Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2021

Junior Section (Round 1)

Wednesday, 2 June 2021 **0930-1200 hrs**

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Enter your answers on the answer sheet provided.*
3. *For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.*
4. *For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.*
5. *No steps are needed to justify your answers.*
6. *Each question carries 1 mark.*
7. *No calculators are allowed.*
8. *Throughout this paper, let* $\lfloor x \rfloor$ *denote the greatest integer less than or equal to* x. *For example,* $\lfloor 2.1 \rfloor = 2$, $\lfloor 3.9 \rfloor = 3$.

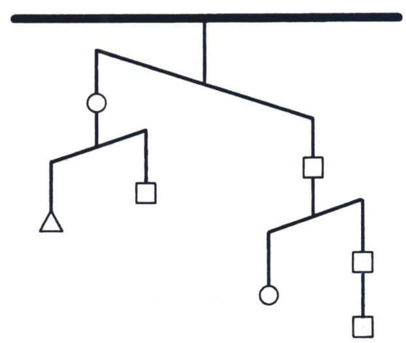
PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO.

Multiple Choice Questions

1. Let a and b be real numbers satisfying $a < 0 < b$. Which of the following is **not** true?

   (A) $a^2b > 0$ (B) $ab^2 < 0$ (C) $\frac{a}{b} > 0$ (D) $b - a > 0$ (E) $|a - b| > 0$

2. The following diagram shows a system of balances hanging from the ceiling with three types of weights. The balances tip down to the heavier side. If we use $\square < \triangle$ to represent $\square$ is lighter than $\triangle$, which of the following is true?

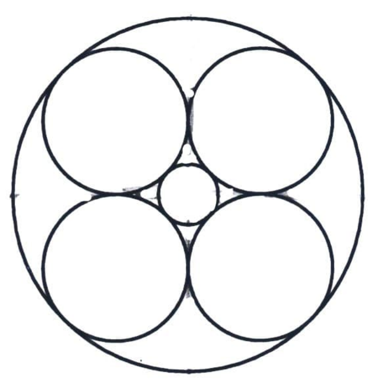


   (A) $\square < \circ < \triangle$ (B) $\square < \triangle < \circ$ (C) $\triangle < \square < \circ$ (D) $\triangle < \circ < \square$
   (E) $\circ < \square < \triangle$

3. Let $x = 2^{20} \cdot 3^5$, $y = 2^5 \cdot 5^{10}$ and $z = 7^{10}$. Which of the following is true?

   (A) $x > y > z$ (B) $x > z > y$ (C) $y > z > x$ (D) $y > x > z$ (E) $z > x > y$

4. In the diagram, six circles are tangent to each other. If the radius of the largest circle is 1 and the radii of the four medium sized circles are equal, what is the radius of the smallest circle?



   (A) $\sqrt{2} - 1$ (B) $3 - 2\sqrt{2}$ (C) $2 - \sqrt{2}$ (D) $6 - 4\sqrt{2}$ (E) None of the above

5. Which of the following is closest to the value of

$$\frac{1}{\sqrt{2}+1} + \frac{1}{\sqrt{3}+\sqrt{2}} + \frac{1}{\sqrt{4}+\sqrt{3}} + \cdots + \frac{1}{\sqrt{2021}+\sqrt{2020}}\ ?$$

   (A) 10 (B) 20 (C) 30 (D) 40 (E) 50

Short Questions

6. Let x be a positive integer. Suppose that the lowest common multiple of x and 14 is 42 and the lowest common multiple of x and 33 is 66. What is the value of x?

7. What are the last four digits of the sum

$$1 + 22 + 333 + 4444 + \cdots + \underbrace{999999999}_{\text{nine 9s}}?$$

4205

Give your answer as a 4-digit number.

8. How many distinct triples of positive integers (a, b, c) satisfy $1 \leqslant a \leqslant b \leqslant c$ and

$$\frac{1}{\underset{3}{a}} + \frac{1}{\underset{3}{b}} + \frac{1}{\underset{3}{c}} = 1?$$

9. Given five consecutive positive integers, if the product of the largest and the smallest integer is 2021, what is the sum of the five integers?

