Course : MAT215

Section : 15

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Problems :

* From Complex Variable by Schaum Series → 5.39, 5.62, 5.79, 6.94, 6.98, 7.47, ~~7~~.51.

* From Differential equations Dennis. G. Zill → 7.4.1 → 7, 8, 12, 14.

5.39/ Given, $I = \frac{1}{2\pi i}\oint_C \frac{e^{zt}}{(z^2+1)^2}$ if $t>0$ & $|z|=3$

Here, $(z^2+1)^2=0$

$\Rightarrow z^2+1=0$

$\Rightarrow z=\pm i$

which is pole of order 2.

Now,

residue at $z=i = \left[\frac{d}{dz}\phi(z)\right]_{z=i}$

$\phi(z) = \frac{e^{zt}}{(z+i)^2}$

Now, $\phi'(z) = \frac{d}{dz}\left(\frac{e^{zt}}{(z+i)^2}\right) = \frac{te^{zt}}{(z+i)^2} - \frac{2e^{zt}}{(z+i)^3}$

$\phi'(z) = \left[\frac{t}{(z+i)^2} - \frac{2}{(z+i)^3}\right]e^{zt}$

$[\phi'(z)]_{z=i} = \left(-\frac{t}{4}+\frac{2}{8i}\right)e^{it}$

$= -\left(\frac{t}{4}+\frac{i}{4}\right)e^{it}$

$= \frac{-1}{4}[t+i](\cos t + i\sin t)$

$[\phi'(z)]_{z=i} = -\frac{t\cos t}{4} - \frac{it\sin t}{4} - \frac{i\cos t}{4} + \frac{\sin t}{4}$

Now, residue at $(z=-i) = \left[\frac{d}{dz}\phi(z)\right]_{z=-i}$ $\phi(z) = \frac{e^{zt}}{(z-i)^2}$

$\phi(z) = \frac{d}{dz}\left(\frac{e^{zt}}{(z-i)^2}\right) = \frac{te^{zt}}{(z-i)^2} - \frac{2e^{zt}}{(z-i)^3}$

$[\phi'(z)]_{z=-i} = \left[\frac{t}{(-i-i)^2} - \frac{2}{(-i-i)^3}\right]e^{-it}$

$= -\left(\frac{t}{4}-\frac{2}{8i}\right)e^{-it}$

$= \frac{1}{4}[-t+i]e^{-it} = \frac{1}{4}[-t+i](\cos t - i\sin t)$

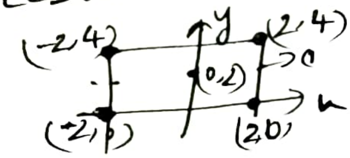
$[\phi'(z)]_{z=-i} = \frac{1}{4}\left[-t\cos t + it\sin t + i\cos t + \sin t\right]$

Therefore, $I = \frac{1}{2\pi i}\int_C \frac{e^{zt}}{(z^2+1)^2}$ = Sum of residue at $z=i$ & $-i$.

$\Rightarrow I = \frac{-t\cos t}{2} + \frac{\sin t}{2}$

$\therefore I = \boxed{\frac{1}{2}(\sin t - t\cos t)}$ Ans

5.79| Given, $\frac{1}{2\pi i}\oint_C \frac{z^2dz}{z^2+4}$, vertices at $\pm 2, \pm 2+4i$.



So, $\frac{1}{2\pi i}\int \frac{z^2dz}{(z+2i)(z-2i)}\,dz$

Let, $f(z)=\frac{z^2dz}{(z+2i)(z-2i)} = \frac{(z^2/z+2i)}{(z-2i)} = \frac{g(z)}{(z-2i)}$

So, $f(z)$ has singularity at $z=2i$ in C. So, $f(z)$ is not analytic at $z=2i$ in C.

By cauchy integral, $\frac{1}{2\pi i}\int_C \frac{(z^2/z+2i)}{(z-2i)}\,dz = \frac{1}{2\pi i}2\pi i\, g(2i)$

$= \frac{(2i)^2}{(2i+2i)}$

$= \frac{-4}{4i}\ \boxed{=i}$

$\therefore f(z)=i$ Ans

5.62| Given, $f(z)=\frac{(z^2+1)^2}{(z^2+2z+2)^3}$, $\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}$, $|z|=4$.

By argument principle, $\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = N-P$ — (1)

$N=$ number of 0's of f inside C

$f(z)=\frac{(z^2+1)^2}{(z^2+2z+2)^3}$

$P=$ poles.

