Harmonic map

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Spherical Harmonic Map

Spherical harmonic map

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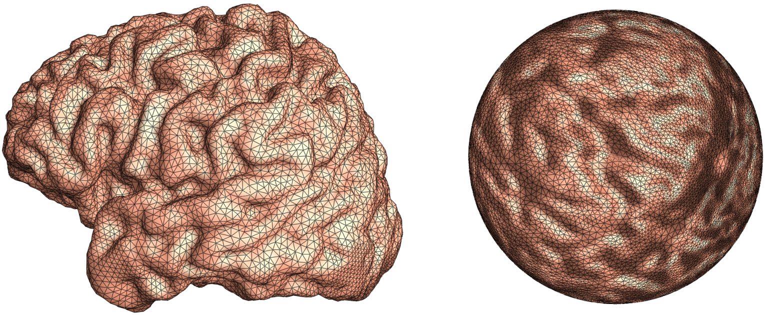

Figure: Spherical harmonic map

Spherical harmonic map

Basic Idea:

We first find a degree one map $\vec{h}$ between M and the unit sphere $\mathbb{S}^2$, the map may not be a homeomorphism. The map will be smoothed out automatically during the process.
Then we evolve $\vec{h}$ to minimize its harmonic energy until it becomes a harmonic map. The evolution of the map is according to a nonlinear heat diffusion process:

$$\frac{d\vec{f}(t)}{dt} = \Delta\vec{f}(t)$$

Spherical harmonic map

Definition (Normal component)

The normal component of the Laplacian is:

$$(\Delta\vec{f}(v))^{\perp} = \langle \Delta\vec{f}(v), \vec{n}(\vec{f}(v)) \rangle \cdot \vec{n}(\vec{f}(v))$$

where $\langle , \rangle$ is the inner product in $\mathbb{R}^3$.

It is obvious that the tangential component of the Laplacian is

$$(\Delta\vec{f}(v))^{\parallel} = \Delta\vec{f}(v) - (\Delta\vec{f}(v))^{\perp}$$

Spherical harmonic map

Definition (Harmonic Map)

A map $\vec{f}: M_1 \to M_2$ is harmonic, if and only if $\Delta_{PL}\vec{f}$ only has a normal component, and its tangential component is zero:

$$\Delta_{PL}(\vec{f}) = (\Delta_{PL}\vec{f})^{\perp}$$

So, if the nonlinear heat diffusion equation is

$$\frac{d\vec{f}(v,t)}{dt} = (\Delta\vec{f}(v))^{\parallel}$$

then $\vec{f}(v,\infty)$ is the harmonic map.

Spherical harmonic map

Any harmonic map from genus zero closed surface to the unit sphere is conformal, all such conformal mappings differ by a Möbius transformation on the sphere, which form a six dimensional group.

Definition (Möbius Transformation)

Mapping $\vec{f} : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ is a Möbius transformation if and only if

$$\phi : z \to \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{C}, ad - bc = 1$$

Spherical harmonic map

To ensure the convergence of the algorithm and the uniqueness of the solution, constrains need to be added. In practice we use the following zero mass-center constraint.

Definition (Zero mass-center condition)

Mapping $\vec{f}: M \to \mathbb{S}^2$ satisfies zero mass-center condition if and only if

$$\int_{\mathbb{S}^2} \vec{f} d\sigma = \vec{0}$$

where $d\sigma$ is the area element on M.

Spherical harmonic map

Definition (Gauss Map)

A Gauss Map $\vec{g}: M \to \mathbb{S}^2$ is defined as

$$\vec{g}(v) = \vec{n}(v), \quad v \in M$$

$\vec{n}(v)$ is the unit normal at v.

We introduce Gauss map here to use it as the initial map.

Question

What is the degree one map? Can you find another degree one map except the Gauss map?

Spherical harmonic map

Algorithm 1 Conformal Spherical Mapping

Require: Mesh M, step length δt, threshold δE

Ensure: A harmonic map $\vec{f}: M \to \mathbb{S}^2$, which satisfies zero mass-center constraint.

