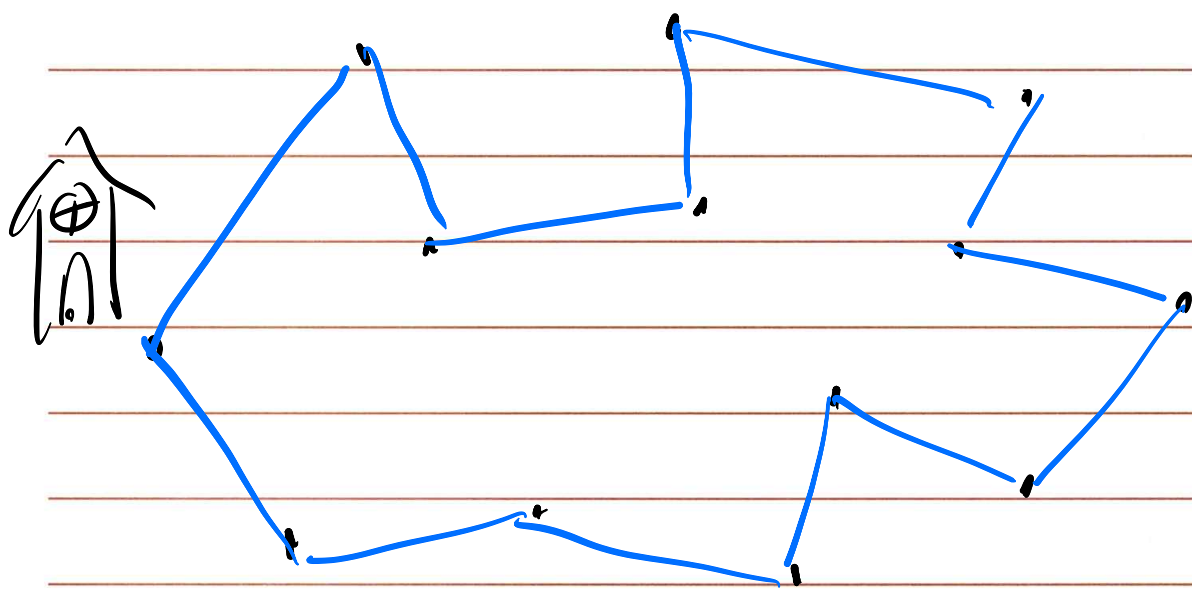
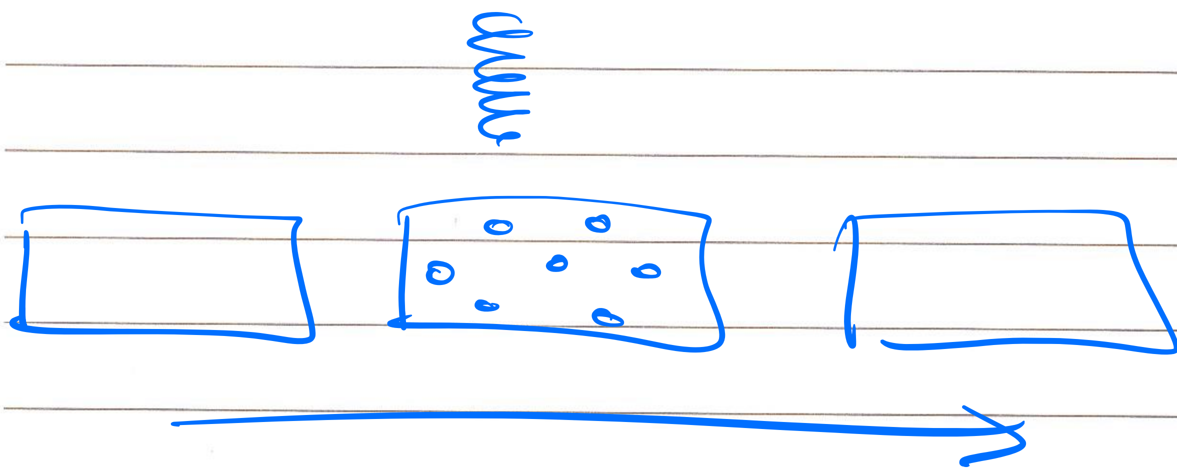
Traveling Salesman Problem (TSP) & Hamiltonian Cycle



Problem Statement

Given the set of distances, order n cities in a tour $V_{i_1}, V_{i_2}, \ldots V_{i_n}$ with $i_1 = 1$, so it minimizes

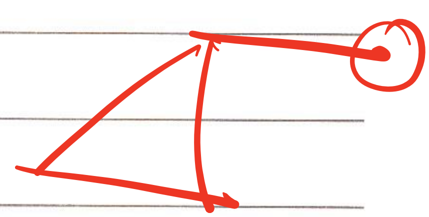
$$\sum d(V_{i_j}, V_{i_{j+1}}) + d(V_{i_n}, V_{i_1})$$



Decision version of TSP:

Given a set of distances on $\underline{n}$ cities and a bound $\underline{D}$, is there a tour of length/cost at most D?

Def. A cycle $\underline{C}$ in G is a Hamiltonian Cycle if it visits each vertex exactly once.



Problem Statement:

Given an undirected graph G, is there a Hamiltonian cycle in G?

Show that the Hamiltonian Cycle Problem is NP-complete

1- Show that The problem is in NP

a. Certificate:

ordered list of nodes on the HC

b. Certifier:

- All nodes appear on the list
- Nodes only appear once
- Every pair of adjacent nodes in the order must have an edge between them.
- There is an edge between last & first node.

