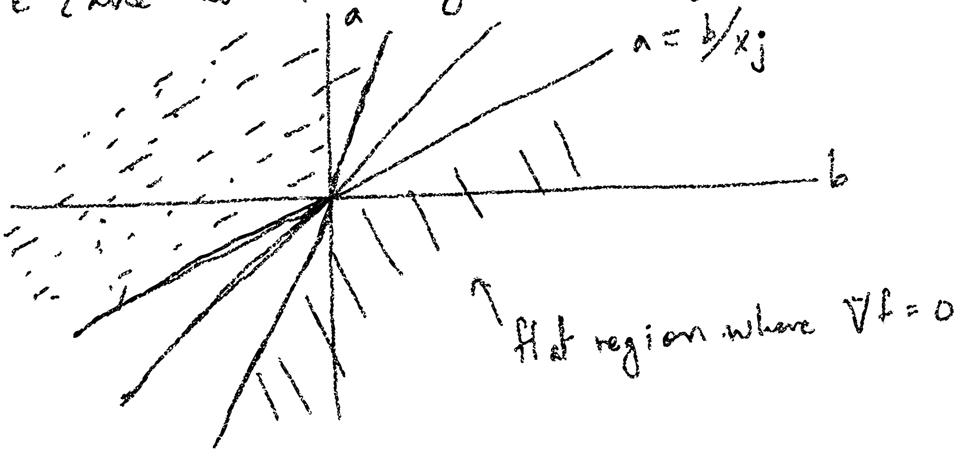
1.1) $f(a,b) = \frac{1}{12} \sum_{j=0}^{5} \left(\overbrace{ReLU(ax_j - b)}^{\frac{ax_j - b + |ax_j - b|}{2}} - 1 + \cos(x_j) \right)^2$

$$\frac{\partial f}{\partial a} = \frac{1}{6} \sum_{j=0}^{5} \left(ReLU(ax_j - b) - 1 + \cos(x_j) \right) \frac{\pi j}{10} \frac{\max(ax_j - b, 0)}{|ax_j - b|}$$

$$\frac{\partial f}{\partial b} = \frac{1}{6} \sum_{j=0}^{5} -\left(ReLU(ax_j - b) - 1 + \cos(x_j) \right) \frac{\max(ax_j - b, 0)}{|ax_j - b|}$$

We care abt the regions $ax_j \gtrless b$:



Solutions to Problem 1

a) Let $f(a,b) = \frac{1}{12} \sum_{j=0}^{5} \left(\mathrm{ReLU}(a\pi j/10 - b) - 1 + \cos(\frac{\pi j}{10})\right)^2$

$$\frac{\partial f}{\partial a} = \frac{1}{6} \sum_{j=0}^{5} \left[\max\left(\frac{a\pi j}{10} - b, 0\right) - 1 + \cos(\pi j/10)\right] \max\left(\mathrm{sgn}\left(\frac{a\pi j}{10} - b\right), 0\right) \frac{\pi j}{10}$$