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Compute a degree one map, such as Gauss map g⃗ : M → S²
Initialize f⃗ ← g⃗, compute harmonic energy E_0
repeat
    for all vertex v ∈ M do
        Compute the Laplacian Δf⃗(v)
        Compute the normal component (Δf⃗(v))^⊥
        Compute the tangential component (Δf⃗(v))^∥
        Update f⃗(v) by f⃗(v) = f⃗(v) + δt · (Δf⃗(v))^∥
    end for
    Normalize(f⃗)
    E_0 ← E
until Harmonic energy difference |E − E_0| is less than δE
return f⃗
```

Spherical harmonic map

The step of normalization using Möbius transformation is non-linear and expensive to compute. In practice we use the following simple procedure instead:

Algorithm 2 Normalization

Require: Mesh M, a mapping to the sphere $\vec{f}: M \to \mathbb{S}^2$

Ensure: Normalized mapping $\vec{f}_{normal}$, whose mass center is at the sphere center

Compute the mass center $\vec{c}$ of $\vec{f}$: $\vec{c} \leftarrow \int_{\mathbb{S}^2} \vec{f} d\delta$.
where $d\delta$ is the area element on the original mesh M.

for all vertex $v \in M$ **do**

  $\vec{f}_{normal}(v) \leftarrow \vec{f}(v) - \vec{c}$

end for

for all vertex $v \in M$ **do**

  $\vec{f}_{normal}(v) \leftarrow \frac{\vec{f}_{normal}(v)}{|\vec{f}_{normal}|}$

end for

return $\vec{f}_{normal}$

Spherical harmonic map

Remark

We define $d\delta(v_i)$ *as* $\frac{area(v_i)}{\sum_i^n area(v_i)}$ *here, but we can also redefine it as* $\frac{1}{n}$, *where n is the number of vertices on M, and* $area(v) = \frac{1}{3}\sum_i area(f_i)$

Some More Skills

Algorithm 3 Initial map construction

Require: A close mesh M with genus 0.

Ensure: An initial mapping from M to the unit sphere S.

1: Construct the Laplacian Matrix of M, compute the first non-zero eigen value, and the corresponding eigen vector, which defines a real number f_i on each vertex v_i.

2: Find a loop γ on Mesh M such that $\sum_{v_i \in \gamma} f_i^2$ is minimal.

3: Slice Mesh M along γ to get two topological disks M_1 and M_2.

4: Homonically map M_1 to unit disk D_1 and M_2 to D_2 with consistent boundary condition.

5: Compute inverse Stereo-graphic projection from D_1 and D_2 to semi-sphere SS_1 and SS_2.

6: Glue SS_1 and SS_2 along equator to form a sphere S.

Double Covering

Algorithm: Double Cover

Input: An open mesh with boundaries. Output: The double covering of the mesh, which is a closed symmetric mesh.

1. construct a copy of the input mesh Σ'
2. reverse the orientation of the copy, each $[v_0, v_1, v_2]$ is converted to $[v_1, v_0, v_2]$.
3. Identify each boundary vertex in $\partial\Sigma$ with the corresponding one in $\partial\Sigma'$,
4. each boundary halfedge $e \in \partial\Sigma$ has a unique corresponding halfedge $e' \in \Sigma'$. glue the corresponding boundary halfedges to the same boundary edge.

double cover

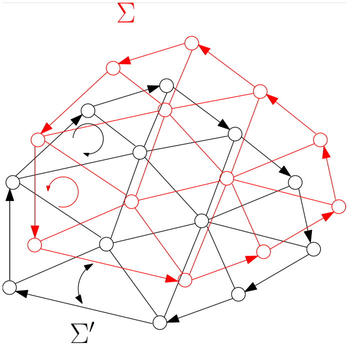

Figure: Double Covering

Conformal Map for Topological Disk

Algorithm 4 Conformal Map for Topological Disk

Require: An open mesh M with genus 0 and 1 boundary.

Ensure: A harmonic map $\vec{f}: M \to \mathbb{D}$

1: Compute the double covering of M.

2: Compute the harmonic map F from the double covering of M to the unit sphere S^2.

3: Slice S^2 along the image of the boundary edges $F(e)$, where $e \in \partial M$, choose one semi-sphere SS.

4: Stereo-graphic projection from SS to the unit disk $\mathbb{D}$.

Conformal Map for Topological Disk

Algorithm 5 Conformal Map for Topological Disk

Require: An open mesh M, a threshold ε.

Ensure: A map $\vec{f} : M \to \mathbb{D}$

Compute an initial harmonic map from M to $\mathbb{D}$ and the corresponding harmonic energy E. Pick p_1, p_2 and p_3 on ∂M.

repeat

  $E_0 = E$

  for all Vertex $v_i \in \{\partial M \setminus \{p_1, p_2, p_3\}\}$ **do**

    $\vec{f}_{v_i}$ += $((\vec{f}'_{v_i} - \vec{f}_{v_i}) - (\vec{f}'_{v_i} - \vec{f}_{v_i}) \cdot \mathbf{n})$, where $\vec{f}'_{v_i} = \sum_{[v_i, v_j] \in M} \frac{k_{ij}\vec{f}(v_j)}{\sum_j k_{ij}}$

  end for

  for all vertex $v_i \notin \partial M$ **do**

    $\vec{f}(v_i) = \sum_{[v_i, v_j] \in M} \frac{k_{ij}\vec{f}(v_j)}{\sum_j k_{ij}}$

  end for

  Recalculate the harmonic energy E.

until $E - E_0 < \varepsilon$

return $\vec{f}$