MA2001
AY25/26 Sem 1

1. Linear Systems

Linear Systems and Their Solutions

- **Linear system** of m linear equations in n variables $x_1, \dots, x_n$:

$$\begin{aligned} a_{11}x_1 + \dots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n &= b_m \end{aligned} \tag{1}$$

- $x_1 = s_1, \dots, x_n = s_n$ is a **solution** to the system if $x_1 = s_1, \dots, x_n = s_n$ is a solution to every equation in the system

  The set of all solutions to the system is the **solution set**

  An expression that gives the entire solution set is a **general solution**
- A linear system is **consistent** if it has at least one solution, **inconsistent** if it has none
  - no solution (consistent)
  - unique solution (consistent)
  - infinitely many solutions (consistent)

Elementary Row Operations

- **Augmented matrix** of (1) is

$$\left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right).$$

- **Elementary row operations** (EROs):
  1. Multiply a row by non-zero constant (kR_i)
  2. Swap two rows ($R_i \leftrightarrow R_j$)
  3. Add a multiple of one row to another row ($R_j + cR_i$)
- Two augmented matrices are **row equivalent** if one can be obtained from the other by a series of EROs.
- **Thm** - If augmented matrices of two systems of linear equations are row equivalent, then the two systems have the same set of solutions.

  Remark. Converse is not true: two linear systems have different # of equations

Row-Echelon Form

- **Row-echelon form** (REF)
  1. Zero rows are at the bottom of the matrix.
  2. For any two successive non-zero rows, leading entry of lower row occurs further to the right than higher row.
- **Pivot point**: leading entry of non-zero row

  Pivot column: column contains a pivot point
- **Reduced row-echelon form** (RREF)
  1. The leading entry of every non-zero row is 1.
  2. In each pivot column, except the pivot point, all other entries are zero.

Gaussian Elimination

- **Gaussian elimination**: augmented matrix $\to$ REF
  1. Find the leftmost non-zero column.
  2. Check the top entry of the column. If it is 0, make it non-zero by swapping rows.
  3. To rows underneath, add a multiple of the top row to make the rest of the column 0.
  4. Cover the top row and repeat until done.
- **Gauss–Jordan elimination**: REF $\to$ RREF
  5. Multiply rows by constants to make all leading entries 1.
  6. Starting from the last non-zero row and working upwards, add multiples of it to rows above to make the rest of the pivot column 0.
- Consistency
  - **No solution**: rightmost column is pivot column (leading entry occurs at the last column)
  - **Unique solution**: every column on the left is a pivot column
  - **Infinitely many solutions**: at least one column on the left is not pivot column

    No. of parameters = no. of non-pivot columns on the left
- Use a **branch diagram** to organise cases (for the values of unknowns) systematically.

Homogeneous Linear Systems

- A linear system (1) is **homogeneous** if $b_1 = \dots = b_n = 0$:

$$\begin{aligned} a_{11}x_1 + \dots + a_{1n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n &= 0 \end{aligned}$$

- **Trivial solution** $x_1 = 0, \dots, x_n = 0$ is a solution to any homogeneous system, so homogeneous system must be consistent:
  - only the trivial solution, or
  - infinitely many solutions in addition to the trivial solution

  Homogeneous system with more unknowns than equations has infinitely many solutions.

2. Matrices

Introduction to Matrices

- **Row matrix**: matrix with only one row

  Column matrix: matrix with only one column
- **Square matrix**: matrix with the same number of rows and columns. An $n \times n$ square matrix is of **order** n.
- The **diagonal** of square matrix $A = (a_{ij})_{n\times n}$ is the sequence of entries $a_{11}, \dots, a_{nn}$.

$$a_{ij} \text{ is a } \begin{cases} \textbf{diagonal entry} & (i = j) \\ \textbf{non-diagonal entry} & (i \neq j) \end{cases}$$

  Diagonal matrix: all non-diagonal entries are 0
- **Scalar matrix**: diagonal matrix with all equal diagonal entries
- **Identity matrix** I: scalar matrix with all diagonal entries 1
- **Zero matrix** $\mathbf{0}$: matrix with all entries 0
- **Symmetric matrix**: square matrix with $a_{ij} = a_{ji}$ for all i, j (symmetric wrt diagonal)
- **Upper-triangular matrix**: square matrix where if $a_{ij} = 0$ whenever $i > j$.

