3^{rd} *Singapore Physics League*

Date: 8 July 2023
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Singapore Physics League (SPhL) is strongly supported by the Institute of Physics Singapore (IPS) and the Singapore Ministry of Education (MOE), and is sponsored by Micron.

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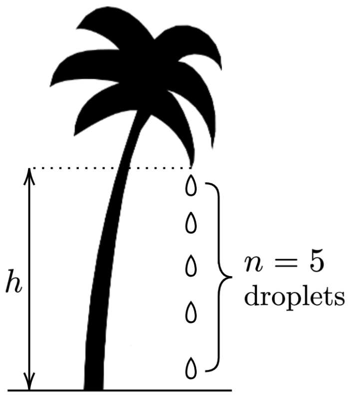
SPhL 2023 was made possible with the assistance of Brian Siew Jiang Yi and Theodore Yoong.

Document version: 1.1 (Last modified: July 21, 2023)

Problem 1: Wet Tree (3 points)

A tree is wet after a rain and slowly drips water, with one droplet falling from rest every $t = 1$ s. At any time, exactly $n = 5$ droplets can be observed mid-air. Determine the height h of the tree. Neglect air resistance.

Leave your answer to 2 significant figures in units of m.



Solution: Consider the falling motion of a single droplet. Let the time taken for the droplet to reach the ground be T. From kinematics, we have:

$$h = \frac{1}{2}gT^2$$

Throughout the duration of its fall, an additional n droplets must have fallen from the tree, such that the n^{th} additional droplet falls exactly when the initial droplet hits the ground. This condition is necessary to ensure that there are always n droplets falling mid-air. Hence, T must be related to t by:

$$T = nt$$

We can then solve for h:

$$h = \frac{1}{2}g(nt)^2 \approx \boxed{120 \text{ m}}$$

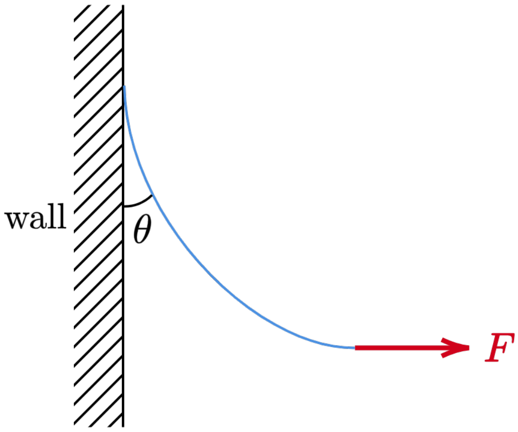
Setter: Christopher Ong, chris.ong@sgphysicsleague.org

Problem A: Mysterious Rope (3 points)

Note: These alphabet problems are additional problems for SPhL (Junior) teams only. In contrast, numbered problems are for both SPhL (Junior) and SPhL (Senior) teams.

A rope is connected to a vertical wall at one end, and a horizontal external force $F = 15.0$ N pulls on the other end. The rope is in equilibrium and makes an angle $\theta = 25.0°$ with the wall. What is the weight W of the rope?

Leave your answer to 3 significant figures in units of N.



Solution: The puzzling aspect of this problem is the apparent lack of information. However, we need only to make an astute observation to solve the problem.

Since the rope makes an angle θ with the wall, and the tension in the rope acts along the rope, we can write the following force balance equations for the horizontal and vertical axes respectively:

$$T \sin\theta = F$$
$$T \cos\theta = W$$

Hence:

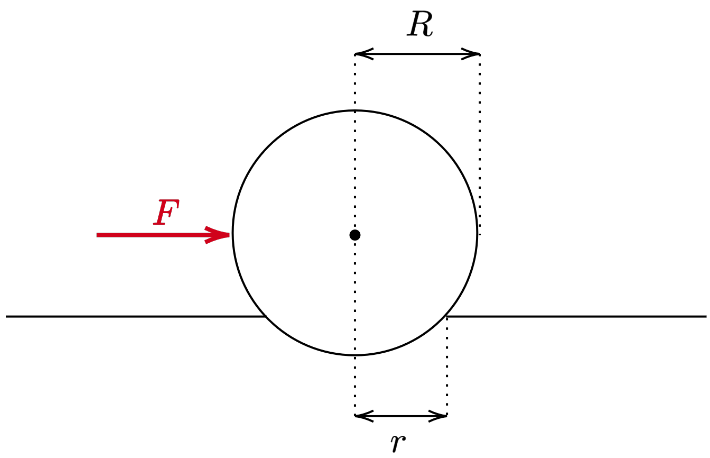
$$\begin{aligned} W &= \frac{F\cos\theta}{\sin\theta} \\ &= F\cot\theta \\ &\approx \boxed{32.2 \text{ N}} \end{aligned}$$

Setter: Roger Zhang, roger.zhang@sgphysicsleague.org

Problem B: Stuck Sphere (3 points)

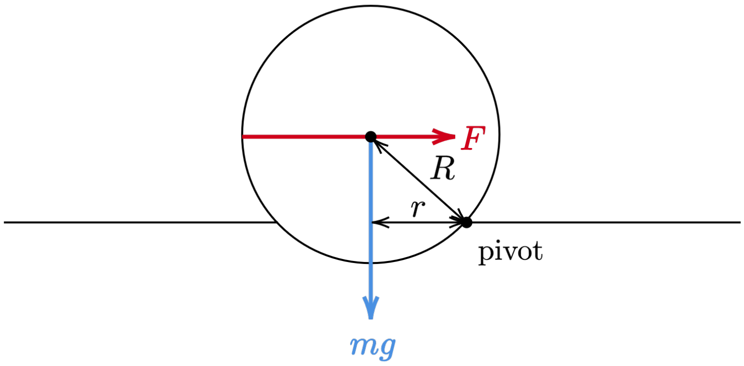
Note: These alphabet problems are additional problems for SPhL (Junior) teams only. In contrast, numbered problems are for both SPhL (Junior) and SPhL (Senior) teams.

A uniform sphere of mass $m = 1$ kg and radius $R = 5$ cm is partially lodged within a circular hole of radius $r = 3$ cm on a flat horizontal surface. We apply a constant horizontal force F towards the sphere's centre. What is the minimum F required to remove the sphere from the hole? Assume that all surfaces are sufficiently rough such that the sphere never slips.



Leave your answer to 2 significant figures in units of N.

Solution: Assume without loss of generality that the applied force F is directed rightwards. For the sphere to roll towards the right, it must lose contact with the hole's left edge.



Hence, consider the torque on the sphere about the hole's right edge. The hole's left edge exerts no torque since the contact force there is zero. The hole's right edge also exerts no torque since it is taken to be the pivot. F results in a clockwise torque of $F\sqrt{R^2 - r^2}$, while the sphere's weight results in an anticlockwise torque of mgr, as drawn below.

The sphere starts rolling when there is a net clockwise torque, which occurs when:

$$F\sqrt{R^2 - r^2} > mgr \implies F > mg\frac{r}{\sqrt{R^2 - r^2}} \approx \boxed{7.4 \text{ N}}$$

Once it begins rolling, it will continue to roll; the torque due to F increases while the torque due to weight decreases as the sphere rotates about the pivot. Hence, as long as F exceeds this minimum value, the sphere will certainly exit the hole.

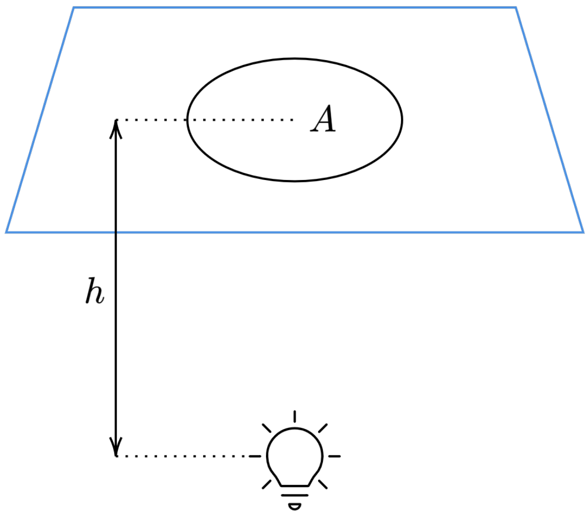
Setter: Christopher Ong, chris.ong@sgphysicsleague.org

Problem C: Underwater Lamp (3 points)

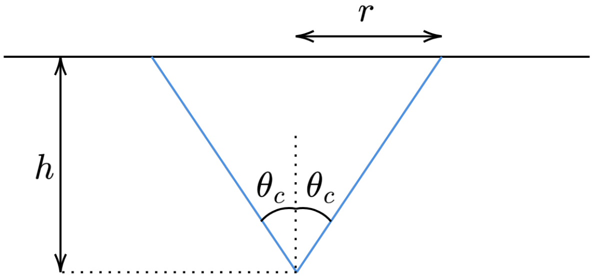
Note: These alphabet problems are additional problems for SPhL (Junior) teams only. In contrast, numbered problems are for both SPhL (Junior) and SPhL (Senior) teams.

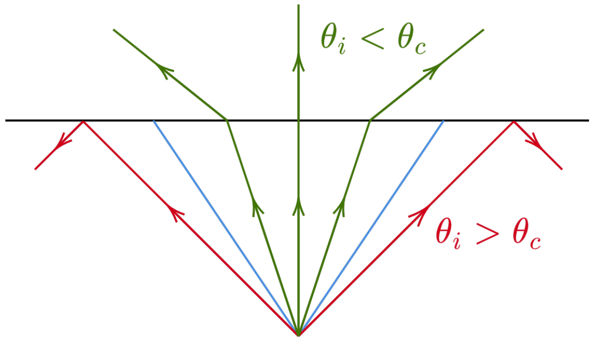
A lit lamp is immersed in water of refractive index $n = 1.33$ on a dark night. It is placed a height $h = 5.0$ m below the horizontal water surface. When viewed from directly above, a bright circular patch of area A is visible on the water surface. Find A. Assume that the lamp emits light in all directions.

Leave your answer to 3 significant figures in units of m^2.



Solution: Consider a light ray travelling from the lamp to the water-air interface. If its angle of incidence θ_i is smaller than the critical angle θ_c, the ray will undergo refraction and emerge into the air, subsequently entering the eyes of an observer from above. Conversely, if $\theta_i > \theta_c$, the ray will be totally internally reflected and will not enter the air, and hence will not be visible to the observer.





As such, consider a vertical cone of height h and half-angle θ_c centred on the lamp. All light rays within this cone satisfy $\theta_i < \theta_c$ and will be seen by the observer, whereas all light rays outside this cone have $\theta_i > \theta_c$ and will not reach the observer. The illuminated region will thus be the base area of this cone.

Given the water's refractive index n, its critical angle θ_c is given by:

$$\theta_c = \sin^{-1}\left(\frac{1}{n}\right)$$

Based on the cone's geometry, the radius r of the base of the cone can be written as:

$$r = h\tan\theta_c = \frac{h}{\sqrt{n^2-1}}$$

We can thus determine the area A of the bright circular patch, which is the cone's base area:

$$A = \pi r^2 = \frac{\pi h^2}{n^2-1} \approx \boxed{102\ \mathrm{m}^2}$$

Setter: Christopher Ong, chris.ong@sgphysicsleague.org

Problem 2: Relative Work

A block of mass $m = 1.0$ kg is initially at rest on horizontal frictionless ground. A horizontal non-constant force $F(t)$ is exerted on the block for some time, after which the block has a final speed $v = 4.0\ \text{m s}^{-1}$ relative to the ground.

(a) From the perspective of an observer that is stationary relative to the ground, find the net work done W by force F on the block.

Leave your answer to 2 significant figures in units of J. (2 points)

(b) From the perspective of another observer that travels at constant horizontal speed u relative to the ground, the net work done W' by force F on the block is zero. Find u.

Leave your answer to 2 significant figures in units of m s^{-1}. (2 points)

Solution:

(a) The block is initially stationary, so its initial kinetic energy is zero. The block has a final speed of v, so its final kinetic energy is $\frac{1}{2}mv^2$. The change in kinetic energy is thus $\frac{1}{2}mv^2$, and by the work-energy theorem, this is equal to the work done W by force F:

$$W = \frac{1}{2}mv^2 = \boxed{8.0\ \text{J}}$$

(b) $\boxed{u = 2.0\ \text{m s}^{-1}}$. In the reference frame travelling at $u = \frac{v}{2}$, the block has an initial velocity $-\frac{v}{2}$, and a final velocity $\frac{v}{2}$; the initial and final velocities are of equal magnitude but in opposite directions. However, kinetic energy only depends on the magnitude and not the direction of velocity, so the initial and final kinetic energies are both $\frac{1}{2}m\left(\frac{v}{2}\right)^2$. As such, the change in kinetic energy is zero, so the work done W' is also zero by the work-energy theorem.

In general, the work done by a force depends on which reference frame the observer is in. One way to see this is using the work-energy theorem, as illustrated in this problem. Alternatively, you could consider the more fundamental definition $W = \vec{F} \cdot \vec{x}$. In a different reference frame, there is no change to force $\vec{F}$ (at least according to classical mechanics), but there is a change to the perceived displacement $\vec{x}$ over which the force is applied.

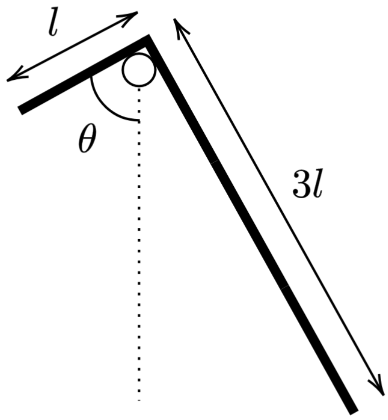
Setter: Christopher Ong, chris.ong@sgphysicsleague.org

Problem D: L (3 points)

Note: These alphabet problems are additional problems for SPhL (Junior) teams only. In contrast, numbered problems are for both SPhL (Junior) and SPhL (Senior) teams.

A thin uniform L-shaped bar with arm lengths l and $3l$ is hung on a frictionless pin of negligible radius at its right-angled corner as shown. What is the angle θ that the shorter arm makes with the vertical?

Leave your answer to 2 significant figures in units of degrees.



Solution: Let the mass of the shorter arm be m, so the mass of the longer arm is $3m$. Balance torques about the pivot point (which is the pin) to obtain:

$$mg\frac{l}{2}\sin\theta = 3mg\frac{3l}{2}\sin(90^\circ - \theta)$$

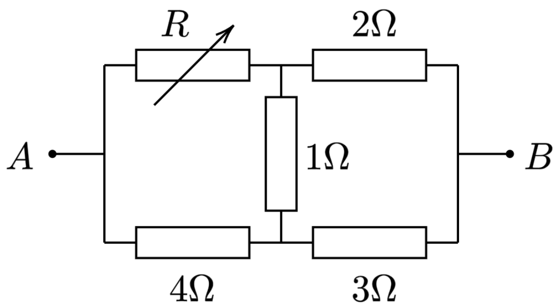
Simplify the equation and use $\sin(90^\circ - \theta) = \cos\theta$ to obtain:

$$\begin{aligned} \sin\theta &= 9\cos\theta \\ \tan\theta &= 9 \\ \theta &\approx \boxed{84^\circ} \end{aligned}$$

Setter: Roger Zhang, roger.zhang@sgphysicsleague.org

Problem 3: Variable Resistor

Chris creates an arrangement of 5 resistors as shown in the diagram, one of which is a variable resistor R. Here, we let the effective resistance across A and B be R_{AB}.



(a) What value does R_{AB} approach as the value of R approaches 0?

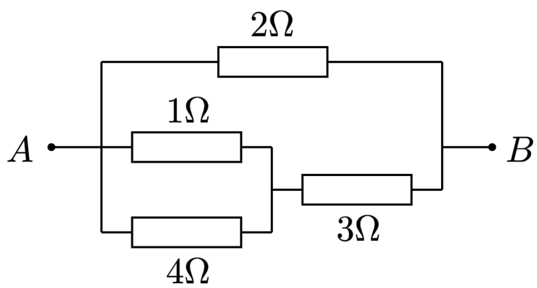
Leave your answer to 2 significant figures in units of Ω. (2 points)

(b) What value does R_{AB} approach as the value of R approaches ∞?

Leave your answer to 2 significant figures in units of Ω. (2 points)

Solution:

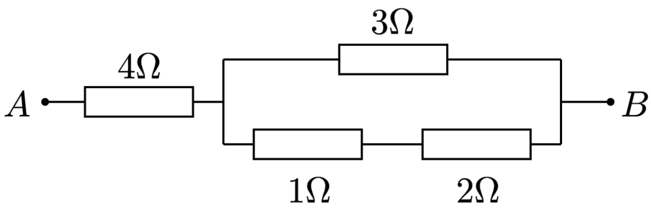
(a) Since $R \to 0$ can be represented as a short circuit, we can redraw the diagram:



Hence:

$$R_{AB} = \left(\frac{1}{3 + \left(\frac{1}{1} + \frac{1}{4}\right)^{-1}} + \frac{1}{2} \right)^{-1}$$
$$\approx \boxed{1.3\ \Omega}$$

(b) Similarly, $R \to \infty$ can be represented as an open circuit; we can once again redraw the diagram:



Hence:

$$
\begin{aligned}
R_{AB} &= 4 + \left(\frac{1}{3} + \frac{1}{1+2}\right)^{-1} \\
&= \boxed{5.5\ \Omega}
\end{aligned}
$$

Setter: Paul Seow, paul.seow@sgphysicsleague.org

Problem 4: A Sinking Feeling (3 points)

A worker uniformly mixes two materials, one with density $\rho_1 = 600$ kg m^{-3} and the other with density $\rho_2 = 1900$ kg m^{-3}, and shapes them into a cube.

He then places the cube underwater, such that it is fully submerged with its top surface at a depth $d = 4.4$ m below the surface of the water, and releases it from rest. Surprisingly, the cube stays in position there.

Find η, the proportion (by volume) of the cube that is made with the material of density ρ_1.

Leave your answer to 2 significant figures.

Your answer should be between 0 and 1.

Solution: Let the side length of the cube be h. The weight of the cube is $Mg = [\rho_1\eta + \rho_2(1-\eta)]gh^3$. The buoyant force is $F = \rho_w gh^3$, given by the difference in water pressure acting on the top and bottom surfaces.

At equilibrium, these forces are equal, therefore:

$$\rho_w = \rho_1\eta + \rho_2(1-\eta) = (\rho_1 - \rho_2)\eta + \rho_2$$

Intuitively, this also makes sense, as the average density of the cube would be equal to the density of water for the cube to be in equilibrium when fully submerged. Solving this for η, we obtain:

$$\eta = \frac{\rho_2 - \rho_w}{\rho_2 - \rho_1} \approx \boxed{0.69}$$

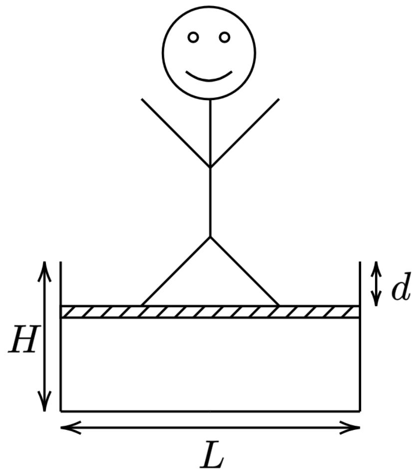
Setter: Galen Lee, galen.lee@sgphysicsleague.org

Problem E: Gas Weighing Scale (3 points)

Note: These alphabet problems are additional problems for SPhL (Junior) teams only. In contrast, numbered problems are for both SPhL (Junior) and SPhL (Senior) teams.

Physicist S has designed a weighing scale that uses gas pressure. It consists of a cuboidal container, which has a square base of side length $L = 20$ cm. The top face of the container is light and able to freely slide up and down without resistance. The container is filled with an ideal gas such that the top face is at initial height $H = 10$ cm. When Physicist S steps onto the top face of the container, it lowers by $d = 1.3$ cm. Assuming that the container is a perfect thermal conductor and airtight, find her mass m.

Leave your answer to 2 significant figures in units of kg.



Solution: Given that the container is perfectly conducting, the ideal gas inside remains at room temperature throughout, and given that it is airtight, the number of moles n of gas inside the container remains constant. Hence, PV must be a constant, where P is the pressure of the ideal gas and V is the volume of the container. Since $V = L^2 h$, where h is the height of the top face, and L is a constant, Ph must be a constant. Letting the initial and final pressures of the ideal gas be P_0 and P_1 respectively, we thus have:

$$P_1(H - d) = P_0 H \Rightarrow P_1 = \frac{H}{H - d} P_0$$

Initially, $P_0 = P_{atm}$ to ensure that the top face of the container remains in equilibrium. When Physicist S steps on the scale, the force exerted by the gas on the top face of the container balances out both the force exerted by the atmosphere and the weight of Physicist S. Thus, balancing forces and dividing by the base area throughout, we have:

$$P_1 = P_0 + \frac{mg}{L^2}$$

$$\frac{H}{H-d}P_0 = P_0 + \frac{mg}{L^2} \Rightarrow m = \frac{L^2}{g}\left(\frac{d}{H-d}P_0\right) \approx \boxed{62\ \text{kg}}$$

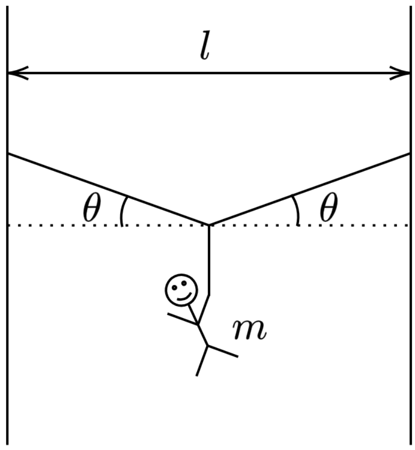
Setter: Shen Xing Yang, xingyang.shen@sgphysicsleague.org

Problem F: Help! (3 points)

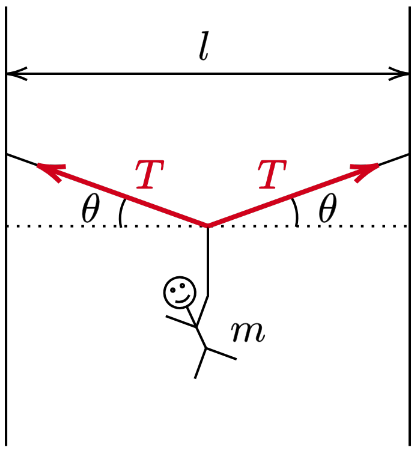
Note: These alphabet problems are additional problems for SPhL (Junior) teams only. In contrast, numbered problems are for both SPhL (Junior) and SPhL (Senior) teams.

Roger is trying to learn how to tightrope walk, on a light horizontal elastic rope with force constant $k = 650.0$ Nm^{-1} and unstretched length $l = 20.0$ m that is connected to two walls a distance l apart. Suddenly, he slips and falls halfway along the rope. Luckily, he grabs onto the rope and manages to hang there. The rope reaches an equilibrium position bent at an angle $\theta = 20.0°$ from the horizontal. Calculate Roger's mass m.

Leave your answer to 3 significant figures in units of kg.



Solution:



Consider each half of the rope with unstretched length $\frac{l}{2}$. Since the full rope has force constant k, each half of the rope on its own has force constant $2k$. Now let the extension of each half of the rope be x. We can now write the following equations:

$$T = 2kx \tag{1}$$

$$\left(\frac{l}{2}+x\right)\cos\theta=\frac{l}{2} \tag{2}$$

$$2T\sin\theta=mg \tag{3}$$

Equation (1) is the expression for the tension in each half of the rope based on their extension. Equation (2) is the geometric relationship between the stretched rope and unstretched rope. Equation (3) is the force balance equation for the mass m. We have 3 equations with 3 unknowns: T, x and m. Thus we can solve for m:

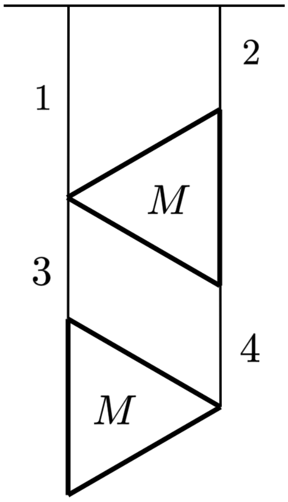
$$\begin{aligned} m &= \frac{2kl\sin\theta(\sec\theta-1)}{g} \\ &\approx \boxed{58.2\text{ kg}} \end{aligned}$$

P.S. That is actually Roger's mass.

Setter: Roger Zhang, roger.zhang@sgphysicsleague.org

Problem 5: Suspended Triangle

Two identical uniform equilateral triangles of mass M are suspended by 4 vertical strings (as shown in the diagram). Strings 1 and 2 can withstand twice the amount of tension as strings 3 and 4 before breaking.



(a) Which string will break first as M increases? (2 points)

(1) String 1

(2) String 2

(3) String 3

(4) String 4

(5) All will break at the same point.

(b) Let T_n be the tension in string n. If $M = 5.0$ kg, find the value of $T_2 + T_4$.

Leave your answer to 2 significant figures in units of N. (3 points)

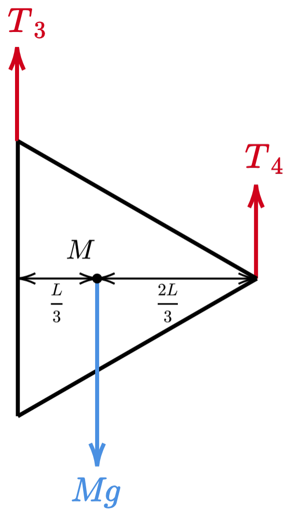
Solution:

(a) Let us consider the system of the two masses, with only strings 1 and 2 as external interactions. Since the centre of mass is in the middle (by symmetry), $T_1 = T_2 = \frac{1}{2}(2Mg) = Mg$.

Now, comparing between strings 3 and 4, the weight of M is distributed more towards the left. Hence, $T_3 > T_4$; in particular, since $T_3 + T_4 = Mg$, we can say that $T_3 > Mg/2$.

Since strings 1 and 2 can take twice the tension of strings 3 and 4, the first string to break will be $\boxed{\text{(3) String 3}}$.

(b) In an equilateral triangle, the centre of mass is at its geometric centre — by some geometry, we can work out that this lies 2/3 along the horizontal in its current orientation, being closer to its thicker end.



We have previously found T_2; to find T_4, we consider moments about the centre of mass of the bottom triangle. Letting the horizontal distance between strings 3 and 4 be L, we solve the following pair of equations simultaneously:

$$\frac{1}{3}LT_3 - \frac{2}{3}LT_4 = 0$$
$$T_3 + T_4 = Mg$$

We can see that $T_4 = \frac{1}{3}Mg$. Hence, $T_2 + T_4 = \frac{4}{3}Mg \approx \boxed{65\text{ N}}$.

Setter: Paul Seow, paul.seow@sgphysicsleague.org

Problem 6: Video Misinformation (3 points)

A fan blade rotates clockwise at a constant angular velocity ω. A fixed camera records a video of it with a frame rate of $f = 30$ frames per second. In that video, the fan blade *appears* to be rotating clockwise at angular velocity $\omega' = 10 \text{ rad s}^{-1}$. However, the actual value of ω differs from ω'. Find the smallest possible value of ω.

Leave your answer to 2 significant figures in units of rad s^{-1}.

Solution: Consider two consecutive frames in the video. The video has frame rate f, so the time interval between the two frames is $T = \frac{1}{f}$. Since the fan blade looks like it is rotating clockwise at ω', its clockwise angular displacement from the first frame to the second frame appears to be $\theta' = \omega' T = \frac{\omega'}{f}$.

Notice that if the fan blade had rotated a complete revolution plus angle θ' within the time T between the two frames, its angular displacement would only appear to be θ'. In other words, an angular displacement of $\theta = \theta' + 2\pi$ would still show up as θ' in the video. In that case, the actual value of ω would be given by:

$$\omega = \frac{\theta}{T} = \omega' + 2\pi f \approx \boxed{200 \text{ rad s}^{-1}}$$

The same logic applies when any integer n revolutions plus angle θ' is covered between two frames. In general, ω is given by $\omega = \omega' + n(2\pi f)$. Since the question only asks for the smallest ω, only the $n = 1$ case needs to be considered. Nevertheless, it is interesting to know that there are infinitely many possible ω values! n can even be negative if the complete revolutions were in the anticlockwise direction, so the fan could appear to be rotating in the direction opposite to its actual rotation.

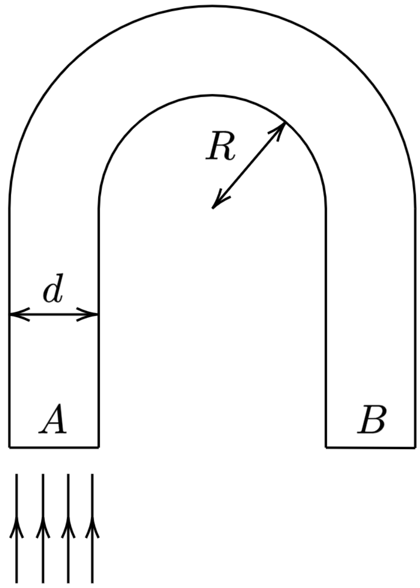
This technically proves that rotation cannot be determined from video – there are infinitely many possible values for ω. This phenomenon is also similar to the working principle of a stroboscope: read more here.

Setter: Christopher Ong, chris.ong@sgphysicsleague.org

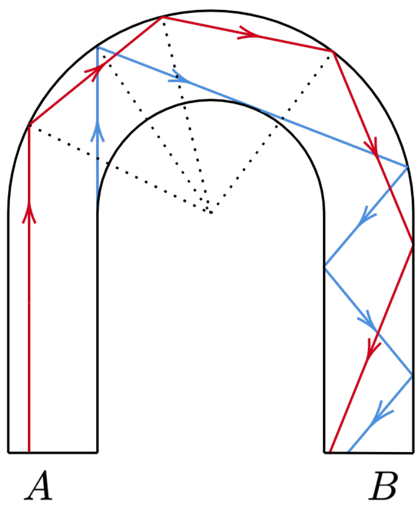
Problem 7: YouTube (3 points)

A glass U-tube with refractive index $n = 1.52$ is constructed by bending a glass rod into a U-shape, where the curved parts of the tube are two arcs of a semicircle. A collimated beam of light falls perpendicularly on the flat surface A. Determine the minimum value of the ratio $\frac{R}{d}$ for which all light entering the glass through surface A will emerge from the glass through surface B.

Leave your answer to 3 significant figures.



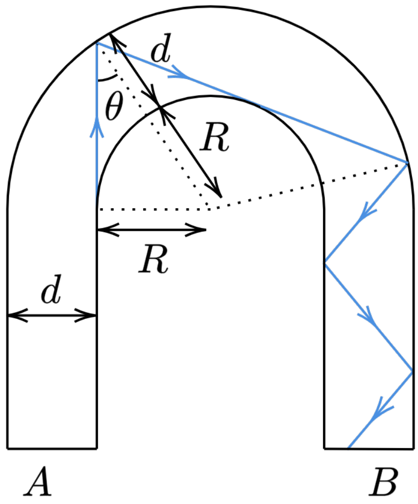
Solution: Consider all the rays passing through surface A. The ray with the smallest angle of incidence θ on the outer bend of the U-tube will clearly be the rightmost ray.



We then consider the conditions under which the rightmost ray will undergo total internal reflection before reaching B. If $\theta > \theta_c$, the critical angle beyond which total internal reflection occurs, all the the rays entering through surface A will emerge through the surface B. Hence we require:

$$\sin\theta > \frac{1}{n}$$

The geometry of the U-tube gives:



$$\sin\theta = \frac{R}{R+d}$$

Hence:

$$\begin{aligned} \frac{R}{R+d} &\geq \frac{1}{n} \\ \left(\frac{R}{d}\right)_{min} &= \frac{1}{n-1} \\ &\approx \boxed{1.92} \end{aligned}$$

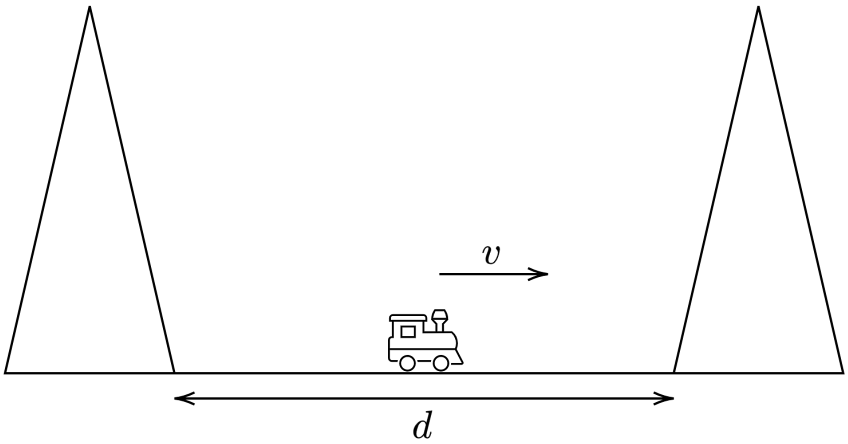
Setter: Roger Zhang, roger.zhang@sgphysicsleague.org

Problem 8: Choo Choo (3 points)

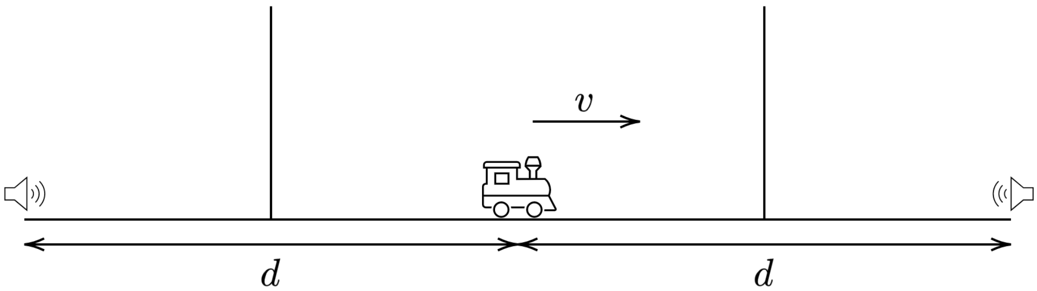
Thomas the Train is parked at the midpoint of two steep mountains of distance $d = 4000$ m apart. He blows his horn and immediately begins travelling with constant velocity v towards one of the mountains. Given that Thomas hears the echo of the horn off the two mountains with a time difference of $\Delta t = 2.0$ s, find v.

