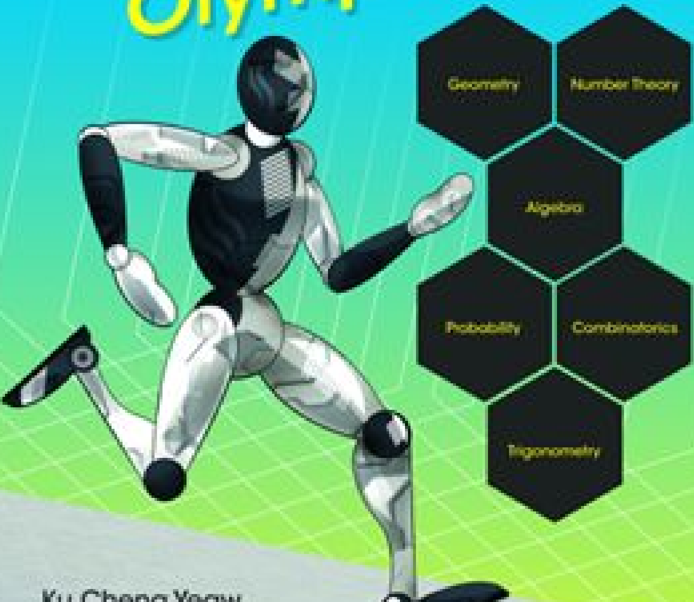
Singapore Mathematical Olympiads 2014



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Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2014

Junior Section (First Round)

Tuesday, 3 June 2014 **0930-1200 hrs**

Instructions to contestants

1. *Answer ALL 35 questions.*
2. *Enter your answers on the answer sheet provided.*
3. *For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.*
4. *For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.*
5. *No steps are needed to justify your answers.*
6. *Each question carries 1 mark.*
7. *No calculators are allowed.*
8. *Throughout this paper, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. For example, $\lfloor 2.1 \rfloor = 2$, $\lfloor 3.9 \rfloor = 3$.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

Multiple Choice Questions

1. Let x, y and z be real numbers satisfying $x > y > 0$ and $z \neq 0$. Which of the inequalities below is not always true?

   (A) $x + z > y + z$ (B) $x - z > y - z$ (C) $xz > yz$ (D) $\frac{1}{y} + z > \frac{1}{x} + z$
   (E) $xz^2 > yz^2$

2. If the radius of a circle is increased by 100%, the area is correspondingly increased by how many percent?

   (A) 50% (B) 100% (C) 200% (D) 300% (E) 400%

3. If $a = \sqrt{7}$, $b = \sqrt{90}$, find the value of $\sqrt{6.3}$.

   (A) $\dfrac{7b}{a\sqrt{10}}$ (B) $\dfrac{b - 7a}{10}$ (C) $\dfrac{10a}{b}$ (D) $\dfrac{ab}{100}$ (E) None of the above

4. Find the value of $\dfrac{1}{1 - \sqrt[4]{5}} + \dfrac{1}{1 + \sqrt[4]{5}} + \dfrac{2}{1 + \sqrt{5}}$.

   (A) -1 (B) 1 (C) $-\sqrt{5}$ (D) $\sqrt{5}$ (E) None of the above

5. Andrew, Catherine, Michael, Nick and Sally ordered different items for lunch. These are (in no particular order): cheese sandwich, chicken rice, duck rice, noodles and steak. Find out what Catherine had for lunch if we are given the following information:

   1. Nick sat between his friend Sally and the person who ordered steak.
   2. Michael does not like noodles.
   3. The person who ate noodles is Sally's cousin.
   4. Neither Catherine, Michael nor Nick likes rice.
   5. Andrew had duck rice.

   (A) Cheese sandwich (B) Chicken rice (C) Duck rice (D) Noodles (E) Steak

6. At 2:40 pm, the angle formed by the hour and minute hands of a clock is $x°$, where $0 < x < 180$. What is the value of x?

   (A) 60° (B) 80° (C) 100° (D) 120° (E) 160°

7. In the figure below, each distinct letter represents a unique digit such that the arithmetic sum holds. If the letter L represents 9, what is the digit represented by the letter T?

$$\begin{array}{rcccccccc} & \mathrm{T} & \mathrm{E} & \mathrm{R} & \mathrm{R} & \mathrm{I} & \mathrm{B} & \mathrm{L} & \mathrm{E} \\ + & & & & \mathrm{N} & \mathrm{U} & \mathrm{M} & \mathrm{B} & \mathrm{E} \; \mathrm{R} \\ \hline \mathrm{T} & \mathrm{H} & \mathrm{I} & \mathrm{R} & \mathrm{T} & \mathrm{E} & \mathrm{E} & \mathrm{N}. & \end{array}$$

   (A) 4 (B) 5 (C) 6 (D) 7 (E) 8

8. A regular cube is to have 2 faces coloured red, 2 faces coloured blue and 2 faces coloured orange. We consider two colourings to be the same if one can be obtained by a rotation of the cube from another. How many different colourings are there?

(A) 4 (B) 5 (C) 6 (D) 8 (E) 9

9. In $\triangle ABC$, $AB = AC$, $\angle BAC = 120°$, D is the midpoint of BC, and E is a point on AB such that DE is perpendicular to AB. Find the ratio $AE : BD$.

(A) $1:2$ (B) $2:3$ (C) $1:\sqrt{3}$ (D) $1:2\sqrt{3}$ (E) $2:3\sqrt{3}$

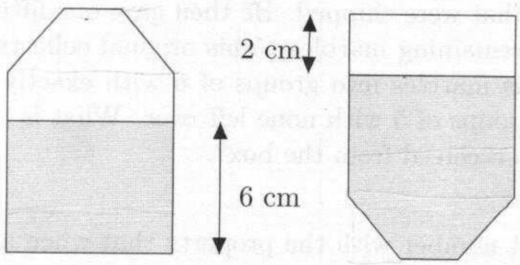
10. How many ways are there to add four positive odd numbers to get a sum of 22?

(A) 14 (B) 15 (C) 16 (D) 17 (E) 18

Short Questions

11. Successive discounts of 10% and 20% are equivalent to a single discount of x%. What is the value of x?

12. The diagram below shows the front view of a container with a rectangular base. The container is filled with water up to a height of 6 cm. If the container is turned upside down, the height of the empty space is 2 cm. Given that the total volume of the container is 28 cm^3, find the volume of the water in cm^3.



13. Let A be the solution of the equation

$$\frac{x-7}{x-8} - \frac{x-8}{x-9} = \frac{x-10}{x-11} - \frac{x-11}{x-12}.$$

Find the value of $6A$.

14. The sum of the two smallest positive divisors of an integer N is 6, while the sum of the two largest positive divisors of N is 1122. Find N.

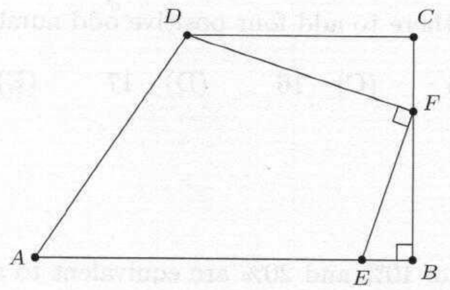
15. Let D be the absolute value of the difference of the two roots of the equation $3x^2 - 10x - 201 = 0$. Find $\lfloor D \rfloor$.

16. If m and n are positive real numbers satisfying the equation

$$m + 4\sqrt{mn} - 2\sqrt{m} - 4\sqrt{n} + 4n = 3,$$

find the value of $\dfrac{\sqrt{m} + 2\sqrt{n} + 2014}{4 - \sqrt{m} - 2\sqrt{n}}$.

17. In the diagram below, $ABCD$ is a trapezium with $AB \parallel DC$ and $\angle ABC = 90°$. Points E and F lie on AB and BC respectively such that $\angle EFD = 90°$. If $CD + DF = BC = 4$, find the perimeter of $\triangle BFE$.



18. If p, q and r are prime numbers such that their product is 19 times their sum, find $p^2 + q^2 + r^2$.

19. John received a box containing some marbles. Upon inspecting the marbles, he immediately discarded 7 that were chipped. He then gave one-fifth of the marbles to his brother. After adding the remaining marbles to his original collection of 14, John discovered that he could divide his marbles into groups of 6 with exactly 2 left over or he could divide his marbles into groups of 5 with none left over. What is the smallest possible number of marbles that John received from the box?

20. Let N be a 4-digit number with the property that when all the digits of N are added to N itself, the total equals 2019. Find the sum of all the possible values of N.

21. There are exactly two ways to insert the numbers 1, 2 and 3 into the circles

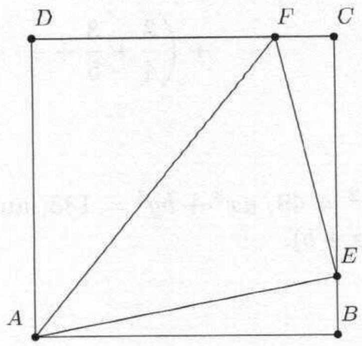
$$\bigcirc < \bigcirc > \bigcirc$$

such that every order relation $<$ or $>$ between numbers in adjacent circles is satisfied. The two ways are ① $<$ ③ $>$ ② and ② $<$ ③ $>$ ①.

Find the total number of possible ways to insert the numbers 3, 14, 15, 9, 2 and 6 into the circles below, such that every order relation $<$ or $>$ between the numbers in adjacent pairs of circles is satisfied.

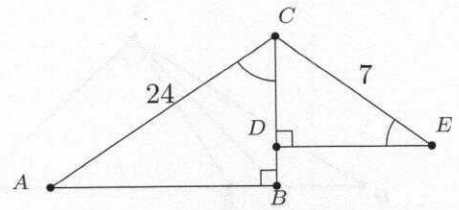
$$\bigcirc > \bigcirc > \bigcirc > \bigcirc < \bigcirc < \bigcirc.$$

22. Let $ABCD$ be a square of sides 8 cm. If E and F are variable points on BC and CD respectively such that $BE = CF$, find the smallest possible area of the triangle $\triangle AEF$ in cm^2.



23. If a, b and c are non-zero real numbers satisfying $a+2b+3c = 2014$ and $2a+3b+2c = 2014$, find the value of $\dfrac{a^2+b^2+c^2}{ac+bc-ab}$.

24. In the diagram below, $\triangle ABC$ and $\triangle CDE$ are two right-angled triangles with $AC = 24$, $CE = 7$ and $\angle ACB = \angle CED$. Find the length of the line segment AE.



25. The hypotenuse of a right-angled triangle is 10 and the radius of the inscribed circle is 1. Find the perimeter of the triangle.

26. Let x be a real number satisfying $\left(x+\dfrac{1}{x}\right)^2 = 3$. Evaluate $x^3+\dfrac{1}{x^3}$.

27. For $2 \le x \le 8$, we define $f(x) = |x-2| + |x-4| - |2x-6|$. Find the sum of the largest and smallest values of $f(x)$.

28. If both n and $\sqrt{n^2+204n}$ are positive integers, find the maximum value of n.

29. Let $N = \overline{abcd}$ be a 4-digit perfect square that satisfies $\overline{ab} = 3 \cdot \overline{cd} + 1$. Find the sum of all possible values of N.

(The notation $n = \overline{ab}$ means that n is a 2-digit number and its value is given by $n = 10a+b$.)

30. Find the following sum:

$$\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{29}\right)+\left(\frac{2}{3}+\frac{2}{4}+\frac{2}{5}+\cdots+\frac{2}{29}\right)$$
$$+\left(\frac{3}{4}+\frac{3}{5}++\cdots+\frac{3}{29}\right)+\cdots+\left(\frac{27}{28}+\frac{27}{29}\right)+\frac{28}{29}.$$

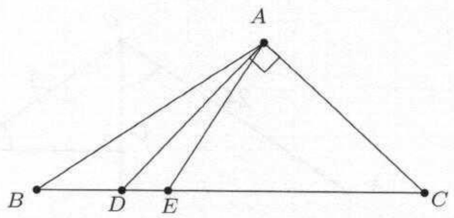
31. If $ax+by=7$, $ax^2+by^2=49$, $ax^3+by^3=133$, and $ax^4+by^4=406$, find the value of $2014(x+y-xy)-100(a+b)$.

32. For $a \geq \frac{1}{8}$, we define

$$g(a)=\sqrt[3]{a+\frac{a+1}{3}\sqrt{\frac{8a-1}{3}}}+\sqrt[3]{a-\frac{a+1}{3}\sqrt{\frac{8a-1}{3}}}.$$

Find the maximum value of $g(a)$.

33. In the diagram below, AD is perpendicular to AC and $\angle BAD = \angle DAE = 12^\circ$. If $AB + AE = BC$, find $\angle ABC$.



34. Define S to be the set consisting of positive integers n, such that the inequalities

$$\frac{9}{17}<\frac{n}{n+k}<\frac{8}{15},$$

hold for *exactly one* positive integer k. Find the largest element of S.

35. The number 2^{29} has exactly 9 distinct digits. Which digit is missing?

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2014

Junior Section (First Round Solutions)

Multiple Choice Questions

1. Answer: (C)

   If $z < 0$, we have $xz < yz$.

2. Answer: (D)

   Assume that a circle has its radius r. If its radius is increased 100%, then its radius becomes $2r$. Thus its area is changed from πr^2 to $\pi(2r)^2 = 4\pi r^2$, i.e. the area is increased by $3\pi r^2$ from πr^2, hence 300%.

3. Answer: (E)

   $\sqrt{6.3} = \sqrt{\frac{630}{100}} = \frac{\sqrt{630}}{10} = \frac{\sqrt{7}\sqrt{90}}{10} = \frac{ab}{10}$. So (D) and (C) are out since $\frac{10a}{b} = \frac{\sqrt{70}}{3}$. (A) is also not possible as $\frac{7b}{a\sqrt{10}} = 3\sqrt{7}$ and finally (B) is impossible since $\frac{b-7a}{10} < 0$.

4. Answer: (A)

$$\begin{aligned}
\frac{1}{1-\sqrt[4]{5}} + \frac{1}{1+\sqrt[4]{5}} + \frac{2}{1+\sqrt{5}} &= \frac{1+\sqrt[4]{5}+1-\sqrt[4]{5}}{(1+\sqrt[4]{5})(1-\sqrt[4]{5})} + \frac{2}{1+\sqrt{5}} \\
&= \frac{2}{1-\sqrt{5}} + \frac{2}{1+\sqrt{5}} \\
&= \frac{2(1+\sqrt{5}) + 2(1-\sqrt{5})}{(1-\sqrt{5})(1+\sqrt{5})} \\
&= \frac{4}{-4} = -1.
\end{aligned}$$

5. Answer: (D)

   Andrew had duck rice and since neither Catherine, Michael nor Nick likes rice, we conclude Sally had chicken rice. Nick did not order steak. Furthermore, he is not related to Sally, so he did not take noodles. Thus Nick had the cheese sandwich. Since Michael did not like noodles, he must have had the steak, leaving noodles for Catherine.

6. Answer: (E)

   At 12:00 pm, the two hands are together. From 12:00 pm to 2:40 pm the minute hand moves $2 \times 360^\circ + 240^\circ = 960^\circ$. Note that the minute hand's speed is 12 times of the hour hand. Thus the hour hand moves $960^\circ/12 = 80^\circ$. So the angle formed by the two hands is $240^\circ - 80^\circ = 160^\circ$.

7. Answer: (A)

The fifth column from the left leads to U=0 and the third column from the left then allows one to deduce that E must be odd. Continuing from there, the only possible solution is the following:

$$\begin{array}{r} 4\ 5\ 8\ 8\ 1\ 7\ 9\ 5 \\ +\quad 3\ 0\ 2\ 7\ 5\ 8 \\ \hline 4\ 6\ 1\ 8\ 4\ 5\ 5\ 3. \end{array}$$

8. Answer: (C)

If the two faces of a certain colour are adjacent (or non-adjacent), we abbreviate by saying that colour is adjacent (or non-adjacent). There is one unique colouring where all three colours are non-adjacent. If only one colour is non-adjacent, we get three distinct colourings. Suppose now that all three colours are adjacent. We can rotate the cube until the red colour is occupying the top and front face. There are now two possibilities for the blue faces: either they occupy the left and back faces or the left and bottom faces. (If the right face is blue, it is possible to rotate it to the left by swopping the top red with the front red.) So in total, we have $1+3+2$ unique colourings.