10. The numbers from 1 to 2021 are concatenated from left to right and the result is read as an integer

$$12345678910111213 \cdots 201920202021.$$

What is the remainder when this number is divided by 6?

11. In the figure below, each distinct letter represents a unique distinct digit such that the arithmetic holds. If S represents 6 and E represents 8, what number does SIX represent?

$$\begin{array}{cccccc}
 & {}^{6}\mathrm{S} & {}^{8}\mathrm{E} & {}^{7}\mathrm{V} & {}^{8}\mathrm{E} & {}^{2}\mathrm{N} \\
 & {}^{6}\mathrm{S} & {}^{8}\mathrm{E} & {}^{7}\mathrm{V} & {}^{8}\mathrm{E} & {}^{2}\mathrm{N} \\
+ & & & {}^{6}\mathrm{S} & {}^{4}\mathrm{I} & {}^{5}\mathrm{X} \\
\hline
{}^{1}\mathrm{T} & {}^{3}\mathrm{W} & {}^{8}\mathrm{E} & {}_{2}\mathrm{N} & {}^{1}\mathrm{T} & {}^{9}\mathrm{Y}
\end{array}$$

12. What is the value of

$$\sqrt{(219)(220)(221)(222) + 1}?$$

13. Let $A, B, \ldots, I$ be unknowns satisfying

$$\begin{aligned}
A + B + C &= 1, \\
B + C + D &= 2, \\
C + D + E &= 3, \\
D + E + F &= 4, \\
E + F + G &= 5, \\
F + G + H &= 6, \\
G + H + I &= 7.
\end{aligned}$$

What is the value of $A + E + I$?

14. If x is a 3-digit number, we define $M(x)$ and $m(x)$ respectively as the largest and smallest positive number that can be formed by rearranging the three digits of x. For example, if $x = 123$, then $M(123) = 321$ and $m(123) = 123$. If $y = 898$, then $M(898) = 988$ and $m(898) = 889$.

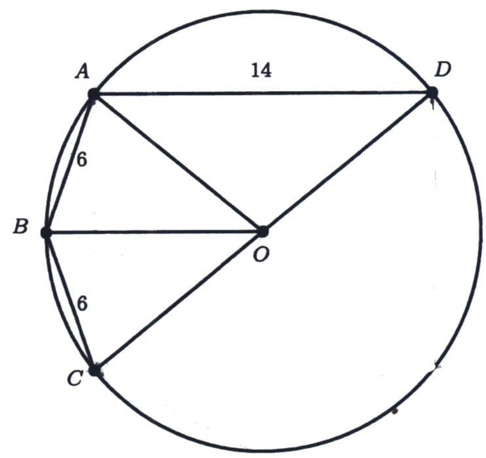
Given that z is a 3-digit number that satisfies $z = M(z) - m(z)$, what is the value of z?

15. How many integers k are there such that the quadratic equation

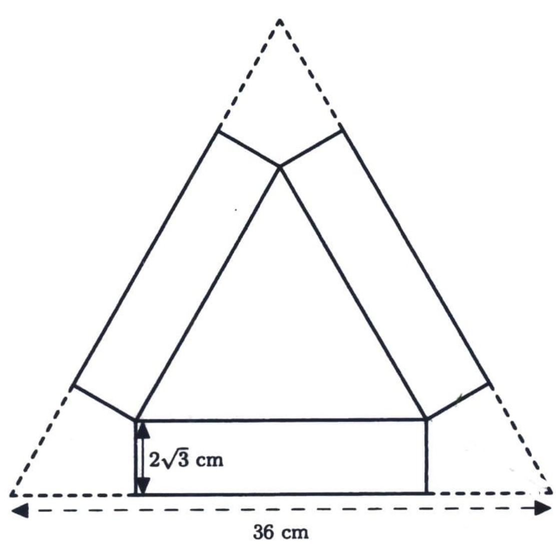
$$kx^2 + 20x + 20 - k = 0$$

has only integer solutions?

16. In the following diagram, $ABCD$ is a quadrilateral inscribed in a circle with centre O. If $|AB| = |BC| = 6$, $|AD| = 14$ and CD is a diameter, what is the length of $|CD|$?



