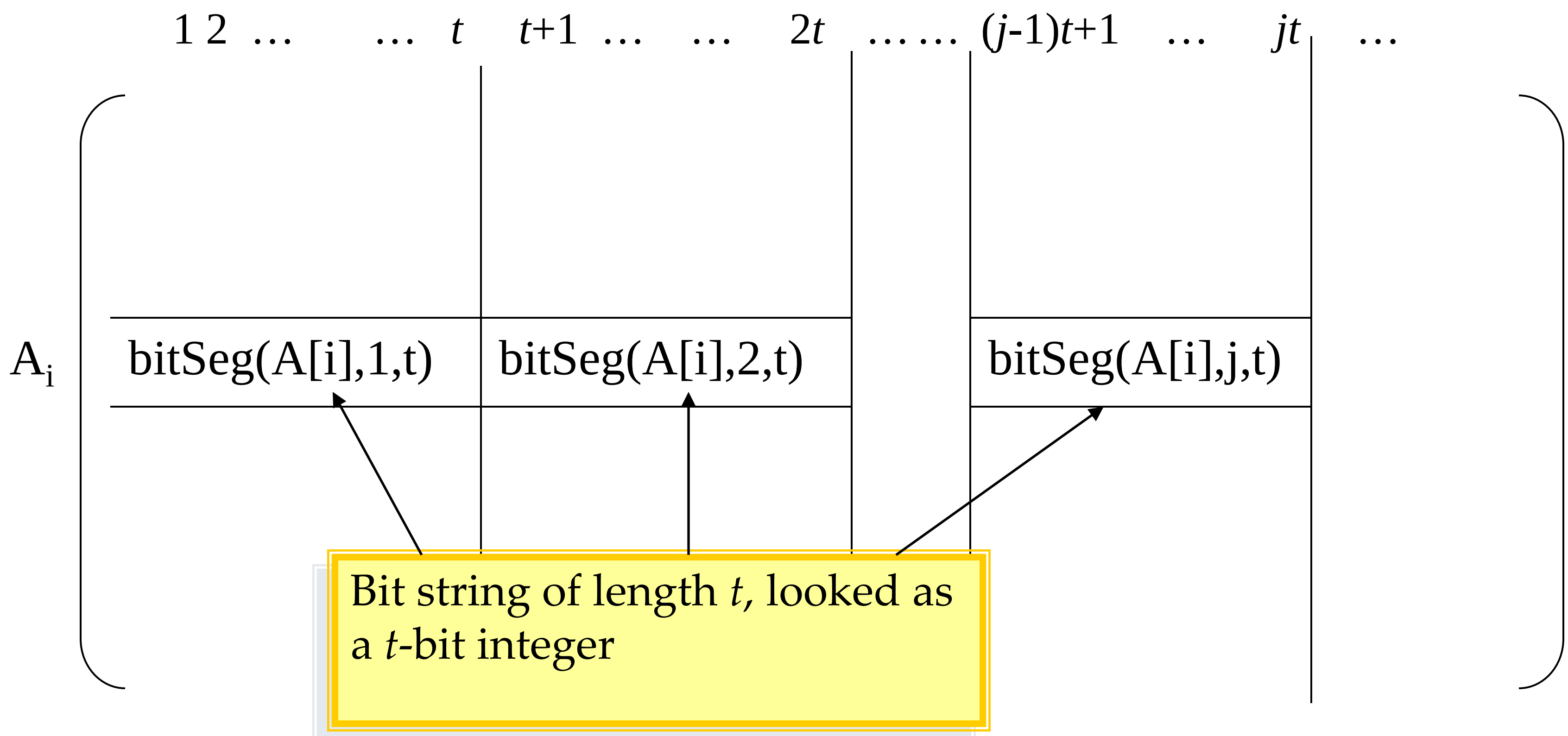
Segment

Length t

- 12 rows of B are divided evenly into 3 groups, with rows 1-4 in group 1, etc.
- With each group, all possible unions of different rows are pre-computed. (This can be done with 11 unions if suitable order is assumed.)
- When the first row of AB is computed, ($B[1] \cup B[3] \cup B[4]$) is used in stead of 3 different unions, and this combination is used in computing the 3rd and 7th rows as well.