Take the speed of sound in air to be $v_s = 340$ m s^{-1}.

Leave your answer to 2 significant figures in units of m s^{-1}.



Solution: Without loss of generality, assume Thomas travels to the right. We can model the sound echoing off each mountain by mirroring the initial position of Thomas across each mountain to act as two sound sources:



Now, we consider the problem in the frame where Thomas is stationary. In this frame, the sound approaching from the left source travels at a velocity of $v_s - v$, and that from the right travels at a velocity of $v_s + v$. As the source only emits sound at the very beginning, each sound traverses a distance d. Hence, we have:

$$\begin{aligned}\Delta t &= \frac{d}{v_s - v} - \frac{d}{v_s + v} \\ &= \frac{2dv}{v_s^2 - v^2}\end{aligned}$$

Rearranging this yields the quadratic equation:

$$\Delta t v^2 + 2dv - \Delta t v_s^2 = 0$$

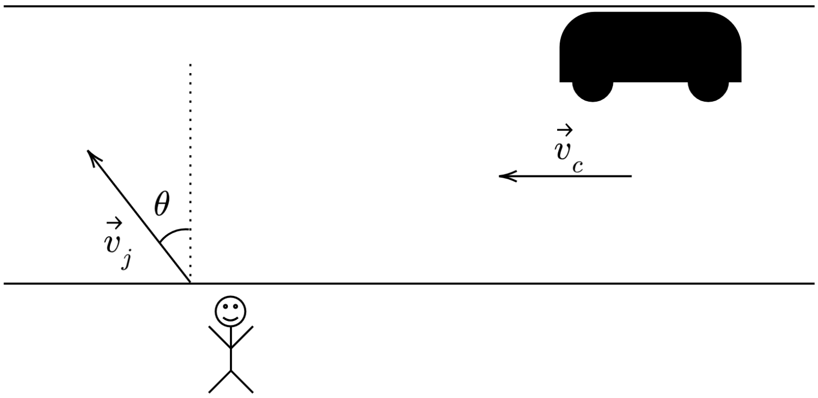
Solving, we obtain $v \approx \boxed{29\ \mathrm{m\ s^{-1}}}$.

Setter: Paul Seow, paul.seow@sgphysicsleague.org

Problem 9: Jaywalking (3 points)

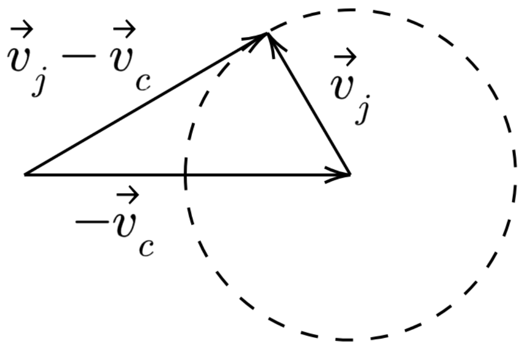
Jay stands at the south edge of a road which runs from east to west, intending to cross. He notices that a car is travelling west at $v_c = 30$ km h^{-1}, keeping to the opposite edge of the road. Given that Jay can run at a speed $v_j = 10$ km h^{-1}, what angle θ (measured counterclockwise from north) should he run at to maximise the distance between him and the car upon reaching the other side of the road? You can assume that the car has not passed Jay when he reaches the other side of the road.

Leave your answer to 3 significant figures in units of degrees.



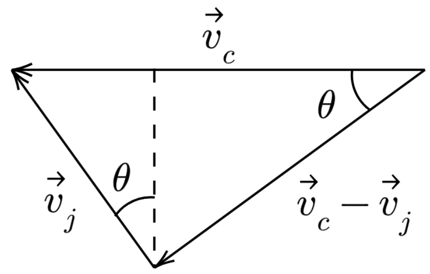
Solution: We consider this problem in the frame of the car. In this frame, the road travels to the east at a velocity of $-\vec{v}_c$. Hence, we see that Jay's velocity in that frame is given by the vector sum $\vec{v}_j - \vec{v}_c$.

As Jay wants to travel as little to the east as possible in the frame of the car, he wants to maximise the angle of inclination of his velocity in that frame. Since the locus of points that represents his range of potential velocities is a circle that does not contain the origin, this maximum will be given by a tangent line to the circle from the origin.



Since $\vec{v}_j$ must be perpendicular to $\vec{v}_j - \vec{v}_c$, we have:

$$\theta = \sin^{-1}\left(\frac{v_j}{v_c}\right) \approx \boxed{19.5^\circ}$$



Alternative solution: We could instead let the width of the road be h. Running at an angle of θ, Jay would take a time of $t = \frac{h}{v_j \cos\theta}$ to cross the road, moving west by distance $v_j t \sin\theta$. At the same time, the car would also have moved a distance of $v_c t$. Hence, we want to maximise the difference in distances:

$$\begin{aligned}\Delta x &= (v_c - v_j \sin\theta)\frac{h}{v_j \cos\theta} \\ &= h\left(\frac{v_c}{v_j \cos\theta} - \tan\theta\right)\end{aligned}$$

By differentiation (or plotting the graph on Desmos), we also find that the maximum occurs at $\sin^{-1}\left(\frac{1}{3}\right) \approx \boxed{19.5°}$.

Setter: Ariana Goh, ariana.goh@sgphysicsleague.org

Problem 10: Unfreezable (4 points)

Consider an isolated system of pure water that is supercooled to a temperature of $T_i = -15°\text{C}$, such that it remains completely liquid. When the supercooled water is disturbed, some but not all of the water freezes. Determine the percentage by mass of water that gets frozen.

Assume the specific heat capacity of ice to be equal to the specific heat capacity of water, which can be treated to be constant at $c_w = 4.19 \times 10^3$ J kg^{-1} K^{-1}. The specific latent heat of fusion of water can also be treated to be constant at $l_f = 3.34 \times 10^5$ J kg^{-1}.

Leave your answers to 2 significant figures as a percentage. (For example, if you think the final answer should be 51%*, input your answer as* 51*)*

Solution: When water freezes, it releases heat. This heat raises the temperature of the surroundings, which might rise above the freezing point of water, causing the freezing process to halt.

Let the total mass of water be M, and the mass of water that eventually freezes be m. The total heat Q released upon freezing mass m of water is given by:

$$Q = ml_f$$

This heat Q causes the entire system's temperature to rise by ΔT:

$$Q = Mc_w\Delta T$$

No more water can freeze once the temperature of the system exceeds freezing point T_f. This occurs when $\Delta T = T_f - T_i$. Hence, we can solve for $\frac{m}{M}$:

$$\frac{m}{M} = \frac{c_w}{l_f}(T_f - T_i) \approx \boxed{19\%}$$

Note that this assumed the specific heat capacity of ice, c_i, to be equal to the specific heat capacity of water, c_w. Had we not made this assumption, the problem would have been a lot harder to solve. The temperature increase would have to be considered in infinitesimal steps, i.e. with every infinitesimal mass dm of water that gets frozen. The fraction $\frac{m}{M}$ would then be given by:

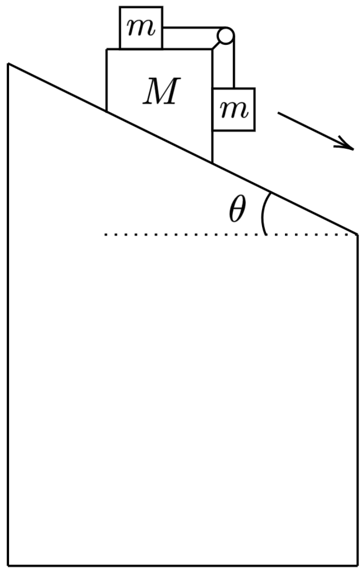
$$\frac{m}{M} = \frac{c_w}{c_i - c_w}\left[e^{\frac{c_i - c_w}{l_f}(T_f - T_i)} - 1\right]$$

Using the conventional value $c_i = 2.09 \times 10^3$ J kg^{-1} K^{-1}, this works out to be $\frac{m}{M} \approx 18\%$. Not a big physical difference, just a lot tougher to work out mathematically. For the mathematically inclined, you are recommended to derive this expression, and prove that it is equal to the previous result when we set $c_i \approx c_w$.

Setter: Christopher Ong, chris.ong@sgphysicsleague.org

Problem 11: Falling Chimney

A chimney-shaped block of mass $M = 3.0$ kg slides down a rooftop (beginning from rest) tilted at an angle θ, with a slanted base and its top face perfectly horizontal. A massless pulley is attached to a corner of the block and a light inextensible string is run over the pulley, with a smaller block of mass $m = 1.0$ kg attached to each side of the string. Take all surfaces to be frictionless.



(a) Find the value of θ such that the blocks slide down sticking together (i.e. they do not have any movement relative to each other).

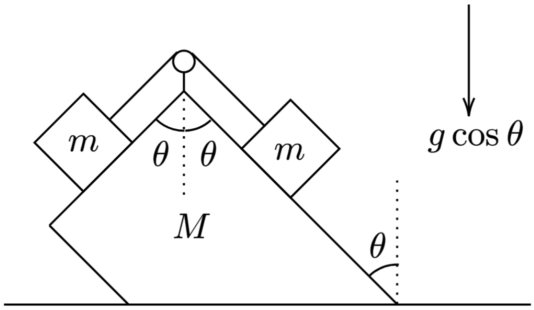
Leave your answer to 3 significant figures in units of degrees. (2 points)

(b) Suppose now that $\theta = 0°$. Find A, the magnitude of the initial acceleration of the large block of mass M when the blocks are released from rest.

Leave your answer to 2 significant figures in units of m s^{-2}. (4 points)

Solution:

(a) If all three stick together as they slide downwards, we can say that they all have the same acceleration. Hence, we analyse the problem from the frame of reference of the blocks, where all three are stationary, i.e. in equilibrium.

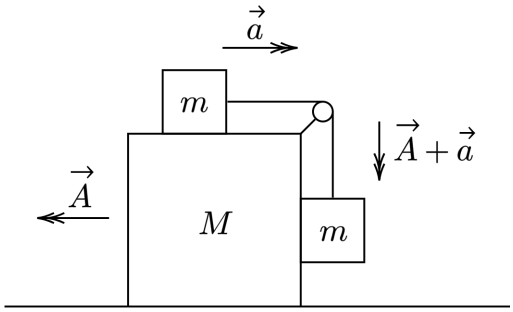


In this frame, the remaining component of gravity is perpendicular to the surface.

We now observe that our angle θ must allow the two smaller blocks to be in equilibrium. By symmetry, this occurs when the blocks' line of symmetry is also perpendicular to the surface of the roof.

Hence, $2\theta = 90°$, and as a result, $\boxed{\theta = 45.0°}$.

(b) We redraw the diagram in a more informative manner:



Now, let the acceleration of the mass M be A, and the acceleration of the upper mass m be a. Then, by considering the conservation of the string length, the downward acceleration of the lower mass m will be $a + A$.

By Newton's 2nd Law, we can write the following for each block:

$$\begin{aligned} MA &= T \\ ma &= T \\ m(a+A) &= mg - T \end{aligned}$$

Solving these simultaneously, we can see that

$$\begin{aligned} A &= \frac{m}{m+2M}g \\ &= \frac{g}{7} \\ &\approx \boxed{1.4 \text{ m s}^{-2}} \end{aligned}$$

Setter: Paul Seow, paul.seow@sgphysicsleague.org

Problem 12: Spring Collision

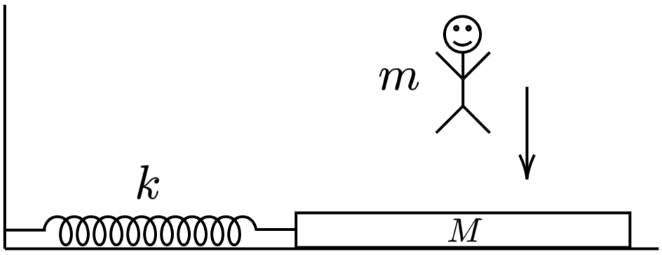
A playground ride is made of a platform of mass M connected to a light spring of force constant k, that oscillates in simple harmonic motion with amplitude x_0. When the platform is at maximum displacement from equilibrium, Roger of mass $m = \frac{M}{3}$ jumps such that he lands perfectly vertically on the platform. Subsequently, Roger sticks on and remains at rest relative to the platform, and the combined body oscillates in simple harmonic motion with amplitude x_1.

(a) What is $\frac{x_1}{x_0}$?

Leave your answer to 3 significant figures. (2 points)

(b) Now consider the case where Roger jumps onto the platform when the platform is at displacement $x = \frac{3}{4}x_0$ instead. The combined body then oscillates with amplitude x_2. What is $\frac{x_2}{x_0}$?

Leave your answer to 3 significant figures. (3 points)



Solution:

(a) The relationship between total energy of the system and amplitude of a mass in SHM is given by $E = \frac{1}{2}kx_0^2$, which is essentially the elastic potential energy stored in the spring when the mass is at maximum displacement from equilibrium. Since k is a constant, the amplitude of a mass in SHM is solely determined by the total energy of the system.

When mass m is dropped onto mass M, we can model the collision as an inelastic collision in the horizontal axis with m initially having zero velocity. We also ignore any momentum considerations in the vertical axis. Hence, we can write the following equations to determine the final velocity v_2 of the spring just after the collision:

$$Mv = (M + m)v_2$$
$$v_2 = \frac{M}{M + m}v$$

where v is the velocity of mass M just before the collision. For (a), m lands on

M when it is instantaneously at rest, hence all the energy is stored in the spring. The inelastic collision yields no loss of kinetic energy, so the total energy of the system remains constant. Hence, the amplitude remains constant.

$$\frac{x_1}{x_0} = \boxed{1.00}$$

(b) The velocity of M before the collision v is found by the conservation of energy during SHM.

$$\frac{1}{2}Mv^2 + \frac{1}{2}k\left(\frac{3}{4}x_0\right)^2 = \frac{1}{2}kx_0^2 \Rightarrow v = \sqrt{\frac{7}{16}\frac{k}{M}}x_0$$

So the final total energy E_f of the system after the inelastic collision is given by:

$$\begin{aligned}
E_f &= E_p + E_k \\
&= \frac{1}{2}(M+m){v_2}^2 + \frac{1}{2}k\left(\frac{3}{4}x_0\right)^2 \\
&= \frac{1}{2}(M+m)\frac{M^2}{(m+M)^2}\left(\frac{7}{16}\frac{k}{M}{x_0}^2\right) + \frac{9}{32}k{x_0}^2 \\
&= \left(\frac{7M}{32(m+M)} + \frac{9}{32}\right)k{x_0}^2
\end{aligned}$$

The amplitude of the oscillating system is proportional to the square root of its total energy, provided that k is constant. Hence substituting $M : m = 3 : 1$, the ratio of amplitudes $\frac{x_2}{x_0}$ is given by:

$$\begin{aligned}
\frac{x_2}{x_0} &= \sqrt{\frac{E_f}{E_i}} \\
&= \sqrt{\frac{\frac{7M}{32(m+M)} + \frac{9}{32}}{\frac{1}{2}}} \\
&\approx \boxed{0.944}
\end{aligned}$$

Setter: Roger Zhang, roger.zhang@sgphysicsleague.org

Problem 13: Balanced Plates (4 points)

A capacitor consists of one plate of area $A = 2.0\ \mathrm{m}^2$ and mass $m = 2$ g attached to the ceiling, and a second plate of identical mass and dimensions floating freely at distance $d = 2.0$ cm below the top plate. What voltage V should be applied across the capacitor such that the bottom plate remains stationary? You may neglect edge effects.

Leave your answer to 2 significant figures in units of V.

Solution: Without loss of generality, suppose that the top plate has charge Q and the bottom plate has charge $-Q$. The electric field due to the top plate is $E = \frac{Q}{2A\epsilon_0}$, pointing downward at the position of the bottom plate. (This result is derived from Gauss' Law.)

Therefore, the bottom plate experiences an upward electrostatic force

$$F = QE = Q\frac{Q}{2A\epsilon_0} = \frac{Q^2}{2A\epsilon_0}$$

Since Q is related to V and the capacitance C by $Q = CV$ and the capacitance of two parallel plates is given by $C = \frac{\epsilon_0 A}{d}$,

$$F = \frac{(CV)^2}{2A\epsilon_0} = \frac{\left(\frac{\epsilon_0 A}{d}V\right)^2}{2A\epsilon_0} = \frac{\epsilon_0 AV^2}{2d^2}$$

Balancing this force with the weight of the bottom plate,

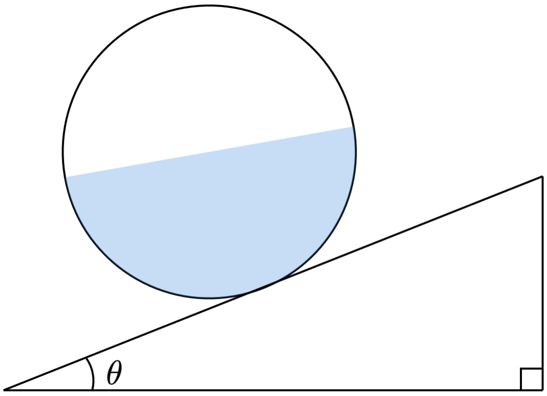
$$\frac{\epsilon_0 AV^2}{2d^2} = mg \implies V = \sqrt{\frac{2mgd^2}{\epsilon_0 A}} \approx \boxed{940\ \mathrm{V}}$$

Setter: Galen Lee, galen.lee@sgphysicsleague.org

Problem 14: Strange Sphere (4 points)

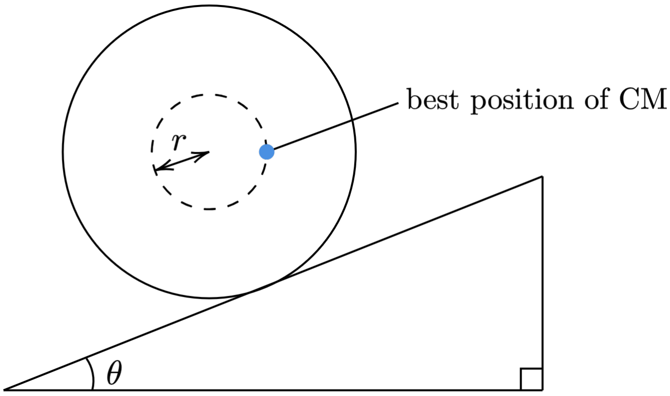
A sphere is made up of two uniform solid hemispheres of different densities joined together. It is placed on a slope inclined at angle θ from the horizontal. Given that the hemispheres' densities and the sphere's rotation can be freely varied, what is the maximum angle θ for which the sphere can rest in equilibrium on the slope? Assume the coefficient of static friction is sufficiently large for the sphere to remain in translational equilibrium.

Leave your answer to 3 significant figures in units of degrees.



Solution: For the sphere to rest in equilibrium on a slope, the line of action of its weight must pass through the contact point between the sphere and the slope. For uniform spheres, this is clearly not possible since the centre of mass is always in the geometric centre of the sphere. However, since this special sphere is made with two hemispheres of different densities, the position of its centre of mass can lie at some distance r from the geometric centre, towards the heavier half.

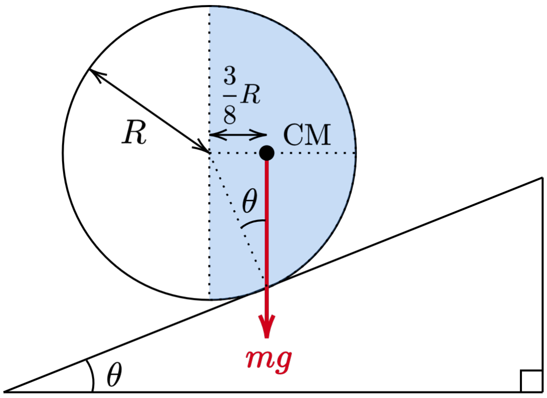
The sphere can be placed on the slope in any orientation, such that the possible positions of the centre of mass trace a circle of radius r. For a slope angled downwards to the left with angle of inclination θ, the centre of mass should be as far to the right of the sphere as possible, so that the line of action of the sphere's weight still passes through the point of contact for greatest possible θ.



This is done by maximising r, which occurs when one hemisphere has infinitely large density and the other hemisphere has negligibly small density, so the setup is essentially made up of only the heavier hemisphere. The sphere is then oriented such that the heavier hemisphere is on the right, for r to lie as far horizontally right as possible.

For a uniform hemisphere, the distance of its center of mass from the flat face R_{cm} is:

$$R_{cm} = \frac{3}{8}R$$



From the diagram, we can write:

$$\begin{aligned} \sin\theta &= \frac{\frac{3}{8}R}{R} \\ &= \frac{3}{8} \\ \theta &= \arcsin\left(\frac{3}{8}\right) \\ &\approx \boxed{22.0^\circ} \end{aligned}$$

Setter: Roger Zhang, roger.zhang@sgphysicsleague.org

Problem 15: Nuclear Fusion

(a) In a fusion power experiment, a mobile deuterium nucleus is fired at a stationary deuterium nucleus such that they have relative velocity v_1 right before collision. The two undergo nuclear fusion to form **only** a helium-4 nucleus. There are no other particles in the reaction chamber for them to interact with. Find v_1.

Leave your answer to 2 significant figures in units of km s^{-1}. (2 points)

If there exists either zero or multiple possible values of v_1*, input your answer as* -1*.*

(b) In another fusion power experiment, a mobile helium-4 nucleus is fired at a stationary helium-4 nucleus such that they have relative velocity v_2 right before collision. The two undergo nuclear fusion to form **only** a beryllium-8 nucleus. There are no other particles in the reaction chamber for them to interact with. Find v_2.

Leave your answer to 2 significant figures in units of km s^{-1}. (3 points)

If there exists either zero or multiple possible values of v_1*, input your answer as* -1*.*

You should assume that all nuclei involved, including the nuclei produced by the reaction, are in their ground states.

Data:
Rest mass of deuterium nucleus: $m_D = 2.01410178$ u
Rest mass of helium-4 nucleus: $m_{He} = 4.00260325$ u
Rest mass of beryllium-8 nucleus: $m_{Be} = 8.00530510$ u

Solution: At first glance, the problem appears ludicrous: you are given only the knowledge that a perfectly inelastic collision occurs, and asked to determine the initial velocity with no other information! However, the masses of the nuclei in the problem should provide a hint that a relativistic energy conservation relation can be expressed here.

In a classical inelastic collision, energy can be lost to "the surroundings" in the form of heat and sound. This is actually the conversion of the kinetic energy of bulk motion to kinetic energy of the surrounding air molecules, as well as random motion kinetic energy of the individual particles of the objects colliding. On the level of individual particles, there can be no distinction between bulk motion and random motion, and in these experiments there are no surrounding air molecules to transfer energy to, so even an inelastic collision must conserve energy. Using the mass-energy equivalence equation $E = mc^2$, we thus see that the relativistic masses[1] of the reactants and products must be the same.

(a) In the rest frame of the helium-4 nucleus produced, i.e. the centre of mass frame of this system, it has mass m_{He}. Hence, each deuterium nucleus must have relativistic mass $\frac{1}{2}m_{He} = 2.00130163$ u. However, this is lower than m_D, the rest mass of a deuterium nucleus. This hints to us that even if the deuterium nuclei start off both stationary, they must lose energy in order to fuse into a helium-4 nucleus, which they cannot do in this problem. Hence, there is no possible value of v_1, and the answer is $\boxed{-1}$.

(b) In the rest frame of the beryllium-8 nucleus produced, i.e. the centre of mass frame of this system, it has mass m_{Be}. Hence, each helium-4 nucleus must have relativistic mass $\frac{1}{2}m_{Be} = 4.00265255$ u. This is more than the rest mass m_{He} of helium-4, and the difference between the two gives us the kinetic energy of each helium-4 nucleus in the frame of the beryllium-8 by the mass-energy equivalence. Since this difference in mass is small, we can solve the rest of this problem using classical mechanics.

For each helium nucleus moving at v in the centre of mass frame:

$$\left(\frac{1}{2}m_{Be} - m_{He}\right)c^2 = \frac{1}{2}m_{He}v^2$$

Thus, $v = 1500$ km s^{-1}. Since the two helium-4 nuclei are both moving with $v = 1500$ km s^{-1} towards each other, $v_2 = 2v = \boxed{3000 \text{ km s}^{-1}}$.

The problem author would like to note that she feels that the relativistic method is more straightforward and originally solved the problem with that approach. Unfortunately, relativity is out of scope for H2/H3 physics, so the within-scope solution has been presented as the intended one. The relativistic solution uses the fact that $m = \gamma m_0$ to find $\gamma = \frac{m_{Be}}{2m_{He}}$, finds v from γ, and then uses relativistic velocity addition to find $v_2 = \frac{2v}{1+\frac{v^2}{c^2}}$. The two answers differ by only about 1 part in 30000, which is only a small deviation from the classical solution.

Setter: Shen Xing Yang, xingyang.shen@sgphysicsleague.org

[1] The relativistic mass m is given by the relation $m = \gamma m_0$ for a particle of rest mass m_0 and velocity v, where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ is the Lorentz factor.

Problem 16: Public Nuisance (4 points)

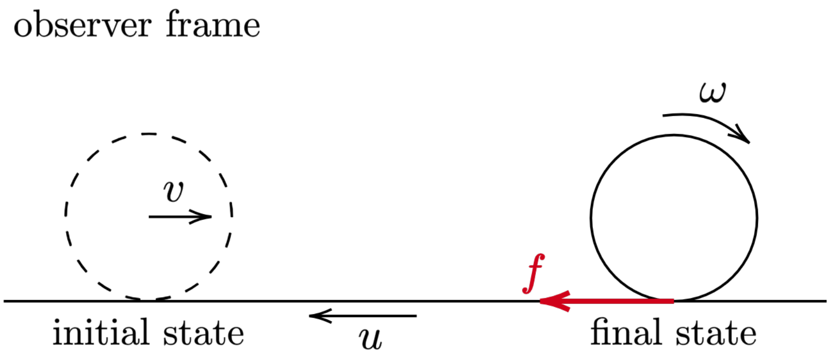
Bobbins is trying to set a football on a travelator that is moving leftward at constant velocity $u = 5.0\ \text{cm s}^{-1}$, such that the ball appears to be rolling on the spot to a stationary observer.

As such, he imparts an initial velocity v (relative to himself) to the ball, without imparting any rotation, as he places it on the travelator. Modelling the ball as a uniform solid sphere, find v such that he succeeds in his goal.

Leave your answer to 2 significant figures in units of cm s^{-1}.

Leave your answer as positive if you think the velocity should be rightward, and as negative if you think the velocity should be leftward.

Solution: For the ball to appear to be rolling on the spot to a stationary observer, the ball must rotate clockwise. This rotation is caused when the friction from the travelator on the ball "converts" its translational velocity to angular velocity. Since the ball starts off not rotating, the frictional force must be leftward to provide the clockwise torque that rotates the ball. This leftward frictional force also decreases the ball's translational velocity, which must therefore be initially v rightward (positive in this context).



Suppose the ball has mass m and radius r, and hence moment of inertia $I = \frac{2}{5}mr^2$. Let the time taken for the ball to reach the state of rolling on the spot be t and consider a frictional force f. Then, the translational impulse on the ball due to friction is ft and the rotational impulse on the ball due to friction is frt.

The initial velocity of the ball is v and the final velocity is 0. Hence, by the impulse-momentum theorem,

$$mv - ft = 0 \implies v = \frac{ft}{m}$$

The initial angular velocity of the ball is $\omega_0 = 0$ and the final angular velocity of the ball is $\omega_1 = u/r$. By a similar rotational impulse-momentum theorem, it follows that:

$$0 + frt = I\omega_1 \implies \frac{frt}{\frac{2}{5}mr^2} = \frac{u}{r} \implies \frac{ft}{m} = \frac{2u}{5}$$

Therefore,

$$v = \frac{2u}{5} = \boxed{2.0 \text{ cm s}^{-1}}$$

Setter: Galen Lee, galen.lee@sgphysicsleague.org

Problem 17: Unknown Motion (3 points)

A charged particle is in a magnetic field within a three-dimensional space described by the standard Cartesian axes x, y, z. The magnetic field strength and direction varies with time and space. Initially, the particle has velocity $v_x = 200$ m s^{-1} and $v_y = 210$ m s^{-1}. After travelling in the magnetic field for some time, the particle has travelled a distance of $d = 500$ m and has $v_z = 290$ m s^{-1}. Find $|\langle\vec{a}\rangle|$, the magnitude of the average acceleration of the particle. Assume there is no gravity.

Leave your answer to 3 significant figures in units of m s^{-2}.

Solution: Consider the particle's velocity $\vec{v} = \vec{v}_x + \vec{v}_y + \vec{v}_z$. Since $\vec{F}_B = q\vec{v} \times \vec{B}$, $\vec{F}_B$ always acts perpendicularly to $\vec{v}$. Hence, no work is done on the particle, so its kinetic energy, and thus the magnitude of its velocity, remains constant.

Initially, $|\vec{v}| = \sqrt{200^2 + 210^2} = 290$ m s^{-1}. This means that when the particle has $\vec{v}_z = 290$ m s^{-1}, both $\vec{v}_x$ and $\vec{v}_y$ must be 0. Any non-zero value would violate the fact that the magnitude of the resultant velocity does not change.

Next, we can determine the time taken Δt for the particle's journey. Since its velocity remains constant at 290 m s^{-1} and the particle is always travelling in the direction of its resultant velocity, $\Delta t = \frac{500}{290}$ s.

Hence, the average acceleration is:

$$
\begin{aligned}
|\langle\vec{a}\rangle| &= \left|\begin{pmatrix} \langle\vec{a}_x\rangle \\ \langle\vec{a}_y\rangle \\ \langle\vec{a}_z\rangle \end{pmatrix}\right| \\
&= \sqrt{\left(\frac{|\Delta\vec{v}_x|}{\Delta t}\right)^2 + \left(\frac{|\Delta\vec{v}_y|}{\Delta t}\right)^2 + \left(\frac{|\Delta\vec{v}_z|}{\Delta t}\right)^2} \\
&= \frac{\sqrt{(-200)^2 + (-210)^2 + (290)^2}}{500/290} \\
&\approx \boxed{238 \text{ m s}^{-2}}
\end{aligned}
$$

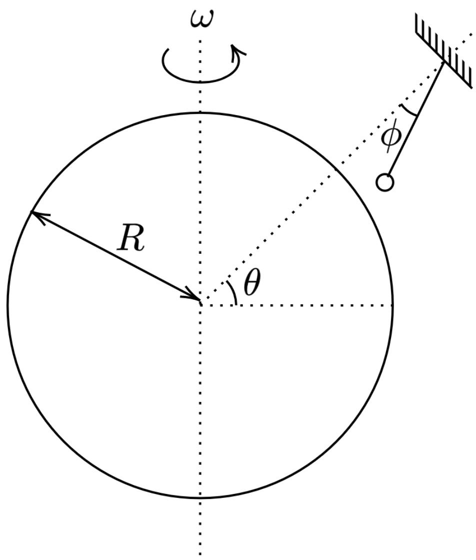
Setter: Gerrard Tai, gerrard.tai@sgphysicsleague.org

Problem 18: Curious Pendulum (4 points)

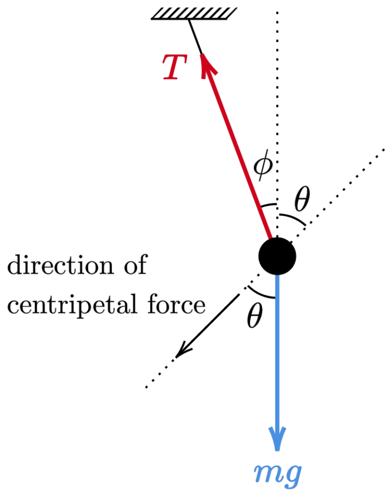
Guangyuan is standing at a point on Earth with latitude $\theta = 45°$. He suspends a simple pendulum at rest from a ceiling. Surprisingly, he claims that the pendulum does not hang completely vertical but instead is offset by an angle ϕ from the vertical, where the vertical axis is defined normal to the ground. Calculate ϕ.

Take the radius of Earth to be $R = 6370$ km, and the period of Earth's rotation to be $T = 24.0$ h.

Leave your answer to 3 significant figures in units of degrees.



