CSE481

Tutorial 05

1) (a) $\langle e_1| = [\cos(\frac{\pi}{7}) \quad \sin(\frac{\pi}{7})]$

$\langle e_2| = [-\sin(\frac{\pi}{7}) \quad \cos(\frac{\pi}{7})]$

(b) $\langle e_1|e_1\rangle = [\cos(\frac{\pi}{7}) \quad \sin(\frac{\pi}{7})]\begin{bmatrix}\cos(\frac{\pi}{7}) \\ \sin(\frac{\pi}{7})\end{bmatrix} = \cos^2(\frac{\pi}{7}) + \sin^2(\frac{\pi}{7}) = 1$

$\langle e_1|e_2\rangle = [\cos(\frac{\pi}{7}) \quad \sin(\frac{\pi}{7})]\begin{bmatrix}-\sin(\frac{\pi}{7}) \\ \cos(\frac{\pi}{7})\end{bmatrix} = -\cos(\frac{\pi}{7})\sin(\frac{\pi}{7}) + \sin(\frac{\pi}{7})\cos(\frac{\pi}{7})$

$= 0$

$\langle e_2|e_2\rangle = [-\sin(\frac{\pi}{7}) \quad \cos(\frac{\pi}{7})]\begin{bmatrix}\sin(\frac{\pi}{7}) \\ \cos(\frac{\pi}{7})\end{bmatrix} = \sin^2(\frac{\pi}{7}) + \cos^2(\frac{\pi}{7}) = 1$

$\langle e_2|e_1\rangle = [-\sin(\frac{\pi}{7}) \quad \cos(\frac{\pi}{7})]\begin{bmatrix}\cos(\frac{\pi}{7}) \\ \sin(\frac{\pi}{7})\end{bmatrix} = -\sin(\frac{\pi}{7})\cos(\frac{\pi}{7}) + \sin(\frac{\pi}{7})\cos(\frac{\pi}{7})$

$= 0$

(c) $|e_1| = \sqrt{\langle e_1|e_1\rangle} = 1$

$|e_2| = \sqrt{\langle e_2|e_2\rangle} = 1$

Yes, they are orthogonal to each other, because their inner product is 0.

~~They~~ For e_1, $\cancel{\langle e_1|e_1\rangle}$ $\cos^2(\frac{\pi}{7}) + \sin^2(\frac{\pi}{7}) = 1$

For e_2, $\sin^2(\frac{\pi}{7}) + \cos^2(\frac{\pi}{7}) = 1$

So, they ~~form~~ satisfy Postulate 1 of quantum mechanics. ~~as they both can be par~~

(d) Yes, we can write any arbitrary state vector in the Hilbert space $|\psi\rangle$ because $\langle e_1|e_2\rangle = \langle e_2|e_1\rangle = 0$ ~~which~~ means they are linearly independent of each other.

(e) $\langle 0|\psi\rangle = \frac{1-7i}{10}\langle 0|e_1\rangle + \frac{1+i}{2}\langle 0|e_2\rangle = \frac{1-7i}{10}[1 \quad 0]\begin{bmatrix}\cos(\frac{\pi}{7}) \\ \sin(\frac{\pi}{7})\end{bmatrix} + \frac{1+i}{2}[1 \quad 0]\begin{bmatrix}-\sin(\frac{\pi}{7}) \\ \cos(\frac{\pi}{7})\end{bmatrix}$

$= \frac{1-7i}{10}\cos(\frac{\pi}{7}) - \frac{1+i}{2}\sin(\frac{\pi}{7})$

$$= \frac{1-7i}{10}\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\cos(\frac{\pi}{7})\\ \sin(\frac{\pi}{7})\end{bmatrix} + \frac{1+i}{2}\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}-\sin(\frac{\pi}{7})\\ \cos(\frac{\pi}{7})\end{bmatrix}$$

$$= \frac{1-7i}{10}\cos\left(\frac{\pi}{7}\right) + \frac{1+i}{2}\left(-\sin\left(\frac{\pi}{7}\right)\right)$$

$$\langle 1|\psi\rangle = \frac{1-7i}{10}\langle 1|e_1\rangle + \frac{1+i}{2}\langle 1|e_2\rangle = \frac{1-7i}{10}\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\cos(\frac{\pi}{7})\\ \sin(\frac{\pi}{7})\end{bmatrix} + \frac{1+i}{2}\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}-\sin(\frac{\pi}{7})\\ \cos(\frac{\pi}{7})\end{bmatrix}$$

$$= \frac{1-7i}{10}\sin\left(\frac{\pi}{7}\right) + \frac{1+i}{2}\cos\left(\frac{\pi}{7}\right)$$

$$\therefore |\psi\rangle = \left(\frac{1-7i}{10}\cos\left(\frac{\pi}{7}\right) - \frac{1+i}{2}\sin\left(\frac{\pi}{7}\right)\right)|0\rangle + \left(\frac{1-7i}{10}\sin\left(\frac{\pi}{7}\right) + \frac{1+i}{2}\cos\left(\frac{\pi}{7}\right)\right)|1\rangle$$

$$= (-0.12684 - 0.84762i)|0\rangle + (\cancel{0.4071}\ 0.49387 + 0.14677i)|1\rangle$$