9. Answer: (D)

Since $\triangle ABC$ is an isosceles triangle and D is the midpoint of BC, AD is perpendicular to BC and bisects $\angle BAC$. Thus $\angle BAD = 60°$. Let $AE = x$. As $\cos\angle BAD = \frac{AE}{AD}$, we have $AD = 2AE = 2x$. We have $\tan\angle BAD = \frac{BD}{AD}$, so $BD = AD\tan 60° = 2\sqrt{3}x$. It follows that $AE : BD = x : 2\sqrt{3}x = 1 : 2\sqrt{3}$.

10. Answer: (E)

Note that $(2x_1+1)+(2x_2+1)+(2x_3+1)+(2x_4+1) = 22$ is equivalent to $x_1+x_2+x_3+x_4 = 9$ where $x_i \geq 0$. There are exactly 18 ways to do this, namely
$9+0+0+0 = 8+1+0+0 = 7+2+0+0 = 7+1+1+0 = 6+3+0+0 = 6+2+1+0$
$= 6+1+1+1 = 5+4+0+0 = 5+3+1+0 = 5+2+2+0 = 5+2+1+1 = 4+4+1+0$
$= 4+3+2+0 = 4+3+1+1 = 4+2+2+1 = 3+3+3+0 = 3+3+2+1 = 3+2+2+2.$

Short Questions

11. Answer: 28

If Y is the original price, the new price is $0.9 \times 0.8Y = 0.72Y$. So the discount is 28%.

12. Answer: 21

Let A be the base area and the volume of water is $6A$. On the other hand, there is $2A$ volume of empty space. Hence $8A = 28$ and so $6A = 21$ cm^3.

13. Answer: 60

$$\begin{aligned}
&\frac{x-7}{x-8}-\frac{x-8}{x-9}=\frac{x-10}{x-11}-\frac{x-11}{x-12}\\
\iff\ & 1+\frac{1}{x-8}-1-\frac{1}{x-9}=1+\frac{1}{x-11}-1-\frac{1}{x-12}\\
\iff\ & (x-11)(x-12)=(x-8)(x-9)\\
\iff\ & 6x=60.
\end{aligned}$$

14. Answer: 935

Since 1 and N are the smallest and largest positive divisors of N, we know that 5 must be the second smallest positive divisor. Write $N = 5k$, then we have $1122 = 5k + k$ and so $k = 187$. Thus $N = 935$.

15. Answer: 16

Using the quadratic formula, the absolute difference of the two roots,

$$D=\frac{2\sqrt{10^2+4(3)(201)}}{6}=\frac{\sqrt{2512}}{3}.$$

Since $50 < \sqrt{2512} < 51$, we have $16 < D < 17$. So $\lfloor D \rfloor = 16$.

16. Answer: 2017

The equation can be written as

$$(\sqrt{m}+2\sqrt{n})^2-2(\sqrt{m}+2\sqrt{n})-3=0.$$

Let $x=\sqrt{m}+2\sqrt{n}>0$, then $x^2-2x-3=(x-3)(x+1)=0$ which gives us $x=3$. The required value is $\dfrac{x+2014}{4-x}=2017$.

17. Answer: 8

Let $BF = x$ and $CD = y$, then the Pythagoras theorem on $\triangle CDF$ gives us

$$(4-y)^2=y^2+(4-x)^2 \implies x(8-x)=8y.$$

Since $\triangle BFE$ is similar to $\triangle CDF$, we have

$$\begin{aligned}
\text{perimeter of } \triangle BFE &= \frac{x}{y}\times \text{perimeter of } \triangle CDF\\
&=\frac{x}{y}(4-x+y+4-y)\\
&=\frac{x(8-x)}{y}=8.
\end{aligned}$$

18. Answer: 491

We have $pqr = 19(p+q+r)$. Since p, q, r are prime numbers, one of them has to be 19. We may let $r = 19$. Then the equation reduces to $pq = p+q+19$, which gives $(p-1)(q-1) = 20$. This equation only holds for $p = 3, q = 11$ or $p = 11, q = 3$ since p and q are prime numbers. In either case, we have

$$p^2 + q^2 + r^2 = 3^2 + 11^2 + 19^2 = 491.$$

19. Answer: 52

Let x be the number of marbles. Then John would have $y = \frac{4}{5}(x-7) + 14$ marbles finally. We know that $y = 6m + 2 = 5n$ for positive integers m and n. Clearly $m = 3$ and $n = 4$ gives a feasible solution and the general solution is $y = 20 + 30k$ for some nonnegative integer k. Hence

$$20 + 30k = \frac{4}{5}(x-7) + 14 \implies x = 7 + \frac{5}{4}(6 + 30k).$$

The smallest integer $x = 52$ when $k = 1$.

20. Answer: 4008

Let $N = \overline{abcd}$, then

$$1000a + 100b + 10c + d + (a+b+c+d) = 1001a + 101b + 11c + 2d = 2019.$$

Now a can only be 1 or 2. If $a = 1$, we have $101b + 11c + 2d = 1018$. As the maximum value of $11c + 2d = 99 + 18 = 117$, we must have $101b \geq 901$, i.e. $b = 9$ which forces $c = 9$ and $d = 5$. So 1995 is a possible value.

If $a = 2$, we have $101b + 11c + 2d = 17$ which means $b = 0$, $c = 1$ and $d = 3$. Thus 2013 is the other possible value. Thus the sum of all possible values is $1995 + 2013 = 4008$.

21. Answer: 10

The fourth circle must contain the smallest number. We must then choose three out of the remaining five numbers to fill up the first three circles. These have to be entered in decreasing order. The remaining two numbers then go into the fifth and sixth circle in increasing order. The total number of possible ways is then $\binom{5}{3} = 10$. (Note that the actual numbers do not matter as long as they are distinct.)

22. Answer: 24

Let $BE = CF = x$, then $CE = DF = 8 - x$. Then the area of $\triangle AEF$ is given by

$$64 - \frac{x(8-x)}{2} - \frac{8(8-x)}{2} - \frac{8x}{2} = 32 - 4x - \frac{x^2}{2} = \frac{1}{2}(x-4)^2 + 24.$$

The minimum area occurs at $x = 4$ and is 24 cm^2.

23. Answer: 2

The difference of the two equations give $a+b-c=0$ and so

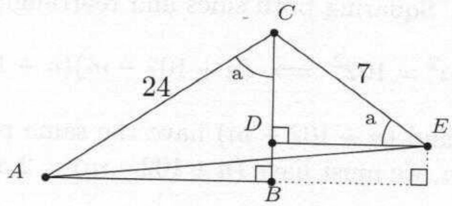
$$0=(a+b-c)^2=a^2+b^2+c^2-2(ac+bc-ab).$$

Thus $\frac{a^2+b^2+c^2}{ac+bc-ab}=2$.

24. Answer: 25

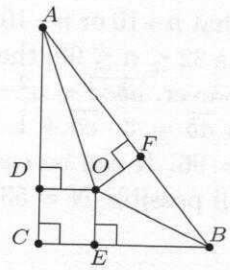
We have $AB=24\sin a$, $DE=7\cos a$ and $BD=24\cos a-7\sin a$. By Pythagoras theorem,

$$\begin{aligned} AE^2 &= (24\sin a+7\cos a)^2+(24\cos a-7\sin a)^2 \\ &= 24^2(\sin^2 a+\cos^2 a)+7^2(\sin^2 a+\cos^2 a)=25^2. \end{aligned}$$



25. Answer: 22

Let $\triangle ABC$ be the right-angled triangle, where $\angle C$ is the right angle. Let O be the center of the inscribed circle, and D, E and F be the foot of the perpendiculars from O to the three sides. (See Figure.) Thus $OD=OE=OF=1$, and $\triangle ADO\cong\triangle AFO$ and so $AD=AF$. Similarly, we have $BE=BF$. Since $AF+BF=AB=10$, we have $AD+BE=10$. Hence the perimeter of the triangle is $10+10+2=22$.



26. Answer: 0

We have $x^2+\frac{1}{x^2}=(x+\frac{1}{x})^2-2=1$. Hence

$$x^3+\frac{1}{x^3}=\left(x^2+\frac{1}{x^2}\right)\left(x+\frac{1}{x}\right)-\left(x+\frac{1}{x}\right)=0.$$

27. Answer: 2

We have

$$f(x) = \begin{cases} x-2-2(3-x)+(4-x), & 2 \le x \le 3; \\ x-2-2(x-3)+(4-x), & 3 < x \le 4; \\ x-2-2(x-3)+(x-4), & 4 < x \le 8; \end{cases}$$
$$= \begin{cases} 2x-4, & 2 \le x \le 3; \\ -2x+8, & 3 < x \le 4; \\ 0, & 4 < x \le 8. \end{cases}$$

The largest and smallest values of $f(x)$ are 2 and 0 respectively.

28. Answer: 2500

Let $m = \sqrt{n^2+204n}$. Squaring both sides and rearranging, we get

$$(n+102)^2 - m^2 = 102^2 \implies (n+102-m)(n+102+m) = 2^2 \times 3^2 \times 17^2.$$

Since $(n+102-m)$ and $(n+102+m)$ have the same parity, we conclude that both are even. For maximum n, we must have $(n+102-m) = 2$ and $(n+102+m) = 2 \times 3^2 \times 17^2$. So

$$n+102+n+100 = 5202 \implies n = 2500.$$

29. Answer: 2809

We have $N = \overline{abcd} = n^2$, where $32 \le n \le 99$. Let $x = \overline{cd}$. Then $\overline{ab} = 3x+1$, and we have $100(3x+1)+x = n^2$, which gives $301x = n^2-100$. That is,

$$43 \cdot 7 \cdot x = (n+10)(n-10).$$

Since 43 is prime, this implies that $n+10$ or $n-10$ is a multiple of 43. Suppose $n+10 = 43k$, where k is a positive integer. As $32 \le n \le 99$, the equation leads to $n+10 = 43$ or $n+10 = 86$, i.e. $n = 33$ or $n = 76$. However, $\overline{abcd} = n^2 = 33^2 = 1089$ or $\overline{abcd} = n^2 = 76^2 = 5776$ does not satisfy the condition $\overline{ab} = 3 \cdot \overline{cd} + 1$. If $n-10 = 43k$, then $n-10 = 43$ or $n-10 = 86$, i.e. $n = 53$ or $n = 96$. It can be easily verified that only $n = 53$ satisfies the condition. Hence the sum of all possible $N = 53^2 = 2809$.

30. Answer: 203

The sum is equal to

$$\frac{1}{2} + \left(\frac{1}{3}+\frac{2}{3}\right) + \left(\frac{1}{4}+\frac{2}{4}+\frac{3}{4}\right) + \left(\frac{1}{5}+\frac{2}{5}+\frac{3}{5}+\frac{4}{5}\right) + \cdots + \left(\frac{1}{29}+\frac{2}{29}+\frac{3}{29}+\cdots+\frac{28}{29}\right),$$

which equals $\displaystyle\sum_{n=1}^{28} \frac{1+\cdots+n}{n+1} = \sum_{n=1}^{28} \frac{n(n+1)}{2(n+1)} = \frac{28 \times 29}{4} = 203.$

31. Answer: 5956

Note that $(ax^2+by^2)(x+y) = ax^3+by^3+xy(ax+by)$. Thus by the hypotheses, we obtain

$$49(x+y) = 133 + 7xy. \tag{1}$$

Similarly, from the identity $(ax^3+by^3)(x+y) = ax^4+by^4+xy(ax^2+by^2)$, we obtain

$$133(x+y) = 406 + 49xy. \tag{2}$$

Solving simultaneous equations (1) and (2), we obtain $x+y = 2.5$ and $xy = -1.5$. Now the identity $(ax+by)(x+y) = ax^2+by^2+xy(a+b)$ gives

$$7 \times 2.5 = 49 - 1.5(a+b).$$

It follows that $a+b = 21$. Hence $2014(x+y-xy) - 100(a+b) = 5956$.

32. Answer: 1

Let $b = \sqrt{\frac{8a-1}{3}}$, then $a = \frac{3b^2+1}{8}$. Thus

$$a + \frac{a+1}{3}\sqrt{\frac{8a-1}{3}} = \frac{3b^2+1}{8} + \frac{\frac{3b^2+1}{8}+1}{3}b = \frac{b^3+3b^2+3b+1}{8}.$$

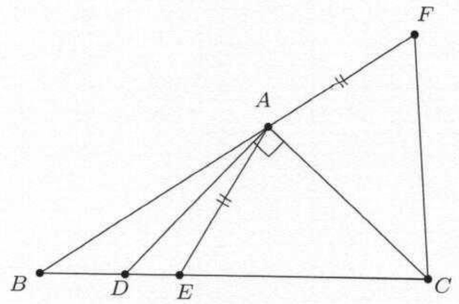
Similarly,

$$a - \frac{a+1}{3}\sqrt{\frac{8a-1}{3}} = \frac{3b^2+1}{8} - \frac{\frac{3b^2+1}{8}+1}{3}b = \frac{-b^3+3b^2-3b+1}{8}.$$

Hence $g(a) = \dfrac{b+1}{2} + \dfrac{-b+1}{2} = 1$ for all $a \geq \dfrac{1}{8}$.

33. Answer: 44

Let F be a point on BA extended such that $AF = AE$.



Then $BF = BA + AF = AB + AE = BC$, so $\triangle BCF$ is isosceles. We have

$$\angle FAC = 90^\circ - \angle BAD = 90^\circ - \angle DAE = \angle EAC.$$

Thus it follows from SAS postulate that $\triangle CAE$ is congruent to $\triangle CAF$. Consequently, we have $\angle ACF = \angle ACE$. Let $\alpha = \angle ACF = \angle ACE$. Since $\triangle BCF$ is an isosceles triangle, we have $\angle BFC = \angle BCF = 2\alpha$. Therefore $\angle ABC = 180^\circ - 4\alpha$. Note that

$$\angle ADC = 90^\circ - \alpha = \angle BAD + \angle ABC = 12^\circ + 180^\circ - 4\alpha.$$

Solving the above equation gives $\alpha = 34^\circ$. Hence $\angle ABC = 180^\circ - 4 \times 34^\circ = 44^\circ$.

34. Answer: 144

Since $\frac{9}{17} < \frac{n}{n+k} < \frac{8}{15}$, we have $\frac{15}{8} < \frac{n+k}{n} < \frac{17}{9}$, which gives $\frac{7}{8} < \frac{k}{n} < \frac{8}{9}$. Thus the given inequalities are equivalent to

$$\frac{7n}{8} < k < \frac{8n}{9}.$$

Since for all pairs (n, k) that satisfy the inequalities, there is only one k for each n, it follows that

$$\frac{8n}{9} - \frac{7n}{8} \leq 2,$$

that is, $\frac{n}{72} \leq 2$. Consequently, we have $n \leq 144$. When $n = 144$, $126 < k < 128$, so $k = 127$. Hence the largest possible value of n is 144.

35. Answer: 4

If we write a 9 digit number m as $\overline{a_1 a_2 \dots a_9}$, then

$$m \equiv a_1 + a_2 + \cdots + a_9 \pmod{9}.$$

We can check that

$$2^{29} = 2^{3\times 9+2} \equiv (-1)^9 \times 4 \equiv -4 \pmod{9}.$$

The missing digit must be 4 since $0 + 1 + 2 + \cdots + 9 \equiv 0 \pmod{9}$. Note that $2^{29} = 536870912$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2014

(Junior Section, Second Round)

Saturday, 28 June 2014 **0930-1230**

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1. Consider the integers formed using the digits $0, 1, 2, 3, 4, 5, 6$, without repetition. Find the largest multiple of 55. Justify your answer.