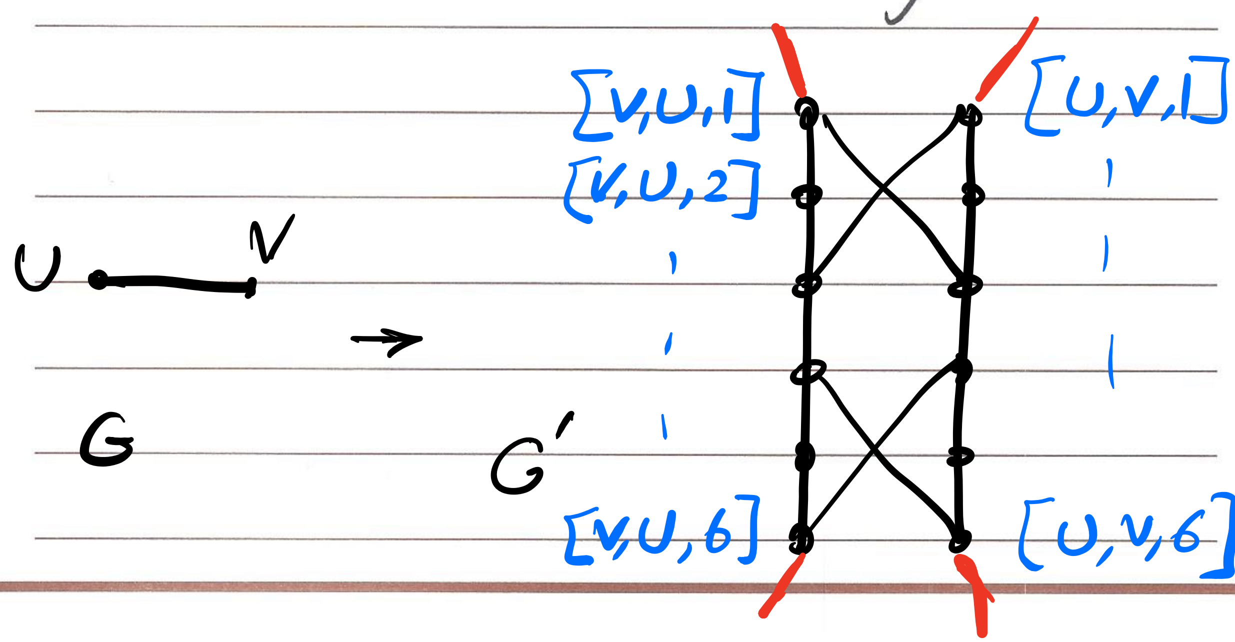
- 2 choose vertex Cover

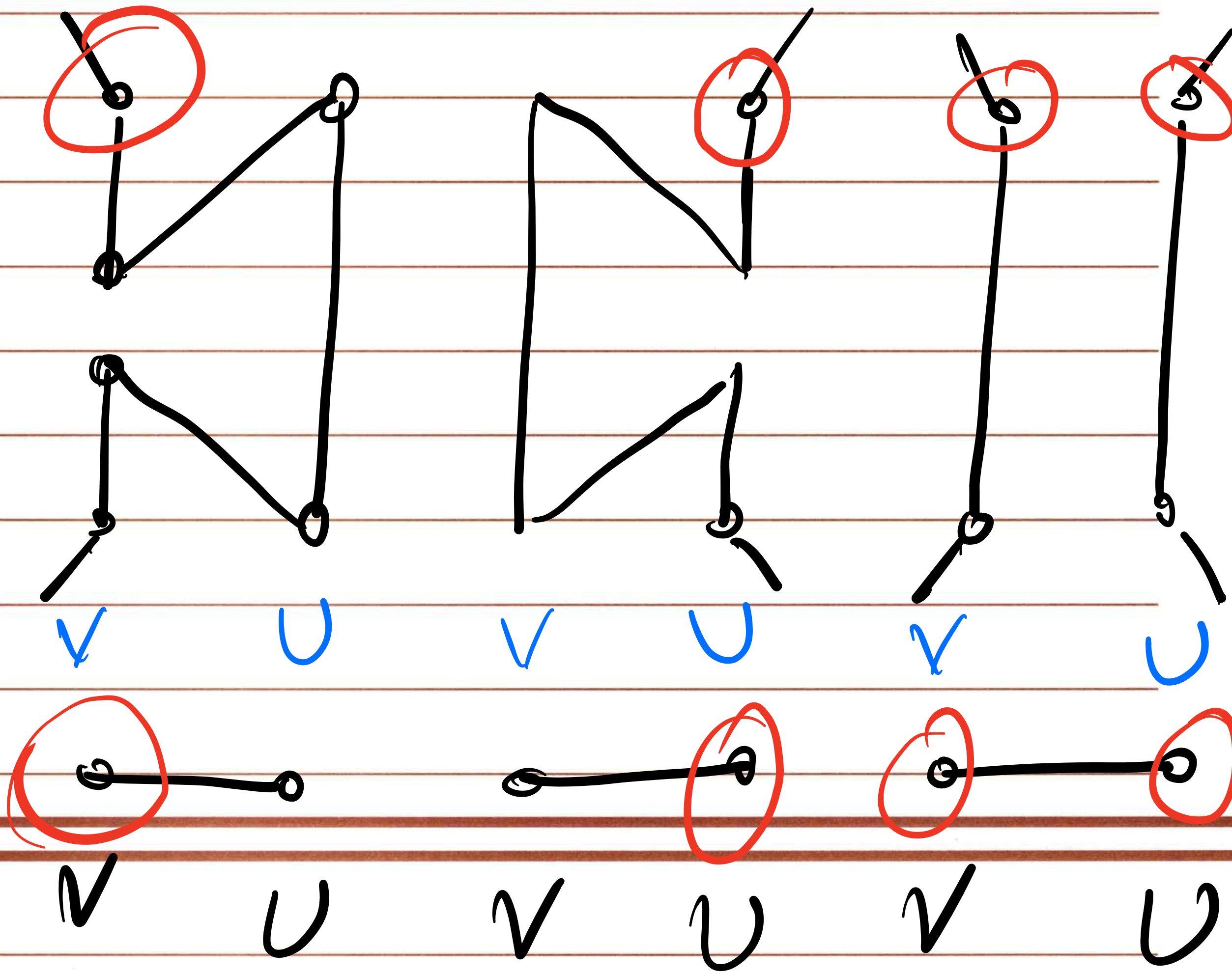
- 3 Show vertex Cover $\leq_p$ HC

Plan: Given an undirected graph $G = (V, E)$ and an integer $\underline{k}$, we construct $G' = (V', E')$ that has a Hamiltonian Cycle iff G has a vertex cover of size at most $\underline{k}$.

<u>Construction of G'</u>

For each edge (V, U) in G, G' will have one gadget W_{VU} with following node labeling:



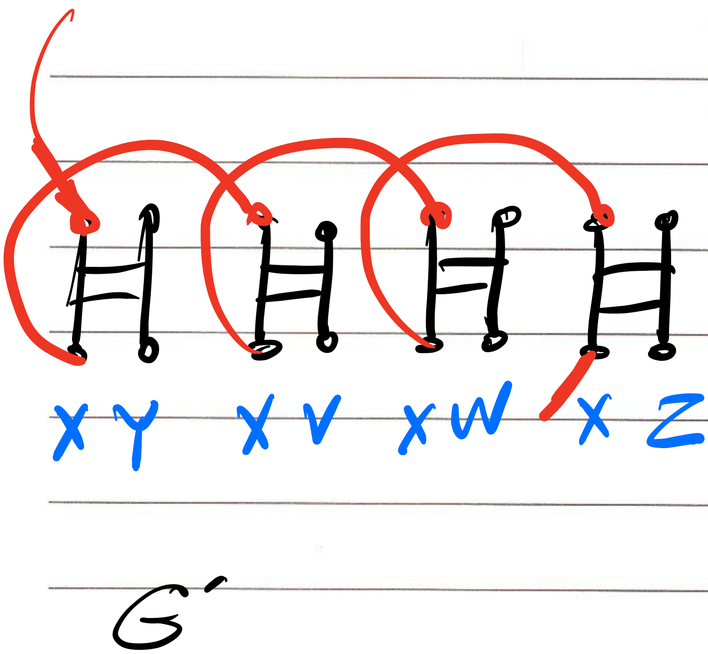

V U V U V U
V U V U V U

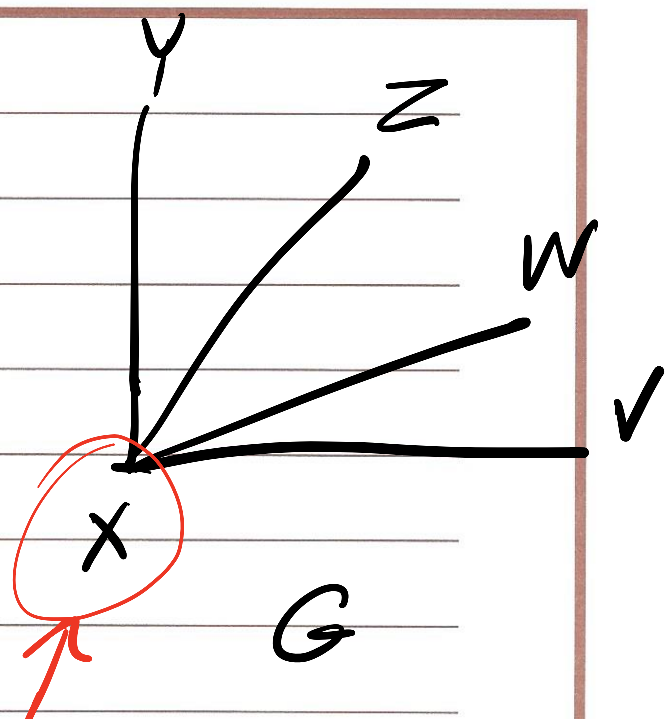
Other vertices in G'

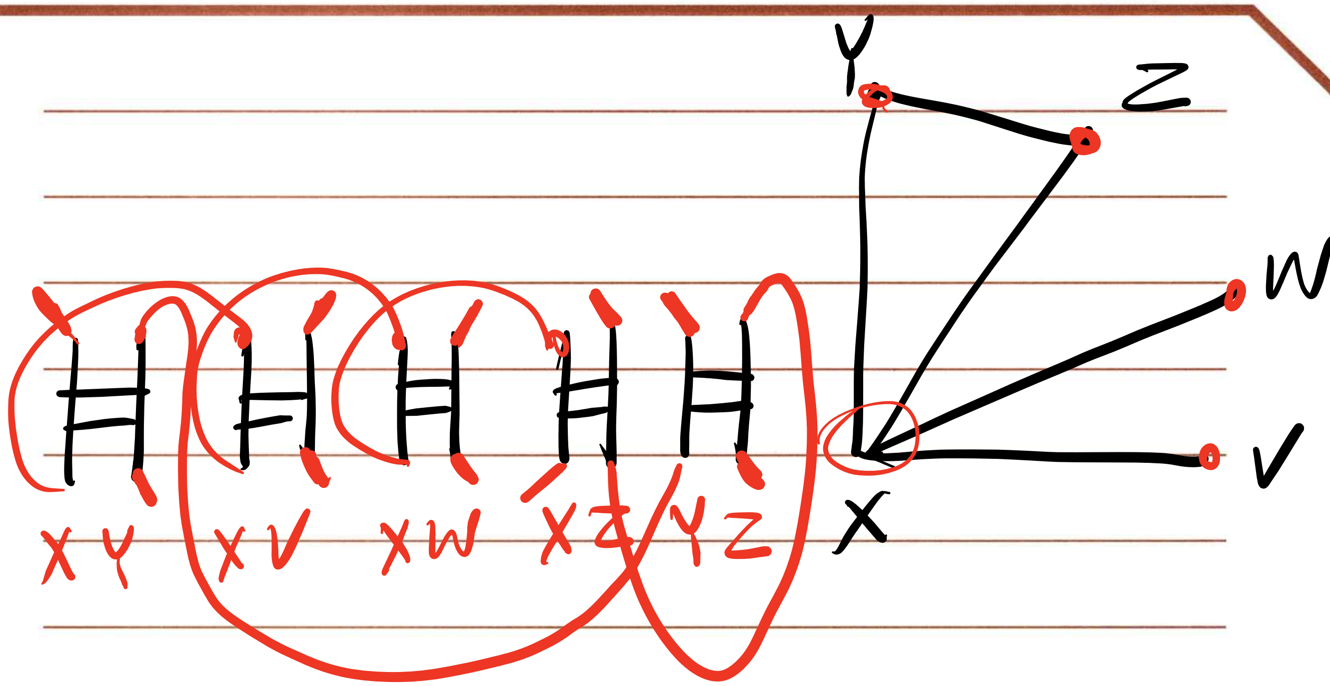
- Selector vertices: There are k selector vertices in G', $s_1, \ldots, s_k$

Other edges in G'

1. For each vertex $v \in V$ we add edges to join pairs of gadgets in order to form a path going through all the gadgets corresponding to edges incident on v in G.