Solution: The key reason why the pendulum does not hang straight is due to the rotation of the Earth. A centripetal force, directed at angle θ from the vertical, is required to keep the pendulum rotating about Earth's axis. This is provided by components of tension and weight parallel to the direction of the centripetal force required. Let the mass of the pendulum bob be m. We can draw a free body diagram on the pendulum bob as shown below.



Balancing forces along the axes parallel and perpendicular to the direction of centripetal force, we obtain:

$$mg\cos\theta - T\cos(\theta+\phi) = m(R\cos\theta)\omega^2 \tag{1}$$

$$mg\sin\theta = Tsin(\theta+\phi) \tag{2}$$

Rearranging (1), we obtain an expression for T:

$$T = \frac{m(g\cos\theta - (R\cos\theta)\omega^2)}{\cos(\theta+\phi)}$$

Substituting this expression into (2), we can obtain an expression for ϕ:

$$\begin{aligned}\phi &= \arctan\left(\frac{g\tan\theta}{g-R\omega^2}\right) - \theta \\ &\approx \boxed{0.0985°}\end{aligned}$$

As you can see, the angle of deviation from the vertical is very small and pretty much undetectable to Guangyuan's naked eye. He was probably capping.

Setter: Roger Zhang, roger.zhang@sgphysicsleague.org

Problem 19: Rainbow

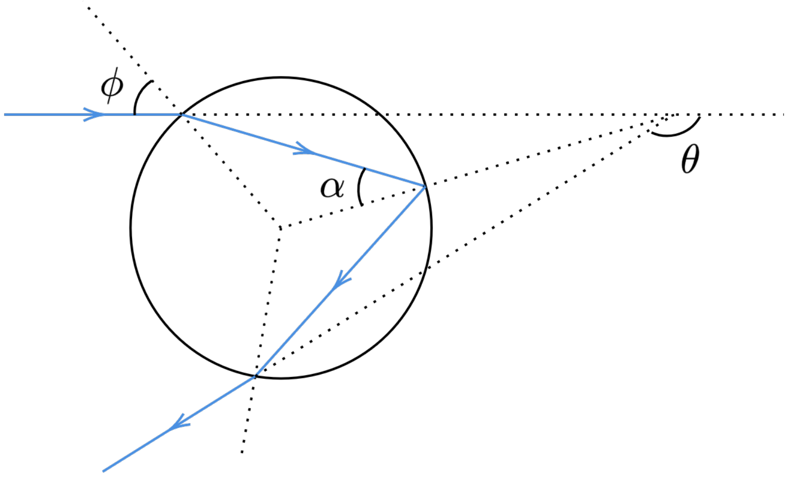
A ray of light in air enters a spherical drop of water of index $n = 1.33$ at an angle $\phi = 50°$ to the normal of the water surface.

(a) What is the angle of incidence α of the ray on the droplet's back surface?

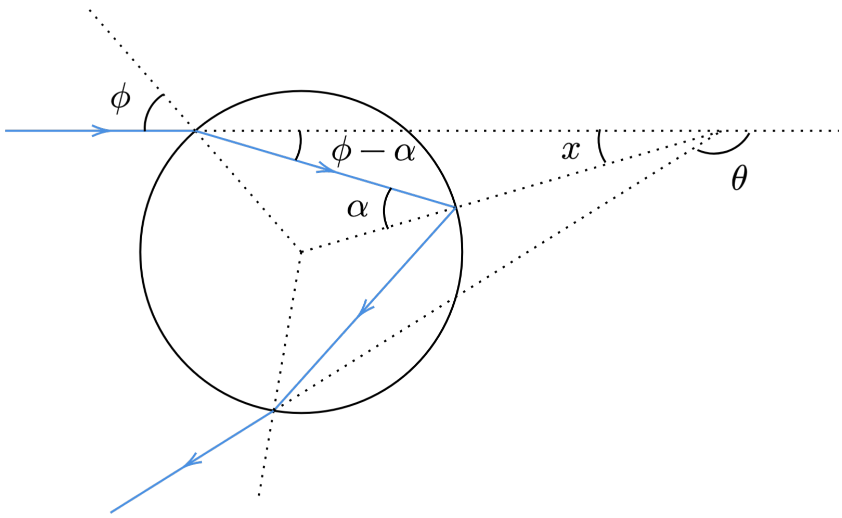
Leave your answer to 3 significant figures in units of degrees. (2 points)

(b) The light ray is partially reflected off the back surface, before exiting the drop at some angle of deflection θ from its initial direction. For such a light ray that reflects exactly once, find the angle ϕ that minimises θ. (3 points)

Leave your answer to 3 significant figures in units of degrees.



Solution:



(a) Using Snell's Law,

$$\begin{aligned}\sin\phi &= n\sin\alpha\\ \alpha &= \arcsin\left(\frac{1}{n}\sin\phi\right)\\ &\approx \boxed{35.2^\circ}\end{aligned}$$

(b) We define angle x as labelled in the diagram. Using geometry,

$$\begin{aligned}\alpha &= \phi-\alpha+x\\ x &= 2\alpha-\phi\\ \theta &= 180^\circ-2x\\ &= 180^\circ-4\alpha+2\phi\end{aligned}$$

To find minimum θ, set $\frac{d\theta}{d\phi}=0$.[2]

$$\begin{aligned}\frac{d\theta}{d\phi} &= -4\frac{d\alpha}{d\phi}+2=0\\ \frac{d\alpha}{d\phi} &= \frac{1}{2}\\ &= \frac{1}{\sqrt{1-\left(\frac{1}{n}\sin\phi\right)^2}}\left(\frac{1}{n}\cos\phi\right)\\ 1-\frac{1}{n^2}\sin^2\phi &= \frac{4}{n^2}\cos^2\phi\\ \cos\phi &= \sqrt{\frac{n^2-1}{3}}\\ \phi &\approx \boxed{59.6^\circ}\end{aligned}$$

Setter: Roger Zhang, roger.zhang@sgphysicsleague.org

[2]You can also set $\frac{dx}{d\phi}=0$ to obtain the same result, since minimising θ is equivalent to maximising x.

Problem 20: Rolling Ring

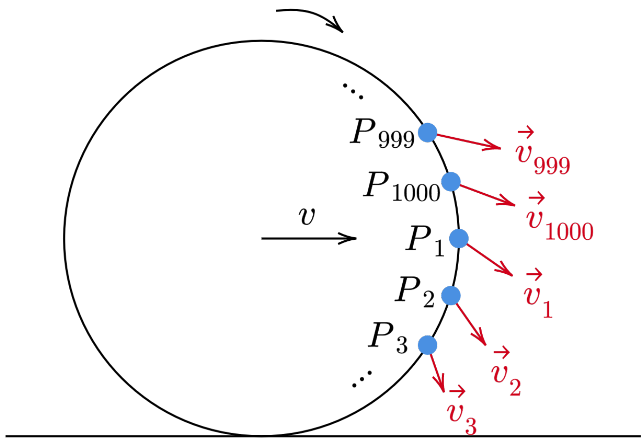
A rigid ring rolls without slipping along horizontal ground, with constant translational velocity $v = 5.0$ m s^{-1} towards the right. Consider 1000 points $P_1, P_2, \ldots, P_{1000}$ on the ring, evenly spaced across the ring's full circumference, with the first point P_1 taken to be the ring's rightmost point. Denote their velocities as $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{1000}$ respectively.

(a) Find $|\vec{v}_1|$, the magnitude of point P_1's velocity.

Leave your answers to 2 significant figures in units of m s^{-1}. (2 points)

(b) Find $|\vec{v}_1 + \vec{v}_2 + \ldots + \vec{v}_{1000}|$, the magnitude of the sum of velocities of all 1000 points.

Leave your answers to 2 significant figures in units of m s^{-1}. (2 points)



Solution: The motion of every point on the ring can be treated to be the combination of (1) translational motion – a uniform translation with rightward velocity v; (2) a rotational motion – a clockwise rotation with angular velocity ω. Since the ring does not slip, we have the relation $\omega = \frac{v}{r}$, where r is the ring's radius.

(a) At point P_1, the rotational motion results in a downward velocity of $r\omega$, while the translational motion results in a rightward velocity of v. As $\vec{v}_1$ is the vector sum of these two components, its magnitude is given by:

$$|\vec{v}_1| = \sqrt{v^2 + (r\omega)^2} = \sqrt{2}v \approx \boxed{7.1 \text{ m s}^{-1}}$$

(b) Consider the vector sum $\vec{v}_1 + \vec{v}_2 + \ldots + \vec{v}_{1000}$ in terms of both the translational parts and the rotational parts.

The translational parts sum to $1000v$ rightwards, since the translation is uniform across all points.

The rotational parts sum to a zero vector; the 1000 equal-length and equally spaced tangential velocity vectors form the sides of a regular 1000-gon, so if you align all of their velocity vectors tip-to-tail, they will form a closed loop and thus produce a vector sum of zero.

Hence, summing the velocities' translational and rotational parts, we have $|\vec{v}_1 + \vec{v}_2 + \ldots + \vec{v}_{1000}| = 1000v = \boxed{5000 \text{ m s}^{-1}}$.

Incidentally, based on this result, the velocity averaged across all 1000 points works out to be $\frac{|\vec{v}_1+\vec{v}_2+\ldots+\vec{v}_{1000}|}{1000} = v$. Hence, this average velocity is equal to the centre-of-mass velocity v! In fact, this comes quite intuitively — recall that the velocity of the centre of mass of any system can be expressed as the mass-weighted average of velocities of the masses in the system!

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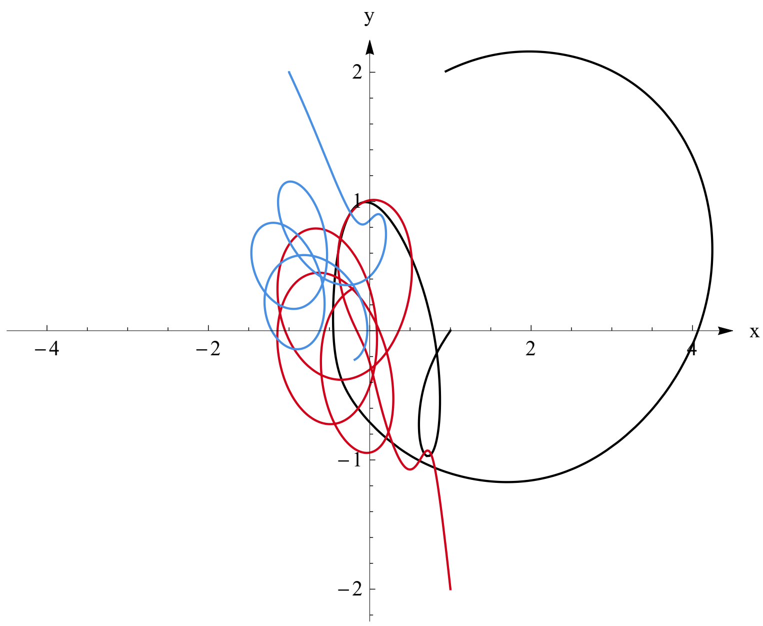
Problem 21: Three-Body Problem (4 points)

Three planets of mass m_1, m_2 and m_3 begin at rest with position vectors $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$ respectively. The masses are then released, and move only in the 2D plane of the diagram below in a chaotic manner. After some time t, the position vectors of m_1 and m_2 are $\vec{r}_1'$ and $\vec{r}_2'$ respectively. Find $|\vec{r}_3'|$, the distance from m_3 to the origin at that time.

Data:
$m_1 = 1.00 \times 10^{24}$ kg
$m_2 = 2.00 \times 10^{24}$ kg
$m_3 = 3.00 \times 10^{24}$ kg
$\vec{r}_1 = (1.00, 0.00)$
$\vec{r}_2 = (1.00, -2.00)$
$\vec{r}_3 = (-1.00, 2.00)$
$\vec{r}_1' = (3.93, -0.20)$
$\vec{r}_2' = (-1.06, 0.59)$

All position vectors are given in units of AU.

Leave your answer to 2 significant figures in units of AU.



Solution: The key to solving this problem is realising that there are no external forces acting on the system and the initial velocity of the centre of mass is zero, so the position of the centre of mass remains constant. Recall that the position of the centre of mass can be given by the formula:

$$(x_{CM}, y_{CM}) = \frac{1}{\sum m_i}\left(\sum m_i x_i, \sum m_i y_i\right)$$

We equate the position of the centre of mass at the beginning to the end. Multiplying throughout by $\sum m_i$ gives us:

$$\left(\sum m_i x_i, \sum m_i y_i\right) = \left(\sum m_i x_i', \sum m_i y_i'\right)$$

Upon solving the equation, we have:

$$(x_3', y_3') \approx (-0.60, 0.34)$$

Finally, we get:

$$|\vec{r}_3'| = \sqrt{x_3'^2 + y_3'^2} \approx \boxed{0.69 \text{ AU}}$$

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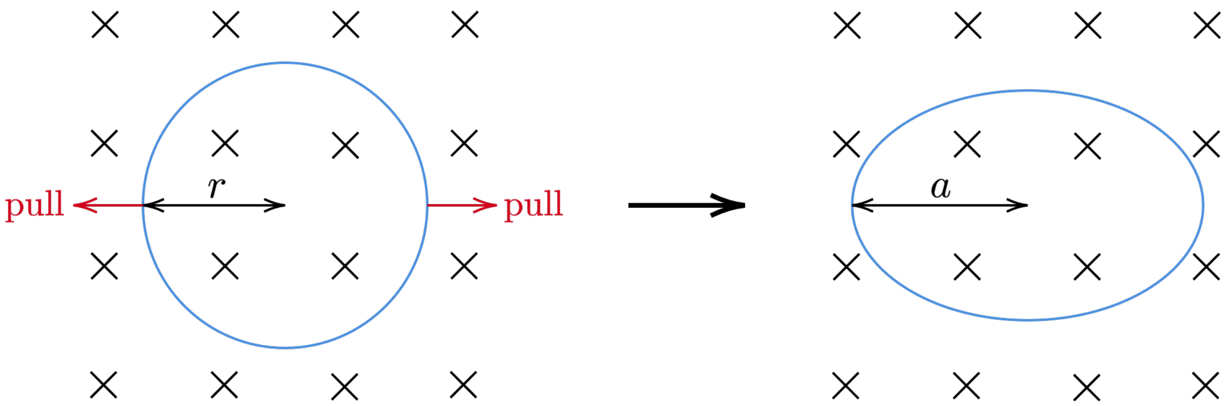
Problem 22: Wire Distortion (4 points)

A circular loop of wire with radius $r = 10.0$ cm and resistance $R = 5.00$ mΩ is placed in a uniform magnetic field $B = 3.00$ mT perpendicular to the plane of the loop. The wire is pulled at opposite ends outwards such that it now forms an ellipse with semi-major axis $a = 12.0$ cm. How much charge Q flows through the wire during this process?

Leave your answer to 3 significant figures in units of C.

Use the formula $p \approx 2\pi\sqrt{\frac{a^2+b^2}{2}}$ for the perimeter p of an ellipse with semi-major axis a and semi-minor axis b.



Solution: Using Faraday's Law of Electromagnetic Induction, the induced emf ϵ in the loop is:

$$\begin{aligned}\epsilon &= \frac{\mathrm{d}\phi}{\mathrm{d}t} \\ &= B\frac{\mathrm{d}A}{\mathrm{d}t}\end{aligned}$$

The total charge Q that flows through the wire during the distortion is given by:

$$Q = \int I \, \mathrm{d}t$$

where I is the instantaneous current throughout the distortion process. But $I = \frac{\epsilon}{R}$ and $\epsilon = B\frac{\mathrm{d}A}{\mathrm{d}t}$, so we have:

$$\begin{aligned}Q &= \int \frac{B\frac{\mathrm{d}A}{\mathrm{d}t}}{R}\, \mathrm{d}t \\ &= \int \frac{B}{R}\, \mathrm{d}A \\ &= \frac{B}{R}\triangle A\end{aligned}$$

Thus we only need to find the change in area $\triangle A$ of the shape formed by the wire throughout the distortion process. $\triangle A$ is given by:

$$\triangle A = \pi r^2 - \pi ab$$

where πr^2 is the area of the original circular wire loop and πab is the area of the final elliptical wire loop. Using the formula given in the problem, we can solve for b in terms of p and a:

$$p = 2\pi\sqrt{\frac{a^2+b^2}{2}} \Rightarrow b = \sqrt{\frac{p^2}{2\pi^2} - a^2}$$

The length of the wire remains constant no matter its shape. Hence, $p = 2\pi r$. Combining all the above results:

$$\begin{aligned} Q &= \frac{B\pi}{R}\left(r^2 - a\sqrt{2r^2 - a^2}\right) \\ &\approx \boxed{0.00192\ \mathrm{C}} \end{aligned}$$

Setter: Roger Zhang, roger.zhang@sgphysicsleague.org

Problem 23: Interplanetary Bridge

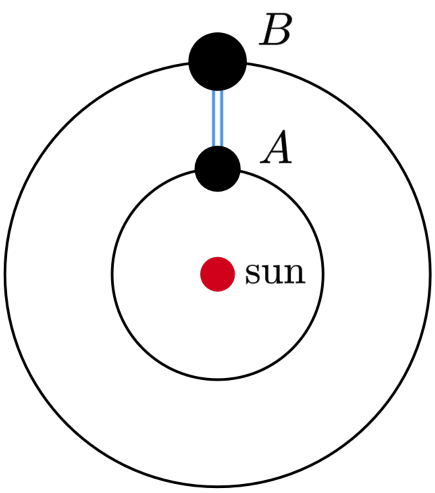
Consider two planets, A and B, of masses $M_A = 1.0 \times 10^{24}$ kg and $M_B = 2.0 \times 10^{24}$ kg, in circular orbits of radius $R_A = 1.0 \times 10^{11}$ m and $R_B = 1.3 \times 10^{11}$ m around a common star of mass $M_s = 2.0 \times 10^{30}$ kg. The two planets are orbiting in the same direction. You may assume that the planets are point masses and have no rotation about their own individual axes.

(a) The planets have angular momentum L_A and L_B about the star. Determine the ratio L_A/L_B.

Leave your answer to 2 significant figures. (2 points)

(b) The inhabitants of the planets have constructed a light indestructible link bridge, which they connect when planets A and B are closest to each other. This joins the two planets instantaneously. Determine the angular velocity ω of the two planets about their centre of mass at the instant just after the connection is made.

Leave your answer to 2 significant figures in units of rad year^{-1}. (3 points)



Solution:

(a) The gravitational force acting on each planet provides its centripetal force for its rotation:

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

$$v = \sqrt{\frac{GM}{r}}$$

Since $L = mvr = m\sqrt{GMr}$, we can see that $L \propto m\sqrt{r}$. Hence:

$$\frac{L_A}{L_B} = \frac{M_A\sqrt{R_A}}{M_B\sqrt{R_B}} \approx \boxed{0.44}$$

(b) First, we find the position of the centre of mass relative to the two planets, $R_{\text{A,CM}}$ and $R_{\text{B,CM}}$:

$$M_A R_{\text{A,CM}} = M_B R_{\text{B,CM}}$$
$$R_{\text{A,CM}} + R_{\text{B,CM}} = R_B - R_A$$

Hence, $R_{\text{A,CM}} = 2.0 \times 10^{10}$ m and $R_{\text{B,CM}} = 1.0 \times 10^{10}$ m. We shall now use the centre of mass as our **reference point** about which we consider the bodies' angular momentum, as the subsequent rotation is about this point. We also take the motion to remain in the **world frame**.

The total angular momentum of the two planets about their centre of mass is conserved after joining them together, as there is no external torque during that instant. Hence, equating their total angular momentum about their centre of mass before and after the bridge is formed:

$$M_A R_{\text{A,CM}}\sqrt{\frac{GM_s}{R_A}} - M_B R_{\text{B,CM}}\sqrt{\frac{GM_s}{R_B}} = M_A \omega R_{\text{A,CM}}^2 + M_B \omega R_{\text{B,CM}}^2$$
$$\omega = \frac{M_A R_{\text{A,CM}}\sqrt{\frac{GM_s}{R_A}} - M_B R_{\text{B,CM}}\sqrt{\frac{GM_s}{R_B}}}{M_A R_{\text{A,CM}}^2 + M_B R_{\text{B,CM}}^2} = \boxed{4.7 \text{ rad year}^{-1}}$$

The conservation of angular momentum will apply regardless of our choice of reference point. Hence, we can choose a point of convenience (in this case, the centre of mass) as it allows us to directly calculate ω.

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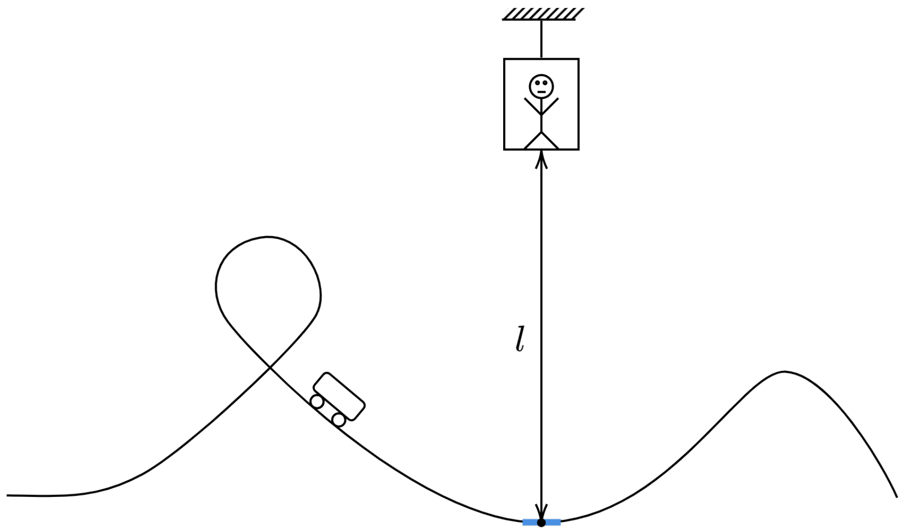
Problem 24: Killer Coaster (3 points)

Roger is a worker at a theme park who dislikes his boss. His job is to maintain a roller coaster with carts of mass $m = 100$ kg. One day, he paints a section of the roller coaster near the bottom of its loop with reflective paint. At midday, his boss (who sits in an office $l = 75$ m above the bottom of the roller coaster) was fried to a crisp, bringing Roger great joy. If the roller coaster carts have a velocity of $v = 35$ m s^{-1} when they travel across that section, what is the normal force N exerted by the track on the cart at the bottom? Assume that the rays from the sun are normal to the track surface at the point where it is painted.

Leave your answer to 2 significant figures in units of N.



Solution: The reflective surface of the roller coaster acts as a mirror which focuses light at a distance l away. The radius of a curvature of a lens is twice its focal length, so the radius of curvature $r = 2l$. Now, we can write the expression for the centripetal force

$$\frac{mv^2}{r} = N - mg$$

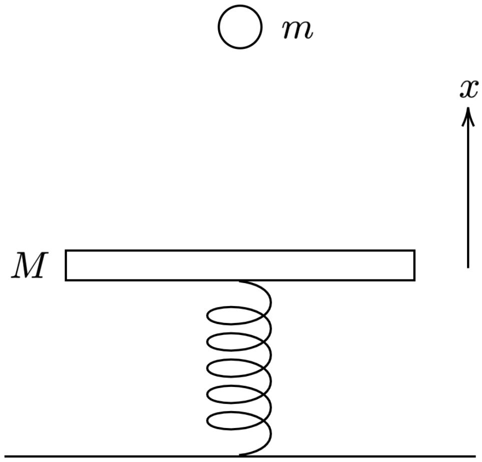
$$N = \frac{mv^2}{r} + mg = \frac{mv^2}{2l} + mg \approx \boxed{1800 \text{ N}}$$

The moral of the story is to treat your employees well.

Setter: Chen Guangyuan, guangyuan.chen@sgphysicsleague.org

Problem 25: Boing Boing (5 points)

Roger watches a video of table tennis players juggling a ball on their bats, seemingly indefinitely. Inspired, Roger tries his luck at replicating this trick with a physical model.



A large plate of mass $M = 0.50$ kg is mounted on top of a spring of spring constant k as shown in the diagram. The position $x = 0$ is defined as the equilibrium position of the spring, with x being positive upwards. The spring is stretched by a distance $x_0 = +0.070$ m and released. Assume that the damping of the spring oscillations is negligible.

At a certain time $t = t_0$, a ball of mass $m = 0.050$ kg is dropped onto the centre of the plate and collides elastically with the plate while the plate is at position $x_1 = +0.040$ m. To Roger's delight, the ball continues to collide with the plate indefinitely, each collision appearing identical to the last. Determine the smallest possible value of k.

Leave your answer to 3 significant figures in units of N m^{-1}.

Solution: Firstly, we try to develop a qualitative understanding. The plate is undergoing simple harmonic oscillation, while the ball is in free fall under gravity. In order for the collision to repeat itself in an identical manner, the position and velocities of both objects before collision have to be constant for every collision. This implies that the amplitude of oscillation of the plate remains constant at x_0. However, the velocity of a body in simple harmonic oscillation can only take two values at a particular position, and clearly the plate's velocity cannot be unchanged after the collision. We conclude that the plate's velocity is reversed after the collision, while its magnitude remains constant. Similarly, the ball's velocity is also reversed after the collision while its magnitude remains constant.

For an elastic collision of two bodies, their velocities are reversed exactly in the frame of their centre of mass[3]. This happens to be just what we want! The lab frame must thus be the centre of mass frame right before the collision. If we were to define the velocity of M and m right before the collision as V and $-v$ respectively, then:

$$MV = mv$$

Next, we can apply conservation of energy for the plate to find V:

$$\frac{1}{2}kx_0^2 = \frac{1}{2}kx_1^2 + \frac{1}{2}MV^2$$

We also equate the time taken for the plate and ball to return to the position of collision respectively. Since we are told to find the minimum value of k, we choose the case where the plate completes less than one full oscillation before returning to its collision point. Due to symmetry, we simply have to equate the time taken for the plate and the ball to reach their maximal points respectively. For the ball, this time t can be obtained easily:

$$t = \frac{v}{g}$$

For the plate, we consider the simple harmonic motion equation, $x = x_0 \cos \omega t$, where $\omega = \frac{k}{M}$ is the angular frequency of oscillation of the spring-plate system. At the point of collision, $x = x_1$ and at the lowest point, $x = -x_0$. Solving for the argument of the cosine term at these two points, we obtain:

$$t = \frac{1}{\omega}\left(\pi - \cos^{-1}\left(\frac{x_1}{x_0}\right)\right)$$

We can manipulate the equations above to obtain:

$$\frac{1}{g}\frac{M}{m}\sqrt{\frac{k}{M}(x_0^2 - x_1^2)} = \frac{1}{\omega}\left(\pi - \cos^{-1}\left(\frac{x_1}{x_0}\right)\right)$$

Recalling that $\omega = \sqrt{\frac{k}{M}}$, we have:

$$k = \frac{mg}{\sqrt{x_0^2 - x_1^2}}\left(\pi - \cos^{-1}\left(\frac{x_1}{x_0}\right)\right) \approx \boxed{18.6 \text{ N m}^{-1}}$$

Setter: Chen Guangyuan, guangyuan.chen@sgphysicsleague.org

[3] We see that $MU + mu = MV + mv = 0$ for two masses M and m with initial velocities U and u and final velocities V and v. Hence we have $\frac{v}{V} = \frac{u}{U} = -\frac{m}{M}$, and if the velocities change then either both increase or both decrease, which would violate energy conservation.

Problem 26: Pulling a Rope

A uniform inelastic rope of linear mass density $\lambda = 0.120$ kg m^{-1} and length $l = 2.00$ m is hung over a pole of negligible radius, such that both ends start at the same height. Roger pulls one end downwards with a constant velocity $v = 0.500$ m s^{-1}. The pole exerts a constant frictional force $f = 3.00$ N as the rope slides over the pole.

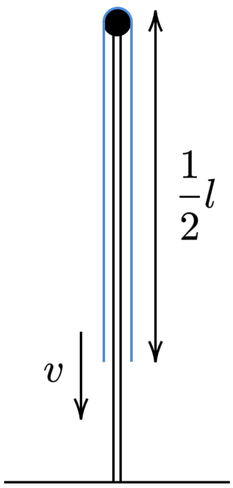
(a) Find the acceleration a_{cm} of the centre of mass of the rope.

Leave your answer to 3 significant figures in units of m s^{-2}. (3 points)

If you think the the centre of mass of the rope is not accelerating, input your answer as $a_{cm} = 0.00$ m s^{-2}.

(b) Find the pulling force F exerted by Roger when the end being pulled is a vertical distance $h = 0.400$ m below the other end.

Leave your answer to 2 significant figures in units of N. (2 points)



Solution:

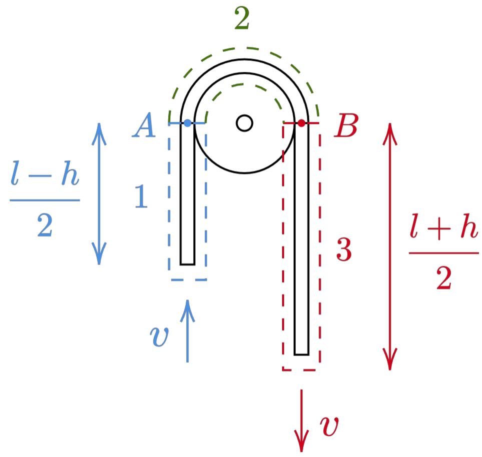
(a) Define s_{cm}, v_{cm}, and a_{cm} as the displacement, velocity and acceleration of the centre of mass of the rope respectively. Let the rope have moved a distance x, meaning one end has risen by x and one end has fallen by x. In a small time interval $\mathrm{d}t$, the rope will have moved a small distance $v\,\mathrm{d}t$. The small displacement of the centre of mass of the rope $\mathrm{d}s_{\text{cm}}$ can then be modelled as due to the small segment of rope at one end with mass $\lambda v\,\mathrm{d}t$ being 'transferred' a distance $2x$ downwards to the other end of the rope.

$$\begin{aligned}\mathrm{d}s_{\text{cm}} &= \frac{\lambda v\,\mathrm{d}t}{\lambda l}2x \\ &= \frac{2vx\,\mathrm{d}t}{l}\end{aligned}$$

Therefore:

$$\begin{aligned}
v_{\mathrm{cm}} &= \frac{\mathrm{d}s_{\mathrm{cm}}}{\mathrm{d}t} \\
&= \frac{2vx}{l} \\
a_{\mathrm{cm}} &= \frac{\mathrm{d}v_{\mathrm{cm}}}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} \\
&= \frac{2v}{l}v \\
&= \frac{2v^2}{l} \\
&= \boxed{0.250 \text{ m s}^{-2}}
\end{aligned}$$

(b) Consider splitting the rope into 3 sections as shown below. The left end of the rope has length $\frac{l-h}{2}$ while the right end has length $\frac{l+h}{2}$. The sections are defined such that they are fixed in space and do not move along with the rope.



Considering section 1, its velocity is upward and its mass is decreasing, hence its rate of change of momentum is downward. Using Newton's Second Law, we have

$$\lambda\frac{l-h}{2}g - T_A = \lambda v^2$$

where T_A is the tension at point A.

Now consider section 2. The tangential acceleration must be zero since the rope travels at constant speed. Since friction opposes the direction of motion, the friction on the rope by the pole acts anticlockwise. The tension T_B at point B is thus given by:

$$T_B = T_A + f$$

Finally, looking at section 3, its velocity is downward and its mass is increasing, so the rate of change of momentum is downward. We can thus apply Newton's

Second Law as follows to find the applied force F:

$$\lambda\frac{l+h}{2}g + F - T_B = \lambda v^2 \implies F = f - \lambda hg \approx \boxed{2.5\text{ N}}$$

Setter: Roger Zhang, roger.zhang@sgphysicsleague.org

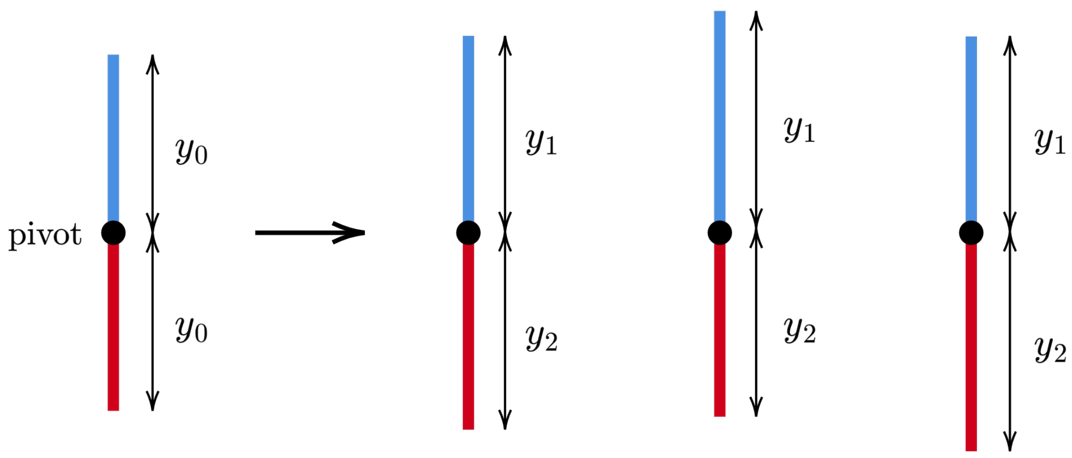
Problem 27: Vertical Expansion

A uniform vertical rod is pivoted at its centre. This way, the top of the rod is at distance y_0 above the pivot, while the bottom of the rod is at the same distance y_0 below the pivot.

