Notes

Singapore Physics Olympiad 2023

Topic 1: Kinematics of Motion

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Scalars and vectors

Notes

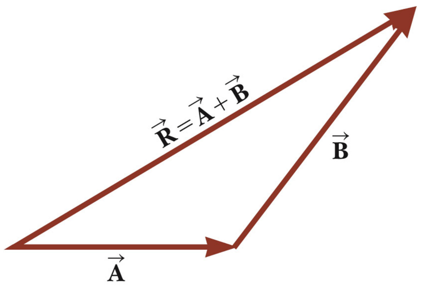
- A **scalar** is a physical quantity which is characterized by its *magnitude*, for instance, mass, speed, energy, etc

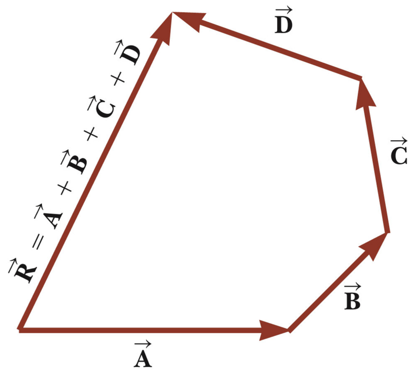
- A **vector** is a physical quantity with both *magnitude* and *direction*, for instance, position, force, momentum, etc

- A vector, denoted by $\vec{\mathbf{A}}$, is a quantity that can be graphically represented as a directed line segment
  - Direction of the vector is given by the arrow
  - Magnitude of the vector is *proportional* to the length of the line segment

Vector addition (tail-to-tip)

- **Resultant vector** is the vector drawn from the tail of the first vector to the tip of the last vector



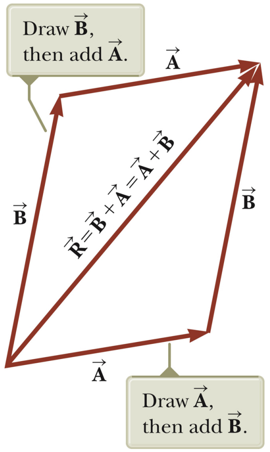


Notes

Vector addition is commutative

- When two vectors are added, the sum is independent of the order of the addition

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$

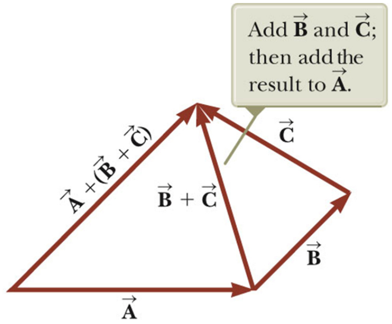


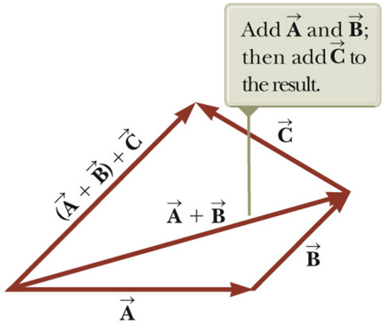
Notes

Vector addition is associative

- When three or more vectors are added, their sum is independent of the way in which the individual vectors are grouped together

$$\vec{\mathbf{A}} + \left(\vec{\mathbf{B}} + \vec{\mathbf{C}}\right) = \left(\vec{\mathbf{A}} + \vec{\mathbf{B}}\right) + \vec{\mathbf{C}}$$



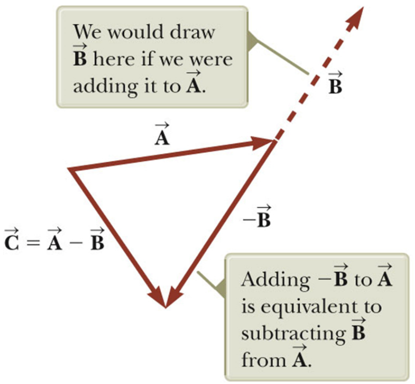


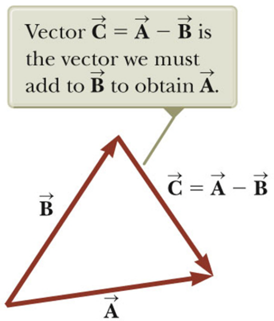
Notes

Vector subtraction

- When a vector is multiplied by -1, the magnitude of the vector remains the same but the direction of the vector is reversed

$$\vec{\mathbf{A}} - \vec{\mathbf{B}} = \vec{\mathbf{A}} + \left(-\vec{\mathbf{B}}\right)$$