Let J^+ be the set of indices s.t. $ax_j - b > 0$. Then

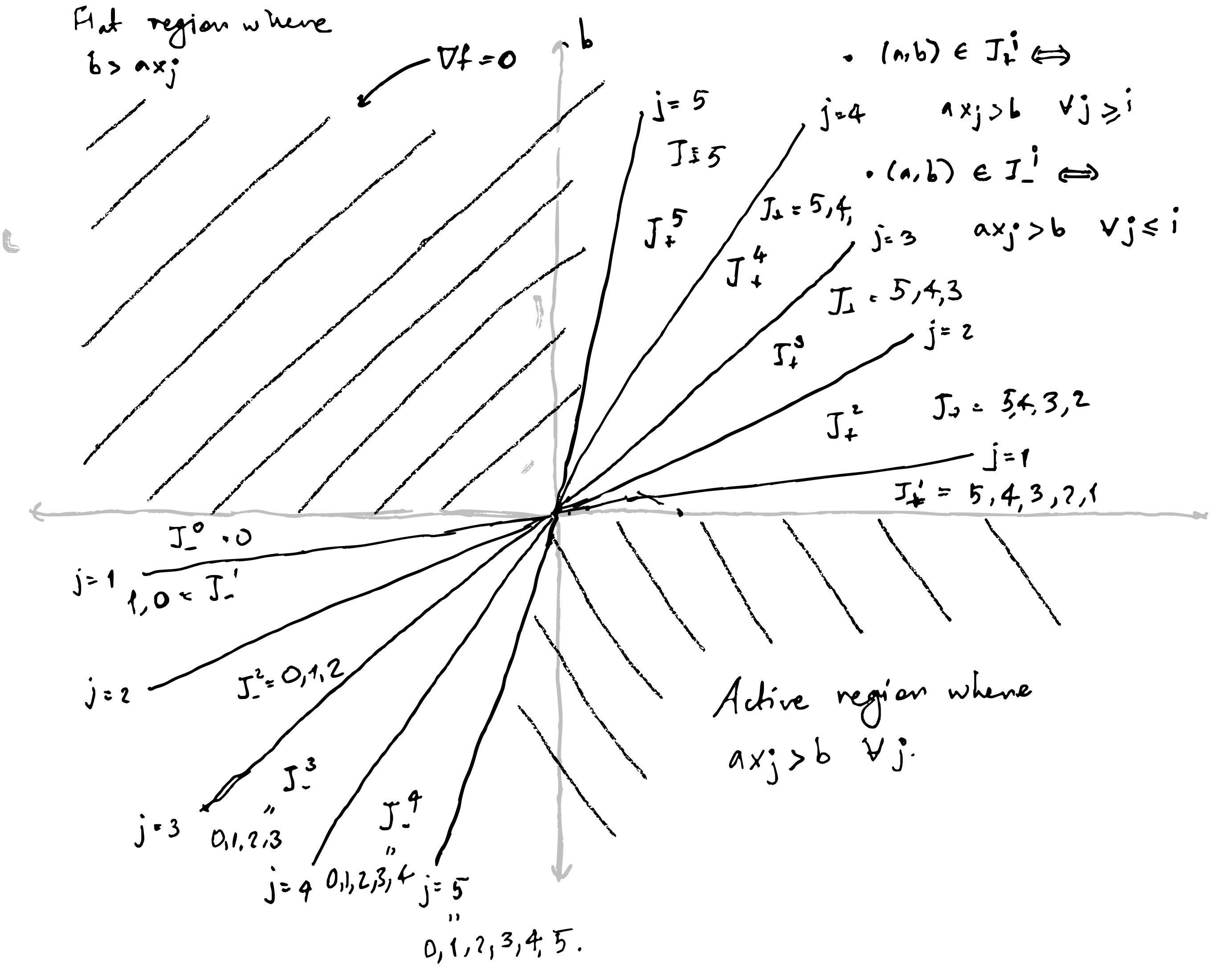
$$\frac{\partial f}{\partial a} = \pi/60 \sum_{j \in J^+} \left[\frac{a\pi j}{10} - b - 1 + \cos(\pi j/10)\right] j =$$

$$= \pi/60 \sum_{j \in J^+} \left[\frac{a\pi j^2}{10} - bj - j + j\cos(\pi j/10)\right] = 0$$

$$\Leftrightarrow \sum_{j \in J^+} \frac{a\pi}{10} j^2 - bj - j + j\cos(\pi j/10) = 0.$$

Similarly,

$$\frac{\partial f}{\partial b} = -1/6 \overbrace{\sum_{j \in J^+} \left[\frac{a\pi j}{10} - b - 1 + \cos \pi j/10\right]}^{A} = 0 \Leftrightarrow A = 0. \quad \text{— (1)}$$


Flat region where
$b > ax_j$
$\nabla f = 0$
b
a
• $(a,b) \in J_+^i \iff$
$ax_j > b \quad \forall j \geq i$
• $(a,b) \in J_-^i \iff$
$ax_j > b \quad \forall j \leq i$
$j=5$
$J_+ = 5$
J_+^5
$j=4$
$J_+ = 5,4,$
J_+^4
$j=3$
$J_+ = 5,4,3$
J_+^3
$j=2$
$J_+ = 5,4,3,2$
J_+^2
$j=1$
$J_+^1 = 5,4,3,2,1$
$J_-^0 = 0$
$j=1$
$1,0 = J_-^1$
$j=2$
$J_-^2 = 0,1,2$
$j=3$
J_-^3
"
$0,1,2,3$
$j=4$
J_-^4
"
$0,1,2,3,4$
$j=5$
"
$0,1,2,3,4,5.$
Active region where
$ax_j > b \quad \forall j.$

- Note that in the flat region, none of the constraints are active i.e $b > ax_j^\circ \; \forall j$ so $\max(\mathrm{sgn}(ax_j^\circ - b), 0) = 0$ so $\nabla f(a,b)$ is trivially 0. Here $f = \frac{1}{12}\sum_{j=0}^{5} (1 - \cos(\pi j/10))^2$.

