Singapore Mathematical Olympiads 2015

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Singapore Mathematical Olympiad (SMO) 2015

Junior Section (Round 1)

Wednesday, 3 June 2015 **0930-1200 hrs**

Instructions to contestants

1. *Answer ALL 35 questions.*
2. *Enter your answers on the answer sheet provided.*
3. *For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.*
4. *For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.*
5. *No steps are needed to justify your answers.*
6. *Each question carries 1 mark.*
7. *No calculators are allowed.*
8. *Throughout this paper, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. For example, $\lfloor 2.1 \rfloor = 2$, $\lfloor 3.9 \rfloor = 3$.*
9. *Throughout this paper, let $\overline{a_{n-1}a_{n-2}\dots a_0}$ denote an n-digit number with the digits a_i in the corresponding position, i.e. $\overline{a_{n-1}a_{n-2}\dots a_0} = a_{n-1}10^{n-1} + a_{n-2}10^{n-2} + \dots + a_0 10^0$.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

Multiple Choice Questions

1. Among the five numbers $\frac{5}{9}, \frac{4}{7}, \frac{3}{5}, \frac{6}{11}$ and $\frac{13}{21}$, which one has the smallest value?

   (A) $\frac{5}{9}$ (B) $\frac{4}{7}$ (C) $\frac{3}{5}$ (D) $\frac{6}{11}$ (E) $\frac{13}{21}$

2. Adrian, Billy, Christopher, David and Eric are the five starters of a school's basketball team. Two among the five shoot with their left hand while the rest shoot with their right hand. Among the five, only two are more than 1.8 metres in height. Adrian and Billy shoot with the same hand, but Christopher and David shoot with different hands. Billy and Christopher are respectively the shortest and tallest member of the team, while Adrian and David have the same height. Who is more than 1.8 metres tall and shoots with his left hand?

   (A) None (B) Only Christopher (C) Only Eric
   (D) Christopher and Eric (E) Not enough information to ascertain

3. How many ways are there to arrange 3 identical blue balls and 2 identical red balls in a row if the two red balls must always be next to each other?

   (A) 2 (B) 4 (C) 5 (D) 10 (E) 20

4. If a, b and c are positive real numbers such that

$$\frac{a}{a+b} = \frac{a+b}{a+b+c} = \frac{c}{b+c},$$

   then $\frac{a}{b}$ equals

   (A) $\frac{\sqrt{3}-1}{2}$ (B) 1 (C) $\sqrt{2}$ (D) $\frac{1+\sqrt{5}}{2}$ (E) None of the above

5. In the figure below, each distinct letter represents a unique digit such that the arithmetic sum holds. What is the digit represented by the letter B?

$$\begin{array}{ccccc} & M & A & T & H \\ + & M & A & T & H \\ \hline H & A & B & I & T. \end{array}$$

   (A) 0 (B) 2 (C) 4 (D) 6 (E) 8

6. Find the minimum value of the function $2015 - \frac{10}{x^2 - 4x + 5}$.

   (A) 2000 (B) 2005 (C) 2010 (D) 2013 (E) None of the above

7. It is known that 99900009 is the product of four consecutive odd numbers. Find the sum of squares of these four odd numbers.

   (A) 40000 (B) 40010 (C) 40020 (D) 40030 (E) 40040

8. The lengths of the sides of a triangle are x^2, $22-x$ and $x-2$. The total number of possible integer values of x is

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

9. Find the value of $\left(4\sqrt{4+2\sqrt{3}}-\sqrt{49+8\sqrt{3}}\right)^2$.

(A) $3\sqrt{3}$ (B) 6 (C) $4\sqrt{3}$ (D) 9 (E) $4\sqrt{3}+3$

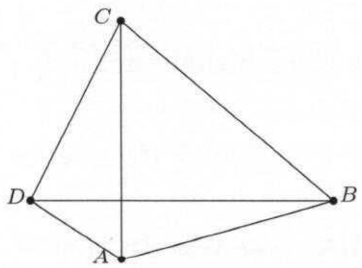
10. If x and y satisfy the equation $2x^2+3y^2=4x$, the maximum value of $10x+6y^2$ is

(A) 2 (B) $\dfrac{9}{2}$ (C) 20 (D) $\dfrac{81}{4}$ (E) None of the above

Short Questions

11. In a shop, the price of a particular type of toy is a whole number greater than \$100. The total sales of this type of toy on a particular Saturday and Sunday were \$1518 and \$2346 respectively. Find the total number of toys sold on these two days.

12. A quadrilateral $ABCD$ has perpendicular diagonals AC and BD with lengths 8 and 10 respectively. Find the area of the quadrilateral.



13. Find the smallest positive integer that is divisible by every integer from 1 to 12.

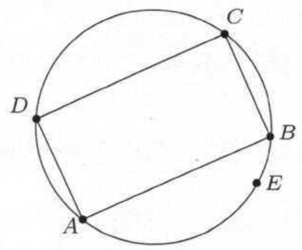
14. Results of a school wide vote for the president of the student council showed that two-fifths of the vote went to Alice, five-twelfths of the vote went to Bobby and the remaining 33 votes went to Charlie. If every student voted, how many students were there in the school?

15. Evaluate the sum $\left\lfloor\dfrac{2^2}{3}\right\rfloor+\left\lfloor\dfrac{3^2}{4}\right\rfloor+\left\lfloor\dfrac{4^2}{5}\right\rfloor+\cdots+\left\lfloor\dfrac{99^2}{100}\right\rfloor$.

16. A quiz was given to a class in which one quarter of the students are male. The class average score on the quiz was 16.5. Excluding three male students whose total score was 21, the average score of all the other students would be 17 while the average score of all the other male students would be 13.25. Find the average score of female students.

17. If $N = 1001^4$, find the sum of all the digits of N.

18. In the diagram below, a rectangle $ABCD$ is inscribed in a circle and E is a point on the circumference of the circle. Given that $|AE|^2 + |BE|^2 + |CE|^2 + |DE|^2 = 450$ and $|AB| \times |BC| = 108$, find the perimeter of the rectangle.



19. Find the largest prime factor of 999936.

20. Find the value of $p + q$, where p and q are two positive integers such that p and q have no common factor larger than 1 and

$$\frac{1}{20} + \frac{1}{30} + \frac{1}{42} + \frac{1}{56} + \frac{1}{72} + \frac{1}{90} = \frac{p}{q}.$$

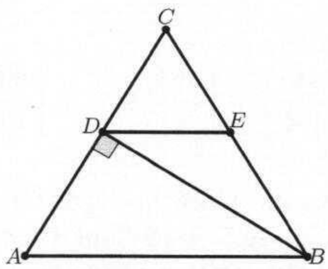
21. Find the value of $\sqrt{(98 \times 100 + 2)(100 \times 102 + 2) + (100 \times 2)^2}$.

22. Find the value of $x^3 - x^2 - 3x + 2015$ if $x = \sqrt{2} + 1$.

23. In the diagram below, ABC is an isosceles triangle with $|AC| = |BC|$. The point D lies on AC such that BD is perpendicular to AC. The point E lies on BC such that DE is parallel to AB. If $|AD| = 3$, $|AB| = 5$ and

$$\frac{\text{Area } \triangle CDE}{\text{Area } \triangle CAB} = \frac{m}{n},$$

where m and n are positive integers with no common factor larger than 1, find the value of $m + n$.



24. Find the remainder when 2015^{2015} is divided by 7.

25. Find the total number of integers in the sequence $20, 21, 22, 23, \cdots, 2014, 2015$ which are multiples of 3 but not multiples of 5.

26. In the quadrilateral $ABCD$, $|AB| = 8$, $|BC| = 1$, $\angle DAB = 30°$ and $\angle ABC = 60°$. If the area of the quadrilateral is $5\sqrt{3}$, find the value of $|AD|^2$.

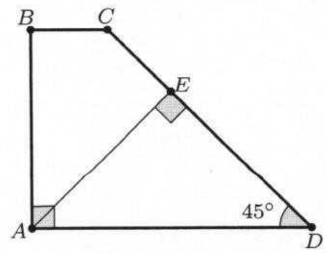
27. Find the total number of six-digit integers of the form $\overline{x2015y}$ which are divisible by 33.

28. The line whose equation is $2x+y = 100$ meets the y-axis at A. B is the point on the x-axis such that AB is perpendicular to the line and C lies on AB such that OC is perpendicular to AB, where O is the origin. D is the foot of perpendicular from C to the x-axis. Find the area of triangle OCD.

29. If $xy < 0$, $\frac{1}{x^2} + \frac{1}{y^2} = 40$ and $x + y = \frac{1}{3}$, find the value of $\frac{1}{x^4} + \frac{1}{y^4}$.

30. Let n be a positive integer. Assume that the sum of n and 7 is a multiple of 8 but the difference of n and 7 is a multiple of 14. Find the largest possible value of n such that $n < 10000$.

31. In the diagram below, $ABCD$ is a trapezium with $\angle D = 45°$, $\angle A = 90°$, $|BC| = 1$ and $|CD| = 2\sqrt{2}$. E is a point on CD such that AE is perpendicular to CD, find the value of $4|AE|^2$.



32. A donut shop sells its donuts *only* in packs of 6 (half-dozen) or in packs of 13 (a baker's dozen). So it is impossible to purchase exactly 14 donuts from this shop, since 14 cannot be written as an integral combination of 6 and 13. Find the largest number of donuts that cannot be purchased from this shop.

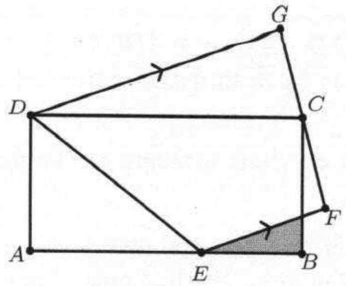
33. If n is a positive integer such that $n^2 - 7n + 17$ is equal to the product of two consecutive odd integers, find the sum of these two consecutive odd integers.

34. Evaluate the sum

$$\left\lfloor\frac{1}{1}\right\rfloor + \left\lfloor\frac{2}{1}\right\rfloor + \left\lfloor\frac{1}{2}\right\rfloor + \left\lfloor\frac{2}{2}\right\rfloor + \left\lfloor\frac{3}{2}\right\rfloor + \left\lfloor\frac{4}{2}\right\rfloor + \left\lfloor\frac{1}{3}\right\rfloor + \left\lfloor\frac{2}{3}\right\rfloor + \left\lfloor\frac{3}{3}\right\rfloor + \left\lfloor\frac{4}{3}\right\rfloor + \left\lfloor\frac{5}{3}\right\rfloor + \left\lfloor\frac{6}{3}\right\rfloor + \cdots$$

up to the 2015^{th} term.

35. In the diagram below, the area of the rectangle $ABCD$ is 80. The point E lies on the side AB. $DEFG$ is a trapezium with parallel sides EF and DG. C is the midpoint of the side FG. Find the area of the trapezium $DEFG$ if the area of $\triangle AED$ is 23 and the area of the shaded triangle is 5.



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Junior Section (Round 1 Solutions)

Multiple Choice Questions

1. Answer: (D)

   It is easy to verify that $\frac{6}{11}$ is the smallest.

2. Answer: (E)

   Since Adrian and David have the same height, they together with Billy are the three members less than 1.8 metres tall. So both Christopher and Eric are more than 1.8 metres tall. As Christopher and David shoot with different hands, one of them together with Adrian and Billy are the three who shoot with their right hand. So Eric certainly shoots with his left hand. However, there is not enough information to determine if Christopher shoots with his left hand.

3. Answer: (B)

   Treat the two red balls as one object. There are exactly 4 ways to arrange one red object and three identical blue balls in a row.

4. Answer: (D)

   $\dfrac{a}{a+b} = \dfrac{a+b}{a+b+c}$ simplifies to $ac = ab + b^2$ while $\dfrac{a}{a+b} = \dfrac{c}{b+c}$ simplifies to $a = c$. Hence

$$\begin{aligned} a^2 - ab - b^2 = 0 &\implies a = \frac{b \pm \sqrt{b^2 + 4b^2}}{2} \\ &\implies \frac{a}{b} = \frac{1+\sqrt{5}}{2}, \end{aligned}$$

   since both a and b are positive.

5. Answer: (A)

   The fourth and fifth columns from the right means H=1 and M is between 5 and 9. Since H=1, the first column gives T=2. Consequently, I=4. It is now easy to check that

$$\begin{array}{r} 7\ 5\ 2\ 1 \\ +\quad 7\ 5\ 2\ 1 \\ \hline 1\ 5\ 0\ 4\ 2. \end{array}$$

   is the only possible solution where all distinct letters represent distinct digits.

6. Answer: (B)

Note that $2015 - \frac{10}{x^2-4x+5}$ has a minimum value when $x^2 - 4x + 5$ is minimum. Since $x^2 - 4x + 5 = (x-2)^2 + 1$ achieves the minimum value 1 at $x = 2$. Thus the answer is $2015 - 10/1 = 2005$.

7. Answer: (C)

Let $n-3$, $n-1$, $n+1$ and $n+3$ be the four consecutive odd numbers, for some even $n > 3$. So
$$(n-3)(n-1)(n+1)(n+3) = (n^2-1)(n^2-9) = 99900009.$$
Simplifying, we have
$$n^4 - 10n^2 - 99900000 = (n^2 - 10000)(n^2 + 9990) = 0.$$
Thus $n = 100$ and the four odd numbers are $97, 99, 101$ and 103. Hence the sum of squares of the four consecutive odd numbers is
$$97^2 + 99^2 + 101^2 + 103^2 = 40020.$$

8. Answer: (A)

For positive lengths, we have $2 < x < 22$. The sides of a triangle must satisfy the triangle inequality. We thus have the following three inequalities
$$\begin{aligned} x^2 + (22-x) > x - 2 &\implies (x-1)^2 + 23 > 0; \\ x^2 + (x-2) > 22 - x &\implies (x+1)^2 > 25; \\ (22-x) + (x-2) > x^2 &\implies x^2 < 20. \end{aligned}$$
The first inequality always holds. The second inequality reduces to $x > 4$ and the last inequality gives $x < \sqrt{20} < 5$. Hence there are no integer values of x satisfying the constraints.

9. Answer: (D)
$$\begin{aligned} 4\sqrt{4+2\sqrt{3}} - \sqrt{49+8\sqrt{3}} &= 4\sqrt{(\sqrt{3}+1)^2} - \sqrt{(4\sqrt{3}+1)^2} \\ &= 4(\sqrt{3}+1) - (4\sqrt{3}+1) = 3. \end{aligned}$$
So $\left(4\sqrt{4+2\sqrt{3}} - \sqrt{49+8\sqrt{3}}\right)^2 = 9$.

10. Answer: (C)

Since $3y^2 = 2x(2-x)$, we have $0 \le x \le 2$. Now
$$\begin{aligned} 10x + 6y^2 &= 10x + 8x - 4x^2 \\ &= -4\left(x - \frac{9}{4}\right)^2 + \frac{81}{4} \end{aligned}$$
This expression achieves maximum at $x = \frac{9}{4}$ but that lies outsides the range $0 \le x \le 2$. So the maximum is achieved at $x = 2$ and equals 20.

Short Questions

11. Answer: 28

$$1518 = 2 \times 3 \times 11 \times 23;$$
$$2346 = 2 \times 3 \times 17 \times 23.$$

The only common factor above 100 is $2 \times 3 \times 23$. Thus there were 11 and 17 toys sold on these two days and 28 in total.

12. Answer: 40

Let O be the point of intersection of AC and BD. If $|CO| = x$, then the area of $ABCD$ is

$$\frac{10x}{2} + \frac{10(8-x)}{2} = \frac{10(8)}{2} = 40.$$

13. Answer: 27720

Let N be the required integer. We require, $5 \mid N$, $7 \mid N$, and $11 \mid N$. In addition, we also need $3^2 \mid N$ and $2^3 \mid N$. Thus the smallest possible N is

$$2^3 \times 3^2 \times 5 \times 7 \times 11 = 27720.$$

14. Answer: 180

Let X be the number of students. We have

$$33 = X - \frac{2}{5}X - \frac{5}{12}X = \frac{11}{60}X.$$

Hence $X = 180$.

15. Answer: 4851

First note that $\left\lfloor \frac{(n-1)^2}{n} \right\rfloor = \left\lfloor n - 2 + \frac{1}{n} \right\rfloor = n - 2$. Hence

$$\begin{aligned} \left\lfloor \frac{2^2}{3} \right\rfloor + \left\lfloor \frac{3^2}{4} \right\rfloor + \left\lfloor \frac{4^2}{5} \right\rfloor + \cdots + \left\lfloor \frac{99^2}{100} \right\rfloor &= 1 + 2 + 3 + \cdots + 98 \\ &= \frac{98 \times 99}{2} = 4851. \end{aligned}$$

16. Answer: 18

Let N be the total number students, x and y be the respective average score of male and female students. Then we have

$$17(N-3) = 16.5N - 21 \implies N = 60.$$

Furthermore

$$\left(\frac{N}{4} - 3\right) \times 13.25 = \frac{N}{4}x - 21 \implies 15x = 12 \times 13.25 + 21.$$

So $x = 12$ and finally we have

$$\frac{N}{4}x + \frac{3N}{4}y = N \times 16.5 \implies 15 \times 12 + 45y = 60 \times 16.5,$$

giving us the answer $y = 18$.

17. Answer: 16

$$N^4 = (1000+1)^4 = (1000)^4 + 4(1000)^3 + 6(1000)^2 + 4(1000) + 1.$$

So the sum of digits equals $1 + 4 + 6 + 4 + 1 = 16$.

