MA2002 Calculus

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1 Functions

Mostly omitted.

- f is **increasing** if $f(x_1) < f(x_2)$ for all x_1, x_2 where $x_1 < x_2$.

  f is **decreasing** if $f(x_1) > f(x_2)$ for all x_1, x_2 where $x_1 < x_2$.

- f is **even** if $f(-x) = f(x)$ for all x.

  f is **odd** if $f(-x) = -f(x)$ for all x.

  Graph of even function is **symmetric about y-axis**; graph of odd function is **symmetric about the origin**.

  Remark. To show that f is not even, we need to provide some x such that $f(-x) \neq f(x)$. Similar remarks hold for odd functions.

- **Theorem** - Any function is the unique sum of an even and odd function.

$$f(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2}.$$

2 Limits

Let $f(x)$ be defined on an open interval about a, except possibly at a itself.

- The **limit** of $f(x)$ as x approaches a equals L, denoted by $\lim_{x\to a} f(x) = L$, if

$$(\forall \varepsilon > 0) \quad (\exists \delta > 0) \quad 0 < |x-a| < \delta \implies |f(x) - L| < \varepsilon. \tag{1}$$

  (1) means that by taking x sufficiently close to a (but not equal to a), $f(x)$ is arbitrarily close to L.

  The process of finding a suitable $\delta > 0$ is as follows:

  1. Solve $|f(x)-L| < \varepsilon$ to find an open interval (α, β) containing a on which the inequality holds for all $x \neq a$.
     Useful to express $|f(x)-L|$ in terms of $|x-a|$; factorising helps.
     Triangle inequality is useful to deal with inequalities: $|a+b| \leq |a| + |b|$.
  2. Find a value of $\delta > 0$ that places the open interval $(a-\delta, a+\delta)$ centered at a inside the interval (α, β). Then the inequality $|f(x)-L| < \varepsilon$ will hold for all $x \neq a$ in this δ-interval.

- **Limit Laws** - If $L, M, a, c \in \mathbb{R}$ and $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then

  (i) Constant: $\lim_{x\to a} c = c$, where c is a constant.

(ii) Scalar multiplication rule: $\lim_{x\to a}(cf(x)) = cL$

(iii) Sum and difference rule: $\lim_{x\to a}(f(x) \pm g(x)) = L \pm M$

(iv) Product rule: $\lim_{x\to a} f(x)g(x) = LM$

(v) Quotient rule: $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, provided that $M \neq 0$

(vi) Power rule: $\lim_{x\to a}[f(x)]^n = L^n$, $n \in \mathbb{Z}^+$

(vii) Root rule: $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{\frac{1}{n}}$, $n \in \mathbb{Z}^+$
(If n is even, we assume that $\lim_{x\to a} f(x) = L > 0$.)

Remark. We can only apply limit laws when the limits of f and g exist.

- When dealing with **rational functions** $\dfrac{P(x)}{Q(x)}$,

  1. **Factorise** $P(x)$ and $Q(x)$.
  2. Cancel the common factors of $P(x)$ and $Q(x)$.
  3. Evaluate the limit of the simplified function.

- When dealing with **algebraic functions** (involving square roots),

  1. Multiply the algebraic **conjugate** to numerator and denominator.
  2. Cancel the common factors.
  3. Evaluate the limit of the simplified function.

- The **right-hand limit** of $f(x)$ as x approaches a equals L, denoted by $\lim_{x\to a^+} f(x) = L$, if

$$(\forall \varepsilon > 0) \quad (\exists \delta > 0) \quad a < x < a + \delta \implies |f(x) - L| < \varepsilon. \tag{2}$$

  The **left-hand limit** of $f(x)$ as x approaches a equals L, denoted by $\lim_{x\to a^-} f(x) = L$, if

$$(\forall \varepsilon > 0) \quad (\exists \delta > 0) \quad a - \delta < x < a \implies |f(x) - L| < \varepsilon. \tag{3}$$

  Remark. We consider half intervals.

- **Theorem** - One-sided limits also satisfy the limit laws.

- **Theorem** - $\lim_{x\to a} f(x) = L \iff \lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^-} f(x) = L$.
  To prove a limit does not exist, show that left and right-hand limits are not equal.

- $f(x)$ approaches **infinity** as x approaches a, denoted by $\lim_{x\to a} f(x) = \infty$, if

$$(\forall M > 0) \quad (\exists \delta > 0) \quad 0 < |x - a| < \delta \implies f(x) > M. \tag{4}$$

  $f(x)$ approaches **negative infinity** as x approaches a, denoted by $\lim_{x\to a} f(x) = -\infty$, if

$$(\forall M < 0) \quad (\exists \delta > 0) \quad 0 < |x - a| < \delta \implies f(x) < M. \tag{5}$$

  (4) means that by taking x sufficiently close to a, the value of $f(x)$ is arbitrarily large.
  Remark. We can replace "$M > 0$" with "$M \in \mathbb{R}$".

The one-sided infinite limits can be defined similarly:

$$\lim_{x\to a^+} f(x) = \infty, \quad \lim_{x\to a^+} f(x) = -\infty, \quad \lim_{x\to a^-} f(x) = \infty, \quad \lim_{x\to a^-} f(x) = -\infty.$$

Remark. Since $\pm\infty$ are not real numbers, the infinite limit *does not exist*.

- To determine infinite limits,
  1. Determine whether $f(x)$ is **large**, i.e., $|f(x)| \to \infty$
     $f(x) = \dfrac{P(x)}{Q(x)}$ with $P(x) \to L \neq 0$ while $Q(x) \to 0$.
  2. Determine whether $f(x)$ is **positively large** or **negatively large**
     As $x \to a$ (or a^+, a^-), check whether $f(x) > 0$ or $f(x) < 0$.

- **Theorem** - Suppose $f(x) = g(x)$ for all x in an open interval containing a, except at a. If $\lim_{x\to a} f(x) = L$, then $\lim_{x\to a} g(x) = L$.

- **Lemma** - Suppose $f(x) \geq 0$ for all x in an open interval containing a, except possibly at a. If $\lim_{x\to a} f(x) = L$, then $L \geq 0$.

- **Theorem** - Suppose $f(x) \geq g(x)$ for all x in an open interval containing a, except possibly at a. If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then $L \geq M$.
  Proof: Let $h(x) = f(x) - g(x)$. Then $h(x) \geq 0$ near a, and $\lim_{x\to a} f(x) = L - M$. Apply above lemma.

- **Squeeze Theorem** - Suppose that $f(x) \leq g(x) \leq h(x)$ for all x near a (except possibly at a). If $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x)$ exists, and $\lim_{x\to a} g(x) = L$.
  In practice, g is a relatively complicated function. We want to *bound* g between two simple functions f and h whose limits are easy to compute.

- The limit of $f(x)$ as x approaches infinity is L, denoted by $\lim_{x\to\infty} f(x) = L$, if

  $$(\forall \varepsilon > 0) \quad (\exists N \in \mathbb{R}) \quad x > N \implies |f(x) - L| < \varepsilon.$$

  The limit of $f(x)$ as x approaches negative infinity is L, denoted by $\lim_{x\to-\infty} f(x) = L$, if

  $$(\forall \varepsilon > 0) \quad (\exists N \in \mathbb{R}) \quad x < N \implies |f(x) - L| < \varepsilon.$$

  We are concerned with the behaviour of f at the right and left tail respectively.

3 Continuity

- f is **continuous** at a if $\lim_{x\to a} f(x) = f(a)$; otherwise, f is **discontinuous** at a.
  The definition of continuity consists of three parts:
  1. $f(a)$ is well-defined, i.e., a is in the domain of f;
  2. $\lim_{x\to a} f(x)$ exists;
  3. $\lim_{x\to a} f(x) = f(a)$.

- Types of discontinuities

1. f has **removable discontinuity** at a if $\lim_{x\to a} f(x)$ exists, but $f(a)$ is undefined, or $f(a)$ is well-defined but $f(a) \neq \lim_{x\to a} f(x)$.
   We can remove the discontinuity of f at a by redefining $f(a)$:
   $$f^*(x) = \begin{cases} f(x) & (x \neq a) \\ \lim_{x\to a} f(x) & (x = a) \end{cases}$$
   is the **continuous extension** of f at a.
2. f has a **jump continuity** at a if $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist but not equal.
3. f has an **infinite discontinuity** at a if $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$.
   The vertical line $x = a$ is a **vertical asymptote** of $y = f(x)$.
   E.g. Let $f(x) = \dfrac{P(x)}{Q(x)}$. Then f has an infinite discontinuity at $a \iff Q(a) = 0$.

- f is **continuous** from the **right** at a if $\lim_{x\to a^+} f(x) = f(a)$.
  f is **continuous** from the **left** at a if $\lim_{x\to a^-} f(x) = f(a)$.
- **Theorem** - f is continuous at $a \iff f$ is continuous from the left and right at a.
  Proof: $\lim_{x\to a} f(x) = f(a) \iff \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)$.
- **Properties** - If f and g are continuous at a, then the following are continuous at a:
  $$cf \quad f \pm g \quad fg \quad f/g \quad f(x)^n \quad \sqrt[n]{f(x)}.$$
  Proof: Use limit laws.
- Common continuous functions:
  - Polynomials
  - Rational functions
  - Root functions
  - Trigonometric functions (cont. on the domain)
- **Theorem** - Suppose $\lim_{x\to a} f(x) = b$ and $\lim_{y\to b} g(y) = c$, and $f(x) \neq b$ for all x in an open interval containing a except at a. Then $\lim_{x\to a} g(f(x)) = c$.
- **Theorem** - Suppose g is continuous. Then $\lim_{x\to a} g(f(x)) = g\left(\lim_{x\to a} f(x)\right)$.
  The limit operator lim commutes with continuous function.
- **Theorem** - If f is continuous at a, and g is continuous at $f(a)$, then $g \circ f$ is continuous at a.
  Proof: $\lim_{x\to a} g(f(x)) = g\left(\lim_{x\to a} f(x)\right) = g(f(a))$.
- **Intermediate Value Theorem** - If f is continuous on $[a,b]$, and y_0 is between $f(a)$ and $f(b)$, then there exists $c \in [a,b]$ such that $y_0 = f(c)$.
  Corollary - If f is continuous on $[a,b]$ and $f(a)f(b) < 0$, then there exists $c \in [a,b]$ such that $f(c) = 0$ (i.e., f has a root in (a,b)).
  Remark. Closed interval.

4 Derivatives

- The **derivative** of f at a is $f'(a) := \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$.

  If $f'(a)$ exists, f is **differentiable** at a.

  The **tangent line** to $y=f(x)$ at $x=a$ is $y=f'(a)(x-a)+f(a)$.

- The **derivative** of f is the function f' defined by $f'(x) := \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$, if the limit exists.

  Alternatively, $f'(x) = \lim_{z\to x} \frac{f(z)-f(x)}{z-x}$, if the limit exists.

  Let $y=f(x)$. We write $f'(x) = \frac{\mathrm{d}y}{\mathrm{d}x}$ and $f'(a) = \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=a}$.

  Also write $f'(x) = \frac{\mathrm{d}}{\mathrm{d}x} f(x)$, where $\frac{\mathrm{d}}{\mathrm{d}x}$ is the **differentiation operator**.

- f is **differentiable** on an open interval I if f is differentiable at every point in I.

- **Theorem** - Differentiability implies continuity

  The converse is not true.

- **Differentiation formulae**

  $(cf)' = cf' \qquad (f+g)' = f'+g' \qquad (fg)' = f'g+fg' \qquad \left(\frac{f}{g}\right)' = \frac{gf'-fg'}{g^2}$ if $g(x)\neq 0$

- Examples:

$$\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x \qquad \frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x \qquad \frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\cot x = -\csc^2 x \qquad \frac{\mathrm{d}}{\mathrm{d}x}\sec x = \sec x\tan x \qquad \frac{\mathrm{d}}{\mathrm{d}x}\csc x = -\csc x\cot x$$

  Proof: use differentiation formulae.

  Proof of $\lim_{h\to 0}\frac{\sin h}{h} = 1$: Geometrically $\sin h < h < \tan h \Rightarrow \cos h < \frac{\sin h}{h} < 1$, then Squeeze theorem.

  Proof of $\lim_{h\to 0}\frac{\cos h - 1}{h} = 0$: Squeeze theorem, $-\frac{\sin^2 h}{h} < \frac{\cos h - 1}{h} < 0$.

- **Chain rule** - If f is differentiable at x, g is differentiable at $f(x)$, then $g\circ f$ is differentiable at x,

$$(g\circ f)'(x) = g'(f(x))f'(x). \tag{6}$$

  Let $y=f(x)$ and $z=g(y)$, then $\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x}$.

- Let $f(x,y)=0$ be an equation in x and y. If y can be expressed in terms of x near a point on $f(x,y)=0$, we call y an **implicit function** of x near the point.

  Suppose y is an implicit function of x such that $\frac{\mathrm{d}y}{\mathrm{d}x}$ exists. Evaluate $\frac{\mathrm{d}y}{\mathrm{d}x}$ by **implicit differentiation**:

  1. Differentiate $f(x,y)=0$ with respect to x, regarding y as a differentiable function in x.
  2. Solve $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y.

Remark. In order to use implicit differentiation, it is necessary to assume y as an implicit differentiable function in x.

Remark. Implicit differentiation *cannot* be used in proving differentiability.

- If f' is differentiable, then $(f')'$ is a function, denoted by f''.

  f'' is called the **second order derivative** of f. If $f''(a)$ exists, f is **twice differentiable** at a.

  Let $y = f(x)$. Then $f'(x) = \frac{\mathrm{d}y}{\mathrm{d}x}$ and $f''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

- Higher order derivatives: For $n \in \mathbb{N}$, the **n-th order derivative** of f is $f^{(n)}$. f is **n times differentiable** if $f^{(n)}$ exists.

  Let $y = f(x)$. Then $f^{(n)}(x) = \frac{\mathrm{d}^n y}{\mathrm{d}x^n}$.

5 Applications of Derivatives

- Let $f\colon D \subseteq \mathbb{R} \to \mathbb{R}$. f has **absolute maximum** value at $c \in D$ if $f(c) \geq f(x)$ for every $x \in D$.

  f has **absolute minimum** value at $c \in D$ if $f(c) \leq f(x)$ for every $x \in D$.

  The absolute maximum and minimum are both called the **extreme values**.

- f has **local maximum** at $c \in D$ if $f(c) \geq f(x)$ for all x in an <u>open interval</u> containing c.

  f has **local minimum** at $c \in D$ if $f(c) \leq f(x)$ for all x in an <u>open interval</u> containing c.

  The local maximum and minimum are both called the **local extreme values**.

  Remark. By definition, local extreme values cannot occur at the endpoints – only at the interior.

- **Extreme Value Theorem** - Suppose f is continuous on $[a,b]$. Then f attains extreme values on $[a,b]$, i.e., $\exists c, d \in [a,b]$ such that $f(c) \leq f(x) \leq f(d)$ for every $x \in [a,b]$.

  Proof requires compactness. Bonus: compactness implies completeness (IVT).

- **Fermat's Theorem** - If f has a local extreme value at c, and f is differentiable at c, then $f'(c) = 0$.

  Proof: Suppose $f(c) \leq f(x)$ for all x in open interval containing c. Two cases: $x > c$ or $x < c$.

- Let c be an interior point of the domain of f. We call c a **stationary point** of f if $f'(c) = 0$.

  We call c a **critical point** of f, if either c is a stationary point, or $f'(c)$ does not exist.

- **Closed Interval Method** - Let f be continuous on $[a,b]$.

  1. Evaluate the values of f at **endpoints**: $f(a)$ and $f(b)$.
  2. Evaluate the values of f at **critical points** on (a,b).
  3. Compare the values obtained in Steps 1 and 2:
     - Largest = **absolute maximum**
     - Smallest = **absolute minimum**

- **Rolle's Theorem** - Suppose f is continuous on $[a,b]$, differentiable on (a,b). If $f(a) = f(b)$, there exists $c \in (a,b)$ such that $f'(c) = 0$.

  Proof: Use Extreme Value Theorem on f, then Fermat's Theorem on extreme values.

  Use to show a function has exactly n roots.

- **Mean Value Theorem** - Suppose f is continuous on $[a,b]$, differentiable on (a,b). There exists $c \in (a,b)$ such that
$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$
Proof: Let $h(x)$ be the difference between the curve $y = f(x)$ and the line segment joining $(a, f(a))$ and $(b, f(b))$; apply Rolle's Theorem on $h(x)$.

- **Cauchy's Mean Value Theorem** - Suppose f is continuous on $[a,b]$, differentiable on (a,b). If $g'(x) \neq 0$ for any $x \in (a,b)$, then there exists $c \in (a,b)$ such that
$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

- **Increasing, Decreasing Test** - Suppose f is continuous on interval I, differentiable on interior I' of I.
  - $f'(x) = 0$ for every $x \in I' \Rightarrow f$ is constant on I
  - $f'(x) > 0$ for every $x \in I' \Rightarrow f$ is increasing on I
  - $f'(x) < 0$ for every $x \in I' \Rightarrow f$ is decreasing on I

  Converse is not true: consider derivative of $f(x) = x^3$ at $x = 0$

  Proof: Mean Value Theorem, consider the sign of $f'(c) = \frac{f(b)-f(a)}{b-a}$.

- **First Derivative Test** - Suppose f is continuous at a critical point c, and differentiable on open interval containing c, except at c.
  - If f' changes from negative to positive at c, then f has **local minimum** at c
  - If f' changes from positive to negative at c, then f has **local maximum** at c
  - If f' does not change sign at c, then f does not have local extreme value at c (saddle point)

- **Second Derivative Test** - Suppose $f'(c) = 0$.
  - $f''(c) > 0 \Rightarrow$ **local minimum** at c
  - $f''(c) < 0 \Rightarrow$ **local maximum** at c
  - $f''(c) = 0 \Rightarrow$ **inconclusive**

  Proof: Use First Derivative Test

- Suppose f is differentiable on open interval I.
  - f is **concave up** on I if $f(x) > f'(a)(x-a) + f(a)$ for any $a \neq x$ in I. (above tangent lines)
  - f is **concave down** on I if $f(x) < f'(a)(x-a) + f(a)$ for any $a \neq x$ in I. (below tangent lines)

- **Theorem** - Suppose f is differentiable on open interval I.
  - f' is increasing on $I \Leftrightarrow f$ is concave up on I
  - f' is decreasing on $I \Leftrightarrow f$ is concave down on I

- **Concavity Test** - Suppose f be twice differentiable on open interval I.
  - $f''(x) > 0$ for all $x \in I \Rightarrow f$ is concave up on I
  - $f''(x) < 0$ for all $x \in I \Rightarrow f$ is concave down on I

- If f is continuous at c, and changes concavity at c, then f has an **inflection point** at c.

  Note: differentiability is not required.

- **Theorem** - Suppose f has inflection point at c. If f is twice differentiable at c, then $f''(c) = 0$.

- **Curve Sketching**

  1. Find critical points of $f(x)$ (either $f'(x) = 0$ or $f'(x)$ DNE)
  2. Partition domain into subintervals, consider signs of $f'(x)$ respectively - monotonicity (increasing/decreasing)
  3. Local max / min
  4. Concavity - concave up/down

- **Optimisation Problem**

  1. Understand the problem
  2. Introduce notations
  3. Find relations among the variables
  4. Express the problem as finding **extreme value** (absolute max/min) of a single-variable function on interval I
     - If $I = [a,b]$, use **closed interval method**
     - For arbitrary interval I, use **increasing and decreasing tests** to find the intervals of increasing and decreasing.

  Remark. Do not use 2nd derivative test.

- **L'Hopital's Rule** - Suppose f and g are differentiable at a. If $\lim_{x\to a} \frac{f(x)}{g(x)}$ is in indeterminate form ($\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$), then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}. \tag{7}$$

  Remark. Here $x \to a$ can be replaced by $x \to a^-$, $x \to a^+$, $x \to \infty$ or $x \to -\infty$.

  Evaluate $\lim_{x\to a} \frac{f(x)}{g(x)}$ by l'Hopital's rule:

  1. Continue to differentiate f and g, as long as numerator and denominator *both* tend to 0 or $\pm\infty$.
  2. Stop if one of numerator/denominator does not tend to 0 or $\pm\infty$.

6 Integrals

- Let f be continuous on $[a,b]$.

  1. Divide $[a,b]$ into n equal subintervals, each of length $\Delta x = \frac{b-a}{n}$:

$$[x_0, x_1], \quad [x_1, x_2], \quad \ldots, \quad [x_{n-1}, x_n] \qquad \text{where } x_i = a_i \Delta x.$$

2. Choose **sample points** $x_1^*,\dots,x_n^*$ from these subintervals:

$$x_i^* \in [x_{i-1},x_i] \qquad (i=1,\dots,n).$$

3. Compute the **Riemann sum** $\sum_{i=1}^n f(x_i^*)\Delta x$.
4. Taking $n\to\infty$, the **definite integral** of f from a to b is

$$\int_a^b f(x)\,\mathrm{d}x := \lim_{n\to\infty}\sum_{i=1}^n f(x_i^*)\Delta x.$$

We call x the **variable**, $f(x)$ the **integrand**, a the **lower limit**, b the **upper limit**, and $\int$ the **integral sign**.

- Let (a_n) be a sequence. Then $\lim\limits_{n\to\infty} a_n = L \in \mathbb{R}$ means

$$(\forall \varepsilon > 0) \quad (\exists N\in\mathbb{N}) \quad n>N \implies |a_n - L| < \varepsilon.$$

- Let $f\geq 0$ continuous on $[a,b]$. Geometrically, $\int_a^b f(x)\,\mathrm{d}x$ represents the **area** of the region between $y=f(x)$ and x-axis from a to b.

  If $f\geq 0$, then consider $-f$. Geometrically, $\int_a^b f(x)\,\mathrm{d}x$ represents the **negative** of the **area** of region between $y=f(x)$ and x-axis from a to b.

  In general, let A_1 be area of region below $y=f(x)$ and above x-axis, A_2 be area of region above $y=f(x)$ and below x-axis. Then $\int_a^b f(x)\,\mathrm{d}x = A_1 - A_2$ represents the **net area** of region between $y=f(x)$ and x-axis from a to b.