- In the rest of the regions, J^+ constraints are active so

Here the critical point equations become

$$\partial f/\partial a = \frac{1}{6}\sum_{J^+}\left(\frac{a\pi j^\circ}{10} - b + g_j\right)\frac{\pi j}{10} = 0 \quad \text{— (1)}$$

$$\partial f/\partial b = -\frac{1}{6}\sum_{J^+}\left(\frac{a\pi j}{10} - b + g_j^\circ\right) = 0 \quad \text{— (2)}$$

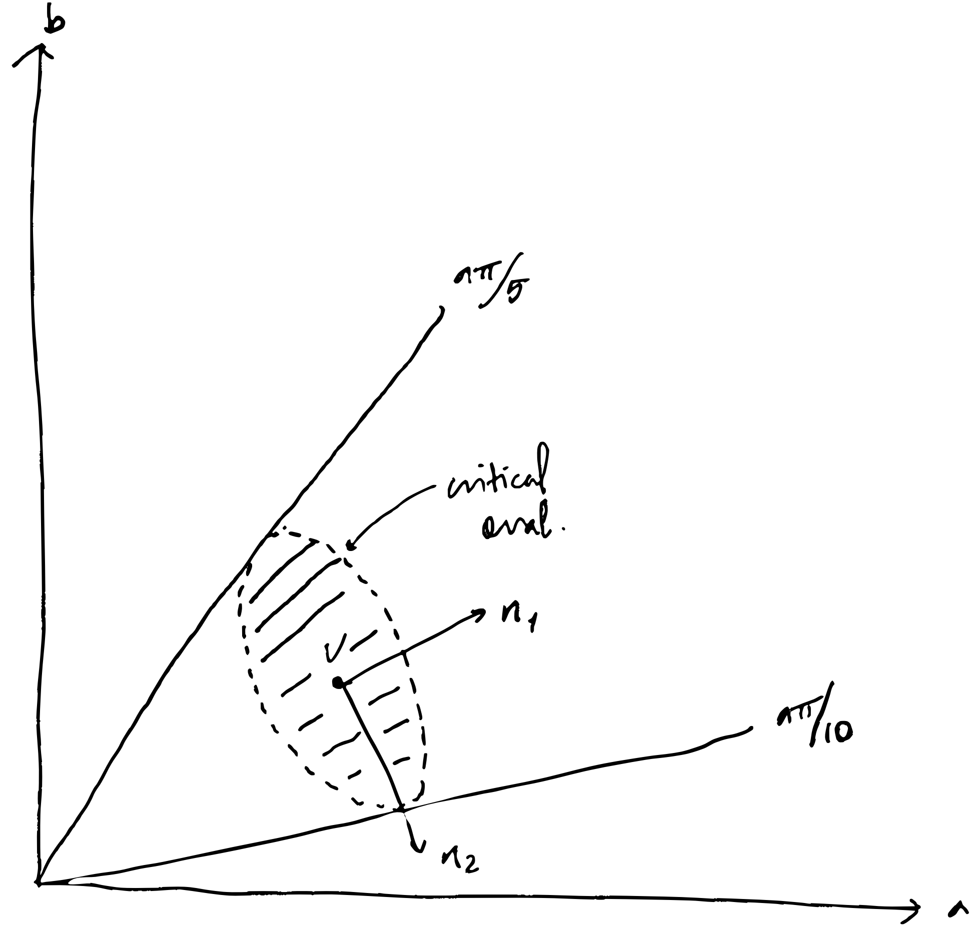
$$\frac{a\pi j^\circ}{10} - b > 0 \quad \text{for } j \in J.$$

$$\& \; \frac{a\pi j^\circ}{10} \le b \quad \text{for } j \in J^c.$$

Equations (1) & (2) reduce to the following linear system:

$$\begin{bmatrix} \pi/10 \sum_J j^2 & -\sum_J j \\ \pi/10 \sum_J j & -\sum_J 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum_J (1-\cos(\pi j/10))\, j \\ \sum_J (1-\cos(\pi j/10)) \end{bmatrix}$$

Solving these in each region: Flat, Active, J_i^+ & J_i^-: we find that $\underline{\underline{\text{only}}}$ J_2^+ admits a valid solution (a^*, b^*) moreover, $|\nabla^2 f(a^*, b^*)| < 0$ so it is indeed a minimum. Since it is the only critical point outside the flat region & $f(a^*, b^*) < f(0,0)$ so (a^*, b^*) is the global min.


b
a
nπ/5
nπ/10
critical
oval.
v
n_1
n_2