Introduction to

Algorithm Design and Analysis

[07] Selection

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In the last class …

- **MergeSort**
  - Worst-case analysis of MergeSort
- **Lower Bounds for comparison-based sorting**
  - Worst-case
  - Average-case

The Selection

- **Selection - warm-ups**
  - Finding max and min
  - Finding the second largest key
- **Adversary argument and lower bound**
- **Selection - select the median**
  - Expected linear time
  - Worst-case linear time
- **A Lower Bound for Finding the Median**

The Selection Problem

- **Problem Definition**
  - Suppose E is an array containing n elements with keys from some linearly order set, and let k be an integer such that 1<=k<=n. The selection problem is to find an element with the k^{th} smallest key in E.
- **Special cases**
  - Find the max/min -k=n or k=1
  - Find the median (k=n/2)

Selection
v.s.
Searching

Lower Bound of Finding the Max

- **For any algorithm A that can compare and copy numbers exclusively, in the worst case, A cannot do fewer than n-1 comparisons to find the largest entry in an array with n entries.**

  - Proof: an array with n distinct entries is assumed. We can exclude a specific entry from being the largest entry only after it is determined to be “loser” to at least one entry. So, n-1 entries must be “losers” in comparisons done by the algorithm. However, each comparison has only one loser, so at least n-1 comparisons must be done.

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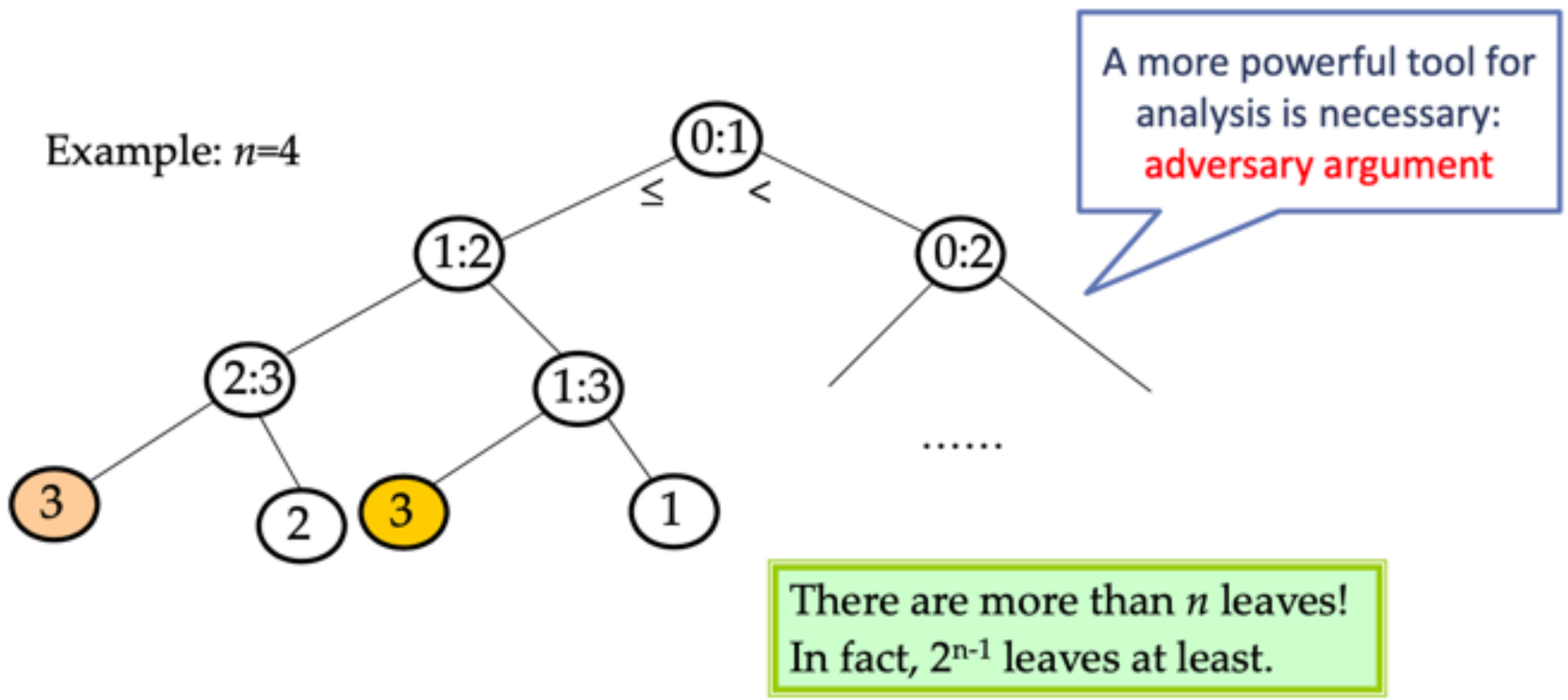
Decision Tree and Lower Bound

