Properties of Wishart Distribution

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1. Definition

Model: $W \sim W_p(r, \Sigma) \iff W \overset{d}{=} \sum_{j=1}^{n} Z_j {Z_J}^\top$

- Z_j: multivariate standard normal distribution
  $Z_j \overset{iid}{\sim} N_p(0, \Sigma), (j = 1, \ldots, r).$

2. Basic Properties of Wishart Distribution

- Assuming $Z_j \overset{iid}{\sim} N_p(0, \Sigma), (j = 1, \ldots, r), X_i \overset{iid}{\sim} N_p(0, \Sigma), (i = 1, \ldots, n), X : n \times p$data matrix.
  - (quadratic form)
    $$X^\top A X = \sum_{j=1}^{r} Z_j {Z_J}^\top \sim W_p(r, \Sigma) \iff A^2 = A,$$
    where r=rank(A)=tr(A)
  - (closure under convolution)
    $$W_1 \sim W_p(r_1, \Sigma) \overset{d}{=} \sum_{j=1}^{r_1} Z_j {Z_J}^\top, W_2 \sim W_p(r_2, \Sigma) \overset{d}{=} \sum_{j=1}^{r_2} Z_j {Z_J}^\top$$
    $W_1 \perp\!\!\!\perp W_2, W_1 + W_2 \sim W_p(r_1 + r_2, \Sigma)$
  - (scaled form)
    $W \sim W_p(r, \Sigma) \Rightarrow CWC^\top \sim W_q(r, C\Sigma C^\top) \forall C : q \times p$

2. Basic Properties of Wishart Distribution : proof(1)

- (quadratic form)

$$(\Leftarrow)\mathsf{A}^2 = A, X^\top AX = \sum_{j=1}^{r} X^\top P_j {P_j}^\top X = \sum_{j=1}^{r} Z_j {Z_J}^\top \sim W_p(r, \Sigma)$$

$$(\Rightarrow)\mathsf{X}^\top AX \sim W_p(r, \Sigma) \overset{d}{=} \sum_{j=1}^{r} Z_j {Z_J}^\top \to \forall d \in R^p, d^\top X^\top AXd =$$

$$d^\top(\sum_{j=1}^{r} Z_j {Z_J}^\top)d = \sum_{j=1}^{r} d^\top Z_j {Z_j}^\top d \overset{d}{=} (d^\top \Sigma d){\chi^2}_r$$

- (closure under convolution)

$$W_1 + W_2 = \sum_{j=1}^{r_1} Z_{1j} {Z_{1j}}^\top + \sum_{k=1}^{r_2} Z_{2k} {Z_{2k}}^\top \overset{d}{=}$$

$$\sum_{j=1}^{r_1} Z_{1j} {Z_{1j}}^\top + \sum_{j=r_1+1}^{r_1+r_2} Z_{1j} {Z_{1j}}^\top = \sum_{j=1}^{r_1+r_2} Z_{1j} {Z_{1j}}^\top \sim W_p(r_1 + r_2, \Sigma)$$

- (scaled form)
$CWC^\top = C[\sum_{j=1}^{r} Z_j {Z_J}^\top]C^\top = \sum_{j=1}^{r} CZ_jC{Z_j}^\top = \sum_{j=1}^{r} Y_j{Y_j}^\top \sim W_q(r, C\Sigma C^\top), \because Y_j = CZ_j \sim N_q(0, C\Sigma C^\top)$

3. Additional properties of Wishart Distribution(1)

- Assume $S \sim W_p(r, \Sigma)(r > p)$
- $$S = \begin{pmatrix} S_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$$
- $$\Sigma = \begin{pmatrix} \Sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$
- $$S^{-1} = \begin{pmatrix} S^{11} & s^{12} \\ s^{21} & s^{22} \end{pmatrix}$$
- $$\Sigma^{-1} = \begin{pmatrix} \Sigma^{11} & \sigma^{12} \\ \sigma^{21} & \sigma^{22} \end{pmatrix}$$

