THE RECURSIVE STRUCTURE OF THE DISTRIBUTION OF PRIMES

CAROLIN ZÖBELEIN

Abstract. In this work I look at the distribution of primes by calculation of an infinite number of intersections. For this I use the set of all numbers which are not elements of a certain times table in each case. I am able to show that it exists a recursive relationship between primes of different ranges and so to describe some inner structure of this special set of numbers.

1. Introduction

Primes are a big secret and an ark of fascination for us for a long time. Until now a lot of people have tried to find out the key to this distribution and its properties. One of them is Stanisław Marcin Ulam with his well known Ulam spiral which he presented 1964 in the work *A Visual Display of Some Properties of the Distribution of Primes* [2] for the first time. This geometric presentation provides us a small but amazing insight into this special set. In my work we will use an other angle of view which will show us a recursive structure of primes.

2. Integer divisible numbers

To study the properties of primes, we have to look at the set of integer divisible numbers at first. Be $n \in \mathbb{N} : n > 1$ the set of all natural numbers larger than one. We are able to split it into the disjoint subsets

$$E := \{2\gamma\}, \qquad O_3 := \{6\gamma - 3\}, \qquad O_- := \{6\gamma - 1\}, \qquad O_+ := \{6\gamma + 1\} \tag{2.1}$$

with $\gamma \in \mathbb{N}$. We see that 2 is the only prime of the set E and 3 is the only prime of the set O_3. All other primes $p \geq 5$ are elements of O_- or O_+. Hence we will only discuss this two sets and will ignore the numbers 2 and 3 in this work. Now the question is, for which γ we receive primes p and for which γ' we receive integer divisible numbers $p' \in \mathbb{N} \setminus \mathcal{P}$ in the set O_- respectively O_+. At first we search γ'_- for O_-. We can it easy find by the ansatz

$$(6\alpha + 1)(6\beta - 1) = 36\alpha\beta - 6\alpha + 6\beta - 1 = 6\gamma'_- - 1 = p'_- \tag{2.2}$$

$$\gamma'_{-,1} = 6\alpha\beta - \alpha + \beta \tag{2.3}$$

respectively we can also write

$$\gamma'_{-,2} = 6\alpha\beta + \alpha - \beta \tag{2.4}$$

Date: October 31, 2014.
2010 *Mathematics Subject Classification.* Primary 11N05.
Key words and phrases. Distribution of primes.
Thanks: 1820010913 & 71081013.

with $\alpha, \beta \in \mathbb{N}$. Usually we will use the first version (2.3), $\gamma'_- := \gamma'_{-,1}$, in this work. In the same way we receive all γ'_+ for O_+ by the ansatz

$$(6\alpha + 1)(6\beta + 1) = 36\alpha\beta + 6\alpha + 6\beta + 1 = 6\gamma'_{+,1} + 1 = p'_{+,1} \tag{2.5}$$

$$\gamma'_{+,1} = 6\alpha\beta + \alpha + \beta \tag{2.6}$$

and

$$(6\alpha - 1)(6\beta - 1) = 36\alpha\beta - 6\alpha - 6\beta + 1 = 6\gamma'_{+,2} + 1 = p'_{+,2} \tag{2.7}$$

$$\gamma'_{+,2} = 6\alpha\beta - \alpha - \beta. \tag{2.8}$$

A short look at the relationships between this equations gives us for (2.3) and the substitution $\beta \to -\beta$

$$\begin{aligned} \gamma'_{-,1}(\alpha, -\beta) &= -6\alpha\beta - \alpha - \beta \\ &= -(6\alpha\beta + \alpha + \beta) \\ &= -\gamma'_{+,1}(\alpha, \beta) \end{aligned} \tag{2.9}$$

and the second case for (2.4)

$$\begin{aligned} \gamma'_{-,2}(\alpha, -\beta) &= -6\alpha\beta + \alpha + \beta \\ &= -(6\alpha\beta - \alpha - \beta) \\ &= -\gamma'_{+,2}(\alpha, \beta). \end{aligned} \tag{2.10}$$

3. Pseudoprimes for one constant parameter

After we have found our equations for integer divisible numbers we will search one possibility to describe all pseudoprimes. Pseudoprimes means in this context, all numbers which are not solutions of one of the equations (2.3), (2.6) and (2.8) if one parameter is constant.

3.1. **Considerations for the set O_-.** For the set O_- we start with our equation (2.3) from the previous section

$$\gamma'_- = 6\alpha\beta - \alpha + \beta$$

and use the angle of view

$$\gamma'_- = (6\alpha + 1)\beta - \alpha. \tag{3.1}$$

If we have a look at our equation we can see the following properties:

(1) Before the case $\beta = 1$ we always have 5α numbers.
(2) Between the case β and $\beta + 1$ we always have 6α numbers.

Now we want to find an equation $\tilde{\gamma}_\alpha$ which describes all numbers for our α which are not generated by (2.3). At first we expand the set $\beta \in \mathbb{N}$ to $\beta \in \mathbb{N}_0$ which gives us the chance to involve also the numbers before the first γ'_- solution. With the gap 6α we can find very easy our new equation and we can write

$$\begin{aligned} \tilde{\gamma}_\alpha &= (6\alpha + 1)\beta - \alpha + 6\alpha + 1 - \chi \\ &= (6\alpha + 1)\beta + 5\alpha + 1 - \chi \end{aligned} \tag{3.2}$$

with $\chi \leq 6\alpha$, $\chi \in \mathbb{N}$. Here we get the wrong numbers $\tilde{\gamma}_\alpha = -\alpha + 1, \ldots, 0$ but we will see that this will be not a real problem for the next steps of our analysis of primes. Now we will change to the other angle of view

