MA2001
AY25/26 Sem 1

2. Matrices

Matrix Operations

- **Matrix multiplication**: $\sum_{k=1}^{p} a_{ik}b_{kj} = a_{i1}b_{1j} + \cdots + a_{in}b_{nj}$
- **Block multiplication** - $\boldsymbol{A} = (a_{ij})_{m\times p}$ has i-th row $\boldsymbol{a}_i$, $\boldsymbol{B} = (a_{ij})_{p\times n}$ has j-th column $\boldsymbol{b}_j$

$$\boldsymbol{AB} = \begin{pmatrix} \boldsymbol{a}_1\boldsymbol{b}_1 & \cdots & \boldsymbol{a}_1\boldsymbol{b}_n \\ \vdots & & \vdots \\ \boldsymbol{a}_m\boldsymbol{b}_1 & \cdots & \boldsymbol{a}_m\boldsymbol{b}_n \end{pmatrix} = \begin{pmatrix} \boldsymbol{a}_1\boldsymbol{B} \\ \vdots \\ \boldsymbol{a}_m\boldsymbol{B} \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}\boldsymbol{b}_1 & \cdots & \boldsymbol{A}\boldsymbol{b}_n \end{pmatrix}.$$

- **Thm** - Linear system with > 1 solution $\implies$ infinite solutions
  Proof: If $\boldsymbol{Ax} = \boldsymbol{b}$ has two distinct solutions $\boldsymbol{u}_1, \boldsymbol{u}_2$, then $\boldsymbol{u}_2 + t(\boldsymbol{u}_1 - \boldsymbol{u}_2)$ is a solution $\forall t \in \mathbb{R}$
- **Transpose**: $\boldsymbol{A}^\top$ whose (i,j)-entry is a_{ji}
- **Properties:** $(\boldsymbol{A}^\top)^\top = \boldsymbol{A}$, $(\boldsymbol{A}+\boldsymbol{B})^\top = \boldsymbol{A}^\top + \boldsymbol{B}^\top$, $(c\boldsymbol{A})^\top = c\boldsymbol{A}^\top$, $(\boldsymbol{AB})^\top = \boldsymbol{B}^\top\boldsymbol{A}^\top$, $\boldsymbol{A}$ is symmetric $\iff \boldsymbol{A} = \boldsymbol{A}^\top$

Inverses

- $\boldsymbol{A}$ is **invertible** if $\exists \boldsymbol{B}$ s.t. $\boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{I}$, $\boldsymbol{B}$ is **inverse** of $\boldsymbol{A}$
  Singular: no inverse (use proof by contradiction)
- **Properties:** $(c\boldsymbol{A})^{-1} = \frac{1}{c}\boldsymbol{A}^{-1}$, $(\boldsymbol{A}^\top)^{-1} = (\boldsymbol{A}^{-1})^\top$, $(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}$, $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$, $\boldsymbol{A}^{-n} = (\boldsymbol{A}^{-1})^n = (\boldsymbol{A}^n)^{-1}$

