Tutorial: 05

BRAC University

Semster: Fall 2023
Course No: CSE481
Course Title: QUANTUM COMPUTING
Section: 01
Faculty: JMD, RMT

Lecture: 1
Week: 5
Time: 1 hour 30 minutes
Date: November 22, 2023

Unless otherwise explicitly stated, for all purposes, $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, where they respectively represent state "0" and state "1" (computational basis). We will also use $|+\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ and $|-\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$ to denote the diagonal basis set.

Question 1

From Postulate 1 of QM, we know that the state of a single qubit system can be described by a normalized vector in Hilbert Space. Two such vectors are $|e_1\rangle = \begin{bmatrix} \cos\left(\frac{\pi}{7}\right) \\ \sin\left(\frac{\pi}{7}\right) \end{bmatrix}$ and $|e_2\rangle = \begin{bmatrix} -\sin\left(\frac{\pi}{7}\right) \\ \cos\left(\frac{\pi}{7}\right) \end{bmatrix}$. Answer the following questions for these vectors.

(a) What would be the values of $\langle e_1|$ and $\langle e_2|$?

(b) What are the values of the inner products between them?

- $\langle e_1|e_1\rangle$
- $\langle e_1|e_2\rangle$
- $\langle e_2|e_2\rangle$
- $\langle e_2|e_1\rangle$

(c) Based on the answers of the previous question, what are the norms of these vectors ($|e_1\rangle, |e_2\rangle$)? And are they orthogonal to each other? Additionally, confirm that they do satisfy Postulate 1 of QM.

(d) Can you write any arbitrary state vector in the Hilbert space $|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, in terms of $|e_1\rangle$ and $|e_2\rangle$? In other words, do $|e_1\rangle$ and $|e_2\rangle$ form a basis set for a single qubit Hilbert space?

(e) Take a qubit with a state $|\psi\rangle = \frac{1-7i}{10}|e_1\rangle + \frac{1+i}{2}|e_2\rangle$. It is written in $\{e_1, e_2\}$ basis. But if you want, you can change it to other basis. Express $|\psi\rangle$ in terms of computational basis $\{0, 1\}$

(f) Now express $|\psi\rangle$ in terms of diagonal basis $\{+, -\}$.

Question 2

This time, let's say we have 3 qubits. Each of the qubit in an isolated environment had state vectors like: $|u\rangle = a|e_1\rangle + b|e_2\rangle$, $|v\rangle = m|e_1\rangle + n|e_2\rangle$, $|w\rangle = p|e_1\rangle + q|e_2\rangle$. Answer the following questions for them:

(a) Since $|u\rangle$ is a state vector in Hilbert space, is there any condition that a and b needs to satisfy? (similarly for $|v\rangle$ and $|w\rangle$)

(b) Now if we create a composite system containing all 3 qubits, what will be the vector that describes their state? In other words, what's the value of $|\phi\rangle = |u\rangle \otimes |v\rangle \otimes |w\rangle$?

(c) Does the $|\phi\rangle$ satisfy Postulate 1 of QM? i.e. can this vector be used to describe the composite state of the 3 qubits?

(d) Does the $|\phi\rangle$ form an entangled state?

Question 3

(a) Given that, a state vector in 2 qubit system,

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|e_1\rangle \otimes |e_1\rangle) + \frac{1}{\sqrt{2}}(|e_2\rangle \otimes |e_2\rangle)$$

Is $|\psi\rangle$ an entangled state?

(b) Given that, a state vector in 2 qubit system,

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|e_1\rangle \otimes |e_1\rangle) + \frac{1}{\sqrt{2}}(|e_1\rangle \otimes |e_2\rangle)$$

Is $|\psi\rangle$ an entangled state?

(c) Show that for any arbitrary state vector in a 2 qubit system,

$$|\psi\rangle = \alpha\,|e_1\rangle \otimes |e_1\rangle + \beta\,|e_1\rangle \otimes |e_2\rangle + \gamma\,|e_2\rangle \otimes |e_1\rangle + \delta\,|e_2\rangle \otimes |e_2\rangle$$

is an entangled state **iff** $\alpha\delta - \beta\gamma \neq 0$