

Bayesian approaches to designing replication studies

Supplementary materials

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In this document we provide additional information on computing the predictive distribution of the replication effect estimate when a prior is assigned to the heterogeneity variance τ^2 . Moreover, we provide additional information on methods for analyzing replication data. For each method we also derive the *success region* in terms of the effect estimate of the replication study $\hat{\theta}_r$, which is required for sample size determination as illustrated in the main manuscript. For the two-trials rule and the replication Bayes factor methods we additionally provide derivations on how these methods can be generalized to the multisite replication setting.

1 Predictive distribution with prior on heterogeneity variance

When also a prior is assigned to the heterogeneity variance τ^2 , the predictive distribution of the replication effect estimate $\hat{\theta}_r$ is given by

$$f(\hat{\theta}_r | \hat{\theta}_o, \sigma_o, \sigma_r) = \int_0^{+\infty} f(\hat{\theta}_r | \sigma_r, \hat{\theta}_o, \sigma_o, \tau^2) f(\tau^2 | \hat{\theta}_o, \sigma_o) d\tau^2.$$

That is, it is the predictive distribution of the replication effect estimate $\hat{\theta}_r$ integrated with respect to the marginal posterior of τ^2 based on the original data $x_o = \{\hat{\theta}_o, \sigma_o^2\}$. If the initial prior for θ is formulated conditional on τ^2 and taken as in the main manuscript, and the initial prior for τ^2 is formulated marginally, we have

$$\begin{aligned} f(\tau^2 | \hat{\theta}_o, \sigma_o) &= \int_{-\infty}^{+\infty} f(\theta, \tau^2 | \hat{\theta}_o, \sigma_o) d\theta \\ &= \frac{\int f(\hat{\theta}_r | \theta, \tau^2, \sigma_r^2) f(\theta | \tau^2) f(\tau^2) d\theta}{\int_0^{+\infty} \int_{-\infty}^{+\infty} f(\hat{\theta}_r | \theta_*, \tau_*^2, \sigma_r^2) f(\theta_* | \tau_*^2) f(\tau_*^2) d\theta_* d\tau_*^2} \\ &= \frac{N(\hat{\theta}_r | \mu_{\hat{\theta}_r}, \sigma_{\hat{\theta}_r}^2) f(\tau^2)}{\int_0^{+\infty} N(\hat{\theta}_r | \mu_{\hat{\theta}_r}, \sigma_{\hat{\theta}_r}^2) f(\tau_*^2) d\tau_*^2} \end{aligned}$$

with $\mu_{\hat{\theta}_r} = \hat{\theta}_o/(1+1/g) + \mu_\theta/(1+g)$, $\sigma_{\hat{\theta}_r}^2 = \sigma_r^2 + \tau^2 + (\sigma_o^2 + \tau^2)/(1+1/g)$, and $g = \sigma_\theta^2/(\tau^2 + \sigma_o^2)$. To compute the marginal posterior density of τ^2 one numerical integration is hence required. The probability

of replication success can thus be written

$$\begin{aligned}\Pr(\hat{\theta}_r \in S \mid \hat{\theta}_o, \sigma_o, \sigma_r) &= \int_S \int_0^{+\infty} f(\hat{\theta}_r \mid \hat{\theta}_o, \sigma_o, \sigma_r, \tau^2) f(\tau^2 \mid \hat{\theta}_o, \sigma_o) d\hat{\theta}_r d\tau^2 \\ &= \int_0^{+\infty} \Pr(\hat{\theta}_r \in S \mid \hat{\theta}_o, \sigma_o, \sigma_r, \tau^2) f(\tau^2 \mid \hat{\theta}_o, \sigma_o) d\tau^2\end{aligned}$$

2 The two-trials rule

The two-trials rule is the most common analysis approach for replication studies. Replication success is declared if both original and replication study achieve statistical significance at some level α (and both estimates go in the same direction which can be taken into account by using one-sided p -values). We will study the two-trial under normality using the data model $\hat{\theta}_i \mid \theta \sim N(\theta, \sigma_i^2)$ with $\hat{\theta}_i$ the estimate of the unknown effect size θ from study i and σ_i is the corresponding standard error (assumed to be known). The p -values for testing $H_0: \theta = 0$ versus $H_1: \theta > 0$ are then $p_i = 1 - \Phi(\hat{\theta}_i/\sigma_i)$ whereas for the alternative $H_1: \theta < 0$ they are $p_i = \Phi(\hat{\theta}_i/\sigma_i)$. Suppose the original effect estimate was statistically significant at level α , i. e., $p_o \leq \alpha$. Replication success at level α is then established if the replication effect estimate $\hat{\theta}_r$ is also statistically significant at level α , i. e., $p_r \leq \alpha$. By applying some algebraic manipulations to the success condition, one can show that this implies that replication success is achieved if the replication effect estimate $\hat{\theta}_r$ is contained in the success region