18. Answer: 42

Let d be the diameter of the circle, then

$$\begin{aligned} |AE|^2 + |CE|^2 &= |AC|^2 = d^2; \\ |BE|^2 + |DE|^2 &= |BD|^2 = d^2. \end{aligned}$$

So $2d^2 = 450$ which means $d = 15$. Hence

$$\begin{aligned} |AB|^2 + |BC|^2 &= 225; \\ |AB| \times |BC| &= 108. \end{aligned}$$

Assuming without loss of generality that $|BC|$ is shorter, these can be solved to give $|AB| = 12$ and $|BC| = 9$, i.e. the perimeter is $2 \times 21 = 42$.

19. Answer: 31

$$\begin{aligned} 999936 &= 10^6 - 2^6 \\ &= 2^6(5^6 - 1) \\ &= 2^6(5^2 - 1)(5^4 + 5^2 + 1) \\ &= 2^6 \times 24 \times 651 \\ &= 2^9 \times 3^2 \times 7 \times 31. \end{aligned}$$

20. Answer: 23

$$\begin{aligned}&\frac{1}{20}+\frac{1}{30}+\frac{1}{42}+\frac{1}{56}+\frac{1}{72}+\frac{1}{90}\\&=\frac{1}{4}-\frac{1}{5}+\frac{1}{5}-\frac{1}{6}+\frac{1}{6}-\frac{1}{7}+\frac{1}{7}-\frac{1}{8}+\frac{1}{8}-\frac{1}{9}+\frac{1}{9}-\frac{1}{10}\\&=\frac{10-4}{40}=\frac{3}{20}.\end{aligned}$$

21. Answer: 10002

Let $a=100$,

$$\begin{aligned}\sqrt{((a-2)a+2)(a(a+2)+2)+(2a)^2}&=\sqrt{a^2(a^2-4)+2a^2+4a+2a^2-4a+4+4a^2}\\&=\sqrt{a^4+4a^2+4}\\&=\sqrt{(a^2+2)^2}=a^2+2=10002.\end{aligned}$$

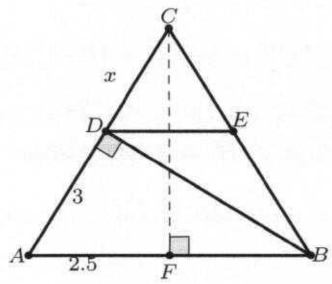
22. Answer: 2016

We have $x^2=3+2\sqrt{2}$. So

$$\begin{aligned}x^3-x^2-3x+2015&=x(x^2-3)-x^2+2015\\&=(\sqrt{2}+1)(2\sqrt{2})-(3+2\sqrt{2})+2015\\&=2016.\end{aligned}$$

23. Answer: 674

Let $|CD|=x$. Construct F the foot of the perpendicular from C to AB. Since $\triangle ABC$ is isosceles, $|AF|=2.5$.



We also have $\triangle BDA$ is similar to $\triangle CFA$, thus

$$\frac{x+3}{2.5}=\frac{5}{3}\implies x=\frac{7}{6}.$$

So

$$\frac{m}{n}=\frac{x^2}{(x+3)^2}=\frac{7^2}{25^2}.$$

Thus $m+n=49+625=674$.

24. Answer: 6

Note that $2015 = 2009 + 6$, where 2009 is a multiple of 7. Thus

$$2015^{2015} \equiv 6^{2015} \pmod 7.$$

Furthermore $6^2 = 36 \equiv 1 \pmod 7$, so

$$6^{2015} \equiv 36^{1007} \times 6 \equiv 6 \pmod 7.$$

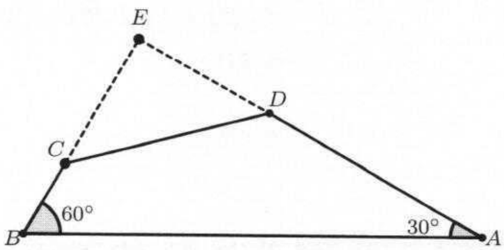
25. Answer: 532

Let A and B be the set of multiples of 3 and 5 respectively in the sequence 20, 21, 22, $23, \cdots, 2015$. We wish to determine $|A - B|$. Now

$$\begin{aligned}|A - B| &= |A| - |A \cap B| \\ &= \lfloor 2015/3 \rfloor - \lfloor 19/3 \rfloor - (\lfloor 2015/15 \rfloor - \lfloor 19/15 \rfloor) \\ &= 671 - 6 - (134 - 1) = 532.\end{aligned}$$

26. Answer: 12

Extend the sides BC and AD to meet at point E. $\triangle AEB$ is a right-angled triangle.



So $|BE| = \frac{1}{2}|AB| = 4$ and

$$|AE|^2 = |AB|^2 - |BE|^2 = 64 - 16 = 48.$$

The area of $\triangle AEB$ is $\frac{1}{2}|BE| \times |AE| = 2\sqrt{48} = 8\sqrt{3}$. Hence the area of $\triangle CED = 3\sqrt{3}$. Combined with the fact that $|CE| = 3$, we deduce that $|ED| = 2\sqrt{3}$. Finally,

$$|AD|^2 = (|AE| - |ED|)^2 = (4\sqrt{3} - 2\sqrt{3})^2 = 12.$$

27. Answer: 3

Note that $\overline{x2015y}$ is divisible by 33 if and only if it is divisible by both 3 and 11, i.e. these two conditions both hold:

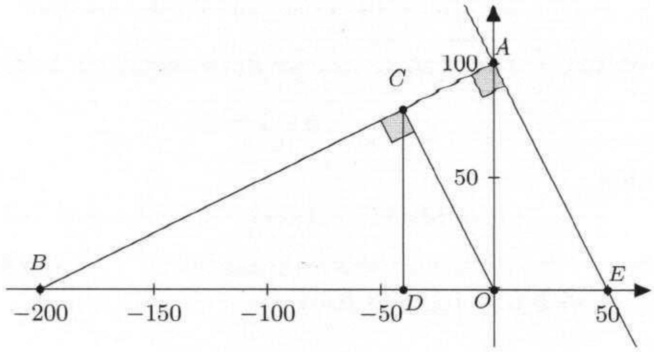
(a) $x + 2 + 0 + 1 + 5 + y = x + y + 8$ is a multiple of 3;

(b) $y - 5 + 1 - 0 + 2 - x = -x + y - 2$ is a multiple of 11.

We can then check that the only possibilities for (x, y) that satisfies both conditions are $(1, 3)$, $(4, 6)$ and $(7, 9)$.

28. Answer: 1600

Let E be the point of intersection of the line $2x + y = 100$ and the x-axis.



We then have $\triangle ABE$ is similar to $\triangle CBO$. Since AB is perpendicular to $2x + y = 100$, we can work out the equation of the line AB to be $y = \frac{1}{2}x + 100$. This implies that B has coordinates $(-200, 0)$ and thus $|BE| = 250$. As $|BO| = 200$, we have the ratio of $|AB| : |CB| = 5 : 4$.

Now observe that CD is parallel to AO, so $\triangle BCD$ is similar to $\triangle BAO$. Thus

$$\frac{|BD|}{|BO|} = \frac{|BC|}{|BA|} \implies |BD| = \frac{4}{5} \times 200 = 160.$$

Similarly

$$\frac{|CD|}{|AO|} = \frac{|BC|}{|BA|} \implies |CD| = \frac{4}{5} \times 100 = 80.$$

Finally, the area of $\triangle OCD$ is

$$\frac{1}{2}|CD| \times |DO| = \frac{1}{2} \times 80 \times (200 - 160) = 1600.$$

29. Answer: 1312

$$40 = \frac{1}{x^2} + \frac{1}{y^2} = \frac{(x+y)^2 - 2xy}{(xy)^2} = \frac{\frac{1}{9} - 2t}{t^2},$$

where we substituted $t = xy$. This quadratic in t gives us $t = -\frac{1}{12}$ since the other solution $t = \frac{1}{30}$ is positive. Now

$$\begin{aligned} \frac{1}{x^4} + \frac{1}{y^4} &= \left(\frac{1}{x^2} + \frac{1}{y^2}\right)^2 - \frac{2}{(xy)^2} \\ &= 40^2 - \frac{2}{t^2} = 1312. \end{aligned}$$

30. Answer: 9961

We have $n+7=8a$ and $n-7=14b$ for some integers a and b. Thus

$$14b+14=8a \implies 7(b+1)=4a.$$

So $7|a$. If we write $a=7k$ for some integer k, the equation becomes

$$b+1=4k.$$

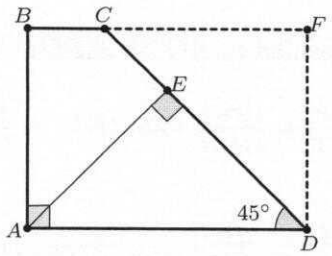
We conclude that

$$n=14b+7=14(4k-1)+7=56k-7.$$

The largest integer k satisfying the condition $56k-7<10000$ is $k=178$. Hence the largest value of n with $n<10000$ is 9961.

31. Answer: 18

Construct a line through D parallel to AB, and extend BC to meet this line at F. $ABFD$ is a rectangle and $\triangle DFC$ is a right-angled isosceles triangle.



Using Pythagoras' Theorem, we have $|DF|^2=|CF|^2=\frac{1}{2}|CD|^2=4$. Thus $|CF|=2$ and $BF=3$ which means rectangle $ABFD$ has area 6.

We then have area of $\triangle ACD=3$ giving us

$$\frac{1}{2}|AE|\times|CD|=3 \implies |AE|=\frac{6}{2\sqrt{2}} \implies 4|AE|^2=18.$$

32. Answer: 59

We are interested in the largest n such that

$$n=6a+13b,$$

has no solutions for nonnegative integers a and b. One can check that $n=59$ has no possible solutions. To show that it is the largest, it suffices to show that $60,61,62,63,64,65$ all have solutions and consequently, every integer greater than 65 has a solution.

$$\begin{aligned}60&=6\times 10\\61&=6\times 8+13\times 1\\62&=6\times 6+13\times 2\\63&=6\times 4+13\times 3\\64&=6\times 2+13\times 4\\65&=6\times 0+13\times 5.\end{aligned}$$

33. Answer: 12

Let

$$n^2 - 7n + 17 = (2k-1)(2k+1) = 4k^2 - 1.$$

By completing the square and simplifying, we obtain

$$4k^2 - \left(n - \frac{7}{2}\right)^2 = \frac{23}{4} \implies (4k)^2 - (2n-7)^2 = 23.$$

Since 23 is prime, we must have

$$4k - (2n-7) = 1 \qquad \text{and} \qquad 4k + (2n-7) = 23.$$

We then have $k = 3$, $n = 9$, which means the product is 5×7 and the required sum is 12.

34. Answer: 1078

First, we determine the largest integer k such that

$$2 + 4 + 6 + \cdots + 2k \leq 2015.$$

It can be found that the largest integer k is 44, and

$$2 + 4 + 6 + \cdots + 2 \times 44 = 44 \times 45 = 1980.$$

For any integer i with $1 \leq i \leq 44$, the above sum includes the following $2i$ terms with denominator i:

$$\left\lfloor \frac{1}{i} \right\rfloor, \left\lfloor \frac{2}{i} \right\rfloor, \left\lfloor \frac{3}{i} \right\rfloor, \ldots, \left\lfloor \frac{2i}{i} \right\rfloor,$$

and the sum of these $2i$ terms is equal to

$$i + 2.$$

As $2015 - 1980 = 35$, there are exactly 35 terms with denominator 45:

$$\left\lfloor \frac{1}{45} \right\rfloor, \left\lfloor \frac{2}{45} \right\rfloor, \left\lfloor \frac{3}{45} \right\rfloor, \ldots, \left\lfloor \frac{35}{45} \right\rfloor,$$

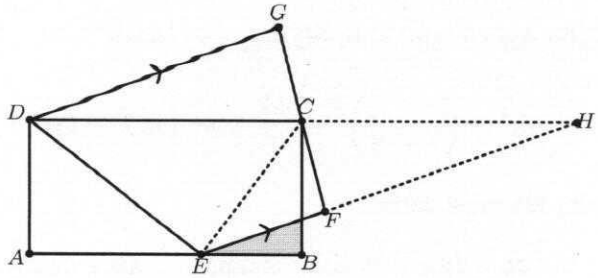
whose sum is 0.

Hence the answer is

$$\sum_{i=1}^{44} (i+2) = 3 + 4 + 5 + \cdots + 46 = 1078.$$

35. Answer: 80

Join CE and extend DC and EF to meet at H.



The area of $\triangle CDE = 40 = \frac{1}{2}$ the area of $ABCD$. Since EH and DG are parallel and $|CG| = |CF|$, $\triangle DCG$ is congruent to $\triangle HCF$ and they have the same area.

So the area of trapezium $DEFG$ = area of $\triangle DEH$. Since $|CD| = |CH|$, the area of $\triangle DEH$ equals twice the area of $\triangle CDE = 80$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2015

(Junior Section, Round 2)

Saturday, 27 June 2015 **0930-1230**

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1. Let $n = \overline{30x070y03}$ be a 9-digit integer. Find all possible values of the pair (x, y), so that n is a multiple of 37.

2. In a convex hexagon $ABCDEF$, AB is parallel to DE, BC is parallel EF and CD is parallel to FA. Prove that the triangles ACE and BDF have the same area.

3. There are 30 children, $a_1, a_2, \ldots, a_{30}$, seated clockwise in a circle on the floor. The teacher walks behind the children in the clockwise direction with a box of 1000 candies. She drops a candy behind the first child a_1. She then skips one child and drops a candy behind the third child, a_3. Now she skips two children and drops a candy behind the next child, a_6. She continues this way, at each stage skipping one child more than at the preceding stage before dropping a candy behind the next child. How many children will never receive a candy? Justify your answer.

4. Let A be a set of numbers chosen from $1, 2, \ldots, 2015$ with the property that any two distinct numbers, say x and y, in A determine a unique isosceles triangle (which is non equilateral) whose sides are of length x or y. What is the largest possible size of A?

5. Find all positive integers k such that $k^k + 1$ is divisible by 30. Justify your answer.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2015

(Junior Section, Round 2 solutions)

1. We have

$$n = 300070003 + 10^6 x + 10^2 y = 37(8110000 + 27027x + 3y) + (3 + x - 11y).$$

Since $0 \le x, y \le 9$, we have $-96 \le 3 + x - 11y \le 12$. Also $37 \mid 3 + x - 11y$. Thus $3 + x - 11y = 0, -37, -74$.

(1) $3 + x - 11y = 0$. Then $x = 8, y = 1$.

(2) $3 + x - 11y = -37$. Then $y = 4$ and $x = 4$.

(3) $3 + x - 11y = -74$. Then $y = 7, x = 0$.

The above are the only three solutions.

2. Let R be the point inside the hexagon such that $ABCR$ is a parallelogram. Let P be the point on the line CR such that $CDEP$ is a parallelogram. P must be inside the hexagon. Let the line AR intersect the line PE at the point Q inside the hexagon. Thus $AQEF$ is also a parallelogram. See Figure 1. Similarly, we may construct parallelograms $ABYF$, $BCDZ$ and $DEFY$ as in Figure 2.

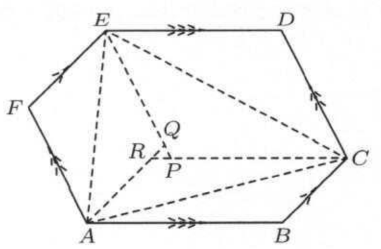


Figure 1

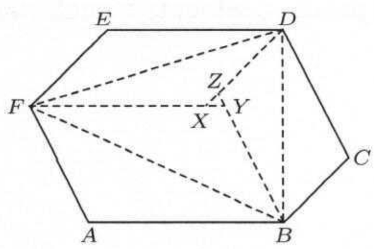


Figure 2

Let's use $[UV \cdots W]$ to denote the area of a polygon $UV \cdots W$. In figure 1, we have $[AQE] = \frac{1}{2}[AQEF]$, $[ACR] = \frac{1}{2}[ABCR]$, $[PCE] = \frac{1}{2}[PCDE]$. Hence $[ACE] = \frac{1}{2}[AQEF] + \frac{1}{2}[ABCR] + \frac{1}{2}[PCDE] + [PQR] = \frac{1}{2}[ABCDEF] + \frac{1}{2}[PQR]$. Similarly, in figure 2, we have $[BDF] = \frac{1}{2}[ABCDEF] + \frac{1}{2}[XYZ]$. Note that the triangles PQR

and XYZ are similar since their sides are parallel to the respective sides of the hexagon. As $PQ = CD - AF = YZ$, they are in fact congruent so that $[PQR] = [XYZ]$. Consequently, $[ACE] = [BDF]$. (Note: The argument works even when $P = Q = R$ as this implies $X = Y = Z$.)

Solution 2. Let the coordinates of A be $(0,0)$, B be $(1,0)$, D be (c,b), E be (a,b), F be (e,f) and C be (x,y). Then $AB \parallel ED$. For $AF \parallel CD$, we need $(b-y)/(c-x) = f/e$. For $EF \parallel CB$ we need $(x-1)/y = (a-e)/(b-f)$. From these equations, we get

$$xb - ay = b - f + cf - be. \qquad (*)$$

Thus

$$[ACE] = [BDF] \quad \Leftrightarrow \quad \begin{vmatrix} 0 & 0 & 1 \\ x & y & 1 \\ a & b & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ c & b & 1 \\ e & f & 1 \end{vmatrix} \quad \Leftrightarrow \quad xb - ay = b - f + cf - be.$$

The proof is now complete since the last equation is true by $(*)$.