- Since the decision tree for the selection problem must have at least *n* leaves, the height of the tree is at least $\lceil \log n \rceil$. It's not a good lower bound.



Finding max and min

- **The strategy**
  - Pair up the keys, and do *n*/2 comparisons(if *n* odd, having E[*n*] uncompared);
  - Doing findMax for larger key set and findMin for small key set respectively (if *n* odd, E[*n*] included in both sets)
- **Number of comparisons**
  - For even *n*:$n/2 + 2(n/2 - 1) = 3n/2 - 2$
  - For odd *n*:$(n-1)/2 + 2((n-1)/2 + 1 - 1) = \lceil 3n/2 \rceil - 2$

How to prove this lower bound? **Adversary Argument!**

Unit of Information

- **Max and Min**
  - That x is *max* can only be known when it is sure that every key other than x has lost some comparison.
  - That y is *min* can only be known when it is sure that every key other than y has win some comparison.
- **Each win or loss is counted as one unit of information**
  - *Any* algorithm must have at least $2n$-2 units of information to be sure of specifying the *max* and *min*.

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Adversary Strategy

Status of keys x and y Compared by an algorithm	Adversary response	New status	Units of new information
N,N	$x>y$	W,L	2
W,N or WL,N	$x>y$	W,L or WL,L	1
L,N	$x<y$	L,W	1
W,W	$x>y$	W,WL	1
L,L	$x>y$	WL,L	1
W,L or WL,L or W,WL	$x>y$	No change	0
WL,WL	Consistent with Assigned values	No change	0

The principle: let the key win if it never lose, or, let the key lose if it never win, and change one value if necessary.

