Halfedge Mesh Representation

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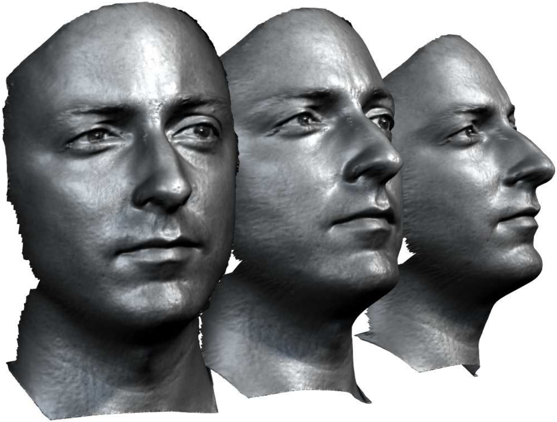
Halfedge Data Structure

Discrete Surfaces

Acquired using 3D scanner.



Discrete Surfaces

Our group has developed high speed 3D scanner, which can capture dynamic surfaces 180 frames per second.

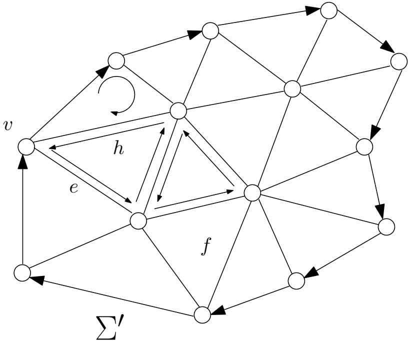
Generic Surface Model - Triangular Mesh

- Surfaces are represented as polyhedron triangular meshes.
- Isometric gluing of triangles in $\mathbb{E}^2$.
- Isometric gluing of triangles in $\mathbb{H}^2, \mathbb{S}^2$.

Discrete structures

- Topology - Simplicial Complex , combinatorics
- Conformal Structure - Corner angles (and other variant definitions)
- Riemannian metrics - Edge lengths
- Embedding - Vertex coordinates

Generic Surface Model - Triangular Mesh



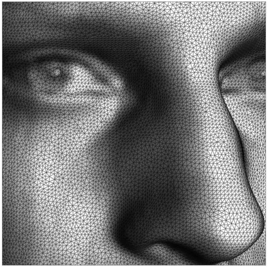
Triangle mesh

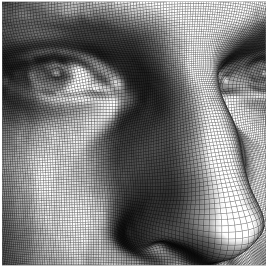
Definition (Mesh)

A triangle mesh is an oriented two dimensional simplical complex, generally embedded in $\mathbb{R}^3$.

Our goal is to design a data structure to efficiently represent general meshes.

Generic Surface Model - Triangular Mesh





halfedge data structure

fundamental classes

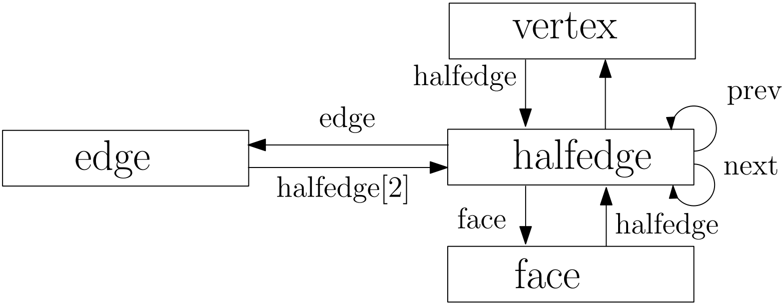
- Vertex
- Halfedge, oriented edge
- Edge, non-oriented edge
- Face, oriented

Links

All objects are linked together through pointers, such that

1. The local Euler operation can be easily performed
2. The memory cost is minimized

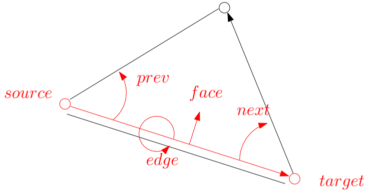
Generic Surface Model - Triangular Mesh



Halfedge class

Pointers

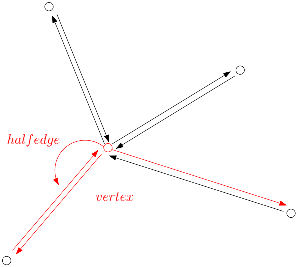
- Halfedge pointers: prev, next halfedge;
- Vertex pointers: target vertex, source vertex;
- Edge pointer: the adjacent edge;
- face pointer: the face it belongs to;



Vertex class

Pointers

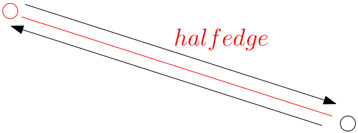
- Halfedge pointers: the first in halfedge



Edge class

Pointers

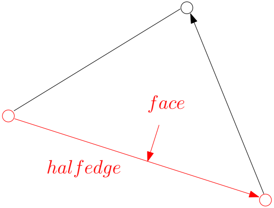
- Halfedge pointers: to the adjacent two halfedges.
- if the edge is on the boundary, then the second halfedge pointer is null.