$$\gamma'_- = (6\beta - 1)\,\alpha + \beta. \tag{3.3}$$

In this case we can see that we now always have $7\beta - 2$ numbers before the $\alpha = 1$ result and between α and $\alpha + 1$ a gab with $6\beta - 2$ numbers. We expand the set $\alpha \in \mathbb{N}$ to $\alpha \in \mathbb{N}_0 \cup \{-1\}$. If we only add $\alpha = 0$ we don't get the numbers $\tilde{\gamma}_\beta = 1, \ldots, \beta - 1$. With the gab $6\beta - 2$ we can find

$$\begin{aligned} \tilde{\gamma}_\beta &= (6\beta - 1)\,\alpha + \beta + 6\beta - 2 + 1 - \chi \\ &= (6\beta - 1)\,\alpha + 7\beta - 1 - \chi \end{aligned} \tag{3.4}$$

with $\chi \leq 6\beta - 2$, $\chi \in \mathbb{N}$. We get the wrong numbers $\tilde{\gamma}_\beta = -5\beta - 2, \ldots, 0$. In addition we have one pseudoprime which is not solution of equation (3.4) caused by $\alpha = 0$. This is always the case $\tilde{\gamma}_\beta = \beta$. With the substitution

$$\alpha = \tilde{\alpha} - 1 \tag{3.5}$$

$\tilde{\alpha} \in \mathbb{N}_0$ we can also write

$$\tilde{\gamma}_\beta = (6\beta - 1)\,\tilde{\alpha} + \beta - \chi. \tag{3.6}$$

3.2. **Considerations for the set O_+.** Now we will do the same for the equations of O_+. We start with (2.6)

$$\gamma'_{+,1} = 6\alpha\beta + \alpha + \beta$$

and the angle of view

$$\gamma'_{+,1} = (6\alpha + 1)\,\beta + \alpha. \tag{3.7}$$

We will expand the set of β to the set $\beta \in \mathbb{N}_0 \cup \{-1\}$. Without $\beta = -1$ we don't get the numbers $\tilde{\gamma}_{\alpha,1} = 1, \ldots, \alpha - 1$. We have a gab of 6α numbers which gives us

$$\begin{aligned} \tilde{\gamma}_{\alpha,1} &= (6\alpha + 1)\,\beta + \alpha + 6\alpha + 1 - \chi \\ &= (6\alpha + 1)\,\beta + 7\alpha + 1 - \chi \end{aligned} \tag{3.8}$$

with $\chi \leq 6\alpha$, $\chi \in \mathbb{N}$ and the wrong solutions $\tilde{\gamma}_{\alpha,1} = -5\alpha, \ldots, 0$. We also have one pseudoprime which is not solution of (3.8) caused by $\beta = 0$. This is always the case $\tilde{\gamma}_{\alpha,1} = \alpha$. Since for the next sections we want to have a consistent definition area for β for all used equations, we make the substitution

$$\beta = \tilde{\beta} - 1 \tag{3.9}$$

with $\tilde{\beta} \in \mathbb{N}_0$ and receive

$$\tilde{\gamma}_{\alpha,1} = (6\alpha + 1)\,\tilde{\beta} + \alpha - \chi. \tag{3.10}$$

For equation (2.8)

$$\gamma'_{+,2} = 6\alpha\beta - \alpha - \beta$$

we have the angle of view

$$\gamma'_{+,2} = (6\alpha - 1)\,\beta - \alpha. \tag{3.11}$$

Here we have $7\alpha - 2$ numbers before the first solution and a size of the gab with $6\alpha - 2$. We expand the set $\beta \in \mathbb{N}$ to $\beta \in \mathbb{N}_0$ and receive

$$\begin{aligned} \tilde{\gamma}_{\alpha,2} &= (6\alpha - 1)\,\beta - \alpha + 6\alpha - 2 + 1 - \chi \\ &= (6\alpha - 1)\,\beta + 5\alpha - 1 - \chi \end{aligned} \tag{3.12}$$

with $\chi \leq 6\alpha - 2$, $\chi \in \mathbb{N}$ and the wrong solutions $\tilde{\gamma}_{\alpha,2} = -\alpha + 1, \ldots, 0$.

3.3. **Relationships between pseudoprime equations.** Let's have a look at the properties and relationships between pseudoprime equations. We start with (3.2)

$$\tilde{\gamma}_{\alpha} = (6\alpha + 1)\,\beta + 5\alpha + 1 - \chi$$

with the wrong solutions $\tilde{\gamma}_{\alpha} = -\alpha + 1, \ldots, 0$ and (3.10)

$$\tilde{\gamma}_{\alpha,1} = (6\alpha + 1)\,\tilde{\beta} + \alpha - \chi$$

with the wrong solutions $\tilde{\gamma}_{\alpha,1} = -5\alpha, \ldots, 0$, the missing pseudoprime α and $\chi \leq 6\alpha$ for both equations. If we also allow negative values for β and $\tilde{\beta}$, we can follow from (2.9)

$$\begin{aligned} \tilde{\gamma}_{\alpha}\,(\alpha, -\beta, \chi) &= -\tilde{\gamma}_{\alpha,1}\,(\alpha, \beta, 6\alpha + 1 - \chi) \\ \Leftrightarrow \qquad -\tilde{\gamma}_{\alpha}\,(\alpha, \beta, 6\alpha + 1 - \chi) &= \tilde{\gamma}_{\alpha,1}\,(\alpha, -\beta, \chi)\,. \end{aligned} \tag{3.13}$$