Lower Bound by the Adversary Argument

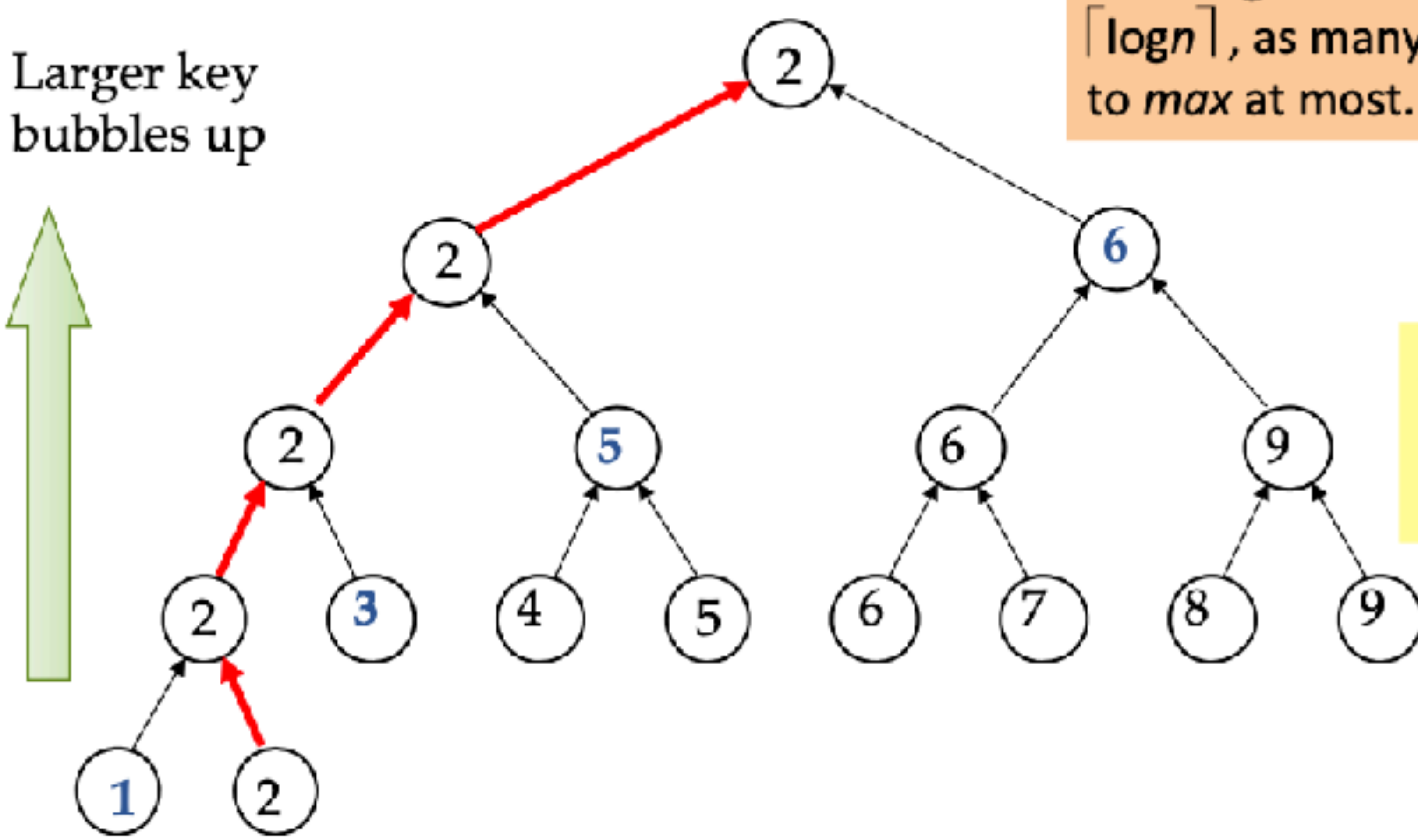
- **Construct an input to force *the* algorithm to do more comparisons as possible**
  - To give away as few as possible units of new information with each comparison.
    - It can be achieved that 2 units of new information are given away only when the status is N,N.
    - It is ***always*** possible to give adversary response for other status so that at most one new unit of information is given away, ***without any inconsistencies***.
- **So, the *Lower Bound* is *n*/2+n−2(for even *n*)**

$$\frac{n}{2} \times 2 + (n-2) \times 1 = 2n - 2$$

Find the 2^{nd} Largest Key

- **Brute force – using FindMax twice**
  - Need 2n-3 comparisons.
- **For a better algorithm**
  - Collect some useful information from the first FindMax
- **Observations**
  - The key which loses to a key other than max cannot be the 2^{nd} largest key.
  - To check “whether you lose to max?”

Tournament for the 2nd Largest Key



The length of the longest path is $\lceil \log n \rceil$, as many as those compared to *max* at most.

x_2 is *max*

Only x_1, x_3, x_5, x_6 may be the second largest key.

Analysis of Finding the 2nd

- Any algorithm that finds *secondLargest* must also find *max* before. $(n-1)$
- The *secondLargest* can only be in those which lose directly to *max*.
- On its path along which bubbling up to the root of tournament tree, *max* beat $\lceil \log n \rceil$ keys at most.
- Pick up secondLargest $(\lceil \log n \rceil - 1)$
- Totalcost: $n + \lceil \log n \rceil - 2$

Lower Bound by Adversary

- **Theorem**
  - Any algorithm (that works by comparing keys) to find the second largest in a set of *n* keys must do at least $n + \lceil \log n \rceil - 2$ comparisons in the worst case.
- **Proof**
  - There is an adversary strategy that can force any algorithm that finds *secondLargest* to compare *max* to $\lceil \log n \rceil$ distinct keys.

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Weighted Key

- **Assigning a weight *w*(*x*) to each key**
  - The initial values are all 1.
- Adversary strategy

Note: for one comparison, the weight increasing is no more than doubled.