Notes

Vector and coordinate system

Notes

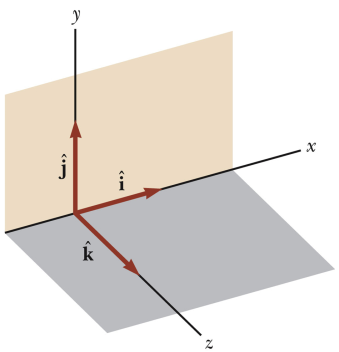
- Vectors are geometrical quantities; however, it is always convenient to specify a vector with respect to a **coordinate system**
- Three numbers are needed to specify a vector in three-dimensional space and these numbers depend on the choice of a coordinate system
- Laws of Physics do not require the specification of a coordinate system and must hold irrespective of the coordinate system
- Solutions of physical problems, however, require that the relations derived from these laws be expressed in a coordinate system appropriate to the geometry of the given problems

Unit vectors

Notes

- A *unit vector* is a dimensionless vector with a magnitude of exactly 1
- Unit vectors are used to specify a direction and have no other physical significance
- Unit vectors $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ are pointing in the *positive* x, y and z directions respectively

$$|\hat{\boldsymbol{\imath}}| = |\hat{\boldsymbol{\jmath}}| = \left|\hat{\boldsymbol{k}}\right| = 1$$



- $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ form a set of mutually perpendicular vectors in a *right-handed* coordinate system

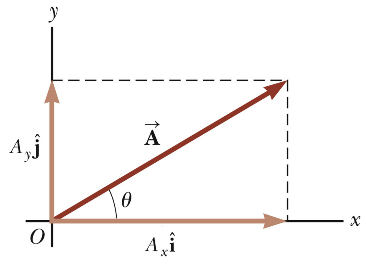
Components of a vector in two dimensions

- Projections of vector along coordinate axes are called the **components** of the vector

- A vector $\vec{\mathbf{A}}$ lying in the xy-plane can be expressed as the sum of two **component vectors** $\vec{\mathbf{A}}_x$ and $\vec{\mathbf{A}}_y$:

$$\vec{\mathbf{A}} = \vec{\mathbf{A}}_x + \vec{\mathbf{A}}_y = A_x\,\hat{\imath} + A_y\,\hat{\jmath} = A\cos\theta\,\hat{\imath} + A\sin\theta\,\hat{\jmath}$$

$$A \equiv \left|\vec{\mathbf{A}}\right| = \sqrt{A_x^2 + A_y^2}\,, \qquad \theta = \tan^{-1}\left(\frac{A_y}{A_x}\right)$$



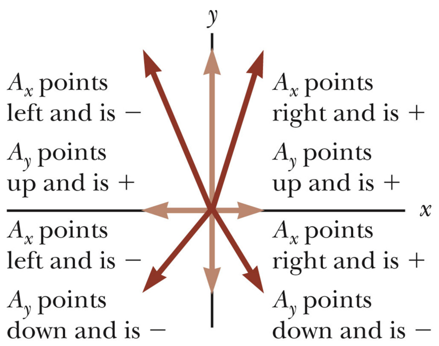
The angle θ is measured counter-clockwise from the positive x-axis.

Notes

Components are SIGNED scalar quantities

- Components of vector can be either positive or negative

- Component A_x (A_y) is positive if the component vector $\vec{\mathbf{A}}_x$ ($\vec{\mathbf{A}}_y$) points in the positive x (y) direction and is negative if $\vec{\mathbf{A}}_x$ ($\vec{\mathbf{A}}_y$) points in the negative x (y) direction



