Solutions to Assignment 1

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1 Problem 1

Note that given the existence of matrices X and S we can make the following computation:

$$\begin{bmatrix} G & A^\top \\ A^\top & \mathbf{0} \end{bmatrix} = \begin{bmatrix} I & \mathbf{0} \\ X & I \end{bmatrix} \begin{bmatrix} G & \mathbf{0} \\ \mathbf{0} & S \end{bmatrix} \begin{bmatrix} I & X^\top \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} I & \mathbf{0} \\ X & I \end{bmatrix} \begin{bmatrix} G & GX^\top \\ \mathbf{0} & S \end{bmatrix} = \begin{bmatrix} G & GX^\top \\ XG & XGX^\top + S \end{bmatrix}$$

Consequently, we have $XG = A$ (as a sanity check we also have $(XG)^\top = G^\top X^\top = GX^\top$). Since G is symmetric and positive definite, we can set $G = Q\Lambda Q^\top$ and $X = AQ\Lambda^{-1}Q^\top$, Λ^{-1} being well defined as it is a diagonal matrix with strictly positive entries. Furthermore, we also have $XGX^\top + S = 0$ so $S = -XGX^\top = -(QX)$. We can now work the equalities backwards to get the desired decomposition. Now, $-\Lambda^{-1}$ is congruent to the diagonal matrix $-\Lambda^{-1} = PSP^\top$ where P is invertible (since A has full rank) so S has m negative eigenvalues. Then, the block matrix $\begin{bmatrix} G & \mathbf{0} \\ \mathbf{0} & S \end{bmatrix}$ has precisely d positive (from G) and m negative (from S) eigenvalues. Moreover, $\begin{bmatrix} I & X^\top \\ 0 & I \end{bmatrix}$ is invertible so from Sylvester's law of inertia we have that $\begin{bmatrix} G & A^\top \\ A^\top & \mathbf{0} \end{bmatrix}$ has d positive and m negative eigenvalues.

2 Problem 2

a) The Lagrangian for the problem is $L(x, \Lambda) = (1/2)x^\top Gx + c^\top x - \Lambda^\top(Ax - b) = (1/2)x^\top Gx + c^\top x - (x^\top A^\top - b^\top)\Lambda$ so taking the gradient with respect to x and Λ respectively we get $\nabla_x L(x, \Lambda) = Gx + c - A^\top \Lambda = 0$ and $\nabla_\Lambda L(x, \Lambda) = Ax - b = 0$ so the KKT system for this problem is

$$\begin{bmatrix} G & -A^\top \\ A & \mathbf{0} \end{bmatrix} \begin{bmatrix} x \\ \Lambda \end{bmatrix} = \begin{bmatrix} -c \\ b \end{bmatrix}$$

b) Let $K = \begin{bmatrix} G & -A^\top \\ A & \mathbf{0} \end{bmatrix}$. Suppose $\exists z = [x^\top y^\top]^\top$ such that $Kz = 0$. Consequently, $\begin{bmatrix} G & -A^\top \\ A & \mathbf{0} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Gx - A^\top y \\ Ax \end{bmatrix} = 0^\top$. Thus, $Ax = 0$. Furthermore, $0 = z^\top Kz =$

$[x^\top y^\top] \begin{bmatrix} Gx - A^\top y \\ Ax \end{bmatrix} = x^\top Gx - x^\top Gy + y^\top Gx = x^\top Gx$. But $Ax = 0$ so $x^\top Gx = 0 \iff x = 0$. Since $x = 0$ we have that $0 = Gx - A^\top y = -A^\top y$ so $y = 0$ since $A^\top \in M_{d\times m}(\mathbb{R})$ has full rank.

c) The solution to the system $Kz = M$ is given by $z = \mathrm{Ker}(K) + z^*$ where z^* is some particular solution. Since $\mathrm{Ker}(K) = \{0\}$ we have that $z = z^*$ is the only solution to the KKT system.