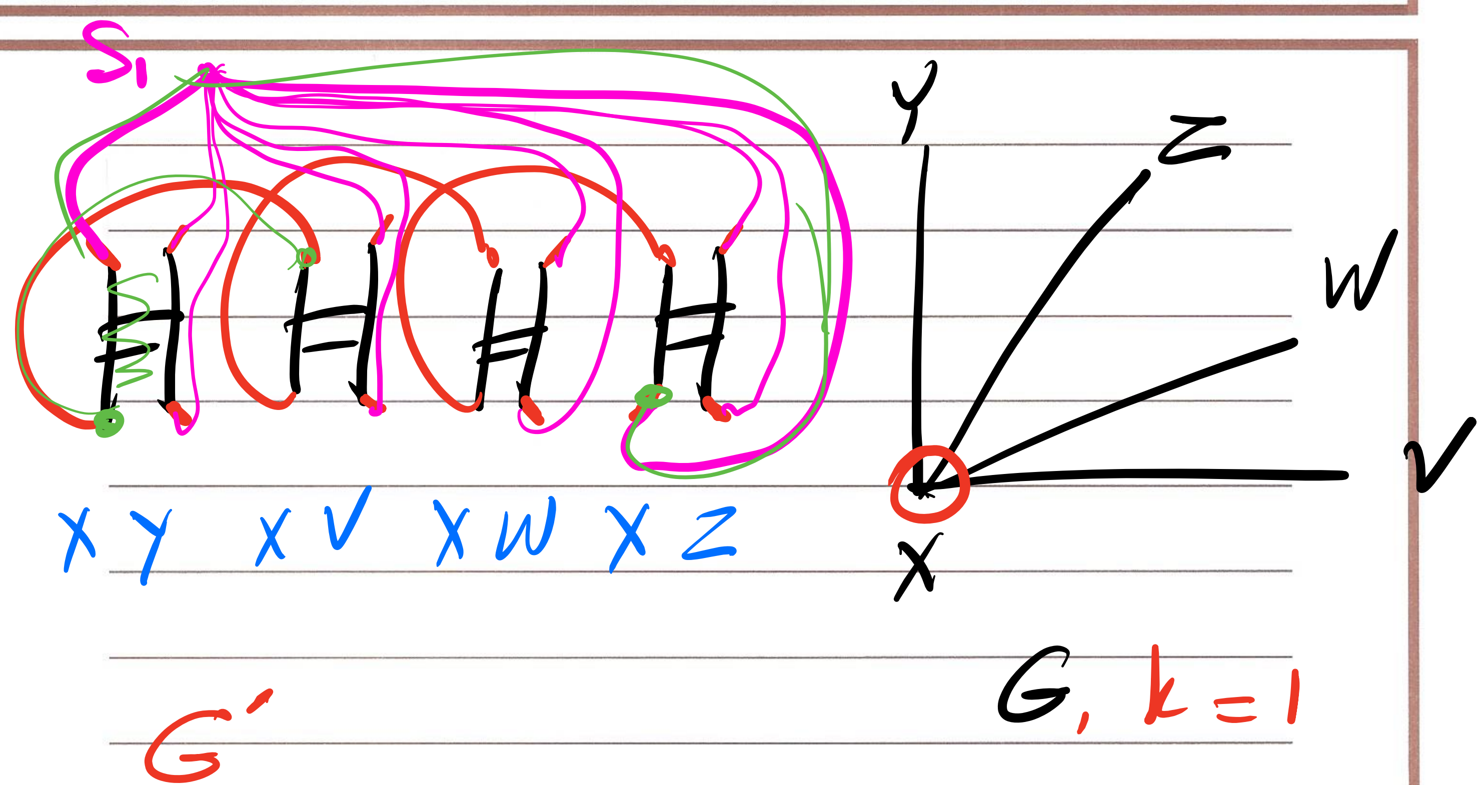


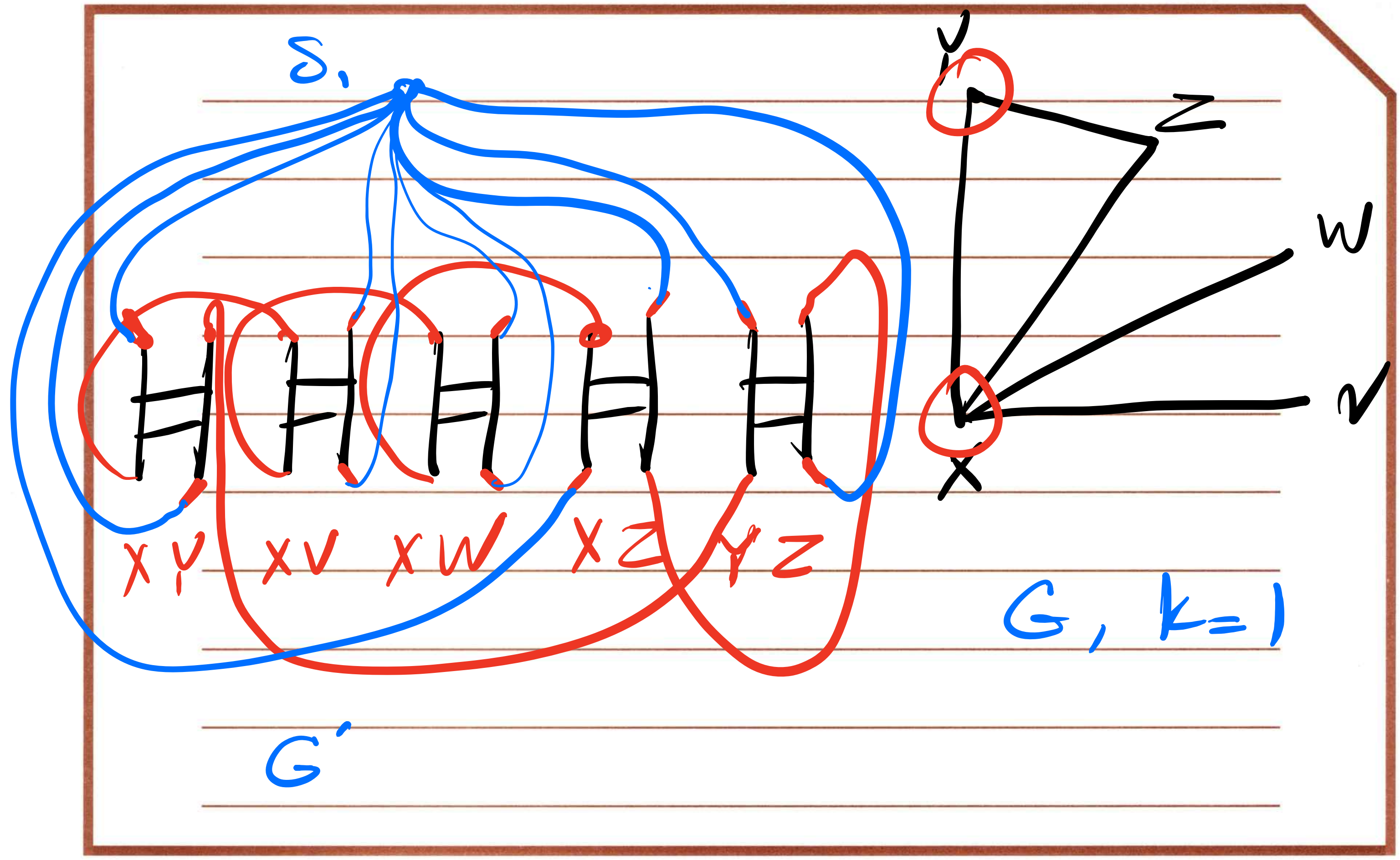


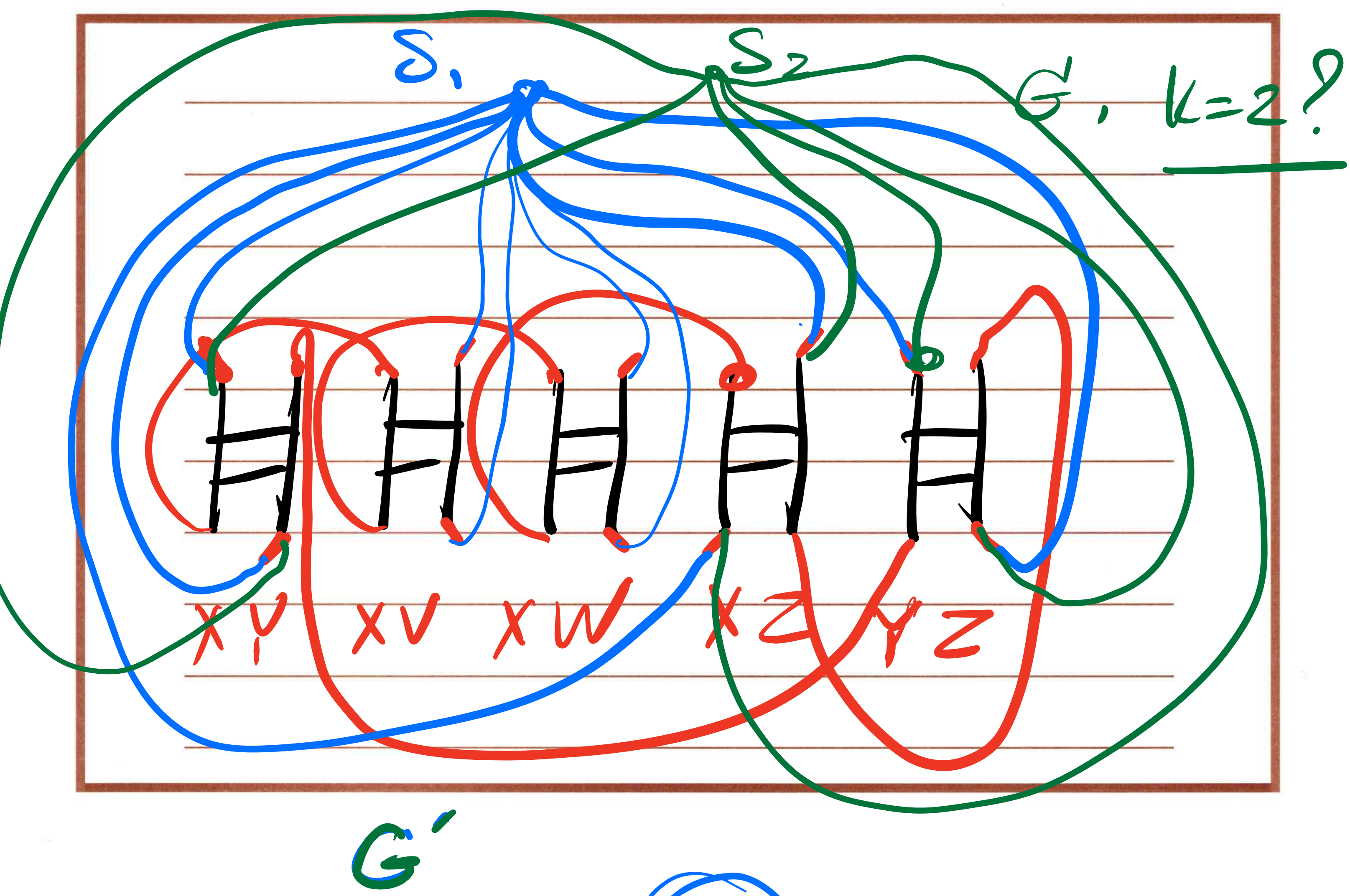


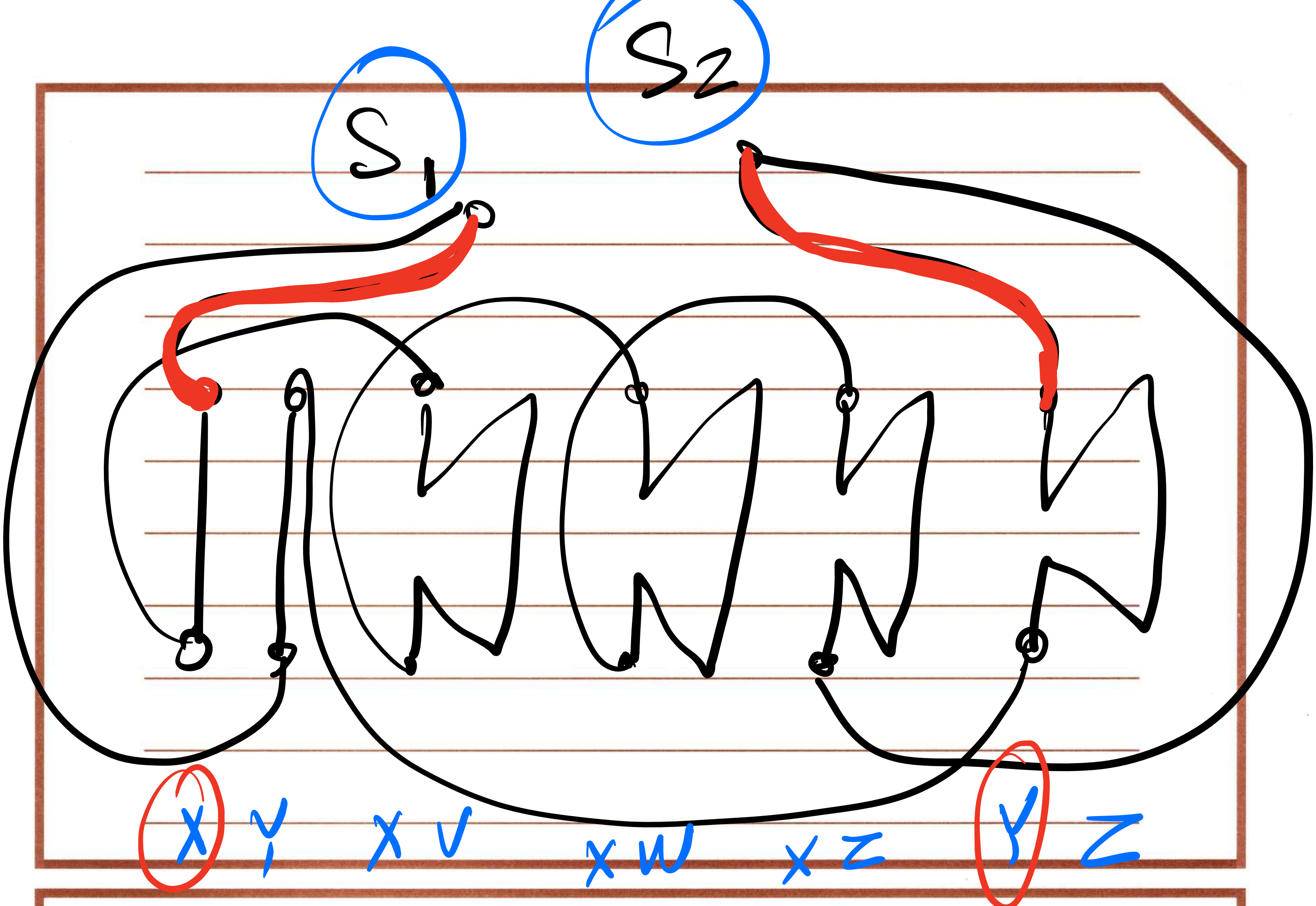
G'

2- Final set of edges in G' join the first vertex [X, Y, 1] and last vertex [X, Y(deg(X)), 6] of each of these paths to each of the selector vertices.




S1
y
z
w
v
x
XY XV XW XZ YZ
G, k=1
G'


S1
S2
G, k=2?
XY XV XW XZ YZ
G'

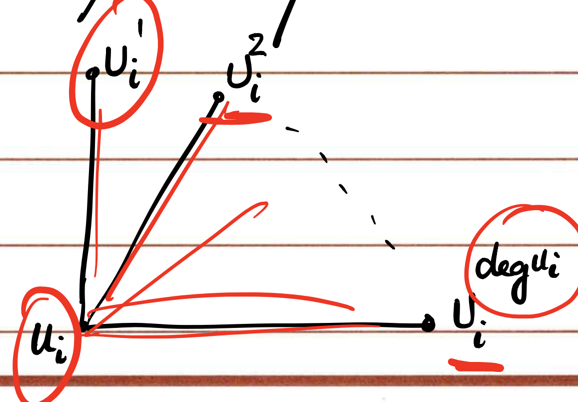


S1
S2
X Y X V X W X Z Y Z

Proof: A) Suppose that $G = (V, E)$ has a vertex cover of size $\underline{k}$. Let the vertex cover set be

$$\underline{S} = \{ \underline{U_1}, \underline{U_2}, \ldots \underline{U_k} \}$$

We will identify neighbors of U_i as shown here:



Form a Ham. Cycle in G' by following the nodes in G in this order:

start at s_1 and go to

$[\underline{U_1}, \underline{U_1^1}, 1]$ ...... $[U_1, U_1^1, 6]$

$[\underline{U_1}, U_1^2, 1]$ ...... $[U_1, U_1^2, \underline{6}]$

$\vdots$

$[U_1, U_1^{\deg u_1}, 1]$ ...... $[\underline{U_1}, U_1^{\deg u_1}, \underline{6}]$

Then go to S_2 and follow the nodes

$[U_2, U_2^1, 1]$ .... $[U_2, U_2^1, 6]$

$[U_2, U_2^2, 1]$ .... $[U_2, U_2^2, 6]$

$\vdots$

$[U_2, U_2^{\deg U_2}, 1]$ .... $[U_2, U_2^{\deg U_2}, 6]$

Then go to S_3 ....

