CSE 481
Jishnu Mahmud
jishnu.mahmud@bracu.ac.bd

Observables and the Second Postulate of Quantum Mechanics

Observable is any physical quantity that we can measure out of a particle. It can be Energy, Momentum, Angular Momentum, Position, etc.
These observables can be represented as linear operators in the vector space.

Recap of Linear Operators: Linear operators acting on a vector space preserve the linear structure of the vector space. This means that,

1. Addition remains Addition:
$$\boldsymbol{M}(|\psi\rangle + |\varphi\rangle) = \boldsymbol{M}|\psi\rangle + \boldsymbol{M}|\varphi\rangle$$

2. Scalar multiplication remains Scalar Multiplication:
$$\boldsymbol{M}(c|\psi\rangle) = c\,\boldsymbol{M}|\psi\rangle$$

- Where ***M*** is the Linear Operator

A linear operator is an abstract mapping of a particular vector in a vector space. It can be represented as a matrix if we decide on a particular basis. The reason that we have represented linear operators as matrices is that we have assumed a basis for them. For example, while representing some of the quantum gates we have used matrices by considering their effect on the $|0\rangle$, $|1\rangle$ vectors.

It is important to remember that Quantum Mechanics does not have a standard basis (although we do use a lot of $\{|0\rangle, |1\rangle\}$ basis in our examples!) abstract *operators* are always used in formal definitions.

Now suppose we want to measure the Position, (an observable), of an isolated particle. Once we perform the measurement, and observe a value Ω, we know that the state of the particle collapses to a particular quantum state for which the position value is Ω with 100% certainty. This is consistent with what we have learned before; that once measurement has been performed the new state can no longer be in a superposition of the measured quantity.

Now the next question arises, what are the possible values that we can get upon measuring Position?
In other words, we are interested in the set of values, (of which Ω is a member), that we can observe upon measuring the Position of the particle. Let us call this set ***Æ***.
It must be realized that each of the values, $\boldsymbol{v_i}$, of ***Æ***, must have a corresponding state vector in which the position is deemed to be $\boldsymbol{v_i}$ with full certainty. These are definite states, where the states have a definite value of Position corresponding to the value $\boldsymbol{v_i}$.

The values $\boldsymbol{v_i}$ happen to be the eigenvalues of the Position operator and their corresponding vectors are the eigenvectors. Moreover, the operator corresponding to any observable (Position in our example) is a Hermitian operator.

We will try to give a small intuition as to why the eigenvalues correspond to the observable quantities and why operators corresponding to observables are Hermitian.

Suppose we want to find out the expectation value of an observable A. Now A can be any observable from above. Thus, the expectation value of A,

$$\mathbb{E}(A) = \sum a_i \mathrm{P}(a_i)$$

Where a_i are the possible outcomes from i orthogonal states and $\mathrm{P}(a_i)$ is the probability corresponding to a particular outcome. If $|a1\rangle$ and $|a2\rangle$ are two possible states, then $|\psi\rangle = \alpha|a1\rangle + \beta|a2\rangle$ is also another possible state. We know that $\mathrm{P}(a1) = |\alpha|^2$ & $\mathrm{P}(a2) = |\beta|^2$. We can also see that $\alpha = \langle a1|\psi\rangle$ and $\alpha * = \langle\psi|a1\rangle$.

Hence, the expectation value will be,

$$\begin{aligned}\mathbb{E}(A) &= |\alpha|^2 a_1 + |\beta|^2 a_2 \\ &= \alpha_* \alpha\, a_1 + \beta_* \beta\, a_2 \\ &= \langle\psi|a_1\rangle\langle a_1|\psi\rangle a_1 + \langle\psi|a_2\rangle\langle a_2|\psi\rangle a_2 \\ &= \langle\psi|\left(\sum_i |a_i\rangle\langle a_i| a_i\right)|\psi\rangle\end{aligned}$$

The i in the last line stems from the fact that the vector $|\psi\rangle$ could also be a state which is the summation of i orthogonal states. According to the spectral theorem, any Hermitian Operator can be written as $A = \sum i|\lambda_i\rangle\langle\lambda_i|\lambda_i$ where λ_i is the eigenvalues of the operator and $|\lambda_i\rangle$ is the eigenvectors corresponding to the eigenvalue. It must also be noted that the eigenvectors form an orthonormal basis. Since $|a_i\rangle\langle a_i|$ is a projection along $|a_i\rangle$, only the eigenvectors can be the state of the particle after measurement.

$$\therefore\ \mathbb{E}(A) = \langle\psi|\mathbf{A}|\psi\rangle$$

where A is a Hermitian operator.

Recap of a Hermitian Matrix: A matrix whose complex conjugate + transpose (dagger operation) is equal to itself. This means that the eigenvalues of the matrix are real. *(This makes sense as the observable outcomes of observables such as position cannot be imaginary or complex numbers).* Just as we have mentioned earlier, the abstract observable Hermitian operator can be represented as a matrix upon selecting a basis.

These concepts are summarized in the Second Postulate of Quantum Mechanics which states:

a. To each observable, there is a Hermitian operator.
b. The eigenvalues of the Hermitian operator, Ω, associated with the physical observable are the only possible values the observable can take upon measurement on any given state. Furthermore, the eigenvectors of Ω form an orthogonal basis in the Hilbert Space.
c. $\langle\psi|\Omega|\psi\rangle$ is the expectation value of observing Ω repeatedly in the same state $|\psi\rangle$.

References:

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