Elementary Matrices

- **Elementary matrix**: square matrix obtained from the identity matrix by performing a single ERO
- **Thm** - If $\boldsymbol{E}$ is the elementary matrix obtained by performing an ERO to $\boldsymbol{I}_m$, then for any $m \times n$ matrix $\boldsymbol{A}$, $\boldsymbol{EA}$ obtained by performing the same ERO to $\boldsymbol{A}$.
- **Thm** - Every elementary matrix has an inverse that is also elementary.
- **Thm** - $\boldsymbol{A}$ and $\boldsymbol{B}$ are row equiv $\iff \exists$ elementary matrices $\boldsymbol{E}_1, \ldots, \boldsymbol{E}_k$ such that $\boldsymbol{E}_k \cdots \boldsymbol{E}_1\boldsymbol{A} = \boldsymbol{B}$.
- **Thm** - Augmented matrices of two linear systems are row equiv $\Rightarrow$ same solution set.
- **Invertibility Equivalences** -
  1. $\boldsymbol{A}$ is invertible
  2. $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution
  3. RREF of $\boldsymbol{A}$ is identity matrix
  4. $\boldsymbol{A}$ can be expressed as a product of elementary matrices
  5. $\det(\boldsymbol{A}) \neq 0$
  6. Rows of $\boldsymbol{A}$ form a basis for $\mathbb{R}^n$
  7. Columns of $\boldsymbol{A}$ form a basis for $\mathbb{R}^n$
  8. $\operatorname{rank}(\boldsymbol{A}) = n$
  9. 0 is not an eigenvalue of $\boldsymbol{A}$
- Find $\boldsymbol{A}^{-1}$ by Gaussian elimination: $\text{RREF}(\boldsymbol{A} \mid \boldsymbol{I}) = (\boldsymbol{I} \mid \boldsymbol{A}^{-1})$
- **Half-price Thm** - If $\boldsymbol{AB} = \boldsymbol{I}$, then $\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible, $\boldsymbol{A}^{-1} = \boldsymbol{B}$, $\boldsymbol{B}^{-1} = \boldsymbol{A}$.

Determinant

- Let $\boldsymbol{M}_{ij}$ be submatrix of $\boldsymbol{A}$ obtained by deleting i-th row and j-th column of $\boldsymbol{A}$. **(i,j)-cofactor** of $\boldsymbol{A}$ is $A_{ij} := (-1)^{i+j}\det(\boldsymbol{M}_{ij})$.
- **Determinant** $\det(\boldsymbol{A}) := \sum_{k=1}^{n} a_{1k}A_{1k} = a_{11}A_{11} + \cdots + a_{1n}A_{1n}$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

- **Cofactor Expansion** - Along row/column with many 0s

$$\det(\boldsymbol{A}) = \sum_{k=1}^{n} a_{ik}A_{ik} = a_{i1}A_{i1} + \cdots + a_{in}A_{in} \qquad [i\text{-th row}]$$
$$= \sum_{k=1}^{n} a_{kj}A_{kj} = a_{1j}A_{1j} + \cdots + a_{nj}A_{nj} \qquad [j\text{-th column}]$$

- **Thm** - If $\boldsymbol{A}$ is triangular, then $\det(\boldsymbol{A}) = a_{11} \cdots a_{nn}$.
- **Thm** - $\det(\boldsymbol{A}) = \det(\boldsymbol{A}^\top)$
- **Thm** - Two identical rows/ columns $\implies$ det = 0
- **EROs** -
  - $\boldsymbol{A} \xrightarrow{cR_i} \boldsymbol{B} \implies \det(\boldsymbol{B}) = c\det(\boldsymbol{A})$
  - $\boldsymbol{A} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{B} \implies \det(\boldsymbol{B}) = -\det(\boldsymbol{A})$
  - $\boldsymbol{A} \xrightarrow{R_i + cR_j} \boldsymbol{B} \implies \det(\boldsymbol{B}) = \det(\boldsymbol{A})$
- **Thm** - For elementary matrix $\boldsymbol{E}$, $\det(\boldsymbol{EA}) = \det(\boldsymbol{E})\det(\boldsymbol{A})$.
- To find $\det(\boldsymbol{A})$:
  1. Perform Gaussian elimination on $\boldsymbol{A}$, reduce it to REF (upper-triangular)
  2. $\det(\boldsymbol{R})$ = product of diagonal entries
  3. $\boldsymbol{E}_k \cdots \boldsymbol{E}_1\boldsymbol{A} = \boldsymbol{R} \Rightarrow \det(\boldsymbol{E}_k) \cdots \det(\boldsymbol{E}_1)\det(\boldsymbol{A}) = \det(\boldsymbol{R})$
- **Properties:** $\det(c\boldsymbol{A}) = c^n\det(\boldsymbol{A})$, $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$, $\det(\boldsymbol{A}^{-1}) = \dfrac{1}{\det(\boldsymbol{A})}$
- **Adjoint** $\operatorname{adj}(\boldsymbol{A}) = (A_{ji})_{n\times n} = (A_{ij})^\top$
- **Method of Adjoints** - $\boldsymbol{A}^{-1} = \dfrac{1}{\det(\boldsymbol{A})}\operatorname{adj}(\boldsymbol{A})$
- **Cramer's Rule** - Suppose $\boldsymbol{Ax} = \boldsymbol{b}$, where $\boldsymbol{A}$ is $n \times n$. Let $\boldsymbol{A}_i$ be matrix obtained from $\boldsymbol{A}$ by replacing i-th column of $\boldsymbol{A}$ by $\boldsymbol{b}$. If $\boldsymbol{A}$ is invertible, then linear system has unique solution $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$:

$$\boldsymbol{x} = \frac{1}{\det(\boldsymbol{A})}\begin{pmatrix} \det(\boldsymbol{A}_1) \\ \vdots \\ \det(\boldsymbol{A}_n) \end{pmatrix}.$$

3. Vector Spaces

Linear Combinations and Linear Spans

- **Span** is set of all linear combinations of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$
- Check if $\boldsymbol{w} \in \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$: solve
$$\begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_k & \mid & \boldsymbol{w} \end{pmatrix}$$
- Check if $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m \in \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$: solve
$$\begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_k & \mid & \boldsymbol{w}_1 & \cdots & \boldsymbol{w}_m \end{pmatrix}$$
- To check if $\operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\} = \mathbb{R}^n$, check
$$\begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_k & \mid & \boldsymbol{x} \end{pmatrix} \text{ is consistent for } all\ \boldsymbol{x} \in \mathbb{R}^n$$
$\iff$ REF($\boldsymbol{A}$) has no zero rows
$\iff$ $\boldsymbol{A}$ is invertible
- **Thm** - Let $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\} \subseteq \mathbb{R}^n$. If $k < n$, then $\operatorname{span}(S) \neq \mathbb{R}^n$.
- **Thm** - For any $S \subseteq \mathbb{R}^n$, $\operatorname{span}(S)$ is closed under addition and scalar multiplication, and $\boldsymbol{0} \in \operatorname{span}(S)$.
- **Thm** - Given $S_1 = \{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k\}$, $S_2 = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\} \subseteq \mathbb{R}^n$,
$$\operatorname{span}(S_1) \subseteq \operatorname{span}(S_2) \iff$$
$$\text{every } \boldsymbol{u}_i \text{ is a linear combination of } \boldsymbol{v}_1, \ldots, \boldsymbol{v}_m.$$
  (Show $\operatorname{span}(S_1) = \operatorname{span}(S_2)$)
- **Redundancy** - If $\boldsymbol{v}_k$ is a linear combination of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k-1}$, then $\operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k-1}\} = \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$.

Subspaces

- $V \subseteq \mathbb{R}^n$ is a **subspace** of $\mathbb{R}^n$ if $\exists \boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \in \mathbb{R}^n$ s.t. $V = \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$.
  V is the **subspace spanned** by $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$.
  $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ **spans** the subspace V.
- Contains $\boldsymbol{0}$, closed under addition, scalar multiplication
- **Subspace criterion** - $\boldsymbol{u} + d\boldsymbol{v} \in V$ for all $\boldsymbol{u}, \boldsymbol{v} \in V$, $c, d \in \mathbb{R}$
- **Solution space** - Solution set of homogeneous linear system of n variables is a subspace of $\mathbb{R}^n$

