Matrix Computations

CPSC 5006-EL

Assignment2

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Q1

```
A = [36 30 24;30 34 26;24 26 21];
B = [1 -1 1 -1 1;-1 2 -2 2 -2;1 -2 3 -3 3;-1 2 -3 4 -4;1 -2 3 -4 5];
C = [1 2 3;2 5 10;3 10 16];
```

1)

Please refer to cholesky_innerproduct.m

```
cholesky_innerproduct(A)
```

```
ans = 3×3
     6     5     4
     0     3     2
     0     0     1
```

```
cholesky_innerproduct(B)
```

```
ans = 5×5
     1    -1     1    -1     1
     0     1    -1     1    -1
     0     0     1    -1     1
     0     0     0     1    -1
     0     0     0     0     1
```

```
cholesky_innerproduct(C)
```

```
警告: This is not a positive definite matrix
ans =

     []
```

2)

Please refer to cholesky_outerproduct.m

```
cholesky_outerproduct(A)
```

```
ans = 3×3
     6     5     4
     0     3     2
     0     0     1
```

```
cholesky_outerproduct(B)
```

```
ans = 5×5
     1    -1     1    -1     1
     0     1    -1     1    -1
     0     0     1    -1     1
```

```
     0     0     0     1    -1
     0     0     0     0     1
```

```
cholesky_outerproduct(C)
```

```
警告: This is not a positive definite matrix
ans =

     []
```

Q2

1)

Please refer to forward.m and backward.m

```
A = [36 -30 24;-30 34 -26;24 -26 21];
b = [0;12;-7];
R = cholesky_innerproduct(A)
```

```
R = 3×3
     6    -5     4
     0     3    -2
     0     0     1
```

```
y = forward(R', b)
```

```
y = 3×1
     0
     4
     1
```

```
x = backward(R, y)
```

```
x = 3×1
     1
     2
     1
```

2)

```
A = [1 1 1 1 1;1 2 2 2 2;1 2 3 3 3;1 2 3 4 4;1 2 3 4 5];
b = [5;9;12;14;15];
R = cholesky_innerproduct(A)
```

```
R = 5×5
     1     1     1     1     1
     0     1     1     1     1
     0     0     1     1     1
     0     0     0     1     1
     0     0     0     0     1
```

```
y = forward(R', b)
```

```
y = 5×1
     5
     4
     3
     2
     1
```

```
x = backward(R, y)
```

```
x = 5×1
     1
     1
     1
     1
     1
```

Q3

$$\text{number of upper triangle members} = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$$\text{number of envelope members} = n - 2 + n - 1 = 2n - 3$$

$$\text{fraction} = \frac{2 * (2n-3)}{n(n+1)} = \frac{4n-6}{n^2+n}$$

Q4

backward

for each n rows

b(i) = b(i) - a(i,j)*b(j);

There are at most s multiplication and minus operations, other a(i, j>i+s)=0

So, there are at most 2s flops

end

Therefore, there are at most 2ns flops

Q5

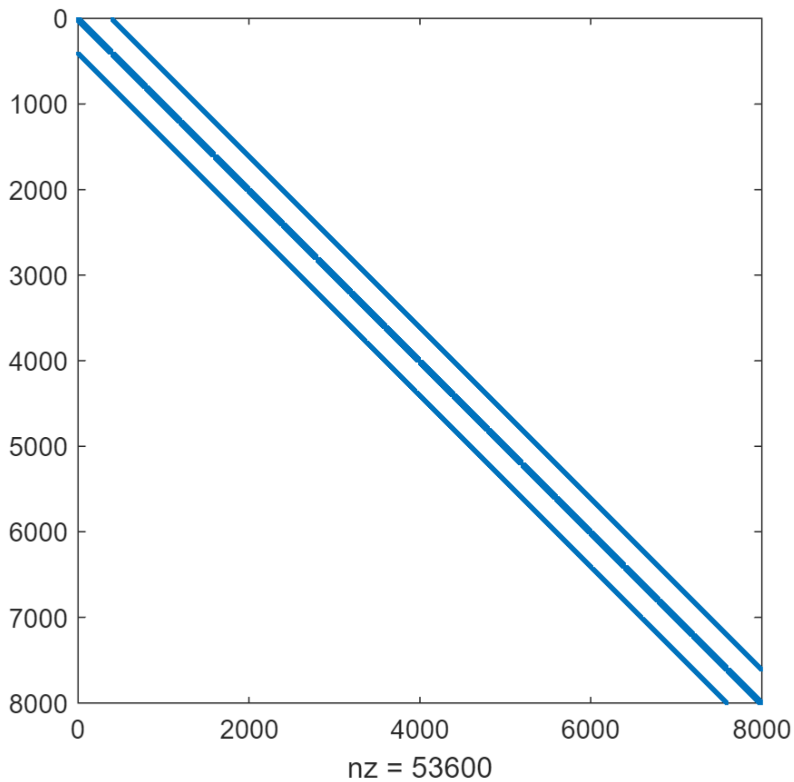
```
m = 20;
r = zeros(1,m); r(1)=2; r(2)=-1;
B = sparse(toeplitz(r));
C = speye(m);
A = kron(kron(B,C),C) + ...
    kron(kron(C,B),C) + ...
    kron(kron(C,C),B);

size(A)
```

```
ans = 1×2
        8000        8000
```

```
issparse(A)
```

```
ans = logical
   1
```

```
spy(A)
```




```
nnz(A)
```