- **Properties** - Let f and g be continuous on $[a,b]$, $c\in\mathbb{R}$.
  - Constant: $\displaystyle\int_a^b c\,\mathrm{d}x = c(b-a)$
  - Monotonicity: If $f(x)\geq g(x)$ for all $x\in[a,b]$, then $\displaystyle\int_a^b f(x)\,\mathrm{d}x \geq \int_a^b g(x)\,\mathrm{d}x$
  - Boundedness: $\displaystyle m(b-a)\leq \int_a^b f(x)\,\mathrm{d}x \leq M(b-a)$
  - Split interval: $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$
  - Swap limits: $\displaystyle\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$
  - Linearity: $\displaystyle\int_a^b cf(x)\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x \qquad \int_a^b (f(x)+g(x))\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$

- **Fundamental Theorem of Calculus (I)** - Suppose f is continuous on $[a,b]$. Define $g(x)=\int_a^x f(t)\,\mathrm{d}t$. Then g is continuous on $[a,b]$, differentiable on (a,b), and $g'(x)=f(x)$ for every $x\in(a,b)$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x). \tag{8}$$

- **Mean Value Theorem for Definite Integrals** - Suppose f is continuous on $[a,b]$. There exists $c\in(a,b)$ such that

$$\int_a^b f(x)\,\mathrm{d}x = (b-a)f(c).$$

- If $F'(x)=f(x)$ for all x, we say that F is *an* **anti-derivative** of f.

- **Fundamental Theorem of Calculus (II)** - Suppose f is continuous on $[a,b]$. Suppose F is an anti-derivative of f continuous on $[a,b]$, differentiable on (a,b). Then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - f(a). \tag{9}$$

- By FTC (II), evaluation of definite integrals may be reduced to finding anti-derivatives.

  Let f continuous on an interval. If F is an antiderivative of f, we call F an **indefinite integral** of f.

  The **indefinite integral** of f is the entire family of antiderivatives

$$\int f(x)\,\mathrm{d}x = F(x) + C \qquad \text{where } C \text{ is an arbitrary constant.}$$

  By definition,

$$\int f(x)\,\mathrm{d}x = F(x) + C \iff \frac{\mathrm{d}}{\mathrm{d}x}F(x) = f(x).$$

- **Properties** -
  - $\int k\,\mathrm{d}x = kx + C,\ k \in \mathbb{R}$
  - $\int x^n\,\mathrm{d}x = \dfrac{x^{n+1}}{n+1} + C,\ n \neq -1$
  - $\int (f(x)+g(x))\,\mathrm{d}x = \int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x$
  - $\int \alpha f(x)\,\mathrm{d}x = \alpha \int f(x)\,\mathrm{d}x$

- **Substitution rule** (indefinite integral) - Suppose $u = g(x)$ is differentiable, whose range is an interval I. Suppose g' is continuous, and f is continuous on I. Then

$$\int f(g(x))g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u\,. \tag{10}$$

  Use this method when the derivative of the substitution appears as a factor.

  Substitution rule (definite integral) - Suppose g' is continuous on $[a,b]$, f continuous on the range of g. Then

$$\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u\,. \tag{11}$$

- **Theorem** - Let f be continuous on $[-a,a]$.
  - If f is odd, then $\int_{-a}^{a} f(x)\,\mathrm{d}x = 0$.
  - If f is even, then $\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_0^a f(x)\,\mathrm{d}x$.

- Let f be continuous on $[a,b)$, discontinuous at b from the left. The **improper integral**

$$\int_a^b f(x)\,\mathrm{d}x := \lim_{t\to b^-} \int_a^t f(x)\,\mathrm{d}x\,.$$

  Let f be continuous on $(a,b]$, discontinuous at a from the right. The **improper integral**

$$\int_a^b f(x)\,\mathrm{d}x := \lim_{t\to a^+} \int_t^b f(x)\,\mathrm{d}x\,.$$

Let f be continuous on $[a,c)\cup(c,b]$, discontinuous at $c\in(a,b)$. The **improper integral**

$$\int_a^b f(x)\,\mathrm{d}x := \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x.$$

The improper integral is **convergent** if the limit exists, **divergent** if the limit does not exist.

- **Theorem** - Let f be continuous on (a,b), and $\lim_{x\to a^+} f(x)$ and $\lim_{x\to b^-} f(x)$ exist. Let f_1 be the continuous extension of f. Then

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^b f_1(x)\,\mathrm{d}x.$$

- Let f be continuous on $[a,\infty)$. If $\int_a^t f(x)\,\mathrm{d}x$ exists for every $t\ge a$, the **improper integral**

$$\int_a^\infty f(x)\,\mathrm{d}x := \lim_{t\to\infty}\int_a^t f(x)\,\mathrm{d}x.$$

  Let f be continuous on $(-\infty,b]$. If $\int_t^b f(x)\,\mathrm{d}x$ exists for every $t\le b$, the **improper integral**

$$\int_{-\infty}^b f(x)\,\mathrm{d}x := \lim_{t\to-\infty}\int_t^b f(x)\,\mathrm{d}x.$$

- The **improper integral** of f on $(-\infty,\infty)$ is

$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \int_{-\infty}^a f(x)\,\mathrm{d}x + \int_a^\infty f(x)\,\mathrm{d}x \quad (a\in\mathbb{R}).$$

  It is **convergent** if both improper integrals on the right are convergent, and it is **divergent** otherwise.

7 Inverse Functions and Transcendental Functions

- Let f be a function with domain D. f is called a **one-to-one** function if

$$a\neq b \implies f(a)\neq f(b) \qquad \text{for any } a,b\in D.$$

  Equivalently, by taking the contrapositive,

$$f(a)=f(b) \implies a=b \qquad \text{for any } a,b\in D.$$

- Let f be a one-to-one function. Let A be the domain of f, and B the range of f. For each $y\in B$, there is a unique $x\in A$ such that $f(x)=y$.

  The inverse function of f, denoted by f^{-1}, is defined by

$$f^{-1}(y)=x \iff y=f(x) \qquad \text{for any } x\in A,\ y\in B.$$

  B is the domain of f^{-1}, and A is the range of f^{-1}.

- To find inverse function of f,

  1. Let $y=f(x)$.
  2. Express x in terms of y: $\quad x=f^{-1}(y)$.
  3. Interchange x and y to express f^{-1} as a function in variable x: $\quad y=f^{-1}(x)$.

- In $\mathbb{R}^2$, interchanging x and y has the same effect as reflection wrt $y = x$.

  The graph of f and the graph of f^{-1} are **symmetric** wrt $y = x$.

- **Theorem** - Let f be continuous on interval I. Then f is one-to-one if and only if f is monotonic (increasing/decreasing).

- **Theorem** - Let f be one-to-one, continuous on interval I. Then f^{-1} is continuous.

- **Theorem** - Let f be one-to-one, continuous on interval I.

  If f is differentiable at interior point a of I, and $f'(a) \neq 0$, then f' is differentiable at $b = f(a)$, and

$$(f^{-1})'(b) = \frac{1}{f'(a)}. \tag{12}$$

  Remark. Product of slopes equals 1.

- **Inverse sine function** $\sin^{-1} : [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ is $y = \sin^{-1} x \iff \sin y = x$

$$\frac{\mathrm{d}}{\mathrm{d}x} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

- **Inverse cosine function** $\cos^{-1} : [-1,1] \to [0,\pi]$ is $y = \cos^{-1} x \iff \cos y = x$

$$\frac{\mathrm{d}}{\mathrm{d}x} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}.$$

- **Inverse tangent function** $\tan^{-1} : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is $y = \tan^{-1} x \iff \tan y = x$

$$\frac{\mathrm{d}}{\mathrm{d}x} \tan^{-1} x = \frac{1}{1+x^2}.$$

- **Inverse cotangent function** $\cot^{-1} : \mathbb{R} \to (0,\pi)$ is $y = \cot^{-1} y \iff \cot y = x$

$$\frac{\mathrm{d}}{\mathrm{d}x} \cot^{-1} x = -\frac{1}{1+x^2}.$$

- **Inverse secant function** $\sec^{-1} : (-\infty,1] \cup [1,\infty) \to [0,\frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$ is $y = \sec^{-1} x \iff \sec y = x$.

$$\frac{\mathrm{d}}{\mathrm{d}x} \sec^{-1} x = \frac{1}{x\sqrt{x^2-1}}.$$

- **Inverse cosecant function** $\csc^{-1} : (-\infty,1] \cup [1,\infty) \to (0,\frac{\pi}{2}] \cup (\pi, \frac{3\pi}{2}]$ is $y = \csc^{-1} x \iff \csc y = x$

$$\frac{\mathrm{d}}{\mathrm{d}x} \sec^{-1} x = -\frac{1}{x\sqrt{x^2-1}}.$$

- **Natural logarithmic function** ln is defined by

$$\ln x := \int_1^x \frac{1}{t}\,\mathrm{d}t \qquad (x > 0)$$

  Properties -

  - $\ln x$ is continuous and differentiable on $\mathbb{R}^+$, and $\frac{\mathrm{d}}{\mathrm{d}x} \ln x = \frac{1}{x}$

- $\ln x$ is increasing and concave down on $\mathbb{R}^+$, since $\dfrac{d^2}{dx^2}\ln x = -\dfrac{1}{x^2} < 0$
- $\lim\limits_{x\to 0^+} \ln x = -\infty$, $\lim\limits_{x\to\infty} \ln x = \infty$, and the range of $\ln x$ is $\mathbb{R}$
- $\ln(xy) = \ln x + \ln y$
- $\ln(x^r) = r\ln x$
- $\dfrac{d}{dx}\ln|x| = \dfrac{1}{x} \iff \displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$

- **Logarithmic differentiation**: Let $y = f_1(x)^{r_1}\cdots f_n(x)^{r_n}$.
  1. Take absolute value: $|y| = |f_1(x)|^{r_1}\cdots|f_n(x)|^{r_n}$
  2. Take ln on both sides to bring down power: $\ln|y| = r_1\ln|f_1(x)| + \cdots + r_n\ln|f(x)|$
  3. Differentiate both sides with respect to x, use chain rule

- **Euler's number** e is the (unique) number such that $\ln e = 1$.

- Since ln is increasing on $\mathbb{R}^+$ with range $\mathbb{R}$, $\ln\colon \mathbb{R}^+ \to \mathbb{R}$ admits an inverse function $\exp\colon \mathbb{R} \to \mathbb{R}^+$ such that

$$y = \exp x \iff \ln y = x \qquad (x, y > 0)$$

exp is called the **exponential function**. We write $e^x = \exp x$

- $\lim\limits_{x\to-\infty} e^x = 0$, $\lim\limits_{x\to\infty} e^x = \infty$
- $\ln(e^x) = x$ for $x \in \mathbb{R}$, $e^{\ln y} = y$ for $y \in \mathbb{R}^+$
- $\dfrac{d}{dx}e^x = e^x$

- The **exponential function** of base $a > 0$ is defined by

$$a^x := \exp(x\ln a) = e^{x\ln a} \qquad (x \in \mathbb{R}).$$

Properties -

- $\ln(a^x) = x\ln a$
- $a^x a^y = a^{x+y}$
- $a^{-x} = 1/a^x$
- $(a^x)^y = a^{xy}$
- $\dfrac{d}{dx}a^x = a^x\ln a$
- $\dfrac{d}{dx}x^a = ax^{a-1}$

Therefore $\displaystyle\int x^a\,dx = \begin{cases} \ln|x| + C & \text{if } a = -1 \\ \dfrac{x^{a+1}}{a+1} + C & \text{if } a \neq -1 \end{cases}$

If both base and exponent are functions of x, then cannot apply usual differentiation formulae. For example, $\dfrac{d}{dx}x^x = (\ln x + 1)x^x$.

More generally, let $y = f(x)^{g(x)}$, where $f(x) > 0$.

1. Express $f(x)^{g(x)} = \exp[g(x)\ln f(x)]$
2. Take ln on both sides: $\ln y = g(x)\ln f(x)$

3. Differentiate:

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} g(x)\ln f(x).$$

then use chain rule on RHS.

- **Hyperbolic sine function** is $\sinh x = \dfrac{e^x - e^{-x}}{2}$.

  Hyperbolic cosine function is $\cosh x = \dfrac{e^x + e^{-x}}{2}$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\sinh x = \cosh x, \qquad \frac{\mathrm{d}}{\mathrm{d}x}\cosh x = \sinh x.$$

  Inverse hyperbolic sine function $\sinh^{-1} : \mathbb{R} \to \mathbb{R}$

$$\frac{\mathrm{d}}{\mathrm{d}x}\sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}.$$

  Inverse hyperbolic cosine function $\cosh^{-1} : [1,\infty) \to [0,\infty)$

$$\frac{\mathrm{d}}{\mathrm{d}x}\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}.$$

8 Techniques of Integration

- **Inverse Substitution Rule** - Let f be continuous. Suppose $x = g(t)$ is one-to-one and g' is continuous. Then

$$\int f(x)\,\mathrm{d}x = \int f(g(t))g'(t)\,\mathrm{d}t\,. \tag{13}$$

  Remark. Convert the type of the integrand (e.g. root function to rational function, root to trigo), such that integrand on RHS is of a simpler type than LHS.

  Remark. need g one-to-one since replace t by $g^{-1}(x)$ after integrating

- **Integration by Parts** - Let $u = u(x)$, $v = v(x)$ be differentiable with continuous derivatives. Then

$$\int \left(u\frac{\mathrm{d}v}{\mathrm{d}x}\right)\mathrm{d}x = uv - \int \left(\frac{\mathrm{d}u}{\mathrm{d}x}v\right)\mathrm{d}x\,. \tag{14}$$

  In differential form, $\mathrm{d}u = \frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$ and $\mathrm{d}v = \frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x$. Then

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u\,.$$

  Remark. Express the integrand as $u(\frac{\mathrm{d}v}{\mathrm{d}x})$ such that the integration of $v(\frac{\mathrm{d}u}{\mathrm{d}x})$ is relatively easier than the integration of $u(\frac{\mathrm{d}v}{\mathrm{d}x})$.

- **Trigonometric Substitution** - When integrand contains square root of quadratic functions.
  - $\sqrt{a^2-x^2} \overset{x=a\sin t}{=} \sqrt{a^2-(a\sin t)^2} = a\cos t$
  - $\sqrt{a^2+x^2} \overset{x=a\tan t}{=} \sqrt{a^2+(a\tan t)^2} = a\sec t$
  - $\sqrt{x^2-a^2} \overset{x=a\sec t}{=} \sqrt{(a\sec t)^2-a^2} = a\tan t$

- **Rational Functions** - A rational function of the ratio of polynomials $f(x) = \dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials.

- **Theorem** - Every non-constant single-variable polynomial with real coefficients can be uniquely factorised as the product of linear factors and irreducible quadratic factors.

- **Theorem** - Every rational function can be uniquely expressed as the sum of partial fractions.

  Let $f(x) = \frac{A(x)}{B(x)}$, where $A(x)$, $B(x)$ are polynomials. If $\deg A(x) \geq \deg B(x)$, use long division to get

  $$A(x) = B(x)Q(x) + R(x), \qquad \text{where } \deg R(x) < \deg B(x).$$

  Then $f(x) = Q(x) + \dfrac{R(x)}{B(x)}$.

  $f(x) = \frac{A(x)}{B(x)}$ is a **proper rational function** if $A(x)$, $B(x)$ are polynomials with $\deg A(x) < \deg B(x)$.

- Let $f(x) = \frac{A(x)}{B(x)}$ be a proper rational function. Then $f(x)$ is the sum of the following partial fractions: Let $x+a$ be a linear factor of $B(x)$ with multiplicity r:

  $$\frac{A_1}{x+a} + \frac{A_2}{(x+a)^2} + \cdots + \frac{A_r}{(x+a)^r}$$

  Let x^2+bx+c be an irreducible quadratic factor of $B(x)$ with multiplicity s:

  $$\frac{B_1x+C_1}{x^2+bx+c} + \cdots + \frac{B_sx+C_s}{x^2+bx+c^s}$$

  The total number of unknowns A_i, B_j, C_k equals $\deg B(x)$.

- **Universal Trigonometric Substitution** - Let f be a rational expression in two variables. On $x \in (-\pi, \pi)$, $\int f(\sin x, \cos x)\,\mathrm{d}x$ can be evaluated by $t = \tan \frac{x}{2}$.

  Then $x = 2\tan^{-1} t \Rightarrow \dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{2}{1+t^2}$. Thus $\sin x = \dfrac{2t}{1+t^2}$ and $\cos x = \dfrac{1-t^2}{1+t^2}$. Hence

  $$\int f(\sin x, \cos x)\,\mathrm{d}x = \int f\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2}{1+t^2}\,\mathrm{d}t\,. \tag{15}$$

  The integrand becones a rational function in t.

  Remark. This may not be the best way, but it *always* works.

9 Applications of Definite Integrals

For geometry problems, drawing a picture (graph) is important to visualise the area/volume.

- **Area Problem** - Let f and g be continuous such that $f(x) \geq g(x)$ for all $x \in [a,b]$.

  Integrate with respect to x:

  1. Cut the region using vertical line segments (perpendicular to x-axis)
  2. Length of line segment at x is $\ell(x) = f(x) - g(x)$
  3. Area of region bounded between graphs of f and g on $[a,b]$ is

  $$A = \int_a^b \ell(x)\,\mathrm{d}x = \int_a^b [f(x) - g(x)]\,\mathrm{d}x\,.$$

We can also integrate with respect to y: cut the region using horizontal line segments at $y \in [c,d]$; length $L(y)$ = right endpoint – left endpoint. Then area of region is

$$A = \int_c^d L(y)\,\mathrm{d}y\,.$$

- Suppose a solid is placed along x-axis on $[a,b]$. Cut the solid using planes perpendicular to x-axis. Let $A(x)$ be the area of cross-section at $x \in [a,b]$. Then

  $$V = \int_a^b A(x)\,\mathrm{d}x\,.$$

  Suppose a solid is placed along y-axis on $[c,d]$. Cut the solid using planes perpendicular to y-axis. Let $A(y)$ be the area of cross section at $y \in [c,d]$. Then

  $$V = \int_c^d A(y)\,\mathrm{d}y\,.$$

- **Disk Method** - Let f be continuous on $[a,b]$. Consider the region between $y = f(x)$ and x-axis on $[a,b]$.

  1. Cut the region using vertical line segments perpendicular to x-axis
  2. Cross-section at x is a circle of radius $|f(x)|$.
  3. Area of cross-section is $A(x) = \pi|f(x)|^2 = \pi[f(x)]^2$.
  4. Volume of solid formed by rotating the region about x-axis is

  $$V = \int_a^b \pi[f(x)]^2\,\mathrm{d}x\,.$$

- **Washer Method** - More generally, let f and g be continuous such that $f(x) \geq g(x) \geq 0$ on $[a,b]$. Consider the region between $f(x)$ and $g(x)$ on $[a,b]$.

  1. Cut the region using vertical line segments perpendicular to x-axis
  2. Cross-section is difference of two concentric circles of outer radius R, inner radius r
  3. Area of cross-section is $A(x) = \pi(R^2 - r^2)$
  4. Volume of solid formed by rotating the region about x-axis is

  $$\int_a^b \pi[f(x)]^2\,\mathrm{d}x - \int_a^b \pi[f(x)]^2\,\mathrm{d}x = \int_a^b \pi\left([f(x)]^2 - [g(x)]^2\right)\mathrm{d}x\,.$$

- **Cylindrical Shell Method** - Let f be continuous and non-negative on $[a,b]$, where $a \geq 0$. Consider the region between $y = f(x)$ and x-axis on $[a,b]$.

  1. The region is the union of vertical line segments. The rotation of each segment about the y-axis is the shell of a cylinder radius is x, height is $f(x)$
  2. Volume of solid formed by rotating region about y-axis is

  $$\int_a^b 2\pi x f(x)\,\mathrm{d}x\,.$$

  Remark. Use this method instead of washer method for y-axis, if the expression for $y = f(x)$ is difficult to rewrite in terms of y.

More generally, suppose a region is placed along the x-axis on $[a,b]$, where $c \le a$ or $b \le c$.

1. Cut the region using vertical line segments (parallel to axis of rotation), and rotate about $x = c$
2. radius is distance between the segment and $x = c$, i.e., $|x-c|$
   height is length of line segment = upper endpoint – lower endpoint
3. Let $A(x) = 2\pi rh$. Then the volume of the solid is $\int_a^b A(x)\,\mathrm{d}x$.

- Let f be continuously differentiable on $[a,b]$. The **arc length** of the curve $y = f(x)$, $a \le x \le b$, is defined by

$$L = \int_a^b \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \int_a^b \sqrt{1+[f'(x)]^2}\,\mathrm{d}x.$$

- Let f be continuous and non-negative on $[a,b]$. The area of the surface formed by rotating the curve $y = f(x)$, $a \le x \le b$, about the x-axis is

$$\int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,\mathrm{d}x = \int_a^b 2\pi y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x.$$

  This is **surface area of revolution**.

  approximate using rotation of line segment

10 First-Order Differential Equations

- An **ordinary differential equation** (ODE) is an equation of the form

$$F\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, \ldots, \frac{\mathrm{d}^n y}{\mathrm{d}x^n}\right) = 0,$$

  where y is an implicit function in variable x. The highest order of the derivative is the **degree** of the ODE.

  1st order ODE has the form $\dfrac{\mathrm{d}y}{\mathrm{d}x} = F(x,y)$.

- **Separable** - $\boxed{\dfrac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y)}$

  1. Separate the variables: $\dfrac{1}{g(y)}\,\mathrm{d}y = f(x)\,\mathrm{d}x$
  2. Integrate both sides: $\displaystyle\int \frac{1}{g(y)}\,\mathrm{d}y = \int f(x)\,\mathrm{d}x$

- **Homogeneous** - $\dfrac{\mathrm{d}y}{\mathrm{d}x} = F(x,y)$ where $F(x,y)$ is **homogeneous** if $\boxed{F(tx,ty) = F(x,y)}$ for $t \in \mathbb{R}\setminus\{0\}$

  Substitution: $y = vx$. Then the ODE becomes separable:

$$z + x\frac{\mathrm{d}z}{\mathrm{d}x} = F(1,z).$$

- **Linear** - $\boxed{\dfrac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x)}$ (linear in y)

  Multiply **integrating factor** $v(x) = e^{\int p(x)\mathrm{d}x}$ to the equation to obtain

$$v(x)\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)v(x)y = v(x)q(x).$$

By product rule, $\frac{\mathrm{d}}{\mathrm{d}x}[v(x)y] = v(x)q(x)$. Integrating both sides wrt x gives

$$y = \frac{1}{v(x)} \int v(x)q(x)\,\mathrm{d}x\,.$$

- **Bernoulli's Equation** - $\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x)y^n}$ where $n \in \mathbb{R}$

  Substitution: $z = y^{1-n}$. Then $\frac{\mathrm{d}z}{\mathrm{d}x} = (1-n)y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x}$, so

  $$(1-n)y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x} + (1-n)y^{-n}p(x)y = (1-n)y^{-n}q(x)y^n$$

  or

  $$\frac{\mathrm{d}z}{\mathrm{d}x} + (1-n)p(x)z = (1-n)q(x)$$

  which is linear.