2. Let a be a positive integer such that the last two digits of a^2 are both non-zero. When the last two digits of a^2 are deleted, the resulting number is still a nonzero perfect square. Find, with justification, all possible values of a.

3. In the triangle ABC, the bisector of $\angle A$ intersects the bisection of $\angle B$ at the point I; D is the foot of the perpendicular from I onto BC. Prove that the bisector of $\angle BIC$ is perpendicular to the bisector of $\angle AID$.

4. Find, with justification, all positive real numbers a, b, c satisfying the system of equations:

$$a\sqrt{b} = a + c, \quad b\sqrt{c} = b + a, \quad c\sqrt{a} = c + b.$$

5. In an 8×8 grid, n disks, numbered 1 to n are stacked, with random order, in a pile in the bottom left corner. The disks can be moved one at a time to a neighbouring cell either to the right or above. The aim is to move all the disks to the cell at the top right corner and stack them in the order $1, 2, \ldots, n$ from the bottom. Each cell, except the bottom left and top right cell, can have at most one disk at any given time. Find the largest value of n so that the aim can be achieved.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2014

(Junior Section, Second Round Solutions)

1. Let A be the sum of the 4 digits that occupy the odd positions and B the sum of the other 3 digits. Then $A+B=21$ and, since the integer is divisible by 11, $|A-B|=11k$ where k is an integer. Since $A+B$ is odd, $A-B\neq 0$. Also since A,B are positive integers $\leq 3+4+5+6=18$, $|A-B|<18$. Thus $k=1$. Therefore A and B are $(21+11)/2=16$ and $(21-11)/2=5$. Since $A\geq 0+1+2+3=6$, we have $A=16$ and $B=5$. There are only two triples whose sum is 5, namely $(0,2,3)$ and $(0,1,4)$. Thus we have two cases:

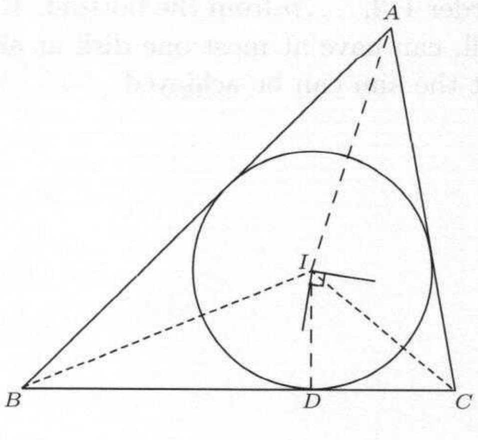
$$A=2+3+5+6, B=0+1+4; \quad \text{and} \quad A=1+4+5+6, B=0+2+3.$$

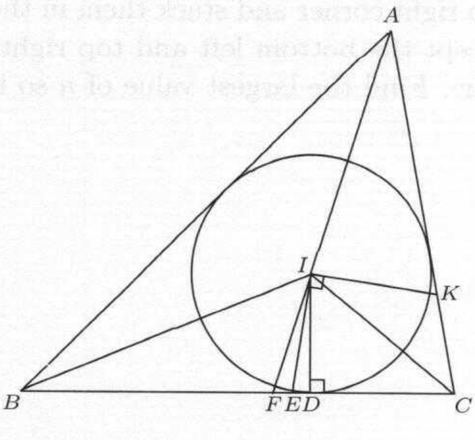
So the largest multiple of 55 is 6431205.

2. Let the number formed by the last two digits of a^2 be b and c^2 be the number obtained after the last two digits of a^2 are deleted. Then $a^2=100c^2+b$. Let $a+10c=m$ and $a-10c=n$. Then m and n are positive integers and $a=(m+n)/2$, $c=(m-n)/20$ and $b=mn$. Since b is a two digit number, we have $99\geq m>n\geq 1$. Also $m-n$ is a multiple of 20. Thus $c=1,2,3,4$. Also $b=n(20c+n)$. When $c=1$, $b=n(20+n)\leq 99$. Thus $n=1,2,3,4$. The corresponding values of m, b and a are: $m=21,22,23,24$, $b=21,44,69,96$, and $a=11,12,13,14$.

The cases $c=2,3,4$ can be considered in the same way. The answers are 11, 12, 13, 14, 21, 22, 31, 41.

3.





Note that IC bisects $\angle C$. Let the bisector of $\angle BIC$ intersect BC at E, and the bisector of $\angle AID$ intersect AC at K. Extend AI meeting BC at F. We have $\angle BIF = \frac{1}{2}(\angle A + \angle B) = 90^\circ - \frac{1}{2}\angle C = \angle CID$. Thus IE bisects $\angle FID$. Therefore IE is perpendicular to IK since

$$\angle EIK = \frac{1}{2}(\angle FID + \angle AID) = 90^\circ.$$

4. Without loss of generality, we may suppose $0 < c \le a$ and $0 < b \le a$, i.e. a is the largest number among a, b, c. The first equation gives $a(\sqrt{b} - 1) = c \le a$ so that $b \le 4$. From the second equation we have $b \le a = b(\sqrt{c} - 1)$ so that $c \ge 4$. Thus $b \le 4 \le c \le a$. From the third equation, we have $c(\sqrt{a} - 1) = b \le c$ so that $a \le 4$. Consequently, the only solution is $a = b = c = 4$.

5. Answer is $n = 50$. If there are 51 disks, for disk 51 to end up as the bottom disk in the final destination, before it is moved, the other disks must already occupy 50 cells other than the final destination. Then there is no clear path to move disk 51 to the final destination as a clear path would require 15 cells, include the start and end leaving a total of 49 cells for the first 50 disks.

We shall prove that for $n = 50$, it can be done. Label the columns and rows from 1 to 8 starting bottom left.

The idea is to keep column 8 and row 1 clear and move disks $2, 3, \ldots, 8$ to column 7, disks $9, \ldots, 15$ to column 6, disks $16, \ldots, 22$ to column 5, disk $23, \ldots, 29$ to column 4, disks $30, \ldots, 36$ to column 3, disks $37, \ldots, 43$ to column 2 and disks $44, \ldots, 50$ to column 1. Disk 1 is move directly to the destination cell.

This can certainly be done by moving the disk along row 1 until it reaches the appropriate column and then move upwards along the column until it hits an occupied cell.

When this is achieved, we can then move $2, 3, \ldots, 50$ in succession to column 8 and then upwards to the destination.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2014

Senior Section (First Round)

Tuesday, 3 June 2014 **0930 – 1200 hrs**

Instructions to contestants

1. *Answer ALL 35 questions.*
2. *Enter your answers on the answer sheet provided.*
3. *For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.*
4. *For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.*
5. *No steps are needed to justify your answers.*
6. *Each question carries 1 mark.*
7. *No calculators are allowed.*
8. *Throughout this paper, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. For example, $\lfloor 2.1 \rfloor = 2$, $\lfloor 3.9 \rfloor = 3$.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO.

Multiple Choice Questions

1. If α, β are the roots of the equation $3x^2 + x - 1 = 0$, where $\alpha > \beta$, find the value of $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$.

   (A) $\frac{7}{9}$ (B) $-\frac{7}{9}$ (C) $\frac{7}{3}$ (D) $-\frac{7}{3}$ (E) $-\frac{1}{9}$

2. Find the value of

$$\frac{2014^3 - 2013^3 - 1}{2013 \times 2014}.$$

   (A) 3 (B) 5 (C) 7 (D) 9 (E) 11

3. Find the value of

$$\frac{\log_5 9 \log_7 5 \log_3 7}{\log_2 \sqrt{6}} + \frac{1}{\log_9 \sqrt{6}}.$$

   (A) 2 (B) 3 (C) 4 (D) 6 (E) 7

4. Find the smallest number among the following numbers:

   (A) $\sqrt{55} - \sqrt{52}$ (B) $\sqrt{56} - \sqrt{53}$ (C) $\sqrt{77} - \sqrt{74}$
   (D) $\sqrt{88} - \sqrt{85}$ (E) $\sqrt{70} - \sqrt{67}$

5. Find the largest number among the following numbers:

   (A) 30^{30} (B) 50^{10} (C) 40^{20} (D) 45^{15} (E) 5^{60}

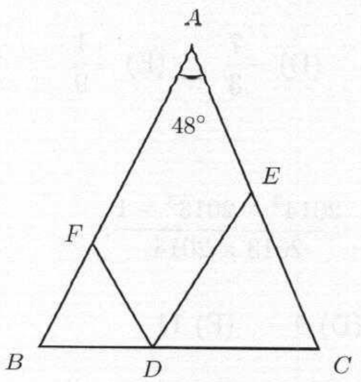
6. Given that $\tan A = \dfrac{12}{5}$, $\cos B = -\dfrac{3}{5}$ and that A and B are in the same quadrant, find the value of $\cos(A - B)$.

   (A) $-\frac{63}{65}$ (B) $-\frac{64}{65}$ (C) $\frac{63}{65}$ (D) $\frac{64}{65}$ (E) $\frac{65}{63}$

7. Find the largest number among the following numbers:

   (A) $\tan 47^\circ + \cos 47^\circ$ (B) $\cot 47^\circ + \sqrt{2}\sin 47^\circ$ (C) $\sqrt{2}\cos 47^\circ + \sin 47^\circ$
   (D) $\tan 47^\circ + \cot 47^\circ$ (E) $\cos 47^\circ + \sqrt{2}\sin 47^\circ$

8. In the diagram below, $\triangle ABC$ is a triangle and D, E, F are points on BC, CA, AB respectively. It is given that $BF = BD$, $CD = CE$ and $\angle BAC = 48°$. Find the angle $\angle EDF$.



(A) 64° (B) 66° (C) 68° (D) 70° (E) 72°

9. Find the number of real numbers which satisfy the equation
$$x|x-1| - 4|x| + 3 = 0.$$

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

10. If $f(x) = \dfrac{1}{x} - \dfrac{4}{\sqrt{x}} + 3$ where $\dfrac{1}{16} \leq x \leq 1$, find the range of $f(x)$.

(A) $-2 \leq f(x) \leq 4$ (B) $-1 \leq f(x) \leq 3$ (C) $0 \leq f(x) \leq 3$
(D) $-1 \leq f(x) \leq 4$ (E) None of the above

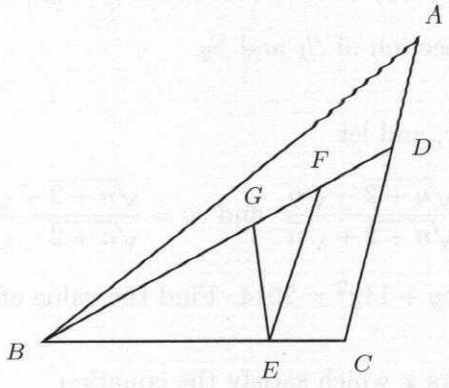
Short Questions

11. Suppose x is real number such that $\dfrac{27 \cdot 9^x}{4^x} = \dfrac{3^x}{8^x}$. Find the value of $2^{-(1+\log_2 3)x}$.

12. Evaluate $50\left(\cos 39° \cos 21° + \cos 129° \cos 69°\right)$.

13. Suppose a and b are real numbers such that the polynomial $x^3 + ax^2 + bx + 15$ has a factor of $x^2 - 2$. Find the value of a^2b^2.

14. In the triangle $\triangle ABC$ below, $AC = 3AD$, $BC = 4EC$, $BD = 5GF = 5FD$. Suppose the area of $\triangle ABC$ is 900 meter^2. Find the area of the triangle $\triangle EFG$ in meter^2.



15. Let x, y be real numbers such that $y = |x-1|$. What is the smallest value of $(x-1)^2 + (y-2)^2$?

16. Evaluate the sum

$$\frac{3!+4!}{2(1!+2!)} + \frac{4!+5!}{3(2!+3!)} + \cdots + \frac{12!+13!}{11(10!+11!)}$$

17. Let n be a positive integer such that $12n^2 + 12n + 11$ is a 4-digit number with all 4 digits equal. Determine the value of n.

18. Given that in the expansion of $(2+3x)^n$, the coefficients of x^3 and x^4 are in the ratio $8:15$. Find the value of n.

19. In a triangle $\triangle ABC$, it is given that

$$(\sin A + \sin B) : (\sin B + \sin C) : (\sin C + \sin A) = 9 : 10 : 11.$$

Find the value of $480 \cos A$.

20. Let $x = \sqrt{37 - 20\sqrt{3}}$. Find the value of

$$\frac{x^4 - 9x^3 + 5x^2 - 7x + 68}{x^2 - 10x + 19}.$$

21. Let n be an integer, and let $\triangle ABC$ be a right-angled triangle with right angle at C. It is given that $\sin A$ and $\sin B$ are the roots of the quadratic equation

$$(5n+8)x^2 - (7n-20)x + 120 = 0.$$

Find the value of n.

22. Let S_1 and S_2 be sets of points on the coordinate plane $\mathbb{R}^2$ defined as follows:

$$S_1 = \{(x, y) \in \mathbb{R}^2 \ : \ |x + |x|| + |y + |y|| \leq 2\},$$

$$S_2 = \{(x, y) \in \mathbb{R}^2 \ : \ |x - |x|| + |y - |y|| \leq 2\}.$$

Find the area of the intersection of S_1 and S_2.

23. Let n be a positive integer, and let

$$x = \frac{\sqrt{n+2} - \sqrt{n}}{\sqrt{n+2} + \sqrt{n}} \quad \text{and} \quad y = \frac{\sqrt{n+2} + \sqrt{n}}{\sqrt{n+2} - \sqrt{n}}.$$

It is given that $14x^2 + 26xy + 14y^2 = 2014$. Find the value of n.

24. Find the number of integers x which satisfy the equation

$$(x^2 - 5x + 5)^{x+5} = 1.$$

25. Find the number of ordered pairs of integers (p, q) satisfying the equation

$$p^2 - q^2 + p + q = 2014.$$

26. Suppose x is measured in radians. Find the maximum value of

$$\frac{\sin 2x + \sin 4x + \sin 6x}{\cos 2x + \cos 4x + \cos 6x}$$

for $0 \leq x \leq \dfrac{\pi}{16}$.

27. Determine the number of ways of colouring a 10×10 square board using two colours black and white such that each 2×2 subsquare contains 2 black squares and 2 white squares.

28. In the isosceles triangle ABC with $AB = AC$, D and E are points on AB and AC respectively such that $AD = CE$, and $DE = BC$. Suppose $\angle AED = 18^\circ$. Find the size of $\angle BDE$ in degree.

29. Find the number of ordered triples of real numbers (x, y, z) that satisfy the following system of equations:

$$\begin{aligned} x^2 &= 4y - 4, \\ y^2 &= 4z - 4, \\ z^2 &= 4x - 4. \end{aligned}$$

30. Let $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $A = \{1, 2, 3, 4\}$. Find the number of 4-element subsets Y of X such that $10 \in Y$ and the intersection of Y and A is not empty.

31. Find the number of ways that 7 different guests can be seated at a round table with exactly 10 seats, without removing any empty seats. Here, two seatings are considered to be the same if they can be obtained from each other by a rotation.

32. Determine the maximum value of

$$\frac{8(x+y)(x^3+y^3)}{(x^2+y^2)^2}$$

for all $(x,y) \neq (0,0)$.