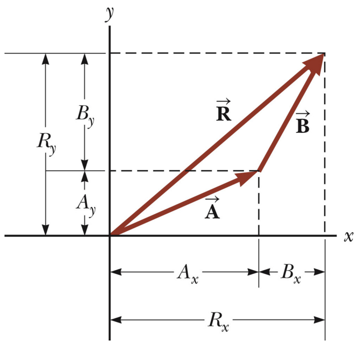
Notes

Vector addition with components

Notes

- If A_x and A_y are the components of vector $\vec{\mathbf{A}}$, and B_x and B_y are the components of vector $\vec{\mathbf{B}}$, the components R_x and R_y of the resultant vector $\vec{\mathbf{R}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$ are given by $R_x = A_x + B_x$ and $R_y = A_y + B_y$

$$\begin{aligned}\vec{\mathbf{R}} &= \vec{\mathbf{A}} + \vec{\mathbf{B}} = (A_x\,\hat{\boldsymbol{\imath}} + A_y\,\hat{\boldsymbol{\jmath}}) + (B_x\,\hat{\boldsymbol{\imath}} + B_y\,\hat{\boldsymbol{\jmath}}) \\ &= (A_x + B_x)\,\hat{\boldsymbol{\imath}} + (A_y + B_y)\,\hat{\boldsymbol{\jmath}} = R_x\,\hat{\boldsymbol{\imath}} + R_y\,\hat{\boldsymbol{\jmath}}\end{aligned}$$



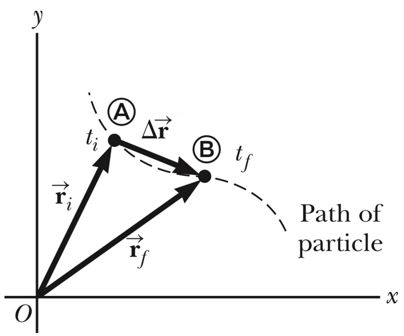
Vector	x-component	y-component
$\vec{\mathbf{A}}$	A_x	A_y
$\vec{\mathbf{B}}$	B_x	B_y
$\vec{\mathbf{R}}$	R_x	R_y

Position vectors

Notes

- Describing motion requires some convenient *coordinate system* and a specific *origin* – **frame of reference**

- **Position vector** describes the position of an object in the space at an instant of time



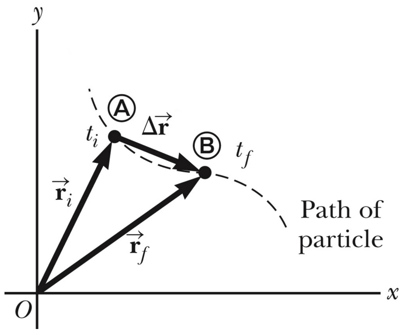
$$\vec{\mathbf{r}}(t) = x(t)\,\hat{\boldsymbol{\imath}} + y(t)\,\hat{\boldsymbol{\jmath}}$$

$$\begin{cases}\vec{\mathbf{r}}_i \equiv \vec{\mathbf{r}}(t_i) = x_i\,\hat{\boldsymbol{\imath}} + y_i\,\hat{\boldsymbol{\jmath}} \\ \vec{\mathbf{r}}_f \equiv \vec{\mathbf{r}}(t_f) = x_f\,\hat{\boldsymbol{\imath}} + y_f\,\hat{\boldsymbol{\jmath}}\end{cases}$$

Displacement vectors

Notes

- Motion involves the change in position of an object from one point in space and time to another
- **Displacement**: change of position vectors



$$\boxed{\Delta\vec{\mathbf{r}} \equiv \vec{\mathbf{r}}_f - \vec{\mathbf{r}}_i}$$

SI unit: meter (m)

Distance travelled by the particle along the path is not given by $|\Delta\vec{\mathbf{r}}|$.

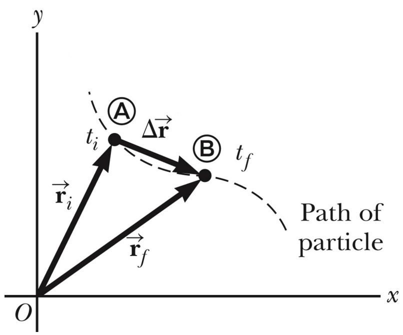
Average velocity

Notes

- **Average velocity**: ratio of the displacement to the time interval for the displacement

$$\boxed{\vec{\mathbf{v}}_{\mathsf{av}} \equiv \frac{\Delta\vec{\mathbf{r}}}{\Delta t}}$$