We now apply a small amount of heat uniformly across the rod, causing it to expand. As such, the top of the rod is now at distance y_1 above the pivot, while the bottom of the rod is at distance y_2 below the pivot.

Assume that there is no heat flow within the rod and no heat loss to the surroundings.

(a) Select the correct relation between y_1 and y_2. (You may refer to the diagram below to visualise the physical setup illustrated by each option.) (1 point)



(1) $y_1 = y_2$ (2) $y_1 > y_2$ (3) $y_1 < y_2$

(b) The rod has specific heat capacity $c = 50$ J kg^{-1} K^{-1}, linear expansion coefficient $\alpha = 8.0 \times 10^{-3}$ K^{-1}, and $y_0 = 10$ m. Calculate the ratio $\dfrac{y_1 - y_0}{y_2 - y_0}$.

Leave your answer to 3 significant figures. (4 points)

Solution: Throughout our analysis, we consider the upper portion of the rod (above the pivot) and the lower portion of the rod (below the pivot) separately.

(a) $\boxed{\text{(3) } y_1 < y_2}$. The upper portion is constrained at its bottom by the pivot, so its centre rises upon expansion. Hence, the heat applied to the upper portion goes partially into raising its GPE, and the rest is used to raise its temperature. On the other hand, the lower portion is constrained at its top, so its centre drops upon expansion. This means the heat applied to the lower portion, in addition to the decrease in its GPE, is used to raise its temperature. The lower portion thus gains a higher temperature than the upper portion, causing the lower portion to expand further than the upper portion.

(b) Denote the heat applied to each portion of the rod as Q, and the mass of each portion as m. Considering the energy conversion taking place in the upper portion of the rod as described above, we have

$$Q = mg\left(\frac{y_1}{2} - \frac{y_0}{2}\right) + mc\Delta T_1$$

where ΔT_1 is the change in temperature of the upper portion. We can relate ΔT_1 to the change in length using the linear expansion formula:

$$y_1 - y_0 = \alpha y_0 \Delta T_1$$

Combining both equations, we have:

$$Q = (y_1 - y_0)\left(\frac{mc}{\alpha y_0} + \frac{mg}{2}\right)$$

Let us now perform the same analysis for the lower portion of the rod, letting its change in temperature be ΔT_2.

$$\begin{cases} Q + mg\left(\frac{y_2}{2} - \frac{y_0}{2}\right) = mc\Delta T_2 \\ y_2 - y_0 = \alpha y_0 \Delta T_2 \end{cases}$$
$$\implies Q = (y_2 - y_0)\left(\frac{mc}{\alpha y_0} - \frac{mg}{2}\right)$$

The heat is applied uniformly, so the Q for both portions of the rod is identical. As such, we may equate both expressions for Q, and derive the required ratio:

$$\frac{y_1 - y_0}{y_2 - y_0} = \frac{\frac{2c}{\alpha y_0 g} - 1}{\frac{2c}{\alpha y_0 g} + 1}$$

Right away, we can see that this ratio is smaller than 1, which agrees with our qualitative reasoning in (a). Plugging in the numbers, we obtain $\frac{y_1 - y_0}{y_2 - y_0} \approx \boxed{0.984}$.

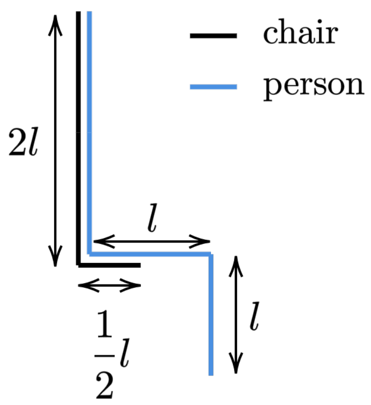
Setter: Christopher Ong, chris.ong@sgphysicsleague.org

Problem 28: Bad Driver (4 points)

Paul is sitting on a fixed bus seat with his feet off the floor. Here, he is modelled as 3 thin, rigid rods of uniform mass density joined to each other, with dimensions as shown in the figure. His seat consists of a base and a backrest. The coefficient of static friction between Paul and the seat is $\mu = 0.5$. The bus driver suddenly brakes at a constant deceleration and Paul finds himself crashing into the seat in front of him. Fuming, Paul decides to calculate the maximum bus deceleration a at which he would have remained stationary. What is the value of a?

Leave your answer to 2 significant figures in units of $\mathrm{m\ s^{-2}}$.

Your answer should be positive.

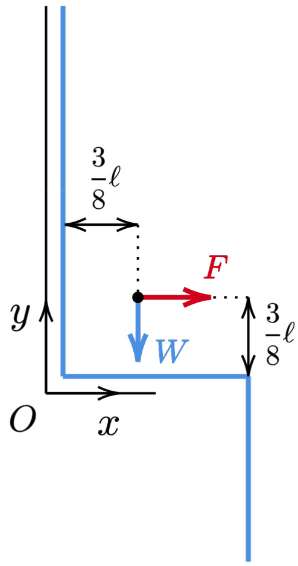


Solution: Since the bus accelerates backwards, Paul experiences a fictitious force F forwards in the frame of the bus. One might naively try to determine when Paul slips forward in his seat, that is, when F exceeds the maximum force provided by the static friction f, i.e. $a = \mu g$.

However, it is possible that Paul moves without slipping by rotating about the edge of the seat. When Paul is about to rotate, his back loses contact with the back of the seat, and his bottom lifts off the bottom of the seat. The contact force (normal and frictional force) thus acts at the edge of the seat. Considering the torques about the edge of the seat, we have only the clockwise torque due to F acting on Paul's centre of mass, which is equal to the counterclockwise torque due to his weight W acting on his centre of mass.

To balance these torques, we first find the position of Paul's centre of mass. We set the bottom end of the rod $2l$ as the origin. Considering the centre of mass of each rod, and using the centre of mass formula, Paul's overall centre of mass is

$$(x_{CM}, y_{CM}) = \left(\frac{3}{8}l, \frac{3}{8}l\right)$$



Therefore,

$$F\left(\frac{3l}{8}\right) = W\left(\frac{l}{8}\right)$$
$$F = \frac{1}{3}W$$

When this happens, by balancing horizontal forces, $F = f$, and by balancing vertical forces, $W = N$. Therefore, $f = \frac{1}{3}N \leq \mu N$. This rotation toppling happens **before** Paul slips horizontally, therefore, the maximum acceleration a is given by

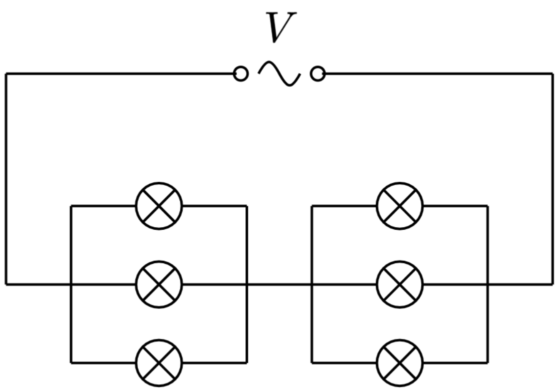
$$a = \frac{1}{3}g \approx \boxed{3.3\ \mathrm{m\ s^{-2}}}$$

Setter: Chen Guangyuan, guangyuan.chen@sgphysicsleague.org

Problem 29: Power Saving Mode (4 points)

Paul is trying to save the Earth by replacing bulbs with diodes. In his setup, there are six bulbs, each of resistance R, connected to an external voltage source as shown in the diagram below. The voltage source is alternating, creating an electromotive force in the form $V = V_0 \cos \omega t$. The average power drawn by this setup over a long period of time is P_0. Now, one of the bulbs is replaced with an ideal diode, and the average power drawn over a long period of time is P_1. Determine the ratio $\frac{P_1}{P_0}$.

Leave your answers to 2 significant figures.



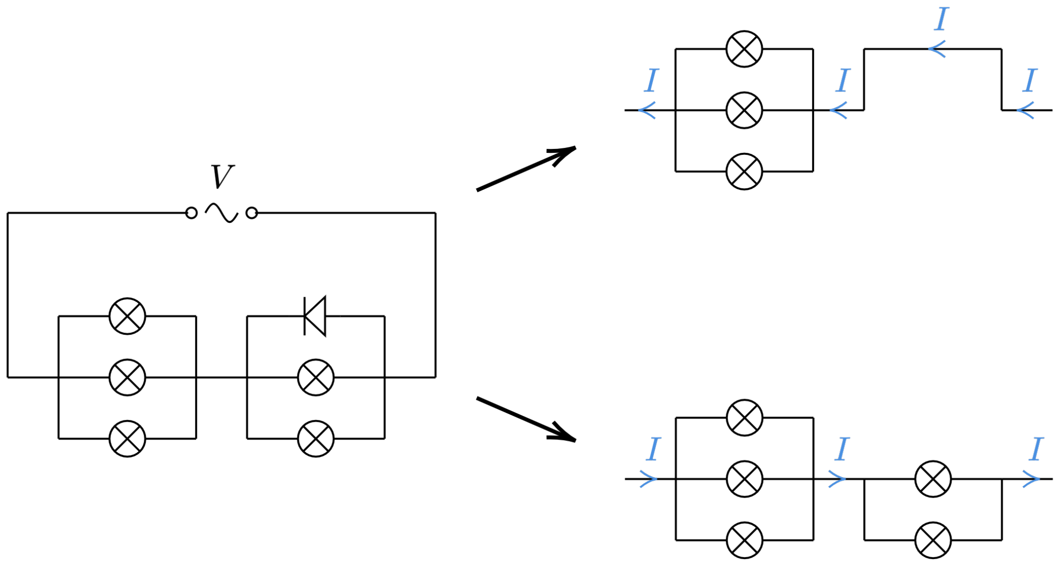
Solution: In the original circuit, there is a potential difference of $\frac{V}{2}$ across each bulb, so the average total power drawn by all six bulbs is

$$P_0 = 6\frac{\left(\frac{V_{rms}}{2}\right)^2}{R} = \frac{3}{2}\frac{V_{rms}^2}{R}$$

where V_{rms} is the root-mean-square voltage from the source.

When a diode replaces a bulb, the behaviour of the current in the circuit is different in the forward and backward directions.

When the current flows forward across the diode, it acts as a path of zero resistance. Hence, the remaining set of three bulbs in parallel each have a potential difference of V across them, while the other two bulbs have no current passing through them.



When the current is flowing backwards against the diode, it acts as a path of infinite resistance. Effectively, we now have a set of two bulbs in parallel, which is in series with a set of three bulbs in parallel. The potential difference across the set of two bulbs is $\frac{3V}{5}$, and the potential difference across the set of three bulbs is $\frac{2V}{5}$.

The current flows forwards half the time and backwards half the time, so we can simply take the average power drawn in the forward and backward cycles. The average total power drawn is then

$$P_1 = \frac{1}{2}\left(3\frac{V_{rms}^2}{R} + 2\frac{\left(\frac{3V_{rms}}{5}\right)^2}{R} + 3\frac{\left(\frac{2V_{rms}}{5}\right)^2}{R}\right) = \frac{21}{10}\frac{V_{rms}^2}{R}$$

$$\frac{P_1}{P_0} = \boxed{1.4}$$

It appears that trying to save power by replacing a bulb with a diode fails.

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Problem 30: Infinite Energy

$2n$ point charges are arranged in a straight line, each separated from its neighbour by distance $l = 5.0$ mm. The charges have alternating signs, but the same magnitude $q = 1.5 \times 10^{-6}$ C. The total potential energy of this system of charges is denoted by U_n. Take the potential energy of the system to be zero when the charges are far apart.

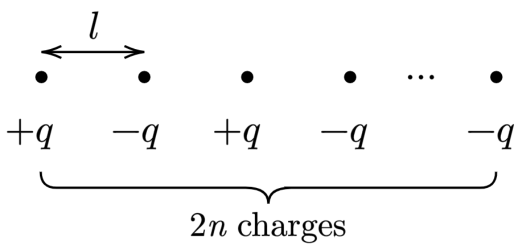
(a) Determine $\frac{U_n}{2n}$ for $n = 2$. This is the average potential energy contribution due to each of the $2n$ charges.

Leave your answers to 2 significant figures in units of J. (3 points)

(b) Determine $\lim\limits_{n \to \infty} \frac{U_n}{2n}$.

Leave your answers to 2 significant figures in units of J. (3 points)

If the limit tends towards positive or negative infinity, input your answer as 1000 *or* −1000 *respectively.*



Solution:

(a) The formula for the potential energy between two point charges, q_1 and q_2, separated by a distance r is:

$$U = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r}$$

We can repeatedly apply this formula to each pair of charges in the system of four charges, ensuring we only count each unique pair once:

$$U_2 = \frac{1}{4\pi\varepsilon_0} \frac{q^2}{l} \left(-3(1) + 2\left(\frac{1}{2}\right) - 1\left(\frac{1}{3}\right) \right) = -\frac{7}{3} \frac{1}{4\pi\varepsilon_0} \frac{q^2}{l}$$

Substituting the numerical values given, we get:

$$\frac{U_2}{2n} \approx \boxed{-2.4 \text{ J}}$$

(b) As n tends towards infinity, the contribution of each charge to the total potential energy of the system tends to the same value. This is because from the perspective of any charge, there are infinitely many charges on either side of it. Furthermore, the relative signs of the other charges from the perspective of any charge is always the same. The contribution by a single charge to the total potential energy is what we are looking for in our answer. Hence, our limit is:

$$\lim_{n\to\infty} \frac{U_n}{2n} = \frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{q^2}{l}\left(2\sum_{i=1}^{\infty}(-1)^i\frac{1}{i}\right)$$

The factor of $\frac{1}{2}$ is included to ensure there is no over-counting, as each pair of charges must only be accounted for once. The factor of 2 is due to the fact that charges to the left and right of a point charge both contribute to the potential energy. To evaluate the sum to infinity, we can simply sum the first ten terms, which yields the same numerical answer to 1 significant figure. Alternatively, we can use the series expansion of $\ln(1+x)$ to find an exact value:

$$\ln(1+x) = \sum_{i=1}^{\infty}(-1)^{i+1}\frac{x^i}{i}$$

Substituting $x = 1$, we get:

$$\begin{aligned}
\ln 2 &= \sum_{i=1}^{\infty}(-1)^{i+1}\frac{1}{i} \\
-\ln 2 &= \sum_{i=1}^{\infty}(-1)^{i}\frac{1}{i} \\
\lim_{n\to\infty} \frac{U_n}{2n} &= -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{l}\ln 2
\end{aligned}$$

Substituting the numerical values given, we get:

$$\lim_{n\to\infty} \frac{U_n}{2n} \approx \boxed{-2.8\ \mathrm{J}}$$

Setter: Chen Guangyuan, guangyuan.chen@sgphysicsleague.org

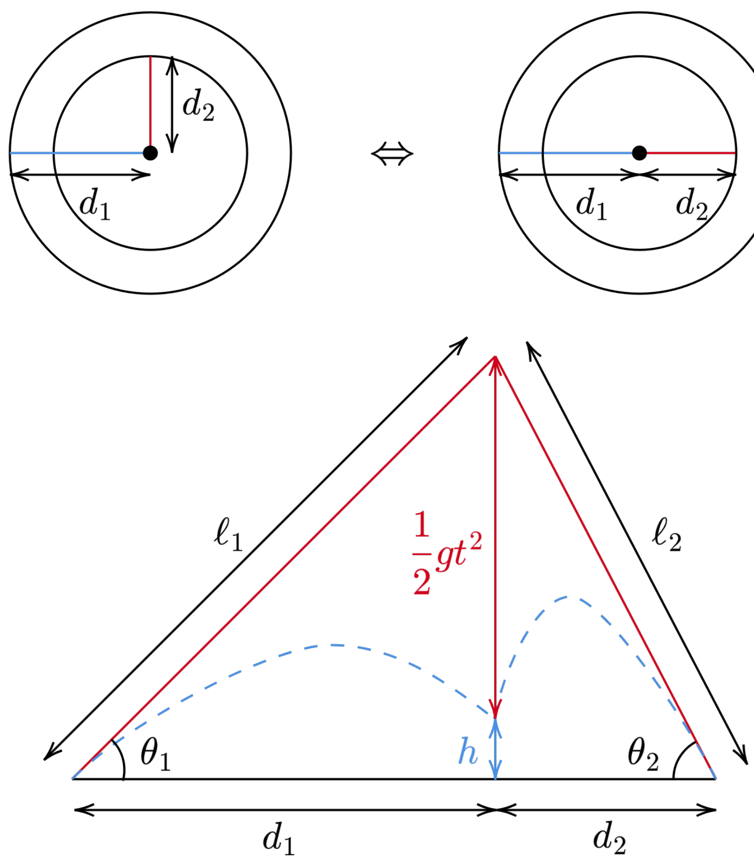
Problem 31: High Level Golf (4 points)

Brian and Chris are playing a game of golf. Brian's ball is at a distance d_1 from the hole and Chris's ball is at a distance d_2 from the hole. Aiming for the hole and hitting each of their balls simultaneously, they each drive their ball too high; Brian's ball flies with speed v_1 angled $\theta_1 = 45°$ above the ground, while Chris's flies with speed v_2 angled $\theta_2 = 60°$ above the ground. Miraculously, the two balls collide at a height $h = 1$ m perfectly above the hole. Given that $d_1 + d_2 = 10$ m, find $v_1 + v_2$.

Leave your answer to 3 significant figures in units of m s^{-1}.

Solution: As the balls both have the same acceleration due to gravity, we can instead consider what happens when there is no gravity, i.e. the two balls travel with constant velocity. The balls now travel distances of $\ell_1 = v_1 t$ and $\ell_2 = v_2 t$ respectively before colliding, where t is the time of the collision.

Due to the radial symmetry around the hole, we line up the balls' trajectories along a single flat plane to simplify our work:



As seen in the second diagram, the collision point (where the lines ℓ_1 and ℓ_2 meet) must fulfil the geometric constraints set by the angles of launch. Hence, by considering its height above the ground:

$$\begin{aligned} d_1 \tan 45^\circ &= d_2 \tan 60^\circ \\ d_1 &= \sqrt{3} d_2 \\ &= \frac{\sqrt{3}}{1+\sqrt{3}}(d_1 + d_2) \\ &= 15 - 5\sqrt{3} \\ d_2 &= \frac{1}{1+\sqrt{3}}(d_1 + d_2) \\ &= 5\sqrt{3} - 5 \end{aligned}$$

From the geometry, we also deduce that:

$$\begin{aligned} d_1 &= \frac{1}{2} g t^2 + h \\ t &= \sqrt{\frac{2(d_1 - h)}{g}} \\ &\approx 1.043 \text{ s} \end{aligned}$$

Consequently:

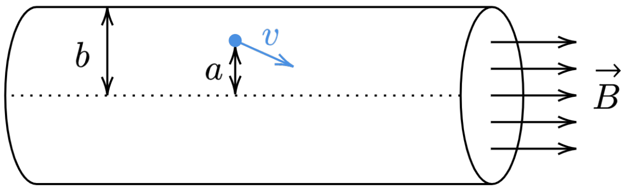
$$\begin{aligned} v_1 + v_2 &= \frac{\ell_1}{t} + \frac{\ell_2}{t} \\ &= \frac{1}{t}\left(\frac{d_1}{\cos 45^\circ} + \frac{d_2}{\cos 60^\circ}\right) \\ &\approx \boxed{15.6 \text{ m s}^{-1}} \end{aligned}$$

Setter: Paul Seow, paul.seow@sgphysicsleague.org

Problem 32: So Close Yet So Far (5 points)

A uniform magnetic field of field strength $B = 0.500$ T runs parallel to the axis of a long insulating cylindrical shell of radius $b = 35.0$ m. A charged particle with mass $m = 0.0500$ kg and charge $q = 0.100$ C is initially positioned at a distance $a = 10.0$ m away from the axis of the cylinder. The particle is launched with speed $v = 20.0$ m s^{-1} in an arbitrary direction. What is the minimum time taken t for the particle to reach the wall of the cylinder?

Leave your answer to 3 significant figures in units of s.



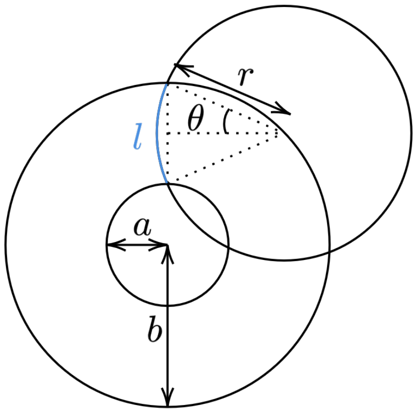
Solution: Firstly, we note that the velocity of the particle should be perpendicular to the cylinder axis to minimise the time taken to reach the wall of the cylinder. This restricts the motion of the particle to a 2D plane, where it is launched from some position inside a circle of radius b and its path has to intersect the circle. The only variable we can vary is the direction of launch, since just about everything else is fixed.

It might be tempting to think that the shortest path is simply a straight line. However, due to the presence of the uniform magnetic field perpendicular to the plane of motion, the particle will undergo circular motion and its path is an arc of a circle. A brute-force approach of expressing the coordinates of the circular path and determining where it intersects with the outer circle is possible, but would be extremely tedious. Thus, let us try to geometrically determine the optimal path.

We notice that since velocity is constant, the time taken increases with the length of the arc. Since the radius of the circular path is constant, the length of the arc will increase with the length of the chord connecting the initial and final point in the path. Hence, we see that the path of shortest time would simply be the arc for which the chord is the shortest straight line from the particle's starting position to the wall of the cylinder.

We equate the magnetic force to the centripetal force to obtain the radius of curvature of the path.

$$\frac{mv^2}{r} = Bvq$$
$$r = \frac{mv}{Bq}$$



Now, we will obtain the length of the path the ball travels, l.

$$\begin{aligned}\sin\theta &= \frac{b-a}{2r}\\ l &= 2r\theta\\ &= 2r\sin^{-1}\left(\frac{b-a}{2r}\right)\\ &= \frac{2mv}{Bq}\sin^{-1}\left(\frac{Bq(b-a)}{2mv}\right)\end{aligned}$$

The time taken is therefore

$$t = \frac{l}{v} = \frac{2m}{Bq}\sin^{-1}\left(\frac{Bq(b-a)}{2mv}\right) \approx \boxed{1.35\text{ s}}$$

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Problem 33: Problematic Proton (5 points)

Alice carries out experiments in a spherical charged gas cloud with radius $R = 5.0$ m and uniform volume charge density $+\rho$. She releases an electron at rest at a distance of $r_0 = 2.0$ m from the centre of the cloud, and notices that it performs oscillatory motion with period $T_1 = 0.60$ s.

However, one day she accidentally releases a proton from the same position, and notices that it reaches the surface of the gas cloud in a time T_2. Determine T_2.

Neglect any gravitational effects and collisions between the electron/proton and gas particles, and assume that the gas particles remain stationary.

Leave your answer to 2 significant figures in units of s.

Solution: We can determine the electric field inside a charged sphere using Gauss' Law.

$$4\pi r^2 E = \frac{\rho \frac{4}{3}\pi r^3}{\varepsilon_0}$$
$$E = \frac{\rho r}{3\varepsilon_0}$$

The acceleration experienced by the electron is:

$$\ddot{r} = -\frac{e\rho}{3\varepsilon_0 m_e} r$$

This equation is in the form $\ddot{r} = -\Omega^2 r$, so the electron performs simple harmonic motion about the centre of the sphere. The angular frequency Ω is given by:

$$\Omega = \sqrt{\frac{e\rho}{3\varepsilon_0 m_e}}$$

If a proton is used, the acceleration experienced is away from the centre of the sphere.

$$\ddot{r} = \frac{e\rho}{3\varepsilon_0 m_p} = \Omega^2 \frac{m_e}{m_p} r$$

This equation is very similar to the equation for simple harmonic motion. Although the negative sign is absent, we can use the fact that $i = \sqrt{-1}$ and proceed with solving the equation. Using the initial conditions, $r = r_0, \dot{r} = 0$, we can write the general solution:

$$r = r_0 \cos\left(i\Omega\sqrt{\frac{m_e}{m_p}} t\right)$$

Using the identity that $\cos ix = \cosh x$, we have:

$$r = r_0 \cosh\left(\Omega\sqrt{\frac{m_e}{m_p}}t\right)$$

Then, substituting $r = R$ and $t = T_2$, we have:

$$T_2 = \frac{1}{\Omega}\sqrt{\frac{m_p}{m_e}}\cosh^{-1}\left(\frac{R}{r_0}\right) = \frac{T_1}{2\pi}\sqrt{\frac{m_p}{m_e}}\cosh^{-1}\left(\frac{R}{r_0}\right) \approx \boxed{6.4\text{ s}}$$

Alternative solution: For those unfamiliar with hyperbolic trigonometry, we can solve the equation with other methods, one of which is presented here. A differential equation of the form $\ddot{r} = \omega^2 r$ is solved by the general solution, $x = Ae^{\omega t} + Be^{-\omega t}$. The initial conditions for the proton are $r = r_0, \dot{r} = 0$, so we have $A = B = \frac{1}{2}r_0$. Then, taking $\omega = \sqrt{\frac{e\rho}{3\varepsilon_0 m_p}}$:

$$r = \frac{1}{2}r_0\left(e^{\omega t} + e^{-\omega t}\right)$$

Substituting $r = R$, $t = T_2$, and solving a quadratic equation, we obtain:

$$e^{\omega T_2} = \frac{R}{r_0} + \sqrt{\left(\frac{R}{r_0}\right)^2 - 1}$$

$$T_2 = \frac{1}{\omega}\ln\left(\frac{R + \sqrt{R^2 - r_0^2}}{r_0}\right)$$

Notice that

$$\omega = \Omega\sqrt{\frac{m_e}{m_p}} = \frac{2\pi}{T_1}\sqrt{\frac{m_e}{m_p}}$$

We can substitute this expression into our previous equation to obtain

$$T_2 = \frac{T_1}{2\pi}\sqrt{\frac{m_p}{m_e}}\ln\left(\frac{R + \sqrt{R^2 - r_0^2}}{r_0}\right) \approx \boxed{6.4\text{ s}}$$

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Problem 34: Triple Bounce

(a) A ball is projected horizontally from the top of an edge of a square pit with side length ℓ. Consider the following scenarios:

A: an initial velocity v_A makes the ball bounce three times only on the base, before reaching the top of the opposite edge.

B: an initial velocity v_B makes the ball bounce exactly once on each of the three sides (including the base), before reaching the top of the opposite edge.

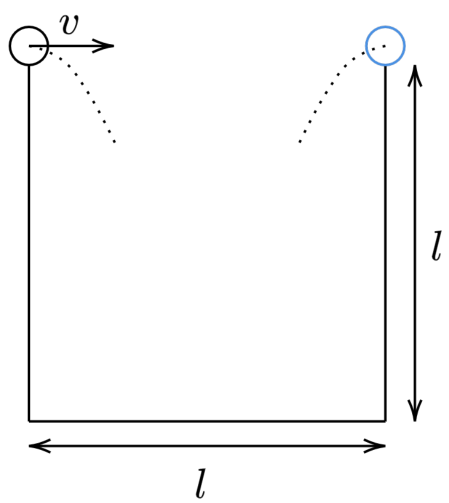
Find the ratio v_A/v_B.

Leave your answer to 3 significant figures. (3 points)

(b) Scenario B is now modified such that the ball is projected at an angle θ below the horizontal, bouncing off each of the three sides exactly once before reaching the top of the opposite edge. In addition, its maximum height during its journey exceeds its initial height by $\ell/2$. Find θ.

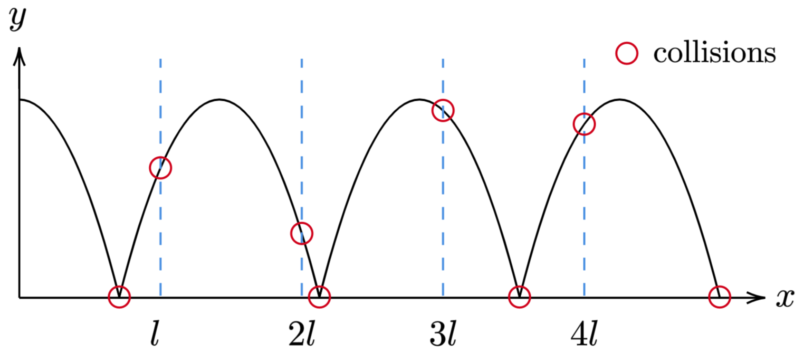
Leave your answer to 3 significant figures in units of degrees. (3 points)

Assume all collisions are elastic, and no spin is imparted to the ball.



Solution:

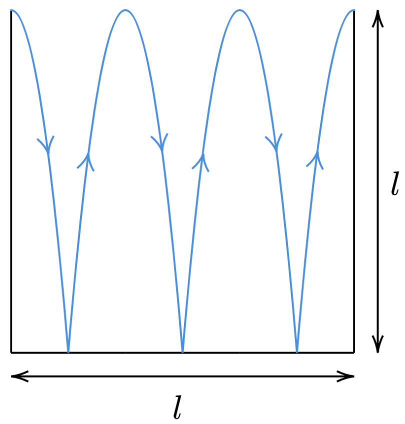
(a) Since all collisions are elastic, the direction of the x-velocity flips while the y-velocity remains unchanged. Hence, we may analyse the trajectory of the ball as if it were a series of identical parabolas. The velocity will then affect the width of the parabola, with $x = \ell, 2\ell, \ldots$ being subsequent collisions with vertical walls. Taking the bottom of our pit as $y = 0$, we can plot the ball's trajectory in the following manner:



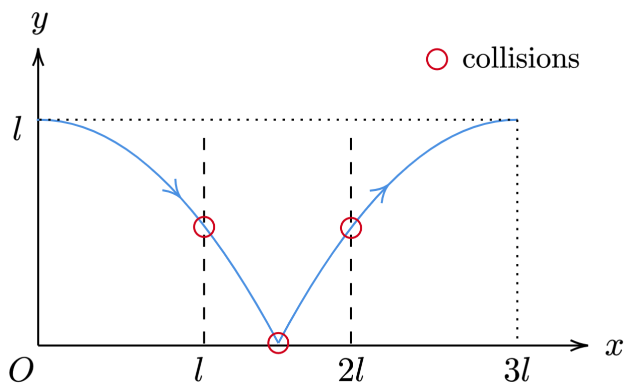
Since the ball is projected horizontally, i.e. with no vertical component of velocity, the time it takes to travel from the top of the pit to the base is constant, which we shall define as T (representing a single half-parabola). Hence, each parabola (from root to root) defines a time period of $2T$.

