Mathematical demonstration of that “Monotone heuristic is admissible”

Heuristic:

A heuristic is any criteria, method, or principle to decide between a set of alternative paths which could be the most effective to reach the goal.

$h(s)$ = heuristic estimate from s to the goal.

$c(s)$ = optimal cost to the goal s

Admissible Heuristic (H):

H is admissible $\leftrightarrow$ H never overestimates true cost to the goal

H admissible $\leftrightarrow$ for all $s \in S$: $h(s) \leq c(s)$

Consistent/Monotone Heuristic (H):

h (consistent) $\leftrightarrow$ 1. $h(Goal)=0$

2. $h(s) \leq c(s,n+1) + h(n+1)$ for all $s \in S$ and all $n \in$ neighbors(S)

- $c(s,n)$ = step cost from s to n

- neighbors(s) = set of all states one step from s

In other words, heuristic value of the goal is 0 and for all states the heuristic value of each state is less than or equal than the cost to reach the neighbour plus the heuristic value of that neighbour.

Demonstration:

In the following lines I will demonstrate that h(consistent) $\rightarrow$ h (admissible) trough induction.

Base Case:

We begin by considering the $n-1$th node in any path where n denotes the goal state.

$$h(n-1) \leq c(n-1, n) + h(n)$$

Because n is the goal state, by definition, $h(n) = h*(n)$. Therefore, we can rewrite the above as

$$h(n-1) \leq c(n-1, n) + h*(n)$$

and given that $c(n-1, n) + h*(n) = h*(n-1)$, we can see:

$$h(n-1) \leq h*(n-1)$$

Here we see the definition of admissibility.

To check if always happens the same, we consider the $n-2$nd node in any of the paths we considered above (where there is precisely one node between it and the goal state). The cost to get from this node to the goal state can be written as

$$h(n-2) \leq c(n-2, n-1) + h(n-1)$$

From our base case above, we know that

$$h(n-2) \leq c(n-2, n-1) + h(n-1) \leq c(n-2, n-1) + h*(n-1)$$

$$h(n-2) \leq c(n-2, n-1) + h*(n-1)$$

And again, we know that $c(n-2, n-1) + h*(n-1) = h*(n-2)$, so we can see:

$$h(n-2) \leq h*(n-2)$$

By the this inductive hypothesis it is proved that consistency does imply admissibility.