The Segmentation for Matrix A

The $n \times n$ array



An Example

bitSeg(A[7], 1, t)

$= 1011_2 = 11$

	Group 1 $(1\ldots t)$	Group 2 $(t+1\ldots 2t)$	Group 3 $(2t+1\ldots 3t)$

$$
\begin{array}{c}
A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \\ A_8 \\ A_9 \\ A_{10} \\ A_{11} \\ A_{12}
\end{array}
\begin{bmatrix}
1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}
$$

$t=4$

B_1, B_2, B_3, B_4	any union combination of $\{B_1,B_2,B_3,B_4\}$, totaling 16, 11 computed	Group 1 $(1\ldots t)$
B_5, B_6, B_7, B_8	any union combination of $\{B_5,B_6,B_7,B_8\}$, totaling 16, 11 computed	Group 2 $(t+1\ldots 2t)$
B_9, B_{10}, B_{11}, B_{12}	any union combination of $\{B_9,B_{10},B_{11},B_{12}\}$, totaling 16, 11 computed	Group 3 $(2t+1\ldots 3t)$

Where to store?

Storage of the Row Combinations

- **Using one large 2-dimensional array**
- **Goals**
  - o keep all unions generated
  - o provide indexing for using
- **Coding within a group**
  - o One-to-one correspondence between a bit string of length t and one union for a subset of a set of t elements
- **Establishing indexing for union required**
  - o When constructing a row of AB, a segment can be notated as a integer. Use it as index.

Storage the Unions

allUnion

one row for
one group

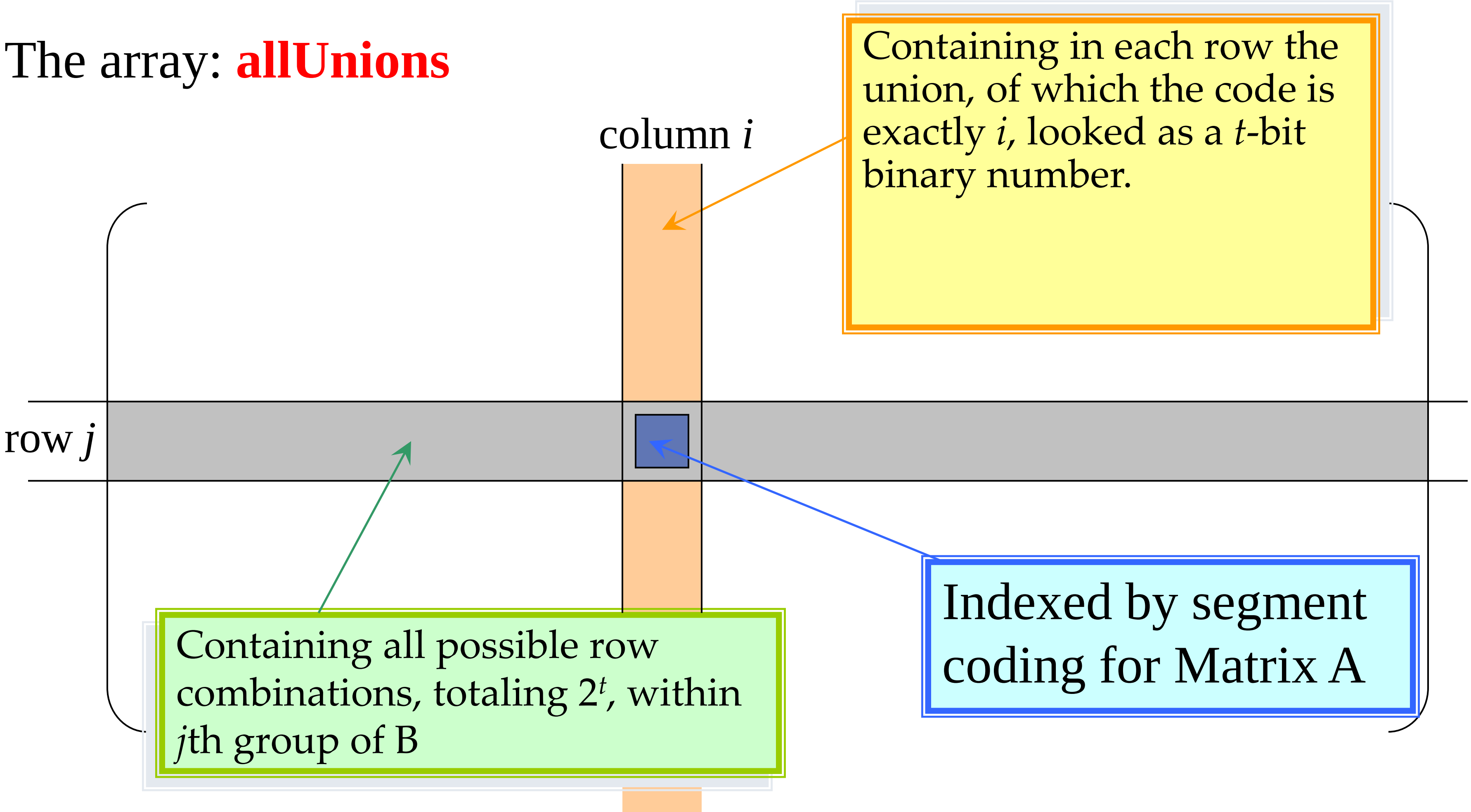
column indexed by bitSeg(A[i],j,t)