In **Scenario A**, the ball's trajectory looks like the following image. As it takes 6 half-parabolas to reach the other edge,

$$v_A = \frac{\ell}{6T}$$



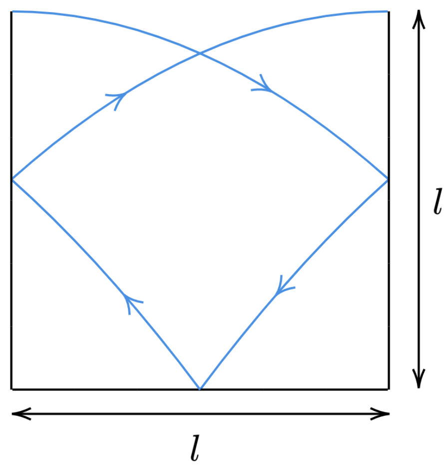
In **Scenario B**, the ball collides with two vertical walls; hence, the ball must have travelled 3ℓ horizontally. As it may only collide with the ground once, we first suppose that the ball collides with the ground prior to the opposite wall — this means it must collide with the ground again before reaching 3ℓ, which is a contradiction. If it instead collides with the original wall before the ground, the ground collision point will overshoot the halfway point and hence the ball will not reach the top at 3ℓ. Hence, we may deduce the following trajectory:



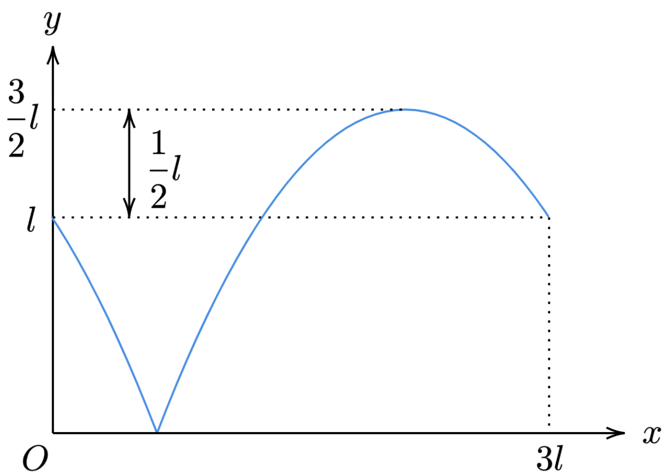
Since this consists of two half-parabolas, It represents a time taken of $2T$; hence,

$$v_B = \frac{3\ell}{2T}$$
$$\frac{v_A}{v_B} = \frac{1}{9} \approx \boxed{0.111}$$

The path may be visualised in the following manner:



(b) Similarly to Scenario B, the ball must travel 3ℓ horizontally. However, the parabola must now have height $3\ell/2$ above the bottom of the pit:



We now wish to derive expressions for v_x and v_y at $(x, y) = (0, \ell)$. Notice that $|v_y|$ is the same at $(0, \ell)$ and $(3\ell, \ell)$, and that $v_y = 0$ at $y = 3\ell/2$. Hence,

$$|v_y| = \sqrt{0^2 + 2g\frac{\ell}{2}} = \sqrt{g\ell}$$

Letting T be the time taken for the ball to travel its full motion and v_y' the vertical velocity at the bottom of the pit, we see that

$$\begin{aligned} v_y' &= \sqrt{0^2 + 2g\frac{3\ell}{2}} \\ &= \sqrt{3g\ell} \\ T &= \frac{v_y'}{g} \\ &= \sqrt{\frac{3\ell}{g}} \end{aligned}$$

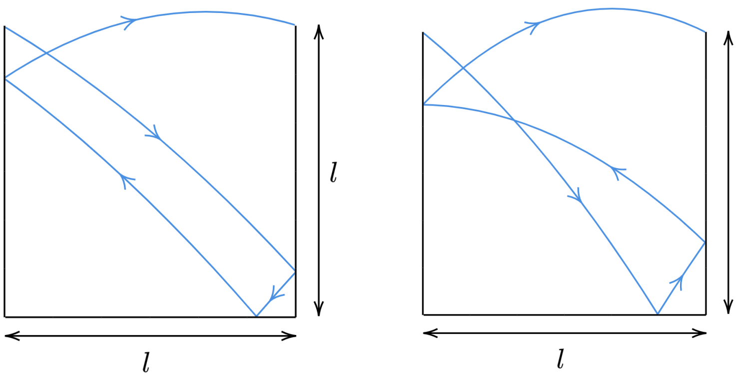
Since the ball completes two half-parabolas,

$$\begin{aligned} v_x &= \frac{3\ell}{2T} \\ &= \frac{\sqrt{3g\ell}}{2} \end{aligned}$$

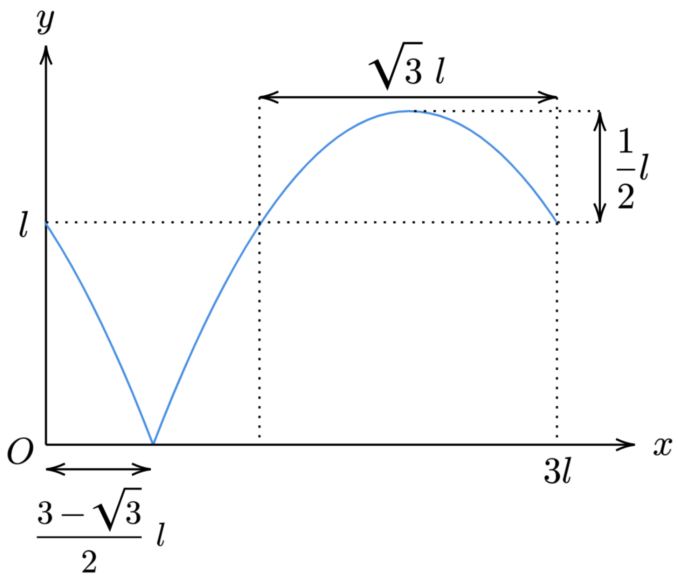
Hence,

$$\begin{aligned} \theta &= \tan^{-1}\frac{v_y}{v_x} \\ &= \tan^{-1}\frac{2}{\sqrt{3}} \\ &\approx \boxed{49.1^\circ} \end{aligned}$$

Note: For part (b), two possible paths exist - the ball may collide with the ground first, or the opposite wall first. This does not affect the solution of (b), as we are able to approach it identically regardless of the path and later verify its correctness.



We can deduce that P2 is the correct path, by geometrically finding the collision point with the ground.



Since $\frac{3-\sqrt{3}}{2} < 1$, the ball collides with the ground before reaching the opposite wall, and hence P2 is the correct path.

Setter: Gerrard Tai, gerrard.tai@sgphysicsleague.org

Problem 35: Hard Work (5 points)

Bob is trying to pump a ball. Model the pump as a vessel with initial volume $V_0 = 2.5 \times 10^{-5}$ m^3 and the ball as a vessel with constant volume $V_1 = 7.0 \times 10^{-3}$ m^3.

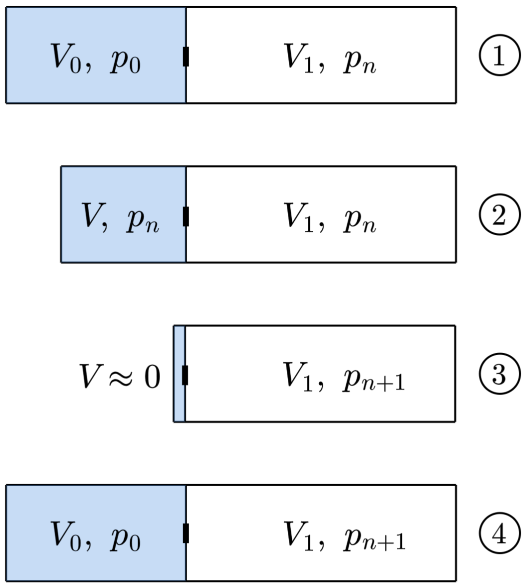
Initially, the pump is empty, and the ball is at atmospheric pressure p_0. During each pumping process,

1. Air from the atmosphere first fills up the pump to pressure p_0.
2. Bob then pushes down on the pump handle, causing the pump's volume to contract until the pressure in the pump is equal to the pressure in the ball.
3. A valve connecting the pump to the ball then opens, and Bob pushes down again until the volume of the pump is zero.
4. The valve is closed, and the pump volume is restored to V_0 by letting air from the atmosphere fill it up. The cycle then repeats itself.

Assume that air molecules in the atmosphere are diatomic, and all compressions are adiabatic. Determine the ratio $\frac{p_{10}}{p_1}$.

Leave your answers to 3 significant figures.

Hint: The pressure in the ball after the n-th cycle can be expressed as $p_n = p_0 a_n^\gamma$ where $a_n(n)$ is some function of n and $\gamma = 1.4$ is the heat capacity ratio of atmospheric air.



Solution: We can summarise the pumping process in the n-th cycle into three key steps. First, the air in the pump undergoes an adiabatic compression from volume V_0 to some volume V_n, such that the pressure in the pump goes from p_0 to p_{n-1}.

$$p_0 V_0^\gamma = p_{n-1} V_n^\gamma \tag{1}$$

Second, a valve opens, allowing the air from the two vessels to mix. The pressure in the ball and pump system remains unchanged at p_{n-1}.

Lastly, the air in the pump and ball system undergoes an adiabatic compression from volume $V_n + V_1$ to volume V_1, resulting in the pressure going from p_{n-1} to p_n.

$$p_{n-1}(V_n + V_1)^\gamma = p_n V_1^\gamma \tag{2}$$

Equations (1) and (2) can be solved to obtain a recurrence relation for p_n.

$$p_n = p_{n-1}\left(1 + \frac{V_0}{V_1}\left(\frac{p_{n-1}}{p_0}\right)^{-\frac{1}{\gamma}}\right)^\gamma$$

To solve this, we can use the hint given in the question. Substituting the hint into the recurrence relation,

$$a_n^\gamma = a_{n-1}^\gamma\left(1 + \frac{V_0}{V_1 a_{n-1}}\right)^\gamma$$

$$a_n = a_{n-1} + \frac{V_0}{V_1}$$

Remarkably, a_n is a linear function of n. Since $a_0 = 1$, we have

$$a_n = 1 + \frac{V_0}{V_1}n$$

$$p_n = p_0\left(1 + \frac{V_0}{V_1}n\right)^\gamma$$

$$\frac{p_{10}}{p_1} = \frac{\left(1 + 10\frac{V_0}{V_1}\right)^\gamma}{\left(1 + \frac{V_0}{V_1}\right)^\gamma} \approx \boxed{1.05}$$

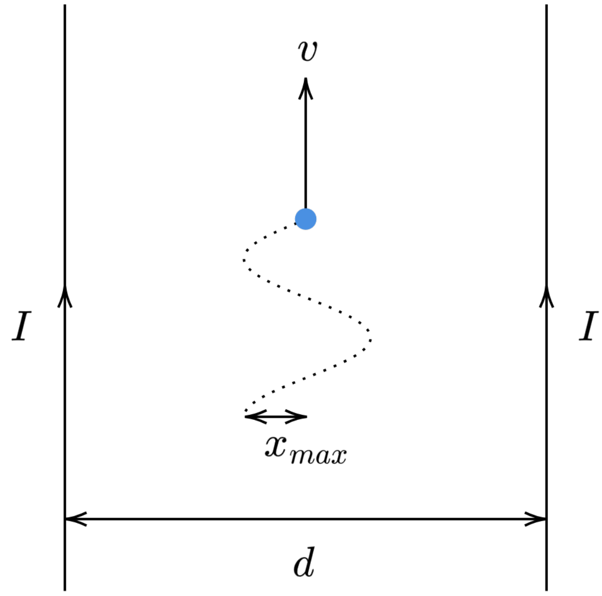
Setter: Chen Guangyuan, guangyuan.chen@sgphysicsleague.org

Problem 36: Approximate Magnetic Oscillation (5 points)

Two parallel, infinitely long wires carrying current $I = 3.40$ mA upwards are fixed at a large distance $d = 1.50$ m apart. An electron is in the same plane, halfway between the two wires, with velocity $v = 5.00$ m s^{-1} upwards. The electron is given a slight horizontal displacement such that it exhibits simple harmonic motion along the horizontal axis. Find the period of small oscillations, T.

Assume that both gravity and the vertical magnetic force are negligible. Additionally, as d is extremely large, assume the amplitude of oscillations $x_{\max} \ll d$.

Leave your answer to 3 significant figures in units of s.



Solution: Let us define the electron's horizontal displacement as $\vec{x}$, where its equilibrium position (also its starting position) is $x = 0$, taking rightward as positive. The magnetic field at a point due to an infinitely long wire is

$$B = \frac{\mu_0 I}{2\pi r}$$

where r is the perpendicular distance from the wire to the point. The direction of $\vec{B}$ can be determined via the Right Hand Grip Rule.

Since the magnetic fields contributed by each wire act in opposite directions, we consider the difference of the two fields after a small displacement x is introduced to the right. The magnetic field becomes:

$$\begin{aligned} B_{\text{net}} &= \frac{\mu_0 I}{2\pi}\left(\frac{1}{\frac{d}{2}-x}-\frac{1}{\frac{d}{2}+x}\right) \\ &= \frac{\mu_0 I}{\pi d}\left(\frac{1}{1-\frac{2x}{d}}-\frac{1}{1-\frac{2x}{d}}\right) \end{aligned}$$

and it points out of the page. Applying a first order approximation of $(1+x)^{-1} \approx 1-x$:

$$\begin{aligned} B_{\text{net}} &\approx \frac{\mu_0 I}{\pi d}\left(1+\frac{2x}{d}-1+\frac{2x}{d}\right) \\ &= \frac{4\mu_0 I x}{\pi d^2} \end{aligned}$$

Since $\vec{F}_{\text{net}} = -e\vec{v} \times \vec{B}_{\text{net}}$, we can use Fleming's Left Hand Rule to determine that the force will point to the left (i.e. opposite to x). Hence,

$$m_e \ddot{x} = -\frac{4ev\mu_0 I}{\pi d^2}x$$

This is a simple harmonic motion equation as it is of the form $\ddot{x} = -\omega^2 x$; hence, the particle's angular frequency is:

$$\omega = \sqrt{\frac{4ev\mu_0 I}{m_e \pi d^2}}$$

$$T = \frac{2\pi}{\omega} \approx \boxed{0.136 \text{ s}}$$

Remarks: Trust me, everything is a harmonic oscillator.

Setter: Gerrard Tai, gerrard.tai@sgphysicsleague.org

Problem 37: Daredevil Paul

Paul the daredevil is trying to impress his girlfriend with his newest stunt — running off a tall tower! Of course, he has no intention of dying, so he reassures her he will run fast enough such that he can make it around the Earth and come back without colliding with the ground. He stands on a tower of height $h = R$ above the surface of the Earth, where R is the radius of the Earth.

Take the mass of Earth to be $M = 5.97 \times 10^{24}$ kg and the radius of Earth to be $R = 6370$ km.

(a) Assuming that he runs off the tower horizontally, what is the minimum velocity u which he needs to run at to survive?

Leave your answer to 2 significant figures in units of m s^{-1}. (4 points)

(b) Sadly, due to a skill issue, Paul only runs at $u' \equiv \eta u = 0.6u$. Compute v_n, the normal component of the velocity with which he hits the ground.

Leave your answer to 2 significant figures in units of m s^{-1}. (3 points)

Solution:

(a) When Paul runs at his minimum speed, he barely grazes the surface of the Earth on the other side of his orbit (see Fig. 1). Call Paul's speed at perigee[4] v. We consider the conservation of energy and angular momentum, evaluating the values of both integrals of motion at perigee and apogee:

$$\underbrace{\frac{1}{2}\cancel{m}u^2 - \frac{GM\cancel{m}}{h+R}}_{\text{energy at apogee}} = \underbrace{\frac{1}{2}\cancel{m}v^2 - \frac{GM\cancel{m}}{R}}_{\text{energy at perigee}}$$

$$\underbrace{\cancel{m}u(h+R)}_{\text{angular momentum at apogee}} = \underbrace{\cancel{m}vR}_{\text{angular momentum at perigee}}$$

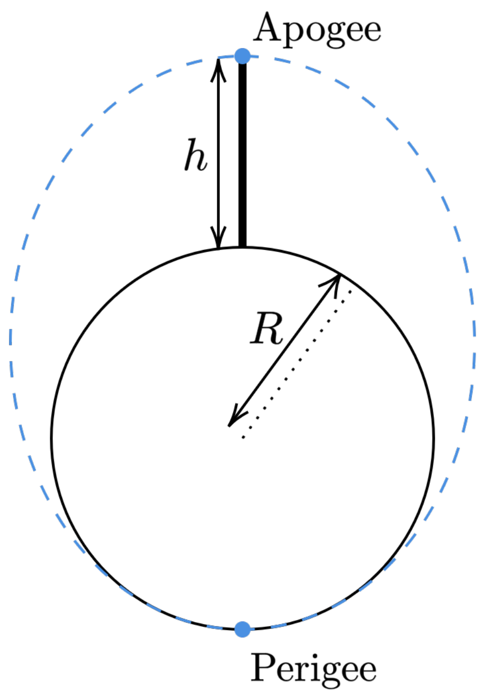
We use the second equation to eliminate v in the first, and multiplying by 2 throughout yields:

$$u^2 - \frac{2GM}{h+R} = u^2\left(\frac{h+R}{R}\right)^2 - \frac{2GM}{R}$$

Grouping terms once again yields:

$$2GM\left(\frac{1}{R} - \frac{1}{h+R}\right) = u^2\left[\left(\frac{h+R}{R}\right)^2 - 1\right]$$

[4]The perigee in an orbit around Earth is the point at which the object is closest to the center of the Earth, and the apogee is the point at which it is the furthest. In general, we use the words periapsis and apoapsis for general orbits; perigee and apogee apply only to orbits around Earth.



Substituting $h = R$ into the general case, we see a great simplication in our equation:

$$\frac{GM}{R} = 3u^2$$

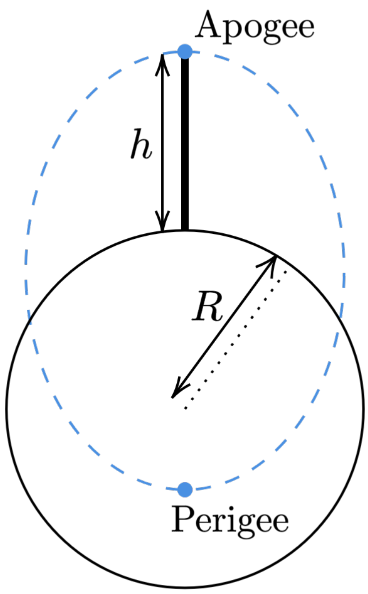
$$u = \sqrt{\frac{GM}{3R}} = \sqrt{\frac{(6.67 \times 10^{-11}\ \text{m}^3\ \text{kg}^{-1}\ \text{s}^{-2})(5.97 \times 10^{24}\ \text{kg})}{3(6370000\ \text{m})}} \approx \boxed{4600\ \text{m s}^{-1}}$$

(b) Part (b) is solved using very much the same concepts. However, now we do not compare perigee and apogee. The point where Paul starts still remains the apogee of the orbit, as the velocity then is perpendicular to the position vector from the center of the Earth to Paul. However, the second point we are interested in is the point at which Paul collides with the Earth.

While we cannot easily find the position of this point this is not required. We know that at this point, Paul is a distance R away from the center of the Earth. We break his unknown velocity into its normal component v_n and tangential component v_t. Since we assume the Earth is spherical (as opposed to, say, flat) the normal component is also the radial component. Our equations for the conservation of energy and angular momentum are now:

$$\frac{1}{2}\cancel{m}u'^2 - \frac{GM\cancel{m}}{h+R} = \frac{1}{2}\cancel{m}(v_t^2 + v_n^2) - \frac{GM\cancel{m}}{R}$$

$$\cancel{m}u'(h+R) = \cancel{m}v_t R$$



Again, we use the second equation to eliminate v_t, and multiply by two throughout:

$$u'^2 - \frac{2GM}{h+R} = v_n^2 + u'^2\left(\frac{h+R}{R}\right)^2 - \frac{2GM}{R}$$

$$v_n^2 = 2GM\left(\frac{1}{R} - \frac{1}{h+R}\right) + \eta^2 u^2\left[1 - \left(\frac{h+R}{R}\right)^2\right]$$

Applying the special value $h = R$, we get:

$$v_n^2 = \frac{GM}{R} - 3\eta^2 u^2$$

$$v_n \approx \boxed{6300 \text{ m s}^{-1}}$$

Alternatively, we could simplify the above expression to:

$$v_n^2 = \frac{GM}{R}(1 - \eta^2)$$

Some further insights: One cool fact that was noticed in the setting of this problem was that the answer is sensitive to the value of η. If we take the derivative of v_n with respect to η, we get:

$$\frac{\mathrm{d}v_n}{\mathrm{d}\eta} = \frac{-6\eta u^2}{\sqrt{\frac{GM}{R} - 3\eta^2 u^2}}$$

The denominator is equal to v_n and we have:

$$\lim_{\eta \to 0} v_n = 0$$

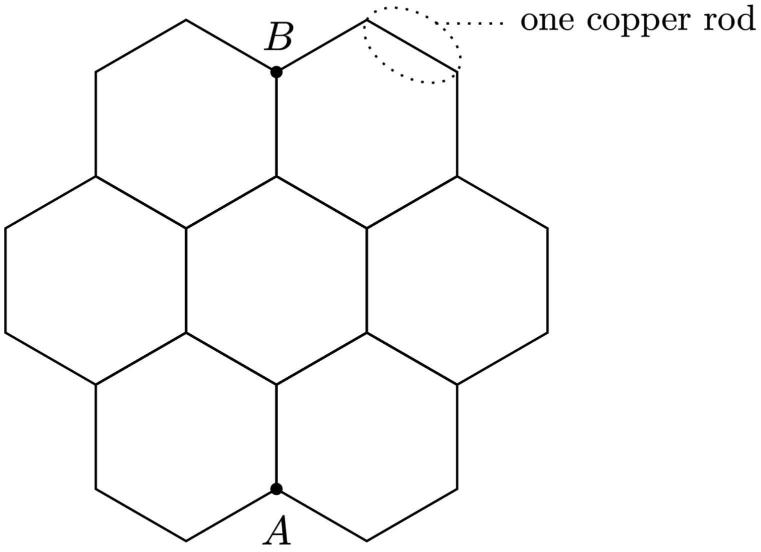
Hence, the derivative explodes as $\eta \to 1$. In fact, if $\eta = 0.9$ (which was the original value in the question), the rounding errors can exceed the second significant figure. prompting a modification to $\eta = 0.6$. Fret not, this is still an underestimate of Paul's skill issues.

Setter: Tan Jun Wei, junwei.tan@sgphysicsleague.org

Problem 38: Hexagonmania (5 points)

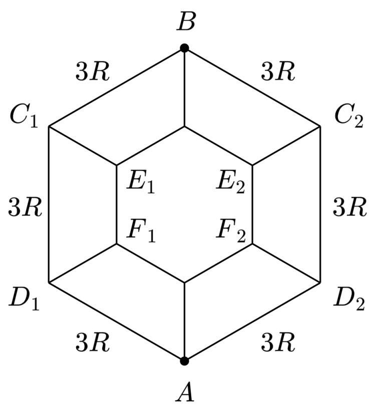
Roger is bored, so he decides to use his collection of uniform thin copper rods, each of resistance $R = 1.00\ \Omega$, to create a rigid compound shape shown below. The copper rods form seven regular hexagons. Calculate the effective resistance R_{AB} between points A and B.

Leave your answer to 3 significant figures in units of Ω.



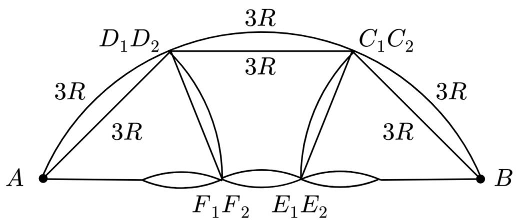
Solution:

Although intimidating on first sight, if we patiently simplify the diagram step by step, the answer can be obtained. We begin by adding up the outer copper rods in series, obtaining the following setup. Note that all unlabelled segments have resistance R.

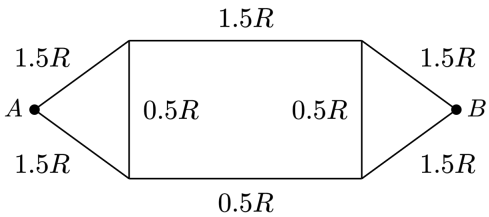


Next, we need to make a few observations. We know that different points in a circuit with the same potential can be connected or disconnected to our convenience. Observing our above setup, and taking into account its symmetry, we deduce that points

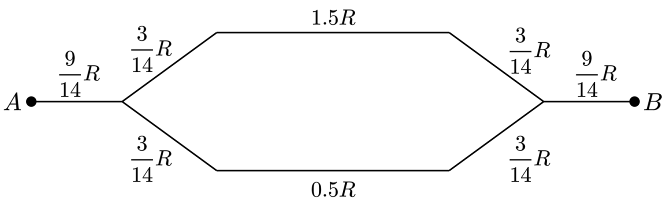
C_1 and C_2 have the same potential, and similarly for points D_1 and D_2, E_1 and E_2, and F_1 and F_2. We then connect each pair of equipotential points, which "folds" the setup in half:



After collapsing parallel segments and adding up series segments again, we obtain a remarkably simple setup:



However, simple as it looks, we are actually unable to simplify the setup further through connecting or disconnecting equipotential points because no points in the above setup are at the same potential. This is where the $\Delta - Y$ transformation comes in. Applying the prescribed formula for the transformation, we obtain the following setup:



Finally, through some hard work and determination, we are left with something we can solve by hand. After simplification and substituting in $R = 1.00\ \Omega$, we obtain:

$$R_{AB} = \frac{153}{80}\Omega \approx \boxed{1.91\Omega}$$

Setter: Roger Zhang, roger.zhang@sgphysicsleague.org

Problem 39: Quantum Tunnelling (4 points)

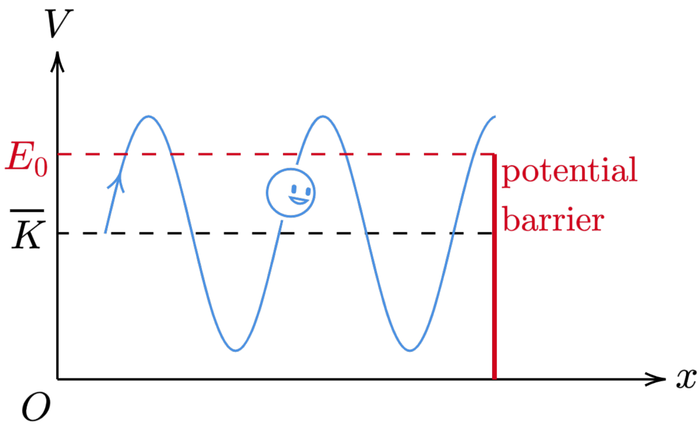
A particle of mass $m = 100$ g is projected at velocity v_0 towards a potential barrier of height $E_0 = 0.07000$ J. When the particle is near the potential barrier, Paul turns on his oscillator, and the particle begins to oscillate such that its velocity is given by $v(t) = v_0 + \epsilon\omega\cos(\omega t + \varphi)$, where $\omega = 440$ Hz and $\epsilon\omega = 0.5$ m s^{-1}.

We measure the average kinetic energy and find it to be $\bar{K} = 0.05625$ J. However, we cannot determine the exact kinetic energy of the particle when it hits the barrier, and thus cannot know for certain if it will pass through. What we can determine is the probability that the particle passes through the barrier, p, which you should give as your answer.

You may assume that ϵ is small, but you may not assume that $\epsilon\omega$ is. However, you may take $\epsilon\omega < v_0$ (the particle does not reverse direction due to Paul's oscillator).

Leave your answer to 2 significant figures.

Your answer should range between 0 and 1.



Solution: The kinetic energy is given by $\frac{1}{2}m(v_0 + \epsilon\omega\cos\xi)^2$, where $\xi = \omega t + \varphi$. A period corresponds to a motion of ξ through 2π. Expanding this gives

$$K = \frac{1}{2}mv_0^2 + \frac{1}{2}m\epsilon^2\omega^2\cos^2(\xi) + mv_0\epsilon\omega\cos\xi$$

First, we seek to find the average kinetic energy $\bar{K}$. Under a time average, the last term vanishes, while the first term is unaffected. The average of $\cos^2$ is just $\frac{1}{2}$, from which we get

$$\bar{K} = \frac{1}{2}mv_0^2 + \frac{1}{4}m\omega^2\epsilon^2 \tag{1}$$

This allows us to solve for the value of v_0 from $\bar{K}$, which we will use as if given for the rest of this solution.

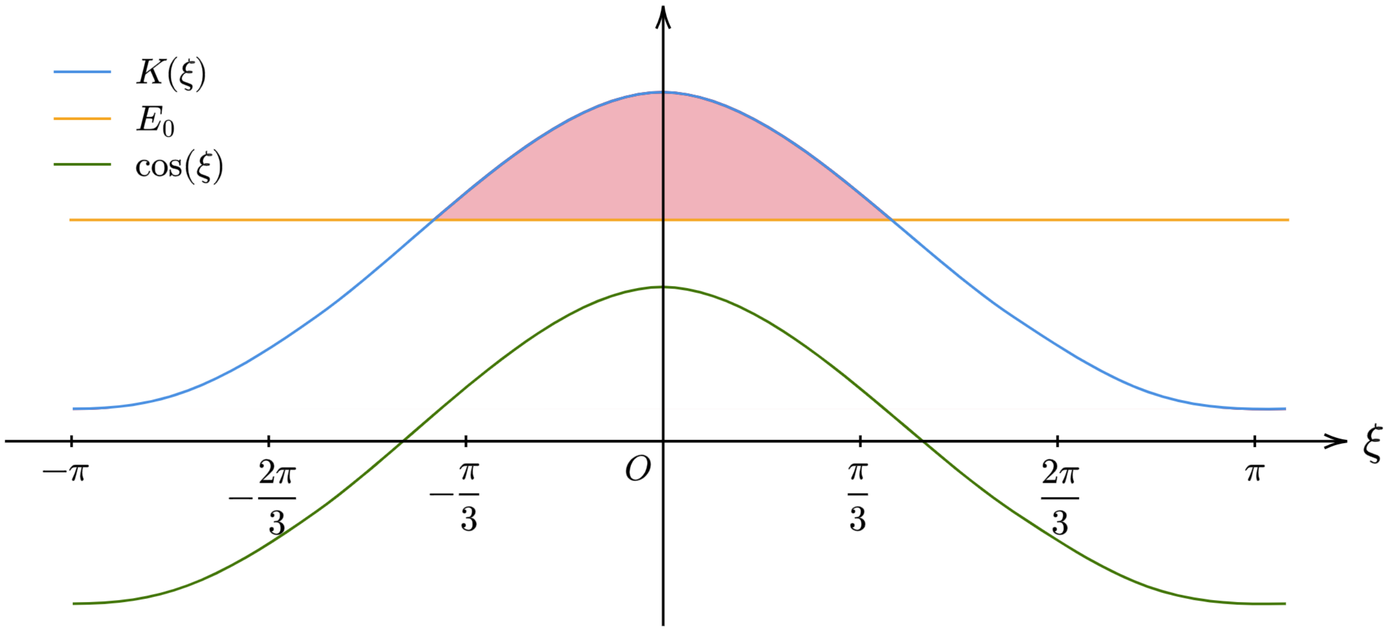


Figure 1: Graphs of $K(\xi)$, E_0 and $\cos\xi$ are plotted here for the parameters given in the problem. The region for which $K(\xi) > E_0$ is shaded in red. Notice that despite there being two ξ values where $K(\xi) = E_0$ there is only one $\cos\xi$ value. The possible existence of more values of $\cos\xi$ is left as an exercise to the reader.

The particle tunnels through if and only if the kinetic energy is greater than E_0. As the magnitude of oscillation ϵ is small, the probability of colliding with the barrier is independent of phase angle, and thus we only need to determine the range of phase angles for which the energy is higher than E_0. The total measure of the intervals where $K(\xi) \geq E_0$, divided by 2π, is then the probability of tunnelling. A graph illustrating this is given in Fig 1.

We seek to find ξ such that $K = E_0$. This yields a quadratic equation in $\cos\xi$:

$$\epsilon^2\omega^2\cos^2\xi + 2v_0\epsilon\omega\cos\xi + v_0^2 - \frac{2E_0}{m} = 0$$

This can be solved for the two roots:

$$\cos\xi = -\frac{v_0}{\epsilon\omega} \pm \frac{1}{2\epsilon\omega}\sqrt{\frac{8E_0}{m}}$$

We now have a problem - if you plotted a figure like Fig. 1, we should only have one root (corresponding to two values of ξ). However, we see two roots here. The only possible conclusion is that one root does not lead to real values of ξ. It can be shown that if only one root is in $[-1, 1]$, then it must be the positive root. Determining the conditions under which there are two roots in $[-1, 1]$ is identical to the exercise left to the reader in the caption of Fig. 1. Thus, the interval in which $K(\xi) > E_0$ is given by $[-\theta, \theta]$, where:

$$\theta = \arccos\left(-\frac{v_0}{\epsilon\omega} + \frac{1}{2\epsilon\omega}\sqrt{\frac{8E_0}{m}}\right)$$

Hence, the probability p is given by $\frac{\cancel{2}\theta}{\cancel{2}\pi} = \frac{\theta}{\pi}$. Numerically, we have $v_0 = 1\ \mathrm{m\ s^{-1}}$, $\theta \approx 1.196$ and $p \approx \boxed{0.38}$.

At this point, the author wishes to lament about a more formal approach that ultimately failed. Formally, the probability that the particle is found within an interval of energy E and $E + \mathrm{d}E$, is given by $\frac{1}{T}\sum \frac{1}{\mathrm{d}E/\mathrm{d}t}$, where the sum runs over all points with the same E. Integrating this will give the probability that the particle is found with a certain energy. However, the inversion of the (multivalued) functions found in this problem is too difficult.

The essential idea that was supposed to be expressed using this problem is that quantum tunnelling and superposition are not sufficient to establish quantum mechanics as different from classical mechanics. While the most unintuitive idea of quantum mechanics is allegedly its stochastic nature, this problem shows that a quantum system cannot effectively be distinguished from a classical one which is unpredictable due to our lack of knowledge over its state at any time t. Instead, it is entanglement and the Bell's Inequalities that underlie what is "truly" quantum.

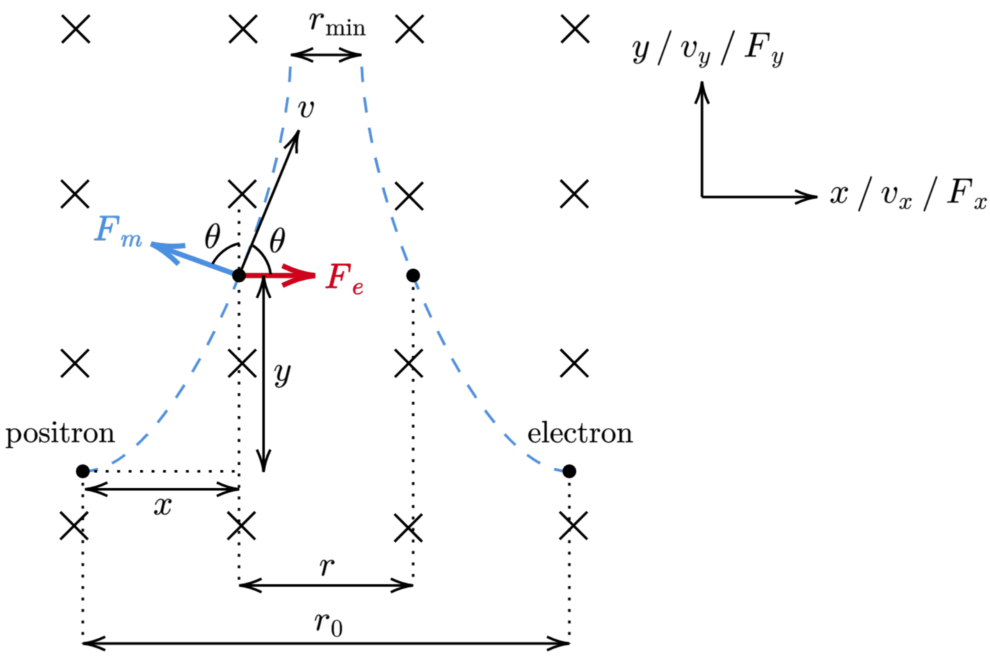
Setter: Tan Jun Wei, junwei.tan@sgphysicsleague.org

Problem 40: Crazy Electron (6 points)

An electron and a positron are separated by distance $r_0 = 100\ \mu\text{m}$ in a region of uniform magnetic field $B = 1.00$ mT that is perpendicular to the line joining both charges. Given that the two charges are released simultaneously from rest, find the minimum distance r_{min} achieved between them throughout their motion.