3. Additional properties of Wishart Distribution (2)

- Then following can be shown
- (a) $\sigma^{22}/s^{22} \sim {\chi^2}_({r - (p-1)}),$
  where $S \overset{d}{=} \sum_{i=1}^{r} Z_i {Z_i}^\top, Z_i = [{Z_{i1}}^\top, {z_{i2}}^\top]^\top, Z_i \overset{iid}{\sim} N_p(0, \Sigma)$
- (b) $\forall d, (d^\top \Sigma^{-1} d)/(d^\top S^{-1} d) \sim {\chi^2}_({r - (p-1)})$
- (c) $S_{22\cdot 1} = S_{22} - S_{21} {S_{11}}^{-1} S_{12} \sim W_{p_2}(r - p_1, \Sigma_{22\cdot 1})$

3. Additional properties of Wishart Distribution (proof of (a)-1)

- $s^{22} = (s_{22} - s_{21}{S_{11}}^{-1}s_{12})^{-1},\ \sigma^{22} = (\sigma_{22} - \sigma_{21}{\Sigma_{11}}^{-1}\sigma_{12})^{-1}$
- $Y = \begin{bmatrix} Z_{12} \\ \vdots \\ Z_{r2} \end{bmatrix}, X = \begin{bmatrix} {Z_{11}}^{\top} \\ \vdots \\ {Z_{r1}}^{\top} \end{bmatrix}$
- put $\epsilon = Y - X\beta,$ where $\beta = {\Sigma_{11}}^{-1}\sigma_{12}$
- $\epsilon = Y - X\beta = Y - X{\Sigma_{11}}^{-1}\sigma_{12} \sim N(0, \sigma_{22 \cdot 1} I_r),\ \epsilon \perp\!\!\!\perp X$
- keypoint : $Z_{i1} \perp\!\!\!\perp Z_{i2} - \sigma_{21}{\Sigma_{11}}^{-1}Z_{i1} \sim N(0, \sigma_{22 \cdot 1})$

3. Additional properties of Wishart Distribution (proof of (a)-2)

- $$\begin{aligned} S &= \sum_{i=1}^{r} Z_i {Z_i}^\top && (1) \\ &= \sum_{i=1}^{r} \begin{bmatrix} Z_{i1} \\ z_{i2} \end{bmatrix} [{Z_{i1}}^\top, {z_{i2}}^\top] && (2) \\ &= \begin{bmatrix} \sum_{i=1}^{r} Z_{i1} {Z_{i1}}^\top & \sum_{i=1}^{r} Z_{i1} {z_{i2}}^\top \\ \sum_{i=1}^{r} z_{i2} {Z_{i1}}^\top & \sum_{i=1}^{r} z_{i2} {z_{i2}}^\top \end{bmatrix} && (3) \end{aligned}$$
- $(s^{22})^{-1} = 1/s^{22} = (s_{22} - s_{21} {S_{11}}^{-1} s_{12})$

3. Additional properties of Wishart Distribution (proof of (a)-3)

- $$\begin{aligned}
(s^{22})^{-1} &= (s_{22} - s_{21}{S_{11}}^{-1}s_{12}) && (4)\\
&= \left[\sum_{i=1}^{r} z_{i2}{z_{i2}}^\top - \sum_{i=1}^{r} z_{i2}{Z_{i1}}^\top (\sum_{i=1}^{r} Z_{i1}{Z_{i1}}^\top)^{-1} \sum_{i=1}^{r} Z_{i1}{z_{i2}}^\top\right] && (5)\\
&= [Y^\top Y - Y^\top X(X^\top X)^{-1}X^\top Y] && (6)\\
&= Y^\top [I_r - X(X^\top X)^{-1}X^\top]Y && (7)\\
:=&\ Y^\top A Y && (8)
\end{aligned}$$

3. Additional properties of Wishart Distribution (proof of (a)-4)

- $$\begin{aligned}(s^{22})^{-1} &= Y^\top[I_r - X(X^\top X)^{-1}X^\top]Y \\ &= (Y - X\beta)^\top[I_r - X(X^\top X)^{-1}X^\top](Y - X\beta) \\ &= (Y - X\beta)^\top A(Y - X\beta) \sim \sigma_{22\cdot 1}{\chi^2}_{r-(p-1)}\end{aligned}$$
- $\because (Y - X\beta) \sim N_r(0, (\sigma_{22\cdot 1})I_r) \to (\sigma_{22\cdot 1})^{-1/2}(Y - X\beta) \sim N_r(0, I_r)$
- $\because rank(A) = tr(A) = tr(I_r - X(X^\top X)^{-1}X^\top) = r - tr[I_{p-1}] = r - (p-1)$
- $\therefore \frac{1}{s^{22}} \cdot (\sigma_{22\cdot 1})^{-1} = \frac{\sigma^{22}}{s^{22}} \sim {\chi^2}_{r-(p-1)}$