  Lower-triangular matrix: square matrix where $a_{ij} = 0$ whenever $i < j$.

  Triangular matrix: upper/lower-triangular

Matrix Operations

- Two matrices are **equal** if 1. same size 2. corresponding entries are equal

  Addition: $(a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$

  Scalar multiplication: $c(a_{ij}) = (ca_{ij})$
- Let $A = (a_{ij})_{m\times p}$, $B = (b_{ij})_{p\times n}$. The **product** AB is the $m \times n$ matrix whose (i, j)-entry is

$$\sum_{k=1}^{p} a_{ik}b_{kj} = a_{i1}b_{1j} + \dots + a_{in}b_{nj}.$$

- Let $A = (a_{ij})_{m\times n}$ and $a_i = \begin{pmatrix} a_{i1} & \cdots & a_{in} \end{pmatrix}$ denote the i-th row. Then $A = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}$.

  If $b_j = \begin{pmatrix} b_{1j} \\ \vdots \\ b_{pj} \end{pmatrix}$ is the j-th column, then $A = \begin{pmatrix} b_1 & \cdots & b_n \end{pmatrix}$.
- If $A = (a_{ij})_{m\times p}$ with i-th row a_i, $B = (a_{ij})_{p\times n}$ with j-th column b_j, then

$$AB = \begin{pmatrix} a_1b_1 & \cdots & a_1b_n \\ \vdots & & \vdots \\ a_mb_1 & \cdots & a_mb_n \end{pmatrix} = \begin{pmatrix} a_1B \\ \vdots \\ a_mB \end{pmatrix} = \begin{pmatrix} Ab_1 & \cdots & Ab_n \end{pmatrix}.$$

- The linear system (1) can be written as $Ax = b$:

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}.$$

  A is **coefficient matrix**, x is **variable matrix**, b is **constant matrix** of the linear system.
- **Thm** - Linear system with > 1 solution $\Rightarrow$ infinite solutions

  Proof: If $Ax = b$ has two distinct solutions u_1, u_2, then $u_2 + t(u_1 - u_2)$ is a solution $\forall t \in \mathbb{R}$.
- The **transpose** of $A = (a_{ij})_{m\times n}$ is the $n \times m$ matrix $A^\top$ whose (i, j)-entry is a_{ji}.
- **Properties** -
  - $(A^\top)^\top = A$
  - $(A + B)^\top = A^\top + B^\top$
  - $(cA)^\top = cA^\top$
  - $(AB)^\top = B^\top A^\top$
  - A is symmetric $\Leftrightarrow A = A^\top$

Inverses of Square Matrices

- A square matrix A order n is **invertible** if there exists a square matrix B of order n such that $AB = BA = I$. Such a matrix B is called an **inverse** of A.

  Singular: no inverse (use proof by contradiction)
- **Thm** - An invertible matrix has a unique inverse.

  The inverse of an invertible matrix A is denoted by A^{-1}.
- **Properties** -
  - $(cA)^{-1} = \frac{1}{c}A^{-1}$
  - $(A^\top)^{-1} = (A^{-1})^\top$
  - $(A^{-1})^{-1} = A$
  - $(AB)^{-1} = B^{-1}A^{-1}$.
  - $A^{-n} = (A^{-1})^n = (A^n)^{-1}$

Elementary Matrices

- **Elementary matrix**: square matrix obtained from the identity matrix by performing a single ERO
- **Thm** - If E is the elementary matrix obtained by performing an ERO to I_m, then for any $m \times n$ matrix A, EA obtained by performing the same ERO to A.
- **Thm** - Every elementary matrix has an inverse that is also elementary.
- **Thm** - A and B are row equiv $\Leftrightarrow \exists$ elementary matrices $E_1, \dots, E_k$ such that $E_k \cdots E_1 A = B$.
- **Thm** - Augmented matrices of two linear systems are row equiv $\Rightarrow$ same solution set.
- **Invertibility Equivalences** - If A is a square matrix, TFAE:
  1. A is invertible
  2. $Ax = 0$ has only the trivial solution
  3. RREF of A is an identity matrix
  4. A can be expressed as a product of elementary matrices
  5. $\det(A) \neq 0$
- **Thm** - Let A be invertible. To find A^{-1},

$$\text{RREF of } (A \mid I) \text{ is } (I \mid A^{-1}).$$

- **Half-price Thm** - Let A and B be square matrices of same size. If $AB = I$, then A and B are invertible, $A^{-1} = B$, $B^{-1} = A$.
- **Elementary column operations** (ECOs) - EROs but on columns
- If E is obtained from I_n by a single elementary column operation, then E is an elementary matrix.
- **Thm** - If E is the elementary matrix obtained by performing an ECO to I_n, then for any $m \times n$ matrix A, AE can be obtained by performing the same ECO to A.