Here we have a symmetrical situation with the same range for χ with $\{\tilde{\gamma}_{\alpha}\,(\alpha, -\beta)\} = \{-\tilde{\gamma}_{\alpha,1}\,(\alpha, \beta)\}$ and $\{-\tilde{\gamma}_{\alpha}\,(\alpha, \beta)\} = \{\tilde{\gamma}_{\alpha,1}\,(\alpha, -\beta)\}$. The second case we have for (3.6)

$$\tilde{\gamma}_{\beta} = (6\beta - 1)\,\tilde{\alpha} + \beta - \chi$$

$\chi \leq 6\beta - 2$, with the wrong solutions $\tilde{\gamma} = -5\beta - 2, \ldots, 0$, the missing pseudoprime β and (3.12)

$$\tilde{\gamma}_{\alpha,2} = (6\alpha - 1)\,\beta + 5\alpha - 1 - \chi$$

$\chi \leq 6\alpha - 2$ with the wrong solutions $\tilde{\gamma}_{\alpha,2} = -\alpha + 1, \ldots, 0$. With negative values for $\tilde{\alpha}$ and β, we also can follow from (2.10)

$$\begin{aligned} \tilde{\gamma}_{\beta}\,(-\alpha, \beta, \chi) &= -\tilde{\gamma}_{\alpha,2}\,(\beta, \alpha, 6\beta - 1 - \chi) \\ \Leftrightarrow \qquad -\tilde{\gamma}_{\beta}\,(\beta, \alpha, 6\alpha - 1 - \chi) &= \tilde{\gamma}_{\alpha,2}\,(\alpha, -\beta, \chi) \end{aligned} \tag{3.14}$$

with $\chi \leq 6\beta - 2$ respectively $\chi \leq 6\alpha - 2$. So we also have a symmetrical situation for the same distance factor and its χ with $\{\tilde{\gamma}_{\beta}\,(-\alpha, \beta)\} = \{-\tilde{\gamma}_{\alpha,2}\,(\beta, \alpha)\}$ and $\{-\tilde{\gamma}_{\beta}\,(\beta, \alpha)\} = \{\tilde{\gamma}_{\alpha,2}\,(\alpha, -\beta)\}$.

4. Solutions for intersections of arbitrary α's

In the final step of our analysis of primes we will need the solutions for intersections of the equations (3.2), (3.10) and (3.12). Here we have to differ between two possible main cases which we will solve in the following.

4.1. **Intersections for equations with the same factor sign.** At first we look at the case

$$0 = (6\alpha_i \pm 1)\,\beta_i - (6\alpha_j \pm 1)\,\beta_j + \kappa_i - \kappa_j \tag{4.1}$$

with $\kappa_i, \kappa_j \in \mathbb{Z}$, $\alpha_j = \alpha_i + \Delta\alpha$, $\Delta\alpha \in \mathbb{N} : \Delta\alpha < \alpha_j$, $i, j \in \mathbb{N}$ and the choice of the same sign for number one for both factors. For a linear Diophantine equation we know that this is only solvable if and only if

$$\gcd\,(6\alpha_i \pm 1, 6\alpha_j \pm 1) \mid (\kappa_i - \kappa_j)\,. \tag{4.2}$$

So we receive three different subcases.

4.1.1. *Subcase (1): Integer divisibility.* Be

$$6\alpha_j \pm 1 = (6\alpha_i \pm 1)\left(6\alpha_j' + 1\right) \tag{4.3}$$

with $\alpha_j' \in \mathbb{N}$ and it follows

$$0 = \beta_i - \left(6\alpha_j' + 1\right)\beta_j + \frac{\kappa_i - \kappa_j}{6\alpha_i \pm 1} \tag{4.4}$$

with the constraint

$$(6\alpha_i \pm 1) \mid (\kappa_i - \kappa_j)\,. \tag{4.5}$$

We receive this case for

$$\begin{aligned} \alpha_j &= 6\alpha_i\alpha_j' + \alpha_i \pm \alpha_j' \\ \Leftrightarrow \quad \Delta\alpha &= \alpha_j'\,(6\alpha_i \pm 1) \end{aligned} \tag{4.6}$$

and the solutions

$$\begin{aligned} \beta_i &= \left(6\alpha_j' + 1\right) Y_{j\circ i} - \frac{\kappa_i - \kappa_j}{6\alpha_i \pm 1} \\ \beta_j &= Y_{j\circ i} \end{aligned} \tag{4.7}$$

with $Y_{j\circ i} \in \mathbb{Z}$.

4.1.2. *Subcase (2): One common factor.* Be at first

$$6\alpha_{i,j} + 1 = \left(6\alpha_{i,j}' \pm 1\right)(6\beta' \pm 1) \tag{4.8}$$

with $\alpha_{i,j}', \beta' \in \mathbb{N}$ it follows

$$0 = (6\alpha_i' \pm 1)\,\beta_i - \left(6\alpha_j' \pm 1\right)\beta_j + \frac{\kappa_i - \kappa_j}{6\beta' \pm 1} \tag{4.9}$$

with the constraint

$$(6\beta' \pm 1) \mid (\kappa_i - \kappa_j) \tag{4.10}$$

for

$$\begin{aligned} \alpha_i &= 6\alpha_i'\beta' \pm \alpha_i' \pm \beta' \\ \alpha_j &= 6\alpha_j'\beta' \pm \alpha_j' \pm \beta' \\ \Leftrightarrow \quad \Delta\alpha &= (6\beta' \pm 1)\left(\alpha_j' - \alpha_i'\right). \end{aligned} \tag{4.11}$$

The second possibility we have for

$$6\alpha_{i,j} - 1 = \left(6\alpha_{i,j}' \pm 1\right)(6\beta' \mp 1)\,. \tag{4.12}$$

Here it follows

$$0 = (6\alpha_i' \pm 1)\,\beta_i - \left(6\alpha_j' \pm 1\right)\beta_j + \frac{\kappa_i - \kappa_j}{6\beta' \mp 1} \tag{4.13}$$

with the constraint

$$(6\beta' \mp 1) \mid (\kappa_i - \kappa_j) \tag{4.14}$$

for

$$\begin{aligned} \alpha_i &= 6\alpha_i'\beta' \mp \alpha_i' \pm \beta' \\ \alpha_j &= 6\alpha_j'\beta' \mp \alpha_j' \pm \beta' \\ \Leftrightarrow \quad \Delta\alpha &= (6\beta' \mp 1)\left(\alpha_j' - \alpha_i'\right). \end{aligned} \tag{4.15}$$

For the equations (4.9) and (4.13) we now have subcase (3).

