MA2002

AY25/26 Sem 1

Limits

- **Rational functions** - factorise, L'Hopital
- **Algebraic functions** - multiply algebraic **conjugate** to numerator and denominator
- **Function as power** - take ln to bring down power

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} \exp\left(f(x)\ln g(x)\right)$$
$$= \exp\left(\lim_{x\to a} f(x)\ln g(x)\right) = \exp\left(\lim_{x\to a} \frac{\ln g(x)}{1/f(x)}\right)$$

  then apply L'Hopital
- **Limit** $\lim_{x\to a} f(x) = L$:
$$(\forall\epsilon>0) \quad (\exists\delta>0) \quad 0<|x-a|<\delta \implies |f(x)-L|<\epsilon$$
  Choose $\delta = \min\{\frac{1}{2}, f(\epsilon)\}$; use $f(\epsilon)$ to deal with other terms, then use $\frac{1}{2}$ to deal with $|x-a|$
- **Limit Laws** - Only hold when the limits of f and g exist
- **Right-hand limit** $\lim_{x\to a^+} f(x) = L$:
$$(\forall\epsilon>0) \quad (\exists\delta>0) \quad a<x<a+\delta \implies |f(x)-L|<\epsilon$$
  Left-hand limit $\lim_{x\to a^-} f(x) = L$:
$$(\forall\epsilon>0) \quad (\exists\delta>0) \quad a-\delta<x<a \implies |f(x)-L|<\epsilon$$
- **Theorem** - One-sided limits satisfy limit laws.
- **Theorem** - $\lim_{x\to a} f(x) = L \iff \lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^-} f(x) = L$.

  To prove a limit DNE, show that left and right-hand limits are not equal.
- **Infinite limit** $\lim_{x\to a} f(x) = \infty$:
$$(\forall M>0) \quad (\exists\delta>0) \quad 0<|x-a|<\delta \implies f(x)>M$$
  Infinite limit $\lim_{x\to a} f(x) = -\infty$:
$$(\forall M<0) \quad (\exists\delta>0) \quad 0<|x-a|<\delta \implies f(x)<M$$
  To determine infinite limits,
  1. Determine whether $f(x)$ is **large**, i.e., $|f(x)|\to\infty$
  2. Determine whether $f(x)$ is **positively/negatively large**
     As $x\to a$ (or a^+, a^-), check whether $f(x)>0$ or $f(x)<0$.
- **Squeeze Theorem** - If $f(x)\le g(x)\le h(x)$ and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x)$ exists, and $\lim_{x\to a} g(x) = L$
- **Limit at infinity** $\lim_{x\to\infty} f(x) = L$:
$$(\forall\epsilon>0) \quad (\exists N\in\mathbb{R}) \quad x>N \implies |f(x)-L|<\epsilon$$
  Limit at negative infinity $\lim_{x\to-\infty} f(x) = L$:
$$(\forall\epsilon>0) \quad (\exists N\in\mathbb{R}) \quad x<N \implies |f(x)-L|<\epsilon$$
- **Continuity**: $\lim_{x\to a} f(x) = f(a)$
- **Removable discontinuity**: $\lim_{x\to a} f(x)$ exists, but $f(a)$ is undefined, or $f(a)$ is well-defined but $f(a)\neq\lim_{x\to a} f(x)$

  Jump continuity: $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist but not equal

  Infinite discontinuity: $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$
- **Theorem** - f cont. at $a \iff f$ cont. from left and right at a
- **Theorem** - If g cont. then $\lim_{x\to a} g(f(x)) = g\left(\lim_{x\to a} f(x)\right)$
- **IVT** - If f cont. on $[a,b]$, and y_0 is between $f(a)$ and $f(b)$, then there exists $c\in[a,b]$ such that $y_0 = f(c)$.