SI unit: meter per second $(\mathrm{m/s})$



$$\begin{aligned}\vec{\mathbf{v}}_{\mathsf{av}} &= \frac{x_f - x_i}{t_f - t_i}\,\hat{\boldsymbol{\imath}} + \frac{y_f - y_i}{t_f - t_i}\,\hat{\boldsymbol{\jmath}} \\ &= \frac{\Delta x}{\Delta t}\,\hat{\boldsymbol{\imath}} + \frac{\Delta y}{\Delta t}\,\hat{\boldsymbol{\jmath}}\end{aligned}$$

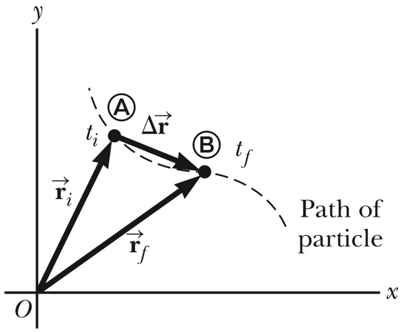
The average velocity between points is *independent* of the path taken.

Average speed

- **Average speed**: ratio of the length of the path travelled to the time interval

$$\boxed{v_{\text{av}} \equiv \frac{\text{path length}}{\Delta t}}$$

SI unit: meter per second $(\mathrm{m/s})$



Average speed is not given by the magnitude of the average velocity $|\vec{\mathbf{v}}_{\text{av}}|$.

Notes

Instantaneous velocity and speed

- **Instantaneous velocity**: limit of the average velocity as Δt approaches zero

$$\boxed{\vec{\mathbf{v}}(t) \equiv \lim_{\Delta t \to 0} \frac{\Delta \vec{\mathbf{r}}}{\Delta t} = \frac{\mathrm{d}\vec{\mathbf{r}}(t)}{\mathrm{d}t}}$$

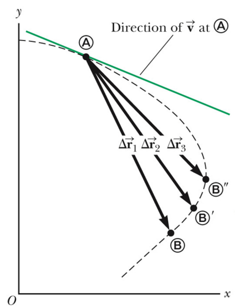
SI unit: meter per second $(\mathrm{m/s})$

$$\vec{\mathbf{v}}(t) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}\,\hat{\boldsymbol{\imath}} + \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}\,\hat{\boldsymbol{\jmath}} = \frac{\mathrm{d}x(t)}{\mathrm{d}t}\,\hat{\boldsymbol{\imath}} + \frac{\mathrm{d}y(t)}{\mathrm{d}t}\,\hat{\boldsymbol{\jmath}} = v_x(t)\,\hat{\boldsymbol{\imath}} + v_y(t)\,\hat{\boldsymbol{\jmath}}$$

- **Instantaneous speed**: magnitude of the instantaneous velocity

$$v(t) \equiv |\vec{\mathbf{v}}(t)| = \sqrt{v_x^2(t) + v_y^2(t)}$$

The direction of the instantaneous velocity is along a line that is *tangent* to the path of the object and in the direction of motion.



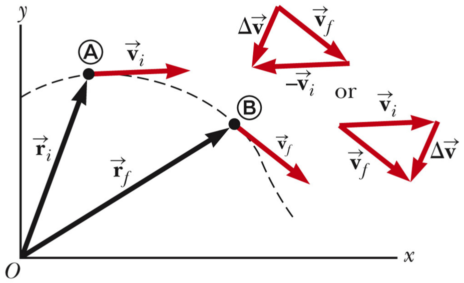
Notes

Average acceleration

- **Average acceleration**: ratio of the change of instantaneous velocity to the time interval for the change

$$\boxed{\vec{\mathbf{a}}_{\mathsf{av}} \equiv \frac{\Delta \vec{\mathbf{v}}}{\Delta t}}$$

SI unit: meter per second squared $(\mathrm{m/s^2})$



$$\vec{\mathbf{a}}_{\mathsf{av}} = \frac{v_{xf} - v_{xi}}{t_f - t_i}\,\hat{\boldsymbol{\imath}} + \frac{v_{yf} - v_{yi}}{t_f - t_i}\,\hat{\boldsymbol{\jmath}}$$

$$= \frac{\Delta v_x}{\Delta t}\,\hat{\boldsymbol{\imath}} + \frac{\Delta v_y}{\Delta t}\,\hat{\boldsymbol{\jmath}}$$

$$= a_{\mathsf{av},x}\,\hat{\boldsymbol{\imath}} + a_{\mathsf{av},y}\,\hat{\boldsymbol{\jmath}}$$