3. When the k^{th} candy is dropped, the teacher has skipped $1 + 2 + \cdots + (k-1) = k(k-1)/2$ children. Then it is dropped behind a_i, where $i \equiv k + k(k-1)/2 = k(k+1)/2 \pmod{30}$. Thus $2i \equiv k(k+1) \pmod{60}$. From here it is clear that the i^{th} and j^{th} candies are given to the same child if $i \equiv j \pmod{60}$. Thus the sequence of children receiving candies is periodic with period 60. Note that $k(k+1) \equiv (60-k-1)(60-k) \pmod{60}$. Thus the k^{th} and $(60-k-1)^{\text{th}}$ are given to the same child. Thus we only need to compute $k(k+1)/2 \pmod{30}$ for $k = 1, \ldots, 29$. This yields the following sequence of children receiving the first 29 candies. Thus 18 children never receive any candy.

$$1,\ 3,\ 6,\ 10,\ 15,\ 21,\ 28,\ 6,\ 15,\ 25,\ 6,\ 18,\ 1,\ 15,\ 30$$
$$16,\ 3,\ 21,\ 10,\ 30,\ 21,\ 13,\ 6,\ 30,\ 25,\ 21,\ 18,\ 16,\ 15$$

(The triangular numbers $T_k = k(k+1)/2$ can be computed easily by using $T_{k+1} = k + 1 + T_k$.)

4. Let $x < y$ be two numbers in A. For them to determine a unique isosceles triangle, we must have $2x \le y$. If $A = \{2^0, 2^1, 2^2, \ldots, 2^{10}\}$, then any two of the numbers $x < y$ satisfy $2x \le y$. So the maximum size is ≥ 11.

Now suppose that there is a set A with $|A| = 12$ that has the property. Let $a_1, a_2, \ldots.$, in ascending order, be the members of A. Then we have $a_2 \ge 2a_1$, $a_3 \ge 2a_2 \ge 2^2 a_1$, ..., $a_{12} \ge 2a_{11} \ge 2^{11} a_1 \ge 2^{11} = 2048$, a contradiction.

Thus the maximum size is 11.

5. An integer is divisible by 30 iff it is divisible by 2, 3 and 5.

Note that $2 \mid k^k + 1$ iff k is odd. Thus we may assume that k is odd. Write $k = 2t + 1$.

If $k \equiv 0$ or $1 \pmod 3$, then $3 \nmid k^k + 1$. If $k \equiv 2 \equiv -1 \pmod 3$, then $3 \mid k^k + 1$ iff k is odd, i.e. iff $k = 6t + 5$.

If $k \equiv 0$ or $1 \pmod 5$, $5 \nmid k^k + 1$.

If $k \equiv 2$ or $3 \pmod 5$, then $k \equiv \pm 2 \pmod 5$. Therefore

$$\begin{aligned} k^k + 1 &\equiv (\pm 2)^k + 1 \equiv (\pm 2)^{2t+1} + 1 \equiv (\pm 2)4^t + 1 \\ &\equiv (\pm 2)(-1)^t + 1 \not\equiv 0 \pmod 5. \end{aligned}$$

If $k \equiv 4 \equiv -1 \pmod 5$, then $5 \mid k^k + 1$ iff k is odd. Thus $30 \mid k^k + 1$ iff k is odd and $k \equiv 5 \pmod 6$ and $k \equiv 9 \pmod{10}$. Thus $k = 30n + 29$, $n = 0, 1, 2 \ldots$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2015

Senior Section (Round 1)

Wednesday, 3 June 2015 **0930 – 1200 hrs**

Instructions to contestants

1. *Answer ALL 35 questions.*
2. *Enter your answers on the answer sheet provided.*
3. *For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.*
4. *For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.*
5. *No steps are needed to justify your answers.*
6. *Each question carries 1 mark.*
7. *No calculators are allowed.*
8. *Throughout this paper, let* $\lfloor x \rfloor$ *denote the greatest integer less than or equal to* x*. For example,* $\lfloor 2.1 \rfloor = 2$, $\lfloor 3.9 \rfloor = 3$.

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO.

Multiple Choice Questions

1. Find the exact value of $\dfrac{\sqrt{50}+7}{\sqrt{50}-7}+\dfrac{\sqrt{50}-7}{\sqrt{50}+7}$.

   (A) 197 (B) 198 (C) 199 (D) 200 (E) 201

2. Simplify $\dfrac{3^x+63}{21^{x-2}+7^{x-1}}$.

   (A) $\dfrac{3}{7^x}$ (B) $\dfrac{9\times 49}{7^x}$ (C) $\dfrac{9}{7^x}$ (D) $\dfrac{9\times 7}{7^x}$ (E) $\dfrac{3\times 7}{7^x}$

3. Suppose m and n are real numbers such that the roots of the equation $2x^2-mx+8=0$ are α and β while the roots of the equation $5x^2-10x+5n=0$ are $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$. Find the value of mn.

   (A) $\dfrac{1}{4}$ (B) 4 (C) 8 (D) 12 (E) 16

4. Find the largest number among the following numbers:

   (A) $\sqrt{8}+\sqrt{8}$ (B) $\sqrt{7}+\sqrt{9}$ (C) $\sqrt{6}+\sqrt{10}$ (D) $\sqrt{5}+\sqrt{11}$ (E) $\sqrt{4}+\sqrt{12}$

5. Which of the following is true?

   (A) $\cos 221^\circ > \sin 319^\circ$ (B) $\cos 139^\circ > \sin 221^\circ$ (C) $\sin 139^\circ > \cos 41^\circ$
   (D) $\sin 221^\circ > \cos 139^\circ$ (E) $\sin 41^\circ > \cos 319^\circ$

6. Find the smallest number among the following numbers:

   (A) $\log_{2015} 2016$ (B) $\log_{2016} 2017$ (C) $\log_{2017} 2018$
   (D) $\log_{2018} 2019$ (E) $\log_{2019} 2020$

7. If x and y are non-zero numbers such that $x > y$, which of the following is always true?

   (A) $\dfrac{1}{x}<\dfrac{1}{y}$ (B) $\dfrac{x}{y}>1$ (C) $|x|>|y|$ (D) $\dfrac{1}{xy^2}>\dfrac{1}{x^2y}$ (E) $\dfrac{x}{y}>\dfrac{y}{x}$

8. If $f(x)=2|x^2-1|-x-1$, where $0\le x\le \dfrac{3}{2}$, find the range of $f(x)$.

   (A) $-3\le f(x)\le 1$ (B) $-3\le f(x)\le \dfrac{3}{2}$ (C) $-2\le f(x)\le 1$
   (D) $-2\le f(x)\le \dfrac{3}{2}$ (E) $-2\le f(x)\le 2$

9. How many prime numbers p satisfy the inequality

$$\left|2+\log_{3/p}(3p^2)\right| > \frac{3}{2} ?$$

(A) 6 (B) 7 (C) 8 (D) 9 (E) 10

10. When a polynomial $f(x)$ is divided by $(x-1)$ and $(x+5)$, the remainders are -6 and 6 respectively. Let $r(x)$ be the remainder when $f(x)$ is divided by x^2+4x-5. Find the value of $r(-2)$.

(A) 0 (B) 1 (C) 2 (D) 3 (E) 5

Short Questions

11. Find the value of

$$\frac{2+\log_2 3}{1+\log_2 3}+\frac{3+\log_3 4}{1+\log_3 2}.$$

12. Find the absolute value of the coefficient of $\frac{1}{x}$ in the expansion of

$$\left(2x^2-\frac{1}{x}\right)^{10}.$$

13. If $a+b=\frac{25}{4}$ and $(1+\sqrt{a})(1+\sqrt{b})=\frac{15}{2}$, find the value of ab.

14. Find the value of p if there is a unique value of x satisfying the equation $p^{2x+1}+1=\sqrt{44}e^{x\ln p}$.

15. Suppose x and y are real numbers such that x^2 and y^2 are positive integers. Find the maximum value of x^2-xy if

$$(3x^2-y^2)^2+(x^2+y^2)^2=72.$$

16. Find the smallest positive integer x (measured in degrees) such that

$$\tan(x-160^\circ)=\frac{\cos 50^\circ}{1-\sin 50^\circ}.$$

17. Find the smallest positive integer k such that

$$\frac{1}{\log_{3^k} 2015!}+\frac{1}{\log_{4^k} 2015!}+\cdots+\frac{1}{\log_{2015^k} 2015!} > 2015.$$

18. Let $a < b < c$ be the three real roots of the equation

$$3x^3 - 35x^2 + 500 = 0.$$

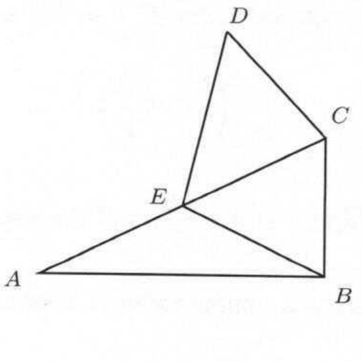
Find $\dfrac{500^2}{(bc)^2} + \dfrac{500^2}{(ac)^2} + \dfrac{500^2}{(ab)^2}$.

19. A function f satisfies $f(x) + f(3x) = x^2 + 1$ for all real numbers x. If $f(2) + f(18) = 6$, determine the value of $f(6)$.

20. It is given that n consecutive positive even integers, the smallest of which is x, have a sum of 154. Find the smallest possible value of x.

21. Find the largest nonnegative integer n such that 2^n is a factor of $\lfloor\sqrt{2}\rfloor \times \lfloor\sqrt{3}\rfloor \times \lfloor\sqrt{4}\rfloor \times \cdots \times \lfloor\sqrt{99}\rfloor$.

22. In the diagram below, $\angle CBA = 90°$, $\angle DCE = 2\angle CAB$ and $AC = 2CD = 2CE =$ 20 meter. Find the maximum possible area (in meter^2) of the quadrilateral $BCDE$.



23. Let $N = 1 + 2 + 2^2 + 2^3 + \cdots + 2^{2015}$. Find the last digit of the number

$$(9 + N)^N.$$

24. Consider an equilateral triangle in which each side has length 1 centimetre. What is the smallest number of points that must be chosen from the region enclosed by the triangle (including the boundary) so that at least two of these points have distance of at most $\frac{1}{2}$ centimetre between them.

25. Find the value of

$$\frac{\cot^3 75° + \tan^3 75°}{\cot 75° + \tan 75°}.$$

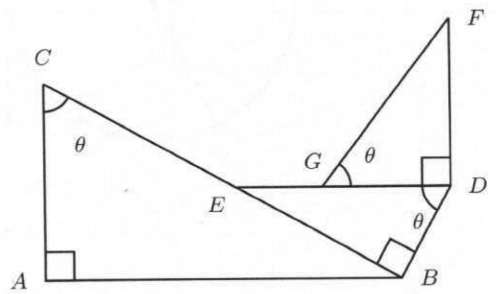
26. Find a 5-digit number so that its digits is completely reversed when multiplying it with some integer n, where $2 \leq n \leq 8$.

27. Consider the following equations:

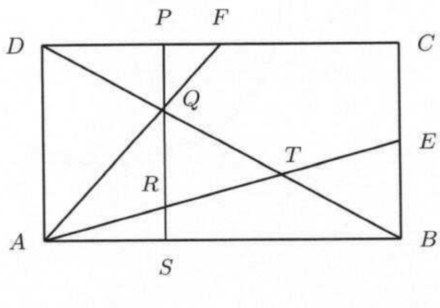
$$\begin{aligned}\frac{1}{x^2} &= \frac{1}{y}+\frac{1}{z},\\ \frac{1}{y^2} &= \frac{1}{x}+\frac{1}{z},\\ \frac{1}{z^2} &= \frac{1}{x^2}+\frac{1}{y^2}.\end{aligned}$$

Find the sum of $(x+1)^4(y+1)^4$ over all possible ordered triples (x,y,z) that satisfy the above three equations simultaneously.

28. The diagram below shows three right-angled triangles, where $BC = 14$, $GF = 10$, $DE = 7$ and $\angle BCA = \angle BDE = \angle FGD = \theta$. Find the maximum possible value of $AB + BD + DF$.

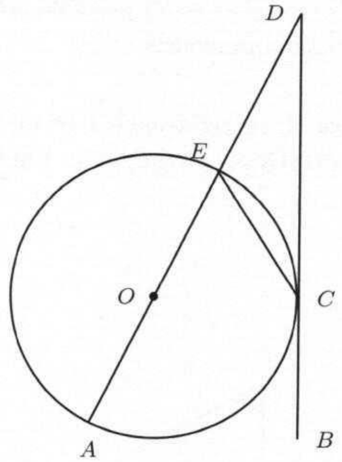


29. The diagram below shows a rectangle $ABCD$ such that E is the midpoint of BC and F is the midpoint of CD. The diagonal BD intersects AF and AE at Q and T respectively. The vertical line PS passing through Q is perpendicular to AB and intersects AE at R. It is also given that $AB = CD = 12$ cm and $BC = AD = 6$ cm. Find the area of the triangle $\triangle QRT$ in cm^2.

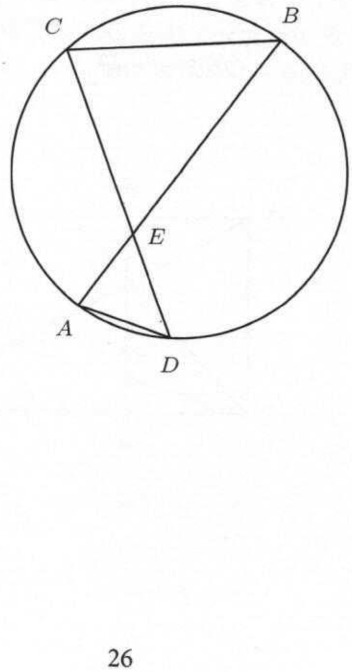


30. Find the minimum value of $13\sec\theta - 9\sin\theta\tan\theta$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

31. In the figure below, the line BD is tangent to the circle at C. The line AD passes through the centre O of the circle and intersects the circle at E. It is given that $\angle CDE = 34^\circ$ and $\angle DCE = x^\circ$. Find the value of x.



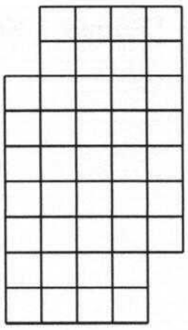
32. In the figure below, A, B, C and D are points on the circle such that the straight lines AB and CD intersect at E. Let $[BCE]$ and $[ADE]$ denote the areas of the triangles $\triangle BCE$ and $\triangle ADE$ respectively. If $\frac{[BCE]}{[ADE]} = 25$ and $AE = 1$ cm, find the length of the line CE in cm.



trendyline

33. In how many ways can a group of 8 different guests (consisting of 4 males and 4 females) be seated at a round table with 8 seats such that there are exactly 3 males who are seated next to each other?

34. Find the number of rectangles that can be formed from the gridlines of the board as shown in the figure below.



35. Let $X = \{1, 2, 3, 4, 5, 6, 7\}$, and let $\mathcal{A} = \{F_1, F_2, \ldots, F_n\}$ be a collection of distinct subsets of X such that the intersection $F_i \cap F_j$ contains exactly one element whenever $i \neq j$. For each $i \in X$, let r_i be the number of elements in $\mathcal{A}$ which contains i. Suppose $r_1 = r_2 = 1$, $r_3 = r_4 = r_5 = r_6 = 2$ and $r_7 = 4$. Find the value of $n^2 - n$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2015

Senior Section (Round 1 Solutions)

Multiple Choice Questions

1. **Answer.** (B)

   Solution.

$$\begin{aligned}\frac{\sqrt{50}+7}{\sqrt{50}-7}+\frac{\sqrt{50}-7}{\sqrt{50}+7} &= \frac{(\sqrt{50}+7)^2+(\sqrt{50}-7)^2}{(\sqrt{50}-7)(\sqrt{50}+7)}\\ &= \frac{50+14\sqrt{50}+49+50-14\sqrt{50}+49}{50-49}\\ &= 198.\end{aligned}$$

2. **Answer.** (B)

   Solution.

$$\begin{aligned}\frac{3^x+63}{21^{x-2}+7^{x-1}} &= \frac{3^x+3^2\times 7}{3^{x-2}\times 7^{x-2}+7^{x-1}}\\ &= \frac{3^2(3^{x-2}+7)}{7^{x-2}(3^{x-2}+7)}\\ &= \frac{9}{7^{x-2}}\\ &= \frac{9\times 7^2}{7^x}=\frac{9\times 49}{7^x}.\end{aligned}$$

3. **Answer.** (B)

   Solution. Since α and β are the roots of the equation $2x^2-mx+8=0$, we have

$$\alpha+\beta=\frac{m}{2},\quad \alpha\beta=4.$$

   Similarly, since $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ are the roots of the equation $5x^2-10x+5n=0$, we have

$$\frac{1}{\alpha}+\frac{1}{\beta}=2,\quad \frac{1}{\alpha\beta}=n.$$

Therefore,

$$\begin{aligned} mn &= \frac{2(\alpha+\beta)}{\alpha\beta} \\ &= 2\left(\frac{1}{\alpha}+\frac{1}{\beta}\right) \\ &= 4. \end{aligned}$$

4. **Answer.** (A)

   Solution. Each of the possible options has the form $\sqrt{a}+\sqrt{b}$, where $a+b=16$. Squaring it results in a number of the form $16+2\sqrt{ab}$. Since $\sqrt{a}+\sqrt{b}$ is maximum if and only if its square is maximum, we pick the one with the largest value of ab, which is given by option (A).

5. **Answer.** (D)

   Solution. Let $x=41$. Then

$$\begin{aligned} \cos 139^\circ &= \cos(180-x)^\circ = -\cos x, \\ \cos 221^\circ &= \cos(180+x)^\circ = -\cos x, \\ \cos 319^\circ &= \cos(360-x)^\circ = \cos x, \\ \sin 139^\circ &= \sin(180-x)^\circ = \sin x, \\ \sin 221^\circ &= \sin(180+x)^\circ = -\sin x, \\ \sin 319^\circ &= \sin(360-x)^\circ = -\sin x. \end{aligned}$$

   (A) $\cos 221^\circ > \sin 319^\circ \iff \sin x > \cos x$
   (B) $\cos 139^\circ > \sin 221^\circ \iff \sin x > \cos x$
   (C) $\sin 139^\circ > \cos 41^\circ \iff \sin x > \cos x$
   (D) $\sin 221^\circ > \cos 139^\circ \iff \cos x > \sin x$
   (E) $\sin 41^\circ > \cos 319^\circ \iff \sin x > \cos x$

   Only option D is correct.