$(z^2+1)^2=0$ So, $z=|i|=1<4$

$\Rightarrow z^2+1=0$ $z=|-i|=1<4$; Both lies inside C.

$\Rightarrow z=\pm i$

The zeros $z=i$, $z=-i$ occurs two times.

$\therefore$ Number of zeros $=4=N$.

Poles of $f(z)$ are, $(z^2+2z+2)^3=0$

$\Rightarrow (z^2+2z+2)=0$

$\Rightarrow z=\frac{-2\pm\sqrt{2^2-4\cdot 1\cdot 2}}{2\cdot 1}=-1\pm i$

$\therefore -1+i, -1-i$ have multiplicity three, $|-1+i|=\sqrt{(-1)^2+1^2}$

$=\sqrt{2}$

$\therefore z=-1+i$, $z=-1-i$ lies inside C. $\Rightarrow \sqrt{2}<4$

$\therefore$ Number of poles inside $C=3+3=6=P$

$\therefore N=4$ & $P=6$.

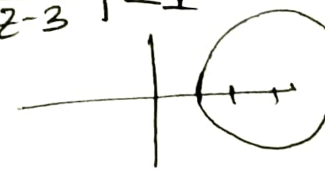
Therefore, $\frac{1}{2\pi i}\int_C \frac{f'(z)\,dz}{f(z)} = N-P = 4-6\ \boxed{=-2}$

Ans

6.94 Given,

$$f(z) = \frac{z}{(z^2+1)}$$ valid for $|z-3|>2$

$$\Rightarrow \left|\frac{2}{z-3}\right| < 1$$



$$f(z) = \frac{z}{z^2+1} = \frac{z}{(z+i)(z-i)}$$

$$\frac{z}{z^2+1} = \frac{A}{z+i} + \frac{B}{z i}$$

So,

$$z = A(z-i) + B(z+i)$$

$$z = (A+B)z + (B-A)i$$

Here, $A+B=1$

$-A+B=1$

$\Rightarrow A = B = \frac{1}{2}$

$$\frac{z}{z^2+1} = \frac{1}{2}\left[\frac{1}{z+i} + \frac{1}{z-i}\right]$$

Now, with laurent series expansion,

$$f(z) = \frac{1}{2}\left[\frac{1}{(z-3)+(3+i)} + \frac{1}{(z-3)+(3-i)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{z-3}\left[\frac{1}{1+\left(\frac{3+i}{z-3}\right)}\right] + \frac{1}{z-3}\left[\frac{1}{1+\left(\frac{3-i}{z-3}\right)}\right]\right]$$

$$= \frac{1}{2}\left[\frac{1}{(z-3)}\left(1+\left(\frac{3+i}{z-3}\right)\right)^{-1} + \frac{1}{(z-3)}\left(1+\left(\frac{3-i}{z-3}\right)\right)^{-1}\right]$$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{z-3}\right)\left[\sum_{n=0}^{\infty}(-1)^n\left(\frac{3+i}{z-3}\right)^n + \sum(-1)^n\left(\frac{3-i}{z-3}\right)^n\right]$$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{z-3}\right)\sum_{0}^{\infty}(-1)^n\left[\frac{(3+i)^n+(3-i)^n}{(z-3)^n}\right]$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{2}\,\frac{(3+i)^n+(3-i)^n}{(z-3)^{n+1}}$$

Thus, above sereie is valid for $|z-3|>2$.

Ans

6.098 (a) Given,

$f(z) = 1/(2\sin z - 1)^2$

$\therefore \frac{1}{f(z)} = 0 \Rightarrow 2\sin z = 1$

$\Rightarrow \sin z = (1/2)$

$\Rightarrow z = 2k\pi + \frac{\pi}{6} + \ldots k = 0 \pm 1 \pm 2 \ldots$

Now, $\lim f(z)$ doesn't exists; $k \in 4$, $z \to \left(2k\pi + \frac{\pi}{6}\right)$

Again, $k \in 4$, $\lim_{z \to \boxed{(2\pi k + \frac{\pi}{6})}} z^2 f(z)$

$= \lim_{z \to (2k\pi + \pi/6)} \cdot \frac{1}{\left(2\frac{\sin z}{z} - \frac{1}{z}\right)^2}$

$= \frac{1}{\left(2 - \frac{1}{2k\pi + \pi/6}\right)^2} \neq 0$

So, $z = 2k\pi + \frac{\pi}{6}$, $k \in 4$ are pole singularity of f of $\boxed{\text{order } 2}$. Ans