Face class

Pointers

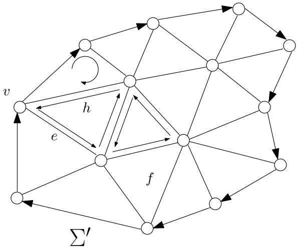
- Halfedge pointers: to the first halfedge.



Mesh class

Data members

- A list of vertices;
- A list of halfedges;
- A list of edges;
- A list of faces;



Euler Operation

circulating neighbors of a vertex $v \to v/e/f/h$

- iterate out-halfedges counter-clock-wisely
- iterate in-halfedges counter-clock-wisely
- iterate neighboring faces CCWly
- iterate neighboring vertices CCWly

Rotate a halfedge about its target vertex clwly:

$$he = he \to next() \to dual();$$

Rotate a halfedge about its target vertex ccwly:

$$he = he \to dual() \to prev();$$

Euler Operation

circulating neighbors of a face $f \to v/e/f/h$

- iterate halfedges ccwly
- iterate edges ccwly
- iterate vertices ccwly
- iterate faces ccwly

Circulate halfedges of a face ccwly:

$$he = he \to next()$$

circulate halfedge of a face clwly:

$$he = he \to prev();$$

Attributes

Attributes

Each object stores attributes (traits) which defines other structures on the mesh:

- metric structure: edge length
- angle structure: halfedge
- curvature : vertex
- conformal factor: vertex
- Laplace-Beltrami operator: edge
- Ricci flow edge weight; edge
- holomorphic 1-form: halfedge

Surface Topology

Philosophy

Associate groups with manifolds, study the topology by analyzing the group structures.

$$
\begin{aligned}
\mathfrak{C}_1 &= \{\textit{Topological Spaces}, \textit{Homeomorphisms}\} \\
\mathfrak{C}_2 &= \{\textit{Groups}, \textit{Homomorphisms}\} \\
\mathfrak{C}_1 &\to \mathfrak{C}_2
\end{aligned}
$$

Functor between categories.

Surface Topological Classification

Genus

Figure: Topological deformation between two genus zero surfaces.

Surface Topological Classification

Genus

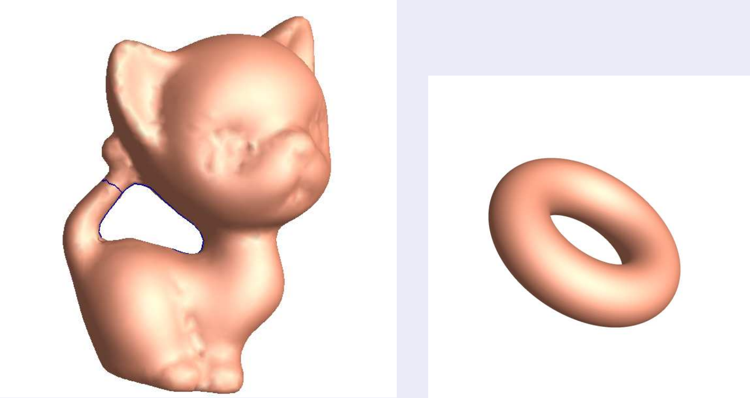


Figure: Topological deformation between two genus one surfaces.

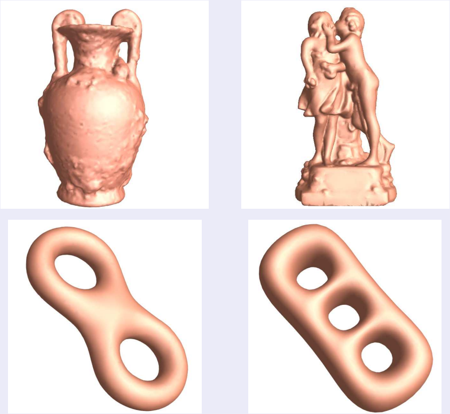
Surface Topological Classification

Genus

The number of handles of a surface is called the “genus” of the surface, which is the major topological invariants.

Surface Topological Classification

Genus



genus 2 genus 3

Figure: Topological invariants: genus of surfaces.