- **Initial Value Problem** - An **initial value problem** is an ODE together with initial conditions.

  The solution to an initial value problem is a **particular solution**.

  Remark. Don't touch initial conditions until you have got general solution.

- **Exponential Growth and Decay** - Quantity grows or decays at a rate proportional to the amount.

  Let $y(t)$ be the value of such a quantity at time t. Then $\frac{\mathrm{d}y}{\mathrm{d}t} = ky$ for a fixed constant $k \neq 0$. This is a linear ODE; solving gives general solution

  $$y = Ce^{kt}.$$

  growth when $k > 0$, decay when $k < 0$.

- **Logistic Population Growth** - Resources are limited; only a maximum population M can be accomodated. M is the **limiting population** or **carrying capacity**.

  Let $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$ with limiting population M. If $P(t) > M$, we want $k < 0$; if $P(t) < M$, then $k > 0$. Take $k = r(M-P)$, $r > 0$ is a constant. Then

  $$\frac{\mathrm{d}P}{\mathrm{d}t} = r(M-P)P. \tag{16}$$

  (16) is a Bernoulli differential equation; rearrange to get $\frac{\mathrm{d}P}{\mathrm{d}t} + (-rM)P = (-r)P^2$. Let $z = P^{1-2} = P^{-1}$. Then $\frac{\mathrm{d}z}{\mathrm{d}t} + (rM)z = r$, which is linear.

  Integrating factor is $v(t) = e^{\int rM\mathrm{d}t} = e^{Mrt}$.

  $$z = \frac{1}{e^{Mrt}} \int (e^{Mrt}r)\,\mathrm{d}t = e^{-Mrt} \cdot \frac{e^{Mrt} + C}{M} = \frac{1 + Ce^{-Mrt}}{M}$$

  Thus the general solution is

  $$P(t) = \frac{M}{1 + Ce^{-Mrt}}.$$

  Notice that as $t \to \infty$, $P(t) \to M$. This is known as a **logistic function**.

- **Newton's Law of Cooling** - The rate of heat loss of an object is proportional to the difference of temperatures of the object and its surroundings.

Let $T(t)$ denote the temperature of an object at time t. Let T_S be the surrounding temperature. Then $\frac{\mathrm{d}T}{\mathrm{d}t} = k(T - T_s)$, where k is a constant. If $T > T_S$, this is heat loss, so $\frac{\mathrm{d}T}{\mathrm{d}t} < 0$, so $k < 0$. Write $k = -r$, where $r > 0$.

Heat transfer model:

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -r(T - T_s). \tag{17}$$

Let $A(t) = T(t) - T_S$, then $\frac{\mathrm{d}A}{\mathrm{d}t} = -rA$, so general solution is $A(t) = A(0)e^{-rt}$. Thus $T(t) - T_S = (T(0) - T_S)e^{-rt}$.

Let $T_0 = T(0)$ be the initial temperature of the object. Then

$$T(t) = T_S + (T_0 - T_S)e^{-rt}.$$

(The constant is the initial difference.)

As $t \to \infty$, $T(t) \to T_S$.

A Lecture Problems

Lecture 1

1. For the functions $f(x) = 1 - x^3$ and $g(x) = \dfrac{1}{x}$, find following functions and their domains.

   (i) $f \circ g$, (ii) $g \circ f$, (iii) $f \circ f$, (iv) $g \circ g$.

   Solution.

   (i) $1 - \dfrac{1}{x^3}$, domain is $\mathbb{R} \setminus \{0\}$
   (ii) $\dfrac{1}{1-x^3}$, domain is $\mathbb{R} \setminus \{1\}$
   (iii) $1 - (1-x^3)^3$, domain is $\mathbb{R}$
   (iv) x, domain is $\mathbb{R} \setminus \{0\}$

   $\square$

2. For the functions $f(x) = \dfrac{2}{x+1}$, $g(x) = \cos x$ and $h(x) = \sqrt{x+3}$, find $f \circ g \circ h$.

   Solution. $(f \circ g \circ h)(x) = \dfrac{2}{\cos\sqrt{x+3}+1}$. $\square$

3. For each of the following functions, determine whether it is even, odd, or neither.

   (a) $f(x) = x^{-3}$, (b) $f(x) = |\sin x| - 4x^2$, (c) $f(x) = 3x^3 + 2x + 1$.

   Solution.

   (a) For all x, $f(-x) = (-x)^{-3} = -x^{-3} = -f(x)$. Hence f is odd.
   (b) For all x, $f(-x) = |\sin(-x)| - 4(-x)^2 = |\sin x| - 4x^2 = f(x)$. Hence f is even.
   (c) Since $f(1) = 6$ and $f(-1) = -4$, we have $f(-1) \neq f(1)$ and $f(-1) \neq -f(1)$. Hence f is neither odd nor even.

   $\square$

4. For each of the following, evaluate the limit, if it exists.

   (a) $\displaystyle\lim_{x \to -4} (x+3)^{2024}$, (b) $\displaystyle\lim_{u \to 1} \frac{u^4-1}{u^3-1}$, (c) $\displaystyle\lim_{x \to 1} \frac{x-1}{\sqrt{x+3}-2}$,

   (d) $\displaystyle\lim_{x \to 4} \frac{x^2-4x}{x^2-3x-4}$, (e) $\displaystyle\lim_{x \to -1} \frac{x^2-4x}{x^2-3x-4}$, (f) $\displaystyle\lim_{h \to 0} \frac{\sqrt{1+h}-1}{h}$,

   (g) $\displaystyle\lim_{x \to 1} \frac{\sqrt{x}-x^2}{1-\sqrt{x}}$.

   Solution.

   (a) Since polynomials are continuous,

   $$\lim_{x \to -4} (x+3)^{2024} = (-4+3)^{2024} = (-1)^{2024} = 1.$$

(b) By factorising numerator and denominator of the rational function,

$$\lim_{u\to 1}\frac{u^4-1}{u^3-1}=\lim_{u\to 1}\frac{(u-1)(u+1)(u^2+1)}{(u-1)(u^2+u+1)}=\lim_{u\to 1}\frac{(u+1)(u^2+1)}{u^2+u+1}=\frac{(2)(2)}{3}=\frac{4}{3}.$$

(c) By multiplying the conjugate of the denominator on numerator and denominator,

$$\begin{aligned}\lim_{x\to 1}\frac{x-1}{\sqrt{x+3}-2}&=\lim_{x\to 1}\frac{(x-1)(\sqrt{x+3}+2)}{(\sqrt{x+3}-2)(\sqrt{x+3}+2)}=\lim_{x\to 1}\frac{(x-1)(\sqrt{x+3}+2)}{(x+3)-4}\\&=\lim_{x\to 1}\frac{(x-1)(\sqrt{x+3}+2)}{x-1}=\lim_{x\to 1}\left(\sqrt{x+3}+2\right)=\sqrt{1+3}+2=4.\end{aligned}$$

(d) Similar to (b),

$$\lim_{x\to 4}\frac{x^2-4x}{x^2-3x-4}=\lim_{x\to 4}\frac{x(x-4)}{(x+1)(x-4)}=\lim_{x\to 4}\frac{x}{x+1}=\frac{4}{5}.$$

(e) At $x=-1$, the numerator is 5, denominator is 0. Hence the limit does not exist.

(f) Similar to (c),

$$\begin{aligned}\lim_{h\to 0}\frac{\sqrt{1+h}-1}{h}&=\lim_{h\to 0}\frac{(\sqrt{1+h}-1)(\sqrt{1+h}+1)}{h(\sqrt{1+h}+1)}=\lim_{h\to 0}\frac{h}{h(\sqrt{1+h}+1)}\\&=\lim_{h\to 0}\frac{1}{\sqrt{1+h}+1}=\frac{1}{2}.\end{aligned}$$

(g) Similar to (c),

$$\begin{aligned}\lim_{x\to 1}\frac{\sqrt{x}-x^2}{1-\sqrt{x}}&=\lim_{x\to 1}\frac{\sqrt{x}(1-\sqrt{x}^3)}{1-\sqrt{x}}=\lim_{x\to 1}\frac{\sqrt{x}(1-\sqrt{x})(1+\sqrt{x}+x)}{1-\sqrt{x}}\\&=\lim_{x\to 1}\sqrt{x}(1+\sqrt{x}+x)=3.\end{aligned}$$

□

Lecture 2

1. Determine the following infinite limits.

(a) $\displaystyle\lim_{x\to 5^-}\frac{6}{x-5}$, (b) $\displaystyle\lim_{x\to 0}\frac{x-1}{x^2(x+2)}$, (c) $\displaystyle\lim_{x\to\pi^-}\csc x$, (d) $\displaystyle\lim_{x\to 1^+}\frac{x+1}{x\sin\pi x}$.

Solution.

(a) As $x\to 5^-$, $6\to 6\neq 0$ and $x-5\to 0$, so $\left|\dfrac{6}{x-5}\right|\to\infty$.

As $x\to 5^-$, $6>0$ and $x-5<0$, so $\dfrac{6}{x-5}<0$.

Hence $\displaystyle\lim_{x\to 5^-}\frac{6}{x-5}=-\infty$.

(b) As $x\to 0$, $x-1\to -1\neq 0$ and $x^2(x+2)\to 0$, so $\left|\dfrac{x-1}{x^2(x+2)}\right|\to\infty$.

As $x\to 0$, $x-1<0$, $x^2>0$ and $x+2>0$, so $\dfrac{x-1}{x^2(x+2)}<0$.

Hence $\displaystyle\lim_{x\to 0}\frac{x-1}{x^2(x+2)}=-\infty$.

(c) As $x \to \pi^-$, $1 \to 1 \neq 0$ and $\sin x \to \sin \pi = 0$, so $\left|\dfrac{1}{\sin x}\right| \to \infty$.

As $x \to \pi^-$, $1 > 0$ and $\sin x > 0$, so $\dfrac{1}{\sin x} > 0$.
Hence $\lim\limits_{x \to \pi^-} \csc x = \infty$.

(d) As $x \to 1^+$, $x+1 \to 2 \neq 0$ and $x \sin \pi x \to 1 \cdot \sin \pi = 0$, so $\left|\dfrac{x+1}{x \sin \pi x}\right| \to \infty$.

As $x \to 1^+$, $x+1 > 0$, $x > 0$ and $\sin \pi x < 0$, so $\dfrac{x+1}{x \sin \pi x} < 0$.
Hence $\lim\limits_{x \to 1^+} \dfrac{x+1}{x \sin \pi x} = -\infty$.

□

2. Is there a real number a such that

$$\lim_{x \to 1} \frac{ax^2 + a^2x - 2}{x^3 - 3x + 2}$$

exists? If so, find the value of a and the value of the limit.

Solution. Suppose $\lim\limits_{x \to 1} \dfrac{ax^2 + a^2x - 2}{x^3 - 3x + 2} = L \in \mathbb{R}$. We can write

$$ax^2 + a^2x - 2 = \frac{ax^2 + a^2x - 2}{x^3 - 3x + 2}(x^3 - 3x + 2).$$

Since the limits of all three terms exist, by the product rule,

$$\lim_{x \to 1}(ax^2 + a^2x - 2) = \lim_{x \to 1} \frac{ax^2 + a^2x - 2}{x^3 - 3x + 2} \cdot \lim_{x \to 1}(x^3 - 3x + 2)$$

which simplifies to $a + a^2 - 2 = L \cdot 0 = 0$. Thus $a = 1$ or $a = -2$. If $a = 1$, then

$$\lim_{x \to 1} \frac{ax^2 + a^2x - 2}{x^3 - 3x + 2} = \lim_{x \to 1} \frac{x^2 + x - 2}{x^3 - 3x + 2} = \lim_{x \to 1} \frac{(x-1)(x+2)}{(x-1)^2(x+2)} = \lim_{x \to 1} \frac{1}{x-1}.$$

This limit does not exist. If $a = -2$, then

$$\lim_{x \to 1} \frac{ax^2 + a^2x - 2}{x^3 - 3x + 2} = \lim_{x \to 1} \frac{-2x^2 + 4x - 2}{x^3 - 3x + 2} = \lim_{x \to 1} \frac{-2(x-1)^2}{(x-1)^2(x+2)} = \lim_{x \to 1} \frac{-2}{x+2} = -\frac{2}{3}.$$

□

3. Find $\lim\limits_{x \to 0^+} \sqrt{x} \sin\left(\dfrac{1}{x}\right)$.

Solution. For $x > 0$, $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$, so $-\sqrt{x} \leq \sqrt{x}\sin\left(\frac{1}{x}\right) \leq \sqrt{x}$.
We have $\lim\limits_{x \to 0^+} (-\sqrt{x}) = -\sqrt{0} = 0$ and $\lim\limits_{x \to 0^+} \sqrt{x} = \sqrt{0} = 0$.
By Squeeze Theorem, $\lim\limits_{x \to 0^+} \sqrt{x}\sin\left(\frac{1}{x}\right)$ exists and equals 0. □

4. Let

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Prove that $\lim\limits_{x \to 0} f(x) = 0$.

Solution. If $x \in \mathbb{Q}$, then $f(x) = x^2$, so $0 \le f(x) = x^2$. If $x \in \mathbb{R} \setminus \mathbb{Q}$, then $f(x) = 0$, so $0 = f(x) \le x^2$. In both cases, $0 \le f(x) \le x^2$.

We have $\lim\limits_{x \to 0} 0 = 0$ and $\lim\limits_{x \to 0} x^2 = 0$.

By Squeeze Theorem, $\lim\limits_{x \to 0} f(x)$ exists and equals 0. $\square$

Remark. Note that 0 is rational. Then $f(0) = 0^2 = 0 = \lim\limits_{x \to 0} f(x)$, so f is continuous at 0.

Lecture 3

1. Find the following limits.

   (a) $\lim\limits_{x \to 1} \left(\dfrac{1}{x-1} + \dfrac{1}{x^2 - 3x + 2} \right)$, (b) $\lim\limits_{x \to 9^-} \left(\sqrt{9-x} + \lfloor x+1 \rfloor \right)$.

   (The floor function $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

   Solution.

   (a) Note that $\lim\limits_{x \to 1} \dfrac{1}{x-1}$ and $\lim\limits_{x \to 1} \dfrac{1}{x^2 - 3x + 2}$ do not exist. Hence we cannot apply the limit law. By factorising,

   $$\begin{aligned} \lim_{x \to 1} \left(\frac{1}{x-1} + \frac{1}{x^2 - 3x + 2} \right) &= \lim_{x \to 1} \left(\frac{1}{x-1} + \frac{1}{(x-1)(x-2)} \right) \\ &= \lim_{x \to 1} \frac{x-1}{(x-1)(x-2)} = \lim_{x \to 1} \frac{1}{x-2} = \frac{1}{1-2} = -1. \end{aligned}$$

   (b) Let $y = x+1$. As $x \to 9^-$, $y \to 10^-$. Hence $\lim\limits_{x \to 9^-} \lfloor x+1 \rfloor = \lim\limits_{y \to 10^-} \lfloor y \rfloor = 10 - 1 = 9$.

   Let $z = 9 - x$. As $x \to 9^-$, $z \to 0^+$. Hence $\lim\limits_{x \to 9^-} \sqrt{9-x} = \lim\limits_{z \to 0^+} \sqrt{z} = \sqrt{0} = 0$. Therefore

   $$\begin{aligned} \lim_{x \to 9^-} \left(\sqrt{9-x} + \lfloor x+1 \rfloor \right) &= \lim_{x \to 9^-} \sqrt{9-x} + \lim_{x \to 9^-} \lfloor x+1 \rfloor \\ &= 0 + 9 = 9. \end{aligned}$$

   $\square$

2. Use the graph of $f(x)$ to find a number $\delta > 0$ such that

   $$0 < |x-2| < \delta \implies \left| \frac{1}{x} - 0.5 \right| > 0.2.$$

   Solution. It is required that

   $$\left| \frac{1}{x} - 0.5 \right| < 0.2 \iff \frac{10}{7} < x < \frac{10}{3}.$$

   We need to choose $\delta > 0$ such that $2 - \delta \ge \frac{10}{7}$ and $2 + \delta \le \frac{10}{3}$, i.e., $\delta \le \frac{4}{7}$ and $\delta \le \frac{4}{3}$. Hence δ can be any number satisfying $0 < \delta \le \frac{4}{7}$. $\square$

3. Prove the following limits using the ε, δ-definition.

   (a) $\lim\limits_{x \to 3} \left(\dfrac{4}{3}x - 2 \right) = 2$, (b) $\lim\limits_{x \to -1} \left(2x^2 - x - 1 \right) = 2$,

   (c) $\lim\limits_{x \to 1} x^3 = 1$, (d) $\lim\limits_{x \to 1} \dfrac{2x^2 + 3x - 2}{x+2} = 1$.

Solution.

(a) Let $\varepsilon > 0$. Choose $\delta = \frac{3}{4}\varepsilon$. Then $0 < |x-3| < \delta$ implies

$$\left|\left(\frac{4}{3}x - 2\right) - 2\right| = \frac{4}{3}|x-3| < \frac{4}{3}\delta = \varepsilon.$$

(b) Let $\varepsilon > 0$. Choose $\delta = \min\left\{\frac{1}{2}, \frac{\varepsilon}{6}\right\}$. Then $0 < |x+1| < \delta$ implies

$$\begin{aligned}\left|\left(2x^2 - x - 1\right) - 2\right| &= |x+1|\,|2x-3| \\ &= |x+1|\,|2(x+1) + (-5)| \\ &\leq |x+1|\,\left(2|x+1| + 5\right) \\ &< \delta(2\delta + 5) \leq \delta \cdot 6 \leq \varepsilon.\end{aligned}$$

(c) Let $\varepsilon > 0$. Choose $\delta = \min\left\{1, \frac{\varepsilon}{7}\right\}$. Then $0 < |x-1| < \delta$ implies

$$\begin{aligned}|x^3 - 1| &= |x-1|\,|x^2 + x + 1| \\ &= |x-1|\,\left|(x-1)^2 + 3(x-1) + 3\right| \\ &\leq |x-1|\,\left(|x-1|^2 + 3|x-1| + 3\right) \\ &< \delta(\delta^2 + 3\delta + 3) \leq \delta \cdot 7 \leq \varepsilon.\end{aligned}$$

(d) Let $\varepsilon > 0$. Choose $\delta = \min\left\{3, \frac{\varepsilon}{2}\right\}$.
(We need $\dfrac{2x^2 + 3x - 2}{x+2}$ to be well-defined, i.e., $x \neq -2$, so we want to choose δ such that the open interval around x does not contain -2.)
If $0 < |x-1| < \delta$, then $x - 1 > -\delta \leq -3 \implies x > -2$, so

$$\begin{aligned}\left|\frac{2x^2 + 3x - 2}{x+2} - 1\right| &= \left|\frac{(x+2)(2x-1)}{x+2} - 1\right| \\ &= |(2x-1) - 1| = 2|x-1| < 2\delta \leq \varepsilon.\end{aligned}$$

□

Lecture 4

1. Use the ε, δ-definition of limit and one-sided limits to show that

$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L.$$

Solution.

($\Rightarrow$) Suppose $\lim_{x \to a} f(x) = L$. Let $\varepsilon > 0$ be given. There exists $\delta > 0$ such that

$$|x - a| < \delta \implies |f(x) - L| < \varepsilon.$$

Choosing the same $\delta > 0$,

$$\begin{aligned}a - \delta < x < a &\implies |x - a| < \delta \implies |f(x) - L| < \varepsilon \\ a < x < a + \delta &\implies |x - a| < \delta \implies |f(x) - L| < \varepsilon.\end{aligned}$$

($\Leftarrow$) Suppose $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$. Let $\varepsilon > 0$ be given. There exist $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned} a - \delta_1 < x < a &\implies |f(x) - L| < \varepsilon, \\ a < x < a + \delta_2 &\implies |f(x) - L| < \varepsilon. \end{aligned}$$

Take $\delta = \min\{\delta_1, \delta_2\}$. Suppose $|x - a| < \delta$. We consider cases:

- If $x < a$, then $a - \delta_1 \le a - \delta < x < a \implies |f(x) - L| < \varepsilon$.
- If $x > a$, then $a < x < a + \delta \le a + \delta_2 \implies |f(x) - L| < \varepsilon$.

In both cases, $|x - a| < \delta \implies |f(x) - L| < \varepsilon$. Hence $\lim_{x\to a} f(x) = L$. $\square$

2. Suppose that $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = c$, where c is a real number. Using the precise definition of limit and infinite limit, prove that

$$\lim_{x\to a} (f(x) + g(x)) = \infty.$$

Solution. Let $\varepsilon > 0$ be given. There exists $\delta_1 > 0$ such that

$$\begin{aligned} |x - a| < \delta_1 &\implies |g(x) - c| < \varepsilon \\ &\implies g(x) > c - \varepsilon. \end{aligned}$$

Let $M > 0$ be given. There exists $\delta_2 > 0$ such that

$$|x - a| < \delta_2 \implies f(x) > M - (c - \varepsilon).$$

Choose $\delta = \min\{\delta_1, \delta_2\}$. Then $|x - a| < \delta \implies f(x) + g(x) > M$. $\square$

3. **Definition of Limit at Infinity.** We write $\lim_{x\to\infty} f(x) = L$ if $f(x)$ is arbitrarily close to L by taking x sufficiently large. Precisely, it means that for every $\varepsilon > 0$ there is a number N such that

$$x > N \implies |f(x) - L| < \varepsilon.$$

One can verify that the limit laws for $x \to a$ still hold for $x \to \infty$. From now on, we will use the limit laws at infinity without proof unless stated otherwise.

(a) Write down the precise definitions of
(i) $\lim_{x\to -\infty} f(x) = L$, (ii) $\lim_{x\to\infty} f(x) = \infty$.