4.1.3. *Subcase (3): No common factors.* Now we have no common factors between $6\alpha_i \pm 1$ and $6\alpha_j \pm 1$, so that we have the situation (4.1) and no constraint. With the substitution $\alpha_j = \alpha_i + \Delta\alpha$ we receive

$$0 = (6\alpha_i \pm 1)(\beta_i - \beta_j) - 6\Delta\alpha\beta_j + \kappa_i - \kappa_j. \tag{4.16}$$

At first we solve it for the case $\Delta\alpha = 1$, for which we can directly see the solution

$$\begin{aligned} \beta_i^1 &= \mp(\kappa_i - \kappa_j)(\alpha_i + 1) \\ \beta_j^1 &= \mp(\kappa_i - \kappa_j)\alpha_i. \end{aligned} \tag{4.17}$$

Now we will make the crossover

$$\begin{aligned} \Delta\alpha &\to \Delta\alpha + 1 \\ \beta_{i,j}^{\Delta\alpha+1} &= \beta_{i,j}^{\Delta\alpha} + \Delta\beta_{i,j}^{\Delta\alpha+1} \end{aligned} \tag{4.18}$$

to

$$0 = (6\alpha_i \pm 1)\left(\Delta\beta_i^{\Delta\alpha+1} - \Delta\beta_j^{\Delta\alpha+1}\right) - 6(\Delta\alpha + 1)\Delta\beta_j^{\Delta\alpha+1} - 6\beta_j^{\Delta\alpha}. \tag{4.19}$$

For step $\Delta\alpha = 2$, it follows the recursive solution

$$\begin{aligned} \Delta\beta_j^2 &= \pm\frac{6}{2}\beta_j^1\alpha_i \\ &= -\frac{6}{2}(\kappa_i - \kappa_j)\alpha_i^2 \\ \Rightarrow \quad \beta_j^2 &= \mp(\kappa_i - \kappa_j)\alpha_i\left(1 \pm \frac{6}{2}\alpha_i\right). \end{aligned} \tag{4.20}$$

If we do it in the same way for further $\Delta\alpha$, finally we receive

$$\beta_j^{\Delta\alpha} = (6\alpha_i \pm 1)Y_{j\circ i} \mp (\kappa_i - \kappa_j)\alpha_i\prod_{k=2}^{\Delta\alpha}\left(1 \pm \frac{6}{k}\alpha_i\right) \tag{4.21}$$

for $\Delta\alpha > 1$, the constraint

$$\Delta\alpha < 6\alpha_i \pm 1 \tag{4.22}$$

and the facts $\beta_j^{\Delta\alpha} = \beta_j^{\Delta\alpha+n(6\alpha_i\pm1)}$, $\beta_i^{\Delta\alpha+n(6\alpha_i\pm1)} = \beta_i^{\Delta\alpha} + 6n\beta_j^{\Delta\alpha}$, $n \in \mathbb{N}$, which follows from (4.6). In the same way we can receive a solution for $\beta_i^{\Delta\alpha}$ if we substitute $\alpha_i = \alpha_j - \Delta\alpha$, $\Delta\alpha < \alpha_j$, in (4.1) and it follows

$$0 = (6\alpha_j \pm 1)(\beta_i - \beta_j) - 6\Delta\alpha\beta_i + \kappa_i - \kappa_j, \tag{4.23}$$

$$\beta_i^{\Delta\alpha} = (6\alpha_j \pm 1)Y_{j\circ i} \mp (\kappa_i - \kappa_j)\alpha_j\prod_{k=2}^{\Delta\alpha}\left(1 \pm \frac{6}{k}\alpha_j\right) \tag{4.24}$$

and the constraint

$$\Delta\alpha < 6\alpha_j \pm 1. \tag{4.25}$$

But attention! It doesn't belong to $\beta_j^{\Delta\alpha}$ from (4.21). Now we simplify the product of (4.21). For the positive case we can write

$$\begin{aligned}\prod_{k=2}^{\Delta\alpha}\left(1+\frac{6}{k}\alpha_{i,j}\right) &= \frac{1}{\Delta\alpha!}\left(6\alpha_{i,j}+2\right)\cdot\dots\cdot\left(6\alpha_{i,j}+\Delta\alpha\right)\\ &= \frac{1}{\Delta\alpha!}\frac{\left(6\alpha_{i,j}+\Delta\alpha\right)!}{\left(6\alpha_{i,j}+1\right)!}\\ &= \frac{\Gamma\left(6\alpha_{i,j}+\Delta\alpha+1\right)}{\Gamma\left(\Delta\alpha+1\right)\Gamma\left(6\alpha_{i,j}+2\right)}\end{aligned} \tag{4.26}$$

and for the negative case

$$\begin{aligned}\prod_{k=2}^{\Delta\alpha}\left(1-\frac{6}{k}\alpha_{i,j}\right) &= (-1)^{\Delta\alpha+1}\frac{1}{\Delta\alpha!}\left(6\alpha_{i,j}-\Delta\alpha\right)\cdot\dots\cdot\left(6\alpha_{i,j}-2\right)\\ &= (-1)^{\Delta\alpha+1}\frac{1}{\Delta\alpha!}\frac{\left(6\alpha_{i,j}-2\right)!}{\left(6\alpha_{i,j}-\Delta\alpha-1\right)!}\\ &= (-1)^{\Delta\alpha+1}\frac{\Gamma\left(6\alpha_{i,j}-1\right)}{\Gamma\left(\Delta\alpha+1\right)\Gamma\left(6\alpha_{i,j}-\Delta\alpha\right)}\end{aligned} \tag{4.27}$$

with the Gamma function $\Gamma(x)$.

4.2. **Intersections for equations with opposite factor signs.** Now we look at the case

$$0=\left(6\alpha_i\pm 1\right)\beta_i-\left(6\alpha_j\mp 1\right)\beta_j+\kappa_i-\kappa_j. \tag{4.28}$$

Since we assume $6\alpha_i\pm 1<6\alpha_j\mp 1$, it follows $\alpha_i<\alpha_j$, hence $\Delta\alpha\in\mathbb{N}$, for the sign choice plus and minus and $\alpha_i\leq\alpha_j$, hence $\Delta\alpha\in\mathbb{N}_0$, for the sign choice minus and plus.