Linear Independence

- **Linearly independent**: $c_1\boldsymbol{v}_1 + \cdots + c_k\boldsymbol{v}_k = \boldsymbol{0} \Rightarrow c_1 = \cdots = c_k = 0$
  Linearly dependent: $\exists c_1, \ldots, c_k \in \mathbb{R}$, not all zero, such that $c_1\boldsymbol{v}_1 + \cdots + c_k\boldsymbol{v}_k = \boldsymbol{0}$
- To check if $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ is linearly independent, $c_1\boldsymbol{v}_1 + \cdots + c_k\boldsymbol{v}_k = \boldsymbol{0}$ only has trivial solution, so
$$\begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_k & \mid & \boldsymbol{0} \end{pmatrix}$$
  has unique solution (only the trivial solution)
- **Thm** - If $\boldsymbol{0} \in S$, then S is linearly dependent.
- **Thm** - Let $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\} \subseteq \mathbb{R}^n$, $k \geq 2$. Then
  - S is linearly dependent $\iff \exists \boldsymbol{v}_i$ such that it is a linear combination of the other vectors
  - S is linearly independent $\iff$ no vector in S is the linear combination of the other vectors
- **Too many vectors** - Let $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\} \subseteq \mathbb{R}^n$. If $k > n$, then S is linearly dependent.
- **Extend linearly independent set** - Let $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ be linearly independent. If $\boldsymbol{v}_{k+1} \notin \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$, then $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k, \boldsymbol{v}_{k+1}\}$ is linearly independent.

Bases

- V is **vector space** if $V = \mathbb{R}^n$ or V is a subspace of $\mathbb{R}^n$
  If $W \subseteq V$, then W is **subspace** of V
- **Basis** 1. linearly independent 2. spanning set
- A basis for a vector space is the **smallest** spanning set, the **largest** linearly independent set.
  Any basis for $\mathbb{R}^n$ has exactly n vectors (if $< n$ then not spanning set, if $> n$ then linearly dependent).
- Let $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\} \subseteq V$ be a basis for V. Let $\boldsymbol{v} \in V$, then $\boldsymbol{v} = \sum_{i=1}^{k} c_i\boldsymbol{v}_i$ uniquely. The coefficients $c_1, \ldots, c_k$ are *the* **coordinates** of $\boldsymbol{v}$ relative to the basis S.
  The **coordinate vector** of $\boldsymbol{v}$ relative to the basis S is
$$(\boldsymbol{v})_S = (c_1, \ldots, c_k) \in \mathbb{R}^k.$$
- **Thm** - Let S be a basis of V.
  - $\boldsymbol{u} = \boldsymbol{v} \iff (\boldsymbol{u})_S = (\boldsymbol{v})_S$
  - $(c_1\boldsymbol{v}_1 + \cdots + c_r\boldsymbol{v}_r)_S = c_1(\boldsymbol{v}_1)_S + \cdots + c_r(\boldsymbol{v}_r)_S$
- **Thm** - Let S be a basis of V, where $|S| = k$.
  - $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r$ are linearly independent in $V \iff (\boldsymbol{v}_1)_S, \ldots, (\boldsymbol{v}_r)_S$ are linearly independent in $\mathbb{R}^k$
  - $\operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\} = V \iff \operatorname{span}\{(\boldsymbol{v}_1)_S, \ldots, (\boldsymbol{v}_r)_S\} = \mathbb{R}^k$

Dimensions

- **Thm** - Let V have a basis of k vectors.
  - Any subset of V with $> k$ vectors is linearly dependent;
  - Any subset of V with $< k$ vectors cannot span V.
- **Dimension**: length of basis
- **Thm** - Let $S \subseteq V$ and $\dim(V) = k$. TFAE:
  - S is a basis for V.
  - S is linearly independent and $|S| = k$.
  - S spans V and $|S| = k$.
- **Subspace is whole space** - Let U be a subspace of V. Then $U = V \iff \dim(U) = \dim(V)$.
- **Dim(subspace)** - If U is subspace of V, then $\dim(U) \leq \dim(V)$. Further if $U \neq V$, then $\dim(U) < \dim(V)$.