$$S_{2\text{TR}} = \begin{cases} [z_\alpha \sigma_r, \infty) & \text{for } \hat{\theta}_o > 0 \\ [-\infty, -z_\alpha \sigma_r) & \text{for } \hat{\theta}_o < 0. \end{cases}$$

TODO: polish

2.1 The multisite two-trials rule

If multiple replication studies are conducted for one original study (a *multisite* replication), the two-trials rule is typically modified by meta-analyzing the effect estimates from all replications and then using the combined estimate as usual in the two-trials rule (see e. g., the “Many labs” projects from [Klein et al., 2014, 2018](#)). Suppose m replication studies are conducted and produce m effect estimates $\hat{\theta}_{r1}, \dots, \hat{\theta}_{rm}$ with standard errors $\sigma_{r1}, \dots, \sigma_{rm}$. Subsequently, a weighted average $\hat{\theta}_{r*} = \{\sum_{i=1}^m \hat{\theta}_{ri}/(\sigma_{ri}^2 + \tau_r^2)\} \sigma_{r*}^2$ with standard error $\sigma_{r*} = 1/\sqrt{\{\sum_{i=1}^m 1/(\sigma_{ri}^2 + \tau_r^2)\}}$ can be computed. If the between-replication heterogeneity variance τ_r^2 is set to zero this corresponds to the fixed effects estimate of θ , while estimating τ_r^2 from the data corresponds to the random effects estimate. Replication success at level α is then established if the replication p -value is smaller than α , i. e., $p_{r*} = 1 - \Phi(\hat{\theta}_{r*}/\sigma_{r*}) \leq \alpha$. With some algebra one can show that this implies a success region for the weighted average replication effect estimate $\hat{\theta}_{r*}$ given by

$$S_{2\text{TR}} = \begin{cases} [z_\alpha \sigma_{r*}, \infty) & \text{for } \hat{\theta}_o > 0 \\ [-\infty, -z_\alpha \sigma_{r*}) & \text{for } \hat{\theta}_o < 0. \end{cases}$$

3 Fixed effects meta-analysis

Assume again the data model $\hat{\theta}_i | \theta \sim N(\theta, \sigma_i^2)$ where $\hat{\theta}_i$ is an estimate of the effect size θ from study $i \in \{o, r\}$ and σ_i is the corresponding standard error (assumed to be known). In the fixed effects meta-analysis approach replicability is assessed in terms of the pooled effect estimate $\hat{\theta}_m$ and standard error σ_m which are

$$\hat{\theta}_m = \left(\hat{\theta}_o / \sigma_o^2 + \hat{\theta}_r / \sigma_r^2 \right) \sigma_m^2 \quad \text{and} \quad \sigma_m = (1/\sigma_o^2 + 1/\sigma_r^2)^{-1/2},$$

which are also equivalent to the mean and standard deviation of a posterior distribution for the effect size θ based on the data from original and replication study and an initial flat prior for θ . Fixed effects meta-analysis is typically used because estimating a heterogeneity variance from two studies is highly unstable. Replication success at level α is established if the one-sided meta-analytic p -value (in the direction of the original effect estimate $\hat{\theta}$) is significant at level α , i. e., $p_m = 1 - \Phi(\hat{\theta}_m / \sigma_m) \leq \alpha$ for $\hat{\theta}_o > 0$ and $p_m = \Phi(\hat{\theta}_m / \sigma_m) \leq \alpha$ for $\hat{\theta}_o < 0$. With some algebraic manipulations one can show that this criterion implies a success region S_{MA} for the replication effect estimate $\hat{\theta}_r$ given by

$$S_{MA} = \begin{cases} [\sigma_r z_\alpha \sqrt{1 + \sigma_r^2 / \sigma_o^2} - (\hat{\theta}_o \sigma_r^2) / \sigma_o^2, \infty) & \text{for } \hat{\theta}_o > 0 \\ (-\infty, -\sigma_r z_\alpha \sqrt{1 + \sigma_r^2 / \sigma_o^2} - (\hat{\theta}_o \sigma_r^2) / \sigma_o^2] & \text{for } \hat{\theta}_o < 0. \end{cases}$$