4.2.1. *Subcase (1): Integer divisibility.* Be

$$6\alpha_j\pm 1=\left(6\alpha_i\mp 1\right)\left(6\alpha_j'-1\right) \tag{4.29}$$

it follows

$$0=\beta_i-\left(6\alpha_j'-1\right)\beta_j+\frac{\kappa_i-\kappa_j}{6\alpha_i\mp 1} \tag{4.30}$$

with the constraint

$$\left(6\alpha_i\mp 1\right)\mid\left(\kappa_i-\kappa_j\right). \tag{4.31}$$

We receive it for

$$\begin{aligned} &\alpha_j=6\alpha_i\alpha_j'-\alpha_i\mp\alpha_j'\\ \Leftrightarrow\quad &\Delta\alpha=\alpha_j'\left(6\alpha_i\mp 1\right)-2\alpha_i\end{aligned} \tag{4.32}$$

with the solutions

$$\begin{aligned}\beta_i&=\left(6\alpha_j'-1\right)Y_{j\circ i}-\frac{\kappa_i-\kappa_j}{6\alpha_i\mp 1}\\ \beta_j&=Y_{j\circ i}.\end{aligned} \tag{4.33}$$

4.2.2. *Subcase (2): One common factor.* Be at first

$$\begin{aligned} 6\alpha_i + 1 &= (6\alpha_i' \pm 1)(6\beta' \pm 1) \\ 6\alpha_j - 1 &= \left(6\alpha_j' \mp 1\right)(6\beta' \pm 1)\,. \end{aligned} \tag{4.34}$$

It follows

$$0 = (6\alpha_i' \pm 1)\,\beta_i - \left(6\alpha_j' \mp 1\right)\beta_j + \frac{\kappa_i - \kappa_j}{6\beta' \pm 1} \tag{4.35}$$

with the constraint

$$(6\beta' \pm 1) \mid (\kappa_i - \kappa_j) \tag{4.36}$$

for

$$\begin{aligned} \alpha_i &= 6\alpha_i'\beta' \pm \alpha_i' \pm \beta' \\ \alpha_j &= 6\alpha_j'\beta' \pm \alpha_j' \mp \beta' \\ \Leftrightarrow \quad \Delta\alpha &= (6\beta' \pm 1)\left(\alpha_j' - \alpha_i'\right) \mp 2\beta'. \end{aligned} \tag{4.37}$$

And as second

$$\begin{aligned} 6\alpha_i - 1 &= (6\alpha_i' \pm 1)(6\beta' \mp 1) \\ 6\alpha_j + 1 &= \left(6\alpha_j' \mp 1\right)(6\beta' \mp 1) \end{aligned} \tag{4.38}$$

it follows

$$0 = (6\alpha_i' \pm 1)\,\beta_i - \left(6\alpha_j' \mp 1\right)\beta_j + \frac{\kappa_i - \kappa_j}{6\beta' \mp 1} \tag{4.39}$$

with the constraint

$$(6\beta' \mp 1) \mid (\kappa_i - \kappa_j) \tag{4.40}$$

for

$$\begin{aligned} \alpha_i &= 6\alpha_i'\beta' \mp \alpha_i' \pm \beta' \\ \alpha_j &= 6\alpha_j'\beta' \mp \alpha_j' \mp \beta' \\ \Leftrightarrow \quad \Delta\alpha &= (6\beta' \mp 1)\left(\alpha_j' - \alpha_i'\right) \mp 2\beta'. \end{aligned} \tag{4.41}$$

For the equations (4.35) and (4.39) we now have subcase (3), too.

4.2.3. *Subcase (3): No common factors.* For different signs we have the issue

$$\begin{aligned} 0 &= (6\alpha_i \pm 1)\,\beta_i - (6\alpha_i \mp 1)\,\beta_j - 6\Delta\alpha\beta_j + \kappa_i - \kappa_j \\ &= (6\alpha_i \pm 1)(\beta_i - \beta_j) - (6\Delta\alpha \mp 2)\,\beta_j + \kappa_i - \kappa_j \end{aligned} \tag{4.42}$$

for $\alpha_j = \alpha_i + \Delta\alpha$. Be $\beta_j := 6\mathcal{B}_j$, $\mathcal{B}_j \in \mathbb{Z}$ and $\Delta\mathcal{A} := 6\Delta\alpha \mp 2$. We also can write

$$0 = (6\alpha_i \pm 1)(\beta_i - 6\mathcal{B}_j) - 6\Delta\mathcal{A}\mathcal{B}_j + \kappa_i - \kappa_j. \tag{4.43}$$

We see that we receive our solution

$$\beta_j^{\Delta\mathcal{A}} = 6\mathcal{B}_j^{\Delta\mathcal{A}} \tag{4.44}$$

with $\mathcal{B}_j^{\Delta\mathcal{A}}$ from the same factor sign case (4.21) for $\Delta\mathcal{A}$. From (4.37) respectively (4.41) follows with $\alpha_j' \in \{0,1\}$, $\alpha_i' = 0$ and $\beta' = \alpha_i$ the constraint

$$-2\beta' < \Delta\alpha < 4\beta' \pm 1. \tag{4.45}$$

But in contrast to the previous section, the set of $\Delta\alpha$ is additional constricted and hence the using of the resulting equation, caused by the constraint $\Delta\mathcal{A} < 6\alpha_i \pm 1$. Since $\Delta\alpha \mapsto 6\Delta\alpha \mp 2$, we have to map $\Delta\mathcal{A}$ on the range $[1, 6\alpha_i \pm 1 - 1]$ with

$$\Delta\mathcal{A}_{[1,6\alpha_i\pm1-1]} = \Delta\mathcal{A} - f(\alpha_i, \Delta\alpha)(6\alpha_i \pm 1), \tag{4.46}$$