(b) Use the precise definition to show that $\lim_{x\to\infty} \dfrac{x}{2x+1} = \dfrac{1}{2}$.

Solution.

(a) (i) For every $\varepsilon > 0$, there exists N such that $x < N \implies |f(x) - L| < \varepsilon$.
(ii) For every $M > 0$, there exists N such that $x > N \implies f(x) > M$.

(b) Let $\varepsilon > 0$ be given.
Choose $N = \max\left\{-\frac{1}{2}, \frac{1}{4\varepsilon} - \frac{1}{2}\right\} = \frac{1}{4\varepsilon} - \frac{1}{2}$.
(We need to ensure $\frac{x}{2x+1}$ is well-defined for $x > N$, so $x \ne -\frac{1}{2}$; hence take $N \ge -\frac{1}{2}$.)
(Since we want $N > -\frac{1}{2}$ and $N > \frac{1}{4\varepsilon} - \frac{1}{2}$, we need to take their maximum.)
If $x > N$, then $2x + 1 > 2\left(-\frac{1}{2}\right) + 1 = 0$, so

$$\left|\frac{x}{2x+1} - \frac{1}{2}\right| = \left|\frac{2x - (2x+1)}{2(2x+1)}\right| = \frac{1}{2(2x+1)} < \frac{1}{2(2N+1)} = \varepsilon.$$

$\square$

Lecture 5

1. Consider the function $f(x) = \begin{cases} 2x & \text{if } 0 \le x < 1, \\ 1 & \text{if } x = 1, \\ -2x+4 & \text{if } 1 < x < 2. \end{cases}$

   (i) Determine whether $f(1)$ and $\lim\limits_{x\to 1} f(x)$ exist. Is f continuous at $x = 1$?

   (ii) Determine whether $f(2)$ and $\lim\limits_{x\to 2^-} f(x)$ exist. Is f continuous at $x = 2$?

Solution.

(i) $f(1) = 1$, which exists.

Since f is a piecewise function, we consider left and right-hand limits:

$$\begin{aligned} \lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} 2x = 2 \\ \lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+} (-2x+4) = 2 \end{aligned}$$

Since $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^+} f(x) = 2$, $\lim\limits_{x\to 1} f(x)$ exists and equals 2.

Since $\lim\limits_{x\to 1} f(x) \neq f(1)$, f is not continuous at $x = 1$ (removable discontinuity).

(ii) f is not defined at 2, so $f(2)$ does not exist.

We have $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} (-2x+4) = 0$. Since f is undefined for $x > 2$, $\lim\limits_{x\to 2^+} f(x)$ does not exist.

Thus $\lim\limits_{x\to 2} f(x)$ does not exist.

Hence f is not continuous at $x = 2$.

$\square$

2. For each of the following functions, determine whether it is continuous at $x = 1$. Give a proof if it is continuous, and state the type of discontinuity otherwise.

   (a) $f(x) = \begin{cases} \dfrac{1}{x-1} & \text{if } x \neq 1, \\ 2 & \text{if } x = 1. \end{cases}$

   (b) $f(x) = \begin{cases} \dfrac{x^2-x}{x^2-1} & \text{if } x \neq 1, \\ 1 & \text{if } x = 1. \end{cases}$

   (c) $f(x) = \begin{cases} x & \text{if } x \text{ is rational}, \\ 1 & \text{if } x \text{ is irrational}. \end{cases}$

Solution.

(a) Since $\lim\limits_{x\to 1^+} f(x) = \infty$ and $\lim\limits_{x\to 1^-} f(x) = -\infty$, f has an infinite discontinuity at $x = 1$.

(b) Since

$$1 = f(1) \neq \lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x(x-1)}{(x+1)(x-1)} = \lim_{x\to 1} \frac{x}{x+1} = \frac{1}{2}$$

f has a removable discontinuity at $x = 1$.

(c) Consider one-sided limits.

- If $x > 1$, then $1 \le f(x) \le x$. By Squeeze Theorem, $\lim\limits_{x\to 1^+} f(x)$ exists and equals 1.
- If $x < 1$, then $x \le f(x) \le 1$. By Squeeze Theorem, $\lim\limits_{x\to 1^-} f(x)$ exists and equals 1.

Since $f(1) = \lim_{x\to 1} f(x) = 1$, f is continuous at $x = 1$.

$\square$

3. Show that each of the following equations has at least one real root.

(a) $\sin x + x + 1 = 0$. (b) $\sqrt{x-3} = \dfrac{10}{x-5}$.

Solution.

(a) Let $f(x) = \sin x + x + 1$.
We have $f(-\frac{\pi}{2}) = -\frac{\pi}{2} < 0$ and $f(0) = 1 > 0$. f is continuous on $\mathbb{R}$ (in particular, $[-\frac{\pi}{2}, 0]$). By IVT, there exists $c \in (-\frac{\pi}{2}, 0)$ such that $f(c) = 0$.

(b) Let $f(x) = \sqrt{x-3} - \dfrac{10}{x-5}$.
$f(7) = -3 < 0$, $f(10) = \sqrt{7} - 2$. f is continuous on $[3, \infty) \setminus \{5\}$ (in particular, $[7, 10]$).
By IVT, there exists $c \in (7, 10)$ such that $f(c) = 0$.

$\square$

4. For each of the following functions, use the definition of derivative to find the slope of the graph at the given point. Then find an equation for the tangent line of the graph there.

(a) $f(x) = 4 - x^2$, at $(-1, 3)$. (b) $f(x) = x^3$, at $(2, 8)$.

Solution.

(a)

$$f'(-1) = \lim_{h\to 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h\to 0} \frac{\left(4 - (-1+h)^2\right) - 3}{h} = 2.$$

The tangent line is $y = 2(x+1) + 3$.

(b)

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{(2+h)^3 - 8}{h} = 12.$$

The tangent line is $y = 12(x-2) + 8$.

$\square$

Lecture 6

1. Is there a straight line which is tangent to both the curves $y = x^2$ and $y = x^2 - 2x + 2$? If so, find its equation. If not, why not?

Solution. Let L be a line tangent to $f(x) = x^2$ at $x = a$. The slope of L is $f'(a) = 2a$. Thus L has equation $y = 2a(x-a) + a^2 = 2ax - a^2$.

Let L be a line tangent to $g(x) = x^2$ at $x = b$. The slope of L is $g'(b) = 2b - 2$. Thus L has equation $y = (2b-2)(x-b) + (b^2 - 2b + 2) = (2b-2)x + (-b^2 + 2)$.

These two equations must be equal, so

$$\begin{cases} 2a = 2b - 2 \\ a^2 = b^2 - 2 \end{cases} \implies \begin{cases} a = \frac{1}{2} \\ b = \frac{3}{2} \end{cases}$$

Hence the equation of L is $y = x - \frac{1}{4}$. $\square$

2. Show that the function $f(x) = |x+2|$ is not differentiable at $x = -2$.

Solution. We need to show the limit

$$\lim_{h\to 0}\frac{f(-2+h)-f(-2)}{h} = \lim_{h\to 0}\frac{|h|}{h}$$

does not exist. Consider one-sided limits:

$$\begin{aligned}\lim_{h\to 0^+}\frac{|h|}{h} &= \lim_{h\to 0^+}\frac{h}{h} = 1\\ \lim_{h\to 0^+}\frac{|h|}{h} &= \lim_{h\to 0^-}\frac{-h}{h} = -1.\end{aligned}$$

Since one-sided limits are not equal, the limit does not exist. □

3. Let

$$f(x) = \begin{cases} x^2 & \text{if } x \le 2,\\ mx+b & \text{if } x > 2.\end{cases}$$

Find the values of m and b that make f differentiable everywhere.

Solution. Check differentiability at $x = 2$:

$$\begin{aligned}\lim_{x\to 2^-}\frac{f(x)-f(2)}{x-2} &= \lim_{x\to 2^-}\frac{x^2-4}{x-2} = \lim_{x\to 2^-}\frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2^-}(x+2) = 4\\ \lim_{x\to 2^+}\frac{f(x)-f(2)}{x-2} &= \lim_{x\to 2^+}\frac{mx+b-4}{x-2}\end{aligned}$$

Since $\lim_{x\to 2^+}\frac{mx+b-4}{x-2}$ exists, we must have $2m+b-4=0$, i.e., $b-4=-2m$. Substituting this back gives

$$\lim_{x\to 2^+}\frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^+}\frac{mx+b-4}{x-2} = \lim_{x\to 2^+}\frac{mx-2m}{x-2} = m.$$

Since f is differentiable at 2, $\lim_{x\to 2}\frac{f(x)-f(2)}{x-2}$ exists. Thus $4 = m$ and $2m+b-4 = 0$, i.e., $m = 4$ and $b = -4$. □

4. Find the derivatives of the following functions.

(a) $y = (x^2+1)\left(x+5+\frac{1}{x}\right)$,

(b) $g(x) = \dfrac{x^2-0.4}{x+0.5}$,

(c) $v = \dfrac{1+x-4\sqrt{x}}{x}$,

(d) $f(x) = \dfrac{x^3+x}{x^4-2}$.

Solution.

(i) By the product rule,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= (2x)(x+5+x^{-1}) + (x^2+1)(1-x^{-2})\\ &= (2x^2+10x+2)+(x^2-x^{-2})\\ &= 3x^2+10x+2-\frac{1}{x^2}\end{aligned}$$

(ii) By the quotient rule,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(x+0.5)(2x)-(x^2-0.4)(1)}{(x+0.5)^2}\\ &= \frac{(2x^2+x)-(x^2-0.4)}{(x+0.5)^2}\\ &= \frac{x^2+x+0.4}{(x+0.5)^2}\end{aligned}$$

(iii) Instead of using the quotient rule,

$$\begin{aligned}\frac{\mathrm{d}v}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}+1-\frac{4}{\sqrt{x}}\right)\\ &= -\frac{1}{x^2}+\frac{2}{x\sqrt{x}}\end{aligned}$$

(iv) By the quotient rule,

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}x} &= \frac{(3x^2+1)(x^4-2)-(x^3+x)(4x^3)}{(x^4-2)^2}\\ &= \frac{-x^6-3x^4-6x^2-2}{(x^4-2)^2}.\end{aligned}$$

□

Lecture 7

1. Using $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$, find the following limits.

(a) $\lim_{x\to 1}\frac{\sin(x-1)}{x^2+x-2}$,

(b) $\lim_{x\to 0}\frac{\sin ax}{\sin bx}$ ($a\neq 0$, $b\neq 0$).

Solution.

(a) Let $y=x-1$. Then

$$\begin{aligned}\lim_{x\to 1}\frac{\sin(x-1)}{(x+2)(x-1)} &= \lim_{y\to 0}\frac{\sin y}{y(y+3)}\\ &= \lim_{y\to 0}\frac{\sin y}{y}\cdot\lim_{y\to 0}\frac{1}{y+3}\\ &= 1\cdot\frac{1}{3}=\frac{1}{3}.\end{aligned}$$

(b) As $x\to 0$, $ax\to 0$ and $bx\to 0$. Thus

$$\begin{aligned}\lim_{x\to 0}\frac{\sin ax}{\sin bx} &= \lim_{x\to 0}\frac{\frac{\sin ax}{ax}\cdot ax}{\frac{\sin bx}{bx}\cdot bx}\\ &= \lim_{x\to 0}\frac{a}{b}\cdot\frac{\frac{\sin ax}{ax}}{\frac{\sin bx}{bx}}\\ &= \frac{a}{b}\cdot\frac{\lim_{x\to 0}\frac{\sin ax}{ax}}{\lim_{x\to 0}\frac{\sin bx}{bx}}\\ &= \frac{a}{b}\cdot\frac{1}{1}=\frac{a}{b}.\end{aligned}$$

$\square$

2. Find the derivatives of the functions using differentiation formulas.

(a) $f(x) = \dfrac{(x-1)^4}{(x^2+2x)^5}$, (b) $f(x) = \dfrac{1}{(x+1/x)^2}$,

(c) $f(x) = \sin(\sin(\sin x))$.

Solution.

(a) Using quotient rule and then chain rule,

$$\begin{aligned} f'(x) &= \frac{(x^2+2x)^5 \cdot 4(x-1)^3 - (x-1)^4 \cdot 5(x^2+2x)^4(2x+2)}{[(x^2+2x)^5]^2} \\ &= \frac{(x^2+2x) \cdot 4(x-1)^3 - (x-1)^4 \cdot 5(2x+2)}{(x^2+2x)^6} \\ &= \frac{2(x-1)^3}{(x^2+2x)^6}\left[2(x^2+2x) - 5(x-1)(x+1)\right] \\ &= \frac{(x-1)^3(-6x^2+8x+10)}{(x^2+2x)^6} \end{aligned}$$

(b)

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(x+\frac{1}{x}\right)^{-2} \\ &= -2\left(x+\frac{1}{x}\right)^{-3}\left(1-\frac{1}{x^2}\right) \\ &= -\frac{2x(x^2-1)}{(x^2+1)^3} \end{aligned}$$

Alternatively, use quotient rule.

(c) Applying chain rule repeatedly,

$$\begin{aligned} f'(x) &= \cos(\sin(\sin x)) \cdot \frac{\mathrm{d}}{\mathrm{d}x}\sin(\sin x) \\ &= \cos(\sin(\sin x)) \cdot \cos(\sin x) \cdot \cos x. \end{aligned}$$

Alternatively, use $\dfrac{\mathrm{d}w}{\mathrm{d}x} = \dfrac{\mathrm{d}w}{\mathrm{d}z}\dfrac{\mathrm{d}z}{\mathrm{d}y}\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

$\square$

3. Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ by implicit differentiation.

(a) $\sin x + \cos y = \sin x \cos y$, (b) $\tan(x-y) = \dfrac{y}{1+x^2}$.

Solution.

(a) Differentiate the equation with respect to x:

$$\cos x - \sin y \frac{\mathrm{d}y}{\mathrm{d}x} = \sin x(-\sin y)\frac{\mathrm{d}y}{\mathrm{d}x} + \cos x \cos y.$$

Then express $\frac{dy}{dx}$ in terms of x and y:

$$\frac{dy}{dx} = \frac{\cos x(\cos y - 1)}{\sin y(\sin x - 1)}.$$

(b) Differentiate the equation with respect to x:

$$\left(1 - \frac{dy}{dx}\right)\sec^2(x-y) = \frac{(1+x^2)\frac{dy}{dx} - y(2x)}{(1+x^2)^2}.$$

Then express $\frac{dy}{dx}$ in terms of x and y:

$$\frac{dy}{dx} = \frac{\sec^2(x-y) + 2xy/(1+x^2)^2}{\sec^2(x-y) + 1/(1+x^2)}.$$

□

4. Find an equation of the tangent line to the curve $x^2 + 2xy - y^2 + x = 2$ at the point $(1,2)$.

Solution. Differentiate the equation with respect to x:

$$2x + 2x\frac{dy}{dx} + 2y - 2y\frac{dy}{dx} + 1 = 0.$$

Substituting $x = 1$ and $y = 2$ gives $\frac{dy}{dx} = \frac{7}{2}$. Hence the equation of the tangent line is $y = \frac{7}{2}(x-1) + 2$. □

Remark. This is a hyperbola.

5. Let (x_0, y_0) be a point on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Show that the tangent line to the ellipse passing through (x_0, y_0) is

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1.$$

Solution. The tangent line is vertical $\Leftrightarrow (x_0, y_0) = (\pm a, 0)$.

- Let $(x_0, y_0) = (a, 0)$. The tangent line is $x = a$, which has the form $\frac{ax}{a^2} + \frac{0y}{b^2} = 1$.
- Let $(x_0, y_0) = (-a, 0)$. The tangent line is $x = -a$, which has the form $\frac{(-a)x}{a^2} + \frac{0y}{b^2} = 1$.

Suppose $y_0 \neq 0$. Differentiate the ellipse with respect to x:

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0.$$

Substituting $x = x_0$ and $y = y_0$ gives

$$\frac{2x_0}{a^2} + \frac{2y_0}{b^2}\frac{dy}{dx} = 0$$

so

$$\frac{dy}{dx} = -\frac{2x_0/a^2}{2y_0/b^2} = -\frac{x_0/a^2}{y_0/b^2}.$$

The tangent line L at (x_0, y_0) is

$$\frac{y - y_0}{x - x_0} = -\frac{x_0/a^2}{y_0/b^2}.$$

Finally apply $\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1$ (since (x_0, y_0) is a point on the ellipse). □

Lecture 8

1. Use the closed interval method to find the absolute maximum and minimum values of each function f on the given interval and the values of x where they occur.

   (i) $f(x) = x^3 - 6x^2 + 9x + 2, \qquad [-1,4]$,

   (ii) $f(x) = \sqrt[3]{x}(8-x), \qquad [-1,8]$.

   Solution.

   (i) The endpoints are -1 and 4. Find the critical points of f on $(-1,4)$:
   $$f'(x) = 3x^2 - 12x + 9$$
   always exists. Then
   $$f'(x) = 3(x-1)(x-3) = 0 \implies x = 1 \text{ or } x = 3.$$
   Compare $f(x)$ at the points obtained: $f(-1) = -14$, $f(4) = 6$, $f(1) = 6$, $f(3) = 2$.
   Hence absolute maximum is 6, absolute minimum is -14.

   (ii) The endpoints are -1 and 8. Find the critical points of f on $(-1,8)$:
   $$\begin{aligned} f'(x) &= \frac{8}{3}x^{-2/3} - \frac{4}{3}x^{1/3} \\ &= \frac{4}{3} \cdot \frac{2-x}{x^{2/3}} \end{aligned}$$
   $f'(x)$ does not exist when $x = 0$. When $f'(x) = 0$, $2 - x = 0 \implies x = 2$.
   Compare $f(x)$ at the points obtained: $f(-1) = -9$, $f(8) = 0$, $f(0) = 0$, $f(2) = 6\sqrt[3]{2}$.
   Hence absolute maximum is $6\sqrt[3]{2}$, absolute minimum is -9.
   (Note: $x = 0$ is called a *saddle point*, at which f is not differentiable since the slope of the tangent line goes to infinity.)

   $\square$

2. Let $r > 1$ be a rational number. Prove that for any $x \in [0,1]$,
   $$\frac{1}{2^{r-1}} \leq x^r + (1-x)^r \leq 1.$$

   Solution. Let $f(x) = x^r + (1-x)^r$. The given equation means that $\frac{1}{2^{r-1}}$ is the absolute minimum, and 1 is the absolute maximum of f. We shall use the closed interval method to find the extreme values.

   The endpoints are 0 and 1. Find the critical points of f on $(0,1)$:
   $$\begin{aligned} f'(x) &= rx^{r-1} - r(1-x)^{r-1} \\ &= r\left[x^{r-1} - (1-x)^{r-1}\right] \end{aligned}$$
   always exists. Then
   $$f'(x) = 0 \implies x^{r-1} = (1-x)^{r-1} \implies x = 1 - x \implies x = \frac{1}{2}.$$
   Compare $f(x)$ at the points obtained: $f(0) = 1$, $f(1) = 1$, $f\left(\frac{1}{2}\right) = \frac{1}{2^r} + \frac{1}{2^r} = \frac{1}{2^{r-1}}$. Since $r > 1$, $r - 1 > 0 \Rightarrow 2^{r-1} > 2^0 = 1 \Rightarrow \frac{1}{2^{r-1}} < 1$.

   Hence absolute maximum is 1, absolute minimum is $\frac{1}{2^{r-1}}$. $\square$

3. Prove that the function $f(x) = x^{101} + x^{51} + x + 1$ has no local extreme values.

   Remark. To show something does not exist, we shall assume it exists, then derive a *contradiction*.

   Solution. Suppose, for a contradiction, that f has a local extreme value at x_0. Since f is a polynomial, f is differentiable at x_0. By Fermat's Theorem, $f'(x_0) = 0$, so x_0 is a root to

   $$f'(x) = 101x^{100} + 51x^{50} + 1.$$

   But $f'(x) \geq 1 > 0$, so $f'(x) \neq 0$ for all $x \in \mathbb{R}$. This is a contradiction. □

4. Let $x^3 + bx^2 + cx + d$ be a cubic function. Show that if $b^2 < 3c$, then f has no local extreme values.

   Solution. Suppose, for a contradiction, that f has a local extreme value at x_0. Since f is a polynomial, f is differentiable at x_0. By Fermat's Theorem, $f'(x_0) = 0$, so x_0 is a root to

   $$f'(x) = 3x^2 + 2bx + c = 0$$

   which is a quadratic. Since the quadratic has a real root, the discriminant $\Delta = (2b)^2 - 4(3)(c) = 4(b^2 - 3c) \geq 0$, so $b^2 \geq 3c$. This contradicts the given condition $b^2 < 3c$. □

Lecture 9

1. Show that $f(x) = 2x - \sin x$ has *exactly one* zero in $\mathbb{R}$.

   Remark. "Exactly one" means "at least one" and "at most one".

   Solution.

   $\boxed{\text{Existence}}$ Since $f(0) = 0$, 0 is a root of $f(x)$. Hence $f(x)$ has at least one zero.

   (If the function is complicated, we may need to use the Intermediate Value Theorem – choose one positive point and one negative point).

   $\boxed{\text{Uniqueness}}$ Suppose, for a contradiction, that $f(x)$ has two zeros x_1 and x_2. WLOG assume $x_1 < x_2$. Then $f(x_1) = f(x_2) = 0$. Since f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2), by Rolle's theorem, there exists $c \in (x_1, x_2)$ such that $f'(c) = 0$:

   $$f'(c) = 2 - \cos c = 0 \Rightarrow \cos c = 2$$

   which is absurd. □

2. Show that $x^4 - 4x + 1 = 0$ has *exactly two* real roots.

   Solution.

   $\boxed{\text{Existence}}$ Let $f(x) = x^4 - 4x + 1$. It is continuous on $\mathbb{R}$. Apply Intermediate Value Theorem twice:

   - $f(0) = 1 > 0$, $f(1) = -2 < 0$, f is continuous on $[0, 1]$. By IVT, $\exists c_1 \in (0, 1)$ such that $f(c_1) = 0$.
   - $f(1) = -2 < 0$, $f(2) = 9 > 0$, f is continuous on $[1, 2]$. By IVT, $\exists c_2 \in (1, 2)$ such that $f(c_2) = 0$.

   $\boxed{\text{Uniqueness}}$ Suppose, for a contradiction, that $f(x) = 0$ has three real roots $d_1 < d_2 < d_3$.