Leave your answer to 3 significant figures in units of μm.

Solution: The motion paths of the electron and positron are shown in in the diagram below. Since both particles have equal but opposite charges, and the same mass, the forces acting on them are symmetrical, and so their paths are symmetrical.



Without loss of generality, we consider the motion of the positron. Let the positron have charge q and mass m. The two forces acting on the positron are the electrostatic force F_e and the magnetic force F_m. They are each given by:

$$F_e = \frac{kq^2}{r^2}$$
$$F_m = Bqv$$

Because the magnetic force acting on the positron is always perpendicular to its velocity, the magnetic force does no work on the positron. Hence the gain in kinetic energy of the positron is only due to the loss of electric potential energy:

$$\begin{aligned} EPE_i &= EPE_f + KE_f \\ -\frac{q^2}{4\pi\varepsilon_0 r_0} &= -\frac{q^2}{4\pi\varepsilon_0 r} + 2\left(\frac{1}{2}mv^2\right) \\ &= -\frac{q^2}{4\pi\varepsilon_0 r} + m\left({v_x}^2 + {v_y}^2\right) \end{aligned}$$

When the distance between the positron and the electron is a minimum, $v_x = 0$. Hence, we need to find v_y, which is done by considering the force along the y-axis (due only to the magnetic force):

$$\begin{aligned} F_y &= F_m \cos\theta \\ &= Bqv_x \\ m\frac{\mathrm{d}v_y}{\mathrm{d}t} &= Bq\frac{\mathrm{d}x}{\mathrm{d}t} \\ m\int \mathrm{d}v_y &= Bq\int \mathrm{d}x \\ mv_y &= Bqx \\ v_y &= \frac{Bq}{m}\left(\frac{r_0 - r}{2}\right) \end{aligned}$$

Therefore, we substitute the above expression for v_y to obtain:

$$\begin{aligned} -\frac{q^2}{4\pi\varepsilon_0 r_0} &= -\frac{q^2}{4\pi\varepsilon_0 r} + m{v_y}^2 \\ &= -\frac{q^2}{4\pi\varepsilon_0 r} + \frac{B^2q^2(r_0 - r)^2}{4m} \\ \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{r_0}\right) &= \frac{B^2(r_0 - r)^2}{4m} \\ r^2 - r_0 r + \frac{m}{\pi\varepsilon_0 B^2 r_0} &= 0 \end{aligned}$$

The solution to this quadratic equation is:

$$r = \frac{r_0 - \sqrt{{r_0}^2 - \frac{4m}{\pi\varepsilon_0 B^2 r_0}}}{2}$$

and we take the negative square root of the discriminant since we want the minimum distance. Substituting numerical values, the minimum distance is:

$$r_{\min} \approx \boxed{3.39\ \mu\mathrm{m}}$$

Alternative solution: We can also solve this problem by integrating the force on the positron to get a function for its velocity. The force acting on the positron along the x-axis F_x is:

$$\begin{aligned}
F_x &= \frac{kq^2}{r^2} - F_m \sin\theta \\
&= \frac{kq^2}{r^2} - Bqv_y \\
m\frac{\mathrm{d}v_x}{\mathrm{d}t} &= \frac{kq^2}{r^2} - Bq\left(\frac{Bqx}{m}\right)
\end{aligned}$$

Using $\frac{\mathrm{d}v_x}{\mathrm{d}t} = v_x\frac{\mathrm{d}v_x}{\mathrm{d}x}$ and $r = r_0 - 2x$, we obtain the differential equation:

$$\begin{aligned}
v_x\frac{\mathrm{d}v_x}{\mathrm{d}x} &= \frac{kq^2}{(r_0-2x)^2 m} - \frac{B^2q^2x}{m^2} \\
\int_0^{v_x} v_x \,\mathrm{d}_x &= \int_0^x \left(\frac{kq^2}{(r_0-2x)^2 m} - \frac{B^2q^2x}{m^2}\right)\mathrm{d}x \\
\frac{{v_x}^2}{2} &= \left[\frac{kq^2}{m}\frac{1}{2(r_0-2x)}\right]_0^x - \left[\frac{B^2q^2}{m^2}\frac{x^2}{2}\right]_0^x \\
&= \frac{kq^2}{2m(r_0-2x)} - \frac{kq^2}{2mr_0} - \frac{B^2q^2x^2}{2m^2}
\end{aligned}$$

When the distance between the positron and the electron is a minimum, $v_x = 0$. Hence:

$$\begin{aligned}
\frac{kq^2}{2m(r_0-2x)} - \frac{kq^2}{2mr_0} - \frac{B^2q^2x^2}{2m^2} &= 0 \\
kr_0m - k(r_0-2x)(m) - B^2x^2r_0(r_0-2x) &= 0 \\
2km - B^2{r_0}^2x + 2B^2r_0x^2 &= 0 \\
2x^2 - r_0x + \frac{m}{\pi\varepsilon_0B^2r_0} &= 0
\end{aligned}$$

The solution to this quadratic equation is

$$x = \frac{r_0 + \sqrt{{r_0}^2 - \frac{4m}{\pi\varepsilon_0B^2r_0}}}{4}$$

and we take the positive square root of the discriminant since when r is a minimum,

x is a maximum. Hence

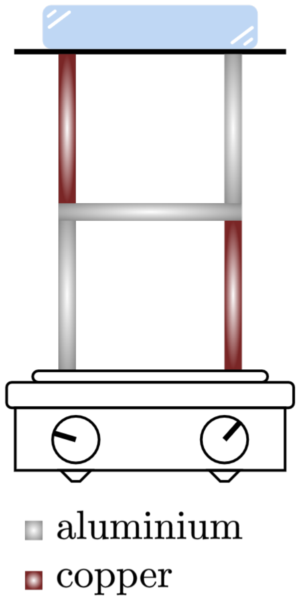
$$\begin{aligned} r_{\text{min}} &= r_0 - 2x \\ &= r_0 - 2\left(\frac{r_0 + \sqrt{{r_0}^2 - \frac{4m}{\pi\varepsilon_0 B^2 r_0}}}{4}\right) \\ &= \frac{r_0 - \sqrt{{r_0}^2 - \frac{4m}{\pi\varepsilon_0 B^2 r_0}}}{2} \\ &\approx \boxed{3.39\ \mu\text{m}} \end{aligned}$$

It is interesting to note that the minimum distance between the two point charges is independent of their charge magnitude.

Setter: Roger Zhang, roger.zhang@sgphysicsleague.org

Problem 41: Thermal H Bar (5 points)

5 bars are joined together to form a H-shape (as pictured below). It comprises two metals: aluminium, of thermal conductivity $k_a = 200$ W m^{-1} K^{-1}, and copper, of thermal conductivity $k_c = 400$ W m^{-1} K^{-1}. Each bar has length $\ell = 0.10$ m, and has a square cross section with width $w = 0.0010$ m.



The system is placed on a hot plate of constant temperature $T_h = 100°$C. A thin conducting sheet with an ice block of constant temperature $T_c = 0°$C is then placed on top of it. What is the rate of heat flow from the hot plate to the ice?

You may assume that no heat is transferred to the air and that the bar's width is negligible compared to its length.

Leave you answer to 2 significant figures in units of W.

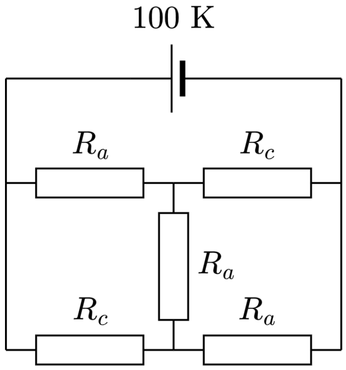
Solution: For a single conducting bar, the thermal conductivity is related to the rate of heat transfer by:

$$\frac{dQ}{dt} = \frac{kA\Delta T}{\ell}$$

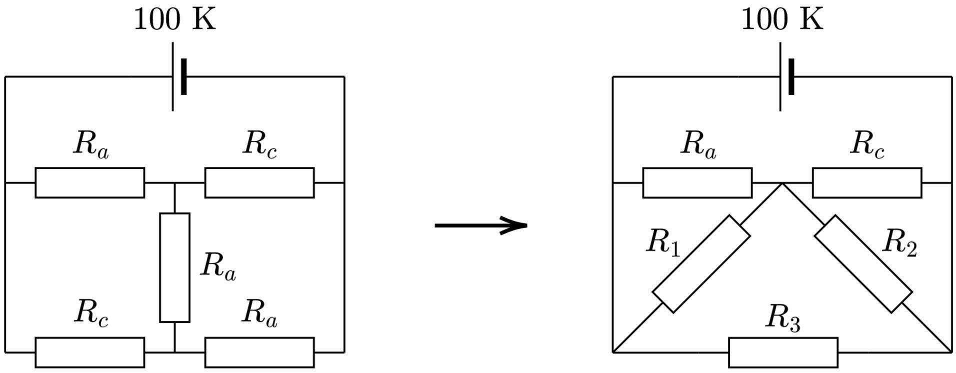
Notice that we are relating a rate of heat flow to a temperature difference — this can be compared to a **circuit analogy**, where a similar relation between current (rate of flow of charges) and potential difference can be established. As the form we wish to look for is $V = IR$ by Ohm's Law, we can obtain an equivalent resistance R_{equiv} to transform the problem to a circuit problem:

$$\begin{aligned}\Delta T &= \frac{\ell}{kA}\frac{dQ}{dt}\\ R_{\text{equiv}} &= \frac{\ell}{kA}\\ &= \frac{\ell}{kw^2}\end{aligned}$$

This gives us the following circuit, where $R_a = \dfrac{\ell}{k_a w^2}$ and $R_c = \dfrac{\ell}{k_c w^2}$ are the equivalent resistances of the aluminium and copper bars respectively:



One approach is to perform a Y to delta transformation to simplify our calculations:



$$\begin{aligned}
R_1 &= \frac{R_a^2 + 2R_aR_c}{R_a} \\
&= \frac{\ell}{w^2}\left(\frac{\dfrac{1}{k_a^2} + \dfrac{2}{k_ak_c}}{\dfrac{1}{k_a}}\right) \\
&= \frac{\ell}{w^2}\left(\frac{1}{k_a} + \frac{2}{k_c}\right) \\
R_2 &= \frac{R_a^2 + 2R_aR_c}{R_c} \\
&= \frac{\ell}{w^2}\left(\frac{k_c}{k_a^2} + \frac{2}{k_a}\right) \\
R_3 &= \frac{\ell}{w^2}\left(\frac{1}{k_a} + \frac{2}{k_c}\right) \text{ (same as } R_1\text{)}
\end{aligned}$$

We can now find the effective resistance:

$$\begin{aligned}
R_{\text{eff}} &= \left(\left(\frac{R_aR_1}{R_a + R_1} + \frac{R_cR_2}{R_c + R_2}\right)^{-1} + \frac{1}{R_3}\right)^{-1} \\
&= \frac{\ell}{w^2}\frac{3k_a + k_c}{k_a(k_a + 3k_c)}
\end{aligned}$$

Hence, we can find the rate of heat flow:

$$\begin{aligned}
\frac{dQ}{dt} &= \frac{\Delta T}{R_{\text{eff}}} \\
&\approx \boxed{0.28 \text{ W}}
\end{aligned}$$

Alternative solution: A more direct method would be to apply Kirchhoff's Voltage Rule on each loop. A very convenient method is typically mesh analysis, where we assign each loop a current and look at their superpositions to obtain the following equations:

$$\begin{aligned}
(R_a + R_c)I_1 - R_aI_2 - R_cI_3 &= \Delta T \\
-R_aI_1 + (2R_a + R_c)I_2 - R_aI_3 &= 0 \\
-R_cI_1 - R_aI_2 + (2R_a + R_c)I_3 &= 0
\end{aligned}$$

This can be solved conveniently with a computational tool of your choice, such as the simultaneous equation solving mode on your calculator. If using this method, a helpful way to check its correctness is to see if the coefficients are diagonally symmetric, and only the line of symmetry has positive coefficients.

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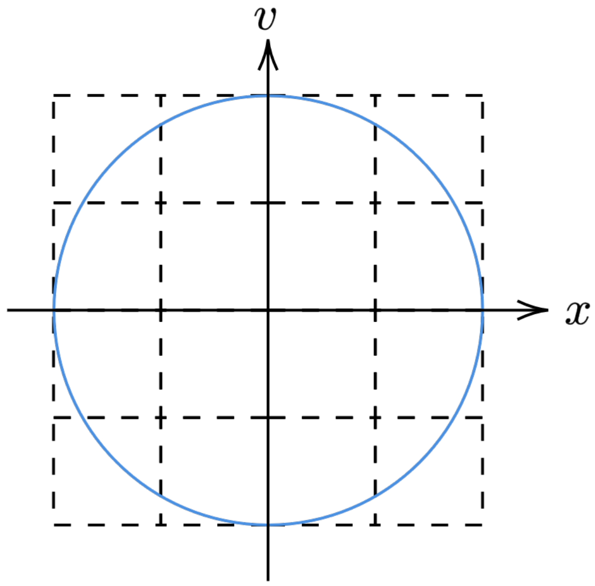
Problem 42: Oscillating Particles

Consider the two oscillations shown in the figure below. Both graphs show the velocity v (in m s^{-1}) of the oscillating particle at position x (in m). Each small dotted square has a side length of 1 unit.



(a) The phase portrait on the left is a circle of radius 2 units, centred on the origin. What is the period of this oscillation?

Leave your answer to 3 significant figures in units of s. (2 points)

(b) The phase portrait on the right is made up of 8 quarter circles, each with radius of curvature 1 unit. What is the period of this oscillation?

Leave your answer to 3 significant figures in units of s. (4 points)

Solution:

(a) Recall that the defining equation of SHM is $\ddot{x} = -\omega^2 x$, which has the general solution of:

$$x = A\cos(\omega t + \phi)$$

We can then differentiate it to obtain the particle's velocity:

$$v \equiv \dot{x} = -A\omega \sin(\omega t + \phi)$$

Using the trigonometric identity $\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi) = 1$, we obtain:

$$\left(\frac{v}{\omega A}\right)^2 + \left(\frac{x}{A}\right)^2 = 1$$

Hence, we can deduce that the phase portrait for any SHM is either a circle (for $\omega = 1$) or an ellipse (for $\omega \neq 1$).

The given phase portrait is a circle, implying that the oscillation is analogous to SHM with $\omega = 1$ rad s^{-1}. Therefore, the period is given by:

$$T = \frac{2\pi}{\omega} \approx \boxed{6.28 \text{ s}}$$

Alternative solution: Strictly speaking, the first method lacks mathematical rigour. We deduced that the phase portrait is a circle if the oscillation is SHM with $\omega = 1$ rad s^{-1} and then proceeded to draw an analogy between SHM and the given oscillation to determine its period. However, to be rigorous, we should instead be proving that the converse is true — if the phase portrait is a circle, then the oscillation is SHM with $\omega = 1$ rad s^{-1}. As shown below, this can indeed be proven.

From the given phase portrait, we know that:

$$\dot{x}^2 + x^2 = 4$$

We can differentiate this with respect to x to obtain:

$$2\dot{x}\frac{\mathrm{d}\dot{x}}{\mathrm{d}x} + 2x = 0$$

Using $\ddot{x} = \dot{x}\frac{\mathrm{d}\dot{x}}{\mathrm{d}x}$, we can re-write the above equation as:

$$\ddot{x} = -x$$

which is the defining equation of SHM with $\omega = 1$ rad s^{-1}. Therefore, the period is:

$$T = \frac{2\pi}{\omega} \approx \boxed{6.28 \text{ s}}$$

(b) In this case, we can no longer rely on the properties of SHM. This, however, does not mean that the problem is unsolvable. Since $v = \frac{\mathrm{d}x}{\mathrm{d}t}$:

$$T = \int dt = \int \frac{dx}{v}$$

over the required interval.[5] With the knowledge that the phase portrait is made up of 8 quarter circles, we can express v as a function of x. By symmetry, we

[5] This method can also be used to solve part (a), but making the connection to SHM is easier.

only need to find the time taken for the particle to complete one quarter of an oscillation. WLOG, let us consider the interval $x \in [0, 2]$.

Before we proceed, it would be helpful to recall that the standard form for the equation of a circle is $(x-h)^2 + (v-k)^2 = r^2$, which can be re-expressed in the form:

$$v = k \pm \sqrt{r^2 - (x-h)^2}$$

Using the above expression for v, we can write the following equation to describe the complete circle containing the quarter circle in the interval $x \in [0, 1]$:

$$v = 2 \pm \sqrt{1 - (x-1)^2} = 2 \pm \sqrt{2x - x^2}$$

However, we only want the quarter circle with $v \leq 2$, so we reject the positive square root:

$$v = 2 - \sqrt{2x - x^2}$$

Similarly, the equation for the complete circle containing the quarter circle in the interval $x \in (1, 2]$ is given by:

$$v = \pm\sqrt{1 - (x-1)^2} = \pm\sqrt{2x - x^2}$$

However, we only want the quarter circle with $v \geq 0$. Thus, we can re-write v as:

$$v = \sqrt{2x - x^2}$$

Combining both expressions, the velocity v for the interval $x \in [0, 2]$ can be written as the following piecewise function:

$$v = \begin{cases} 2 - \sqrt{2x - x^2}, & 0 \leq x \leq 1, \\ \sqrt{2x - x^2}, & 1 < x \leq 2. \end{cases}$$

Now that we have expressed v as a function of x, we can solve for the time taken t for the particle to move from $x = 0$ to $x = 2$:

$$t = \int_0^1 \frac{1}{2 - \sqrt{2x - x^2}} \, \mathrm{d}x + \int_1^2 \frac{1}{\sqrt{2x - x^2}} \, \mathrm{d}x \approx 2.4184 \text{ s}$$

Hence, the period is $T = 4t \approx \boxed{9.67 \text{ s}}$.

Note: Phase portraits are often used by physicists and applied mathematicians to study dynamical systems. Participants who want to learn more about phase portraits and the broader field of nonlinear dynamics may be interested in reading the following:

1. Strogatz, S. H. (2018). *Nonlinear dynamics and chaos: With applications to physics, biology, chemistry, and engineering*. CRC Press.
2. Raviola, L. A., Véliz, M. E., Salomone, H. D., Olivieri, N. A., & Rodríguez, E. E. (2016). The bead on a rotating hoop revisited: an unexpected resonance. *European Journal of Physics*, 38(1), 015005.
3. Baker, T. E. & Bill, A. (2012). Jacobi elliptic functions and the complete solution to the bead on the hoop problem. *American Journal of Physics*, 80(6), 506-514.
4. Glane, S. & Müller, W. H. (2019). The sliding ladder problem revisited in phase space. *American Journal of Physics*, 87(6), 444-448.
5. Bissell, J. J. (2022). Bifurcation, stability, and critical slowing down in a simple mass–spring system. *Mechanics Research Communications*, 125, 103967.

Note that the given list is by no means exhaustive.

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Problem 43: Falling Into a Plane (5 points)

A point charge $q = 3.00$ mC with mass $m = 5.00 \times 10^{-6}$ kg is held above a large grounded conducting plane at a distance $d_0 = 10.0$ m from it and released from rest. How much time t will it take for the point charge to reach the plane? Ignore gravity.

Leave your answer to 3 significant figures in units of ms.

Solution: Using the principle of image charges, the charge distribution induced on the conducting plane by the charge q produces (in the region above the plane) an electrostatic field identical to that of a charge $-q$ situated below the plane at the point which is the mirror image of the body's position, as if the plane were a mirror. Thus, the (attractive) electric force acting on the point charge when it is at distance d from the plane can be calculated using Coulomb's law as:

$$F = \frac{kq^2}{(2d)^2}$$
$$= \frac{\frac{k}{4}q^2}{d^2}$$

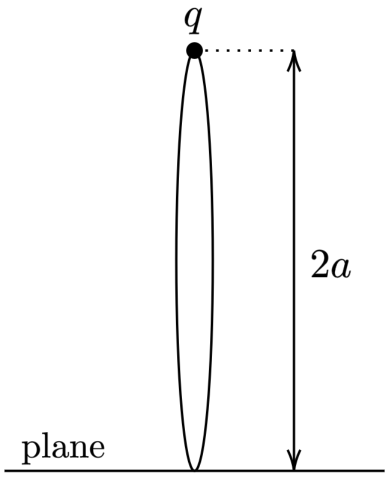
Notice that this expression is analogous to the expression for Newton's Law of Gravitation, both of which follow an inverse square law.

$$\frac{\frac{k}{4}q^2}{d^2} \equiv \frac{GMm}{r^2}$$
$$\frac{kq^2}{4m} \equiv GM$$

Using this analogy, we can apply Kepler's Third Law which states that:

$$T^2 = \frac{4\pi^2 a^3}{GM}$$

where T is the orbital period and a is the semi-major axis of the orbit. In this case, the orbit is an extremely flat ellipse that approaches a straight line, that is, with eccentricity approaching 1.



One focus of the ellipse is the center of the electrostatic or gravitational force, which, in our case, is the point on the plane where d is measured from. Since the focus approaches the edge of the ellipse, the major axis of the ellipse equals d, thus $a = \frac{d}{2}$. Substituting in our earlier expression for GM, we obtain:

$$T^2 = \frac{4\pi^2 \left(\frac{d}{2}\right)^3}{\frac{kq^2}{4m}}$$
$$T = \frac{\pi}{q}\sqrt{\frac{2md^3}{k}}$$

The desired time to fall into the plane is simply half a period. Hence:

$$\begin{aligned} t &= \frac{T}{2} \\ &= \frac{\pi}{q}\sqrt{\frac{md^3}{2k}} \\ &\approx \boxed{0.552 \text{ ms}} \end{aligned}$$

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Problem 44: Dying Photon (6 points)

Alex is doing a physics problem about solar sails. He notices that when photons are incident on a reflective object, the object gains kinetic energy, while the photons seem to be reflected with the same energy. Convinced that this will allow him to create a perpetual motion machine, he carries out an experiment.

A perfectly reflecting block of mass $m = 1.0 \times 10^{-21}$ kg is placed some distance away from a fixed perfectly reflecting wall. The wall is vertical while the ground is horizontal. A single photon is shot from a laser calibrated at wavelength $\lambda_0 = 1.0 \times 10^{-12}$ m. The photon then travels back and forth between the block and the fixed wall. The block moves only along the horizontal, frictionless ground. What is the velocity v of the block after $n = 5000$ collisions between the photon and the wall? Assume that $v \ll c$ and $\frac{h}{\lambda_0} \ll mc$.

Leave your answers to 2 significant figures in units of m s^{-1}.

Solution: The momentum of a photon is directly proportional to its energy. When the photon is reflected the wall, it retains exactly the same energy, as the wall gains no momentum. However, when it is reflected from the block, it loses energy as some momentum is imparted into the block.

Suppose that the momentum of the photon before and after a collision is p_1 and p_2 respectively, while the velocity of the block before and after a collision is v_1 and v_2 respectively. We define p_1 to be positive in the $+x$ direction, and p_2 to be positive in the $-x$ direction. We can write down equations for conservation of momentum and energy.

$$p_1 + mv_1 = -p_2 + mv_2$$
$$p_1 c + \frac{1}{2}mv_1^2 = p_2 c + \frac{1}{2}mv_2^2$$

After some effort, we solve the equations for p_2 and v_2 to obtain

$$p_2 = -p_1 - mv_1 - mc + \sqrt{m^2c^2 + 4mcp_1 + 2m^2cv_1 + m^2v_1^2}$$

$$v_2 = -c + \sqrt{c^2 + 4\frac{p_1 c}{m} + 2v_1 c + v_1^2}$$

Here, we already have done all the physics, and it is possible to write a program to iterate the value of v and p after each successive collision (in fact, this is probably the quickest method to proceed). However, we can also solve the problem analytically. We use the fact that $v \ll c$ and $p \ll mc$ to carry out a first-order Taylor expansion on v_2:

$$v_2 = -c + c\sqrt{1 + 4\frac{p_1}{mc} + 2\frac{v_1}{c} + \frac{v_1^2}{c^2}}$$
$$v_2 \approx -c + c\left(1 + 2\frac{p_1}{mc} + \frac{v_1}{c}\right)$$
$$v_2 - v_1 \approx \frac{2p_1}{m}$$

This is in fact the same as if we just assumed $p_1 = p_2$! This is because from the conservation of energy equation, $p_1 - p_2 = \frac{1}{c}\left(\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2\right) = \frac{m}{2c}(v_2 + v_1)(v_2 - v_1)$, and since $v \ll c$, $p_1 \approx p_2$.

Assuming that the change in v are small during each collision, we can express the discrete number of collisions n as well as the photon momentum p and block velocity v as continuous variables to obtain a differential equation for v.

$$\frac{\mathrm{d}v}{\mathrm{d}n} = \frac{2p}{m}$$

Here, p is still dependent on n. We can express p in terms of v by using the fact that energy is conserved. Let the initial momentum of the photon be p_0.

$$p_0c = pc + \frac{1}{2}mv^2$$

$$p = p_0 - \frac{mv^2}{2c}$$

Substituting,

$$\frac{\mathrm{d}v}{\mathrm{d}n} = \frac{2p_0}{m}\left(1 - \frac{mv^2}{2p_0c}\right)$$
$$\int_0^v \frac{1}{1 - \frac{mv^2}{2p_0c}}\mathrm{d}v = \int_0^n \frac{2p_0}{m}\mathrm{d}n$$

This is a standard integral that can be solved to obtain

$$\frac{1}{2A}\ln\left(\frac{1 + Av}{1 - Av}\right) = \frac{2p_0}{m}n$$

where $A = \sqrt{\frac{m}{2p_0c}}$. After solving for n in terms of v, we have

$$v = \frac{1}{A}\frac{e^{4A\frac{p_0n}{m}} - 1}{e^{4A\frac{p_0n}{m}} + 1} = \frac{1}{A}\tanh\left(\frac{2Ap_0n}{m}\right) = \sqrt{\frac{2p_0c}{m}}\tanh\left(\sqrt{\frac{2p_0}{mc}}n\right)$$

We know that $p_0 = \frac{h}{\lambda_0}$, so

$$v = \sqrt{\frac{2hc}{m\lambda_0}} \tanh\left(\sqrt{\frac{2h}{mc\lambda_0}} n\right) \approx \boxed{6400 \text{ m s}^{-1}}$$

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Problem 45: Broken Water Cooler

Roger places a large water bottle in a broken water cooler and fills it up to $V_0 = 800$ ml with $T_h = 90$°C water. Carelessly, he forgets to take back the bottle, and the broken water cooler continues to drip water at a constant rate $\frac{dV}{dt} = 1.00$ ml s^{-1}. The heat transferred per unit time between the water and the surroundings is proportional to their difference in temperature, with proportionality constant $k = 8.0$ J s^{-1} °C^{-1}. The surrounding temperature is $T_0 = 25$°C. Assume the surface area of water exposed to the surroundings during the entire process remains constant, and that the water bottle is large enough that it will never overflow.

(a) Find the equilibrium temperature T_f of the water after a long period of time.

Leave your answer to 3 significant figures in units of °C. (4 points)

(b) Find the time t taken for the water in the bottle to reach $T = 60$°C.

Leave your answer to 3 significant figures in units of s. (4 points)

Solution:

(a) Let ρ and c be the density and specific heat capacity of water respectively. The rate of change of heat $\frac{dQ}{dt}$ in the system is the heat added to the system by the constant dripping of hot water into the bottle, minus the heat loss to the surroundings given by Newton's Law of Cooling:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \rho c \frac{\mathrm{d}V}{\mathrm{d}t} T_H - kT$$

Note that T_H and T are defined relative to the surrounding temperature to simplify the equation i.e. $T_H = T_h - 25 = 90 - 25 = 65$ and $T = T_{actual} - 25$ at any point in time.

Since $Q = mcT$, we can also write $\frac{\mathrm{d}Q}{\mathrm{d}t}$ as follows, using the chain rule:

$$\begin{aligned}\frac{\mathrm{d}Q}{\mathrm{d}t} &= \frac{\mathrm{d}(mcT)}{\mathrm{d}t} \\ &= cT\frac{\mathrm{d}m}{\mathrm{d}t} + mc\frac{\mathrm{d}T}{\mathrm{d}t} \\ &= \rho c T\frac{\mathrm{d}V}{\mathrm{d}t} + \left(m_0 + \rho\frac{\mathrm{d}V}{\mathrm{d}t}t\right)c\frac{\mathrm{d}T}{\mathrm{d}t}\end{aligned}$$

where m is the mass of water in the bottle at any point in time and m_0 is the initial mass of water. Let $\rho c\frac{\mathrm{d}V}{\mathrm{d}t} = B$ to simplify both expressions, then equate them:

$$
\begin{aligned}
BT_H - kT &= BT + \left(m_0 + \rho\frac{\mathrm{d}V}{\mathrm{d}t}t\right)c\frac{\mathrm{d}T}{\mathrm{d}t} \\
&= BT + m_0c\frac{\mathrm{d}T}{\mathrm{d}t} + Bt\frac{\mathrm{d}T}{\mathrm{d}t}
\end{aligned}
$$

For (a), we take $\frac{\mathrm{d}T}{\mathrm{d}t} = 0$ and solve the simplified equation to obtain:

$$
\begin{aligned}
T_f &= \frac{BT_H}{B+k} + 25 \\
&\approx \boxed{47.3\ \text{°C}}
\end{aligned}
$$

A likely mistake for (a) is to set $\frac{\mathrm{d}Q}{\mathrm{d}t} = 0$ to erroneously obtain $T_f = \frac{BT_H}{k} + 25$ as the answer. While $\frac{\mathrm{d}Q}{\mathrm{d}t} = 0$ is true for most equilibrium temperature systems, hot water is constantly dripping into this system so heat is still being added even when the system reaches equilibrium temperature.

(b) For (b), we solve the differential equation using separation of variables. With the boundary condition that $T = T_H$ when $t = 0$:

$$
\begin{aligned}
(m_0c + Bt)\frac{\mathrm{d}T}{\mathrm{d}t} + (B+k)T &= BT_H \\
\int_{T_H}^{T} \frac{\mathrm{d}T}{BT_H - (B+k)T} &= \int_0^t \frac{\mathrm{d}t}{m_0c + Bt} \\
(m_0c)^{\frac{B+k}{B}} \frac{BT_H - (B+k)T_H}{BT_H - (B+k)T} &= (m_0c + Bt)^{\frac{B+k}{B}} \\
\text{or } T &= \frac{BT_H}{B+k} + \frac{(m_0c)^{\frac{B+k}{B}}}{(m_0c+Bt)^{\frac{B+k}{B}}}\frac{kT_H}{B+k}
\end{aligned}
$$

Solving for t when $T = 60 - 25 = 35$ and substituting numerical values, we obtain:

$$
t \approx \boxed{415\ \text{s}}
$$

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Problem 46: Rotating Spring

One end of a massless spring of natural length $l = 0.100$ m and spring constant $k = 50.0$ N m^{-1} is fixed to a point O. The other end of the spring can freely rotate about O and is attached to a particle of mass $m = 1.00$ kg. It is also known that the spring breaks when its length exceeds $l_{\text{max}} = 3l$. Initially, the spring is straight, at rest, and at natural length. An instantaneous impulse is then imparted to the particle such that it moves at an initial velocity $\vec{v}$ of arbitrary magnitude and direction. The spring-mass system is placed on a frictionless flat surface such that the particle's motion is constrained to a horizontal plane.

(a) Find v_{min}, the minimum magnitude of $\vec{v}$ required to break the spring.

Leave your answer to 3 significant figures in units of m s^{-1}. (2 points)

(b) Find v_{max}, the maximum magnitude of $\vec{v}$ for which the spring does not break.

Leave your answer to 3 significant figures in units of m s^{-1}. (3 points)

Solution:

Since the required extension l_{max} for the spring to break is fixed, the elastic potential energy EPE_{f} when the spring is about to break is also fixed. By conservation of energy:

$$\begin{aligned} \text{initial energy} &= \text{final energy} \\ \frac{1}{2}m|\vec{v}|^2 &= EPE_{\text{f}} + KE_{\text{rad,f}} + KE_{\text{tan,f}} \end{aligned}$$

(a) To minimise $|\vec{v}|$, we minimise the right-hand side of the above equation, setting $KE_{\text{rad,f}} = 0$ and $KE_{\text{tan,f}} = 0$. This means that the final velocity is 0, and all of the initial kinetic energy goes into extending the spring until it snaps at the maximum extension $l_{max} - l = 2l$. Considering conservation of energy:

$$\begin{aligned} \frac{1}{2}mv_{\text{min}}^2 &= \frac{1}{2}k(2l)^2 \\ \implies v_{\text{min}} &= \sqrt{\frac{4kl^2}{m}} \\ &\approx \boxed{1.41 \text{ m s}^{-1}} \end{aligned}$$

Note that by conservation of angular momentum, $KE_{\text{tan,f}} = 0$ implies $KE_{\text{tan,i}} = 0$. Thus, $\vec{v}$ must be purely radial.