6. **Answer.** (E)

   Solution. All the numbers can be expressed in the form $\log_a(a+1)$ where a is a positive integer. Let $x=\log_a(a+1)$. Note that $x>1$. Then $a+1=a^x$. Dividing both sides by a, we have

$$1+\frac{1}{a}=a^{x-1}.$$

   Since the left-hand side decreases when a increases, the value of x on the right-hand side must decrease as a increases. Thus, the smallest number is $\log_{2019} 2020$.

7. **Answer.** (D)

Solution. The option (A), (B), (C) and (E) are not true if $x = 1$ and $y = -1$. Since $x > y$, we have $x - y > 0$, and

$$\frac{1}{xy^2} - \frac{1}{x^2y} = \frac{x-y}{x^2y^2} > 0.$$

So (D) is always true.

8. **Answer.** (C)

Solution. For $0 \leq x \leq 1$, we have $f(x) = 2(1-x^2) - x - 1 = -(2x-1)(x+1)$ which is decreasing on the interval $[0, 1]$, whence $-2 = f(1) \leq f(x) \leq f(0) = 1$. On the other hand, for $1 \leq x \leq \dfrac{3}{2}$, we have $f(x) = 2(x^2-1) - x - 1 = 2x^2 - x - 3 = (2x-3)(x+1)$ which is increasing on the interval $\left[1, \dfrac{3}{2}\right]$, whence $-2 = f(1) \leq f(x) \leq f(3/2) = 0$. Hence, $-2 \leq f(x) \leq 1$ for $0 \leq x \leq \dfrac{3}{2}$.

9. **Answer.** (C)

Solution. Let $t = \log_3 p$. Then

$$2 + \log_{3/p}(3p^2) = 2 + \frac{1+2t}{1-t} = \frac{3}{1-t}.$$

Hence $\left|\frac{3}{1-t}\right| > \frac{3}{2}$ if and only if $-1 < t < 3$, where $t \neq 1$. So $\frac{1}{3} < p < 27$, where $p \neq 3$. Hence, prime number solutions for the given inequality are $p = 2, 5, 7, 11, 13, 17, 19, 23$.

10. **Answer.** (A)

Solution. We can write $f(x) = (x-1)(x+5)g(x) + r(x)$, where $g(x)$ is some polynomial and $r(x) = Ax + B$ for some constants A and B. Thus

$$-6 = f(1) = r(1) = A + B,$$
$$6 = f(-5) = r(-5) = -5A + B.$$

Solving the above equations, we obtain $r(x) = -2x - 4$. So $r(-2) = 0$.

Short Questions

11. **Answer.** 4

Solution.

$$\begin{aligned}\frac{2+\log_2 3}{1+\log_2 3}+\frac{3+\log_3 4}{1+\log_3 2} &= \frac{2+\log_2 3}{1+\log_2 3}+\frac{3+\log_3 4}{1+\frac{1}{\log_2 3}}\\ &= \frac{2+\log_2 3}{1+\log_2 3}+\frac{3\log_2 3+(\log_2 3)(2\log_3 2)}{1+\log_2 3}\\ &= \frac{4(1+\log_2 3)}{1+\log_2 3}\\ &= 4.\end{aligned}$$

12. **Answer.** 960

Solution. By the Binomial Theorem, a term containing $\frac{1}{x}$ has the form

$$\binom{10}{a}(2x^2)^a\left(-\frac{1}{x}\right)^b,$$

where $a+b=10$ and $2a-b=-1$. It follows that $a=3$, $b=7$ so that the coefficient is

$$\binom{10}{3}2^3(-1)^7=-960.$$

Thus, the absolute value of the coefficient is 960.

13. **Answer.** 9

Solution.

$$\begin{aligned}(1+\sqrt{a}+\sqrt{b})^2 &= 1+a+b+2\sqrt{ab}+2(\sqrt{a}+\sqrt{b})\\ &= 2(1+\sqrt{a})(1+\sqrt{b})+a+b-1\\ &= 15+\frac{25}{4}-1\\ &= \frac{81}{4}.\end{aligned}$$

Thus, $1+\sqrt{a}+\sqrt{b}=\frac{9}{2}$.

$$\begin{aligned}\sqrt{ab} &= (1+\sqrt{a})(1+\sqrt{b})-(1+\sqrt{a}+\sqrt{b})\\ &= \frac{15}{2}-\frac{9}{2}\\ &= 3.\end{aligned}$$

So $ab=9$.

14. **Answer.** 11

Solution. Let $y=p^x$. Then $py^2+1-\sqrt{44}y=0$. Since there is a unique solution of y for the quadratic equation, we must have $(-\sqrt{44})^2-4p(1)=0$, whence $p=11$.

15. **Answer.** 6

Solution. Since 72 is a sum of two squares of integers, the only possibilities are $3x^2 - y^2 = \pm 6$ and $x^2 + y^2 = 6$.

Suppose $3x^2 - y^2 = 6$ and $x^2 + y^2 = 6$. Then $4x^2 = 12$ and so $x = \pm\sqrt{3}$. This implies that $y = \pm\sqrt{3}$. In this case, the maximum value of $x^2 - xy$ is achieved by taking x and y with different signs, giving $x^2 - xy = 3 + 3 = 6$.

Suppose $3x^2 - y^2 = -6$ and $x^2 + y^2 = 6$. Then $4x^2 = 0$ and so $x = 0$ and $y = \pm\sqrt{6}$. The value of $x^2 - xy$ is always 0 in this case.

Hence, the maximum value of $x^2 - xy$ is 6.

16. **Answer.** 50

Solution.

$$\begin{aligned}
\tan(x - 160^\circ) &= \frac{\sin 40^\circ}{1 - \cos 40^\circ} \\
&= \frac{2\sin 20^\circ \cos 20^\circ}{2\sin^2 20^\circ} \\
&= \cot 20^\circ \\
&= \tan 70^\circ
\end{aligned}$$

Therefore,

$$x - 160^\circ = (70 + 180n)^\circ,$$

where n is an integer. The smallest positive value of x is 50, which occurs when $n = -1$.

17. **Answer.** 2016

Solution.

$$\frac{1}{\log_{3^k} 2015!} + \frac{1}{\log_{4^k} 2015!} + \cdots + \frac{1}{\log_{2015^k} 2015!}$$

$$\begin{aligned}
&= \log_{2015!} 3^k + \log_{2015!} 4^k + \cdots + \log_{2015!} 2015^k \\
&= \log_{2015!} 3^k \times 4^k \times \cdots \times 2015^k \\
&= \log_{2015!} \frac{(1 \cdot 2 \cdot 3 \cdot 4 \cdots 2015)^k}{2^k} \\
&= \log_{2015!} \frac{2015!^k}{2^k}
\end{aligned}$$

Hence

$$\log_{2015!} \frac{2015!^k}{2^k} > 2015$$

$$\iff \quad \frac{2015!^k}{2^k} > 2015!^{2015}$$

$$\iff \quad 2015!^k > 2^k 2015!^{2015}$$

The smallest such k is 2016, since $2015! > 2^{2016}$.

18. **Answer.** 1225

Solution. Since a, b and c are the three roots of the equation, we can write

$$(x-a)(x-b)(x-c)=0,$$

which, after expanding, becomes

$$x^3-(a+b+c)x^2+(bc+ab+ac)x-abc=0.$$

Thus,

$$(a+b+c)=\frac{35}{3}, \quad (bc+ab+ac)=0, \quad abc=-\frac{500}{3}.$$

It follows that

$$\begin{aligned}\left(\frac{35}{3}\right)^2 &= (a+b+c)^2\\ &= a^2+b^2+c^2+2(ab+bc+ac)\\ &= a^2+b^2+c^2.\end{aligned}$$

Dividing by $(abc)^2 = \left(\frac{500}{3}\right)^2$, we have

$$\frac{1}{(bc)^2}+\frac{1}{(ac)^2}+\frac{1}{(ab)^2}=\frac{a^2+b^2+c^2}{(abc)^2} = \frac{35^2}{500^2}.$$

Hence,

$$\frac{500^2}{(bc)^2}+\frac{500^2}{(ac)^2}+\frac{500^2}{(ab)^2}=35^2=1225.$$

19. **Answer.** 18

Solution. Substituting $x=6$ and $x=2$ respectively into the equation $f(x)+f(3x)=x^2+1$, we have

$$\begin{aligned} f(6)+f(18) &= 37, \quad (1)\\ f(2)+f(6) &= 5. \quad (2)\end{aligned}$$

Substracting (2) from (1), we have

$$f(18)-f(2)=32.$$

Adding the preceding equation to $f(2)+f(18)=6$, we deduce that $2\cdot f(18)=38$, whence $f(18)=19$. Hence, $f(6)=37-f(18)=37-19=18$.

20. **Answer.** 4

Solution.

$$\begin{aligned} x+(x+2)+(x+4)+\cdots+(x+2n-2) &= 154, \\ n(x+n-1) &= 2\times 7\times 11. \end{aligned}$$

Possible pairs of n and x are:

$$(n,x)=(1,154),(2,76),(7,16),(11,4).$$

So the smallest possible value of x is 4.

21. **Answer.** 87

Solution.

$$\begin{aligned} & \lfloor\sqrt{2}\rfloor\times\lfloor\sqrt{3}\rfloor\times\lfloor\sqrt{4}\rfloor\times\cdots\times\lfloor\sqrt{99}\rfloor \\ = & \left(\lfloor\sqrt{2}\rfloor\times\lfloor\sqrt{3}\rfloor\right)\times\left(\lfloor\sqrt{4}\rfloor\times\lfloor\sqrt{5}\rfloor\times\cdots\times\lfloor\sqrt{8}\rfloor\right)\times\left(\lfloor\sqrt{9}\rfloor\times\lfloor\sqrt{10}\rfloor\times\cdots\times\lfloor\sqrt{15}\rfloor\right) \\ & \times\cdots\times\left(\lfloor\sqrt{81}\rfloor\times\lfloor\sqrt{82}\rfloor\times\cdots\times\lfloor\sqrt{99}\rfloor\right) \\ = & (1\times 1)\times\underbrace{(2\times 2\times\cdots\times 2)}_{5\text{ terms}}\times\underbrace{(3\times 3\times\cdots\times 3)}_{7\text{ terms}}\times\cdots\times\underbrace{(9\times 9\times\cdots\times 9)}_{19\text{ terms}} \\ = & 2^5\times 3^7\times 4^9\times 5^{11}\times 6^{13}\times 7^{15}\times 8^{17}\times 9^{19} \\ = & 2^{5+18+13+51}\times m \\ = & 2^{87}m \end{aligned}$$

where m is some odd integer.

22. **Answer.** 100

Solution. Let $\theta=\angle CAB$. Then the area of the triangle $\triangle BCE$ is given by

$$\begin{aligned} \frac{1}{2}(CE)(CB)\sin(\pi/2-\theta) &= \frac{1}{2}(CE)(AC\sin\theta)\sin(\pi/2-\theta) \\ &= \frac{1}{2}(CE)(AC)\sin\theta\cos\theta \\ &= \frac{1}{4}(CE)(AC)\sin 2\theta \\ &= 50\sin 2\theta. \end{aligned}$$

On the other hand, the area of the triangle $\triangle CDE$ is $\frac{1}{2}(CE)(CD)\sin 2\theta = 50\sin 2\theta$. Hence, the area of the quadrilateral is $100\sin 2\theta$ which is maximised when $\theta=\frac{\pi}{4}$. The maximum area is 100 meter^2.

23. **Answer.** 7

Solution. Note that for $k \geq 0$, we have

$$\begin{aligned} 2^{4k+1} &\equiv 2 \pmod{10} \\ 2^{4k+2} &\equiv 4 \pmod{10} \\ 2^{4k+3} &\equiv 8 \pmod{10} \\ 2^{4k+4} &\equiv 6 \pmod{10} \end{aligned}$$

Since $2015 \equiv 3 \pmod 4$, we have

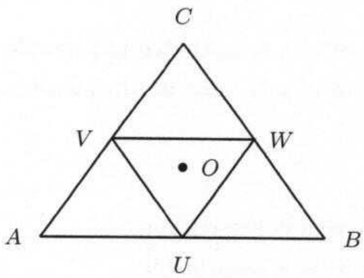
$$\begin{aligned} 9+N &\equiv 9+(2+4+8+6)+\cdots+(2+4+8+6)+(2+4+8) \pmod{10} \\ &\equiv 9+(2+4+8) \pmod{10} \\ &\equiv 3 \pmod{10} \end{aligned}$$

On the other hand, we have $3^{2^m} \equiv 1 \pmod{10}$ for all positive integers $m \geq 2$. It follows that

$$\begin{aligned} (9+N)^N &\equiv 3^N \pmod{10} \\ &\equiv 3^{1+2} \pmod{10} \\ &\equiv 27 \equiv 7 \pmod{10}. \end{aligned}$$

24. **Answer.** 5

Solution. Let U, V , W be the midpoint of each side of the triangle and let O be the centroid of the triangle as shown in the following diagram:



If the points O, A, B and C were chosen, then their pairwise distances are all more than $\frac{1}{2}$ cm. Thus, four points are not enough to ensure that at least two of the points will have distance at most $\frac{1}{2}$ cm. Suppose five points are chosen. By the pigeonhole principle, one of the triangles $\triangle AUV$, $\triangle BUW$, $\triangle UVW$ and $\triangle CVW$ will contain at least two of the points. It is now clear that any two points in such a triangle will have distance at most $\frac{1}{2}$ cm.

25. **Answer.** 13

Solution. Let $t = \tan 75^\circ$. Then

$$\begin{aligned}
\frac{\cot^3 75^\circ + \tan^3 75^\circ}{\cot 75^\circ + \tan 75^\circ} &= \frac{(t+\frac{1}{t})(t^2 - 1 + \frac{1}{t^2})}{t + \frac{1}{t}} \\
&= t^2 - 1 + \frac{1}{t^2} \\
&= \frac{\sin^4 75^\circ + \cos^4 75^\circ}{\sin^2 75^\circ \cos^2 75^\circ} - 1 \\
&= \frac{(\sin^2 75^\circ + \cos^2 75^\circ)^2 - 2\sin^2 75^\circ \cos^2 75^\circ}{\sin^2 75^\circ \cos^2 75^\circ} - 1 \\
&= \frac{1 - \frac{1}{2}\sin^2 150^\circ}{\frac{1}{4}\sin^2 150^\circ} - 1 \\
&= \frac{1 - \frac{1}{8}}{\frac{1}{16}} - 1 \\
&= 13.
\end{aligned}$$

26. **Answer.** 21978

Solution. Write $N = \overline{abcde}$. Then $\overline{abcde} \times n = \overline{edcba}$.

If $n \geq 5$, then $a = 1$. Then n must be odd and $n \neq 5$. So $n = 7$. But this would imply that $e = 3$, which is impossible.

If $n = 2$, then a is even and $a < 5$; so $a = 2$ or $a = 4$.

(i) If $a = 2$, then $e = 1$ or $e = 6$; both are impossible.
(ii) If $a = 4$, then $e = 2$ or $e = 7$; both are impossible.

If $n = 3$, then $a \leq 3$.

(i) If $a = 1$, then $e = 7$; this is impossible.
(ii) If $a = 2$, then $e = 4$; this is impossible.
(iii) If $a = 3$, then $e = 1$; this is impossible.

If $n = 4$, then $a \leq 2$ and a is even. So $a = 2$. It follows that $e = 8$ ($e = 3$ is rejected). So

$$\overline{bcd} \times 4 + 3 = \overline{dcb}.$$

In particular, b is odd and $b \leq 2$; so $b = 1$, and thus $d = 7$ ($d = 2$ is rejected). We have

$$c \times 4 + 3 = \overline{3c} \Rightarrow c = 9.$$

Therefore, $N = \overline{abcde} = 21978$.

27. **Answer.** 32

Solution.

$$\begin{aligned} \frac{1}{x^2} &= \frac{1}{y}+\frac{1}{z}, & (3)\\ \frac{1}{y^2} &= \frac{1}{x}+\frac{1}{z}, & (4)\\ \frac{1}{z^2} &= \frac{1}{x^2}+\frac{1}{y^2}. & (5) \end{aligned}$$

Clearly none of x, y, z are 0. Subtracting (4) from (3), we have

$$\begin{aligned} \frac{1}{x^2}-\frac{1}{y^2} &= \frac{1}{y}-\frac{1}{x},\\ \frac{y^2-x^2}{x^2y^2} &= \frac{x-y}{xy},\\ (y-x)(x+y+xy) &= 0,\\ (y-x)\left(\frac{1}{y}+\frac{1}{x}+1\right) &= 0. & (6) \end{aligned}$$

So either $x=y$ or $\frac{1}{x}+\frac{1}{y}=-1$. In the latter, by substituting (3) and (4) into (5), we have $\frac{1}{z^2}=\frac{2}{z}-1$, which implies that $z=1$. By substituting $\frac{1}{y}=-1-\frac{1}{x}$ and $z=1$ into (3), we have $x=-1$. Now, by (4), $\frac{1}{y^2}=-1+1=0$, which is impossible.