$$\begin{bmatrix} \phi & 4 & 3 & 3,4 & 2 & 2,4 & 2,3 & 2,3,4 & 1 & 1,4 & 1,3 & 1,3,4 & 1,2 & 1,2,4 & 1,2,3 & 1,2,3,4 \\ \phi & 8 & 7 & 7,8 & 6 & 6,8 & & & & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \phi & 12 & 11 & 11,12 & 10 & 10,12 & & & & & & & & & & \end{bmatrix}$$

i,j,k stands for $B_i \cup B_j \cup B_k$

Array for Row Combinations

The array: **allUnions**



Cost as Function of Group Size

- **Cost for the pre-computation**
  - There are 2^t different combination of rows in one group, including an empty and t singleton. Note, in a suitable order, each combination can be made using only one union. So, the total number of union is **$g[2^t-(t+1)]$**, where $g=n/t$ is the number of group.
- **Cost for the generation of the product**
  - In computing one of n rows of AB, at most one combination from each group is used. So, the total number of union is **$n(g-1)$**

Selecting Best Group Size

- **The total number of union done is:**

  $$g[2^t-(t+1)]+n(g-1) \approx (n2^t)/t+n^2/t \quad \textbf{(Note: } g=n/t \textbf{)}$$

- **Trying to minimize the number of union**
  - o Assuming that the first term is of higher order:
    - Then $t \geq \lg n$, and the least value is reached when $t=\lg n$.
  - o Assuming that the second term is of higher order:
    - Then $t \leq \lg n$, and the least value is reached when $t=\lg n$.
- **So, when $t \approx \lg n$, the number of union is roughly $2n^2/\lg n$, which is of lower order than n^2. We use $t=\lfloor \lg n \rfloor$**

For symplicity, exact power for n is assumed

Sketch for the Procedure

- $t=\lfloor \lg n \rfloor$; $g=\lceil n/t \rceil$;
- **<Compute and store in allUnions unions of all combinations of rows of *B*>**
- for (i=1; i≤*n*; i++)
-     **<Initialize C[i] to 0>**
-     for (j=1; j≤*g*; j++)
-         C[i] = C[i] ∪ **allUnions**[j][bitSeg(A[i],j,t)]

Kronrod Algorithm

- **Input: A,B and n, where A and B are n×n bit matrices.**
- **Output: C, the Boolean matrix product.**
- **Procedure**
  - The processing order has been changed, from "row by row" to "group by group", resulting the reduction of storage space for unions.

Complexity of the Kronrod Algorithm

- **For computing all unions within a group, 2^t-1 union operations are done.**
- **One union is bitwiseOR'ed to n row of C**
- **So, altogether, $(n/t)(2^t\text{-}1+n)$ row unions are done.**
- **The cost of row union is $\lceil n/w \rceil$ bitwise or operations, where w is word size of bitwise or instruction dependent constant.**

Thank you!

Q & A

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