Notes

Instantaneous acceleration

- **Instantaneous acceleration**: limit of the average acceleration as Δt approaches zero

$$\boxed{\vec{\mathbf{a}}(t) \equiv \lim_{\Delta t \to 0} \frac{\Delta \vec{\mathbf{v}}}{\Delta t} = \frac{\mathrm{d}\vec{\mathbf{v}}(t)}{\mathrm{d}t} = \frac{\mathrm{d}^2\vec{\mathbf{r}}(t)}{\mathrm{d}t^2}}$$

SI unit: meter per second squared $(\mathrm{m/s^2})$

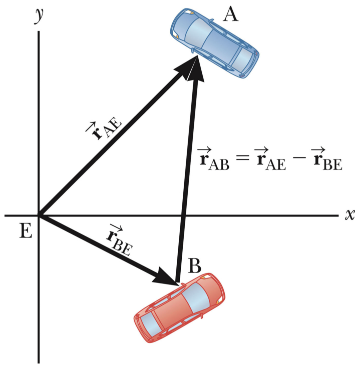
$$\vec{\mathbf{a}}(t) = \lim_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t}\,\hat{\boldsymbol{\imath}} + \lim_{\Delta t \to 0} \frac{\Delta v_y}{\Delta t}\,\hat{\boldsymbol{\jmath}} = \frac{\mathrm{d}v_x(t)}{\mathrm{d}t}\,\hat{\boldsymbol{\imath}} + \frac{\mathrm{d}v_y(t)}{\mathrm{d}t}\,\hat{\boldsymbol{\jmath}} = a_x(t)\,\hat{\boldsymbol{\imath}} + a_y(t)\,\hat{\boldsymbol{\jmath}}$$

An object can accelerate in several ways: (1) speed may change with time; (2) direction of velocity may change with time; and (3) both magnitude and direction of velocity may change at the same time.

Notes

Relative position

- Position of object 1 with respect to object 2: $\vec{\mathbf{r}}_{12}(t)$
- Position of object 2 with respect to object 1: $\vec{\mathbf{r}}_{21}(t) = -\vec{\mathbf{r}}_{12}(t)$



$$\boxed{\vec{\mathbf{r}}_{AB}(t) = \vec{\mathbf{r}}_{AE}(t) + \vec{\mathbf{r}}_{EB}(t)}$$

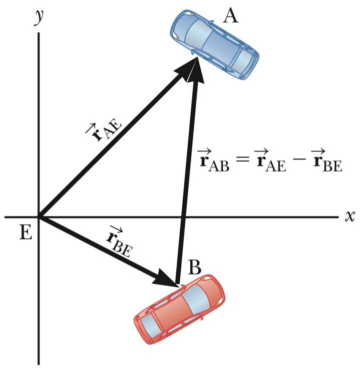
E : *common* reference frame

$$\vec{\mathbf{r}}_{EB}(t) = -\vec{\mathbf{r}}_{BE}(t)$$
$$\Rightarrow \quad \vec{\mathbf{r}}_{AB}(t) = \vec{\mathbf{r}}_{AE}(t) - \vec{\mathbf{r}}_{BE}(t)$$

Notes

Relative velocity and acceleration

- Concept of time is **ABSOLUTE** in classical Physics



$$\vec{\mathbf{r}}_{AB}(t) = \vec{\mathbf{r}}_{AE}(t) + \vec{\mathbf{r}}_{EB}(t)$$
$$\Rightarrow \quad \boxed{\vec{\mathbf{v}}_{AB}(t) = \vec{\mathbf{v}}_{AE}(t) + \vec{\mathbf{v}}_{EB}(t)}$$
$$\Rightarrow \quad \boxed{\vec{\mathbf{a}}_{AB}(t) = \vec{\mathbf{a}}_{AE}(t) + \vec{\mathbf{a}}_{EB}(t)}$$

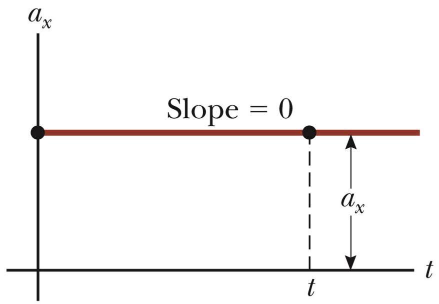
The above are valid only if the speed is *much slower* than the speed of light!