(+) $$\langle +|\psi\rangle = \frac{1-7i}{10}\langle +|e_1\rangle + \frac{1+i}{2}\langle +|e_2\rangle = \frac{1-7i}{10}\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}\cos(\frac{\pi}{7})\\ \sin(\frac{\pi}{7})\end{bmatrix} + \frac{1+i}{2}\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}-\sin(\frac{\pi}{7})\\ \cos(\frac{\pi}{7})\end{bmatrix}$$

$$= \frac{1-7i}{10}\times 0.94388 + \frac{1+i}{2}\times 0.33028$$

$$= 0.2595 - 0.4956i$$

$$\langle -|\psi\rangle = \frac{1-7i}{10}\langle -|e_1\rangle + \frac{1+i}{2}\langle -|e_2\rangle = \frac{1-7i}{10}\begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}\begin{bmatrix}\cos(\frac{\pi}{7})\\ \sin(\frac{\pi}{7})\end{bmatrix} + \frac{1+i}{2}\begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}\begin{bmatrix}-\sin(\frac{\pi}{7})\\ \cos(\frac{\pi}{7})\end{bmatrix}$$

$$= \frac{1-7i}{10}\times 0.33028 + \frac{1+i}{2}\times -0.94388$$

$$= -0.4389 - 0.7031i$$

$$\therefore |\psi\rangle = (0.2595 - \cancel{0.4896}\ 0.4956i)|+\rangle - (\cancel{0.48}\ 0.4389 + 0.7031i)|-\rangle$$

2(a) $a^2+b^2=1$ will have to be followed by a and b

(b) $$\begin{bmatrix}a\\ b\end{bmatrix}\otimes\begin{bmatrix}m\\ n\end{bmatrix}\otimes\begin{bmatrix}p\\ q\end{bmatrix} = \begin{bmatrix}am\\ an\\ bm\\ bn\end{bmatrix}\otimes\begin{bmatrix}p\\ q\end{bmatrix} = \begin{bmatrix}amp\\ amq\\ anp\\ anq\\ bmp\\ bmq\\ bnp\\ bnq\end{bmatrix}$$

(c) Following postulate 1 of quantum mechanics,

$$\sqrt{(amp)^2+(amq)^2+(anp)^2+(anq)^2+(bmp)^2+(bmq)^2+(bnp)^2+(bnq)^2}^{\,2} = (1)^2$$

$$a^2m^2p^2+a^2m^2q^2+a^2n^2p^2+a^2n^2q^2+b^2m^2p^2+b^2m^2q^2+b^2n^2p^2+b^2n^2q^2 = 1^2$$

$$a^2(m^2p^2+m^2q^2+n^2p^2+n^2q^2)+b^2(m^2p^2+m^2q^2+n^2p^2+n^2q^2) = 1$$

$$(a^2+b^2)(m^2p^2+m^2q^2+n^2p^2+n^2q^2) = 1$$

Since $|u\rangle$ is a state vector in Hilbert space, $a^2+b^2=1$

So, $m^2p^2+m^2q^2+n^2p^2+n^2q^2=1$

$m^2(p^2+q^2) + n^2(p^2+q^2) = 1$

$(m^2+n^2)(p^2+q^2) = 1$

Similarly, $|u\rangle$ and $|v\rangle$ are state vectors in a Hilbert space.

$\therefore 1 \times 1 = 1$

$1 = 1$

This is true

So, this satisfies postulate 1 of quantum mechanics

d) Let $|\phi\rangle$ be produced from states $\begin{bmatrix} c \\ d \end{bmatrix}, \begin{bmatrix} e \\ f \end{bmatrix}, \begin{bmatrix} g \\ h \end{bmatrix}$

$$\begin{bmatrix} c \\ d \end{bmatrix} \otimes \begin{bmatrix} e \\ f \end{bmatrix} \otimes \begin{bmatrix} g \\ h \end{bmatrix} = \begin{bmatrix} amp \\ amq \\ anp \\ anq \\ bmp \\ bmq \\ bnp \\ bnq \end{bmatrix}$$

$$\begin{bmatrix} ceg \\ ceh \\ cfg \\ cfh \\ deg \\ deh \\ dfg \\ dfh \end{bmatrix} = \begin{bmatrix} amp \\ amq \\ anp \\ anq \\ bmp \\ bmq \\ bnp \\ bnq \end{bmatrix}$$

Let $c = a$, so $eg = mp$

$eh = mq$

$fg = np$

$fh = nq$

Let $e = m$, so $g = p$

Therefore, $h = q$

$f = n$

$d\cancel{e}g = bmp$

$\cancel{d m g} = \cancel{bmp}$

$\therefore d = b$

For these values, the rest of the product space is valid. So, $|\phi\rangle$

not form an entangled state

a) ~~Let~~ $\cancel{|e_1\rangle = \begin{bmatrix} a \\ b \end{bmatrix}}$ ~~and~~ $\cancel{|e_2\rangle = \begin{bmatrix} c \\ d \end{bmatrix}}$