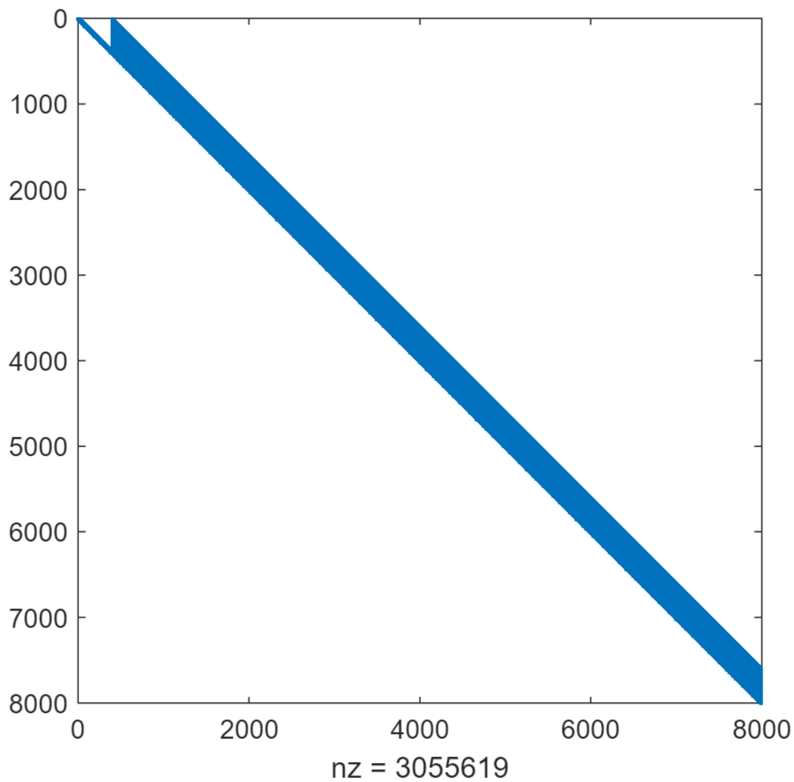
```
ans = 53600
```

a)

```
tic, R = chol(A); toc
```

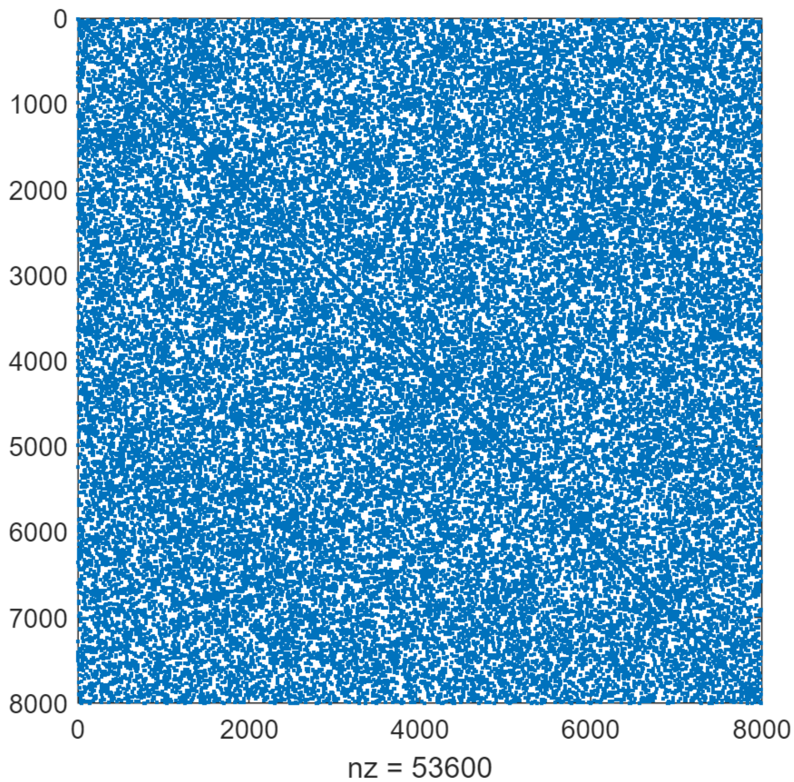
```
历时 0.133699 秒。
```

```
spy(R)
```



b)

```
p = randperm(size(A,1));
arnd = A(p,p);
spy(arnd)
```



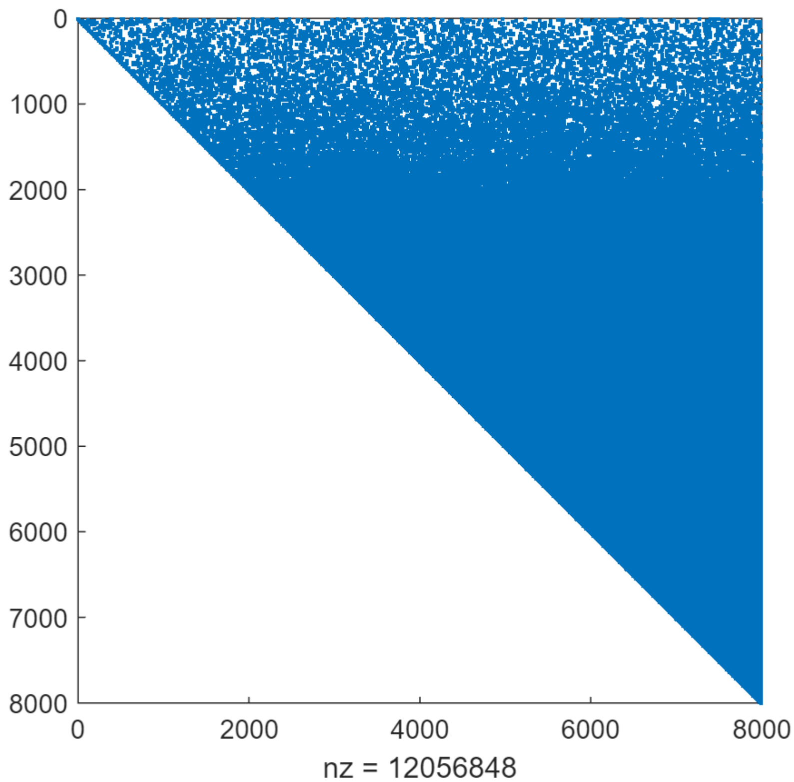