$\vdots$

$\vdots$

$[U_k, U_k^1, 1]$ .... $[U_k, U_k^1, 6]$

$[U_k, U_k^2, 1]$ .... $[U_k, U_k^2, 6]$

$\vdots$

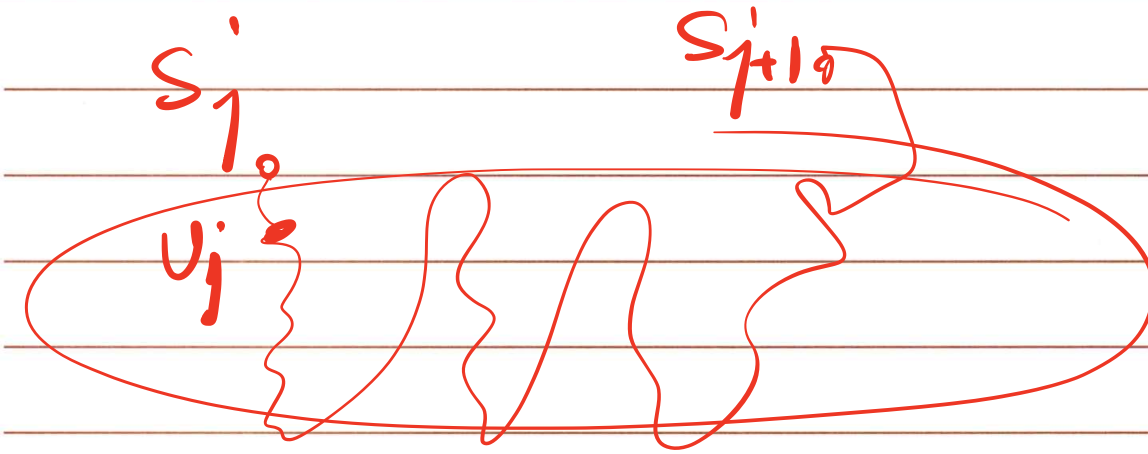
$[U_k, U_k^{\deg U_k}, 1]$ .... $[U_k, U_k^{\deg U_k}, 6]$

Then return back to S_1.

B) Suppose G' has a Hamiltonian cycle C, then the set

$$S = \{ u_j \in V : (s_j, [u_j, u_j', 1]) \in C \text{ for some } 1 \leq j \leq k \}$$

will be a vertex cover set in G.



We Prove that TSP is NP-Complete

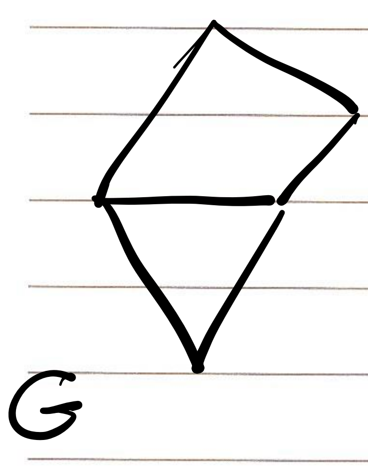
1- Show that TSP $\in$ NP

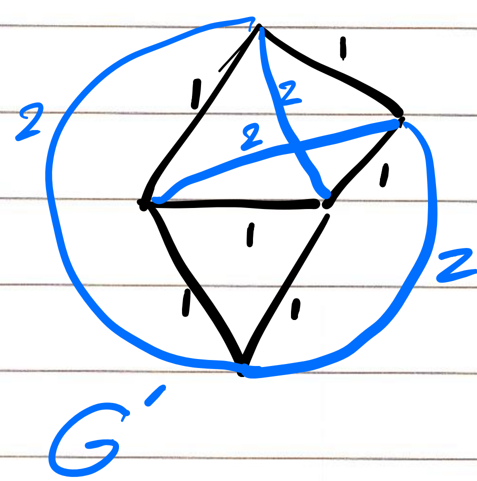
a. Certificate:
a tour of cost at most D.

b. Certifier:
- all checks we did for HC
- + check that cost of tour $\leq D$

2- Choose an NP-Complete problem:
Hamiltonian Cycle.

3- Prove that Ham. Cycle $\leq_p$ TSP

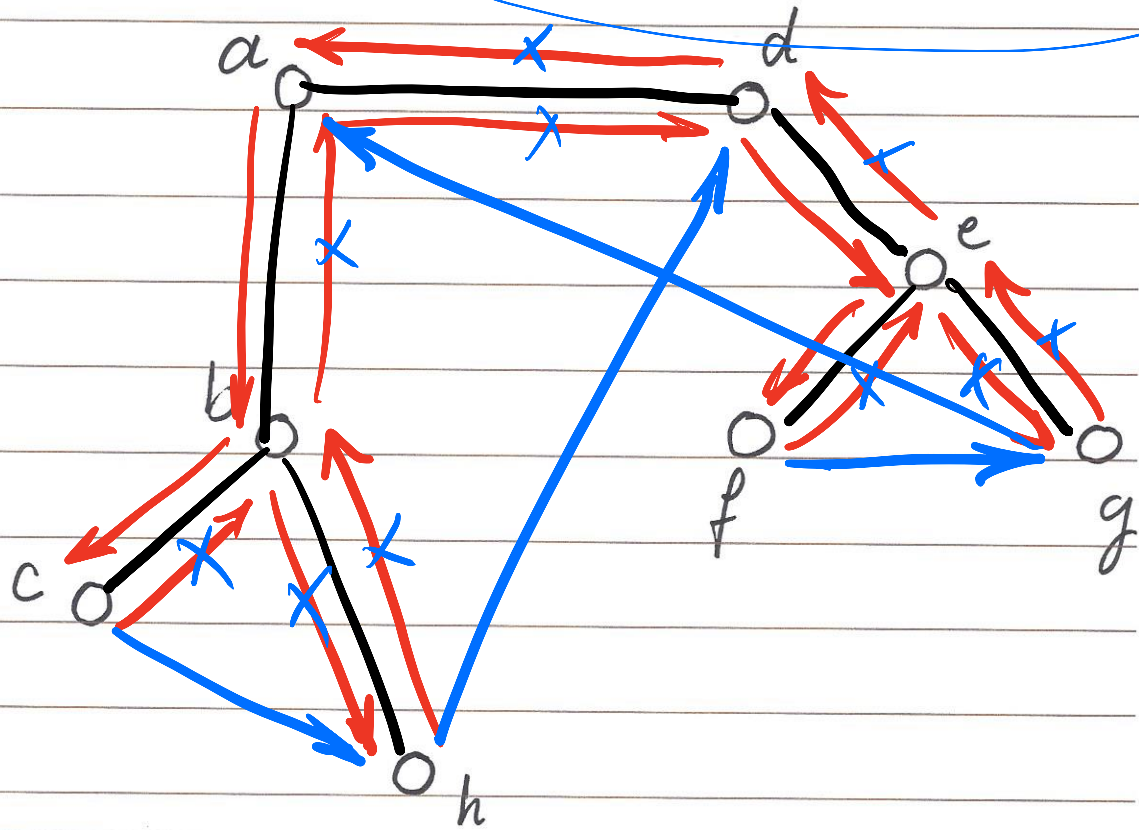




Problems to choose from:

- 3SAT
- indep set
- vertex Cover
- Set Cover
- Set packing
- Ham. Cycle + Ham. Path
- TSP
- 0-1 knapsack
- subset sum