Notes

1D constantly accelerated motion: $a_x(t)$ versus t

Notes

- When an object moves with constant acceleration, the instantaneous acceleration at *any instant of time* is equal to the value of the average acceleration over the *entire time interval*



$$\boxed{a_x(t) = a_{\mathsf{av},x} = a_x}$$

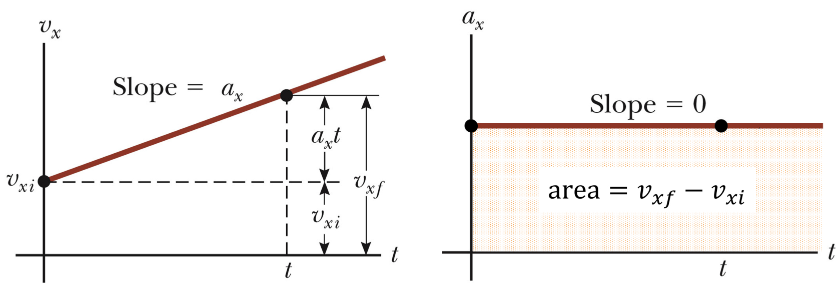
1D constantly accelerated motion: $v_x(t)$ versus t

Notes

$$a_x(t) = \frac{\mathrm{d}v_x(t)}{\mathrm{d}t} = a_x \quad \Rightarrow \quad \mathrm{d}v_x = a_x\,\mathrm{d}t \quad \Rightarrow \quad \int_{v_{xi}}^{v_{xf}} \mathrm{d}v_x = a_x \int_{t_i}^{t_f} \mathrm{d}t$$

$$\Rightarrow \quad v_{xf} = v_{xi} + a_x\,(t_f - t_i)$$

$$\Rightarrow \quad \boxed{v_x(t) = v_{xi} + a_x\,(t - t_i)}$$



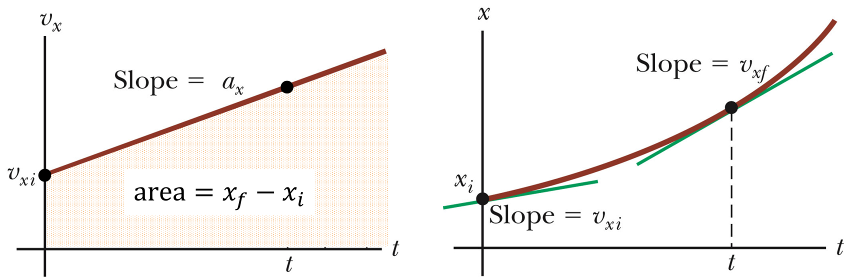
1D constantly accelerated motion: $x(t)$ versus t

Notes

$$v_x(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t} \quad \Rightarrow \quad \mathrm{d}x = v_x(t)\,\mathrm{d}t \quad \Rightarrow \quad \int_{x_i}^{x_f} \mathrm{d}x = \int_{t_i}^{t_f} \left[v_{xi} + a_x\,(t - t_i)\right]\,\mathrm{d}t$$

$$\Rightarrow \quad x_f = x_i + v_{xi}\,(t_f - t_i) + \frac{1}{2}\,a_x\,(t_f - t_i)^2$$

$$\Rightarrow \quad \boxed{x(t) = x_i + v_{xi}\,(t - t_i) + \frac{1}{2}\,a_x\,(t - t_i)^2}$$



1D constantly accelerated motion: $v_x(x)$ versus x

Notes

- Eliminating time dependence:

$$a_x(t) = \frac{\mathrm{d}v_x(t)}{\mathrm{d}t} = \frac{\mathrm{d}v_x(x)}{\mathrm{d}x}\,\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \frac{\mathrm{d}v_x(x)}{\mathrm{d}x}\,v_x(x) = a_x$$