Thus, we may assume that $x=y$. It follows from (3) that $\frac{1}{z}=\frac{1-x}{x^2}$. On the other hand, from (5) $\frac{1}{z^2}=\frac{2}{x^2}$. Combining these equations, we obtain

$$\frac{2}{x^2}=\frac{(1-x)^2}{x^4},$$

which implies that $x+1=\pm\sqrt{2}$. Since $x=y$, we conclude that the sum of $(x+1)^4(y+1)^4$ over all possible solution (x,y,z) is

$$4\times 4+4\times 4=32.$$

28. **Answer.** 25

Solution. Note that

$$AB+BD+DF=14\sin\theta+7\cos\theta+10\sin\theta=24\sin\theta+7\cos\theta.$$

Writing $24\sin\theta+7\cos\theta=R\sin(\theta+\alpha)=R\sin\theta\cos\alpha+R\cos\theta\sin\alpha$, we deduce that

$$\begin{aligned} R &= \sqrt{24^2+7^2}=25,\\ \alpha &= \tan^{-1}\frac{7}{24}. \end{aligned}$$

So the maximum length of $AB+BD+DF$ is 25 and this occurs when $\theta=90^\circ-\alpha$.

29. **Answer.** 6

Solution. Placing the rectangle on the xy-plane gives the following coordinates: $A(0,0)$, $B(12,0)$, $C(12,6)$, $D(0,6)$, $E(6,3)$, $F(6,6)$. The lines AF, AE, BD can be represented by the following equations:

$$\begin{aligned} \text{line } AF: \ y &= x && (7)\\ \text{line } AE: \ y &= \frac{1}{4}x && (8)\\ \text{line } BD: \ y &= 6 - \frac{1}{2}x && (9) \end{aligned}$$

Therefore, $Q(4,4)$ and $T(8,2)$. Since the line PS passes through Q, the equation for the line PS is $x = 4$. Thus, we have $R(4,1)$. It follows that

$$\begin{aligned} \text{The area of } \triangle QRT &= \frac{1}{2}\begin{vmatrix} 4 & 8 & 4 & 4 \\ 4 & 2 & 1 & 4 \end{vmatrix} \\ &= \frac{1}{2}\left|(4(2) + 8(1) + 4(4)) - (8(4) + 4(2) + 4(1))\right| \\ &= \frac{1}{2}\left|32 - 44\right| \\ &= 6. \end{aligned}$$

30. **Answer.** 12

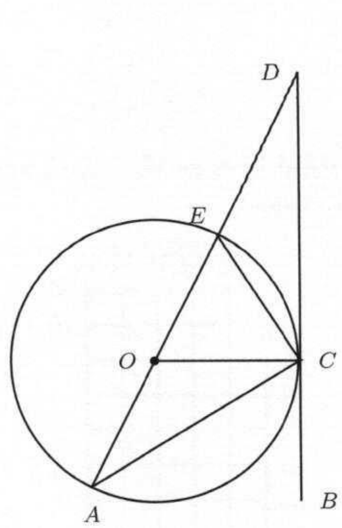
Solution.

$$\begin{aligned} 13\sec\theta - 9\sin\theta\tan\theta &= \frac{13 - 9\sin^2\theta}{\cos\theta} \\ &= \frac{13 - (9 - 9\cos^2\theta)}{\cos\theta} \\ &= \frac{4}{\cos\theta} + 9\cos\theta \\ &\geq 2\sqrt{\left(\frac{4}{\cos\theta}\right)(9\cos\theta)} = 12, \end{aligned}$$

with the minimum value attained when $\frac{4}{\cos\theta} = 9\cos\theta$, i.e. $\cos\theta = \frac{2}{3}$.

31. **Answer.** 28

Solution. As shown in the diagram below, we have $\angle OCD = 90°$. So in the right-angled triangle $\triangle OCD$, $\angle DOC = 90° - \angle CDE = 90° - 34° = 56°$. Note that $\angle OAC + \angle OCA = \angle DOC = 56°$. Since $\angle OAC = \angle OCA$, we deduce that $\angle OAC = \angle OCA = \frac{56}{2}^\circ = 28°$. Now, since $\angle ACE = 90°$, we have $\angle OCE = 90° - \angle OCA = 90° - 28° = 62°$. Also, since $\angle OCD = 90°$, we deduce that $\angle DCE = 90° - \angle OCE = 90° - 62° = 28°$.



32. **Answer.** 5

Solution. By the Intersecting Chord Theorem, we have

$$\begin{aligned} AE \cdot EB &= CE \cdot ED \\ \frac{CE}{AE} &= \frac{EB}{ED} \end{aligned}$$

Let $\theta = \angle BEC = \angle AED$. Then

$$\begin{aligned} 25 &= \frac{[BCE]}{[ADE]} \\ &= \frac{\frac{1}{2}EB \cdot CE \sin\theta}{\frac{1}{2}AE \cdot ED \sin\theta} \\ &= \frac{EB \cdot CE}{AE \cdot ED} \\ &= \left(\frac{CE}{AE}\right)^2. \end{aligned}$$

Since $AE = 1$, we deduce that $CE = 5$.

33. **Answer.** 1728

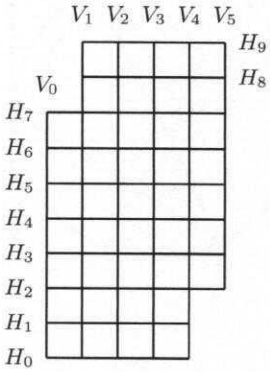
Solution. First, we count the number of ways to arrange two females and three males in a row $F_1M_1M_2M_3F_2$, where F_1, F_2 are different females, and M_1, M_2, M_3 are different males selected from the group. There are $\binom{4}{3} = 4$ ways to choose $\{M_1, M_2, M_3\}$ while there are $\binom{4}{2} = 6$ ways to choose $\{F_1, F_2\}$. Having chosen these sets, there are $3! = 6$ ways to arrange the males in a row, and there are $2! = 2$ ways to put the females at the sides. Hence, the total number of such arrangements is $4 \times 6 \times 6 \times 2 = 288$.

Now, using the sequence $F_1M_1M_2M_3F_2$ as the reference for our circular arrangement, the number of ways such that only exactly 3 males siting next to each other is

$$288 \times (4-1)! = 288 \times 6 = 1728.$$

34. **Answer.** 509

Solution. Label the vertical lines as V_0, ..., V_5 and the horizontal lines as H_0, ..., H_9, as shown in the figure below.



Every rectangle uses two distinct horizontal lines (top and bottom) and two distinct vertical lines (left and right). Suppose that R is such a rectangle. Exactly one of the following possibilities must hold for R:

- The left vertical line is V_0, and the right vertical line is one of V_1, V_2, V_3, V_4. There are 4 choices for the right vertical line. The horizontal lines must come from H_0, H_1, H_2, H_3, H_4, H_5, H_6, H_7. There are $\binom{8}{2} = 28$ choices of the horizontal lines. So the total number in this case is $4 \times 28 = 112$.
- The left vertical line is V_0, and the right vertical line is V_5. The horizontal lines must come from H_2, H_3, H_4, H_5, H_6, H_7. There are $\binom{6}{2} = 15$ choices of the horizontal lines. So the total number in this case is 15.
- The left and right vertical lines come from V_1, V_2, V_3, V_4. There are $\binom{4}{2} = 6$ choices of the vertical lines. The horizontal lines must come from H_0, H_1, H_2, H_3, H_4, H_5, H_6, H_7, H_8, H_9. There are $\binom{10}{2} = 45$ choices of the horizontal lines. So the total number in this case is $6 \times 45 = 270$.
- The left vertical line is one of V_1, V_2, V_3, V_4, and the right vertical line is V_5. There are 4 choices for the left vertical line. The horizontal lines must come from H_2, H_3, H_4, H_5, H_6, H_7, H_8, H_9. There are $\binom{8}{2} = 28$ choices of the horizontal lines. So the total number in this case is $4 \times 28 = 112$.

Hence, the total number of rectangles is $112 + 15 + 270 + 112 = 509$.

35. **Answer.** 20

Solution. Let ϕ denote the number of ordered triples (i, F_j, F_k), $j \neq k$, such that $i \in X$ and $i \in F_j \cap F_k$. For every fixed $i \in X$, there are precisely $r_i(r_i - 1)$ choices for the ordered pair (F_j, F_k). Thus, $\phi = \sum_{i=1}^{7} r_i(r_i - 1)$.

On the other hand, there are exactly $n(n-1)$ ways we can form an ordered pair (F_j, F_k)

using the sets from $\mathcal{A}$. Since every such pair must intersect in exactly one point, there is a unique i such that $i \in F_j \cap F_k$. Thus, $\phi = 1 \times n(n-1)$.

Hence,

$$\begin{aligned} n(n-1) &= \sum_{i=1}^{7} r_i(r_i - 1) \\ &= 1(0) + 1(0) + 2(1) + 2(1) + 2(1) + 2(1) + 4(3) \\ &= 20. \end{aligned}$$

Such a collection $\mathcal{A}$ is possible by taking $\mathcal{A} = \{\{1,3,7\},\{2,4,7\},\{5,7\},\{6,7\},\{3,4,5,6\}\}$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2015

(Senior Section, Round 2)

Saturday, 27 June 2015 **0900-1300**

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1. In an acute-angled triangle ABC, M is a point on the side BC, the line AM meets the circumcircle ω of ABC at the point Q distinct from A. The tangent to ω at Q intersects the line through M perpendicular to the diameter AK of ω at the point P. Let L be the point on ω distinct from Q such that PL is tangent to ω at L. Prove that L, M and K are collinear.

2. There are $n = 1681$ children, $a_1, a_2, \ldots, a_n$, seated clockwise in a circle on the floor. The teacher walks behind the children in the clockwise direction. She drops a candy behind the first child a_1. She then skips one child and drops a candy behind the third child, a_3. Now she skips two children and drops a candy behind the next child, a_6. She continues this way, at each stage skipping one child more than at the preceding stage before dropping a candy behind the next child. How many children will never receive a candy? Justify your answer.

3. Let $n \geq 3$ be an integer. Prove that there exist positive integers ≥ 2, $a_1, a_2, \ldots, a_n$, such that $a_1 a_2 \cdots \hat{a}_i \cdots a_n \equiv 1 \pmod{a_i}$, for $i = 1, \ldots, n$. Here $\hat{a}_i$ means the term a_i is omitted.

4. Is it possible to color each square on a 9×9 board so that each 2×3 or 3×2 block contains exactly 2 black squares? If so, what is/are the possible total number(s) of black squares?

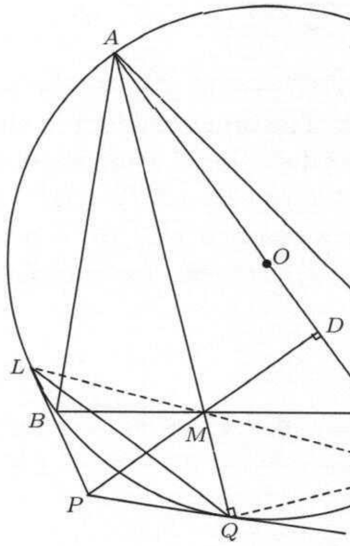
5. Let A be a point on the circle ω centred at B and Γ a circle centred at A. For $i = 1, 2, 3$, a chord P_iQ_i of ω is tangent to Γ at S_i and another chord P_iR_i of ω is perpendicular to AB at M_i. Let Q_iT_i be the other tangent from Q_i to Γ at T_i, and N_i the intersection of AQ_i with M_iT_i. Prove that N_1, N_2, N_3 are collinear.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2015

(Senior Section, Round 2 solutions)

1.



Let D be the foot of the perpendicular from M onto the diameter AK. Since AK is a diameter of ω, we have $\angle MQK = 90^\circ = \angle MDK$ so that M, Q, K, D are concyclic. Join QK. Then $\angle PMQ = \angle QKD = \angle QKA = \angle PQA = \angle PQM$. Thus the triangle PQM is isosceles with $PQ = PM$. Therefore, $PQ = PM = PL$ and P is the circumcentre of the triangle QML. It follows that $\angle QLM = \frac{1}{2}\angle QPM = 90^\circ - \angle PQM = 90^\circ - \angle QKA = \angle QAK = \angle QLK$. This shows that L, M and K are collinear.

Remark. If M is the midpoint of BC, then PM is tangent to the 9-point circle of ABC and the line LK passes through the orthocentre of ABC.

2. Note that $n = 41^2$ where 41 is a prime number. We shall consider the general case where $n = p^2$ where p is an odd prime. When the k^{th} candy is dropped, the teacher has skipped $1 + 2 + \cdots + (k-1) = k(k-1)/2$ children. Therefore it is dropped behind a_i, where $i \equiv k + k(k-1)/2 = k(k+1)/2 \pmod{p^2}$. From here it is clear that the i^{th} and j^{th} candies are given to the same child if $i \equiv j \pmod{p^2}$. Thus the sequence of children receiving candies is periodic with period p^2. Note that $k(k+1) \equiv (p^2-k-1)(p^2-k) \pmod{p^2}$. Thus the k^{th} and $(p^2-k-1)^{\text{th}}$ are given to the same child. Therefore the sequence $d_1, d_2, \ldots$, where d_i is the child who receive the i^{th} candy, is periodic with period p^2. Thus we may add d_0, where $d_0 = d_{p^2}$ and we only need to investigate the sequence $d_0, d_1, \ldots, d_{(p^2-1)/2}$. Now let $0 \le i < j \le (p^2-1)/2$. We have

$$d_i = d_j \Leftrightarrow i(i+1) \equiv j(j+1) \Leftrightarrow (j-i)(j+i+1) \equiv 0 \pmod{p^2}$$

Since $0 \le i < j \le (p^2-1)/2$, we have

$$\begin{aligned} & j - i = sp, \quad j+i+1 = tp \quad \text{for some integers } s, t \text{ of opposite parity.} \\ \Leftrightarrow \quad & j = \frac{(s+t)p-1}{2}, \quad i = \frac{(t-s)p-1}{2}. \end{aligned}$$

Thus we see that d_i, where $i \equiv (p-1)/2 \pmod{p^2}$, are the same child and there are $(p+1)/2$ such i from 0 to $(p^2-1)/2$. The other children in the sequence are distinct. Since $(p^2+1)/2 = (p^2+1)/2$ candies have been distributed, the number of children who receive at least one candy is $(p^2+1)/2-(p-1)/2 = (p^2-p+2)/2$. Therefore the number of children who never receive a candy is $p^2-(p^2-p+2)/2 = (p^2+p-2)/2$. In our case, the answer is $(41^2+41-2)/2 = 860$. Incidentally, the child who receives the most candies is $a_{(p^2-1)/8}$

3. Let $a_1 = 2, a_2 = 3$. For $i = 3, \ldots, n-1$, let $a_i = a_1a_2\cdots a_{i-1}+1$. Let $a_n = a_1a_2\cdots a_{n-1}-1$. Clearly, $a_1a_2\cdots a_{n-1} \equiv 1 \pmod{a_n}$. Also $a_{i+1} \equiv a_{i+2} \equiv \cdots \equiv a_{n-1} \equiv 1 \pmod{a_i}$. For $i = 1, \ldots, n-1$, we have

$$a_1a_2\cdots \hat{a}_i \cdots a_n = (a_1\cdots a_{i-1})(a_{i+1}\cdots a_{n-1})a_n \equiv (-1)(1)(-1) \equiv 1 \pmod{a_i}.$$

4. We assume each nonblack square is white. Repeat the 3×3 block with black squares down the main diagonal 9 times. This is a possible construction with $3 \times 9 = 27$ black squares.

We shall prove that this is the only answer.

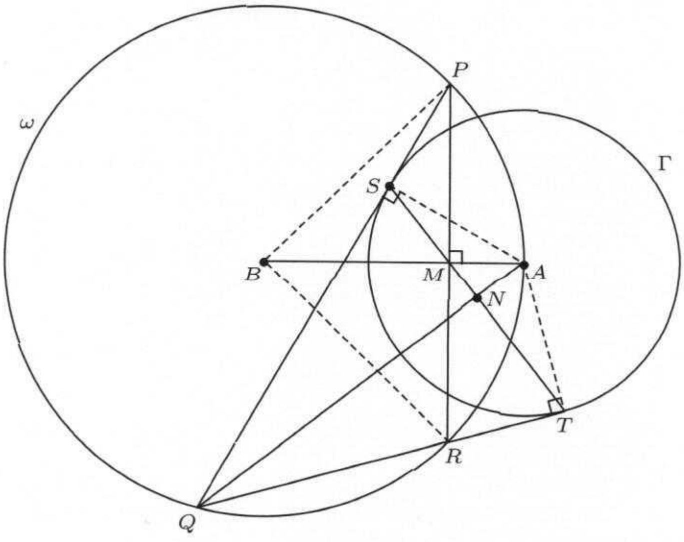
Suppose there are 26 black squares. We name the bottom left square as $(1,1)$. Remove the squares in the block $\{(1,1),(1,2),(1,3)\}$. The remaining squares can be partitioned into 3×2 or 2×3 blocks which contain 26 black squares. Thus the removed squares are white. Repeat the argument with the blocks $\{(3,1),(3,2),(3,3)\}$, $\{(1,1),(2,1),(3,1)\}$, $\{(1,3),(2,3),(3,3)\}$, we conclude that all these squares are white. Thus we have a 3×3 block with at most 1 black square (see figure below). This is a contradiction.

$$\begin{array}{ccc} w & w & w \\ w & b & w \\ w & w & w \end{array}$$

Suppose there are 28 black squares. Using the same argument as above, we conclude that there are 2 black squares in each of the blocks $\{(1,1),(1,2),(1,3)\}$, $\{(3,1),(3,2),(3,3)\}$, $\{(1,1),(2,1),(3,1)\}$ and $\{(1,3),(2,3),(3,3)\}$. Thus all the squares in the three blocks $\{(2,1),(2,2),(2,3)\}$, $\{(1,2),(2,2),(3,2)\}$, $\{(1,4),(2,4),(3,4)\}$ are white as shown in the following diagram, again leading to a contradiction.

$$\begin{array}{ccc} w & w & w \\ b & w & b \\ w & w & w \\ b & w & b \end{array}$$

5. We shall prove that N_i lies on the radical axis of Γ and ω. As such we shall drop the subscript i. Join BR and BP. As PR is perpendicular to the radius BA of ω, the point A is the midpoint of the arc PR. Since $\angle RQA = \frac{1}{2}\angle RBA = \frac{1}{2}PBA = \angle PQA$ and QP is tangent to Γ, we must have QR is also tangent to Γ so that Q, R, T are collinear. By Simson's theorem applied to the triangle PQR and the point A on the circumcircle of the triangle PQR, the points S, M, T are collinear. Therefore, N is the intersection of AQ and ST. It is easy to show that $\angle ANT$ is a right angle. Thus $SN \cdot NT = AN \cdot NQ$ so that N is of equal power with respect to the circles ω and Γ.