(b) Since we are finding the maximum $|\vec{v}|$ for which the spring does not break, we want the mass to **just** reach its maximum extension, with zero radial velocity, so $KE_{\text{rad,f}} = 0$. From the conservation of energy equation, which is now $TE = \frac{1}{2}m|\vec{v}|^2 = EPE_{\text{f}} + KE_{\text{tan,f}}$, $v_{\max}$ is attained when $KE_{\text{tan,f}}$ is maximised.

By conservation of angular momentum, the final tangential velocity must be a fixed fraction f of the initial tangential velocity. Thus, $v_{\max}$ is achieved when $KE_{\text{tan,i}}$ is maximised. Intuitively, this occurs when $\vec{v}$ is purely tangential.[6]

Since angular momentum is conserved, we have:

$$lv_{\text{tan,i}} = 3lv_{\text{tan,f}} \implies v_{\text{tan,f}} = \frac{v_{\max}}{3}$$

By conservation of energy, we may now solve for $v_{\max}$:

$$\begin{aligned}
\frac{1}{2}mv_{\max}^2 &= \frac{1}{2}k(2l)^2 + \frac{1}{2}mv_{\text{tan,f}}^2 \\
&= 2kl^2 + \frac{1}{18}mv_{\max}^2 \\
\implies \frac{4}{9}mv_{\max}^2 &= 2kl^2 \\
\implies v_{\max} &= \sqrt{\frac{9kl^2}{2m}} \\
&= \boxed{1.50 \text{ m s}^{-1}}
\end{aligned}$$

Alternative solution: We employ a more direct mathematical approach. Since the spring force is a conservative force, the total mechanical energy of the system is conserved:

$$\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{1}{2}k(r-l)^2 = \frac{1}{2}m|\vec{v}|^2$$

The particle's angular momentum can be expressed as $L = r^2\dot{\theta}$, so we can rewrite the above equation as:

$$\frac{1}{2}m\left(\dot{r}^2 + \frac{L^2}{r^2}\right) + \frac{1}{2}k(r-l)^2 = \frac{1}{2}m|\vec{v}|^2$$

Let $\dot{r}_i$ and $\dot{\theta}_i$ be the initial radial and angular velocities of the particle. Considering the initial velocity of the particle, we know that $|\vec{v}|^2 = \dot{r}_i^2 + l^2\dot{\theta}_i^2 = \dot{r}_i^2 + L^2/l^2$, where

[6] Strictly speaking, this explanation lacks rigour. To show that $\vec{v}$ must be purely tangential, we should prove that all of the initial (kinetic) energy is tangential, that is, the **ratio** $KE_{\text{tan,i}}/TE$ is maximised. By conservation of angular momentum, $KE_{\text{tan,i}}/KE_{\text{tan,f}}$ is some constant (depending on initial and final extensions). Thus, maximising the ratio of $KE_{\text{tan,i}}/TE$ is equivalent to maximising $\eta = KE_{\text{tan,f}}/TE = KE_{\text{tan,f}}/(KE_{\text{tan,f}}+EPE_{\text{f}})$, which can be done by maximising $KE_{\text{tan,f}}$ (as η is a strictly increasing function) or, equivalently, $KE_{\text{tan,i}}$.

the last equality holds due to the conservation of angular momentum (since the spring force is a central force). Therefore, we obtain the following expression for L:

$$L^2 = l^2(|\vec{v}|^2 - \dot{r}_i^2)$$

Now, let us consider the system when the spring is at its maximum extension. At this instant, $r = 3l$ and $\dot{r} = 0$. Thus, by the conservation of energy:

$$\begin{aligned}\frac{m(|\vec{v}|^2 - \dot{r}_i^2)}{18} + 2kl^2 &= \frac{1}{2}m|\vec{v}|^2 \\ \implies \frac{4}{9}m|\vec{v}|^2 &= 2kl^2 - \frac{1}{18}m\dot{r}_i^2\end{aligned}$$

We also know that depending on the direction of $\vec{v}$, $0 \leq \dot{r}_i^2 \leq |\vec{v}|^2$, which implies that:

$$2kl^2 - \frac{1}{18}m|\vec{v}|^2 \leq \frac{4}{9}m|\vec{v}|^2 \leq 2kl^2$$

(a) The minimum magnitude of $\vec{v}$ is:

$$v_{\text{min}} = \sqrt{\frac{4kl^2}{m}} \approx \boxed{1.41\ \text{m s}^{-1}}$$

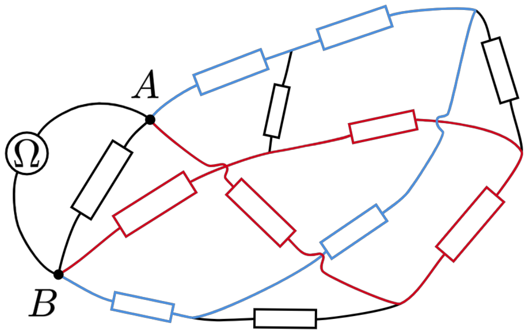
(b) The maximum magnitude of $\vec{v}$ is:

$$v_{\text{max}} = \sqrt{\frac{9kl^2}{2m}} = \boxed{1.50\ \text{m s}^{-1}}$$

Setter: Robert Frederik Uy, robert.uy@sgphysicsleague.org

Problem 47: Möbius Strip (5 points)

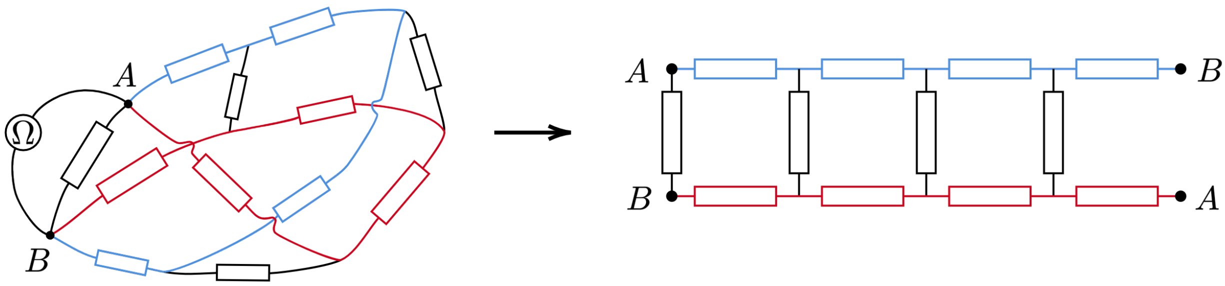
Two wires are each strung with 4 resistors, along with another 4 resistors that bridge pairs of resistors across both wires. The wires are then twisted together to form a *Möbius strip*, as shown below.



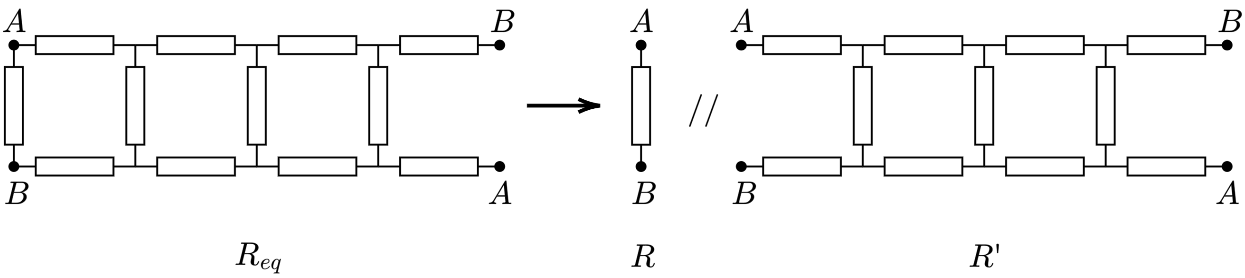
Every resistor has identical resistance $R = 1.0\ \Omega$. Determine the equivalent resistance R_{eq} between the points A and B in the Möbius strip.

Leave your answer to 2 significant figures in units of Ω.

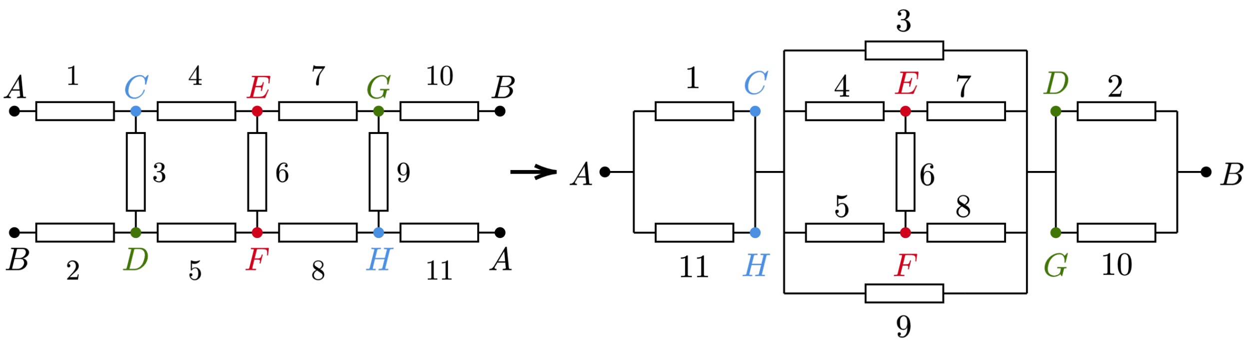
Solution: Imagine "slicing" the Möbius strip along line AB and then opening up and untwisting the circuit. The circuit can then be redrawn in its deconstructed form:



To determine the resistance R_{eq} across AB, we can treat the 1 bridging resistor, of resistance R, to be in parallel with the rest of the circuit, of unknown resistance R'.



Let us now focus on finding this unknown resistance R'. Notice that due to the symmetry of this circuit, there are 3 pairs of equipotential points as marked below (each pair takes a different colour). As such, we obtain the following equivalent circuit after combining points A and B on both ends:



From here, we can determine the value of R' (noting that resistor 6 can be disregarded since its two ends are equipotential):

$$R' = \frac{1}{\frac{1}{R} + \frac{1}{R}} + \frac{1}{\frac{1}{R} + \frac{1}{2R} + \frac{1}{2R} + \frac{1}{R}} + \frac{1}{\frac{1}{R} + \frac{1}{R}} = \frac{4}{3}R$$

We can hence calculate the resistance R_{eq} of the complete circuit:

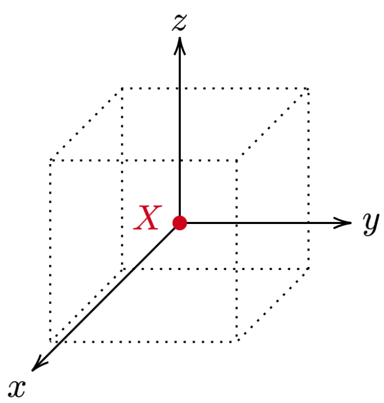
$$R_{\text{eq}} = \frac{1}{\frac{1}{R} + \frac{1}{R'}} = \frac{4}{7}R \approx \boxed{0.57\ \Omega}$$

Setter: Theodore Yoong, theodore.yoong@sgphysicsleague.org

Problem 48: Two-Dimensional Gas

Let us model a world as a three-dimensional space with three orthogonal axes x, y and z. There are two types of gas particles in this world, helium and **X**. The two types of particles have the same mass $m = 6.6465 \times 10^{-27}$ kg, and both may be assumed to display ideal gas behaviour. However, helium is able to move freely through all three dimensions, while the **X** particle is confined to the yz-plane ($x = 0$) and is unable to move along the x-axis. Assume that the laws of conservation of momentum and energy continue to hold true in this world, and that all collisions are elastic.

(a) An **X** particle is placed at $(0, 0, 0)$ with velocity $v = 3.00 \times 10^5$ m s^{-1} in a direction chosen uniformly at random in the yz-plane. It is contained within a heavy cube with side length $a = 1.00 \times 10^{-3}$ m centred at the origin, with axis-aligned edges.



The expected value of the time-averaged magnitude of force that the **X** particle exerts on the cube through collision with the sides is F, which is given as a numerical quantity in units of N. Find $\ln F$.

Leave your answer to 3 significant figures. (3 points)

(b) We now consider a collection of $n = 1000$ particles of **X**. The i^{th} **X** particle starts at a random position on the yz-plane and is given an initial velocity of $v_i = i$ m s^{-1} in a direction chosen uniformly at random in the yz-plane. The world is filled with helium gas at temperature T, which is allowed to interact with the collection of **X** particles. Calculate the value of T such that the expected total energy of the **X** particles stays constant over time.

Leave your answer to 3 significant figures in units of K. (3 points)

(c) We consider another collection of N particles of **X**, placed inside the cube from part (a). This collection stays in thermodynamic equilibrium with helium gas at temperature T, where T is the solution to part (b). The average pressure exerted on the cube by the **X** particles is $P = 3.00$ Pa. Find $\ln N$.

Leave your answer to 3 significant figures. (2 points)

Solution:

(a) First, we observe that the movements of the particle in the two directions perpendicular to the walls are entirely independent. As collision with the walls are elastic, their speed in each direction is conserved. Hence, we will only consider a single wall and motion perpendicular to that wall.

Let the velocity perpendicular to the wall be v_n. We observe that the change in momentum for each collision is $2mv_n$, and that collisions happen when the particle has travelled to the opposite wall and back, which takes $\frac{2a}{v_n}$ time. Hence, the average force on a single wall is $\frac{m\langle v_n^2\rangle}{a}$, and the total force on the four relevant walls must therefore be $\frac{4m\langle v_n^2\rangle}{a}$.

We need to find $\langle v_n^2\rangle$. We notice that $v_p^2+v_n^2 = v^2$ where v_p is the velocity parallel to the wall. Due to symmetry, we have $\langle v_n^2\rangle = \langle v_p^2\rangle$, so $\langle v_n^2\rangle = \frac{1}{2}v^2$.

Hence:

$$F = \frac{2mv^2}{a} = \frac{2 \times (6.6465 \times 10^{-27}) \times (3.00 \times 10^5)^2}{1.00 \times 10^{-3}} \approx 1.20 \times 10^{-12} \text{ N}$$

We thus have $\ln F \approx \boxed{-27.5}$.

(b) Since conservation of momentum must be obeyed, whenever a helium particle collides with an **X** particle, the **X** particle must still have zero velocity along the x-axis, thus the velocity of the helium particle along the x-axis cannot change. Hence, we can "collapse" the motion of helium into the yz-plane.

In classical thermodynamics, two gases are in thermodynamic equilibrium (and will not, in aggregate, transfer energy to each other) if and only if their particles have the same average kinetic energy. We can apply a similar argument to two gases constrained to move in two dimensions, as we have above. Hence, we simply need the average kinetic energy of the projection of the helium gas in the yz-plane to be equal to the average kinetic energy of the **X** particles.

Using the same symmetry argument we employed in part (a), we can show that since $U = \frac{3}{2}kT$, the average kinetic energy of the projection of the helium gas in the yz-plane is $E = \frac{2}{3}U = kT$. The average kinetic energy of the **X** particles, on the other hand, is given by $\frac{1}{2n}m\sum v_i^2 = \frac{m}{12}(n+1)(2n+1)$.

Hence, we have:

$$T = \frac{m}{12k}(n+1)(2n+1) = \frac{6.6465 \times 10^{-27}}{12(1.38 \times 10^{-23})}(1001)(2001) \approx \boxed{80.4 \text{ K}}$$

(c) Using $\frac{1}{2n}m\sum v_i^2 = \frac{m}{12}(n+1)(2n+1)$, we can deduce that:

$$\begin{aligned}\langle v^2\rangle &= \frac{\sum v_i^2}{n}\\ &= \frac{(n+1)(2n+1)}{6}\end{aligned}$$

Hence, to find the expected force F exerted by an $\mathbf{X}$ particle on the cube:

$$F = \frac{2mv^2}{a} = \frac{m}{3a}(n+1)(2n+1)$$

Divided over the six faces of the cube, each with area a^2, we can calculate that it amounts to a pressure of $\frac{m}{18a^3}(n+1)(2n+1)$ for each particle. Thus:

$$\begin{aligned}N &= \frac{P}{\frac{m}{18a^3}(n+1)(2n+1)}\\ &= \frac{18Pa^3}{m(n+1)(2n+1)}\\ &= \frac{18(3.00)(1.00\times10^{-3})^3}{(6.6465\times10^{-27})(1001)(2001)}\\ &\approx 4.06\times10^{12}\end{aligned}$$

after which we simply get $\ln N \approx \boxed{29.0}$.

Note: There is actually another way to find N that is much more direct. It can be shown from the expressions for part (a) and (b) that the equation $P'V' = NkT$ still holds in two dimensions for our specific setup, assuming that P' and V' are defined appropriately (in fact, it can be proven to hold true for ideal gases in any container in any number of dimensions, but the general proof is a little harder). P' must be defined as the force per unit *length* of the curve enclosing the two-dimensional surface containing the gas, for which V' is the *area*. As such, we see that P' is not the P given in the question, but in fact $P' = \frac{6a^2}{4a}P = \frac{3}{2}Pa$. Then, since $V' = a^2$, $NkT = \frac{3}{2}Pa^3$. Substituting the value of T from part (b), we get:

$$N = \frac{3Pa^3}{2k\frac{m}{12k}(n+1)(2n+1)} = \frac{18Pa^3}{m(n+1)(2n+1)}$$

which is in fact the same expression as we got using the more complicated method.

Setter: Shen Xing Yang, xingyang.shen@sgphysicsleague.org

Half Hour Rush M1: Clogged Bathtub (3 points)

Galen is taking a bath. His bathtub is shaped in the form of a cuboid, with length $L = 2.0$ m, width $W = 1.0$ m and vertical height $H = 1.0$ m. On the base of the bathtub is a drainage pipe, which is circular with inner radius $r = 0.050$ m. Much to Galen's dismay, there is a blockage at the pipe's entrance which requires a downward force of $F = 50$ N to be cleared. What is the required depth h of water in the bathtub to clear the blockage?

Leave your answer to 2 significant figures in units of m.

Solution: We first note that the bathtub's geometry is irrelevant, as what matters is the pressure exerted on the blockage itself. The hydrostatic pressure at the base of the tub is given by:

$$P = \rho_w g h$$

By the definition of pressure, $F = PA = P\pi r^2$. Hence, we can solve for the required value of h:

$$h = \frac{F}{\rho_w g \pi r^2} \approx \boxed{0.65 \text{ m}}$$

Setter: Paul Seow, paul.seow@sgphysicsleague.org

Half Hour Rush M2: Accidental Exposure (3 points)

We've all been there: you're texting with your phone in the shower, when suddenly some water accidentally hits your phone's touchscreen, coincidentally tapping the video call button. And then it gets awkward...

Suppose that your phone registers a touch input when a minimum contact pressure $P_c = 25$ kPa is applied on the screen, and that the showerhead emits a water jet of uniform velocity v perpendicular to the screen. Find the minimum value of v for which you risk starting a video call in the shower. Assume that the water is brought to rest instantaneously upon contact with the screen, and neglect any accumulation of water on the screen.

Leave your answer to 2 significant figures in units of m s^{-1}.

Solution:

The risk of video call occurs when the water jet hits the call button on the phone screen. In time $\mathrm{d}t$, the mass of water $\mathrm{d}m$ that hits an area A of the phone screen is given by:

$$\mathrm{d}m = \rho A v \,\mathrm{d}t$$

The momentum carried by $\mathrm{d}m$ of water, $\mathrm{d}p$, is thus given by:

$$\mathrm{d}p = \rho A v^2 \,\mathrm{d}t$$

Since the water is brought to rest immediately after contacting the screen, the incoming water loses all of its momentum after the collision. This means that within time $\mathrm{d}t$, the change in momentum of water is given by $\mathrm{d}p$. Hence, the force F exerted by the screen on the water is:

$$F = \frac{\mathrm{d}p}{\mathrm{d}t} = \rho A v^2$$

By Newton's Third Law, the force exerted by the water on the screen is also F. As such, the pressure P exerted by the water on the screen is:

$$P = \frac{F}{A} = \rho v^2$$

The water activates the touchscreen if $P \geq P_c$. Hence, the minimum v is given by:

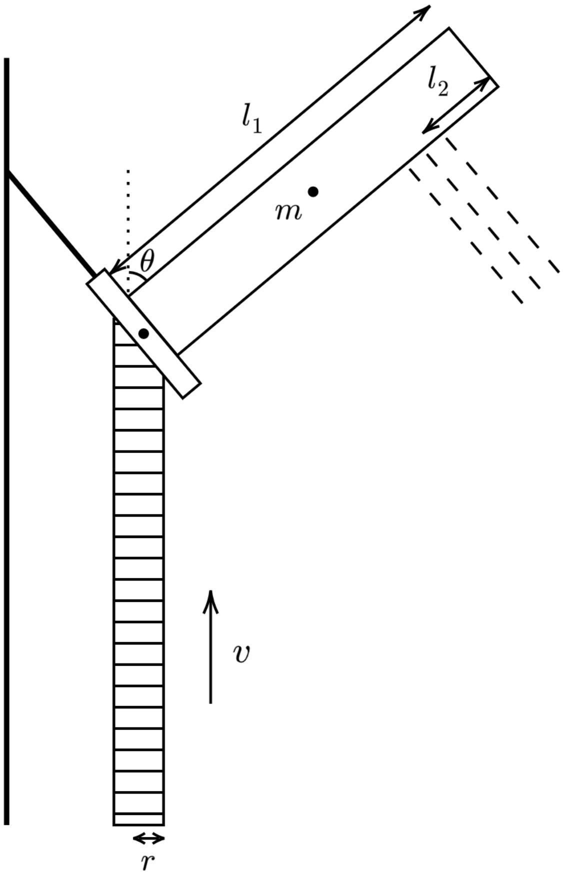
$$v \geq \sqrt{\frac{P_c}{\rho}} = \boxed{5.0 \text{ m s}^{-1}}$$

For further learning, touch-sensitive screens can actually use many different methods to register touch input. Here, we assumed pressure sensitivity, but many other technologies exist! Read more here.

Setter: Christopher Ong, chris.ong@sgphysicsleague.org

Half Hour Rush M3: Suspended Showerhead (4 points)

A uniform showerhead is connected to a flexible pipe, and is freely pivoted at this connection point (in a manner that does not disrupt the flow of water). The showerhead is a cylinder with a dry mass $m = 0.40$ kg and length $\ell_1 = 20.0$ cm. Water is supplied through the pipe which extends through the length of the handle, with a uniform inner radius of $r = 0.50$ cm:



When Robert turns on the showerhead, water begins to travel at a uniform speed $v = 5.0$ m s^{-1} in the pipe, and emerges perpendicularly from a point $l_2 = 5.0$ cm from the tip of the showerhead. Robert then notices the showerhead suspends itself at an acute angle θ from the vertical. Find θ.

Leave your answer to 3 significant figures in units of degrees.

Solution: Let us first establish how the showerhead is able to suspend itself. Recall that Newton's Second Law states that the net force on an object is equal to the rate of change of its momentum; in the case of a mass flow at constant velocity:

$$F = \frac{dp}{dt} = v\frac{dm}{dt}$$

For the water to change direction abruptly by a right angle, the showerhead must act on it with a force F; the water must hence act on the showerhead with an identical force, providing a force to suspend the showerhead. By considering the water flowing through the pipe, we can find the rate at which water is expelled from the showerhead, which is the mass flow rate:

$$\frac{dm}{dt} = \pi r^2 \rho v$$

Finally, we equate the clockwise torque due to gravity to the counter-clockwise torque from the water:

$$\begin{aligned} \tau_{\text{grav}} &= \tau_{\text{water}} \\ mg\frac{\ell_1}{2}\sin\theta &= \pi r^2 \rho v^2 (\ell_1 - \ell_2) \\ \theta &= \sin^{-1}\left(\frac{2\pi r^2 \rho v^2 (\ell_1 - \ell_2)}{mg\ell_1}\right) \\ &\approx \boxed{48.6°} \end{aligned}$$

Notice that since the water travels through the pivot point, the force acting on the pipe due to the bend at the pivot produces no torque; hence, can disregard it entirely.

Note to readers: the angle in the diagram is actually an unstable equilibrium, but was chosen due to its visual simplicity. In real life, you may instead notice an angle of $\theta = 180° - 48.6° = 131.4°$, which is a stable equilibrium.

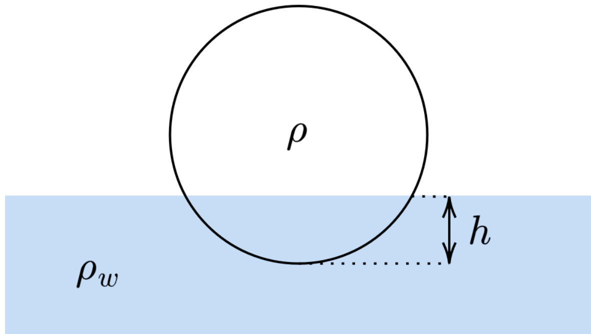
Setter: Paul Seow, paul.seow@sgphysicsleague.org

Half Hour Rush M4: Bath Fun! (5 points)

While taking a bath in water, Roger plays with his bath toy, a uniform spherical rubber ball of density $\rho = 110$ kg m^{-3}. Initially the ball is at rest on the water surface such that the bottom of the ball is a vertical height $h = 2.00$ cm below the water surface. Roger then gives the ball a small vertical displacement, causing it to oscillate with period T. Ignoring resistive forces, calculate T.

Leave your answer to 3 significant figures in units of s.

Hint: The volume V of a spherical cap with height h on a sphere with radius r is given by $V = \frac{\pi h^2}{3}(3r - h)$.



Solution: Since the problem asks for the period of oscillation, we suspect the motion to be simple harmonic. To confirm this we need to verify that the restoring force on the ball is indeed proportional to its vertical displacement. First let us use the information in the problem to determine the radius of the ball. Let the radius of the ball be r, volume of ball be V, and volume of ball submerged be V_s. By Archimedes' Principle, the buoyant force on the ball, which balances its weight, equals the weight of water displaced.

$$\rho g V = \rho_w g V_s$$
$$\frac{V_s}{V} = \frac{\rho}{\rho_w} = \frac{110}{1000} = 0.11$$

where V is the volume of a sphere with radius r, which is $\frac{4}{3}\pi r^3$, while V_s is the volume of a spherical cap with height h, which is $\frac{\pi h^2}{3}(3r - h)$, hence:

$$\frac{\frac{\pi h^2}{3}(3r - h)}{\frac{4}{3}\pi r^3} = 0.11$$
$$0.44r^3 - 3h^2 r + h^3 = 0$$

Solving the above cubic equation for r in terms of h yields:

$$r \approx 2.4251h$$

Consider the ball being displaced vertically downwards by height Δh. There is now a resultant force f on the ball due to additional volume of water displaced ΔV. Using the hint in the problem again, ΔV is given by:

$$\Delta V = \frac{\pi(h+\Delta h)^2}{3}(3r-h-\Delta h) - \frac{\pi h^2}{3}(3r-h)$$

Since Δh is small compared to h and r, we perform a first order approximation by cancelling out terms where Δh is raised to the power of 2 or higher. Simplifying the expression yields:

$$\Delta V \approx \pi h(2r-h)\Delta h$$

Since $f = \rho_w g \Delta V$:

$$f \approx -\rho_w g \pi h(2r-h)\Delta h$$

You can check that giving the ball a small upwards displacement yields the same expression for f after performing a first order approximation. Since $\rho_w g \pi h(2r-h)$ is a constant, f is proportional to Δh and acts in the opposite direction of the displacement, and so the ball oscillates in simple harmonic motion in the vertical axis. From $a = -\omega^2 \Delta h$:

$$\begin{aligned} a &= -\frac{\rho_w g \pi h(2r-h)}{\frac{4}{3}\pi r^3 \rho}\Delta h \\ \omega &= \sqrt{\frac{3\rho_w g h(2r-h)}{4r^3\rho}} \\ T &= \frac{2\pi}{\omega} \\ &= 2\pi\sqrt{\frac{4r^3\rho}{3\rho_w g h(2r-h)}} \\ &\approx \boxed{0.209 \text{ s}} \end{aligned}$$

Setter: Roger Zhang, roger.zhang@sgphysicsleague.org

Half Hour Rush E1: Simp (3 points)

Amy has a crush on Jake. She hopes to get together with him through the charm of electrostatic attraction. From a distance of $r = 5.0$ m, she channels her superpowers and secretly transfers $N = 100$ million electrons from Jake's body to her own body. What is the magnitude of the attractive force F that she achieves?

Assume that both of them are point particles that are initially neutral and do not exchange charge with the environment.

Leave your answer to 2 significant figures in units of pN.

Solution: By conservation of charge, the charge of Jake's body upon losing N electrons (each of charge $-e$) is given by $+Ne$, whereas the charge of Amy's body upon gaining these N electrons is given by $-Ne$. As such, by Coulomb's Law, the attractive force F is given by:

$$F = \frac{1}{4\pi\varepsilon_0}\frac{(+Ne)(-Ne)}{r^2} \implies |F| \approx \boxed{0.092 \text{ pN}}$$

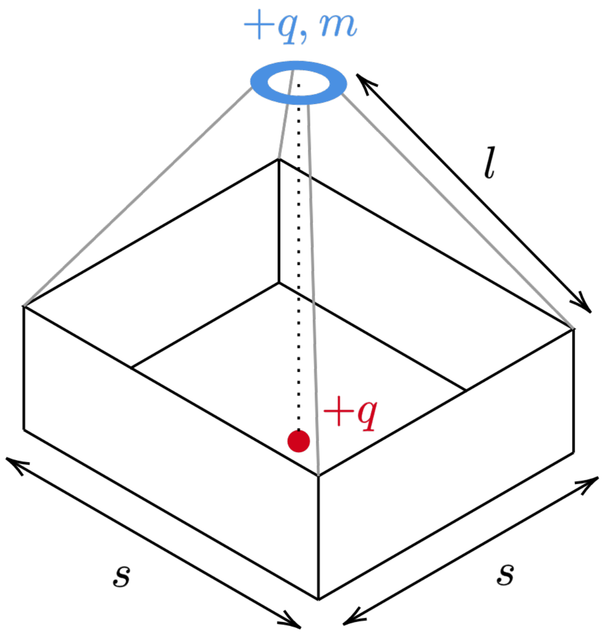
Based on how small F is, Amy has L rizz.

Setter: Christopher Ong, chris.ong@sgphysicsleague.org

Half Hour Rush E2: A Simple Proposal (4 points)

Josiah bought a small engagement ring of mass $m = 1.00 \times 10^{-3}$ kg, which he wanted to present to his fiancée in a box with a square base of side length $s = 0.100$ m and negligible height. On opening the box, he wanted the ring to hover a short distance above its centre. To achieve this, he hid a positive point charge $+q$ under the centre of the box and applied the same positive charge $+q$ to the ring. To constrain the ring to hover directly above the centre of the box, he tied four thin inextensible strings of length $l = 0.120$ m to the ring and secured them to the four corners of the box. Suppose the ring is small enough to be approximated by a point charge. What is the minimum charge q required to ensure the four strings remain taut while the ring hovers above the box?

Leave your answer to 3 significant figures in units of μC.

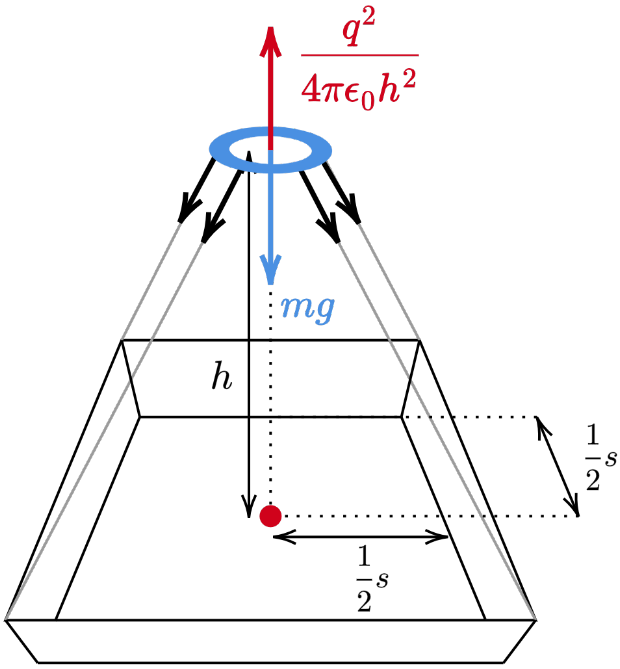


Solution: Three types of forces act on the hovering ring: the electrostatic repulsion from the hidden point charge, the weight of the ring, and the tension from the strings. These forces must cancel for the ring to hover in place.

Let h be the height above the box at which the ring hovers, and let us define the downwards direction to be positive. As shown in the diagram, the electrostatic repulsion is $-\frac{q^2}{4\pi\epsilon_0 h^2}$ while the weight from the ring is $+mg$.