17. The diagram below shows a piece of cardboard in the shape of an equilateral triangle with side length 36 cm. Six perpendicular cuts of length $2\sqrt{3}$ cm are made to remove the corners in order to fold the cardboard into a tray whose base is an equilateral triangle and height is $2\sqrt{3}$ cm. What is the volume of the tray in cm^3?



18. What is the value of

$$\left\lfloor \sqrt{45+\sqrt{2021}} - \sqrt{45-\sqrt{2021}} \right\rfloor ?$$

19. Let x be the positive real number that satisfies

$$\sqrt{x^2-4x+5}+\sqrt{x^2+4x+5}=3x.$$

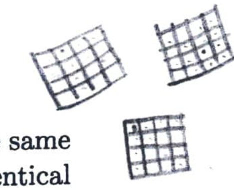
What is the value of $\lfloor 10^4 x^2 \rfloor$?

20. What is the number of positive integers c such that the equation

$$x^2-2021x+100c=0$$

has real roots?

12+24+48
= 42×2
= 84



21. In chess, two queens are said to be attacking each other if they are positioned in the same row, column or diagonal on a chessboard. How many ways are there to place two identical queens in a 4×4 chessboard such that they do not attack each other?

$\frac{1}{2} \times \frac{1}{4} \times 401 \times 403 \times \cdots \times 801 =$

22. Let $A = \frac{3}{2} \times \frac{5}{4} \times \frac{7}{6} \times \cdots \times \frac{801}{800}$. What is the value of $\left\lfloor \frac{A}{10} \right\rfloor$?

23. A 3×3 grid is filled with the integers 1 to 9. An arrangement is *nicely ordered* if the integers in each horizontal row is increasing from left to right and the integers in each vertical column is increasing from top to bottom. Two examples of nicely ordered arrangements are given in the diagram below. What is the total number of distinct nicely ordered arrangements?

1	3	7
2	4	8
5	6	9

1	2	6
3	5	8
4	7	9

24. A class has exactly 50 students and it is known that 40 students scored A in English, 45 scored A in Mathematics and 42 scored A in Science. What is the minimum number students who scored A in all three subjects?

40 EMS

25. Suppose a positive integer x satisfies the following equation

$$\sqrt[5]{x+76638}-\sqrt[5]{x-76637}=5.$$

What is the value of x?

SMO 2021 (Junior Section) Answers

1. C
2. B
3. C
4. B
5. D
6. 6
7. 3685
8. 3
9. 225
10. 3
11. 650
12. 48619
13. 4
14. 495
15. 12 (or 13 if k=0 is allowed)
16. 18
17. 864
18. 9
19. 25777
20. 10211
21. 44
22. 2
23. 42
24. 27
25. 84413

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2021

Senior Section (Round 1)

Wednesday, 2 June 2021 **0930 – 1200 hrs**

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Multiple Choice Questions

1. Let p be a real number such that the equation $x^2 - 10x = p$ has no real solution. of the following is true?

   (A) $0 < p < 25$ (B) $p = 25$ (C) $p > 25$ (D) $p < -25$ (E) $-25 < p < 0$

2. Which of the following is the largest?

   (A) $\tan 50° + \sin 50°$ (B) $\tan 50° + \cos 50°$ (C) $\sin 50° + \cos 50°$

   (D) $\tan 50° + \sin^2 50°$ (E) $\sin^2 50° + \cos^2 50°$

3. Find the value of $2021^{(\log_{2021} 2020)(\log_{2020} 2019)(\log_{2019} 2018)}$.

   (A) 2018 (B) 2019 (C) 2020 (D) 2021 (E) None of the above

4. Suppose $\sin\theta = \dfrac{n-3}{n+5}$ and $\cos\theta = \dfrac{4-2n}{n+5}$ for some integer n. Find the maximum value of $160\tan^2\theta$.

