Advanced Mathematics for Engineers, Laboratory Problems

November 9, 2017

Linear Algebra

Problem 1.1 Implement a tool that reads a matrix and right hand side vector of a system of linear equations from a file and does the following:

a) Write programs to compute the relevant information about the matrix using built-in functions: determinant, inverse, rank and the eigenvalues and eigenvectors, whether it is symmetric and positive definite.

b) Solve the linear system using a built in function (do not use the inverse matrix).

c) Program the Gaussian Elimination method described in section 5.2.1 for the matrix and print the resultant matrix (built-in function not to be used).

If any of the above does not exist/cannot be calculated, print the appropriate reason.

Problem 1.2

a) Write a program that multiplies two arbitrary matrices. Don't forget to check the dimensions of the two matrices before multiplying. The formula is

$$C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}.$$

Do not use built-in functions for matrix manipulation.

b) Write a program that computes the transpose of a matrix.

Calculus – Selected Topics

Problem 3.1 Given the function $f : \mathbb{N}_0 \to \mathbb{N}$ with

$$f(x) = x! \equiv \begin{cases} x \cdot (x-1) \cdot (x-2) \cdot \ldots \cdot 2 \cdot 1 & \text{if } x \geq 1 \\ 1 & \text{if } x = 0. \end{cases}$$

Give a recursive definition of the function.

a) Implement an iterative and a recursive method for calculating $f(x)$.

b) The upper function can be generalized, such that $\Gamma : \mathbb{C} \to \mathbb{R}$ with

$$\Gamma(z) \equiv \int_0^1 \left[\ln\left(\frac{1}{t}\right)\right]^{z-1} \mathrm{d}t.$$

Implement this function using a built-in function for the definite integral and show empirically that $\Gamma(n) = (n-1)!$.

c) Write a recursive program that calculates the Fibonacci function

$$\mathrm{Fib(n)} = \begin{cases} \mathrm{Fib}(n-1) + \mathrm{Fib}(n-2) & \text{if } n > 1 \\ 1 & \text{if } n = 0, 1 \end{cases}$$

and test it for $n = 1 \ldots 20$. Report about your results!

d) Plot the computing time of your program as a function of n.

Problem 3.2

a) Write a short program to show that the series $\sum_{k=0}^{\infty} \frac{1}{k!}$ converges to the Euler number e and plot the graph.

b) Calculate the Taylor polynomials of the function $sin(x)$ at $x_0 = 0$ from degree 0 to 6. Plot all the polynomials and the $sin(x)$ in the interval $[-2\pi, 2\pi]$ in one diagram.

c) Implement a program to calculate e^x with a precision of 10 digits. Utilize the Maclaurin series of the function (Taylor expansion in $x_0 = 0$). Use the Lagrangian form of the remainder term to find a proper polynomial degree for the approximation.
Test your program for different values of x.

Problem 3.3 In a bucket with capacity v there is a poisonous liquid with volume αv. The bucket has to be cleaned by repeatedly diluting the liquid with a fixed amount $(\beta - \alpha)v$ $(0 < \beta \leq 1)$ of water and then emptying the bucket. After emptying, the bucket always keeps αv of its liquid. Cleaning stops when the concentration c_n of the poison after n iterations is reduced from 1 to $c_n < \epsilon > 0$, where $\alpha < 1$.

a) Assume $\alpha = 0.01$, $\beta = 1$ and $\epsilon = 10^{-9}$. Compute the number of cleaning-iterations.

b) Compute the total volume of water required for cleaning.

c) Can the total volume be reduced by reducing β? If so, determine the optimal β.

d) Give a formula for the time required for cleaning the bucket.

e) How can the time for cleaning the bucket be minimized?

Statistics and Probability

Problem 4.1

a) Implement the mentioned linear congruential generator of the form $x_n = (ax_{n-1} + b) \bmod m$ with $a = 7141$, $b = 54773$ and $m = 259200$ in a programming language of your choice.

b) Test this generator on symmetry and periodicity.

c) Repeat the test after applying the Neumann Filter.

d) Experiment with different parameters a, b, m. Will the quality of the bits be better?