4 Effect size equivalence

The effect size equivalence approach (Anderson and Maxwell, 2016) defines replication success via comparability of the effect estimates from both studies. Under normality we may assume the data model $\hat{\theta}_i | \theta_i \sim N(\theta_i, \sigma_i^2)$ for study $i \in \{o, r\}$, and we are interested in the true effect size difference $\delta = \theta_r - \theta_o$. A $(1 - \alpha)$ confidence interval for δ is then given by

$$C_\alpha = \left[\hat{\theta}_r - \hat{\theta}_o - z_{\alpha/2} \sqrt{\sigma_r^2 + \sigma_o^2}, \hat{\theta}_r - \hat{\theta}_o + z_{\alpha/2} \sqrt{\sigma_r^2 + \sigma_o^2} \right]$$

Effect size equivalence is established if the confidence interval is fully included in an equivalence region $C_\alpha \subseteq [-\Delta, \Delta]$ with $\Delta > 0$ a pre-specified margin. Applying some algebraic manipulations to the success conditions one can show that the equivalence test replication success criterion implies a success region S_E for the replication estimate $\hat{\theta}_r$ given by

$$S_E = \left[\hat{\theta}_o - \Delta + z_{\alpha/2} \sqrt{\sigma_o^2 + \sigma_r^2}, \hat{\theta}_o + \Delta - z_{\alpha/2} \sqrt{\sigma_o^2 + \sigma_r^2} \right].$$

5 The replication Bayes factor

The replication Bayes factor approach uses the replication data x_r to quantify the evidence for the null hypothesis $H_0: \theta = 0$ relative to the alternative hypothesis $H_1: \theta \sim f(\theta | x_o)$, which postulates that the effect size θ is distributed according to its posterior distribution based on the original data x_o . Assume again a normal model $\hat{\theta}_i | \theta \sim N(\theta, \sigma_i^2)$ with $\hat{\theta}_i$ an estimate of the effect size θ from study $i \in \{o, r\}$ and σ_i the corresponding standard error (assumed to be known), and that we use the alternative $H_1: N(\hat{\theta}_o, \sigma_o^2)$ which arises from updating an initial flat prior for θ the original data $x_o = \{\hat{\theta}_o, \sigma_o\}$. The replication Bayes

factor is then

$$\text{BF}_R = \frac{f(\hat{\theta}_r | H_0)}{f(\hat{\theta}_r | H_1)} = \sqrt{1 + \sigma_o^2/\sigma_r^2} \exp \left[-\frac{1}{2} \left\{ \frac{\hat{\theta}_r^2}{\sigma_r^2} - \frac{(\hat{\theta}_r - \hat{\theta}_o)^2}{\sigma_o^2 + \sigma_r^2} \right\} \right]. \quad (1)$$

Replication success at level $\gamma \in (0, 1)$ is achieved if $\text{BF}_R \leq \gamma$. By applying some algebra to $\text{BF}_R \leq \gamma$, one can show that it is equivalent to the replication effect estimate $\hat{\theta}_r$ falling in the success region

$$S_{\text{BF}_R} = \left(-\infty, -\sqrt{A} - (\hat{\theta}_o \sigma_r^2)/\sigma_o^2 \right] \cup \left[\sqrt{A} - (\hat{\theta}_o \sigma_r^2)/\sigma_o^2, \infty \right)$$

where $A = \sigma_r^2(1 + \sigma_r^2/\sigma_o^2)\{\hat{\theta}_o^2/\sigma_o^2 - 2 \log \gamma + \log(1 + \sigma_r^2/\sigma_o^2)\}$.

5.1 The multisite replication Bayes factor

The generalization of the replication Bayes factor to the multisite setting is straightforward. The data are represented by vector of replication effect estimates $\hat{\theta}_r = (\hat{\theta}_{r1}, \dots, \hat{\theta}_{rm})^\top$ with corresponding standard error vector $\sigma_r = (\sigma_{r1}, \dots, \sigma_{rm})^\top$, and we assume the data model $\hat{\theta}_r | \theta \sim N_m\{\theta \mathbf{1}_m, \text{diag}(\sigma^2 + \tau_r^2 \mathbf{1}_m)\}$ where $\mathbf{1}_m$ is a vector of m ones and τ_r^2 is a heterogeneity variance for the replication effect sizes (not to be confused with the heterogeneity variance τ^2 used in the design prior).