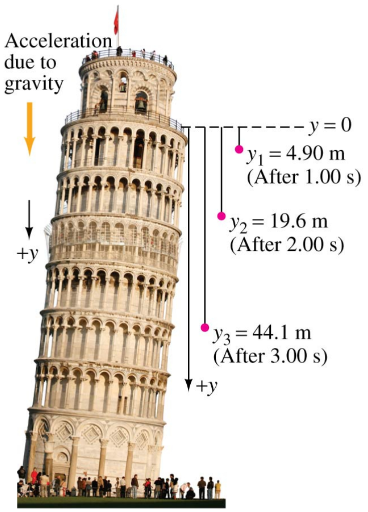
$$\Rightarrow \quad \int_{v_{xi}}^{v_{xf}} v_x\,\mathrm{d}v_x = a_x \int_{x_i}^{x_f} \mathrm{d}x \quad \Rightarrow \quad v_{xf}^2 = v_{xi}^2 + 2a_x\,(x_f - x_i)$$

$$\Rightarrow \quad \boxed{v_x^2(x) = v_{xi}^2 + 2a_x\,(x - x_i)}$$

- It gives final velocity in terms of acceleration and displacement *but* it does not give any information about the time

Free falling objects

- A free falling object is any object under the influence of **gravity** alone
- Ignoring air resistance and assuming the object is falling near the Earth's surface, free fall is constantly accelerated motion
- The acceleration is called the **acceleration due to gravity** and indicated by g with magnitude $\approx 9.8\,\mathrm{m/s^2}$
- $\vec{g}$ is always directed downwards, i.e., toward the center of the Earth



Notes

Kinematic equations for free falling objects

- One dimensional motion along the y-direction with constant acceleration a_y
- Acceleration: $+y$-direction is vertically upward

$$a_y(t) = -g = -9.8\,\mathrm{m/s^2}$$

- Kinematic equations: $t_i = 0$, $y_i = y_0$, $v_{yi} = v_{0y}$

$$\begin{cases} v_y(t) = v_{0y} - gt \\ y(t) = y_0 + v_{0y}t - \dfrac{1}{2}gt^2 \\ v_y^2(y) = v_{0y}^2 - 2g\,(y - y_0) \end{cases}$$

- The highest point of the motion is the *turning point* with zero instantaneous vertical velocity, i.e., $v_y = 0$. However, the acceleration throughout the entire motion is $a_y = -g$

Notes

Two dimensional motion with constant a_x and a_y

- Horizontal and vertical motions are completely *independent* from each other
- Convention: $+x$-direction is horizontally rightward, $+y$-direction is vertically upward
- Kinematic equations: $t_i = 0$, $x_i = x_0$, $y_i = y_0$, $v_{xi} = v_{0x}$, $v_{yi} = v_{0y}$

Horizontal Motion

$$v_x(t) = v_{0x} + a_x t$$

$$x(t) = x_0 + v_{0x} t + \frac{1}{2} a_x t^2$$

$$v_x^2(x) = v_{0x}^2 + 2a_x (x - x_0)$$

Vertical Motion

$$v_y(t) = v_{0y} + a_y t$$

$$y(t) = y_0 + v_{0y} t + \frac{1}{2} a_y t^2$$

$$v_y^2(y) = v_{0y}^2 + 2a_y (y - y_0)$$

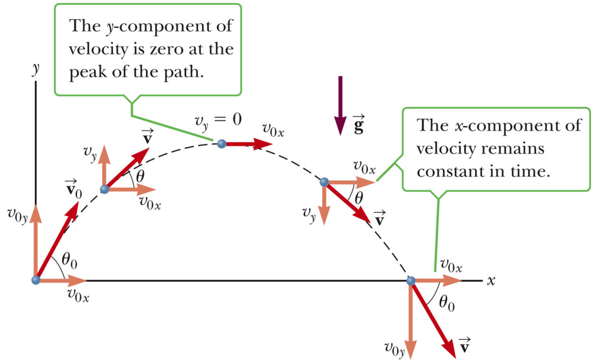
Notes

Two dimensional free falling motion

- Acceleration:

$$a_x(t) = a_x = 0\,, \qquad a_y(t) = a_y = -g$$

- Superposition of motions in two mutually perpendicular directions: constant velocity horizontal motion and free fall vertical motion



Notes

Kinematics for two dimensional free falling motion

- Initial conditions:

$$v_{0x} = v_0 \cos\theta_0 \,, \qquad v_{0y} = v_0 \sin\theta_0$$

- Velocities: $a_x(t) = 0$, $a_y(t) = -g$

$$v_x(t) = v_{0x} = v_0 \cos\theta_0 \,, \qquad v_y(t) = v_{0y} - gt = v_0 \sin\theta_0 - gt$$