$f(\alpha_i, \Delta\alpha)$ a step function. With the estimation $\frac{6\Delta\alpha\mp2}{6\alpha_i\pm1} \approx \frac{6\Delta\alpha}{6\alpha_i\pm1} \approx \frac{6\Delta\alpha}{6\alpha_i} \approx \frac{\Delta\alpha}{\alpha_i}$ we get the width for each step $\mathcal{W}(\alpha_i, \Delta\alpha) = \alpha_i$ and the position for the first using of the point one, as reference point, we receive $f(\alpha_i, \Delta\alpha = \alpha_i + 0.5 \pm 0.5) = 1$. Like in the same sign case, we also can follow $\beta_i^{\Delta\mathcal{A}} = 6\mathcal{B}_i^{\Delta\mathcal{A}}$ which not belongs to $\beta_j^{\Delta\mathcal{A}}$ from the equation above, too. Additional it has the advantage that the constraint $\Delta\mathcal{A} < 6\alpha_j \pm 1 \Rightarrow \Delta\alpha < \alpha_j \pm \frac{1}{2}$ is always fulfilled for the allowed restricted domain $\Delta\alpha < \alpha_j$, so that we have no step function, here.

5. Points of prime calculation

Now we are able to construct our primes with the results of the last sections. We will discuss the important properties by doing the first steps by hand.

5.1. **The first numbers which we know.** Let us look again at (2.3), (2.6) and (2.8) and its solutions from the beginning (see table 1). If we look at case $\alpha = 1$,

Table 1. γ-values which belongs to the first integer divisible numbers for the sets O_- respectively O_+.

	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$
$\gamma'_- = (6\alpha + 1)\beta - \alpha$	$6, 13, 20, \ldots$	$11, 24, 37, \ldots$	$16, 35, 54, \ldots$
$\gamma'_{+,1} = (6\alpha + 1)\beta + \alpha$	$8, 15, 22, \ldots$	$15, 28, 41, \ldots$	$22, 41, 60, \ldots$
$\gamma'_{+,2} = (6\alpha - 1)\beta - \alpha$	$4, 9, 14, \ldots$	$9, 20, 31, \ldots$	$14, 31, 48, \ldots$

which we called step $s = 0$, we see that we already know sure which numbers are primes and which are not primes for γ'_- in the range $[1, 6]$. Strictly speaking we know all numbers in the range $[1, 11]$ since the second number of $\alpha + 1$ always sits between the first and the second number of α but it will be easier to take the smaller range here and in the further steps. For the set O_+ we know the two ranges of pseudoprimes with $[1, 8]$ for $\gamma'_{+,1}$ and $[1, 4]$ for $\gamma'_{+,2}$. Even the combination of this two part solutions gives us the range, the smaller one of both, in which we know sure the primes. Until now we can write for step $s = 0$, for the set O_-

$$\begin{aligned} \gamma_-^0 &= 6 - \chi \\ \gamma_-^{0\prime} &= 6 \end{aligned} \tag{5.1}$$

with $\chi \leq 6$ and the wrong solution $\gamma_-^0 = 0$. For the set O_+

$$\begin{aligned} \gamma_+^0 &= 4 - \chi \\ \gamma_+^{0\prime} &= 4 \end{aligned} \tag{5.2}$$

with $\chi \leq 4$ and the wrong solution $\gamma_+^0 = 0$, too.

5.2. **Steps of calculation.** With the results from $s = 0$ we are able to find our next in size range. Until now, we know all primes γ_- in the range $\mathcal{R}_0^- := [1, 6]$ and γ_+ in the range $\mathcal{R}_0^+ := [1, 4]$. In step $s = 1$ we will make the intersections calculations for all primes $\alpha_{-,+} := \gamma_{-,+}$ within this ranges. Section 4 shows us that we have the easiest case if we have no common factors, since we have consider no constraint. Fortunately, we only need this case, caused by the uniqueness of prime factorization of all divisible numbers. If we look at $\tilde{\gamma}_{\alpha,1}$, for which we have $\alpha_+ = 1, 2, 3$, we don't need the integer divisible number $6 \cdot 4 + 1 = 25$, since we also have the solutions for its prime factor 5 by $\tilde{\gamma}_{1,2}$. The two equations $\tilde{\gamma}_{\alpha,1,2}$ for the set O_+ gives the prime factors for themselves and for each other, too. Equation $\tilde{\gamma}_\alpha$ solves it for all α_+ itself since for the set O_- every divisible number has always one factor of each of the sets O_- and O_+ at least. We look at the general case

$$\tilde{\gamma}_1 = p_1\beta_1 + \kappa_1$$
$$\tilde{\gamma}_2 = p_2\beta_2 + \kappa_2,$$

$p_{1,2} \in \mathcal{P} : p_1 \neq p_2$ and $\kappa_{1,2} \in \mathbb{Z}$. It follows

$$\begin{aligned} \tilde{\gamma}_1 &= p_1 \cdot p_2 \cdot Y_{2\circ 1} + p_1\mathcal{F}_1(\kappa_1 - \kappa_2, \alpha_1, \alpha_2) + \kappa_1 \\ \text{and} \quad \tilde{\gamma}_2 &= p_1 \cdot p_2 \cdot Y_{2\circ 1} + p_2\mathcal{F}_2(\kappa_1 - \kappa_2, \alpha_1, \alpha_2) + \kappa_2 \end{aligned} \tag{5.3}$$

with $p_{i,j} := 6\alpha_{i,j} \pm 1$, $\mathcal{F}_{i,j}(\kappa_i - \kappa_j, \alpha_i, \alpha_j)$, the second term of the solution for $\beta_{i,j}$ from section 4. Additional we have

$$\tilde{\gamma}_3 = p_3\beta_3 + \kappa_3$$

with $p_1 \neq p_2 \neq p_3$. Now we can take two possible ways. The first one, if we choose $\tilde{\gamma}_1$