3. Additional properties of Wishart Distribution (sketch of (c))

- $S_{22\cdot 1} = S_{22} - S_{21}{S_{11}}^{-1}S_{12} \sim W_{p_2}(r - p_1, \Sigma_{22\cdot 1})$
- It is multivariate version of (a), where Z_{i1} is $p_1$dimension, Z_{i2} is $p_2$dimension.

4. Additional topics of Wishart Distribution (expectation of inverse wishart (1))

Theorem 1

$M \sim W_p(r, \Sigma)\ (r > p),$ *then* $E(M^{-1}) = \frac{\Sigma^{-1}}{r-p-1}$

Proof Let $\Sigma = CC^\top$, where C is nonsingular. Then $M = CBC^\top \sim W_p(r, \Sigma)$, where $B \sim W_p(r, I) \rightarrow$ E(M^{-1}) $= (C^\top)^{-1}E(B^{-1})C^{-1}$, by symmetry, the diagonal elements of $E(B^{-1})$are same and off-diagonal elements are same $\rightarrow E(B^{-1}) = k_1 I_p + k_2 \epsilon\epsilon^\top$.

4. Additional topics of Wishart Distribution (expectation of inverse wishart (2))

Proof (continued) For every orthogonal matrix Q, $QBQ^\top \sim W_p(r, I)$
$\rightarrow$ $\mathrm{E}[(\mathrm{QBQ}^\top)^{-1}] = (Q^\top)^{-1}E[B^{-1}]Q^{-1} = E(B^{-1})$. $\therefore k_2 = 0$

Let $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & B_{22} \end{bmatrix}, B^{-1} = \begin{bmatrix} b^{11} & \dots & b^{1p} \\ b^{21} & \dots & b^{2p} \\ \vdots & & \vdots \\ b^{p1} & \dots & b^{pp} \end{bmatrix}$

Let $Z_i = \begin{bmatrix} z_{i1} \\ Z_{i2} \end{bmatrix},\ Z_i \sim N_p(0, I)$

Put $Z_1 = \begin{bmatrix} z_{11} \\ \vdots \\ z_{r1} \end{bmatrix}, Z_2 = \begin{bmatrix} {Z_{12}}^\top \\ \vdots \\ {Z_{r2}}^\top \end{bmatrix}$

4. Additional topics of Wishart Distribution (expectation of inverse wishart (3))

Proof (continued)

$$B = \sum_{i=1}^{r} Z_i {Z_i}^\top \tag{9}$$

$$= \sum_{i=1}^{r} \begin{bmatrix} z_{i1} \\ Z_{i2} \end{bmatrix} [{z_{i1}}^\top, {Z_{i2}}^\top] \tag{10}$$

$$= \begin{bmatrix} \sum_{i=1}^{r} z_{i1}{z_{i1}}^\top & \sum_{i=1}^{r} z_{i1}{Z_{i2}}^\top \\ \sum_{i=1}^{r} Z_{i2}{z_{i1}}^\top & \sum_{i=1}^{r} Z_{i2}{Z_{i2}}^\top \end{bmatrix} \tag{11}$$

$$= \begin{bmatrix} z_1{z_1}^\top & z_1{Z_2}^\top \\ Z_2{z_1}^\top & Z_2{Z_2}^\top \end{bmatrix} \tag{12}$$

4. Additional topics of Wishart Distribution (expectation of inverse wishart (4))

Proof (continued)

$$
\begin{aligned}
(b^{11})^{-1} &= {z_1}^\top z_1 - {z_1}^\top [Z_2({Z_2}^\top Z_2){Z_2}^\top] z_1 && (13) \\
&= {z_1}^\top [I_r - Z_2({Z_2}^\top Z_2){Z_2}^\top] z_1 && (14) \\
&= z_1^\top V z_1 \sim {\chi^2}_{r-(p-1)} && (15)
\end{aligned}
$$