   - Since f is continuous on $[d_1, d_2]$, differentiable on (d_1, d_2), and $f(d_1) = f(d_2) = 0$, by Rolle's Theorem, $\exists e_1 \in (d_1, d_2)$ such that $f'(e_1) = 0$.

- Since f is continuous on $[d_2, d_3]$, differentiable on (d_2, d_3), and $f(d_2) = f(d_3) = 0$, by Rolle's Theorem, $\exists e_2 \in (d_2, d_3)$ such that $f'(e_2) = 0$.

Since $e_1 < e_2$ are roots to $f'(x) = 0$,

$$f'(x) = 4x^3 - 4 = 0 \Rightarrow x = 1$$

so $e_1 = e_2 = 1$. This is a contradiction. □

3. It is known that a quadratic equation has at most two real roots. How many real roots can a cubic equation $x^3 + bx^2 + cx + d = 0$ have? Justify your answer, and give examples to illustrate *all* possibilities.

Solution. At most 3 roots. Let $f(x) = x^3 + bx^2 + cx + d$.

Suppose, for a contradiction, that $f(x)$ has 4 roots x_1, x_2, x_3, x_4. WLOG assume $x_1 < x_2 < x_3 < x_4$.

- Since f is continuous on $[x_1, x_2]$, differentiable on (x_1, x_2), and $f(x_1) = f(x_2) = 0$, by Rolle's Theorem, $\exists c_1 \in (x_1, x_2)$ such that $f'(c_1) = 0$.
- Since f is continuous on $[x_2, x_3]$, differentiable on (x_2, x_3), and $f(x_2) = f(x_3) = 0$, by Rolle's Theorem, $\exists c_2 \in (x_2, x_3)$ such that $f'(c_2) = 0$.
- Since f is continuous on $[x_3, x_4]$, differentiable on (x_3, x_4), and $f(x_3) = f(x_4) = 0$, by Rolle's Theorem, $\exists c_3 \in (x_3, x_4)$ such that $f'(c_3) = 0$.

Hence $c_1 < c_2 < c_3$ are roots of $f'(x)$. But $f'(x)$ is a quadratic with at most two real roots, a contradiction.

□

4. A number c is said to be a **fixed point** of a function f if $f(c) = c$.

Suppose that f is differentiable on $\mathbb{R}$ and that $f'(x) \neq 1$ for all $x \in \mathbb{R}$. Prove that f has *at most one* fixed point.

Solution. Suppose, for a contradiction, that f has two fixed point $c_1 < c_2$. Then $f(c_1) = c_1$ and $f(c_2) = c_2$. (Draw a graph to visualise.)

Since f is continuous on $[c_1, c_2]$ and differentiable on (c_1, c_2), by Mean Value Theorem, there exists $c \in (c_1, c_2)$ such that

$$f'(c) = \frac{f(c_2) - f(c_1)}{c_2 - c_1} = \frac{c_2 - c_1}{c_2 - c_1} = 1.$$

This contradicts the given condition that $f'(x) \neq 1$ for all $x \in \mathbb{R}$. □

Lecture 10

1. (a) Let f be a function whose derivative is given by $f'(x) = (x-1)(x+2)(x-3)$.
   (b) Let $g(x) = x^{1/5}(x+6)$.

   For each of f and g, find the following:

   (i) Critical points.
   (ii) Open intervals on which the function is increasing and decreasing.
   (iii) Points where the function assumes local maximum and minimum values.

Solution.

(a) Solve $f'(x)=0$ to get $x=1$, $x=-2$ and $x=3$.

Thus f has critical points -2, 1, and 3 (arrange from small to large).

The critical points divide the real line into subintervals:

$$\begin{array}{ccccc} & (-\infty,-2) & (-2,1) & (1,3) & (3,\infty) \\ f'(x) & - & + & - & + \\ f(x) & \searrow & \nearrow & \searrow & \nearrow \end{array}$$

Hence f has local maximum at $x=1$, local minimum at $x=-2$ and $x=3$.

(b) $g'(x)=\dfrac{6}{5}x^{1/5}+\dfrac{6}{5}x^{-4/5}=\dfrac{6}{5}\dfrac{x+1}{x^{4/5}}$.

$g'(x)$ does not exist when $x=0$.

When $g'(x)=0$, $x=-1$.

Thus g has critical points at $x=-1$ and $x=0$.

The critical points -1 and 0 divide the real line into subintervals:

$$\begin{array}{cccc} & (-\infty,-1) & (-1,0) & (0,\infty) \\ g'(x) & - & + & + \\ g(x) & \searrow & \nearrow & \nearrow \end{array}$$

Hence g attains a local minimum at $x=-1$ (in fact, this is a global minimum).

□

2. For each of the functions

(a) $f(x)=2+3x-x^3$ on $\mathbb{R}$, (b) $g(x)=4x-\tan x$ on $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$,

(i) Find the open intervals on which it is increasing and decreasing;

(ii) Find the coordinates of all its local maximum and minimum points;

(iii) Find the open intervals on which it is concave up and concave down;

(iv) Find the coordinates of all its inflection points;

(v) Use the information from parts (i)–(iv) to sketch its graph.

Solution.

(a) $f'(x)=3-3x^2$. Then $f'(x)=0\Rightarrow x=-1$ or $x=1$.

$$\begin{array}{cccc} & (-\infty,-1) & (-1,1) & (1,\infty) \\ f'(x) & - & + & - \\ f(x) & \searrow & \nearrow & \searrow \end{array}$$

Hence $f(-1)=0$ is a local maximum, and $f(1)=4$ is a local minimum.

$f''(x)=-6x$. Then $f''(x)=0\Rightarrow x=0$.

$$\begin{array}{ccc} & (\infty,0) & (0,\infty) \\ f''(x) & + & - \\ f(x) & \text{concave up} & \text{concave down} \end{array}$$

Hence f has an inflection point $(0,2)$.

(b) $g'(x) = 4 - \sec^2 x$. Solve $g'(x) = 0$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ to get $x = \pm\frac{\pi}{3}$.

	$\left(-\frac{\pi}{2}, -\frac{\pi}{3}\right)$	$\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$	$\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$
$g'(x)$	$-$	$+$	$-$
$g(x)$	$\searrow$	$\nearrow$	$\searrow$

$g\left(-\frac{\pi}{3}\right) = -\frac{4}{3}\pi + \sqrt{3}$ is a local minimum, and $g\left(\frac{\pi}{3}\right) = \frac{4}{3}\pi - \sqrt{3}$ is a local maximum.
$g''(x) = -2\tan x \sec^2 x$. Solve $g''(x) = 0$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ to get $x = 0$.

	$\left(-\frac{\pi}{2}, 0\right)$	$\left(0, \frac{\pi}{2}\right)$
$g''(x)$	$+$	$-$
$g(x)$	concave up	concave down

Hence g has an inflection point $(0,0)$.

$\square$

3. Suppose that f is a function such that $f'(x) = 1/x$ for all $x > 0$. Prove that if $f(1) = 0$, then $f(xy) = f(x) + f(y)$ for all $x > 0$ and $y > 0$.

Solution. Rewrite the given statement: we need to prove that $f(xa) = f(x) + f(a)$ for all $x > 0$ and $a > 0$.
Let $g(x) = f(xa) - f(x)$, where $a > 0$ is a fixed constant.
Then g is differentiable and thus continuous on $(0, \infty)$. By the chain rule,

$$\begin{aligned} g'(x) &= f'(ax) \cdot a - f'(x) \\ &= \frac{1}{ax} \cdot a - \frac{1}{x} = 0. \end{aligned}$$

Hence g is a constant function, i.e., thre exists $C \in \mathbb{R}$ such that $g(x) = C$ for any $x > 0$.
In particular, using the given condition that $f(1) = 0$,

$$C = g(1) = f(1 \cdot a) - f(1) = f(a) - 0 = f(a).$$

For any $x > 0$, $g(x) = C = f(a)$. That is, $f(xa) = f(x) + f(a)$. $\square$

Lecture 11

1. What is the largest possible area for a right triangle whose hypotenuse is 5 cm long?

Solution. Let x and y be the two legs of the triangle. By Pythagoras' theorem, we have $x^2 + y^2 = 5^2$. Thus $y^2 = 25 - x^2$, so $y = \sqrt{25 - x^2}$ since $x, y > 0$.
The area of the triangle is

$$A = \frac{1}{2}xy = \frac{1}{2}x\sqrt{25 - x^2}.$$

The domain is $0 < x < 5$. Extend the domain at the two endpoints to obtain a closed interval $[0,5]$ (we can do so since $A(x)$ is continuous at $x = 0$ and $x = 5$). This allows us to apply the closed interval method.
Our goal is to maximise $A(x) = \frac{1}{2}x\sqrt{25 - x^2}$ on $[0,5]$.

$$A'(x) = \frac{1}{2}\sqrt{25 - x^2} - \frac{x^2}{2\sqrt{25 - x^2}} = \frac{25 - 2x^2}{2\sqrt{25 - x^2}} \qquad x \in (0,5).$$

Note: We are only interested in differentiability on the open interval $(0,5)$.

Solve $A'(x)$ on $(0,5)$ to get $x = \frac{5}{\sqrt{2}}$.

Compare $A(0) = 0$, $A(5) = 0$, $A\left(\frac{5}{\sqrt{2}}\right) = \frac{25}{4}$.

Hence the maximum value of A on $[0,5]$ (and thus $(0,5)$) is $\frac{25}{4}$. Therefore the largest area is $\frac{25}{4}$ cm^2, when the two legs are $\frac{5}{\sqrt{2}}$ cm. □

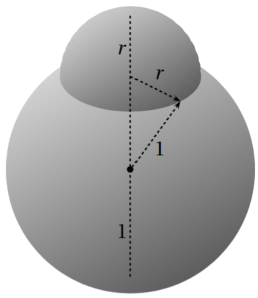
Alternative solution. Let θ be the angle between one leg and the hypotenuse. Then area of triangle is

$$A(\theta) = \frac{1}{2} 5\sin\theta \cdot 5\cos\theta = \frac{25}{4}\sin 2\theta.$$

where $0 < \theta < \frac{\pi}{2}$. It attains the maximum value when $\sin 2\theta = 1$ (i.e., $\theta = \frac{\pi}{4}$).

Hence the largest area of the triangle is $\frac{25}{4}$ cm^2. □

2. A hemisphere bubble is placed on a spherical bubble of radius 1. Find the maximum height of the bubble tower.



Solution. Let r denote the radius of the hemisphere bubble. The height of the bubble tower is

$$h(r) = r + \sqrt{1-r^2} + 1.$$

Domain is $0 < r \leq 1$. Since $h(r)$ is continuous, we can extend continuity to the endpoint $x = 0$.

It is convenient to maximise $h(r)$ on $[0,1]$, using closed interval method.

$$h'(r) = 1 - \frac{r}{\sqrt{1-r^2}} \qquad r \in (0,1).$$

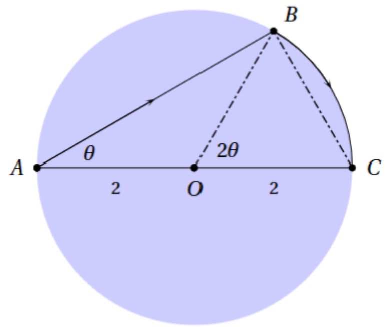
Solve $h'(r) = 0$ on $(0,1)$ to get $r = \frac{1}{\sqrt{2}}$.

Compare $h(0) = 2$, $h(1) = 2$, $h\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2}+1$.

Hence the maximum value of h on $[0,1]$ is $h\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2}+1$. Therefore the maximum height of the bubble tower is $\sqrt{2}+1$. □

3. A woman at a point A on the shore of a circular lake with radius 2 km wants to arrive at the point C diametrically opposite A on the other side of the lake in the shortest possible time.

   She can row a boat at 2 km/h and walk at the rate of 4 km/h. How should she proceed?

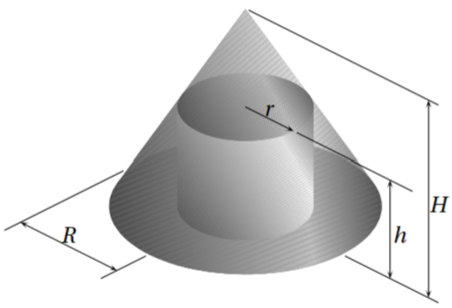


Solution. Choose a point B on the upper semicircle. Row a boat from A to B, and walk from B to C.
Let $\theta = \angle ABC$. Then $|AB| = 4\cos\theta$, $BC = 4\theta$. Travel time is $|AB|/2 + BC/4 = 2\cos\theta + \theta$, $0 \leq \theta \leq \frac{\pi}{2}$.
Find the minimum value of $T(\theta) = 2\cos\theta + \theta$ on $[0, \frac{\pi}{2}]$.

$$T'(\theta) = -2\sin\theta + 1 \qquad \theta \in (0, \frac{\pi}{2})$$

Solve $T'(\theta) = 0$ on $\left(0, \frac{\pi}{2}\right)$ to get $\theta = \frac{\pi}{6}$.
Compare $T(0) = 2$, $T\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$ (min), $T\left(\frac{\pi}{6}\right) = \sqrt{3} + \frac{\pi}{6}$ (max).
The minimum value of T on $[0, \frac{\pi}{2}]$ is $T\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$.
The travel time is minimised if $\theta = \frac{\pi}{2}$. The limiting position of B as $\theta \to \frac{\pi}{2}^-$ is A, so BC $\to$ AC. Hence travel time is minimised if the woman walks around the lake. □

4. A right circular cylinder is inscribed in a cone with height H and base radius R. Find the largest possible volume of such a cylinder.



Solution. Considering projection from the front, by similar triangles,

$$\frac{H}{R} = \frac{h}{R-r} \implies h = \frac{H}{R}(R-r).$$

We need to maximise

$$V = \pi r^2 h = \frac{\pi H}{R} r^2 (R-r)$$

where $0 < r < R$. Extend continuity to the endpoints 0 and R.
Find the maximum value of $V(r) = \frac{\pi H}{R} r^2 (R-r)$ on $[0, R]$.

$$V'(r) = \frac{\pi H}{R} r(2R - 3r) \qquad r \in (0, R).$$

Solve $V'(r) = 0$ on $(0, R)$ to get $r = \frac{2}{3}R$.
Compare $V(0) = 0$, $V(R) = 0$, and $V\left(\frac{2}{3}R\right) = \frac{4}{27}\pi R^2 H$.
Hence the largest volume of the inscribed cylinder is $\frac{4}{27}\pi R^2 H$. □

Alternative solution. Express V in terms of h. $\square$

5. A $216\,\mathrm{m}^2$ rectangular plot is to be enclosed by a fence and divided into two equal parts by another fence parallel to one of the sides. What dimensions for the outer rectangle will require the smallest total length of fence? How much fence will be needed?

Solution. Let side lengths of rectangle by x and y. It is given that $xy = 216$, $x, y > 0$.

Note: We cannot extend by continuity to 0.

Minimise total length $L = 3x + 2y = 3x + \frac{432}{x}$, $x > 0$.

Note: $(0, \infty)$ is an infinite open interval, so we cannot use closed interval method.

Find the minimum value of $L(x) = 3x + \frac{432}{x}$ on $(0, \infty)$.

$$L'(x) = 3 - \frac{432}{x^2} = \frac{3(x^2 - 144)}{x^2} \qquad x \in (0, \infty).$$

Solve $L'(x) = 0$ on $(0, \infty)$ to get $x = 12$. By increasing/decreasing test,

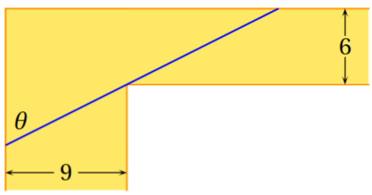
	$(0, 12)$	$(12, \infty)$
$L'(x)$	$-$	$+$
$L(x)$	$\searrow$	$\nearrow$

Hence L has minimum value 72 when $x = 12$, and $y = 18$.

When rectangle is $12\,\mathrm{m} \times 18\,\mathrm{m}$, total fence has minimum length $72\,\mathrm{m}$. $\square$

Lecture 12

1. A steel pipe is being carried down a hallway 9 m wide. At the end of the hall there is a right-angled turn into a narrower hallway 6 m wide. What is the approximate length (in feet) of the longest pipe that can be carried horizontally around the corner? Round your answer down to the nearest metre.



Solution. Let L be length of pipe at θ. Then

$$L(\theta) = \frac{9}{\sin\theta} + \frac{6}{\cos\theta}, \qquad \theta \in (0, \frac{\pi}{2}).$$

Note: We cannot extend continuity at endpoints, since the denominators would be $\sin 0 = 0$ and $\cos\frac{\pi}{2} = 0$.

$$L'(\theta) = -\frac{9\cos\theta}{\sin^2\theta} + \frac{6\sin\theta}{\cos^2\theta} = \frac{6\sin^3\theta - 9\cos^3\theta}{\sin^2\theta\cos^2\theta}.$$

Solve $L'(\theta) = 0$ on $(0, \frac{\pi}{2})$ to get $\tan\theta = \sqrt[3]{3/2}$, so $\theta = \tan^{-1}\sqrt[3]{3/2} = \theta_0$.

	$(0, \theta_0)$	$\theta = \theta_0$	$(\theta_0, \frac{\pi}{2})$
$L'(\theta)$	$-$	0	$+$
$L(\theta)$	$\searrow$	min	$\nearrow$

$L(\theta)$ is an *upper bound* for the length of the pipe at angle θ. Suppose a pipe of length k can be carried around the corner; then $k \le L(\theta)$ for any $\theta \in (0, \frac{\pi}{2})$. Thus $k \le \min_{0<\theta<\frac{\pi}{2}} L(\theta)$.

Hence the length of the longest pipe that can be carried around corner is minimum value of $L(\theta)$ on $(0, \frac{\pi}{2})$, which is 21 m. $\square$

2. Find the following limits.

(a) $\lim_{x\to -3} \frac{t^3-4t+15}{t^2-t-12}$,

(b) $\lim_{x\to a} \frac{x^m-a^m}{x^n-a^n} \quad (a \neq 0,\, m,n \in \mathbb{Z}^+)$,

(c) $\lim_{x\to 1} \frac{\sqrt{3-x}-\sqrt{1+x}}{x^2+x-2}$,

(d) $\lim_{x\to\infty} \frac{\sqrt{9x+1}}{\sqrt{x+1}}$,

(e) $\lim_{x\to 0} \frac{8x^2}{\cos x - 1}$,

(f) $\lim_{t\to 0} \frac{t(1-\cos t)}{t-\sin t}$,

(g) $\lim_{x\to \frac{\pi}{2}^-} \frac{\sec x}{\tan x}$,

(h) $\lim_{x\to \frac{\pi}{2}} (\sec x - \tan x)$.

Solution.

(a) Let $f(t) = t^3-4t+15$ and $g(t) = t^2-t-12$.
Then $f(-3) = g(-3) = 0$, so

$$\lim_{x\to -3} \frac{t^3-4t+15}{t^2-t-12} = \frac{f'(-3)}{g'(-3)} = -\frac{23}{7}.$$

(b) Let $f(x) = x^m - a^m$ and $g(x) = x^n - a^n$.
Then $f(a) = g(a) = 0$, so

$$\lim_{x\to a} \frac{x^m-a^m}{x^n-a^n} = \frac{f'(a)}{g'(a)} = \frac{ma^{m-1}}{na^{n-1}} = \frac{m}{n}a^{m-n}.$$

Note: Need to check that $g'(a) \neq 0$ for baby version of L'Hopital's Rule.

(c) Let $f(x) = \sqrt{3-x}-\sqrt{1+x}$ and $g(x) = x^2+x-2$.
Then $f(1) = g(1) = 0$, so

$$\lim_{x\to 1} \frac{\sqrt{3-x}-\sqrt{1+x}}{x^2+x-2} = \frac{f'(1)}{g'(1)} = -\frac{1}{3\sqrt{2}}.$$

(d) In this case, l'Hopital's rule leads to nowhere:

$$\lim_{x\to\infty} \frac{\sqrt{9x+1}}{\sqrt{x+1}} = \lim_{x\to\infty} \frac{\frac{9}{2\sqrt{9x+1}}}{\frac{1}{2\sqrt{x+1}}} = \lim_{x\to\infty} \frac{9\sqrt{x+1}}{\sqrt{9x+1}} = \lim_{x\to\infty} \frac{\frac{9}{2\sqrt{9x+1}}}{\frac{9}{2\sqrt{x+1}}} = \lim_{x\to\infty} \frac{\sqrt{9x+1}}{\sqrt{x+1}}.$$

Instead,

$$\begin{aligned}
\lim_{x\to\infty}\frac{\sqrt{9x+1}}{\sqrt{x+1}} &= \lim_{x\to\infty}\frac{\sqrt{9x+1}/\sqrt{x}}{\sqrt{x+1}/\sqrt{x}} \\
&= \lim_{x\to\infty}\frac{\sqrt{9+\frac{1}{x}}}{\sqrt{1+\frac{1}{x}}} \\
&= \frac{\lim_{x\to\infty}\sqrt{9+\frac{1}{x}}}{\lim_{x\to\infty}\sqrt{1+\frac{1}{x}}} = 3.
\end{aligned}$$

(e) Applying l'Hopital's rule and recalling that $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$,

$$\begin{aligned}
\lim_{x\to 0}\frac{8x^2}{\cos x - 1} &= \lim_{x\to 0}\frac{16x}{-\sin x} \\
&= -16\lim_{x\to 0}\frac{x}{\sin x} = -16.
\end{aligned}$$

(f) Applying l'Hopital's rule and recalling that $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$,

$$\begin{aligned}
\lim_{t\to 0}\frac{t(1-\cos t)}{t-\sin t} &= \lim_{t\to 0}\frac{(1-\cos t)+t(\sin t)}{-\cos t} \\
&= \lim_{t\to 0}\frac{2\sin t + t\cos t}{\sin t} \\
&= \lim_{t\to 0}\left(2+\frac{t}{\sin t}\cdot\cos t\right) = 2+1\cdot 1 = 3.
\end{aligned}$$

(g) L'Hopital rule leads to nowhere. Instead,

$$\begin{aligned}
\lim_{x\to\frac{\pi}{2}^-}\frac{\sec x}{\tan x} &= \lim_{x\to\frac{\pi}{2}^-}\frac{1/\cos x}{\sin x/\cos x} \\
&= \lim_{x\to\frac{\pi}{2}^-}\frac{1}{\sin x} = 1
\end{aligned}$$

(h) Rewriting in quotient form, we can apply l'Hopital's rule:

$$\begin{aligned}
\lim_{x\to\frac{\pi}{2}}(\sec x - \tan x) &= \lim_{x\to\frac{\pi}{2}}\frac{1-\sin x}{\cos x} \\
&= \lim_{x\to\frac{\pi}{2}}\frac{-\cos x}{-\sin x} = \frac{0}{1} = 0.
\end{aligned}$$

□

3. Let $a > 0$. Use Riemann sums to compute $\int_0^a x^3\,\mathrm{d}x$.

Solution.