Transition Matrices

- **Transition matrix** from S to T is $P := ([\boldsymbol{u}_1]_T \quad \cdots \quad [\boldsymbol{u}_n]_T)$
- For all $\boldsymbol{w} \in V$, $[\boldsymbol{w}]_T = \boldsymbol{P}[\boldsymbol{w}]_S$
- To find transition matrix from bases $S = \{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\}$ to $T = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$:
  1. Perform Gauss-Jordan elimination on $\begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n & \mid & \boldsymbol{u}_1 & \cdots & \boldsymbol{u}_n \end{pmatrix}$
  2. Then we obtain coefficients to express $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ as linear combinations of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$
- **Thm** - Let S and T be bases for a vector space V, and $\boldsymbol{P}$ be transition matrix from S to T. Then $\boldsymbol{P}$ is invertible,
$$\boldsymbol{P}^{-1} \text{ is transition matrix from } T \text{ to } S.$$

4. Vector Spaces Associated with Matrices

Row Spaces and Column Spaces

- **Row space** is span of rows, **column space** is span of columns
- $\operatorname{Row}(\boldsymbol{A}) = \operatorname{Col}(\boldsymbol{A}^\top)$, $\operatorname{Col}(\boldsymbol{A}) = \operatorname{Row}(\boldsymbol{A}^\top)$
- **EROs preserve row space** - If $\boldsymbol{A}$ and $\boldsymbol{B}$ are row equivalent, then $\operatorname{Row}(\boldsymbol{A}) = \operatorname{Row}(\boldsymbol{B})$.
- Basis for $\operatorname{Row}(\boldsymbol{A})$ = non-zero rows of $\boldsymbol{R}$
  Basis for $\operatorname{Col}(\boldsymbol{A})$ = columns of $\boldsymbol{A}$ that correspond to pivot columns of $\boldsymbol{R}$
- Find a basis of a subspace of $\mathbb{R}^n$ spanned by $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$:
  1. Put $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ as rows, find row space of matrix
  2. Put $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ as columns, find column space of matrix
- **Thm** $\operatorname{Col}(\boldsymbol{A}) = \{\boldsymbol{Au} : \boldsymbol{u} \in \mathbb{R}^n\}$
$$\boldsymbol{Ax} = \boldsymbol{b} \text{ is consistent} \iff \boldsymbol{b} \in \operatorname{Col}(\boldsymbol{A}).$$

Ranks

- **Row rank equals column rank** - $\dim \mathrm{Row}(A) = \dim \mathrm{Col}(A)$
- **Rank** $\mathrm{rank}(A)$ is dimension of row/column space of A

  $\mathrm{rank}(\boldsymbol{A}) = \#$ non-zero rows of $\boldsymbol{R} = \#$ pivot columns in $\boldsymbol{R}$
- **Properties:**
  - $\mathrm{rank}(A) = \mathrm{rank}(A^\top)$
  - $\mathrm{rank}(A) = 0 \iff A = 0$
  - $\mathrm{rank}(A) \le \min\{m,n\}$ (largest no. of pivot columns in R)
    * A is **full rank** if $\mathrm{rank}(A) = \min\{m,n\}$
  - A square matrix A is full rank $\iff$ A is invertible
- **Thm** - $\mathrm{rank}(AB) \le \min\{\mathrm{rank}(A), \mathrm{rank}(B)\}$

Nullspaces and Nullities

- **Nullspace** $\mathrm{NS}(A) := \{u \in \mathbb{R}^n \mid Au = 0\}$ (solution space of $Ax = 0$)
- **Nullity** $\mathrm{nullity}(A)$ is dimension of nullspace

  If A is $m \times n$, then $\mathrm{nullity}(A) \le n$ ($\because$ $\mathrm{NS}(A)$ is subspace of $\mathbb{R}^n$)