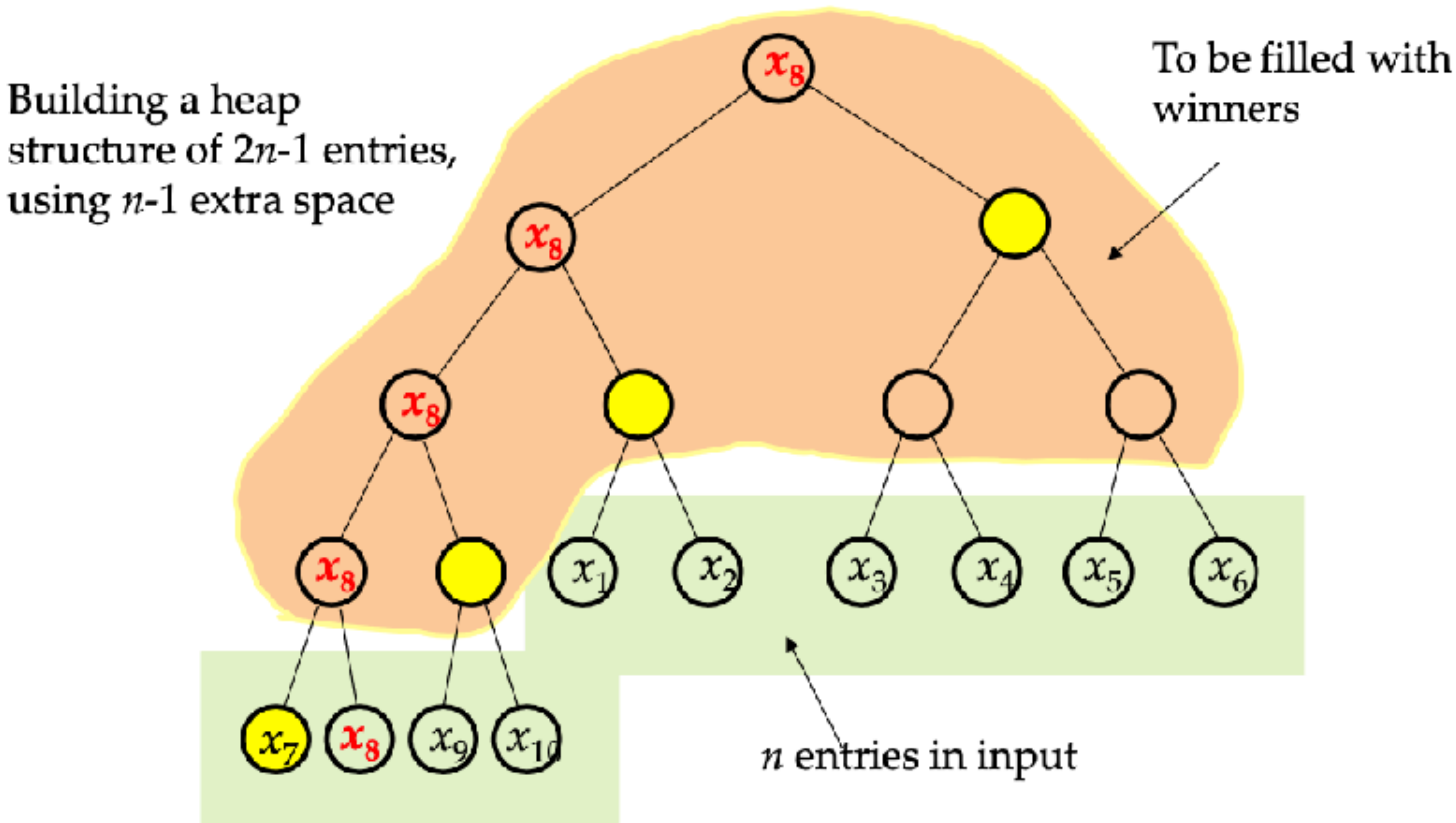
Case	Adversary reply	Updating of weights
$w(x)>w(y)$	x>y	$w(x):=w(x)+w(y)$; $w(y):=0$
$w(x)=w(y)>0$	$x>y$	$w(x):=w(x)+w(y)$; $w(y):=0$
$w(y)>w(x)$	$y>x$	$w(y):=w(x)+w(y)$; $w(x):=0$
$w(x)=w(y)=0$	Consistent with previous replies	No change