Problem 4.2 Download the file from [1]. It contains greyscale numbers from the Google Streetview house number dataset and a Octave program skeleton. Complete the following tasks in the program skeleton:

a) Run PCA and visualize the first 100 eigenvectors. (PCA is provided in pca.m, the visualisation is provided in displayData.m, each function provides help via *help commandname*)

b) Project the data down to 100 dimensions.

c) Recover the data.

d) Experiment with different (smaller) number of principal components.

e) Compare the effects you observe on the restored data with effects you know from JPEG images. Can you observe similarities and explain them?

The program shows you the eigenvectors as images and the difference between the original numbers and the numbers using just 100 eigenvectors. Now you can try different numbers of eigenvectors and see how PCA works on images.

Numerical Mathematics Fundamentals

Problem 5.1 Roots of Nonlinear Equations

Given the fixed-point equation $\frac{1}{2}e^{-x^2} = x$

a) Plot the left- and right-hand side of the equation in one diagram to approximate the solution graphically.

b) Analyse if fixed-point iteration is applicable and find a contraction interval!

c) Determine the Lipschitz constant L!

d) How many iteration steps are necessary to reach a precision of 12 decimal digits starting from $x_0 = 0$ (a priori estimation)?

e) Implement fixed-point iteration, interval bisection and newton's method and apply the programs for solving the equation! Which method is the fastest, slowest (give reasons)? Does the expected speed of convergence meet the real speed?

f) How many steps does fixed-point iteration take to a precision of 12 digits on the above equation? Why is the number less than estimated?

Function Approximation

Problem 6.1

a) Write a program that calculates a table of all coefficients of the interpolating polynomial of degree n for any function f in any interval $[a, b]$. Pass the function name, the degree of the polynomial and the value table as parameters to the program.

b) Apply the program to the interpolation of the function $f(x) := e^{-x^2}$ in the interval $[-2, 10]$ and calculate the polynomial up to the 10th degree. The given points are to be distributed "equidistant". Plot both funtions together in one diagram.

c) Calculate the maximum norm of the deviation between the interpolation polynomial p and f from exercise b) on an equidistant grid with 100 given points.

d) Compare the equidistant interpolation with the Taylor series of f of degree 10 (expanded around $x_0 = 0$ and $x_0 = 4$), with respect to maximum norm of the approximation error.

[1] http://www.hs-weingarten.de/~ertel/vorlesungen/mae/exercise7_13.zip

Problem 6.2

a) Write an octave function to determine the coefficients $a_1 \dots a_k$ of a function

$$f(x) = a_1 f_1(x) + a_2 f_2(x) + \cdots + a_k f_k(x)$$

with the method of least squares. The two parameters of the function are an $m \times 2$ matrix of data points as well as a cell array with the handles of the base functions $f_1, \dots, f_k$ (see www.gnu.org/software/octave/doc/v4.0.1/Function-Handles.html for function handles).

b) Test the function by creating a linear equation with 100 points on a line, and then use your function to determine the coefficients of the line. Repeat the test with slightly noisy data (add a small random number to the data values). Plot the approximated straight line in red together with the data points in blue.

c) Determine the polynomial of degree 4, which minimizes the sum of the squared errors of the following value table (see: http://www.hs-weingarten.de/~ertel/vorlesungen/mathi/mathi-ueb15.txt):

8	-16186.1	18	8016.53	28	10104.	38	41046.6
9	-2810.82	19	7922.01	29	15141.8	39	37451.1
10	773.875	20	4638.39	30	15940.5	40	37332.2
11	7352.34	21	3029.29	31	19609.5	41	29999.8
12	11454.5	22	2500.28	32	22738.	42	24818.1
13	15143.3	23	6543.8	33	25090.1	43	10571.6
14	13976.	24	3866.37	34	29882.6	44	1589.82
15	15137.1	25	2726.68	35	31719.7	45	-17641.9
16	10383.4	26	6916.44	36	38915.6	46	-37150.2
17	14471.9	27	8166.62	37	37402.3		

d) Calculate to c) the sum of the squared erors. Determine the coefficients of a parabola and calculate again the sum of the squared errors. What difference do you see? Plot both curves together with the data points in one diagram.