   (A) 80 (B) 90 (C) 100 (D) 120 (E) None of the above

5. Select all the inequalities which hold for **all** real values of x and y.

   (i) $x \le x^2 + y^2$ $0<x<1,\ y=0$

   ✓ (ii) $xy \le x^2 + y^2$

   (iii) $x - y \le x^2 + y^2$ $x=0,\ -1<y<0$

   (iv) $y + xy \le x^2 + y^2$ $x=0,\ 0<y<1$

   ✓ (v) $x + y - 1 \le x^2 + y^2$ $(x-\frac{1}{2})^2+(y-\frac{1}{2})^2+\frac{1}{2} \ge 0$

   (A) (i) (B) (i) & (iii) (C) (iii) & (iv) (D) (ii) (E) (ii) & (v)

Short Questions

6. Let x be the integer such that $x = 5^{\sqrt{2+4\log_x 5}}$. Determine the value of x.

7. If $\cos A - \cos B = \dfrac{1}{2}$ and $\sin A - \sin B = -\dfrac{1}{4}$, find the value of $100\sin(A+B)$.

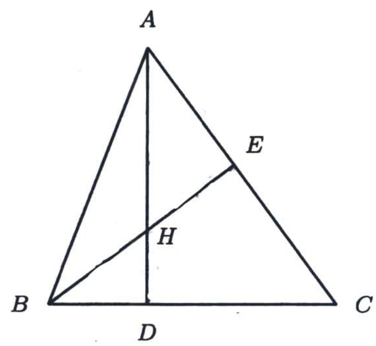
8. Find the constant in the expansion of $\left(\sqrt[3]{x} + \frac{1}{\sqrt{x}}\right)^6 \left(\sqrt{x} + \frac{1}{x}\right)^{10}$.

9. A quadratic polynomial $P(x) = ax^2 + bx + c$, where $a \neq 0$, has the following properties:

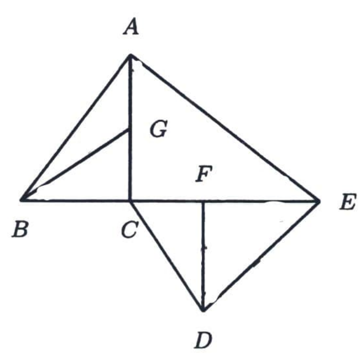
$$P(n) = \frac{1}{n^2} \text{ for all } n = -1, 2, 3.$$

Determine the smallest positive value of k, where $k \neq 2, 3$, such that $P(k) = \frac{1}{k^2}$.

10. The figure below shows a triangle ABC such that AD and BE are altitudes to the sides BC and CA respectively. The lines AD and BE intersect at H. Determine the area (in cm^2) of the triangle ABC if $AH = 50$ cm, $DH = 18$ cm and $BH = EH$.



11. In the figure below, $\angle GCB = \angle ACE = \angle DFE = 90°$, and $\angle GBC = \angle EAC = \angle EDF = \theta°$. Also, $GB = 6$ cm, $AE = 10$ cm and $DE = 8$ cm. Let S denote the sum of the areas of the triangles ABC and CDE. Find the maximum possible value of S (in cm^2).



12. Find the sum of all the solutions to the equation

$$\sqrt[3]{x - 110} - \sqrt[3]{x - 381} = 1.$$

13. If $f(x) = \left(2x + 4 + \frac{x-2}{x+3}\right)^2$, where $-2 \leq x \leq 2$, find the maximum value of $f(x)$.

14. Given that

$$D = \sqrt{\sqrt{x^2 + (y-1)^2} + \sqrt{(x-1)^2 + y^2}}$$

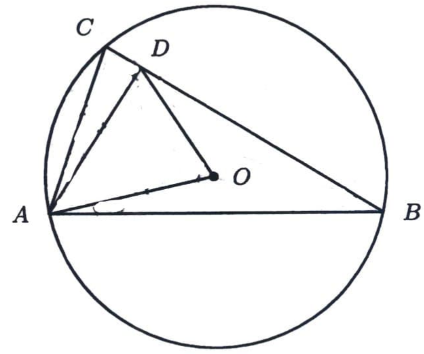
for real values of x and y, find the minimum value of D^8.