Remark. Note that N is also the inverse of Q with respect to the inversion in the circle Γ. As the circle ω is inverted to the radical axis of Γ and ω, the point N being the inverse of Q is on the radical axis of Γ and ω.

Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2015
(Open Section, Round 1)

Thursday, 4 June 2015 **0930-1200 hrs**

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.*
3. *No steps are needed to justify your answers.*
4. *Each question carries 1 mark.*
5. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

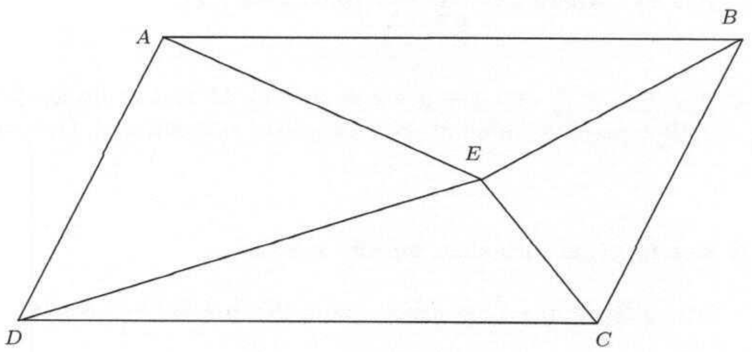
In this paper, let $\lfloor x \rfloor$ denote the greatest integer not exceeding x. For examples, $\lfloor 5 \rfloor = 5$, $\lfloor 2.8 \rfloor = 2$, and $\lfloor -2.3 \rfloor = -3$.

1. Find the largest positive integer N for which $n^5 - 5n^3 + 4n$ is divisible by N for all positive integers n.

2. Consider all sequences of numbers with distinct terms which follow a geometric progression such that the first, second and fourth terms of the sequence are three consecutive terms of an arithmetic progression. Find the sum of squares of all the possible common ratios of these sequences.

3. Suppose that a given sequence $\{x_n\}$ satisfies the conditions that $x_1 = 1$ and, for $n \geq 1$,
$$x_{n+1} = \frac{1}{16}(1 + 4x_n + \sqrt{1 + 24x_n}).$$
Determine $\lim_{n \to \infty} 3x_n$.

4. In the figure below, E is a point inside the parallelogram $ABCD$ such that $\angle DAE = \angle DCE = 50^\circ$ and $\angle ABE = 30^\circ$. Find $\angle ADE$ in degrees.



5. It is given that $S = \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \cdots + \frac{1}{2014 \times 2015}$. Find the value of $\lfloor 4030 \times S \rfloor$.

6. Let $ABCD$ be a trapezium with AB parallel to DC, and that $AB = 20$ cm, $CD = 30$ cm, $BD = 40$ cm and $AC = 30$ cm. Let E be the intersection of AC and BD. Find the area of triangle DEC in cm^2.

7. How many distinct integers are there in the following sequence:
$$\left\lfloor \frac{1^2}{2015} \right\rfloor, \left\lfloor \frac{2^2}{2015} \right\rfloor, \left\lfloor \frac{3^2}{2015} \right\rfloor, \ldots, \left\lfloor \frac{2015^2}{2015} \right\rfloor?$$

8. Let S be the sum of all the positive solutions of the equation $\sqrt{\frac{4 - x^2}{3}} + \sqrt{\frac{x^2 - 1}{3}} = 1$. Find $\lfloor S \rfloor$.

9. Given that $\dfrac{(1+10)(1+10^2)(1+10^4)\cdots(1+10^{2^m})}{1+10+10^2+10^3+10^4+\cdots+10^{127}} = 1$, find the value of m.

10. Let a_n be the nth term of a geometric progression, where $a_1 = 1$ and $a_3 = 3$. Find

$$\left(\sum_{i=0}^{10}\binom{10}{i}a_{i+1}\right)\cdot\left(\sum_{j=0}^{10}(-1)^j\binom{10}{j}a_{j+1}\right).$$

11. Let $y = \sqrt{38x-152}+\sqrt{2015-403x}$ be a real function. Find the largest possible value of y.

12. Find the coefficient of x^{50} in the expansion of $(x+1)(x+2)(x+3)\cdots(x+50)(x+51)(x+52)$.

13. Fifty numbers from the set $\{1,2,\ldots,100\}$ are chosen and another fifty numbers from the set $\{101,102,\ldots,200\}$ are chosen. It is known that no two chosen numbers differ by 0 or 100. Determine the sum of all the 100 chosen numbers.

14. Let $H = \left(x^3-x^2+x\right)^9$, where $x = \dfrac{2}{\sqrt{5}-1}$. Determine $\lfloor H \rfloor$.

15. Assume that $x \geq y \geq z \geq \frac{\pi}{12}$ and $x+y+z = \frac{\pi}{2}$. Let M and m be the largest possible value and the smallest possible value of $\cos x \sin y \cos z$ respectively. Determine the value of $\lfloor \frac{M}{m} \rfloor$.

16. Let $\mathrm{f} : \mathbb{R} \longrightarrow \mathbb{R}$ be a function such that, for any $x, y \in \mathbb{R}$,

$$(x-y)\mathrm{f}(x+y)-(x+y)\mathrm{f}(x-y) = (6x^2y+2y^3)(x^2-y^2).$$

Suppose $\mathrm{f}(1) = -999$. Determine the value of $\mathrm{f}(10)$.

17. Let b, c, d and e be real numbers such that the following equation

$$x^5-20x^4+bx^3+cx^2+dx+e = 0$$

has real roots only. Find the largest possible value of b.

18. Let $\mathbb{N}$ denote the set of all positive integers. Suppose that $\mathrm{f} : \mathbb{N} \to \mathbb{N}$ satisfies

(a) $\mathrm{f}(1) = 1$,

(b) $3\mathrm{f}(n)\mathrm{f}(2n+1) = \mathrm{f}(2n)(1+3\mathrm{f}(n))$ for all $n \in \mathbb{N}$,

(c) $\mathrm{f}(2n) < 6\mathrm{f}(n)$ for all $n \in \mathbb{N}$.

Determine $\mathrm{f}(2015)$.

19. Let x_1, x_2, x_3, x_4 and x_5 be positive integers such that

$$x_1 + x_2 + x_3 + x_4 + x_5 = x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5.$$

Find the largest possible value of x_5.

20. Ah Meng is going to pick up 2015 peanuts on the ground in several steps according to the following rules. In the first step, he picks up 1 peanut. For each next step, he picks up either the same number of peanuts or twice the number of peanuts of the previous step. What is the minimum number of steps that he can complete the task?

21. Determine the number of integers in the set $S = \{1, 2, 3, \cdots, 10000\}$ which are divisible by exactly one of integers in $\{2, 3, 5, 7\}$.

22. Determine the largest integer n such that

$$\sum_{i=1}^{n} x_i^2 \geq x_n \sum_{i=1}^{n-1} x_i$$

for all real numbers $x_1, x_2, \ldots, x_n$.

23. A circle ω_1 centred at O_1 intersects another circle ω_2 centred at O_2 at two distinct points P and Q. Points A and B are on ω_1 and ω_2 respectively such that AB is an external common tangent to ω_1 and ω_2. The line through PQ intersects the segments AB and O_1O_2 at M and N respectively. Suppose the radius of ω_1 is 143 cm, the radius of ω_2 is 78 cm and $O_1O_2 = 169$ cm. Determine the length of MN in centimetres.

24. Let XY be a diameter of a circle ω of radius 10 cm centered at O. Let A and B be the points on XY such that X, A, O, B, Y are in this order and $AO = OB = 4$ cm. Suppose that P is a point on ω such that the lines PA and PB intersect ω at C and D respectively with C and D distinct from P. Given $\frac{PB}{BD} = \frac{16}{21}$, determine the ratio $\frac{PA}{AC}$.

25. In a triangle ABC, the incircle ω centred at I touches the sides BC, CA and AB at D, E and F respectively, Q is the point on ω diametrically opposite to D, and P is the intersection of the lines FQ and DE. Suppose that $BC = 50$ cm, $CA = 49$ cm and $PQ = QF$. Determine the length of AB in centimetres.

Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2015
Open Section (Round 1 Solutions)

1. **Answer.** 120

   Solution. It is clear that $n^5 - 5n^3 + 4n = (n-2)(n-1)n(n+1)(n+2)$, which is a product of five consecutive integers. Hence, the largest possible value of N is 120. $\square$

2. **Answer.** 3

   Solution. Let the first term and the common ratio be a and r respectively. Since the first, second and fourth terms of the sequence are three consecutive terms of an arithmetic progression, we have

$$ar^3 - ar = ar - a.$$

   The equation reduces to

$$r^3 - 2r + 1 = (r-1)(r^2 + r - 1) = 0.$$

   As $r \neq 1$ (since the terms of the sequence are distinct), we must have $r^2 + r - 1 = 0$. So there are only two distinct values of the common ratios, denoting them by r_1 and r_2 respectively. Hence

$$r_1^2 + r_2^2 = (r_1 + r_2)^2 - 2r_1r_2 = 1 - 2(-1) = 3,$$

   which is our answer. $\square$

3. **Answer.** 1

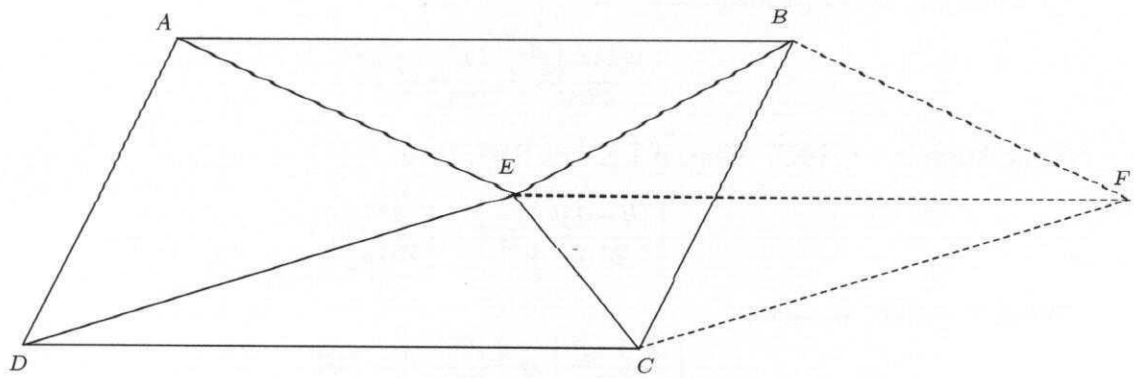
   Assume that the limit $l = \lim_{n\to\infty} x_n$ exists. By taking limit on both sides of the given equation, we obtain

$$\begin{aligned} l &= \frac{1}{16}\left(1 + 4l + \sqrt{1+24l}\right) \\ 16l &= 1 + 4l + \sqrt{1+24l} \\ (12l-1)^2 &= \left(\sqrt{1+24l}\right)^2, \end{aligned}$$

   which ends up with $l = 0$ or $l = \frac{1}{3}$. Since it is clear that the sequence is nondecreasing, the required limit is $3l = 1$. $\square$

4. **Answer.** 30

   Solution. We choose the point F such that $BF = AE$ and $CF = DE$. Hence triangle ADE is congruent to triangle BCF, hence DE is parallel to CF. So $DEFC$ is a parallelogram, so that EF, AB and DC are parallel.



We thus know that $\angle FBC = \angle EAD = \angle ECD = \angle CEF$ (alternate interior angles), hence the points C, E, B and F are concyclic. Hence

$$\begin{aligned}\angle ADE &= \angle FCB \text{ (corresponding angles of congruent triangles)}\\ &= \angle BEF \text{ (angles in the same segment)}\\ &= \angle EBA \text{ (alternate interior angles)}\\ &= 30^\circ,\end{aligned}$$

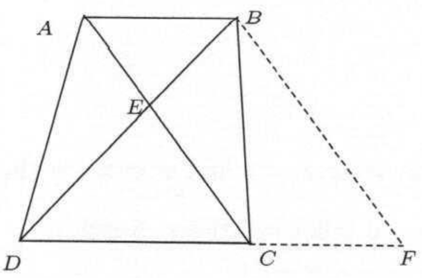
which yields the answer. □

5. **Answer.** 4028

Solution. By using the method of difference, it is easy to find that $S = \dfrac{2014}{2015}$. Hence $4030 \times S = 4028$. □

6. **Answer.** 216.

Let F be the point on the extension of DC such that BF is parallel to AC.



We first claim that $\angle AEB = 90^\circ$. Since $ABFC$ is a parallelogram, $CF = AB = 20$ cm, $BF = AC = 30$ cm and $DF = DC + CF = 50$ cm. Since $DF^2 = DB^2 + BF^2$, triangle DBF is a right-angled triangle, where $\angle DBF = 90^\circ$. Since EC is parallel to BF, we must also have $\angle DEC = \angle DBF = 90^\circ$. Hence, $\angle AEB = \angle DEC = 90^\circ$. Note that triangle DEC is similar to BEA, so that $EC = \dfrac{3}{5}AC = 18$ cm and $DE = \dfrac{3}{5}DB = 24$ cm. Hence area of triangle DEC equals $\dfrac{1}{2} \times 18 \times 24 = 216$ cm^2. □

7. **Answer.** 1512

Consider the following inequality:

$$\frac{(k+1)^2}{2015} - \frac{k^2}{2015} \le 1.$$

Its solution is $k \le 1007$. Thus, if $1 \le k \le 1007$, then

$$\left\lfloor \frac{(k+1)^2}{2015} \right\rfloor \le 1 + \left\lfloor \frac{k^2}{2015} \right\rfloor.$$

When $k = 1007$, we have

$$\left\lfloor \frac{(k+1)^2}{2015} \right\rfloor = \left\lfloor \frac{1008^2}{2015} \right\rfloor = 504.$$

Thus the following sequence

$$\left\lfloor \frac{1^2}{2015} \right\rfloor, \left\lfloor \frac{2^2}{2015} \right\rfloor, \left\lfloor \frac{3^2}{2015} \right\rfloor, \ldots, \left\lfloor \frac{1008^2}{2015} \right\rfloor$$

contains 505 different integers: $0, 1, 2, \cdots, 504$.

Note that for $k \ge 1008$, we have

$$\frac{(k+1)^2}{2015} - \frac{k^2}{2015} > 1$$

and so

$$\left\lfloor \frac{(k+1)^2}{2015} \right\rfloor \ge 1 + \left\lfloor \frac{k^2}{2015} \right\rfloor.$$

Thus all numbers below are different:

$$\left\lfloor \frac{1009^2}{2015} \right\rfloor, \left\lfloor \frac{1010^2}{2015} \right\rfloor, \left\lfloor \frac{1011^2}{2015} \right\rfloor, \ldots, \left\lfloor \frac{2015^2}{2015} \right\rfloor.$$

Hence the answer is

$$505 + (2015 - 1008) = 1512.$$

□

8. **Answer.** 3

Solution. It is clear that $x = 1$ and $x = 2$ are the solutions of the given equation. We claim that there are no other solutions. Suppose $\sqrt{\frac{4-x^2}{3}} \ne 0$ and $\sqrt{\frac{x^2-1}{3}} \ne 0$. Then, as $\left(\sqrt{\frac{4-x^2}{3}}\right)^2 + \left(\sqrt{\frac{x^2-1}{3}}\right)^2 = 1$, the numbers $\sqrt{\frac{4-x^2}{3}}$, $\sqrt{\frac{x^2-1}{3}}$ and 1 form the three sides of a right-angled triangle, thus $\sqrt{\frac{4-x^2}{3}} + \sqrt{\frac{x^2-1}{3}} > 1$. Hence such a choice of x is not a solution of the given equation. Hence the only solutions are $x = 1$ and $x = 2$, so that the required value equals 3. □