```
tic, rrnd = chol(arnd); toc
```

历时 3.704967 秒。

```
nz = nnz(rrnd)
```

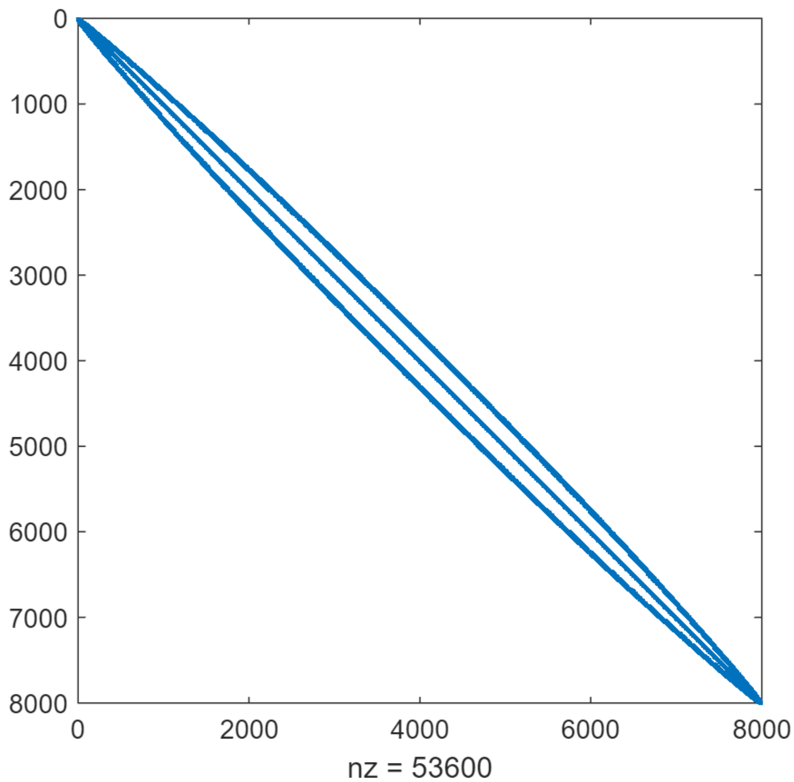
nz = 12056848

```
spy(rrnd)
```



c)

```
p = symrcm(A);
arcm = A(p,p);
spy(arcm)
```



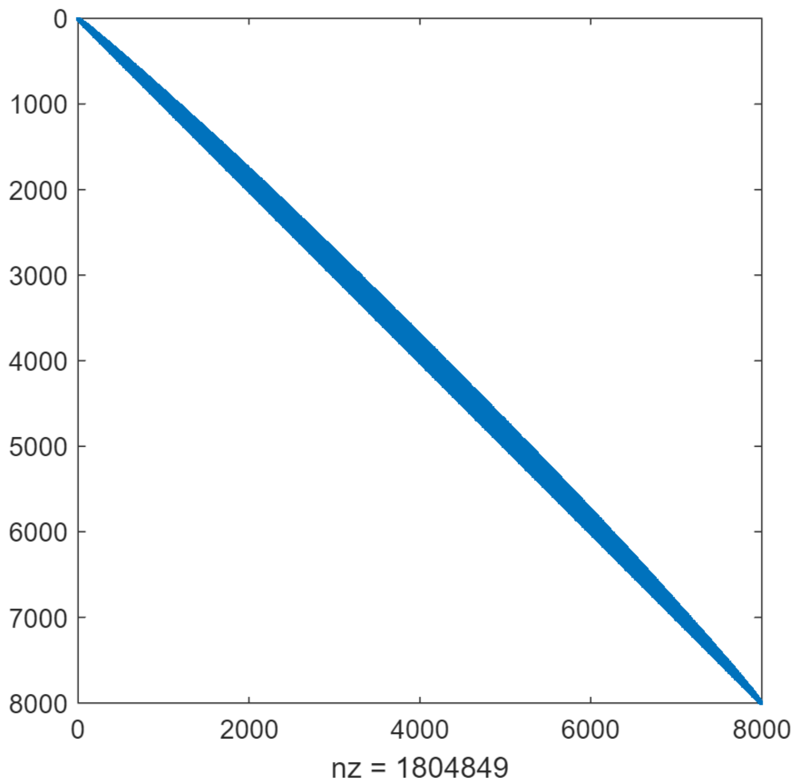

```
tic, rrcm = chol(arcm); toc