33. Find the value of

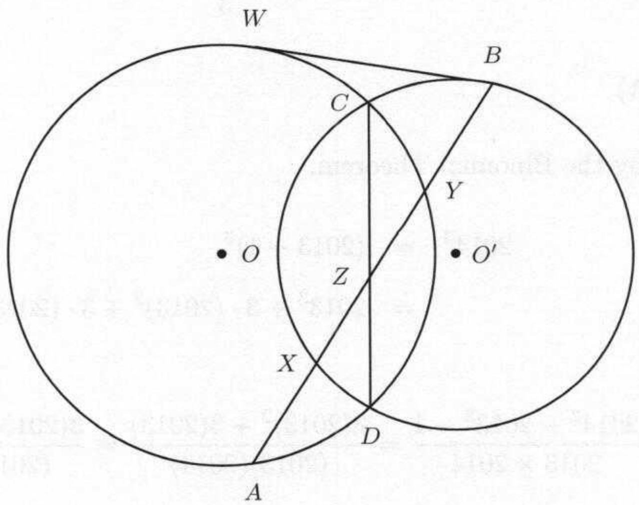
$$2\left(\sin 2^\circ \tan 1^\circ + \sin 4^\circ \tan 1^\circ + \sin 6^\circ \tan 1^\circ + \cdots + \sin 178^\circ \tan 1^\circ\right).$$

34. Let $x_1, x_2, \ldots, x_{100}$ be real numbers such that

$$|x_1| = 63, \quad \text{and} \quad |x_{n+1}| = |x_n + 1| \quad \text{for } n = 1, 2, \ldots, 99.$$

Find the largest possible value of $(-x_1 - x_2 - \cdots - x_{100})$.

35. As shown in the figure below, two circles ω, ω', with centers O and O' respectively, intersect at the points C and D. The straight lines CD and $BYXA$ intersect at the point Z. Moreover, the straight line WB is tangent to both of the circles. Suppose $ZX = ZY$ and $AB \cdot AX = 100$. Find the value of BW.



Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2014

Senior Section (First Round Solutions)

Multiple Choice Questions

1. **Answer.** (D).

   Solution. Note that $\alpha + \beta = -\frac{1}{3}$ and $\alpha\beta = -\frac{1}{3}$. Thus

$$
\begin{aligned}
\frac{\alpha}{\beta} + \frac{\beta}{\alpha} &= \frac{\alpha^2+\beta^2}{\alpha\beta} \\
&= \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} \\
&= \frac{(-1/3)^2 - 2(-1/3)}{-1/3} \\
&= -\frac{7}{3}
\end{aligned}
$$

2. **Answer.** (A)

   Solution. By the Binomial Theorem,

$$
\begin{aligned}
2014^3 &= (2013+1)^3 \\
&= 2013^3 + 3\cdot(2013)^2 + 3\cdot(2013) + 1.
\end{aligned}
$$

   Thus,

$$
\frac{2014^3 - 2013^3 - 1}{2013 \times 2014} = \frac{3(2013)^2 + 3(2013)}{(2013)(2014)} = \frac{3(2013)(2013+1)}{(2013)(2014)} = 3.
$$

3. **Answer.** (C)

Solution.

$$\begin{aligned}
\frac{\log_5 9 \log_7 5 \log_3 7}{\log_2 \sqrt{6}} + \frac{1}{\log_9 \sqrt{6}} &= 2\frac{\log_5 3 \log_7 5 \log_3 7}{\log_2 \sqrt{6}} + \frac{1}{\log_9 \sqrt{6}} \\
&= 2\frac{\frac{\log 3}{\log 5}\frac{\log 5}{\log 7}\frac{\log 7}{\log 3}}{\log_2 \sqrt{6}} + \frac{1}{\log_9 \sqrt{6}} \\
&= 2\log_{\sqrt{6}} 2 + \log_{\sqrt{6}} 9 \\
&= \log_{\sqrt{6}}(2^2 \times 9) \\
&= \log_{\sqrt{6}} \sqrt{6}^4 \\
&= 4.
\end{aligned}$$

4. **Answer.** (D)

Solution. Each of the number has the form $f(x) = \sqrt{x+3} - \sqrt{x}$ for some positive integer x. We will show that $f(x)$ decreases when x increases, i.e. if $m > n$ then $f(m) < f(n)$. Suppose $m > n$. Then

$$\begin{aligned}
2\sqrt{m(n+3)} &> 2\sqrt{n(m+3)} \\
m + (n+3) + 2\sqrt{m(n+3)} &> n + (m+3) + 2\sqrt{n(m+3)} \\
\left(\sqrt{m} + \sqrt{n+3}\right)^2 &> \left(\sqrt{n} + \sqrt{m+3}\right)^2 \\
\sqrt{n+3} - \sqrt{n} &> \sqrt{m+3} - \sqrt{m} \\
f(n) &> f(m).
\end{aligned}$$

Hence, the smallest number is $f(85) = \sqrt{88} - \sqrt{85}$.

5. **Answer.** (A)

Solution. Note that

$$\begin{aligned}
\text{(A)} &= 30^{30} = (2\cdot 3\cdot 5)^{30} = 2^{30}3^{30}5^{30} = 6^{30}5^{30} \\
\text{(B)} &= 50^{10} = (2\cdot 5^2)^{10} = 2^{10}5^{20} \\
\text{(C)} &= 40^{20} = (2^3\cdot 5)^{20} = 2^{60}5^{20} = 4^{30}5^{20} \\
\text{(D)} &= 45^{15} = (3^2\cdot 5)^{15} = 3^{30}5^{15} \\
\text{(E)} &= 5^{60} = 5^{30}5^{30}
\end{aligned}$$

By comparing the exponents, it is easy to see that 30^{30} is the largest number.

6. **Answer.** (C)

Solution. Since $\tan A > 0$ and $\cos B < 0$ and they are in the same quadrant, we deduce that A and B are in the third quadrant. Hence,

$$\sin A = -\frac{12}{13}, \quad \cos A = -\frac{5}{13}, \quad \sin B = -\frac{4}{5}, \quad \cos B = -\frac{3}{5}.$$

It follows that

$$\begin{aligned}\cos(A-B) &= \cos A \cos B + \sin A \sin B \\ &= \left(-\frac{5}{13}\right)\left(-\frac{3}{5}\right) + \left(-\frac{12}{13}\right)\left(-\frac{4}{5}\right) \\ &= \frac{63}{65}.\end{aligned}$$

7. **Answer.** (D)

Solution. First note that

$$(\mathrm{E}) - (\mathrm{C}) = (\sqrt{2}-1)(\sin 47^\circ - \cos 47^\circ) > 0.$$

Hence, we have (C) < (E). Also, we have

$$(\mathrm{B}) - (\mathrm{E}) = \cot 47^\circ - \cos 47^\circ = \frac{\cos 47^\circ}{\sin 47^\circ} - \cos 47^\circ > 0.$$

Hence, we have (E) < (B). Also,

$$\begin{aligned}(\mathrm{D}) - (\mathrm{B}) &= \tan 47^\circ - \sqrt{2}\sin 47^\circ \\ &= \sqrt{2}\tan 47^\circ \left(\frac{1}{\sqrt{2}} - \cos 47^\circ\right) \\ &= \sqrt{2}\tan 47^\circ (\cos 45^\circ - \cos 47^\circ) > 0.\end{aligned}$$

Hence, we have (B) < (D). Summarizing the above, we have

$$(\mathrm{C}) < (\mathrm{E}) < (\mathrm{B}) < (\mathrm{D}).$$

Finally, $(\mathrm{D}) - (\mathrm{A}) = \cot 47^\circ - \cos 47^\circ > 0$. Hence, (A) < (D). Therefore, (D) is the largest number.

8. **Answer.** (B)

Solution. Since $\angle BAC = 48^\circ$, we have

$$\angle ABC + \angle ACB = 180^\circ - 48^\circ = 132^\circ.$$

Since $BF = BD$, it follows that

$$\angle BDF = \frac{180^\circ - \angle ABC}{2}.$$

Similarly, since $CD = CE$, we have

$$\angle CDE = \frac{180^\circ - \angle ACB}{2}.$$

Now,

$$\begin{aligned}
\angle EDF &= 180^\circ - \angle BDF - \angle CDE \\
&= 180^\circ - \frac{180^\circ - \angle ABC}{2} - \frac{180^\circ - \angle ACB}{2} \\
&= \frac{1}{2}(\angle ABC + \angle ACB) \\
&= \frac{132^\circ}{2} = 66^\circ.
\end{aligned}$$

9. **Answer.** (D)

Solution. There are three cases to consider.

(i) When $x \geq 1$, the equation becomes $x(x-1) - 4x + 3 = 0$. So $x = \frac{5\pm\sqrt{13}}{2}$. Since $\frac{5+\sqrt{13}}{2} > 1$ and $\frac{5-\sqrt{13}}{2} < 1$, only $\frac{5+\sqrt{13}}{2}$ is an admissible solution when $x \geq 1$.

(ii) When $0 \leq x < 1$, the equation becomes $x(1-x) - 4x + 3 = 0$. So $x = \frac{-3\pm\sqrt{21}}{2}$. Since $\frac{-3-\sqrt{21}}{2} < 0$ and $0 < \frac{-3+\sqrt{21}}{2} < 1$, only $\frac{-3+\sqrt{21}}{2}$ is an admissible solution when $0 \leq x < 1$.

(iii) When $x < 0$, the equation becomes $x(1-x) + 4x + 3 = 0$. So $x = \frac{5\pm\sqrt{37}}{2}$. Since $\frac{5+\sqrt{37}}{2} > 0$ and $\frac{5-\sqrt{37}}{2} < 0$, only $\frac{5-\sqrt{37}}{2}$ is an admissible solution when $x < 0$.

Combining the three cases, there are exactly three real solutions.

10. **Answer.** (B)

Solution. Let $y = \frac{1}{\sqrt{x}}$. Then $f(x) = g(y) = y^2 - 4y + 3 = (y-1)(y-3)$ for $1 \leq y \leq 4$. The minimum value of $g(y)$ occurs at $y = 2$, and the maximum value of $g(y)$ occurs at $y = 4$. Thus

$$-1 = (2-1)(2-3) = g(2) \leq f(x) \leq g(4) = (4-1)(4-3) = 3.$$

Short Questions

11. **Answer.** 27

Solution.

$$
\begin{aligned}
\frac{27\cdot 9^x}{4^x} &= \frac{3^x}{8^x}\\
\frac{3^{3+2x}}{2^{2x}} &= \frac{3^x}{2^{3x}}\\
3^{3+x} &= 2^{-x}\\
(3+x)\log_2 3 &= -x\\
3\log_2 3 + x\log_2 3 &= -x\\
3\log_2 3 &= -(1+\log_2 3)x\\
\log_2 27 &= -(1+\log_2 3)x\\
27 &= 2^{-(1+\log_2 3)x}
\end{aligned}
$$

12. **Answer.** 25

Solution.

$$
\begin{aligned}
50\,(\cos 39^\circ\cos 21^\circ + \cos 129^\circ\cos 69^\circ) &= 50\,(\cos 39^\circ\cos 21^\circ + (-\cos 51^\circ)\sin 21^\circ)\\
&= 50\,(\cos 39^\circ\cos 21^\circ - \sin 39^\circ\sin 21^\circ)\\
&= 50\,(\cos(39^\circ + 21^\circ))\\
&= 25.
\end{aligned}
$$

13. **Answer.** 225

Solution. Note that

$$x^3 + ax^2 + bx + 15 = (x^2-2)(x+a) + (b+2)x + (15+2a).$$

Since x^2-2 is a factor, $b=-2$ and $a=-15/2$. So

$$a^2b^2 = 15^2 = 225.$$

14. **Answer.** 90

Solution. Since $CD = \frac{2}{3}AC$, we have Area$(\triangle BCD) = \frac{2}{3}$Area$(\triangle ABC)$. Since $BE = \frac{3}{4}BC$, we have Area$(\triangle BED) = \frac{3}{4}$Area$(\triangle BCD)$. Since $GF = \frac{1}{5}BD$, we have Area$(\triangle EFG) =$

$\frac{1}{5}$Area($\triangle BED$). Hence,

$$\begin{aligned}
\text{Area}(\triangle EFG) &= \frac{1}{5}\text{Area}(\triangle BED) \\
&= \frac{1}{5}\cdot\frac{3}{4}\text{Area}(\triangle BCD) \\
&= \frac{1}{5}\cdot\frac{3}{4}\cdot\frac{2}{3}\text{Area}(\triangle ABC) \\
&= \frac{1}{5}\cdot\frac{3}{4}\cdot\frac{2}{3}\cdot 900 \\
&= 90.
\end{aligned}$$

15. **Answer.** 2

Solution. Let $P(x,y)$ be a point on the curve $y = |x-1|$. The smallest value s of $(x-1)^2+(y-2)^2$ is the square of the smallest distance d between $P(x,y)$ and the point $Q(1,2)$. This happens precisely when the line PQ is perpendicular to $y = x-1$ or $y = -(x-1)$, that is either $(x,y) = (0,1)$ or $(x,y) = (2,1)$. Therefore,

$$s = d^2 = (0-1)^2 + (1-2)^2 = (2-1)^2 + (1-2)^2 = 2.$$

16. **Answer.** 95

Solution. Let $S = \dfrac{3!+4!}{2(1!+2!)} + \cdots + \dfrac{12!+13!}{11(10!+11!)}$. Note that

$$\frac{(n+2)!+(n+3)!}{(n+1)(n!+(n+1)!)} = \frac{(n+2)!(n+4)}{(n+1)n!(n+2)} = n+4.$$

Therefore,

$$S = \sum_{n=1}^{10}(n+4) = 95.$$

17. **Answer.** 21

Solution. It is required that $12n^2+12n+11 = \overline{mmmm}$, where $\overline{mmmm}$ denotes the decimal representation of the 4 digit number $12n^2+12n+11$. It follows that the first two digits of $12n^2+12n$ must be equal to $m-1$. Since $12n^2+12n$ is even, $m-1$ has to be even. Thus the possibilities for $12n^2+12n$ are $1100, 3322, 5544, 7766$ and 9988. Note that $12n^2+12n$ is divisible by 3. The only number among $1100, 3322, 5544, 7766, 9988$ which is divisible by 3 is 5544. Therefore, $n^2+n = 5544/12 = 462$. Solving for n, we obtain $n = 21$ or -22. Since n is positive, we have $n = 21$.

18. **Answer.** 8

Solution. By the Binomial Theorem, the terms containing x^3 and x^4 are

$$\binom{n}{3}(3x)^3 2^{n-3}, \quad \binom{n}{4}(3x)^4 2^{n-4},$$

respectively. Therefore,

$$\begin{aligned}
\frac{\binom{n}{3}3^3 2^{n-3}}{\binom{n}{4}3^4 2^{n-4}} &= \frac{8}{15} \\
\frac{\frac{n(n-1)(n-2)}{6}\cdot 2}{\frac{n(n-1)(n-2)(n-3)}{24}\cdot 3} &= \frac{8}{15} \\
\frac{8}{3(n-3)} &= \frac{8}{15} \\
n-3 &= 5 \\
n &= 8.
\end{aligned}$$

19. **Answer.** 270.

Solution. By the Sine Rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

where a, b, c are the lengths of the sides of the triangle, and A, B, C are the corresponding opposite angles. It follows that we can write

$$\sin A + \sin B = \frac{a+b}{b}\sin B,$$

$$\sin B + \sin C = \frac{b+c}{c}\sin C,$$

$$\sin C + \sin A = \frac{c+a}{a}\sin A.$$

Thus, the given equation can be written as

$$(a+b) : (b+c) : (c+a) = 9 : 10 : 11.$$

Using the identity $c = (a+b+c) - (a+b)$ etc., one sees that

$$c : a : b = (15-9) : (15-10) : (15-11) = 6 : 5 : 4.$$

By the Cosine Rule,

$$\begin{aligned}
\cos A &= \frac{b^2+c^2-a^2}{2bc} \\
&= \frac{4^2+6^2-5^2}{2(4)(6)} = \frac{9}{16}.
\end{aligned}$$

Hence, we have

$$480\cos A = 270.$$

20. **Answer.** 7

Solution. Let $N = x^4 - 9x^3 + 5x^2 - 7x + 68$ and $D = x^2 - 10x + 19$.

Observe that $x = \sqrt{37 - 20\sqrt{3}} = 5 - 2\sqrt{3}$. This can be seen by solving the equation $(a + b\sqrt{3})^2 = 37 - 20\sqrt{3}$ with the unknowns $a, b \in \mathbb{Q}$.