9. **Answer.** 6

Solution. Note that

$$
\begin{aligned}
& \frac{(1+10)(1+10^2)(1+10^4)\cdots(1+10^{2^m})}{1+10+10^2+10^3+10^4+\cdots+10^{127}} \\
= & \frac{(1-10)(1+10)(1+10^2)(1+10^4)\cdots(1+10^{2^m})}{(1-10)(1+10+10^2+10^3+10^4+\cdots+10^{127})} \\
= & \frac{(1-10^2)(1+10^2)(1+10^4)\cdots(1+10^{2^m})}{(1-10^{128})} \\
\vdots & \quad \vdots \quad \vdots \\
= & \frac{1-10^{2^{m+1}}}{1-10^{128}} = 1,
\end{aligned}
$$

so that we must have $2^{m+1} = 128$, so that $m = 6$. □

10. **Answer.** 1024

Solution. We note that the common ratio $r = \pm\sqrt{3}$. Substituting $a_{i+1} = a_1 r^i$ into both series and using the binomial theorem, we get

$$
\begin{aligned}
& \left(\sum_{i=0}^{10}\binom{10}{i}a_{i+1}\right)\left(\sum_{j=0}^{10}(-1)^j\binom{10}{j}a_{j+1}\right) \\
= & \left(\sum_{i=0}^{10}\binom{10}{i}a_1 r^i\right)\cdot\left(\sum_{j=0}^{10}\binom{10}{j}a_1(-r)^j\right) \\
= & {a_1}^2\left(\sum_{i=0}^{10}\binom{10}{i}r^i\right)\cdot\left(\sum_{j=0}^{10}\binom{10}{j}(-r)^j\right) \\
= & {a_1}^2(1+r)^{10}(1-r)^{10} \\
= & {a_1}^2(1-r^2)^{10} \\
= & (-2)^{10} \\
= & 1024,
\end{aligned}
$$

which completes the solution. □

11. **Answer.** 21

Solution. Note that

$$\sqrt{38x-152}+\sqrt{2015-403x} = \sqrt{38(x-4)}+\sqrt{403(5-x)}.$$

Thus the domain is $4 \le x \le 5$. Let $x = 4+\sin^2\theta$, where $0 \le \theta \le \pi/2$. Then

$$
\begin{aligned}
y &= \sqrt{38\sin^2\theta}+\sqrt{403(\cos^2\theta)} \\
&= \sqrt{38}\sin\theta+\sqrt{403}\cos\theta \\
&= 21\times\left(\frac{\sqrt{38}}{21}\sin\theta+\frac{\sqrt{403}}{21}\cos\theta\right) \\
&= 21\times\sin(\theta+\beta),
\end{aligned}
$$

where $0 < \beta < \frac{\pi}{2}$ with $\cos\beta = \frac{\sqrt{38}}{21}$. Thus the maximum value of y is 21. □

12. **Answer.** 925327

Solution. The coefficient of x^{50} in the expansion of $(x+1)(x+2)(x+3)\cdots(x+50)(x+51)(x+52)$ is

$$\begin{aligned}\sum_{1\le i<j\le 52} i\times j &= \frac{1}{2}(1+2+3+\cdots+52)^2-\frac{1}{2}(1^2+2^2+3^2+\cdots+52^2)\\ &= \frac{1}{2}(52\times\frac{53}{2})^2-\frac{1}{2}\left(52\times 53\times\frac{(2\times 52+1)}{6}\right)\\ &= 925327.\end{aligned}$$

□

13. **Answer.** 10050

Solution. Subtract 100 from each chosen number exceeding 100. Since no two chosen numbers differ by 0 or 100, each of the resulting differences is distinct from any of 50 chosen numbers not exceeding 100. Thus these numbers and the differences form a set of 100 distinct positive integers not exceeding 100. Therefore it contains all positive integers from 1 to 100. Their sum equals 101×50. Hence the sum of the 100 chosen numbers equals $101\times 50+100\times 50=10050$. □

14. **Answer.** 38918

Solution. Note that $x=\dfrac{\sqrt{5}+1}{2}$ and $x^2-x-1=0$, hence $x^2-x+1=2$. Note also that when the polynomial t^9 is divided by t^2-t-1, it has the remainder $34t+21$, i.e,

$$t^9=(t^2-t-1)q(t)+34t+21$$

for some polynomial $q(t)$. Thus

$$\begin{aligned}H &= x^9(x^2-x+1)^9\\ &= (34x+21)2^9\\ &= (34\times\frac{\sqrt{5}+1}{2}+21)\times 2^9\\ &= (38+17\sqrt{5})\times 512\\ &= 19456+8704\sqrt{5}=38918.735\cdots,\end{aligned}$$

which yields the answer 38918.

Alternative Solution: we can easily check that $W=(x^3-x^2+x)=1+\sqrt{5}$ when $x=\dfrac{2}{\sqrt{5}-1}$. Let $G=1-\sqrt{5}$. Then W,G are roots of $x^2-2x-4=0$. Let $a_n=W^n+G^n$. Then $a_{n+2}=2a_{n+1}+4a_n$. With $a_0=a_1=2$, we can compute $a_9=38912$. Now $W^9G^9=-4^9=-262144$. Thus the quadratic equation satisfied by W^9 and G^9 is $F(x)=x^2-38912x-262144=0$. Now $F(-7)>0$ and $F(-6)<0$. Thus $\lceil G^9\rceil=-6$. Therefore $\lfloor H\rfloor=\lfloor W^9\rfloor=38912+6=38918$.

□

15. **Answer.** 3

Solution. As $x \geq y \geq z \geq \frac{\pi}{12}$ and $x+y+z=\frac{\pi}{2}$, we have

$$\frac{\pi}{6} \leq x \leq \frac{\pi}{3}, \quad \sin(x-y) \geq 0, \quad \sin(y-z) \geq 0.$$

Observe that

$$\begin{aligned}
&\cos x \sin y \cos z \\
= \ & \frac{1}{2}\cos x(\sin(y+z)+\sin(y-z)) \\
\geq \ & \frac{1}{2}\cos x \sin(y+z) \\
= \ & \frac{1}{2}\cos^2 x \\
\geq \ & \frac{1}{2}\cos^2\frac{\pi}{3} = \frac{1}{8}.
\end{aligned}$$

When $x=\frac{\pi}{3}$ and $y=z=\frac{\pi}{12}$, we have $\cos x \sin y \cos z = \frac{1}{8}$. Thus $m=\frac{1}{8}$.

Again observe that

$$\begin{aligned}
&\cos x \sin y \cos z \\
= \ & \frac{1}{2}\cos z(\sin(x+y)-\sin(x-y)) \\
\leq \ & \frac{1}{2}\cos z \sin(x+y) \\
= \ & \frac{1}{2}\cos^2 z \\
\leq \ & \frac{1}{2}\cos^2\frac{\pi}{12} \\
= \ & \frac{1}{4}(1+\cos\frac{\pi}{6}) = \frac{2+\sqrt{3}}{8}.
\end{aligned}$$

When $x=y=\frac{5\pi}{24}$ and and $z=\frac{\pi}{12}$, we have $\cos x \sin y \cos z = \frac{2+\sqrt{3}}{8}$. Thus $M=\frac{2+\sqrt{3}}{8}$.
Hence $\lfloor \frac{M}{m} \rfloor = 3$. □

16. **Answer.** 0

Solution. Let $a=x+y$ and $b=x-y$. Then the given functional equation is equivalent to $b\mathrm{f}(a)-a\mathrm{f}(b)=(a^3-b^3)ab$. This holds for all real numbers a and b. For nonzero a and b, this can be rewritten as

$$\frac{\mathrm{f}(a)}{a}-a^3=\frac{\mathrm{f}(b)}{b}-b^3.$$

Hence, for any nonzero real number x, $\frac{\mathrm{f}(x)}{x}-x^3=\mathrm{f}(1)-1=-1000$. Therefore, $\mathrm{f}(x)=x^4-1000x$, for all $x \neq 0$. So $\mathrm{f}(10)=0$. □

17. **Answer.** 160

Solution. Let x_1, x_2, x_3, x_4 and x_5 be the roots of the equation below

$$x^5-20x^4+bx^3+cx^2+dx+e=0.$$

Then

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20.$$

$$\sum_{1 \le i < j \le 5} x_i x_j = b.$$

Let $a = x_1 + x_2 + x_3 + x_4 + x_5 = 20$. If $5b > 2a^2 = 800$, i.e., $b > 160$, then

$$\begin{aligned} 0 > 2a^2 - 5b &= 2(x_1 + x_2 + \cdots + x_5)^2 - 5 \sum_{1 \le i < j \le 5} x_i x_j \\ &= \frac{1}{2} \sum_{1 \le i < j \le 5} (x_i - x_j)^2 \ge 0, \end{aligned}$$

a contradiction. Thus $b \le 160$.

Note that

$$\begin{aligned} (x-4)^5 &= x^5 - 20x^4 + 4^2\binom{5}{2}x^3 - 4^3\binom{5}{3}x^2 + 4^4\binom{5}{4}x^1 - 4^5\binom{5}{5} \\ &= x^5 - 20x^4 + 160x^3 - 640x^2 + 1380x^1 - 1024 \end{aligned}$$

has all five real roots.

Hence the largest possible value of b is 160. □

18. **Answer.** 88330

Solution. Note that $\gcd(3\mathrm{f}(n), 1 + 3\mathrm{f}(n)) = 1$ so that $3\mathrm{f}(n)$ divides $\mathrm{f}(2n)$. But $\mathrm{f}(2n) < 6\mathrm{f}(n)$, hence $\mathrm{f}(2n) = 3\mathrm{f}(n)$ and $\mathrm{f}(2n+1) = 3\mathrm{f}(n)+1$. One can compute that $\mathrm{f}(2015) = 88330$ using these 2 relations recursively in 10 iterations. Alternatively, we can work out a formula for $\mathrm{f}(n)$ as follow.

Claim: If $n = (b_1 b_2 \cdots b_{\ell(n)})_2$ in base 2, then $\mathrm{f}(n) = (b_1 b_2 \cdots b_{\ell(n)})_3$ in base 3.

This can be proved using induction on n. Since $\mathrm{f}(1) = 1$ and $\mathrm{f}(2) = 3\mathrm{f}(1) = 3$. The claim is true for $n = 1$ and 2. Suppose the claim is true for $k = 1, 2, \cdots, n$ where $n \ge 2$. If n is odd, then $n + 1 = 2m$ is even with $m = (b_1 b_2 \cdots b_{\ell(m)})_2$. Thus $n + 1 = (b_1 b_2 \cdots b_{\ell(m)}0)_2$ and $\mathrm{f}(n+1) = \mathrm{f}(2m) = 3\mathrm{f}(m) = 3(b_1 b_2 \cdots b_{\ell(m)})_3 = (b_1 b_2 \cdots b_{\ell(m)}0)_3$, where the second last equality is by induction hypothesis applied to m. If n is even, then $n + 1 = 2m + 1$ is odd with $m = (b_1 b_2 \cdots b_{\ell(m)})_2$. Thus $n + 1 = (b_1 b_2 \cdots b_{\ell(m)}1)_2$ and $\mathrm{f}(n+1) = \mathrm{f}(2m+1) = 3\mathrm{f}(m)+1 = 3(b_1 b_2 \cdots b_{\ell(m)})_3+1 = (b_1 b_2 \cdots b_{\ell(m)}1)_3$. This completes the proof of the claim. Thus $\mathrm{f}(2015) = \mathrm{f}\left((11111011111)_2\right) = (11111011111)_3 = 3^{10} + 3^9 + 3^8 + 3^7 + 3^6 + 3^4 + 3^3 + 3^2 + 3 + 1 = 88330$.

19. **Answer.** 5

Solution. Assume that $x_1 \le x_2 \le x_3 \le x_4 \le x_5$. First, we can show that $x_4 \ge 2$. Otherwise, $x_1 = x_2 = x_3 = x_4 = 1$ and so $4 + x_5 = x_5$, which is impossible. Note also that

$$\begin{aligned} x_5 = \frac{x_1 + x_2 + x_3 + x_4}{x_1 x_2 x_3 x_4 - 1} &\le \frac{x_1 x_2 x_3 + x_1 x_2 x_3 + x_1 x_2 x_3 + x_1 x_2 x_3 x_4}{x_1 x_2 x_3 x_4 - x_1 x_2 x_3} \\ &= \frac{3 + x_4}{x_4 - 1} = 1 + \frac{4}{x_4 - 1} \le 5. \end{aligned}$$

When $x_1 = x_2 = x_3 = 1$, $x_4 = 2$, condition $x_1 + x_2 + \cdots + x_5 = x_1 x_2 \cdots x_5$ implies that $x_5 = 5$. Hence the answer is 5.

20. **Answer.** 15

Solution. Note that the number of peanuts that he picks up in each step must be a power of 2. If at some step he picks up 2^n peanuts, then he must have picked up $1, 2, 2^2, \ldots, 2^{n-1}$ peanuts at certain previous steps. That means he has picked up at least $1+2+2^2+\cdots+2^n = 2^{n+1}-1$ peanuts. If $n=10$, then $2^{11}-1=2047>2015$. Thus, in each step, he picks up at most $2^9=512$ peanuts. If Ah Meng is to minimize the total number of steps, then the number of steps in which he picks up 2^k peanuts is less than or equal to 2. This is because if the number of such steps is more than 2, then he can reduce the number of steps by combining two such steps into 1 step by taking 2^{k+1} peanuts instead. For $i=0,\cdots,9$, let a_i be the number of steps in which he picks up 2^i peanuts. Thus $a_i=1$ or 2, and $\sum_{i=0}^{9} a_i 2^i = 2015$. Let $b_i = a_i - 1$. Then $b_i = 0$ or 1, and $\sum_{i=0}^{9} b_i 2^i = 2015 - 1023 = 992$. Using the binary representation of the number 992, we obtain $b_0=b_1=b_2=b_3=b_4=0$, $b_5=b_6=b_7=b_8=b_9=1$. That is $a_0=a_1=a_2=a_3=a_4=1$, $a_5=a_6=a_7=a_8=a_9=2$. The number of steps is thus 15. Since the binary representation of a number is unique, the minimum number of steps is 15. □

21. **Answer.** 4383

Solution. Let P_1 be the property that an integer is divisible by 2; P_2 be the property that an integer is divisible by 3; P_3 be the property that an integer is divisible by 5; P_4 be the property that an integer is divisible by 7.

For any distinct numbers $i_1, i_2, \cdots, i_k$ in the set $\{1,2,3,4\}$, where $1 \le k \le 4$, let $\omega(P_{i_1}P_{i_2}\cdots P_{i_k})$ denote the number of integers in S which have properties $P_{i_1}, P_{i_2}, \cdots, P_{i_k}$. Let

$$\omega(k) = \sum_{1\le i_1<i_2<\cdots<i_k} \omega(P_{i_1}P_{i_2}\cdots P_{i_k}).$$

Then, the number of integers in the set S which are divisible by exactly one of integers in $\{2,3,5,7\}$ is equal to

$$\omega(1) - 2\omega(2) + 3\omega(3) - 4\omega(4).$$

Note that

$$\begin{aligned}
\omega(1) &= \omega(P_1)+\omega(P_2)+\omega(P_3)+\omega(P_4)\\
&= \left\lfloor \frac{10000}{2} \right\rfloor + \left\lfloor \frac{10000}{3} \right\rfloor + \left\lfloor \frac{10000}{5} \right\rfloor + \left\lfloor \frac{10000}{7} \right\rfloor = 11761;\\
\omega(2) &= \omega(P_1P_2)+\omega(P_1P_3)+\omega(P_1P_4)+\omega(P_2P_3)+\omega(P_2P_4)+\omega(P_3P_4)\\
&= \left\lfloor \frac{10000}{6} \right\rfloor + \left\lfloor \frac{10000}{10} \right\rfloor + \left\lfloor \frac{10000}{14} \right\rfloor + \left\lfloor \frac{10000}{15} \right\rfloor\\
&\qquad + \left\lfloor \frac{10000}{21} \right\rfloor + \left\lfloor \frac{10000}{35} \right\rfloor\\
&= 4807;\\
\omega(3) &= \omega(P_1P_2P_3)+\omega(P_1P_2P_4)+\omega(P_1P_3P_4)+\omega(P_2P_3P_4)\\
&= \left\lfloor \frac{10000}{30} \right\rfloor + \left\lfloor \frac{10000}{42} \right\rfloor + \left\lfloor \frac{10000}{70} \right\rfloor + \left\lfloor \frac{10000}{105} \right\rfloor\\
&= 808;\\
\omega(4) &= \omega(P_1P_2P_3P_4) = \left\lfloor \frac{10000}{210} \right\rfloor = 47.
\end{aligned}$$

Hence the answer is

$$\begin{aligned}
& \omega(1) - 2\omega(2) + 3\omega(3) - 4\omega(4) \\
= \; & 11761 - 2 \times 4807 + 3 \times 808 - 4 \times 47 = 4383,
\end{aligned}$$

which is the required answer. □

22. **Answer.** 5

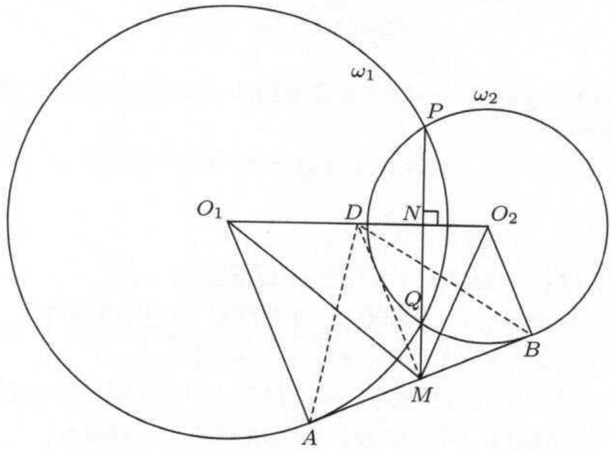
Solution. For $n \geq 6$, take $x_{n-5} = x_{n-4} = x_{n-3} = x_{n-2} = x_{n-1} = 1/2$ and $x_n = 1$ and zero for other x_i. Then the left hand side of the inequality is $9/4$, while the right hand side is $5/2$. So the inequality is not valid for $n \geq 6$. We just need to prove that the inequality is valid for $n = 5$. In fact the inequality holds for $n = 2, 3, 4, 5$. The cases $n = 2$ and 3 can be verified easily. Let's consider the case $n = 5$. (The case $n = 4$ can be proved in a similar way.) The inequality to be proved is equivalent to

$$x_5^2 - (x_1 + x_2 + x_3 + x_4)x_5 + (x_1^2 + x_2^2 + x_3^2 + x_4^2) \geq 0.$$

Regard this as a quadratic equation in x_5. It suffices to prove that its discriminant is less than or equal to zero. The discriminant is equal to $(x_1+x_2+x_3+x_4)^2 - 4(x_1^2+x_2^2+x_3^2+x_4^2)$ which can be simplified to $-[(x_1 - x_2)^2 + (x_1 - x_3)^2 + (x_1 - x_4)^2 + (x_2 - x_3)^2 + (x_2 - x_4)^2 + (x_3 - x_4)^2]$. It is obviously less than or equal to zero. □

23. **Answer.** 102

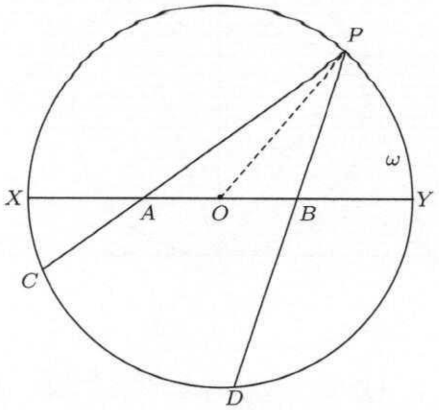
Solution.