```

历时 0.140366 秒。

```
nz = nnz(rrcm)
```

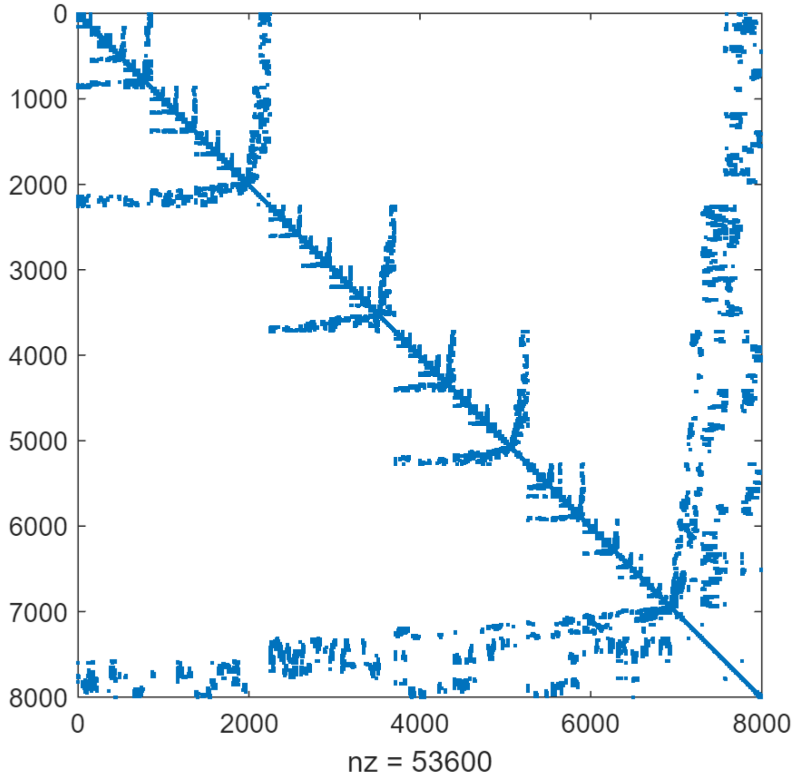
nz = 1804849

```
spy(rrcm)
```



d)

```
p = symamd(A);
aamd = A(p,p);
spy(aamd)
```



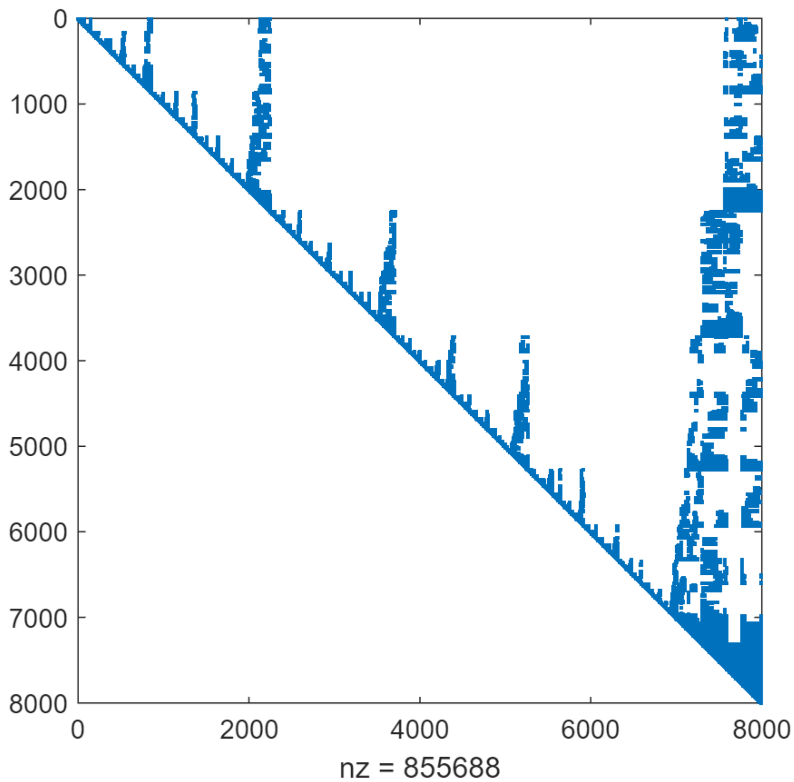

```
tic, ramd = chol(aamd); toc
```

历时 0.060410 秒。

```
nz = nnz(ramd)
```

nz = 855688

```
spy(ramd)
```



e)

Like the situation when m=5, the random ordering has much worse fill-in than the others. The reverse Cuthill-McKee ordering has less fill than the original banded ordering, the approximate minimum-degree ordering is the best.