Note that $(x-5)^2 = (-2\sqrt{3})^2 = 12$ which is $x^2 - 10x + 13 = 0$. It follows that

$$D = (x^2 - 10x + 13) + 6 = 0 + 6 = 6.$$

On the other hand,

$$\begin{aligned} N &= (x^2 - 10x + 13)(x^2 + x + 2) + 42 \\ &= 0 + 42 \\ &= 42. \end{aligned}$$

Hence we have

$$\frac{x^4 - 9x^3 + 5x^2 - 7x + 68}{x^2 - 10x + 19} = \frac{N}{D} = \frac{42}{6} = 7.$$

21. **Answer.** 66.

Solution. Since $A + B = 90^\circ$, it follows that $\cos A = \sin B$. Thus we have $\sin A + \cos A = \frac{7n-20}{5n+8}$ and $\sin A \cos A = \frac{120}{5n+8}$. Now we have

$$\begin{aligned} (\sin A + \cos A)^2 &= 1 + 2 \sin A \cos A \\ \left(\frac{7n-20}{5n+8}\right)^2 &= 1 + 2 \times \frac{120}{5n+8} \\ (7n-20)^2 &= (5n+8)^2 + 240(5n+8) \\ n^2 - 65n - 66 &= 0 \end{aligned}$$

So either $n = 66$ or $n = -1$. Note that A is acute, and so $\sin A + \cos A > 0$. Thus $n \neq -1$; otherwise, $\sin A + \cos A = \frac{7(-1)-20}{5(-1)+8} = -9$. Hence $n = 66$.

22. **Answer.** 3

Solution. Label the inequality $|x + |x|| + |y + |y|| \leq 2$ by (1), and the inequality $|x - |x|| + |y - |y|| \leq 2$ by (2). Consider S_1.

If $x \geq 0, y \geq 0$, then (1) is equivalent to $x + y \leq 1$.

If $x \geq 0, y \leq 0$, then (1) is equivalent to $x \leq 1$.

If $x \leq 0, y \geq 0$, then (1) is equivalent to $y \leq 1$.

If $x \leq 0, y \leq 0$, then (1) is equivalent to $0 \leq 2$ which is always true.

Therefore $S_1 = \{(x, y) \in \mathbb{R}^2 \ : \ x \leq 1, x + y \leq 1, y \leq 1\}$.

Note that $(x, y) \in S_1$ if and only if $(-x, -y) \in S_2$. Thus $S_2 = \{(x, y) \in \mathbb{R}^2 \ : \ x \geq -1, x+y \geq -1, y \geq -1\}$. Consequently, $S_1 \cap S_2$ is the hexagon with vertices $(1, 0)$, $(0, 1)$, $(-1, 1)$, $(-1, 0)$, $(0, -1)$, $(1, -1)$, whose area is 3.

23. **Answer.** 5.

Solution. Observe that

$$x + y = \frac{(\sqrt{n+2} - \sqrt{n})^2 + (\sqrt{n+2} + \sqrt{n})^2}{(\sqrt{n+2} + \sqrt{n})(\sqrt{n+2} - \sqrt{n})} = \frac{4n+4}{2} = 2n + 2,$$

$$xy = 1.$$

Hence we have

$$\begin{aligned} 14x^2 + 26xy + 14y^2 &= 14(x+y)^2 - 2xy \\ &= 14(2n+2)^2 - 2. \end{aligned}$$

Therefore,

$$\begin{aligned} 14(2n+2)^2 - 2 &= 2014 \\ (2n+2)^2 &= \frac{2016}{14} \\ (2n+2)^2 &= 144. \end{aligned}$$

So $2n + 2 = 12$ or $2n + 2 = -12$. But $2n + 2 = -12$ is inadmissible since $n > 0$. Hence, $2n + 2 = 12$ and so $n = 5$.

24. **Answer.** 4

Solution. Note that if x is an integer, so are $x^2 - 5x + 5$ and $x + 5$. Observe that for two integers a, b satisfying $a^b = 1$, one must have (i) $b = 0$ and $a \neq 0$; (ii) $a = 1$; or (iii) $a = -1$ and b is even.

- Case (i): $x + 5 = 0$. So $x = -5$ and $x^2 - 5x + 5 = 55$.

- Case (ii): $x^2 - 5x + 5 = 1$. So $x = 1$ or $x = 4$.
- Case (iii): $x^2 - 5x + 5 = -1$ and $x + 5$ is even. The equation $x^2 - 5x + 5 = -1$ implies that $x = 2, 3$. If $x = 2$ then $x + 5 = 7$ is odd, which is inadmissible. If $x = 3$, then $x + 5 = 8$, which is admissible.

In summary, there are 4 solutions for x: $x = -5, 1, 4, 3$.

25. **Answer.** 16

Solution. Observe that $p^2 - q^2 + p + q = (p+q)(p-q+1)$. Write 2014 as a product of two integers a and b, i.e. $2014 = ab$. Such an ordered pair (a, b) corresponds to a pair (p, q) given by

$$p + q = a, \; p - q + 1 = b$$
$$\text{i.e. } (p, q) = \left(\frac{a+b-1}{2}, \frac{a-b+1}{2}\right).$$

Since $2014 = 2 \times 19 \times 53$ is divisible by 2 (but not by 4), it follows that one of the numbers a, b is even, and the other must be odd. Hence, $a + b$ and $a - b$ are both odd. It follows that (p, q) is a pair of integers.

There are 16 integers that divide 2014, namely 1, 2, 19, 53, 38, 106, 1007, 2014, -1, -2, -19, -53, -38, -106, -1007, -2014. Thus, there are 16 ordered pairs of integers (p, q) satisfying the equation.

26. **Answer.** 1.

Solution. Note that

$$\begin{aligned}
\frac{\sin 2x + \sin 4x + \sin 6x}{\cos 2x + \cos 4x + \cos 6x} &= \frac{\sin 4x + 2\sin\left(\frac{6x+2x}{2}\right)\cos\left(\frac{6x-2x}{2}\right)}{\cos 4x + 2\cos\left(\frac{6x+2x}{2}\right)\cos\left(\frac{6x-2x}{2}\right)} \\
&= \frac{\sin 4x(1 + 2\cos 2x)}{\cos 4x(1 + 2\cos 2x)} \\
&= \tan 4x
\end{aligned}$$

The function $f(x) = \tan 4x$ is increasing on $0 \leq x \leq \frac{\pi}{16}$. The maximum occurs at $x = \frac{\pi}{16}$, i.e. $f(\pi/16) = \tan \pi/4 = 1$.

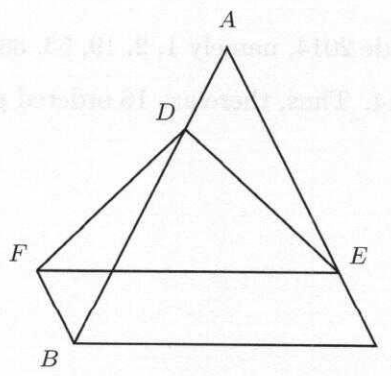
27. **Answer.** 2046.

Solution. Consider the colours of the first row. There are 2 cases: (1) the squares are coloured alternately with the 2 colours, (2) some adjacent squares are coloured with the same colours. There are 2 such colourings in case (1), and thus there are $2^{10} - 2$ colourings in case (2). For case (1), each row can either be coloured exactly as the previous row or in alternate colours as the previous row. Thus case (1) gives rise to $2 \times 2^9 = 2^{10}$ colourings. For case (2), starting with the two squares of the the same colours in the first row, the colourings of the remaining rows are all determined. Thus there are $2^{10} - 2$ colourings for case (2). Consequently there are altogether $2^{11} - 2$ such colourings.

In general, the number of such colourings for a $n \times n$ square board is $2^{n+1} - 2$.

28. **Answer.** 42

Solution. Let F be the point such that $BCEF$ is a parallelogram. The point F is necessarily outside the triangle ABC since $\angle C$ is acute. Join FB and FD. Then $\angle FBD = \angle DAE$, $BF = CE = AD$, $BD = AE$. Thus the triangles FBD and DAE are congruent. Hence $FD = DE = BC = FE$. That is the triangle DEF is equilateral. Hence $\angle EDF = 60°$. Consequently, $\angle BDE = \angle EDF - \angle BDF = \angle EDF - \angle AED = 60° - 18° = 42°$.



29. **Answer.** 1.

Solution. If $x = y = z = t$, then it is easy to see that there is exactly one solution, i.e. $t = 2$.

Suppose $x > y$. Then $y = \frac{x^2+4}{4} > \frac{y^2+4}{4} = z$. So $y > z$. Likewise, we have $z > x$. So $x > y > z > x$, which is a contradiction.

Similarly, if $x < y$, then $x < y < z < x$, a contradiction. Hence, $x = y = z = 2$.

30. **Answer.** 74

Solution. Let Z be the set of all the 4-element subsets of X that contains 10, and W be the set of all the 4-element subsets of X that contains 10 and is disjoint from A. Clearly, $|Z| = \binom{9}{3} = 84$, and $|W| = \binom{5}{3} = 10$. Hence, the number of 4-element subsets Y of X such that $10 \in Y$ and the intersection of Y and A is not empty is

$$|Z| - |W| = 84 - 10 = 74.$$

31. **Answer.** 60480

Solution. Suppose we introduce 3 extra guests A, B, C. Then the total number of ways to arrange all the guests would be $(10-1)! = 9!$. By removing A, B and C from any such arrangement P, we obtain an arrangement P' that we originally wanted. However, there are $3!$ different arrangements P that would produce the same P' in this way (since there are $3!$ ways to arrange the extra three guests). Thus, the required number is $\frac{9!}{3!} = 60480$.

32. **Answer.** 9.

Solution. $9(x^2+y^2)^2 - 8(x+y)(x^3+y^3) = (x^2-4xy+y^2)^2 \geq 0$. Thus for all $(x,y) \neq (0,0)$, $\frac{8(x+y)(x^3+y^3)}{(x^2+y^2)^2} \leq 9$, with equality if and only if $x^2 - 4xy + y^2 = 0$, which is equivalent to $(x,y) = k(1, 2 \pm \sqrt{3})$ for $k \neq 0$.

33. **Answer.** 2

Solution.

$$2\left(\sin 2^\circ \tan 1^\circ + \sin 4^\circ \tan 1^\circ + \sin 6^\circ \tan 1^\circ + \cdots + \sin 178^\circ \tan 1^\circ\right)$$

$$\begin{aligned}
&= \frac{1}{\cos 1^\circ}\left(2\sin 2^\circ \sin 1^\circ + 2\sin 4^\circ \sin 1^\circ + 2\sin 6^\circ \sin 1^\circ + \cdots + 2\sin 178^\circ \sin 1^\circ\right) \\
&= \frac{1}{\cos 1^\circ}\left((\cos 1^\circ - \cos 3^\circ) + (\cos 3^\circ - \cos 5^\circ) + (\cos 5^\circ - \cos 7^\circ) + \cdots + (\cos 177^\circ - \cos 179^\circ)\right) \\
&= \frac{1}{\cos 1^\circ}\left(\cos 1^\circ - \cos 179^\circ\right) \\
&= \frac{1}{\cos 1^\circ}\left(\cos 1^\circ + \cos 1^\circ\right) \\
&= 2.
\end{aligned}$$

Here, we have used the fact that

$$\begin{aligned}
\cos(2n-1)^\circ - \cos(2n+1)^\circ &= \cos 2n^\circ \cos 1^\circ + \sin 2n^\circ \sin 1^\circ - (\cos 2n^\circ \cos 1^\circ - \sin 2n^\circ \sin 1^\circ) \\
&= 2\sin 2n^\circ \sin 1^\circ.
\end{aligned}$$

34. **Answer.** 2034

Solution. Let $S = -(x_1 + x_2 + \cdots + x_{100})$. Note that x_i is an integer for all $i = 1, \ldots, 100$. Thus S is also an integer. Then

$$\begin{aligned} x_1^2 + x_2^2 + \cdots + x_{100}^2 &= 63^2 + (x_1+1)^2 + (x_2+1)^2 + \cdots + (x_{99}+1)^2 \\ &= x_1^2 + \cdots + x_{99}^2 + 2(x_1 + \cdots + x_{99}) + 99 + 63^2 \\ &= x_1^2 + \cdots + x_{99}^2 - 2(S + x_{100}) + 4068 \end{aligned}$$

Hence $S = \frac{1}{2}(4068 - (x_{100}+1)^2 + 1)$ which is an integer if and only if x_{100} is even. Consequently, $S \le \frac{1}{2}(4068 - 0) = 2034$ with equality if and only if $x_{100} = 0$ or -2.

Consider the following sequence $x_1, \ldots, x_{100}$ of 100 integers.

$$-63, -62, \ldots, -2, -1, \underbrace{0, -1, 0, -1, \ldots, 0, -1}_{\text{18 pairs of 0 and } -1}, 0.$$

It satisfies $|x_{n+1}| = |x_n + 1|$ for $n = 1, 2, \ldots, 99$. For this sequence of 100 integers, $S = 2034$. Thus the largest possible value of $S = -(x_1 + x_2 + \cdots + x_{100})$ is 2034.

35. **Answer.** 10

Solution. By the Intersecting Chord Theorem,

$$AZ \cdot ZY = CZ \cdot ZD = BZ \cdot ZX.$$

Write $AZ = AX + XZ$ and $BZ = BY + YZ$. It follows that

$$\begin{aligned} (AX + XZ) \cdot ZY &= (BY + YZ) \cdot ZX && (1) \\ AX \cdot ZY &= BY \cdot ZX && (2) \\ AX &= BY \quad \text{(since it is given that } ZY = ZX). && (3) \end{aligned}$$

On the other hand, by the Tangent Secant Theorem,

$$\begin{aligned} BW^2 &= AB \cdot BY \\ &= AB \cdot AX \\ &= 100. && (4) \end{aligned}$$

Hence, $BW = 10$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2014

(Senior Section, Second Round)

Saturday, 28 June 2014 **0900-1300**

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1. In the triangle ABC, the excircle opposite to the vertex A with centre at I touches the side BC at D. (The circle also touches the sides AB, AC extended.) Let M be the midpoint of BC and N the midpoint of AD. Prove that I, M, N are collinear.

2. Find, with justification, all positive real numbers a, b, c satisfying the system of equations:

$$a\sqrt{b} = a + c, \quad b\sqrt{c} = b + a, \quad c\sqrt{a} = c + b.$$

3. Some blue and red circular disks of identical size are packed together to form a triangle. The top level has one disk and each level has 1 more disk than the level above it. Each disk not at the bottom level touches two disks below it and its colour is blue if these two disks are of the same colour. Otherwise its colour is red.

Suppose the bottom level has 2048 disks of which 2014 are red. What is the colour of the disk at the top?

4. For each positive integer n, let

$$x_n = p_1 + \cdots + p_n$$

where $p_1, \ldots, p_n$ are the first n primes. Prove that for each positive integer n, there is an integer k_n such that $x_n < k_n^2 < x_{n+1}$.

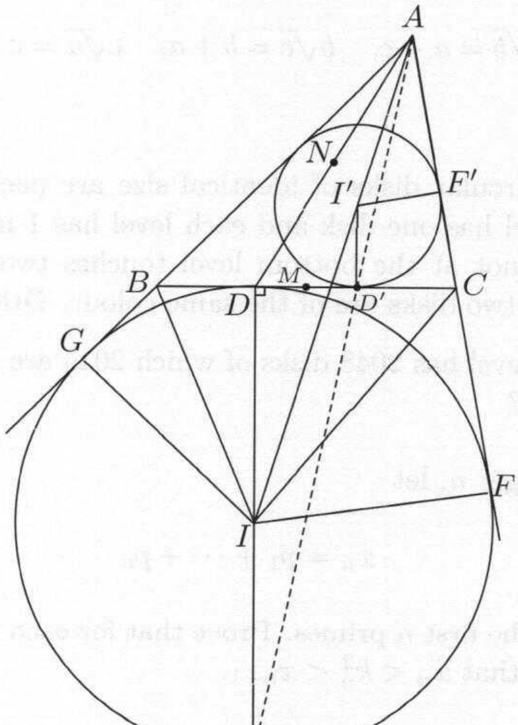
5. Alice and Bob play a number game. Starting with a positive integer n, they take turns changing the number with Alice going first. Each player may change the current number k to either $k - 1$ or $\lceil k/2 \rceil$. The person who changes 1 to 0 wins. Determine all n such that Alice has a winning strategy. (For each real number x, its ceiling $\lceil x \rceil$ is the smallest integer $\geq x$.)