  Nullity = # arbitrary parameters = # non-pivot columns in REF
- **Rank-nullity theorem** - $\mathrm{rank}(A) + \mathrm{nullity}(A) = n$
- **Thm** - Suppose $Ax = b$ has a solution u. Then solution set of $Ax = b$ is

  $$\{u + v \mid v \in \text{nullspace of } A\}.$$

  That is, $Ax = b$ has general solution

  $$x = (\text{particular soln to } Ax = b) + (\text{general soln for } Ax = 0)$$

  Consistent $Ax = b$ has only one soln $\iff$ $\mathrm{NS}(A) = \{0\}$

5. Orthogonality

Dot Product

- **Dot product** $u \cdot v := \sum_{i=1}^n u_i v_i$

  Norm $\|v\| := \sqrt{v \cdot v} = \sqrt{\sum_{i=1}^n v_i{}^2}$

  Distance $d(u,v) := \|u - v\| = \sqrt{\sum_{i=1}^n (u_i - v_i)^2}$

  Angle $\theta = \cos^{-1}\left(\frac{u \cdot v}{\|u\|\|v\|}\right)$
- If u and v are column vectors, then $u \cdot v = u^\top v$

  If u and v are row vectors, then $u \cdot v = uv^\top$

Orthogonal and Orthonormal Bases

- $u, v \in \mathbb{R}^n$ are **orthogonal** if $u \cdot v = 0$

  S is **orthogonal** if $v_i \cdot v_j = 0 \ \forall i \ne j$

  S is **orthonormal** if S is orthogonal, every vector is unit vector
- **Thm** - Orthogonal **non-zero** vectors is linearly independent.
- Basis is **orthogonal basis** if orthogonal, **orthonormal basis** if orthonormal
- **Thm** - $\{u_1, \ldots, u_k\}$ orthogonal basis

  $$w = \frac{w \cdot u_1}{u_1 \cdot u_1} u_1 + \cdots + \frac{w \cdot u_k}{u_k \cdot u_k} u_k$$

  $\{v_1, \ldots, v_k\}$ orthonormal basis

  $$w = (w \cdot v_1) v_1 + \cdots + (w \cdot v_k) \cdot v_k$$
- $u \in \mathbb{R}^n$ is **orthogonal** to V if $u \cdot v = 0$ for all $v \in V$.
- Let $V = \mathrm{span}\{u_1, \ldots, u_k\}$ be a subspace of $\mathbb{R}^n$. For $v \in \mathbb{R}^n$,

  $$v \text{ is orthogonal to } V \iff v \cdot u_i = 0 \qquad (i = 1, \ldots, k).$$
- Every $u = n + p$, where n is orthogonal to V, $p \in V$

  p is **orthogonal projection** of u onto V
- **Formula** - $\{u_1, \ldots, u_k\}$ orthogonal basis

  $$\mathrm{proj}_V(w) = \frac{w \cdot u_1}{u_1 \cdot u_1} u_1 + \cdots + \frac{w \cdot u_k}{u_k \cdot u_k} u_k$$
- **Gram-Schmidt process** - Convert basis to orthogonal basis

  $$v_1 = u_1 \qquad v_2 = u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1} v_1 \qquad v_3 = u_3 - \frac{u_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{u_3 \cdot v_2}{v_2 \cdot v_2} v_2$$

Best Approximations

- **Thm** - $p = \mathrm{proj}_V(u)$ is the **best approximation** of u in V:

  $$d(u,p) \le d(u,v) \qquad (v \in V).$$
- $u \in \mathbb{R}^n$ is **least squares solution** to $Ax = b$ if

  $$\|b - Au\| \le \|b - Av\| \qquad (v \in \mathbb{R}^n).$$
- **Thm** - u is a least squares solution to $Ax = b$ $\iff$ u is a solution to $Ax = p$
- To find least squares solution to $Ax = b$:
  1. Find an orthogonal basis for $\mathrm{Col}(A)$.
  2. Find p, the orthogonal projection of b onto $\mathrm{Col}(A)$.
  3. Solve $Ax = p$.
- u is a least squares solution to $Ax = b$ $\iff$ u is a solution to $A^\top Ax = A^\top b$.
- **Least squares method**: find best approximation of a function
  1. Substitute given data points to form $Ax = b$.
  2. Solve $A^\top Ax = A^\top b$ to find coefficients.