15. Find the minimum value of $\dfrac{8}{\sin 2\theta} + 12\tan\theta$, where $0 < \theta < \frac{\pi}{2}$.

16. Determine the largest angle θ (in degree), where $0^\circ \leq \theta \leq 360^\circ$, such that

$$\sin(\theta+18^\circ)+\sin(\theta+162^\circ)+\sin(\theta+234^\circ)+\sin(\theta+306^\circ) = 1+\cos(\theta+60^\circ)+\cos(\theta+300^\circ).$$

17. Let O be the circumcentre of the triangle ABC and that $\angle ABC = 30^\circ$. Let D be a point on the side BC such that the length of AD is the same as the radius of the circle. Determine the value of $\angle ADO$ (in degree) if $\angle OAB = 10^\circ$.



18. A function f satisfies $f(x)f(x+1) = x^2+3x$ for all real numbers x. If $f(1)+f(2) = \dfrac{25}{6}$ and $0 < f(1) < 2$, determine the value of $11 \times f(10)$.

19. Find the value of

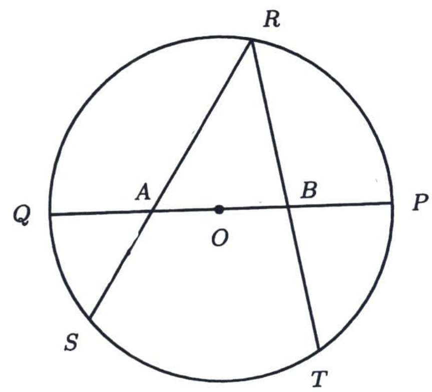
$$\frac{1}{\sin^2 0.5^\circ} - \tan^2 0.5^\circ + \frac{1}{\sin^2 1.5^\circ} - \tan^2 1.5^\circ + \frac{1}{\sin^2 2.5^\circ} - \tan^2 2.5^\circ + \cdots + \frac{1}{\sin^2 179.5^\circ} - \tan^2 179.5^\circ.$$

20. Let a_1, a_2, a_3 be three distinct integers where $1000 > a_1 > a_2 > a_3 > 0$. Suppose there exist real numbers x, y, z such that

$$\begin{aligned}
(a_1 - a_2)y + (a_1 - a_3)z &= a_1 + a_2 + a_3 \\
(a_1 - a_2)x + (a_2 - a_3)z &= a_1 + a_2 + a_3 \\
(a_1 - a_3)x + (a_2 - a_3)y &= a_1 + a_2 + a_3.
\end{aligned}$$

Find the largest possible value of $x + y + z$.

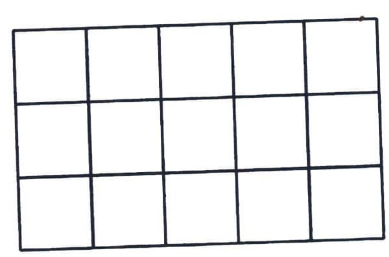
21. The figure below shows a circle centred at O with radius 555 cm. If $OA = OB$ and $\frac{RA}{AS} + \frac{RB}{BT} = \frac{13}{6}$, find OA (in cm).



22. Find the number of real solutions (x, y) of the system of equations

$$\begin{aligned} x^3 + y^3 + y^2 &= 0, \\ x^2 + x^2y + xy^2 &= 0. \end{aligned}$$

23. The following 3×5 rectangle consists of 15 1×1 squares. Determine the number of ways in which 9 out of the 15 squares are to be coloured in black such that every row and every column has an odd number of black squares.



24. Let n be a positive integer such that

$$\frac{2021n}{2021 + n}$$

is also a positive integer. Determine the smallest possible value of n.

25. Determine the number of 5-digit numbers with the following properties:

(i) All the digits are non-zero;
(ii) The digits can be repeated;
(iii) The difference between consecutive digits is exactly 1.