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2014

(Senior Section, Second Round Solutions)

1. Let I' be the incentre. Extend DI to E so that DE is a diameter of the excircle. Let the incircle touches the side BC at D'. By dropping perpendiculars IF and $I'F'$ from I and I' to the line AC, we see that $AI'/AI = I'F'/IF = I'D'/IE$. Since $I'D' \parallel IE$ (they are both perpendicular to BC), we conclude that A, D', E are collinear. It is well-known that $BD = CD'$. (They are both equal to $s - c$, where s is the semiperimeter and a, b, c are the lenghts of the sides of $\triangle ABC$.) Thus M is the midpoint of DD'. Now I is the midpoint of ED and N is the midpoint of AD. Thus we conclude that E, M, N are collinear.



Second solution. First, we note that the locus of points P such that $[PBD]+[PCA] = K$ is a straight line, where K is a constant, $[PBD]$ and $[PCA]$ denote the oriented areas of the triangles PBD and PCA respectively. (The area is positive if the orientation is counterclockwise.) To see this, we use the area formula for a triangle in coordinate geometry. Let the coordinates of the points A, B, C, D be $(a_1, a_2), (b_1, b_2), (c_1, c_2), (d_1, d_2)$ respectively. Then

$$2[PBD] = \begin{vmatrix} 1 & x & y \\ 1 & b_1 & b_2 \\ 1 & d_1 & d_2 \end{vmatrix}, \quad 2[PCA] = \begin{vmatrix} 1 & x & y \\ 1 & c_1 & c_2 \\ 1 & a_1 & a_2 \end{vmatrix}.$$

Thus the locus of points P such that $[PBD]+[PCA]=K$ has the equation

$$\begin{vmatrix} 1 & x & y \\ 1 & b_1 & b_2 \\ 1 & d_1 & d_2 \end{vmatrix} + \begin{vmatrix} 1 & x & y \\ 1 & c_1 & c_2 \\ 1 & a_1 & a_2 \end{vmatrix} = 2K.$$

This is a linear equation in x and y, and thus represents a straight line.

Let $2K=[ABC]$. Then the locus of points P such that $[PBD]+[PCA]=K$ is a straight line ℓ. We shall show M,N,I all satisfy this equation and thus lie on ℓ. As N is the midpoint of AD, and M is the midpoint of BC, we have $[NBD]+[NCA]=[ABC]/2=K$ and $[MBD]+[MCA]=[ABC]/2=K$.

As the orientation of IBD is clockwise as in the diagram, the area $[IBD]$ is negative. Thus the sum of the oriented areas

$$[IBD]+[ICA]=[ICA]-[IDB]=[IFA]-([IFC]+[IDB]).$$

As $\triangle IFA \cong \triangle IGA$, $[IFA]=[IFAG]/2$. It is obvious that $[IBG]=[IDB]$ and $[ICD]=[IFC]$. Thus $[IDB]+[IFC]=[IFCBG]/2$. Thus

$$[IFA]-([IFC])+[IDB])=([IFAG]-[IFCBG])/2=[ABC]/2=K.$$

Since $[NBD]+[NCA]=[MBD]+[MCA]=[IBD]+[ICA]=K$, the points N,M,I are collinear.

Note: In general, for a quadrilateral $ABCD$, the locus of points P such that $[PBD]+[PCA]=[ABCD]/2$ is a straight line called the Newton line of the quadrilateral. Let AB intersect DC at E, and BC intersect AD at F. Let M,N,L be the midpoints of AC,BD,EF respectively. Then M,N,L lie on the Newton line of $ABCD$. Furthermore, if the sides of $ABCD$ touch a circle centred at I, then I also lies on the Newton line. In this problem, the triangle ABC can be considered as a degenerate quadrilateral $ABDC$ where all four sides touch the excircle centerd at I. Thus M,N,I lie on the Newton line of the degenerate quadrilateral $ABDC$.

2. Without loss of generality, we may suppose $0<c\le a$ and $0<b\le a$, i.e. a is the largest number among a,b,c. The first equation gives $a(\sqrt{b}-1)=c\le a$ so that $b\le 4$. From the second equation we have $b\le a=b(\sqrt{c}-1)$ so that $c\ge 4$. Thus $b\le 4\le c\le a$. From the third equation, we have $c(\sqrt{a}-1)=b\le c$ so that $a\le 4$. Consequently, the only solution is $a=b=c=4$.

3. Since 2048 is a power of 2, we'll solve the more general case where the bottom level has 2^n disks. Represent a blue disk by 1 and a red disk by -1. The top disk is labelled T. In general, if a is a disk and b,c are two disks below it, then $a=bc$.

It is easy to prove by induction on m that if the bottom layer has m disks, $b_1, b_2, \ldots, b_m$, then

$$T = b_1^{\binom{m-1}{0}} b_2^{\binom{m-1}{1}} b_3^{\binom{m-1}{2}} \cdots b_m^{\binom{m-1}{m-1}}.$$

The base case when $m = 2$ is obvious. Suppose the result holds for $m-1$. If A is the left disk and B is the right below the disk at the top, then the disks at the bottom of A are $b_1, \ldots, b_{m-1}$. Thus, by the induction hypothesis, $A = b_1^{\binom{m-2}{0}} b_2^{\binom{m-2}{1}} \cdots b_{m-1}^{\binom{m-2}{m-2}}$. Similarly, $B = b_2^{\binom{m-2}{0}} b_3^{\binom{m-2}{1}} \cdots b_m^{\binom{m-2}{m-2}}$. Then $T = AB$ and the result follows.

We next prove that the binomial coefficients $\binom{2^n-1}{i}$ is odd for $0 \le i \le 2^n - 1$. Note that

$$\binom{2^n-1}{i} = \frac{2^n-1}{1}\frac{2^n-2}{2} \cdots \frac{2^n-i}{i}.$$

If $m = 2^k a$ where a is odd and $0 \le m \le i \le 2^n - 1$, then $\frac{2^n - m}{m} = \frac{2^{n-k} - a}{a}$ which is odd. Thus the binomial coefficients $\binom{2^n-1}{i}$ are all odd.

Hence

$$T = b_1 b_2 \cdots b_{2^n} = 1$$

since the number of terms which are -1 is 2014. The top disk is thus blue.

4. Let

$$2 < 3 < 5 < 7 < q_5 < q_6 < \cdots$$

be a sequence of positive odd integers satisfying the condition

$$q_{n+1} > 2n + 1 \quad \text{for} \quad n \ge 4. \tag{1}$$

Putting $y_n = q_1 + \cdots + q_n$ we have $y_i > i^2$ for $i = 1, 2, 3, 4$ and from (1) we infer by mathematical induction that this is true for all positive integers i. Now we suppose

$$(n + k + 1)^2 > y_n \ge (n + k)^2, \tag{2}$$

where k is a nonnegative integer. From (1) and (2), it follows that, for $k = 0$ and $n \ge 4$,

$$y_{n+1} > (n + 1)^2 > y_n. \tag{3}$$

This is true also or $n = 1, 2, 3$. If $k \ne 0$, (2) implies

$$q_{n+1} > 2(n + k) + 1; \tag{4}$$

otherwise we would have $q_{n-j} \le 2(n + k) - (2j + 1)$ for $j = 0, 1, 2, \cdots, n - 1$. therefore, in opposition to (2),

$$y_n \le n^2 + 2nk < (n + k)^2.$$

Now we get from (2) and (4)

$$y_{n+1} > (n+k+1)^2 > y_n. \tag{5}$$

By (3) and (5) we see that the assertion of the problem is valid for our generalized sequence.

5. Let each positive integer be a *winning* or *losing* number, depending on whether Alice can force a win. trivially, $n = 1$ is a winning. Consider $n \geq 2$. Note that n is a winning number if and only if $n - 1$ or $\lceil n/2 \rceil$ is a losing number.

Every $n \geq 2$ has a unique expression of the form $n = (2m+1)2^k + 1$ with m and k being nonnegative integers. We show that n is a winning number if and only if $m > 0$ and k is even, or $m = 0$ and k is odd.

We use induction on k. First consider $k = 0$. That is n is even. If $m = 0$, then n is the losing number 2. Hence 4 is a winning number. For $2d$ to be a losing number, both d and $2d - 2$ must be winning numbers. This makes $2d - 1$ a losing number, but then $2d$ is a winning number. We conclude that an even number is a winning number if and only if it exceeds 2.

Now consider $k \geq 1$. That is n is odd. If $n > 3$, then $n - 1$ is a winning number, while $n - 1 = \lceil n/2 \rceil$ when $n = 3$. Thus n is a winning number if and only if $\lceil n/2 \rceil$ is a losing number. Since $\lceil n/2 \rceil = (2m+1)2^{k-1} + 1$, the claim now follows from the induction hypothesis.

Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2014
(Open Section, First round)

Wednesday, 4 June 2014 **0930-1200 hrs**

Instructions to contestants

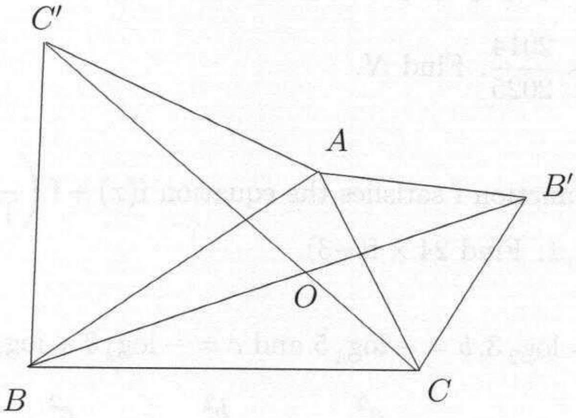
1. *Answer ALL 25 questions.*
2. *Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.*
3. *No steps are needed to justify your answers.*
4. *Each question carries 1 mark.*
5. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

In this paper, $\lfloor x \rfloor$ denotes the greatest integer not exceeding x. For examples, $\lfloor 5 \rfloor = 5$; $\lfloor 2.8 \rfloor = 2$; and $\lfloor -2.3 \rfloor = -3$.

1. Find the sum
$$1^2 \times 3 + 2^2 \times 4 + 3^2 \times 5 + 4^2 \times 6 + \cdots + 20^2 \times 22.$$

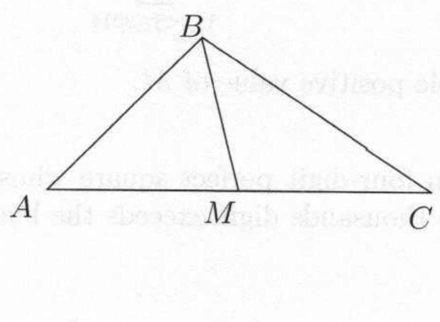
2. In the following figure, ABC is a triangle and both ABC' and $AB'C$ are equilateral triangles. Let O be the meeting point of lines CC' and BB'. Find $\angle BOC$ in degrees.



3. Consider the function $g(x) = Ax^2 + Bx$, where A and B are constants. Assume that u, v are two numbers such that
$$g(u-3) = g(v+3), \quad \text{and } u - v \neq 6.$$
Find the largest possible value of $A\left(g(u+v)\right)^2 + Bg(u+v)$.

4. Let $a_1, a_2, \cdots$ and $b_1, b_2, \cdots$ be two arithmetic progressions such that $a_1 = 10$ and $b_1 = 24$, and that $a_{100} + b_{100} = 2014$. Find the sum of the first twenty terms of the sequence
$$a_1 + b_1, a_2 + b_2, a_3 + b_3, \cdots.$$

5. The figure below (not drawn to scale) shows a triangle ABC with $BC = 8$ cm, $BA = 4$ cm and $AC = 2\sqrt{31}$ cm. The point M is the midpoint of AC. Find the length of BM in centimetres.



6. If x, y and z are real numbers satisfying the equation $x^2+y^2+z^2-xy-yz-zx=27$, find the maximum value of $|y-z|$.

7. The set A is a non-empty subset of the set $\{1,2,3,4,5,6,7,8,9\}$ with the property that whenever $a \in A$, then $10-a \in A$. How many possible subsets A are there?

8. In the sequence

$$\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{9}, \frac{2}{9}, \frac{3}{9}, \cdots, \frac{8}{9}, \frac{1}{16}, \frac{2}{16}, \cdots, \frac{15}{16}, \frac{1}{25}, \frac{2}{25}, \cdots, \frac{24}{25}, \cdots$$

the N^{th} term is $\dfrac{2014}{2025}$. Find N.

9. A real-valued function f satisfies the equation $\mathrm{f}(x)+\mathrm{f}\left(\dfrac{1}{1-x}\right)=\dfrac{x}{x-1}$ for all real numbers $x \neq 0, 1$. Find $24 \times \mathrm{f}(-3)$.

10. Given that $a=\log_2 3, b=-\log_4 5$ and $c=-\log_2 3+\log_4 5$, find the value of

$$\frac{a^2}{2a^2+bc}+\frac{b^2}{2b^2+ac}+\frac{c^2}{2c^2+ab}.$$

11. Let m be the smallest value of the function $y=\sqrt{x^2+4x+7}+\sqrt{x^2-2x+5}$. Determine $\lfloor m \rfloor$.

12. Find the number of triples (a,b,c) such that a,b,c are numbers in the set $\{1,2,3,\cdots,15\}$ satisfying the conditions $a<b-1$ and $b<c-2$.

13. The first term of an arithmetic progression is an integer and the common difference is 2. If the sum of the first n terms $(n>1)$ of the arithmetic progression is 2014, find the sum of all the possible values of n.

14. Let $a_i \in \{1,-1\}$ for all $i=1,2,3,\cdots,2014$ and

$$M=\sum_{1\leq i<j\leq 2014} a_i a_j.$$

Find the least possible positive value of M.

15. Find the largest even four-digit perfect square whose units digit exceeds the tens digit by 1 and whose thousands digit exceeds the hundreds digit by 1.

16. Let ABC be a triangle with $a = BC, b = AC$ and $c = AB$. Assume that $3a^2 + 3b^2 = 5c^2$. Find the value of
$$\frac{\cot A + \cot B}{\cot C}.$$

17. Find the number of integers k in the set $S = \{1, 2, 3, \cdots, 12014\}$ such that k is of exactly one of the following forms
$$n^2, n^3, n^5,$$
where n is an integer.

(Note: For example, $100, 1000$ are such numbers but $64 = 8^2 = 4^3$ is not.)

18. Let a, b, c be positive real numbers such that $abc = 1$. Find the least possible value of
$$\frac{2014}{a^3(b+c)} + \frac{2014}{b^3(a+c)} + \frac{2014}{c^3(a+b)}.$$

19. A sequence $a_0, a_1, a_2, a_3, \cdots$, with $a_1 = 1$, is defined such that for any positive integers m and n, where $m \geq n$, the terms of the sequence satisfy the relation
$$a_{m+n} + a_{m-n} + m - n = \frac{1}{2}(a_{2m} + a_{2n}) + 1.$$
Find the value of $\left\lfloor \frac{a_{2014}}{2014} \right\rfloor$.

20. Let n be a positive integer such that, for each of the digits $0, 1, \ldots, 9$, there exists a factor of n ending in that digit. What is the smallest possible value of n?

21. Let $a_1, a_2, a_3, \cdots, a_{2001}, \cdots$ be an arithmetic progression such that $a_1^2 + a_{1001}^2 \leq 10$. Find the largest possible value of the following expression
$$a_{1001} + a_{1002} + a_{1003} + \cdots + a_{2001}.$$

22. It is given that w, a, b and c are positive integers that satisfy the equation
$$w! = a! + b! + c!.$$
Find the largest possible value of $w + a + b + c$.