Orthogonal Matrices

- **Orthogonal**: $A^{-1} = A^\top \iff A^\top A = AA^\top = I$
- **Properties:**
  - If A orthogonal, so is $A^\top$
  - If A, B orthogonal, then AB is orthogonal
- **Thm** - A is orthogonal

  $\iff$ rows of A form orthonormal basis for $\mathbb{R}^n$

  $\iff$ columns of A form orthonormal basis for $\mathbb{R}^n$
- **Thm** - Let S and T be orthonormal bases. Transition matrix P from S to T is orthogonal,

  $P^\top$ is transition matrix from T to S

6. Diagonalisation

Eigenvalues and Eigenvectors

- **Eigenvector**, **eigenvalue**: $Au = \lambda u$ where $u \in \mathbb{R}^n$, $u \ne 0$

  $\iff (\lambda I - A)u = 0$ for some $u \in \mathbb{R}^n$, $u \ne 0$

  $\iff (\lambda I - A)x = 0$ has non-trivial solutions (Gaussian)

  $\iff (\lambda I - A)x$ is singular

  $\iff \det(\lambda I - A) = 0.$ (find det)
- **Characteristic polynomial** $\det(\lambda I - A)$

  Characteristic equation $\det(\lambda I - A) = 0$

  Roots of characteristic equation are eigenvalues of A
- **Properties:** Same char eqn $\implies$ same set of eigenvalues
  - A and $A^\top$ have same char poly
  - If P is invertible, A and $P^{-1}AP$ have same char poly
  - If char poly of A is $p(\lambda) = a_0 + \cdots + a_{n-1}\lambda^{n-1} + \lambda^n$, then $a_0 = p(0) = \det(0I - A) = \det(-A) = (-1)^n \det(A)$
- **Thm** - If A is triangular, then eigenvalues are diagonal entries
- **Eigenspace**: solution space of $(\lambda I - A)x = 0$

  $$E_\lambda := \{x \in \mathbb{R}^n \mid (\lambda I - A)x = 0\}$$

  If $u \in E_\lambda$ is non-zero, then u is eigenvector of A associated with eigenvalue λ

  Solve $(\lambda I - A)x = 0$ by Gaussian elimination to get general solution, to write E_λ explicitly

Diagonalisation

- **Diagonalisable**: $P^{-1}AP = D \iff A = PDP^{-1}$ for some invertible P, diagonal D
- **Properties:** If A, B diagonalisable,
  - $A^\top$ diagonalisable
  - A^2 diagonalisable
  - If same inv P diagonalises A, B, then AB is diagonalisable
  - $I + A$ is diagonalisable
- **Thm** - Diagonalisable $\iff$ n linearly independent eigenvectors ($\mathbb{R}^n$ needs to have a basis of eigenvectors)
- **Alg** - Determine if A is diagonalisable
  1. Find all distinct eigenvalues (solve char eqn)
  2. Diagonalisable $\iff$ algebraic = geometric multiplicities
  3. For each eigenvalue, find a basis for eigenspace
  4. Put basis vectors together to form $P = \begin{pmatrix} u_1 & \cdots & u_n \end{pmatrix}$
- **Thm** - n distinct eigenvalues $\implies$ diagonalisable

Orthogonal Diagonalisation

- **Orthogonally diagonalisable**: $P^\top AP = D \iff A = PDP^\top$ for some orthogonal P, diagonal D
- **Spectral thm** - Orthogonally diagonalisable $\iff$ symmetric
- **Properties:**
  - $A + B$ is orthogonally diagonalisable
  - If $AB = BA$, then AB is orthogonally diagonalisable
  - If u and v are eigenvectors of A associated with eigenvalues λ and μ, then $u \cdot v = 0$:

    $$\lambda(u \cdot v) = (\lambda u)^\top v = (Au)^\top v = u^\top A^\top v = u^\top Av = u^\top(\mu v) = \mu(u \cdot v)$$
- **Alg** - Given symmetric matrix A, find orthogonal matrix P
  1. Find all distinct eigenvalues
  2. For each eigenvalue, find a basis for eigenspace

     Convert basis into orthonormal basis (Gram-Schmidt)
  3. Put basis vectors together to form $P = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$