Q6

1.7.10

a)

$$\det(2) = 2 > 0$$

$$\det\left(\begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix}\right) = 2 \neq 0$$

$$\det\left(\begin{bmatrix} 2 & 1 & -1 \\ -2 & 0 & 0 \\ 4 & 1 & -2 \end{bmatrix}\right) = -2 \neq 0$$

$$\det\left(\begin{bmatrix} 2 & 1 & -1 & 3 \\ -2 & 0 & 0 & 0 \\ 4 & 1 & -2 & 6 \\ -6 & -1 & 2 & -3 \end{bmatrix}\right) = -6 \neq 0$$

So, A can be transformed.

b)

$$\begin{bmatrix} 2 & 1 & -1 & 3 & 13 \\ -2 & 0 & 0 & 0 & -2 \\ 4 & 1 & -2 & 6 & 24 \\ -6 & -1 & 2 & -3 & -14 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & -1 & 3 & 13 \\ 0 & 1 & -1 & 3 & 11 \\ 0 & -1 & 0 & 0 & -2 \\ 0 & 2 & -1 & 6 & 25 \end{bmatrix}, m_{21} = -1, m_{31} = 2, m_{41} = -3$$

$$\begin{bmatrix} 2 & 1 & -1 & 3 & 13 \\ 0 & 1 & -1 & 3 & 11 \\ 0 & 0 & -1 & 3 & 9 \\ 0 & 0 & 1 & 0 & 3 \end{bmatrix}, m_{32} = -1, m_{42} = 2$$

$$\begin{bmatrix} 2 & 1 & -1 & 3 & 13 \\ 0 & 1 & -1 & 3 & 11 \\ 0 & 0 & -1 & 3 & 9 \\ 0 & 0 & 0 & 3 & 12 \end{bmatrix}, m_{43} = -1$$

So, $\begin{bmatrix} 2 & 1 & -1 & 3 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 3 \end{bmatrix} x = \begin{bmatrix} 13 \\ 11 \\ 9 \\ 12 \end{bmatrix}$

c)

$$b_4 = \frac{12}{3} = 4$$

$$b_3 = \frac{9 - 3 * 4}{-1} = 3$$

$$b_2 = \frac{11 - 3 * 4 - (-1) * 3}{1} = 2$$

$$b_1 = \frac{13 - 3 * 4 - (-1) * 3 - 1 * 2}{2} = 1$$

So, $x = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

```
A = [2 1 -1 3;-2 0 0 0;4 1 -2 6;-6 -1 2 -3];
b = [13;-2;24;-14];
A\b
```

```
ans = 4×1
    1.0000
    2.0000
    3.0000
    4.0000
```

1.7.18

$$U = \begin{bmatrix} 2 & 1 & -1 & 3 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 3 \end{bmatrix}, L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ -3 & 2 & -1 & 1 \end{bmatrix}, \widehat{b} = \begin{bmatrix} 12 \\ -8 \\ 21 \\ -26 \end{bmatrix}$$

Forward

$\mathrm{Ly} = \widehat{b}$

$y_1 = 12$

$y_2 = -8 + 12 = 4$

$y_3 = 21 - 2 * 12 + 4 = 1$

$y_4 = -26 + 3 * 12 - 2 * 4 + 1 = 3$

$$y = \begin{bmatrix} 12 \\ 4 \\ 1 \\ 3 \end{bmatrix}$$

Backward

$\mathrm{Ux} = y$

$x_4 = \frac{3}{3} = 1$

$x_3 = \frac{1 - 3 * 1}{-1} = 2$

$x_2 = 4 + 2 - 3 * 1 = 3$

$x_1 = \frac{12 - 3 + 2 - 3 * 1}{2} = 4$

$$x = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

```
A = [2 1 -1 3;-2 0 0 0;4 1 -2 6;-6 -1 2 -3];
b = [12;-8;21;-26];
A\b
```

```
ans = 4×1
    4.0000
    3.0000
    2.0000
    1.0000
```

Q7

a)

$$M = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & m & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$$

consider MA.

Suppose that m is in ith row, jth column.

Using the definition of matrix multiplication,

for each row other than ith row, $\widehat{a_{k \neq i,l}} = a_{k \neq i,l}$

for the ith row, $\widehat{a_{i,l}} = m * a_{j,l} + a_{i,l}$

Therefore, $\text{MA} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ m * a_{\text{j}1} + a_{\text{i}1} & m * a_{\text{j}2} + a_{\text{i}2} & \cdots & m * a_{\text{jn}} + a_{\text{jn}} \\ \cdots & \cdots & \cdots & \cdots \\ a_{\text{n}1} & a_{\text{n}2} & \cdots & a_{\text{nn}} \end{bmatrix}$

This is exactly $\widehat{A}$. Therefore $\widehat{A} = \text{MA}$.

b)

$$\det(M) = \det\left(\begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & m & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}\right) = 1 * 1 * \cdots * 1 * \det\left(\begin{bmatrix} 1 & & & \\ & \ddots & & \\ m & & 1 & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}\right) = 1 * \det(I) \pm m * 0 = 1$$

Since each first j-1 column has only one nonzero 1, others are zero, we can generate *1 until jth row.