Zero loss

Lower Bound by Adversary: Details

- Note: the sum of weights is always *n*.
- Let *x* is *max*, then *x* is the only nonzero weighted key, that is *w*(*x*)=*n*.
- By the adversary rules: $w_k(x) \leq 2w_{k-1}(x)$
- Let *K* be the number of comparisons *x* wins against previously undefeated keys:
  $n = w_K(x) \leq 2^K w_0(x) = 2^K$
- So, $K \leq \lceil \log n \rceil$

Tracking the Losers to MAX



Finding the Median: the Strategy

- **Observation**
  - If we can partition the problem set of keys into 2 subsets: S1, S2, such that any key in S1 is smaller that that of S2, the median must located in the set with more elements.
- **Divide-and-Conquer**
  - Only one subset is needed to be processed recursively.

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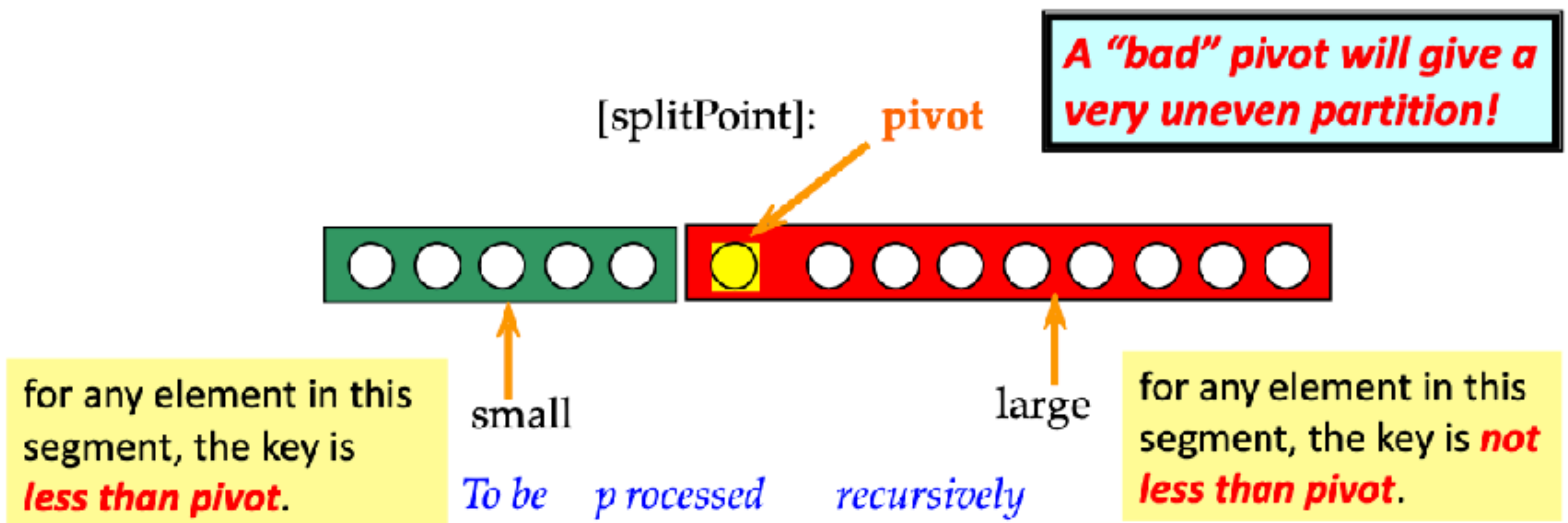
Adjusting the Rank

- **The rank of the median (of the original set) in the subset considered can be evaluated easily.**
- **An example**
  - Let n=255
  - The rank of median we want is 128
  - Assuming $|S_1|$=96, $|S_2|$=159
  - Then, the original median is in S_2, and the new rank is 128-96=32

Partitioning: Larger and Smaller

- Dividing the array to be considered into two subsets: “small” and “large”, the one with more elements will be processed recursively.