Since the net (downwards) force from the tension of the four strings, T, balances the gravitational and electrostatic forces on the ring, we have:

$$T = \frac{q^2}{4\pi\epsilon_0 h^2} - mg$$



For the strings to remain taut, the tensions in the string must be non-negative, which implies that:

$$T = \frac{q^2}{4\pi\epsilon_0 h^2} - mg \geq 0 \implies q^2 \geq 4\pi\epsilon_0 mgh^2$$

From the same diagram above and Pythagoras's theorem, we can infer that:

$$l^2 = \left(\frac{s}{2}\right)^2 + \left(\frac{s}{2}\right)^2 + h^2 \implies h = \sqrt{l^2 - \frac{s^2}{2}}$$

Substituting this expression for h into the previous equation and isolating q implies that the charge must be at least:

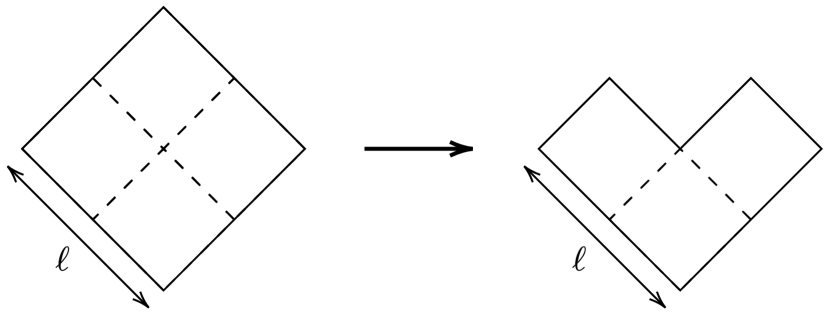
$$q \geq \sqrt{4\pi\epsilon_0 mg\left(l^2 - \frac{s^2}{2}\right)} \approx \boxed{0.101\ \mu\text{C}}$$

Setter: Tian Shuhao, shuhao.tian@sgphysicsleague.org

Half Hour Rush E3: Inducing Love (4 points)

Donghang is planning to propose to his girlfriend, so he bought a square ring of side length $\ell = 2.00$ cm. To make the ring seem more special, he deformed it into the shape of a heart. As shown in the figure below, the heart-shaped ring consists of three of the four smaller squares that make up the original square ring. Given that the self-inductance of the square ring is $L_\square = 0.100$ H, find the self-inductance $L_\heartsuit$ of the heart-shaped ring.

Leave your answer to 3 significant figures in units of H.



square ring heart-shaped ring

Solution:

The self-inductance of a system depends on its geometry. The key to solving this problem is determining, by dimensional analysis, that the self-inductance L_S of any square loop is proportional to its side length ℓ. The relevant quantities for determining L_S are the permeability of free space μ_0 and the side length ℓ. These physical quantities have the following dimensions:

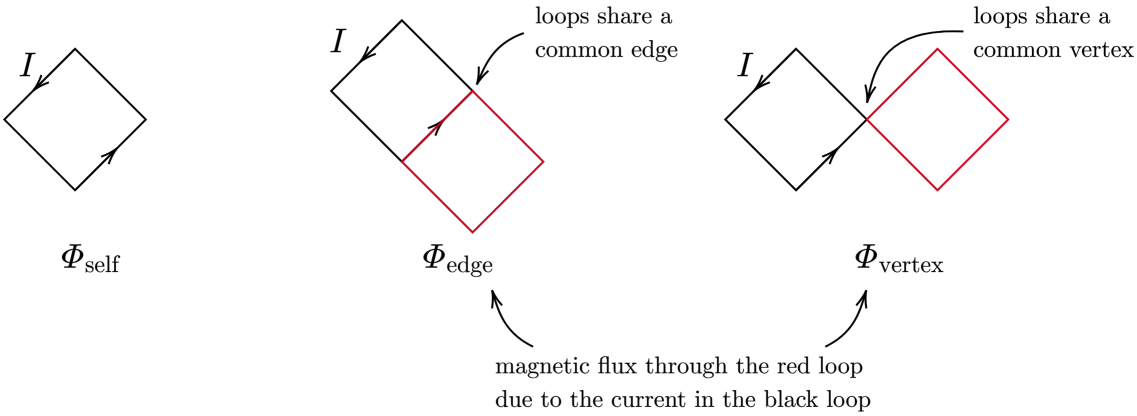
$$[L_S] = ML^2T^{-2}I^{-2}, \qquad [\mu_0] = MLT^{-2}I^{-2}, \qquad [\ell] = L,$$

where M, L, T and I denote dimensions of mass, length, time and current, respectively. Notice that it is impossible to form a dimensionless quantity with just μ_0 and ℓ, so the only way to express L_S in terms of the given quantities is:

$$L_S = k\mu_0\ell,$$

where k is a dimensionless constant. Indeed, $L_S \propto \ell$. Applying this to the problem, we can then deduce that the self-inductance of a square loop of side length $\ell/2$ is $L_\square/2$.

Let Φ_{self} be the magnetic flux through a square loop of side length $\ell/2$ due to an anticlockwise current I flowing through itself, Φ_{edge} be the magnetic flux through a square loop of side length $\ell/2$ due to an anticlockwise current I flowing through an identical loop that shares a common edge with it, and Φ_{vertex} be the magnetic flux through a square loop of side length $\ell/2$ due to an anticlockwise current I flowing through an identical loop that shares a common vertex with it.



A square loop of side length ℓ with an anticlockwise current I flowing through it can be interpreted as four smaller square loops of side length $\ell/2$ with an anticlockwise current I flowing through each of them. Through each smaller square loop, there is flux Φ_{self} due to itself, $2\Phi_{\text{edge}}$ due to its 2 edge-sharing neighbours, and Φ_{vertex} due to its vertex-sharing neighbour. Thus, the magnetic flux flowing through the whole square loop of side length ℓ can be expressed as $4\Phi_{\text{self}} + 8\Phi_{\text{edge}} + 4\Phi_{\text{vertex}}$. Recalling that the magnetic flux through any loop is simply the product of its self-inductance and the current flowing through it, we obtain:

$$\begin{aligned}
L_{\square} I &= 4\Phi_{\text{self}} + 8\Phi_{\text{edge}} + 4\Phi_{\text{vertex}} \\
\implies -L_{\square} I &= 8\Phi_{\text{edge}} + 4\Phi_{\text{vertex}} \qquad \because \ \Phi_{\text{self}} = \frac{L_{\square} I}{2} \\
\implies -\frac{L_{\square} I}{2} &= 4\Phi_{\text{edge}} + 2\Phi_{\text{vertex}}.
\end{aligned}$$

Using a similar analysis as above, the magnetic flux through the heart-shaped loop when an anticlockwise current I flows through it is $3\Phi_{\text{self}} + 4\Phi_{\text{edge}} + 2\Phi_{\text{vertex}}$.

Hence, the self-inductance of the heart-shaped loop is:

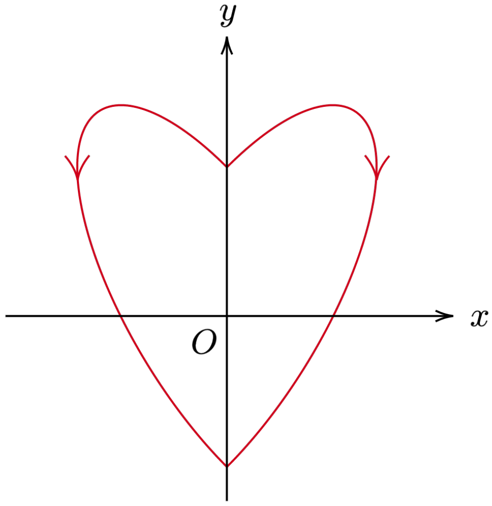
$$L_{\heartsuit} = \frac{3\Phi_{\text{self}} + 4\Phi_{\text{edge}} + 2\Phi_{\text{vertex}}}{I} = L_{\square} = \boxed{0.100 \text{ H}}.$$

Setter: Robert Frederik Uy, robert.uy@sgphysicsleague.org

Half Hour Rush E4: An Elaborate Proposal (5 points)

Paul wants to draw a heart in an elaborate manner to impress his crush. To do this, he carefully creates a region with spatially varying magnetic field $B(\vec{r})$ in a direction perpendicular to the plane of the page. The maximum magnitude of magnetic field he is capable of producing is B_{max}. He then ejects two identical charged particles of charge $q = 0.50$ C and mass $m = 0.020$ kg with equal speeds $v = 3.0$ m s^{-1}. These particles travelled in mirrored paths, tracing out a heart-shaped figure with equation $x^2 + (y - |x|)^2 = 1$, where x and y are lengths in units of metres. What is the minimum value of B_{max} for this to be possible? Neglect the forces exerted by the charged particles on each other.

Leave your answer to 2 significant figures in units of T.



Solution: The force exerted by the magnetic field on the particle is always perpendicular to the direction of the particle's motion, so the particle's speed is constant at v. The acceleration $a(\vec{r})$ of the particle is always centripetal, and can be related to the radius of curvature of its path $R_c(\vec{r})$ by the following equation:

$$a(\vec{r}) = \frac{v^2}{R_c(\vec{r})}$$

We can apply Newton's second law to relate the particle's acceleration to the magnetic field, using scalar quantities:

$$qvB = ma$$
$$B = \frac{mv}{qR_c}$$

The radius of curvature of a Cartesian equation is given by:

$$R_c = \left| \frac{(1 + [y']^2)^{\frac{3}{2}}}{y''} \right|$$

We only consider positive values of x value due to symmetry, so we can simply remove the modulus in the equation of the curve. Upon implicit differentiation of the heart equation and some rearrangement, we have:

$$y' = 1 + \frac{x}{y-x}$$

We differentiate the expression again and use substitutions from before to obtain:

$$y'' = -\frac{1}{(y-x)^3}$$

Substituting into our equation for radius of curvature, we have:

$$R_c = \left[(y-x)^2 + (y-2x)^2\right]^{\frac{3}{2}}$$

Now, we can express y purely as a function of x by rearranging the original equation of the curve. Upon substitution, we obtain R_c as a function of x.

$$R_c = \left[1 - x^2 + (\pm\sqrt{1-x^2} - x)^2\right]^{\frac{3}{2}}$$

To maximise the magnetic field, we must minimise R_c. Naturally, we will choose the positive square-root upon inspection of our expression. Then, we can find the minimum value of R_c, using differentiation or a graphical method. This occurs at:

$$(x, R_c) = (0.85065, 0.23607)$$

Substituting the numerical values into our earlier equation for the magnetic field, we get:

$$B_{max} \approx \boxed{0.51\ \mathrm{T}}$$

Alternative solution: The minimum radius of curvature of the particle's path can be found without the use of calculus. Consider the equation of the path, taking only positive values of x without loss of generality. Upon close inspection, we have the equation for an ellipse centered at and rotated about the origin. Only the part of the ellipse with $x > 0$ is included in the path, and reflected about the y-axis to form the full heart shape.[7]

$$x^2 + (y-x)^2 = 1$$
$$2x^2 + y^2 - 2xy = 1 \tag{1}$$

[7] At this stage, we can directly find the value of the semi-major and semi-minor axis of the ellipse using the coefficients of the general equation.

The equation for an ellipse centered at the origin with semi-major axis a, semi-minor axis b, and rotation angle θ between the ellipse's major axis and horizontal can be found with a rotational transformation of the usual equation for an ellipse.

$$\frac{(x\cos\theta + y\sin\theta)^2}{a^2} + \frac{(x\sin\theta - y\cos\theta)^2}{b^2} = 1 \tag{2}$$

Comparing coefficients for x^2, y^2 and xy between equations (1) and (2), we have:

$$\left(\frac{\cos\theta}{a}\right)^2 + \left(\frac{\sin\theta}{b}\right)^2 = 2 \tag{3}$$

$$\left(\frac{\sin\theta}{a}\right)^2 + \left(\frac{\cos\theta}{b}\right)^2 = 1 \tag{4}$$

$$2\sin\theta\cos\theta\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = 2 \tag{5}$$

Adding equations (3) and (4):

$$\frac{1}{a^2} + \frac{1}{b^2} = 3 \tag{6}$$

Subtracting equation (4) from equation (3):

$$(\cos^2\theta - \sin^2\theta)\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = 1 \tag{7}$$

We apply the double angle formulae to equations (5) and (7), then square and add them to obtain:

$$\left(\frac{1}{a^2} - \frac{1}{b^2}\right)^2 = 5 \tag{8}$$

We can substitute equation (6) into equation (8), and simplify to obtain a.

$$\left(\frac{2}{a^2} - 3\right)^2 = 5 \tag{9}$$

$$\frac{2}{a^2} = 3 \pm \sqrt{5} \tag{10}$$

$$a = \sqrt{\frac{2}{3 \pm \sqrt{5}}} \tag{11}$$

Due to the symmetry in equation (8), where a and b can take on each other's values and the equation still holds, we deduce that these two possible values of a obtained are simply a and b, where $a > b$. The minimum radius of curvature in an ellipse occurs at the vertices on the major axis, and is given by $R_c = \frac{b^2}{a}$. Substituting the values of a and b obtained:

$$R_c = \frac{\frac{2}{3+\sqrt{5}}}{\sqrt{\frac{2}{3-\sqrt{5}}}} \approx 0.23607 \tag{12}$$

which gives us the same result of $B_{max} \approx \boxed{0.51\,\mathrm{T}}$ obtained earlier.

Setter: Chen Guangyuan, guangyuan.chen@sgphysicsleague.org

Half Hour Rush X1: Diabolus in Musica (3 points)

Johannes, an organ student, hears of a "cursed interval" in music, which comprises two notes whose frequencies have a ratio of $\sqrt{2}$. Messing around, he accidentally presses two notes on the organ to form this interval, summoning his music professor (who he believes to be the devil). The lower and higher pitched notes are produced with separate organ pipes of lengths L_a and L_b respectively. Find the ratio L_b/L_a.

You may assume that the sound heard is the fundamental frequency and that both pipes are open on both ends.

Leave your answer to 3 significant figures.

Solution: Firstly, we note that for a pipe of length L open on both ends, the wavelength of sound at the fundamental frequency is given by $\lambda = 2L$. Since $v = f\lambda$ and the speed of sound is identical for both organ pipes, the length of the pipe and the fundamental frequency of the pipe are inversely proportional.

Let the frequencies produced by the pipes of lengths L_a and L_b be f_a and f_b respectively. Hence:

$$\frac{L_b}{L_a} = \frac{f_a}{f_b}$$

Since f_b is the higher pitched note, it must have the higher frequency. Consequently:

$$\frac{L_b}{L_a} = \frac{1}{\sqrt{2}} \approx \boxed{0.707}$$

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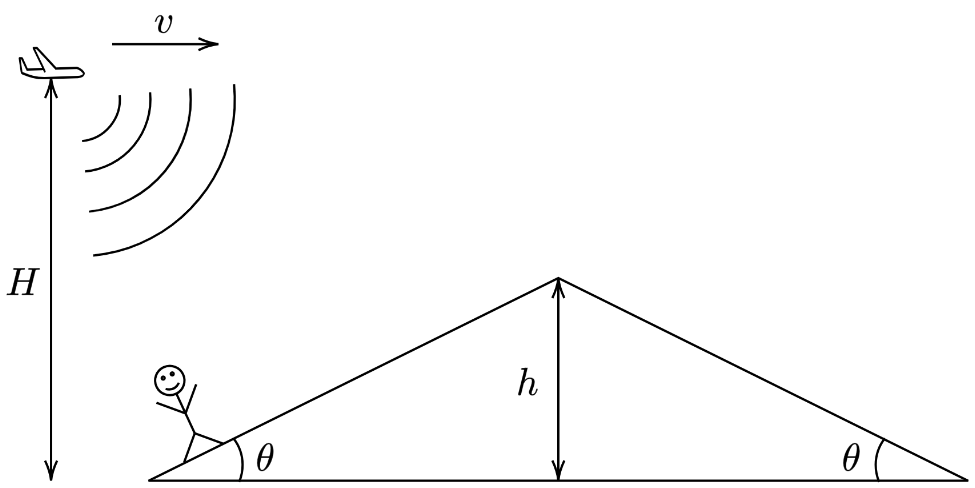
Half Hour Rush X2: Sonic Boom (3 points)

A military jet is flying over a hill of vertical height $h = 500$ m and angle of inclination $\theta = 30°$, from west to east. Chris is standing on a point along the west side of the hill to get a picture of the jet. Approaching from a distance away, the jet flies overhead at height $H = 1000$ m with a horizontal velocity of $v = 680$ m s^{-1}. As this is greater than the speed of sound in air, a shockwave is generated — a sonic boom.

Let the time at which the jet is directly on top of the western base of the hill be $t = 0$. Let the time at which Chris is hit by the sonic boom be $t = T$. Find the maximum value of T.

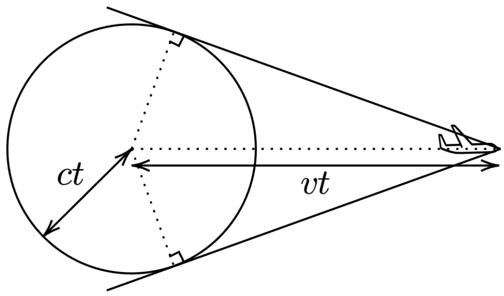
Take the speed of sound in air to be $c = 340$ m s^{-1}.

Leave your answer to 2 significant figures in units of s.

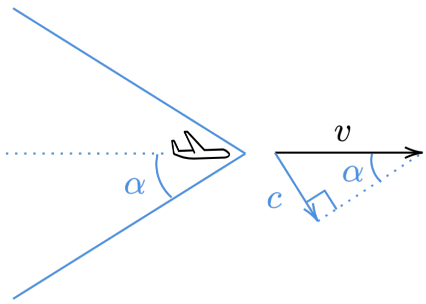


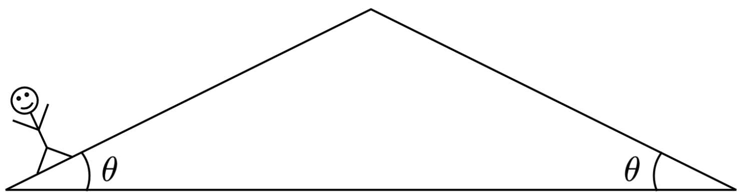
Solution: A sonic boom creates a series of pressure waves in its wake. Consider a single point of sound emission. Since the sound is emitted in all directions, the wavefront of the sound after a time t is a circle of radius ct. During this time, the jet has travelled a distance $vt > ct$ from the point of emission. The greatest angle between the jet's trajectory and the circular wavefront is given by a tangent line from the jet to the wavefront. This line subtends an angle α from the trajectory of the jet, given by:

$$\alpha = \sin^{-1}\frac{ct}{vt} = \sin^{-1}\frac{c}{v} = 30°$$



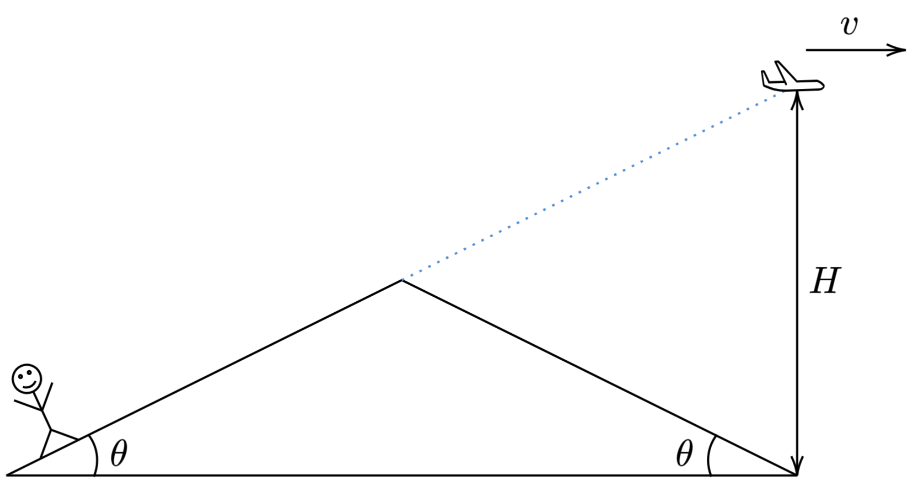
Note that as c and v are constant, α is also constant. Hence, the sound waves generated across the whole trajectory of the jet form a conical shape with cone angle α.





We see that α is the same as θ! In other words, the wavefront is parallel to the surface of the hill. Hence, the time taken for the pressure wave of the sonic boom to reach Chris will be **identical regardless of his position**.

To calculate this, we can find the time taken for the jet to reach a point that lies along the same line as the west side of the hill:



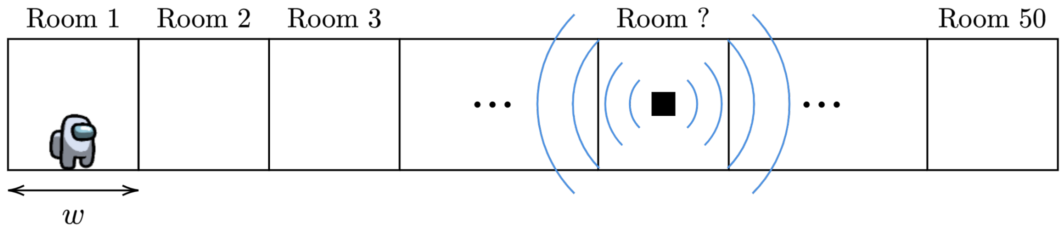
Hence, the time taken T can be found:

$$T = \frac{d}{v} = \frac{H}{v \tan\theta} \approx \boxed{2.5 \text{ s}}$$

Setter: Paul Seow, paul.seow@sgphysicsleague.org

Half Hour Rush X3: An Impostor Among Us (4 points)

Gray is a crewmate aboard a spaceship consisting of a long row of 50 adjacent rooms, each of width $w = 10.0$ m, labelled sequentially from Room 1 to Room 50.



One day, Gray picked up a suspicious encrypted signal and tried to narrow down its source using an intensity meter. He took a measurement at the centre of Room 1 and another at the centre of Room 2, and deduced that the source is at the centre of Room 39. However, Room 39 turned out to be empty.

This is because Gray's intensity meter is suffering from a zero error that leads it to output measurements that are $\Delta = 2.000 \times 10^{-5}$ W m^{-2} lower than the actual wave intensities detected. If Gray's intensity meter showed $I_1 = 1.000 \times 10^{-4}$ W m^{-2} at the centre of Room 1, deduce the actual room number of the signal's source.

Gray had assumed that the waves came from a point source and that the reflection or absorption of these waves by the intervening walls was negligible. Suppose Gray's assumptions and calculations were valid, barring the incorrect measurements.

Leave your answer as an integer from 1 to 50.

Solution: We will introduce our coordinate system such that the origin is $w/2$ left of Room 1 and arrange the rooms along the positive x axis, with Room 1 spanning $w/2 \leq x \leq 3w/2$, Room 2 spanning $3w/2 \leq x \leq 5w/2$, and so on. The centre of Room n will therefore be at $x = nw$. Let the x coordinate of the source be s, and its intensity distribution be $I(x)$.

The assumptions on the radio waves imply that their intensity obeys the inverse-square law:

$$I(x) \propto \frac{1}{(x-s)^2}$$

This is because negligible absorption and reflection imply that radio waves transmit fully in all directions. Just like in free space, a sphere of radius r centered on the source receives constant power. Distributing this over a surface area proportional to r^2 results in intensities proportional to $1/r^2$.

In particular, the inverse square law implies that if I_1 is the intensity at x_1 and I_2 is the intensity at x_2, then:

$$I_1(s-x_1)^2 = I_2(s-x_2)^2$$

In Gray's calculation, the source's location is $s = 39w$. The inverse square law tells us that:

$$I_1(39w-w)^2 = I_2(39w-2w)^2$$

and therefore the measurement Gray recorded from the centre of Room 2 must have been:

$$I_2 = \frac{38^2}{37^2} I_1 \approx 1.055 \times 10^{-4} \text{ W m}^{-2}$$

We can now redo the calculations with the corrected values $I_1' = I_1 + \Delta, I_2' = I_2 + \Delta$, in order to find the corrected source coordinate s'. Since the inverse square law still applies, we have:

$$I_1'(s'-w)^2 = I_2'(s'-2w)^2 \implies \left(\frac{s'-w}{s'-2w}\right)^2 = \frac{I_2'}{I_1'} = \frac{I_2+\Delta}{I_1+\Delta}$$

Taking square roots (noting that $s'-2w, s'-w > 0$ are both positive), we have:

$$\begin{aligned} \frac{s'-w}{s'-2w} = \sqrt{\frac{I_2+\Delta}{I_1+\Delta}} &\implies s'-w = \sqrt{\frac{I_2+\Delta}{I_1+\Delta}}(s'-2w) \\ &\implies \left(\sqrt{\frac{I_2+\Delta}{I_1+\Delta}}-1\right)s' = \left(2\sqrt{\frac{I_2+\Delta}{I_1+\Delta}}-1\right)w \end{aligned}$$

which implies:

$$s' = \left(\frac{2\sqrt{I_2+\Delta}-\sqrt{I_1+\Delta}}{\sqrt{I_2+\Delta}-\sqrt{I_1+\Delta}}\right) w \approx 46.3024\ w$$

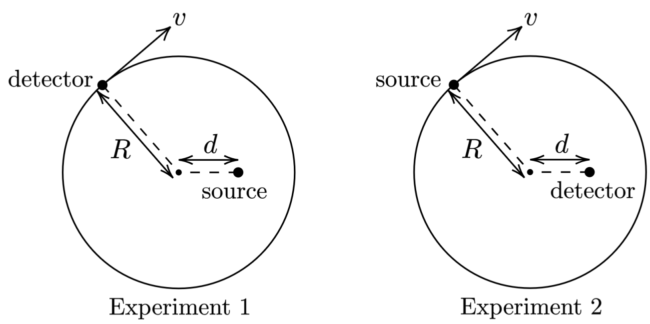
Since room n covers $(n-\frac{1}{2})w \le x \le (n+\frac{1}{2})w$, s' lies within room $\boxed{46}$.

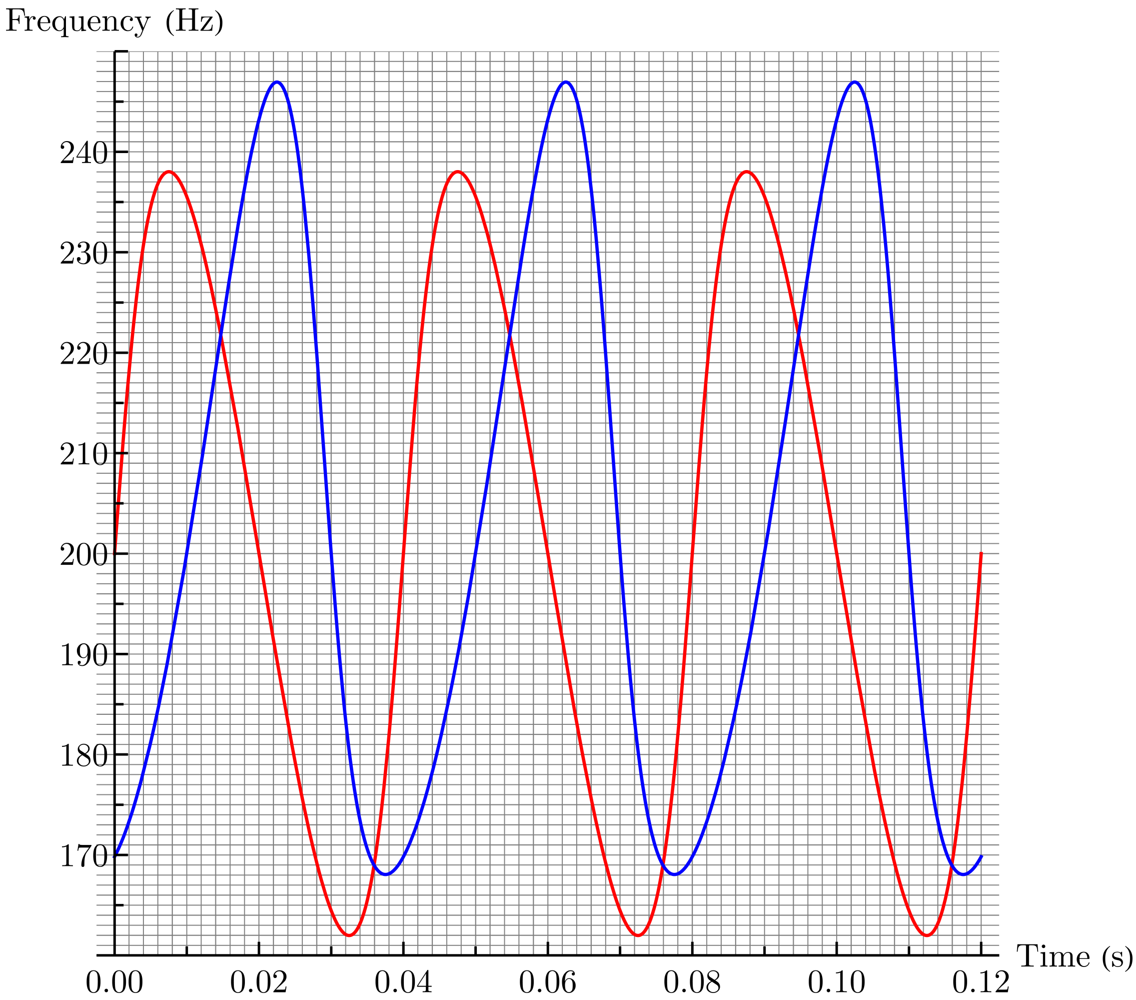
Setter: Tian Shuhao, shuhao.tian@sgphysicsleague.org

Half Hour Rush X4: Doppler's Confusion (5 points)

In one of his late night musings, Doppler came up with a sophisticated way to measure the speed of sound. The following day, he went to the lab to conduct two experiments. In the first one, a sound detector moves at a constant speed v along a circular path of radius $R = 1.00$ m. A speaker is then placed a distance $d = 0.380$ m away from the centre of the circular path. In the second experiment, the detector and the speaker swap positions, as shown in the figure below. The frequency data collected by the detector in both experiments is shown in the graph below. Unfortunately, he forgot to note down which curve corresponds to each experiment. Find c, the speed of sound.

Leave your answer to 3 significant figures in units of m s^{-1}.

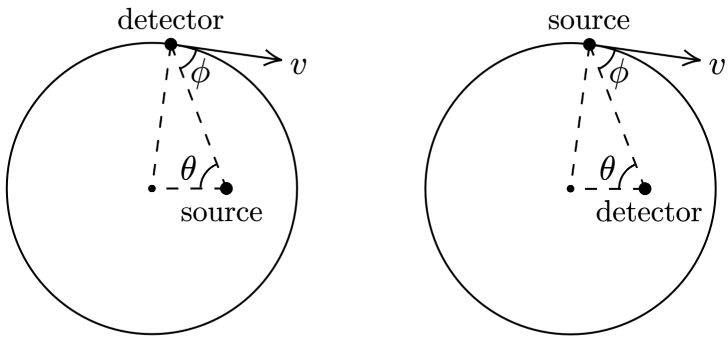




Solution: Consider a detector and a source both moving. Along the line connecting the two, if the source is moving at velocity v_s and the detector is moving at velocity v_d, then the detected frequency f' is related to the source's frequency f by:

$$f' = \frac{v_d \pm c}{v_s \pm c} f,$$

where the signs are determined by whether the objects are moving towards or away from each other.



In the first experiment, the detector is in uniform circular motion around the source. If the velocity of the detector is making an angle ϕ with the vector pointing from the detector to the source, then the detected frequency f' is related to the source's frequency f by:

$$f' = \left(1 + \frac{v}{c}\cos\phi\right) f.$$

By Sine Rule:

$$\frac{\sin\theta}{R} = \frac{\sin(\pi/2 - \phi)}{d} = \frac{\cos\phi}{d}.$$

Thus, we can re-write the expression for f' as:

$$f' = \left(1 + \frac{vd}{cR}\sin\theta\right) f.$$

Since $-1 \leq \sin\theta \leq 1$:

$$f'_{\text{min}} = \left(1 - \frac{vd}{cR}\right) f,$$

$$f'_{\text{max}} = \left(1 + \frac{vd}{cR}\right) f.$$

Meanwhile, in the second experiment, the source is in uniform circular motion around the detector. If the velocity of the source is making an angle ϕ with the vector pointing from the source to the detector, then the detected frequency f'' is related to the source's frequency f by:

$$f'' = \frac{f}{1 + \frac{v}{c}\cos\phi} = \frac{f}{1 + \frac{vd}{cR}\sin\theta}.$$

Therefore:

$$f''_{\text{min}} = \frac{f}{1 + \frac{vd}{cR}},$$
$$f''_{\text{max}} = \frac{f}{1 - \frac{vd}{cR}}.$$

It is easy to see that $f'_{\text{min}} < f''_{\text{min}}$ and $f'_{\text{max}} < f''_{\text{max}}$ since f, v, d, c and R are positive. This implies that the red curve corresponds to the first experiment and the blue curve corresponds to the second one.

We can see from the red curve that $f'_{\text{min}} = 162$ Hz and $f'_{\text{max}} = 238$ Hz. Taking the average of these two values, we can deduce that $f = 200$ Hz.

Recall that v is related to the period T of the circular motion by:

$$v = \frac{2\pi R}{T}.$$

Based on the red curve, the period is $T = 0.04$ s, implying that $v = 50\pi$ m s^{-1}.

By rearranging the equation defining f'_{min}, we obtain:

$$c = \frac{vfd}{R(f - f'_{\text{min}})}.$$

Plugging in the given values then yields $c = \boxed{314 \text{ m s}^{-1}}$.

Setter: Robert Frederik Uy, robert.uy@sgphysicsleague.org