  Remark. *Post-multiply* E to A, instead of pre-multiplying it.

Determinant

- Let M_{ij} be the submatrix of A obtained by deleting the i-th row and j-th column of A. The **(i,j)-cofactor** of A is
$$A_{ij} := (-1)^{i+j}\det(M_{ij}).$$
- Let $A = (a_{ij})_{n\times n}$. The **determinant** of A is
$$\det(A) := \sum_{k=1}^{n} a_{1k}A_{1k} = a_{11}A_{11} + \cdots + a_{1n}A_{1n}$$
if $n > 1$, and $\det(A) := a_{11}$ if $n = 1$.
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$
$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$
- **Cofactor Expansion** - Let $A = (a_{ij})_{n\times n}$. Then for all i, j,
$$\det(A) = \sum_{k=1}^{n} a_{ik}A_{ik} = a_{i1}A_{i1} + \cdots + a_{in}A_{in} \qquad [i\text{-th row}]$$
$$\det(A) = \sum_{k=1}^{n} a_{kj}A_{kj} = a_{1j}A_{1j} + \cdots + a_{nj}A_{nj} \qquad [j\text{-th column}]$$
Perform cofactor expansion along row/column with many 0s
- **Thm** - If $A = (a_{ij})_{n\times n}$ is triangular, then $\det(A) = a_{11}\cdots a_{nn}$.
- **Thm** - $\det(A) = \det(A^\top)$
- **Lemma** - The determinant of any square matrix with two identical rows/columns is zero.

  Lemma - If two square matrices of order n differ at the i-th row only, then their $(i,1),\dots,(i,n)$ cofactors are the same.
- **Determinants Under EROs** -
  - $A \xrightarrow{cR_i} B \implies \det(B) = c\det(A)$
  - $A \xrightarrow{R_i \leftrightarrow R_j} B \implies \det(B) = -\det(A)$
  - $A \xrightarrow{R_i + cR_j} B \implies \det(B) = \det(A)$
- **Thm** - For elementary matrix E, $\det(EA) = \det(E)\det(A)$.

  To find $\det(A)$:
  1. Perform Gaussian elimination on A reduce it to REF (upper-triangular)
  2. $\det(R)$ = product of diagonal entries
  3. $E_k \cdots E_1 A = R \Rightarrow \det(E_k)\cdots\det(E_1)\det(A) = \det(R)$
- **Thm** - Let A and B be $n\times n$ matrices.
  - $\det(cA) = c^n \det(A)$
  - $\det(AB) = \det(A)\det(B)$
  - If A is invertible, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$
- Let A be $n\times n$ matrix. The **adjoint** of A is the transpose of cofactor matrix:
$$\operatorname{adj}(A) = (A_{ji})_{n\times n} = (A_{ij})^\top.$$
- **Method of Adjoints** - $A^{-1} = \dfrac{1}{\det(A)}\operatorname{adj}(A)$.
- **Cramer's Rule** - Suppose $Ax = b$ is a linear system where A is an $n\times n$ matrix. Let A_i be the matrix obtained from A by replacing the i-th column of A by b. If A is invertible, then the linear system has unique solution $x = A^{-1}b$:
$$x = \frac{1}{\det(A)}\begin{pmatrix} \det(A_1) \\ \vdots \\ \det(A_n) \end{pmatrix}.$$

3. Vector Spaces

Euclidean n-Spaces

- An **n-vector** of real numbers is $\boldsymbol{v} = (v_1,\dots,v_n)$
- We can identify an n-vector $(v_1,\dots,v_n)$ with row matrix $\begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$ or column matrix $\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$
- **Zero vector** $\mathbf{0} = (0,\dots,0)$
- **Euclidean n-space** $\mathbb{R}^n$ is the set of n-vectors of real numbers