($\because z_1 \sim N_r(0, I), V = I_r - Z_2({Z_2}^\top Z_2){Z_2}^\top$is idempotent with
rank(V) = tr(V) = r-(p-1))

$\Rightarrow$b$^{11} \sim [{\chi^2}_{r-(p-1)}]^{-1}$

4. Additional topics of Wishart Distribution (expectation of inverse wishart (5))

Proof (continued) Back to problem :

$E(B^{-1}) = k_1 I_p,\ E(b^{11}) = E[({\chi^2}_{r-p+1})^{-1}] = \frac{1}{r-p-1} = k_1$

$\therefore E(M^{-1}) = (C^\top)^{-1}(\frac{1}{r-p-1} I_p)(C^{-1}) = \frac{1}{r-p-1}(CC^\top)^{-1} = \frac{1}{r-p-1}\Sigma^{-1}$

4. Additional topics of Wishart Distribution (conditional wishart (1))

Theorem 2

Let $W \sim W_p(n, \Sigma)$ *and* W *and* Σ *be partitioned as*

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}, \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

respectively, where $_{ij} : p_i \times p_j$ *and* $\Sigma_{ij} : p_i \times p_j$*. Then:*

- $W_{11 \cdot 2} \perp\!\!\!\perp \{W_{21}, W_{22}\}$
- *The conditional distribution of* $W_{12} = (B_1, \ldots, B_{p_2})$ *is normal with mean* $E(W_{12}|W_{22}) = \Sigma_{12}{\Sigma_{22}}^{-1}W_{22}$ *and*

$$Cov(B_i, B_j|W_{22}) = (W_{22})_{ij}\Sigma_{11 \cdot 2},$$

where $(A)_{ij}$ *denotes the* (i, j)*th element of* A*.*

4. Additional topics of Wishart Distribution (conditional wishart (2))

Proof We may write W as

$$W = \sum_{i=1}^{n} Z_i {Z_i}^\top = \sum_{i=1}^{n} \begin{bmatrix} Z_{i1} \\ Z_{i2} \end{bmatrix} [{Z_{i1}}^\top, {Z_{i2}}^\top], \; Z_i \sim N_p(0, I), i = 1, \ldots, n$$

Consider orthogonal matrix $H = [H_1 \;\, H_2]$ such that
$H_1 = X_2({X_2}^\top X_2)^{-1/2}$. Then it can be shown that :

- $H^\top H = HH^\top = I_n, \; {H_1}^\top H_2 = 0.$
- ${H_1}^\top H_1 = I_{p_2}, \; H_1 {H_1}^\top = Z_2({Z_2}^\top Z_2)^{-1} {Z_2}^\top$
- $H_2 = Z_{1|2}({Z_{1|2}}^\top Z_{1|2}) {Z_{1|2}}^\top,$ where $Z_{1|2} = (I - Z_2({Z_2}^\top Z_2)^{-1} {Z_2}^\top) Z_1$

4. Additional topics of Wishart Distribution (conditional wishart (3))

Proof (contiunued)

$$\Rightarrow Consider\ \mathsf{X} = (\mathsf{X}_1, \ldots, X_n)^\top = H^\top Z_1 = \begin{bmatrix} H_1{}^\top \\ H_2{}^\top \end{bmatrix} Z_1 = \begin{bmatrix} H_1{}^\top Z_1 \\ H_2{}^\top Z_1 \end{bmatrix},$$

shortly $X = \begin{bmatrix} X_1{}^\top \\ \vdots \\ X_{p_2}{}^\top \\ X_{p_2+1}{}^\top \\ \vdots \\ X_n{}^\top \end{bmatrix}, \Rightarrow Z_1{}^\top H_1 H_1{}^\top Z_1 = [X_1, \ldots, X_{p_2}] \begin{bmatrix} X_1{}^\top \\ \vdots \\ X_{p_2}{}^\top \end{bmatrix}$