$\cancel{\therefore |\Psi\rangle = \begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} c \\ d \end{bmatrix}}$

$\cancel{\begin{bmatrix} ac \\ ad \\ bc \\ bd \end{bmatrix}} = \cancel{\begin{bmatrix} \; \end{bmatrix}}$

$|\Psi\rangle = (a|e_1\rangle + b|e_2\rangle) \otimes (c|e_1\rangle + d|e_2\rangle)$

$= ac(|e_1\rangle \otimes |e_1\rangle) + ad(|e_1\rangle \otimes |e_2\rangle) + bc(|e_1\rangle \otimes |e_2\rangle) + bd(|e_1\rangle \otimes |e_2\rangle)$

$\therefore \begin{bmatrix} ac \\ ad \\ bc \\ bd \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$

Let $a = 1$, $\therefore c = \frac{1}{\sqrt{2}}$

$1 \times ad = 0$

$\therefore d = 0$

$b \times \frac{1}{\sqrt{2}} = 0$

$\therefore b = 0$

$b \times d = \frac{1}{\sqrt{2}}$

$0 \times 0 \neq \frac{1}{\sqrt{2}}$

$0 \neq \frac{1}{\sqrt{2}}$

$\therefore |\Psi\rangle$ is an entangled state

b) $|\Psi\rangle = (a|e_1\rangle + b|e_2\rangle) . (c|e_1\rangle + d|e_2\rangle)$

$= ac(|e_1\rangle \otimes |e_2\rangle) + ad(|e_1\rangle \otimes |e_2\rangle) + bc(|e_2\rangle \otimes |e_1\rangle) + bd(|e_2\rangle \otimes |e_2\rangle)$

$\therefore \begin{bmatrix} ac \\ ad \\ bc \\ bd \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}$

Let $a = 1$, $ac = 1/\sqrt{2}$, $1 \times c = 1/\sqrt{2}$, $\therefore c = 1/\sqrt{2}$

$ad = 1/\sqrt{2}$, $1 \times d = 1/\sqrt{2}$, $\therefore d = 1/\sqrt{2}$

$b \times c = 0$, $b \times 1/\sqrt{2} = 0$, $\therefore b = 0$

$bd = 0$, $0 \times 1/\sqrt{2} = 0$, $0 = 0$

$\therefore$ It is not an entangled state

(c) $\cancel{u|e_1\rangle \otimes |e_1\rangle}$

$$u\begin{bmatrix} e \\ f \end{bmatrix} \otimes \begin{bmatrix} g \\ h \end{bmatrix} = \begin{bmatrix} ue \\ uf \end{bmatrix} \otimes \begin{bmatrix} g \\ h \end{bmatrix}$$

$$= \begin{bmatrix} ueg \\ ueh \\ ufg \\ ufh \end{bmatrix}$$

$$= u\begin{bmatrix} eg \\ eh \\ fg \\ fh \end{bmatrix}$$

$$= u\left(\begin{bmatrix} e \\ f \end{bmatrix} \otimes \begin{bmatrix} g \\ h \end{bmatrix}\right)$$

$\therefore |\psi\rangle = (a|e_1\rangle + b|e_2\rangle).(c|e_1\rangle + d|e_2\rangle)$

$= ac|e_1\rangle \otimes |e_1\rangle + ad|e_1\rangle \otimes |e_2\rangle + bc|e_2\rangle \otimes |e_1\rangle + bd|e_2\rangle \otimes |e_2\rangle$

$$\begin{bmatrix} ac \\ ad \\ bc \\ bd \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix}$$

$ac = \alpha$

$ad = \beta$

$bc = \gamma$

$bd = \delta$

$c = \frac{\alpha}{a}$

$d = \frac{\beta}{a}$

$bc = \gamma$

$\therefore b \times \frac{\alpha}{a} = \gamma$

$b = \frac{a\gamma}{\alpha}$

$b \times \frac{\beta}{a} = \delta$

$b = \frac{a\delta}{\beta}$

~~$\therefore a = \frac{\alpha}{\alpha}$~~

~~Let $a = 1$,~~

~~$\therefore c = \alpha$~~

~~$d = \beta$~~

~~$b\alpha = \gamma$~~

~~$\therefore b = \frac{\gamma}{\alpha}$~~

In order for $|\psi\rangle$ to be a product space, the value of b must be same for both bc and bd,

$\therefore \frac{\alpha\gamma}{\alpha} = \frac{\alpha\delta}{\beta}$

$\beta\gamma = \alpha\delta$

$\alpha\delta - \beta\gamma = 0$

$\therefore |\psi\rangle$ can only be an ~~pte~~ entangled state iff $\alpha\delta - \beta\gamma \neq 0$