TSP w/ Triangle inequalities



Cost of our initial tour = 2 * Cost of MST

Cost of our approx tour $\leq$ 2 * Cost of MST

Cost of opt. tour $\geq$ cost of MST

Cost of our approx $\leq$ 2 * Cost of opt. tour

This ~~is~~ is a 2-approximation

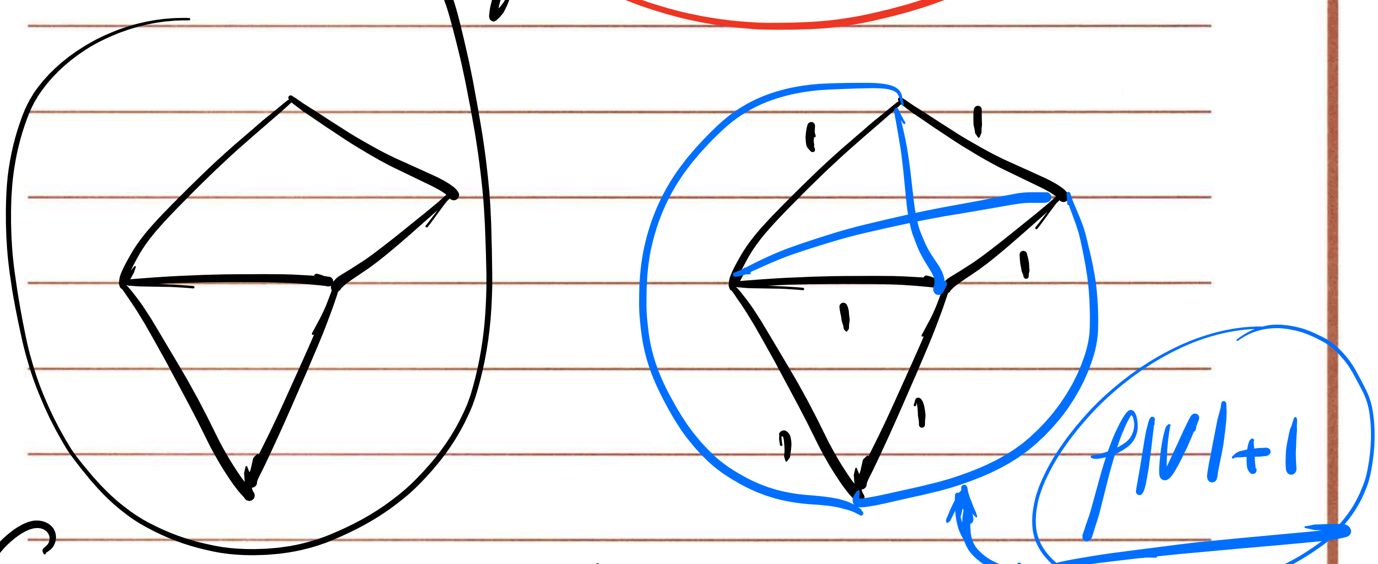
General TSP

Theorem: if $P \neq NP$, then for any constant $\rho \geq 1$, there is no polynomial time approximation algorithm with approximation ratio ρ for the general TSP

Plan: We will assume that such an approximation algorithm exists. We will then use it to solve the HC problem.

Given an instance of the HC problem on graph G, we will construct G' as follows:

- G' has the same set nodes as in G
- G' is a fully connected graph.
- Edges in G' that are also in G have a cost of 1.
- Other edges in G' have a cost of $\rho|V|+1$



G

G'

if we have a HC in G

$\Longrightarrow$ a tour of cost $|V|$ in G'

$|V| < \rho|V|$

if we have a tour of cost $\leq \rho|V|$ in G'

$\Longrightarrow$ There is a HC in G

Discussion 11

1. In the *Min-Cost Fast Path* problem, we are given a directed graph $G=(V,E)$ along with positive integer times t_e and positive costs c_e on each edge. The goal is to determine if there is a path P from s to t such that the total time on the path is at most T and the total cost is at most C (both T and C are parameters to the problem). Prove that this problem is **NP**-complete.

2. We saw in lecture that finding a Hamiltonian Cycle in a graph is **NP**-complete. Show that finding a Hamiltonian Path -- a path that visits each vertex exactly once, and isn't required to return to its starting point -- is also **NP**-complete.

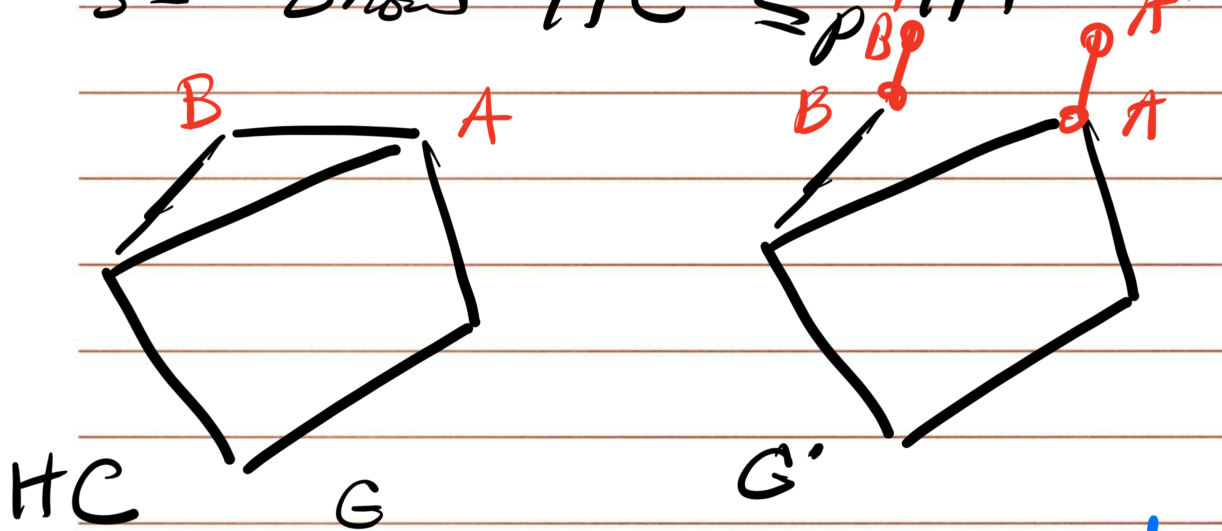
3. Some **NP**-complete problems are polynomial-time solvable on special types of graphs, such as bipartite graphs. Others are still **NP**-complete.
Show that the problem of finding a Hamiltonian Cycle in a bipartite graph is still **NP**-complete.

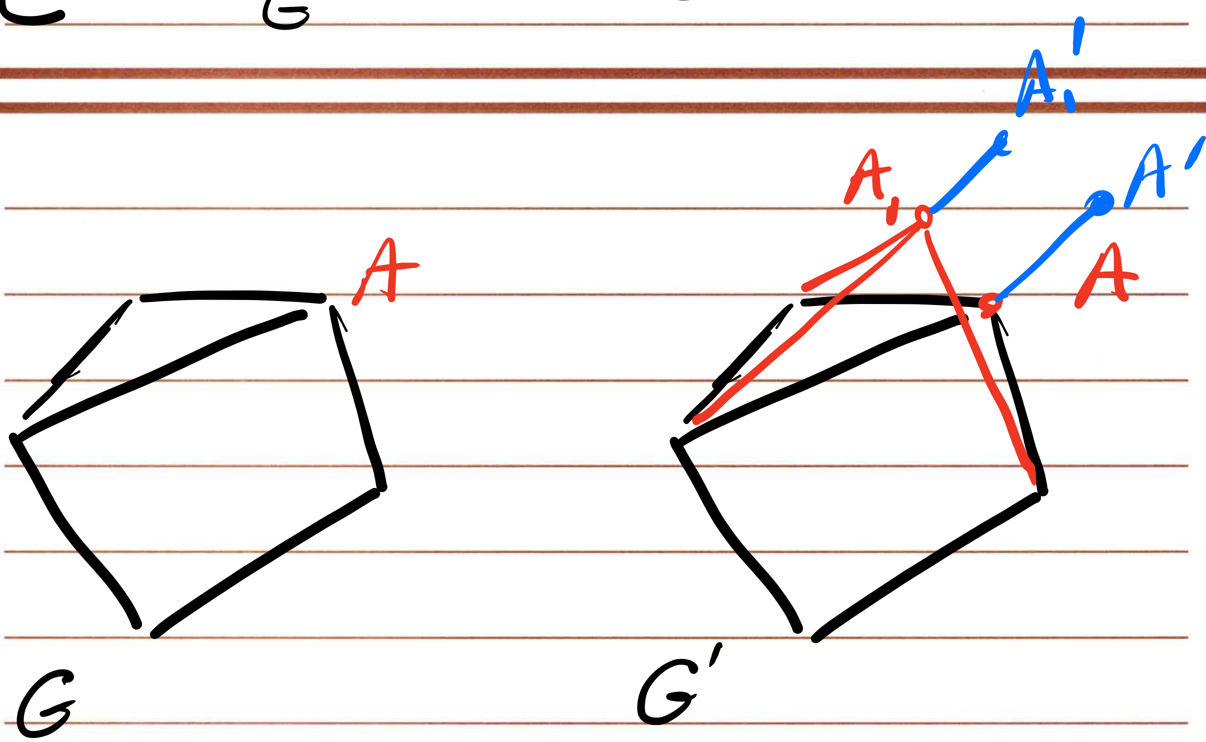
2. We saw in lecture that finding a Hamiltonian Cycle in a graph is **NP**-complete. Show that finding a Hamiltonian Path -- a path that visits each vertex exactly once, and isn't required to return to its starting point -- is also **NP**-complete.