As in the singlsite case, the replication Bayes factor quantifies the evidence that the data provide for the null hypothesis $H_0: \theta = 0$ relative to the alternative hypothesis $H_1: \theta \sim N(\hat{\theta}_o, \sigma_o^2)$. The marginal density of the replication data under the null hypothesis is simply $\hat{\theta}_r | H_0 \sim N_m\{0 \mathbf{1}_m, \text{diag}(\sigma^2 + \tau_r^2 \mathbf{1}_m)\}$, whereas the marginal likelihood under the alternative H_1 is obtained from integrating the likelihood with respect to the prior distribution of θ under the alternative H_1 . Let $N(x; m, v)$ denote the normal density function mean m and variance v evaluated at x . Define also $\hat{\theta}_{r*} = \left\{ \sum_{i=1}^n \hat{\theta}_{ri}/(\sigma_{ri}^2 + \tau_r^2) \right\} \sigma_{r*}^2$ and $\sigma_{r*}^2 = 1/\left\{ \sum_{i=1}^n 1/(\sigma_{ri}^2 + \tau_r^2) \right\}$, i. e., the weighted average of the replication effect estimates based on the heterogeneity τ_r^2 and its variance. The marginal density is then

$$\begin{aligned} f(\hat{\theta}_r | H_1) &= \int f(\hat{\theta}_r | \theta) f(\theta | H_1) d\theta \\ &= \int \frac{\exp \left[-\frac{1}{2} \left\{ \sum_{i=1}^n \frac{(\hat{\theta}_{ri} - \theta)^2}{\sigma_{ri}^2 + \tau_r^2} + \frac{(\theta - \hat{\theta}_o)^2}{\sigma_o^2} \right\} \right]}{\left\{ 2\pi\sigma_o^2 \prod_{i=1}^n 2\pi(\sigma_{ri}^2 + \tau_r^2) \right\}^{1/2}} d\theta \\ &= \int \frac{\exp \left[-\frac{1}{2} \left\{ \sum_{i=1}^n \frac{(\hat{\theta}_{ri} - \hat{\theta}_{r*})^2}{\sigma_{ri}^2 + \tau_r^2} + \frac{(\hat{\theta}_{r*} - \theta)^2}{\sigma_{r*}^2} + \frac{(\theta - \hat{\theta}_o)^2}{\sigma_o^2} \right\} \right]}{\left\{ 2\pi\sigma_o^2 \prod_{i=1}^n 2\pi(\sigma_{ri}^2 + \tau_r^2) \right\}^{1/2}} d\theta \\ &= \frac{\exp \left[-\frac{1}{2} \left\{ \sum_{i=1}^n \frac{(\hat{\theta}_{ri} - \hat{\theta}_{r*})^2}{\sigma_{ri}^2 + \tau_r^2} \right\} \right]}{\left\{ 2\pi\sigma_o^2 \prod_{i=1}^n 2\pi(\sigma_{ri}^2 + \tau_r^2) \right\}^{1/2}} \underbrace{\int \exp \left[-\frac{1}{2} \left\{ \frac{(\hat{\theta}_{r*} - \theta)^2}{\sigma_{r*}^2} + \frac{(\theta - \hat{\theta}_o)^2}{\sigma_o^2} \right\} \right] d\theta}_{=N(\hat{\theta}_{r*}; m, \sigma_o^2 + \sigma_{r*}^2) 2\pi\sigma_o\sigma_{r*}} \\ &= \left\{ (1 + \sigma_o^2/\sigma_{r*}^2) \prod_{i=1}^n 2\pi(\sigma_{ri}^2 + \tau_r^2) \right\}^{-1/2} \exp \left[-\frac{1}{2} \left\{ \sum_{i=1}^n \frac{(\hat{\theta}_{ri} - \hat{\theta}_{r*})^2}{\sigma_{ri}^2 + \tau_r^2} + \frac{(\hat{\theta}_{r*} - \hat{\theta}_o)^2}{\sigma_{r*}^2 + \sigma_o^2} \right\} \right]. \end{aligned}$$

Dividing the marginal density of $\hat{\theta}_r$ under H_0 by the marginal density of $\hat{\theta}_r$ under H_1 leads to cancelation

of several terms, and produces the replication Bayes factor

$$\text{BF}_{01}(\hat{\theta}_r) = \frac{f(\hat{\theta}_r | H_0)}{f(\hat{\theta}_r | H_1)} = \sqrt{1 + \sigma_o^2 / \sigma_{r*}^2} \exp \left[-\frac{1}{2} \left\{ \frac{\hat{\theta}_{r*}^2}{\sigma_{r*}^2} - \frac{(\hat{\theta}_{r*} - \hat{\theta}_o)^2}{\sigma_{r*}^2 + \sigma_o^2} \right\} \right].$$

The multisite replication Bayes factor is therefore equivalent to the singlesite replication Bayes factor from (1) but using the weighted average $\hat{\theta}_{r*}$ and its standard error σ_{r*} as the replication effect estimate $\hat{\theta}_r$ and standard error σ_r .