$$\begin{aligned} \tilde{\gamma}_1 =& p_1 \cdot p_2 \cdot p_3 \cdot Y_{3\circ 2\circ 1} + p_1 \cdot p_2\mathcal{F}_{2\circ 1}(\kappa_{2\circ 1} - \kappa_3, \alpha_{2\circ 1}, \alpha_3) \\ &+ p_1\mathcal{F}_1(\kappa_1 - \kappa_2, \alpha_1, \alpha_2) + \kappa_1 \\ \text{and} \quad \tilde{\gamma}_3 =& p_1 \cdot p_2 \cdot p_3 \cdot Y_{3\circ 2\circ 1} + p_3\mathcal{F}_3(\kappa_{2\circ 1} - \kappa_3, \alpha_{2\circ 1}, \alpha_3) + \kappa_3 \end{aligned} \tag{5.4}$$

with $p_1 \cdot p_2 := 6\alpha_{2\circ 1} \pm 1$, $\kappa_{2\circ 1} := p_1\mathcal{F}_1(\kappa_1 - \kappa_2, \alpha_1, \alpha_2) + \kappa_1$, and the second one, if we choose $\tilde{\gamma}_2$

$$\begin{aligned} \tilde{\gamma}_2 =& p_1 \cdot p_2 \cdot p_3 \cdot Y_{3\circ 2\circ 1} + p_1 \cdot p_2\mathcal{F}_{2\circ 1}(\kappa_{2\circ 1} - \kappa_3, \alpha_{2\circ 1}, \alpha_3) \\ &+ p_2\mathcal{F}_2(\kappa_1 - \kappa_2, \alpha_1, \alpha_2) + \kappa_2 \\ \text{and} \quad \tilde{\gamma}_3 =& p_1 \cdot p_2 \cdot p_2 \cdot Y_{3\circ 2\circ 1} + p_3\mathcal{F}_3(\kappa_{2\circ 1} - \kappa_3, \alpha_{2\circ 1}, \alpha_3) + \kappa_3 \end{aligned} \tag{5.5}$$

with $\kappa_{2\circ 1} := p_2\mathcal{F}_2(\kappa_1 - \kappa_2, \alpha_1, \alpha_2) + \kappa_2$. And so on for further substeps. We have to remark some points. Firstly, the final $\tilde{\gamma}_3$ seems to be easier than $\tilde{\gamma}_1$ or $\tilde{\gamma}_2$ in relating to the last terms. Secondly, it is easier to find the β which belongs to the smaller factor caused by the constraints $\Delta\alpha < 6\alpha_i \pm 1$ and $\Delta\alpha < 6\alpha_j \pm 1$ (the second one is always fulfilled) from section 4.1.3 for the same sign case. The same we have for the opposite sign case, for which we have no step function if we choose this point of view. Now, to find our final results for the next step, we have to do the following. At first, we make the calculation above for $\tilde{\gamma}_{\alpha,1}$ and $\tilde{\gamma}_\alpha$ for all primes $\alpha_+ \in \mathcal{R}_0^+$. After this, we already have our equation for the new range $\mathcal{R}_1^-$. To find our final solution for O_+, we have to calculate the new $\tilde{\gamma}_{\alpha,2}$ for all primes $\alpha_- \in \mathcal{R}_0^-$ and find out the section of this two part solutions. This is our final equation for the range $\mathcal{R}_1^+$. In the same way we do it for further steps to receive always the next in size ranges.

5.3. **Ranges of calculation.** If we had used all $\alpha_+ \in \mathcal{R}_0^+$, we receive our new next in size range for pseudoprimes relating to $\tilde{\gamma}_{\alpha,1}$ with $\mathcal{R}_1^{+,1} := [1, \gamma'_{+,1}(4,1)] = [1, 29]$. In the same way we receive for $\tilde{\gamma}_{\alpha,2}$ and $\alpha_- \in \mathcal{R}_0^-$ the range $\mathcal{R}_1^{+,2} := [1, \gamma'_{+,2}(6,1)] = [1, 29]$. To find the primes of the set O_+, we have to make the section of this two part solutions and receive the final new range $\mathcal{R}_1^+ := \min\left(\mathcal{R}_1^{+,1}, \mathcal{R}_1^{+,2}\right)$. We have to watch, that we always have the missing number $\tilde{\gamma}_{\alpha,1} = \alpha$, so that we don't receive all possible solutions after all sections of the step in the subrange $[1,4]$ in one equation. The solution for the whole range $\mathcal{R}^{+,1}$ consists of two equations. The equation from step $s = 0$ for the subrange and the equation from step $s = 1$ for the rest of the new range. For O_- we have different kinds to receive our searched numbers. Either we can take the solution $\tilde{\gamma}_{\alpha,1}$ for the same κ-values as yet, but for negative solutions, or we take the positive solutions for $\kappa := 5\alpha + 1 - \chi$, $\chi \leq 6\alpha$, which follows from the relationships from section 3.3. Here we have the new range $\mathcal{R}_1^- := [1, \gamma'_-(4,1)] = [1, 21]$. In general, we can write for the ranges as a function of $s - 1$

$$\begin{aligned} \mathcal{R}_s^{+,1} &:= [1, \gamma'_{+,1}\left(r_{s-1}^+, 1\right)], \\ \mathcal{R}_s^{+,2} &:= [1, \gamma'_{+,2}\left(r_{s-1}^-, 1\right)], \\ \Rightarrow \quad \mathcal{R}_s^+ &:= [1, \min\left(r_s^{+,1}, r_s^{+,2}\right)] \\ \text{and} \quad \mathcal{R}_s^- &:= [1, \gamma'_-\left(r_{s-1}^+, 1\right)]. \end{aligned} \tag{5.6}$$