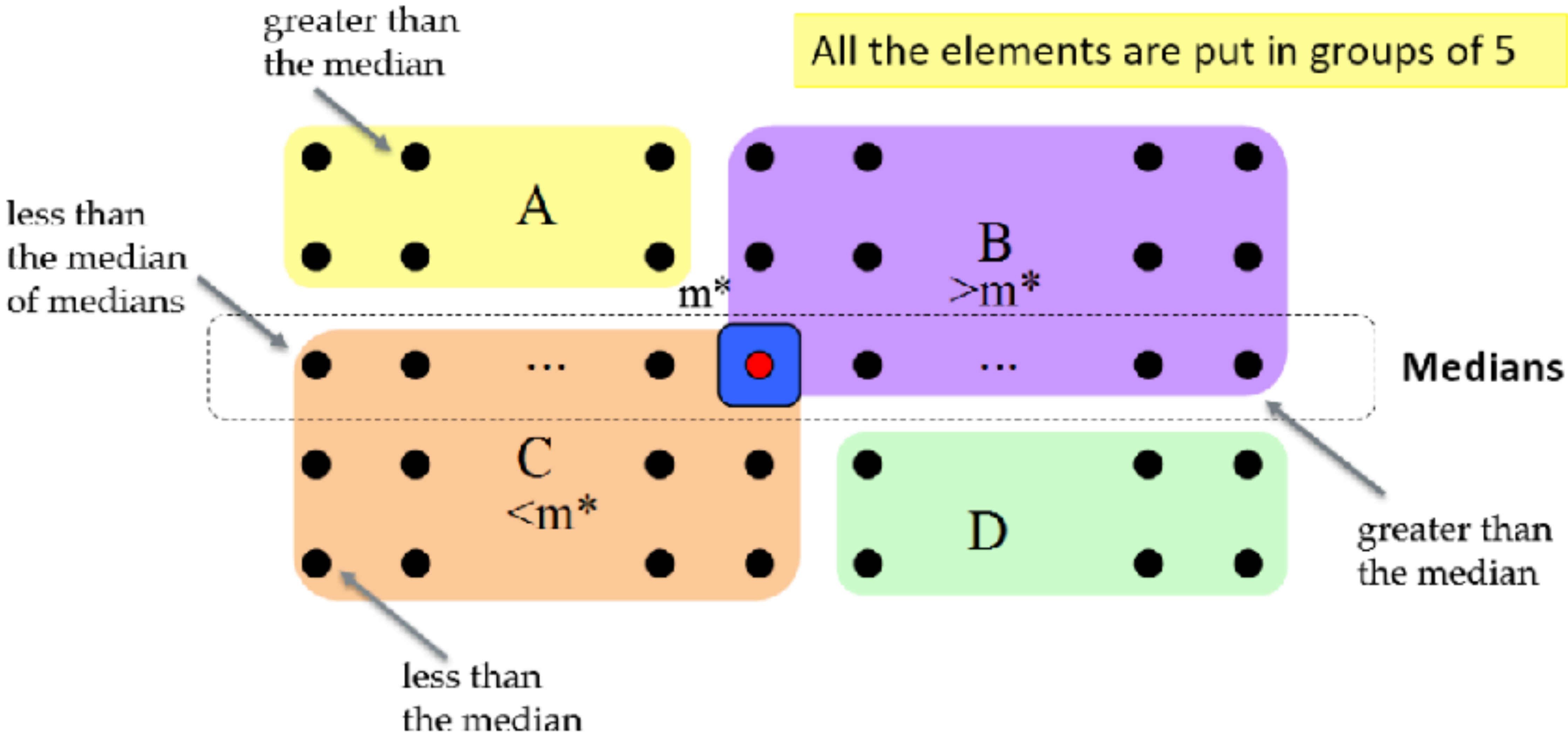


Selection: the Algorithm

- **Input: *S*, a set of *n* keys; and *k*, an integer such that $1 \leq k \leq n$.**
- **Output: The *k*th smallest key in *S*.**
- **Note: Median selection is only a special case of the algorithm, with $k = \lceil n/2 \rceil$.**
- **Procedure**
- **Element select(SetOfElements *S*, int *k*)**
  - **if** |S|<=5 **return** direct solution; **else**
  - Constructing the subsets S_1 and $S_{2;}$
  - Processing one of $\boldsymbol{S_1,S_2}$ with more elements, recursively.

Key issue:
How to construct the partition?