(b) Given, $f(z) = \frac{z}{e^{1/z} - 1}$

Now, $e^{1/z} = 1$, $z = \frac{1}{2k\pi i}$, $k \neq 0$, $k \in 4$

So, $\lim_{z \to \frac{1}{2k\pi i}} \left(\frac{z}{e^{1/z} - 1}\right)$ $\left[\frac{0}{0} \text{ form}\right]$

$= \lim_{z \to \frac{1}{2k\pi i}} \frac{z}{e^{1/z} \cdot (-1/z^2)}$

$= \lim_{z \to \frac{1}{2k\pi i}} \frac{-z^2}{e^{1/z}}$

$= -\left(\frac{1}{2\pi k i}\right)^2$

$= \frac{1}{4k^2\pi^2}$, exists.

Again, $\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{z}{e^{1/z} - 1} = 0$, exists.

Thus, $f(z)$ has regular singular points at $z = 0$, $\boxed{\frac{1}{2k\pi i}}$ $(k \in 4, k \neq 0)$, Ans

simple poles, $z = \infty$, pole of $\boxed{\text{order } 2}$.

(c) Given, $f(z) = \cos(z^2+z^{-2})$

$$= 1 - \left(\frac{(z^2+z^{-2})^2}{2!}\right) + \left(\frac{(z^2+z^{-2})^4}{4!}\right) - \left(\frac{(z^2+z^{-2})^6}{6!}\right) + \cdots$$

It's of the form, $f(z) = (a_0 + a_1 z + a_2 z^2 + \cdots a_n z^n + \cdots \infty)$
$+ \left(\frac{b_1}{z} + \frac{b_2}{z^2} + \cdots + \frac{b_n}{z^n} + \cdots \infty\right)$

So, $z=0$ is an isolated essential, singularity of $f(z)$, ∞.

Ans

(d) Given, $f(z) = \tan^{-1}(z^2+2z+2)$

$$= (z^2+2z+2) - \frac{(z^2+2z+2)^3}{3} + \frac{(z^2+2z+2)^5}{5} \cdots \quad \text{for } 0 \le |z| \le 1$$

$\therefore \tan^{-1}(z^2+2z+2)$ can be written in the form $\sum_{k=0}^{\infty} a_k (z-0)^k$

for $0 \le |z| \le 1$

In the laurent series expansion of $\tan^{-1}(z^2+2z+2)$,

there is no principal part.

Hence, $\tan^{-1}(z^2+2z+2)$ has $\boxed{-1 \pm i}$ branch points.

Ans

(e) Let, $g(z) = \frac{z}{e^z-1}$ $\quad \boxed{z = 2k\pi i}$

$$= \lim_{z \to 0} \frac{z}{e^z - 1}$$

$$= \frac{1}{\lim_{z \to 0} \frac{e^z - 1}{z}}$$

$$= \frac{1}{1}$$

$$= 1 \in \mathbb{C}$$

$\therefore \lim_{z \to 0} \frac{z}{e^z-1}$ exists in $\mathbb{C}$. Hence, at $z=0$, $g(z)$ has removeable singularity. $z = \infty$. essential singularity.

Ans

7.471 Given Function, $F(z) = \dfrac{2+3\sin \pi z}{z(z-1)^2}$

Here,

$F(z)$ has pole of order 1 at 0 & pole of order 2 at 1. These 2 poles are inside C. Next, let's calculate residues,

Residue of F at 0 is $= 2$

Residue of F at 1 is $= g'(1)$ where $g(z) = \dfrac{2+3\sin\pi z}{z}$

$= -3\pi - 2$

If, F has a pole of order m at $z = a$ & n at $g(z) = (z-a)^m f(z)$, thus residue of F at $a = \dfrac{1}{(m-1)!} z^{(n-1)}(a)$