SMO 2021 (Senior Section) Answers

1. D
2. A
3. A
4. B
5. E
6. 25
7. 80
8. 2400
9. 6
10. 2550
11. 40
12. 491
13. 64
14. 4
15. 16
16. 240
17. 70
18. 120
19. 180
20. 2994
21. 111
22. 3
23. 60
24. 188
25. 106

Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2021
(Open Section, Round 1)

Thursday, 3 June 2021 **0930-1200 hrs**

Instructions to contestants

1. *Answer ALL 25 questions.*
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3. *No steps are needed to justify your answers.*
4. *Each question carries 1 mark.*
5. *No calculators are allowed.*

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In this paper, let $\mathbb{R}$ denote the set of all real numbers, and $\lfloor x \rfloor$ denote the greatest integer not exceeding x and let $\lceil x \rceil$ denote the smallest integer not less than x. For examples, $\lfloor 5 \rfloor = 5$, $\lfloor 2.8 \rfloor = 2$, and $\lfloor -2.3 \rfloor = -3$; $\lceil 5 \rceil = 5$, $\lceil 2.8 \rceil = 3$, and $\lceil -2.3 \rceil = -2$.

1. It is given that $\frac{\pi}{2} < \beta < \alpha < \frac{3\pi}{4}$, $\cos(\alpha - \beta) = \frac{12}{13}$ and $\sin(\alpha + \beta) = -\frac{3}{5}$. Find $\lfloor |2021 \sin(2\alpha)| \rfloor$.

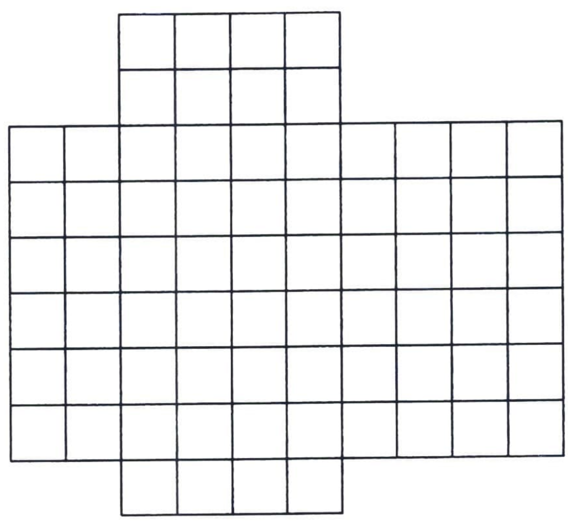
2. Find the number of solutions of the equation $|x - 3| + |x - 5| = 2$.

   (Note: If you think that there are infinitely many solutions, enter your answer as "99999".)

3. Evaluate $1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + \cdots + 10 \times 11 \times 12$.

4. It is given that the solution of the inequality $\sqrt{81 - x^4} \le kx + 1$ is $a \le x \le b$ with $b - a = 2$, where $k > 0$. Determine $\lfloor k \rfloor$.

5. The figure below shows a cross that is cut out from a 10×9 rectangular board.



   Find the total number of rectangles in the above figure.

   (Note: A square is a rectangle.)

6. Consider all the polynomials $P(x, y)$ in two variables such that $P(0, 0) = 2020$ and for all x and y, $P(x, y) = P(x + y, y - x)$. Find the largest possible value of $P(1, 1)$.

7. In the three dimensional Cartesian space with $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ denoting the unit vectors along three perpendicular directions in a clockwise manner, the line l with equation given by $\mathbf{r} \times (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = 5\mathbf{i} - 13\mathbf{j} + 7\mathbf{k}$ intersects the plane Π with equation $x + y + z = 16$ at the point (a, b, c). Find the value of $a + b + c$.

8. Find the minimum value of $(x+7)^2 + (y+2)^2$ subject to the constraint $(x-5)^2 + (y-7)^2 = 4$.

9. Find the largest possible value of $\alpha^4+\beta^4+\gamma^4$ among all possible sets of numbers (α, β, γ) that satisfy the equations

$$\begin{aligned} \alpha+\beta+\gamma &= 2 \\ \alpha^2+\beta^2+\gamma^2 &= 14 \\ \alpha^3+\beta^3+\gamma^3 &= 20. \end{aligned}$$

10. If p is the product of all the non-zero real roots of the equation

$$\sqrt[9]{x^7+30x^5} = \sqrt[7]{x^9-30x^5},$$

find $\lfloor |p| \rfloor$.