Partition improved: the Strategy



Constructing the Partition

- **Find the *m**, the median of medians of all the groups of 5, as illustrated previously.**
- **Compare each key in sections A and D to *m**, and**
    - Let $S_1 = C \cup \{x \mid x \in A \cup D and x < m^*\}$
    - Let$S_2 = B \cup \{x \mid x \in A \cup D and x > m^*\}$*(m** is to be used as the pivot for the partition*)*

Divide and Conquer

- if (k=$|S_1|$+1) return m^*;
- else if (k<=$|S_1|$) return select(S_1,k); //recursion
- else return select(S_2,$k-|S_1|-1$); //recursion

Analysis

- **For simplicity:**
  - Assuming n=5(2r+1) for all calls of *select*.
- $$W(n) \le 6\left(\frac{n}{5}\right) + W\left(\frac{n}{5}\right) + 4r + W(7r+2)$$

The extreme case: all the elements in A∪D in one subset.

Finding the median in every group of 5

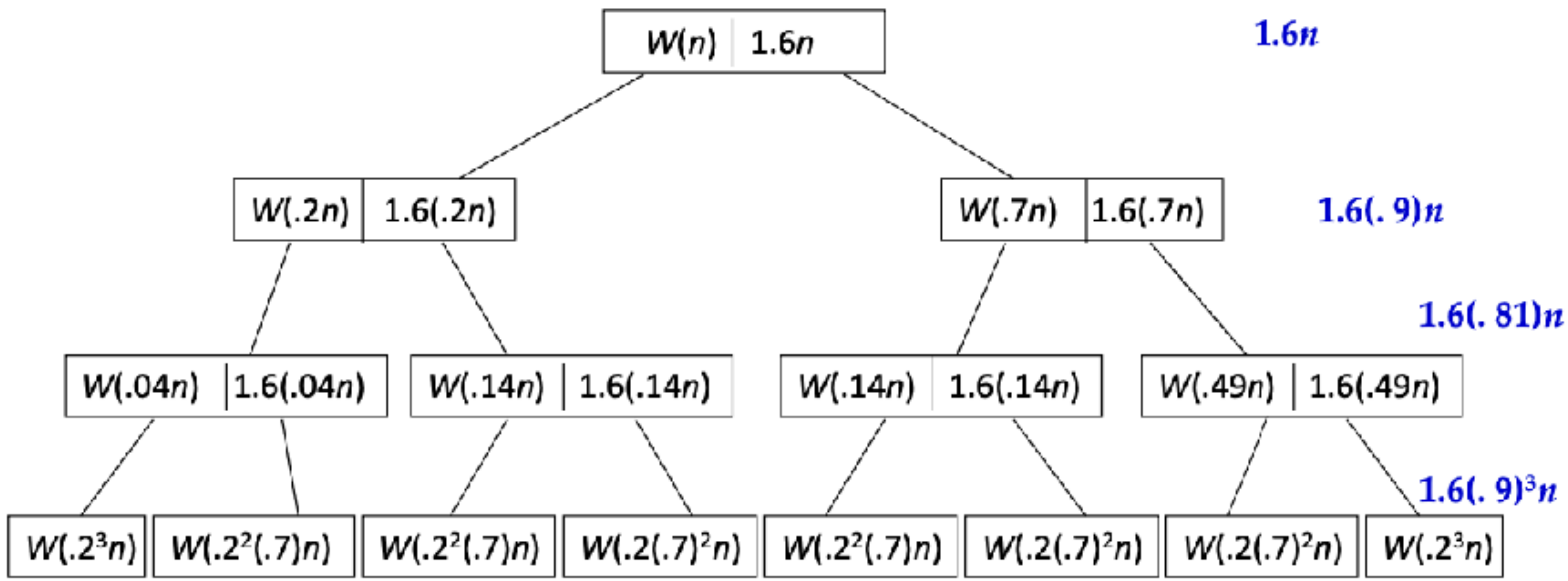
Finding the median of the medians

Comparing all the elements in A∪D with ***m****

- ***Note: r is about n/10, and 0.7n+2 is about 0.7n,*** **so**

$$W(n) \le 1.6n + W(0.2n) + W(0.7n)$$

Worst Case Complexity of Select

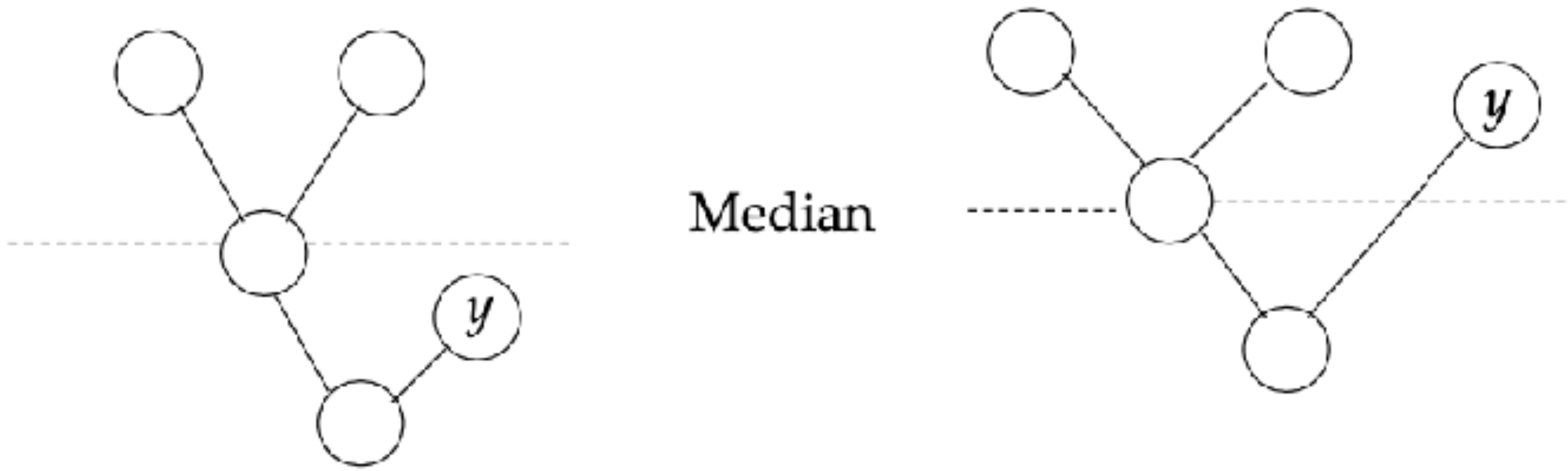


Note: Row sums is a decreasing geometric series, so

$W(n) \in \Theta(n)$

Relation to Median

- **Observation**
  - Any algorithm of selection must know the relation of every element to the *median*.



The adversary makes you wrong in either case

Crucial Comparison

- **A crucial comparison**
  - Establishing the relation of some x to the median.
- **Definition (for a comparison involving a key x)**
  - Crucial comparison for x: the first comparison where $x>y$, for some $y=>$median, or $x<y$ for some $y<=$median