Linear Combinations and Linear Spans

- $\sum_{i=1}^{k} c_i\boldsymbol{v}_i$ is a **linear combination** of $\boldsymbol{v}_1,\dots,\boldsymbol{v}_k$
- Let $S = \{\boldsymbol{v}_1,\dots,\boldsymbol{v}_k\} \subseteq \mathbb{R}^n$. The **span** of S is the set of all linear combinations of $\boldsymbol{v}_1,\dots,\boldsymbol{v}_k$:
$$\operatorname{span}(S) := \{c_1\boldsymbol{v}_1 + \cdots + c_k\boldsymbol{v}_k \mid c_1,\dots,c_k \in \mathbb{R}\}.$$
- Let $S = \{\boldsymbol{v}_1,\dots,\boldsymbol{v}_k\} \subseteq \mathbb{R}^n$. To check if $\boldsymbol{w} \in \operatorname{span}(S)$, show that the following vector equation is consistent:
$$c_1\boldsymbol{v}_1 + \cdots + c_k\boldsymbol{v}_k = \boldsymbol{w} \Leftrightarrow \begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_k \end{pmatrix}\begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = \boldsymbol{w} \Leftrightarrow A\boldsymbol{x} = \boldsymbol{w}.$$
Do so by solving linear system $\left(\begin{array}{ccc|c} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_k & \boldsymbol{w} \end{array}\right)$ and checking if it is consistent.
  - Inconsistent $\Rightarrow \boldsymbol{w} \notin \operatorname{span}\{\boldsymbol{v}_1,\dots,\boldsymbol{v}_k\}$
  - Unique solution $\Rightarrow$ unique linear combination
  - Infinitely many solutions $\Rightarrow$ non-unique linear combination
- To check if $\boldsymbol{w}_1,\dots,\boldsymbol{w}_m \in \operatorname{span}\{\boldsymbol{v}_1,\dots,\boldsymbol{v}_k\}$, check
$\left(\begin{array}{ccc|ccc} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_k & \boldsymbol{w}_1 & \cdots & \boldsymbol{w}_m \end{array}\right)$ is consistent.
- To check if $\operatorname{span}\{\boldsymbol{v}_1,\dots,\boldsymbol{v}_k\} = \mathbb{R}^n$, check
$\left(\begin{array}{ccc|c} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_k & \boldsymbol{x} \end{array}\right)$ is consistent for **all** $\boldsymbol{x} \in \mathbb{R}^n$
$\Leftrightarrow$ REF(A) has no zero rows
$\Leftrightarrow$ A is invertible
- **Thm** - Let $S = \{\boldsymbol{v}_1,\dots,\boldsymbol{v}_k\} \subseteq \mathbb{R}^n$. If $k < n$, then $\operatorname{span}(S) \neq \mathbb{R}^n$.
- **Thm** - For any $S \subseteq \mathbb{R}^n$, $\operatorname{span}(S)$ is closed under addition and scalar multiplication, and $\mathbf{0} \in \operatorname{span}(S)$.
- **Thm** - Given $S_1 = \{\boldsymbol{u}_1,\dots,\boldsymbol{u}_k\}$, $S_2 = \{\boldsymbol{v}_1,\dots,\boldsymbol{v}_m\} \subseteq \mathbb{R}^n$,
$$\operatorname{span}(S_1) \subseteq \operatorname{span}(S_2) \Leftrightarrow$$
every $\boldsymbol{u}_i$ is a linear combination of $\boldsymbol{v}_1,\dots,\boldsymbol{v}_m$.

Remark. To show $\operatorname{span}(S_1) = \operatorname{span}(S_2)$, need to show $\operatorname{span}(S_1) \subseteq \operatorname{span}(S_2)$ and $\operatorname{span}(S_2) \subseteq \operatorname{span}(S_1)$; use Gaussian elimination above.
- **Redundancy** - If $\boldsymbol{v}_k$ is a linear combination of $\boldsymbol{v}_1,\dots,\boldsymbol{v}_{k-1}$, then $\operatorname{span}\{\boldsymbol{v}_1,\dots,\boldsymbol{v}_{k-1}\} = \operatorname{span}\{\boldsymbol{v}_1,\dots,\boldsymbol{v}_k\}$.