1. Divide $[0,a]$ into n equal subintervals, each of length $\Delta x = \frac{a}{n}$:

$$\left[0,\frac{a}{n}\right],\quad \left[\frac{a}{n},\frac{2a}{n}\right],\quad \ldots,\quad \left[\frac{(n-1)a}{n},a\right].$$

2. Choose sample points $x_1^*, \ldots, x_n^*$ from these subintervals.
   In particular, we take the right-hand point, i.e., $x_i^* = x_i = ia/n$ for $i = 1, \ldots, n$.
3. Compute the Riemann sum:

$$\begin{aligned}\sum_{i=1}^{n} f(x_i^*)\Delta x_i &= \sum_{i=1}^{n}\left(\frac{ia}{n}\right)^3\left(\frac{a}{n}\right)\\ &= \frac{a^4}{n^4}\sum_{i=1}^{n} i^3\\ &= \frac{a^4}{n^4}\left[\frac{n(n+1)}{2}\right]^2\\ &= \frac{a^4}{4}\frac{(n+1)^2}{n^2} = \left(1+\frac{1}{n}\right)^2\frac{a^4}{4}.\end{aligned}$$

4. Find the definite integral:

$$\int_0^a x^3\,\mathrm{d}x = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^2\frac{a^4}{4}\right] = \frac{a^4}{4}.$$

□

Lecture 13

1. Suppose that h is continuous such that $\int_{-1}^{1} h(r)\,\mathrm{d}r = 0$ and $\int_{-1}^{3} h(r)\,\mathrm{d}r = 6$. Find $\int_{3}^{1} h(r)\,\mathrm{d}r$.

Solution. We have

$$\begin{aligned}\int_3^1 h(r)\,\mathrm{d}r &= \int_3^{-1} h(r)\,\mathrm{d}r + \int_{-1}^{1} h(r)\,\mathrm{d}r\\ &= -\int_{-1}^{3} h(r)\,\mathrm{d}r + \int_{-1}^{1} h(r)\,\mathrm{d}r\\ &= -6+0 = 0.\end{aligned}$$

□

2. Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ of the following functions.

   (a) $y = \displaystyle\int_0^{x^2} \cos\left(t^{1/3}\right)\mathrm{d}t$

   (b) $y = \displaystyle\int_\pi^{\sqrt{x}} \sin t\,\mathrm{d}t$

   (c) $y = \displaystyle\int_{\tan x}^{0} \frac{\mathrm{d}t}{(1+t^2)^2}$

   (d) $y = \displaystyle\int_{\cos x}^{5x} \cos\left(t^2\right)\mathrm{d}t$

Solution. Note that the upper limit is not the same as the variable we differentiate wrt.

(a) Let $u = x^2$. Using the chain rule,

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x^2} \cos\left(t^{1/3}\right)\mathrm{d}t \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\int_0^{u} \cos\left(t^{1/3}\right)\mathrm{d}t \\
&= \frac{\mathrm{d}u}{\mathrm{d}x}\cdot\frac{\mathrm{d}}{\mathrm{d}u}\int_0^{u} \cos\left(t^{1/3}\right)\mathrm{d}t \\
&= 2x\cdot\cos\left(x^{1/3}\right) = 2x\cos\left(x^{2/3}\right)
\end{aligned}$$

In some sense, the du's cancel each other out.

(b) Let $u = \sqrt{x}$.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\int_\pi^{\sqrt{x}} \sin t\,\mathrm{d}t \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\int_\pi^{u} \sin t\,\mathrm{d}t \\
&= \frac{\mathrm{d}u}{\mathrm{d}x}\cdot\frac{\mathrm{d}}{\mathrm{d}u}\int_\pi^{u} \sin t\,\mathrm{d}t \\
&= \frac{1}{2\sqrt{x}}\cdot\sin(u) = \frac{1}{2\sqrt{x}}\sin\sqrt{x}
\end{aligned}$$

The lower limit does not matter, as long as it is a constant.

(c) Let $u = \tan x$. Swap the upper and lower limits:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\int_{\tan x}^{0} \frac{\mathrm{d}t}{(1+t^2)^2} \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\int_{u}^{0} \frac{\mathrm{d}t}{(1+t^2)^2} \\
&= -\frac{\mathrm{d}}{\mathrm{d}x}\int_{0}^{u} \frac{\mathrm{d}t}{(1+t^2)^2} \\
&= -\frac{\mathrm{d}u}{\mathrm{d}x}\cdot\frac{\mathrm{d}}{\mathrm{d}u}\int_{0}^{u} \frac{\mathrm{d}t}{(1+t^2)^2} \\
&= -\sec^2 x\cdot\frac{1}{(1+u^2)^2} = -\cos^2 x.
\end{aligned}$$

(d) Split the integral into two:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\int_{\cos x}^{5x} \cos\left(t^2\right)\mathrm{d}t \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^{5x} \cos\left(t^2\right)\mathrm{d}t - \int_0^{\cos x} \cos\left(t^2\right)\mathrm{d}t\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\int_0^{5x} \cos\left(t^2\right)\mathrm{d}t - \frac{\mathrm{d}}{\mathrm{d}x}\int_0^{\cos x} \cos\left(t^2\right)\mathrm{d}t\,.
\end{aligned}$$

For the first one, let $u = 5x$. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{u} \cos\left(t^2\right)\mathrm{d}t = \frac{\mathrm{d}u}{\mathrm{d}x}\cdot\frac{\mathrm{d}}{\mathrm{d}u}\int_0^{u} \cos\left(t^2\right)\mathrm{d}t = 5\cos\left(u^2\right) = 5\cos\left(25x^2\right).$$

For the second one, let $v = \cos x$. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^u \cos\left(t^2\right)\mathrm{d}t = \frac{\mathrm{d}v}{\mathrm{d}x}\cdot\frac{\mathrm{d}}{\mathrm{d}v}\int_0^v \cos\left(t^2\right)\mathrm{d}t = -\sin x\cos\left(v^2\right) = -\sin x\cos\left(\cos^2 x\right).$$

□

3. Let f be a continuous function on $\mathbb{R}$. Define

$$F(x) = \int_a^x f(t)(x-t)\,\mathrm{d}t\,.$$

Evaluate $F''(x)$.

Solution. Since the integrand contains x, it is not purely a function of t. Hence we cannot apply FTC (I) directly. Thus we attempt to split x and t:

$$\begin{aligned}
F'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)(x-t)\,\mathrm{d}t \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(x\int_a^x f(t)\,\mathrm{d}t - \int_a^x tf(t)\,\mathrm{d}t\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(x\int_a^x f(t)\,\mathrm{d}t\right) - \frac{\mathrm{d}}{\mathrm{d}x}\int_a^x tf(t)\,\mathrm{d}t && [x \text{ is a constant relative to } t] \\
&= \int_a^x f(t)\,\mathrm{d}t + x\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t - \frac{\mathrm{d}}{\mathrm{d}x}\int_a^x tf(t) && [\text{chain rule}] \\
&= \int_a^x f(t)\,\mathrm{d}t + xf(x) - xf(x) \\
&= \int_a^x f(t).
\end{aligned}$$

By FTC (I), $F''(x) = f(x)$. □

Lecture 14

1. Evaluate the following indefinite integrals.

(a) $\displaystyle\int \frac{\cos(\pi/x)}{x^2}\,\mathrm{d}x$

(b) $\displaystyle\int (2+\tan^2\theta)\,\mathrm{d}\theta$

(c) $\displaystyle\int \cos\theta(\tan\theta+\sec\theta)\,\mathrm{d}\theta$

(d) $\displaystyle\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,\mathrm{d}x$

(e) $\displaystyle\int \frac{\sec^2 y}{\sqrt{1-\tan^2 y}}\,\mathrm{d}y$

(f) $\displaystyle\int \csc x\,\mathrm{d}x$

(g) $\displaystyle\int \sqrt{1+\sin^2(x-1)}\,\sin(x-1)\cos(x-1)\,\mathrm{d}x$

Solution.

(a) Let $u = \pi/x$. Then $\frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{\pi}{x^2}$, so $\mathrm{d}u = -\frac{\pi}{x^2}\,\mathrm{d}x$. By the substitution rule,

$$\begin{aligned}\int \frac{\cos(\pi/x)}{x^2}\,\mathrm{d}x &= -\frac{1}{\pi}\int \cos(\pi/x)\left(-\frac{\pi}{x^2}\right)\mathrm{d}x \\ &= -\frac{1}{\pi}\int \cos u\,\mathrm{d}u \\ &= -\frac{1}{\pi}\sin u + C = -\frac{1}{\pi}\sin\frac{\pi}{x} + C.\end{aligned}$$

(b) Note that there is not an obvious anti-derivative of $\tan^2\theta$. But there is a known anti-derivatives of $\sec^2\theta$, and $\tan^2\theta$ and $\sec^2\theta$ differ by 1. Thus

$$\begin{aligned}\int (2+\tan^2\theta)\,\mathrm{d}\theta &= \int (1+\sec^2\theta)\,\mathrm{d}\theta \\ &= \theta + \tan\theta + C.\end{aligned}$$

(c) Writing the given expression in terms of sine and cosine,

$$\begin{aligned}\int \cos\theta(\tan\theta + \sec\theta)\,\mathrm{d}\theta &= \int (\sin\theta + 1)\,\mathrm{d}\theta \\ &= -\cos\theta + \theta + C.\end{aligned}$$

(d) Let $u = 1+\sqrt{x}$. Then $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2\sqrt{x}}$, or $\frac{1}{2\sqrt{x}}\,\mathrm{d}x = \mathrm{d}u$. Thus

$$\begin{aligned}\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,\mathrm{d}x &= 2\int \frac{1}{(1+\sqrt{x})^2}\cdot\frac{1}{2\sqrt{x}}\,\mathrm{d}x \\ &= 2\int \frac{1}{u^2}\,\mathrm{d}u \\ &= 2\left(-\frac{1}{u}\right) + C = -\frac{2}{1+\sqrt{x}} + C.\end{aligned}$$

(e) Let $u = \tan y$. Then $\frac{\mathrm{d}u}{\mathrm{d}y} = \sec^2 y$, or $\sec^2 y\,\mathrm{d}y = \mathrm{d}u$. Thus

$$\begin{aligned}\int \frac{\sec^2 y}{\sqrt{1-\tan^2 y}}\,\mathrm{d}y &= \int \frac{1}{\sqrt{1-\tan^2 y}}\cdot\sec^2 y\,\mathrm{d}y \\ &= \int \frac{1}{\sqrt{1-u^2}}\,\mathrm{d}u \\ &= \sin^{-1} u + C = \sin^{-1}(\tan y) + C.\end{aligned}$$

(f) Let $u = \csc x + \cot x$. Then $\frac{\mathrm{d}u}{\mathrm{d}x} = -\csc x\cot x - \csc^2 x$. Thus

$$\begin{aligned}\int \csc x\,\mathrm{d}x &= \int \frac{\csc x(\csc x + \cot x)}{\csc x + \cot x}\,\mathrm{d}x \\ &= \int \frac{\csc^2 x + \csc x\cot x}{\csc x + \cot x}\,\mathrm{d}x \\ &= -\int \frac{1}{u}\,\mathrm{d}u = -\ln|u| + C \\ &= -\ln|\csc x + \cot x| + C\end{aligned}$$

Note: The technique is to multiply numerator and denominator by some term.

Recall finding $\int \sec x \, dx$: Let $u = \sec x + \tan x$. Then $\frac{du}{dx} = \sec x \tan x + \sec^2 x$. Thus

$$\begin{aligned}
\int \sec x \, dx &= \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} \, dx \\
&= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx \\
&= \int \frac{1}{\sec x + \tan x} (\sec^2 x + \sec x \tan x) \, dx \\
&= \int \frac{1}{u} \, du = \ln|u| + C \\
&= \ln|\sec x + \tan x| + C
\end{aligned}$$

(g) Let $u = 1 + \sin^2(x-1)$. Then $\frac{du}{dx} = 2\sin(x-1)\cos(x-1)$.

$$\begin{aligned}
\int \sqrt{1+\sin^2(x-1)} \sin(x-1)\cos(x-1) \, dx &= \frac{1}{2} \int \sqrt{1+\sin^2(x-1)} 2\sin(x-1)\cos(x-1) \, dx \\
&= \frac{1}{2} \int \sqrt{u} \, du \\
&= \frac{1}{2} \cdot \frac{2}{3} u^{3/2} + C = \frac{1}{3} \left[1+\sin^2(x-1)\right]^{3/2} + C.
\end{aligned}$$

Note: When dealing with square roots, it is useful to use substitution.

□

2. Evaluate the following definite integrals.

(a) $\displaystyle\int_0^1 x^2(x+1)^2 \, dx$

(b) $\displaystyle\int_0^4 \left|\sqrt{x} - 1\right| \, dx$

(c) $\displaystyle\int_{-1}^{-1/2} t^{-2} \sin^2\left(1 + \frac{1}{t}\right) dt$

(d) $\displaystyle\int_2^4 \frac{dx}{x(\ln x)^2}$

Solution.

(a) Expand and integrate term-by-term:

$$\begin{aligned}
\int x^2(x+1)^2 \, dx &= \int x^4 + 2x^3 + x^2 \, dx \\
&= \frac{1}{5}x^5 + \frac{1}{2}x^4 + \frac{1}{3}x^3 + C.
\end{aligned}$$

Hence $\displaystyle\int_0^1 x^2(x+1)^2 \, dx = \left[\frac{1}{5}x^5 + \frac{1}{2}x^4 + \frac{1}{3}x^3\right]_{x=0}^{x=1} = \frac{31}{30}$.

(b) Since the integrand is a piecewise function, consider where the function is positive and negative, and then integrate separately.

$$\left|\sqrt{x} - 1\right| = \begin{cases} 1 - \sqrt{x} & (0 \le x \le 1) \\ \sqrt{x} - 1 & (1 \le x \le 4) \end{cases}$$

Then

$$\begin{aligned}\int_0^4 \left|\sqrt{x}-1\right| \mathrm{d}x &= \int_0^1 \left|\sqrt{x}-1\right| \mathrm{d}x + \int_1^4 \left|\sqrt{x}-1\right| \mathrm{d}x \\ &= \int_0^1 (1-\sqrt{x})\, \mathrm{d}x + \int_1^4 (\sqrt{x}-1)\, \mathrm{d}x \\ &= \left[x - \frac{2}{3}x^{3/2}\right]_{x=0}^{x=1} + \left[\frac{2}{3}x^{3/2} - x\right]_{x=1}^{x=4} = 2.\end{aligned}$$

(c) Let $u = 1 + \frac{1}{t}$. Then $\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{1}{t^2}$, or $-\frac{1}{t^2}\,\mathrm{d}t = \mathrm{d}u$. Thus

$$\begin{aligned}\int t^{-2}\sin^2\left(1+\frac{1}{t}\right) \mathrm{d}t &= -\int \sin^2\left(1+\frac{1}{t}\right)\left(-\frac{1}{t^2}\right) \mathrm{d}t \\ &= -\int \sin^2 u\, \mathrm{d}u = -\int \frac{1-\cos 2u}{2}\, \mathrm{d}u \\ &= -\frac{1}{2}u + \frac{1}{4}\sin 2u + C \\ &= -\frac{1}{2}\left(1+\frac{1}{t}\right) + \frac{1}{4}\sin\left(2+\frac{2}{t}\right) + C.\end{aligned}$$

Note: To evaluate $\sin^2 u$, use double angle formula.

Hence $\int_{-1}^{-1/2} t^{-2}\sin^2\left(1+\frac{1}{t}\right) \mathrm{d}t = \left[-\frac{1}{2}\left(1+\frac{1}{t}\right) + \frac{1}{4}\sin\left(2+\frac{2}{t}\right)\right]_{t=-1}^{t=-1/2} = \frac{1}{2} - \frac{1}{4}\sin 2.$

(d) Let $u = \ln x$. Then $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$, or $\frac{1}{x}\,\mathrm{d}x = \mathrm{d}u$.

$$\begin{aligned}\int \frac{\mathrm{d}x}{x(\ln x)^2} &= \int \frac{1}{(\ln x)^2}\cdot\frac{1}{x}\,\mathrm{d}x \\ &= \int \frac{1}{u^2}\,\mathrm{d}u = -\frac{1}{u} + C = -\frac{1}{\ln x} + C\end{aligned}$$

Hence $\int_2^4 \frac{\mathrm{d}x}{x(\ln x)^2} = \left[-\frac{1}{\ln x}\right]_{x=2}^{x=4} = \frac{1}{2\ln 2}.$

□

3. For each of the following functions, find the *area* of the region between the graph of the function and the x-axis.

   (a) $y = x^3 - 3x^2 + 2x, \quad x \in [0, 2].$

   (b) $y = x\sqrt{4-x^2}, \qquad x \in [-2, 2].$

   need to consider positive/negative when dealing Geometrically - for area problems * Need graph for geometric problems

   Solution.

   (a) Note that $y = x(x-1)(x-2)$. Considering sign of y on $[0, 2]$:

	$(0,1)$	$(1,2)$
y	$+$	$-$

Integrate separately:

$$\begin{aligned}
A &= \int_0^2 |y|\,\mathrm{d}x = \int_0^1 |y|\,\mathrm{d}x + \int_1^2 |y|\,\mathrm{d}x \\
&= \int_0^1 y\,\mathrm{d}x + \int_1^2 (-y)\,\mathrm{d}x \\
&= \int_0^1 (x^3 - 3x^2 + 2x)\,\mathrm{d}x + \int_1^2 (-x^3 + 3x^2 - 2x)\,\mathrm{d}x \\
&= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.
\end{aligned}$$

Note: The graph is symmetric about $(1,0)$. (In fact, any cubic function is symmetric about its inflection point.)

(b) Considering sign of y on $[-2,2]$:

$$\begin{array}{ccc}
 & (-2,0) & (0,2) \\
y & - & +
\end{array}$$

Since the function is an odd function, the area of the positive part equals the area of the negative part. By symmetry, it suffices to integrate positive part, then multiply by 2 to obtain the total area:

$$A = \int_{-2}^2 |y|\,\mathrm{d}x = 2\int_0^2 |y|\,\mathrm{d}x = 2\int_0^2 x\sqrt{4-x^2}\,\mathrm{d}x.$$

Let $u = 4 - x^2$. Then $\frac{\mathrm{d}u}{\mathrm{d}x} = -2x$.

$$\begin{aligned}
\int 2x\sqrt{4-x^2}\,\mathrm{d}x = -\int \sqrt{u}\,\mathrm{d}u &= -\frac{2}{3}u^{3/2} + C \\
&= -\frac{2}{3}(4-x^2)^{3/2} + C.
\end{aligned}$$

Thus $A = \displaystyle\int_0^2 2x\sqrt{4-x^2} = \frac{16}{3}$.

□

Lecture 15

1. Evaluate the following improper integrals.

   (a) $\displaystyle\int_{-\infty}^{\infty} \frac{2x}{(x^2+1)^2}\,\mathrm{d}x$,

   (b) $\displaystyle\int_0^{\infty} \frac{16\tan^{-1}x}{1+x^2}\,\mathrm{d}x$,

   (c) $\displaystyle\int_0^a \frac{1}{\sqrt[5]{x}}\,\mathrm{d}x \quad (a > 0)$.

   Solution.

   (a) Let $u = x^2 + 1$. Then $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$, or $2x\,\mathrm{d}x = \mathrm{d}u$.

$$\int \frac{2x}{(x^2+1)^2} = \int \frac{1}{u^2}\,\mathrm{d}u = -\frac{1}{u} + C = -\frac{1}{1+x^2} + C.$$

Thus

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{2x}{(x^2+1)^2}\,\mathrm{d}x &= \int_{-\infty}^{a} \frac{2x}{(x^2+1)^2}\,\mathrm{d}x + \int_{a}^{\infty} \frac{2x}{(x^2+1)^2}\,\mathrm{d}x \\
&= \lim_{t\to-\infty} \int_{t}^{a} \frac{2x}{(x^2+1)^2}\,\mathrm{d}x + \lim_{t\to\infty} \int_{a}^{t} \frac{2x}{(x^2+1)^2}\,\mathrm{d}x \\
&= \lim_{t\to-\infty} \left(\frac{1}{1+t^2} - \frac{1}{1+a^2} \right) + \lim_{t\to\infty} \left(\frac{1}{1+a^2} - \frac{1}{1+t^2} \right) \\
&= -\frac{1}{1+a^2} + \frac{1}{1+a^2} = 0.
\end{aligned}$$

Note: the function is odd.

(b) Let $u = \tan^{-1} x$. Then $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{1+x^2}$, or $\frac{1}{1+x^2}\,\mathrm{d}x = \mathrm{d}u$. Thus

$$\int 16\tan^{-1} x \cdot \frac{1}{1+x^2}\,\mathrm{d}x = \int 16u\,\mathrm{d}u = 8u^2 + C = 8(\tan^{-1} x)^2 + C.$$

so

$$\int_0^{\infty} \frac{16\tan^{-1} x}{1+x^2}\,\mathrm{d}x = \lim_{t\to\infty} 8(\tan^{-1} t)^2 = 8\left(\frac{\pi}{2}\right)^2 = 2\pi^2.$$

(c) Note that $\frac{1}{\sqrt[5]{x}}$ is discontinuous at 0.

$$\int_0^a \frac{1}{\sqrt[5]{x}}\,\mathrm{d}x = \lim_{t\to 0^+} \int_t^a \frac{1}{\sqrt[5]{x}}\,\mathrm{d}x = \lim_{t\to 0^+} \frac{5}{4}\left(a^{4/5} - t^{4/5}\right) = \frac{5}{4}a^{4/5}.$$

□

2. (i) Let f be a continuous function on $[0,1]$. Use the substitution $x = \pi - t$ to show that

$$\int_0^{\pi} x f(\sin x)\,\mathrm{d}x = \frac{\pi}{2} \int_0^{\pi} f(\sin x)\,\mathrm{d}x.$$

(ii) Evaluate the integral $\displaystyle\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,\mathrm{d}x$.