1- Skip

2- Choose HC

3- Show HC $\leq_p$ HP





1. In the *Min-Cost Fast Path* problem, we are given a directed graph $G=(V,E)$ along with positive integer times t_e and positive costs c_e on each edge. The goal is to determine if there is a path P from s to t such that the total time on the path is at most T and the total cost is at most C (both T and C are parameters to the problem). Prove that this problem is **NP**-complete.

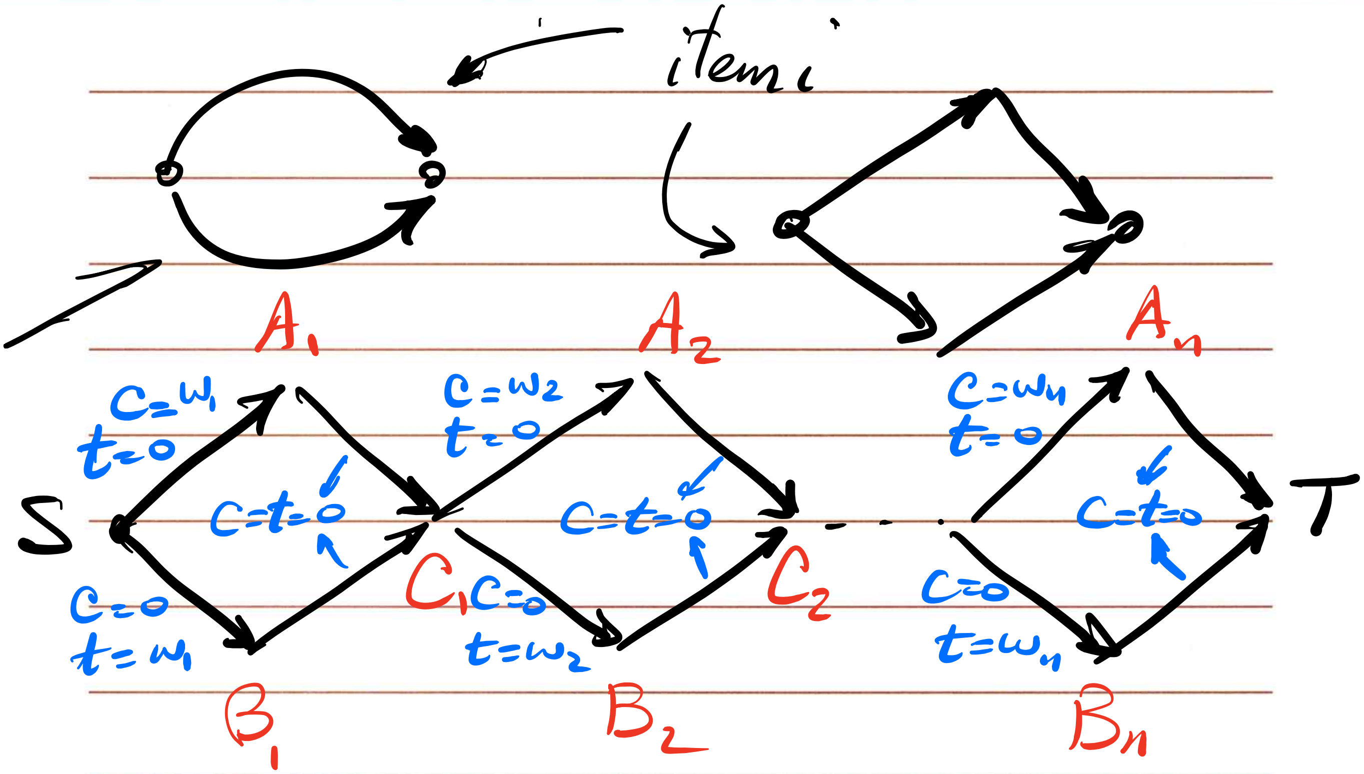
1- Skip

2- Choose Subset Sum

3- Show Subset Sum $\leq_p$ MCFP

<u>Decision version of Subset sum:</u>

Given a set of n items with weights $w_1 \ldots w_n$, is there a subset of them w/ total weight X where $M \leq X \leq W$



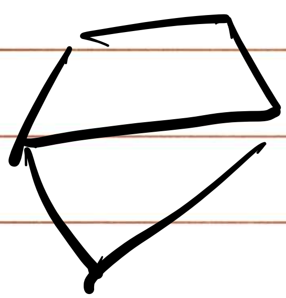
Is there an s-t path with total cost $\leq W$

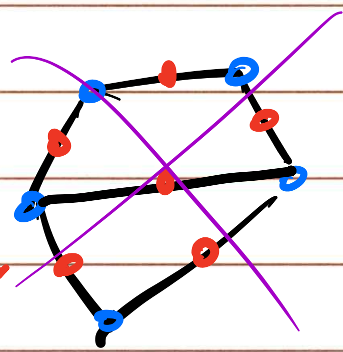
& " time $\leq \sum w_i - M$

3. Some **NP**-complete problems are polynomial-time solvable on special types of graphs, such as bipartite graphs. Others are still **NP**-complete.
Show that the problem of finding a Hamiltonian Cycle in a bipartite graph is still **NP**-complete.

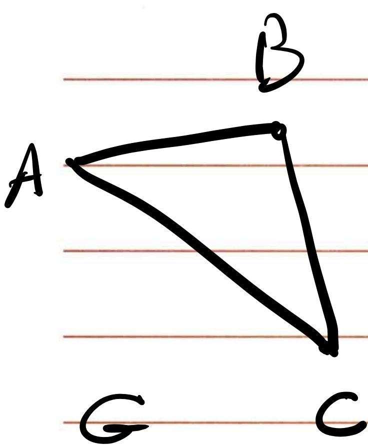
1- Skip
2- Choose Ham. Cycle.
3- Show Ham. Cycle $\leq_P$ Ham. Cycle in bipartite graphs.

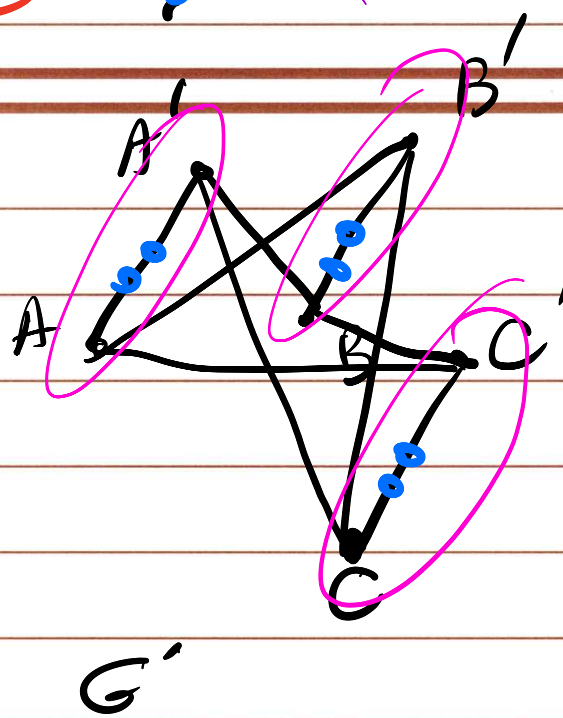
G





G′





ABCA

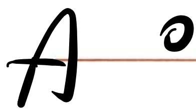
AA′BB′CC′A

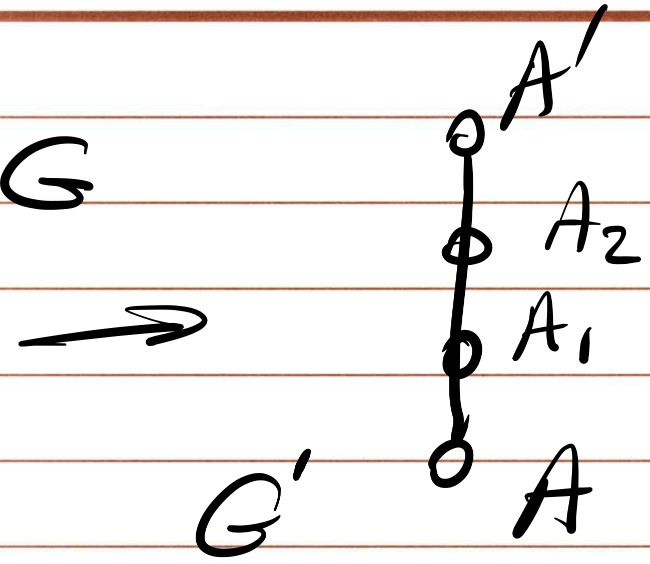
BCAB ← BB′CC′AA′B

AB′CA′B . . . A

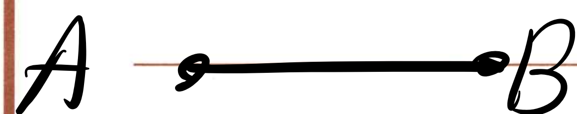
?