  - Non-crucial comparison: the comparison between x and y where $x>$median and $y<$median, or vise versa

Adversary for Lower Bound

- **Status of the key during the running of the Algorithm:**
  - L: Has been assigned a value larger than median
  - S: Has been assigned a value smaller than median
  - N: Has not yet been in a comparison
- **Adversary rule:**

Comparands	Adversary's action
N,N	one L, the another S
L,N or N,L	change N to S
S,N or N,S	change N to L

(In all other cases, just keep consistency)

Notes on the Adversary Arguments

- **All actions explicitly specified above make the comparisons un−crucial.**
  - At least, $(n\text{-}1)/2$ *L* or *S* can be assigned freely.
  - If there are already $(n\text{-}1)/2$ *S*, a value larger than median must be assigned to the new key, and if there are already $(n\text{-}1)/2$ *L*, a value smaller than median must be assigned to the new key. The last assigned value is the median.
- **So, an adversary can force the algorithm to do (n−1)/2 un−crucial comparisons at least(In the case that the algorithm start out by doing $(n-1)/2$ comparisons involving two *N*.**

Lower Bound for Selection Problem

- **Theorem:**
  - Any algorithm to find the median of n keys(for odd n) by comparison of keys must do at least 3n/2-3/2 comparisons in the worst case.
- **Argument:**
  - There must be done n-1 crucial comparisons at least.
  - An adversary can force the algorithm to perform as many as (n-1)/2 uncrucial comparisons.
    - Note: the algorithm can always start out by doing (n-1)/2 comparisons involving 2 N-keys, so, only (n-1)/2 L or S left for the adversary to assign freely as the adversary rule.

Thank you!

Q & A