Thus by residue therem,

$$\int_C \frac{2+3\sin\pi z}{z(z-1)^2} = 2\pi i\,\big(2 + (-3\pi) + (-2)\big)$$

$= \boxed{-6\pi i}$ Ans

7.51 Given, $\int_0^{2\pi} \frac{\sin 3\theta}{5-3\cos\theta}\,d\theta$

$$= \int \frac{\sin(3\theta)}{3\cos(\theta)-5}\,d\theta$$

$$= \left[\int -\frac{\sin 3\theta}{3\cos\theta-5}\,d\theta\right] \text{ — (III)}$$

$$= \left[\int -\left(-\frac{4\cos^2\theta-1}{3\cos\theta-5}\right)\sin\theta\,d\theta\right] \quad \left[\begin{array}{l}\sin 3\theta = 3\cos^2(\theta)\sin\theta) - \sin 3(\theta) \\ \sin^2\theta = 1-\cos^2(\theta)\end{array}\right]$$

$$= \left[-\int \frac{4u^2-1}{3u-5}\right] \text{ — (II)} \quad \left[\text{Substitute } u=\cos\theta \rightarrow du = -\sin\theta\right]$$

$$= \int \frac{4u^2-1}{3u-5}\,du$$

$$\left[= \frac{1}{27}\int \frac{(2v+7)(2v+13)}{v}\,dv \text{ — (I)}\right] \quad \left[\text{Substitute } v = 3u-5 \rightarrow dv = 3du,\ u^2 = \frac{(v+5)^2}{9}\right]$$

Now, $= \int \frac{(2v+7)(2v+13)}{v}\,dv$

$$= \int \left(4v + \frac{91}{v} + 40\right) dv$$

$$= 4\int v\,dv + 91\int \frac{1}{v}\,dv + 40\int 1\,dv$$

$$= 4\int \frac{v^2}{2} + 91\ln v + 40v$$

$$= 2v^2 + 91\ln v + 40v$$

Also, puttin these in (I),

$$\frac{1}{27}\int \frac{(2v+7)(2v+13)}{v}\,dv$$

$$= \frac{91\ln v}{2} + \frac{2v^2}{27} + \frac{40v}{27} + C$$

$$= \frac{91(3u-5)}{27} + \frac{2(3u-5)^2}{27} + \frac{40(3u-5)}{27} + C$$

$$= \frac{91\ln(3u-5)}{27} - \frac{2(3u-5)^2}{27} - \frac{40(3u-5)}{27} + C$$

[Plugged in (II)] $= -\frac{91\ln(3\cos\theta-5)}{27} + \frac{2(3\cos\theta-5)^2}{27}$

[Plugged in (III)] $= \frac{91\ln(3\cos\theta-5)}{27} + \frac{40(3\cos\theta-5)}{27} + C$

$$= \frac{31\ln(5-3\cos\theta) + 2(3\cos\theta+5)^2}{27} + c$$

$$= \frac{16\ln(5-3\cos\theta)}{27} + 3\left(\frac{25\ln(5-3\cos\theta)}{27} + \frac{3\cos^2\theta+10\cos\theta}{18}\right)$$

$$+ \frac{3\cos^2\theta + 10\cos\theta}{18} + c$$

$\boxed{\text{computed by maxima}}$

$$= \left[\frac{31\ln(5-3\cos\theta + 18\cos^2\theta + 60\cos\theta}{27} + c\right] = f$$

$$= \int_0^{2\pi} f\,\theta\,d\theta = \boxed{0}$$

Ans

⑦ $L\{te^{2t}\sin 6t\}$

$$t(s) = L\{e^{2t}\sin 6t\}$$

$$= \frac{6}{(s-2)^2+36}$$

$$\frac{d}{ds}\left(\frac{6}{(s-2)^2+36}\right) = \frac{24-12s}{(s-2)^2+36}$$

$$(-1)\left(\frac{24-12s}{((s-2)^2+36)^2}\right) = \boxed{\frac{24+12s}{((s-2)^2+36)^2}}$$ Ans

⑧ $L\{te^{-3t}\cos 3t\}$

$$F(s) = L\{e^{-3t}\cos 3t\} = \frac{s+3}{(s+3)^2+9}$$

$$\frac{d}{ds}\left(\frac{s+3}{(s+3)^2+9}\right) = \frac{(s+3)^2+9(s+3)(2s+6)}{((s+3)^2+9)^2}$$

$$(-1)\ \frac{(s+3)^2+9-(s+3)(2s+6)}{((s+3)^2+9)^2}$$

$$= \boxed{\frac{s^2+6s}{((s+3)^2+9)^2}}$$ Ans

12 Given, $y'' + y = \sin t$, $y(0) = 1$ & $y'(0) = 1$

Here,

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{1}{s^2+1}$$

$$\Rightarrow Y(s)(s^2+1) = \frac{1}{s^2+1} + s - 1$$

$$\Rightarrow Y(s) = \frac{1}{(s^2+1)^2} + \frac{s}{s^2+1} - \frac{1}{s^2+1}$$

$$= \frac{1}{2s} \cdot -\frac{d}{ds} \cdot \frac{1}{s^2+1} + \frac{1}{s^2+1} - \frac{1}{s^2+1}$$