Solution.

(i) Let $x = \pi - t$. Then $t = \pi - x$, so $\frac{\mathrm{d}t}{\mathrm{d}x} = -1$, or $\mathrm{d}x = -\mathrm{d}t$.

$$\begin{aligned}
\int_0^{\pi} x f(\sin x)\,\mathrm{d}x &= \int_{\pi}^{0} (\pi - t) f(\sin(\pi - t))(-1)\,\mathrm{d}t \\
&= \int_0^{\pi} (\pi - t) f(\sin t)\,\mathrm{d}t \\
&= \pi \int_0^{\pi} f(\sin t)\,\mathrm{d}t - \int_0^{\pi} t f(\sin t)\,\mathrm{d}t \\
&= \pi \int_0^{\pi} f(\sin x)\,\mathrm{d}x - \int_0^{\pi} x f(\sin x)\,\mathrm{d}x
\end{aligned}$$

Rearranging gives

$$\pi \int_0^{\pi} f(\sin x)\,\mathrm{d}x = 2\int_0^{\pi} x f(\sin x)\,\mathrm{d}x$$

which is the desired result.

Note: Remember to change upper and lower limits, when doing substitution.

(ii) Write $f(\sin x) = \dfrac{\sin x}{1+\cos^2 x} = \dfrac{\sin x}{2-\sin^2 x}$. By (i), we have

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,\mathrm{d}x = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,\mathrm{d}x.$$

Let $u = \cos x$. Then $\frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x$, or $\sin x\,\mathrm{d}x = -\mathrm{d}u$. Thus

$$\int \frac{\sin x}{1+\cos^2 x} = \int \frac{1}{1+u^2}(-1)\,\mathrm{d}u = -\tan^{-1} u + C = -\tan^{-1}(\cos x) + C.$$

Hence

$$\frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,\mathrm{d}x = -\frac{\pi}{2}[\tan^{-1}(\cos\pi) - \tan^{-1}(\cos 0)] = \frac{\pi^2}{4}.$$

□

3. Evaluate the derivative of f^{-1} at 2. (You may assume that the function is one-to-one in its domain without proof.)

   (a) $f(x) = x^5 - x^3 + 2x$,

   (b) $f(x) = \sqrt{x^3+x^2+x+1}$.

   Solution.

   (a) By observation, we see that $f(1) = 2$.
   Then $f'(x) = 5x^4 - 3x^2 + 2$, so $f'(1) = 4$. Hence

   $$(f^{-1})'(2) = \frac{1}{f'(1)} = \frac{1}{4}.$$

   (b) By observation, we see that $f(1) = 2$.
   Then $f'(x) = \dfrac{3x^2+2x+1}{2\sqrt{x^3+x^2+x+1}}$, so $f'(1) = \frac{3}{2}$. Hence

   $$(f^{-1})'(2) = \frac{1}{f'(1)} = \frac{2}{3}.$$

   □

   Remark. To show a function is one-to-one, we show that it is monotonic (use increasing test) and continuous, instead of using the definition.

Lecture 16

1. Use logarithmic differentiation to find the derivative of y with respect to x.

   (a) $y = \sqrt[3]{\dfrac{x(x-2)}{x^2+1}}$,

   (b) $y = \dfrac{x\sin x}{\sqrt{\sec x}}$,

   (c) $y = \dfrac{x^2 2^x}{\sqrt[3]{\sin 3x}}$,

   (d) $y = x^{(x^x)}, \quad x > 0$.

Avoid use of chain rule, product rule, quotient rule, etc. Use ln to simplify functions.

Note: This tool is often forgotten.

Solution.

(a) $|y| = |x|^{1/3}|x-2|^{1/3}(x^2+1)^{-1/3}$
$\ln|y| = 1/3\ln|x| + 1/3\ln|x-2| - \frac{1}{3}\ln\left(x^2+1\right)$
Differentiate with respect to x:

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{3}\frac{1}{x} + \frac{1}{3}\frac{1}{x-2} - \frac{1}{3}\frac{2x}{x^2+1} = \frac{1}{3}\left(\frac{1}{x} + \frac{1}{x-2} - \frac{2x}{x^2+1}\right)$$

Therefore

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{3}\left(\frac{1}{x} + \frac{1}{x-2} - \frac{2x}{x^2+1}\right)\cdot y \\ &= \frac{1}{3}\left(\frac{1}{x} + \frac{1}{x-2} - \frac{2x}{x^2+1}\right)\sqrt[3]{\frac{x(x-2)}{x^2+1}}\end{aligned}$$

(b) $|y| = |x|\cdot|\sin x|\cdot|\sec x|^{-1/2}$
$\ln|y| = \ln|x| + \ln|\sin x| - \frac{1}{2}\ln|\sec x|$
Differentiate

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x} + \frac{\cos x}{\sin x} - \frac{1}{2}\frac{\sec x\tan x}{\sec x} = \frac{1}{x} + \cot x - \frac{1}{2}\tan x.$$

Therefore

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{1}{x} + \cot x - \frac{1}{2}\tan x\right)\frac{x\sin x}{\sqrt{\sec x}}$$

(c) $|y| = |x|^2\cdot 2^x\cdot|\sin 3x|^{1/3}$
$\ln|y| = 2\ln|x| + x\ln 2 - \frac{1}{3}\ln|\sin 3x|$
Differentiate

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = 2\cdot\frac{1}{x} + \ln 2 - \frac{1}{3}\frac{3\cos 3x}{\sin 3x} = \frac{2}{x} - \ln 2 - \cot 3x.$$

Therefore

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{2}{x} - \ln 2 - \cot 3x\right)\frac{x^2 2^x}{\sqrt[3]{\sin 3x}}$$

(d) $\ln y = (x^x)\ln x$
Differentiate

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = (\frac{\mathrm{d}}{\mathrm{d}x}x^x)\ln x + x^x\frac{\mathrm{d}}{\mathrm{d}x}\ln x = x^x(\ln x + 1)\ln x + x^{x-1} = \left((\ln x + 1)\ln x + \frac{1}{x}\right)x^x.$$

Therefore

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left((\ln x + 1)\ln x + \frac{1}{x}\right)x^x x^{x^x}.$$

□

2. Find the following limits.

(a) $\lim_{x\to\infty}\frac{e^x}{x^4}$,

(b) $\lim_{x\to 0}(e^{2x}+2x)^{1/x}$,

(c) $\lim_{x\to\infty} x^{1/x}$,

(d) $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2}$.

Solution.

(a) Repeatedly apply L'opital's rule:

$$\lim_{x\to\infty}\frac{e^x}{x^4}=\lim_{x\to\infty}\frac{e^x}{4x^3}=\lim_{x\to\infty}\frac{e^x}{12x^2}=\lim_{x\to\infty}\frac{e^x}{24x}=\lim_{x\to\infty}\frac{e^x}{24}=\infty.$$

(b) Recall the formula

$$f(x)^{g(x)}=\exp\left(g(x)\ln f(x)\right).$$

If the function is continuous, we can bring the limit in:

$$\begin{aligned}\lim_{x\to 0}(e^{2x}+2x)^{1/x}&=\lim_{x\to 0}\exp\left(\frac{1}{x}\ln\left(e^{2x}+2x\right)\right)\\&=\exp\left(\lim_{x\to 0}\frac{\ln\left(e^{2x}+2x\right)}{x}\right)\\&=\exp\left(\lim_{x\to 0}\frac{\frac{2e^{2x}+2}{e^{2x}+2x}}{1}\right)\\&=\exp(4)=e^4.\end{aligned}$$

(c) Standard question:

$$\begin{aligned}\lim_{x\to\infty}x^{1/x}&=\lim_{x\to\infty}\exp\left(\frac{1}{x}\ln x\right)\\&=\exp\left(\lim_{x\to\infty}\frac{\ln x}{x}\right)\\&=\exp\left(\lim_{x\to\infty}\frac{1/x}{1}\right)\\&=\exp(0)=1.\end{aligned}$$

(d) Note that $\lim_{x\to 0}\frac{\tan x}{x}=\lim_{x\to 0}\frac{1}{\cos x}\frac{\sin x}{x}=1$.

$$
\begin{aligned}
\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2} &= \lim_{x\to 0}\exp\left(\frac{1}{x^2}\ln\frac{\tan x}{x}\right)\\
&= \exp\left(\lim_{x\to 0}\frac{\ln\frac{\tan x}{x}}{x^2}\right)\\
&= \exp\left(\lim_{x\to 0}\frac{\frac{x\sec^2 x-\tan x}{x\tan x}}{2x}\right) = \exp\left(\lim_{x\to 0}\frac{x\sec^2 x-\tan x}{2x^3}\cdot\frac{x}{\tan x}\right)\\
&= \exp\left(\lim_{x\to 0}\frac{x\sec^2 x-\tan x}{2x^3}\cdot\lim_{x\to 0}\frac{x}{\tan x}\right)\\
&= \exp\left(\lim_{x\to 0}\frac{x\sec^2 x-\tan x}{2x^3}\right)\\
&= \exp\left(\lim_{x\to 0}\frac{\sec^2 x+x\cdot 2\sec x\cdot\sec x\tan x-\sec^2 x}{6x^2}\right)\\
&= \exp\left(\lim_{x\to 0}\frac{\sec^2 x\tan x}{3x}\right)\\
&= \exp\left(\lim_{x\to 0}\frac{\sec^2 x}{3}\cdot\lim_{x\to 0}\frac{\tan x}{x}\right)\\
&= \exp(1/3) = e^{1/3}.
\end{aligned}
$$

□

3. (i) Show that the inverse hyperbolic sine function $\sinh^{-1} x$ is given by

$$\sinh^{-1} x = \ln\left(x+\sqrt{x^2+1}\right), \qquad x\in\mathbb{R}.$$

(ii) Verify that the derivative of $\sinh^{-1} x$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}x}\sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}, \qquad x\in\mathbb{R}.$$

Solution.

(i) Let $y=\sinh^{-1} x$. We want to express x in terms of y:

$$x=\sinh y = \frac{e^y-e^{-y}}{2} = \frac{e^y-e^{1/y}}{2} = \frac{z-1/z}{2}$$

where we let $z=e^y$. Rearranging gives a quadratic in z:

$$z^2-2xz-1=0$$

Solving for z,

$$z=\frac{2x\pm\sqrt{(2x)^2+4}}{2} = x\pm\sqrt{x^2+1}.$$

Since $z=e^y>0$, we reject the minus sign.

(ii) By logarithmic differentiation,

$$\frac{\mathrm{d}}{\mathrm{d}x}\sinh^{-1} x = \frac{\mathrm{d}}{\mathrm{d}x}\ln\left(x+\sqrt{x^2+1}\right) = \frac{1+\frac{x}{\sqrt{x^2+1}}}{x+\sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1}}.$$

This gives the integral $\displaystyle\int \frac{1}{\sqrt{x^2+1}} = \sinh^{-1} x + C = \ln\left(x + \sqrt{x^2+1}\right) + C.$

$\square$

Lecture 17

1. Evaluate the following indefinite integrals.

   (a) $\displaystyle\int \frac{\mathrm{d}x}{x^2\sqrt{x^2+4}}$,

   (b) $\displaystyle\int \frac{x^2}{\sqrt{6x-x^2}}\,\mathrm{d}x$,

   (c) $\displaystyle\int \frac{\ln x}{x^2}\,\mathrm{d}x$,

   (d) $\displaystyle\int \tan^{-1}\left(\frac{1}{x}\right)\mathrm{d}x$,

   (e) $\displaystyle\int \cos(\ln x)\,\mathrm{d}x$,

   (f) $\displaystyle\int e^{\sqrt{x}}\,\mathrm{d}x$,

   (g) $\displaystyle\int \frac{4(x+1)}{x^2(x^2+4)}\,\mathrm{d}x$,

   (h) $\displaystyle\int \frac{\sin\theta}{\cos^2\theta+\cos\theta-2}\,\mathrm{d}\theta$,

   (i) $\displaystyle\int \frac{1}{x^{2002}-x}\,\mathrm{d}x$,

   (j) $\displaystyle\int \frac{1}{2-2\sin x+3\cos x}\,\mathrm{d}x$.

   Solution.

   (a) Substitution: Let $x = 2\tan t$, $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then $\frac{\mathrm{d}x}{\mathrm{d}t} = 2\sec^2 t$, so $\mathrm{d}x = 2\sec^2 t\,\mathrm{d}t$.

$$\begin{aligned}\int \frac{\mathrm{d}x}{x^2\sqrt{x^2+4}} &= \int \frac{2\sec^2 t}{(2\tan t)^2 \cdot 2\sec t}\,\mathrm{d}t \\ &= \frac{1}{4}\int \frac{\cos t}{\sin^2 t}\,\mathrm{d}t \\ &= \frac{1}{4}\int \frac{1}{u^2}\,\mathrm{d}u && [u = \sin t] \\ &= -\frac{1}{4u} + C = -\frac{1}{4\sin t} + C = -\frac{\sqrt{x^2+4}}{4x} + C.\end{aligned}$$

   where $\sin t = \frac{x}{\sqrt{x^2+4}}$ can be obtained by drawing a right triangle with angle t.

   (b) Complete the square: $6x - x^2 = 3^3 - (x-3)^2$.

Substitution: Let $x = 3 + 3\sin t$, $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Then $\frac{\mathrm{d}x}{\mathrm{d}t} = 3\cos t$.

$$\begin{aligned}
\int \frac{x^2}{\sqrt{6x - x^2}}\,\mathrm{d}x &= \int \frac{(3 + 3\sin t)^2}{3\cos t} \cdot 3\cos t\,\mathrm{d}t \\
&= 9\int (\sin^2 t + 2\sin t + 1)\,\mathrm{d}t \\
&= 9\int \left(\frac{1 - \cos 2t}{2} + 2\sin t + 1\right)\mathrm{d}t \\
&= \frac{27}{2}t - 18\cos t - \frac{9}{4}\sin 2t + C \\
&= \frac{27}{2}t - 18\cos t - \frac{9}{2}\sin t\cos t + C \\
&= \frac{27}{2}\sin^{-1}\left(\frac{x-3}{3}\right) - 18 \cdot \frac{\sqrt{6x - x^2}}{3} - \frac{9}{2} \cdot \frac{x-3}{3} \cdot \frac{\sqrt{6x - x^2}}{3} + C \\
&= \frac{27}{2}\sin^{-1}\left(\frac{x-3}{3}\right) - \frac{x+9}{2}\sqrt{6x - x^2} + C.
\end{aligned}$$

(c) Integrate by parts: Let $u = \ln x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{x^2}$. Then $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$ and $v = -\frac{1}{x}$.

$$\begin{aligned}
\int \frac{\ln x}{x^2}\,\mathrm{d}x &= -\frac{\ln x}{x} - \int \left(-\frac{1}{x^2}\right)\mathrm{d}x \\
&= -\frac{\ln x}{x} - \frac{1}{x} + C.
\end{aligned}$$

(d) Integrate by parts: Let $u = \tan^{-1}\left(\frac{1}{x}\right)$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = 1$. Then $\frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{1}{1+x^2}$ and $v = x$.

$$\begin{aligned}
\int \tan^{-1}\left(\frac{1}{x}\right)\mathrm{d}x &= x\tan^{-1}\left(\frac{1}{x}\right) - \int \frac{-x}{1+x^2}\,\mathrm{d}x \\
&= x\tan^{-1}\left(\frac{1}{x}\right) + \frac{1}{2}\ln\left(1 + x^2\right) + C.
\end{aligned}$$

(e) Integrate by parts twice:

$$\begin{aligned}
\int \cos(\ln x)\,\mathrm{d}x &= x\cos(\ln x) + \int \sin(\ln x)\,\mathrm{d}x \\
&= x\cos(\ln x) + x\sin(\ln x) - \int \cos(\ln x)\,\mathrm{d}x.
\end{aligned}$$

Rearranging gives $\displaystyle\int \cos(\ln x)\,\mathrm{d}x = \frac{1}{2}x\cos(\ln x) + \frac{1}{2}x\sin(\ln x) + C$.

Remark. We can also solve for $\displaystyle\int \sin(\ln x)\,\mathrm{d}x = \frac{1}{2}x\cos(\ln x) - \frac{1}{2}x\sin(\ln x) + C$.

(f) Substitution: Let $t = \sqrt{x}$. Then $x = t^2$, and $\frac{\mathrm{d}x}{\mathrm{d}t} = 2t$.

$$\begin{aligned}
\int e^{\sqrt{x}}\,\mathrm{d}x &= \int e^t 2t\,\mathrm{d}t \\
&= e^t 2t - \int 2e^t\,\mathrm{d}t \\
&= 2te^t - 2e^t + C = 2(\sqrt{x} - 1)e^{\sqrt{x}} + C.
\end{aligned}$$

(g) The integrand is already a proper fraction. By partial fraction decomposition,

$$\frac{4(x+1)}{x^2(x^2+4)} = \frac{1}{x} + \frac{1}{x^2} - \frac{x+1}{x^2+4}.$$

Note: Find coefficients of partial fractions by comparing coefficients.

Then integrate term-by-term:

$$\begin{aligned}\int \frac{4(x+1)}{x^2(x^2+4)} &= \int \left(\frac{1}{x} + \frac{1}{x^2} - \frac{x+1}{x^2+4}\right) \mathrm{d}x \\ &= \int \left(\frac{1}{x} + \frac{1}{x^2} - \frac{x}{x^2+4} - \frac{1}{x^2+4}\right) \mathrm{d}x \\ &= \ln|x| - \frac{1}{x} - \frac{1}{2}\ln\left(x^2+4\right) - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C\end{aligned}$$

where we used trigonometric substitution to deal with $\int \frac{1}{x^2+4}\,\mathrm{d}x$.

(h) Let $x = \cos\theta$. Then $\frac{\mathrm{d}x}{\mathrm{d}\theta} = -\sin\theta$. By substitution then partial fractions,

$$\begin{aligned}\int \frac{\sin\theta}{\cos^2\theta + \cos\theta - 2}\,\mathrm{d}\theta &= \int \frac{-1}{x^2+x-2}\,\mathrm{d}x = \int \frac{-1}{(x+2)(x-1)}\,\mathrm{d}x \\ &= \int \left(\frac{1/3}{x+2} + \frac{-1/3}{x-1}\right) \mathrm{d}x \\ &= \frac{1}{3}\ln|x+2| - \frac{1}{3}|x-1| + C \\ &= \frac{1}{3}\ln|\cos\theta+2| - \frac{1}{3}|\cos\theta-1| + C \\ &= \frac{1}{3}\ln(\cos\theta+2) - \frac{1}{3}(1-\cos\theta) + C.\end{aligned}$$

(i) Let $x = \frac{1}{t}$. Then $\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{1}{t^2}$.

$$\begin{aligned}\int \frac{1}{x^{2002}-x}\,\mathrm{d}x &= \int \frac{t^{2000}}{t^{2001}-1}\,\mathrm{d}t = \frac{1}{2001}\int \frac{2001t^{2000}}{t^{2001}-1}\,\mathrm{d}t \\ &= \frac{1}{2001}\ln\left|t^{2001}-1\right| + C = \frac{1}{2001}\ln\left|x^{-2001}-1\right| + C.\end{aligned}$$

Note: We cannot deal with the denominator directly (using partial fractions) because the degree of the numerator is too large.

Since the degrees of denominator and numerator are very unbalanced, we can try to flip over numerator and denominator: let $x = \frac{1}{t}$. (We might not know if the substitution works; just try it.)

(j) Let $t = \tan\frac{x}{2}$. Then $x = 2\tan^{-1}t$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2}{1+t^2}$.

Recall that $\sin x = \frac{2t}{1+t^2}$ and $\cos x = \frac{1-t^2}{1+t^2}$.

$$\begin{aligned}\int \frac{1}{2-2\sin x+3\cos x}\,\mathrm{d}x &= \int \frac{-2}{(t+5)(t-1)}\,\mathrm{d}t \\ &= \int \left(\frac{1/3}{t+5} + \frac{-1/3}{t-1}\right) \mathrm{d}t \\ &= \frac{1}{3}\ln|t+5| - \frac{1}{3}|t-1| + C \\ &= \frac{1}{3}\ln\left|\tan\frac{x}{2}+5\right| - \frac{1}{3}\left|\tan\frac{x}{2}-1\right| + C.\end{aligned}$$

□

Lecture 18

1. Find the area of the region enclosed by the curve $y = x^4$ and the line $y = 8x$.

   Solution. graph the region, get the bounds of the region (integration bounds)

   Let $x^4 = 8x$. Then $x = 0$ or $x = 2$. (two intersection points - so the area is between 0 and 2)

   Cut the region by vertical segment at x, $\ell(x) = 8x - x^4$

   $$A = \int_0^2 (8x - x^4)\,\mathrm{d}x = \frac{48}{5}.$$

   Alternatively, integrate wrt y. Then $y = 0$ or $y = 16$. Cut the region by horizontal segment at y, $L(y) = \sqrt[4]{y} - 7/8$ (perpendicular to the axis that you want to integrate) Rewrite as $x = \frac{y}{8}$, $x = \sqrt[4]{y}$. right endpoint minus left endpoint

   $$A = \int_0^{16} (\sqrt[4]{y} - \frac{1}{8}y)\,\mathrm{d}y = \frac{48}{5}.$$

   □

2. (i) Let $a > 0$ and $b > 0$. Compute the area of the ellipse

   $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

   (ii) Let $a > 0$, $b > 0$ and $c > 0$. Compute the volume of the ellipsoid

   $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

   Solution.

   (i) Write y in terms of x $y^2 = b^2\left(1 - \frac{x^2}{a^2}\right) = \frac{b^2}{a^2}(a^2 - x^2)$, so $y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$ (if +, above x-axis, if -, below x-axis) split ellipse into two curves

   Cut ellipse by vertical line segment at x: $L(x) = \frac{2b}{a}\sqrt{a^2 - x^2}$. Then

   $$A = \int_{-a}^{a} \frac{2b}{a}\sqrt{a^2 - x^2}\,\mathrm{d}x = \frac{2b}{a}\int_{-a}^{a}\sqrt{a^2 - x^2}\,\mathrm{d}x = \frac{2b}{a}\cdot\frac{1}{2}\pi a^2 = \pi ab.$$

   where $\int_{-a}^{a}\sqrt{a^2 - x^2}\,\mathrm{d}x = \frac{1}{2}\pi a^2$ is the area of semicircle of radius a.