23. In the triangle ABC, $AB = 63$ cm, $BC = 56$ cm, $CA = 49$ cm, M is the midpoint of BC, and the extension of AM meets the circumcircle ω of the triangle ABC at P. The circle through P and tangent to BC at M intersects ω at Q distinct from P. Find the length of MQ in centimetres.

24. Let M and C be two distinct points on the arc AB of a circle such that M is the midpoint of the arc AB. If D is the foot of the perpendicular from M onto the chord AC such that $AD = 100$ cm and $DC = 36$ cm, find the length of the chord BC in centimetres.

25. Let AD be the bisector of $\angle A$ of the triangle ABC. Let M and N be points on AB and AC, respectively such that $\angle MDA = \angle ABC$ and $\angle NDA = \angle ACB$. Let P be the intersection between AD and MN. Suppose $AD = 14$ cm. Find the value of $AB \times AC \times AP$ in cm^3.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2014

(Open Section, First Round Solutions)

Wednesday, 4 June 2014 **0930-1200 hrs**

1. **Answer.** 49840

Solution. Using sigma notation, the sum can be expressed and simplified as

$$\begin{aligned}\sum_{r=1}^{20} r^2(r+2) &= \sum_{r=1}^{20} r^3 + 2\sum_{r=1}^{20} r^2 \\ &= \left(\frac{20(21)}{2}\right)^2 + 2\left(\frac{20(21)(41)}{6}\right) \\ &= 44100 + 5740 \\ &= 49840,\end{aligned}$$

which is the answer. $\square$

2. **Answer.** 120

Solution. It is clear that $\triangle BAB' \cong \triangle C'AC$. Thus $\angle ACO = \angle AB'O$, implying that points O, A, B', C are concyclic. Thus $\angle B'OC = \angle B'AC = 60^\circ$, and so

$$\angle BOC = 180^\circ - \angle B'OC = 120^\circ,$$

thereby completing the solution. $\square$

3. **Answer.** 0

Solution. By the given condition, we have

$$A(u-3)^2 + B(u-3) = A(v+3)^2 + B(v+3).$$

$$\begin{aligned}A\left((u-3)^2 - (v+3)^2\right) + B\left((u-3) - (v+3)\right) &= 0 \\ A\left((u-v-6)(u+v)\right) + B\left(u-v-6\right) &= 0 \\ (u-v-6)\left(A(u+v) + B\right) &= 0.\end{aligned}$$

Since $u - v - 6 \neq 0$, we have $A(u+v) + B = 0$.

If $A = 0$, then $B = 0$ so that $\mathrm{g}(u+v) = 0$.

If $A \neq 0$, then $u + v = -\dfrac{B}{A}$, so that $\mathrm{g}(u+v) = \mathrm{g}\left(-\dfrac{B}{A}\right) = 0$.

Thus, in either case, $A\left(\mathrm{g}(u+v)\right)^2 + B\mathrm{g}(u+v) = 0$, thereby completing our solution. $\square$

4. **Answer.** 4480

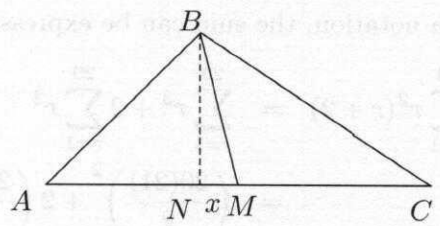
Solution. Let $a_n = a_1 + (n-1)c$ and $b_n = b_1 + (n-1)d$. Then $a_n + b_n = (a_1 + b_1) + (n-1)(c+d)$, showing that the sequence $a_1 + b_1, a_2 + b_2, a_3 + b_3, \cdots$ is also an arithmetic progression. Using the condition $a_{100} + b_{100} = 2014$, we obtain

$$\begin{aligned} 2014 &= (10+24) + 99(c+d) \\ c+d &= 20. \end{aligned}$$

Then the sum of the first twenty terms of the sequence is $\frac{20}{2}(2 \times 34 + 19 \times 20) = 4480$. □

5. **Answer.** 3

Solution. In triangle ABC, draw the height BN to the side AC and let $NM = x$.



Then

$$BN^2 + (\sqrt{31} - x)^2 = AB^2 = 4^2. \quad (1)$$
$$BN^2 + (\sqrt{31} + x)^2 = BC^2 = 8^2. \quad (2)$$

Subtracting (1) from (2), we obtain $x = \frac{12}{\sqrt{31}}$. Hence

$$\begin{aligned} BM^2 &= BN^2 + x^2 \\ &= 8^2 - 2x\sqrt{31} - 31 \\ &= 9, \end{aligned}$$

showing that $BM = 3$ cm. □

6. **Answer.** 6

Solution. The given equation can be re-arranged as $x^2 - (y+z)x + y^2 + z^2 - yz - 27 = 0$. Since x is real,

$$(y+z)^2 - 4(y^2 + z^2 - yz - 27) \geq 0,$$

which can be simplified into $y^2 + z^2 - 2yz - 36 \leq 0$, which is equivalent to $(y-z)^2 \leq 36$, or $|y-z| \leq 6$. Hence the maximum value of $|y-z|$ is 6. Note that explicitly, $y = 6, x = 3, z = 0$ is one such solution. □

7. **Answer.** 31

Solution. Let us partition the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ into the following:

$$B_1 = \{5\},\ B_2 = \{1, 9\},\ B_3 = \{2, 8\},\ B_4 = \{3, 7\},\ B_5 = \{4, 6\}.$$

Note that elements of A must come from the entire set of elements of $B_r, r = 1, 2, 3, 4, 5$. Hence the number of subsets A is $2^5 - 1 = 31$. □

8. **Answer.** 31340

Solution. Note that $45^2 = 2025$. The number of terms between $\frac{1}{4}$ and $\frac{44^2-1}{44^2}$ inclusive is

$$\begin{aligned}&\sum_{r=1}^{44} r^2 - 44\\ &= \frac{1}{6}(44)(45)(89) - 44 = 29326.\end{aligned}$$

hence $N = 29326 + 2014 = 31340$. □

9. **Answer.** 61

Solution. We are given the equation

$$\mathrm{f}(x) + \mathrm{f}\left(\frac{1}{1-x}\right) = \frac{x}{x-1}. \tag{3}$$

Note that since $x \neq 0, 1$, we also have $\frac{x-1}{x} \neq 0, 1$. Replacing x by $\frac{x-1}{x}$ in equation (3), we obtain

$$\mathrm{f}\left(\frac{x-1}{x}\right) + \mathrm{f}(x) = 1 - x. \tag{4}$$

From equations (3) and (4), we obtain

$$\mathrm{f}\left(\frac{x-1}{x}\right) - \mathrm{f}\left(\frac{1}{1-x}\right) = -x - \frac{1}{x-1}. \tag{5}$$

Replacing x by $\frac{x-1}{x}$ in (5), we obtain

$$\mathrm{f}\left(\frac{1}{1-x}\right) - \mathrm{f}(x) = -\frac{x-1}{x} + x. \tag{6}$$

Solving equations (3) and (6), we obtain

$$2\mathrm{f}(x) = \frac{x}{x-1} + \frac{x-1}{x} - x,$$

so that we obtain $24 \times \mathrm{f}(-3) = 61$, thereby completing our solution. □

10. **Answer.** 1

Solution. Note that $c = -a - b$, and so

$$\begin{aligned}\frac{a^2}{2a^2+bc} &= \frac{a^2}{2a^2+b(-a-b)} = \frac{a^2}{2a^2-ab-b^2}\\ &= \frac{a^2}{(a-b)(2a+b)} = \frac{a^2}{(a-b)(a-c)}.\end{aligned}$$

Similarly, we can show that

$$\frac{b^2}{2b^2+ac} = \frac{b^2}{(b-a)(b-c)}$$

and

$$\frac{c^2}{2c^2+ab} = \frac{c^2}{(c-a)(c-b)}.$$

Thus

$$\begin{aligned}
&\frac{a^2}{2a^2+bc} + \frac{b^2}{2b^2+ac} + \frac{c^2}{2c^2+ab} \\
= &\frac{a^2}{(a-b)(a-c)} + \frac{b^2}{(b-a)(b-c)} + \frac{c^2}{(c-a)(c-b)} \\
= &\frac{a^2(b-c)+b^2(c-a)+c^2(a-b)}{(a-b)(a-c)(b-c)} \\
= &\frac{(a-b)(a-c)(b-c)}{(a-b)(a-c)(b-c)} = 1,
\end{aligned}$$

thereby completing the solution. □

11. **Answer.** 4

Solution. Note that

$$y = \sqrt{(x+2)^2+3} + \sqrt{(x-1)^2+4}.$$

Let A be the point $(-2, -\sqrt{3})$, B be the point $(1, 2)$, and P be the point $(x, 0)$. Thus

$$y = |AP| + |BP|,$$

where $|AP|$ represents the distance between A and P. Thus y has the smallest values if and only if P is the meeting point of the line passing A and B with the x-axis. Hence the smallest value of y is the distance between A and B:

$$m = |AB| = \sqrt{(-2-1)^2 + (-\sqrt{3}-2)^2} = 2\sqrt{4+\sqrt{3}}.$$

Note that

$$m^2 = 16 + 4\sqrt{3} < 25.$$

Thus the answer is 4. □

12. **Answer.** 220

Solution. Let $x = b-a-2$, $y = c-b-3$ and $z = 15-c$. Then (a, b, c) is such a triple if and only if $a-1, x, y, z$ are non-negative integers such that

$$(a-1) + x + y + z = 9.$$

Thus number of non-negative integer solutions of the following equation

$$w_1 + w_2 + w_3 + w_4 = 9$$

is

$$\binom{9+3}{3} = 220.$$

Hence the answer is 220. □

13. **Answer.** 3239

Solution. Let a, an integer, be the first term of the arithmetic progression. Then,

$$\frac{n}{2}(2a + 2(n-1)) = 2014 \Rightarrow 2014 = n(a+n-1),$$

implying that n must be a factor of 2014 and $n > 1$. Consider all possible cases:

When $n = 2$, $a = 1006$;

When $n = 19$, $a = 88$;

When $n = 38$, $a = 1$;

When $n = 53$, $a = -14$;

When $n = 106$, $a = -86$;

When $n = 1007$, $a = -1004$;

When $n = 2014$, $a = -2012$.

By adding up all the possiblee values of n, we obtain 3239. □

14. **Answer.** 51

Solution. Assume that among $a_1, a_2, \cdots, a_{2014}$, x of them are 1 and y of them are -1. Thus

$$x + y = 2014$$

and

$$a_1 + a_2 + \cdots + a_{2014} = x - y.$$

Note that

$$(a_1 + a_2 + \cdots + a_{2014})^2 = a_1^2 + a_2^2 + \cdots + a_{2014}^2 + 2M = 2014 + 2M$$

we have

$$(x-y)^2 = 2014 + 2M.$$

Note that $x - y$ must be even. So $2014 + 2M$ must be even and a perfect square, and the smallest positive number M to make this property hold is 51.

When $M = 51$, we have $(x-y)^2 = 46^2$ and so $x - y = 46$ or $x - y = -46$, i.e., $(x, y) = (1030, 984)$ or $(x, y) = (984, 1030)$. Thus the answer is 51. □

15. **Answer.** 4356

Solution. Let $(x+1)$ be the thousands digit and $(y+1)$ be the units digit. Then the required number N has the value

$$\begin{aligned} & 1000(x+1) + 100x + 10y + y + 1 \\ = & \ 1100x + 11y + 101 \\ = & \ 11(100x + y + 91) \\ = & \ 11(99x + 88 + x + y + 3), \end{aligned}$$

which shows that 11 divides N. Since the number N is a perfect square, we must have 11 divides $x + y + 3$. On the other hand, since $0 \leq x \leq 8$ and $0 \leq y \leq 8$, we can conclude that $3 \leq x + y + 3 \leq 19$. Hence $x + y = 8$, and so $N = 11(99x + 99) = 121(9x + 9)$. For $0 \leq x \leq 8$, only $x = 0, 3, 8$ make $9x+9$ a perfect square, and among them only $x = 3$ yields an even number. Thus, there is only ONE possible choice of N, which is $66^2 = 4356$. □.

16. **Answer.** 3

Solution. By using the cosine rule and the sine rule, we have

$$\begin{aligned} \cos C &= \frac{a^2+b^2-c^2}{2ab} \\ &= \frac{2c^2/3}{2ab} = \frac{c^2}{3ab} \\ &= \frac{1}{3} \times \frac{\sin^2 C}{\sin A \sin B}. \end{aligned}$$

Hence,

$$\begin{aligned} \cot C &= \frac{1}{3} \times \frac{\sin C}{\sin A \sin B} = \frac{1}{3} \times \frac{\sin(A+B)}{\sin A \sin B} \\ &= \frac{1}{3} \times \frac{\sin A \cos B + \sin B \cos A}{\sin A \sin B} \\ &= \frac{1}{3} \times (\cot A + \cot B). \end{aligned}$$

Hence

$$\frac{\cot A + \cot B}{\cot C} = 3,$$

which is the answer. □

17. **Answer.** 126

Solution. Let P_1, P_2, P_3 be the following properties for integers:
P_1: an integer is of the form n^2;
P_2: an integer is of the form n^3;
P_3: an integer is of the form n^5.
Let $W(P_i)$ be the number of integers in S which have property P_i, $W(P_iP_j)$ be the number of integers in S which have properties P_i and P_j and $W(P_1P_2P_3)$ be the number of integers in S which have properties P_1, P_2 and P_3.
Then

$$\begin{aligned} W_1 &= W(P_1) + W(P_2) + W(P_3) \\ &= \lfloor 12014^{1/2} \rfloor + \lfloor 12014^{1/3} \rfloor + \lfloor 12014^{1/5} \rfloor \\ &= 109 + 22 + 6 = 137; \\ W_2 &= W(P_1P_2) + W(P_1P_3) + W(P_2P_3) \\ &= \lfloor 12014^{1/6} \rfloor + \lfloor 12014^{1/10} \rfloor + \lfloor 12014^{1/15} \rfloor \\ &= 4 + 2 + 1 = 7 \end{aligned}$$

and

$$W_3 = W(P_1P_2P_3) = \lfloor 12014^{1/30} \rfloor = 1.$$

Thus the number of integers in S which have exactly one of the properties P_1, P_2 and P_3 is equal to

$$W_1 - 2W_2 + 3W_3 = 137 - 2 \times 7 + 3 \times 1 = 126,$$

thus completing the solution of this problem, □

18. **Answer.** 3021

Solution. Let
$$f(a,b,c)=\frac{1}{a^3(b+c)}+\frac{1}{b^3(a+c)}+\frac{1}{c^3(a+b)}.$$
Note that $f(1,1,1)=3/2$. We will show that $f(a,b,c)\geq 3/2$ for all positive real numbers a,b,c such that $abc=1$.

