CS 138: Formal languages and Automota
Homework 1 solutions

1.
   - $S_1 \times S_2 = \{(2,2),(2,4),(2,5),(2,8),(2,9),(3,2),(3,4),(3,5),(3,8),(3,9),(5,2),(5,4),(5,5),(5,8),(5,9),(7,2),(7,4),(7,5),(7,8),(7,9)\}$
   - $S_2 \times S_1 = \{(2,2),(2,3),(2,5),(2,7),(4,2),(4,3),(4,5),(4,7),(5,2),(5,3),(5,5),(5,7),(8,2),(8,3),(8,5),(8,7),(9,2),(9,3),(9,5),(9,7)\}$

2. $\mid S \cap T \mid + \mid S \cup T \mid = \mid \{2,6,8\} \mid + \mid \{2,4,5,6,8\} \mid = 3+5=8$

3. $S_1 \subseteq S_2 \xRightarrow{\text{by definition}} x \in S_1 \rightarrow x \in S_2 \xRightarrow{\text{contrapositive}} x \notin S_2 \rightarrow x \notin S_1 \Rightarrow x \in \bar{S_2} \rightarrow x \in \bar{S_1} \xRightarrow{\text{by definition}} \bar{S_2} \subseteq \bar{S_1}$

4. $(S_1 \cap \bar{S_2}) \cup (\bar{S_1} \cap S_2) = \emptyset \Leftrightarrow S_1 \cap \bar{S_2} = \emptyset$ and $\bar{S_1} \cap S_2 = \emptyset \Leftrightarrow S_1 - S_2 = \emptyset$ and $S_2 - S_1 = \emptyset \Leftrightarrow S_1 = S_2$

5. Induction of $|v|$

   Basis step: $|v| = 0$, $(u\lambda)^R = u^R$ and $\lambda^R u^R = u^R$.

   Inductive hypothesis: $(uv)^R = v^R u^R$ holds for all v of length $1, 2, ..., n$.

   Inductive step: We want to show that $(uv)^R = v^R u^R$ where $|v| = n+1$. Let $v = \omega a$ where $|\omega| = n$, so $(uv)^R = (u\omega a)^R = ((u\omega)a)^R =$ (using the question hint) $a^R(u\omega)^R =$ (using induction hypothesis) $a^R\omega^R u^R = v^R u^R$

6.
   - abaabaaabaa, aaaabaaaa, baaaaabaa
   - aaaabaaaa, baaaaabaa

7. $G = (\{S\}, \{a\}, S, P)$ and P is $S \rightarrow aaS$, $S \rightarrow \lambda$.

8.
   - 1st: λ, 2nd: a, 3rd: aa, ..., 129st: a^{128}
   - If $\mid \omega \mid = k$, we have k^n words with length ω, e.g., if $\omega = 0$, there is $2^0 = 1$ word (ω) and if $\omega = 1$, there are totally $2^1 = 2$ words (a and b). On the other hand $513 = 2^0 + 2^1 + ... + 2^6 + 2$, so 127st word is b^6, 128st word is a^7, and 129st word is a^6b.

9. **Basis step:** if $|u| = 1$ then u is a single symbol in Σ and it's obvious that $|u^2| = 2|u|$.

   Inductive hypothesis: $|u^2| = 2|u|$ holds for all u of length $1, 2, ..., n$.

   Inductive step: We want to show that $|u^2| = 2|u|$ where $|u| = n+1$. Let write u as ωa where $|\omega| = n$ and a is a single symbol, then $|u^2| = |\omega a \omega a| = |\omega^2| + |aa| = 2n+2 = 2|u|$

10. We first show that all sentential forms must have the form $\omega_i = a^{2i}Sb^i$. Suppose that it holds for all sentential forms ω_i of length $3i+1$ or less. The only way to get another sentential form is to apply the production $S \to aaSb$. It gets us $a^{2i}Sb^i \Rightarrow a^{2i+2}Sb^{i+1}$, so that every sentential form of length $3i+4$ is also of form $\omega_i = a^{2i}Sb^i$.

    Finally, to get a senstence, we must apply the production $S \to \lambda$, so $S \stackrel{*}{\Rightarrow} a^{2n}Sb^n \Rightarrow a^{2n}b^n$ represents all possible derivations. Thus, G can derive only strings of the form $a^{2n}b^n$. To show that all strings of this form can be derived we simply apply $S \to aaSb$ as many times as needed, followed by $S \to \lambda$.