- Displacements: $a_x(t) = 0$, $a_y(t) = -g$

$$x(t) = x_0 + v_{0x} t = x_0 + v_0 \cos\theta_0 \, t$$

$$y(t) = y_0 + v_{0y} t - \frac{1}{2} g t^2 = y_0 + v_0 \sin\theta_0 \, t - \frac{1}{2} g t^2$$

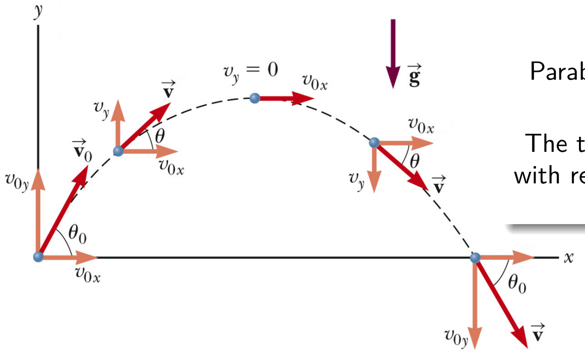
Notes

Trajectory of two dimensional free falling motion

$$x(t) = x_0 + v_0 \cos\theta_0 \, t \quad \Rightarrow \quad t = \frac{x(t) - x_0}{v_0 \cos\theta_0}$$

$$y(t) = y_0 + v_0 \sin\theta_0 \, t - \frac{1}{2} g t^2$$

$$\Rightarrow \quad y(t) = y_0 + \tan\theta_0 \left[x(t) - x_0\right] - \left(\frac{g}{2 v_0^2 \cos^2\theta_0}\right) \left[x(t) - x_0\right]^2$$



Parabolic: $y = ax^2 + bx + c$

The trajectory is symmetrical with respect to the peak of the motion.

Notes

Problem 1

Notes

The acceleration of a marble in a certain fluid is proportional to the speed of the marble squared and is given by $a = -kv^2$ where $k = 3.00\,\mathrm{m}^{-1}$ for $v > 0$. If the marble enters this fluid with a speed of $1.50\,\mathrm{m/s}$, how long will it take before the marble's speed is half of its initial value?

Problem 2

Notes

Ship A is $10\,\mathrm{km}$ due west of ship B. Ship A is heading directly north at a speed of $30\,\mathrm{km/h}$ while ship B is heading in a direction 60° west of north at a speed of $20\,\mathrm{km/h}$. What will be their distance of closest approach?

Problem 3

Notes

An elevator starts from rest and maintains a constant upward acceleration of $1.20\,\mathrm{m/s^2}$. A bolt in the elevator ceiling $2.70\,\mathrm{m}$ above the elevator floor works loose and falls out $2.00\,\mathrm{s}$ after the start.

(a) How long does it take for the bolt to fall to the elevator floor?

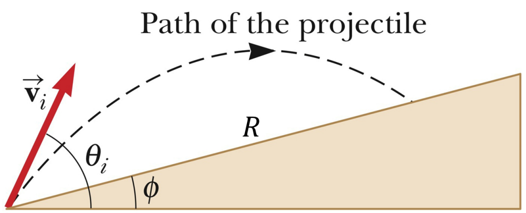
(b) What are the speeds of the bolt just as it hits the elevator floor **(i)** according to an observer in the elevator and **(ii)** according to an observer standing on one of the floor landings of the building respectively?

(c) What are the displacement and distance traveled by the bolt during the free fall according to both observers in **(b)** respectively?

Problem 4

Notes

A projectile is fired up an incline (incline angle ϕ) with an initial speed v_i at an angle θ_i with respect to the horizontal ($\theta_i > \phi$). Find the direction in which it should be aimed to achieve the maximum range along the incline. What is this maximum range?



Problem 5

At $t = 0$ on the planet X, a projectile is fired with speed v_0 at an angle θ above the horizontal. This planet is a strange one, in that the acceleration due to gravity increases linearly with time, starting with a value of zero when the projectile is fired from the ground, i.e., $g(t) = \alpha t$. What horizontal distance does the projectile travel? What should θ be to maximize this distance?

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