   (ii) Evidently $-a \le x \le a$. Cut ellipsoid by planes perpendicular to x-axis.

   Fix $x \in [a, b]$. Boundary of cross-section is $\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 - \frac{x^2}{a^2} = K^2$ ($K \ge 0$)

   If $x \in (a, a)$, $K > 0$, so $\frac{y^2}{(bK)^2} + \frac{z^2}{(cK)^2} = 1$ is ellipse of semi-major and semi-minor axes bK and cK. By (i), area of this ellipse is

   $$A(x) = \pi(bK)(cK) = \pi bcK^2 = \pi bc\left(1 - \frac{x^2}{a^2}\right)$$

   If $x = \pm a$, cross-section is a point $(\pm a, 0, 0)$, which has area 0. Its area can also be evaluated by $A(x) = \pi bc\left(1 - \frac{x^2}{a^2}\right)$.

   For $x \in [-a, a]$, area of cross-section at x is $A(x) = \pi bc\left(1 - \frac{x^2}{a^2}\right)$.

   $$V = \int_{-a}^{a} \pi bc\left(1 - \frac{x^2}{a^2}\right)\mathrm{d}x = \frac{4\pi}{3}abc.$$

□

3. For each of the following, using the *washer method* or *disc method* to find the volume of the solid generated by revolving the region enclosed by the given curves about the specified line.

   (a) $y = x^3$, $y = 0$ and $x = 2$; about the x-axis.
   (b) $y = \sin x$ $(0 \le x \le \pi)$ and $y = 0$; about the y-axis.
   (c) $y = x^2$, the x-axis and $x = 1$; about $x = -1$.
   (d) $x = (y-3)^2$ and $x = 4$; about $y = 1$.

   Solution.

   (a) Area of cross section is $A(x) = \pi(x^3)^2 = \pi x^6$
   $$V = \int_0^2 \pi x^6 \,\mathrm{d}x = \frac{128}{7}\pi.$$
   (b) cut using horizontal lines **perpendicular** to y-axis
   integrate from 0 to 1 Choose any $y \in [0,1]$, cut the region $y = \sin x$, so $x = \sin^{-1} y$ (this is only between $\pi/2$ and $-\pi/2$) by symmetry, the other point is $\pi - \sin^{-1} y$ (outer radius) $x = \sin^{-1} y$ is the inner radius (nearer to y-axis)
   $r = \sin^{-1} y$, $R = \pi - \sin^{-1} y$. Then $A(y) = \pi(R^2 - r^2) = \pi^2(\pi - 2\sin^{-1} y)$.
   $$V = \int_0^1 \pi^2(\pi - 2\sin^{-1} y)\,\mathrm{d}y = 2\pi^2.$$
   Include working for integral of sin-1
   construction of the figure is important
   (c) Integrate with respect to y, from $y = 0$ to $y = 1$.
   write $x = \sqrt{y}$
   Outer radius $R = 1 - (-1) = 2$, inner radius $r = \sqrt{y} - (-1) = \sqrt{y} + 1$ Then $A(y) = \pi(R^2 - r^2) = \pi(3 - y - 2\sqrt{y})$.
   $$V = \int_0^1 \pi(3 - y - 2\sqrt{y})\,\mathrm{d}y = \frac{7\pi}{6}.$$
   (d) Integrate with respect to x, from $x = 0$ to $x = 4$.
   Rewrite $\pm x = y - 3$, so $y = 3 + \sqrt{x}$ or $y = 3 - \sqrt{x}$
   Outer radius $R = (3 + \sqrt{x}) - 1 = 2 + \sqrt{x}$, inner radius is $r = (3 - \sqrt{x}) - 1 = 2 - \sqrt{x}$. Then $A(x) = \pi(R^2 - r^2) = 8\pi\sqrt{x}$
   $$V = \int_0^4 8\pi\sqrt{x}\,\mathrm{d}x = \frac{128\pi}{3}.$$

□

Lecture 19

1. For each of the following, using the *cylindrical shell method* to find the volume of the solid generated by revolving the region enclosed by the given curves about the specified line.

   (a) $y = x^3$, $y = 0$ and $x = 2$; about the x-axis.
   (b) $y = \sin x$ $(0 \le x \le \pi)$ and $y = 0$; about the y-axis.

(c) $y = x^2$, the x-axis and $x = 1$; about $x = -1$.

(d) $x = (y-3)^2$ and $x = 4$; about $y = 1$.

Solution.

(a) Cut the region using line segments parallel to the axis that the region is rotated about Then the variable is y, so we integrate wrt y

Rewrite $x = y^{1/3}$

For any $y \in [0,8]$, radius $r = y$, height $h = 2 - y^{1/3}$. Area $A(y) = 2\pi rh = 2\pi y(2 - y^{1/3}) = 2\pi(2y - y^{4/3})$.

$$V = \int_0^8 2\pi(2y - y^{4/3})\,\mathrm{d}y = \frac{128\pi}{7}.$$

(b) Cut region using vertical line segments Radius $r = x$, height $h = \sin x$. Area $A(x) = 2\pi rh = 2\pi x \sin x$.

$$V = \int_0^\pi 2\pi x \sin x\,\mathrm{d}x = 2\pi\left[-x\cos x + \sin x\right]_{x=0}^{x=\pi} = 2\pi^2.$$

(c) Cut region using vertical line segments Radius $r = x - (-1) = x + 1$, height $h = x^2$ Area $A(x) = 2\pi(x^2 + x^3)$.

$$V = \int_0^1 2\pi(x^2 + x^3)\,\mathrm{d}x = 2\pi\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_{x=0}^{x=1} = \frac{7\pi}{6}.$$

(d) Cut the region using horizontal line segments, so variable is y. Range of y is $[1,5]$. Radius $r = y - 1$ (distance between line segment is axis of rotation), height $h = 4 - (y-3)^2$. Area $A(x) = 2\pi(5 - 11y + 7y^2 - y^3)$.

$$V = \int_1^5 2\pi(5 - 11y + 7y^2 - y^3)\,\mathrm{d}y = 2\pi\left[5y - \frac{11}{2}y^2 + \frac{7}{3}y^3 - \frac{1}{4}y^4\right]_{y=1}^{y=5} = \frac{128\pi}{3}.$$

□

2. Find the lengths of the following curves.

(a) $y = \sqrt{2 - x^2}$ from $x = 0$ to $x = 1$.

(b) $y = \ln(\cos x)$ from $x = 0$ to $x = \pi/3$.

(c) $x = \dfrac{y^3}{6} + \dfrac{1}{2y}$ from $y = 2$ to $y = 3$.

Solution.

(a) We first simplify:

$$\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \sqrt{1 + \frac{x^2}{2 - x^2}} = \sqrt{\frac{2}{2 - x^2}}$$

Then

$$\int \sqrt{\frac{2}{2 - x^2}}\,\mathrm{d}x = \int \frac{1}{\sqrt{1 - (x/\sqrt{2})^2}}\,\mathrm{d}x = \int \frac{\sqrt{u}}{\sqrt{1 - u^2}} = \sqrt{2}\sin^{-1} u + C = \sqrt{2}\sin^{-1}\frac{x}{\sqrt{2}} + C$$

so $\frac{\sqrt{2}\pi}{4}$.

In fact, the curve is an arc of $x^2 + y^2 = 2$, angle $45°$. so L is 1/8 of circle of radius $\sqrt{2} = \frac{1}{8} \cdot 2\pi\sqrt{2} = \frac{\sqrt{2}\pi}{4}$

(b)

$$\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}=\sqrt{1+(-\tan x)^2}=\sqrt{\sec^2 x}=|\sec x|=\sec x$$

because $\sec x = 1/\cos x > 0$ for $x \in [0,\pi/3]$.

$$L=\int_0^{\pi/3}\sec x\,\mathrm{d}x=\ln|\sec x+\tan x|=\ln\left(2+\sqrt{3}\right).$$

(c) Change: $L=\int\sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)}\,\mathrm{d}y$

$$\sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}=\sqrt{\left(\frac{y^2}{2}+\frac{1}{2y^2}\right)^2}=\frac{y^2}{2}+\frac{1}{2y^2}.$$

Then

$$L=\int_2^3\left(\frac{y^2}{2}+\frac{1}{2y^2}\right)\mathrm{d}y=[]_{y=2}^{y=3}=\frac{13}{4}.$$

□

3. Find the areas of the surfaces generated by revolving the following curves about the indicated axes.

(a) $y=\sqrt{2x-x^2}$, $1/2\le x\le 3/2$; about the x-axis.

(b) $x=2\sqrt{4-y}$, $0\le y\le 15/4$; about the y-axis.

(c) $x^{2/3}+y^{2/3}=1$, $y\ge 0$; about the x-axis.

Solution.

(a) Simplify: we have

$$\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}=\sqrt{1+\left(\frac{1-x}{\sqrt{2x-x^2}}\right)}=\sqrt{\frac{1}{2x-x^2}}$$

so

$$2\pi\sqrt{2x-x^2}\sqrt{\frac{1}{2x-x^2}}=2\pi.$$

Note: Simplify before integration.

$$A=\int_{1/2}^{3/2}2\pi\,\mathrm{d}x=2\pi\left(\frac{3}{2}-\frac{1}{2}\right)=2\pi.$$

Similarly, the area on $[a,b]$ is $2\pi(b-a)$ ($a\ge 0$, $b\le 2$) (this only depends on the length of the interval)

(b) y as variable, x as function - interchange roles of x and y

$$A=\int_a^b 2\pi x\sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y.$$

$\frac{\mathrm{d}x}{\mathrm{d}y}=-\frac{1}{\sqrt{4-y}}$, so $1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2=1+\frac{1}{4-y}=\frac{5-y}{4-y}$

$2\pi x\sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}=2\pi\cdot 2\sqrt{4-y}\sqrt{\frac{5-y}{4-y}}=4\pi\sqrt{5-y}$

$$A=\int_0^{15/4}4\pi\sqrt{5-y}\,\mathrm{d}y=\frac{35\sqrt{5}}{3}\pi$$

(c) $y=(1-x^{2/3})^{3/2}$

The curve is symmetric about y-axis, so it suffices to find the area formed by rotating the portion in the 1st quadrant. Let $x \geq 0$ and $y \geq 0$. to find dy/dx without breaking the balance of powers of x and y, use implicit differentiation to get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x^{-1/3}}{y^{-1/3}} = -\frac{y^{1/3}}{x^{1/3}}.$$

Then

$$\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \sqrt{\frac{x^{2/3}+y^{2/3}}{x^{2/3}}} = \frac{1}{x^{1/3}}$$

so

$$A = 2\int_0^1 \left(2\pi y \cdot \frac{1}{x^{1/3}}\right)\mathrm{d}x = 4\pi\int_0^1 y\frac{1}{x^{1/3}}\,\mathrm{d}x = 4\pi\int_1^0 y\frac{-1}{y^{1/3}}\,\mathrm{d}y = 4\pi\,()_{y=0}^{y=1} = \frac{12\pi}{5}.$$

□

Lecture 20

1. Solve the following differential equations.

   (a) $2\sqrt{xy}\frac{\mathrm{d}y}{\mathrm{d}x} = 1 \qquad x>0, y>0$

   (b) $\sqrt{x}\frac{\mathrm{d}y}{\mathrm{d}x} = e^{y+\sqrt{x}} \qquad x>0$

   (c) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2(e^y)^{1/x}+y^2}{xy} \qquad x>0, y>0$

   (d) $x\frac{\mathrm{d}y}{\mathrm{d}x}+3y = \frac{\sin x}{x^2} \qquad x>0$

   (e) $(t-1)^3\frac{\mathrm{d}s}{\mathrm{d}t}+4(t-1)^2 s = t+1 \qquad t>1$

   (f) $\frac{\mathrm{d}y}{\mathrm{d}x} - y = -y^2$

Solution.

(i) This ODE is **separable**:

$$\int \sqrt{y}\,\mathrm{d}y = \int \frac{1}{2\sqrt{x}}\,\mathrm{d}x$$

$$\frac{2}{3}y^{3/2} = \sqrt{x}+C.$$

(ii) This ODE is **separable**:

$$\int e^{-y}\,\mathrm{d}y = \int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x.$$

$$\int e^{-y}\,\mathrm{d}y = -e^{-y}$$

Let $u=\sqrt{x}$. Then $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2\sqrt{x}}$.

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x = \int 2e^u\,\mathrm{d}u = 2e^u = 2e^{\sqrt{x}}$$

Hence $-e^{-y} = 2e^{\sqrt{x}}+C$.

(iii) Rewrite as $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{e^{y/x} + (y/x)^2}{y/x}$. This ODE is **homogeneous**: Let $z = y/x$.

Then $z + x\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{e^z + z^2}{z} = \frac{e^z}{z} + z$, so $x\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{e^z}{z}$, which is **separable**:

$$\int ze^{-z}\,\mathrm{d}z = \int \frac{1}{x}\,\mathrm{d}x$$

Integrating by parts,

$$\int ze^{-z}\,\mathrm{d}z = -ze^{-z} + \int e^{-z}\,\mathrm{d}z = -ze^{-z} - e^{-z}$$

Hence $\ln x = -(z+1)e^{-z} + C = -\left(\frac{y}{x} + 1\right)e^{-y/x} + C$.

(iv) Write in standard form: $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{3}{x} \cdot y = \frac{\sin x}{x^3}$. This ODE is **linear**. Integrating factor:

$$\int \frac{3}{x}\,\mathrm{d}x = 3\ln x \implies v(x) = e^{3\ln x} = x^3.$$

General solution is

$$y = \frac{1}{v(x)} \int v(x)q(x)\,\mathrm{d}x = \frac{1}{x^3} \int \sin x\,\mathrm{d}x = \frac{1}{x^3}(C - \cos x).$$

(v) Write in standard form: $\frac{\mathrm{d}s}{\mathrm{d}t} + \frac{4}{t-1} \cdot s = \frac{t+1}{(t-1)^3}$. This ODE is **linear**. Integrating factor:

$$\int \frac{4}{t-1}\,\mathrm{d}t = 4\ln(t-1) \implies v(t) = e^{4\ln(t-1)} = (t-1)^4.$$

General solution is

$$y = \frac{1}{v(t)} \int v(t)q(t)\,\mathrm{d}t = \frac{1}{(t-1)^4} \int (t^2 - 1)\,\mathrm{d}t = \frac{1}{(t-1)^4}\left(\frac{1}{3}t^3 - t + C\right).$$

(vi) This ODE is **Bernoulli** differential equation. Let $z = y^{1-2} = y^{-1}$. Then

$$\frac{\mathrm{d}z}{\mathrm{d}x} + (-1)(-1)z = (-1)(-1) \implies \frac{\mathrm{d}z}{\mathrm{d}x} + z = 1$$

which is **linear**. Integrating factor:

$$\int 1\,\mathrm{d}x = x \implies v(x) = e^x.$$

General solution is

$$z = \frac{1}{e^x} \int e^x \cdot 1\,\mathrm{d}x = e^{-x}(e^x + C) = 1 + Ce^{-x}$$
$$y = \frac{1}{z} = \frac{1}{1 + Ce^{-x}}.$$

□

2. Solve the following initial value problems.

(a) $\frac{\mathrm{d}y}{\mathrm{d}t} = e^t \sin\left(e^t - 2\right) \qquad y(\ln 2) = 0$

(b) $x\frac{\mathrm{d}y}{\mathrm{d}x} = y + x^2 \sin x \qquad y(\pi) = 0$

(c) $(x+1)\dfrac{\mathrm{d}y}{\mathrm{d}x} - 2(x^2+x)y = \dfrac{e^{x^2}}{x+1} \qquad y(0) = 5$

Solution.

(a) Integrate directly. Substitution: $u = e^t - 2$. Then $\frac{\mathrm{d}u}{\mathrm{d}t} = e^t$.
General solution is

$$y = \int e^t \sin\left(e^t - 2\right) \mathrm{d}t = \int \sin u \,\mathrm{d}u = C - \int \cos u = C - \cos\left(e^t - 2\right).$$

Plug in initial conditions: $0 = C - \cos\left(e^{\ln 2} - 2\right) \Rightarrow C = 1$.
Hence particular solution is $y = 1 - \cos(e^t - 2)$.

(b) Linear Write in standard form: $\frac{\mathrm{d}y}{\mathrm{d}x} + \left(-\frac{1}{x}\right) y = x \sin x$
Integrating factor:

$$\int -\frac{1}{x} = -\ln x \implies v(x) = e^{-\ln x} = \frac{1}{x}$$

General solution is

$$y = x \int \sin x \,\mathrm{d}x = x(C - \cos x).$$

Plug in initial conditions: $0 = \pi(C - \cos\pi) \Rightarrow C = -1$.
Hence particular solution is $y = x(-1-\cos x)$.

(c) Write in standard form: $\frac{\mathrm{d}y}{\mathrm{d}x} + (-2x)y = \frac{e^{x^2}}{(x+1)^2}$ This is linear

$$\int (-2x)\,\mathrm{d}x = -x^2 \implies v(x) = e^{-x^2}$$

so

$$y = e^{x^2} \int \frac{1}{(x+1)^2}\,\mathrm{d}x = e^{x^2}\left(C - \frac{1}{x+1}\right).$$

Plug in initial conditions: $5 = e^{0^2}(C-1) = C - 1 \Rightarrow C = 6$.
Hence particular solution is $e^{x^2}\left(6 - \frac{1}{x+1}\right)$.

$\square$

this is the last lesson

Lecture 21

1. Find an equation of the curve that satisfies $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 4x^3 y$ and whose y-intercept is 7.

   Solution. The type of this ODE is linear: $\frac{\mathrm{d}y}{\mathrm{d}x} + (-4x^3)y = 0$
   Integrating factor:

   $$\int (-4x^3)\,\mathrm{d}x = -x^4 + C \implies v(x) = e^{-x^4}.$$

   Hence $y = \frac{1}{e^{-x^4}} \int e^{-x^4} \cdot 0 \,\mathrm{d}x = Ce^{x^4}$.
   y-intercept is $7 \Rightarrow y = 7$ at $x = 0$. Plug in initial conditions to get $7 = Ce^{0^4} = C$. Hence $y = 7e^{x^4}$. $\square$

2. In some chemical reactions, the rate at which the amount of a substance changes with time is proportional to the amount of the substance present. For the change of δ-glucono lactone to gluconic acid, for example,

   $$\frac{\mathrm{d}y}{\mathrm{d}t} = -0.6y$$

when t is measured in hours. If there are 100 grams of δ-glucono lactone present when $t = 0$, how many grams will be left after the first hour?

Note: Take note of units.

Solution. Since this is exponential growth/decay, general solution is $y = Ce^{-0.6t}$.

Plug in initial condition $y(0) = 100$ to get $C = 100$. Hence $y = 100e^{-0.6t}$.

When $t = 1$, $y = 100e^{-0.6} \approx 54.88$ grams. □

3. The intensity $L(x)$ of light x metres beneath the surface of the ocean satisfies the differential equation

$$\frac{\mathrm{d}L}{\mathrm{d}x} = -kL.$$

A diver knows from experience that diving to 10 metres around Bahamas cuts the intensity in half (*half-intensity depth*). He cannot work without artificial light when the intensity falls below one-tenth of the surface value. About how deep can he expect to work without artificial light?

Solution. The initial condition is $L(10) = \frac{1}{2}L(0)$. Our goal is to find x such that $L(x) = \frac{1}{10}L(0)$.

Since this is exponential decay, the general solution $L(x) = Ce^{-kx}$.

Plug in initial conditions: $Ce^{-10k} = \frac{1}{2}C \Rightarrow e^{-10k} = \frac{1}{2} \Rightarrow k = \frac{1}{10}\ln 2$.

Hence $L(x) = \frac{1}{10}L(0) \Rightarrow Ce^{-kx} = \frac{1}{10}C \Rightarrow x = \frac{1}{k}\ln 10 = 10 \cdot \frac{\ln 10}{\ln 2} \approx 33.22$ m. □

4. A tank can support no more than 150 guppies. Six guppies are introduced into the tank. Assume that the rate of growth of the population is

$$\frac{\mathrm{d}P}{\mathrm{d}t} = 0.0015P(150 - P),$$

where time t is in weeks.

(i) Find a formula for the guppy population in terms of t.

(ii) How long will it take for the guppy population to be 100? to be 125?

Solution.

(i) Recall that the general solution is $P(t) = \dfrac{M}{1 + Ce^{-Mrt}}$. Thus $P(t) = \dfrac{150}{1 + Ce^{-150 \cdot 0.0015t}} = \dfrac{150}{1 + Ce^{-0.225t}}$.

Plug in initial condition $P(0) = 6$ to get $C = 24$.

Hence $P(t) = \dfrac{150}{1 + 24e^{-0.225t}}$.

(ii) When $P = 100$, $e^{0.225t} = 48$, so $t = \frac{1}{0.225}\ln 48 \approx 17.21$ weeks.

When $P = 125$, $e^{0.225t} = 120$, so $t = \frac{1}{0.225}\ln 120 \approx 21.28$ weeks.

□

5. A pan of warm water (46°C) was put in a refrigerator. Ten minutes later, the water's temperature was 39°C; ten minutes after that, it was 33°C. Use Newton's Law of Cooling to estimate how cold the refrigerator was.

Solution. Recall that the general solution is $T(t) = T_S + (T_0 - T_S)e^{-rt}$. Thus $T(t) = T_S + (46 - T_S)e^{-rt}$.

Plug in initial conditons $T(10) = 39$ and $T(20) = 33$ to get

$$39 = T_S + (46 - T_S)e^{-10r}$$
$$33 = T_S + (46 - T_S)e^{-20r}$$

Since we want to find T_S, we want to cancel out r, by writing r in terms of T_S then do substitution: $e^{-10r} = \dfrac{39 - T_S}{46 - T_S}$ and $e^{-20r} = \dfrac{33 - T_S}{46 - T_S}$. Then

$$\left(\frac{39 - T_S}{46 - T_S}\right)^2 = \frac{33 - T_S}{46 - T_S}.$$

Solving (by hand) gives $T_S = -3$. $\square$

B Selected Problems from Past Year Papers