Quadratic Forms

- **Quadratic form** $Q(x_1, \ldots, x_n) := \sum_{i=1}^n \sum_{j=1}^n q_{ij} x_i x_j$

  $$\begin{aligned} & q_{11}x_1{}^2 + q_{12}x_1x_2 + \cdots + q_{1n}x_1x_n \\ & + q_{22}x_2{}^2 + \cdots + q_{2n}x_2x_n \\ & + \cdots\cdots \\ & + q_{nn}x_n{}^2 \end{aligned}$$

  $$Q(x_1, \ldots, x_n) = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} q_{11} & \frac{1}{2}q_{12} & \cdots & \frac{1}{2}q_{1n} \\ \frac{1}{2}q_{12} & q_{22} & \cdots & \frac{1}{2}q_{2n} \\ \vdots & \vdots & & \vdots \\ \frac{1}{2}q_{1n} & \frac{1}{2}q_{2n} & \cdots & q_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x^\top Ax$$

  (Diagonal terms of A are coefficients for square terms; non-diagonal terms are half of coefficients for cross terms.)

  Thus quadratic form is a mapping $Q\colon \mathbb{R}^n \to \mathbb{R}$, $Q(x) = x^\top Ax$
- **Quadratic equation** $ax^2 + bxy + cy^2 + dx + ey = f$

  $$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} d & e \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = f$$
- **Alg** - Make A diagonal
  1. Since A is symmetric, find orthogonal P that orthogonally diagonalises A
  2. Let $y = P^\top x$. Then quadratic form becomes

     $$x^\top Ax = x^\top(PDP^\top)x = (P^\top x)^\top D(P^\top x) = y^\top Dy$$

  In the case of two variables, let $\begin{pmatrix} x' \\ y' \end{pmatrix} = P^\top \begin{pmatrix} x \\ y \end{pmatrix}$.

7. Linear Transformations

Linear Transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$

- **Linear transformation** $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is $T(u) = Au$

  A is the **standard matrix** for T
- **Linearity** - $T(0) = 0$,

  $$T(c_1u_1 + \cdots + c_ku_k) = c_1T(u_1) + \cdots + c_kT(u_k)$$
- Standard matrix $A = \begin{pmatrix} T(e_1) & \cdots & T(e_n) \end{pmatrix}$
- **Composition** $(T \circ S)(u) = T(S(u))$

  Standard matrix for $T \circ S$ is BA

Ranges and Kernels

- **Range** $R(T) = \{T(u) : u \in \mathbb{R}^n\}$
- **Thm** - $\{u_1, \ldots, u_n\}$ a basis for $\mathbb{R}^n$, then

  $$R(T) = \mathrm{span}\{T(u_1), \ldots, T(u_n)\}$$

  If we choose standard basis for $\mathbb{R}^n$, then

  $$R(T) = \mathrm{span}\{T(e_1), \ldots, T(e_n)\} = \mathrm{Col}(A)$$

  since $T(e_i)$ is the i-th column of A.
- **Rank** $\mathrm{rank}(T) := \dim R(T) \implies \mathrm{rank}(T) = \mathrm{rank}(A)$
- **Kernel** $\mathrm{Ker}(T) := \{u \in \mathbb{R}^n \mid T(u) = 0\} \implies \mathrm{Ker}(T) = \mathrm{NS}(A)$
- **Nullity** $\mathrm{nullity}(T) := \dim \mathrm{Ker}(T) \implies \mathrm{nullity}(T) = \mathrm{nullity}(A)$
- **Rank-nullity theorem** - $\mathrm{rank}(T) + \mathrm{nullity}(T) = n$
- Inj: $\exists$ linear trans $T\colon \mathbb{R}^n \to \mathbb{R}^m$ with $\ker T = \{0\} \iff m \ge n$

  Surj: $\exists$ linear trans $T\colon \mathbb{R}^n \to \mathbb{R}^m$ with $R(T) = \mathbb{R}^m \iff m \le n$