Surface Topological Classification

Genus

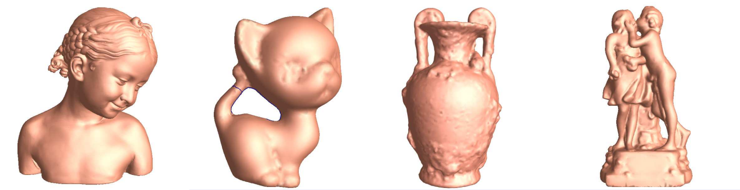


Figure: Surface topological classification

Surface Topological Classification

Boundary

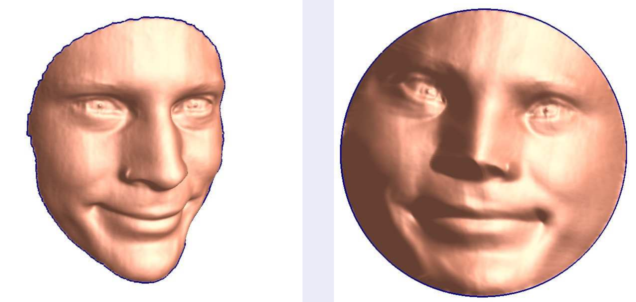


Figure: A human face surface is with one boundary and topologically equivalent to a planar disk.

A surface with boundaries is called an *open surface*. The number of boundaries is also a topological invariant.

Surface Topological Classification

Boundary

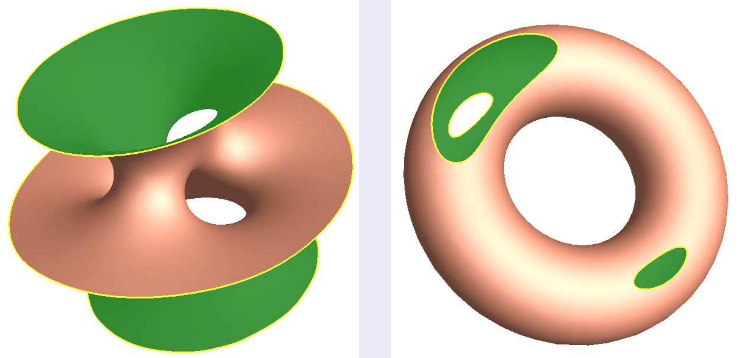


Figure: A minimal surface is topologically equivalent to a torus with three holes.

Orientability

Intuitively, if a closed surface embedded in $\mathbb{R}^3$, then it separates $\mathbb{R}^3$ into two parts. One is finite, called the interior, the other is infinite, called the exterior. Then the surface has two sides, one is inside, the other is outside. We say a surface is *orientable*, if we can differentiate its inside and outside.

Surface Topological Classification

Orientability

M.c. Escher

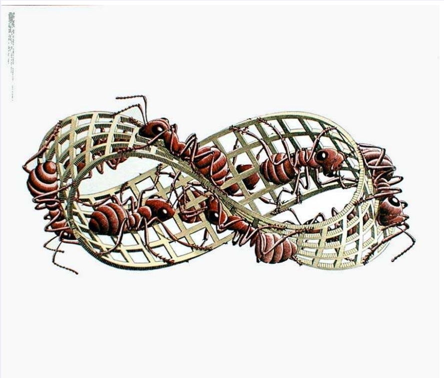


Figure: A Möbius Band is a non-orientable surface with one boundary.

Surface Topological Classification

Orientability

Figure: A Klein bottle is a non-orientable closed surface.

Connected Sum

A Genus eight Surface, constructed by connected sum.

Euler Characteristic

Euler Characteristic (Orientable surface)

Euler characteristic (or Euler number) is a topological invariant,

$$\chi = V - E + F = 2 - 2g - b$$

where V, E, F are the numbers of vertices, edges and faces, respectively. b is the number of boundaries. g is the genus.

Name	Image	Euler characteristic
Interval		1
Circle		0
Disk		1
Sphere		2
Torus (Product of two circles)		0
Double torus		−2
Triple torus		−4

Discrete Curvature

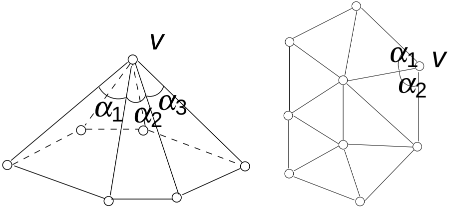
Definition (Discrete Curvature)

Discrete curvature: $K : V = \{vertices\} \to \mathbb{R}^1$.

$$K(v) = 2\pi - \sum_i \alpha_i, v \notin \partial M; K(v) = \pi - \sum_i \alpha_i, v \in \partial M$$

Theorem (Discrete Gauss-Bonnet theorem)

$$\sum_{v \notin \partial M} K(v) + \sum_{v \in \partial M} K(v) = 2\pi\chi(M).$$



Assignment

Assignment 1

Verify the Discrete Gauss-Bonnet theorem.

(Send to shawnxpzheng@gmail.com, due on September 22.)