So,

$$Y(t) = \frac{1}{2}(1 \propto t\sin t) + \cos t - \sin t$$

$$= \frac{1}{2}(t\sin t \propto 1) + \cos t - \sin t$$

$$= \frac{1}{2}\int_0^t u\sin u\,(t-u)^0\,du + \cos t - \sin t$$

$w = u$, $\frac{dv}{du} = \sin u$, $v = -\cos u$, $du = du$

$$= \frac{1}{2}\left(-u\cos u + \int_0^t \cos u\,du\right) + \cos t - \sin t$$

$$= \frac{1}{2}(-u\cos u + \sin t)\Big|_0^t + \cos t - \sin t$$

$$= \frac{1}{2}(-t\cos t + \sin t) + \cos t - \sin t$$

$$\therefore y(t) = \boxed{-\frac{1}{2}t\cos t + \cos t - \frac{1}{2}\sin t}$$

Ans

14) Given, $y''+y=f(t)$, $y(0)=1$, $y'(0)=0$ where — (I)

$$f(t)=\begin{cases}1, & 0\le t<\pi/2\\ \sin t, & t\ge \pi/2\end{cases}$$

$$=1\left(u(t)-u(t-\pi/2)\right)+\sin t\left(u(t-\pi/2)\right)$$

$$=1-u(t-\pi/2)+\cos(t-\pi/2)\,u(t-\pi/2)$$

using laplace, $F(s)=\mathcal{L}[F(t)]=\frac{1}{s}-\frac{e^{-\pi/2 s}}{s}+\frac{se^{-\pi/2 s}}{s^2+1}$ ... (II)

Now, using laplace on (I), $s^2Y(s)-sy(0)-y'(0)+Y(s)=F(s)$

$$\Rightarrow (s^2+1)Y(s)-s=\frac{1}{s}-\frac{e^{-\pi/2 s}}{s}+\frac{se^{-\pi/2 s}}{s^2+1},\ \left[\begin{matrix}y(0)=1\\ y'(0)=0\end{matrix}\right]$$

$$\text{or, } Y(s)=\frac{s}{s^2+1}+\frac{1}{s(s^2+1)}-\frac{e^{-\pi/2 s}}{s(s^2+1)}+\frac{se^{-\pi/2 s}}{(s^2+1)^2}$$

By taking inverse laplace transformation,

$$y(t)=L^{-1}\left\{\frac{s}{s^2+1}\right\}+L^{-1}\left\{\frac{1}{s(s^2+1)}\right\}-L^{-1}\left\{\frac{e^{-\pi/2 s}}{s(s^2+1)}\right\}+\left\{\frac{se^{-\pi/2 s}}{(s^2+1)^2}\right\}$$ — (III)

Now, $L^{-1}\left\{\frac{s}{s^2+1}\right\}=\cos t$

$$L^{-1}\left\{\frac{1}{s(s^2+1)}\right\}=L^{-1}\left\{\frac{1}{s}-\frac{s}{s^2+1}\right\}=1-\cos t$$

$$\therefore L^{-1}\left\{\frac{e^{-\pi/2 s}}{s(s^2+1)}\right\}=\left\{1-\cos(t-\pi/2)\right\}u(t-\pi/2)$$

$$=(1-\sin t)\,u(t-\pi/2)$$

$$\Rightarrow L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}=\frac{1}{2}t\sin t$$

$$\therefore L^{-1}\left\{\frac{se^{-\pi/2 s}}{(s^2+1)^2}\right\}=\frac{1}{2}(t-\pi/2)\sin(t-\pi/2)\,u(t-\pi/2)$$

$$=\frac{1}{2}(t-\pi/2)(-\cos t)\cdot u(t-\pi/2)$$

$$=\frac{1}{2}(\pi/2-t)\cos t\ u(t-\pi/2)$$

Putting in (III),

$$y(t)=\cos t+1-\cos t+\left(\sin t-1+\frac{1}{2}(\pi/2-t)\cos t\right)u(t-\pi/2)$$

$$=\boxed{1+\left(\sin t-1+\frac{1}{2}(\pi/2-t)\cos t\right)u(t-\pi/2)}$$ Ans