6 The sceptical p -value

Held (2020) proposed a reverse-Bayes approach for assessing replicability. One assumes again the data model $\hat{\theta}_i | \theta \sim N(\theta, \sigma_i^2)$ with $i \in \{o, r\}$, along with a zero-mean “sceptical” prior $\theta \sim N(0, \sigma_s^2)$ for the effect size. In a first step, a level $\alpha \geq p_o = 1 - \Phi(|\hat{\theta}_o|/\sigma_o)$ is fixed and the “sufficiently sceptical” prior variance σ_s^2 is computed

$$\sigma_s^2 = \frac{\sigma_o^2}{(z_o^2/z_\alpha^2) - 1}$$

where $z_o = \hat{\theta}_o/\sigma_o$. The sufficiently sceptical prior variance σ_s^2 has the property that it renders the resulting posterior of θ no longer “credible” at level α , that is, the posterior tail probability is fixed to $\Pr(\theta \geq 0 | \hat{\theta}_o, \sigma_o, \sigma_s) = 1 - \alpha$ for positive estimates and $\Pr(\theta \leq 0 | \hat{\theta}_o, \sigma_o, \sigma_s) = 1 - \alpha$ for negative estimates. In a second step, the conflict between the sceptical prior and the observed replication data is quantified, larger conflict indicating a higher degree of replication success. For doing so, a prior predictive tail probability

$$p_{\text{Box}} = \begin{cases} 1 - \Phi \left\{ \hat{\theta}_r / (\sigma_r^2 + \sigma_s^2) \right\} & \text{if } \hat{\theta}_o > 0 \\ \Phi \left\{ \hat{\theta}_r / (\sigma_r^2 + \sigma_s^2) \right\} & \text{if } \hat{\theta}_o < 0 \end{cases}$$

is computed and replication success at level α is declared if $p_{\text{Box}} \leq \alpha$. The smallest level α at which replication success is achieved is called the *the sceptical p -value* p_s and replication success at level α is equivalent with $p_s \leq \alpha$ (see Held, 2020; Held et al., 2022, for more details on p_s). By applying some algebraic manipulations to the condition $p_{\text{Box}} \leq \alpha$, one can show that it is equivalent to the replication effect estimate $\hat{\theta}_r$ falling in the success region

$$S_{p_s} = \begin{cases} [z_\alpha \sqrt{\{\sigma_r^2 + \frac{\sigma_o^2}{(z_o^2/z_\alpha^2) - 1}\}}, \infty) & \text{if } \hat{\theta}_o > 0 \\ (-\infty, -z_\alpha \sqrt{\{\sigma_r^2 + \frac{\sigma_o^2}{(z_o^2/z_\alpha^2) - 1}\}}] & \text{if } \hat{\theta}_o < 0. \end{cases}$$

7 The sceptical Bayes factor

Pawel and Held (2022) modified the reverse-Bayes assessment of replication success from Held (2020) to use Bayes factors (Jeffreys, 1961; Kass and Raftery, 1995) instead of tail probabilities as measures of evidence and prior data conflict. The procedure assumes again the data model $\hat{\theta}_i | \theta \sim N(\theta, \sigma_i^2)$ for study $i \in \{o, r\}$. In the first step the original data are used to contrast the evidence for the point null hypothesis