  Corollary - If f cont. on $[a,b]$ and $f(a)f(b)<0$, then $\exists c\in[a,b]$ s.t. $f(c)=0$ (root)

$$\lim_{x\to0}\frac{\sin x}{x} = 1 \qquad \lim_{h\to0}\frac{\cos h-1}{h} = 0$$

Derivatives

- **Derivative**: $f'(a) := \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{h\to0}\frac{f(a+h)-f(a)}{h}$

  Differentiable: derivative exists

  Tangent line: $y = f'(a)(x-a)+f(a)$

  Derivative: $f'(x) := \lim_{h\to0}\frac{f(x+h)-f(x)}{h}$
- **Theorem** - Differentiability $\implies$ continuity
- **Product rule, quotient rule, chain rule** - $\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$
- **Implicit differentiation** - Diff. both sides wrt x
  (Find dy/dx without making y the subject)
- **L'Hopital** - $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$
- **Logarithmic differentiation** - Take ln on both sides, then diff.
- **Fermat** - At local extreme value, $f'(c)=0$
- **Critical point**: c is stationary point, or $f'(c)$ does not exist
- **Closed Interval Method** - Find absolute max/min
  1. Evaluate values at **endpoints**: $f(a)$ and $f(b)$
  2. Evaluate values at **critical points** on (a,b)
  3. Compare the values
- **Rolle** - If f cont. on $[a,b]$, diff. on (a,b), and $f(a)=f(b)$, then $\exists c\in(a,b)$ s.t. $f'(c)=0$
- **MVT** - If f cont. on $[a,b]$, diff. on (a,b), then $\exists c\in(a,b)$ s.t.
$$f'(c) = \frac{f(b)-f(a)}{b-a}$$
- **Cauchy's MVT** - If f cont. on $[a,b]$, diff. on (a,b), $g'(x)\neq0$, then $\exists c\in(a,b)$ s.t.
$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$
- **Increasing, Decreasing Test** -
  - $f'(x)=0$ for every $x\in I' \Rightarrow f$ is constant on I
  - $f'(x)>0$ for every $x\in I' \Rightarrow f$ is increasing on I
  - $f'(x)<0$ for every $x\in I' \Rightarrow f$ is decreasing on I
- **First Derivative Test** - determine **local minimum/maximum**
- **Concavity Test** -
  - $f''(x)>0$ for all $x\in I \implies f$ is **concave up** on I
  - $f''(x)<0$ for all $x\in I \implies f$ is **concave down** on I
- **Inflection point**: change in concavity
- **Optimisation Problem** - Express the problem as finding **extreme value** (absolute max/min) of a single-variable function on interval I
  - If $I=[a,b]$, use **closed interval method**
  - For arbitrary interval I, use **increasing and decreasing tests** to find the intervals of increasing and decreasing.
- **Inverse function** - $(f^{-1})'(b) = \frac{1}{f'(a)}$, i.e. product of slopes $=1$
- Hyperbolic: $\sinh x = \frac{e^x-e^{-x}}{2}$, $\cosh x = \frac{e^x+e^{-x}}{2}$

$$\begin{array}{ll}
\frac{d}{dx}x^n = nx^{n-1} & \\
\frac{d}{dx}\sin x = \cos x & \frac{d}{dx}\cos x = -\sin x \\
\frac{d}{dx}\tan x = \sec^2 x & \frac{d}{dx}\cot x = -\csc^2 x \\
\frac{d}{dx}\sec x = \sec x\tan x & \frac{d}{dx}\csc x = -\csc x\cot x \\
\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}} & \frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}} \\
\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2} & \frac{d}{dx}\cot^{-1} x = -\frac{1}{1+x^2} \\
\frac{d}{dx}\sec^{-1} x = \frac{1}{x\sqrt{x^2-1}} & \frac{d}{dx}\csc^{-1} x = -\frac{1}{x\sqrt{x^2-1}} \\
\frac{d}{dx}\ln x = \frac{1}{x} & \frac{d}{dx}e^x = e^x \\
\frac{d}{dx}a^x = a^x\ln a & \frac{d}{dx}x^a = ax^{a-1} \\
\frac{d}{dx}\sinh x = \cosh x & \frac{d}{dx}\cosh x = \sinh x \\
\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{1+x^2}} & \frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}
\end{array}$$

Integrals

- **Riemann sum**: $\int_a^b f(x)\,dx := \lim_{n\to\infty}\sum_{i=1}^n f(x_i^*)\Delta x$