Let $x=1/a$, $y=1/b$ and $z=1/c$. Then $xyz=1$ and
$$\begin{aligned}
f(a,b,c) &= \frac{x^3yz}{y+z}+\frac{y^3xz}{x+z}+\frac{z^3xy}{x+y}\\
&= \frac{x^2}{y+z}+\frac{y^2}{x+z}+\frac{z^2}{x+y}\\
&= \frac{x^2}{y+z}+\frac{y+z}{4}+\frac{y^2}{x+z}+\frac{x+z}{4}+\frac{z^2}{x+y}+\frac{x+y}{4}-\frac{x+y+z}{2}\\
&\geq x+y+z-\frac{x+y+z}{2}\\
&= \frac{x+y+z}{2}\\
&\geq \frac{3}{2}\sqrt[3]{xyz}=3/2.
\end{aligned}$$
Hence $f(a,b,c)$ has the smallest possible value $3/2$. Thus the answer follows. □

19. **Answer.** 2013

Solution. Substituting $n=0$ into the above equation, we obtain
$$a_{2m}=4a_m+2m-2-a_0.$$
By substituting $m=n$ into the equation we obtain
$$a_{2m}+a_0=\frac{1}{2}(a_{2m}+a_{2m})+1,$$
which implies $a_0=1$. Thus, we have
$$a_{2m}=4a_m+2m-3.$$
Let $m=1$, we have $a_2=4a_1+2-3=3$.
Let $n=1$, we have
$$\begin{aligned}
a_{m+1}+a_{m-1}+m-1 &= \frac{1}{2}(a_{2m}+a_2)+1\\
a_{m+1}+a_{m-1}+m-1 &= 2a_m+m+1\\
a_{m+1}-a_m &= a_m-a_{m-1}+2.
\end{aligned}$$
Thus, we have $\sum_{m=1}^{k}(a_{m+1}-a_m)=\sum_{m=1}^{k}(a_m-a_{m-1})+2k,\ k=1,2,3,\cdots$. We thus have
$$a_{k+1}-a_1=a_k-a_0+2k,$$
that is, $a_{k+1}-a_k=2k$. Adding up again,
$$\sum_{k=1}^{N-1}(a_{k+1}-a_k)=\sum_{k=1}^{N-1}2k,$$

that is, $a_N - 1 = N(N-1)$ for $N = 1, 2, 3, \cdots$, which simplifies to

$$\frac{a_N}{N} = N - 1 + \frac{1}{N}.$$

When $N = 2014$, we thus have $\dfrac{a_{2014}}{2014} = 2013 + \dfrac{1}{2014}$, thus we have $\left\lfloor \dfrac{a_{2014}}{2014} \right\rfloor = 2013$, thereby completing our solution. □.

20. **Answer.** 270

Solution. Clearly 10 must be a factor of n. If 9 is also a factor of n, then n is a multiple of 90. By inspection, 90 and 180 do not have factors that end in 7. On the other hand, 270 is divisible by 1, 2, 3, 54, 5, 6, 27, 18, 9, and 10. Hence the smallest possible value of n in this case is 270.

If 19 is a factor of n then n cannot be 190 since 190 has no factor ending in 4. The next possible n is 380 which is bigger than 270. □

21. **Answer.** 5005

Solution. Let

$$S = a_{1001} + a_{1002} + a_{1003} + \cdots + a_{2001}.$$

Then

$$S = 1001 \times \frac{a_{1001} + a_{2001}}{2}.$$

Note that $a_{2001} - a_{1001} = a_{1001} - a_1$. Thus $a_{2001} = 2a_{1001} - a_1$ and

$$S = 1001 \times \frac{a_{1001} + 2a_{1001} - a_1}{2} = \frac{1001}{2}(3a_{1001} - a_1).$$

Note that by Cauchy inequality,

$$(3a_{1001} - a_1)^2 \le (3^2 + 1^2)(a_{1001}^2 + a_1^2) \le 10 \times 10 = 100,$$

where the equality holds if and only if $a_{1001}/3 = -a_1$. Thus the largest possible value of S is

$$\frac{1001}{2} \times 10 = 5005,$$

which can be obtained if a_1 and a_{1001} satisfy the condition that $a_{1001}/3 = -a_1$ and $a_{1001}^2 + a_1^2 = 10$. □

22. **Answer.** 9

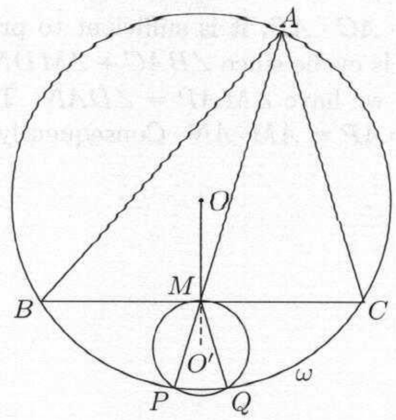
Solution. Without loss of generality, we may assume that $a \le b \le c$. It is clear that $w \ge c + 1$, so that

$$(c+1)c! = (c+1)! \le w! = a! + b! + c! \le 3c!,$$

showing that $c \le 2$. It is easily checked that $a = b = c = 2, w = 3$ is the only solution. Hence $w + a + b + c = 3 + 2 + 2 + 2 = 9$. □

23. **Answer.** 16

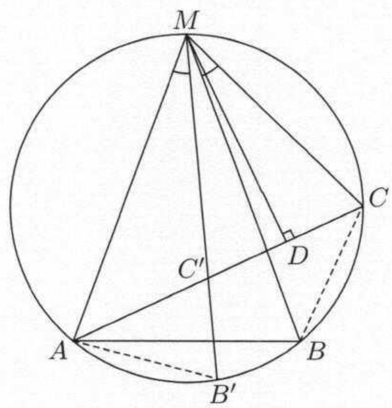
Solution.



Let O and O' be the centres of ω and the circumcircle ω' of the triangle MPQ. Then O, M, O' are collinear. Hence ω and ω' are symmetric about the line OO'. Thus the two points P and Q of intersection between ω and ω' are symmetric about the line OO'. Hence $MQ = MP$. By Pappus' theorem, $AM = \frac{1}{2}\sqrt{2AB^2 + 2AC^2 - BC^2} = 49$. Thus by the intersecting chords theorem, $MP = BM \times MC/AM = 28^2/49 = 16$. Therefore $MQ = 16$ cm. $\square$

24. **Answer.** 64

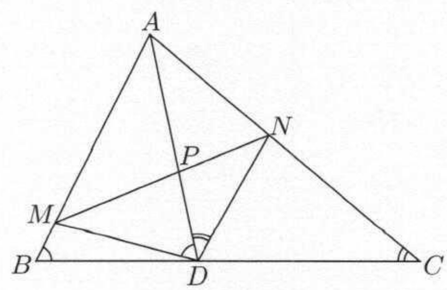
Solution.



Let C' be the point on AD such that $C'D = DC$. Let the extension of MC' intersect the circle at B'. Then the isosceles triangles MAB, $MC'C$ and $AB'C'$ are similar. Thus $\angle AMB = \angle C'MC$ so that $\angle AMB' = \angle BMC$. This implies that $AB' = BC$. Therefore, $AD = AC' + C'D = AB' + C'D = BC + DC$. From this we get $BC = AD - DC = 100 - 36 = 64$. $\square$

25. **Answer.** 2744

Solution. We shall prove $AB \cdot AC \cdot AP = AD^3$. First triangle ADN and ACD are similar so that $AD^2 = AN \cdot AC$. Similarly, $AD^2 = AM \cdot AB$. Therefore, $AD^4 = AB \cdot AC \cdot AM \cdot AN$.



To prove $AD^3 = AB \cdot AC \cdot AP$, it is sufficient to prove $AD \cdot AP = AM \cdot AN$. But the quadrilateral $AMDN$ is cyclic since $\angle BAC + \angle MDN = 180^\circ$, so that $\angle PMA = \angle NDA$. Since DA bisects $\angle A$, we have $\angle MAP = \angle DAN$. Therefore, triangles APM and AND are similar. Thus, $AD \cdot AP = AM \cdot AN$. Consequently, $AB \cdot AC \cdot AP = AD^3 = 14^3 = 2744$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2014

(Open Section, Second Round)

Saturday, 5 July 2014 **0900-1300**

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1. The quadrilateral $ABCD$ is inscribed in a circle which has diameter BD. Points A' and B' are symmetric to A and B with respect to the line BD and AC respectively. If the lines $A'C$, BD intersects at P and AC, $B'D$ intersect at Q, prove that PQ is perpendicular to AC.

2. Let $\mathbb{R}$ be the set of real numbers. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$f(xf(y) + x) = xy + f(x).$$

3. Let $0 < a_1 < a_2 < \cdots < a_n$ be real numbers. Prove that

$$\left(\frac{1}{1+a_1} + \frac{1}{1+a_2} + \cdots + \frac{1}{1+a_n}\right)^2 \leq \frac{1}{a_1} + \frac{1}{a_2 - a_1} + \frac{1}{a_3 - a_2} + \cdots + \frac{1}{a_n - a_{n-1}}.$$

4. In a 50×50 grid, an integer is written in each of the 2500 cells. Let G be the configuration of 8 cells formed by removing the central cell of a 3×3 grid. It is given that for any group of 8 cells in the 50×50 grid forming the configuration G, the sum of the numbers written in the 8 cells is positive. Prove that there is a 2×2 grid so that the sum of the numbers in the 4 cells is positive.

5. Determine the largest odd positive integer N such that every odd integer k with $1 < k < N$ and $(k, N) = 1$ is a prime.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2014

(Open Section, Second Round Solutions)

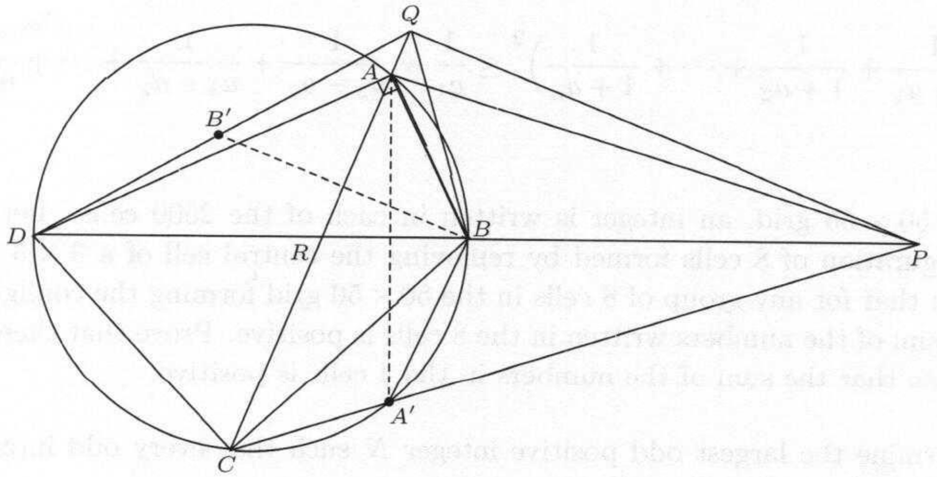
1. Let AC intersect BD at R. Then $\angle BAR = \angle BAC = \angle BA'P = \angle BAP$. That is AB bisects $\angle PAR$. As $\angle BAD = 90°$, we also have AD is the external bisector of $\angle PAR$. By the angle bisector theorem, we have

$$\frac{BR}{BP} = \frac{DR}{DP} = \frac{AR}{AP} \qquad (1)$$

As B and B' are symmetric with respect to the line AC, we have $\angle BQR = \angle B'QR = \angle DQR$. Thus QR bisects $\angle BQD$. By the angle bisector theorem and (1), we have

$$\frac{QD}{QB} = \frac{RD}{RB} = \frac{PD}{PB}.$$

Thus QP is the external bisector of $\angle BQD$. Hence $\angle RQP = 90°$.



2. Let $x = 1$, we have $f(f(y)+1) = y + f(1)$. From this we conclude that f is a bijection.

Let $y = 0$, we have $f(xf(0)+x) = f(x)$. Since f is injective, we have $xf(0)+x = x$, i.e., $xf(0) = 0$ for all x. Hence $f(0) = 0$.

For $x \neq 0$, take $y = -f(x)/x$, then $f(xf(y)+x) = 0$. Thus $xf(y)+x = 0$ or $f(y) = -1$. Thus $f(-f(x)/x) = -1$. Hence $-f(x)/x = c$, where c is the constant satisfying $f(c) = -1$. Thus $f(x) = -cx$ for $x \neq 0$. Since $f(0) = 0$, $f(x) = -cx$ for all $x \in \mathbb{R}$.

Substitute this into the original equation, we get $f(-xcy+x) = -c(-xcy+x) = xy-cx$. Thus $c^2xy = xy$. Hence $c^2 = 1$ or $c = \pm 1$. It is easy to check that both $f(x) = x$ and $f(x) = -x$ satisfy the functional equation.

3. By Cauchy -Schwarz inequality,

$$\text{LHS} \le \left(\frac{1}{a_1} + \frac{1}{a_2 - a_1} + \cdots + \frac{1}{a_n - a_{n-1}}\right)\left(\frac{a_1}{(1+a_1)^2} + \frac{a_2 - a_1}{(1+a_2)^2} + \cdots + \frac{a_n - a_{n-1}}{(1+a_n)^2}\right).$$

Note that $\frac{a_1}{(1+a_1)^2} \le \frac{a_1}{1+a_1}$ and for $i = 2, \ldots, n$,

$$\frac{a_i - a_{i-1}}{(1+a_i)^2} \le \frac{a_i - a_{i-1}}{(1+a_{i-1})(1+a_i)} = \frac{1}{1+a_{i-1}} - \frac{1}{1+a_i}.$$

Thus, after telescoping,

$$\frac{a_1}{(1+a_1)^2} + \frac{a_2 - a_1}{(1+a_2)^2} + \cdots + \frac{a_n - a_{n-1}}{(1+a_n)^2} \le \frac{a_1}{1+a_1} + \frac{1}{1+a_1} - \frac{1}{1+a_n} < 1$$

and we are done.

4. Consider the 4×4 grid. Place 4 overlapping copies of G as shown (left figure), where the number $i = 1, 2, 3, 4$ indicates the cells of the i^{th} copy of G. The same grid is also covered by 8 overlapping copies of 2×2 grid (right), with each cell covered the same number of times. (Note that in the first figure the top left cells of the 4 copies of G form a 2×2 grid while in the second figure, the top left cells of the 8 2×2 grid form the configuration G. This is important for the general case.)

1	12	12	2
13	234	134	24
13	124	123	24
3	34	34	4

1	12	23	3
14	124	235	35
46	467	578	58
6	67	78	8

Thus the sum of the sums of the numbers in the cells of the 4 copies of G is equal to that of the 8 copies of the 2×2 grid. Since the former is positive, one of the 2×2 grid must be positive as well.

It is easy to see that the result holds for any two configurations G_1, G_2, provided the grid is large enough. Choose a pair of coordinate axes. Let the number of cells in

G_1 and G_2 be m and n respectively. Place G_1 and G_2 in some position. Let $a_1, \ldots, a_m$ be the centres of the cells covered by G_1 and let $b_1, \ldots b_n$ be the centres of the cells covered by G_2. Let $G_1(b_i)$ be the position of G_1 after it has been translated by b_i and let $P_1(b_i)$ be the corresponding sum. The sum $P_2(a_i)$ is similarly defined. It is clear that the cells, counting multiplicity, covered by G_1 when it is translated by the vectors $b_1, b_2, \ldots, b_n$ are the same as the cells, counting multiplicity, covered by G_2 when it is translated by the vectors $a_1, a_2, \ldots, a_m$. Thus

$$\sum P_1(b_i) = \sum P_2(a_j).$$

Since the LHS is positive, at least one of the terms on the RHS is positive.

5. The largest such integer N is 105. Let p_i denote the ith prime, i.e. $p_1 = 2, p_2 = 3$, etc. We call a positive integer N *admissible* if it is odd and has the property stated in the problem. Any admissible number exceeding p_i^2 must clearly contain the factor p_i for $i \geq 2$. By Bertrand's postulate, $p_{n+1} < 2p_n$. Thus for any $n \geq 5$, we have $p_{n+1}^2 < 4p_n^2 < 8p_{n-1}p_n$. Since $p_2p_3 = 15 > 8$,

$$p_{n+1}^2 < p_2p_3 \cdots p_n \qquad (n \geq 5) \tag{1}$$

If there were an admissible number N such that $p_{n+1}^2 \geq N > p_n^2$ for some $n \geq 5$, then N would be divisible by $p_2p_3 \cdots p_n$ and so $N \geq p_2p_3 \cdots p_n > p_{n+1}^2$ by (1) which gives a contradiction. Therefore no admissible number can exceed $p_5^2 = 121$.

The number 105 can be checked to be admissible. Also since any admissible number exceeding 49 must be divisible by $3 \times 5 \times 7 = 105$, there can be no admissible number N between 105 and 121.