node in G

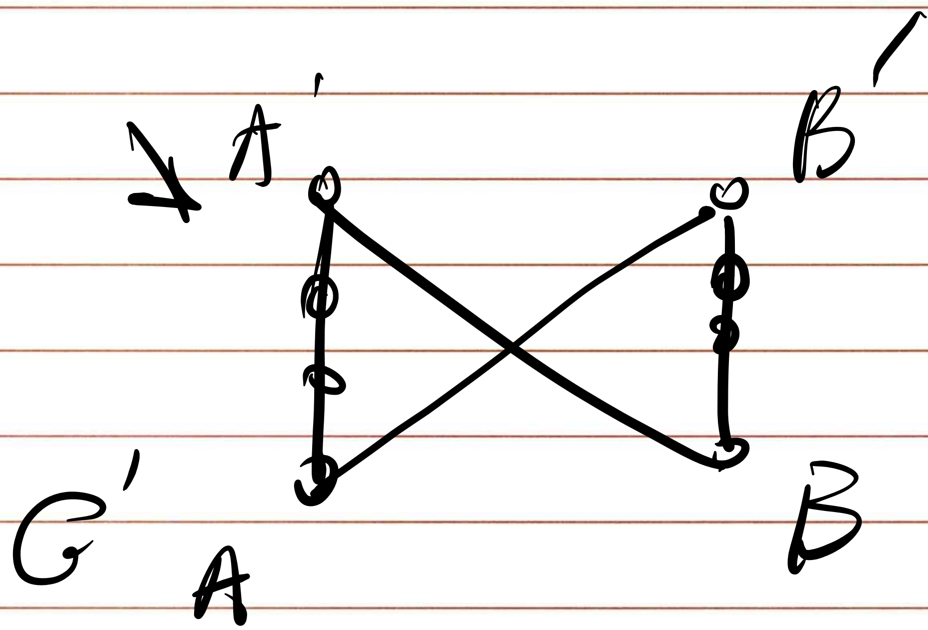




edge between two nodes



G



CSCI 570 - Fall 2022 - HW 10

Due: November 16, 2022

Problem 1 (25pts)

Consider the partial satisfiability problem, denoted as 3-Sat(α). We are given a collection of k clauses, each of which contains exactly three literals, and we are asked to determine whether there is an assignment of true/false values to the literals such that at least αk clauses will be true. Note that 3-Sat(1) is exactly the 3-SAT problem from lecture.

Prove that 3-Sat(15/16) is **NP**-complete.

Hint: If x, y, and z are literals, there are eight possible clauses containing them: $(x \vee y \vee z), (!x \vee y \vee z), (x \vee !y \vee z), (x \vee y \vee !z), (!x \vee !y \vee z), (!x \vee y \vee !z), (x \vee !y \vee !z), (!x \vee !y \vee !z)$

8 clauses

Add these 8 clauses → is there a truth assignment that satisfies at least 15/16 of the clauses?

$\underbrace{8}\underbrace{8}$

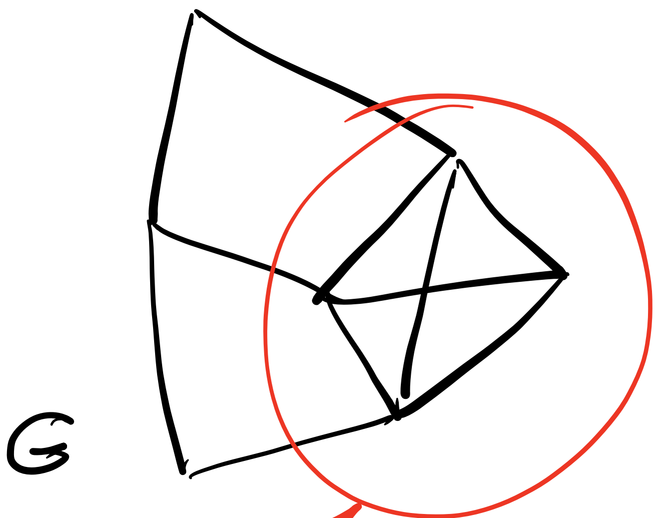
add 2 sets of 8 clauses →

if we have m*8 clauses
we will add m sets of
those dummy 8 clauses.

send to
blackbox.

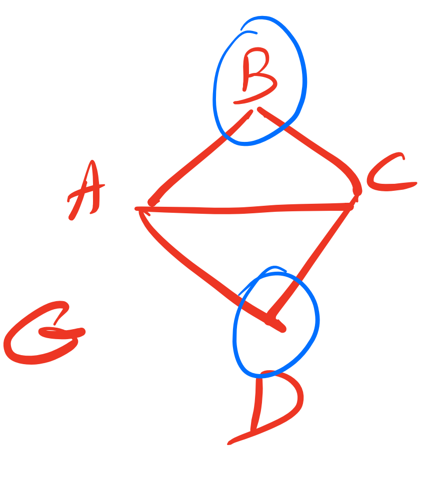
Problem 2 (25 pts)

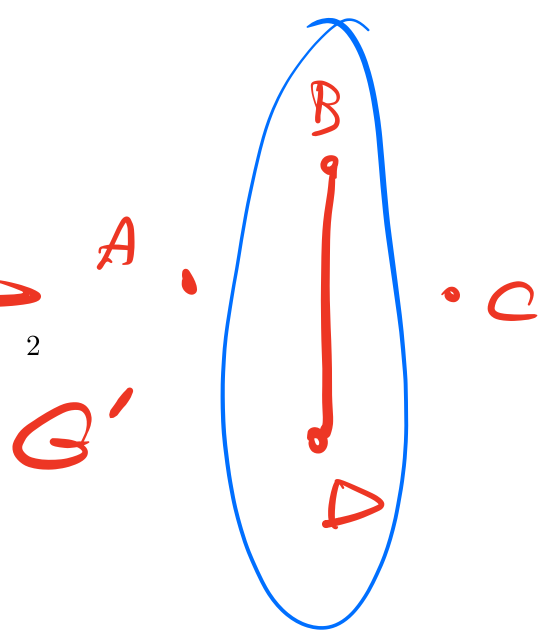
Given a graph $G = (V, E)$ and two integers k, m, the Dense Subgraph Problem is to find a subset V' of V, whose size is at most k and are connected by at least m edges. Prove that the Dense Subgraph Problem is NP-Complete.

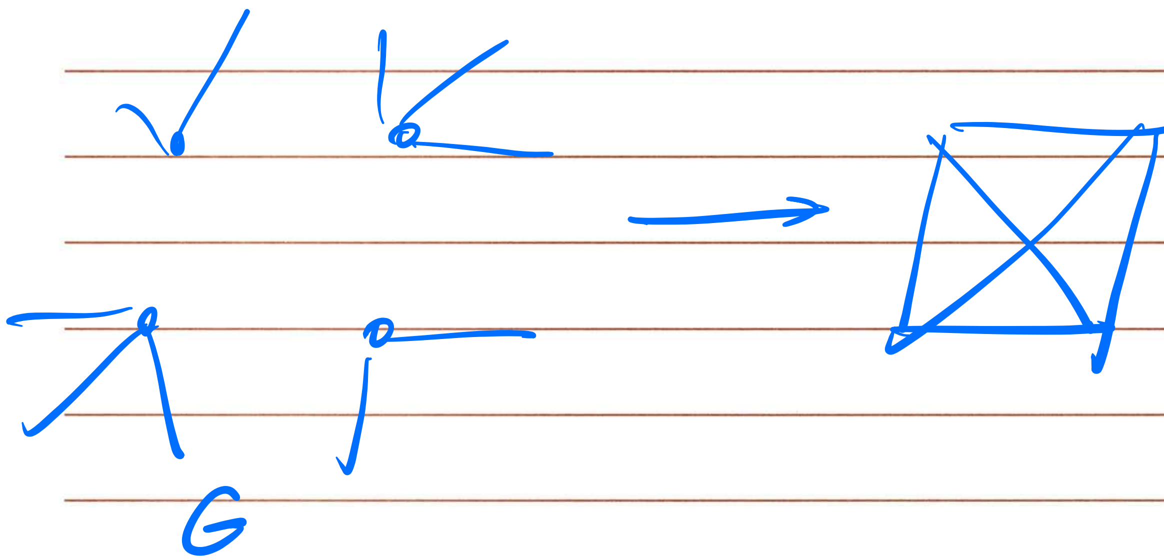


a clique of size 4

a clique of size k will have $\frac{k(k-1)}{2}$







→ is there a subgraph w/ k nodes w/ at least $\frac{k(k-1)}{2}$ edges?

Problem 4 (25 pts)

Show that Vertex Cover is still NP-complete even when all vertices in the graph are restricted to have even degree.