$r_{s-1}^{-,+}$ be the upper bound of $\mathcal{R}_{s-1}^{-,+}$. Now, be $r_1^{+,1} = r_1^{+,2}$, which we have for $r_0^+ = 5n - 1$ and $r_0^- = 7n - 1$, with $n \in \mathbb{N}$. It follows

$$\begin{aligned} r_1^+ &= \min\left(7\left(5n-1\right)+1, 5\left(7n-1\right)-1\right) \\ &= \min\left(5\cdot 7n - 7 + 1, 5\cdot 7n - 5 - 1\right) \\ &= 5\cdot 7n - 5 - 1 \\ \text{and} \quad r_1^- &= 5\left(5n-1\right)+1 \\ &= 5^2 n - 5 + 1. \end{aligned} \tag{5.7}$$

For $s = 2$ we receive

$$\begin{aligned} r_2^+ &= \min\left(7\left(5\left(7n-1\right)-1\right)+1, 5\left(5\left(5n-1\right)+1\right)-1\right) \\ &= \min\left(5\cdot 7^2 n - 5\cdot 7 - 7 + 1, 5^3 n - 5^2 + 5 - 1\right) \\ &= 5^3 n - 5^2 + 5 - 1 \\ \text{and} \quad r_2^- &= 5\left(5\left(7n-1\right)-1\right)+1 \\ &= 5^2\cdot 7n - 5^2 - 5 + 1. \end{aligned} \tag{5.8}$$

If we look at it for further s, we can also write for all odd $s := 2k-1$, $k \in \mathbb{N}$,

$$
\begin{aligned}
r^{+}_{2k-1} &= 5^{2k-1}\left(7n-1\right) + \sum_{i=1}^{2k-1} \left(-1\right)^{i} 5^{2k-1-i} \\
&= 5^{2k-1}\left(7n-1\right) - \frac{1+5^{2k-1}}{1+5} \\
\text{and} \quad r^{-}_{2k-1} &= 5^{2k} n + \sum_{i=1}^{2k} \left(-1\right)^{i} 5^{2k-i} \\
&= 5^{2k} n + \frac{1-5^{2k}}{1+5}
\end{aligned}
\tag{5.9}
$$

and for all even $s := 2k$,

$$
\begin{aligned}
r^{+}_{2k} &= 5^{2k+1} n + \sum_{i=1}^{2k+1} \left(-1\right)^{i} 5^{2k+1-i} \\
&= 5^{2k+1} n - \frac{1+5^{2k+1}}{1+5} \\
\text{and} \quad r^{-}_{2k} &= 5^{2k}\left(7n-1\right) + \sum_{i=1}^{2k} \left(-1\right)^{i} 5^{2k-i} \\
&= 5^{2k}\left(7n-1\right) + \frac{1-5^{2k}}{1+5}.
\end{aligned}
\tag{5.10}
$$

If we now choose $n = 1$, we have found our final ranges as function of s. In step $s = 1$ and the further steps, we took not the highest prime but the highest divisible number from $\gamma'\left(\alpha, 1\right)$. This is not a problem, since its prime factors are always elements of the already known ranges, too. It is trivial to see, that the estimation $0 < r^{-,+} - \frac{1}{6}\left(\sqrt{6r^{+,-} \pm 1} + 1\right)$ is always fulfilled. Finally, we have to ensure that we always receive one new prime at least, for every recursion step. The *Bertrand-Chebyshev theorem* [1] shows that it always exists at least one prime p between m and $2m-2$, $m \in \mathbb{N} : m > 1$. It is also trivial to see that the constraint $2\left(6r^{+,-}_{s} \pm 1\right) - 2 \leq 6r^{+,-}_{s+1} \pm 1$ is always fulfilled, too.

6. A short résumé

Now we want to give a short résumé about the final calculation steps. It is easy to find our starting ranges $\mathcal{R}^{-,+}_{0}$ for $s = 0$. Here we have one equation of primes for each set. If we go to $s = 1$ we can use it, at first, to find $\tilde{\gamma}_{\alpha,1}$ for all numbers of the new range $\mathcal{R}^{+,1}_{1}$ and $\mathcal{R}^{-}_{1}$. But for $\mathcal{R}^{+,1}_{1}$ we have now two equations. One for the range $\mathcal{R}^{+}_{0}$ and the second for the range $[r^{+}_{0} + 1, r^{+,1}_{1}]$. If we use the second one for section with the solution for $\tilde{\gamma}_{\alpha,2}$, we receive the final equation, which is only good in the range $\mathcal{R}^{+}_{1} = [r^{+}_{0} + 1, r^{+}_{1}]$. For the first part we still need the solution from step $s = 0$. So, for every further step we always receive one new additional equation to describe all allowed primes for the set O_{+}. This recursion is possible for all numbers $\mathbb{N}$. At last, we have to choose a reasonable order for calculation respectively choose of function, preferred this one which brought the new factor to calculation, for next sub steps. Although, we receive expressions which make it difficult to say something about the whole set of primes, we are able to see that there exists some structures caused by the development of the second part of equation over all steps. Here we have, for example, always the α from the β-factor of the last supstep as a factor in the new term of the current step. This gives them a relationship which explains some structures of the Ulam spiral.

7. Conclusion

We have seen that there exists a relationship between primes of different ranges. Although we are not able to eliminate it two a closed equation, we are however able to identify an interesting angle of view on the set of primes which gives us a background for further works in future.

References

1. P. Chebyshev, *Mémoire sur les nombres premiers*, Mém. Acad. Sci. St. Pétersbourg **7** (1850) 1854, 17–33; Reprinted in A. Markoff, N. Sonin (Ed.): *Oeuvres de P. L. Tschebychef, Tome I*, Com. Acad. Sci. St. Pétersbourg (1899), 51-64.
2. S. M. Ulam, M. L. Stein and M. B. Wells, *A visual display of some properties of the distribution of primes*, Amer. Math. Monthly **71** (1964), no. 5, 516–520.

Josephsplatz 8, 90403 Nürnberg (*int.*: Nuremberg), Germany
E-mail address: contact@carolin-zoebelein.de, PGP Key-ID: 0xde47e13b
URL: http://research.carolin-zoebelein.de