11. Let S be the sum of a convergent geometric series with first term 1. If the third term of the series is the arithmetic mean of the first two terms, find $\lfloor 3S+4 \rfloor$.

12. Given that $\sin\alpha+\sin\beta = \frac{1}{10}$, and $\cos\alpha+\cos\beta = \frac{1}{9}$, find $\lfloor \tan^2(\alpha+\beta) \rfloor$.

13. Determine the number of positive integers that are divisible by 2021 and has exactly 2021 divisors (including 1 and itself).

14. Let $S = \sum_{k=0}^{25} \binom{100}{4k} - 2^{98}$. Find $\left\lfloor \left| \frac{S}{2^{48}} \right| \right\rfloor$.

15. Assume that ABC is an acute triangle with $\sin(A+B) = \frac{3}{5}$ and $\sin(A-B) = \frac{1}{5}$. If $AB = 2022(\sqrt{6}-2)$, determine $\lfloor h \rfloor$, where h is the height of the triangle from C on AB.

16. Let $a_1, a_2, \cdots$ be a sequence with $a_1 = 1$ and $a_{n+1} = \frac{n+2}{n} S_n$ for all $n = 1, 2, \cdots$, where $S_n = a_1 + a_2 + \cdots + a_n$. Determine the minimum integer n such that $a_n \geq 2021$.

17. Each card of a stack of 101 cards has one side colored red and the other colored blue. Initially all cards have the red side facing up and stacked together in a deck. On each turn, Ah Meng takes 8 cards on the top, flip them over, and place them to the bottom deck. Determine the minimum number of turns required so that all the cards have the red sides facing up again.

18. Let ABC be a triangle with $AB = 10$ and $\frac{\cos A}{\cos B} = \frac{AC}{BC} = \frac{4}{3}$. Let P be a point on the inscribed circle of triangle ABC. Find the largest possible value of $PA^2+PB^2+PC^2$.

19. A basket contains 19 apples labeled by the numbers 2,3,...,20, and 19 bananas labeled by the numbers 2,3,...,20. Ah Beng picks m apples and n bananas from the basket. However he needs to ensure that for any apple labeled a and any banana labeled b that he picks, a and b are relatively prime. Determine the largest possible value of mn.

20. Let $p(x) = ax^2 - bx + c$ be a polynomial where a, b, c are positive integers and $p(x)$ has two distinct roots in $(0, 1)$. Determine the least possible value of abc.

21. In the triangle ABC, $\angle A > 90°$, the incircle touches the side BC and AC at A_1 and B_1 respectively. The line A_1B_1 meets the extension of BA at X such that $\angle CXB = 90°$. Suppose $BC^2 = AB^2 + BC \cdot AC$. Find the size of $\angle A$ in degrees.

22. Find the number of positive integers n such that $7n - 16$ divides $n \cdot 13^{2019}$.

23. In the acute triangle ABC, P is a point on AB, Q is a point on AC such that $BP + CQ = PQ$. The bisector of $\angle A$ meets the circumcircle of the triangle ABC at the point R distinct from A. Suppose $\angle PRQ = 52.5°$. Find the size of $\angle BAC$ in degrees.

24. Let $S = \displaystyle\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,dx$. Determine the value of $\lfloor S^2 \rfloor$.

25. Let p, q, r be positive numbers with $p - r = 4q$, and $a_1, a_2, \cdots$ and $b_1, b_2, \cdots$ be two sequences defined by $a_1 = p, b_1 = q$ and for $n \geq 2$,

$$a_n = pa_{n-1}, b_n = qa_{n-1} + rb_{n-1}.$$

Find the value of $\displaystyle\lim_{n\to\infty} \frac{\sqrt{a_n^2 + (3b_n)^2}}{b_n}$.

SMO 2021 (Open Section) Answers

1. 1741
2. 99999
3. 4290
4. 7
5. 1395
6. 2020
7. 16
8. 169
9. 98
10. 6
11. 6
12. 89
13. 2
14. 2
15. 1348
16. 10
17. 101 (or 312 if the 8 cards are flipped as a stack of 8)
18. 88
19. 65
20. 25
21. 108
22. 2021
23. 75
24. 6
25. 5