Problem 6.3

a) From the data in problem 6.2 c), extract all data points with an even x within one line of code.

b) Divide these data points randomly in two matrices X and T of size 10×2 (each consisting of 10 data points).

c) X is called the training data, T the test data. Determine the polynomials of degree 2-8, each minimizing the sum of the squared errors on the training data X. Plot the seven polynomial functions together with the data points in one diagram. The points in X have to be plotted in blue and those in T with red. Plot each function in a different color using the following cell array of color characters: `{'b','c','r','g','m','y','k'}`

d) For each polynomial, calculate the sum of the squared errors on the training data X and the test data T. Plot both squared error sums in dependence of degree n in one diagram (using again blue for X and red for T).

e) Since we divide the data randomly to X and T, the outcome might change if we run the program again. Run the program from b) to c) several times. What is the decisive difference in the plots of the squared error sums for X and T and can you explain this difference? At what degree do we have the smallest error on the test data in most of the cases?

f) The method you applied is called cross validation. In practice, we would choose the degree which minimizes the error on the test data. Why does it make sense, to use different data subsets for the approximation and for the test?

Numerical Integration and Solution of Ordinary Differential Equations

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Problem 7.1

a) Compute the area of a unit circle using both presented Monte-Carlo methods (naive and mean of function values) to an accuracy of at least 10^{-3}.

b) Produce for both methods a table of the deviations of the estimated value depending on the number of trials (random number pairs) and draw this function. What can you say about the convergence of this method?

c) Compute the volume of the four dimensional unit sphere to a relative accuracy of 10^{-3}. How much more running time do you need?

Problem 7.2

a) Write programs that implement the Euler-, Heun- and Runge Kutta methods for solving first order initial value problems.

b) Implement the Richardson extrapolation scheme for these methods.

Problem 7.3 The initial value problem

$$\frac{dy}{dx} = \sin(xy) \qquad y_0 = y(0) = 1$$

is to be solved numerically for $x \in [0, 10]$.

a) Compare the Euler-, Heun- and Runge Kutta methods on this example. Use $h = 0.1$.

b) Apply Richardson extrapolation to improve the results in $x = 5$ for all methods. (attention: use the correct p_k for each method.)

Problem 7.4 Apply the Runge Kutta method to the predator-prey example **??** and experiment with the parameter α and the initial values. Try to explain the population results biologically.

Problem 7.5 Use Runge Kutta to solve the initial value problem

$$\frac{dy}{dx} = x \sin(xy) \qquad y_0 = y(0) = 1$$

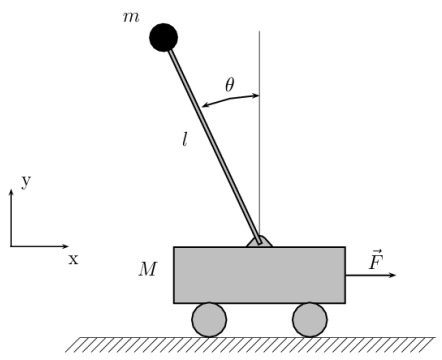
for $x \in [0, 20]$. Report about problems and possible solutions.

Problem 7.6 (challenging)

The dynamics of the inverted pendulum – also called cart pole – system as shown beside can be described by the following two differential equations of second order. Here $\dot{x}$ $\ddot{x}$, etc. are the first and second derivatives wrt. the time t. A derivation of these equations can be found on Wikipedia (not required here).

$$(M + m)\,\ddot{x} - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2 \sin\theta = 0 \qquad (7.1)$$

$$ml(-g\sin\theta - \ddot{x}\cos\theta + l\ddot{\theta}) = 0 \qquad (7.2)$$



a) Use the substitution $y_1 = x,\ y_2 = \dot{x},\ y_3 = \theta,\ y_4 = \dot{\theta}$ to obtain a system of 4 first order ODEs of the form $\dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y})$. (hint: make sure, the right hand sides of the differential equations contain no derivatives!)

b) Apply the Runge Kutta method to solve the system for $g = 9.81$, $m = 1$, $m = 1$ with the initial condition $y_1(0) = 0,\ y_2(0) = 0,\ y_3(0) = 0.01,\ y_4(0) = 0$.

c) Plot the functions $y_1(t),\ y_2(t),\ y_3(t),\ y_4(t)$ and try to understand them.

d) Experiment with other initial conditions and other masses, e.g. $m = 1$, $M = 100000$ or $M = 1$, $m = 100000$.