Let the radii of ω_1 and ω_2 be r_1 and r_2 respectively. Let $O_1O_2 = d$. First note that M lies on the radical axis of ω_1 and ω_2. Thus M is the midpoint of AB. Let D be the midpoint of O_1O_2. Then MD is parallel to AO_1 and BO_2. Thus MD is perpendicular to AB. Consider the area of the triangle O_1O_2M. We have $[O_1O_2M] = [O_1MD] + [O_2MD] = [AMD] + [BMD] = [ABD]$. Thus $\frac{1}{2}d \times MN = \frac{1}{2}AB \times MD$. Note that $AB = (d^2 - (r_1 - r_2)^2)^{\frac{1}{2}}$ and $MD = \frac{1}{2}(r_1 + r_2)$. From this we get $MN = \frac{1}{2d}(r_1 + r_2)(d^2 - (r_1 - r_2)^2)^{\frac{1}{2}}$. Substituting the values $r_1 = 143$, $r_2 = 78$ and $d = 169$, we obtain $MN = 102$ cm. □

24. **Answer.** 2

Solution.



Let the radius of ω be R and $AO = OB = a$. By Intersecting Chord Theorem, we have $PA \cdot AC = XA \cdot AY = (R-a)(R+a) = R^2 - a^2$. Thus we have

$$\frac{PA}{AC} = \frac{PA^2}{PA \cdot AC} = \frac{PA^2}{R^2 - a^2}.$$

Similarly,

$$\frac{PB}{BD} = \frac{PB^2}{R^2 - a^2}.$$

Thus

$$\frac{PA}{AC} + \frac{PB}{BD} = \frac{PA^2 + PB^2}{R^2 - a^2}.$$

Using cosine rule on the triangles OPA and OBP, we have $PA^2 = R^2 + a^2 - 2Ra\cos\angle AOP$ and $PB^2 = R^2 + a^2 - 2Ra\cos\angle BOP$. Since $\cos\angle AOP = -\cos\angle BOP$, we have $PA^2 + PB^2 = 2(R^2 + a^2)$. Consequently,

$$\frac{PA}{AC} + \frac{PB}{BD} = \frac{2(R^2 + a^2)}{R^2 - a^2}.$$

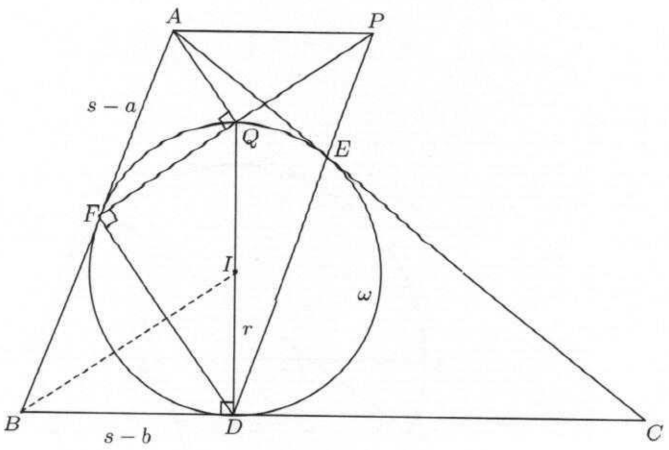
As $R = 10, a = 4$ and $\dfrac{PB}{BD} = \dfrac{16}{21}$, we have

$$\frac{PA}{AC} = \frac{2((10^2 + 4^2)}{(10^2 - 4^2)} - \frac{16}{21} = 2.$$

Note that direct calculation using cosine rule gives $PB = 8$ cm, $PA = \sqrt{168}$ cm. □

25. **Answer.** 33

Solution.



Let $BC = a, CA = b, AB = c$ and $s = \dfrac{a+b+c}{2}$, the semi-perimeter of the triangle ABC. Let r be the radius of ω. Since DQ is a diameter of ω, we have $\angle PFD = 90°$. As $\angle FDE = \dfrac{B}{2} + \dfrac{C}{2}$, we have $\angle FPE = 90° - \angle FDE = \dfrac{A}{2}$. Note that A and P are on the same side of the line EF. Thus the circle centred at A with radius $AF = AE$ passes through P so that $AP = AE = AF$. (The angle subtended by the arc FE is half the angle subtended at the centre A). Given also that Q is the midpoint of PF, we have $\angle AQF = 90°$. As $\angle AFQ = \angle IBD = \dfrac{B}{2}$, the triangles AFQ and IBD are similar. Thus $\dfrac{AF}{IB} = \dfrac{FQ}{BD}$. That is $\dfrac{s-a}{r/\sin\frac{B}{2}} = \dfrac{2r\sin\frac{B}{2}}{s-b}$. Or equivalently, $(s-a)(s-b) = 2r^2$. Multiplying both sides by $s(s-a)$, we get $s(s-a)(s-b)(s-c) = 2s(s-c)r^2$. By Heron's formula, $s(s-a)(s-b)(s-c) = r^2s^2$ is the square of the area of the triangle ABC. Therefore, $r^2s^2 = 2s(s-c)r^2$. That is $s = 2c$, or equivalently $c = \dfrac{a+b}{3}$. Since $a = 50$, $b = 49$, we get $c = 33$.

$\square$

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2015

(Open Section, Round 2)

Saturday, 4 July 2015 **0900-1300**

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1. In an acute-angled triangle ABC, D is the point on BC such that AD bisects $\angle BAC$, E and F are the feet of the perpendiculars from D onto AB and AC respectively. The segments BF and CE intersect at K. Prove that AK is perpendicular to BC.

2. A boy lives in a small island in which there are three roads at every junction. He starts from his home and walks along the roads. At each junction he would choose to turn to the road on his right or left alternately, i.e., his choices would be ..., left, right, left, ... Prove that he will eventually return to his home.

3. Find all functions $f : \mathbb{R} \to \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers, such that

$$f(x)f(yf(x)-1) = x^2 f(y) - f(x) \quad \text{for all } x, y \in \mathbb{R}.$$

4. Let $f_0, f_1, \ldots$ be the Fibonacci sequence: $f_0 = f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$ if $n \geq 2$. Determine all possible positive integers n so that there is a positive integer a such that $f_n \leq a \leq f_{n+1}$ and that

$$a\Big(\frac{1}{f_1} + \frac{1}{f_1 f_2} + \frac{1}{f_1 f_2 f_3} + \cdots + \frac{1}{f_1 f_2 \cdots f_n}\Big)$$

is an integer.

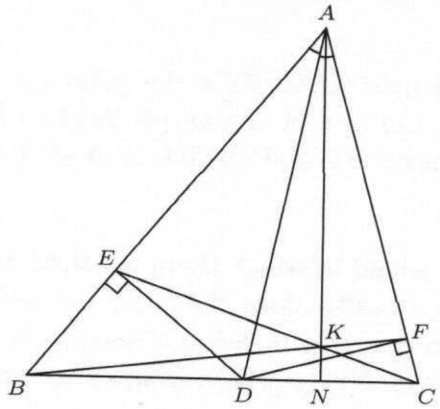
5. Let $n > 3$ be a given integer. Find the largest integer d (in terms of n) such that for any set S of n integers, there are four distinct (but not necessarily disjoint) nonempty subsets, the sum of the elements of each of which is divisible by d.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2015

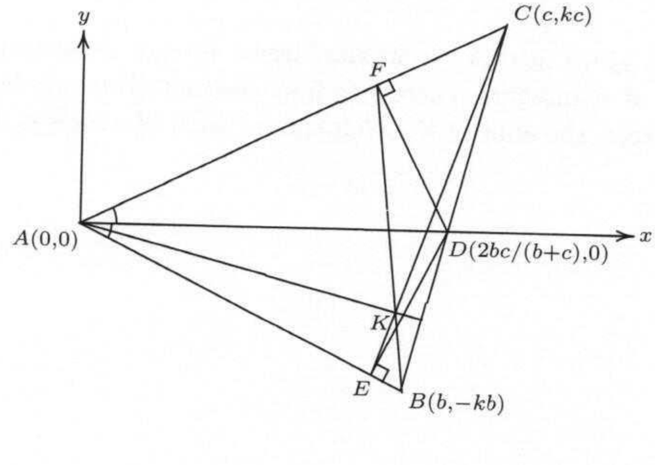
(Open Section, Round 2 solutions)

1.



Let the extension of AK intersect BC at N. Note that $AE = AF$ and $BD : DC = c : b$, where $b = AC$ and $c = AB$. The cevians AN, BF, CE concur at K. By Ceva's theorem, we have $(BN/NC)(CF/FA)(AE/EB) = 1$. Thus $BN/NC = EB/CF$. On the other hand, $EB = BD\cos B$ and $CF = DC\cos C$ so that $EB/CF = (BD\cos B)/(DC\cos C) = (c\cos B)/(b\cos C)$. Therefore, $BN/NC = (c\cos B)/(b\cos C)$. This shows that N is the foot of the perpendicular from A onto BC.

Second solution. We set up a coordinate system where A is the origin, and the bisector AD is the x-axis. Let the coordinates of B and C be $(b, -kb)$ and (c, kc) respectively. Then $D = (\frac{2bc}{b+c}, 0)$. The coordinates of E and F are found to be $E = \left(\frac{2bc}{(1+k^2)(b+c)}, \frac{-2bck}{(1+k^2)(b+c)}\right)$ and $F = \left(\frac{2bc}{(1+k^2)(b+c)}, \frac{2bck}{(1+k^2)(b+c)}\right)$ respectively. Note that E and F are symmetric with respect to the x-axis. From this, we get the coordinate of K, which is equal to $K = \left(\frac{4bc(b+c)k^2}{k^2(k^2+2)(b+c)^2+(b-c)^2}, \frac{4bc(b-c)k}{k^2(k^2+2)(b+c)^2+(b-c)^2}\right)$.



The slope of OK is $\frac{b-c}{k(b+c)}$ and the slope of BC is $\frac{k(c+b)}{c-b}$. Their product is -1, and so AK is perpendicular to BC.

2. After walking for a long time, he must walk along a certain road 6 times and in the same direction at least 3 times. When he reaches the junction along this direction, he must choose the same direction twice. Both times, tracing backwards, we conclude that the trails are the same. Since first one traces back to his home, we conclude that the second time, the trails also go back to his home. Thus the boy must return to his home.

3. The constant function $f(x) = 0$ is a solution.

Let f be a solution that is not identically 0. We shall show that $f(x) = x$ for all x. Letting $x = 0$ in the given equation, we get

$$f(0)[f(yf(0) - 1) + 1] = 0.$$

Suppose $f(0) \neq 0$. Let $x = yf(0) - 1$. As y ranges over all real numbers, so does x. Thus we get $f(x) = -1$ for all x. But this does not satisfy the given equation. So $f(0) = 0$.

Now suppose that $f(a) = 0$ for some $a \neq 0$. Then the original equation becomes $0 = a^2 f(y)$ for all y, implying that $f(y) = 0$ for all y. This contradicts our assumption that f is not identically 0. Thus $f(x) = 0$ iff $x = 0$.

Letting $x = y = 1$ in the given equation, we have $f(f(1) - 1) = 0$. Thus $f(1) = 1$. When $x = 1$, the original equation becomes

$$f(y-1) = f(y) - 1. \tag{1}$$

Let $y = 1$ in the given equation and use (1) to obtain

$$x^2 - f(x) = f(x)[f(f(x) - 1)] = f(x)[f(f(x)) - 1] = f(x)f(f(x)) - f(x).$$

Thus

$$f(x)f(f(x)) = x^2 \tag{2}$$

Now replace x by $x - 1$ in (2) and apply (1) 3 times, and finally apply (2)

$$\begin{aligned}(x-1)^2 &= f(x-1)f(f(x-1)) = (f(x) - 1)[f(f(x) - 1)] \\ x^2 - 2x + 1 &= (f(x) - 1)[f(f(x)) - 1] = f(x)f(f(x)) - f(x) - f(f(x)) + 1 \\ &= x^2 - f(x) - f(f(x)) + 1\end{aligned}$$

Therefore

$$f(x) + f(f(x)) = 2x. \tag{3}$$

Eliminating $f(f(x))$ from (2) and (3) gives

$$[x - f(x)]^2 = 0,$$

so that $f(x) = x$, as claimed. It is clear that this is also a solution.

4. The number may be rewritten as

$$\frac{a}{f_1 \cdots f_n}(f_2 \cdots f_n + f_3 \cdots f_n + \cdots + f_{n-2}f_{n-1}f_n + f_{n-1}f_n + 1).$$

If this is an integer, then

$$f_n \mid a(f_2 \cdots f_n + f_3 \cdots f_n + \cdots + f_{n-2}f_{n-1}f_n + f_{n-1}f_n + 1).$$

Thus $f_n \mid a$. First consider the case where $n \geq 2$. Then

$$f_n \leq a \leq f_{n+1} < 2f_n.$$

Hence $a = f_n$. Then

$$\frac{1}{f_1 \cdots f_{n-1}}(f_2 \cdots f_n + f_3 \cdots f_n + \cdots + f_{n-2}f_{n-1}f_n + f_{n-1}f_n + 1)$$

is an integer. Thus $f_{n-1} \mid 1$. Hence $n - 1 = 0$ or 1. This shows that the only possible values of n are 1 and 2.

For $n = 1$, $1 = f_1 \leq 1 \leq f_2$ and $a/f_1 = 1/1 = 1$ is an integer.

For $n = 2$, $f_2 \leq 2 \leq f_3$ and

$$2(\frac{1}{f_1} + \frac{1}{f_1 f_2}) = 2(1 + \frac{1}{2})$$

is an integer.

5. It is not possible for $d \geq n$. To see this, take a set S of n integers so that each element of S is equal to $1 \pmod{n}$. The sum of any nonempty subset T of S is equal to $|T| \pmod{d}$. Since $d \geq n$, the only possibility for this to hold is if $d = n$ and $|T| = n$, i.e., $T = S$. This proves that $d \geq n$ fails the given condition.

It is also not possible for $d = n - 1$. In this case, take a set $S = \{a_1, \ldots, a_n\}$ of n integers so that $a_1 = 0$ and $a_i = 1 \pmod{d}$ if $1 < i \leq n$. The only subsets of S whose sums are divisible by d are:

$$\{a_1\}, \{a_2, \ldots, a_n\} \quad \text{and} \quad S.$$

We will show that $d = n - 2$ satisfies the given condition. So the largest d is $n - 2$.

Lemma. (Erdős): Any set of n integers has nonempty subset whose sum is divisible by n.

Proof of lemma: Consider a set of n integers $\{a_1, \ldots, a_n\}$. Consider the set of n numbers $\{a_1, a_1 + a_2, \ldots, a_1 + \cdots + a_n\}$. If one of these numbers is equal to $0 \pmod{n}$, then we are done. Otherwise, there are two of them that are equal $\pmod{n}$. Say

$$a_1 + \cdots + a_i = a_1 + \cdots + a_j \pmod{n}, i < j.$$

Then $a_{i+1} + \cdots + a_j = 0 \pmod{n}$.

We return to the proof that $n-2$ satisfies the given condition. Let $S = \{a_1, \ldots, a_n\}$ be a set of n integers. By the lemma, there is a subset T_1 of $\{a_1, \ldots, a_{n-2}\}$ whose sum is divisible by $n - 2$. We may also assume that T_1 is minimal in the sense that removal of any element from T_1 would either render it empty, or make the sum of its elements indivisible by $n - 2$. Take $a_i \in T_1$. Choose nonempty subset T_2 of $S \backslash \{a_i, a_{n-1}\}$ whose sum is divisible by $n - 2$. Clearly, $T_1 \neq T_2$.

If $T_1 \cap T_2 = \emptyset$, pick $a_i \in T_1, a_j \in T_2$ and let $I = S \backslash \{a_i, a_j\}$. By the lemma, there exists a subset T_3 of I whose sum is divisible by $n - 2$. Clearly, $T_1, T_2, T_1 \cup T_2$ and T_3 are four distinct sets, the sum of the elements of each of which is divisible by $n - 2$.

The previous paragraph shows that if there are two disjoint nonempty subsets of S whose sums are both divisible by $n - 2$, then we are done. Assume that this does not occur. Let T_1 and T_2 be two distinct subsets of S whose sums are divisible by $n - 2$. Choose $a_i \neq a_j$ so that $a_i \in T_1$ and $a_j \in T_2$. By the Lemma, there exists a subset T_3 of $S \backslash \{a_i, a_j\}$ whose sum is divisible by $n - 2$. In particular, T_3 is different from both T_1 and T_2. Moreover, by the previous paragraph, we may assume that $T_i \cap T_j \neq \emptyset$ if $1 \leq i < j \leq 3$. Then we can choose two distinct points a_r and a_s so that $\{a_r, a_s\} \cap T_i \neq \emptyset$ for $i = 1, 2, 3$. Use the lemma again to find $T_4 \subseteq S \backslash \{a_r, a_s\}$ so that the sum of the elements in T_4 is divisible by $n - 2$. By construction T_4 is different from T_1, T_2 and T_3.