4. Additional topics of Wishart Distribution (conditional wishart (4))

Proof (contiunued)

$$\Rightarrow \mathrm{W}_{11\cdot 2} = {Z_1}^\top Z_1 - {Z_1}^\top H_1 {H_1}^\top Z_1 = \sum_{i=p_2+1}^{n} X_i {X_i}^\top \sim W_{p_1}(n-p_2, \Sigma_{11\cdot 2})$$

Since $W_{11\cdot 2}$ does not depend on Z_2, and $W_{12} = Z_1 H_1 ({Z_2}^\top Z_2)^{1/2}$,

hence, the conditional distributions of $W_{11\cdot 2}$ and $W_{12}|W_{22}$ are independent, which implies $W_{11\cdot 2} \perp\!\!\!\perp \{W_{21}, W_{22}\}$.

Now, All we need to check is : $X_i \overset{iid}{\sim} N_{p_1}(0, \Sigma_{11\cdot 2})$
$\Rightarrow \mathrm{Z}_{i1}|Z_{i2} \sim N_{p_1}(\Sigma_{12}{\Sigma_{22}}^{-1} Z_{i2}, \Sigma_{11\cdot 2})$. For checking above, we need another theorem.

4. Additional topics of Wishart Distribution (conditional wishart (5))

Theorem 3

Let $X = (X_1, \ldots, X_n)^\top$ be an $n \times p$ random matrix whose rows are independent normal variates with mean $E(X) = M$ and the same covariance matrix Σ. Let $Y = HX$ be the random matrix obtained by an $n \times n$ orthogonal transformation H. Then the rows of Y are independent normal variates with mean $E(Y) = HM$ and the same covariance matrix Σ.

Proof (SKIP)

4. Additional topics of Wishart Distribution (conditional wishart (6))

Proof (continued)

If $X_i \overset{iid}{\sim} N_{p_1}(0, \Sigma_{11\cdot 2})$, then $\displaystyle\sum_{i=p_2+1}^{n} X_i {X_i}^\top \sim W_{p_1}(n-p_2, \Sigma_{11\cdot 2})$.

We need to check for distribution of rows of ${H_2}^\top Z_1$. Since this is too difficult, instead, we will think of rows of $X = {H_2}^\top Z_1 | Z_2$.

$\Rightarrow$ rows of $\mathsf{H}_2^\top Z_1|Z_2$: $Z_{p_2+1}, \ldots, Z_n \overset{iid}{\sim} N_{p_1}({H_2}^\top E(Z_1|Z_2), \Sigma_{11\cdot 2})$
($\because Since \mathsf{H}_2$ is orthonormal, ${H_2}^\top \Sigma_{11\cdot 2} H_2 = \Sigma_{11\cdot 2}$)
$\Rightarrow \mathsf{H}_2^\top E(Z_1|Z_2) = {H_2}^\top Z_2 {\Sigma_{22}}^{-1} \Sigma_{21} = 0$. ($\because \mathsf{H}_2^\top Z_2 = 0$)
$\therefore X_i \overset{iid}{\sim} N_{p_1}(0, \Sigma_{11\cdot 2})$.

4. Additional topics of Wishart Distribution (conditional wishart (7))

Proof (continued)
Since $W_{12} = {Z_1}^\top Z_2 = {Z_1}^\top H_1({Z_2}^\top Z_2)^{1/2}$, $W_{11\cdot 2} = ({Z_1}^\top H_2)({Z_1}^\top H_2)^\top$,
And we know that ${H_1}^\top H_2 = 0 \rightarrow \quad \therefore W_{11\cdot 2} \perp\!\!\!\perp W_{12}|Z_2, W_{11\cdot 2} \perp\!\!\!\perp W_{22}$
Also,

$$
\begin{aligned}
E[W_{12}|Z_2] = E[{Z_1}^\top|Z_2]Z_2 \qquad &= [E(Z_{11}|Z_2), \ldots, E(Z_{n1}|Z_2)]Z_2 && (16)\\
= \Sigma_{12}{\Sigma_{22}}^{-1}{Z_2}^\top Z_2 \qquad &= \Sigma_{12}{\Sigma_{22}}^{-1}W_{22} && (17)
\end{aligned}
$$

We skip the covariance part.

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Thank you