  In particular, $\int_0^1 f(x)\,dx = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n f\left(\frac{k}{n}\right)$
- **FTC (I)** - $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$
- **FTC (II)** - $\int_a^b f(x)\,dx = F(b)-F(a)$
- **Substitution** - $\int f(g(x))g'(x)\,dx \overset{u=g(x)}{=} \int f(u)\,du$
  (derivative of the substitution is as a factor)
- **Improper integral** convergent if the limit exists, else divergent
- **Inverse Substitution** - $\int f(x)\,dx \overset{x=g(t)}{=} \int f(g(t))g'(t)\,dt$
- **Integration by Parts** - $\int u\,dv = uv - \int v\,du$
- **Trigo Substitution** - When integrand contains square root of quadratic functions
  - $\sqrt{a^2-x^2} \overset{x=a\sin t}{=} \sqrt{a^2-(a\sin t)^2} = a\cos t$
  - $\sqrt{a^2+x^2} \overset{x=a\tan t}{=} \sqrt{a^2+(a\tan t)^2} = a\sec t$
  - $\sqrt{x^2-a^2} \overset{x=a\sec t}{=} \sqrt{(a\sec t)^2-a^2} = a\tan t$
- **Partial Fractions** - Deal with proper rational functions
$$\frac{f(x)}{(x+a)(x^2+bx+c)} = \frac{A}{x+a} + \frac{Bx+C}{x^2+bx+c}$$
- **Universal Trigo Substitution** - $f(\sin x,\cos x)$

  Sub. $t=\tan\frac{x}{2}$, then $x = 2\tan^{-1}t \Rightarrow \frac{dx}{dt} = \frac{2}{1+t^2}$
$$\sin x = \frac{2t}{1+t^2}, \qquad \cos x = \frac{1-t^2}{1+t^2}$$
- **Area** - Cut region using vertical line segments, find length of line segment at x
$$\boxed{A = \int_a^b [f(x)-g(x)]\,dx}$$
  if $f(x)\ge g(x)$
- **Disk/Washer Method** (about x-axis) - Cut region using vertical line segments, find area of cross-section (disk)
$$\boxed{V = \int_a^b \pi[f(x)]^2\,dx}$$
- **Cylindrical Shell Method** (about y-axis) - Rotation of each segment about y-axis is the shell of a cylinder with radius x, height $f(x)$
$$\boxed{V = \int_a^b 2\pi x\, f(x)\,dx}$$
  More generally, suppose a region is placed along the x-axis on $[a,b]$, where $c\le a$ or $b\le c$.
  1. Cut the region using vertical line segments (parallel to axis of rotation), and rotate about $x=c$
  2. Radius = distance between the segment and $x=c$, i.e., $|x-c|$
     Height = length of line segment = upper − lower endpoint
  3. Let $A(x)=2\pi rh$. Then volume of solid is $\int_a^b A(x)\,dx$.
- **Arc length**: $dL^2 = dx^2+dy^2$
$$\boxed{L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}$$
- **Surface area of revolution** (rotate about x-axis):
$$\boxed{\int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}$$

$$\int\sec x\,dx = \ln|\sec x+\tan x| \qquad \int\csc x\,dx = -\ln|\csc x+\cot x|$$

Differential Equations

- **Separable** - $\boxed{\frac{dy}{dx} = f(x)g(y)}$
- **Homogeneous** - $\frac{dy}{dx} = F(x,y)$ where $F(x,y)$ is **homogeneous** if $\boxed{F(tx,ty)=F(x,y)}$ for $t\in\mathbb{R}\setminus\{0\}$

  Substitution: $y=vx$. Then the ODE becomes separable:
$$z + x\frac{dz}{dx} = F(1,z)$$
- **Linear** - $\boxed{\frac{dy}{dx}+p(x)y = q(x)}$ is linear in y

  Integrating factor $v(x) = e^{\int p(x)dx}$
$$y = \frac{1}{v(x)}\int v(x)q(x)\,dx$$
- **Bernoulli** - $\boxed{\frac{dy}{dx}+p(x)y = q(x)y^n}$ where $n\in\mathbb{R}$

  Substitution: $z=y^{1-n}$. Then $\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$, so
$$(1-n)y^{-n}\frac{dy}{dx} + (1-n)y^{-n}p(x)y = (1-n)y^{-n}q(x)y^n$$
  or
$$\frac{dz}{dx} + (1-n)p(x)z = (1-n)q(x)$$
  which is linear.
- **Exponential Growth and Decay** - $y = Ce^{kt}$
- **Logistic Population Growth** - $P(t) = \frac{M}{1+Ce^{-Mrt}}$
- **Newton's Law of Cooling** - $T(t) = T_S + (T_0-T_S)e^{-rt}$