$H_0: \theta = 0$ relative to the “sceptical” alternative $H_S: \theta \sim N(0, \sigma_s^2)$ with the Bayes factor

$$\text{BF}_{0S} = \frac{f(\hat{\theta}_o | H_0)}{f(\hat{\theta}_o | H_S)} = \sqrt{1 + \sigma_s^2 / \sigma_o^2} \exp \left\{ -\frac{z_o^2}{2(1 + \sigma_s^2 / \sigma_o^2)} \right\}.$$

where $z_o = \hat{\theta}_o / \sigma_o$. One then determines the sufficiently sceptical prior variance σ_s^2 so that the Bayes factor is fixed to a level $\gamma \in (0, 1)$ meaning that there is no longer evidence against the null hypothesis at level γ . The sufficiently sceptical prior variance can be computed by

$$\sigma_s^2 = \begin{cases} -\frac{\hat{\theta}_o^2}{q} - \sigma_o^2 & \text{if } -\frac{\hat{\theta}_o^2}{q} \geq \sigma_o^2 \\ \text{undefined} & \text{else} \end{cases} \quad (2)$$

$$\text{where } q = W_{-1} \left\{ -\frac{z_o^2}{\gamma^2} \exp(-z_o^2) \right\} \quad (3)$$

with $W_{-1}(\cdot)$ the branch of the Lambert W function with $W(y) \leq -1$ for $y \in [-1/e, 0)$.

In a second step the conflict between the sceptical prior and the replication data is quantified. To do so, the sceptic is contrasted to the “advocacy” alternative $H_A: \theta \sim N(\hat{\theta}_o, \sigma_o^2)$ which represents the position of an advocate as the prior corresponds to the posterior distribution based on the original data $\{\hat{\theta}_o, \sigma_o\}$ and a flat prior for the effect size θ . This is done by computing the Bayes factor

$$\text{BF}_{SA} = \frac{f(\hat{\theta}_r | H_S)}{f(\hat{\theta}_r | H_A)} = \sqrt{\frac{\sigma_o^2 + \sigma_r^2}{\sigma_s^2 + \sigma_r^2}} \exp \left[-\frac{1}{2} \left\{ \frac{\hat{\theta}_r^2}{\sigma_s^2 + \sigma_r^2} - \frac{(\hat{\theta}_r - \hat{\theta}_o)^2}{\sigma_o^2 + \sigma_r^2} \right\} \right]$$

and replication success at level γ is defined by $\text{BF}_{SA} \leq \gamma$ as the data favor the advocate over the sceptic at a higher level than the sceptic’s initial objection to the null hypothesis. The smallest level γ at which replication success is achievable is then called *the sceptical Bayes factor* BF_s , and replication success at level γ is equivalent to $\text{BF}_s \leq \gamma$ (see [Pawel and Held, 2022](#), for details on how to compute BF_s). To derive the success region of the sceptical Bayes factor one can apply algebraic manipulations to $\text{BF}_{SA} \leq \gamma$, the condition for replication success at level γ , which leads to

$$S_{\text{BF}_s} = \begin{cases} (-\infty, -\sqrt{B} - M] \cup [\sqrt{B} - M, \infty) & \text{for } \sigma_s^2 < \sigma_o^2 \\ [\hat{\theta}_o - \{(\sigma_o^2 + \sigma_r^2) \log \gamma\} / \hat{\theta}_o, \infty) & \text{for } \sigma_s^2 = \sigma_o^2 \\ [-\sqrt{B} - M, \sqrt{B} - M] & \text{for } \sigma_s^2 > \sigma_o^2 \end{cases} \quad (4)$$

with

$$B = \left\{ \frac{\hat{\theta}_o^2}{\sigma_o^2 - \sigma_s^2} + 2 \log \left(\frac{\sigma_o^2 + \sigma_r^2}{\sigma_s^2 + \sigma_r^2} \right) - 2 \log \gamma \right\} \frac{(\sigma_s^2 + \sigma_r^2)(\sigma_o^2 + \sigma_r^2)}{\sigma_o^2 - \sigma_s^2}$$

$$M = \frac{\hat{\theta}_o(\sigma_s^2 + \sigma_r^2)}{\sigma_o^2 - \sigma_s^2}$$

and the sufficiently sceptical prior variance σ_s^2 computed by (2).

8 Optimal number of sites

The total cost of the design are $K = m(K_c n_r + K_s)$ so that we can write the number of sites m for a given total cost as

$$m = K(K_c n_r + K_s)^{-1}. \quad (5)$$

We now want to minimize the predictive variance of the weighted average $\hat{\theta}_{r*}$ which, for a balanced design, is given by

$$\sigma_{\hat{\theta}_{r*}}^2 = \frac{\sigma_r^2 + \tau^2}{m} + \frac{\tau^2 + \sigma_o^2}{1 + 1/g}. \quad (6)$$

Plugging in (5) into (6) and minimizing it with respect to n_r , leads to the optimal sample size

$$n_r^* = \frac{\lambda}{\tau} \sqrt{\frac{K_s}{K_c}}$$

for a given cost ratio K_s/K_c .

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