The sub-determinant $\det\left(\begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ m & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}\right)$ shows that the sub-determinant of 1 is still 1, the sub-determinant of m is singlar matrix.

Therefore, $\det(M) = 1$.

Since $\det(\mathrm{AB}) = \det(A) \cdot \det(B)$

So $\det\left(\widehat{A}\right) = \det(\mathrm{MA}) = \det(M) \cdot \det(A) = \det(A)$.

c)

$$M^{-1} = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & -m & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$$

M^{-1} means minusing m times the jth row to the ith row.

Q8

1.8.12

Please refer to Gauss.m

1.8.15

Please refer to SolveG.m

```
A = [2 10 8 8 6;1 4 -2 4 -1;0 2 3 2 1;3 8 3 10 9;1 4 1 2 1];
b = [52;14;12;51;15];
c = [50;4;12;48;12];
[flag, intch, a] = Gauss(A)
```

```
flag = 0
intch = 1×5
     4     4     4     5     5
a = 5×5
    3.0000    8.0000    3.0000   10.0000    9.0000
    0.6667    4.6667    6.0000    1.3333         0
    0.3333    0.2857   -4.7143    0.2857   -4.0000
    0.3333    0.2857    0.3636   -1.8182   -0.5455
         0    0.4286   -0.0909   -0.8000    0.2000
```

```
[flag, x] = SolveG(a, b, intch)
```

```
flag = 0
x = 5×1
    1.0000
    2.0000
    1.0000
    2.0000
    1.0000
```

```
[flag, x] = SolveG(a, c, intch)
```

```
flag = 0
x = 5×1
    2.0000
    1.0000
```

```
    2.0000
    1.0000
    2.0000
```

Q9

Without pivoting

for k=1:n-1

for j=k+1:n

L(j,k) = U(j,k) / U(k,k)

U(j,k:n) = U(j,k:n) - L(j,k)U(k,k:n)

GE without pivoting, for each diagnal element a_{ss}, we have 2 operations * and -, and operate for s^2 times.

$$\text{flops} = 2 * \left(n^2 + (n-1)^2 + (n-2)^2 + \cdots + 1\right) = \frac{2(n+1)n(2n+1)}{6} \approx \frac{2n^3}{3}.$$

Solving Ly = b and Ux = y requires about $2n^2$ flops.

Partial pivoting $\text{flops} = (n-1) + (n-2) + \cdots + 1 = \frac{n(n-1)}{2} \approx \frac{n^2}{2}$.

Complete pivoting requires $\text{flops} = n^2 + (n-1)^2 + (n-2)^2 + \cdots + 1 = \frac{(n+1)n(2n+1)}{6} \approx \frac{n^3}{3}$.

Q10

2.2.28

```
for n = [3 6 9]
    A = hilb(n);
    condest(A)
    cond(A, 1)
    fprintf('------------------------------\n')
end
```

```
ans = 748.0000
ans = 748.0000
------------------------------
ans = 2.9070e+07
ans = 2.9070e+07
------------------------------
ans = 1.0997e+12
ans = 1.0996e+12
------------------------------
```

2.6.6

```
for n = [4 8 12]
    z = ones(n, 1);
    H = hilb(n);
    b = H * z;
```

```
    xhat = H \ b;
    k = cond(H, 2)
    deltax = norm(xhat - z, 2)
    rhat = norm(b - H * xhat, 2)
    fprintf('------------------------------\n')
end
```

```
k = 1.5514e+04
deltax = 7.7236e-13
rhat = 1.1102e-16
------------------------------
k = 1.5258e+10
deltax = 6.0931e-07
rhat = 4.5776e-16
------------------------------
警告: 矩阵接近奇异值, 或者缩放不良。结果可能不准确。RCOND =  2.609829e-17。
k = 1.6212e+16
deltax = 0.5757
rhat = 7.0217e-16
------------------------------
```